

1 General equations

Define $N(t)$ as the population size at time t , with t in units of $2N(0)$. Let $\lambda(t)$ be the relative population size, scaled by $N(0)$, such that $N(t) = N(0)\lambda(t)$. Define $\Omega(u, v)$ as the cumulative coalescent rate between times u and v :

$$\Omega(u, v) = \int_u^v \frac{dt}{\lambda(t)}. \quad (1)$$

The state of the TSMC at each point along the genome is described by the vector $\mathbf{s} = (s_3, s_2)$, where s_3 is the time of the first coalescence event and s_2 is the time of the second coalescence event amongst the three lineages in a triploid genome. The equilibrium joint distribution of (t_3, t_2) is

$$\pi(t_3, t_2) = \frac{3}{\lambda(t_3)\lambda(t_2)} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)}. \quad (2)$$

Let $q(\mathbf{t}|\mathbf{s})$ be the transition kernel at recombination sites along the genome. Then

$$q(\mathbf{t}|\mathbf{s}) =$$

For $t_3 = s_3; t_2 > s_2$:

$$\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} + 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} e^{-2\Omega(u, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)}$$

For $t_3 = s_3; t_2 < s_2$:

$$\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3, t_2)} + 2 \int_{s_3}^{t_2} \frac{du}{2s_2 + s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u, t_2)}$$

For $t_3 < s_3; t_2 = s_3$:

$$\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)}$$

For $t_3 < s_3; t_2 = s_2$:

$$2 \int_0^{t_3} \frac{du}{2s_2 + s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u, t_3)}$$

For $t_3 > s_3; t_2 = s_2$:

$$2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)}$$

For $t_3 = s_2; t_2 > s_2$:

$$2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)}$$

For $t_3 = s_3; t_2 = s_2$:

$$3 \int_0^{s_3} \frac{du}{2s_2 + s_3} \frac{1}{3} \left[1 - e^{-3\Omega(u, s_3)} \right] + \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{2} \left[1 - e^{-2\Omega(s_3, s_2)} \right] + 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} \frac{1}{2} \left[1 - e^{-2\Omega(u, s_2)} \right]. \quad (3)$$

Each part is implicitly multiplied by a delta function to limit the density to points where the parameters are assumed to be equal to each other. For example, the first part of $q(\mathbf{t}|\mathbf{s})$ is implicitly multiplied by $\delta(t_3 - s_3)$, and the last part is multiplied by $\delta(t_3 - s_3)\delta(t_2 - s_2)$.

2 Piecewise constant transition probabilities

Suppose that the population changes size at times (T_1, \dots, T_n) and that the size between T_i and T_{i+1} is a constant $2N\lambda_i$. Define $T_0 = 0$, $T_{n+1} = \infty$ and $\Delta_i = T_{i+1} - T_i$. Let $\alpha(t)$ be the index of the time interval to which t belongs, *i.e.*, $\alpha(t) = \max_i \{i : T_i \leq t\}$.

Then the cumulative coalescent rate between u and v can be written

$$\Omega(u, v) = \begin{cases} \frac{v-u}{\lambda_{\alpha(u)}} & \alpha(u) = \alpha(v) \\ \frac{T_{\alpha(u)+1}-u}{\lambda_{\alpha(u)}} + \sum_{i=\alpha(u)+1}^{\alpha(v)-1} \frac{\Delta_i}{\lambda_i} + \frac{v-T_{\alpha(v)}}{\lambda_{\alpha(v)}} & \alpha(u) < \alpha(v). \end{cases} \quad (4)$$

The equilibrium joint density of (t_3, t_2) is now approximately

$$\pi(t_3, t_2) = \frac{3}{\lambda_{\alpha(t_3)}\lambda_{\alpha(t_2)}} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)}. \quad (5)$$

There are several integrals of the form $\int_x^y e^{-k\Omega(u, y)} du$ in Equation (??). This integral can be written

$$\begin{aligned}
\int_x^y e^{-k\Omega(u,y)} du &= \int_x^{T_{\alpha(x)+1}} e^{-k\Omega(u,y)} du + \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \int_{T_i}^{T_{i+1}} e^{-k\Omega(u,y)} du + \int_{T_{\alpha(y)}}^y e^{-k\Omega(u,y)} du \\
&= \int_x^{T_{\alpha(x)+1}} \exp \left(-k \left[(T_{\alpha(u)+1} - u) \frac{1}{\lambda_{\alpha(u)}} + \sum_{j=\alpha(u)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) du + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \int_{T_i}^{T_{i+1}} \exp \left(-k \left[(T_{\alpha(u)+1} - u) \frac{1}{\lambda_{\alpha(u)}} + \sum_{j=\alpha(u)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) du + \\
&\quad \int_{T_{\alpha(y)}}^y \exp \left(-k(y - u) \frac{1}{\lambda_{\alpha(y)}} \right) du \\
&= \int_x^{T_{\alpha(x)+1}} \exp \left(-k \left[(T_{\alpha(x)+1} - u) \frac{1}{\lambda_{\alpha(x)}} + \sum_{j=\alpha(x)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) du + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \int_{T_i}^{T_{i+1}} \exp \left(-k \left[(T_{i+1} - u) \frac{1}{\lambda_i} + \sum_{j=i+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) du + \\
&\quad \int_{T_{\alpha(y)}}^y \exp \left(-k(y - u) \frac{1}{\lambda_{\alpha(y)}} \right) du \\
&= \exp \left(-k \left[\sum_{j=\alpha(x)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) \int_x^{T_{\alpha(x)+1}} \exp \left(-k (T_{\alpha(x)+1} - u) \frac{1}{\lambda_{\alpha(x)}} \right) du + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \exp \left(-k \left[\sum_{j=i+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) \int_{T_i}^{T_{i+1}} \exp \left(-k (T_{i+1} - u) \frac{1}{\lambda_i} \right) du + \\
&\quad \int_{T_{\alpha(y)}}^y \exp \left(-k(y - u) \frac{1}{\lambda_{\alpha(y)}} \right) du \\
&= \exp \left(-k \left[\sum_{j=\alpha(x)+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) \left[1 - \exp \left(-\frac{k (T_{\alpha(x)+1} - x)}{\lambda_{\alpha(x)}} \right) \right] \frac{\lambda_{\alpha(x)}}{k} + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \exp \left(-k \left[\sum_{j=i+1}^{\alpha(y)-1} \frac{\Delta_j}{\lambda_j} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}} \right] \right) \left[1 - \exp \left(-\frac{k \Delta_i}{\lambda_i} \right) \right] \frac{\lambda_i}{k} + \\
&\quad \left[1 - \exp \left(-\frac{k (y - T_{\alpha(y)})}{\lambda_{\alpha(y)}} \right) \right] \frac{\lambda_{\alpha(y)}}{k} \\
&= \exp (-k\Omega(T_{\alpha(x)+1}, y)) \left[1 - \exp \left(-\frac{k (T_{\alpha(x)+1} - x)}{\lambda_{\alpha(x)}} \right) \right] \frac{\lambda_{\alpha(x)}}{k} + \\
&\quad \sum_{i=\alpha(x)+1}^{\alpha(y)-1} \exp (-k\Omega(T_{i+1}, y)) \left[1 - \exp \left(-\frac{k \Delta_i}{\lambda_i} \right) \right] \frac{\lambda_i}{k} + \\
&\quad \left[1 - \exp \left(-\frac{k (y - T_{\alpha(y)})}{\lambda_{\alpha(y)}} \right) \right] \frac{\lambda_{\alpha(y)}}{k} \\
&= e^{-k\Omega(T_{\alpha(x)+1}, y)} \left[1 - e^{-\frac{k (T_{\alpha(x)+1} - x)}{\lambda_{\alpha(x)}}} \right] \frac{\lambda_{\alpha(x)}}{k} + \sum_{i=\alpha(x)+1}^{\alpha(y)-1} e^{-k\Omega(T_{i+1}, y)} \left[1 - e^{-\frac{k \Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{k} + \left[1 - e^{-\frac{k (y - T_{\alpha(y)})}{\lambda_{\alpha(y)}}} \right] \frac{\lambda_{\alpha(y)}}{k}
\end{aligned} \tag{6}$$

