SIXTH EDITION

MECHANICS of MATERIALS

Ferdinand P. Beer
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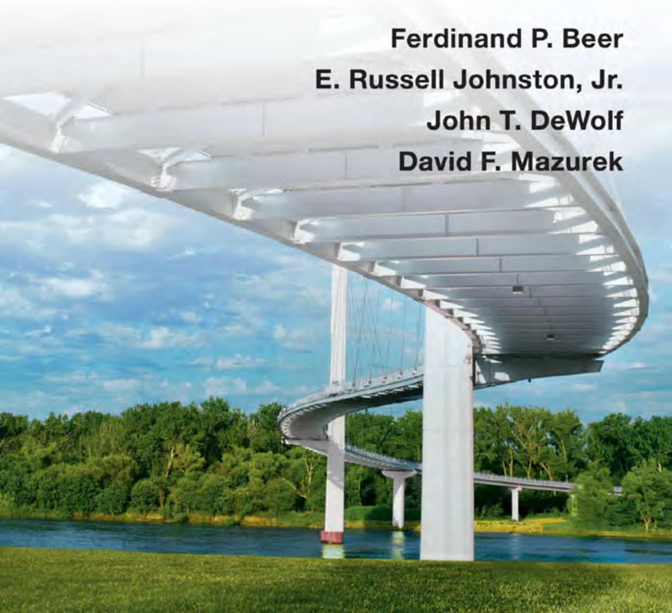

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SI Prefixes

Multiplication Factor	Prefix†	Symbol
1 000 000 000 000 = 10^{12}	tera	T
1 000 000 000 = 10^{9}	giga	G
1 000 000 = 10^{6}	mega	M
1 000 = 10^{3}	kilo	k
100 = 10^{2}	hecto‡	h
10 = 10^{1}	deka‡	da
0.1 = 10^{-1}	deci‡	d
0.01 = 10^{-2}	centi‡	c
0.001 = 10^{-3}	milli	m
0.000 001 = 10^{-6}	micro	μ
0.000 000 001 = 10^{-9}	nano	n
0.000 000 000 001 = 10^{-12}	pico	p
0.000 000 000 000 001 = 10^{-15}	femto	f
0.000 000 000 000 000 001 = 10^{-18}	atto	a

† The first syllable of every prefix is accented so that the prefix will retain its identity. Thus, the preferred pronunciation of kilometer places the accent on the first syllable, not the second.

‡ The use of these prefixes should be avoided, except for the measurement of areas and volumes and for the nontechnical use of centimeter, as for body and clothing measurements.

Principal SI Units Used in Mechanics

Quantity	Unit	Symbol	Formula
Acceleration	Meter per second squared	···	m/s^2
Angle	Radian	rad	†
Angular acceleration	Radian per second squared	···	rad/s^2
Angular velocity	Radian per second	···	rad/s
Area	Square meter	···	m^2
Density	Kilogram per cubic meter	···	kg/m^3
Energy	Joule	J	$N \cdot m$
Force	Newton	N	$kg \cdot m/s^2$
Frequency	Hertz	Hz	s^{-1}
Impulse	Newton-second	···	$kg \cdot m/s$
Length	Meter	m	‡
Mass	Kilogram	kg	‡
Moment of a force	Newton-meter	···	$N \cdot m$
Power	Watt	W	J/s
Pressure	Pascal	Pa	N/m^2
Stress	Pascal	Pa	N/m^2
Time	Second	s	‡
Velocity	Meter per second	···	m/s
Volume, solids	Cubic meter	···	m^3
Liquids	Liter	L	10^{-3} m^3
Work	Joule	J	$N \cdot m$

† Supplementary unit (1 revolution = 2π rad = 360°).

‡ Base unit.

U.S. Customary Units and Their SI Equivalents

Quantity	U.S. Customary Units	SI Equivalent
Acceleration	ft/s^2	$0.3048\ m/s^2$
	$in./s^2$	$0.0254\ m/s^2$
Area	ft^2	$0.0929\ m^2$
	in^2	$645.2\ mm^2$
Energy	ft · lb	1.356 J
Force	kip	4.448 kN
	lb	4.448 N
	oz	0.2780 N
Impulse	lb · s	4.448 N · s
Length	ft	0.3048 m
	in.	25.40 mm
	mi	1.609 km
Mass	oz mass	28.35 g
	lb mass	0.4536 kg
	slug	14.59 kg
	ton	907.2 kg
Moment of a force	lb · ft	1.356 N · m
	lb · in.	0.1130 N · m
Moment of inertia		
Of an area	in^4	$0.4162 \times 10^6\ mm^4$
Of a mass	$lb \cdot ft \cdot s^2$	$1.356\ kg \cdot m^2$
Power	ft · lb/s	1.356 W
	hp	745.7 W
Pressure or stress	lb/ft^2	47.88 Pa
	lb/in^2 (psi)	6.895 kPa
Velocity	ft/s	0.3048 m/s
	in./s	0.0254 m/s
	mi/h (mph)	0.4470 m/s
	mi/h (mph)	1.609 km/h
Volume, solids	ft^3	$0.02832\ m^3$
	in^3	$16.39\ cm^3$
Liquids	gal	3.785 L
	qt	0.9464 L
Work	ft · lb	1.356 J

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Sixth Edition

MECHANICS OF MATERIALS

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MECHANICS OF MATERIALS, SIXTH EDITION

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The photos on the front and back cover show the Bob Kerrey Pedestrian Bridge, which spans the Missouri River between Omaha, Nebraska, and Council Bluffs, Iowa. This S-curved structure utilizes a cable-stayed design, and is the longest pedestrian bridge to connect two states.

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About the Authors

As publishers of the books written by Ferd Beer and Russ Johnston, we are often asked how did they happen to write the books together, with one of them at Lehigh and the other at the University of Connecticut.

The answer to this question is simple. Russ Johnston's first teaching appointment was in the Department of Civil Engineering and Mechanics at Lehigh University. There he met Ferd Beer, who had joined that department two years earlier and was in charge of the courses in mechanics. Born in France and educated in France and Switzerland (he held an M.S. degree from the Sorbonne and an Sc.D. degree in the field of theoretical mechanics from the University of Geneva), Ferd had come to the United States after serving in the French army during the early part of World War II and had taught for four years at Williams College in the Williams-MIT joint arts and engineering program. Born in Philadelphia, Russ had obtained a B.S. degree in civil engineering from the University of Delaware and an Sc.D. degree in the field of structural engineering from MIT.

Ferd was delighted to discover that the young man who had been hired chiefly to teach graduate structural engineering courses was not only willing but eager to help him reorganize the mechanics courses. Both believed that these courses should be taught from a few basic principles and that the various concepts involved would be best understood and remembered by the students if they were presented to them in a graphic way. Together they wrote lecture notes in statics and dynamics, to which they later added problems they felt would appeal to future engineers, and soon they produced the manuscript of the first edition of *Mechanics for Engineers*. The second edition of *Mechanics for Engineers* and the first edition of *Vector Mechanics for Engineers* found Russ Johnston at Worcester Polytechnic Institute and the next editions at the University of Connecticut. In the meantime, both Ferd and Russ had assumed administrative responsibilities in their departments, and both were involved in research, consulting, and supervising graduate students—Ferd in the area of stochastic processes and random vibrations, and Russ in the area of elastic stability and structural analysis and design. However, their interest in improving the teaching of the basic mechanics courses had not subsided, and they both taught sections of these courses as they kept revising their texts and began writing together the manuscript of the first edition of *Mechanics of Materials*.

Ferd and Russ's contributions to engineering education earned them a number of honors and awards. They were presented with the Western Electric Fund Award for excellence in the instruction of engineering students by their respective regional sections of the American Society for Engineering Education, and they both received the Distinguished Educator Award from the Mechanics Division of the same society. In 1991 Russ received the Outstanding Civil Engineer Award from the Connecticut Section of the American Society of Civil Engineers, and in 1995 Ferd was awarded an honorary Doctor of Engineering degree by Lehigh University.

John T. DeWolf, Professor of Civil Engineering at the University of Connecticut, joined the Beer and Johnston team as an author on the second edition of *Mechanics of Materials.* John holds a B.S. degree in civil engineering from the University of Hawaii and M.E. and Ph.D. degrees in structural engineering from Cornell University. His research interests are in the area of elastic stability, bridge monitoring, and structural analysis and design. He is a registered Professional Engineering and a member of the Connecticut Board of Professional Engineers. He was selected as the University of Connecticut Teaching Fellow in 2006.

David F. Mazurek, Professor of Civil Engineering at the United States Coast Guard Academy, joined the team in the fourth edition. David holds a B.S. degree in ocean engineering and an M.S. degree in civil engineering from the Florida Institute of Technology, and a Ph.D. degree in civil engineering from the University of Connecticut. He is a registered Professional Engineer. He has served on the American Railway Engineering & Maintenance of Way Association's Committee 15—Steel Structures for the past seventeen years. Professional interests include bridge engineering, structural forensics, and blast-resistant design.

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Preface

OBJECTIVES

The main objective of a basic mechanics course should be to develop in the engineering student the ability to analyze a given problem in a simple and logical manner and to apply to its solution a few fundamental and well-understood principles. This text is designed for the first course in mechanics of materials—or strength of materials—offered to engineering students in the sophomore or junior year. The authors hope that it will help instructors achieve this goal in that particular course in the same way that their other texts may have helped them in statics and dynamics.

GENERAL APPROACH

In this text the study of the mechanics of materials is based on the understanding of a few basic concepts and on the use of simplified models. This approach makes it possible to develop all the necessary formulas in a rational and logical manner, and to clearly indicate the conditions under which they can be safely applied to the analysis and design of actual engineering structures and machine components.

Free-body Diagrams Are Used Extensively. Throughout the text free-body diagrams are used to determine external or internal forces. The use of "picture equations" will also help the students understand the superposition of loadings and the resulting stresses and deformations.

Design Concepts Are Discussed Throughout the Text Whenever Appropriate. A discussion of the application of the factor of safety to design can be found in Chap. 1, where the concepts of both allowable stress design and load and resistance factor design are presented.

A Careful Balance Between SI and U.S. Customary Units Is Consistently Maintained. Because it is essential that students be able to handle effectively both SI metric units and U.S. customary units, half the examples, sample problems, and problems to be assigned have been stated in SI units and half in U.S. customary units. Since a large number of problems are available, instructors can assign problems using each system of units in whatever proportion they find most desirable for their class.

Optional Sections Offer Advanced or Specialty Topics. Topics such as residual stresses, torsion of noncircular and thin-walled members, bending of curved beams, shearing stresses in non-symmetrical members, and failure criteria, have been included in optional sections for use in courses of varying emphases. To preserve the integrity of the subject, these topics are presented in the proper sequence, wherever they logically belong. Thus, even when not covered in the course, they are highly visible and can be easily referred to by the students if needed in a later course or in engineering practice. For convenience all optional sections have been indicated by asterisks.

CHAPTER ORGANIZATION

It is expected that students using this text will have completed a course in statics. However, Chap. 1 is designed to provide them with an opportunity to review the concepts learned in that course, while shear and bending-moment diagrams are covered in detail in Secs. 5.2 and 5.3. The properties of moments and centroids of areas are described in Appendix A; this material can be used to reinforce the discussion of the determination of normal and shearing stresses in beams (Chaps. 4, 5, and 6).

The first four chapters of the text are devoted to the analysis of the stresses and of the corresponding deformations in various structural members, considering successively axial loading, torsion, and pure bending. Each analysis is based on a few basic concepts, namely, the conditions of equilibrium of the forces exerted on the member, the relations existing between stress and strain in the material, and the conditions imposed by the supports and loading of the member. The study of each type of loading is complemented by a large number of examples, sample problems, and problems to be assigned, all designed to strengthen the students' understanding of the subject.

The concept of stress at a point is introduced in Chap. 1, where it is shown that an axial load can produce shearing stresses as well as normal stresses, depending upon the section considered. The fact that stresses depend upon the orientation of the surface on which they are computed is emphasized again in Chaps. 3 and 4 in the cases of torsion and pure bending. However, the discussion of computational techniques—such as Mohr's circle—used for the transformation of stress at a point is delayed until Chap. 7, after students have had the opportunity to solve problems involving a combination of the basic loadings and have discovered for themselves the need for such techniques.

The discussion in Chap. 2 of the relation between stress and strain in various materials includes fiber-reinforced composite materials. Also, the study of beams under transverse loads is covered in two separate chapters. Chapter 5 is devoted to the determination of the normal stresses in a beam and to the design of beams based on the allowable normal stress in the material used (Sec. 5.4). The chapter begins with a discussion of the shear and bending-moment diagrams (Secs. 5.2 and 5.3) and includes an optional section on the use of singularity functions for the determination of the shear and bending moment in a beam (Sec. 5.5). The chapter ends with an optional section on nonprismatic beams (Sec. 5.6).

Chapter 6 is devoted to the determination of shearing stresses in beams and thin-walled members under transverse loadings. The formula for the shear flow, $q = VQ/I$, is derived in the traditional way. More advanced aspects of the design of beams, such as the determination of the principal stresses at the junction of the flange and web of a W-beam, are in Chap. 8, an optional chapter that may be covered after the transformations of stresses have been discussed in Chap. 7. The design of transmission shafts is in that chapter for the same reason, as well as the determination of stresses under combined loadings that can now include the determination of the principal stresses, principal planes, and maximum shearing stress at a given point.

Statically indeterminate problems are first discussed in Chap. 2 and considered throughout the text for the various loading conditions encountered. Thus, students are presented at an early stage with a method of solution that combines the analysis of deformations with the conventional analysis of forces used in statics. In this way, they will have become thoroughly familiar with this fundamental method by the end of the course. In addition, this approach helps the students realize that stresses themselves are statically indeterminate and can be computed only by considering the corresponding distribution of strains.

The concept of plastic deformation is introduced in Chap. 2, where it is applied to the analysis of members under axial loading. Problems involving the plastic deformation of circular shafts and of prismatic beams are also considered in optional sections of Chaps. 3, 4, and 6. While some of this material can be omitted at the choice of the instructor, its inclusion in the body of the text will help students realize the limitations of the assumption of a linear stress-strain relation and serve to caution them against the inappropriate use of the elastic torsion and flexure formulas.

