

Integração Numérica

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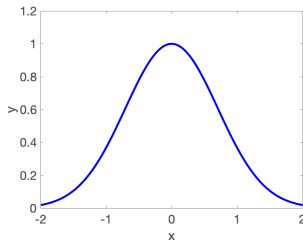


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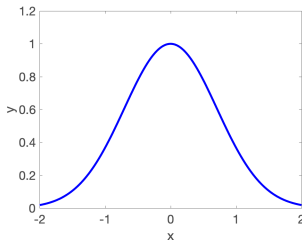
Tente resolver a integral definida a seguir:

$$\int_0^1 e^{-x^2} dx = ?$$



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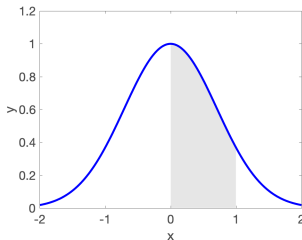
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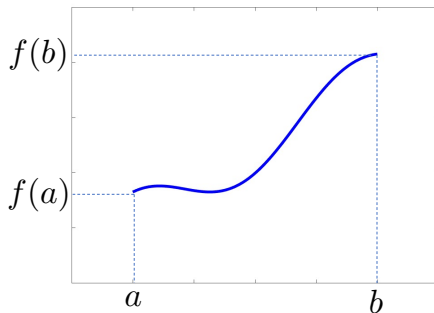


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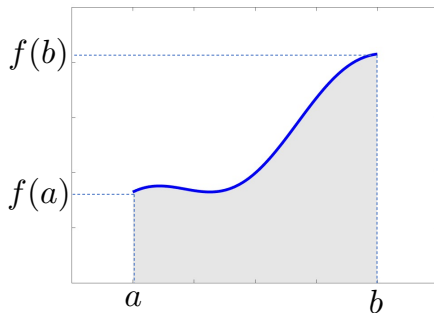
Mas ela tem um valor numérico bem definido, que corresponde à área em cinza acima!



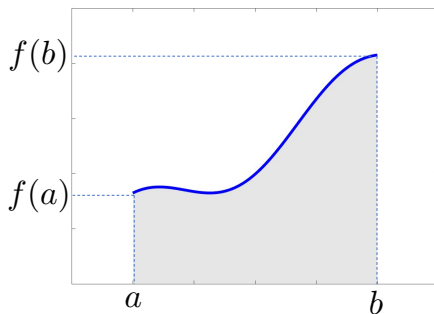
A integral definida de uma função



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A integral definida de uma função



$$\text{Área} = \int_a^b f(x) dx$$



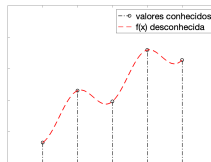
O problema de integração numérica

Nem sempre uma integral definida tem tratamento analítico!

$f(x)$ complicada

$$\int_0^1 e^{-x^2} dx = ?$$

$f(x)$ numérica



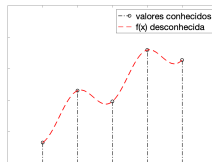
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Esses casos demandam o uso de *regras de quadratura*:

$$\int_a^b f(x) dx \approx \sum_{j=0}^n w_j f(x_j)$$

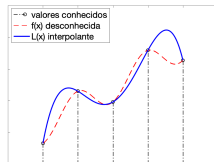


Regras de quadratura

Ideia do método:

- Calcular o polinômio que interpola $f(x)$ em $n + 1$ pontos, i.e.,

$$L(x) = \sum_{j=0}^n f(x_j) L_j(x)$$



- Usar a integral de $L(x)$ em $[a, b]$ como aproximação para a integral definida procurada, i.e.,

$$\int_a^b f(x) dx \approx \int_a^b L(x) dx$$



Regras de quadratura

$$\int_a^b f(x) dx \approx \int_a^b L(x) dx$$



Regras de quadratura

$$\begin{aligned}\int_a^b f(x) dx &\approx \int_a^b L(x) dx \\ &\approx \int_a^b \left(\sum_{j=0}^n f(x_j) L_j(x) \right) dx\end{aligned}$$



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- x_j pontos de quadratura
- w_j pesos da quadratura



