1 General equations

Define N(t) as the population size at time t, with t in units of 2N(0). Let $\lambda(t)$ be the relative population size, scaled by N(0), such that $N(t) = N(0)\lambda(t)$. Define $\Omega(u, v)$ as the cumulative coalescent rate between times u and v:

$$\Omega(u,v) = \int_{u}^{v} \frac{dt}{\lambda(t)}.$$
 (1)

The state of the TSMC at each point along the genome is described by the vector $\mathbf{s} = (s_3, s_2)$, where s_3 is the time of the first coalescence event and s_2 is the time of the second coalescence event amongst the three lineages in a triploid genome. The equilibrium joint distribution of (t_3, t_2) is

$$\pi(t_3, t_2) = \frac{3}{\lambda(t_3)\lambda(t_2)} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)}.$$
 (2)

Let q(t|s) be the transition kernel at recombination sites along the genome. Then

q(t|s) =

For $t_3 = s_3; t_2 > s_2$:

$$\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} + 2 \int_{s_2}^{s_2} \frac{du}{2s_2 + s_3} e^{-2\Omega(u,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)}$$

For $t_3 = s_3; t_2 < s_2$:

$$\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3,t_2)} + 2 \int_{s_3}^{t_2} \frac{du}{2s_2+s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u,t_2)}$$

For $t_3 < s_3$; $t_2 = s_3$:

$$\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)}$$

For $t_3 < s_3; t_2 = s_2$:

$$2\int_0^{t_3} \frac{du}{2s_2 + s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u, t_3)}$$

For $t_3 > s_3$; $t_2 = s_2$:

$$2\int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)}$$

For $t_3 = s_2; t_2 > s_2$:

$$2\int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)}$$

For $t_3 = s_3$; $t_2 = s_2$:

$$3\int_{0}^{s_{3}} \frac{du}{2s_{2}+s_{3}} \frac{1}{3} \left[1-e^{-3\Omega(u,s_{3})}\right] + \int_{0}^{s_{3}} \frac{du}{2s_{2}+s_{3}} e^{-3\Omega(u,s_{3})} \frac{1}{2} \left[1-e^{-2\Omega(s_{3},s_{2})}\right] + 2\int_{s_{3}}^{s_{2}} \frac{du}{2s_{2}+s_{3}} \frac{1}{2} \left[1-e^{-2\Omega(u,s_{2})}\right]. \tag{3}$$

Each part is implicitly multiplied by a delta function to limit the density to points where the parameters are assumed to be equal to each other. For example, the first part of q(t|s) is implicitly multiplied by $\delta(t_3 - s_3)$, and the last part is multiplied by $\delta(t_3 - s_3)\delta(t_2 - s_2)$.

2 Piecewise constant transition probabilities

Suppose that the population changes size at times (T_1, \ldots, T_n) and that the size between T_i and T_{i+1} is a constant $2N\lambda_i$. Define $T_0 = 0$, $T_{n+1} = \infty$ and $\Delta_i = T_{i+1} - T_i$. Let $\alpha(t)$ be the index of the time interval to which t belongs, i.e., $\alpha(t) = \max_i \{i : T_i \leq t\}$.

Then the cumulative coalescent rate between u and v can be written

$$\Omega(u,v) = \begin{cases}
\frac{v-u}{\lambda_{\alpha(u)}} & \alpha(u) = \alpha(v) \\
\frac{T_{\alpha(u)+1}-u}{\lambda_{\alpha(u)}} + \sum_{i=\alpha(u)+1}^{\alpha(v)-1} \frac{\Delta_i}{\lambda_i} + \frac{v-T_{\alpha(v)}}{\lambda_{\alpha(v)}} & \alpha(u) < \alpha(v).
\end{cases}$$
(4)

The equilibrium joint density of (t_3, t_2) is now approximately

$$\pi(t_3, t_2) = \frac{3}{\lambda_{\alpha(t_3)} \lambda_{\alpha(t_2)}} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)}.$$
 (5)

There are several integrals of the form $\int_x^y e^{-k\Omega(u,y)} du$ in Equation (3). This integral can be written

$$\begin{split} \int_{x}^{y} e^{-k\Omega(u,y)} du &= \int_{x}^{T_{u(x)+1}} e^{-k\Omega(u,y)} du + \sum_{i=a(x)+1}^{c(y)-1} \int_{T_{i}}^{T_{i}+1} e^{-k\Omega(u,y)} du + \int_{T_{u}(y)}^{y} e^{-k\Omega(u,y)} du \\ &= \int_{x}^{T_{u(x)+1}} \exp\left(-k\left[\left(T_{a(u)+1} - u\right) \frac{1}{\lambda_{a(u)}} + \sum_{j=a(u)+1}^{a(u)-1} \frac{\lambda_{j}}{\lambda_{j}} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{a(y)}}\right]\right) du + \\ &\sum_{i=a(x)+1}^{C(y)-1} \int_{T_{i}}^{T_{i+1}} \exp\left(-k\left[\left(T_{a(u)+1} - u\right) \frac{1}{\lambda_{a(u)}} + \sum_{j=a(u)+1}^{a(y)-1} \frac{\lambda_{j}}{\lambda_{j}} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{a(y)}}\right]\right) du + \\ &\int_{x}^{D} \exp\left(-k\left[\left(T_{a(x)+1} - u\right) \frac{1}{\lambda_{a(x)}} + \sum_{j=a(x)+1}^{a(y)-1} \frac{\lambda_{j}}{\lambda_{j}} + (y - T_{a(y)}) \frac{1}{\lambda_{a(y)}}\right]\right) du + \\ &\sum_{i=a(x)+1}^{C(y)-1} \int_{T_{i}}^{T_{i+1}} \exp\left(-k\left[\left(T_{i+1} - u\right) \frac{1}{\lambda_{i}} + \sum_{j=i+1}^{a(y)-1} \frac{\lambda_{j}}{\lambda_{j}} + (y - T_{a(y)}) \frac{1}{\lambda_{a(y)}}\right]\right) du + \\ &\int_{x}^{D} \exp\left(-k(y - u) \frac{1}{\lambda_{a(y)}}\right) du \\ &= \exp\left(-k\left[\frac{u(y)-1}{y} \frac{\lambda_{j}}{\lambda_{j}} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}}\right]\right) \int_{x}^{T_{a(x)+1}} \exp\left(-k\left(T_{a(x)+1} - u\right) \frac{1}{\lambda_{a(x)}}\right) du + \\ &\sum_{i=a(x)+1}^{D(y)-1} \exp\left(-k\left(y - u\right) \frac{1}{\lambda_{a(y)}}\right) du \\ &= \exp\left(-k\left[\sum_{j=a(x)+1}^{D(y)-1} \frac{\lambda_{j}}{\lambda_{j}} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}}\right]\right) \int_{x}^{T_{a(x)}} \exp\left(-k\left(T_{i+1} - u\right) \frac{1}{\lambda_{i}}\right) du + \\ &\int_{x}^{D} \exp\left(-k\left(y - u\right) \frac{1}{\lambda_{a(y)}}\right) du \\ &= \exp\left(-k\left[\sum_{j=a(x)+1}^{D(y)-1} \frac{\lambda_{j}}{\lambda_{j}} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}}\right]\right) \left[1 - \exp\left(-k\left(T_{a(x)+1} - x\right) \frac{1}{\lambda_{i}}\right) du + \\ &\sum_{i=a(x)+1}^{D(y)-1} \frac{\lambda_{i}}{\lambda_{j}} + (y - T_{\alpha(y)}) \frac{1}{\lambda_{\alpha(y)}}\right) \left[1 - \exp\left(-k\left(T_{\alpha(x)+1} - x\right) \frac{1}{\lambda_{i}}\right) \frac{\lambda_{i}(x)}{k} + \\ &\sum_{i=a(x)+1}^{D(y)-1} \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \left[1 - \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \frac{\lambda_{i}(x)}{k} + \\ &\sum_{i=a(x)+1}^{D(y)-1} \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \left[1 - \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \frac{\lambda_{i}(x)}{k} + \\ &\sum_{i=a(x)+1}^{D(y)-1} \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \left[1 - \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \frac{\lambda_{i}(x)}{k} + \\ &\sum_{i=a(x)+1}^{D(y)-1} \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \left[1 - \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \frac{\lambda_{i}(x)}{k} + \\ &\sum_{i=a(x)+1}^{D(y)-1} \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \left[1 - \exp\left(-k\left(T_{a(x)+1} - x\right)\right)\right] \frac{\lambda_{i}(x)}{k} + \\ &\sum_{i=a(x)+1}^{D(y)-1} \exp\left(-k\left(T_{a(x)+1} - x\right)\right) \left[1 - \exp\left(-k\left(T$$

With this equation, we can calculate all of the transition probabilities in the transition kernel (3). For $t_3 = s_3$; $t_2 > s_2$:

$$\begin{split} &\frac{1}{2s_2+s_3}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_{\alpha(t_2)}}e^{-\Omega(s_2,t_2)}\left\{\sum_{i=0}^{\alpha(s_3)-1}e^{-3\Omega(T_{i+1},s_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3\left(s_3-T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{3}\right\}+\\ &\frac{2}{2s_2+s_3}\frac{1}{\lambda_{\alpha(t_2)}}e^{-\Omega(s_2,t_2)}\times\\ &\left\{e^{-2\Omega(T_{\alpha(s_3)+1},s_2)}\left[1-e^{-\frac{2\left(T_{\alpha(s_3)+1}-s_3\right)}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{2}+\sum_{i=\alpha(s_3)+1}^{\alpha(s_2)-1}e^{-2\Omega(T_{i+1},s_2)}\left[1-e^{-\frac{2\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{2}\right.\\ &+\left[1-e^{-\frac{2\left(s_2-T_{\alpha(s_2)}\right)}{\lambda_{\alpha(s_2)}}}\right]\frac{\lambda_{\alpha(s_2)}}{2}\right\}\\ &=\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}+2\int_{s_3}^{s_2}\frac{du}{2s_2+s_3}e^{-2\Omega(u,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)} \end{split}$$

For $t_3 = s_3; t_2 < s_2$:

$$\begin{split} \frac{1}{2s_2+s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-2\Omega(s_3,t_2)} \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1},s_3)} \left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right] \frac{\lambda_i}{3} + \left[1-e^{-\frac{3\left(s_3-T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}}\right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ \frac{2}{2s_2+s_3} \frac{1}{\lambda_{\alpha(t_2)}} \\ \left\{ e^{-2\Omega(T_{\alpha(s_3)+1},t_2)} \left[1-e^{-\frac{2\left(T_{\alpha(s_3)+1}-s_3\right)}{\lambda_{\alpha(s_3)}}}\right] \frac{\lambda_{\alpha(s_3)}}{2} + \sum_{i=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{i+1},t_2)} \left[1-e^{-\frac{2\Delta_i}{\lambda_i}}\right] \frac{\lambda_i}{2} \\ + \left[1-e^{-\frac{2\left(t_2-T_{\alpha(t_2)}\right)}{\lambda_{\alpha(t_2)}}}\right] \frac{\lambda_{\alpha(t_2)}}{2} \right\} \\ = \int_0^{s_3} \frac{du}{2s_2+s_3} e^{-3\Omega(u,s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3,t_2)} + 2\int_{s_3}^{t_2} \frac{du}{2s_2+s_3} \frac{1}{\lambda(t_2)} e^{-2\Omega(u,t_2)} \end{split}$$

For $t_3 < s_3$; $t_2 = s_3$:

$$\frac{1}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3) - 1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\}$$

$$= \int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, t_3)} \frac{2}{\lambda(t_3)}$$

For $t_3 < s_3; t_2 = s_2$:

$$\frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} \left\{ \sum_{i=0}^{\alpha(t_3) - 1} e^{-3\Omega(T_{i+1}, t_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(t_3 - T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_{\alpha(t_3)}}{3} \right\}$$

$$= 2 \int_0^{t_3} \frac{du}{2s_2 + s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u, t_3)}$$

For $t_3 > s_3$; $t_2 = s_2$:

$$\frac{2}{2s_2 + s_3} \frac{2}{\lambda_{\alpha(t_3)}} e^{-2\Omega(s_3, t_3)} \left\{ \sum_{i=0}^{\alpha(s_3) - 1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3\left(s_3 - T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\
= 2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)}$$

For $t_3 = s_2$; $t_2 > s_2$:

$$\frac{2}{2s_2 + s_3} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda_{\alpha(t_2)}} e^{-\Omega(s_2, t_2)} \left\{ \sum_{i=0}^{\alpha(s_3) - 1} e^{-3\Omega(T_{i+1}, s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3\left(s_3 - T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \\
= 2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} e^{-2\Omega(s_3, s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2, t_2)}$$

For $t_3 = s_3$; $t_2 = s_2$:

$$\begin{split} &\frac{1}{2s_2+s_3} \left\{ s_3 - \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1},s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} - \left[1 - e^{-\frac{3\left(s_3 - T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ &\frac{1}{2s_2+s_3} \frac{1}{2} \left[1 - e^{-2\Omega(s_3,s_2)} \right] \left\{ \sum_{i=0}^{\alpha(s_3)-1} e^{-3\Omega(T_{i+1},s_3)} \left[1 - e^{-\frac{3\Delta_i}{\lambda_i}} \right] \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3\left(s_3 - T_{\alpha(s_3)}\right)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} + \\ &\frac{1}{2s_2+s_3} \left[s_2 - s_3 - e^{-2\Omega(T_{\alpha(s_3)+1},t_2)} \left(1 - e^{-\frac{2\left(T_{\alpha(s_3)+1} - s_3\right)}{\lambda_{\alpha(s_3)}}} \right) \frac{\lambda_{\alpha(s_3)}}{2} - \sum_{i=\alpha(s_3)+1}^{\alpha(t_2)-1} e^{-2\Omega(T_{i+1},t_2)} \left(1 - e^{-\frac{2\Delta_i}{\lambda_i}} \right) \frac{\lambda_i}{2} - \left(1 - e^{-\frac{2\left(t_2 - T_{\alpha(t_2)}\right)}{\lambda_{\alpha(t_2)}}} \right) \frac{\lambda_{\alpha(t_2)}}{2} \right] \end{split}$$

$$=3\int_{0}^{s_{3}}\frac{du}{2s_{2}+s_{3}}\frac{1}{3}\left[1-e^{-3\Omega(u,s_{3})}\right]+\int_{0}^{s_{3}}\frac{du}{2s_{2}+s_{3}}e^{-3\Omega(u,s_{3})}\frac{1}{2}\left[1-e^{-2\Omega(s_{3},s_{2})}\right]+2\int_{s_{3}}^{s_{2}}\frac{du}{2s_{2}+s_{3}}\frac{1}{2}\left[1-e^{-2\Omega(u,s_{2})}\right].$$

3 Discrete approximation to the triploid SMC' coalescent process

In order to construct a hidden Markov model (HMM) to infer demography, it is necessary to discretize the triploid coalescent process described above.