The determination of the deflection of beams is discussed in Chap. 9. The first part of the chapter is devoted to the integration method and to the method of superposition, with an optional section (Sec. 9.6) based on the use of singularity functions. (This section should be used only if Sec. 5.5 was covered earlier.) The second part of Chap. 9 is optional. It presents the moment-area method in two lessons.

Chapter 10 is devoted to columns and contains material on the design of steel, aluminum, and wood columns. Chapter 11 covers energy methods, including Castigliano's theorem.

PEDAGOGICAL FEATURES

Each chapter begins with an introductory section setting the purpose and goals of the chapter and describing in simple terms the material to be covered and its application to the solution of engineering problems.

Chapter Lessons. The body of the text has been divided into units, each consisting of one or several theory sections followed by sample problems and a large number of problems to be assigned.

Each unit corresponds to a well-defined topic and generally can be covered in one lesson.

Examples and Sample Problems. The theory sections include many examples designed to illustrate the material being presented and facilitate its understanding. The sample problems are intended to show some of the applications of the theory to the solution of engineering problems. Since they have been set up in much the same form that students will use in solving the assigned problems, the sample problems serve the double purpose of amplifying the text and demonstrating the type of neat and orderly work that students should cultivate in their own solutions.

Homework Problem Sets. Most of the problems are of a practical nature and should appeal to engineering students. They are primarily designed, however, to illustrate the material presented in the text and help the students understand the basic principles used in mechanics of materials. The problems have been grouped according to the portions of material they illustrate and have been arranged in order of increasing difficulty. Problems requiring special attention have been indicated by asterisks. Answers to problems are given at the end of the book, except for those with a number set in italics.

Chapter Review and Summary. Each chapter ends with a review and summary of the material covered in the chapter. Notes in the margin have been included to help the students organize their review work, and cross references provided to help them find the portions of material requiring their special attention.

Review Problems. A set of review problems is included at the end of each chapter. These problems provide students further opportunity to apply the most important concepts introduced in the chapter.

Computer Problems. Computers make it possible for engineering students to solve a great number of challenging problems. A group of six or more problems designed to be solved with a computer can be found at the end of each chapter. These problems can be solved using any computer language that provides a basis for analytical calculations. Developing the algorithm required to solve a given problem will benefit the students in two different ways: (1) it will help them gain a better understanding of the mechanics principles involved; (2) it will provide them with an opportunity to apply the skills acquired in their computer programming course to the solution of a meaningful engineering problem. These problems can be solved using any computer language that provide a basis for analytical calculations.

Fundamentals of Engineering Examination. Engineers who seek to be licensed as *Professional Engineers* must take two exams. The first exam, the *Fundamentals of Engineering Examination*, includes subject material from *Mechanics of Materials.* Appendix E lists the topics in *Mechanics of Materials* that are covered in this exam along with problems that can be solved to review this material.

SUPPLEMENTAL RESOURCES

Instructor's Solutions Manual. The Instructor's and Solutions Manual that accompanies the sixth edition continues the tradition of exceptional accuracy and keeping solutions contained to a single page for easier reference. The manual also features tables designed to assist instructors in creating a schedule of assignments for their courses. The various topics covered in the text are listed in Table I, and a suggested number of periods to be spent on each topic is indicated. Table II provides a brief description of all groups of problems and a classification of the problems in each group according to the units used. Sample lesson schedules are also found within the manual.

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Mechanics of Materials 6e also features a companion website (www.mhhe.com/beerjohnston) for instructors. Included on the website are lecture PowerPoints, an image library, and animations. Via the website, instructors can also request access to C.O.S.M.O.S., a complete online solutions manual organization system that allows instructors to create custom homework, quizzes, and tests using end-of-chapter problems from the text. For access to this material, contact your sales representative for a user name and password.

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ACKNOWLEDGMENTS

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John T. DeWolf
David F. Mazurek

List of Symbols

a	Constant; distance
$\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$	Forces; reactions
$A, B, C, \ldots$	Points
$A, \mathcal{A}$	Area
b	Distance; width
c	Constant; distance; radius
C	Centroid
$C_1, C_2, \ldots$	Constants of integration
C_P	Column stability factor
d	Distance; diameter; depth
D	Diameter
e	Distance; eccentricity; dilatation
E	Modulus of elasticity
f	Frequency; function
$\mathbf{F}$	Force
$F.S.$	Factor of safety
G	Modulus of rigidity; shear modulus
h	Distance; height
$\mathbf{H}$	Force
H, J, K	Points
$I, I_x, \ldots$	Moment of inertia
$I_{xy}, \ldots$	Product of inertia
J	Polar moment of inertia
k	Spring constant; shape factor; bulk modulus; constant
K	Stress concentration factor; torsional spring constant
l	Length; span
L	Length; span
L_e	Effective length
m	Mass
$\mathbf{M}$	Couple
$M, M_x, \ldots$	Bending moment
M_D	Bending moment, dead load (LRFD)
M_L	Bending moment, live load (LRFD)
M_U	Bending moment, ultimate load (LRFD)
n	Number; ratio of moduli of elasticity; normal direction
p	Pressure
$\mathbf{P}$	Force; concentrated load
P_D	Dead load (LRFD)
P_L	Live load (LRFD)
P_U	Ultimate load (LRFD)
q	Shearing force per unit length; shear flow
$\mathbf{Q}$	Force
Q	First moment of area

r	Radius; radius of gyration
$\mathbf{R}$	Force; reaction
R	Radius; modulus of rupture
s	Length
S	Elastic section modulus
t	Thickness; distance; tangential deviation
$\mathbf{T}$	Torque
T	Temperature
u, v	Rectangular coordinates
u	Strain-energy density
U	Strain energy; work
$\mathbf{v}$	Velocity
$\mathbf{V}$	Shearing force
V	Volume; shear
w	Width; distance; load per unit length
$\mathbf{W}, W$	Weight, load
x, y, z	Rectangular coordinates; distance; displacements; deflections
$\bar{x}, \bar{y}, \bar{z}$	Coordinates of centroid
Z	Plastic section modulus
α, β, γ	Angles
α	Coefficient of thermal expansion; influence coefficient
γ	Shearing strain; specific weight
γ_D	Load factor, dead load (LRFD)
γ_L	Load factor, live load (LRFD)
δ	Deformation; displacement
ϵ	Normal strain
θ	Angle; slope
λ	Direction cosine
ν	Poisson's ratio
ρ	Radius of curvature; distance; density
σ	Normal stress
τ	Shearing stress
ϕ	Angle; angle of twist; resistance factor
ω	Angular velocity

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MECHANICS OF MATERIALS

This chapter is devoted to the study of the stresses occurring in many of the elements contained in these excavators, such as two-force members, axles, bolts, and pins.

CHAPTER 1

Introduction—Concept of Stress

Chapter 1 Introduction—Concept of Stress

1.1 Introduction
1.2 A Short Review of the Methods of Statics
1.3 Stresses in the Members of a Structure
1.4 Analysis and Design
1.5 Axial Loading; Normal Stress
1.6 Shearing Stress
1.7 Bearing Stress in Connections
1.8 Application to the Analysis and Design of Simple Structures
1.9 Method of Problem Solution
1.10 Numerical Accuracy
1.11 Stress on an Oblique Plane Under Axial Loading
1.12 Stress Under General Loading Conditions; Components of Stress
1.13 Design Considerations

1.1 INTRODUCTION

The main objective of the study of the mechanics of materials is to provide the future engineer with the means of analyzing and designing various machines and load-bearing structures.

Both the analysis and the design of a given structure involve the determination of *stresses* and *deformations*. This first chapter is devoted to the concept of *stress*.

Section 1.2 is devoted to a short review of the basic methods of statics and to their application to the determination of the forces in the members of a simple structure consisting of pin-connected members. Section 1.3 will introduce you to the concept of *stress* in a member of a structure, and you will be shown how that stress can be determined from the *force* in the member. After a short discussion of engineering analysis and design (Sec. 1.4), you will consider successively the *normal stresses* in a member under axial loading (Sec. 1.5), the *shearing stresses* caused by the application of equal and opposite transverse forces (Sec. 1.6), and the *bearing stresses* created by bolts and pins in the members they connect (Sec. 1.7). These various concepts will be applied in Sec. 1.8 to the determination of the stresses in the members of the simple structure considered earlier in Sec. 1.2.

The first part of the chapter ends with a description of the method you should use in the solution of an assigned problem (Sec. 1.9) and with a discussion of the numerical accuracy appropriate in engineering calculations (Sec. 1.10).

In Sec. 1.11, where a two-force member under axial loading is considered again, it will be observed that the stresses on an *oblique* plane include both *normal* and *shearing* stresses, while in Sec. 1.12 you will note that *six components* are required to describe the state of stress at a point in a body under the most general loading conditions.

Finally, Sec. 1.13 will be devoted to the determination from test specimens of the *ultimate strength* of a given material and to the use of a *factor of safety* in the computation of the *allowable load* for a structural component made of that material.

1.2 A SHORT REVIEW OF THE METHODS OF STATICS

In this section you will review the basic methods of statics while determining the forces in the members of a simple structure.

Consider the structure shown in Fig. 1.1, which was designed to support a 30-kN load. It consists of a boom AB with a 30×50-mm rectangular cross section and of a rod BC with a 20-mm-diameter circular cross section. The boom and the rod are connected by a pin at B and are supported by pins and brackets at A and C, respectively. Our first step should be to draw a *free-body diagram* of the structure by detaching it from its supports at A and C, and showing the reactions that these supports exert on the structure (Fig. 1.2). Note that the sketch of the structure has been simplified by omitting all unnecessary details. Many of you may have recognized at this point that AB and BC are *two-force members*. For those of you who have not, we will pursue our analysis, ignoring that fact and assuming that the directions of the reactions at A and C are unknown. Each of these

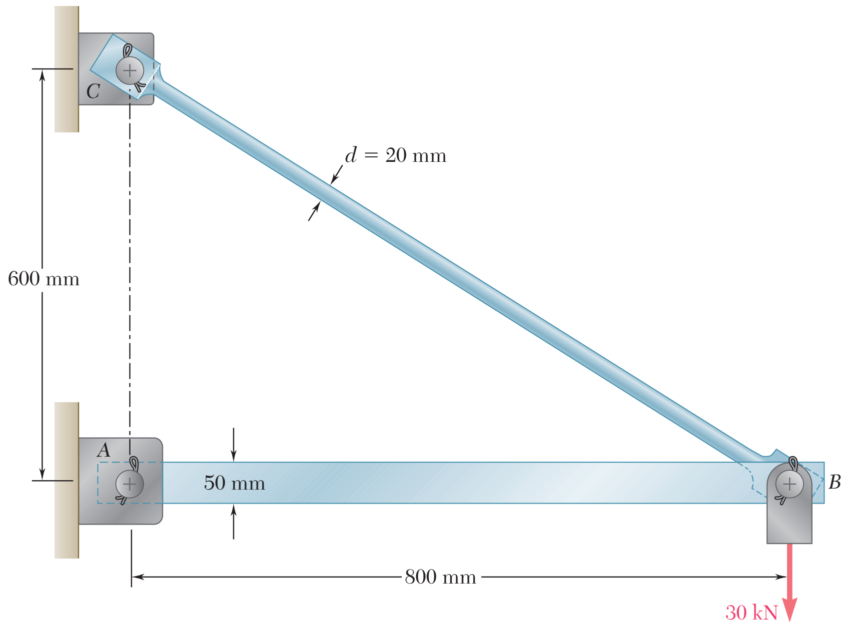


Fig. 1.1 Boom used to support a 30-kN load.

reactions, therefore, will be represented by two components, $\mathbf{A}_x$ and $\mathbf{A}_y$ at A, and $\mathbf{C}_x$ and $\mathbf{C}_y$ at C. We write the following three equilibrium equations:

$$+\curvearrowleft \Sigma M_C = 0: \qquad A_x(0.6 \text{ m}) - (30 \text{ kN})(0.8 \text{ m}) = 0$$

$$A_x = +40 \text{ kN} \tag{1.1}$$

$$\xrightarrow{+} \Sigma F_x = 0: \qquad A_x + C_x = 0$$

$$C_x = -A_x \qquad C_x = -40 \text{ kN} \tag{1.2}$$

$$+\uparrow \Sigma F_y = 0: \qquad A_y + C_y - 30 \text{ kN} = 0$$

$$A_y + C_y = +30 \text{ kN} \tag{1.3}$$

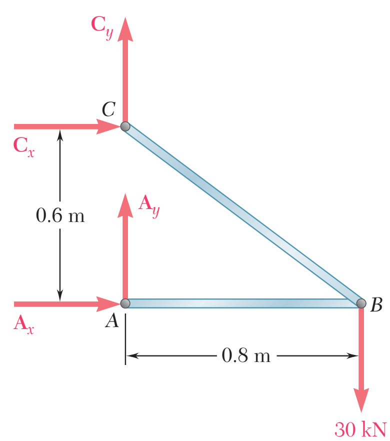


Fig. 1.2

We have found two of the four unknowns, but cannot determine the other two from these equations, and no additional independent equation can be obtained from the free-body diagram of the structure. We must now dismember the structure. Considering the free-body diagram of the boom AB (Fig. 1.3), we write the following equilibrium equation:

$$+\curvearrowleft \Sigma M_B = 0: \qquad -A_y(0.8 \text{ m}) = 0 \qquad A_y = 0 \tag{1.4}$$

Substituting for A_y from (1.4) into (1.3), we obtain $C_y = +30$ kN. Expressing the results obtained for the reactions at A and C in vector form, we have

$$\mathbf{A} = 40 \text{ kN} \rightarrow \qquad \mathbf{C}_x = 40 \text{ kN} \leftarrow, \mathbf{C}_y = 30 \text{ kN} \uparrow$$

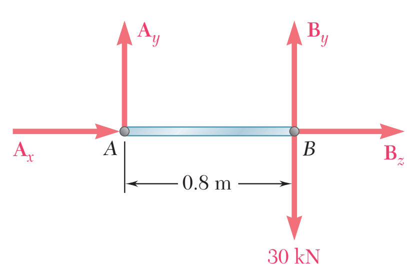


Fig. 1.3

We note that the reaction at A is directed along the axis of the boom AB and causes compression in that member. Observing that the components C_x and C_y of the reaction at C are, respectively, proportional to the horizontal and vertical components of the distance from B to C, we conclude that the reaction at C is equal to 50 kN, is directed along the axis of the rod BC, and causes tension in that member.