Regras de quadratura

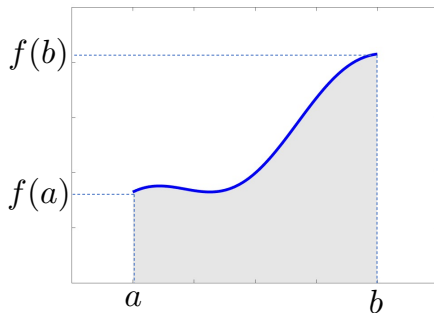
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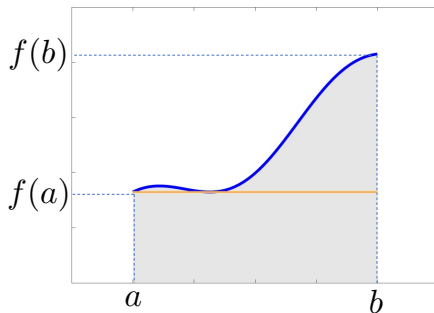
Diferentes escolhas para n e x_j produzem diferentes esquemas de quadratura.



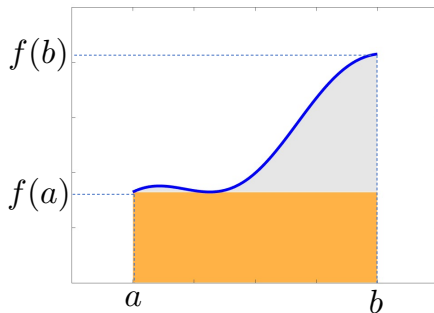
Regra do retângulo (ponto à esquerda)



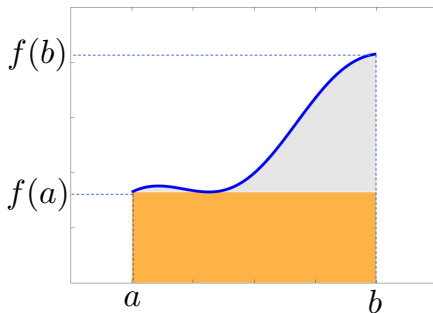
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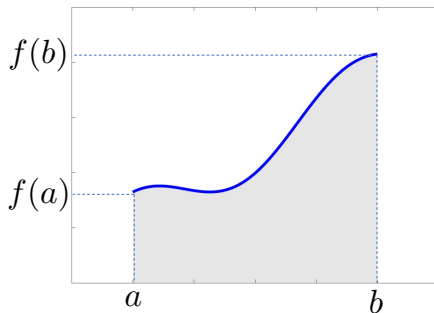


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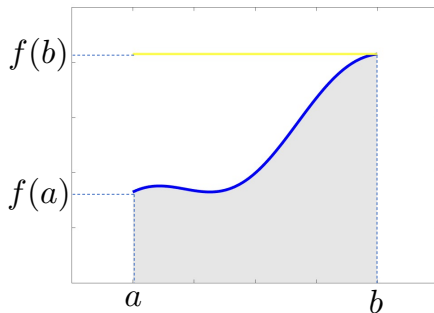


$$\int_a^b f(x) dx \approx (b - a) f(a)$$

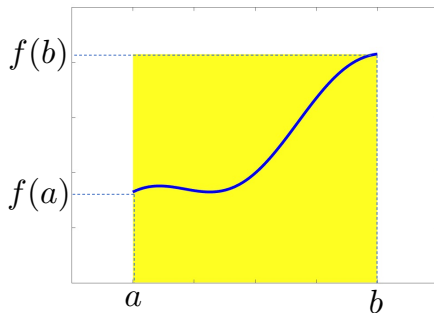
Regra do retângulo (ponto à direita)



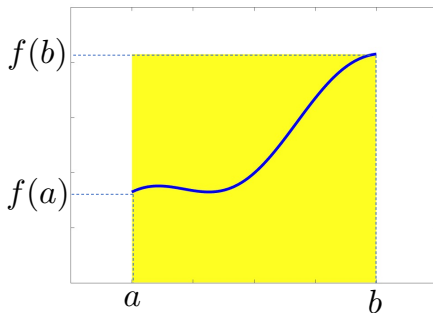
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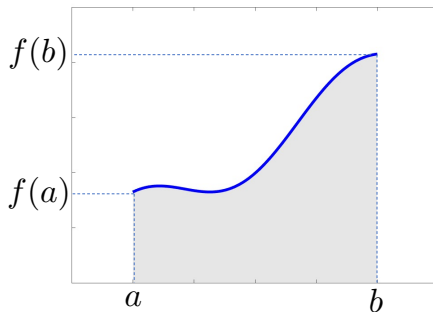