With this equation, we can calculate all of the transition probabilities in the transition kernel (??).

For $t_3 = s_3; t_2 > s_2$:

$$\begin{aligned} & \frac{1}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ & \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \times \\ & \left\{ e^{-2\Omega(T_{\alpha(s_3)+1}, s_2)} \left[1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{i=\alpha(s_3)+1}^{\alpha(s_2)-1} e^{-2\Omega(T_{i+1}, s_2)} \left[1 - e^{-\frac{2\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{2} \right. \\ & \quad \left. + \left[1 - e^{-\frac{2(s_2 - T_{\alpha(s_2)})}{\lambda_{\alpha(s_2)}}} \right] \frac{\lambda_{\alpha(s_2)}}{2} \right\} \\ & \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} + 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} e^{-2\Omega(u, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \end{aligned}$$

For $t_3 = s_3; t_2 < s_2$:

$$\begin{aligned} & \frac{1}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-2\Omega(s_3, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ & \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} \\ & \left\{ e^{-2\Omega(T_{\alpha(s_3)+1}, t_2)} \left[1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{i=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{i+1}, t_2)} \left[1 - e^{-\frac{2\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{2} \right. \\ & \quad \left. + \left[1 - e^{-\frac{2(t_2 - T_{\alpha(t_2)})}{\lambda_{\alpha(t_2)}}} \right] \frac{\lambda_{\alpha(t_2)}}{2} \right\} \\ & \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3, t_2)} + 2 \int_{s_3}^{t_2} \frac{du}{2s_2 + s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u, t_2)} \end{aligned}$$

For $t_3 < s_3; t_2 = s_3$:

$$\begin{aligned} & \frac{1}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\} \\ & \int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \end{aligned}$$

For $t_3 < s_3; t_2 = s_2$:

$$\begin{aligned} & \frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\} \\ & 2 \int_0^{t_3} \frac{du}{2s_2 + s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u, t_3)} \end{aligned}$$

For $t_3 > s_3; t_2 = s_2$:

$$\frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} e^{-2\Omega(s_3, t_3)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\ 2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)}$$

For $t_3 = s_2; t_2 > s_2$:

$$\frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\ 2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)}$$

For $t_3 = s_3; t_2 = s_2$:

$$\frac{1}{2s_2 + s_3} \left\{ s_3 - \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} - \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ \frac{1}{2s_2 + s_3} \frac{1}{2} \left[1 - e^{-2\Omega(s_3, s_2)} \right] \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ \frac{1}{2s_2 + s_3} \left[s_2 - s_3 - e^{-2\Omega(T_{\alpha(s_3)+1}, t_2)} \left(1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right) \frac{\lambda_{\alpha(s_3)}}{2} - \sum_{i=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{i+1}, t_2)} \left(1 - e^{-\frac{2\Delta_i}{\lambda_i}} \right) \frac{\lambda_i}{2} \right. \\ \left. - \left(1 - e^{-\frac{2(t_2 - T_{\alpha(t_2)})}{\lambda_{\alpha(t_2)}}} \right) \frac{\lambda_{\alpha(t_2)}}{2} \right] \\ 3 \int_0^{s_3} \frac{du}{2s_2 + s_3} \frac{1}{3} \left[1 - e^{-3\Omega(u, s_3)} \right] + \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{2} \left[1 - e^{-2\Omega(s_3, s_2)} \right] + 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} \frac{1}{2} \left[1 - e^{-2\Omega(u, s_2)} \right].$$

3 Discrete approximation to the triploid SMC' coalescent process

In order to construct a hidden Markov model (HMM) to infer demography, it is necessary to discretize the triploid coalescent process described above.

Let the discrete state (i, j) , $i < j$, correspond to the continuous states in which $T_i < t_3 < T_{i+1}$ and $T_j < t_2 < T_{j+1}$. We first calculate the equilibrium probability that the coalescent process is in (i, j) , assuming $i < j$:

$$\begin{aligned}
\pi_{i,j} &= \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{T_j}^{T_{j+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\Omega(T_j,t_2)} dt_2 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\frac{t_2-T_j}{\lambda_j}} dt_2 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} e^{-\frac{T_{i+1}-t_3}{\lambda_i}} dt_3 \\
&= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \\
&= \frac{3}{2} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right].
\end{aligned} \tag{7}$$

If $j = n$, we let $\Delta_j = \infty$ and $1 - \exp(-\Delta_j/\lambda_j) = 1$.

It is also necessary to calculate $\pi_{i,i}$:

$$\begin{aligned}
\pi_{i,i} &= \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{3}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{1}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\frac{t_2-t_3}{\lambda_i}} dt_2 dt_3 \\
&= \frac{3}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \left(\lambda_i \left[1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} \right] \right) dt_3 \\
&= \frac{3}{\lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} \right) dt_3 \\
&= \frac{3}{\lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{6} \left(2 - 3e^{-\frac{\Delta_i}{\lambda_i}} + e^{-\frac{3\Delta_i}{\lambda_i}} \right) \\
&= \frac{1}{2} e^{-3\Omega(0,T_i)} \left(2 - 3e^{-\frac{\Delta_i}{\lambda_i}} + e^{-\frac{3\Delta_i}{\lambda_i}} \right)
\end{aligned} \tag{8}$$

For $i = n$, again we let $\Delta_i = \infty$ and thus $\pi_{i,i} = \exp(-3\Omega(0,T_i))$.