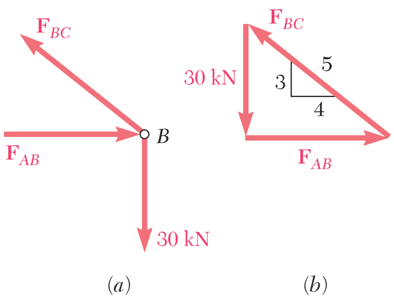


Fig. 1.4

These results could have been anticipated by recognizing that *AB* and *BC* are two-force members, i.e., members that are subjected to forces at only two points, these points being *A* and *B* for member *AB*, and *B* and *C* for member *BC*. Indeed, for a two-force member the lines of action of the resultants of the forces acting at each of the two points are equal and opposite and pass through both points. Using this property, we could have obtained a simpler solution by considering the free-body diagram of pin *B*. The forces on pin *B* are the forces $\mathbf{F}_{AB}$ and $\mathbf{F}_{BC}$ exerted, respectively, by members *AB* and *BC*, and the 30-kN load (Fig. 1.4*a*). We can express that pin *B* is in equilibrium by drawing the corresponding force triangle (Fig. 1.4*b*).

Since the force $\mathbf{F}_{BC}$ is directed along member *BC*, its slope is the same as that of *BC*, namely, 3/4. We can, therefore, write the proportion

$$\frac{F_{AB}}{4} = \frac{F_{BC}}{5} = \frac{30\text{ kN}}{3}$$

from which we obtain

$$F_{AB} = 40\text{ kN} \qquad F_{BC} = 50\text{ kN}$$

The forces $\mathbf{F}'_{AB}$ and $\mathbf{F}'_{BC}$ exerted by pin *B*, respectively, on boom *AB* and rod *BC* are equal and opposite to $\mathbf{F}_{AB}$ and $\mathbf{F}_{BC}$ (Fig. 1.5).

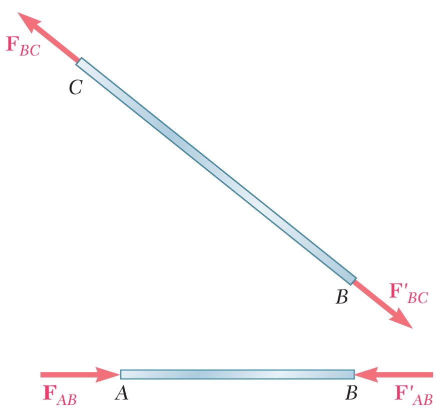


Fig. 1.5

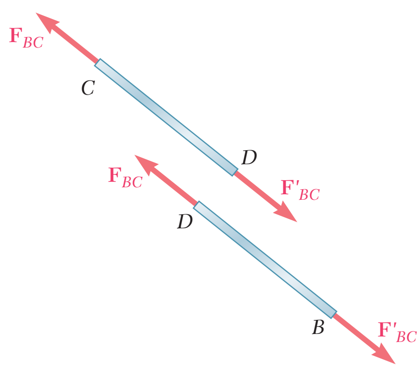


Fig. 1.6

Knowing the forces at the ends of each of the members, we can now determine the internal forces in these members. Passing a section at some arbitrary point *D* of rod *BC*, we obtain two portions *BD* and *CD* (Fig. 1.6). Since 50-kN forces must be applied at *D* to both portions of the rod to keep them in equilibrium, we conclude that an internal force of 50 kN is produced in rod *BC* when a 30-kN load is applied at *B*. We further check from the directions of the forces $\mathbf{F}_{BC}$ and $\mathbf{F}'_{BC}$ in Fig. 1.6 that the rod is in tension. A similar procedure would enable us to determine that the internal force in boom *AB* is 40 kN and that the boom is in compression.

1.3 STRESSES IN THE MEMBERS OF A STRUCTURE

While the results obtained in the preceding section represent a first and necessary step in the analysis of the given structure, they do not tell us whether the given load can be safely supported. Whether rod *BC*, for example, will break or not under this loading depends not only upon the value found for the internal force F_{BC}, but also upon the cross-sectional area of the rod and the material of which the rod is made. Indeed, the internal force F_{BC} actually represents the resultant of elementary forces distributed over the entire area *A* of the cross section (Fig. 1.7) and the average intensity of these distributed forces is equal to the force per unit area, F_{BC}/A, in the section. Whether or not the rod will break under the given loading clearly depends upon the ability of the material to withstand the corresponding value F_{BC}/A of the intensity of the distributed internal forces. It thus depends upon the force F_{BC}, the cross-sectional area *A*, and the material of the rod.

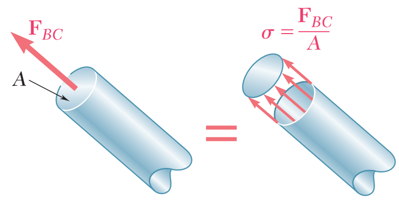


Fig. 1.7

The force per unit area, or intensity of the forces distributed over a given section, is called the *stress* on that section and is denoted by the Greek letter σ (sigma). The stress in a member of cross-sectional area *A* subjected to an axial load **P** (Fig. 1.8) is therefore obtained by dividing the magnitude *P* of the load by the area *A*:

$$\sigma = \frac{P}{A} \tag{1.5}$$

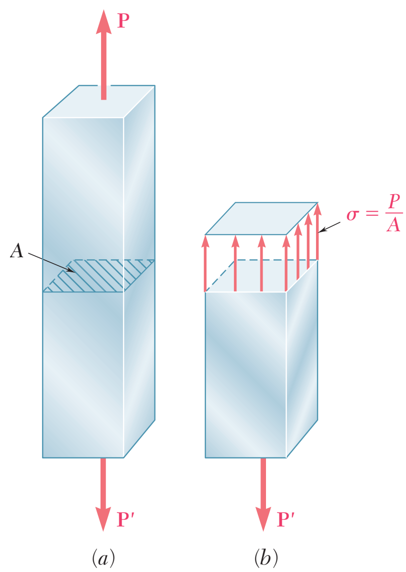


Fig. 1.8 Member with an axial load.

A positive sign will be used to indicate a tensile stress (member in tension) and a negative sign to indicate a compressive stress (member in compression).

Since SI metric units are used in this discussion, with *P* expressed in newtons (N) and *A* in square meters (m^2), the stress σ will be expressed in N/m^2. This unit is called a *pascal* (Pa). However, one finds that the pascal is an exceedingly small quantity and that, in practice, multiples of this unit must be used, namely, the kilopascal (kPa), the megapascal (MPa), and the gigapascal (GPa). We have

$$1 \text{ kPa} = 10^3 \text{ Pa} = 10^3 \text{ N/m}^2$$

$$1 \text{ MPa} = 10^6 \text{ Pa} = 10^6 \text{ N/m}^2$$

$$1 \text{ GPa} = 10^9 \text{ Pa} = 10^9 \text{ N/m}^2$$

When U.S. customary units are used, the force *P* is usually expressed in pounds (lb) or kilopounds (kip), and the cross-sectional area *A* in square inches (in^2). The stress σ will then be expressed in pounds per square inch (psi) or kilopounds per square inch (ksi).†

†The principal SI and U.S. customary units used in mechanics are listed in tables inside the front cover of this book. From the table on the right-hand side, we note that 1 psi is approximately equal to 7 kPa, and 1 ksi approximately equal to 7 MPa.

1.4 ANALYSIS AND DESIGN

Considering again the structure of Fig. 1.1, let us assume that rod *BC* is made of a steel with a maximum allowable stress $\sigma_{\text{all}} = 165$ MPa. Can rod *BC* safely support the load to which it will be subjected? The magnitude of the force F_{BC} in the rod was found earlier to be 50 kN. Recalling that the diameter of the rod is 20 mm, we use Eq. (1.5) to determine the stress created in the rod by the given loading. We have

$$P = F_{BC} = +50 \text{ kN} = +50 \times 10^3 \text{ N}$$

$$A = \pi r^2 = \pi\left(\frac{20 \text{ mm}}{2}\right)^2 = \pi(10 \times 10^{-3} \text{ m})^2 = 314 \times 10^{-6} \text{ m}^2$$

$$\sigma = \frac{P}{A} = \frac{+50 \times 10^3 \text{ N}}{314 \times 10^{-6} \text{ m}^2} = +159 \times 10^6 \text{ Pa} = +159 \text{ MPa}$$

Since the value obtained for σ is smaller than the value σ_{all} of the allowable stress in the steel used, we conclude that rod *BC* can safely support the load to which it will be subjected. To be complete, our analysis of the given structure should also include the determination of the compressive stress in boom *AB*, as well as an investigation of the stresses produced in the pins and their bearings. This will be discussed later in this chapter. We should also determine whether the deformations produced by the given loading are acceptable. The study of deformations under axial loads will be the subject of Chap. 2. An additional consideration required for members in compression involves the *stability* of the member, i.e., its ability to support a given load without experiencing a sudden change in configuration. This will be discussed in Chap. 10.

The engineer's role is not limited to the analysis of existing structures and machines subjected to given loading conditions. Of even greater importance to the engineer is the *design* of new structures and machines, that is, the selection of appropriate components to perform a given task. As an example of design, let us return to the structure of Fig. 1.1, and assume that aluminum with an allowable stress $\sigma_{\text{all}} = 100$ MPa is to be used. Since the force in rod *BC* will still be $P = F_{BC} = 50$ kN under the given loading, we must have, from Eq. (1.5),

$$\sigma_{\text{all}} = \frac{P}{A} \qquad A = \frac{P}{\sigma_{\text{all}}} = \frac{50 \times 10^3 \text{ N}}{100 \times 10^6 \text{ Pa}} = 500 \times 10^{-6} \text{ m}^2$$

and, since $A = \pi r^2$,

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{500 \times 10^{-6} \text{ m}^2}{\pi}} = 12.62 \times 10^{-3} \text{ m} = 12.62 \text{ mm}$$

$$d = 2r = 25.2 \text{ mm}$$

We conclude that an aluminum rod 26 mm or more in diameter will be adequate.

1.5 AXIAL LOADING; NORMAL STRESS

As we have already indicated, rod BC of the example considered in the preceding section is a two-force member and, therefore, the forces $\mathbf{F}_{BC}$ and $\mathbf{F}'_{BC}$ acting on its ends B and C (Fig. 1.5) are directed along the axis of the rod. We say that the rod is under *axial loading*. An actual example of structural members under axial loading is provided by the members of the bridge truss shown in Photo 1.1.

Photo 1.1 This bridge truss consists of two-force members that may be in tension or in compression.

Returning to rod BC of Fig. 1.5, we recall that the section we passed through the rod to determine the internal force in the rod and the corresponding stress was perpendicular to the axis of the rod; the internal force was therefore normal to the plane of the section (Fig. 1.7) and the corresponding stress is described as a *normal stress*. Thus, formula (1.5) gives us the *normal stress in a member under axial loading*:

$$\sigma = \frac{P}{A} \tag{1.5}$$

We should also note that, in formula (1.5), σ is obtained by dividing the magnitude P of the resultant of the internal forces distributed over the cross section by the area A of the cross section; it represents, therefore, the *average value* of the stress over the cross section, rather than the stress at a specific point of the cross section.

To define the stress at a given point Q of the cross section, we should consider a small area ΔA (Fig. 1.9). Dividing the magnitude of $\Delta\mathbf{F}$ by ΔA, we obtain the average value of the stress over ΔA. Letting ΔA approach zero, we obtain the stress at point Q:

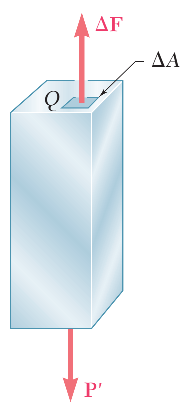


Fig. 1.9

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1.6}$$

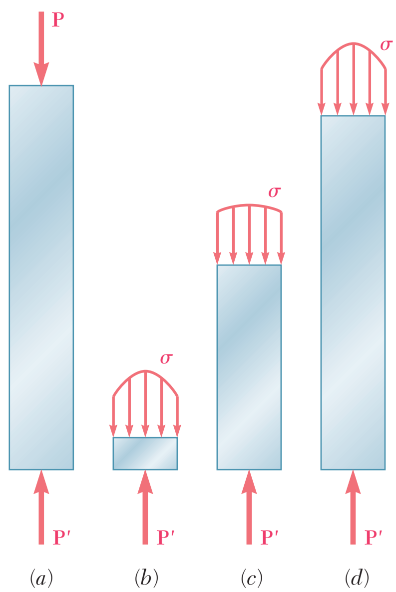


Fig. 1.10 Stress distributions at different sections along axially loaded member.

In general, the value obtained for the stress σ at a given point Q of the section is different from the value of the average stress given by formula (1.5), and σ is found to vary across the section. In a slender rod subjected to equal and opposite concentrated loads **P** and **P**′ (Fig. 1.10*a*), this variation is small in a section away from the points of application of the concentrated loads (Fig. 1.10*c*), but it is quite noticeable in the neighborhood of these points (Fig. 1.10*b* and *d*).

It follows from Eq. (1.6) that the magnitude of the resultant of the distributed internal forces is

$$\int dF = \int_A \sigma \, dA$$

But the conditions of equilibrium of each of the portions of rod shown in Fig. 1.10 require that this magnitude be equal to the magnitude P of the concentrated loads. We have, therefore,

$$P = \int dF = \int_A \sigma \, dA \tag{1.7}$$

which means that the volume under each of the stress surfaces in Fig. 1.10 must be equal to the magnitude P of the loads. This, however, is the only information that we can derive from our knowledge of statics, regarding the distribution of normal stresses in the various sections of the rod. The actual distribution of stresses in any given section is *statically indeterminate*. To learn more about this distribution, it is necessary to consider the deformations resulting from the particular mode of application of the loads at the ends of the rod. This will be discussed further in Chap. 2.