Regra do retângulo (ponto à direita)

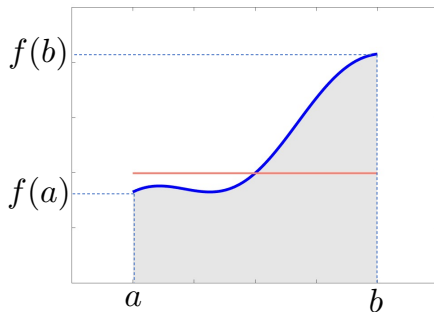


$$\int_a^b f(x) dx \approx (b - a) f(b)$$

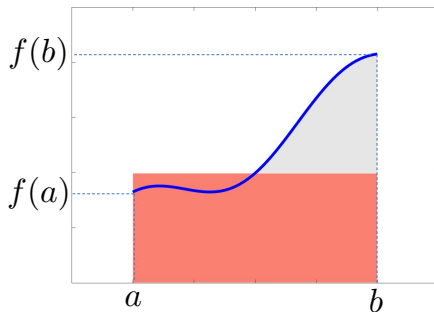
Regra do retângulo (ponto médio)



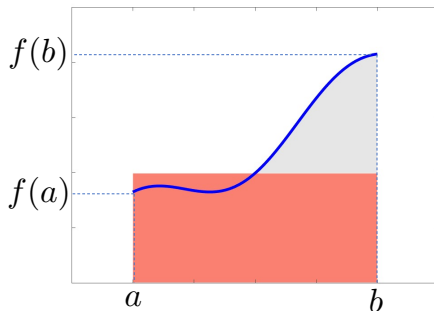
Regra do retângulo (ponto médio)



Regra do retângulo (ponto médio)

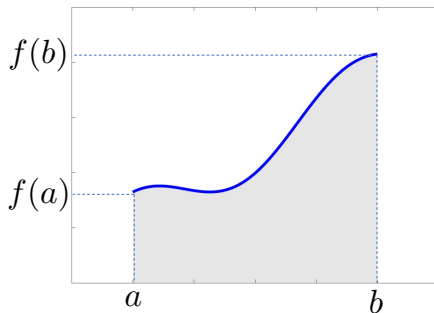


Regra do retângulo (ponto médio)

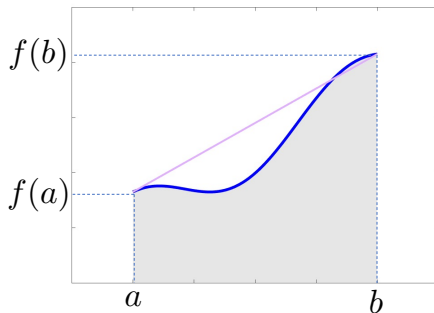


$$\int_a^b f(x) dx \approx (b-a) f\left(\frac{a+b}{2}\right)$$

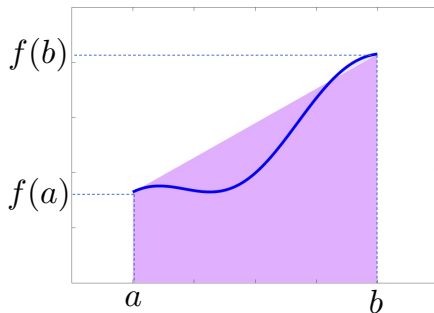
Regra do trapézio



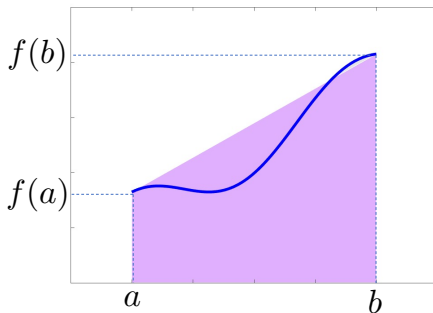
Regra do trapézio



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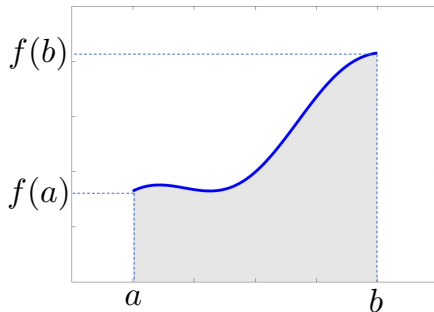