Next, we calculate marginal expectations for t_3 and t_2 given that the continuous process is in interval represented by (i, j) , assuming $i < j$. The marginal expectation of t_3 in the interval (i, j) is

$$\begin{aligned}
E_{i,j}[t_3] &= E[t_3 | t_3 \in [T_i T_{i+1}), t_2 \in [T_j, T_{j+1})] \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3t_3}{\lambda_i \lambda_j} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} \int_{T_j}^{T_{j+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\Omega(T_j,t_2)} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\frac{t_2-T_j}{\lambda_j}} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{4} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}} \right] \\
&= \frac{3}{4\pi_{i,j}} e^{-3\Omega(0,T_i)} \left(1 - e^{-\frac{\Delta_j}{\lambda_j}}\right) e^{-\Omega(T_{i+1},T_j)} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}} \right]
\end{aligned} \tag{9}$$

With $j = n$, this is

$$\begin{aligned}
E_{i,j}[t_3] &= E[t_3 | t_3 \in [T_i T_{i+1}), t_2 \in [T_n, \infty)] \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} \frac{3t_3}{\lambda_i \lambda_n} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} \int_{T_n}^{\infty} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_n)} dt_3 \int_{T_n}^{\infty} e^{-\Omega(T_n,t_2)} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_n)} dt_3 \int_{T_n}^{\infty} e^{-\frac{t_2-T_n}{\lambda_n}} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} \lambda_n \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_n)} dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} \lambda_n e^{-\Omega(T_{i+1},T_n)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} \lambda_n e^{-\Omega(T_{i+1},T_n)} \frac{\lambda_i}{4} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}} \right] \\
&= \frac{3}{4\pi_{i,n}} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \left[(\lambda_i + 2T_i) e^{\frac{-\Delta_i}{\lambda_i}} - (\lambda_i + 2T_{i+1}) e^{\frac{-3\Delta_i}{\lambda_i}} \right]
\end{aligned} \tag{10}$$

The marginal expectation of t_2 in (i, j) is

$$\begin{aligned}
E_{i,j}[t_2] &= E[t_2 | t_3 \in [T_i, T_{i+1}), t_2 \in [T_j, T_{j+1}]] \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} t_2 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3t_2}{\lambda_i \lambda_j} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_{i+1})} dt_3 \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3 - T_i)}{\lambda_i}} e^{-\frac{T_{i+1} - t_3}{\lambda_i}} dt_3 \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j, t_2)} dt_2 \quad (11) \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \int_{T_j}^{T_{j+1}} t_2 e^{-\frac{t_2 - T_j}{\lambda_j}} dt_2 \\
&= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \lambda_j \left(\lambda_j + T_j - (\lambda_j + T_{j+1}) e^{-\frac{\Delta_j}{\lambda_j}} \right) \\
&= \frac{3}{2\pi_{i,j}} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \left(\lambda_j + T_j - (\lambda_j + T_{j+1}) e^{-\frac{\Delta_j}{\lambda_j}} \right).
\end{aligned}$$

With $j = n$, this is

$$\begin{aligned}
E_{i,n}[t_2] &= E[t_2 | t_3 \in [T_i, T_{i+1}), t_2 \in [T_n, \infty)] \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} t_2 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} \frac{3t_2}{\lambda_i \lambda_n} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_{i+1})} dt_3 \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3 - T_i)}{\lambda_i}} e^{-\frac{T_{i+1} - t_3}{\lambda_i}} dt_3 \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n, t_2)} dt_2 \quad (12) \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n, t_2)} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \int_{T_n}^{\infty} t_2 e^{-\frac{t_2 - T_n}{\lambda_n}} dt_2 \\
&= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \lambda_n (\lambda_n + T_n) \\
&= \frac{3}{2\pi_{i,n}} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_n)} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) (\lambda_n + T_n)
\end{aligned}$$

We also calculate the marginal expectations of s_3 and s_2 conditional on the interval (i, i) :

$$\begin{aligned}
E_{i,i}[t_3] &= E[t_3 | t_3 \in [T_i T_{i+1}), t_2 \in [T_i, T_{i+1}]] \\
&= \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{t_3}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-\frac{3(t_3-T_i)}{\lambda_i}} \lambda_i \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right) dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} t_3 e^{-\frac{3(t_3-T_i)}{\lambda_i}} \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right) dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{36} \left(-9e^{-\frac{\Delta_i}{\lambda_i}} (2T_i + \lambda_i) + 4(3T_i + \lambda_i) + e^{-\frac{3\Delta_i}{\lambda_i}} (6T_{i+1} + 5\lambda_i)\right) \\
&= \frac{1}{12\pi_{i,i}} e^{-3\Omega(0,T_i)} \left(4(3T_i + \lambda_i) + e^{-\frac{3\Delta_i}{\lambda_i}} (6T_{i+1} + 5\lambda_i) - 9e^{-\frac{\Delta_i}{\lambda_i}} (2T_i + \lambda_i)\right)
\end{aligned} \tag{13}$$

For $i = n$, the expectation is

$$E_{n,n}[s_3] = T_n + \frac{\lambda_n}{3} \tag{14}$$

Double-checking:

$$\begin{aligned}
E_{n,n}[t_3] &= E[t_3 | t_3 \in [T_n, \infty), t_2 \in [T_n, \infty)] \\
&= \frac{1}{\pi_{n,n}} \int_{T_n}^{\infty} \int_{t_3}^{\infty} t_3 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n^2} \int_{T_n}^{\infty} \int_{t_3}^{\infty} t_3 e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n^2} \int_{T_n}^{\infty} t_3 e^{-3\Omega(0,t_3)} \int_{t_3}^{\infty} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n} \int_{T_n}^{\infty} t_3 e^{-3\Omega(0,t_3)} dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n} e^{-3\Omega(0,T_n)} \int_{T_n}^{\infty} t_3 e^{-3\Omega(T_n,t_3)} dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n} e^{-3\Omega(0,T_n)} \int_{T_n}^{\infty} t_3 e^{-\frac{3(t_3-T_n)}{\lambda_n}} dt_3 \\
&= \frac{3}{\pi_{n,n} \lambda_n} e^{-3\Omega(0,T_n)} \frac{\lambda_n}{9} (3T_n + \lambda_n) \\
&= \frac{1}{3\pi_{n,n}} e^{-3\Omega(0,T_n)} (3T_n + \lambda_n)
\end{aligned} \tag{15}$$

Checks out since $\pi_{n,n} = \exp(-3\Omega(0, T_n))$.

