1 General equations

Define $N(t)$ as the population size at time t, with t in units of $2N(0)$. Let $\lambda(t)$ be the relative population size, scaled by $N(0)$, such that $N(t) = N(0)\lambda(t)$. Define $\Omega(u,v)$ as the cumulative coalescent rate between times u and v:

$$\Omega(u,v) = \int_u^v \frac{dt}{\lambda(t)}. \tag{1}$$

The state of the TSMC at each point along the genome is described by the vector $\boldsymbol{s} = (s_3, s_2)$, where s_3 is the time of the first coalescence event and s_2 is the time of the second coalescence event amongst the three lineages in a triploid genome. The equilibrium joint distribution of (t_3, t_2) is

$$\pi(t_3,t_2) = \frac{3}{\lambda(t_3)\lambda(t_2)} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)}. \tag{2}$$

Let $q(\boldsymbol{t}|\boldsymbol{s})$ be the transition kernel at recombination sites along the genome. Then

$q(\boldsymbol{t}|\boldsymbol{s}) =$

For $t_3 = s_3; t_2 > s_2$:

$$\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} + 2\int_{s_3}^{s_2} \frac{du}{2s_2+s_3} e^{-2\Omega(u,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)}$$

For $t_3 = s_3; t_2 < s_2$:

$$\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3,t_2)} + 2\int_{s_3}^{t_2} \frac{du}{2s_2+s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u,t_2)}$$

For $t_3 < s_3; t_2 = s_3$:

$$\int_0^{t_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,t_3)} \frac{2}{\lambda(t_3)}$$

For $t_3 < s_3; t_2 = s_2$:

$$2\int_0^{t_3} \frac{du}{2s_2+s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u,t_3)}$$

For $t_3 > s_3; t_2 = s_2$:

$$2\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3,t_3)}$$

For $t_3 = s_2; t_2 > s_2$:

$$2\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)}$$

For $t_3 = s_3; t_2 = s_2$:

$$3\int_0^{s_3} \frac{du}{2s_2+s_3} \frac{1}{3}\left[1 - e^{-3\Omega(u,s_3)}\right] + \int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} \frac{1}{2}\left[1 - e^{-2\Omega(s_3,s_2)}\right] + 2\int_{s_3}^{s_2} \frac{du}{2s_2+s_3} \frac{1}{2}\left[1 - e^{-2\Omega(u,s_2)}\right]. \tag{3}$$

Each part is implicitly multiplied by a delta function to limit the density to points where the parameters are assumed to be equal to each other. For example, the first part of $q(\boldsymbol{t}|\boldsymbol{s})$ is implicitly multiplied by $\delta(t_3 - s_3)$, and the last part is multiplied by $\delta(t_3 - s_3)\delta(t_2 - s_2)$.

2 Piecewise constant transition probabilities

Suppose that the population changes size at times $(T_1, \ldots, T_n)$ and that the size between T_i and T_{i+1} is a constant $2N\lambda_i$. Define $T_0 = 0$, $T_{n+1} = \infty$ and $\Delta_i = T_{i+1} - T_i$. Let $\alpha(t)$ be the index of the time interval to which t belongs, *i.e.*, $\alpha(t) = \max_i\{i : T_i \leq t\}$.

Then the cumulative coalescent rate between u and v can be written

$$\Omega(u, v) = \begin{cases} \frac{v-u}{\lambda_{\alpha(u)}} & \alpha(u) = \alpha(v) \\ \frac{T_{\alpha(u)+1} - u}{\lambda_{\alpha(u)}} + \sum_{i=\alpha(u)+1}^{\alpha(v)-1} \frac{\Delta_i}{\lambda_i} + \frac{v - T_{\alpha(v)}}{\lambda_{\alpha(v)}} & \alpha(u) < \alpha(v). \end{cases} \tag{4}$$

The equilibrium joint density of (t_3, t_2) is now approximately

$$\pi(t_3, t_2) = \frac{3}{\lambda_{\alpha(t_3)}\lambda_{\alpha(t_2)}} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)}. \tag{5}$$

There are several integrals of the form $\int_x^y e^{-k\Omega(u,y)}du$ in Equation (3). This integral can be written

$$
\begin{aligned}
\int_x^y e^{-k\Omega(u,y)}du &= \int_x^{T_{\alpha(x)+1}} e^{-k\Omega(u,y)}du + \sum_{i=\alpha(x)+1}^{\alpha(y)-1}\int_{T_i}^{T_i+1} e^{-k\Omega(u,y)}du + \int_{T_{\alpha(y)}}^{y} e^{-k\Omega(u,y)}du \\
&= \int_x^{T_{\alpha(x)+1}} \exp\left(-k\left[\left(T_{\alpha(u)+1}-u\right)\frac{1}{\lambda_{\alpha(u)}} + \sum_{j=\alpha(u)+1}^{\alpha(y)-1}\frac{\Delta_j}{\lambda_j} + (y-T_{\alpha(y)})\frac{1}{\lambda_{\alpha(y)}}\right]\right)du + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1}\int_{T_i}^{T_{i+1}} \exp\left(-k\left[\left(T_{\alpha(u)+1}-u\right)\frac{1}{\lambda_{\alpha(u)}} + \sum_{j=\alpha(u)+1}^{\alpha(y)-1}\frac{\Delta_j}{\lambda_j} + (y-T_{\alpha(y)})\frac{1}{\lambda_{\alpha(y)}}\right]\right)du + \\
&\quad \int_{T_{\alpha(y)}}^{y} \exp\left(-k(y-u)\frac{1}{\lambda_{\alpha(u)}}\right)du \\
&= \int_x^{T_{\alpha(x)+1}} \exp\left(-k\left[\left(T_{\alpha(x)+1}-u\right)\frac{1}{\lambda_{\alpha(x)}} + \sum_{j=\alpha(x)+1}^{\alpha(y)-1}\frac{\Delta_j}{\lambda_j} + (y-T_{\alpha(y)})\frac{1}{\lambda_{\alpha(y)}}\right]\right)du + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1}\int_{T_i}^{T_{i+1}} \exp\left(-k\left[\left(T_{i+1}-u\right)\frac{1}{\lambda_i} + \sum_{j=i+1}^{\alpha(y)-1}\frac{\Delta_j}{\lambda_j} + (y-T_{\alpha(y)})\frac{1}{\lambda_{\alpha(y)}}\right]\right)du + \\
&\quad \int_{T_{\alpha(y)}}^{y} \exp\left(-k(y-u)\frac{1}{\lambda_{\alpha(y)}}\right)du \\
&= \exp\left(-k\left[\sum_{j=\alpha(x)+1}^{\alpha(y)-1}\frac{\Delta_j}{\lambda_j} + (y-T_{\alpha(y)})\frac{1}{\lambda_{\alpha(y)}}\right]\right)\int_x^{T_{\alpha(x)+1}} \exp\left(-k\left(T_{\alpha(x)+1}-u\right)\frac{1}{\lambda_{\alpha(x)}}\right)du + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \exp\left(-k\left[\sum_{j=i+1}^{\alpha(y)-1}\frac{\Delta_j}{\lambda_j} + (y-T_{\alpha(y)})\frac{1}{\lambda_{\alpha(y)}}\right]\right)\int_{T_i}^{T_{i+1}} \exp\left(-k\left(T_{i+1}-u\right)\frac{1}{\lambda_i}\right)du + \\
&\quad \int_{T_{\alpha(y)}}^{y} \exp\left(-k(y-u)\frac{1}{\lambda_{\alpha(y)}}\right)du \\
&= \exp\left(-k\left[\sum_{j=\alpha(x)+1}^{\alpha(y)-1}\frac{\Delta_j}{\lambda_j} + (y-T_{\alpha(y)})\frac{1}{\lambda_{\alpha(y)}}\right]\right)\left[1-\exp\left(-\frac{k\left(T_{\alpha(x)+1}-x\right)}{\lambda_{\alpha(x)}}\right)\right]\frac{\lambda_{\alpha(x)}}{k} + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \exp\left(-k\left[\sum_{j=i+1}^{\alpha(y)-1}\frac{\Delta_j}{\lambda_j} + (y-T_{\alpha(y)})\frac{1}{\lambda_{\alpha(y)}}\right]\right)\left[1-\exp\left(-\frac{k\Delta_i}{\lambda_i}\right)\right]\frac{\lambda_i}{k} + \\
&\quad \left[1-\exp\left(-\frac{k\left(y-T_{\alpha(y)}\right)}{\lambda_{\alpha(y)}}\right)\right]\frac{\lambda_{\alpha(y)}}{k} \\
&= \exp\left(-k\Omega(T_{\alpha(x)+1},y)\right)\left[1-\exp\left(-\frac{k\left(T_{\alpha(x)+1}-x\right)}{\lambda_{\alpha(x)}}\right)\right]\frac{\lambda_{\alpha(x)}}{k} + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \exp\left(-k\Omega(T_{i+1},y)\right)\left[1-\exp\left(-\frac{k\Delta_i}{\lambda_i}\right)\right]\frac{\lambda_i}{k} + \\
&\quad \left[1-\exp\left(-\frac{k\left(y-T_{\alpha(y)}\right)}{\lambda_{\alpha(y)}}\right)\right]\frac{\lambda_{\alpha(y)}}{k} \\
&= e^{-k\Omega(T_{\alpha(x)+1},y)}\left[1-e^{-\frac{k\left(T_{\alpha(x)+1}-x\right)}{\lambda_{\alpha(x)}}}\right]\frac{\lambda_{\alpha(x)}}{k} + \sum_{i=\alpha(x)+1}^{\alpha(y)-1} e^{-k\Omega(T_{i+1},y)}\left[1-e^{-\frac{k\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{k} + \left[1-e^{-\frac{k\left(y-T_{\alpha(y)}\right)}{\lambda_{\alpha(y)}}}\right]\frac{\lambda_{\alpha(y)}}{k}
\end{aligned} \tag{6}
$$

With this equation, we can calculate all of the transition probabilities in the transition kernel (3).

For $t_3 = s_3; t_2 > s_2$:

$$\frac{1}{2s_2+s_3}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_{\alpha(t_2)}}e^{-\Omega(s_2,t_2)}\left\{\sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1},s_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3\left(s_3-T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{3}\right\}+$$
$$\frac{2}{2s_2+s_3}\frac{1}{\lambda_{\alpha(t_2)}}e^{-\Omega(s_2,t_2)}\times$$
$$\left\{e^{-2\Omega(T_{\alpha(s_3)+1},s_2)}\left[1-e^{-\frac{2\left(T_{\alpha(s_3)+1}-s_3\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{2}+\sum_{i=\alpha(s_3)+1}^{\alpha(s_2)-1} e^{-2\Omega(T_{i+1},s_2)}\left[1-e^{-\frac{2\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{2}\right.$$
$$\left.+\left[1-e^{-\frac{2\left(s_2-T_{\alpha(s_2)}\right)}{\lambda_{\alpha(s_2)}}}\right]\frac{\lambda_{\alpha(s_2)}}{2}\right\}$$
$$=\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}+2\int_{s_3}^{s_2}\frac{du}{2s_2+s_3}e^{-2\Omega(u,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}$$

For $t_3 = s_3; t_2 < s_2$:

$$\frac{1}{2s_2+s_3}\frac{1}{\lambda_{\alpha(t_2)}}e^{-2\Omega(s_3,t_2)}\left\{\sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1},s_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3\left(s_3-T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{3}\right\}+$$
$$\frac{2}{2s_2+s_3}\frac{1}{\lambda_{\alpha(t_2)}}$$
$$\left\{e^{-2\Omega(T_{\alpha(s_3)+1},t_2)}\left[1-e^{-\frac{2\left(T_{\alpha(s_3)+1}-s_3\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{2}+\sum_{i=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{i+1},t_2)}\left[1-e^{-\frac{2\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{2}\right.$$
$$\left.+\left[1-e^{-\frac{2\left(t_2-T_{\alpha(t_2)}\right)}{\lambda_{\alpha(t_2)}}}\right]\frac{\lambda_{\alpha(t_2)}}{2}\right\}$$
$$=\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}\frac{1}{\lambda(t_2)}e^{-2\Omega(s_3,t_2)}+2\int_{s_3}^{t_2}\frac{du}{2s_2+s_3}\frac{1}{\lambda(t_2)}e^{-2\Omega(u,t_2)}$$

For $t_3 < s_3; t_2 = s_3$:

$$\frac{1}{2s_2+s_3}\frac{2}{\lambda_{\alpha(t_3)}}\left\{\sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1},t_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3\left(t_3-T_{\alpha(t_3)}\right)}{\lambda_{\alpha(t_3)}}}\right]\frac{\lambda_{\alpha(t_3)}}{3}\right\}$$
$$=\int_0^{t_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,t_3)}\frac{2}{\lambda(t_3)}$$

For $t_3 < s_3; t_2 = s_2$:

$$\frac{2}{2s_2+s_3}\frac{2}{\lambda_{\alpha(t_3)}}\left\{\sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1},t_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3\left(t_3-T_{\alpha(t_3)}\right)}{\lambda_{\alpha(t_3)}}}\right]\frac{\lambda_{\alpha(t_3)}}{3}\right\}$$
$$=2\int_0^{t_3}\frac{du}{2s_2+s_3}\frac{2}{\lambda(t_3)}e^{-3\Omega(u,t_3)}$$