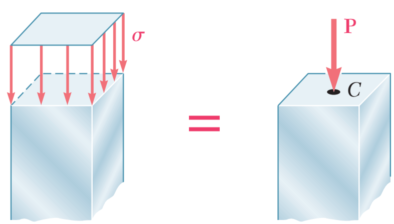


Fig. 1.11

In practice, it will be assumed that the distribution of normal stresses in an axially loaded member is uniform, except in the immediate vicinity of the points of application of the loads. The value σ of the stress is then equal to σ_{ave} and can be obtained from formula (1.5). However, we should realize that, when we assume a uniform distribution of stresses in the section, i.e., when we assume that the internal forces are uniformly distributed across the section, it follows from elementary statics† that the resultant **P** of the internal forces must be applied at the centroid C of the section (Fig. 1.11). This means that *a uniform distribution of stress is possible only if the line of action of the concentrated loads* **P** *and* **P**′ *passes through the centroid of the section considered* (Fig. 1.12). This type of loading is called *centric loading* and will be assumed to take place in all straight two-force members found in trusses and pin-connected structures, such as the one considered in Fig. 1.1. However, if a two-force member is loaded axially, but *eccentrically* as shown in Fig. 1.13*a*, we find from the conditions of equilibrium of the portion of member shown in Fig. 1.13*b* that the internal forces in a given section must be

†See Ferdinand P. Beer and E. Russell Johnston, Jr., *Mechanics for Engineers,* 5th ed., McGraw-Hill, New York, 2008, or *Vector Mechanics for Engineers,* 9th ed., McGraw-Hill, New York, 2010, Secs. 5.2 and 5.3.

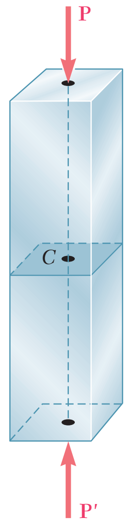


Fig. 1.12

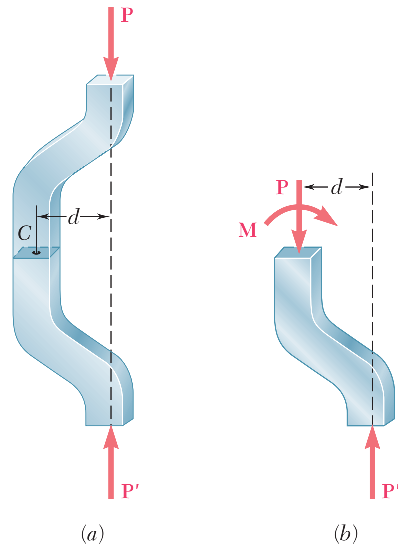


Fig. 1.13 Eccentric axial loading.

equivalent to a force **P** applied at the centroid of the section and a couple **M** of moment $M = Pd$. The distribution of forces—and, thus, the corresponding distribution of stresses—*cannot be uniform*. Nor can the distribution of stresses be symmetric as shown in Fig. 1.10. This point will be discussed in detail in Chap. 4.

1.6 SHEARING STRESS

The internal forces and the corresponding stresses discussed in Secs. 1.2 and 1.3 were normal to the section considered. A very different type of stress is obtained when transverse forces **P** and **P′** are applied to a member AB (Fig. 1.14). Passing a section at C between the points of application of the two forces (Fig. 1.15*a*), we obtain the diagram of portion AC shown in Fig. 1.15*b*. We conclude that internal forces must exist in the plane of the section, and that their resultant is equal to **P.** These elementary internal forces are called *shearing forces*, and the magnitude P of their resultant is the *shear* in the section. Dividing the shear P by the area A of the cross section, we

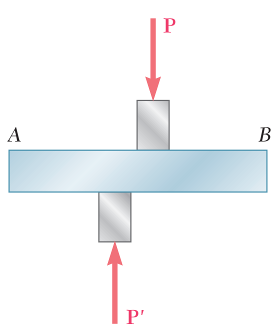


Fig. 1.14 Member with transverse loads.

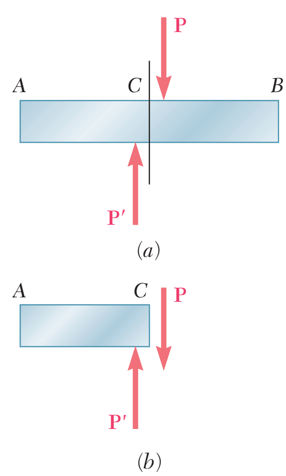


Fig. 1.15

obtain the *average shearing stress* in the section. Denoting the shearing stress by the Greek letter τ (tau), we write

$$\tau_{\text{ave}} = \frac{P}{A} \tag{1.8}$$

It should be emphasized that the value obtained is an average value of the shearing stress over the entire section. Contrary to what we said earlier for normal stresses, the distribution of shearing stresses across the section *cannot* be assumed uniform. As you will see in Chap. 6, the actual value τ of the shearing stress varies from zero at the surface of the member to a maximum value τ_{max} that may be much larger than the average value τ_{ave}.

Photo 1.2 Cutaway view of a connection with a bolt in shear.

Shearing stresses are commonly found in bolts, pins, and rivets used to connect various structural members and machine components (Photo 1.2). Consider the two plates A and B, which are connected by a bolt CD (Fig. 1.16). If the plates are subjected to tension forces of magnitude F, stresses will develop in the section of bolt corresponding to the plane EE'. Drawing the diagrams of the bolt and of the portion located above the plane EE' (Fig. 1.17), we conclude that the shear P in the section is equal to F. The average shearing stress in the section is obtained, according to formula (1.8), by dividing the shear $P = F$ by the area A of the cross section:

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A} \tag{1.9}$$

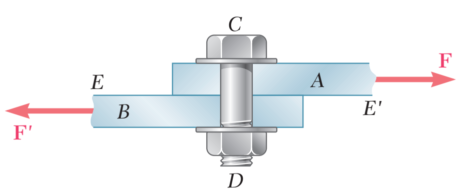


Fig. 1.16 Bolt subject to single shear.

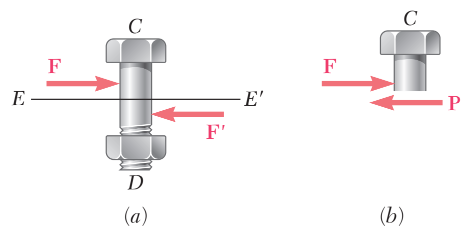


Fig. 1.17

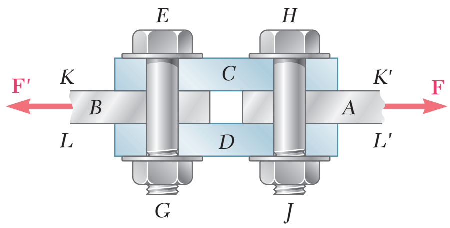


Fig. 1.18 Bolts subject to double shear.

The bolt we have just considered is said to be in *single shear*. Different loading situations may arise, however. For example, if splice plates C and D are used to connect plates A and B (Fig. 1.18), shear will take place in bolt HJ in each of the two planes KK' and LL' (and similarly in bolt EG). The bolts are said to be in *double shear*. To determine the average shearing stress in each plane, we draw free-body diagrams of bolt HJ and of the portion of bolt located between the two planes (Fig. 1.19). Observing that the shear P in each of the sections is $P = F/2$, we conclude that the average shearing stress is

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F/2}{A} = \frac{F}{2A} \tag{1.10}$$

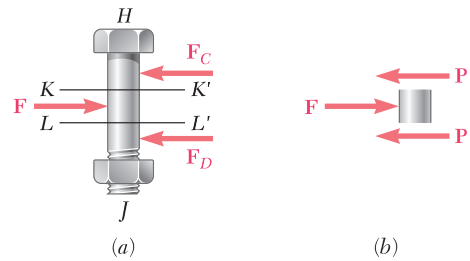


Fig. 1.19

1.7 BEARING STRESS IN CONNECTIONS

Bolts, pins, and rivets create stresses in the members they connect, along the *bearing surface,* or surface of contact. For example, consider again the two plates A and B connected by a bolt CD that we have discussed in the preceding section (Fig. 1.16). The bolt exerts on plate A a force **P** equal and opposite to the force **F** exerted by the plate on the bolt (Fig. 1.20). The force **P** represents the resultant of elementary forces distributed on the inside surface of a half-cylinder of diameter d and of length t equal to the thickness of the plate. Since the distribution of these forces—and of the corresponding stresses—is quite complicated, one uses in practice an average nominal value σ_b of the stress, called the *bearing stress,* obtained by dividing the load P by the area of the rectangle representing the projection of the bolt on the plate section (Fig. 1.21). Since this area is equal to td, where t is the plate thickness and d the diameter of the bolt, we have

$$\boxed{\sigma_b = \frac{P}{A} = \frac{P}{td}} \tag{1.11}$$

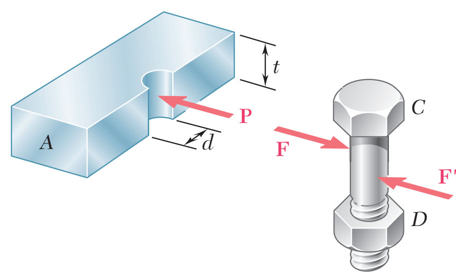


Fig. 1.20

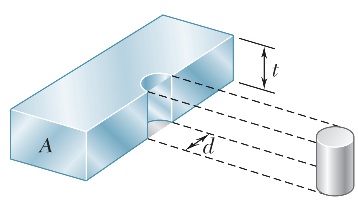


Fig. 1.21

1.8 APPLICATION TO THE ANALYSIS AND DESIGN OF SIMPLE STRUCTURES

We are now in a position to determine the stresses in the members and connections of various simple two-dimensional structures and, thus, to design such structures.

As an example, let us return to the structure of Fig. 1.1 that we have already considered in Sec. 1.2 and let us specify the supports and connections at A, B, and C. As shown in Fig. 1.22, the 20-mm-diameter rod BC has flat ends of 20×40-mm rectangular cross section, while boom AB has a 30×50-mm rectangular cross section and is fitted with a clevis at end B. Both members are connected at B by a pin from which the 30-kN load is suspended by means of a U-shaped bracket. Boom AB is supported at A by a pin fitted into a double bracket, while rod BC is connected at C to a single bracket. All pins are 25 mm in diameter.

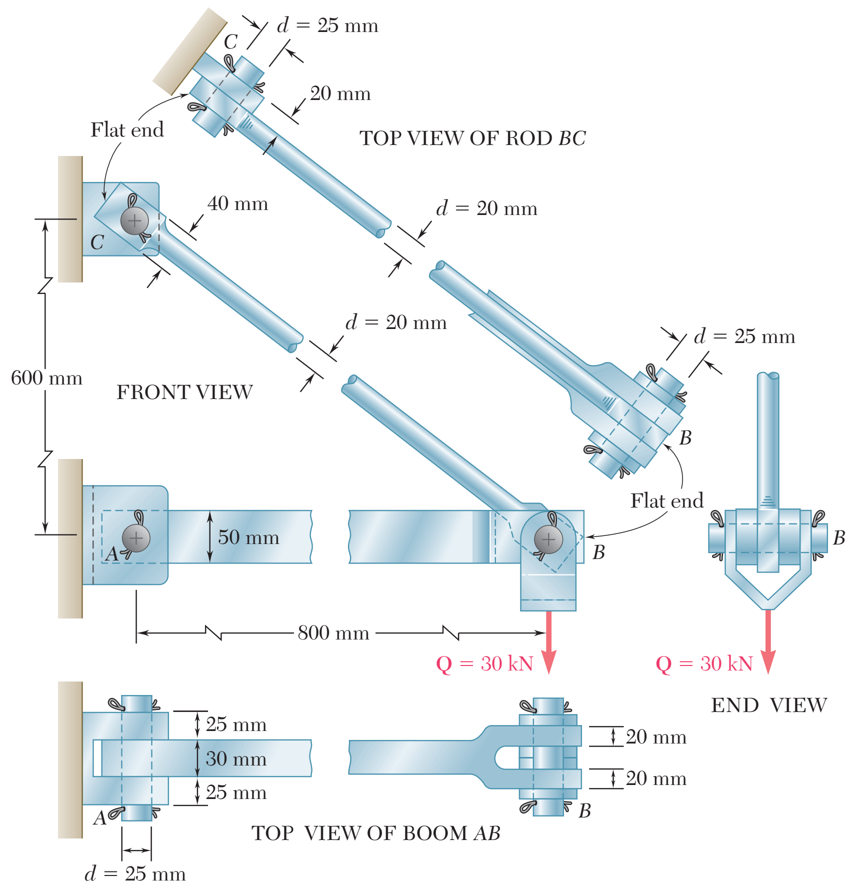


Fig. 1.22

a. Determination of the Normal Stress in Boom *AB* and Rod *BC*. As we found in Secs. 1.2 and 1.4, the force in rod BC is $F_{BC} = 50$ kN (tension) and the area of its circular cross section is $A = 314 \times 10^{-6}$ m^2; the corresponding average normal stress is $\sigma_{BC} = +159$ MPa. However, the flat parts of the rod are also under tension and at the narrowest section, where a hole is located, we have

$$A = (20 \text{ mm})(40 \text{ mm} - 25 \text{ mm}) = 300 \times 10^{-6} \text{ m}^2$$

The corresponding average value of the stress, therefore, is

$$(\sigma_{BC})_{\text{end}} = \frac{P}{A} = \frac{50 \times 10^3 \text{ N}}{300 \times 10^{-6} \text{ m}^2} = 167 \text{ MPa}$$

Note that this is an *average value*; close to the hole, the stress will actually reach a much larger value, as you will see in Sec. 2.18. It is clear that, under an increasing load, the rod will fail near one of the holes rather than in its cylindrical portion; its design, therefore, could be improved by increasing the width or the thickness of the flat ends of the rod.