$$\begin{aligned}
\mathbb{E}_{i,i}[t_2] &= \mathbb{E}[t_2 | t_3 \in [T_i, T_{i+1}), t_2 \in [T_i, T_{i+1})] \\
&= \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_2 \pi(t_3, t_2) dt_2 dt_3 \\
&= \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{3t_2}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_2 e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} t_2 e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} t_2 e^{-\frac{t_2-t_3}{\lambda_i}} dt_2 dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \lambda_i \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} (T_{i+1} + \lambda_i) \right] dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \lambda_i \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} (T_{i+1} + \lambda_i) \right] dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1}-t_3}{\lambda_i}} (T_{i+1} + \lambda_i) \right] dt_3 \\
&= \frac{3}{\pi_{i,i} \lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{18} \left(e^{-\frac{3\Delta_i}{\lambda_i}} (3T_{i+1} + \lambda_i) + 6T_i + 8\lambda_i - 9e^{-\frac{\Delta_i}{\lambda_i}} (T_{i+1} + \lambda_i) \right) \\
&= \frac{1}{6\pi_{i,i}} e^{-3\Omega(0,T_i)} \left(e^{-\frac{3\Delta_i}{\lambda_i}} (3T_{i+1} + \lambda_i) + 6T_i + 8\lambda_i - 9e^{-\frac{\Delta_i}{\lambda_i}} (T_{i+1} + \lambda_i) \right)
\end{aligned} \tag{16}$$

For $i = n$, this expectation is

$$\mathbb{E}_{n,n}[s_2] = T_n + \frac{\lambda_n}{3} + \lambda_n \tag{17}$$

3.1 Discrete $q((k, l) | (i, j))$ transition function

To calculate the discrete-process transition probabilities from (i, j) , to (k, l) , we integrate the transition kernel (??) over the interval corresponding to (k, l) , replacing s_3 and s_2 with their conditional expectations $\mathbb{E}_{i,j}[s_3]$ and $\mathbb{E}_{i,j}[s_2]$ respectively. Thus

$$q((k, l) | (i, j)) = \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} q((t_3, t_2) | (\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])) dt_2 dt_3. \tag{18}$$

Note that in any single transition, either the first or second coalescence time changes, but not both. This simplifies the calculation of these integrals.

3.1.1 Case A

For $i = k < j < l$, ($t_3 = s_3; t_2 > s_2$):

$$\begin{aligned}
&= \frac{1}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\
&\quad \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \times \\
&\quad \left\{ e^{-2\Omega(T_{\alpha(s_3)+1}, s_2)} \left[1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{i=\alpha(s_3)+1}^{\alpha(s_2)-1} e^{-2\Omega(T_{i+1}, s_2)} \left[1 - e^{-\frac{2\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{2} \right. \\
&\quad \left. + \left[1 - e^{-\frac{2(s_2 - T_{\alpha(s_2)})}{\lambda_{\alpha(s_2)}}} \right] \frac{\lambda_{\alpha(s_2)}}{2} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \delta(t_3 - s_3) dt_2 dt_3 + \\
&\quad \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \right. \\
&\quad \left. \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} \delta(t_3 - s_3) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} dt_2 + \\
&\quad \int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \right. \\
&\quad \left. \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} dt_2 + \\
&\quad \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \right. \\
&\quad \left. \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(\mathbb{E}_{i,j}[s_2], t_2)} dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times
\end{aligned}$$

$$\begin{aligned}
& \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \right. \\
& \left. \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 + \\
& = \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
& \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \right. \\
& \left. \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2 \\
& = \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], \mathbb{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
& \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \lambda_l \left(1 - e^{-\frac{\Delta_l}{\lambda_l}} \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \right. \\
& \left. \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \left[1 - e^{-\frac{2(\mathbb{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} e^{-\Omega(\mathbb{E}_{i,j}[s_2], T_l)} \lambda_l \left(1 - e^{-\frac{\Delta_l}{\lambda_l}} \right)
\end{aligned}$$

Here we assume that $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

3.1.2 Case B

For $i = k < l < j$ ($t_3 = s_3; t_2 < s_2$):

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{1}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-2\Omega(s_3, t_2)} \\
&\quad \left\{ \sum_{a=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \delta(t_3 - s_3) dt_2 dt_3 + \\
&\int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} \\
&\quad \left\{ e^{-2\Omega(T_{\alpha(s_3)+1}, t_2)} \left[1 - e^{-\frac{2(T_{\alpha(s_3)+1} - s_3)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{a=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right. \\
&\quad \left. + \left[1 - e^{-\frac{2(t_2 - T_{\alpha(t_2)})}{\lambda_{\alpha(t_2)}}} \right] \frac{\lambda_{\alpha(t_2)}}{2} \right\} \delta(t_3 - s_3) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
&\quad \left\{ e^{-2\Omega(T_{i+1}, t_2)} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right. \\
&\quad \left. + \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] \frac{\lambda_l}{2} \right\} dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(e^{-2\Omega(T_{i+1}, t_2)} \left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right) dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] \frac{\lambda_l}{2} dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} dt_2 + \\
&\frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_{i+1}, t_2)} dt_2 + \\
&\int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right) dt_2 +
\end{aligned}$$

$$\begin{aligned}
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
& \quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_{i+1}, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \int_{T_l}^{T_{l+1}} \left(\sum_{a=i+1}^{l-1} e^{-2\Omega(T_{a+1}, t_2)} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \right) dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
& \quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_{i+1}, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_{a+1}, t_2)} dt_2 \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
& \quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_l)} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_l, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) e^{-2\Omega(T_{i+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_l, t_2)} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} e^{-2\Omega(T_{a+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_l, t_2)} dt_2 \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2 - T_l)}{\lambda_l}} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) e^{-2\Omega(T_{i+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2 - T_l)}{\lambda_l}} dt_2 + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} e^{-2\Omega(T_{a+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2 - T_l)}{\lambda_l}} dt_2 \right) + \\
& \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}} \right] dt_2 \\
& = \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\
& \quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) + \\
& \quad \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbb{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) e^{-2\Omega(T_{i+1}, T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) + \\
& \quad \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left[\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} e^{-2\Omega(T_{a+1}, T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) \right] + \\
& \quad \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \left[\Delta_l - \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) \right]
\end{aligned}$$

3.1.3 Case C

For $k < i = l < j$ ($t_3 < s_3; t_2 = s_3$):

$$\begin{aligned}
&= \frac{1}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\} \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{i=0}^{k-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} \delta(t_2 - \mathbb{E}_{i,j}[s_3]) dt_2 dt_3 \\
&= \int_{T_k}^{T_{k+1}} \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} dt_3 \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right\} \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{1}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left[1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right] \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\}
\end{aligned}$$