For $t_3 > s_3; t_2 = s_2$:

$$\frac{2}{2s_2+s_3}\frac{2}{\lambda_{\alpha(t_3)}}e^{-2\Omega(s_3,t_3)}\left\{\sum_{i=0}^{\alpha(s_3)-1}e^{-3\Omega(T_{i+1},s_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3\left(s_3-T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{3}\right\}$$
$$=2\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}\frac{2}{\lambda(t_3)}e^{-2\Omega(s_3,t_3)}$$

For $t_3 = s_2; t_2 > s_2$:

$$\frac{2}{2s_2+s_3}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_{\alpha(t_2)}}e^{-\Omega(s_2,t_2)}\left\{\sum_{i=0}^{\alpha(s_3)-1}e^{-3\Omega(T_{i+1},s_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3\left(s_3-T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{3}\right\}$$
$$=2\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}$$

For $t_3 = s_3; t_2 = s_2$:

$$\frac{1}{2s_2+s_3}\left\{s_3-\sum_{i=0}^{\alpha(s_3)-1}e^{-3\Omega(T_{i+1},s_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}-\left[1-e^{-\frac{3\left(s_3-T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{3}\right\}+$$
$$\frac{1}{2s_2+s_3}\frac{1}{2}\left[1-e^{-2\Omega(s_3,s_2)}\right]\left\{\sum_{i=0}^{\alpha(s_3)-1}e^{-3\Omega(T_{i+1},s_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3\left(s_3-T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{3}\right\}+$$
$$\frac{1}{2s_2+s_3}\left[s_2-s_3-e^{-2\Omega(T_{\alpha(s_3)+1},t_2)}\left(1-e^{-\frac{2\left(T_{\alpha(s_3)+1}-s_3\right)}{\lambda_{\alpha(s_3)}}}\right)\frac{\lambda_{\alpha(s_3)}}{2}-\sum_{i=\alpha(s_3)+1}^{\alpha(t_2)-1}e^{-2\Omega(T_{i+1},t_2)}\left(1-e^{-\frac{2\Delta_i}{\lambda_i}}\right)\frac{\lambda_i}{2}\right.$$
$$\left.-\left(1-e^{-\frac{2\left(t_2-T_{\alpha(t_2)}\right)}{\lambda_{\alpha(t_2)}}}\right)\frac{\lambda_{\alpha(t_2)}}{2}\right]$$

$$=3\int_0^{s_3}\frac{du}{2s_2+s_3}\frac{1}{3}\left[1-e^{-3\Omega(u,s_3)}\right]+\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}\frac{1}{2}\left[1-e^{-2\Omega(s_3,s_2)}\right]+2\int_{s_3}^{s_2}\frac{du}{2s_2+s_3}\frac{1}{2}\left[1-e^{-2\Omega(u,s_2)}\right].$$

3 Discrete approximation to the triploid SMC' coalescent process

In order to construct a hidden Markov model (HMM) to infer demography, it is necessary to discretize the triploid coalescent process described above.

Let the discrete state (i,j), $i<j$, correspond to the continuous states in which $T_i < t_3 < T_{i+1}$ and $T_j < t_2 < T_{j+1}$. We first calculate the equilibrium probability that the coalescent process is in (i,j), assuming $i<j$:

$$
\begin{aligned}
\pi_{i,j} &= \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{T_j}^{T_{j+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\Omega(T_j,t_2)} dt_2 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\frac{t_2-T_j}{\lambda_j}} dt_2 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}}\right] \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \qquad (7) \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}}\right] e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}}\right] e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} e^{-\frac{T_{i+1}-t_3}{\lambda_i}} dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}}\right] e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) \\
&= \frac{3}{2} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) \left[1 - e^{-\frac{\Delta_j}{\lambda_j}}\right].
\end{aligned}
$$

If $j = n$, we let $\Delta_j = \infty$ and $1 - \exp(-\Delta_j/\lambda_j) = 1$.

It is also necessary to calculate $\pi_{i,i}$:

$$
\begin{aligned}
\pi_{i,i} &= \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{3}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{1}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\frac{t_2-t_3}{\lambda_i}} dt_2 dt_3 \\
&= \frac{3}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \left(\lambda_i \left[1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right]\right) dt_3 \qquad (8) \\
&= \frac{3}{\lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right) dt_3 \\
&= \frac{3}{\lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{6} \left(2 - 3e^{-\frac{\Delta_i}{\lambda_i}} + e^{-\frac{3\Delta_i}{\lambda_i}}\right) \\
&= \frac{1}{2} e^{-3\Omega(0,T_i)} \left(2 - 3e^{-\frac{\Delta_i}{\lambda_i}} + e^{-\frac{3\Delta_i}{\lambda_i}}\right)
\end{aligned}
$$

For $i = n$, again we let $\Delta_i = \infty$ and thus $\pi_{i,i} = \exp(-3\Omega(0, T_i))$.

Next, we calculate marginal expectations for t_3 and t_2 given that the continuous process is in interval represented by (i, j), assuming $i < j$. The marginal expectation of t_3 in the interval (i, j) is

$$
\begin{aligned}
\mathrm{E}_{i,j}[t_3] &= \mathrm{E}\left[t_3 | t_3 \in [T_i T_{i+1}), t_2 \in [T_j, T_{j+1})\right] \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3t_3}{\lambda_i \lambda_j} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} \int_{T_j}^{T_{j+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\Omega(T_j,t_2)} dt_2 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\frac{t_2-T_j}{\lambda_j}} dt_2 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{4} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}}\right] \\
&= \frac{3}{4\pi_{i,j}} e^{-3\Omega(0,T_i)} \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) e^{-\Omega(T_{i+1},T_j)} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}}\right]
\end{aligned} \tag{9}
$$

With $j = n$, this is

$$
\begin{aligned}
\mathrm{E}_{i,j}[t_3] &= \mathrm{E}\left[t_3 | t_3 \in [T_i T_{i+1}), t_2 \in [T_n, \infty)\right] \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} \frac{3t_3}{\lambda_i \lambda_n} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} \int_{T_n}^{\infty} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_n)} dt_3 \int_{T_n}^{\infty} e^{-\Omega(T_n,t_2)} dt_2 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_n)} dt_3 \int_{T_n}^{\infty} e^{-\frac{t_2-T_n}{\lambda_n}} dt_2 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} \lambda_n \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_n)} dt_3 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} \lambda_n e^{-\Omega(T_{i+1},T_n)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} \lambda_n e^{-\Omega(T_{i+1},T_n)} \frac{\lambda_i}{4} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}}\right] \\
&= \frac{3}{4\pi_{i,n}} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}}\right]
\end{aligned} \tag{10}
$$

The marginal expectation of t_2 in (i, j) is

$$
\begin{aligned}
\mathrm{E}_{i,j}[t_2] &= \mathrm{E}\left[t_2 | t_3 \in [T_i T_{i+1}), t_2 \in [T_j, T_{j+1})\right] \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} t_2 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3t_2}{\lambda_i \lambda_j} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j,t_2)} dt_2 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} e^{-\frac{T_{i+1}-t_3}{\lambda_i}} dt_3 \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j,t_2)} dt_2 \qquad (11) \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j,t_2)} dt_2 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) \int_{T_j}^{T_{j+1}} t_2 e^{-\frac{t_2-T_j}{\lambda_j}} dt_2 \\
&= \frac{3}{\pi_{i,j}\lambda_i\lambda_j} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) \lambda_j \left(\lambda_j + T_j - (\lambda_j + T_{j+1}) e^{-\frac{\Delta_j}{\lambda_j}}\right) \\
&= \frac{3}{2\pi_{i,j}} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) \left(\lambda_j + T_j - (\lambda_j + T_{j+1}) e^{-\frac{\Delta_j}{\lambda_j}}\right).
\end{aligned}
$$

With $j = n$, this is

$$
\begin{aligned}
\mathrm{E}_{i,n}[t_2] &= \mathrm{E}\left[t_2 | t_3 \in [T_i T_{i+1}), t_2 \in [T_n, \infty)\right] \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} t_2 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} \frac{3t_2}{\lambda_i \lambda_n} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n,t_2)} dt_2 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} e^{-\frac{T_{i+1}-t_3}{\lambda_i}} dt_3 \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n,t_2)} dt_2 \qquad (12) \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n,t_2)} dt_2 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) \int_{T_n}^{\infty} t_2 e^{-\frac{t_2-T_n}{\lambda_n}} dt_2 \\
&= \frac{3}{\pi_{i,n}\lambda_i\lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) \lambda_n (\lambda_n + T_n) \\
&= \frac{3}{2\pi_{i,n}} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}}\right) (\lambda_n + T_n)
\end{aligned}
$$

We also calculate the marginal expectations of s_3 and s_2 conditional on the interval (i, i):

$$
\begin{aligned}
\mathrm{E}_{i,i}[t_3] &= \mathrm{E}\left[t_3 | t_3 \in [T_i T_{i+1}), t_2 \in [T_i, T_{i+1})\right] \\
&= \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_3 \pi(t_3, t_2) dt_2 \, dt_3 \\
&= \frac{3}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{t_3}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 \, dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i^2} \int_{T_i}^{T_{i+1}} t_3 \, e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 \, dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 \, dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 \, e^{-\frac{3(t_3-T_i)}{\lambda_i}} \lambda_i \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right) dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 \, e^{-\frac{3(t_3-T_i)}{\lambda_i}} \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right) dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{36} \left(-9e^{-\frac{\Delta_i}{\lambda_i}}(2T_i + \lambda_i) + 4(3T_i + \lambda_i) + e^{-\frac{3\Delta_i}{\lambda_i}}(6T_{i+1} + 5\lambda_i)\right) \\
&= \frac{1}{12\pi_{i,i}} e^{-3\Omega(0,T_i)} \left(4(3T_i + \lambda_i) + e^{-\frac{3\Delta_i}{\lambda_i}}(6T_{i+1} + 5\lambda_i) - 9e^{-\frac{\Delta_i}{\lambda_i}}(2T_i + \lambda_i)\right)
\end{aligned}
\tag{13}
$$

For $i = n$, the expectation is

$$
\mathrm{E}_{n,n}[s_3] = T_n + \frac{\lambda_n}{3} \tag{14}
$$

Double-checking:

$$
\begin{aligned}
\mathrm{E}_{n,n}[t_3] &= \mathrm{E}\left[t_3 | t_3 \in [T_n \infty), t_2 \in [T_n, \infty)\right] \\
&= \frac{1}{\pi_{n,n}} \int_{T_n}^{\infty} \int_{t_3}^{\infty} t_3 \pi(t_3, t_2) dt_2 \, dt_3 \\
&= \frac{3}{\pi_{n,n}\lambda_n^2} \int_{T_n}^{\infty} \int_{t_3}^{\infty} t_3 e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 \, dt_3 \\
&= \frac{3}{\pi_{n,n}\lambda_n^2} \int_{T_n}^{\infty} t_3 e^{-3\Omega(0,t_3)} \int_{t_3}^{\infty} e^{-\Omega(t_3,t_2)} dt_2 \, dt_3 \\
&= \frac{3}{\pi_{n,n}\lambda_n} \int_{T_n}^{\infty} t_3 e^{-3\Omega(0,t_3)} dt_3 \\
&= \frac{3}{\pi_{n,n}\lambda_n} e^{-3\Omega(0,T_n)} \int_{T_n}^{\infty} t_3 e^{-3\Omega(T_n,t_3)} dt_3 \\
&= \frac{3}{\pi_{n,n}\lambda_n} e^{-3\Omega(0,T_n)} \int_{T_n}^{\infty} t_3 e^{-\frac{3(t_3-T_n)}{\lambda_n}} dt_3 \\
&= \frac{3}{\pi_{n,n}\lambda_n} e^{-3\Omega(0,T_n)} \frac{\lambda_n}{9} (3T_n + \lambda_n) \\
&= \frac{1}{3\pi_{n,n}} e^{-3\Omega(0,T_n)} (3T_n + \lambda_n)
\end{aligned}
\tag{15}
$$

Checks out since $\pi_{n,n} = \exp(-3\Omega(0, T_n))$.

$$
\begin{aligned}
\mathrm{E}_{i,i}[t_2] &= \mathrm{E}\left[t_2 | t_3 \in [T_i T_{i+1}), t_2 \in [T_i, T_{i+1})\right] \\
&= \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_2 \pi(t_3, t_2) dt_2\, dt_3 \\
&= \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{3t_2}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2\, dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i^2} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_2\, e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2\, dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} t_2\, e^{-\Omega(t_3,t_2)} dt_2\, dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} t_2\, e^{-\frac{t_2 - t_3}{\lambda_i}} dt_2\, dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \lambda_i \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1} - t_3}{\lambda_i}} (T_{i+1} + \lambda_i)\right] dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \lambda_i \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1} - t_3}{\lambda_i}} (T_{i+1} + \lambda_i)\right] dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3 - T_i)}{\lambda_i}} \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1} - t_3}{\lambda_i}} (T_{i+1} + \lambda_i)\right] dt_3 \\
&= \frac{3}{\pi_{i,i}\lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{18} \left(e^{-\frac{3\Delta_i}{\lambda_i}} (3T_{i+1} + \lambda_i) + 6T_i + 8\lambda_i - 9e^{-\frac{\Delta_i}{\lambda_i}} (T_{i+1} + \lambda_i)\right) \\
&= \frac{1}{6\pi_{i,i}} e^{-3\Omega(0,T_i)} \left(e^{-\frac{3\Delta_i}{\lambda_i}} (3T_{i+1} + \lambda_i) + 6T_i + 8\lambda_i - 9e^{-\frac{\Delta_i}{\lambda_i}} (T_{i+1} + \lambda_i)\right)
\end{aligned}
\tag{16}
$$

For $i = n$, this expectation is

$$
\mathrm{E}_{n,n}[s_2] = T_n + \frac{\lambda_n}{3} + \lambda_n \tag{17}
$$

3.1 Discrete $q\big((k,l)|(i,j)\big)$ transition function

To calculate the discrete-process transition probabilities from (i,j), to (k,l), we integrate the transition kernel (6) over the interval corresponding to (k,l), replacing s_3 and s_2 with their conditional expectations $\mathrm{E}_{i,j}[s_3]$ and $\mathrm{E}_{i,j}[s_2]$ respectively. Thus

$$
q\Big((k,l)\,|\,(i,j)\Big) = \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} q\Big((t_3,t_2)|\,(\mathrm{E}_{i,j}[s_3], \mathrm{E}_{i,j}[s_2])\Big) dt_2 dt_3. \tag{18}
$$

Note that in any single transition, either the first or second coalescence time changes, but not both. This simplifies the calculation of these integrals.