Turning now our attention to boom AB, we recall from Sec. 1.2 that the force in the boom is $F_{AB} = 40$ kN (compression). Since the area of the boom's rectangular cross section is $A = 30 \text{ mm} \times 50 \text{ mm} = 1.5 \times 10^{-3} \text{ m}^2$, the average value of the normal stress in the main part of the rod, between pins A and B, is

$$\sigma_{AB} = -\frac{40 \times 10^3 \text{ N}}{1.5 \times 10^{-3} \text{ m}^2} = -26.7 \times 10^6 \text{ Pa} = -26.7 \text{ MPa}$$

Note that the sections of minimum area at A and B are not under stress, since the boom is in compression, and, therefore, *pushes* on the pins (instead of *pulling* on the pins as rod BC does).

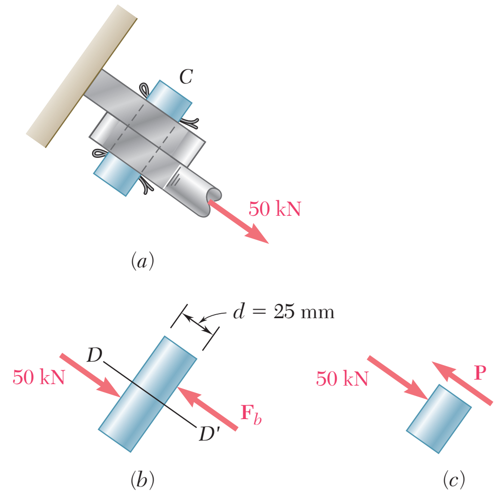


Fig. 1.23

b. Determination of the Shearing Stress in Various Connections. To determine the shearing stress in a connection such as a bolt, pin, or rivet, we first clearly show the forces exerted by the various members it connects. Thus, in the case of pin C of our example (Fig. 1.23*a*), we draw Fig. 1.23*b*, showing the 50-kN force exerted by member BC on the pin, and the equal and opposite force exerted by the bracket. Drawing now the diagram of the portion of the pin located below the plane DD' where shearing stresses occur (Fig. 1.23*c*), we conclude that the shear in that plane is $P = 50$ kN. Since the cross-sectional area of the pin is

$$A = \pi r^2 = \pi\left(\frac{25 \text{ mm}}{2}\right)^2 = \pi(12.5 \times 10^{-3} \text{ m})^2 = 491 \times 10^{-6} \text{ m}^2$$

we find that the average value of the shearing stress in the pin at C is

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{50 \times 10^3 \text{ N}}{491 \times 10^{-6} \text{ m}^2} = 102 \text{ MPa}$$

Considering now the pin at A (Fig. 1.24), we note that it is in double shear. Drawing the free-body diagrams of the pin and of the portion of pin located between the planes DD' and EE' where shearing stresses occur, we conclude that $P = 20$ kN and that

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{20 \text{ kN}}{491 \times 10^{-6} \text{ m}^2} = 40.7 \text{ MPa}$$

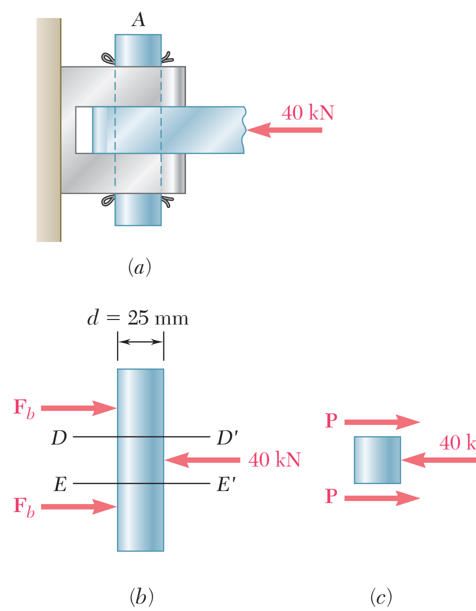


Fig. 1.24

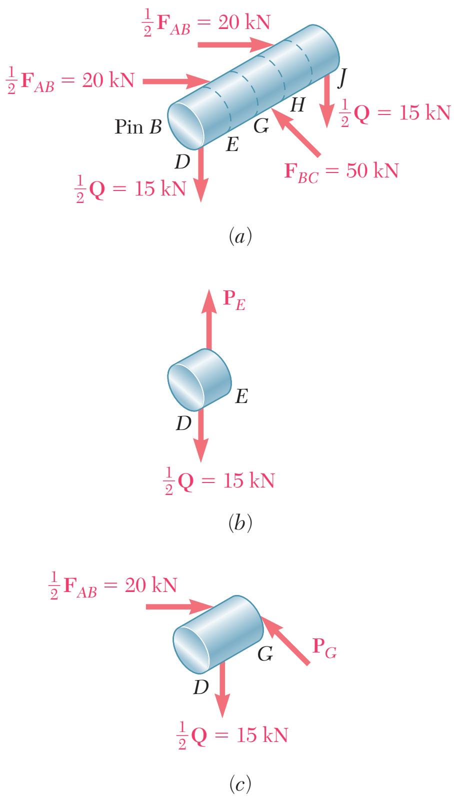


Fig. 1.25

Considering the pin at B (Fig. 1.25*a*), we note that the pin may be divided into five portions which are acted upon by forces exerted by the boom, rod, and bracket. Considering successively the portions DE (Fig. 1.25*b*) and DG (Fig. 1.25*c*), we conclude that the shear in section E is $P_E = 15$ kN, while the shear in section G is $P_G = 25$ kN. Since the loading of the pin is symmetric, we conclude that the maximum value of the shear in pin B is $P_G = 25$ kN, and that the largest shearing stresses occur in sections G and H, where

$$\tau_{\text{ave}} = \frac{P_G}{A} = \frac{25 \text{ kN}}{491 \times 10^{-6} \text{ m}^2} = 50.9 \text{ MPa}$$

c. Determination of the Bearing Stresses. To determine the nominal bearing stress at A in member AB, we use formula (1.11) of Sec. 1.7. From Fig. 1.22, we have $t = 30$ mm and $d = 25$ mm. Recalling that $P = F_{AB} = 40$ kN, we have

$$\sigma_b = \frac{P}{td} = \frac{40 \text{ kN}}{(30 \text{ mm})(25 \text{ mm})} = 53.3 \text{ MPa}$$

To obtain the bearing stress in the bracket at A, we use $t = 2(25 \text{ mm}) = 50$ mm and $d = 25$ mm:

$$\sigma_b = \frac{P}{td} = \frac{40 \text{ kN}}{(50 \text{ mm})(25 \text{ mm})} = 32.0 \text{ MPa}$$

The bearing stresses at B in member AB, at B and C in member BC, and in the bracket at C are found in a similar way.

1.9 METHOD OF PROBLEM SOLUTION

You should approach a problem in mechanics of materials as you would approach an actual engineering situation. By drawing on your own experience and intuition, you will find it easier to understand and formulate the problem. Once the problem has been clearly stated, however, there is no place in its solution for your particular fancy. Your solution must be based on the fundamental principles of statics and on the principles you will learn in this course. Every step you take must be justified on that basis, leaving no room for your "intuition." After an answer has been obtained, it should be checked. Here again, you may call upon your common sense and personal experience. If not completely satisfied with the result obtained, you should carefully check your formulation of the problem, the validity of the methods used in its solution, and the accuracy of your computations.

The *statement* of the problem should be clear and precise. It should contain the given data and indicate what information is required. A simplified drawing showing all essential quantities involved should be included. The solution of most of the problems you will encounter will necessitate that you first determine the *reactions at supports* and *internal forces and couples*. This will require the drawing of one or several *free-body diagrams,* as was done in Sec. 1.2, from which you will write *equilibrium equations*. These equations can be solved for the unknown forces, from which the required *stresses* and *deformations* will be computed.

After the answer has been obtained, it should be *carefully checked*. Mistakes in *reasoning* can often be detected by carrying the units through your computations and checking the units obtained for the answer. For example, in the design of the rod discussed in Sec. 1.4, we found, after carrying the units through our computations, that the required diameter of the rod was expressed in millimeters, which is the correct unit for a dimension; if another unit had been found, we would have known that some mistake had been made.

Errors in *computation* will usually be found by substituting the numerical values obtained into an equation which has not yet been used and verifying that the equation is satisfied. The importance of correct computations in engineering cannot be overemphasized.

1.10 NUMERICAL ACCURACY

The accuracy of the solution of a problem depends upon two items: (1) the accuracy of the given data and (2) the accuracy of the computations performed.

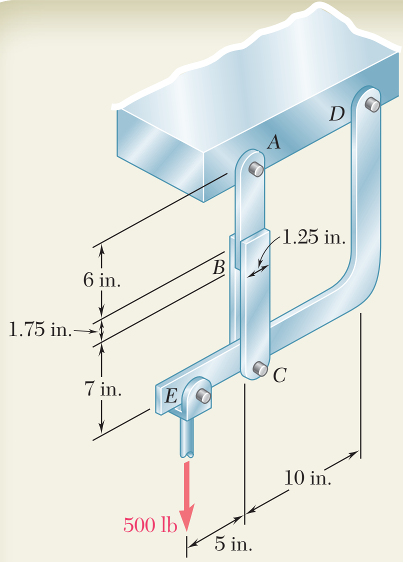
The solution cannot be more accurate than the less accurate of these two items. For example, if the loading of a beam is known to be 75,000 lb with a possible error of 100 lb either way, the relative error which measures the degree of accuracy of the data is

$$\frac{100 \text{ lb}}{75{,}000 \text{ lb}} = 0.0013 = 0.13\%$$

In computing the reaction at one of the beam supports, it would then be meaningless to record it as 14,322 lb. The accuracy of the solution cannot be greater than 0.13%, no matter how accurate the computations are, and the possible error in the answer may be as large as $(0.13/100)(14{,}322 \text{ lb}) \approx 20$ lb. The answer should be properly recorded as $14{,}320 \pm 20$ lb.

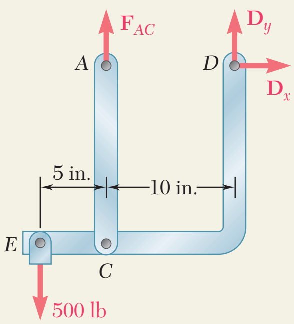
In engineering problems, the data are seldom known with an accuracy greater than 0.2%. It is therefore seldom justified to write the answers to such problems with an accuracy greater than 0.2%. A practical rule is to use 4 figures to record numbers beginning with a "1" and 3 figures in all other cases. Unless otherwise indicated, the data given in a problem should be assumed known with a comparable degree of accuracy. A force of 40 lb, for example, should be read 40.0 lb, and a force of 15 lb should be read 15.00 lb.

Pocket calculators and computers are widely used by practicing engineers and engineering students. The speed and accuracy of these devices facilitate the numerical computations in the solution of many problems. However, students should not record more significant figures than can be justified merely because they are easily obtained. As noted above, an accuracy greater than 0.2% is seldom necessary or meaningful in the solution of practical engineering problems.



SAMPLE PROBLEM 1.1

In the hanger shown, the upper portion of link ABC is $\frac{3}{8}$ in. thick and the lower portions are each $\frac{1}{4}$ in. thick. Epoxy resin is used to bond the upper and lower portions together at B. The pin at A is of $\frac{3}{8}$-in. diameter while a $\frac{1}{4}$-in.-diameter pin is used at C. Determine (*a*) the shearing stress in pin A, (*b*) the shearing stress in pin C, (*c*) the largest normal stress in link ABC, (*d*) the average shearing stress on the bonded surfaces at B, (*e*) the bearing stress in the link at C.

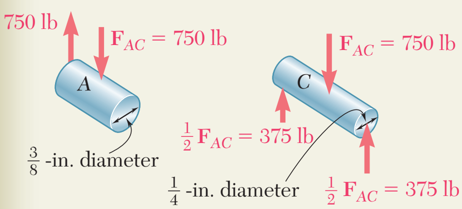


SOLUTION

Free Body: Entire Hanger. Since the link ABC is a two-force member, the reaction at A is vertical; the reaction at D is represented by its components $\mathbf{D}_x$ and $\mathbf{D}_y$. We write

$$+\circlearrowleft \Sigma M_D = 0: \qquad (500 \text{ lb})(15 \text{ in.}) - F_{AC}(10 \text{ in.}) = 0$$

$$F_{AC} = +750 \text{ lb} \qquad F_{AC} = 750 \text{ lb} \qquad \textit{tension}$$

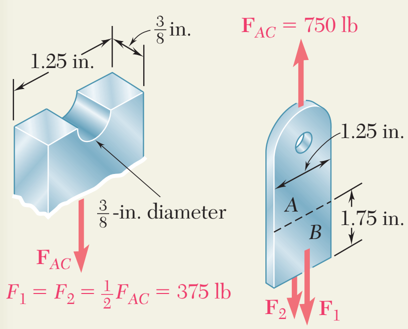


a. Shearing Stress in Pin *A*. Since this $\frac{3}{8}$-in.-diameter pin is in single shear, we write

$$\tau_A = \frac{F_{AC}}{A} = \frac{750 \text{ lb}}{\frac{1}{4}\pi(0.375 \text{ in.})^2} \qquad \tau_A = 6790 \text{ psi} \blacktriangleleft$$

b. Shearing Stress in Pin *C*. Since this $\frac{1}{4}$-in.-diameter pin is in double shear, we write

$$\tau_C = \frac{\frac{1}{2}F_{AC}}{A} = \frac{375 \text{ lb}}{\frac{1}{4}\pi(0.25 \text{ in.})^2} \qquad \tau_C = 7640 \text{ psi} \blacktriangleleft$$

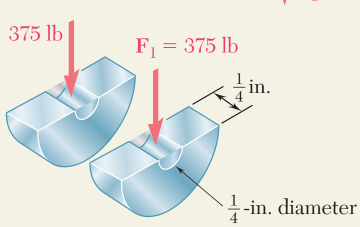


c. Largest Normal Stress in Link *ABC*. The largest stress is found where the area is smallest; this occurs at the cross section at A where the $\frac{3}{8}$-in. hole is located. We have

$$\sigma_A = \frac{F_{AC}}{A_{\text{net}}} = \frac{750 \text{ lb}}{(\frac{3}{8} \text{ in.})(1.25 \text{ in.} - 0.375 \text{ in.})} = \frac{750 \text{ lb}}{0.328 \text{ in}^2} \qquad \sigma_A = 2290 \text{ psi} \blacktriangleleft$$

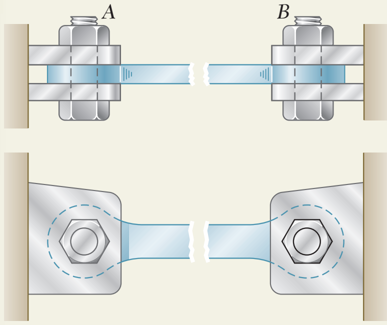
d. Average Shearing Stress at *B*. We note that bonding exists on both sides of the upper portion of the link and that the shear force on each side is $F_1 = (750 \text{ lb})/2 = 375$ lb. The average shearing stress on each surface is thus

$$\tau_B = \frac{F_1}{A} = \frac{375 \text{ lb}}{(1.25 \text{ in.})(1.75 \text{ in.})} \qquad \tau_B = 171.4 \text{ psi} \blacktriangleleft$$



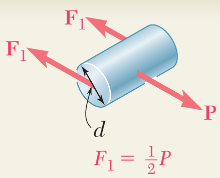
e. Bearing Stress in Link at *C*. For each portion of the link, $F_1 = 375$ lb and the nominal bearing area is $(0.25 \text{ in.})(0.25 \text{ in.}) = 0.0625 \text{ in}^2$.