Let the discrete state (i, j), i < j, correspond to the continuous states in which $T_i < t_3 < T_{i+1}$ and $T_j < t_2 < T_{j+1}$. We first calculate the equilibrium probability that the coalescent process is in (i, j), assuming i < j:

$$\pi_{i,j} = \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3$$

$$= \frac{3}{\lambda_i \lambda_j} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{T_j}^{T_{j+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3$$

$$= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\Omega(T_j,t_2)} dt_2$$

$$= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3 \int_{T_j}^{T_{j+1}} e^{-\frac{t_2-T_j}{\lambda_j}} dt_2$$

$$= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_j)} dt_3$$

$$= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] e^{-\Omega(T_{i+1},T_j)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} e^{-\frac{\Omega(t_3,T_{i+1})}{\lambda_i}} dt_3$$

$$= \frac{3}{\lambda_i \lambda_j} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right)$$

$$= \frac{3}{2} e^{-3\Omega(0,T_i)} \lambda_j \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right] e^{-\Omega(T_{i+1},T_j)} \frac{\lambda_i}{2} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right)$$

$$= \frac{3}{2} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_j)} \left(e^{-\frac{\Delta_i}{\lambda_i}} - e^{-\frac{3\Delta_i}{\lambda_i}} \right) \left[1 - e^{-\frac{\Delta_j}{\lambda_j}} \right].$$

If j = n, we let $\Delta_j = \infty$ and $1 - \exp(-\Delta_j/\lambda_j) = 1$. It is also necessary to calculate $\pi_{i,i}$:

$$\pi_{i,i} = \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{3}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3$$

$$= \frac{1}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\Omega(t_3,t_2)} dt_2 dt_3$$

$$= \frac{3}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \int_{t_3}^{T_{i+1}} e^{-\frac{t_2-t_3}{\lambda_i}} dt_2 dt_3$$

$$= \frac{3}{\lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \left(\lambda_i \left[1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right]\right) dt_3$$

$$= \frac{3}{\lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} \left(1 - e^{-\frac{T_{i+1}-t_3}{\lambda_i}}\right) dt_3$$

$$= \frac{3}{\lambda_i} e^{-3\Omega(0,T_i)} \frac{\lambda_i}{6} \left(2 - 3e^{-\frac{\Delta_i}{\lambda_i}} + e^{-\frac{3\Delta_i}{\lambda_i}}\right)$$

$$= \frac{1}{2} e^{-3\Omega(0,T_i)} \left(2 - 3e^{-\frac{\Delta_i}{\lambda_i}} + e^{-\frac{3\Delta_i}{\lambda_i}}\right)$$

For i = n, again we let $\Delta_i = \infty$ and thus $\pi_{i,i} = \exp(-3\Omega(0, T_i))$.

Next, we calculate marginal expectations for t_3 and t_2 given that the continuous process is in interval represented by (i, j), assuming i < j. The marginal expectation of t_3 in the interval (i, j) is

$$\begin{split} \mathbf{E}_{i,j}[t_{3}] &= \mathbf{E}\left[t_{3}|t_{3} \in [T_{i}T_{i+1}), t_{2} \in [T_{j}, T_{j+1})\right] \\ &= \frac{1}{\pi_{i,j}} \int_{T_{i}}^{T_{i+1}} \int_{T_{j}}^{T_{j+1}} t_{3}\pi(t_{3}, t_{2}) dt_{2} dt_{3} \\ &= \frac{1}{\pi_{i,j}} \int_{T_{i}}^{T_{i+1}} \int_{T_{j}}^{T_{j+1}} \frac{3t_{3}}{\lambda_{i}\lambda_{j}} e^{-3\Omega(0, t_{3})} e^{-\Omega(t_{3}, t_{2})} dt_{2} dt_{3} \\ &= \frac{3}{\pi_{i,j}\lambda_{i}\lambda_{j}} e^{-3\Omega(0, T_{i})} \int_{T_{i}}^{T_{i+1}} t_{3} e^{-3\Omega(T_{i}, t_{3})} \int_{T_{j}}^{T_{j+1}} e^{-\Omega(t_{3}, t_{2})} dt_{2} dt_{3} \\ &= \frac{3}{\pi_{i,j}\lambda_{i}\lambda_{j}} e^{-3\Omega(0, T_{i})} \int_{T_{i}}^{T_{i+1}} t_{3} e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{j})} dt_{3} \int_{T_{j}}^{T_{j+1}} e^{-\Omega(T_{j}, t_{2})} dt_{2} \\ &= \frac{3}{\pi_{i,j}\lambda_{i}\lambda_{j}} e^{-3\Omega(0, T_{i})} \int_{T_{i}}^{T_{i+1}} t_{3} e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{j})} dt_{3} \int_{T_{j}}^{T_{j+1}} e^{-\frac{t_{2} - T_{j}}{\lambda_{j}}} dt_{2} \\ &= \frac{3}{\pi_{i,j}\lambda_{i}\lambda_{j}} e^{-3\Omega(0, T_{i})} \lambda_{j} \left(1 - e^{-\frac{\Delta_{j}}{\lambda_{j}}}\right) \int_{T_{i}}^{T_{i+1}} t_{3} e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{j})} dt_{3} \\ &= \frac{3}{\pi_{i,j}\lambda_{i}\lambda_{j}} e^{-3\Omega(0, T_{i})} \lambda_{j} \left(1 - e^{-\frac{\Delta_{j}}{\lambda_{j}}}\right) e^{-\Omega(T_{i+1}, T_{j})} \int_{T_{i}}^{T_{i+1}} t_{3} e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{i+1})} dt_{3} \\ &= \frac{3}{\pi_{i,j}\lambda_{i}\lambda_{j}} e^{-3\Omega(0, T_{i})} \lambda_{j} \left(1 - e^{-\frac{\Delta_{j}}{\lambda_{j}}}\right) e^{-\Omega(T_{i+1}, T_{j})} \frac{\lambda_{i}}{4} \left[(\lambda_{i} + 2T_{i}) e^{-\frac{\Delta_{i}}{\lambda_{i}}} - (\lambda_{i} + 2T_{i+1}) e^{-\frac{3\Delta_{i}}{\lambda_{i}}}\right] \\ &= \frac{3}{4\pi_{i,j}} e^{-3\Omega(0, T_{i})} \left(1 - e^{-\frac{\Delta_{j}}{\lambda_{j}}}\right) e^{-\Omega(T_{i+1}, T_{j})} \left[(\lambda_{i} + 2T_{i}) e^{-\frac{\Delta_{i}}{\lambda_{i}}} - (\lambda_{i} + 2T_{i+1}) e^{-\frac{3\Delta_{i}}{\lambda_{i}}}\right] \end{aligned}$$

With j = n, this is

$$\begin{split} \mathbf{E}_{i,j}[t_{3}] &= \mathbf{E}\left[t_{3}|t_{3} \in [T_{i}T_{i+1}), t_{2} \in [T_{n}, \infty)\right] \\ &= \frac{1}{\pi_{i,n}} \int_{T_{i}}^{T_{i+1}} \int_{T_{n}}^{\infty} t_{3}\pi(t_{3}, t_{2})dt_{2}dt_{3} \\ &= \frac{1}{\pi_{i,n}} \int_{T_{i}}^{T_{i+1}} \int_{T_{n}}^{\infty} \frac{3t_{3}}{\lambda_{i}\lambda_{n}} e^{-3\Omega(0, t_{3})} e^{-\Omega(t_{3}, t_{2})}dt_{2}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} \int_{T_{n}}^{\infty} e^{-\Omega(t_{3}, t_{2})}dt_{2}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n})}dt_{3} \int_{T_{n}}^{\infty} e^{-\Omega(T_{n}, t_{2})}dt_{2} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n})}dt_{3} \int_{T_{n}}^{\infty} e^{-\frac{t_{2} - T_{n}}{\lambda_{n}}}dt_{2} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \lambda_{n} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n})}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \lambda_{n} e^{-\Omega(T_{i+1}, T_{n})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n})}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \lambda_{n} e^{-\Omega(T_{i+1}, T_{n})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n+1})}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \lambda_{n} e^{-\Omega(T_{i+1}, T_{n})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n+1})}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \lambda_{n} e^{-\Omega(T_{i+1}, T_{n})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n+1})}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \lambda_{n} e^{-\Omega(T_{i+1}, T_{n})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n+1})}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \lambda_{n} e^{-\Omega(T_{i+1}, T_{n})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n+1})}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \lambda_{n} e^{-\Omega(T_{i+1}, T_{n})} \int_{T_{i}}^{T_{i+1}} t_{3}e^{-3\Omega(T_{i}, t_{3})} e^{-\Omega(t_{3}, T_{n})}dt_{3} \\ &= \frac{3}{\pi_{i,n}\lambda_{i}\lambda_{n}} e^{-3\Omega(0, T_{i})} \lambda_{n} e^{-\Omega(T_{i+1}, T_{n})} \int_{T_{i}}^{T_{i+$$

The marginal expectation of t_2 in (i, j) is

$$\begin{split} \mathbf{E}_{i,j}[t_2] &= \mathbf{E} \left[t_2 | t_3 \in [T_i T_{i+1}), t_2 \in [T_j, T_{j+1}) \right] \\ &= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} t_2 \pi(t_3, t_2) dt_2 dt_3 \\ &= \frac{1}{\pi_{i,j}} \int_{T_i}^{T_{i+1}} \int_{T_j}^{T_{j+1}} \frac{3t_2}{\lambda_i \lambda_j} e^{-3\Omega(0, t_3)} e^{-\Omega(t_3, t_2)} dt_2 dt_3 \\ &= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i, t_3)} e^{-\Omega(t_3, T_{i+1})} dt_3 \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j, t_2)} dt_2 \\ &= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3 - T_i)}{\lambda_i}} e^{-\frac{T_{i+1} - t_3}{\lambda_i}} dt_3 \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j, t_2)} dt_2 \\ &= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}} \right) \int_{T_j}^{T_{j+1}} t_2 e^{-\Omega(T_j, t_2)} dt_2 \\ &= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}} \right) \int_{T_j}^{T_{j+1}} t_2 e^{-\frac{t_2 - T_j}{\lambda_j}} dt_2 \\ &= \frac{3}{\pi_{i,j} \lambda_i \lambda_j} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}} \right) \lambda_j \left(\lambda_j + T_j - (\lambda_j + T_{j+1}) e^{-\frac{\Delta_j}{\lambda_j}} \right) \\ &= \frac{3}{2\pi_{i,j}} e^{-3\Omega(0, T_i)} e^{-\Omega(T_{i+1}, T_j)} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}} \right) \left(\lambda_j + T_j - (\lambda_j + T_{j+1}) e^{-\frac{\Delta_j}{\lambda_j}} \right). \end{split}$$

With j = n, this is

$$\begin{split} \mathbf{E}_{i,n}[t_2] &= \mathbf{E} \left[t_2 | t_3 \in [T_i T_{i+1}), t_2 \in [T_n, \infty) \right] \\ &= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} t_2 \pi(t_3, t_2) dt_2 dt_3 \\ &= \frac{1}{\pi_{i,n}} \int_{T_i}^{T_{i+1}} \int_{T_n}^{\infty} \frac{3t_2}{\lambda_i \lambda_n} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\ &= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} e^{-\Omega(t_3,T_{i+1})} dt_3 \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n,t_2)} dt_2 \\ &= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3-T_i)}{\lambda_i}} e^{-\frac{T_{i+1}-t_3}{\lambda_i}} dt_3 \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n,t_2)} dt_2 \\ &= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}} \right) \int_{T_n}^{\infty} t_2 e^{-\Omega(T_n,t_2)} dt_2 \\ &= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}} \right) \int_{T_n}^{\infty} t_2 e^{-\frac{t_2-T_n}{\lambda_n}} dt_2 \\ &= \frac{3}{\pi_{i,n} \lambda_i \lambda_n} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \frac{\lambda_i}{2} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}} \right) \lambda_n \left(\lambda_n + T_n \right) \\ &= \frac{3}{2\pi_{i,n}} e^{-3\Omega(0,T_i)} e^{-\Omega(T_{i+1},T_n)} \left(e^{\frac{-\Delta_i}{\lambda_i}} - e^{\frac{-3\Delta_i}{\lambda_i}} \right) \left(\lambda_n + T_n \right) \end{split}$$

We also calculate the marginal expectations of s_3 and s_2 conditional on the interval (i, i):

$$\begin{split} \mathbf{E}_{i,i}[t_{3}] &= \mathbf{E}\left[t_{3}|t_{3} \in [T_{i}T_{i+1}), t_{2} \in [T_{i}, T_{i+1})\right] \\ &= \frac{1}{\pi_{i,i}} \int_{T_{i}}^{T_{i+1}} \int_{t_{3}}^{T_{i+1}} t_{3}\pi(t_{3}, t_{2}) dt_{2} dt_{3} \\ &= \frac{3}{\pi_{i,i}} \int_{T_{i}}^{T_{i+1}} \int_{t_{3}}^{T_{i+1}} \frac{t_{3}}{\lambda_{i}^{2}} e^{-3\Omega(0, t_{3})} e^{-\Omega(t_{3}, t_{2})} dt_{2} dt_{3} \\ &= \frac{3}{\pi_{i,i} \lambda_{i}^{2}} \int_{T_{i}}^{T_{i+1}} t_{3} e^{-3\Omega(0, t_{3})} \int_{t_{3}}^{T_{i+1}} e^{-\Omega(t_{3}, t_{2})} dt_{2} dt_{3} \\ &= \frac{3}{\pi_{i,i} \lambda_{i}^{2}} e^{-3\Omega(0, T_{i})} \int_{T_{i}}^{T_{i+1}} t_{3} e^{-3\Omega(T_{i}, t_{3})} \int_{t_{3}}^{T_{i+1}} e^{-\Omega(t_{3}, t_{2})} dt_{2} dt_{3} \\ &= \frac{3}{\pi_{i,i} \lambda_{i}^{2}} e^{-3\Omega(0, T_{i})} \int_{T_{i}}^{T_{i+1}} t_{3} e^{-\frac{3(t_{3} - T_{i})}{\lambda_{i}}} \lambda_{i} \left(1 - e^{-\frac{T_{i+1} - t_{3}}{\lambda_{i}}}\right) dt_{3} \\ &= \frac{3}{\pi_{i,i} \lambda_{i}} e^{-3\Omega(0, T_{i})} \int_{T_{i}}^{T_{i+1}} t_{3} e^{-\frac{3(t_{3} - T_{i})}{\lambda_{i}}} \left(1 - e^{-\frac{T_{i+1} - t_{3}}{\lambda_{i}}}\right) dt_{3} \\ &= \frac{3}{\pi_{i,i} \lambda_{i}} e^{-3\Omega(0, T_{i})} \frac{\lambda_{i}}{36} \left(-9 e^{-\frac{\Delta_{i}}{\lambda_{i}}} (2T_{i} + \lambda_{i}) + 4(3T_{i} + \lambda_{i}) + e^{-\frac{3\Delta_{i}}{\lambda_{i}}} (6T_{i+1} + 5\lambda_{i})\right) \\ &= \frac{1}{12\pi_{i,i}} e^{-3\Omega(0, T_{i})} \left(4(3T_{i} + \lambda_{i}) + e^{-\frac{3\Delta_{i}}{\lambda_{i}}} (6T_{i+1} + 5\lambda_{i}) - 9 e^{-\frac{\Delta_{i}}{\lambda_{i}}} (2T_{i} + \lambda_{i})\right) \end{split}$$