3.1.4 Case D

For $k < i < j = l$ ($t_3 < s_3; t_2 = s_2$):

$$\begin{aligned}
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3)-1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} \delta(t_2 - E_{i,j}[s_2]) dt_2 dt_3 \\
&= \int_{T_k}^{T_{k+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right\} dt_3 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right\} \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right\} \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left[1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right] \sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\}
\end{aligned}$$

3.1.5 Case E

For $i < k < j = l$ ($t_3 > s_3; t_2 = s_2$):

$$\begin{aligned}
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} e^{-2\Omega(s_3, t_3)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_3)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \delta(t_2 - \mathbb{E}_{i,j}[s_2]) dt_2 dt_3 \\
&= \int_{T_k}^{T_{k+1}} \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_3)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} dt_3 \\
&= \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_k}^{T_{k+1}} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], t_3)} dt_3 \\
&= \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_k)} \int_{T_k}^{T_{k+1}} e^{-2\Omega(T_k, t_3)} dt_3 \\
&= \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_k)} \int_{T_k}^{T_{k+1}} e^{-\frac{2(t_3 - T_k)}{\lambda_k}} dt_3 \\
&= \frac{2}{2\mathbb{E}_{i,j}[s_2] + \mathbb{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\mathbb{E}_{i,j}[s_3], T_k)} \frac{\lambda_k}{2} \left[1 - e^{-\frac{2\Delta_k}{\lambda_k}} \right]
\end{aligned}$$

3.1.6 Case F

For $i < j = k < l$ ($t_3 = s_2; t_2 > s_2$):

$$\begin{aligned}
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(s_3 - T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(E_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \delta(t_3 - s_2) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} e^{-\Omega(E_{i,j}[s_2], t_2)} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} dt_2 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(E_{i,j}[s_2], t_2)} dt_2 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(E_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(E_{i,j}[s_2], T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2 \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\
&\quad \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-\Omega(E_{i,j}[s_2], T_l)} \lambda_l \left[1 - e^{-\frac{\Delta_l}{\lambda_l}} \right]
\end{aligned}$$

Here again we assume $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

The following cases require special consideration:

3.1.7 Case G

For $i = k = l < j$ and $t_3 = s_3; t_2 < s_2$ (also need to consider $t_3 < s_3; t_2 = s_3$)

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3, t_2)} \delta(t_3 - s_3) + 2 \int_{s_3}^{t_2} \frac{du}{2s_2 + s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u, t_2)} \delta(t_3 - s_3) \right) dt_2 dt_3 \\
&= \int_{s_3}^{T_{k+1}} \left(\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3, t_2)} + 2 \int_{s_3}^{t_2} \frac{du}{2s_2 + s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u, t_2)} \right) dt_2 \\
&= \int_{E_{i,j}[s_3]}^{T_{k+1}} \left(\int_0^{E_{i,j}[s_3]} \frac{du}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} \frac{1}{\lambda_k} e^{-2\Omega(E_{i,j}[s_3], t_2)} + \right. \\
&\quad \left. 2 \int_{E_{i,j}[s_3]}^{t_2} \frac{du}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-2\Omega(u, t_2)} \right) dt_2 \\
&= \int_{E_{i,j}[s_3]}^{T_{k+1}} \left(\int_0^{E_{i,j}[s_3]} \frac{du}{2E_{i,j}[s_2] + E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} \frac{1}{\lambda_k} e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} + \right. \\
&\quad \left. 2 \int_{E_{i,j}[s_3]}^{t_2} \frac{du}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-\frac{2(t_2 - u)}{\lambda_k}} \right) dt_2 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \int_{E_{i,j}[s_3]}^{T_{k+1}} \left(\frac{1}{\lambda_k} e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} \int_0^{E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} du + \right. \\
&\quad \left. \frac{2}{\lambda_k} \int_{E_{i,j}[s_3]}^{t_2} e^{-\frac{2(t_2 - u)}{\lambda_k}} \right) dt_2 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \int_{E_{i,j}[s_3]}^{T_{k+1}} \left(\frac{1}{\lambda_k} e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} \int_0^{E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} du + \right. \\
&\quad \left. \frac{2}{\lambda_k} \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} \right] \right) dt_2 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \left(\frac{1}{\lambda_k} \int_0^{E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} du \int_{E_{i,j}[s_3]}^{T_{k+1}} e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} dt_2 + \right. \\
&\quad \left. \int_{E_{i,j}[s_3]}^{T_{k+1}} \left[1 - e^{-\frac{2(t_2 - E_{i,j}[s_3])}{\lambda_k}} \right] dt_2 \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \left(\frac{1}{\lambda_k} \int_0^{E_{i,j}[s_3]} e^{-3\Omega(u, E_{i,j}[s_3])} du \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1} - E_{i,j}[s_3])}{\lambda_k}} \right] + \right. \\
&\quad \left. T_{k+1} - E_{i,j}[s_3] - \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1} - E_{i,j}[s_3])}{\lambda_k}} \right] \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \left\{ \frac{1}{\lambda_k} \times \right. \\
&\quad \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left(1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right) \frac{\lambda_a}{3} + \left(1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right) \frac{\lambda_i}{3} \right] \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1} - E_{i,j}[s_3])}{\lambda_k}} \right] + \right. \\
&\quad \left. T_{k+1} - E_{i,j}[s_3] - \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(T_{k+1} - E_{i,j}[s_3])}{\lambda_k}} \right] \right\}
\end{aligned}$$

3.1.8 Case G2

For $i = k = l < j$ and $t_3 < s_3; t_2 = s_3$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \delta(t_2 - s_3) \right) dt_2 dt_3 \\
&= \int_{T_k}^{s_3} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \right) dt_3 \\
&= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_3} \frac{2}{\lambda(t_3)} \int_0^{t_3} e^{-3\Omega(u, t_3)} du dt_3 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \int_{T_k}^{E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \int_0^{t_3} e^{-3\Omega(u, t_3)} du dt_3 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \int_{T_k}^{E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{E_{i,j}[s_3]} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{E_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{E_{i,j}[s_3]} e^{-3\Omega(T_k, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{E_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{E_{i,j}[s_3]} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{E_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_k)}{\lambda_k}} \right] + \right. \\
&\quad \left. \frac{\lambda_k}{3} \left(E_{i,j}[s_3] - T_k - \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_k)}{\lambda_k}} \right] \right) \right) \\
&= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_k)}{\lambda_k}} \right] + \right.
\end{aligned}$$

$$\frac{\lambda_k}{3} \left(E_{i,j}[s_3] - T_k - \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_k)}{\lambda_k}} \right] \right)$$

