

1 General equations

Define $N(t)$ as the population size at time t , with t in units of $2N(0)$. Let $\lambda(t)$ be the relative population size, scaled by $N(0)$, such that $N(t) = N(0)\lambda(t)$. Define $\Omega(u, v)$ as the cumulative coalescent rate between times u and v :

$$\Omega(u, v) = \int_u^v \frac{dt}{\lambda(t)}. \quad (1)$$

The state of the TSMC at each point along the genome is described by the vector $\mathbf{s} = (s_3, s_2)$, where s_3 is the time of the first coalescence event and s_2 is the time of the second coalescence event amongst the three lineages in a triploid genome. The equilibrium joint distribution of (t_3, t_2) is

$$\pi(t_3, t_2) = \frac{3}{\lambda(t_3)\lambda(t_2)} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)}. \quad (2)$$

Let $q(\mathbf{t}|\mathbf{s})$ be the transition kernel at recombination sites along the genome. Then

$$q(\mathbf{t}|\mathbf{s}) =$$

For $t_3 = s_3; t_2 > s_2$:

$$\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} + 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} e^{-2\Omega(u, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)}$$

For $t_3 = s_3; t_2 < s_2$:

$$\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3, t_2)} + 2 \int_{s_3}^{t_2} \frac{du}{2s_2 + s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u, t_2)}$$

For $t_3 < s_3; t_2 = s_3$:

$$\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)}$$

For $t_3 < s_3; t_2 = s_2$:

$$2 \int_0^{t_3} \frac{du}{2s_2 + s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u, t_3)}$$

For $t_3 > s_3; t_2 = s_2$:

$$2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)}$$

For $t_3 = s_2; t_2 > s_2$:

$$2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)}$$

For $t_3 = s_3; t_2 = s_2$:

$$3 \int_0^{s_3} \frac{du}{2s_2 + s_3} \frac{1}{3} \left[1 - e^{-3\Omega(u, s_3)} \right] + \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{2} \left[1 - e^{-2\Omega(s_3, s_2)} \right] + 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} \frac{1}{2} \left[1 - e^{-2\Omega(u, s_2)} \right]. \quad (3)$$

Each part is implicitly multiplied by a delta function to limit the density to points where the parameters are assumed to be equal to each other. For example, the first part of $q(\mathbf{t}|\mathbf{s})$ is implicitly multiplied by $\delta(t_3 - s_3)$, and the last part is multiplied by $\delta(t_3 - s_3)\delta(t_2 - s_2)$.

2 Piecewise constant transition probabilities

Suppose that the population changes size at times (T_1, \dots, T_n) and that the size between T_i and T_{i+1} is a constant $2N\lambda_i$. Define $T_0 = 0$, $T_{n+1} = \infty$ and $\Delta_i = T_{i+1} - T_i$. Let $\alpha(t)$ be the index of the time interval to which t belongs, *i.e.*, $\alpha(t) = \max_i \{i : T_i \leq t\}$.

Then the cumulative coalescent rate between u and v can be written

$$\Omega(u, v) = \begin{cases} \frac{v-u}{\lambda_{\alpha(u)}} & \alpha(u) = \alpha(v) \\ \frac{T_{\alpha(u)+1}-u}{\lambda_{\alpha(u)}} + \sum_{i=\alpha(u)+1}^{\alpha(v)-1} \frac{\Delta_i}{\lambda_i} + \frac{v-T_{\alpha(v)}}{\lambda_{\alpha(v)}} & \alpha(u) < \alpha(v). \end{cases} \quad (4)$$

The equilibrium joint density of (t_3, t_2) is now approximately

$$\pi(t_3, t_2) = \frac{3}{\lambda_{\alpha(t_3)}\lambda_{\alpha(t_2)}} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)}. \quad (5)$$

There are several integrals of the form $\int_x^y e^{-k\Omega(u, y)} du$ in Equation (3). This integral can be written

$$\begin{aligned}
\int_x^y e^{-k\Omega(u,y)} du &= \int_x^{T_{\alpha(x)+1}} e^{-k\Omega(u,y)} du + \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \int_{T_i}^{T_{i+1}} e^{-k\Omega(u,y)} du + \int_{T_{\alpha(y)}}^y e^{-k\Omega(u,y)} du \\
&= \int_x^{T_{\alpha(x)+1}} \exp \left(-k \left[(T_{\alpha(u)+1} - u) \frac{1}{\lambda_{\alpha(u)}} + \sum_{j=\alpha(u)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) du + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \int_{T_i}^{T_{i+1}} \exp \left(-k \left[(T_{\alpha(u)+1} - u) \frac{1}{\lambda_{\alpha(u)}} + \sum_{j=\alpha(u)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) du + \\
&\quad \int_{T_{\alpha(y)}}^y \exp \left(-k(y - u) \frac{1}{\lambda_{\alpha(y)}} \right) du \\
&= \int_x^{T_{\alpha(x)+1}} \exp \left(-k \left[(T_{\alpha(x)+1} - u) \frac{1}{\lambda_{\alpha(x)}} + \sum_{j=\alpha(x)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) du + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \int_{T_i}^{T_{i+1}} \exp \left(-k \left[(T_{i+1} - u) \frac{1}{\lambda_i} + \sum_{j=i+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) du + \\
&\quad \int_{T_{\alpha(y)}}^y \exp \left(-k(y - u) \frac{1}{\lambda_{\alpha(y)}} \right) du \\
&= \exp \left(-k \left[\sum_{j=\alpha(x)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) \int_x^{T_{\alpha(x)+1}} \exp \left(-k (T_{\alpha(x)+1} - u) \frac{1}{\lambda_{\alpha(x)}} \right) du + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \exp \left(-k \left[\sum_{j=i+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) \int_{T_i}^{T_{i+1}} \exp \left(-k (T_{i+1} - u) \frac{1}{\lambda_i} \right) du + \\
&\quad \int_{T_{\alpha(y)}}^y \exp \left(-k(y - u) \frac{1}{\lambda_{\alpha(y)}} \right) du \\
&= \exp \left(-k \left[\sum_{j=\alpha(x)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) \left[1 - \exp \left(-\frac{k (T_{\alpha(x)+1} - x)}{\lambda_{\alpha(x)}} \right) \right] \frac{\lambda_{\alpha(x)}}{k} + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \exp \left(-k \left[\sum_{j=i+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) \left[1 - \exp \left(-\frac{k \Delta_i}{\lambda_i} \right) \right] \frac{\lambda_i}{k} + \\
&\quad \left[1 - \exp \left(-\frac{k (y - T_{\alpha(y)})}{\lambda_{\alpha(y)}} \right) \right] \frac{\lambda_{\alpha(y)}}{k} \\
&= \exp (-k\Omega(T_{\alpha(x)+1}, y)) \left[1 - \exp \left(-\frac{k (T_{\alpha(x)+1} - x)}{\lambda_{\alpha(x)}} \right) \right] \frac{\lambda_{\alpha(x)}}{k} + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \exp (-k\Omega(T_{i+1}, y)) \left[1 - \exp \left(-\frac{k \Delta_i}{\lambda_i} \right) \right] \frac{\lambda_i}{k} + \\
&\quad \left[1 - \exp \left(-\frac{k (y - T_{\alpha(y)})}{\lambda_{\alpha(y)}} \right) \right] \frac{\lambda_{\alpha(y)}}{k} \\
&= e^{-k\Omega(T_{\alpha(x)+1}, y)} \left[1 - e^{-\frac{k (T_{\alpha(x)+1} - x)}{\lambda_{\alpha(x)}}} \right] \frac{\lambda_{\alpha(x)}}{k} + \sum_{i=\alpha(x)+1}^{\alpha(y)-1} e^{-k\Omega(T_{i+1}, y)} \left[1 - e^{-\frac{k \Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{k} + \left[1 - e^{-\frac{k (y - T_{\alpha(y)})}{\lambda_{\alpha(y)}}} \right] \frac{\lambda_{\alpha(y)}}{k}
\end{aligned} \tag{6}$$