Regra do trapézio

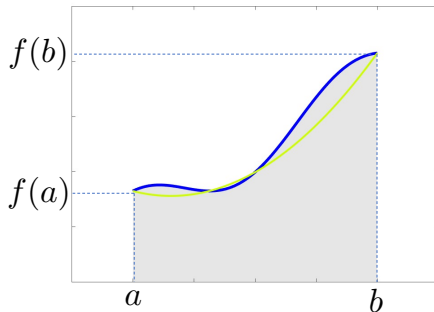


$$\int_a^b f(x) dx \approx \frac{1}{2} (b - a) (f(a) + f(b))$$

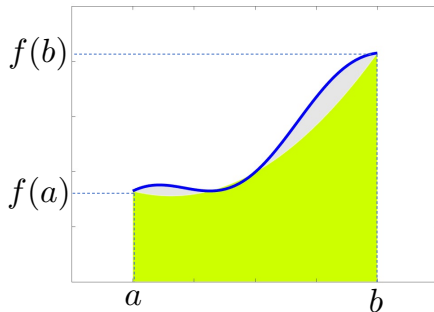
Regra de Simpson



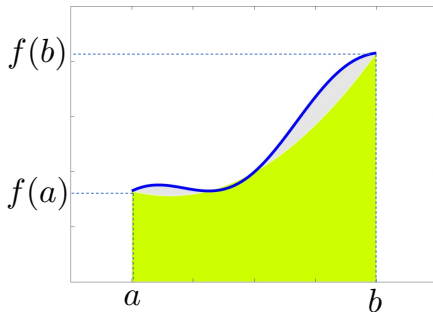
Regra de Simpson



Regra de Simpson



Regra de Simpson



$$\int_a^b f(x) dx \approx \frac{1}{6} (b - a) \left(f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b) \right)$$

Experimento computacional 1

```
1  clc; clear; close all
2
3  f = @(x) exp(-x.^2);
4  a = 0.0;
5  b = 1.0;
6
7  I_L    = (b-a)*f(a)
8  I_R    = (b-a)*f(b)
9  I_M    = (b-a)*f(0.5*(a+b))
10 I_T    = (b-a)*(f(a)+f(b))/2
11 I_S    = (b-a)*(f(a)+4*f(0.5*(a+b))+f(b))/6
12 I_true = quad(f,a,b)
13
14 error_L = abs(I_L-I_true)/abs(I_true)
15 error_R = abs(I_R-I_true)/abs(I_true)
16 error_M = abs(I_M-I_true)/abs(I_true)
17 error_T = abs(I_T-I_true)/abs(I_true)
18 error_S = abs(I_S-I_true)/abs(I_true)
```