For i = n, the expectation is

$$E_{n,n}[s_3] = T_n + \frac{\lambda_n}{3} \tag{14}$$

Double-checking:

$$\begin{split} \mathbf{E}_{n,n}[t_{3}] &= \mathbf{E}\left[t_{3}|t_{3} \in [T_{n}\infty), t_{2} \in [T_{n},\infty)\right] \\ &= \frac{1}{\pi_{n,n}} \int_{T_{n}}^{\infty} \int_{t_{3}}^{\infty} t_{3}\pi(t_{3},t_{2})dt_{2} dt_{3} \\ &= \frac{3}{\pi_{n,n}\lambda_{n}^{2}} \int_{T_{n}}^{\infty} \int_{t_{3}}^{\infty} t_{3}e^{-3\Omega(0,t_{3})}e^{-\Omega(t_{3},t_{2})}dt_{2} dt_{3} \\ &= \frac{3}{\pi_{n,n}\lambda_{n}} \int_{T_{n}}^{\infty} t_{3}e^{-3\Omega(0,t_{3})} \int_{t_{3}}^{\infty} e^{-\Omega(t_{3},t_{2})}dt_{2} dt_{3} \\ &= \frac{3}{\pi_{n,n}\lambda_{n}} \int_{T_{n}}^{\infty} t_{3}e^{-3\Omega(0,t_{3})}dt_{3} \\ &= \frac{3}{\pi_{n,n}\lambda_{n}} e^{-3\Omega(0,T_{n})} \int_{T_{n}}^{\infty} t_{3}e^{-3\Omega(T_{n},t_{3})}dt_{3} \\ &= \frac{3}{\pi_{n,n}\lambda_{n}} e^{-3\Omega(0,T_{n})} \int_{T_{n}}^{\infty} t_{3}e^{-\frac{3(t_{3}-T_{n})}{\lambda_{n}}}dt_{3} \\ &= \frac{3}{\pi_{n,n}\lambda_{n}} e^{-3\Omega(0,T_{n})} \frac{\lambda_{n}}{9} (3T_{n} + \lambda_{n}) \\ &= \frac{1}{3\pi_{n,n}} e^{-3\Omega(0,T_{n})} (3T_{n} + \lambda_{n}) \end{split}$$

Checks out since $\pi_{n,n} = \exp(-3\Omega(0,T_n))$.

$$\begin{split} & \mathrm{E}_{i,i}[t_2] = \mathrm{E}\left[t_2|t_3 \in [T_iT_{i+1}), t_2 \in [T_i, T_{i+1})\right] \\ & = \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_2 \pi(t_3, t_2) dt_2 dt_3 \\ & = \frac{1}{\pi_{i,i}} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} \frac{3t_2}{\lambda_i^2} e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\ & = \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} \int_{t_3}^{T_{i+1}} t_2 e^{-3\Omega(0,t_3)} e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\ & = \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} t_2 e^{-\Omega(t_3,t_2)} dt_2 dt_3 \\ & = \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \int_{t_3}^{T_{i+1}} t_2 e^{-\frac{t_2-t_3}{\lambda_i}} dt_2 dt_3 \\ & = \frac{3}{\pi_{i,i} \lambda_i^2} \int_{T_i}^{T_{i+1}} e^{-3\Omega(0,t_3)} \lambda_i \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1} - t_3}{\lambda_i}} (T_{i+1} + \lambda_i) \right] dt_3 \\ & = \frac{3}{\pi_{i,i} \lambda_i^2} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-3\Omega(T_i,t_3)} \lambda_i \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1} - t_3}{\lambda_i}} (T_{i+1} + \lambda_i) \right] dt_3 \\ & = \frac{3}{\pi_{i,i} \lambda_i} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{3(t_3 - T_i)}{\lambda_i}} \left[(t_3 + \lambda_i) - e^{-\frac{T_{i+1} - t_3}{\lambda_i}} (T_{i+1} + \lambda_i) \right] dt_3 \\ & = \frac{3}{\pi_{i,i} \lambda_i}} e^{-3\Omega(0,T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{3\Delta_i}{\lambda_i}} (3T_{i+1} + \lambda_i) + 6T_i + 8\lambda_i - 9e^{-\frac{\Delta_i}{\lambda_i}} (T_{i+1} + \lambda_i) \right) \\ & = \frac{1}{6\pi_{i,i}}} e^{-3\Omega(0,T_i)} \left(e^{-\frac{3\Delta_i}{\lambda_i}} (3T_{i+1} + \lambda_i) + 6T_i + 8\lambda_i - 9e^{-\frac{\Delta_i}{\lambda_i}} (T_{i+1} + \lambda_i) \right) \end{split}$$

For i = n, this expectation is

$$E_{n,n}[s_2] = T_n + \frac{\lambda_n}{3} + \lambda_n \tag{17}$$

3.1 Discrete q((k,l)|(i,j)) transition function

To calculate the discrete-process transition probabilities from (i, j), to (k, l), we integrate the transition kernel (6) over the interval corresponding to (k, l), replacing s_3 and s_2 with their conditional expectations $E_{i,j}[s_3]$ and $E_{i,j}[s_2]$ respectively. Thus

$$q((k,l)|(i,j)) = \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} q((t_3,t_2)|(\mathbf{E}_{i,j}[s_3],\mathbf{E}_{i,j}[s_2])) dt_2 dt_3.$$
(18)

Note that in any single transition, either the first or second coalescence time changes, but not both. This simplifies the calculation of these integrals.

3.1.1 Case A

For
$$i = k < j < l$$
, $(t_3 = s_3; t_2 > s_2)$:

$$\begin{split} &=\frac{1}{2s_{2}+s_{3}}e^{-2\Omega(s_{3},s_{2})}\frac{1}{\lambda_{\alpha(t_{2})}}e^{-\Omega(s_{2},t_{2})}\left\{\sum_{i=0}^{\alpha(s_{3})-1}e^{-3\Omega(T_{i+1},s_{3})}\left[1-e^{-\frac{3\Delta_{i}}{\lambda_{i}}}\right]\frac{\lambda_{i}}{3}+\left[1-e^{-\frac{3\left(s_{3}-T_{\alpha(s_{3})}\right)}{\lambda_{\alpha(s_{3})}}}\right]\frac{\lambda_{\alpha(s_{3})}}{3}\right\}+\\ &=\frac{2}{2s_{2}+s_{3}}\frac{1}{\lambda_{\alpha(t_{2})}}e^{-\Omega(s_{2},t_{2})}\times\\ &\left\{e^{-2\Omega(T_{\alpha(s_{3})+1},s_{2})}\left[1-e^{-\frac{2\left(T_{\alpha(s_{3})+1}-s_{3}\right)}{\lambda_{\alpha(s_{3})}}}\right]\frac{\lambda_{\alpha(s_{3})}}{2}+\sum_{i=\alpha(s_{3})+1}^{\alpha(s_{2})-1}e^{-2\Omega(T_{i+1},s_{2})}\left[1-e^{-\frac{2\Delta_{i}}{\lambda_{i}}}\right]\frac{\lambda_{i}}{2}\right.\\ &+\left[1-e^{-\frac{2\left(s_{2}-T_{\alpha(s_{2})}\right)}{\lambda_{\alpha(s_{2})}}}\right]\frac{\lambda_{\alpha(s_{2})}}{2}\right\}\\ &=\int_{T_{k}}^{T_{k+1}}\int_{T_{l}}^{T_{l+1}}\frac{1}{2E_{i,j}[s_{2}]+E_{i,j}[s_{3}]}e^{-2\Omega(E_{i,j}[s_{3}],E_{i,j}[s_{2}])}\frac{1}{\lambda_{l}}e^{-\Omega(E_{i,j}[s_{2}],t_{2})}\times\\ &\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},E_{i,j}[s_{3}])}\left[1-e^{-\frac{3\Delta_{a}}{\lambda_{a}}}\right]\frac{\lambda_{a}}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_{3}]-T_{i})}{\lambda_{i}}}\right]\frac{\lambda_{i}}{3}\right\}\delta(t_{3}-s_{3})dt_{2}dt_{3}+\\ &\int_{T_{k}}^{T_{k+1}}\int_{T_{l}}^{T_{l+1}}\frac{2}{2E_{i,j}[s_{2}]+E_{i,j}[s_{3}]}\frac{1}{\lambda_{l}}e^{-\Omega(E_{i,j}[s_{2}],t_{2})}\times\\ &\left\{e^{-2\Omega(T_{i+1},E_{i,j}[s_{2}])}\left[1-e^{-\frac{2(T_{i+1}-E_{i,j}[s_{3}])}{\lambda_{i}}}\right]\frac{\lambda_{i}}{2}+\sum_{a=i+1}^{j-1}e^{-2\Omega(T_{a+1},E_{i,j}[s_{2}])}\left[1-e^{-\frac{2\Delta_{a}}{\lambda_{a}}}\right]\frac{\lambda_{a}}{2}+\\ &\left[1-e^{-\frac{2(E_{i,j}[s_{2}]-T_{j})}{\lambda_{j}}\right]\frac{\lambda_{j}}{2}\right\}\delta(t_{3}-s_{3})dt_{2}dt_{3} \end{split}$$

$$\begin{split} &= \int_{T_{l}}^{T_{l+1}} \frac{1}{2 \operatorname{E}_{i,j}[s_{2}] + \operatorname{E}_{i,j}[s_{3}]} e^{-2\Omega(\operatorname{E}_{i,j}[s_{3}],\operatorname{E}_{i,j}[s_{2}])} \frac{1}{\lambda_{l}} e^{-\Omega(\operatorname{E}_{i,j}[s_{2}],t_{2})} \times \\ & \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\operatorname{E}_{i,j}[s_{3}])} \left[1 - e^{-\frac{3\Delta_{a}}{\lambda_{a}}} \right] \frac{\lambda_{a}}{3} + \left[1 - e^{-\frac{3(\operatorname{E}_{i,j}[s_{3}] - T_{i})}{\lambda_{i}}} \right] \frac{\lambda_{i}}{3} \right\} dt_{2} + \\ & \int_{T_{l}}^{T_{l+1}} \frac{2}{2 \operatorname{E}_{i,j}[s_{2}] + \operatorname{E}_{i,j}[s_{3}]} \frac{1}{\lambda_{l}} e^{-\Omega(\operatorname{E}_{i,j}[s_{2}],t_{2})} \times \\ & \left\{ e^{-2\Omega(T_{i+1},\operatorname{E}_{i,j}[s_{2}])} \left[1 - e^{-\frac{2(T_{i+1} - \operatorname{E}_{i,j}[s_{3}])}{\lambda_{i}}} \right] \frac{\lambda_{i}}{2} + \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1},\operatorname{E}_{i,j}[s_{2}])} \left[1 - e^{-\frac{2\Delta_{a}}{\lambda_{a}}} \right] \frac{\lambda_{a}}{2} + \\ & \left[1 - e^{-\frac{2(\operatorname{E}_{i,j}[s_{2}] - T_{j})}{\lambda_{j}}} \right] \frac{\lambda_{j}}{2} \right\} dt_{2} \end{split}$$

$$\begin{split} &= \frac{1}{2\operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} e^{-2\Omega(\operatorname{E}_{i,j}[s_3],\operatorname{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\ & \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\operatorname{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\operatorname{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(\operatorname{E}_{i,j}[s_2],t_2)} dt_2 + \\ & \frac{2}{2\operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{ e^{-2\Omega(T_{i+1},\operatorname{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2(T_{i+1} - \operatorname{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} + \\ & \sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1},\operatorname{E}_{i,j}[s_2])} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} + \left[1 - e^{-\frac{2(\operatorname{E}_{i,j}[s_2] - T_j)}{\lambda_j}} \right] \frac{\lambda_j}{2} \right\} \int_{T_l}^{T_{l+1}} e^{-\Omega(\operatorname{E}_{i,j}[s_2],t_2)} dt_2 \\ & = \frac{1}{2\operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} e^{-2\Omega(\operatorname{E}_{i,j}[s_3],\operatorname{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times \end{split}$$

$$= \frac{1}{2 \operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} e^{-2\Omega(\operatorname{E}_{i,j}[s_3], \operatorname{E}_{i,j}[s_2])} \frac{1}{\lambda_i}$$

$$\begin{split} &\left\{\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbf{E}_{i,j}[s_3])} \left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(\mathbf{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3}\right\} e^{-\Omega(\mathbf{E}_{i,j}[s_2],T_i)} \int_{T_i}^{T_{i+1}} e^{-\Omega(T_i,t_2)} dt_2 + \\ &\frac{2}{2 + \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{e^{-2\Omega(T_{i+1}, \mathbf{E}_{i,j}[s_2])} \left[1-e^{-\frac{2(T_{i+1}-\mathbf{E}_{i,j}[s_3])}{\lambda_i}}\right] \frac{\lambda_i}{2} + \\ &\sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1}, \mathbf{E}_{i,j}[s_2])} \left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{2} + \left[1-e^{-\frac{2(\mathbf{E}_{i,j}[s_2]-T_j)}{\lambda_j}}\right] \frac{\lambda_j}{2} \right\} e^{-\Omega(\mathbf{E}_{i,j}[s_2],T_i)} \int_{T_i}^{T_{i+1}} e^{-\Omega(T_i,t_2)} dt_2 + \\ &= \frac{1}{2 + \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} e^{-2\Omega(\mathbf{E}_{i,j}[s_3], \mathbf{E}_{i,j}[s_2])} \frac{1}{\lambda_l} \times \\ &\left\{\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbf{E}_{i,j}[s_3])} \left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(\mathbf{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3} \right\} e^{-\Omega(\mathbf{E}_{i,j}[s_2],T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{t_2-T_i}{\lambda_i}} dt_2 + \\ &\frac{2}{2 + \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left\{e^{-2\Omega(T_{i+1},\mathbf{E}_{i,j}[s_2])} \left[1-e^{-\frac{2(T_{i+1}-\mathbf{E}_{i,j}[s_3])}{\lambda_j}}\right] \frac{\lambda_i}{2} + \right. \\ &\sum_{a=i+1}^{j-1} e^{-2\Omega(T_{a+1},\mathbf{E}_{i,j}[s_2])} \left[1-e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{2(\mathbf{E}_{i,j}[s_2]-T_i)}{\lambda_j}}\right] \frac{\lambda_j}{2} \right\} e^{-\Omega(\mathbf{E}_{i,j}[s_2],T_i)} \int_{T_i}^{T_{i+1}} e^{-\frac{t_2-T_i}{\lambda_i}} dt_2 + \\ &\frac{1}{2 + \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]}} e^{-2\Omega(\mathbf{E}_{i,j}[s_3],\mathbf{E}_{i,j}[s_2])} \frac{\lambda_i}{\lambda_i} \times \\ &\left\{\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\mathbf{E}_{i,j}[s_3])} \left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1-e^{-\frac{3(\mathbf{E}_{i,j}[s_2]-T_i)}{\lambda_j}}\right] \frac{\lambda_j}{3} \right\} e^{-\Omega(\mathbf{E}_{i,j}[s_2],T_i)} \lambda_i \left(1-e^{-\frac{\Delta_i}{\lambda_i}}\right) + \\ &\frac{2}{2 + \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left\{e^{-2\Omega(T_{i+1},\mathbf{E}_{i,j}[s_2])} \left[1-e^{-\frac{2(T_{i+1}-\mathbf{E}_{i,j}[s_3]-T_i)}{\lambda_i}}\right] \frac{\lambda_j}{3} \right\} e^{-\Omega(\mathbf{E}_{i,j}[s_2],T_i)} \lambda_i \left(1-e^{-\frac{\Delta_i}{\lambda_i}}\right) + \\ &\frac{2}{2 + \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left\{e^{-2\Omega(T_{i+1},\mathbf{E}_{i,j}[s_2])} \left[1-e^{-\frac{2(T_{i+1}-\mathbf{E}_{i,j}[s_3]-T_i)}{\lambda_i}\right] \frac{\lambda_j}{3} \right\} e^{-\Omega(\mathbf{E}_{i,j}[s_2],T_i)} \lambda_i \left(1-e^{-\frac{\Delta_i}{\lambda_i}}\right) + \\ &\frac{2}{2 + \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} \frac{1}$$