3.1.9 Case H

Another case that requires special consideration is $i < k = l = j$. For $i < k = l = j$ and $t_3 > s_3; t_2 = s_2$:

$$\begin{aligned} &= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} \delta(t_2 - s_2) \right) dt_2 dt_3 \\ &= \int_{T_k}^{s_2} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_2} \left(2 \int_0^{s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} du \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_2} \left(\frac{4}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} \int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) e^{-2\Omega(s_3, T_k)} \int_{T_k}^{s_2} \frac{4}{\lambda(t_3)} e^{-2\Omega(T_k, t_3)} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \int_{T_k}^{s_2} e^{-2\Omega(T_k, t_3)} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \int_{T_k}^{s_2} e^{-\frac{2(t_3 - T_k)}{\lambda_k}} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}} \right] \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}} \right] \\ &= \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{4}{\lambda_k} e^{-2\Omega(E_{i,j}[s_3], T_k)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\ &\quad \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(E_{i,j}[s_2] - T_k)}{\lambda_k}} \right] \end{aligned}$$

3.1.10 Case H2

For $i < k = l = j$ and $t_3 = s_2; t_2 > s_2$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \delta(t_3 - s_2) \right) dt_2 dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \int_{s_2}^{T_{k+1}} \left(e^{-\Omega(s_2, t_2)} \int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) dt_2 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \left(\int_{s_2}^{T_{k+1}} e^{-\Omega(s_2, t_2)} dt_2 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \left(\int_{s_2}^{T_{k+1}} e^{-\Omega(s_2, t_2)} dt_2 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \left(\int_{s_2}^{T_{k+1}} e^{-\frac{t_2 - s_2}{\lambda_k}} dt_2 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3, s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \left(\lambda_k \left[1 - \delta(k - n) e^{-\frac{T_{k+1} - s_2}{\lambda_k}} \right] \right) \\
&= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\
&\quad \lambda_k \left[1 - \delta(k - n) e^{-\frac{T_{k+1} - E_{i,j}[s_2]}{\lambda_k}} \right]
\end{aligned}$$

The delta function is a way to ensure correctness when $k = l = j = n$.

Another special case that requires attention is $i = j = k < l$.

3.1.11 Case I

For $i = j = k < l$ and $t_3 = s_3; t_2 > s_2$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \delta(t_3 - s_3) + \right. \\
&\quad \left. 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} e^{-2\Omega(u, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \delta(t_3 - s_3) \right) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \left(\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} + \right. \\
&\quad \left. 2 \int_{s_3}^{s_2} \frac{du}{2s_2 + s_3} e^{-2\Omega(u, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} + \right. \\
&\quad \left. 2 \int_{s_3}^{s_2} e^{-2\Omega(u, s_2)} du \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \int_0^{s_3} e^{-3\Omega(u, s_3)} du + \right. \\
&\quad \left. \frac{2}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \int_{s_3}^{s_2} e^{-2\Omega(u, s_2)} du \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2, t_2)} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} e^{-\Omega(s_2, t_2)} \int_{s_3}^{s_2} e^{-2\Omega(u, s_2)} du \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2, t_2)} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} e^{-\Omega(s_2, t_2)} \int_{s_3}^{s_2} e^{-\frac{2(s_2 - u)}{\lambda_i}} du \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \int_{T_l}^{T_{l+1}} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2, t_2)} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} e^{-\Omega(s_2, t_2)} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2 - s_3)}{\lambda_i}} \right] \right) dt_2 \\
&= \frac{1}{2s_2 + s_3} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] \int_{T_l}^{T_{l+1}} e^{-\Omega(s_2, t_2)} dt_2 + \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2-s_3)}{\lambda_i}} \right] \int_{T_l}^{T_{l+1}} e^{-\Omega(s_2, t_2)} dt_2 \Bigg) \\
&= \frac{1}{2s_2 + s_3} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3-T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 + \right. \\
&\quad \left. \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2-s_3)}{\lambda_i}} \right] e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 \right) \\
&= \frac{1}{2s_2 + s_3} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3-T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2-T_l}{\lambda_l}} dt_2 + \right. \\
&\quad \left. \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2-s_3)}{\lambda_i}} \right] e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2-T_l}{\lambda_l}} dt_2 \right) \\
&= \frac{1}{2s_2 + s_3} \left(e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3-T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] e^{-\Omega(s_2, T_l)} \lambda_l \left[1 - e^{-\frac{\Delta_l}{\lambda_l}} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(s_2-s_3)}{\lambda_i}} \right] e^{-\Omega(s_2, T_l)} \lambda_l \left[1 - e^{-\frac{\Delta_l}{\lambda_l}} \right] \right) \\
&= \frac{1}{2\mathbb{E}_{i,i}[s_2] + \mathbb{E}_{i,i}[s_3]} \left(e^{-2\Omega(\mathbb{E}_{i,i}[s_3], \mathbb{E}_{i,i}[s_2])} \frac{1}{\lambda_l} \left[\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbb{E}_{i,i}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbb{E}_{i,i}[s_3]-T_l)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right] \times \right. \\
&\quad \left. e^{-\Omega(\mathbb{E}_{i,i}[s_2], T_l)} \lambda_l \left[1 - \delta(l-n)e^{-\frac{\Delta_l}{\lambda_l}} \right] + \right. \\
&\quad \left. \frac{2}{\lambda_l} \frac{\lambda_i}{2} \left[1 - e^{-\frac{2(\mathbb{E}_{i,i}[s_2]-\mathbb{E}_{i,i}[s_3])}{\lambda_i}} \right] e^{-\Omega(\mathbb{E}_{i,i}[s_2], T_l)} \lambda_l \left[1 - \delta(l-n)e^{-\frac{\Delta_l}{\lambda_l}} \right] \right)
\end{aligned}$$

Again, notice the delta function.

3.1.12 Case I2

For $i = j = k < l$ and $t_3 = s_2; t_2 > s_2$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \delta(t_3 - s_2) \right) dt_2 dt_3 \\
&= \int_{T_l}^{T_{l+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)} \right) dt_2 \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \int_{T_l}^{T_{l+1}} \left(e^{-\Omega(s_2, t_2)} \int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) dt_2 \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(\int_{T_l}^{T_{l+1}} e^{-\Omega(s_2, t_2)} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l, t_2)} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{t_2 - T_l}{\lambda_l}} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2, T_l)} \lambda_l \left[1 - \delta(l-n) e^{-\frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \\
&= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2, T_l)} \lambda_l \left[1 - e^{-\delta(l-n) \frac{\Delta_l}{\lambda_l}} \right] \right) \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \\
&= \frac{2}{2E_{i,i}[s_2] + E_{i,i}[s_3]} e^{-2\Omega(E_{i,i}[s_3], E_{i,i}[s_2])} \frac{1}{\lambda_l} \left(e^{-\Omega(E_{i,i}[s_2], T_l)} \lambda_l \left[1 - e^{-\delta(l-n) \frac{\Delta_l}{\lambda_l}} \right] \right) \times \\
&\quad \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,i}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(E_{i,i}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right)
\end{aligned}$$

Another case that requires special consideration: $k < i = j = l$. This includes either $t_3 < s_3; t_2 = s_2$ or $t_3 < s_3; t_2 = s_3$.