3.1.1 Case A

For $i = k < j < l$, $(t_3 = s_3\,; t_2 > s_2)$:

$$= \frac{1}{2s_2+s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2,t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1},s_3)} \left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right] \frac{\lambda_i}{3} + \left[1-e^{-\frac{3\left(s_3-T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} +$$

$$\frac{2}{2s_2+s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2,t_2)} \times$$

$$\left\{ e^{-2\Omega(T_{\alpha(s_3)+1},s_2)} \left[1-e^{-\frac{2\left(T_{\alpha(s_3)+1}-s_3\right)}{\lambda_{\alpha(s_3)}}}\right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{i=\alpha(s_3)+1}^{\alpha(s_2)-1} e^{-2\Omega(T_{i+1},s_2)} \left[1-e^{-\frac{2\Delta_i}{\lambda_i}}\right] \frac{\lambda_i}{2} \right.$$

$$\left. + \left[1-e^{-\frac{2\left(s_2-T_{\alpha(s_2)}\right)}{\lambda_{\alpha(s_2)}}}\right] \frac{\lambda_{\alpha(s_2)}}{2} \right\}$$

$$= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(\mathrm{E}_{i,j}[s_2],t_2)} \times$$

$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} \delta(t_3-s_3)dt_2dt_3 +$$

$$\int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l} e^{-\Omega(\mathrm{E}_{i,j}[s_2],t_2)} \times$$

$$\left\{ e^{-2\Omega(T_{i+1},\mathrm{E}_{i,j}[s_2])} \left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right] \frac{\lambda_i}{2} + \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1},\mathrm{E}_{i,j}[s_2])} \left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{2} + \right.$$

$$\left. \left[1-e^{-\frac{2(\mathrm{E}_{i,j}[s_2]-T_j)}{\lambda_j}}\right] \frac{\lambda_j}{2} \right\} \delta(t_3-s_3)dt_2dt_3$$

$$= \int_{T_l}^{T_{l+1}} \frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(\mathrm{E}_{i,j}[s_2],t_2)} \times$$

$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} dt_2 +$$

$$\int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l} e^{-\Omega(\mathrm{E}_{i,j}[s_2],t_2)} \times$$

$$\left\{ e^{-2\Omega(T_{i+1},\mathrm{E}_{i,j}[s_2])} \left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right] \frac{\lambda_i}{2} + \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1},\mathrm{E}_{i,j}[s_2])} \left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{2} + \right.$$

$$\left. \left[1-e^{-\frac{2(\mathrm{E}_{i,j}[s_2]-T_j)}{\lambda_j}}\right] \frac{\lambda_j}{2} \right\} dt_2$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times$$

$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(\mathrm{E}_{i,j}[s_2],t_2)} dt_2 +$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1},\mathrm{E}_{i,j}[s_2])} \left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right] \frac{\lambda_i}{2} + \right.$$

$$\left. \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1},\mathrm{E}_{i,j}[s_2])} \left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{2} + \left[1-e^{-\frac{2(\mathrm{E}_{i,j}[s_2]-T_j)}{\lambda_j}}\right] \frac{\lambda_j}{2} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(\mathrm{E}_{i,j}[s_2],t_2)} dt_2$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times$$

$$
\begin{aligned}
&\left\{\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\} e^{-\Omega(\mathrm{E}_{i,j}[s_2],T_l)}\int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)}dt_2+\\
&\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left\{e^{-2\Omega(T_{i+1},\mathrm{E}_{i,j}[s_2])}\left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right]\frac{\lambda_i}{2}+\right.\\
&\left.\sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1},\mathrm{E}_{i,j}[s_2])}\left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{2}+\left[1-e^{-\frac{2(\mathrm{E}_{i,j}[s_2]-T_j)}{\lambda_j}}\right]\frac{\lambda_j}{2}\right\} e^{-\Omega(\mathrm{E}_{i,j}[s_2],T_l)}\int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)}dt_2+\\
=&\frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])}\frac{1}{\lambda_l}\times\\
&\left\{\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\} e^{-\Omega(\mathrm{E}_{i,j}[s_2],T_l)}\int_{T_l}^{T_{l+1}} e^{-\frac{t_2-T_l}{\lambda_l}}dt_2+\\
&\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left\{e^{-2\Omega(T_{i+1},\mathrm{E}_{i,j}[s_2])}\left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right]\frac{\lambda_i}{2}+\right.\\
&\left.\sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1},\mathrm{E}_{i,j}[s_2])}\left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{2}+\left[1-e^{-\frac{2(\mathrm{E}_{i,j}[s_2]-T_j)}{\lambda_j}}\right]\frac{\lambda_j}{2}\right\} e^{-\Omega(\mathrm{E}_{i,j}[s_2],T_l)}\int_{T_l}^{T_{l+1}} e^{-\frac{t_2-T_l}{\lambda_l}}dt_2\\
=&\frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])}\frac{1}{\lambda_l}\times\\
&\left\{\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\} e^{-\Omega(\mathrm{E}_{i,j}[s_2],T_l)}\lambda_l\left(1-e^{-\frac{\Delta_l}{\lambda_l}}\right)+\\
&\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left\{e^{-2\Omega(T_{i+1},\mathrm{E}_{i,j}[s_2])}\left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right]\frac{\lambda_i}{2}+\right.\\
&\left.\sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1},\mathrm{E}_{i,j}[s_2])}\left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{2}+\left[1-e^{-\frac{2(\mathrm{E}_{i,j}[s_2]-T_j)}{\lambda_j}}\right]\frac{\lambda_j}{2}\right\} e^{-\Omega(\mathrm{E}_{i,j}[s_2],T_l)}\lambda_l\left(1-e^{-\frac{\Delta_l}{\lambda_l}}\right)
\end{aligned}
$$

Here we assume that $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

3.1.2 Case B

For $i = k < l < j$ ($t_3 = s_3; t_2 < s_2$):

$$= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{1}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-2\Omega(s_3,t_2)}$$

$$\left\{ \sum_{a=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{a+1},s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3\left(s_3 - T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \delta(t_3 - s_3) dt_2 dt_3 +$$

$$\int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}}$$

$$\left\{ e^{-2\Omega(T_{\alpha(s_3)+1},t_2)} \left[1 - e^{-\frac{2\left(T_{\alpha(s_3)+1} - s_3\right)}{\lambda_{\alpha(s_3)}}}\right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{a=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{a+1},t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{2} \right.$$

$$\left. + \left[1 - e^{-\frac{2\left(t_2 - T_{\alpha(t_2)}\right)}{\lambda_{\alpha(t_2)}}}\right] \frac{\lambda_{\alpha(t_2)}}{2} \right\} \delta(t_3 - s_3) dt_2 dt_3$$

$$= \int_{T_l}^{T_{l+1}} \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],t_2)}$$

$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} dt_2 +$$

$$\int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l}$$

$$\left\{ e^{-2\Omega(T_{i+1},t_2)} \left[1 - e^{-\frac{2(T_{i+1} - \mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right] \frac{\lambda_i}{2} + \sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1},t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{2} \right.$$

$$\left. + \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}}\right] \frac{\lambda_l}{2} \right\} dt_2$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l}$$

$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],t_2)} dt_2 +$$

$$\int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(e^{-2\Omega(T_{i+1},t_2)} \left[1 - e^{-\frac{2(T_{i+1} - \mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right] \frac{\lambda_i}{2} \right) dt_2 +$$

$$\int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1},t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{2} \right) dt_2 +$$

$$\int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}}\right] \frac{\lambda_l}{2} dt_2$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l}$$

$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],t_2)} dt_2 +$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right] \frac{\lambda_i}{2} \right) \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_{i+1},t_2)} dt_2 +$$

$$\int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1},t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{2} \right) dt_2 +$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\frac{\lambda_l}{2}\int_{T_l}^{T_{l+1}}\left[1-e^{-\frac{2(t_2-T_l)}{\lambda_l}}\right]dt_2$$

$$=\frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}$$

$$\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}\int_{T_l}^{T_{l+1}}e^{-2\Omega(\mathrm{E}_{i,j}[s_3],t_2)}dt_2+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left(\left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right]\frac{\lambda_i}{2}\right)\int_{T_l}^{T_{l+1}}e^{-2\Omega(T_{i+1},t_2)}dt_2+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\int_{T_l}^{T_{l+1}}\left(\sum_{a=i+1}^{l-1}e^{-2\Omega(T_{a+1},t_2)}\left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{2}\right)dt_2+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\frac{\lambda_l}{2}\int_{T_l}^{T_{l+1}}\left[1-e^{-\frac{2(t_2-T_l)}{\lambda_l}}\right]dt_2$$

$$=\frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}$$

$$\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}\int_{T_l}^{T_{l+1}}e^{-2\Omega(\mathrm{E}_{i,j}[s_3],t_2)}dt_2+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left(\left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right]\frac{\lambda_i}{2}\right)\int_{T_l}^{T_{l+1}}e^{-2\Omega(T_{i+1},t_2)}dt_2+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left(\sum_{a=i+1}^{l-1}\left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{2}\int_{T_l}^{T_{l+1}}e^{-2\Omega(T_{a+1},t_2)}dt_2\right)+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\frac{\lambda_l}{2}\int_{T_l}^{T_{l+1}}\left[1-e^{-\frac{2(t_2-T_l)}{\lambda_l}}\right]dt_2$$

$$=\frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}$$

$$\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}e^{-2\Omega(\mathrm{E}_{i,j}[s_3],T_l)}\int_{T_l}^{T_{l+1}}e^{-2\Omega(T_l,t_2)}dt_2+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left(\left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right]\frac{\lambda_i}{2}\right)e^{-2\Omega(T_{i+1},T_l)}\int_{T_l}^{T_{l+1}}e^{-2\Omega(T_l,t_2)}dt_2+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left(\sum_{a=i+1}^{l-1}\left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{2}e^{-2\Omega(T_{a+1},T_l)}\int_{T_l}^{T_{l+1}}e^{-2\Omega(T_l,t_2)}dt_2\right)+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\frac{\lambda_l}{2}\int_{T_l}^{T_{l+1}}\left[1-e^{-\frac{2(t_2-T_l)}{\lambda_l}}\right]dt_2$$

$$=\frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}$$

$$\left\{\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],T_l)}\int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2-T_l)}{\lambda_l}}dt_2+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left(\left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right]\frac{\lambda_i}{2}\right)e^{-2\Omega(T_{i+1},T_l)}\int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2-T_l)}{\lambda_l}}dt_2+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left(\sum_{a=i+1}^{l-1}\left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{2}e^{-2\Omega(T_{a+1},T_l)}\int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2-T_l)}{\lambda_l}}dt_2\right)+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\frac{\lambda_l}{2}\int_{T_l}^{T_{l+1}}\left[1-e^{-\frac{2(t_2-T_l)}{\lambda_l}}\right]dt_2$$

$$=\frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}$$

$$\left\{\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],T_l)}\frac{\lambda_l}{2}\left(1-e^{-\frac{2\Delta_l}{\lambda_l}}\right)+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left(\left[1-e^{-\frac{2(T_{i+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_i}}\right]\frac{\lambda_i}{2}\right)e^{-2\Omega(T_{i+1},T_l)}\frac{\lambda_l}{2}\left(1-e^{-\frac{2\Delta_l}{\lambda_l}}\right)+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\left[\sum_{a=i+1}^{l-1}\left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{2}e^{-2\Omega(T_{a+1},T_l)}\frac{\lambda_l}{2}\left(1-e^{-\frac{2\Delta_l}{\lambda_l}}\right)\right]+$$

$$\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{1}{\lambda_l}\frac{\lambda_l}{2}\left[\Delta_l-\frac{\lambda_l}{2}\left(1-e^{-\frac{2\Delta_l}{\lambda_l}}\right)\right]$$