Here we assume that $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

3.1.2 Case B

$$\begin{aligned} & \text{For } i = k < l < j \; (t_3 = s_3; t_2 < s_2); \\ & = \int_{T_k}^{T_{k+1}} \int_{T_1}^{T_{k+1}} \frac{1}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} e^{-2\Omega(s_3,t_2)} \\ & \left\{ \sum_{\alpha=0}^{(s_3)-1} e^{-3\Omega(T_{\alpha+1},s_3)} \left[1 - e^{-\frac{3\lambda_3}{\lambda_\alpha}} \right] \frac{\lambda_\alpha}{3} + \left[1 - e^{-\frac{3(s_3-T_{\alpha(s_3)})}{\lambda_{\alpha(t_4)}}} \right] \frac{\lambda_{\alpha(s_3)}}{3} \right\} \delta(t_3 - s_3) dt_2 dt_3 + \int_{T_k}^{T_{k+1}} \int_{T_1}^{T_{t+1}} \frac{2}{2s_2 + s_3} \frac{1}{\lambda_{\alpha(t_2)}} \\ & \left\{ e^{-2\Omega(T_{\alpha(s_3)+1},t_2)} \left[1 - e^{-\frac{2(\tau_{\alpha(s_3)+1}-s_3)}{\lambda_{\alpha(s_3)}}} \right] \frac{\lambda_\alpha(s_3)}{2} + \sum_{\alpha=\alpha(s_3)+1}^{\Omega(t_2)-1} e^{-2\Omega(T_{\alpha+1},t_2)} \left[1 - e^{-\frac{2\lambda_\alpha}{\lambda_\alpha}} \right] \frac{\lambda_\alpha}{2} \\ & + \left[1 - e^{-\frac{2(\tau_{\alpha(s_3)+1}-s_3)}{\lambda_{\alpha(t_3)}}} \right] \frac{\lambda_\alpha(s_3)}{2} \right\} \delta(t_3 - s_3) dt_2 dt_3 \\ & = \int_{T_1}^{T_{t+1}} \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_t} e^{-2\Omega(E_{i,j}|s_3],t_2} \\ & \left\{ \sum_{\alpha=0}^{-2} e^{-3\Omega(T_{\alpha+1},E_{i,j}]s_3)} \left[1 - e^{-\frac{3\lambda_\alpha}{\lambda_\alpha}} \right] \frac{\lambda_\alpha}{3} + \left[1 - e^{-\frac{3(E_{i,j}|s_3|-T_{i+1})}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} dt_2 + \int_{T_1}^{T_{t+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{\lambda_i}{\lambda_i} \\ & \left\{ e^{-2\Omega(T_{t+1},t_2)} \left[1 - e^{-\frac{2(T_{t+1}-E_{i,j}(s_2))}{\lambda_i}} \right] \frac{\lambda_2}{2} + \sum_{\alpha=i+1}^{i-1} e^{-2\Omega(T_{s+1},t_2)} \left[1 - e^{-\frac{2\lambda_\alpha}{\lambda_\alpha}} \right] \frac{\lambda_\alpha}{2} \\ & + \left[1 - e^{-\frac{2(\tau_{i+1}-E_{i,j}(s_2))}{\lambda_i}} \right] \frac{\lambda_i}{2} \right\} dt_2 \\ & = \frac{1}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_i} \left(e^{-2\Omega(T_{t+1},t_2)} \left[1 - e^{-\frac{3(E_{i,j}|s_3|-T_{i+1})}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_1}^{T_{t+1}} e^{-2\Omega(E_{i,j}|s_3|,t_2)} dt_2 + \int_{T_1}^{T_{t+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_i} \left(e^{-2\Omega(T_{t+1},t_2)} \left[1 - e^{-\frac{2(\tau_{s+1}-\tau_{s,j})}{\lambda_s}} \right] \frac{\lambda_i}{2} \right) dt_2 + \int_{T_1}^{T_{t+1}} \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_i} \left(1 - e^{-\frac{2(\tau_{s+1}-T_{s,j})}{\lambda_i}} \right) \frac{\lambda_i}{3} \right\} \int_{T_1}^{T_{t+1}} e^{-2\Omega(E_{i,j}[s_3],t_2)} dt_2 + \\ \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_i} \left(1 - e^{-\frac{2(\tau_{s+1}-T_{s,j})}{\lambda_i}} \right) \frac{\lambda_i}{3} + \left[1 - e^{-\frac{3(E_{i,j}|s_3|-T_{i+1})}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_1}^{T_{t+1}} e^{-2\Omega(E_{i,j}[s_3],t_2)} dt_2 + \\ \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_i} \left(1 - e^{-\frac{2(\tau_{s+1}-T_{s,j})}{\lambda_i}} \right) \frac{\lambda_i}{3} + \left[$$

$$\begin{split} &\frac{2}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \frac{\lambda_j}{2} \int_{T_i}^{T_{i+1}} \left[1 - e^{-\frac{2(s_2 - T_i)}{\lambda_i}} \right] dt_2 \\ &= \frac{1}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \\ &= \frac{1}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left[1 - e^{-\frac{2(s_2 - T_i)}{\lambda_i}} \frac{\lambda_a}{3} + \left[1 - e^{-\frac{2(s_{i,j}[s_2] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} \int_{T_i}^{T_{i+1}} e^{-2\Omega(\mathrm{E}_{i,j}[s_3], t_2)} dt_2 + \\ &= \frac{2}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left(\left[1 - e^{-\frac{2(T_{i+1} - E_{i,j}[s_3])}{\lambda_i}} \frac{\lambda_i}{2} \right] \int_{T_i}^{T_{i+1}} e^{-2\Omega(T_{i+1}, t_2)} dt_2 + \\ &= \frac{2}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \frac{\lambda_i}{2} \int_{T_i}^{T_{i+1}} \left[1 - e^{-\frac{2(s_2 - T_i)}{\lambda_i}} \right] dt_2 \\ &= \frac{2}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left(\left[1 - e^{-\frac{2(s_2 - T_i)}{\lambda_i}} \right] \frac{1}{\lambda_i} \right] dt_2 \\ &= \frac{1}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left(\left[1 - e^{-\frac{2s_{3a}}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{2(s_{3a} - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \int_{T_i}^{T_{i+1}} e^{-2\Omega(\mathrm{E}_{i,j}[s_3], t_2)} dt_2 + \\ &= \frac{2}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left(\left[1 - e^{-\frac{2s_{3a}}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{2(s_{3a} - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \int_{T_i}^{T_{i+1}} e^{-2\Omega(\mathrm{E}_{i,j}[s_3], t_2)} dt_2 + \\ &= \frac{2}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left(\sum_{n=i+1}^{i-1} \left[1 - e^{-\frac{2s_{3a}}{\lambda_a}} \right] \frac{\lambda_a}{2} \int_{T_i}^{T_{i+1}} e^{-2\Omega(T_{i+1}, t_2)} dt_2 \right) + \\ &= \frac{2}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left(\sum_{n=i+1}^{i-1} \left[1 - e^{-\frac{2s_{3a}}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{2(s_{3a} - T_i, t_2)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) e^{-2\Omega(\mathrm{E}_{i,j}[s_3], T_i)} \int_{T_i}^{T_{i+1}} e^{-2\Omega(T_i, t_2)} dt_2 + \\ &= \frac{2}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left(\sum_{n=i+1}^{i-1} \left[1 - e^{-\frac{2s_{3a}}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{2(s_{3a} - T_i, t_2)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) e^{-2\Omega(\mathrm{E}_{i,j}[s_3], T_i)} \int_{T_i}^{T_{i+1}} e^{-2\Omega(T_i, t_2)} dt_2 + \\ &= \frac{2}{2\mathrm{E}_{i,j}[s_2] + \mathrm{E}_{i,j}[s_3]} \frac{1}{\lambda_i} \left(\sum_{n=i+1}^{i-1} \left[1 - e^{-\frac{2s_{3a}}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-2\Omega(T_{i+1}, T_i)} \int_{T_i}^{T_{i+1}} e^{-2\Omega(T_i,$$

$$\begin{split} &\left\{\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, \mathbf{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\mathbf{E}_{i,j}[s_3] - T_i)}{\lambda_i}}\right] \frac{\lambda_i}{3}\right\} e^{-2\Omega(\mathbf{E}_{i,j}[s_3], T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2 - T_l)}{\lambda_l}} dt_2 + \\ &\frac{2}{2 \, \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \mathbf{E}_{i,j}[s_3])}{\lambda_i}}\right] \frac{\lambda_i}{2}\right) e^{-2\Omega(T_{i+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2 - T_l)}{\lambda_l}} dt_2 + \\ &\frac{2}{2 \, \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}}\right] \frac{\lambda_a}{2} e^{-2\Omega(T_{a+1}, T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{2(t_2 - T_l)}{\lambda_l}} dt_2\right) + \\ &\frac{2}{2 \, \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \int_{T_l}^{T_{l+1}} \left[1 - e^{-\frac{2(t_2 - T_l)}{\lambda_l}}\right] dt_2 \\ &= \frac{1}{2 \, \mathbf{E}_{i,j}[s_2] + \mathbf{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\ \end{split}$$

$$\begin{split} &= \frac{1}{2\operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \\ & \left\{ \sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},\operatorname{E}_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(\operatorname{E}_{i,j}[s_3] - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right\} e^{-2\Omega(\operatorname{E}_{i,j}[s_3],T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) + \\ & \frac{2}{2\operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left(\left[1 - e^{-\frac{2(T_{i+1} - \operatorname{E}_{i,j}[s_3])}{\lambda_i}} \right] \frac{\lambda_i}{2} \right) e^{-2\Omega(T_{i+1},T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) + \\ & \frac{2}{2\operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \left[\sum_{a=i+1}^{l-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{2} e^{-2\Omega(T_{a+1},T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) \right] + \\ & \frac{2}{2\operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \left[\Delta_l - \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) \right] \end{split}$$

3.1.3 Case C

For $k < i = l < j \ (t_3 < s_3; t_2 = s_3)$:

$$\begin{split} &=\frac{1}{2s_2+s_3}\frac{2}{\lambda_{\alpha(t_3)}}\left\{\sum_{i=0}^{c(t_3)-1}e^{-3\Omega(T_{i+1},t_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3(t_3-T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}}\right]\frac{\lambda_{\alpha(t_3)}}{3}\right\}\\ &=\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{i=0}^{k-1}e^{-3\Omega(T_{i+1},t_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}\right\}\\ &=\int_{T_k}^{T_{k+1}}\int_{T_1}^{T_{l+1}}\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}e^{-3\Omega(T_{a+1},t_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}\right\}\delta(t_2-\operatorname{E}_{i,j}[s_3])dt_2dt_3\\ &=\int_{T_k}^{T_{k+1}}\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}e^{-3\Omega(T_{a+1},t_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}\right\}dt_3\\ &=\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}\int_{T_k}^{T_{k+1}}e^{-3\Omega(T_{a+1},t_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}dt_3+\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}dt_3\right\}\\ &=\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}\int_{T_k}^{T_{k+1}}e^{-3\Omega(T_{a+1},t_3)}dt_3+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]dt_3\right\}\\ &=\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\int_{T_k}^{T_{k+1}}e^{-3\Omega(T_k,t_3)}dt_3\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]dt_3\right\}\\ &=\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\int_{T_k}^{T_{k+1}}e^{-\frac{3(t_3-T_k)}{\lambda_k}}dt_3\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]dt_3\right\}\\ &=\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\int_{T_k}^{T_{k+1}}e^{-\frac{3(t_3-T_k)}{\lambda_k}}dt_3\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\left[\Delta_k-\frac{\lambda_k}{3}\left(1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right)\right]\right\}\\ &=\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\frac{\lambda_k}{3}\left[1-e^{-\frac{3\Delta_k}{\lambda_k}}\right]\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\left[\Delta_k-\frac{\lambda_k}{3}\left(1-e^{-\frac{3\Delta_k}{\lambda_k}}\right)\right]\right\}\\ &=\frac{1}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_$$

3.1.4 Case D

For k < i < j = l $(t_3 < s_3; t_2 = s_2)$:

$$\begin{split} &=\frac{2}{2s_2+s_3}\frac{2}{\lambda_{\alpha(t_3)}}\left\{\sum_{i=0}^{\alpha(t_3)-1}e^{-3\Omega(T_{i+1},t_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3(t_3-T_{\alpha(t_3)})}{\lambda_{\alpha(t_3)}}}\right]\frac{\lambda_{\alpha(t_3)}}{3}\right\}\\ &=\int_{T_k}^{T_{k+1}}\int_{T_i}^{T_{i+1}}\frac{2}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}e^{-3\Omega(T_{a+1},t_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}\right\}\delta(t_2-\operatorname{E}_{i,j}[s_2])dt_2dt_3\\ &=\int_{T_k}^{T_{k+1}}\frac{2}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}e^{-3\Omega(T_{a+1},t_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}\right\}dt_3\\ &=\frac{2}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}\int_{T_k}^{T_{k+1}}e^{-3\Omega(T_{a+1},t_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}dt_3+\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]\frac{\lambda_k}{3}dt_3\right\}\\ &=\frac{2}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}\int_{T_k}^{T_{k+1}}e^{-3\Omega(T_{a+1},t_3)}dt_3+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]dt_3\right\}\\ &=\frac{2}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\int_{T_k}^{T_{k+1}}e^{-3\Omega(T_k,t_3)}dt_3\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]dt_3\right\}\\ &=\frac{2}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\int_{T_k}^{T_{k+1}}e^{-\frac{3(t_3-T_k)}{\lambda_k}}dt_3\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3(t_3-T_k)}{\lambda_k}}\right]dt_3\right\}\\ &=\frac{2}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\int_{T_k}^{T_{k+1}}e^{-\frac{3(t_3-T_k)}{\lambda_k}}dt_3\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\int_{T_k}^{T_{k+1}}\left[1-e^{-\frac{3\Delta_a}{\lambda_k}}\right]dt_3\right\}\\ &=\frac{2}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\frac{\lambda_k}{3}\left[1-e^{-\frac{3\Delta_k}{\lambda_k}}\right]\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\left[\Delta_k-\frac{\lambda_k}{3}\left(1-e^{-\frac{3\Delta_k}{\lambda_k}}\right)\right]\right\}\\ &=\frac{2}{2\operatorname{E}_{i,j}[s_2]+\operatorname{E}_{i,j}[s_3]}\frac{2}{\lambda_k}\left\{\frac{\lambda_k}{3}\left[1-e^{-\frac{3\Delta_k}{\lambda_k}}\right]\sum_{a=0}^{k-1}\left[1-e^{-\frac{3\Delta_a}{\lambda_k}}\right]\frac{\lambda_a}{3}e^{-3\Omega(T_{a+1},T_k)}+\frac{\lambda_k}{3}\left[\Delta_k-\frac{\lambda_k}{3}\left(1-e^{-\frac{3\Delta_k}{\lambda_k}}\right)\right]\right\}\\ &=\frac{2}$$