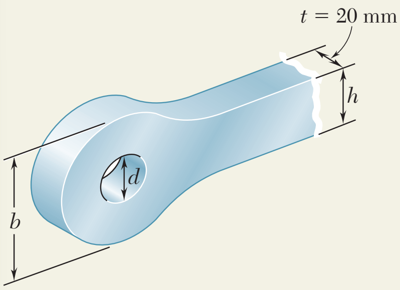
$$\sigma_b = \frac{F_1}{A} = \frac{375 \text{ lb}}{0.0625 \text{ in}^2} \qquad \sigma_b = 6000 \text{ psi} \blacktriangleleft$$

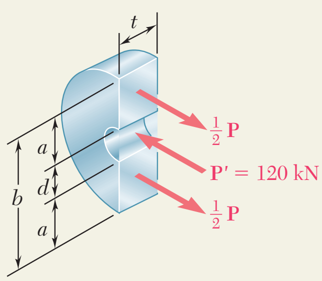


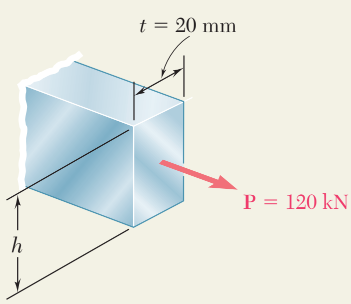
SAMPLE PROBLEM 1.2

The steel tie bar shown is to be designed to carry a tension force of magnitude $P = 120$ kN when bolted between double brackets at A and B. The bar will be fabricated from 20-mm-thick plate stock. For the grade of steel to be used, the maximum allowable stresses are: $\sigma = 175$ MPa, $\tau = 100$ MPa, $\sigma_b = 350$ MPa. Design the tie bar by determining the required values of (a) the diameter d of the bolt, (b) the dimension b at each end of the bar, (c) the dimension h of the bar.









SOLUTION

a. Diameter of the Bolt. Since the bolt is in double shear, $F_1 = \frac{1}{2}P = 60$ kN.

$$\tau = \frac{F_1}{A} = \frac{60 \text{ kN}}{\frac{1}{4}\pi\, d^2} \qquad 100 \text{ MPa} = \frac{60 \text{ kN}}{\frac{1}{4}\pi\, d^2} \qquad d = 27.6 \text{ mm}$$

We will use $d = 28$ mm ◀

At this point we check the bearing stress between the 20-mm-thick plate and the 28-mm-diameter bolt.

$$\tau_b = \frac{P}{td} = \frac{120 \text{ kN}}{(0.020 \text{ m})(0.028 \text{ m})} = 214 \text{ MPa} < 350 \text{ MPa} \qquad \text{OK}$$

b. Dimension b at Each End of the Bar. We consider one of the end portions of the bar. Recalling that the thickness of the steel plate is $t = 20$ mm and that the average tensile stress must not exceed 175 MPa, we write

$$\sigma = \frac{\frac{1}{2}P}{ta} \qquad 175 \text{ MPa} = \frac{60 \text{ kN}}{(0.02 \text{ m})a} \qquad a = 17.14 \text{ mm}$$

$$b = d + 2a = 28 \text{ mm} + 2(17.14 \text{ mm}) \qquad b = 62.3 \text{ mm} \ ◀$$

c. Dimension h of the Bar. Recalling that the thickness of the steel plate is $t = 20$ mm, we have

$$\sigma = \frac{P}{th} \qquad 175 \text{ MPa} = \frac{120 \text{ kN}}{(0.020 \text{ m})h} \qquad h = 34.3 \text{ mm}$$

We will use $h = 35$ mm ◀

PROBLEMS

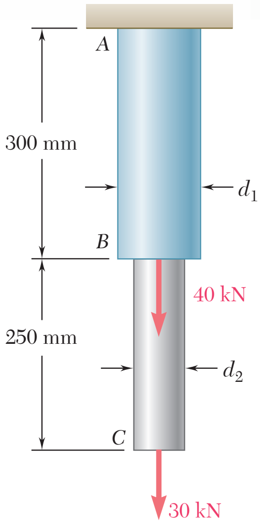


Fig. P1.1 and P1.2

1.1 Two solid cylindrical rods *AB* and *BC* are welded together at *B* and loaded as shown. Knowing that the average normal stress must not exceed 175 MPa in rod *AB* and 150 MPa in rod *BC*, determine the smallest allowable values of d_1 and d_2.

1.2 Two solid cylindrical rods *AB* and *BC* are welded together at *B* and loaded as shown. Knowing that $d_1 = 50$ mm and $d_2 = 30$ mm, find the average normal stress at the midsection of (*a*) rod *AB*, (*b*) rod *BC*.

1.3 Two solid cylindrical rods *AB* and *BC* are welded together at *B* and loaded as shown. Determine the magnitude of the force **P** for which the tensile stress in rod *AB* has the same magnitude as the compressive stress in rod *BC*.

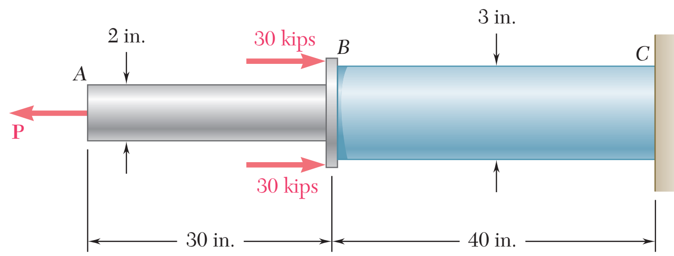


Fig. P1.3

1.4 In Prob. 1.3, knowing that $P = 40$ kips, determine the average normal stress at the midsection of (*a*) rod *AB*, (*b*) rod *BC*.

1.5 Two steel plates are to be held together by means of 16-mm-diameter high-strength steel bolts fitting snugly inside cylindrical brass spacers. Knowing that the average normal stress must not exceed 200 MPa in the bolts and 130 MPa in the spacers, determine the outer diameter of the spacers that yields the most economical and safe design.

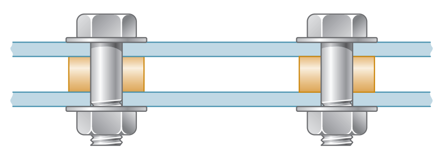


Fig. *P1.5*

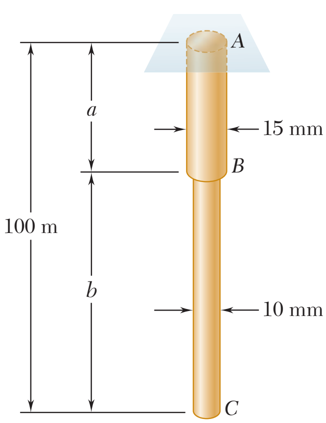


Fig. *P1.6*

1.6 Two brass rods *AB* and *BC*, each of uniform diameter, will be brazed together at *B* to form a nonuniform rod of total length 100 m which will be suspended from a support at *A* as shown. Knowing that the density of brass is 8470 kg/m^3, determine (*a*) the length of rod *AB* for which the maximum normal stress in *ABC* is minimum, (*b*) the corresponding value of the maximum normal stress.

1.7 Each of the four vertical links has an 8 × 36-mm uniform rectangular cross section and each of the four pins has a 16-mm diameter. Determine the maximum value of the average normal stress in the links connecting (*a*) points *B* and *D*, (*b*) points *C* and *E*.

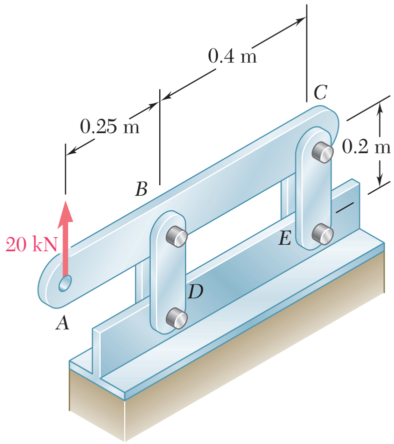


Fig. P1.7

1.8 Knowing that link *DE* is $\frac{1}{8}$ in. thick and 1 in. wide, determine the normal stress in the central portion of that link when (*a*) $\theta = 0$, (*b*) $\theta = 90°$.

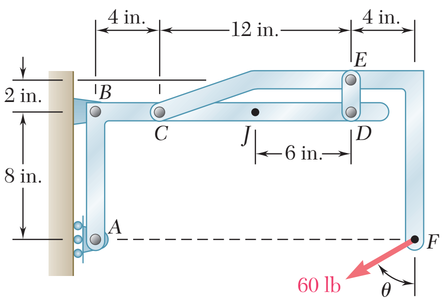


Fig. P1.8

1.9 Link *AC* has a uniform rectangular cross section $\frac{1}{16}$ in. thick and $\frac{1}{4}$ in. wide. Determine the normal stress in the central portion of the link.

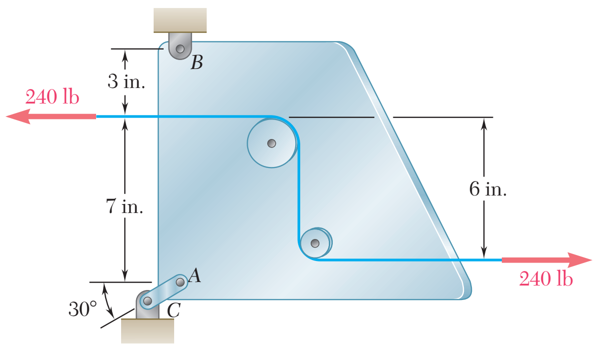


Fig. P1.9

1.10 Three forces, each of magnitude $P = 4$ kN, are applied to the mechanism shown. Determine the cross-sectional area of the uniform portion of rod *BE* for which the normal stress in that portion is +100 MPa.

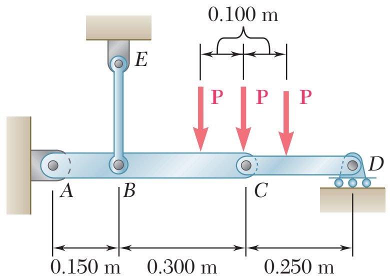


Fig. P1.10

1.11 The frame shown consists of *four* wooden members, *ABC*, *DEF*, *BE*, and *CF*. Knowing that each member has a 2 × 4-in. rectangular cross section and that each pin has a $\frac{1}{2}$-in. diameter, determine the maximum value of the average normal stress (*a*) in member *BE*, (*b*) in member *CF*.

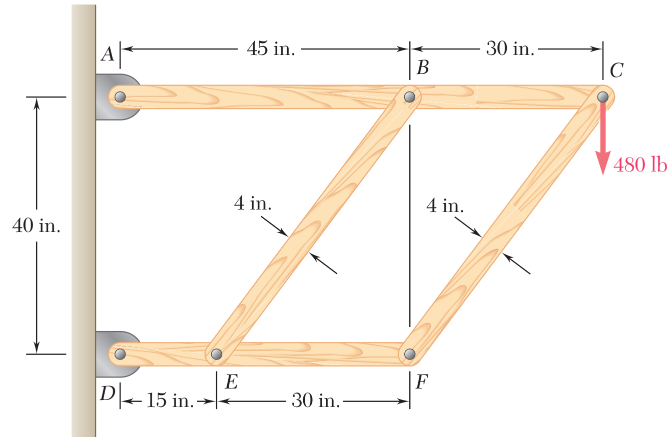


Fig. *P1.11*

1.12 For the Pratt bridge truss and loading shown, determine the average normal stress in member *BE*, knowing that the cross-sectional area of that member is 5.87 in^2.

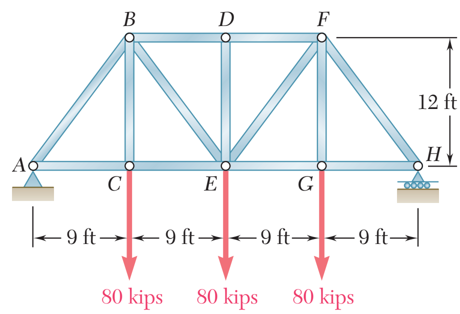


Fig. *P1.12*

1.13 An aircraft tow bar is positioned by means of a single hydraulic cylinder connected by a 25-mm-diameter steel rod to two identical arm-and-wheel units *DEF*. The mass of the entire tow bar is 200 kg, and its center of gravity is located at *G*. For the position shown, determine the normal stress in the rod.