3.1.3 Case C

For $k < i = l < j$ $(t_3 < s_3; t_2 = s_3)$:

$$= \frac{1}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1},t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}}\right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3\left(t_3 - T_{\alpha(t_3)}\right)}{\lambda_{\alpha(t_3)}}}\right] \frac{\lambda_{\alpha(t_3)}}{3} \right\}$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{i=0}^{k-1} e^{-3\Omega(T_{i+1},t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}}\right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] \frac{\lambda_k}{3} \right\}$$

$$= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1},t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] \frac{\lambda_k}{3} \right\} \delta(t_2 - \mathrm{E}_{i,j}[s_3]) dt_2 dt_3$$

$$= \int_{T_k}^{T_{k+1}} \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1},t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] \frac{\lambda_k}{3} \right\} dt_3$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1},t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] \frac{\lambda_k}{3} dt_3 \right\}$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1},t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] dt_3 \right\}$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] dt_3 \right\}$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] dt_3 \right\}$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left[1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right] \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)\right] \right\}$$

3.1.4 Case D

For $k < i < j = l$ $(t_3 < s_3; t_2 = s_2)$:

$$
\begin{aligned}
&= \frac{2}{2s_2+s_3}\frac{2}{\lambda_{\alpha(t_3)}}\left\{\sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1},t_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3\left(t_3-T_{\alpha(t_3)}\right)}{\lambda_{\alpha(t_3)}}}\right]\frac{\lambda_{\alpha(t_3)}}{3}\right\}\\
&= \int_{T_k}^{T_{k+1}}\int_{T_l}^{T_{l+1}}\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}e^{-3\Omega(T_{a+1},t_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}\right\}\delta(t_2-\mathrm{E}_{i,j}[s_2])dt_2dt_3\\
&= \int_{T_k}^{T_{k+1}}\frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}e^{-3\Omega(T_{a+1},t_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}\right\}dt_3\\
&= \frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}\int_{T_k}^{T_{k+1}}e^{-3\Omega(T_{a+1},t_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}dt_3+\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}dt_3\right\}\\
&= \frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}\int_{T_k}^{T_{k+1}}e^{-3\Omega(T_{a+1},t_3)}dt_3+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]dt_3\right\}\\
&= \frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\int_{T_k}^{T_{k+1}}e^{-3\Omega(T_k,t_3)}dt_3\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]dt_3\right\}\\
&= \frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\int_{T_k}^{T_{k+1}}e^{-\frac{3(t_3-T_k)}{\lambda_k}}dt_3\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]dt_3\right\}\\
&= \frac{2}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\frac{\lambda_k}{3}\left[1-e^{-\frac{3\Delta_k}{\lambda_k}}\right]\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\left[\Delta_k-\frac{\lambda_k}{3}\left(1-e^{-\frac{3\Delta_k}{\lambda_k}}\right)\right]\right\}
\end{aligned}
$$

3.1.5 Case E

For $i < k < j = l$ ($t_3 > s_3; t_2 = s_2$):

$$= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} e^{-2\Omega(s_3,t_3)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1},s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}}\right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3\left(s_3 - T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right] \frac{\lambda_{\alpha(s_3)}}{3} \right\}$$

$$= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],t_3)} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} \delta(t_2 - \mathrm{E}_{i,j}[s_2]) dt_2 dt_3$$

$$= \int_{T_k}^{T_{k+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],t_3)} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} dt_3$$

$$= \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} \int_{T_k}^{T_{k+1}} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],t_3)} dt_3$$

$$= \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],T_k)} \int_{T_k}^{T_{k+1}} e^{-2\Omega(T_k,t_3)} dt_3$$

$$= \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],T_k)} \int_{T_k}^{T_{k+1}} e^{-\frac{2(t_3-T_k)}{\lambda_k}} dt_3$$

$$= \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],T_k)} \frac{\lambda_k}{2} \left[1 - e^{-\frac{2\Delta_k}{\lambda_k}}\right]$$

3.1.6 Case F

For $i < j = k < l$ ($t_3 = s_2; t_2 > s_2$):

$$= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2,t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1},s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}}\right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3\left(s_3 - T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right] \frac{\lambda_{\alpha(s_3)}}{3} \right\}$$

$$= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(\mathrm{E}_{i,j}[s_2],t_2)} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} \delta(t_3 - s_2) dt_2 dt_3$$

$$= \int_{T_l}^{T_{l+1}} \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(\mathrm{E}_{i,j}[s_2],t_2)} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} dt_2$$

$$= \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(\mathrm{E}_{i,j}[s_2],t_2)} dt_2$$

$$= \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathrm{E}_{i,j}[s_2],T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)} dt_2$$

$$= \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathrm{E}_{i,j}[s_2],T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2$$

$$= \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times$$
$$\left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathrm{E}_{i,j}[s_2],T_l)} \lambda_l \left[1 - e^{-\frac{\Delta_l}{\lambda_l}}\right]$$

Here again we assume $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

The following cases require special consideration:

3.1.7 Case G

For $i=k=l<j$ and $t_3=s_3; t_2<s_2$ (also need to consider $t_3<s_3; t_2=s_3$)

$$
\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3,t_2)} \delta(t_3-s_3) + 2\int_{s_3}^{t_2} \frac{du}{2s_2+s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u,t_2)} \delta(t_3-s_3) \right) dt_2 dt_3 \\
&= \int_{s_3}^{T_{k+1}} \left(\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3,t_2)} + 2\int_{s_3}^{t_2} \frac{du}{2s_2+s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u,t_2)} \right) dt_2 \\
&= \int_{\mathrm{E}_{i,j}[s_3]}^{T_{k+1}} \left(\int_0^{\mathrm{E}_{i,j}[s_3]} \frac{du}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} e^{-3\Omega(u,\mathrm{E}_{i,j}[s_3])} \frac{1}{\lambda_k} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],t_2)} + \right. \\
&\qquad \left. 2\int_{\mathrm{E}_{i,j}[s_3]}^{t_2} \frac{du}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-2\Omega(u,t_2)} \right) dt_2 \\
&= \int_{\mathrm{E}_{i,j}[s_3]}^{T_{k+1}} \left(\int_0^{\mathrm{E}_{i,j}[s_3]} \frac{du}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} e^{-3\Omega(u,\mathrm{E}_{i,j}[s_3])} \frac{1}{\lambda_k} e^{-\frac{2(t_2-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} + \right. \\
&\qquad \left. 2\int_{\mathrm{E}_{i,j}[s_3]}^{t_2} \frac{du}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-\frac{2(t_2-u)}{\lambda_k}} \right) dt_2 \\
&= \frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \int_{\mathrm{E}_{i,j}[s_3]}^{T_{k+1}} \left(\frac{1}{\lambda_k} e^{-\frac{2(t_2-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} \int_0^{\mathrm{E}_{i,j}[s_3]} e^{-3\Omega(u,\mathrm{E}_{i,j}[s_3])} du + \right. \\
&\qquad \left. \frac{2}{\lambda_k} \int_{\mathrm{E}_{i,j}[s_3]}^{t_2} e^{-\frac{2(t_2-u)}{\lambda_k}} \right) dt_2 \\
&= \frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \int_{\mathrm{E}_{i,j}[s_3]}^{T_{k+1}} \left(\frac{1}{\lambda_k} e^{-\frac{2(t_2-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} \int_0^{\mathrm{E}_{i,j}[s_3]} e^{-3\Omega(u,\mathrm{E}_{i,j}[s_3])} du + \right. \\
&\qquad \left. \frac{2}{\lambda_k} \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(t_2-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} \right] \right) dt_2 \\
&= \frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \left(\frac{1}{\lambda_k} \int_0^{\mathrm{E}_{i,j}[s_3]} e^{-3\Omega(u,\mathrm{E}_{i,j}[s_3])} du \int_{\mathrm{E}_{i,j}[s_3]}^{T_{k+1}} e^{-\frac{2(t_2-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} dt_2 + \right. \\
&\qquad \left. \int_{\mathrm{E}_{i,j}[s_3]}^{T_{k+1}} \left[1 - e^{-\frac{2(t_2-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} \right] dt_2 \right) \\
&= \frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \left(\frac{1}{\lambda_k} \int_0^{\mathrm{E}_{i,j}[s_3]} e^{-3\Omega(u,\mathrm{E}_{i,j}[s_3])} du \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} \right] + \right. \\
&\qquad \left. T_{k+1} - \mathrm{E}_{i,j}[s_3] - \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} \right] \right) \\
&= \frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]} \left\{ \frac{1}{\lambda_k} \times \right. \\
&\qquad \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left(1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right) \frac{\lambda_a}{3} + \left(1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}} \right) \frac{\lambda_i}{3} \right] \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} \right] + \\
&\qquad \left. T_{k+1} - \mathrm{E}_{i,j}[s_3] - \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1}-\mathrm{E}_{i,j}[s_3])}{\lambda_k}} \right] \right\}
\end{aligned}
$$

3.1.8 Case G2

For $i = k = l < j$ and $t_3 < s_3; t_2 = s_3$:

$$= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,t_3)} \frac{2}{\lambda(t_3)} \delta(t_2 - s_3) \right) dt_2\, dt_3$$

$$= \int_{T_k}^{s_3} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,t_3)} \frac{2}{\lambda(t_3)} \right) dt_3$$

$$= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_3} \frac{2}{\lambda(t_3)} \int_0^{t_3} e^{-3\Omega(u,t_3)} du\, dt_3$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \int_{T_k}^{\mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \int_0^{t_3} e^{-3\Omega(u,t_3)} du\, dt_3$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \int_{T_k}^{\mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1},t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] \frac{\lambda_k}{3} \right) dt_3$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} \int_{T_k}^{\mathrm{E}_{i,j}[s_3]} e^{-3\Omega(T_{a+1},t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{\mathrm{E}_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] dt_3 \right)$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{\mathrm{E}_{i,j}[s_3]} e^{-3\Omega(T_k,t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{\mathrm{E}_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] dt_3 \right)$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{\mathrm{E}_{i,j}[s_3]} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{\mathrm{E}_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}}\right] dt_3 \right)$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_k)}{\lambda_k}}\right] + \right.$$
$$\left. \frac{\lambda_k}{3} \left(\mathrm{E}_{i,j}[s_3] - T_k - \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_k)}{\lambda_k}}\right] \right) \right)$$

$$= \frac{1}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_k)}{\lambda_k}}\right] + \right.$$

$$\frac{\lambda_k}{3}\left(\mathrm{E}_{i,j}[s_3]-T_k-\frac{\lambda_k}{3}\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_k)}{\lambda_k}}\right]\right)\Bigg)$$

3.1.9 Case H

Another case that requires special consideration is $i<k=l=j$. For $i<k=l=j$ and $t_3>s_3\,;t_2=s_2$:

$$\begin{aligned}
&=\int_{T_k}^{T_{k+1}}\int_{t_3}^{T_{k+1}}\left(2\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}\frac{2}{\lambda(t_3)}e^{-2\Omega(s_3,t_3)}\delta(t_2-s_2)\right)dt_2\,dt_3\\
&=\int_{T_k}^{s_2}\left(2\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}\frac{2}{\lambda(t_3)}e^{-2\Omega(s_3,t_3)}\right)dt_3\\
&=\frac{1}{2s_2+s_3}\int_{T_k}^{s_2}\left(2\int_0^{s_3}e^{-3\Omega(u,s_3)}\frac{2}{\lambda(t_3)}e^{-2\Omega(s_3,t_3)}du\right)dt_3\\
&=\frac{1}{2s_2+s_3}\int_{T_k}^{s_2}\left(\frac{4}{\lambda(t_3)}e^{-2\Omega(s_3,t_3)}\int_0^{s_3}e^{-3\Omega(u,s_3)}du\right)dt_3\\
&=\frac{1}{2s_2+s_3}\left(\int_0^{s_3}e^{-3\Omega(u,s_3)}du\right)e^{-2\Omega(s_3,T_k)}\int_{T_k}^{s_2}\frac{4}{\lambda(t_3)}e^{-2\Omega(T_k,t_3)}dt_3\\
&=\frac{1}{2s_2+s_3}\frac{4}{\lambda_k}e^{-2\Omega(s_3,T_k)}\left(\int_0^{s_3}e^{-3\Omega(u,s_3)}du\right)\int_{T_k}^{s_2}e^{-2\Omega(T_k,t_3)}dt_3\\
&=\frac{1}{2s_2+s_3}\frac{4}{\lambda_k}e^{-2\Omega(s_3,T_k)}\left(\int_0^{s_3}e^{-3\Omega(u,s_3)}du\right)\int_{T_k}^{s_2}e^{-\frac{2(t_3-T_k)}{\lambda_k}}dt_3\\
&=\frac{1}{2s_2+s_3}\frac{4}{\lambda_k}e^{-2\Omega(s_3,T_k)}\left(\int_0^{s_3}e^{-3\Omega(u,s_3)}du\right)\frac{\lambda_k}{2}\left[1-e^{-\frac{2(s_2-T_k)}{\lambda_k}}\right]\\
&=\frac{1}{2s_2+s_3}\frac{4}{\lambda_k}e^{-2\Omega(s_3,T_k)}\left(\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},s_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right)\frac{\lambda_k}{2}\left[1-e^{-\frac{2(s_2-T_k)}{\lambda_k}}\right]\\
&=\frac{1}{2\,\mathrm{E}_{i,j}[s_2]+\mathrm{E}_{i,j}[s_3]}\frac{4}{\lambda_k}e^{-2\Omega(\mathrm{E}_{i,j}[s_3],T_k)}\left(\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right)\times\\
&\qquad\frac{\lambda_k}{2}\left[1-e^{-\frac{2(\mathrm{E}_{i,j}[s_2]-T_k)}{\lambda_k}}\right]
\end{aligned}$$