With this equation, we can calculate all of the transition probabilities in the transition kernel (3).

For $t_3 = s_3; t_2 > s_2$:

$$\begin{aligned} & \frac{1}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ & \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \times \\ & \left\{ e^{-2\Omega(T_{\alpha(s_3)+1}, s_2)} \left[1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{i=\alpha(s_3)+1}^{\alpha(s_2)-1} e^{-2\Omega(T_{i+1}, s_2)} \left[1 - e^{-\frac{2\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{2} \right. \\ & \quad \left. + \left[1 - e^{-\frac{2(s_2 - T_{\alpha(s_2)})}{\lambda_{\alpha(s_2)}}} \right] \frac{\lambda_{\alpha(s_2)}}{2} \right\} \\ & \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} + 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} e^{-2\Omega(u, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \end{aligned}$$

For $t_3 = s_3; t_2 < s_2$:

$$\begin{aligned} & \frac{1}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-2\Omega(s_3, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ & \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} \\ & \left\{ e^{-2\Omega(T_{\alpha(s_3)+1}, t_2)} \left[1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{i=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{i+1}, t_2)} \left[1 - e^{-\frac{2\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{2} \right. \\ & \quad \left. + \left[1 - e^{-\frac{2(t_2 - T_{\alpha(t_2)})}{\lambda_{\alpha(t_2)}}} \right] \frac{\lambda_{\alpha(t_2)}}{2} \right\} \\ & \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3, t_2)} + 2 \int_{s_3}^{t_2} \frac{du}{2s_2 + s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u, t_2)} \end{aligned}$$

For $t_3 < s_3; t_2 = s_3$:

$$\begin{aligned} & \frac{1}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\} \\ & \int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \end{aligned}$$

For $t_3 < s_3; t_2 = s_2$:

$$\begin{aligned} & \frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\} \\ & 2 \int_0^{t_3} \frac{du}{2s_2 + s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u, t_3)} \end{aligned}$$

For $t_3 > s_3; t_2 = s_2$:

$$\frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} e^{-2\Omega(s_3, t_3)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\ 2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)}$$

For $t_3 = s_2; t_2 > s_2$:

$$\frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\ 2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)}$$

For $t_3 = s_3; t_2 = s_2$:

$$\frac{1}{2s_2 + s_3} \left\{ s_3 - \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} - \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ \frac{1}{2s_2 + s_3} \frac{1}{2} \left[1 - e^{-2\Omega(s_3, s_2)} \right] \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ \frac{1}{2s_2 + s_3} \left[s_2 - s_3 - e^{-2\Omega(T_{\alpha(s_3)+1}, t_2)} \left(1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right) \frac{\lambda_{\alpha(s_3)}}{2} - \sum_{i=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{i+1}, t_2)} \left(1 - e^{-\frac{2\Delta_i}{\lambda_i}} \right) \frac{\lambda_i}{2} \right. \\ \left. - \left(1 - e^{-\frac{2(t_2 - T_{\alpha(t_2)})}{\lambda_{\alpha(t_2)}}} \right) \frac{\lambda_{\alpha(t_2)}}{2} \right] \\ 3 \int_0^{s_3} \frac{du}{2s_2 + s_3} \frac{1}{3} \left[1 - e^{-3\Omega(u, s_3)} \right] + \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{2} \left[1 - e^{-2\Omega(s_3, s_2)} \right] + 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} \frac{1}{2} \left[1 - e^{-2\Omega(u, s_2)} \right].$$

3 Discrete approximation to the triploid SMC' coalescent process

In order to construct a hidden Markov model (HMM) to infer demography, it is necessary to discretize the triploid coalescent process described above.

Let the discrete state (i, j) , $i < j$, correspond to the continuous states in which $T_i < t_3 < T_{i+1}$ and $T_j < t_2 < T_{j+1}$. We first calculate the equilibrium probability that the coalescent process is in (i, j) , assuming $i < j$:

$$\begin{aligned}
\pi_{i,j} &= \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{T_j}^{T_{j+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\Omega(T_j,t_2)} dt_2 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\frac{t_2-T_j}{\lambda_j}} dt_2 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} e^{-\frac{T_{i+1}-t_3}{\lambda_i}} dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \\
&= \frac{3}{2} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right].
\end{aligned} \tag{7}$$

If $j = n$, we let $\Delta_j = \infty$ and $1 - \exp(-\Delta_j/\lambda_j) = 1$.

It is also necessary to calculate $\pi_{i,i}$:

$$\begin{aligned}
\pi_{i,i} &= \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{3}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{1}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\frac{t_2-t_3}{\lambda_i}} dt_2 dt_3 \\
&= \frac{3}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \left(\lambda_i \left[1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} \right] \right) dt_3 \\
&= \frac{3}{\lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} \right) dt_3 \\
&= \frac{3}{\lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{6} \left(2 - 3e^{-\frac{\Delta_i}{\lambda_i}} + e^{-\frac{3\Delta_i}{\lambda_i}} \right) \\
&= \frac{1}{2} e^{-3\Omega(0,T_i)} \left(2 - 3e^{-\frac{\Delta_i}{\lambda_i}} + e^{-\frac{3\Delta_i}{\lambda_i}} \right)
\end{aligned} \tag{8}$$

For $i = n$, again we let $\Delta_i = \infty$ and thus $\pi_{i,i} = \exp(-3\Omega(0,T_i))$.