3.1.5 Case E

For $i < k < j = l \ (t_3 > s_3; t_2 = s_2)$:

$$\begin{split} &=\frac{2}{2s_2+s_3}\frac{2}{\lambda_{\alpha(t_3)}}e^{-2\Omega(s_3,t_3)}\left\{\sum_{i=0}^{\alpha(s_3)-1}e^{-3\Omega(T_{i+1},s_3)}\left[1-e^{-\frac{3\Delta_1}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3(s_3-T_{\gamma_{i(s_3)}})}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{3}\right\}\\ &=\int_{T_k}^{T_{k+1}}\int_{T_i}^{T_{i+1}}\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}\frac{2}{\lambda_k}e^{-2\Omega(E_{i,j}[s_3],t_3)}\times\\ &\left\{\sum_{i=0}^{i-1}e^{-3\Omega(T_{u+1},E_{i,j}[s_3))}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}\delta(t_2-E_{i,j}[s_2])dt_2dt_3\\ &=\int_{T_k}^{T_{k+1}}\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}\frac{2}{\lambda_k}e^{-2\Omega(E_{i,j}[s_3],t_3)}\times\\ &\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{u+1},E_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}dt_3\\ &=\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}\frac{2}{\lambda_k}\times\\ &\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{u+1},E_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}\int_{T_k}^{T_{k+1}}e^{-2\Omega(E_{i,j}[s_3],t_3)}dt_3\\ &=\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}\frac{2}{\lambda_k}\times\\ &\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{u+1},E_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}e^{-2\Omega(E_{i,j}[s_3],T_k)}\int_{T_k}^{T_{k+1}}e^{-2\Omega(T_k,t_3)}dt_3\\ &=\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}\frac{2}{\lambda_k}\times\\ &\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{u+1},E_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}e^{-2\Omega(E_{i,j}[s_3],T_k)}\int_{T_k}^{T_{k+1}}e^{-2\Omega(T_k,t_3)}dt_3\\ &=\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}\frac{2}{\lambda_k}\times\\ &\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{u+1},E_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}e^{-2\Omega(E_{i,j}[s_3],T_k)}\int_{T_k}^{T_{k+1}}e^{-2\frac{2(t_3-T_k)}{\lambda_k}}dt_3\\ &=\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}\frac{2}{\lambda_k}\times\\ &\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{u+1},E_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}e^{-2\Omega(E_{i,j}[s_3],T_k)}\int_{T_k}^{T_{k+1}}e^{-2\frac{2(t_3-T_k)}{\lambda_k}}dt_3\\ &=\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}\frac{2}{\lambda_k}\times\\ &\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{u+1},E_{i,j}[s_3]}\right]\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_3]-T_i)}}{\lambda_i}\right]\frac{\lambda_i}{3}\right\}e^{-2\Omega(E_{i,j$$

3.1.6 Case F

For $i < j = k < l \ (t_3 = s_2; t_2 > s_2)$:

$$\begin{split} &=\frac{2}{2s_2+s_3}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_{\alpha(t_2)}}e^{-\Omega(s_2,t_2)}\left\{\sum_{i=0}^{\alpha(s_3)-1}e^{-3\Omega(T_{i+1},s_3)}\left[1-e^{-\frac{3\Delta_i}{\lambda_i}}\right]\frac{\lambda_i}{3}+\left[1-e^{-\frac{3(s_2-T_{\alpha(s_3)})}{\lambda_{\alpha(s_3)}}}\right]\frac{\lambda_{\alpha(s_3)}}{3}\right\}\\ &=\int_{T_k}^{T_{k+1}}\int_{T_i}^{T_{i+1}}\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}e^{-2\Omega(E_{i,j}[s_3],E_{i,j}[s_2])}\frac{1}{\lambda_i}e^{-\Omega(E_{i,j}[s_2],t_2)}\times\\ &=\left\{\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},E_{i,j}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,j}[s_3]-T_i)}{\lambda_i}}\right]\frac{\lambda_i}{3}\right\}\delta(t_3-s_2)dt_2dt_3\\ &=\int_{T_i}^{T_{i+1}}\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}e^{-2\Omega(E_{i,j}[s_3],E_{i,j}[s_2])}\frac{1}{\lambda_i}e^{-\Omega(E_{i,j}[s_2],t_2)}\times\\ &=\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}e^{-2\Omega(E_{i,j}[s_3],E_{i,j}[s_2])}\frac{1}{\lambda_i}\times\\ &=\frac{2}{2E_{i,j}[s_2]+E_{i,j}[s_3]}e^{-2\Omega(E_{i,j}[s_3],E_{i,j}[s_2])}\frac{1}{\lambda_i$$

Here again we assume $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

The following cases require special consideration:

3.1.7 Case G

For i = k = l < j and $t_3 = s_3; t_2 < s_2$ (also need to consider $t_3 < s_3; t_2 = s_3$)

$$\begin{split} &= \int_{T_{k}}^{T_{k+1}} \int_{t_{3}}^{T_{k+1}} \left(\int_{0}^{s_{3}} \frac{du}{2s_{2} + s_{3}} e^{-3\Omega(u,s_{3})} \frac{1}{\lambda(t_{2})} e^{-2\Omega(s_{3},t_{2})} \delta(t_{3} - s_{3}) + 2 \int_{s_{3}}^{t_{2}} \frac{du}{2s_{2} + s_{3}} \frac{1}{\lambda(t_{2})} e^{-2\Omega(u,t_{2})} \delta(t_{3} - s_{3}) \right) dt_{2} dt_{3} \\ &= \int_{s_{3}}^{T_{k+1}} \left(\int_{0}^{s_{3}} \frac{du}{2s_{2} + s_{3}} e^{-3\Omega(u,s_{3})} \frac{1}{\lambda(t_{2})} e^{-2\Omega(s_{3},t_{2})} + 2 \int_{s_{3}}^{t_{2}} \frac{du}{2s_{2} + s_{3}} \frac{1}{\lambda(t_{2})} e^{-2\Omega(u,t_{2})} \right) dt_{2} \\ &= \int_{E_{l,j}[s_{3}]}^{T_{k+1}} \left(\int_{0}^{E_{l,j}[s_{3}]} \frac{du}{2E_{l,j}[s_{2}] + E_{l,j}[s_{3}]} \frac{ds}{\lambda_{k}} e^{-2\Omega(u,t_{2})} \right) dt_{2} \\ &= \int_{E_{l,j}[s_{3}]}^{t_{2}} \left(\int_{0}^{E_{l,j}[s_{3}]} \frac{du}{2E_{l,j}[s_{2}] + E_{l,j}[s_{3}]} \frac{ds}{\lambda_{k}} e^{-2\Omega(u,t_{2})} \right) dt_{2} \\ &= \int_{E_{l,j}[s_{3}]}^{t_{2}} \left(\int_{0}^{E_{l,j}[s_{3}]} \frac{du}{2E_{l,j}[s_{2}] + E_{l,j}[s_{3}]} \frac{ds}{\lambda_{k}} e^{-2\Omega(u,t_{2})} \right) dt_{2} \\ &= \int_{E_{l,j}[s_{3}]}^{t_{2}} \left(\int_{0}^{E_{l,j}[s_{3}]} \frac{du}{2E_{l,j}[s_{2}] + E_{l,j}[s_{3}]} \frac{ds}{\lambda_{k}} e^{-2\Omega(u,t_{2})} \right) dt_{2} \\ &= \frac{1}{2E_{l,j}[s_{2}] + E_{l,j}[s_{3}]} \int_{E_{l,j}[s_{3}]}^{T_{k+1}} \left(\frac{1}{\lambda_{k}} e^{-\frac{2(t_{2} - u_{l,j}[s_{3}])}{\lambda_{k}} \int_{0}^{E_{l,j}[s_{3}]} e^{-3\Omega(u,E_{l,j}[s_{3}])} du + \frac{2}{\lambda_{k}} \int_{E_{l,j}[s_{3}]}^{t_{2}} \left[1 - e^{-\frac{2(t_{2} - u_{l,j}[s_{3}])}{\lambda_{k}}} \right] dt_{2} \\ &= \frac{1}{2E_{l,j}[s_{2}] + E_{l,j}[s_{3}]} \int_{E_{l,j}[s_{3}]}^{T_{k+1}} \left(\frac{1}{\lambda_{k}} \int_{0}^{E_{l,j}[s_{3}]} e^{-3\Omega(u,E_{l,j}[s_{3}]s_{3}]} e^{-3\Omega(u,E_{l,j}[s_{3}])} du + \frac{2}{\lambda_{k}} \int_{E_{l,j}[s_{3}]}^{T_{k+1}} \left[1 - e^{-\frac{2(t_{2} - u_{l,j}[s_{3}]}{\lambda_{k}}} \right] dt_{2} \right) \\ &= \frac{1}{2E_{l,j}[s_{2}] + E_{l,j}[s_{3}]} \left(\frac{1}{\lambda_{k}} \int_{0}^{E_{l,j}[s_{3}]} e^{-3\Omega(u,E_{l,j}[s_{3}])} du + \frac{2}{\lambda_{k}} \int_{E_{l,j}[s_{3}]}^{T_{k+1}} \left[1 - e^{-\frac{2(t_{2} - u_{l,j}[s_{3}]}{\lambda_{k}}} \right] dt_{2} \right) \\ &= \frac{1}{2E_{l,j}[s_{2}] + E_{l,j}[s_{3}]} \left(\frac{1}{\lambda_{k}} \int_{0}^{E_{l,j}[s_{3}]} e^{-3\Omega(u,E_{l,j}[s_{3}]} du + \frac{2}{\lambda_{k}} \int_{0}^{E_{l,j}[s_{3}]} e^{-3\Omega(u,E_{l,j}[s_{3}]} du + \frac{2}{\lambda_{k}} \int_{0}^{E_{l,j}[s_{3}]} e^{-2(u,E_{l,j}[s_{3}]}) du + \frac{2}{\lambda_{k}} \int_$$

3.1.8 Case G2

For i = k = l < j and $t_3 < s_3; t_2 = s_3$:

$$\begin{split} &= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,t_3)} \frac{2}{\lambda(t_3)} \delta(t_2 - s_3) \right) dt_2 \, dt_3 \\ &= \int_{T_k}^{s_3} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,t_3)} \frac{2}{\lambda(t_3)} \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_3} \frac{2}{\lambda(t_3)} \int_0^{t_3} e^{-3\Omega(u,t_3)} du \, dt_3 \\ &= \frac{1}{2 \operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \int_{T_k}^{\operatorname{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \int_0^{t_3} e^{-3\Omega(u,t_3)} du \, dt_3 \\ &= \frac{1}{2 \operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \int_{T_k}^{\operatorname{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1},t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\ &= \frac{1}{2 \operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{\operatorname{E}_{i,j}[s_3]} e^{-3\Omega(T_{a+1},t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{\operatorname{E}_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\ &= \frac{1}{2 \operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{2\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{\operatorname{E}_{i,j}[s_3]} e^{-3\Omega(T_k,t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{\operatorname{E}_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\ &= \frac{1}{2 \operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{\operatorname{E}_{i,j}[s_3]} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{\operatorname{E}_{i,j}[s_3]} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\ &= \frac{1}{2 \operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{2}{\lambda(t_3)} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] + \frac{\lambda_k}{3} \left(\operatorname{E}_{i,j}[s_3] - T_k - \frac{\lambda_k}{3} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(t_{i,j}+s_{i,j}-T_k)}{\lambda_k}} \right] + \\ &= \frac{1}{2 \operatorname{E}_{i,j}[s_2] + \operatorname{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(t_{i,j}+s_{i,j}-T_k)}{\lambda_k}} \right] + \\ &= \frac{1}{2 \operatorname{E}_{i,j}[s_3] + \operatorname{E}_{i,j}[s_3]} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1$$

$$\frac{\lambda_k}{3} \left(\mathbf{E}_{i,j}[s_3] - T_k - \frac{\lambda_k}{3} \left[1 - e^{-\frac{3(\mathbf{E}_{i,j}[s_3] - T_k)}{\lambda_k}} \right] \right) \right)$$

3.1.9 Case H

Another case that requires special consideration is i < k = l = j. For i < k = l = j and $t_3 > s_3$; $t_2 = s_2$:

$$\begin{split} &= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} \delta(t_2 - s_2) \right) dt_2 dt_3 \\ &= \int_{T_k}^{s_2} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_2} \left(2 \int_0^{s_3} e^{-3\Omega(u, s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} du \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \int_{T_k}^{s_2} \left(\frac{4}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} \int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) dt_3 \\ &= \frac{1}{2s_2 + s_3} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) e^{-2\Omega(s_3, T_k)} \int_{T_k}^{s_2} \frac{4}{\lambda(t_3)} e^{-2\Omega(T_k, t_3)} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \int_{T_k}^{s_2} e^{-2\Omega(T_k, t_3)} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \int_{T_k}^{s_2} e^{-2\Omega(T_k, t_3)} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \int_{T_k}^{s_2} e^{-\frac{2(t_3 - T_k)}{\lambda_k}} dt_3 \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(u, s_3)} du \right) \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}} \right] \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(t_{k+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_t)}{\lambda_k}} \right] \frac{\lambda_i}{3} \right) \frac{\lambda_k}{2} \left[1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}} \right] \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(t_{k+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_t)}{\lambda_k}} \right] \frac{\lambda_i}{3} \right) \frac{\lambda_k}{2} \left[1 - e^{-\frac{3(s_3 - T_t)}{\lambda_k}} \right] \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(t_{k+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_t)}{\lambda_k}} \right] \frac{\lambda_i}{3} \right) \frac{\lambda_k}{2} \left[1 - e^{-\frac{3(s_3 - T_t)}{\lambda_k}} \right] \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(t_{k+1}, s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_k}} \right] \left[1 - e^{-\frac{3\Delta_a}{\lambda_k}} \right] \frac{\lambda_k}{3} + \left[1 - e^{-\frac{3(s_3 - T_t)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right] \\ &= \frac{1}{2s_2 + s_3} \frac{4}{\lambda_k} e^{-2\Omega(s_3, T_k)} \left(\int_0^{s_3} e^{-3\Omega(t_{k+1}, s_3$$

3.1.10 Case H2

For i < k = l = j and $t_3 = s_2$; $t_2 > s_2$:

$$\begin{split} &= \int_{T_k}^{T_{k+1}} \int_{t_3}^{T_{k+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \delta(t_3 - s_2) \right) dt_2 \, dt_3 \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \int_{s_2}^{T_{k+1}} \left(e^{-\Omega(s_2,t_2)} \int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) dt_2 \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \left(\int_{s_2}^{T_{k+1}} e^{-\Omega(s_2,t_2)} dt_2 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\ &\left(\int_{s_2}^{T_{k+1}} e^{-\Omega(s_2,t_2)} dt_2 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\ &\left(\int_{s_2}^{T_{k+1}} e^{-\frac{t_2 - s_2}{\lambda_k}} dt_2 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} e^{-2\Omega(s_3,s_2)} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1},s_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\ &\left(\lambda_k \left[1 - \delta(k - n) e^{-\frac{T_{k+1} - s_2}{\lambda_k}} \right] \right) \\ &= \frac{2}{2E_{i,j}[s_2] + E_{i,j}[s_3]} \frac{1}{\lambda_k} e^{-2\Omega(E_{i,j}[s_3], E_{i,j}[s_2])} \left(\sum_{a=0}^{i-1} e^{-3\Omega(T_{a+1}, E_{i,j}[s_3])} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(s_3 - T_i)}{\lambda_i}} \right] \frac{\lambda_i}{3} \right) \times \\ &\lambda_k \left[1 - \delta(k - n) e^{-\frac{T_{k+1} - E_{i,j}[s_2]}{\lambda_k}} \right] \end{aligned}$$

The delta function is a way to ensure correctness when k = l = j = n. Another special case that requires attention is i = j = k < l.