3.1.13 Case J

For $k < i = j = l$ and $t_3 < s_3; t_2 = s_2$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(2 \int_0^{t_3} \frac{du}{2s_2 + s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u, t_3)} \delta(t_2 - s_2) \right) dt_2 dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{t_3} e^{-3\Omega(u, t_3)} \delta(t_2 - s_2) \right) dt_2 dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} e^{-3\Omega(u, t_3)} \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left[\left(\int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \right) + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right] \\
&= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \right) + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\} \\
&= \frac{2}{2E_{i,i}[s_2] + E_{i,i}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \right) + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\}
\end{aligned}$$

3.1.14 Case J2

For $k < i = j = l$ and $t_3 < s_3; t_2 = s_3$:

$$\begin{aligned}
&= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \delta(t_2 - s_3) \right) dt_2 dt_3 \\
&= \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)} \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} e^{-3\Omega(u, t_3)} du \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1}, t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1}, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\
&= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1}, T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right] + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right)
\end{aligned}$$

This finishes the $q((k, l)|(i, j))$ transitions. Need to give special attention to the boundary cases, or the cases where $i = j$. Also need to consider what happens when k or l is n , since $T_{n+1} = \infty$.

The (diagonal) case of $k = i; l = j$ is calculated by subtracting the sum of the off-diagonal entries from unity.

3.2 Emission probabilities

The genotype at a particular position in a triploid genome can take one of three different values: 0, 1, and 2. The state 0 represents a homozygous site, and 1 (2) represent sites where one (two) of the three chromosomes have a derived (*i.e.*, non-ancestral) copy at that position.

To form our observed chain, we consider all the genotypes in a stretch of b bp and categorize that stretch of the sequence with a state 0, 1, 2, or 3. The state 0 means that the stretch of b bp is completely homozygous. The state 1 means that there was at least one site that had a 1 genotype and none that had a 2 genotype. Likewise, the state 2 means that at least one site had a 2 genotype, and none had a 1 genotype. The state 3 means that at least one site had a 1 genotype and at least one site had a 2 genotype.

With observed states coded this way, the emission probabilities given local coalescence times t_3 and t_2 are

$$e_k(t_3, t_2, T_d) = \begin{cases} e^{-\frac{\theta b(2t_2 + t_3 + 3T_d)}{2}} & k = 0 \\ e^{-\frac{\theta b(t_2 - t_3)}{2}} \left(1 - e^{-\frac{\theta b(2t_3 + t_2 + 3T_d)}{2}} \right) & k = 1 \\ \left(1 - e^{-\frac{\theta b(t_2 - t_3)}{2}} \right) e^{-\frac{\theta b(2t_3 + t_2 + 3T_d)}{2}} & k = 2 \\ \left(1 - e^{-\frac{\theta b(t_2 - t_3)}{2}} \right) \left(1 - e^{-\frac{\theta b(2t_3 + t_2 + 3T_d)}{2}} \right) & k = 3. \end{cases} \quad (19)$$

Here T_d is the divergence time, the time in the past [again measured in units of $2N(0)$ generations] when the asexual lineage was derived from a sexual ancestor. The above probabilities assume that t_3 and t_2 are measured continuously. In practice, we discretize time, so for a particular hidden state (i, j) , we replace t_3 and t_2 with $E_{i,j}[t_3]$ and $E_{i,j}[t_2]$, respectively.

Classifying states and genotypes this way requires that each polymorphism be polarized against an outgroup. If this is not possible, then the states can be recoded as 0 and 1, where 0 is a stretch of b completely homozygous base pairs, and 1 is a stretch of b base pairs with at least one polymorphic position. If no polarization is possible, the emission probabilities become

$$e_k(t_3, t_2, T_d) = \begin{cases} e^{-\frac{\theta b(2t_2+t_3+3T_d)}{2}} & k = 0 \\ 1 - e^{-\frac{\theta b(2t_2+t_3+3T_d)}{2}} & k = 1. \end{cases} \quad (20)$$

A few things to note: The parameter b can be tuned to match the observed polymorphism. If the change in ploidy T_d generations ago also involved a change in mutation rate, this new mutation rate will be unidentifiable, impossible to distinguish from a proportionally scaled T_d . Thus T_d should be viewed as a compound parameter.

4 HMM inference

The above equations for $q((k, l)|(i, j))$ define a transition matrix $\{P_{(i,j),(k,l)}\}$, where

$$P_{(i,j),(k,l)} = \left(1 - e^{-\frac{\rho(2E_{i,j}[t_2] + E_{i,j}[t_3])}{2}}\right) q((k, l)|(i, j)) \quad (21)$$

is the probability of transitioning from state (i, j) to state (k, l) , for $(i, j) \neq (k, l)$. We define a hidden Markov chain $\{X_i\}$ that is governed by this transition matrix. The observed process $\{Y_i\}$ represents the different emissions, 0 through 3. The EM algorithm proceeds by starting with some initial parameters θ and iteratively maximizing the expectation of the full likelihood given the data, which is

$$P(X, Y|\theta) = e_{x_1}(y_1) \pi_{x_1} \prod_{i=1}^{T-1} P_{x_i, x_{i+1}} e_{x_{i+1}}(y_{i+1}), \quad (22)$$

where y_i is the observed state at position i , and x_i is the state of the hidden chain at position i , and T is the length of the sequence. In practice we maximize the log-likelihood:

$$\log P(X, Y|\theta) = \log(e_{x_1}(y_1)) + \log(\pi_{x_1}) + \sum_{i=1}^{T-1} \log(P_{x_i, x_{i+1}}) + \log(e_{x_{i+1}}(y_{i+1})). \quad (23)$$

Since we observe only the observed chain (i.e., the mutation data), we have to have some way of integrating over the states of the hidden chain. We do this by the expectation-maximization (EM) algorithm, paired with the forward-backward algorithm for calculating likelihoods with these chains. In the context of HMM's, the EM algorithm iteratively maximizes

$$\begin{aligned} E_{X|Y}[\log P(X, Y|\theta)] &= E_{X|Y}[\log e_{x_1}(y_1)] + E_{X|Y}[\log \pi_{x_1}] + \\ &E_{X|Y}[\log e_{x_1}(y_1)] + \sum_{i=1}^{T-1} \left(E_{X|Y}[\log P_{x_i, x_{i+1}}] + E_{X|Y}[\log(e_{x_{i+1}}(y_{i+1}))] \right), \end{aligned}$$

updating the parameters of the chains in each iteration.