Next, we calculate marginal expectations for t_3 and t_2 given that the continuous process is in interval represented by (i, j) , assuming $i < j$. The marginal expectation of t_3 in the interval (i, j) is

$$\begin{aligned}
E_{i,j}[t_3] &= E[t_3 | t_3 \in [T_i T_{i+1}), t_2 \in [T_j, T_{j+1})] \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3t_3}{\lambda_i \lambda_j} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} \int_{T_j}^{T_{j+1}} e^{-\Omega(t_3, t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\Omega(T_j, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\frac{t_2 - T_j}{\lambda_j}} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} \lambda_j \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_j)} dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} \lambda_j \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) e^{-\Omega(T_{i+1}, T_j)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_{i+1})} dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} \lambda_j \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{4} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}} \right] \\
&= \frac{3}{4\pi_{i,j}} e^{-3\Omega(0, T_i)} \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) e^{-\Omega(T_{i+1}, T_j)} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}} \right]
\end{aligned} \tag{9}$$

With $j = n$, this is

$$\begin{aligned}
E_{i,j}[t_3] &= E[t_3 | t_3 \in [T_i T_{i+1}), t_2 \in [T_n, \infty)] \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} \frac{3t_3}{\lambda_i \lambda_n} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} \int_{T_n}^{\infty} e^{-\Omega(t_3, t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_n)} dt_3 \int_{T_n}^{\infty} e^{-\Omega(T_n, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_n)} dt_3 \int_{T_n}^{\infty} e^{-\frac{t_2 - T_n}{\lambda_n}} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} \lambda_n \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_n)} dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} \lambda_n e^{-\Omega(T_{i+1}, T_n)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_{i+1})} dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} \lambda_n e^{-\Omega(T_{i+1}, T_n)} \frac{\lambda_i}{4} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}} \right] \\
&= \frac{3}{4\pi_{i,n}} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}} \right]
\end{aligned} \tag{10}$$

The marginal expectation of t_2 in (i, j) is

$$\begin{aligned}
E_{i,j}[t_2] &= E[t_2 | t_3 \in [T_i, T_{i+1}), t_2 \in [T_j, T_{j+1}]] \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} t_2 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3t_2}{\lambda_i \lambda_j} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_{i+1})} dt_3 \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3 - T_i)}{\lambda_i}} e^{-\frac{T_{i+1} - t_3}{\lambda_i}} dt_3 \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j, t_2)} dt_2 \quad (11) \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{-3\Delta_i}{\lambda_i}} \right) \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{-3\Delta_i}{\lambda_i}} \right) \int_{T_j}^{T_{j+1}} t_2 e^{-\frac{t_2 - T_j}{\lambda_j}} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{-3\Delta_i}{\lambda_i}} \right) \lambda_j \left(\lambda_j + T_j - (\lambda_j + T_{j+1}) e^{-\frac{\Delta_j}{\lambda_j}} \right) \\
&= \frac{3}{2\pi_{i,j}} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{-3\Delta_i}{\lambda_i}} \right) \left(\lambda_j + T_j - (\lambda_j + T_{j+1}) e^{-\frac{\Delta_j}{\lambda_j}} \right).
\end{aligned}$$

With $j = n$, this is

$$\begin{aligned}
E_{i,n}[t_2] &= E[t_2 | t_3 \in [T_i, T_{i+1}), t_2 \in [T_n, \infty)] \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} t_2 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} \frac{3t_2}{\lambda_i \lambda_n} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_{i+1})} dt_3 \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3 - T_i)}{\lambda_i}} e^{-\frac{T_{i+1} - t_3}{\lambda_i}} dt_3 \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n, t_2)} dt_2 \quad (12) \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{-3\Delta_i}{\lambda_i}} \right) \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{-3\Delta_i}{\lambda_i}} \right) \int_{T_n}^{\infty} t_2 e^{-\frac{t_2 - T_n}{\lambda_n}} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{-3\Delta_i}{\lambda_i}} \right) \lambda_n (\lambda_n + T_n) \\
&= \frac{3}{2\pi_{i,n}} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{-3\Delta_i}{\lambda_i}} \right) (\lambda_n + T_n)
\end{aligned}$$

We also calculate the marginal expectations of s_3 and s_2 conditional on the interval (i, i) :

$$\begin{aligned}
\mathbb{E}_{i,i}[t_3] &= \mathbb{E}[t_3 | t_3 \in [T_i T_{i+1}), t_2 \in [T_i, T_{i+1}]] \\
&= \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{t_3}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-\frac{3(t_3-T_i)}{\lambda_i}} \lambda_i \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right) dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-\frac{3(t_3-T_i)}{\lambda_i}} \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right) dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{36} \left(-9e^{-\frac{\Delta_i}{\lambda_i}} (2T_i + \lambda_i) + 4(3T_i + \lambda_i) + e^{-\frac{3\Delta_i}{\lambda_i}} (6T_{i+1} + 5\lambda_i)\right) \\
&= \frac{1}{12\pi_{i,i}} e^{-3\Omega(0,T_i)} \left(4(3T_i + \lambda_i) + e^{-\frac{3\Delta_i}{\lambda_i}} (6T_{i+1} + 5\lambda_i) - 9e^{-\frac{\Delta_i}{\lambda_i}} (2T_i + \lambda_i)\right)
\end{aligned} \tag{13}$$

For $i = n$, the expectation is

$$\mathbb{E}_{n,n}[s_3] = T_n + \frac{\lambda_n}{3} \tag{14}$$

Double-checking:

$$\begin{aligned}
\mathbb{E}_{n,n}[t_3] &= \mathbb{E}[t_3 | t_3 \in [T_n, \infty), t_2 \in [T_n, \infty)] \\
&= \frac{1}{\pi_{n,n}} \int_{T_n}^{\infty} \int_{t_3}^{\infty} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n^2} \int_{T_n}^{\infty} \int_{t_3}^{\infty} t_3 e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n^2} \int_{T_n}^{\infty} t_3 e^{-3\Omega(0,t_3)} \int_{t_3}^{\infty} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n} \int_{T_n}^{\infty} t_3 e^{-3\Omega(0,t_3)} dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n} e^{-3\Omega(0,T_n)} \int_{T_n}^{\infty} t_3 e^{-3\Omega(T_n,t_3)} dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n} e^{-3\Omega(0,T_n)} \int_{T_n}^{\infty} t_3 e^{-\frac{3(t_3-T_n)}{\lambda_n}} dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n} e^{-3\Omega(0,T_n)} \frac{\lambda_n}{9} (3T_n + \lambda_n) \\
&= \frac{1}{3\pi_{n,n}} e^{-3\Omega(0,T_n)} (3T_n + \lambda_n)
\end{aligned} \tag{15}$$

Checks out since $\pi_{n,n} = \exp(-3\Omega(0, T_n))$.

