KEB-45251

Numerical Techniques for Process Modeling

Exercise 6 - Boundary value problem

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Problem 1

Write a numerical solver for a temperature profile in a fin with constant circular cross section A_c and constant heat transfer coefficient h. Include radiation. The fin base temperature $T_B > T_\infty$ is given and fin tip is assumed insulated.

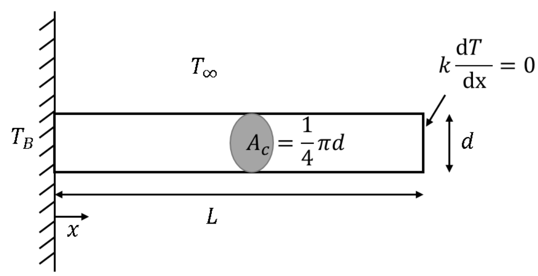


Figure 1: Fin

The governing equation is

$$kA_c\frac{\mathrm{d}^2T}{\mathrm{d}x^2} - hP\left(T - T_\infty\right) - \epsilon\sigma P\left(T^4 - T_\infty^4\right) = 0 \tag{1}$$

where P is fin perimeter and A_c is cross-section area.

If there is no radiation, $\epsilon = 0$, an analytical solution exists

$$\begin{gathered}\frac{T - T_\infty}{T - T_B} = \frac{\cosh\beta\left(L - x\right)}{\cosh\beta L} \\ \beta = \left(\frac{hP}{kA_c}\right)^{1/2}\end{gathered} \tag{2}$$

use the analytical solution to validate your code for convection.

To save time, most of the code is already written in the template file, you only need to implement function *numerical*.

Scipy.integrate includes a numerical boundary value problem solver *solve_ bvp* (see documentation). It requires the ode to be gives as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(x, y\right) \tag{3}$$

with boundary values given as

$$\mathrm{bc}\left(y(a), y(b)\right) = 0 \tag{4}$$

where $a \leq x \leq b$ and y is a vector of first order functions. In our case

$$y = \begin{bmatrix} T \\ T' \end{bmatrix} \tag{5}$$

where $T' = \frac{\mathrm{d}T}{\mathrm{d}x}$.

The solver only requires a few lines of code. However, to write those lines the 2nd order ODE in Eq. 1 needs to be transformed into a system of first order ODEs. **You may wish to start with pen and paper.**

Problem 2

Solve last week's problem with Runge-Kutta (or BDF, LSODA,...) using *solve_ ivp* from Scipy. A template is available. You only need to implement the ode function.

The governing equation is

$$mc_p \frac{dT}{dt} = -hA\left(T - T_\infty\right) - \epsilon\sigma A\left(T^4 - T_\infty^4\right) \tag{6}$$

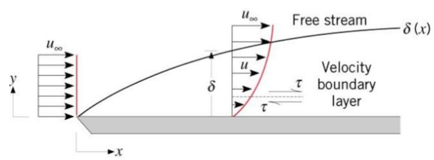
Solve_ ivp expects the problem to be given in a standard form

$$\frac{dy}{dt} = f\left(t, y\right) \tag{7}$$

with an initial value $y\left(t = 0\right) = y_0$. Implement $f\left(t, y\right)$ inside the *ode* function. For more details, see `https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_ivp.html`

Problem 3 - Extra

Flat plate laminar boundary layer flow is a classic example of a real life fluid dynamics problem that can be solved with the methods discussed today.



Starting with the governing equations:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + X$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + Y$$

Figure 2: Laminar boundary layer. Fig: https://www.chegg.com/homework-help/questions-and-answers/1-derive-similarity-solution-flat-plate-laminar-boundary-layer-blasius-solution-show-2-sol-q32968818

The derivation starts from the general 2D equations given in Fig. 2, and after many simplifications and math tricks reduces to

$$\begin{aligned} f''' + 0.5ff'' &= 0 \\ \eta(x,y) &= \frac{y}{\delta(x)} = y\sqrt{\frac{U_\infty}{\nu x}} \\ u(x,y) &= U_\infty \frac{\mathrm{d}f}{\mathrm{d}\eta} = U_\infty f'(\eta) \end{aligned} \tag{8}$$

which can be transformed into a system of first order ODEs and solved numerically. Appropriate boundary conditions are

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ f'(\infty) &= 1 \end{aligned} \tag{9}$$

First, solve the Eq. 8 using *solve_ bvp.* Then plot some velocity profiles.