3.1.11 Case I

For i = j = k < l and $t_3 = s_3; t_2 > s_2$:

$$\begin{split} &=\int_{T_k}^{T_{k+1}}\int_{T_1}^{T_{t+1}}\left(\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}\delta(t_3-s_3)+\\ &\quad 2\int_{s_3}^{s_2}\frac{du}{2s_2+s_3}e^{-2\Omega(u,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}\delta(t_3-s_3)\right)dt_2\,dt_3\\ &=\int_{T_1}^{T_{t+1}}\left(\int_0^{s_3}\frac{du}{2s_2+s_3}e^{-3\Omega(u,s_3)}e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}+\\ &\quad 2\int_{s_3}^{s_2}\frac{du}{2s_2+s_3}e^{-2\Omega(u,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}\right)dt_2\\ &=\frac{1}{2s_2+s_3}\int_{T_1}^{T_{t+1}}\left(\int_0^{s_3}e^{-3\Omega(u,s_3)}du\,e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}\right)dt_2\\ &=\frac{1}{2s_2+s_3}\int_{T_1}^{T_{t+1}}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}\right)dt_2\\ &=\frac{1}{2s_2+s_3}\int_{T_1}^{T_{t+1}}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda(t_2)}e^{-\Omega(s_2,t_2)}\int_0^{s_3}e^{-3\Omega(u,s_3)}du+\\ &\quad \frac{2}{2s_2+s_3}\int_{T_1}^{T_{t+1}}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_t}e^{-\Omega(s_2,t_2)}\int_{s_3}^{t-1}e^{-3\Omega(T_{u+1},s_3)}\left[1-e^{-\frac{3\Delta_u}{\lambda_s}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_1)}{\lambda_t}}\right]\frac{\lambda_t}{3}\right]+\\ &\quad \frac{2}{2s_2+s_3}\int_{T_1}^{T_{t+1}}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_t}e^{-\Omega(s_2,t_2)}\int_{s_3}^{t-1}e^{-3\Omega(T_{u+1},s_3)}\left[1-e^{-\frac{3\Delta_u}{\lambda_s}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_1)}{\lambda_t}}\right]\frac{\lambda_t}{3}\right]+\\ &\quad \frac{2}{2s_2+s_3}\int_{T_1}^{T_{t+1}}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_t}e^{-\Omega(s_2,t_2)}\int_{s_3}^{t-1}e^{-3\Omega(T_{u+1},s_3)}\left[1-e^{-\frac{3\Delta_u}{\lambda_s}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_1)}{\lambda_t}}\right]\frac{\lambda_t}{3}\right]+\\ &\quad \frac{2}{2s_2+s_3}\int_{T_1}^{T_{t+1}}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_t}e^{-\Omega(s_2,t_2)}\int_{s_3}^{t-1}e^{-3\Omega(T_{u+1},s_3)}\left[1-e^{-\frac{3\Delta_u}{\lambda_s}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_1)}{\lambda_t}}\right]\frac{\lambda_t}{3}\right]+\\ &\quad \frac{2}{2s_2+s_3}\int_{T_1}^{T_{t+1}}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_t}e^{-\Omega(s_2,t_2)}\int_{s_3}^{t-1}e^{-3\Omega(T_{u+1},s_3)}\left[1-e^{-\frac{3\Delta_u}{\lambda_s}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_1)}{\lambda_t}}\right]\frac{\lambda_t}{3}\right]+\\ &\quad \frac{2}{2s_2+s_3}\int_{T_1}^{T_{t+1}}\left(e^{-2\Omega(s_3,s_2)}\frac{\lambda_t}{\lambda_t}e^{-\Omega(s_2,t_2)}\int_{s_3}^{t-1}e^{-3\Omega(T_{u+1},s_3)}\left[1-e^{-\frac{3\Delta_u}{\lambda_s}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_1)}{\lambda_t}}\right]\frac{\lambda_t}{3}\right]+\\ &\quad \frac{2}{2s_2+s_3}\left(e^{-2\Omega(s_3,s_2)}\frac{\lambda_t}{\lambda_t}\left[1-e^{-\frac{3\Delta_u}{\lambda_s}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_1)}{\lambda_t}}\right]\frac{\lambda_t}{3}\right]+\\ &\quad \frac{2}{2s_2+s_3}\left(e^{-2\Omega(s_3,s_2)}\frac{\lambda_t}{\lambda_t}\left[1-e^{-\frac{3\Delta_u}{\lambda_t}}\right]\frac{\lambda_t}{3}+\left[1-e^{-\frac{3$$

$$\begin{split} &\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(s_2-s_3)}{\lambda_l}}\right]\int_{T_l}^{T_{l+1}}e^{-\Omega(s_2,t_2)}dt_2 \\ &=\frac{1}{2s_2+s_3}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_l}\left[\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},s_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_l)}{\lambda_l}}\right]\frac{\lambda_i}{3}\right]e^{-\Omega(s_2,T_l)}\int_{T_l}^{T_{l+1}}e^{-\Omega(T_l,t_2)}dt_2 + \\ &\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(s_2-s_3)}{\lambda_l}}\right]e^{-\Omega(s_2,T_l)}\int_{T_l}^{T_{l+1}}e^{-\Omega(T_l,t_2)}dt_2 \\ &=\frac{1}{2s_2+s_3}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_l}\left[\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},s_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_l)}{\lambda_l}}\right]\frac{\lambda_i}{3}\right]e^{-\Omega(s_2,T_l)}\int_{T_l}^{T_{l+1}}e^{-\frac{t_2-T_l}{\lambda_l}}dt_2 + \\ &\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(s_2-s_3)}{\lambda_l}}\right]e^{-\Omega(s_2,T_l)}\int_{T_l}^{T_{l+1}}e^{-\frac{t_2-T_l}{\lambda_l}}dt_2 \\ &=\frac{1}{2s_2+s_3}\left(e^{-2\Omega(s_3,s_2)}\frac{1}{\lambda_l}\left[\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},s_3)}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(s_3-T_l)}{\lambda_l}}\right]\frac{\lambda_i}{3}\right]e^{-\Omega(s_2,T_l)}\lambda_l\left[1-e^{-\frac{\Delta_l}{\lambda_l}}\right] + \\ &\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(s_2-s_3)}{\lambda_l}}\right]e^{-\Omega(s_2,T_l)}\lambda_l\left[1-e^{-\frac{\Delta_l}{\lambda_l}}\right] \\ &=\frac{1}{2E_{i,i}[s_2]+E_{i,i}[s_3]}\left(e^{-2\Omega(E_{i,i}[s_3],E_{i,i}[s_2])}\frac{1}{\lambda_l}\left[\sum_{a=0}^{i-1}e^{-3\Omega(T_{a+1},E_{i,i}[s_3])}\left[1-e^{-\frac{3\Delta_a}{\lambda_a}}\right]\frac{\lambda_a}{3}+\left[1-e^{-\frac{3(E_{i,i}[s_3]-T_l)}{\lambda_l}\right]\frac{\lambda_i}{3}\right] \times \\ &e^{-\Omega(E_{i,i}[s_2],T_l)}\lambda_l\left[1-\delta(l-n)e^{-\frac{\Delta_l}{\lambda_l}}\right] + \\ &\frac{2}{\lambda_l}\frac{\lambda_i}{2}\left[1-e^{-\frac{2(E_{i,i}[s_2]-E_{i,i}[s_3])}}\right]e^{-\Omega(E_{i,i}[s_2],T_l)}\lambda_l\left[1-\delta(l-n)e^{-\frac{\Delta_l}{\lambda_l}}\right] \right) \end{aligned}$$

Again, notice the delta function.

3.1.12 Case I2

For i = j = k < l and $t_3 = s_2; t_2 > s_2$:

$$\begin{split} &= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \delta(t_3 - s_2) \right) dt_2 \, dt_3 \\ &= \int_{T_l}^{T_{l+1}} \left(2 \int_0^{s_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} \right) dt_2 \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \int_{T_l}^{T_{l+1}} \left(e^{-\Omega(s_2,t_2)} \int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) dt_2 \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(\int_{T_l}^{T_{l+1}} e^{-\Omega(s_2,t_2)} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \int_{T_l}^{T_{l+1}} e^{-\frac{i_2-T_l}{\lambda_l}} dt_2 \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - \delta(l - n) e^{-\frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)} du \right) \\ &= \frac{2}{2s_2 + s_3} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \left(e^{-\Omega(s_2,T_l)} \lambda_l \left[1 - e^{-\delta(l - n) \frac{\Delta_l}{\lambda_l}} \right] \right) \left(\int_0^{s_3} e^{-3\Omega(u,s_3)}$$

Another case that requires special consideration: k < i = j = l. This includes either $t_3 < s_3$; $t_2 = s_2$ or $t_3 < s_3$; $t_2 = s_3$.

3.1.13 Case J

For k < i = j = l and $t_3 < s_3; t_2 = s_2$:

$$\begin{split} &= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(2 \int_0^{t_3} \frac{du}{2s_2 + s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(u,t_3)} \delta(t_2 - s_2) \right) dt_2 dt_3 \\ &= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{t_3} e^{-3\Omega(u,t_3)} \delta(t_2 - s_2) \right) dt_2 dt_3 \\ &= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} e^{-3\Omega(u,t_3)} \right) dt_3 \\ &= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1},t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\ &= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1},t_3)} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left[\left(\int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \right) + \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} dt_3 \right] \\ &= \frac{2}{2s_2 + s_3} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \right) + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\} \\ &= \frac{2}{2E_{i,i}[s_2] + E_{i,i}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \right) + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\} \\ &= \frac{2}{2E_{i,i}[s_2] + E_{i,i}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \right) + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right\} \\ &= \frac{2}{2E_{i,i}[s_2] + E_{i,i}[s_3]} \frac{2}{\lambda_k} \left\{ \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \right) + \frac{\lambda_k}$$

3.1.14 Case J2

For k < i = j = l and $t_3 < s_3; t_2 = s_3$:

$$\begin{split} &= \int_{T_k}^{T_{k+1}} \int_{T_l}^{T_{l+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,t_3)} \frac{2}{\lambda(t_3)} \delta(t_2 - s_3) \right) dt_2 dt_3 \\ &= \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} \frac{du}{2s_2 + s_3} e^{-3\Omega(u,t_3)} \frac{2}{\lambda(t_3)} \right) dt_3 \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\int_0^{t_3} e^{-3\Omega(u,t_3)} du \right) dt_3 \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \int_{T_k}^{T_{k+1}} \left(\sum_{a=0}^{k-1} e^{-3\Omega(T_{a+1},t_3)} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} + \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] \frac{\lambda_k}{3} \right) dt_3 \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_{a+1},t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k,t_3)} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{T_{k+1}} e^{-\frac{3(t_3 - T_k)}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{\lambda_k} e^{-\frac{3\Delta_k}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \int_{T_k}^{T_{k+1}} \left[1 - e^{-\frac{3(t_3 - T_k)}{\lambda_k}} \right] dt_3 \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{\lambda_k} e^{-\frac{3\Delta_k}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_k} \left(\sum_{a=0}^{k-1} \left[1 - e^{-\frac{3\Delta_a}{\lambda_a}} \right] \frac{\lambda_a}{3} e^{-3\Omega(T_{a+1},T_k)} \int_{T_k}^{\lambda_k} e^{-\frac{3\Delta_k}{\lambda_k}} dt_3 + \frac{\lambda_k}{3} \left[\Delta_k - \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}} \right) \right] \right) \\ &= \frac{2}{2s_2 + s_3} \frac{1}{\lambda_$$

This finishes the q((k,l)|(i,j)) transitions. Need to give special attention to the boundary cases, or the cases where i=j. Also need to consider what happens when k or l is n, since $T_{n+1}=\infty$.

The (diagonal) case of k = i; l = j is calculated by subtracting the sum of the off-diagonal entries from unity.

3.2 Emission probabilities

The genotype at a particular position in a triploid genome can take one of three different values: 0, 1, and 2. The state 0 represents a homozygous site, and 1 (2) represent sites where one (two) of the three chromosomes have a derived (*i.e.*, non-ancestral) copy at that position.

To form our observed chain, we consider all the genotypes in a stretch of b bp and categorize that stretch of the sequence with a state 0, 1, 2, or 3. The state 0 means that the stretch of b bp is completely homozygous. The state 1 means that there was at least one site that had a 1 genotype and none that had a 2 genotype. Likewise, the state 2 means that at least one site had a 2 genotype, and none had a 1 genotype. The state 3 means that at least one site had a 1 genotype and at least one site had a 1 genotype.

With observed states coded this way, the emission probabilities given local coalescence times t_3 and t_2 are

$$e_{k}(t_{3}, t_{2}, T_{d}) = \begin{cases} e^{-\frac{\theta b(2t_{2} + t_{3} + 3T_{d})}{2}} & k = 0\\ e^{-\frac{\theta b(t_{2} - t_{3})}{2}} \left(1 - e^{-\frac{\theta b(2t_{3} + t_{2} + 3T_{d})}{2}}\right) & k = 1\\ \left(1 - e^{-\frac{\theta b(t_{2} - t_{3})}{2}}\right) e^{-\frac{\theta b(2t_{3} + t_{2} + 3T_{d})}{2}} & k = 2\\ \left(1 - e^{-\frac{\theta b(t_{2} - t_{3})}{2}}\right) \left(1 - e^{-\frac{\theta b(2t_{3} + t_{2} + 3T_{d})}{2}}\right) & k = 3. \end{cases}$$

$$(19)$$

Here T_d is the divergence time, the time in the past [again measured in units of 2N(0) generations] when the asexual lineage was derived from a sexual ancestor. The above probabilities assume that t_3 and t_2 are measured continuously. In practice, we discretize time, so for a particular hidden state (i, j), we replace t_3 and t_2 with $E_{i,j}[t_3]$ and $E_{i,j}[t_2]$, respectively.

Classifying states and genotypes this way requires that each polymorphism be polarized against an outgroup. If this is not possible, then the states can be recoded as 0 and 1, where 0 is a stretch of b completely homozygous base pairs, and 1 is a stretch of b base pairs with at least one polymorphic position. If no polarization is possible, the emission probabilities become

$$e_k(t_3, t_2, T_d) = \begin{cases} e^{-\frac{\theta b(2t_2 + t_3 + 3T_d)}{2}} & k = 0\\ 1 - e^{-\frac{\theta b(2t_2 + t_3 + 3T_d)}{2}} & k = 1. \end{cases}$$
 (20)

A few things to note: The parameter b can be tuned to match the observed polymorphism. If the change in ploidy T_d generations ago also involved a change in mutation rate, this new mutation rate will be unidentifiable, impossible to distinguish from a proportionally scaled T_d . Thus T_d should be viewed as a compound parameter.

4 HMM inference

The above equations for q((k,l)|(i,j)) define a transition matrix $\{P_{(i,j),(k,l)}\}$, where

$$P_{(i,j),(k,l)} = \left(1 - e^{-\frac{\rho(2 \,\mathbb{E}_{i,j}[t_2] + \mathbb{E}_{i,j}[t_3])}{2}}\right) q((k,l)|(i,j)) \tag{21}$$

is the probability of transitioning from state (i, j) to state (k, l), for $(i, j) \neq (k, l)$. We define a hidden Markov chain $\{X_i\}$ that is governed by this transition matrix. The observed process $\{Y_i\}$ represents the different emissions, 0 through 3. The EM algorithm proceeds by starting with some initial parameters θ and iteratively maximizing the expectation of the full likelihood given the data, which is

$$P(X,Y|\theta) = e_{x_1}(y_1) \pi_{x_1} \prod_{i=1}^{T-1} P_{x_i,x_{i+1}} e_{x_{i+1}}(y_{i+1}),$$
(22)

where y_i is the observed state at position i, and x_i is the state of the hidden chain at position i, and T is the length of the sequence. In practice we maximize the log-likelihood:

$$\log P(X, Y | \theta) = \log (e_{x_1}(y_1)) + \log(\pi_{x_1}) + \sum_{i=1}^{T-1} \log (P_{x_i, x_{i+1}}) + \log (e_{x_{i+1}}(y_{i+1})).$$
 (23)

Since we observe only the observed chain (i.e., the mutation data), we have to have some way of integrating over the states of the hidden chain. We do this by the expectation-maximization (EM) algorithm, paired with the forward-backward algorithm for calculating likelihoods with these chains. In the context of HMM's, the EM algorithm iteratively maximizes

$$\begin{split} \mathbf{E}_{X|Y} \left[\log P \left(X, Y | \theta \right) \right] &= \mathbf{E}_{X|Y} \left[\log e_{x_1}(y_1) \right] + \mathbf{E}_{X|Y} \left[\log \pi_{x_1} \right] + \\ &\quad \mathbf{E}_{X|Y} \left[\log e_{x_1}(y_1) \right] + \sum_{i=1}^{T-1} \bigg(\mathbf{E}_{X|Y} \left[\log P_{x_i, x_{i+1}} \right] + \mathbf{E}_{X|Y} \left[\log(e_{x_{i+1}}(y_{i+1})) \right] \bigg), \end{split}$$

updating the parameters of the chains in each iteration.