$$\begin{aligned}
\mathbb{E}_{i,i}[t_2] &= \mathbb{E}[t_2 | t_3 \in [T_i, T_{i+1}), t_2 \in [T_i, T_{i+1}]] \\
&= \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_2 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{3t_2}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_2 e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} t_2 e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} t_2 e^{-\frac{t_2-t_3}{\lambda_i}} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \lambda_i \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} (T_{i+1} + \lambda_i) \right] dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \lambda_i \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} (T_{i+1} + \lambda_i) \right] dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} (T_{i+1} + \lambda_i) \right] dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{18} \left(e^{-\frac{3\Delta_i}{\lambda_i}} (3T_{i+1} + \lambda_i) + 6T_i + 8\lambda_i - 9e^{-\frac{\Delta_i}{\lambda_i}} (T_{i+1} + \lambda_i) \right) \\
&= \frac{1}{6\pi_{i,i}} e^{-3\Omega(0,T_i)} \left(e^{-\frac{3\Delta_i}{\lambda_i}} (3T_{i+1} + \lambda_i) + 6T_i + 8\lambda_i - 9e^{-\frac{\Delta_i}{\lambda_i}} (T_{i+1} + \lambda_i) \right)
\end{aligned} \tag{16}$$

For $i = n$, this expectation is

$$\mathbb{E}_{n,n}[s_2] = T_n + \frac{\lambda_n}{3} + \lambda_n \tag{17}$$

3.1 Discrete $q((k, l) | (i, j))$ transition function

To calculate the discrete-process transition probabilities from (i, j) , to (k, l) , we integrate the transition kernel (6) over the interval corresponding to (k, l) , replacing s_3 and s_2 with their conditional expectations $\mathbb{E}_{i,j}[s_3]$ and $\mathbb{E}_{i,j}[s_2]$ respectively. Thus

$$q((k, l) | (i, j)) = \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} q((t_3, t_2) | (\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])) dt_2 dt_3. \tag{18}$$

Note that in any single transition, either the first or second coalescence time changes, but not both. This simplifies the calculation of these integrals.

3.1.1 Case A

For $i = k < j < l$, ($t_3 = s_3; t_2 > s_2$):

$$\begin{aligned}
&= \frac{1}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\
&\quad \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \times \\
&\quad \left\{ e^{-2\Omega(T_{\alpha(s_3)+1}, s_2)} \left[1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{i=\alpha(s_3)+1}^{\alpha(s_2)-1} e^{-2\Omega(T_{i+1}, s_2)} \left[1 - e^{-\frac{2\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{2} \right. \\
&\quad \left. + \left[1 - e^{-\frac{2(s_2 - T_{\alpha(s_2)})}{\lambda_{\alpha(s_2)}}} \right] \frac{\lambda_{\alpha(s_2)}}{2} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \delta(t_3 - s_3) dt_2 dt_3 + \\
&\quad \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \right. \\
&\quad \left. \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} \delta(t_3 - s_3) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} dt_2 + \\
&\quad \int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \right. \\
&\quad \left. \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} dt_2 + \\
&\quad \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \right. \\
&\quad \left. \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times
\end{aligned}$$

$$\begin{aligned}
& \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \right. \\
& \left. \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 + \\
& = \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
& \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \right. \\
& \left. \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2 \\
& = \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
& \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \lambda_l \left(1 - e^{-\frac{\Delta_l}{\lambda_l}} \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \right. \\
& \left. \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \lambda_l \left(1 - e^{-\frac{\Delta_l}{\lambda_l}} \right)
\end{aligned}$$

Here we assume that $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

3.1.2 Case B

For $i = k < l < j$ ($t_3 = s_3; t_2 < s_2$):

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{1}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-2\Omega(s_3, t_2)} \\
&\quad \left\{ \sum_{a=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \delta(t_3 - s_3) dt_2 dt_3 + \\
&\int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} \\
&\quad \left\{ e^{-2\Omega(T_{\alpha(s_3)+1}, t_2)} \left[1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{a=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right. \\
&\quad \left. + \left[1 - e^{-\frac{2(t_2 - T_{\alpha(t_2)})}{\lambda_{\alpha(t_2)}}} \right] \frac{\lambda_{\alpha(t_2)}}{2} \right\} \delta(t_3 - s_3) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
&\quad \left\{ e^{-2\Omega(T_{i+1}, t_2)} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right. \\
&\quad \left. + \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] \frac{\lambda_l}{2} \right\} dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(e^{-2\Omega(T_{i+1}, t_2)} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right) dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] \frac{\lambda_l}{2} dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} dt_2 + \\
&\frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_{i+1}, t_2)} dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right) dt_2 +
\end{aligned}$$

$$\begin{aligned}
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
& \quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_{i+1}, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \int_{T_l}^{T_{l+1}} \left(\sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right) dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
& \quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_{i+1}, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_{a+1}, t_2)} dt_2 \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
& \quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_l)} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_l, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) e^{-2\Omega(T_{i+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_l, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} e^{-2\Omega(T_{a+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_l, t_2)} dt_2 \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2 - T_l)}{\lambda_l}} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) e^{-2\Omega(T_{i+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2 - T_l)}{\lambda_l}} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} e^{-2\Omega(T_{a+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2 - T_l)}{\lambda_l}} dt_2 \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
& = \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
& \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) e^{-2\Omega(T_{i+1}, T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left[\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} e^{-2\Omega(T_{a+1}, T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) \right] + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \left[\Delta_l - \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) \right]
\end{aligned}$$

3.1.3 Case C

For $k < i = l < j$ ($t_3 < s_3; t_2 = s_3$):

$$\begin{aligned}
&= \frac{1}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\} \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{i=0}^{k-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} \delta(t_2 - E_{i,j}[s_3]) dt_2 dt_3 \\
&= \int_{T_k}^{T_{k+1}} \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} dt_3 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right\} \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left[1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right] \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\}
\end{aligned}$$

3.1.4 Case D

For $k < i < j = l$ ($t_3 < s_3; t_2 = s_2$):

$$\begin{aligned}
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} \delta(t_2 - E_{i,j}[s_2]) dt_2 dt_3 \\
&= \int_{T_k}^{T_{k+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} dt_3 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right\} \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left[1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right] \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\}
\end{aligned}$$

3.1.5 Case E

For $i < k < j = l$ ($t_3 > s_3; t_2 = s_2$):

$$\begin{aligned}
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} e^{-2\Omega(s_3, t_3)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_3)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \delta(t_2 - \mathbb{E}_{i,j}[s_2]) dt_2 dt_3 \\
&= \int_{T_k}^{T_{k+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_3)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} dt_3 \\
&= \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_k}^{T_{k+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_3)} dt_3 \\
&= \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_k)} \int_{T_k}^{T_{k+1}} e^{-2\Omega(T_k, t_3)} dt_3 \\
&= \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_k)} \int_{T_k}^{T_{k+1}} e^{-\frac{2(t_3 - T_k)}{\lambda_k}} dt_3 \\
&= \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_k)} \frac{\lambda_k}{2} \left[1 - e^{-\frac{2\Delta_k}{\lambda_k}} \right]
\end{aligned}$$

3.1.6 Case F

For $i < j = k < l$ ($t_3 = s_2; t_2 > s_2$):

$$\begin{aligned}
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(E_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \delta(t_3 - s_2) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(E_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} dt_2 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(E_{i,j}[s_2], t_2)} dt_2 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(E_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(E_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(E_{i,j}[s_2], T_l)} \lambda_l \left[1 - e^{-\frac{\Delta_l}{\lambda_l}} \right]
\end{aligned}$$