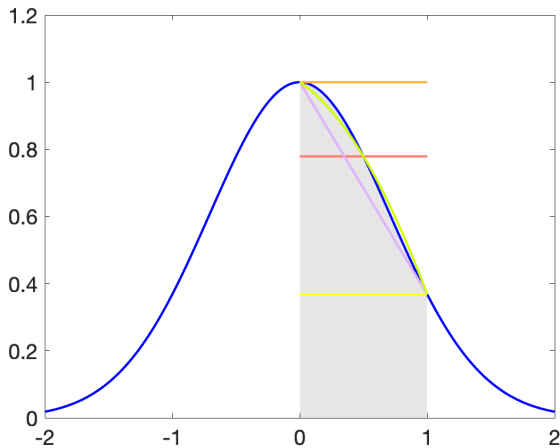


Experimento computacional 1

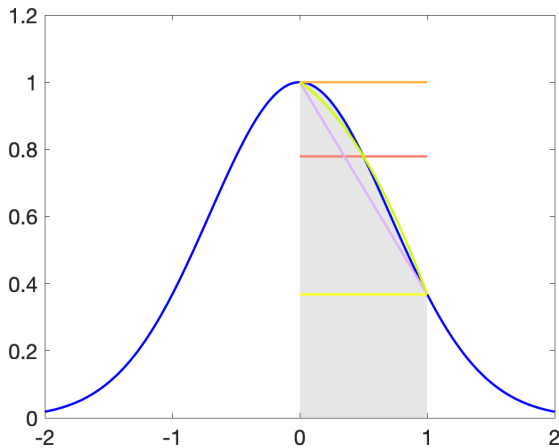
```
1 xtrue = -2:0.01:2;
2 ytrue = f(xtrue);
3 x = a:0.01:b; y = f(x); N = length(x);
4 yL = f(a)*ones(N,1);
5 yR = f(b)*ones(N,1);
6 yM = f(0.5*(a+b))*ones(N,1);
7 coefs = polyfit([a b],[f(a) f(b)],1);
8 yT = polyval(coefs,x);
9 coefs = polyfit([a (a+b)/2 b],[f(a) f((a+b)/2) f(b)],2);
10 yS = polyval(coefs,x);
11
12 figure(1)
13 area(x,y,'FaceColor',[0.9 0.9 0.9],'EdgeColor',[1 1 1]);
14 hold on
15 plot(xtrue,ytrue,'LineWidth',2,'Color','b');
16 plot(x,yL,'LineWidth',2,'Color',[255,179, 71]/255);
17 plot(x,yR,'LineWidth',2,'Color',[254,254, 34]/255);
18 plot(x,yM,'LineWidth',2,'Color',[250,128,114]/255);
19 plot(x,yT,'LineWidth',2,'Color',[224,176,255]/255);
20 plot(x,yS,'LineWidth',2,'Color',[206,255, 0]/255);
21 hold off
22 set(gca,'FontSize',18); xlim([-2 2]); ylim([0 1.2]);
```



Experimento computacional 1



Experimento computacional 1



A precisão varia com o esquema de integração!

Experimento computacional 2

```
1  clc; clear; close all
2
3  f = @(x) x.^3 + 2*x.^2 + 3*x + 4;
4  a = 0.0;
5  b = 1.0;
6
7  I_L    = (b-a)*f(a)
8  I_R    = (b-a)*f(b)
9  I_M    = (b-a)*f(0.5*(a+b))
10 I_T    = (b-a)*(f(a)+f(b))/2
11 I_S    = (b-a)*(f(a)+4*f(0.5*(a+b))+f(b))/6
12 I_true = quad(f,a,b)
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14 error_L = abs(I_L-I_true)/abs(I_true)
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16 error_M = abs(I_M-I_true)/abs(I_true)
17 error_T = abs(I_T-I_true)/abs(I_true)
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```

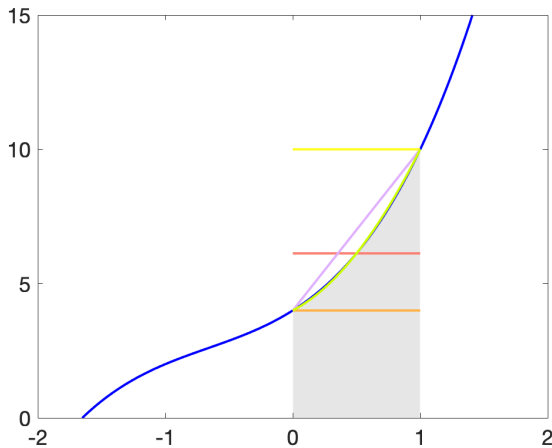


Experimento computacional 2

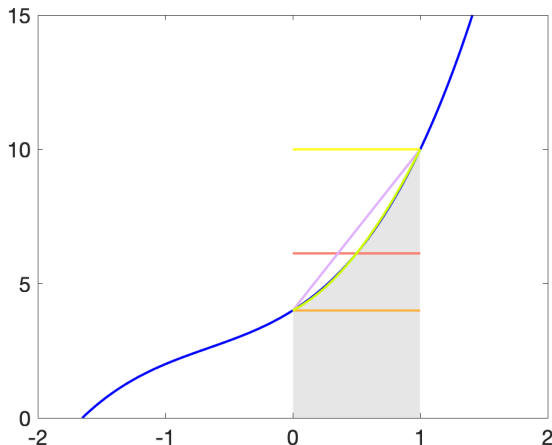
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4 y      = f(x);
5 N      = length(x);
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14 figure(1)
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21 plot(x,yT,'LineWidth',2,'Color',[224,176,255]/255);
22 plot(x,yS,'LineWidth',2,'Color',[206,255, 0]/255);
23 hold off
24 set(gca,'FontSize',18); xlim([-2 2]); ylim([0 15]);
```



Experimento computacional 2



Experimento computacional 2



Por que o resultado da regra de Simpson é exato?