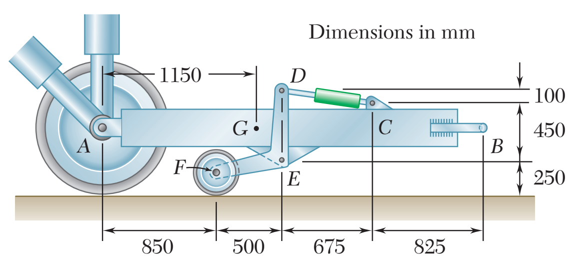


Fig. P1.13

1.14 A couple **M** of magnitude 1500 N · m is applied to the crank of an engine. For the position shown, determine (*a*) the force **P** required to hold the engine system in equilibrium, (*b*) the average normal stress in the connecting rod *BC*, which has a 450-mm^2 uniform cross section.

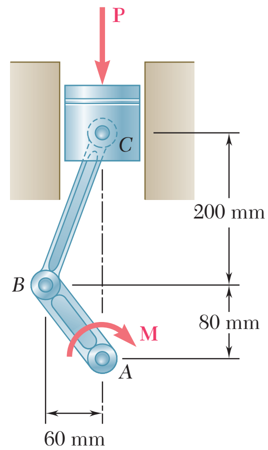


Fig. P1.14

1.15 When the force **P** reached 8 kN, the wooden specimen shown failed in shear along the surface indicated by the dashed line. Determine the average shearing stress along that surface at the time of failure.

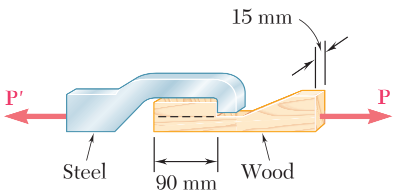


Fig. P1.15

1.16 The wooden members *A* and *B* are to be joined by plywood splice plates that will be fully glued on the surfaces in contact. As part of the design of the joint, and knowing that the clearance between the ends of the members is to be $\frac{1}{4}$ in., determine the smallest allowable length *L* if the average shearing stress in the glue is not to exceed 120 psi.

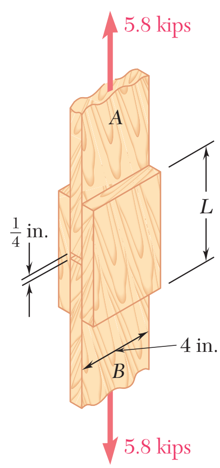


Fig. P1.16

1.17 A load **P** is applied to a steel rod supported as shown by an aluminum plate into which a 0.6-in.-diameter hole has been drilled. Knowing that the shearing stress must not exceed 18 ksi in the steel rod and 10 ksi in the aluminum plate, determine the largest load **P** that can be applied to the rod.

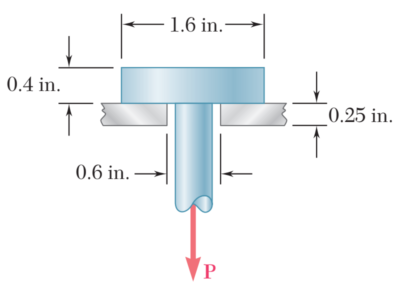


Fig. *P1.17*

1.18 Two wooden planks, each 22 mm thick and 160 mm wide, are joined by the glued mortise joint shown. Knowing that the joint will fail when the average shearing stress in the glue reaches 820 kPa, determine the smallest allowable length *d* of the cuts if the joint is to withstand an axial load of magnitude $P = 7.6$ kN.

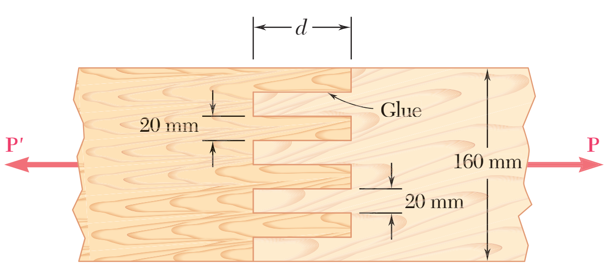


Fig. P1.18

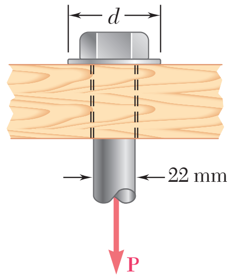


Fig. P1.19

1.19 The load **P** applied to a steel rod is distributed to a timber support by an annular washer. The diameter of the rod is 22 mm and the inner diameter of the washer is 25 mm, which is slightly larger than the diameter of the hole. Determine the smallest allowable outer diameter d of the washer, knowing that the axial normal stress in the steel rod is 35 MPa and that the average bearing stress between the washer and the timber must not exceed 5 MPa.

1.20 The axial force in the column supporting the timber beam shown is $P = 20$ kips. Determine the smallest allowable length L of the bearing plate if the bearing stress in the timber is not to exceed 400 psi.

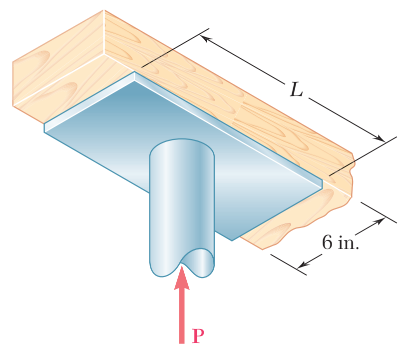


Fig. *P1.20*

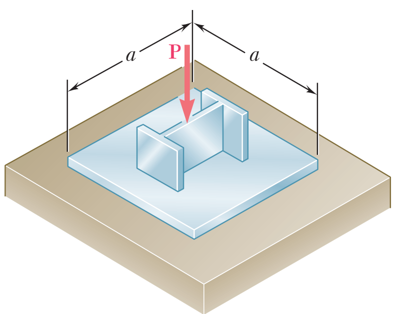


Fig. P1.21

1.21 An axial load **P** is supported by a short W8 × 40 column of cross-sectional area $A = 11.7\ \text{in}^2$ and is distributed to a concrete foundation by a square plate as shown. Knowing that the average normal stress in the column must not exceed 30 ksi and that the bearing stress on the concrete foundation must not exceed 3.0 ksi, determine the side a of the plate that will provide the most economical and safe design.

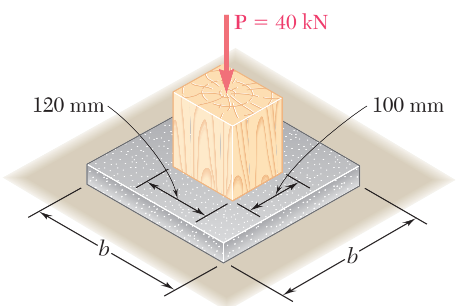


Fig. P1.22

1.22 A 40-kN axial load is applied to a short wooden post that is supported by a concrete footing resting on undisturbed soil. Determine (a) the maximum bearing stress on the concrete footing, (b) the size of the footing for which the average bearing stress in the soil is 145 kPa.

1.23 A $\frac{5}{8}$-in.-diameter steel rod AB is fitted to a round hole near end C of the wooden member CD. For the loading shown, determine (a) the maximum average normal stress in the wood, (b) the distance b for which the average shearing stress is 100 psi on the surfaces indicated by the dashed lines, (c) the average bearing stress on the wood.

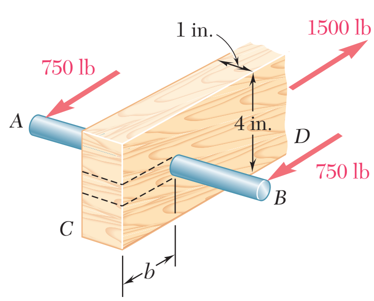


Fig. P1.23

1.24 Knowing that $\theta = 40°$ and $P = 9$ kN, determine (*a*) the smallest allowable diameter of the pin at *B* if the average shearing stress in the pin is not to exceed 120 MPa, (*b*) the corresponding average bearing stress in member *AB* at *B*, (*c*) the corresponding average bearing stress in each of the support brackets at *B*.

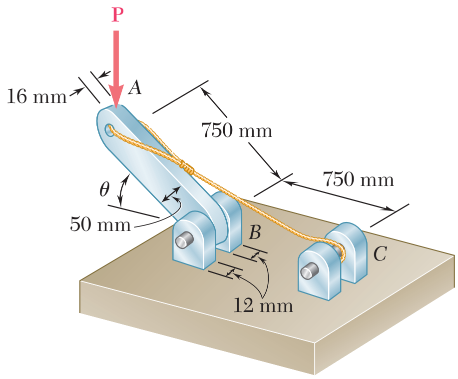


Fig. P1.24 and P1.25

1.25 Determine the largest load **P** that can be applied at *A* when $\theta = 60°$, knowing that the average shearing stress in the 10-mm-diameter pin at *B* must not exceed 120 MPa and that the average bearing stress in member *AB* and in the bracket at *B* must not exceed 90 MPa.

1.26 Link *AB*, of width $b = 50$ mm and thickness $t = 6$ mm, is used to support the end of a horizontal beam. Knowing that the average normal stress in the link is -140 MPa, and that the average shearing stress in each of the two pins is 80 MPa, determine (*a*) the diameter *d* of the pins, (*b*) the average bearing stress in the link.

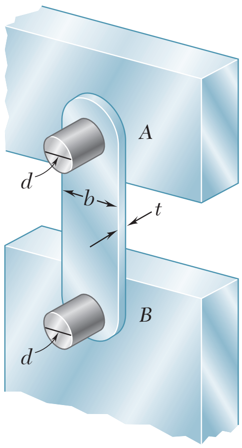


Fig. P1.26

1.27 For the assembly and loading of Prob. 1.7, determine (*a*) the average shearing stress in the pin at *B*, (*b*) the average bearing stress at *B* in member *BD*, (*c*) the average bearing stress at *B* in member *ABC*, knowing that this member has a 10×50-mm uniform rectangular cross section.

1.28 The hydraulic cylinder *CF*, which partially controls the position of rod *DE*, has been locked in the position shown. Member *BD* is $\frac{5}{8}$ in. thick and is connected to the vertical rod by a $\frac{3}{8}$-in.-diameter bolt. Determine (*a*) the average shearing stress in the bolt, (*b*) the bearing stress at *C* in member *BD*.

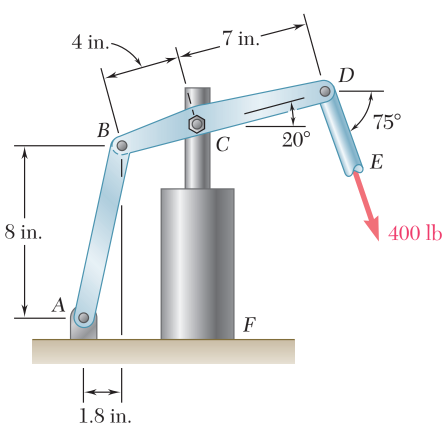


Fig. P1.28

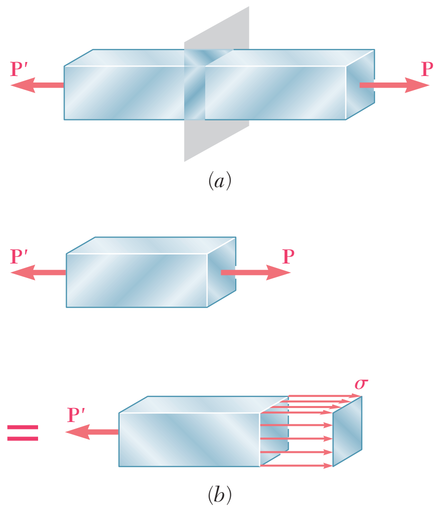


Fig. 1.26 Axial forces.

1.11 STRESS ON AN OBLIQUE PLANE UNDER AXIAL LOADING

In the preceding sections, axial forces exerted on a two-force member (Fig. 1.26*a*) were found to cause normal stresses in that member (Fig. 1.26*b*), while transverse forces exerted on bolts and pins (Fig. 1.27*a*) were found to cause shearing stresses in those connections (Fig. 1.27*b*). The reason such a relation was observed between axial forces and normal stresses on one hand, and transverse forces and shearing stresses on the other, was because stresses were being determined only on planes perpendicular to the axis of the member or connection. As you will see in this section, axial forces cause both normal and shearing stresses on planes which are not perpendicular to the axis of the member. Similarly, transverse forces exerted on a bolt or a pin cause both normal and shearing stresses on planes which are not perpendicular to the axis of the bolt or pin.

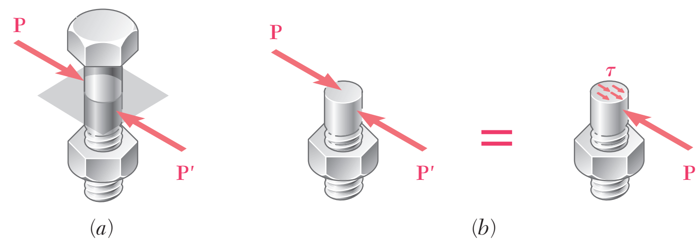


Fig. 1.27 Transverse forces.