Here again we assume $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

The following cases require special consideration:

3.1.7 Case G

For $i = k = l < j$ and $t_3 = s_3; t_2 < s_2$ (also need to consider $t_3 < s_3; t_2 = s_3$)

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3, t_2)} \delta(t_3 - s_3) + 2 \int_{s_3}^{t_2} \frac{du}{2s_2 + s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u, t_2)} \delta(t_3 - s_3) \right) dt_2 dt_3 \\
&= \int_{s_3}^{T_{k+1}} \left(\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3, t_2)} + 2 \int_{s_3}^{t_2} \frac{du}{2s_2 + s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u, t_2)} \right) dt_2 \\
&= \int_{E_{i,j}[s_3]}^{T_{k+1}} \left(\int_0^{E_{i,j}[s_3]} \frac{du}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} \frac{1}{\lambda_k} e^{-2\Omega(E_{i,j}[s_3], t_2)} + \right. \\
&\quad \left. 2 \int_{E_{i,j}[s_3]}^{t_2} \frac{du}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-2\Omega(u, t_2)} \right) dt_2 \\
&= \int_{E_{i,j}[s_3]}^{T_{k+1}} \left(\int_0^{E_{i,j}[s_3]} \frac{du}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} \frac{1}{\lambda_k} e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} + \right. \\
&\quad \left. 2 \int_{E_{i,j}[s_3]}^{t_2} \frac{du}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-\frac{2(t_2 - u)}{\lambda_k}} \right) dt_2 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \int_{E_{i,j}[s_3]}^{T_{k+1}} \left(\frac{1}{\lambda_k} e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} \int_0^{E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} du + \right. \\
&\quad \left. \frac{2}{\lambda_k} \int_{E_{i,j}[s_3]}^{t_2} e^{-\frac{2(t_2 - u)}{\lambda_k}} \right) dt_2 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \int_{E_{i,j}[s_3]}^{T_{k+1}} \left(\frac{1}{\lambda_k} e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} \int_0^{E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} du + \right. \\
&\quad \left. \frac{2}{\lambda_k} \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} \right] \right) dt_2 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \left(\frac{1}{\lambda_k} \int_0^{E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} du \int_{E_{i,j}[s_3]}^{T_{k+1}} e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} dt_2 + \right. \\
&\quad \left. \int_{E_{i,j}[s_3]}^{T_{k+1}} \left[1 - e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} \right] dt_2 \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \left(\frac{1}{\lambda_k} \int_0^{E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} du \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1} - E_{i,j}[s_3])}{\lambda_k}} \right] + \right. \\
&\quad \left. T_{k+1} - E_{i,j}[s_3] - \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1} - E_{i,j}[s_3])}{\lambda_k}} \right] \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \left\{ \frac{1}{\lambda_k} \times \right. \\
&\quad \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left(1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right) \frac{\lambda_a}{3} + \left(1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right) \frac{\lambda_i}{3} \right] \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1} - E_{i,j}[s_3])}{\lambda_k}} \right] + \right. \\
&\quad \left. T_{k+1} - E_{i,j}[s_3] - \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1} - E_{i,j}[s_3])}{\lambda_k}} \right] \right\}
\end{aligned}$$

3.1.8 Case G2

For $i = k = l < j$ and $t_3 < s_3; t_2 = s_3$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \delta(t_2 - s_3) \right) dt_2 dt_3 \\
&= \int_{T_k}^{s_3} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \right) dt_3 \\
&= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_3} \frac{2}{\lambda(t_3)} \int_0^{t_3} e^{-3\Omega(u, t_3)} du dt_3 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \int_{T_k}^{E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \int_0^{t_3} e^{-3\Omega(u, t_3)} du dt_3 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \int_{T_k}^{E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{E_{i,j}[s_3]} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{E_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{E_{i,j}[s_3]} e^{-3\Omega(T_k, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{E_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{E_{i,j}[s_3]} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{E_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_k)}{\lambda_k}} \right] + \right. \\
&\quad \left. \frac{\lambda_k}{3} \left(E_{i,j}[s_3] - T_k - \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_k)}{\lambda_k}} \right] \right) \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_k)}{\lambda_k}} \right] + \right.
\end{aligned}$$

$$\frac{\lambda_k}{3} \left(E_{i,j}[s_3] - T_k - \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_k)}{\lambda_k}} \right] \right)$$

3.1.9 Case H

Another case that requires special consideration is $i < k = l = j$. For $i < k = l = j$ and $t_3 > s_3; t_2 = s_2$:

$$\begin{aligned} &= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} \delta(t_2 - s_2) \right) dt_2 dt_3 \\ &= \int_{T_k}^{s_2} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_2} \left(2 \int_0^{s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} du \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_2} \left(\frac{4}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} \int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) e^{-2\Omega(s_3, T_k)} \int_{T_k}^{s_2} \frac{4}{\lambda(t_3)} e^{-2\Omega(T_k, t_3)} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \int_{T_k}^{s_2} e^{-2\Omega(T_k, t_3)} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \int_{T_k}^{s_2} e^{-\frac{2(t_3 - T_k)}{\lambda_k}} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}} \right] \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}} \right] \\ &= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{4}{\lambda_k} e^{-2\Omega(E_{i,j}[s_3], T_k)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\ &\quad \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(E_{i,j}[s_2] - T_k)}{\lambda_k}} \right] \end{aligned}$$

3.1.10 Case H2

For $i < k = l = j$ and $t_3 = s_2; t_2 > s_2$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \delta(t_3 - s_2) \right) dt_2 dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \int_{s_2}^{T_{k+1}} \left(e^{-\Omega(s_2, t_2)} \int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) dt_2 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \left(\int_{s_2}^{T_{k+1}} e^{-\Omega(s_2, t_2)} dt_2 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \left(\int_{s_2}^{T_{k+1}} e^{-\Omega(s_2, t_2)} dt_2 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \left(\int_{s_2}^{T_{k+1}} e^{-\frac{t_2 - s_2}{\lambda_k}} dt_2 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \left(\lambda_k \left[1 - \delta(k - n) e^{-\frac{T_{k+1} - s_2}{\lambda_k}} \right] \right) \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \lambda_k \left[1 - \delta(k - n) e^{-\frac{T_{k+1} - E_{i,j}[s_2]}{\lambda_k}} \right]
\end{aligned}$$

The delta function is a way to ensure correctness when $k = l = j = n$.

Another special case that requires attention is $i = j = k < l$.