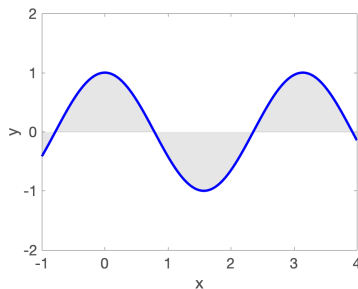
Erro induzido por uma regra de quadratura simples

regra de quadratura	erro de aproximação
ponto à esquerda	$\sim f'(\xi) (b - a)^2$
ponto à direita	$\sim f'(\xi) (b - a)^2$
ponto médio	$\sim \frac{f''(\xi)}{24} (b - a)^3$
trapézio	$\sim \frac{f''(\xi)}{12} (b - a)^3$
Simpson	$\sim \frac{f''''(\xi)}{90} \left(\frac{b - a}{2} \right)^5$



Quadraturas simples são limitadas

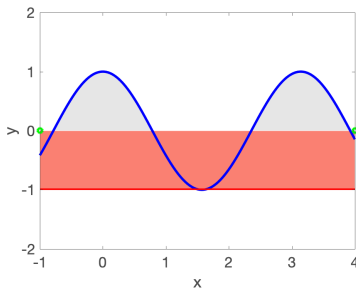
$$\int_{-1}^4 \cos(2x) dx = 0,9493 \dots$$



Quadraturas simples são limitadas

$$\int_{-1}^4 \cos(2x) dx = 0,9493 \dots$$

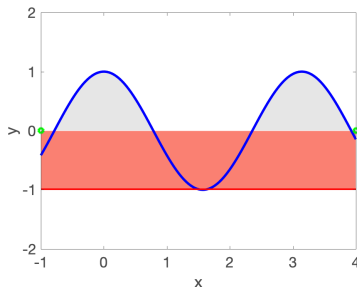
$$I_M = \int_a^b f(x) dx \approx (b-a) f\left(\frac{a+b}{2}\right) = -4,9500$$



Quadraturas simples são limitadas

$$\int_{-1}^4 \cos(2x) dx = 0,9493 \dots$$

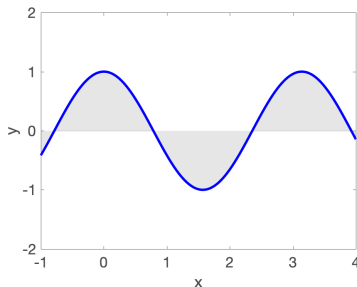
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**Quadraturas simples não fornecem
boas aproximações para intervalos largos!**

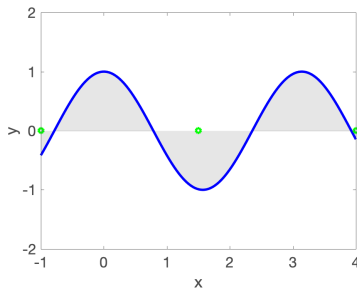
Como aumentar a acurácia de uma quadratura?

$$\int_{-1}^4 \cos(2x) dx = 0,9493 \dots$$



Como aumentar a acurácia de uma quadratura?

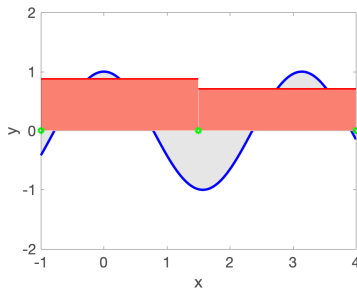
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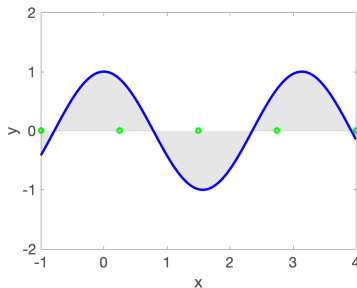
$$I_M = 3,9656$$



Como aumentar a acurácia de uma quadratura?

$$\int_{-1}^4 \cos(2x) dx = 0,9493 \dots$$

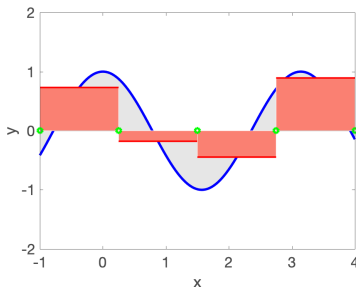
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Como aumentar a acurácia de uma quadratura?