3.1.10 Case H2

For $i < k = l = j$ and $t_3 = s_2\,; t_2 > s_2$:

$$
\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \delta(t_3 - s_2) \right) dt_2\, dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \int_{s_2}^{T_{k+1}} \left(e^{-\Omega(s_2,t_2)} \int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) dt_2 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \left(\int_{s_2}^{T_{k+1}} e^{-\Omega(s_2,t_2)} dt_2 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \left(\int_{s_2}^{T_{k+1}} e^{-\Omega(s_2,t_2)} dt_2 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \left(\int_{s_2}^{T_{k+1}} e^{-\frac{t_2 - s_2}{\lambda_k}} dt_2 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \left(\lambda_k \left[1 - \delta(k-n) e^{-\frac{T_{k+1} - s_2}{\lambda_k}} \right] \right) \\
&= \frac{2}{2\,\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-2\Omega(\mathrm{E}_{i,j}[s_3],\mathrm{E}_{i,j}[s_2])} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \lambda_k \left[1 - \delta(k-n) e^{-\frac{T_{k+1} - \mathrm{E}_{i,j}[s_2]}{\lambda_k}} \right]
\end{aligned}
$$

The delta function is a way to ensure correctness when $k = l = j = n$.

Another special case that requires attention is $i = j = k < l$.

3.1.11 Case I

For $i = j = k < l$ and $t_3 = s_3; t_2 > s_2$:

$$
\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \Bigg(\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \delta(t_3-s_3) + \\
&\qquad\qquad 2\int_{s_3}^{s_2} \frac{du}{2s_2+s_3} e^{-2\Omega(u,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \delta(t_3-s_3) \Bigg) dt_2\, dt_3 \\
&= \int_{T_l}^{T_{l+1}} \Bigg(\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} + \\
&\qquad\qquad 2\int_{s_3}^{s_2} \frac{du}{2s_2+s_3} e^{-2\Omega(u,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \Bigg) dt_2 \\
&= \frac{1}{2s_2+s_3} \int_{T_l}^{T_{l+1}} \Bigg(\int_0^{s_3} e^{-3\Omega(u,s_3)} du\, e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} + \\
&\qquad\qquad 2\int_{s_3}^{s_2} e^{-2\Omega(u,s_2)} du \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \Bigg) dt_2 \\
&= \frac{1}{2s_2+s_3} \int_{T_l}^{T_{l+1}} \Bigg(e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \int_0^{s_3} e^{-3\Omega(u,s_3)} du + \\
&\qquad\qquad \frac{2}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \int_{s_3}^{s_2} e^{-2\Omega(u,s_2)} du \Bigg) dt_2 \\
&= \frac{1}{2s_2+s_3} \int_{T_l}^{T_{l+1}} \Bigg(e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,t_2)} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(s_3-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right] + \\
&\qquad\qquad \frac{2}{\lambda_l} e^{-\Omega(s_2,t_2)} \int_{s_3}^{s_2} e^{-2\Omega(u,s_2)} du \Bigg) dt_2 \\
&= \frac{1}{2s_2+s_3} \int_{T_l}^{T_{l+1}} \Bigg(e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,t_2)} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(s_3-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right] + \\
&\qquad\qquad \frac{2}{\lambda_l} e^{-\Omega(s_2,t_2)} \int_{s_3}^{s_2} e^{-\frac{2(s_2-u)}{\lambda_i}} du \Bigg) dt_2 \\
&= \frac{1}{2s_2+s_3} \int_{T_l}^{T_{l+1}} \Bigg(e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,t_2)} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(s_3-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right] + \\
&\qquad\qquad \frac{2}{\lambda_l} e^{-\Omega(s_2,t_2)} \frac{\lambda_i}{2} \left[1-e^{-\frac{2(s_2-s_3)}{\lambda_i}}\right] \Bigg) dt_2 \\
&= \frac{1}{2s_2+s_3} \Bigg(e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(s_3-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right] \int_{T_l}^{T_{l+1}} e^{-\Omega(s_2,t_2)} dt_2 +
\end{aligned}
$$

$$\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(s_2-s_3)}{\lambda_i}}\right]\int_{T_l}^{T_{l+1}} e^{-\Omega(s_2,t_2)}dt_2\Bigg)$$

$$= \frac{1}{2s_2+s_3}\Bigg(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_l}\left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right]e^{-\Omega(s_2,T_l)}\int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)}dt_2+$$
$$\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(s_2-s_3)}{\lambda_i}}\right]e^{-\Omega(s_2,T_l)}\int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)}dt_2\Bigg)$$

$$= \frac{1}{2s_2+s_3}\Bigg(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_l}\left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right]e^{-\Omega(s_2,T_l)}\int_{T_l}^{T_{l+1}} e^{-\frac{t_2-T_l}{\lambda_l}}dt_2+$$
$$\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(s_2-s_3)}{\lambda_i}}\right]e^{-\Omega(s_2,T_l)}\int_{T_l}^{T_{l+1}} e^{-\frac{t_2-T_l}{\lambda_l}}dt_2\Bigg)$$

$$= \frac{1}{2s_2+s_3}\Bigg(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_l}\left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right]e^{-\Omega(s_2,T_l)}\lambda_l\left[1-e^{-\frac{\Delta_l}{\lambda_l}}\right]+$$
$$\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(s_2-s_3)}{\lambda_i}}\right]e^{-\Omega(s_2,T_l)}\lambda_l\left[1-e^{-\frac{\Delta_l}{\lambda_l}}\right]\Bigg)$$

$$= \frac{1}{2\,\mathrm{E}_{i,i}[s_2]+\mathrm{E}_{i,i}[s_3]}\Bigg(e^{-2\Omega(\mathrm{E}_{i,i}[s_3],\mathrm{E}_{i,i}[s_2])}\frac{1}{\lambda_l}\left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,i}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(\mathrm{E}_{i,i}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right]\times$$
$$e^{-\Omega(\mathrm{E}_{i,i}[s_2],T_l)}\lambda_l\left[1-\delta(l-n)e^{-\frac{\Delta_l}{\lambda_l}}\right]+$$
$$\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(\mathrm{E}_{i,i}[s_2]-\mathrm{E}_{i,i}[s_3])}{\lambda_i}}\right]e^{-\Omega(\mathrm{E}_{i,i}[s_2],T_l)}\lambda_l\left[1-\delta(l-n)e^{-\frac{\Delta_l}{\lambda_l}}\right]\Bigg)$$

Again, notice the delta function.

3.1.12 Case I2

For $i = j = k < l$ and $t_3 = s_2; t_2 > s_2$:

$$
\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \delta(t_3 - s_2) \right) dt_2 \, dt_3 \\
&= \int_{T_l}^{T_{l+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \right) dt_2 \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \int_{T_l}^{T_{l+1}} \left(e^{-\Omega(s_2,t_2)} \int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) dt_2 \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(\int_{T_l}^{T_{l+1}} e^{-\Omega(s_2,t_2)} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - \delta(l-n) e^{-\frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - e^{-\delta(l-n)\frac{\Delta_l}{\lambda_l}} \right] \right) \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \\
&= \frac{2}{2\,\mathrm{E}_{i,i}[s_2] + \mathrm{E}_{i,i}[s_3]} e^{-2\Omega(\mathrm{E}_{i,i}[s_3],\mathrm{E}_{i,i}[s_2])} \frac{1}{\lambda_l} \left(e^{-\Omega(\mathrm{E}_{i,i}[s_2],T_l)} \lambda_l \left[1 - e^{-\delta(l-n)\frac{\Delta_l}{\lambda_l}} \right] \right) \times \\
&\qquad \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathrm{E}_{i,i}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathrm{E}_{i,i}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right)
\end{aligned}
$$

Another case that requires special consideration: $k < i = j = l$. This includes either $t_3 < s_3; t_2 = s_2$ or $t_3 < s_3; t_2 = s_3$.

3.1.13 Case J

For $k < i = j = l$ and $t_3 < s_3; t_2 = s_2$:

$$
\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(2 \int_0^{t_3} \frac{du}{2s_2+s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u,t_3)} \delta(t_2 - s_2) \right) dt_2\, dt_3 \\
&= \frac{2}{2s_2+s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{t_3} e^{-3\Omega(u,t_3)} \delta(t_2 - s_2) \right) dt_2\, dt_3 \\
&= \frac{2}{2s_2+s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} e^{-3\Omega(u,t_3)} \right) dt_3 \\
&= \frac{2}{2s_2+s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1},t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\
&= \frac{2}{2s_2+s_3} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1},t_3)} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right) \\
&= \frac{2}{2s_2+s_3} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right) \\
&= \frac{2}{2s_2+s_3} \frac{2}{\lambda_k} \left[\left(\int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \right) + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right] \\
&= \frac{2}{2s_2+s_3} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \right) + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\} \\
&= \frac{2}{2\,\mathrm{E}_{i,i}[s_2] + \mathrm{E}_{i,i}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \right) + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\}
\end{aligned}
$$

3.1.14 Case J2

For $k < i = j = l$ and $t_3 < s_3; t_2 = s_3$:

$$= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{t_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,t_3)} \frac{2}{\lambda(t_3)} \delta(t_2 - s_3) \right) dt_2\, dt_3$$
$$= \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,t_3)} \frac{2}{\lambda(t_3)} \right) dt_3$$
$$= \frac{2}{2s_2+s_3} \frac{1}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} e^{-3\Omega(u,t_3)} du \right) dt_3$$
$$= \frac{2}{2s_2+s_3} \frac{1}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1},t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right] \frac{\lambda_k}{3} \right) dt_3$$
$$= \frac{2}{2s_2+s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1},t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right] dt_3 \right)$$
$$= \frac{2}{2s_2+s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right] dt_3 \right)$$
$$= \frac{2}{2s_2+s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3-T_k)}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right] dt_3 \right)$$
$$= \frac{2}{2s_2+s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right] + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)\right] \right)$$

This finishes the $q\big((k,l)|(i,j)\big)$ transitions. Need to give special attention to the boundary cases, or the cases where $i = j$. Also need to consider what happens when k or l is n, since $T_{n+1} = \infty$.

The (diagonal) case of $k = i\,; l = j$ is calculated by subtracting the sum of the off-diagonal entries from unity.

3.2 Emission probabilities

The genotype at a particular position in a triploid genome can take one of three different values: 0, 1, and 2. The state 0 represents a homozygous site, and 1 (2) represent sites where one (two) of the three chromosomes have a derived (*i.e.*, non-ancestral) copy at that position.

To form our observed chain, we consider all the genotypes in a stretch of b bp and categorize that stretch of the sequence with a state 0, 1, 2, or 3. The state 0 means that the stretch of b bp is completely homozygous. The state 1 means that there was at least one site that had a 1 genotype and none that had a 2 genotype. Likewise, the state 2 means that at least one site had a 2 genotype, and none had a 1 genotype. The state 3 means that at least one site had a 1 genotype and at least one site had a 2 genotype.

With observed states coded this way, the emission probabilities given local coalescence times t_3 and t_2 are

$$e_k(t_3, t_2, T_d) = \begin{cases} e^{-\frac{\theta b\left(2t_2+t_3+3T_d\right)}{2}} & k=0 \\ e^{-\frac{\theta b(t_2-t_3)}{2}} \left(1 - e^{-\frac{\theta b\left(2t_3+t_2+3T_d\right)}{2}}\right) & k=1 \\ \left(1 - e^{-\frac{\theta b(t_2-t_3)}{2}}\right) e^{-\frac{\theta b\left(2t_3+t_2+3T_d\right)}{2}} & k=2 \\ \left(1 - e^{-\frac{\theta b(t_2-t_3)}{2}}\right) \left(1 - e^{-\frac{\theta b\left(2t_3+t_2+3T_d\right)}{2}}\right) & k=3. \end{cases} \tag{19}$$

Here T_d is the divergence time, the time in the past [again measured in units of $2N(0)$ generations] when the asexual lineage was derived from a sexual ancestor. The above probabilities assume that t_3 and t_2 are measured continuously. In practice, we discretize time, so for a particular hidden state (i, j), we replace t_3 and t_2 with $\mathrm{E}_{i,j}[t_3]$ and $\mathrm{E}_{i,j}[t_2]$, respectively.