3.1.11 Case I

For $i = j = k < l$ and $t_3 = s_3; t_2 > s_2$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \delta(t_3 - s_3) + \right. \\
&\quad \left. 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} e^{-2\Omega(u, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \delta(t_3 - s_3) \right) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \left(\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} + \right. \\
&\quad \left. 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} e^{-2\Omega(u, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} + \right. \\
&\quad \left. 2 \int_{s_3}^{s_2} e^{-2\Omega(u, s_2)} du \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \int_0^{s_3} e^{-3\Omega(u, s_3)} du + \right. \\
&\quad \left. \frac{2}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \int_{s_3}^{s_2} e^{-2\Omega(u, s_2)} du \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2, t_2)} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} e^{-\Omega(s_2, t_2)} \int_{s_3}^{s_2} e^{-2\Omega(u, s_2)} du \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2, t_2)} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} e^{-\Omega(s_2, t_2)} \int_{s_3}^{s_2} e^{-\frac{2(s_2 - u)}{\lambda_i}} du \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2, t_2)} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} e^{-\Omega(s_2, t_2)} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2 - s_3)}{\lambda_i}} \right] \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] \int_{T_l}^{T_{l+1}} e^{-\Omega(s_2, t_2)} dt_2 + \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2-s_3)}{\lambda_i}} \right] \int_{T_l}^{T_{l+1}} e^{-\Omega(s_2, t_2)} dt_2 \Bigg) \\
&= \frac{1}{2s_2 + s_3} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3-T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 + \right. \\
&\quad \left. \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2-s_3)}{\lambda_i}} \right] e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 \right) \\
&= \frac{1}{2s_2 + s_3} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3-T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2-T_l}{\lambda_l}} dt_2 + \right. \\
&\quad \left. \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2-s_3)}{\lambda_i}} \right] e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2-T_l}{\lambda_l}} dt_2 \right) \\
&= \frac{1}{2s_2 + s_3} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3-T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] e^{-\Omega(s_2, T_l)} \lambda_l \left[1 - e^{-\frac{\Delta_l}{\lambda_l}} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2-s_3)}{\lambda_i}} \right] e^{-\Omega(s_2, T_l)} \lambda_l \left[1 - e^{-\frac{\Delta_l}{\lambda_l}} \right] \right) \\
&= \frac{1}{2\mathbb{E}_{i,i}[s_2] + \mathbb{E}_{i,i}[s_3]} \left(e^{-2\Omega(\mathbb{E}_{i,i}[s_3], \mathbb{E}_{i,i}[s_2])} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,i}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,i}[s_3]-T_l)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] \times \right. \\
&\quad \left. e^{-\Omega(\mathbb{E}_{i,i}[s_2], T_l)} \lambda_l \left[1 - \delta(l-n)e^{-\frac{\Delta_l}{\lambda_l}} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(\mathbb{E}_{i,i}[s_2]-\mathbb{E}_{i,i}[s_3])}{\lambda_i}} \right] e^{-\Omega(\mathbb{E}_{i,i}[s_2], T_l)} \lambda_l \left[1 - \delta(l-n)e^{-\frac{\Delta_l}{\lambda_l}} \right] \right)
\end{aligned}$$

Again, notice the delta function.

3.1.12 Case I2

For $i = j = k < l$ and $t_3 = s_2; t_2 > s_2$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \delta(t_3 - s_2) \right) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \right) dt_2 \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \int_{T_l}^{T_{l+1}} \left(e^{-\Omega(s_2, t_2)} \int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) dt_2 \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(\int_{T_l}^{T_{l+1}} e^{-\Omega(s_2, t_2)} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2, T_l)} \lambda_l \left[1 - \delta(l - n) e^{-\frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2, T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \right) \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \\
&= \frac{2}{2E_{i,i}[s_2] + E_{i,i}[s_3]} e^{-2\Omega(E_{i,i}[s_3], E_{i,i}[s_2])} \frac{1}{\lambda_l} \left(e^{-\Omega(E_{i,i}[s_2], T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \right) \times \\
&\quad \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,i}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,i}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right)
\end{aligned}$$

Another case that requires special consideration: $k < i = j = l$. This includes either $t_3 < s_3; t_2 = s_2$ or $t_3 < s_3; t_2 = s_3$.

3.1.13 Case J

For $k < i = j = l$ and $t_3 < s_3; t_2 = s_2$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(2 \int_0^{t_3} \frac{du}{2s_2 + s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u, t_3)} \delta(t_2 - s_2) \right) dt_2 dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{t_3} e^{-3\Omega(u, t_3)} \delta(t_2 - s_2) \right) dt_2 dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} e^{-3\Omega(u, t_3)} \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left[\left(\int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \right) + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right] \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \right) + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\} \\
&= \frac{2}{2E_{i,i}[s_2] + E_{i,i}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \right) + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\}
\end{aligned}$$

3.1.14 Case J2

For $k < i = j = l$ and $t_3 < s_3; t_2 = s_3$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \delta(t_2 - s_3) \right) dt_2 dt_3 \\
&= \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} e^{-3\Omega(u, t_3)} du \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right] + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right)
\end{aligned}$$

This finishes the $q((k, l)|(i, j))$ transitions. Need to give special attention to the boundary cases, or the cases where $i = j$. Also need to consider what happens when k or l is n , since $T_{n+1} = \infty$.

The (diagonal) case of $k = i; l = j$ is calculated by subtracting the sum of the off-diagonal entries from unity.

3.2 Emission probabilities

The genotype at a particular position in a triploid genome can take one of three different values: 0, 1, and 2. The state 0 represents a homozygous site, and 1 (2) represent sites where one (two) of the three chromosomes have a derived (*i.e.*, non-ancestral) copy at that position.

To form our observed chain, we consider all the genotypes in a stretch of b bp and categorize that stretch of the sequence with a state 0, 1, 2, or 3. The state 0 means that the stretch of b bp is completely homozygous. The state 1 means that there was at least one site that had a 1 genotype and none that had a 2 genotype. Likewise, the state 2 means that at least one site had a 2 genotype, and none had a 1 genotype. The state 3 means that at least one site had a 1 genotype and at least one site had a 2 genotype.

With observed states coded this way, the emission probabilities given local coalescence times t_3 and t_2 are

$$e_k(t_3, t_2, T_d) = \begin{cases} e^{-\frac{\theta b(2t_2 + t_3 + 3T_d)}{2}} & k = 0 \\ e^{-\frac{\theta b(t_2 - t_3)}{2}} \left(1 - e^{-\frac{\theta b(2t_3 + t_2 + 3T_d)}{2}} \right) & k = 1 \\ \left(1 - e^{-\frac{\theta b(t_2 - t_3)}{2}} \right) e^{-\frac{\theta b(2t_3 + t_2 + 3T_d)}{2}} & k = 2 \\ \left(1 - e^{-\frac{\theta b(t_2 - t_3)}{2}} \right) \left(1 - e^{-\frac{\theta b(2t_3 + t_2 + 3T_d)}{2}} \right) & k = 3. \end{cases} \quad (19)$$

Here T_d is the divergence time, the time in the past [again measured in units of $2N(0)$ generations] when the asexual lineage was derived from a sexual ancestor. The above probabilities assume that t_3 and t_2 are measured continuously. In practice, we discretize time, so for a particular hidden state (i, j) , we replace t_3 and t_2 with $E_{i,j}[t_3]$ and $E_{i,j}[t_2]$, respectively.