Define the forward variable

$$\alpha_i(t) = P(Y_1 = y_1, Y_2 = y_2, \dots, Y_t = y_t, X_t = i \mid \theta),$$
 (24)

which satisfies the recursion

$$\alpha_j(t+1) = \sum_{i=1}^{N} \alpha_i(t) P_{i,j} e_j(y_{t+1}), \tag{25}$$

where N = (n+1)(n+2)/2 is the number of states in the hidden chain. We define

$$\alpha_i(1) = \pi_i e_i(y_1). \tag{26}$$

Define the backward variables

$$\beta_i(t) = P(Y_{t+1} = y_{t+1}, Y_{t+2} = y_{t+2}, \dots, Y_T = y_T | X_t = i, \theta), \tag{27}$$

which satisfy the recursion

$$\beta_i(t) = \sum_{j=1}^{N} \beta_j(t+1) P_{i,j} e_j(y_{t+1}), \tag{28}$$

defining $\beta_i(T) = 1$.

From these, we can calculate

$$\begin{split} \gamma_i(t) &= \mathrm{P}(X_t = i|Y,\theta) \\ &= \frac{\mathrm{P}(X_t = i,Y|\theta)}{\mathrm{P}(Y)} \\ &= \frac{\alpha_i(t)\beta_i(t)}{\sum_{j=1}^N \alpha_j(t)\beta_j(t)} \end{split}$$

and

$$\begin{split} \xi_{i,j}(t) &= \mathrm{P}(X_t = i, X_{t+1} = j | Y, \theta) \\ &= \frac{\alpha_i(t) P_{i,j} e_j(t+1) \beta_j(t+1)}{\mathrm{P}(Y | \theta)} \\ &= \frac{\alpha_i(t) P_{i,j} e_j(t+1) \beta_j(t+1)}{\sum_{k=1}^{N} \alpha_k(T)} \end{split}$$

With these probabilities, we can calculate $E_{X|Y}[\log(P(X,Y|\theta))]$:

$$\begin{split} \mathbf{E}_{X|Y} \left[\log P\left(X, Y | \theta \right) \right] &= \mathbf{E}_{X|Y} \left[\log e_{x_1}(y_1) \right] + \mathbf{E}_{X|Y} \left[\log \pi_{x_1} \right] + \\ & \sum_{i=1}^{T-1} \left(\mathbf{E}_{X|Y} \left[\log P_{x_i, x_{i+1}} \right] + \mathbf{E}_{X|Y} \left[\log(e_{x_{i+1}}(y_{i+1})) \right] \right) \\ &= \sum_{j=1}^{N} \gamma_j(1) \log \pi_j + \sum_{t=1}^{T} \sum_{j=1}^{N} \log \left[e_j(y_t) \right] \gamma_j(t) + \\ & \sum_{t=1}^{T-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \log(P_{i,j}) \xi_{i,j}(t) \end{split}$$

The expected number of emissions of state $k \in \{0, 1, 2, 3\}$ from (hidden) state i is

$$E_i(k) = \sum_{t=1}^{T} \gamma_i(t) \mathbb{I}(y_t = k),$$
 (29)

and the expected number of transitions from hidden state i to hidden state j is

$$E_{i,j} = \sum_{t=1}^{T-1} \xi_{i,j}(t). \tag{30}$$

Then the expected log-likelihood can be written

$$E_{X|Y} \left[\log P(X, Y | \theta) \right] = \sum_{j=1}^{N} \gamma_j(1) \log \pi_j + \sum_{j=1}^{N} \sum_{k=0}^{3} \log \left[e_j(k) \right] E_j(k) + \sum_{i=1}^{N} \sum_{j=1}^{N} \log(P_{i,j}) E_{i,j}$$

4.1 Underflow

In order to avoid underflow problems, we calculate scaled versions of $\alpha_i(t)$ and $\beta_{i,j}(t)$. We define

$$\tilde{\alpha}_i(t) = \frac{\alpha_i(t)}{\prod_{j=1}^t s_j},\tag{31}$$

for some scaling constants s_i , from which we can see the new recursion

$$\tilde{\alpha}_j(t+1) = \frac{1}{s_{t+1}} e_j(y_{t+1}) \sum_{i=1}^N P_{i,j} \tilde{\alpha}_i(t).$$
(32)

It is convenient to define s_t such that $\sum_{j=1}^N \tilde{\alpha}_j(t) = 1$. This means $s_1 = \sum_{k=1}^N \pi_i e_i(y_1)$ and

$$s_{t+1} = \sum_{j=1}^{N} e_j(y_{t+1}) \sum_{i=1}^{N} P_{i,j} \tilde{\alpha}_i(t)$$
(33)

We scale $\beta_i(t)$ by the same s_i so that

$$\tilde{\beta}_i(t) = \frac{\beta_i(t)}{\prod_{j=t+1}^T s_j} \tag{34}$$

and

$$\tilde{\beta}_i(t) = \frac{1}{s_{t+1}} \sum_{j=1}^N \tilde{\beta}_j(t+1) P_{i,j} e_j(y_{t+1}), \tag{35}$$

with $\tilde{\beta}_i(T) = \beta_i(T) = 1$. Now $\gamma_i(t)$ can be written

$$\gamma_{i}(t) = \frac{\alpha_{i}(t)\beta_{i}(t)}{\sum_{j=1}^{N} \alpha_{j}(t)\beta_{j}(t)}
= \frac{\tilde{\alpha}_{i}(t)\tilde{\beta}_{i}(t) \left(\prod_{j=1}^{T} s_{j}\right)}{\sum_{j=1}^{N} \tilde{\alpha}_{j}(t)\tilde{\beta}_{j}(t) \left(\prod_{j=1}^{T} s_{j}\right)}
= \frac{\tilde{\alpha}_{i}(t)\tilde{\beta}_{i}(t)}{\sum_{j=1}^{N} \tilde{\alpha}_{j}(t)\tilde{\beta}_{j}(t)}$$
(36)

and $\xi_{i,j}(t)$ can now be written

$$\xi_{i,j}(t) = \frac{\alpha_{i}(t)P_{i,j}e_{j}(t+1)\beta_{j}(t+1)}{P(Y|\theta)}
= \frac{\tilde{\alpha}_{i}(t)P_{i,j}e_{j}(t+1)\tilde{\beta}_{j}(t+1)\prod_{k=1}^{T}s_{k}}{P(Y|\theta)s_{t+1}}
= \frac{\tilde{\alpha}_{i}(t)P_{i,j}e_{j}(t+1)\tilde{\beta}_{j}(t+1)\prod_{k=1}^{T}s_{k}}{\sum_{l=1}^{N}\alpha_{l}(t)\beta_{l}(t)s_{t+1}}
= \frac{\tilde{\alpha}_{i}(t)P_{i,j}e_{j}(t+1)\tilde{\beta}_{j}(t+1)}{s_{t+1}\sum_{l=1}^{N}\tilde{\alpha}_{l}(t)\tilde{\beta}_{l}(t)}
= \frac{\gamma_{i}(t)P_{i,j}e_{j}(t+1)\tilde{\beta}_{j}(t+1)}{s_{t+1}\beta_{i}(t)}$$
(37)

Difficult to implement, but efficient and apparently correct!

5 Inferring diploid-to-triploid transition

In this section we consider the possibility that the asexual triploids were generated first by an asexual diploid line that is subsequently fertilized by a a haploid sperm, making a triploid asexual lineage. The goal is to infer the time between this diploid asexual phase and the triploid asexual phase, along with everything else that was considered for inference previously. In this case, the state at each position along the genome is the vector (t_3, t_2, W) , where $W \in \{0, 1\}$ is an indicator for whether the branch introduced by the triploidizing sperm subtends one of the diploid sexual branches coalescing at t_3 (W = 1) or t_2 (W = 0). The transition to diploid asexuality is defined as $D_3 < 0$. See Fig. 1 for an illustration of these states.

Having to keep track of W doubles the number of states, and since the complexity of the forward-backward algorithm is squared in the number of states, the runtime should increase by ~ 4 .

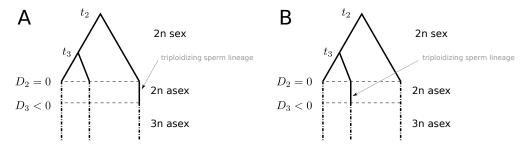


Figure 1: Figure 1. Two states with the same coalescence times t_3 and t_2 but with the triploidizing sperm's lineage subtending a branch that coalescences with the others (in the diploid sexual phase) at t_2 (panel A, W = 0) and at t_3 (panel B, W = 1). $D_2 = 0$ is the time of transition to diploid asexuality, and $D_3 < 0$ is the time of the transition to triploidy.

5.1 Diploid-to-triploid transition probabilities — Summary

The transition probabilities of the hidden model largely remain the same. A few changes occur:

1. In all previous probabilities, the factor $(2t_2 + t_3)^{-1}$ is replaced by the inverse of the new total (sexual) tree length, $(2t_2 + t_3 - D_3)^{-1}$, recalling that $D_3 < 0$. These factors are included unchanged all the way through the derivations and are still present in the final discrete transition probabilities, so they can just be replaced with the new factor.

- 2. Certain transitions will now change W as well. For example, if W = 0, (recall, subtending a branch that coalesces at t_2), the transition $(s_3, s_2) \to (t_3 < s_3, t_2 = s_3)$ implies that now W = 1.
- 3. Additional transition probability must be included, arising from recombination events that occur on the triploidizing sperm's sexual lineage between $t = D_3$ and t = 0.

To do this properly, it will also now be necessary to model the population size changes that occur in the sexual population during the diploid asexual phase, since the triploidizing sperm's lineage will experience those demographic changes. However, the only probabilities that depend on these population sizes are the "healing" probabilities specific to the SMC', meaning that only the "effective recombination rate" for this single lineage will depend on this part of the demographic history. For this reason, we will assume that the diploid sexual population containing the triploidizing sperm is constant in size. [This approximation would be unnecessary under the SMC model (vs. the SMC'), since healing is impossible under the SMC.] It should be a fine approximation and will have to be tested.

In general, we can expect there to be very little information about the length of this hypothesized diploid asexual interval in the triploid asexual's present-day genome. Including this additional period of diploid asexuality doesn't change the emission probabilities at all; it only changes the transition probabilities of the hidden model, and these changes seem like they will be pretty minor. However, it may be possible to rule out long periods of diploid asexuality in the putatively ancient lineages.

5.2 Determining when W changes and when it remains the same

For
$$t_3 = s_3$$
; $t_2 > s_2$:

 $W' = W$

For $t_3 = s_3$; $t_2 < s_2$:

 $W' = W$

For $t_3 < s_3$; $t_2 = s_3$, and $W = 0$:

 $W' = 1$

For $t_3 < s_3$; $t_2 = s_3$, and $W = 1$:

 $W' = 1$ with prob. $1/2$
 $W' = 0$ with prob. $1/2$

For $t_3 < s_3$; $t_2 = s_2$:

 $W' = W$

For $t_3 > s_3$; $t_2 = s_2$:

 $W' = W$

For $t_3 = s_2$; $t_2 > s_2$, $W = 0$:

 $W' = 1$

For $t_3 = s_2$; $t_2 > s_2$, $W = 1$:

 $W' = 1$ with prob. $1/2$
 $W' = 0$ with prob. $1/2$

For $t_3 = s_3$; $t_2 = s_2$:

5.3 Deriving the new probabilities

W' = W

The following are the "supplemental" probabilities of transition at the site of a recombination event, due to recombination events happening along the lineage of the triploidizing sperm in the time interval (D_3, D_2) .

(38)

For
$$t_3 = s_3$$
; $t_2 > s_2$, $W = 0$, and $W' = 0$:

$$\int_{D_2}^0 \frac{1}{2s_3+s_2-D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)}$$

For $t_3 = s_3$; $t_2 < s_2$, W = 0, and W' = 0:

$$\int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3,t_2)}$$

For $t_3 < s_3$; $t_2 = s_3$, W = 0, and W' = 1:

$$\int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} \frac{2}{\lambda(t_3)} e^{-3\Omega(0,t_3)}$$

For $t_3 < s_3$; $t_2 = s_2$, W = 1, and W' = 1:

$$\int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,t_3)} \frac{2}{\lambda(t_3)}$$

For $t_3 > s_3$; $t_2 = s_2$, W = 1, W' = 1:

$$\int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3,t_3)}$$

For $t_3 = s_2$; $t_2 > s_2$, W = 1, and W' = 0:

$$\int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)}$$

5.4 Calculating discrete transition probabilities with diploid-triploid transition

Let the relative population size between D_3 and D_2 be λ_d . Replace s_3 and s_2 with $E_{i,j}[s_3]$ and $E_{i,j}[s_2]$, respectively.

5.4.1 Case A:

For i = k < j < l, $(t_3 = s_3; t_2 > s_2)$, W' = W:

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q(i, j, k, l) + \\ \delta(W) \frac{1}{2s_{2} + s_{3} - D_{3}} \lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right) e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, s_{2})} \frac{1}{\lambda_{l}} \int_{T_{l}}^{T_{l+1}} e^{-\Omega(s_{2}, t_{2})} dt_{2}$$

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q(i, j, k, l) + \\ \delta(W) \frac{1}{2s_{2} + s_{3} - D_{3}} \lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right) e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, s_{2})} \frac{1}{\lambda_{l}} e^{-\Omega(s_{2}, T_{l})} \int_{T_{l}}^{T_{l+1}} e^{-\Omega(T_{l}, t_{2})} dt_{2}$$

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q(i, j, k, l) + \\ \delta(W) \frac{1}{2s_{2} + s_{3} - D_{3}} \lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right) e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, s_{2})} \frac{1}{\lambda_{l}} e^{-\Omega(s_{2}, T_{l})} \lambda_{l} \left(1 - e^{-\frac{\Delta_{l}}{\lambda_{l}}}\right)$$

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q(i, j, k, l) + \\ \delta(W) \frac{1}{2s_{2} + s_{3} - D_{3}} \lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right) e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, s_{2})} e^{-\Omega(s_{2}, T_{l})} \left(1 - e^{-\frac{\Delta_{l}}{\lambda_{l}}}\right)$$

As above we assume that $\Delta_n = \infty$ and thus $e^{-\Delta_n/\lambda_n} = 0$.