Classifying states and genotypes this way requires that each polymorphism be polarized against an outgroup. If this is not possible, then the states can be recoded as 0 and 1, where 0 is a stretch of b completely homozygous base pairs, and 1 is a stretch of b base pairs with at least one polymorphic position. If no polarization is possible, the emission probabilities become

$$e_k(t_3, t_2, T_d) = \begin{cases} e^{-\frac{\theta b(2t_2+t_3+3T_d)}{2}} & k = 0 \\ 1 - e^{-\frac{\theta b(2t_2+t_3+3T_d)}{2}} & k = 1. \end{cases} \tag{20}$$

A few things to note: The parameter b can be tuned to match the observed polymorphism. If the change in ploidy T_d generations ago also involved a change in mutation rate, this new mutation rate will be unidentifiable, impossible to distinguish from a proportionally scaled T_d. Thus T_d should be viewed as a compound parameter.

4 HMM inference

The above equations for $q\big((k,l)|(i,j)\big)$ define a transition matrix $\big\{P_{(i,j),(k,l)}\big\}$, where

$$P_{(i,j),(k,l)} = \left(1 - e^{-\frac{\rho(2\,\mathrm{E}_{i,j}[t_2]+\mathrm{E}_{i,j}[t_3])}{2}}\right) q\big((k,l)|(i,j)\big) \tag{21}$$

is the probability of transitioning from state (i,j) to state (k,l), for $(i,j) \neq (k,l)$. We define a hidden Markov chain $\{X_i\}$ that is governed by this transition matrix. The observed process $\{Y_i\}$ represents the different emissions, 0 through 3. The EM algorithm proceeds by starting with some initial parameters θ and iteratively maximizing the expectation of the full likelihood given the data, which is

$$P(X, Y|\theta) = e_{x_1}(y_1)\,\pi_{x_1} \prod_{i=1}^{T-1} P_{x_i,x_{i+1}} e_{x_{i+1}}(y_{i+1}), \tag{22}$$

where y_i is the observed state at position i, and x_i is the state of the hidden chain at position i, and T is the length of the sequence. In practice we maximize the log-likelihood:

$$\log P(X, Y|\theta) = \log\left(e_{x_1}(y_1)\right) + \log(\pi_{x_1}) + \sum_{i=1}^{T-1} \log\left(P_{x_i,x_{i+1}}\right) + \log\left(e_{x_{i+1}}(y_{i+1})\right). \tag{23}$$

Since we observe only the observed chain (i.e., the mutation data), we have to have some way of integrating over the states of the hidden chain. We do this by the expectation-maximization (EM) algorithm, paired with the forward-backward algorithm for calculating likelihoods with these chains. In the context of HMM's, the EM algorithm iteratively maximizes

$$\begin{aligned} \mathrm{E}_{X|Y}\left[\log P\left(X, Y|\theta\right)\right] = \mathrm{E}_{X|Y}\left[\log e_{x_1}(y_1)\right] + \mathrm{E}_{X|Y}\left[\log \pi_{x_1}\right] + \\ \mathrm{E}_{X|Y}\left[\log e_{x_1}(y_1)\right] + \sum_{i=1}^{T-1}\bigg(\mathrm{E}_{X|Y}\left[\log P_{x_i,x_{i+1}}\right] + \mathrm{E}_{X|Y}\left[\log(e_{x_{i+1}}(y_{i+1}))\right]\bigg), \end{aligned}$$

updating the parameters of the chains in each iteration.

Define the forward variable

$$\alpha_i(t) = \mathrm{P}(Y_1 = y_1, Y_2 = y_2, \ldots, Y_t = y_t, X_t = i \,|\, \theta), \tag{24}$$

which satisfies the recursion

$$\alpha_j(t+1) = \sum_{i=1}^{N} \alpha_i(t) P_{i,j} e_j(y_{t+1}), \tag{25}$$

where $N = (n+1)(n+2)/2$ is the number of states in the hidden chain. We define

$$\alpha_i(1) = \pi_i e_i(y_1). \tag{26}$$

Define the backward variables

$$\beta_i(t) = \mathrm{P}(Y_{t+1} = y_{t+1}, Y_{t+2} = y_{t+2}, \ldots, Y_T = y_T | X_t = i, \theta), \tag{27}$$

which satisfy the recursion

$$\beta_i(t) = \sum_{j=1}^{N} \beta_j(t+1) P_{i,j} e_j(y_{t+1}), \tag{28}$$

defining $\beta_i(T) = 1$.

From these, we can calculate

$$\begin{aligned}\gamma_i(t) &= \mathrm{P}(X_t = i | Y, \theta) \\ &= \frac{\mathrm{P}(X_t = i, Y|\theta)}{\mathrm{P}(Y)} \\ &= \frac{\alpha_i(t)\beta_i(t)}{\sum_{j=1}^{N} \alpha_j(t)\beta_j(t)}\end{aligned}$$

and

$$\begin{aligned}\xi_{i,j}(t) &= \mathrm{P}(X_t = i, X_{t+1} = j | Y, \theta) \\ &= \frac{\alpha_i(t) P_{i,j} e_j(t+1) \beta_j(t+1)}{\mathrm{P}(Y|\theta)} \\ &= \frac{\alpha_i(t) P_{i,j} e_j(t+1) \beta_j(t+1)}{\sum_{k=1}^{N} \alpha_k(T)}\end{aligned}$$

With these probabilities, we can calculate $\mathrm{E}_{X|Y}[\log(\mathrm{P}(X, Y|\theta))]$:

$$\begin{aligned}\mathrm{E}_{X|Y}\left[\log P\left(X, Y|\theta\right)\right] &= \mathrm{E}_{X|Y}\left[\log e_{x_1}(y_1)\right] + \mathrm{E}_{X|Y}\left[\log \pi_{x_1}\right] + \\ &\quad \sum_{i=1}^{T-1} \left(\mathrm{E}_{X|Y}\left[\log P_{x_i, x_{i+1}}\right] + \mathrm{E}_{X|Y}\left[\log(e_{x_{i+1}}(y_{i+1}))\right] \right) \\ &= \sum_{j=1}^{N} \gamma_j(1) \log \pi_j + \sum_{t=1}^{T} \sum_{j=1}^{N} \log\left[e_j(y_t)\right] \gamma_j(t) + \\ &\quad \sum_{t=1}^{T-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \log(P_{i,j}) \xi_{i,j}(t)\end{aligned}$$

The expected number of emissions of state $k \in \{0, 1, 2, 3\}$ from (hidden) state i is

$$E_i(k) = \sum_{t=1}^{T} \gamma_i(t) \mathbb{I}(y_t = k), \tag{29}$$

and the expected number of transitions from hidden state i to hidden state j is

$$E_{i,j} = \sum_{t=1}^{T-1} \xi_{i,j}(t). \tag{30}$$

Then the expected log-likelihood can be written

$$\mathrm{E}_{X|Y}\left[\log P\left(X,Y|\theta\right)\right] = \sum_{j=1}^{N} \gamma_j(1)\log \pi_j + \sum_{j=1}^{N}\sum_{k=0}^{3} \log\left[e_j(k)\right] E_j(k) + \sum_{i=1}^{N}\sum_{j=1}^{N} \log(P_{i,j})E_{i,j}$$

4.1 Underflow

In order to avoid underflow problems, we calculate scaled versions of $\alpha_i(t)$ and $\beta_{i,j}(t)$. We define

$$\tilde{\alpha}_i(t) = \frac{\alpha_i(t)}{\prod_{j=1}^{t} s_j}, \tag{31}$$

for some scaling constants s_i, from which we can see the new recursion

$$\tilde{\alpha}_j(t+1) = \frac{1}{s_{t+1}} e_j(y_{t+1}) \sum_{i=1}^{N} P_{i,j}\tilde{\alpha}_i(t). \tag{32}$$

It is convenient to define s_t such that $\sum_{j=1}^{N} \tilde{\alpha}_j(t) = 1$. This means $s_1 = \sum_{k=1}^{N} \pi_i e_i(y_1)$ and

$$s_{t+1} = \sum_{j=1}^{N} e_j(y_{t+1}) \sum_{i=1}^{N} P_{i,j}\tilde{\alpha}_i(t) \tag{33}$$

We scale $\beta_i(t)$ by the same s_i so that

$$\tilde{\beta}_i(t) = \frac{\beta_i(t)}{\prod_{j=t+1}^{T} s_j} \tag{34}$$

and

$$\tilde{\beta}_i(t) = \frac{1}{s_{t+1}} \sum_{j=1}^{N} \tilde{\beta}_j(t+1) P_{i,j} e_j(y_{t+1}), \tag{35}$$

with $\tilde{\beta}_i(T) = \beta_i(T) = 1$.

Now $\gamma_i(t)$ can be written

$$\begin{aligned}
\gamma_i(t) &= \frac{\alpha_i(t)\beta_i(t)}{\sum_{j=1}^{N} \alpha_j(t)\beta_j(t)} \\
&= \frac{\tilde{\alpha}_i(t)\tilde{\beta}_i(t)\left(\prod_{j=1}^{T} s_j\right)}{\sum_{j=1}^{N} \tilde{\alpha}_j(t)\tilde{\beta}_j(t)\left(\prod_{j=1}^{T} s_j\right)} \\
&= \frac{\tilde{\alpha}_i(t)\tilde{\beta}_i(t)}{\sum_{j=1}^{N} \tilde{\alpha}_j(t)\tilde{\beta}_j(t)}
\end{aligned} \tag{36}$$

and $\xi_{i,j}(t)$ can now be written

$$
\begin{aligned}
\xi_{i,j}(t) &= \frac{\alpha_i(t)P_{i,j}e_j(t+1)\beta_j(t+1)}{\mathrm{P}(Y\,|\,\theta)} \\
&= \frac{\tilde{\alpha}_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)\prod_{k=1}^{T}s_k}{\mathrm{P}(Y\,|\,\theta)s_{t+1}} \\
&= \frac{\tilde{\alpha}_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)\prod_{k=1}^{T}s_k}{\sum_{l=1}^{N}\alpha_l(t)\beta_l(t)s_{t+1}} \\
&= \frac{\tilde{\alpha}_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)}{s_{t+1}\sum_{l=1}^{N}\tilde{\alpha}_l(t)\tilde{\beta}_l(t)} \\
&= \frac{\gamma_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)}{s_{t+1}\beta_i(t)}
\end{aligned}
\tag{37}
$$

Difficult to implement, but efficient and apparently correct!

5 Inferring diploid-to-triploid transition

In this section we consider the possibility that the asexual triploids were generated first by an asexual diploid line that is subsequently fertilized by a a haploid sperm, making a triploid asexual lineage. The goal is to infer the time between this diploid asexual phase and the triploid asexual phase, along with everything else that was considered for inference previously. In this case, the state at each position along the genome is the vector (t_3, t_2, W), where $W \in \{0, 1\}$ is an indicator for whether the branch introduced by the triploidizing sperm subtends one of the diploid sexual branches coalescing at t_3 ($W = 1$) or t_2 ($W = 0$). The transition to diploid asexuality is defined as $D_2 = 0$, and the transition to triploid asexuality is defined as $D_3 < 0$. See Fig. 1 for an illustration of these states.

Having to keep track of W doubles the number of states, and since the complexity of the forward-backward algorithm is squared in the number of states, the runtime should increase by ~ 4.

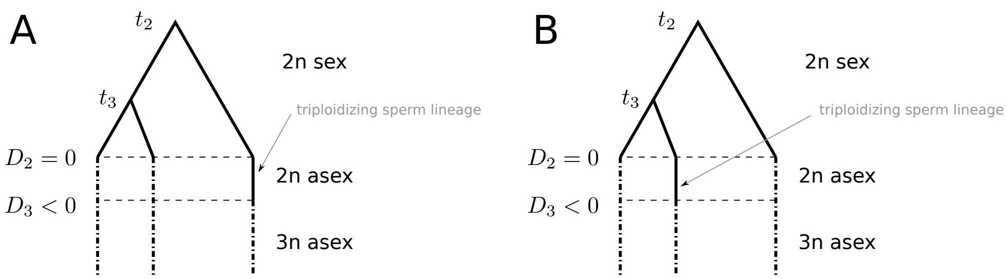

Figure 1: Figure 1. Two states with the same coalescence times t_3 and t_2 but with the triploidizing sperm's lineage subtending a branch that coalesceces with the others (in the diploid sexual phase) at t_2 (panel A, $W = 0$) and at t_3 (panel B, $W = 1$). $D_2 = 0$ is the time of transition to diploid asexuality, and $D_3 < 0$ is the time of the transition to triploidy.

5.1 Diploid-to-triploid transition probabilities — Summary

The transition probabilities of the hidden model largely remain the same. A few changes occur:

1. In all previous probabilities, the factor $(2t_2 + t_3)^{-1}$ is replaced by the inverse of the new total (sexual) tree length, $(2t_2 + t_3 - D_3)^{-1}$, recalling that $D_3 < 0$. These factors are included unchanged all the way through the derivations and are still present in the final discrete transition probabilities, so they can just be replaced with the new factor.