Classifying states and genotypes this way requires that each polymorphism be polarized against an outgroup. If this is not possible, then the states can be recoded as 0 and 1, where 0 is a stretch of b completely homozygous base pairs, and 1 is a stretch of b base pairs with at least one polymorphic position. If no polarization is possible, the emission probabilities become

$$e_k(t_3, t_2, T_d) = \begin{cases} e^{-\frac{\theta b(2t_2+t_3+3T_d)}{2}} & k = 0 \\ 1 - e^{-\frac{\theta b(2t_2+t_3+3T_d)}{2}} & k = 1. \end{cases} \quad (20)$$

A few things to note: The parameter b can be tuned to match the observed polymorphism. If the change in ploidy T_d generations ago also involved a change in mutation rate, this new mutation rate will be unidentifiable, impossible to distinguish from a proportionally scaled T_d . Thus T_d should be viewed as a compound parameter.

4 HMM inference

The above equations for $q((k, l)|(i, j))$ define a transition matrix $\{P_{(i,j),(k,l)}\}$, where

$$P_{(i,j),(k,l)} = \left(1 - e^{-\frac{\rho(2E_{i,j}[t_2] + E_{i,j}[t_3])}{2}}\right) q((k, l)|(i, j)) \quad (21)$$

is the probability of transitioning from state (i, j) to state (k, l) , for $(i, j) \neq (k, l)$. We define a hidden Markov chain $\{X_i\}$ that is governed by this transition matrix. The observed process $\{Y_i\}$ represents the different emissions, 0 through 3. The EM algorithm proceeds by starting with some initial parameters θ and iteratively maximizing the expectation of the full likelihood given the data, which is

$$P(X, Y|\theta) = e_{x_1}(y_1) \pi_{x_1} \prod_{i=1}^{T-1} P_{x_i, x_{i+1}} e_{x_{i+1}}(y_{i+1}), \quad (22)$$

where y_i is the observed state at position i , and x_i is the state of the hidden chain at position i , and T is the length of the sequence. In practice we maximize the log-likelihood:

$$\log P(X, Y|\theta) = \log(e_{x_1}(y_1)) + \log(\pi_{x_1}) + \sum_{i=1}^{T-1} \log(P_{x_i, x_{i+1}}) + \log(e_{x_{i+1}}(y_{i+1})). \quad (23)$$

Since we observe only the observed chain (i.e., the mutation data), we have to have some way of integrating over the states of the hidden chain. We do this by the expectation-maximization (EM) algorithm, paired with the forward-backward algorithm for calculating likelihoods with these chains. In the context of HMM's, the EM algorithm iteratively maximizes

$$\begin{aligned} E_{X|Y}[\log P(X, Y|\theta)] &= E_{X|Y}[\log e_{x_1}(y_1)] + E_{X|Y}[\log \pi_{x_1}] + \\ &E_{X|Y}[\log e_{x_1}(y_1)] + \sum_{i=1}^{T-1} \left(E_{X|Y}[\log P_{x_i, x_{i+1}}] + E_{X|Y}[\log(e_{x_{i+1}}(y_{i+1}))] \right), \end{aligned}$$

updating the parameters of the chains in each iteration.

Define the forward variable

$$\alpha_i(t) = P(Y_1 = y_1, Y_2 = y_2, \dots, Y_t = y_t, X_t = i | \theta), \quad (24)$$

which satisfies the recursion

$$\alpha_j(t+1) = \sum_{i=1}^N \alpha_i(t) P_{i,j} e_j(y_{t+1}), \quad (25)$$

where $N = (n+1)(n+2)/2$ is the number of states in the hidden chain. We define

$$\alpha_i(1) = \pi_i e_i(y_1). \quad (26)$$

Define the backward variables

$$\beta_i(t) = P(Y_{t+1} = y_{t+1}, Y_{t+2} = y_{t+2}, \dots, Y_T = y_T | X_t = i, \theta), \quad (27)$$

which satisfy the recursion

$$\beta_i(t) = \sum_{j=1}^N \beta_j(t+1) P_{i,j} e_j(y_{t+1}), \quad (28)$$

defining $\beta_i(T) = 1$.

From these, we can calculate

$$\begin{aligned} \gamma_i(t) &= P(X_t = i | Y, \theta) \\ &= \frac{P(X_t = i, Y | \theta)}{P(Y)} \\ &= \frac{\alpha_i(t) \beta_i(t)}{\sum_{j=1}^N \alpha_j(t) \beta_j(t)} \end{aligned}$$

and

$$\begin{aligned} \xi_{i,j}(t) &= P(X_t = i, X_{t+1} = j | Y, \theta) \\ &= \frac{\alpha_i(t) P_{i,j} e_j(t+1) \beta_j(t+1)}{P(Y | \theta)} \\ &= \frac{\alpha_i(t) P_{i,j} e_j(t+1) \beta_j(t+1)}{\sum_{k=1}^N \alpha_k(T)} \end{aligned}$$

With these probabilities, we can calculate $E_{X|Y}[\log(P(X, Y | \theta))]$:

$$\begin{aligned} E_{X|Y}[\log P(X, Y | \theta)] &= E_{X|Y}[\log e_{x_1}(y_1)] + E_{X|Y}[\log \pi_{x_1}] + \\ &\quad \sum_{i=1}^{T-1} \left(E_{X|Y}[\log P_{x_i, x_{i+1}}] + E_{X|Y}[\log(e_{x_{i+1}}(y_{i+1}))] \right) \\ &= \sum_{j=1}^N \gamma_j(1) \log \pi_j + \sum_{t=1}^T \sum_{j=1}^N \log[e_j(y_t)] \gamma_j(t) + \\ &\quad \sum_{t=1}^{T-1} \sum_{i=1}^N \sum_{j=1}^N \log(P_{i,j}) \xi_{i,j}(t) \end{aligned}$$

The expected number of emissions of state $k \in \{0, 1, 2, 3\}$ from (hidden) state i is

$$E_i(k) = \sum_{t=1}^T \gamma_i(t) \mathbb{I}(y_t = k), \quad (29)$$

and the expected number of transitions from hidden state i to hidden state j is

$$E_{i,j} = \sum_{t=1}^{T-1} \xi_{i,j}(t). \quad (30)$$

Then the expected log-likelihood can be written

$$\begin{aligned} E_{X|Y} [\log P(X, Y|\theta)] &= \sum_{j=1}^N \gamma_j(1) \log \pi_j + \sum_{j=1}^N \sum_{k=0}^3 \log [e_j(k)] E_j(k) + \\ &\quad \sum_{i=1}^N \sum_{j=1}^N \log(P_{i,j}) E_{i,j} \end{aligned}$$