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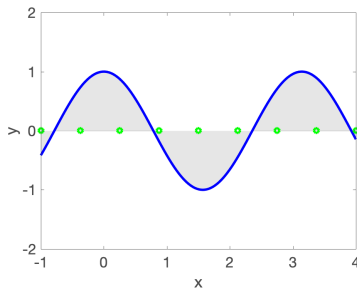
$$I_M = 1,2505$$



Como aumentar a acurácia de uma quadratura?

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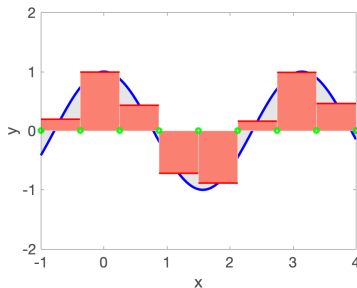
$$I_M = 1,2505$$



Como aumentar a acurácia de uma quadratura?

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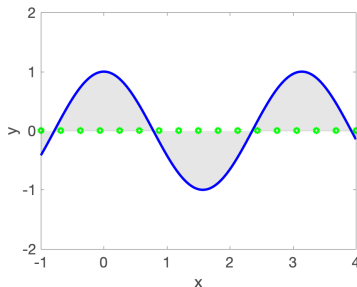
$$I_M = 1,0141$$



Como aumentar a acurácia de uma quadratura?

$$\int_{-1}^4 \cos(2x) dx = 0,9493 \dots$$

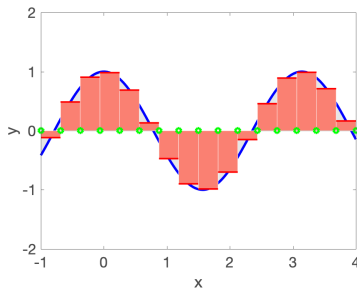
$$I_M = 1,0141$$



Como aumentar a acurácia de uma quadratura?

$$\int_{-1}^4 \cos(2x) dx = 0,9493 \dots$$

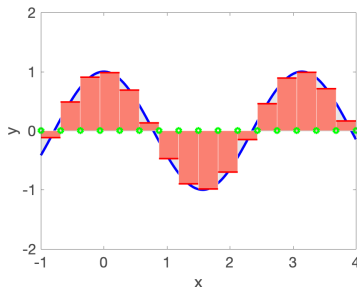
$$I_M = 0,9650$$



Como aumentar a acurácia de uma quadratura?

$$\int_{-1}^4 \cos(2x) dx = 0,9493 \dots$$

$$I_M = 0,9650$$



Refinando a partição do intervalo de integração!

Quadraturas compostas (fórmulas de Newton-Cotes)

O intervalo $[a, b]$ é particionado em N subintervalos uniformes de comprimento $\Delta x = (b - a)/N$.

Ponto médio:

$$I_M = \Delta x \sum_{n=1}^N f(a + (n - 1/2) \Delta x)$$

Trapézio:

$$I_T = \frac{1}{2} \Delta x \left[f(a) + 2 \sum_{n=1}^{N-1} f(a + n \Delta x) + f(b) \right]$$

Simpson (N par):

$$I_S = \frac{1}{3} \Delta x \left[f(a) + 2 \sum_{n=1}^{N/2-1} f(x_{2n}) + 4 \sum_{n=1}^{N/2} f(x_{2n-1}) + f(b) \right]$$



Implementação em GNU Octave (ponto médio)

```
1 function I = quad_mid(f,a,b,N)
2     dx = (b-a)/N;
3     I = 0.0;
4     for n=1:N
5         xm = a + (n-0.5)*dx;
6         I = I + f(xm);
7     end
8     I = dx*I;
9 end
```



Implementação em GNU Octave (trapézio)

```
1 function I = quad_trap(f,a,b,N)
2     dx = (b-a)/N;
3     I = 0.0;
4     for n=1:N
5         x0 = a + (n-1)*dx;
6         x1 = a +      n*dx;
7         I = I + f(x0)+f(x1);
8     end
9     I = 0.5*dx*I;
10 end
```



Implementação em GNU Octave (Simpson)

```
1 function I = quad_simp(f,a,b,N)
2     if mod(N,2) ~= 0
3         error('N deve ser par');
4     end
5     dx = (b-a)/N;
6     I = 0.0;
7     for n=1:N/2
8         x0 = a + (2*n-2)*dx;
9         x1 = a + (2*n-1)*dx;
10        x2 = a + 2*n*dx;
11        I = I + f(x0) + 4*f(x1) + f(x2);
12    end
13    I = (dx/3)*I;
14 end
```