2. Certain transitions will now change W as well. For example, if $W = 0$, (recall, subtending a branch that coalesces at t_2), the transition $(s_3, s_2) \to (t_3 < s_3, t_2 = s_3)$ implies that now $W = 1$.

3. Additional transition probability must be included, arising from recombination events that occur on the triploidizing sperm's sexual lineage between $t = D_3$ and $t = 0$.

To do this properly, it will also now be necessary to model the population size changes that occur in the sexual population during the diploid asexual phase, since the triploidizing sperm's lineage will experience those demographic changes. However, the only probabilities that depend on these population sizes are the "healing" probabilities specific to the SMC', meaning that only the "effective recombination rate" for this single lineage will depend on this part of the demographic history. **For this reason, we will assume that the diploid sexual population containing the triploidizing sperm is constant in size.** [This approximation would be unnecessary under the SMC model (vs. the SMC'), since healing is impossible under the SMC.] It should be a fine approximation and will have to be tested.

In general, we can expect there to be very little information about the length of this hypothesized diploid asexual interval in the triploid asexual's present-day genome. Including this additional period of diploid asexuality doesn't change the emission probabilities at all; it only changes the transition probabilities of the hidden model, and these changes seem like they will be pretty minor. However, it may be possible to rule out long periods of diploid asexuality in the putatively ancient lineages.

5.2 Determining when W changes and when it remains the same

For $t_3 = s_3; t_2 > s_2$:

$W' = W$

For $t_3 = s_3; t_2 < s_2$:

$W' = W$

For $t_3 < s_3; t_2 = s_3$, and $W = 0$:

$W' = 1$

For $t_3 < s_3; t_2 = s_3$, and $W = 1$:

$W' = 1$ with prob. 1/2
$W' = 0$ with prob. 1/2

For $t_3 < s_3; t_2 = s_2$:

$W' = W$

For $t_3 > s_3; t_2 = s_2$:

$W' = W$

For $t_3 = s_2; t_2 > s_2, W = 0$:

$W' = 1$

For $t_3 = s_2; t_2 > s_2, W = 1$:

$W' = 1$ with prob. 1/2
$W' = 0$ with prob. 1/2

For $t_3 = s_3; t_2 = s_2$:

$W' = W$

(38)

5.3 Deriving the new probabilities

The following are the "supplemental" probabilities of transition at the site of a recombination event, due to recombination events happening along the lineage of the triploidizing sperm in the time interval (D_3, D_2).

For $t_3 = s_3; t_2 > s_2$, $W = 0$, and $W' = 0$:

$$\int_{D_3}^{0} \frac{du}{2s_2 + s_3 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)}$$

For $t_3 = s_3; t_2 < s_2$, $W = 0$, and $W' = 0$:

$$\int_{D_3}^{0} \frac{du}{2s_2 + s_3 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3,t_2)}$$

For $t_3 < s_3; t_2 = s_3$, $W = 0$, and $W' = 1$:

$$\int_{D_3}^{0} \frac{du}{2s_2 + s_3 - D_3} e^{-\Omega(u,0)} \frac{2}{\lambda(t_3)} e^{-3\Omega(0,t_3)}$$

For $t_3 < s_3; t_2 = s_2$, $W = 1$, and $W' = 1$:

$$\int_{D_3}^{0} \frac{du}{2s_2 + s_3 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,t_3)} \frac{2}{\lambda(t_3)}$$

For $t_3 > s_3; t_2 = s_2$, $W = 1$, $W' = 1$:

$$\int_{D_3}^{0} \frac{du}{2s_2 + s_3 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3,t_3)}$$

For $t_3 = s_2; t_2 > s_2$, $W = 1$, and $W' = 0$:

$$\int_{D_3}^{0} \frac{du}{2s_2 + s_3 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)}$$

5.4 Calculating discrete transition probabilities with diploid-triploid transition

Let the relative population size between D_3 and D_2 be λ_d. Replace s_3 and s_2 with $\mathrm{E}_{i,j}[s_3]$ and $\mathrm{E}_{i,j}[s_2]$, respectively.

5.4.1 Case A:

For $i = k < j < l$, $(t_3 = s_3\,; t_2 > s_2)$, $W' = W$:

$$\begin{aligned}
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i,j,k,l) + \\
&\quad \delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right) e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \int_{T_l}^{T_{l+1}} e^{-\Omega(s_2,t_2)} dt_2 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i,j,k,l) + \\
&\quad \delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right) e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)} dt_2 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i,j,k,l) + \\
&\quad \delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right) e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \lambda_l \left(1 - e^{-\frac{\Delta_l}{\lambda_l}}\right) \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i,j,k,l) + \\
&\quad \delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right) e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} e^{-\Omega(s_2,T_l)} \left(1 - e^{-\frac{\Delta_l}{\lambda_l}}\right)
\end{aligned} \tag{39}$$

As above we assume that $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

5.4.2 Case B:

For $i = k < l < j$ $(t_3 = s_3; t_2 < s_2)$ and $W = W'$:

$$
\begin{aligned}
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\delta(W) \int_{D_3}^{0} \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} \int_{T_l}^{T_{l+1}} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3,t_2)} dt_2 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right) e^{-3\Omega(0,s_3)} \frac{1}{\lambda_l} e^{-2\Omega(s_3,T_l)} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_l,t_2)} dt_2 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right) e^{-3\Omega(0,s_3)} \frac{1}{\lambda_l} e^{-2\Omega(s_3,T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}}\right) \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right) e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,T_l)} \frac{1}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}}\right)
\end{aligned} \tag{40}
$$

5.4.3 Case C

For $k < i = l < j$ $(t_3 < s_3; t_2 = s_3)$, $W = 0$, $W' = 1$:

$$
\begin{aligned}
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \int_{D_3}^{0} \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} \int_{T_k}^{T_{k+1}} \frac{2}{\lambda(t_3)} e^{-3\Omega(0,t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right) \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{3} e^{-3\Omega(0,T_k)} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)
\end{aligned} \tag{41}
$$

For $k < i = l < j$ $(t_3 < s_3; t_2 = s_3)$, $W = 1$, $W' \in \{0, 1\}$:

$$
= \frac{1}{2} \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) \tag{42}
$$

5.4.4 Case D

For $k < i < j = l$ ($t_3 < s_3; t_2 = s_2$) and $W' = W$:

$$
\begin{aligned}
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \delta(W-1) \int_{D_3}^{0} \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(0,t_3)} \frac{2}{\lambda(t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \delta(W-1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \delta(W-1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right) \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \delta(W-1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{3} e^{-3\Omega(0,T_k)} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)
\end{aligned}
\tag{43}
$$

5.4.5 Case E

For $i < k < j = l$ ($t_3 > s_3; t_2 = s_2$) and $W' = W$:

$$
\begin{aligned}
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \delta(W=1) \int_{D_3}^{0} \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} \int_{T_k}^{T_{k+1}} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3,t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \delta(W-1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} \frac{2}{\lambda_k} e^{-2\Omega(s_3,T_k)} \frac{\lambda_k}{2} \left(1 - e^{-\frac{2\Delta_k}{\lambda_k}}\right) \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q(i,j,k,l) + \\
&\quad \delta(W-1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,T_k)} \left(1 - e^{-\frac{2\Delta_k}{\lambda_k}}\right)
\end{aligned}
\tag{44}
$$

5.4.6 Case F

For $i < j = k < l$ ($t_3 = s_2; t_2 > s_2$) and $W = 1$:

$$
\begin{aligned}
&= \frac{2s_2+s_3}{2s_2+s_3-D_3}\frac{1}{2}\,q(i,j,k,l)+ \\
&\quad \delta(W')\int_{D_3}^{0}\frac{du}{2s_3+s_2-D_3}e^{-\Omega(u,0)}e^{-3\Omega(0,s_3)}e^{-2\Omega(s_3,s_2)}\int_{T_l}^{T_{l+1}}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}dt_2 \\
&= \frac{2s_2+s_3}{2s_2+s_3-D_3}\frac{1}{2}\,q(i,j,k,l)+ \\
&\quad \delta(W')\frac{\lambda_d\left(1-e^{\frac{D_3}{\lambda_d}}\right)}{2s_2+s_3-D_3}e^{-3\Omega(0,s_3)}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_l}e^{-\Omega(s_2,T_l)}\int_{T_l}^{T_{l+1}}e^{-\Omega(T_l,t_2)}dt_2 \\
&= \frac{2s_2+s_3}{2s_2+s_3-D_3}\frac{1}{2}\,q(i,j,k,l)+ \\
&\quad \delta(W')\frac{\lambda_d\left(1-e^{\frac{D_3}{\lambda_d}}\right)}{2s_2+s_3-D_3}e^{-3\Omega(0,s_3)}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_l}e^{-\Omega(s_2,T_l)}\lambda_l\left(1-e^{-\frac{\Delta_l}{\lambda_l}}\right) \\
&= \frac{2s_2+s_3}{2s_2+s_3-D_3}\frac{1}{2}\,q(i,j,k,l)+ \\
&\quad \delta(W')\frac{\lambda_d\left(1-e^{\frac{D_3}{\lambda_d}}\right)}{2s_2+s_3-D_3}e^{-3\Omega(0,s_3)}e^{-2\Omega(s_3,s_2)}e^{-\Omega(s_2,T_l)}\left(1-e^{-\frac{\Delta_l}{\lambda_l}}\right)
\end{aligned}
\tag{45}
$$

For $i<j=k<l$ ($t_3=s_2; t_2>s_2$) and $W=0$, $W'=1$:

$$
= \frac{2s_2+s_3}{2s_2+s_3-D_3}\,q(i,j,k,l) \tag{46}
$$

5.4.7 Case G

For $i=k=l<j$ and $t_3=s_3; t_2<s_2$ and $W=W'$:

$$
\begin{aligned}
&= \frac{2s_2+s_3}{2s_2+s_3-D_3}\,q^G(i,j,k,l)+ \\
&\quad \delta(W)\int_{D_3}^{0}\frac{du}{2s_3+s_2-D_3}e^{-\Omega(u,0)}e^{-3\Omega(0,s_3)}\int_{s_3}^{T_{l+1}}\frac{1}{\lambda(t_2)}e^{-2\Omega(s_3,t_2)}dt_2 \\
&= \frac{2s_2+s_3}{2s_2+s_3-D_3}\,q^G(i,j,k,l)+ \\
&\quad \delta(W)\frac{\lambda_d\left(1-e^{\frac{D_3}{\lambda_d}}\right)}{2s_2+s_3-D_3}e^{-3\Omega(0,s_3)}\frac{1}{\lambda_l}\int_{s_3}^{T_{l+1}}e^{-2\Omega(s_3,t_2)}dt_2 \\
&= \frac{2s_2+s_3}{2s_2+s_3-D_3}\,q^G(i,j,k,l)+ \\
&\quad \delta(W)\frac{\lambda_d\left(1-e^{\frac{D_3}{\lambda_d}}\right)}{2s_2+s_3-D_3}e^{-3\Omega(0,s_3)}\frac{1}{\lambda_l}\frac{\lambda_l}{2}\left(1-e^{-\frac{2(T_{l+1}-s_3)}{\lambda_l}}\right) \\
&= \frac{2s_2+s_3}{2s_2+s_3-D_3}\,q^G(i,j,k,l)+ \\
&\quad \delta(W)\frac{\lambda_d\left(1-e^{\frac{D_3}{\lambda_d}}\right)}{2s_2+s_3-D_3}e^{-3\Omega(0,s_3)}\frac{1}{2}\left(1-e^{-\frac{2(T_{l+1}-s_3)}{\lambda_l}}\right)
\end{aligned}
\tag{47}
$$

Here, $q^G(i,j,k,l)$ is case G without the diploid-to-triploid variables.