4.1 Underflow

In order to avoid underflow problems, we calculate scaled versions of $\alpha_i(t)$ and $\beta_{i,j}(t)$. We define

$$\tilde{\alpha}_i(t) = \frac{\alpha_i(t)}{\prod_{j=1}^t s_j}, \quad (31)$$

for some scaling constants s_i , from which we can see the new recursion

$$\tilde{\alpha}_j(t+1) = \frac{1}{s_{t+1}} e_j(y_{t+1}) \sum_{i=1}^N P_{i,j} \tilde{\alpha}_i(t). \quad (32)$$

It is convenient to define s_t such that $\sum_{j=1}^N \tilde{\alpha}_j(t) = 1$. This means $s_1 = \sum_{k=1}^N \pi_i e_i(y_1)$ and

$$s_{t+1} = \sum_{j=1}^N e_j(y_{t+1}) \sum_{i=1}^N P_{i,j} \tilde{\alpha}_i(t) \quad (33)$$

We scale $\beta_i(t)$ by the same s_i so that

$$\tilde{\beta}_i(t) = \frac{\beta_i(t)}{\prod_{j=t+1}^T s_j} \quad (34)$$

and

$$\tilde{\beta}_i(t) = \frac{1}{s_{t+1}} \sum_{j=1}^N \tilde{\beta}_j(t+1) P_{i,j} e_j(y_{t+1}), \quad (35)$$

with $\tilde{\beta}_i(T) = \beta_i(T) = 1$.

Now $\gamma_i(t)$ can be written

$$\begin{aligned} \gamma_i(t) &= \frac{\alpha_i(t) \beta_i(t)}{\sum_{j=1}^N \alpha_j(t) \beta_j(t)} \\ &= \frac{\tilde{\alpha}_i(t) \tilde{\beta}_i(t) \left(\prod_{j=1}^T s_j \right)}{\sum_{j=1}^N \tilde{\alpha}_j(t) \tilde{\beta}_j(t) \left(\prod_{j=1}^T s_j \right)} \\ &= \frac{\tilde{\alpha}_i(t) \tilde{\beta}_i(t)}{\sum_{j=1}^N \tilde{\alpha}_j(t) \tilde{\beta}_j(t)} \end{aligned} \quad (36)$$

and $\xi_{i,j}(t)$ can now be written

$$\begin{aligned}
\xi_{i,j}(t) &= \frac{\alpha_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)}{P(Y|\theta)} \\
&= \frac{\tilde{\alpha}_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)\prod_{k=1}^T s_k}{P(Y|\theta)s_{t+1}} \\
&= \frac{\tilde{\alpha}_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)\prod_{k=1}^T s_k}{\sum_{l=1}^N \alpha_l(t)\beta_l(t)s_{t+1}} \\
&= \frac{\tilde{\alpha}_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)}{s_{t+1}\sum_{l=1}^N \tilde{\alpha}_l(t)\tilde{\beta}_l(t)} \\
&= \frac{\gamma_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)}{s_{t+1}\beta_i(t)}
\end{aligned} \tag{37}$$

Difficult to implement, but efficient and apparently correct!

5 Inferring diploid-to-triploid transition

In this section we consider the possibility that the asexual triploids were generated first by an asexual diploid line that is subsequently fertilized by a haploid sperm, making a triploid asexual lineage. The goal is to infer the time between this diploid asexual phase and the triploid asexual phase, along with everything else that was considered for inference previously. In this case, the state at each position along the genome is the vector (t_3, t_2, W) , where $W \in \{0, 1\}$ is an indicator for whether the branch introduced by the triploidizing sperm subtends one of the diploid sexual branches coalescing at t_3 ($W = 1$) or t_2 ($W = 0$). The transition to diploid asexuality is defined as $D_2 = 0$, and the transition to triploid asexuality is defined as $D_3 < 0$. See Fig. 1 for an illustration of these states.

Having to keep track of W doubles the number of states, and since the complexity of the forward-backward algorithm is squared in the number of states, the runtime should increase by ~ 4 .

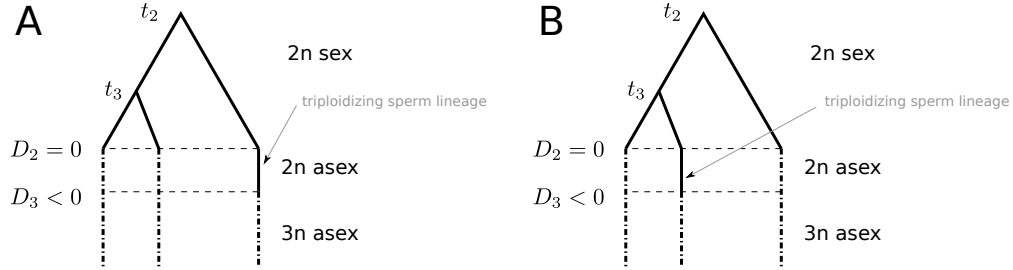


Figure 1: Figure 1. Two states with the same coalescence times t_3 and t_2 but with the triploidizing sperm's lineage subtending a branch that coalesces with the others (in the diploid sexual phase) at t_2 (panel A) and at t_3 (panel B). $D_2 = 0$ is the time of transition to diploid asexuality, and $D_3 < 0$ is the time of the transition to triploidy.

5.1 Diploid-to-triploid transition probabilities

The transition probabilities of the hidden model largely remain the same. A few changes occur:

1. In all previous probabilities, the factor $(2t_2 + t_3)^{-1}$ is replaced by the inverse of the new total (sexual) tree length, $(2t_2 + t_3 - D_3)^{-1}$, recalling that $D_3 < 0$. These factors are included unchanged all the way through the derivations and are still present in the final discrete transition probabilities, so they can just be replaced with the new factor.

2. Certain transitions will now change W as well. For example, if $W = 0$, (recall, subtending a branch that coalesces at t_2), the transition $(s_3, s_2) \rightarrow (t_3 < s_3, t_2 = s_3)$ implies that now $W = 1$.
3. Additional transition probability must be included, arising from recombination events that occur on the triploidizing sperm's sexual lineage between $t = D_3$ and $t = 0$.

To do this properly, it will also now be necessary to model the population size changes that occur in the sexual population during the diploid asexual phase, since the triploidizing sperm's lineage will experience those demographic changes. However, the only probabilities that depend on these population sizes are the “healing” probabilities specific to the SMC’, meaning that only the “effective recombination rate” for this single lineage will depend on this part of the demographic history. **For this reason, we will assume that the diploid sexual population containing the triploidizing sperm is constant in size.** [This approximation would be unnecessary under the SMC model (vs. the SMC’), since healing is impossible under the SMC.] It should be a fine approximation and will have to be tested.

In general, we can expect there to be very little information about the length of this hypothesized diploid asexual interval in the triploid asexual's present-day genome. Including this additional period of diploid asexuality doesn't change the emission probabilities at all; it only changes the transition probabilities of the hidden model, and these changes seem like they will be pretty minor. However, it may be possible to rule out long periods of diploid asexuality in the putatively ancient lineages.