Proof of Max-Flow Min-Cut Theorem

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1 Introduction

1.1 Flow network

A flow network is a $4-$tuple (G, s, t, c), where $G = (V, E)$ is a directed graph with $s, t \in V$, s being the source vertex (no edges coming into s), t, different from s, being the target vertex (no edges going out of t) and c being capacity function for G.

1.2 Capacity and Flow in a Flow network

For a graph $G = (V, E)$ a capacity funciton c is a function $c : E \to \mathbb{R}_{\geq 0}$ and a flow f is a function $f : E \to \mathbb{R}_{\geq 0}$, such that $\forall e \in E : 0 \leq f(e) \leq c(e)$. Last, but not least, the flow conservation should hold in every flow network, i.e. $\forall v \in V \setminus \{s, t\} : \sum_{u \in V} f_{u,v} = \sum_{w \in V} f_{v,w}$.
The value of the flow in the network is $F_{value} = \sum_{v \in V} f_{s,v} - \sum_{u \in V} f_{u,s}$, where $(s, v); (u, s) \in E$.
An $s - t$ cut is a two-set partition of G, A and B, such that $s \in A$, $t \in B$. Then, the capacity of an $s - t$ cut is

$$cap(A, B) = \sum_{u \in A,\ v \in B} c_{u,v},$$

where $u, v \in E$.

1.3 Defining the IN and OUT functions

For convenience, we will define $IN\ f\ s$ in a functional programming language style, for a set s and a function $f : E \to \mathbb{R}$:

$$IN\ f\ s = \sum_{u \in S,\ v \in V \setminus S} f_{u,v}.$$

Analogically,

$$OUT\ f\ s = \sum_{u \in S,\ v \in V \setminus S} f_{v,u}.$$

Hence, we can say that the flow value is

$$F_{value} = OUT\ f\ s - IN\ f\ s,$$

and similarly, the cut capacity is

$$cap(A, B) = OUT\ c\ A,$$

where f is the flow function and c is the capacity function.

1.4 Defining the Max-Flow and Min-Cut problems

The Max-Flow problem is to find the maximum flow we can send from s to t in a given flow network.
The Min-Cut problem is finding the $s - t$ with minimum capacity in a given flow network.

2 Weak Duality

We will prove that for any flow f, for any $s-t$ cut (A, B), we have $F_{value} \leq cap(A, B)$. This result implies that the value of the Max-Flow is less than or equal to the capacity of the min $s-t$ cut (weak duality). For a given flow network (G, s, t, c), we have $s \in A$, thus $\sum_{v\in V} f_{v,s} = 0$, therefore

$$F_{value} = OUT\ f\ s - IN\ f\ s = \sum_{v\in V} f_{s,v} - \sum_{v\in V} f_{v,s} = \sum_{v\in A}(\sum_{w\in V} f_{v,w} - \sum_{u\in V} f_{u,v}).$$

That is because the flow of the edges s, v with $v \in A$ is counted once, and according to flow conservation $\forall v \in A \setminus \{s\} : \sum_{u\in A} f_{u,v} = \sum_{w\in V} f_{v,w}$.
Moreover, the flow for the edges $e = (u, v)$ with $u, v \in A$ will be added for u and then subtracted for v, hence will be counted with coefficient 0.
Therefore,

$$F_{value} = \sum_{u\in A,\ v\in B} f_{u,v} - \sum_{u\in B,\ v\in A} f_{u,v} \leq \sum_{u\in A,\ v\in B} c_{u,v} = OUT\ c\ A = cap(A, B)$$

because $\sum_{u\in B,\ v\in A} f_{u,v} \geq 0$ and $\forall e \in E : 0 \leq f(e) \leq c(e)$. That concludes the weak duality proof.

3 Residual network

Given a flow network, the residual network helps to increase the flow value F_value. We construct the residual network $G_f = (V, E_f)$ with respect to the flow function f in the following way:
For an edge $e = (u, v)$ with flow $f_{u,v}$ and capacity $c_{u,v}$: if $f_{u,v} < c_{u,v}$, then we add a forward edge e with $c_f(e) = c(e) - f(e)$ since we can still push $c(e) - f(e)$ units of flow, and if $f(e) > 0$, we add a backward edge $e^{'} = (v, u)$ with $c_f(e^{'}) = f(e)$ because we can undo the pushed flow.
Then, if f is a Max-Flow, there is no $s-t$ path in the residual network, called also an augmenting path. Otherwise, let P be an $s-t$ path in the residual network. Then, for a flow f, we can increase it with $d = \min_{e\in P} c_f(e)$. That is because we can construct the flow $f^{'}$ on G in the following way:
If $e \in P \implies f'(e) = f(e) + d$;
If $rev(e) \in P \implies f'(e) = f(e) - d$;
Otherwise, $f'(e) = f(e)$.
Now, it is sufficient to show that the construction above is a valid flow for in (G, s, t, c) with flow

$$F'_{value} = F_{value} + d.$$

The only changes in terms of the flow conservation are for $v \in P$, but both the flow in and out of $v \notin \{s, t\}$ increases with d in our construction, so the flow conservation still holds.
As for the capacity constraints, if $e \in P \implies f'(e) = f(e) + d$ and $d \leq c_f(e) = c(e) - f(e) \implies 0 \leq f'(e) \leq f(e) + c(e) - f(e) = c(e)$; if $rev(e) \in P \implies c(e) \geq f'(e) = f(e) - d$ and $d \leq f(e) \implies f'(e) \geq f(e) - f(e) = 0$, otherwise $f'(e) = f(e)$, so there is no overflow.
All other flow network constraints are satisfied because the main network remains the same.
Also, s is incident to exactly one edge in P, so the flow out of s has increased by d and $F'_{value} = F_{value} + d$.

4 Max-Flow Min-Cut Theorem

We will show that if a Max-Flow exists, then its value is equal to the capacity of the Min-Cut.
We have already proven the weak duality, so we just need to show that for a flow of maximum value f, there exists an $s-t$ cut with a capacity equal to F_{value}. We define A to be the set of all vertices in V, such that there exists a path from s to v in the residual network.
We just proved that if the flow is maximal, then an augmenting path doesn't exist in the residual network, thus $s \in A$ and $t \notin A$, i.e. $t \in V \setminus A$.
Moreover, $\forall u \in V \setminus A,\ v \in A : f_{u,v} = 0$, otherwise there is a backward edge $e = (v, u)$ in the residual network, thus there is a path from s to u in the residual network, going through v, which contradicts $u \notin A$.

Finally, $\forall u \in A,\ v \in V \setminus A : f_{u,v} = c_{u,v}$, otherwise there is a forward edge $e = (u, v)$ in the residual network, so $v \in A$, contradiction.
Therefore,

$$F_{value} = \sum_{u \in A,\ v \in V \setminus A} f_{u,v} - \sum_{u \in V \setminus A,\ v \in A} f_{u,v} = \sum_{u \in A,\ v \in V \setminus A} c_{u,v} = cap(A, V \setminus A),$$

thus the Max-Flow is equal to the capacity of the $s - t$ cut $(A, V \setminus A)$, which concludes the proof.