5.4.8 Case G2

For $i = k = l < j$ and $t_3 < s_3; t_2 = s_3$, $W = 0$, and $W' = 1$:

$$\begin{aligned}
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{G2}(i,j,k,l) + \\
&\quad \int_{D_3}^{0} \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} \int_{T_k}^{s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(0,t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{G2}(i,j,k,l) + \\
&\quad \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \int_{T_k}^{s_3} e^{-3\Omega(T_k,t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{G2}(i,j,k,l) + \\
&\quad \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \frac{\lambda_k}{3} \left(1 - e^{-\frac{3(s_3 - T_k)}{\lambda_k}}\right) \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{G2}(i,j,k,l) + \\
&\quad \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{3} e^{-3\Omega(0,T_k)} \left(1 - e^{-\frac{3(s_3 - T_k)}{\lambda_k}}\right)
\end{aligned} \tag{48}$$

For $t_3 < s_3; t_2 = s_3$, $W = 1$, $W' \in \{0, 1\}$:

$$= \frac{1}{2} \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{G2}(i,j,k,l) \tag{49}$$

5.4.9 Case H

For $i < k = l = j$ $(t_3 > s_3\,; t_2 = s_2)$ and $W = W'$:

$$\begin{aligned}
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{H}(i,j,k,l) + \\
&\quad \delta(W-1) \int_{D_3}^{0} \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} \int_{T_k}^{s_2} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3,t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{H}(i,j,k,l) + \\
&\quad \delta(W-1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} \frac{2}{\lambda_k} e^{-2\Omega(s_3,T_k)} \int_{T_k}^{s_2} e^{-2\Omega(T_k,t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{H}(i,j,k,l) + \\
&\quad \delta(W-1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} \frac{2}{\lambda_k} e^{-2\Omega(s_3,T_k)} \frac{\lambda_k}{2} \left(1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}}\right) \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{H}(i,j,k,l) + \\
&\quad \delta(W-1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,T_k)} \left(1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}}\right)
\end{aligned} \tag{50}$$

5.4.10 Case H2

For $i < k = l = j$ and $t_3 = s_2\,; t_2 > s_2$, $W = 0$, $W' = 1$:

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{H2}(i, j, k, l) \tag{51}$$

For $i < k = l = j$ and $t_3 = s_2\,; t_2 > s_2$, $W = 1$, $W' \in \{0, 1\}$:

$$\begin{aligned}
&= \frac{1}{2}\frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{H2}(i, j, k, l) + \\
&\quad \delta(W') \int_{D_3}^{0} \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \int_{s_2}^{T_{l+1}} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} dt_2 \\
&= \frac{1}{2}\frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{H2}(i, j, k, l) + \\
&\quad \delta(W') \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \int_{s_2}^{T_{l+1}} e^{-\Omega(s_2,t_2)} dt_2 \\
&= \frac{1}{2}\frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{H2}(i, j, k, l) + \\
&\quad \delta(W') \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \lambda_l \left(1 - e^{-\frac{T_{l+1} - s_2}{\lambda_l}}\right) \\
&= \frac{1}{2}\frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{H2}(i, j, k, l) + \\
&\quad \delta(W') \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \left(1 - [1 - \delta(l - n)] e^{-\frac{T_{l+1} - s_2}{\lambda_l}}\right)
\end{aligned} \tag{52}$$

5.4.11 Case I

For $i = j = k < l$ and $t_3 = s_3; t_2 > s_2$, $W' = W$:

$$\begin{aligned}
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{(I)}(i, j, k, l) + \\
&\quad \delta(W) \int_{D_3}^{0} \frac{1}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \int_{T_l}^{T_{l+1}} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} dt_2 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{(I)}(i, j, k, l) + \\
&\quad \delta(W) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)} dt_2 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{(I)}(i, j, k, l) + \\
&\quad \delta(W) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \lambda_l \left(1 - e^{-\frac{\Delta_l}{\lambda_l}}\right) \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{(I)}(i, j, k, l) + \\
&\quad \delta(W) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} e^{-\Omega(s_2,T_l)} \left(1 - e^{-\frac{\Delta_l}{\lambda_l}}\right)
\end{aligned} \tag{53}$$

5.4.12 Case I2

For $i = j = k < l$ and $t_3 = s_2; t_2 > s_2$, $W = 0$, $W' = 1$:

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3}\, q^{I2}(i,j,k,l) \tag{54}$$

For $i = j = k < l$ and $t_3 = s_2; t_2 > s_2$, $W = 1$, $W' \in \{0, 1\}$:

$$\begin{aligned}
&= \frac{1}{2}\frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{I2}(i,j,k,l) + \\
&\quad \delta(W') \int_{D_3}^{0} \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \int_{T_l}^{T_{l+1}} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} dt_2 \\
&= \frac{1}{2}\frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{I2}(i,j,k,l) + \\
&\quad \delta(W') \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)} dt_2 \\
&= \frac{1}{2}\frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{I2}(i,j,k,l) + \\
&\quad \delta(W') \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \lambda_l \left(1 - e^{-\frac{\Delta_l}{\lambda_l}}\right) \\
&= \frac{1}{2}\frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{I2}(i,j,k,l) + \\
&\quad \delta(W') \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} e^{-\Omega(s_2,T_l)} \left(1 - e^{-\frac{\Delta_l}{\lambda_l}}\right)
\end{aligned} \tag{55}$$

5.4.13 Case J

For $k < i = j = l$ and $t_3 < s_3; t_2 = s_2$, $W' = W$:

$$\begin{aligned}
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{J}(i,j,k,l) + \delta(W - 1) \int_{D_3}^{0} \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{J}(i,j,k,l) + \delta(W - 1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right) \\
&= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{J}(i,j,k,l) + \delta(W - 1) \frac{\lambda_d \left(1 - e^{\frac{D_3}{\lambda_d}}\right)}{2s_2 + s_3 - D_3} \frac{2}{3} e^{-3\Omega(0,T_k)} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)
\end{aligned} \tag{56}$$

5.4.14 Case J2

For $k < i = j = l$ and $t_3 < s_3; t_2 = s_3$, $W = 0$, $W' = 1$:

$$
\begin{aligned}
&= \frac{2s_2+s_3}{2s_2+s_3-D_3} q^{J2}(i,j,k,l) + \int_{D_3}^{0} \frac{du}{2s_3+s_2-D_3} e^{-\Omega(u,0)} \int_{T_k}^{T_{k+1}} \frac{2}{\lambda(t_3)} e^{-3\Omega(0,t_3)} dt_3 \\
&= \frac{2s_2+s_3}{2s_2+s_3-D_3} q^{J2}(i,j,k,l) + \frac{\lambda_d\left(1-e^{\frac{D_3}{\lambda_d}}\right)}{2s_2+s_3-D_3} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 \\
&= \frac{2s_2+s_3}{2s_2+s_3-D_3} q^{J2}(i,j,k,l) + \frac{\lambda_d\left(1-e^{\frac{D_3}{\lambda_d}}\right)}{2s_2+s_3-D_3} \frac{2}{\lambda_k} e^{-3\Omega(0,T_k)} \frac{\lambda_k}{3}\left(1-e^{-\frac{3\Delta_k}{\lambda_k}}\right) \\
&= \frac{2s_2+s_3}{2s_2+s_3-D_3} q^{J2}(i,j,k,l) + \frac{\lambda_d\left(1-e^{\frac{D_3}{\lambda_d}}\right)}{2s_2+s_3-D_3} \frac{2}{3} e^{-3\Omega(0,T_k)} \left(1-e^{-\frac{3\Delta_k}{\lambda_k}}\right)
\end{aligned}
\tag{57}
$$

For $k < i = j = l$ and $t_3 < s_3; t_2 = s_3$, $W = 1$, $W' \in \{0,1\}$:

$$
= \frac{1}{2} \frac{2s_2+s_3}{2s_2+s_3-D_3} q^{J2}(i,j,k,l) \tag{58}
$$

(Double-checked these.)

6 Harris et al. algorithm for triploid genealogy

Define $f(x_{1:l}, j)$ as the joint forward probability of observing the partial emitted sequence $x_{1:l} = x_1, \ldots, x_l$ and the hidden state (coalescence time) $T_l = j$ at locus l. Let the number of discretized time intervals be d, and define $\phi(j \mid k)$ as $\mathrm{P}(T_{l+1} = j \mid T_l = k)$.

$$
f(x_{1:l+1}, j) = \xi(x_{l+1} \mid j) \sum_{k=1}^{d} f(x_{1:l}, k)\phi(j \mid k), \tag{59}
$$

which contains d terms. Since there are d possibilities for j, the algorithm should take $O(d^2 L)$ time to compute.

Decompose the transition from l to $l+1$ as a sequence of component events. Let R_i be the event that a recombination event occurs during time interval i, and let $\bar{R}$ be the event that no recombination occurs. Then

$$
\phi(j \mid k) = \sum_{i=1}^{\min(j,k)} \mathrm{P}(R_i, T_{l+1} = j \mid T_l = k) + \mathbb{1}_{j=k} \mathrm{P}(\bar{R} \mid T_l = k) \tag{60}
$$

Two events: $C_{>i}$ denotes the event where the recombination event occurs at or before i and floats into $i+1$, and C_i denotes the event where the recombined lineage coalesces back in interval i. Note that $\mathrm{P}(R_i, C_i \mid T_l = i')$ is independent of i' whenever $i < i'$. (Still working between l and $l+1$, here.) This is also true of $\mathrm{P}(R_i, C_{>i} \mid T_l = i')$. Then we can write

$$
\mathrm{P}(R_i, T_{l+1} = j \mid T_l = k) = \begin{cases}
\mathrm{P}(R_i, C_i \mid T_l = i), & i = j = k \\
\mathrm{P}(R_i, C_i \mid T_l > i), & i = j < k \\
\mathrm{P}(R_i, C_{>i} \mid T_l = i)\, \mathrm{P}(C_j \mid C_{>j-1}) \prod_{m=i}^{j-1} \mathrm{P}(C_{>k'+1} \mid C_{>k'}), & i = k < j \\
\mathrm{P}(R_i, C_{>i} \mid T_l > i)\, \mathrm{P}(C_j \mid C_{>j-1}) \prod_{k'=i}^{j-1} \mathrm{P}(C_{>k'+1} \mid C_{>k'}), & i < \min(j,k)
\end{cases} \tag{61}
$$

By combining (60) and (61), we can remove the sum over $T_l = k$ in computing $f(x_{1:l+1}, j)$. Define the additional forward probabilities

$$f(x_{1:l}, T_l = k) := \mathrm{P}(x_{1:l}, T_l = k) \tag{62}$$

$$f(x_{1:l}, T_l > k) := \mathrm{P}(x_{1:l}, T_l > k) = \sum_{k'=k+1}^{d} f(x_{1:l}, T_l = k) \tag{63}$$

$$\begin{aligned} f(x_{1:l}, R_{\le j}, C_{>j}) :=& \sum_{i=1}^{j} \mathrm{P}(x_{1:l}, R_i, C_{>i}, \ldots, C_{>j}) \\ =& \sum_{i=1}^{j} \Bigg\{ \prod_{i'=i+1}^{j} \mathrm{P}(C_{>i'} \mid C_{>i'-1}) \\ &\times \left[f(x_{1:l}, T_l = i)\, \mathrm{P}(R_i, C_{>i} \mid T_l = i) + f(x_{1:l}, T_l > i)\, \mathrm{P}(R_i, C_{>i} | T_l > i) \right] \Bigg\}. \end{aligned} \tag{64}$$

Then (59) can be written as

$$\begin{aligned} f(x_{1:l+1}, j) =& \, \xi(x_{l+1} \mid j) \\ &\times \Bigg[f(x_{1:l}, R_{\le j-1}, C_{>j-1})\, \mathrm{P}(C_j \mid C_{>j-1}) + f(x_{1:l}, T_l > j)\, \mathrm{P}(R_j, C_j \mid T_l > j) + \\ & f(x_{1:l}, T_l = j)\, \mathrm{P}(R_j, C_j \mid T_l = j) + f(x_{1:l}, j)\, \mathrm{P}(\bar{R} \mid T_l = j) \Bigg]. \end{aligned} \tag{65}$$

7 Genotype likelihoods and emission probabilities

When using observed genotypes, the emissions probability, given the hidden state (t_3, t_2) (with T_d), is

$$P_e(G) = e_G(t_3, t_2, T_d) = \begin{cases} e^{-\frac{\theta b(2t_2+t_3+3T_d)}{2}} & G = 0 \\ e^{-\frac{\theta b(t_2-t_3)}{2}} \left(1 - e^{-\frac{\theta b(2t_3+t_2+3T_d)}{2}}\right) & G = 1 \\ \left(1 - e^{-\frac{\theta b(t_2-t_3)}{2}}\right) e^{-\frac{\theta b(2t_3+t_2+3T_d)}{2}} & G = 2 \\ \left(1 - e^{-\frac{\theta b(t_2-t_3)}{2}}\right) \left(1 - e^{-\frac{\theta b(2t_3+t_2+3T_d)}{2}}\right) & G = 3. \end{cases} \tag{66}$$

(Using Vladimir's equations and notation...) Given a window of base pairs ν, with reads $S^{(\nu)}$ covering that window, we want $P_e(\nu)$, the emission probability of the reads in that window. This is

$$P_e(\nu) = P(S^{(\nu)} \mid T) = \prod_{i \in \nu} P(S_i^{(\nu)} \mid T) = \prod_{i \in \nu} \sum_{G_i} P(S_i^{(\nu)} \mid G_i) P(G_i \mid T), \tag{67}$$

where $S_i^{(\nu)}$ is the set of reads in the ith position in window ν. Here, T is the hidden state (t_3, t_2, T_d), and $P(G_i \mid T) = e_G(t_3, t_2, T_d)$.

8 GATK likelihoods for triploids

For each base quality phred score $i \in 0, \ldots, 256$, the probability of an accurate base call is $1 - 10^{-i/10}$, and the probability of an error is $(10^{-i/10})/3$ for each base other than the best. Given genotype $G = \{A_1, A_2, A_3\}$,

$$P(D \mid G) = \prod_{b \in \text{pileup}} P(b \mid G), \tag{68}$$

$$P(b \mid G) = \frac{1}{3}P(b \mid A_1) + \frac{1}{3}P(b \mid A_2) + \frac{1}{3}P(b \mid A_3), \tag{69}$$

and

$$P(b \mid A) = \begin{cases} 1 - 10^{-q_b/10} & b = A \\ \frac{10^{-q_b/10}}{3} & b \neq a \end{cases} \tag{70}$$

A (log) likelihood needs to be calculated for each of the 20 genotypes.