Find DFT directly:

$$A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$W_4 \triangleq e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$$

$$DFT = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$$\boxed{\begin{aligned} W_4 &= -j \\ W_4^2 &= -1 \\ W_4^3 &= -W_4 \\ W_4^4 &= 1 \\ W_4^6 &= -1 \\ W_4^9 &= W_4 \end{aligned}}$$

$$f_1 \triangleq a + b + c + d$$

$$f_2 \triangleq a + W_4 b + W_4^2 c + W_4^3 d$$
$$= (a - c) + W_4(b - d)$$

$$f_3 \triangleq a + W_4^2 b + W_4^4 c + W_4^6 d$$
$$= (a + c) - (b + d)$$

$$f_4 \triangleq a + W_4^3 b + W_4^6 c + W_4^9 d$$
$$= (a - c) - W_4(b + d)$$

Find DFT using Cooley-Tukey:

1 $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \to M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

2 Find dft of rows: $W_2 = e^{-j\frac{2\pi}{2}} = -1$

$$\begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a+c \\ a-c \end{bmatrix} = \begin{bmatrix} a' \\ c' \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} b+d \\ b-d \end{bmatrix} = \begin{bmatrix} b' \\ d' \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} a' & c' \\ b' & d' \end{bmatrix}$

$a' = a + c$ (1)

$b' = b + d$ (2)

$c' = a - c$ (3)

$d' = b - d$ (4)

3 Multiply M by twiddle matrix (element-wise)

$$\text{twiddle matrix} = \begin{bmatrix} 1 & 1 \\ 1 & W_4 \end{bmatrix}$$

Cont'd

3

$$\Rightarrow \begin{bmatrix} a' & c' \\ b' & W_4 d' \end{bmatrix}$$

---

4 Find DFT of Columns

$$\begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} \begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} a'+b' \\ a'-b' \end{bmatrix} = \begin{bmatrix} a'' \\ b'' \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} \begin{bmatrix} c' \\ W_4 d' \end{bmatrix} = \begin{bmatrix} c' + W_4 d' \\ c' - W_4 d' \end{bmatrix} = \begin{bmatrix} c'' \\ d'' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a'' & c'' \\ b'' & d'' \end{bmatrix}$$

$a'' = a' + b'$ (5)

$b'' = a' - b'$ (6)

$c'' = c' + W_4 d'$ (7)

$d'' = c' - W_4 d'$ (8)

---

4 Flatten the previous output:

$$\Rightarrow \begin{bmatrix} a'' \\ c'' \\ b'' \\ d'' \end{bmatrix}$$

Claim:

$$f_1 = a''$$

$$f_2 = c''$$

$$f_3 = b''$$

$$f_4 = d''$$

Let's check:

$$5 \Rightarrow a'' = a' + b' \overset{1, 2}{=} a + c + b + d = f_1$$

$$6 \Rightarrow b'' = a' - b' \overset{1, 2}{=} (a+c) - (b+d) = f_3$$

$$7 \Rightarrow c'' = c' + W_4\, d' \overset{3, 4}{=} (a-c) + W_4\,(b-d) = f_2$$

$$8 \Rightarrow d'' = c' - W_4\, d' \overset{3, 4}{=} (a-c) - W_4\,(b-d) = f_4$$