Define the forward variable

$$\alpha_i(t) = P(Y_1 = y_1, Y_2 = y_2, \dots, Y_t = y_t, X_t = i | \theta), \quad (24)$$

which satisfies the recursion

$$\alpha_j(t+1) = \sum_{i=1}^N \alpha_i(t) P_{i,j} e_j(y_{t+1}), \quad (25)$$

where $N = (n+1)(n+2)/2$ is the number of states in the hidden chain. We define

$$\alpha_i(1) = \pi_i e_i(y_1). \quad (26)$$

Define the backward variables

$$\beta_i(t) = P(Y_{t+1} = y_{t+1}, Y_{t+2} = y_{t+2}, \dots, Y_T = y_T | X_t = i, \theta), \quad (27)$$

which satisfy the recursion

$$\beta_i(t) = \sum_{j=1}^N \beta_j(t+1) P_{i,j} e_j(y_{t+1}), \quad (28)$$

defining $\beta_i(T) = 1$.

From these, we can calculate

$$\begin{aligned} \gamma_i(t) &= P(X_t = i | Y, \theta) \\ &= \frac{P(X_t = i, Y | \theta)}{P(Y)} \\ &= \frac{\alpha_i(t) \beta_i(t)}{\sum_{j=1}^N \alpha_j(t) \beta_j(t)} \end{aligned}$$

and

$$\begin{aligned} \xi_{i,j}(t) &= P(X_t = i, X_{t+1} = j | Y, \theta) \\ &= \frac{\alpha_i(t) P_{i,j} e_j(t+1) \beta_j(t+1)}{P(Y | \theta)} \\ &= \frac{\alpha_i(t) P_{i,j} e_j(t+1) \beta_j(t+1)}{\sum_{k=1}^N \alpha_k(t)} \end{aligned}$$

With these probabilities, we can calculate $E_{X|Y}[\log(P(X, Y | \theta))]$:

$$\begin{aligned} E_{X|Y}[\log P(X, Y | \theta)] &= E_{X|Y}[\log e_{x_1}(y_1)] + E_{X|Y}[\log \pi_{x_1}] + \\ &\quad \sum_{i=1}^{T-1} \left(E_{X|Y}[\log P_{x_i, x_{i+1}}] + E_{X|Y}[\log(e_{x_{i+1}}(y_{i+1}))] \right) \\ &= \sum_{j=1}^N \gamma_j(1) \log \pi_j + \sum_{t=1}^T \sum_{j=1}^N \log[e_j(y_t)] \gamma_j(t) + \\ &\quad \sum_{t=1}^{T-1} \sum_{i=1}^N \sum_{j=1}^N \log(P_{i,j}) \xi_{i,j}(t) \end{aligned}$$

The expected number of emissions of state $k \in \{0, 1, 2, 3\}$ from (hidden) state i is

$$E_i(k) = \sum_{t=1}^T \gamma_i(t) \mathbb{I}(y_t = k), \quad (29)$$

and the expected number of transitions from hidden state i to hidden state j is

$$E_{i,j} = \sum_{t=1}^{T-1} \xi_{i,j}(t). \quad (30)$$

Then the expected log-likelihood can be written

$$\begin{aligned} E_{X|Y} [\log P(X, Y|\theta)] &= \sum_{j=1}^N \gamma_j(1) \log \pi_j + \sum_{j=1}^N \sum_{k=0}^3 \log [e_j(k)] E_j(k) + \\ &\quad \sum_{i=1}^N \sum_{j=1}^N \log(P_{i,j}) E_{i,j} \end{aligned}$$

4.1 Underflow

In order to avoid underflow problems, we calculate scaled versions of $\alpha_i(t)$ and $\beta_{i,j}(t)$. We define

$$\tilde{\alpha}_i(t) = \frac{\alpha_i(t)}{\prod_{j=1}^t s_j}, \quad (31)$$

for some scaling constants s_i , from which we can see the new recursion

$$\tilde{\alpha}_j(t+1) = \frac{1}{s_{t+1}} e_j(y_{t+1}) \sum_{i=1}^N P_{i,j} \tilde{\alpha}_i(t). \quad (32)$$

It is convenient to define s_t such that $\sum_{j=1}^N \tilde{\alpha}_j(t) = 1$. This means $s_1 = \sum_{k=1}^N \pi_i e_i(y_1)$ and

$$s_{t+1} = \sum_{j=1}^N e_j(y_{t+1}) \sum_{i=1}^N P_{i,j} \tilde{\alpha}_i(t) \quad (33)$$

We scale $\beta_i(t)$ by the same s_i so that

$$\tilde{\beta}_i(t) = \frac{\beta_i(t)}{\prod_{j=t+1}^T s_j} \quad (34)$$

and

$$\tilde{\beta}_i(t) = \frac{1}{s_{t+1}} \sum_{j=1}^N \tilde{\beta}_j(t+1) P_{i,j} e_j(y_{t+1}), \quad (35)$$

with $\tilde{\beta}_i(T) = \beta_i(T) = 1$.

Now $\gamma_i(t)$ can be written

$$\begin{aligned} \gamma_i(t) &= \frac{\alpha_i(t) \beta_i(t)}{\sum_{j=1}^N \alpha_j(t) \beta_j(t)} \\ &= \frac{\tilde{\alpha}_i(t) \tilde{\beta}_i(t) \left(\prod_{j=1}^T s_j \right)}{\sum_{j=1}^N \tilde{\alpha}_j(t) \tilde{\beta}_j(t) \left(\prod_{j=1}^T s_j \right)} \\ &= \frac{\tilde{\alpha}_i(t) \tilde{\beta}_i(t)}{\sum_{j=1}^N \tilde{\alpha}_j(t) \tilde{\beta}_j(t)} \end{aligned} \quad (36)$$

and $\xi_{i,j}(t)$ can now be written

$$\begin{aligned}
\xi_{i,j}(t) &= \frac{\alpha_i(t)P_{i,j}e_j(t+1)\beta_j(t+1)}{P(Y|\theta)} \\
&= \frac{\tilde{\alpha}_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)\prod_{k=1}^T s_k}{P(Y|\theta)s_{t+1}} \\
&= \frac{\tilde{\alpha}_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)\prod_{k=1}^T s_k}{\sum_{l=1}^N \alpha_l(t)\beta_l(t)s_{t+1}} \\
&= \frac{\tilde{\alpha}_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)}{s_{t+1}\sum_{l=1}^N \tilde{\alpha}_l(t)\tilde{\beta}_l(t)} \\
&= \frac{\gamma_i(t)P_{i,j}e_j(t+1)\tilde{\beta}_j(t+1)}{s_{t+1}\beta_i(t)}
\end{aligned} \tag{37}$$

Difficult to implement, but efficient and apparently correct!