5.4.2 Case B:

For i = k < l < j $(t_3 = s_3; t_2 < s_2)$ and W = W':

$$\begin{split} &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q(i,j,k,l) + \\ &\delta(W) \int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} \int_{T_l}^{T_{l+1}} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3,t_2)} dt_2 \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q(i,j,k,l) + \\ &\delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right) e^{-3\Omega(0,s_3)} \frac{1}{\lambda_l} e^{-2\Omega(s_3,T_l)} \int_{T_l}^{T_{l+1}} e^{-2\Omega(T_l,t_2)} dt_2 \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q(i,j,k,l) + \\ &\delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right) e^{-3\Omega(0,s_3)} \frac{1}{\lambda_l} e^{-2\Omega(s_3,T_l)} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q(i,j,k,l) + \\ &\delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right) e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,T_l)} \frac{1}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) \\ &\delta(W) \frac{1}{2s_2 + s_3 - D_3} \lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right) e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,T_l)} \frac{1}{2} \left(1 - e^{-\frac{2\Delta_l}{\lambda_l}} \right) \end{split}$$

5.4.3 Case C

For k < i = l < j $(t_3 < s_3; t_2 = s_3), W = 0, W' = 1$:

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l) +$$

$$\int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u, 0)} \int_{T_k}^{T_{k+1}} \frac{2}{\lambda(t_3)} e^{-3\Omega(0, t_3)} dt_3$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l) +$$

$$\frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0, T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l) +$$

$$\frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0, T_k)} \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l) +$$

$$\frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{3} e^{-3\Omega(0, T_k)} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)$$

For k < i = l < j $(t_3 < s_3; t_2 = s_3), W = 1, W' \in \{0, 1\}:$

$$= \frac{1}{2} \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l)$$
(42)

5.4.4 Case D

For k < i < j = l $(t_3 < s_3; t_2 = s_2)$ and W' = W:

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l) + \\ \delta(W - 1) \int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u, 0)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(0, t_3)} \frac{2}{\lambda(t_3)} dt_3$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l) + \\ \delta(W - 1) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0, T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l) + \\ \delta(W - 1) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0, T_k)} \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l) + \\ \delta(W - 1) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{3} e^{-3\Omega(0, T_k)} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)$$

5.4.5 Case E

For i < k < j = l $(t_3 > s_3; t_2 = s_2)$ and W' = W:

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q(i, j, k, l) +$$

$$\delta(W = 1) \int_{D_{3}}^{0} \frac{du}{2s_{3} + s_{2} - D_{3}} e^{-\Omega(u, 0)} e^{-3\Omega(0, s_{3})} \int_{T_{k}}^{T_{k+1}} \frac{2}{\lambda(t_{3})} e^{-2\Omega(s_{3}, t_{3})} dt_{3}$$

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q(i, j, k, l) +$$

$$\delta(W - 1) \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} e^{-3\Omega(0, s_{3})} \frac{2}{\lambda_{k}} e^{-2\Omega(s_{3}, T_{k})} \frac{\lambda_{k}}{2} \left(1 - e^{-\frac{2\Delta_{k}}{\lambda_{k}}}\right)$$

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q(i, j, k, l) +$$

$$\delta(W - 1) \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, T_{k})} \left(1 - e^{-\frac{2\Delta_{k}}{\lambda_{k}}}\right)$$

$$\delta(W - 1) \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, T_{k})} \left(1 - e^{-\frac{2\Delta_{k}}{\lambda_{k}}}\right)$$

5.4.6 Case F

For $i < j = k < l \ (t_3 = s_2; t_2 > s_2)$ and W = 1:

$$\begin{split} &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \frac{1}{2} \, q(i,j,k,l) + \\ &\delta(W') \int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \int_{T_l}^{T_{l+1}} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} dt_2 \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \frac{1}{2} \, q(i,j,k,l) + \\ &\delta(W') \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)} dt_2 \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \frac{1}{2} \, q(i,j,k,l) + \\ &\delta(W') \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \lambda_l \left(1 - e^{-\frac{\Delta_l}{\lambda_l}} \right) \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \frac{1}{2} \, q(i,j,k,l) + \\ &\delta(W') \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} e^{-\Omega(s_2,T_l)} \left(1 - e^{-\frac{\Delta_l}{\lambda_l}} \right) \end{split}$$

For $i < j = k < l \ (t_3 = s_2; t_2 > s_2)$ and W = 0, W' = 1:

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q(i, j, k, l) \tag{46}$$

5.4.7 Case G

For i = k = l < j and $t_3 = s_3; t_2 < s_2$ and W = W':

$$\begin{split} &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q^G(i, j, k, l) + \\ &\delta(W) \int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u, 0)} e^{-3\Omega(0, s_3)} \int_{s_3}^{T_l} \frac{1}{\lambda(t_2)} e^{-2\Omega(s_3, t_2)} dt_2 \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q^G(i, j, k, l) + \\ &\delta(W) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0, s_3)} \frac{1}{\lambda_l} \int_{s_3}^{T_l} e^{-2\Omega(s_3, t_2)} dt_2 \\ &= \frac{2s_2 + s_3}{2s_2 + s_3} \, q^G(i, j, k, l) + \\ &\delta(W) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0, s_3)} \frac{1}{\lambda_l} \frac{\lambda_l}{2} \left(1 - e^{-\frac{2(T_l - s_3)}{\lambda_l}}\right) \\ &= \frac{2s_2 + s_3}{2s_2 + s_3} \, q^G(i, j, k, l) + \\ &\delta(W) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0, s_3)} \frac{1}{2} \left(1 - e^{-\frac{2(T_l - s_3)}{\lambda_l}}\right) \end{split}$$

Here, $q^{G}(i, j, k, l)$ is case G without the diploid-to-triploid variables.

5.4.8 Case G2

For i = k = l < j and $t_3 < s_3; t_2 = s_3, W = 0$, and W' = 1:

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{G2}(i, j, k, l) +$$

$$\int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u, 0)} \int_{T_k}^{s_3} \frac{2}{\lambda(t_3)} e^{-3\Omega(0, t_3)} dt_3$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{G2}(i, j, k, l) +$$

$$\frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0, T_k)} \int_{T_k}^{s_3} e^{-3\Omega(T_k, t_3)} dt_3$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{G2}(i, j, k, l) +$$

$$\frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0, T_k)} \frac{\lambda_k}{3} \left(1 - e^{-\frac{3(s_3 - T_k)}{\lambda_k}}\right)$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{G2}(i, j, k, l) +$$

$$\frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{3} e^{-3\Omega(0, T_k)} \left(1 - e^{-\frac{3(s_3 - T_k)}{\lambda_k}}\right)$$

For $t_3 < s_3$; $t_2 = s_3$, W = 1, $W' \in \{0, 1\}$:

$$= \frac{1}{2} \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{G2}(i, j, k, l)$$
(49)

5.4.9 Case H

For i < k = l = j $(t_3 > s_3; t_2 = s_2)$ and W = W':

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^H(i, j, k, l) + \\ \delta(W - 1) \int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u, 0)} e^{-3\Omega(0, s_3)} \int_{T_k}^{s_2} \frac{2}{\lambda(t_3)} e^{-2\Omega(s_3, t_3)} dt_3$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^H(i, j, k, l) + \\ \delta(W - 1) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0, s_3)} \frac{2}{\lambda_k} e^{-2\Omega(s_3, T_k)} \int_{T_k}^{s_2} e^{-2\Omega(T_k, t_3)} dt_3$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^H(i, j, k, l) + \\ \delta(W - 1) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0, s_3)} \frac{2}{\lambda_k} e^{-2\Omega(s_3, T_k)} \frac{\lambda_k}{2} \left(1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}}\right)$$

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^H(i, j, k, l) + \\ \delta(W - 1) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0, s_3)} e^{-2\Omega(s_3, T_k)} \left(1 - e^{-\frac{2(s_2 - T_k)}{\lambda_k}}\right)$$

5.4.10 Case H2

For i < k = l = j and $t_3 = s_2$; $t_2 > s_2$, W = 0, W' = 1:

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{H2}(i, j, k, l)$$
(51)

For i < k = l = j and $t_3 = s_2$; $t_2 > s_2$, W = 1, $W' \in \{0, 1\}$:

$$= \frac{1}{2} \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q^{H2}(i, j, k, l) +$$

$$\delta(W') \int_{D_{3}}^{0} \frac{du}{2s_{3} + s_{2} - D_{3}} e^{-\Omega(u, 0)} e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, s_{2})} \int_{s_{2}}^{T_{l}} \frac{1}{\lambda(t_{2})} e^{-\Omega(s_{2}, t_{2})} dt_{2}$$

$$= \frac{1}{2} \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q^{H2}(i, j, k, l) +$$

$$\delta(W') \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, s_{2})} \frac{1}{\lambda_{l}} \int_{s_{2}}^{T_{l}} e^{-\Omega(s_{2}, t_{2})} dt_{2}$$

$$= \frac{1}{2} \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q^{H2}(i, j, k, l) +$$

$$\delta(W') \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, s_{2})} \frac{1}{\lambda_{l}} \lambda_{l} \left(1 - e^{-\frac{T_{l} - s_{2}}{\lambda_{l}}}\right)$$

$$= \frac{1}{2} \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q^{H2}(i, j, k, l) +$$

$$\delta(W') \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} e^{-3\Omega(0, s_{3})} e^{-2\Omega(s_{3}, s_{2})} \left(1 - e^{-\frac{T_{l} - s_{2}}{\lambda_{l}}}\right)$$

5.4.11 Case I

For i = j = k < l and $t_3 = s_3; t_2 > s_2, W' = W$:

$$\begin{split} &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q^{(I)}(i,j,k,l) + \\ &\delta(W) \int_{D_3}^0 \frac{1}{2s_3 + s_2 - D_3} e^{-\Omega(u,0)} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \int_{T_l}^{T_{l+1}} \frac{1}{\lambda(t_2)} e^{-\Omega(s_2,t_2)} dt_2 \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q^{(I)}(i,j,k,l) + \\ &\delta(W) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \int_{T_l}^{T_{l+1}} e^{-\Omega(T_l,t_2)} dt_2 \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q^{(I)}(i,j,k,l) + \\ &\delta(W) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} \frac{1}{\lambda_l} e^{-\Omega(s_2,T_l)} \lambda_l \left(1 - e^{-\frac{\Delta_l}{\lambda_l}} \right) \\ &= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} \, q^{(I)}(i,j,k,l) + \\ &\delta(W) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} e^{-\Omega(s_2,T_l)} \left(1 - e^{-\frac{\Delta_l}{\lambda_l}} \right) \\ &\delta(W) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1 \right)}{2s_2 + s_3 - D_3} e^{-3\Omega(0,s_3)} e^{-2\Omega(s_3,s_2)} e^{-\Omega(s_2,T_l)} \left(1 - e^{-\frac{\Delta_l}{\lambda_l}} \right) \end{split}$$

5.4.12 Case I2

For i = j = k < l and $t_3 = s_2; t_2 > s_2, W = 0, W' = 1$:

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^{I2}(i, j, k, l)$$
(54)

For i = j = k < l and $t_3 = s_2; t_2 > s_2, W = 1, W' \in \{0, 1\}$:

$$\begin{split} &=\frac{1}{2}\frac{2s_{2}+s_{3}}{2s_{2}+s_{3}-D_{3}}q^{l2}(i,j,k,l)+\\ &\delta(W')\int_{D_{3}}^{0}\frac{du}{2s_{3}+s_{2}-D_{3}}e^{-\Omega(u,0)}e^{-3\Omega(0,s_{3})}e^{-2\Omega(s_{3},s_{2})}\int_{T_{l}}^{T_{l+1}}\frac{1}{\lambda(t_{2})}e^{-\Omega(s_{2},t_{2})}dt_{2}\\ &=\frac{1}{2}\frac{2s_{2}+s_{3}}{2s_{2}+s_{3}-D_{3}}q^{l2}(i,j,k,l)+\\ &\delta(W')\frac{\lambda_{d}\left(e^{-\frac{D_{3}}{\lambda_{d}}}-1\right)}{2s_{2}+s_{3}-D_{3}}e^{-3\Omega(0,s_{3})}e^{-2\Omega(s_{3},s_{2})}\frac{1}{\lambda_{l}}e^{-\Omega(s_{2},T_{l})}\int_{T_{l}}^{T_{l+1}}e^{-\Omega(T_{l},t_{2})}dt_{2}\\ &=\frac{1}{2}\frac{2s_{2}+s_{3}}{2s_{2}+s_{3}-D_{3}}q^{l2}(i,j,k,l)+\\ &\delta(W')\frac{\lambda_{d}\left(e^{-\frac{D_{3}}{\lambda_{d}}}-1\right)}{2s_{2}+s_{3}-D_{3}}e^{-3\Omega(0,s_{3})}e^{-2\Omega(s_{3},s_{2})}\frac{1}{\lambda_{l}}e^{-\Omega(s_{2},T_{l})}\lambda_{l}\left(1-e^{-\frac{\Delta_{l}}{\lambda_{l}}}\right)\\ &=\frac{1}{2}\frac{2s_{2}+s_{3}}{2s_{2}+s_{3}-D_{3}}q^{l2}(i,j,k,l)+\\ &\delta(W')\frac{\lambda_{d}\left(e^{-\frac{D_{3}}{\lambda_{d}}}-1\right)}{2s_{2}+s_{3}-D_{3}}e^{-3\Omega(0,s_{3})}e^{-2\Omega(s_{3},s_{2})}e^{-\Omega(s_{2},T_{l})}\left(1-e^{-\frac{\Delta_{l}}{\lambda_{l}}}\right) \end{split}$$

5.4.13 Case J

For k < i = j = l and $t_3 < s_3; t_2 = s_2, W' = W$:

$$= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^J(i, j, k, l) + \delta(W - 1) \int_{D_3}^0 \frac{du}{2s_3 + s_2 - D_3} e^{-\Omega(u, 0)} \frac{2}{\lambda_k} e^{-3\Omega(0, T_k)} \int_{T_k}^{T_{k+1}} e^{-3\Omega(T_k, t_3)} dt_3
= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^J(i, j, k, l) + \delta(W - 1) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{\lambda_k} e^{-3\Omega(0, T_k)} \frac{\lambda_k}{3} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right)
= \frac{2s_2 + s_3}{2s_2 + s_3 - D_3} q^J(i, j, k, l) + \delta(W - 1) \frac{\lambda_d \left(e^{-\frac{D_3}{\lambda_d}} - 1\right)}{2s_2 + s_3 - D_3} \frac{2}{3} e^{-3\Omega(0, T_k)} \left(1 - e^{-\frac{3\Delta_k}{\lambda_k}}\right) \tag{56}$$

5.4.14 Case J2

For k < i = j = l and $t_3 < s_3; t_2 = s_3, W = 0, W' = 1$:

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q^{J2}(i, j, k, l) + \int_{D_{3}}^{0} \frac{du}{2s_{3} + s_{2} - D_{3}} e^{-\Omega(u, 0)} \int_{T_{k}}^{T_{k+1}} \frac{2}{\lambda(t_{3})} e^{-3\Omega(0, t_{3})} dt_{3}$$

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q^{J2}(i, j, k, l) + \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} \frac{2}{\lambda_{k}} e^{-3\Omega(0, T_{k})} \int_{T_{k}}^{T_{k+1}} e^{-3\Omega(T_{k}, t_{3})} dt_{3}$$

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q^{J2}(i, j, k, l) + \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} \frac{2}{\lambda_{k}} e^{-3\Omega(0, T_{k})} \frac{\lambda_{k}}{3} \left(1 - e^{-\frac{3\Delta_{k}}{\lambda_{k}}}\right)$$

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q^{J2}(i, j, k, l) + \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} \frac{2}{3} e^{-3\Omega(0, T_{k})} \left(1 - e^{-\frac{3\Delta_{k}}{\lambda_{k}}}\right)$$

$$= \frac{2s_{2} + s_{3}}{2s_{2} + s_{3} - D_{3}} q^{J2}(i, j, k, l) + \frac{\lambda_{d} \left(e^{-\frac{D_{3}}{\lambda_{d}}} - 1\right)}{2s_{2} + s_{3} - D_{3}} \frac{2}{3} e^{-3\Omega(0, T_{k})} \left(1 - e^{-\frac{3\Delta_{k}}{\lambda_{k}}}\right)$$

For k < i = j = l and $t_3 < s_3; t_2 = s_3, W = 1, W' \in \{0, 1\}$:

$$=\frac{1}{2}\frac{2s_2+s_3}{2s_2+s_3-D_3}q^{J2}(i,j,k,l) \tag{58}$$