Experimento computacional 3

```
1  clc; clear; close all
2
3  f = @(x) cos(2*x);
4  a = -1;
5  b = 4;
6  N = 20;
7
8  I_M    = quad_mid(f,a,b,N)
9  I_T    = quad_trap(f,a,b,N)
10 I_S    = quad_simp(f,a,b,N)
11 I_true = quad(f,a,b)
12
13 error_M = abs(I_M-I_true)/abs(I_true)
14 error_T = abs(I_T-I_true)/abs(I_true)
15 error_S = abs(I_S-I_true)/abs(I_true)
```



Características das quadraturas compostas

regra de quadratura	custo computacional	erro de aproximação
ponto médio	$\sim N$	$\sim \frac{f''(\xi)}{24} (b - a) \Delta x^2$
trapézio	$\sim N + 1$	$\sim \frac{f''(\xi)}{12} (b - a) \Delta x^2$
Simpson	$\sim N + 1$	$\sim \frac{f''''(\xi)}{180} (b - a) \Delta x^4$



Quadraturas com malha não uniforme (não Newton-Cotes)

O intervalo $[a, b]$ é dividido em N subintervalos disjuntos $[a, b] = [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{2N-1}, x_{2N}]$, onde $x_1 = a$ e $x_N = b$.

Ponto médio:

$$I_M = \sum_{n=1}^N (x_{n+1} - x_n) f\left(\frac{x_{n+1} + x_n}{2}\right)$$

Trapézio:

$$I_T = \sum_{n=1}^N \frac{1}{2} (x_{n+1} - x_n) (f(x_n) + f(x_{n+1}))$$

Simpson (N par):

$$I_S = \sum_{n=1}^N \frac{1}{6} (x_{n+1} - x_n) \left(f(x_n) + 4 f\left(\frac{x_n + x_{n+1}}{2}\right) + f(x_{n+1}) \right)$$



Para pensar em casa ...

Exercício computacional:

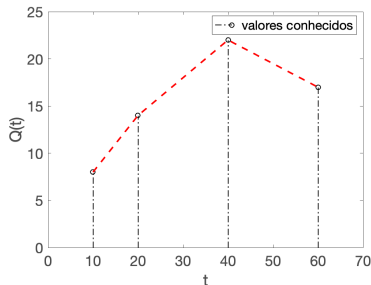
Implemente no ambiente GNU Octave as regras de quadratura com malha não uniforme apresentadas no slide anterior. 



Volume de óleo escoando num duto



tempo (s)	vazão (m ³ /s)
10	8
20	14
40	22
60	17



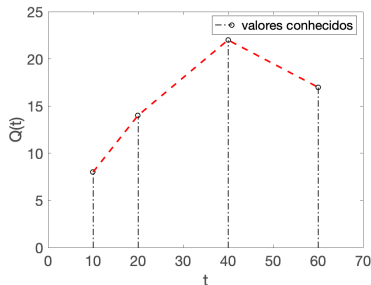
Qual o volume de óleo transportado entre 10 e 60 s ?

Figura em <https://iconarchive.com/show/transport-icons-by-aha-soft/pipe-line-icon.html>

Volume de óleo escoando num duto



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Qual o volume de óleo transportado entre 10 e 60 s ?

$$\text{volume} = \int_{t=10}^{60} \text{vazão}(t) dt$$

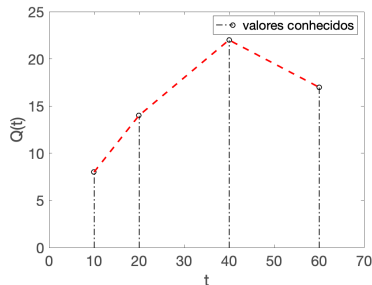
Figura em <https://iconarchive.com/show/transport-icons-by-aha-soft/pipe-line-icon.html>



Volume de óleo escoando num duto



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Qual o volume de óleo transportado entre 10 e 60 s ?

$$\text{volume} = \int_{t=10}^{60} \text{vazão}(t) dt$$



```
>> x = [10 20 40 60];  
>> y = [ 8 14 22 17];  
>> V = trapz(x,y)
```

Figura em <https://iconarchive.com/show/transport-icons-by-aha-soft/pipe-line-icon.html>

Como citar esse material?

A. Cunha, *Integração Numérica*,
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