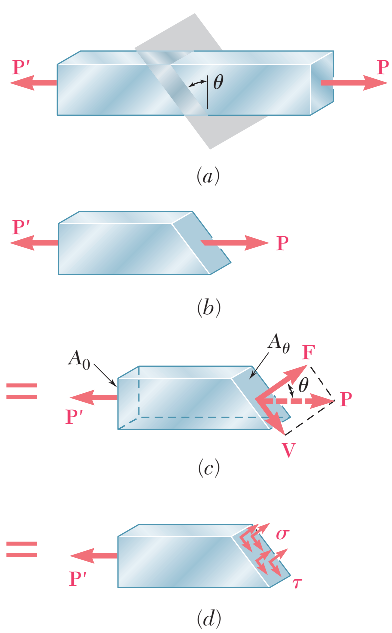


Fig. 1.28

Consider the two-force member of Fig. 1.26, which is subjected to axial forces **P** and **P**′. If we pass a section forming an angle θ with a normal plane (Fig. 1.28*a*) and draw the free-body diagram of the portion of member located to the left of that section (Fig. 1.28*b*), we find from the equilibrium conditions of the free body that the distributed forces acting on the section must be equivalent to the force **P.**

Resolving **P** into components **F** and **V,** respectively normal and tangential to the section (Fig. 1.28*c*), we have

$$F = P\cos\theta \qquad V = P\sin\theta \tag{1.12}$$

The force **F** represents the resultant of normal forces distributed over the section, and the force **V** the resultant of shearing forces (Fig. 1.28*d*). The average values of the corresponding normal and shearing stresses are obtained by dividing, respectively, F and V by the area A_θ of the section:

$$\sigma = \frac{F}{A_\theta} \qquad \tau = \frac{V}{A_\theta} \tag{1.13}$$

Substituting for F and V from (1.12) into (1.13), and observing from Fig. 1.28*c* that $A_0 = A_\theta \cos\theta$, or $A_\theta = A_0/\cos\theta$, where A_0 denotes the area of a section perpendicular to the axis of the member, we obtain

$$\sigma = \frac{P\cos\theta}{A_0/\cos\theta} \qquad \tau = \frac{P\sin\theta}{A_0/\cos\theta}$$

or

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta \tag{1.14}$$

We note from the first of Eqs. (1.14) that the normal stress σ is maximum when $\theta = 0$, i.e., when the plane of the section is perpendicular to the axis of the member, and that it approaches zero as θ approaches 90°. We check that the value of σ when $\theta = 0$ is

$$\sigma_m = \frac{P}{A_0} \tag{1.15}$$

as we found earlier in Sec. 1.3. The second of Eqs. (1.14) shows that the shearing stress τ is zero for $\theta = 0$ and $\theta = 90°$, and that for $\theta = 45°$ it reaches its maximum value

$$\tau_m = \frac{P}{A_0}\sin 45°\cos 45° = \frac{P}{2A_0} \tag{1.16}$$

The first of Eqs. (1.14) indicates that, when $\theta = 45°$, the normal stress σ' is also equal to $P/2A_0$:

$$\sigma' = \frac{P}{A_0}\cos^2 45° = \frac{P}{2A_0} \tag{1.17}$$

The results obtained in Eqs. (1.15), (1.16), and (1.17) are shown graphically in Fig. 1.29. We note that the same loading may produce either a normal stress $\sigma_m = P/A_0$ and no shearing stress (Fig. 1.29*b*), or a normal and a shearing stress of the same magnitude $\sigma' = \tau_m = P/2A_0$ (Fig. 1.29 *c* and *d*), depending upon the orientation of the section.

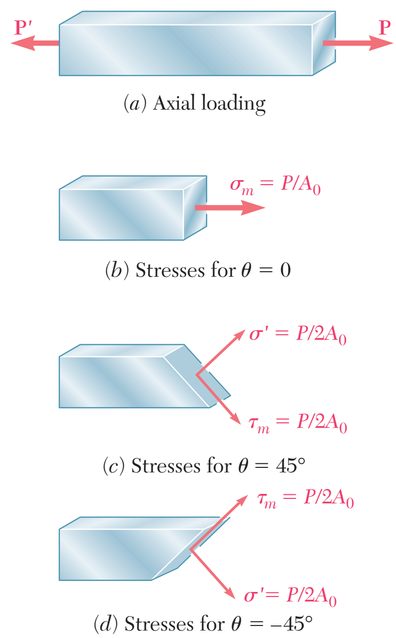


Fig. 1.29

1.12 STRESS UNDER GENERAL LOADING CONDITIONS; COMPONENTS OF STRESS

The examples of the previous sections were limited to members under axial loading and connections under transverse loading. Most structural members and machine components are under more involved loading conditions.

Consider a body subjected to several loads $\mathbf{P}_1$, $\mathbf{P}_2$, etc. (Fig. 1.30). To understand the stress condition created by these loads at some point Q within the body, we shall first pass a section through Q, using a plane parallel to the yz plane. The portion of the body to the left of the section is subjected to some of the original loads, and to normal and shearing forces distributed over the section. We shall denote by $\Delta\mathbf{F}^x$ and $\Delta\mathbf{V}^x$, respectively, the normal and the shearing

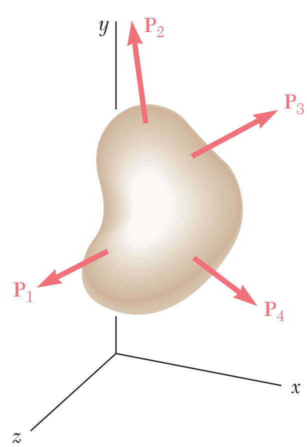


Fig. 1.30

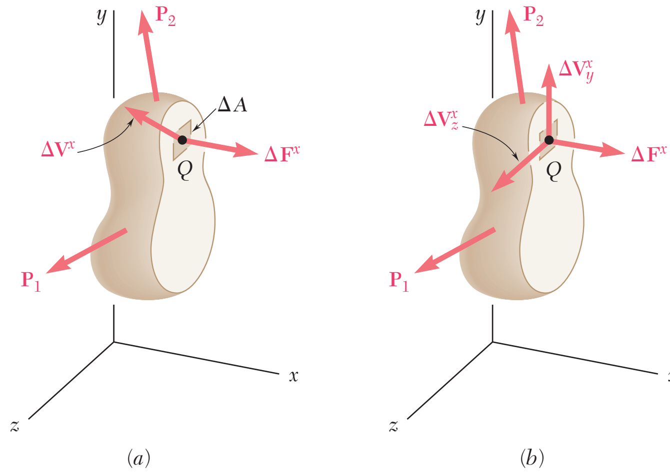


Fig. 1.31

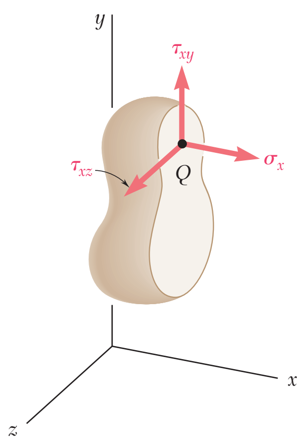


Fig. 1.32

forces acting on a small area ΔA surrounding point Q (Fig. 1.31*a*). Note that the superscript x is used to indicate that the forces $\Delta\mathbf{F}^x$ and $\Delta\mathbf{V}^x$ act on a surface perpendicular to the x axis. While the normal force $\Delta\mathbf{F}^x$ has a well-defined direction, the shearing force $\Delta\mathbf{V}^x$ may have any direction in the plane of the section. We therefore resolve $\Delta\mathbf{V}^x$ into two component forces, $\Delta\mathbf{V}_y^x$ and $\Delta\mathbf{V}_z^x$, in directions parallel to the y and z axes, respectively (Fig. 1.31*b*). Dividing now the magnitude of each force by the area ΔA, and letting ΔA approach zero, we define the three stress components shown in Fig. 1.32:

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F^x}{\Delta A}$$
$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_y^x}{\Delta A} \qquad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta V_z^x}{\Delta A} \tag{1.18}$$

We note that the first subscript in σ_x, τ_{xy}, and τ_{xz} is used to indicate that the stresses under consideration are exerted *on a surface perpendicular to the x axis.* The second subscript in τ_{xy} and τ_{xz} identifies *the direction of the component.* The normal stress σ_x is positive if the corresponding arrow points in the positive x direction, i.e., if the body is in tension, and negative otherwise. Similarly, the shearing stress components τ_{xy} and τ_{xz} are positive if the corresponding arrows point, respectively, in the positive y and z directions.

The above analysis may also be carried out by considering the portion of body located to the right of the vertical plane through Q (Fig. 1.33). The same magnitudes, but opposite directions, are obtained for the normal and shearing forces $\Delta\mathbf{F}^x$, $\Delta\mathbf{V}_y^x$, and $\Delta\mathbf{V}_z^x$. Therefore, the same values are also obtained for the corresponding stress components, but since the section in Fig. 1.33 now faces the *negative x axis,* a positive sign for σ_x will indicate that the corresponding arrow points *in the negative x direction.* Similarly, positive signs for τ_{xy} and τ_{xz} will indicate that the corresponding arrows point, respectively, in the negative y and z directions, as shown in Fig. 1.33.

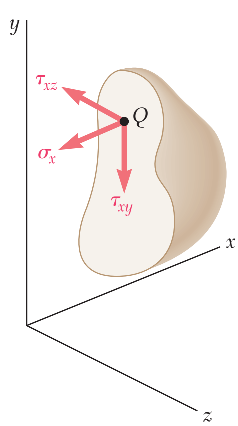


Fig. 1.33

Passing a section through Q parallel to the zx plane, we define in the same manner the stress components, σ_y, τ_{yz}, and τ_{yx}. Finally, a section through Q parallel to the xy plane yields the components σ_z, τ_{zx}, and τ_{zy}.

To facilitate the visualization of the stress condition at point Q, we shall consider a small cube of side a centered at Q and the stresses exerted on each of the six faces of the cube (Fig. 1.34). The stress components shown in the figure are σ_x, σ_y, and σ_z, which represent the normal stress on faces respectively perpendicular to the x, y, and z axes, and the six shearing stress components τ_{xy}, τ_{xz}, etc. We recall that, according to the definition of the shearing stress components, τ_{xy} represents the y component of the shearing stress exerted on the face perpendicular to the x axis, while τ_{yx} represents the x component of the shearing stress exerted on the face perpendicular to the y axis. Note that only three faces of the cube are actually visible in Fig. 1.34, and that equal and opposite stress components act on the hidden faces. While the stresses acting on the faces of the cube differ slightly from the stresses at Q, the error involved is small and vanishes as side a of the cube approaches zero.

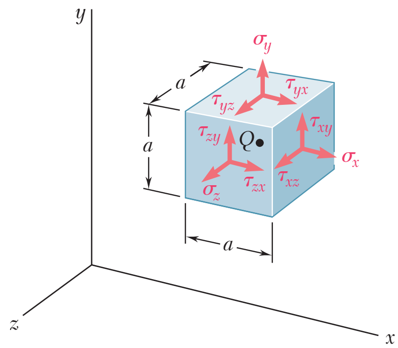


Fig. 1.34

Important relations among the shearing stress components will now be derived. Let us consider the free-body diagram of the small cube centered at point Q (Fig. 1.35). The normal and shearing forces acting on the various faces of the cube are obtained by multiplying the corresponding stress components by the area ΔA of each face. We first write the following three equilibrium equations:

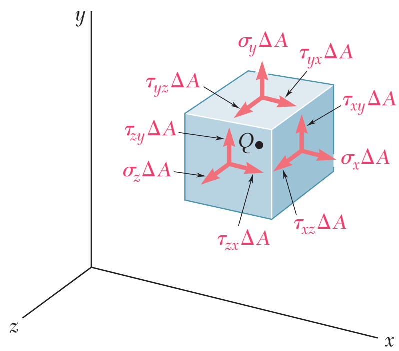


Fig. 1.35

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{1.19}$$

Since forces equal and opposite to the forces actually shown in Fig. 1.35 are acting on the hidden faces of the cube, it is clear that Eqs. (1.19) are satisfied. Considering now the moments of the forces about axes x', y', and z' drawn from Q in directions respectively parallel to the x, y, and z axes, we write the three additional equations

$$\Sigma M_{x'} = 0 \qquad \Sigma M_{y'} = 0 \qquad \Sigma M_{z'} = 0 \tag{1.20}$$

Using a projection on the $x'y'$ plane (Fig. 1.36), we note that the only forces with moments about the z axis different from zero are the shearing forces. These forces form two couples, one of counterclockwise (positive) moment $(\tau_{xy}\ \Delta A)a$, the other of clockwise (negative) moment $-(\tau_{yx}\ \Delta A)a$. The last of the three Eqs. (1.20) yields, therefore,

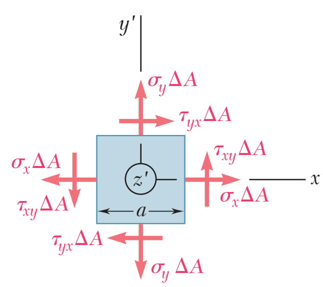


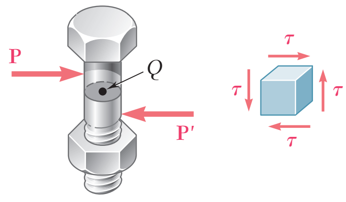
Fig. 1.36

$$+\circlearrowleft \Sigma M_z = 0: \qquad (\tau_{xy}\ \Delta A)a - (\tau_{yx}\ \Delta A)a = 0$$

from which we conclude that

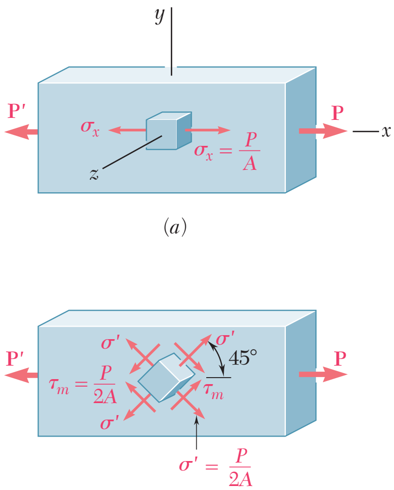
$$\boxed{\tau_{xy} = \tau_{yx}} \tag{1.21}$$

The relation obtained shows that the y component of the shearing stress exerted on a face perpendicular to the x axis is equal to the x



(a) (b)

Fig. 1.37



(a)

(b)

Fig. 1.38

component of the shearing stress exerted on a face perpendicular to the y axis. From the remaining two equations (1.20), we derive in a similar manner the relations

$$\tau_{yz} = \tau_{zy} \qquad \tau_{zx} = \tau_{xz} \tag{1.22}$$

We conclude from Eqs. (1.21) and (1.22) that only six stress components are required to define the condition of stress at a given point Q, instead of nine as originally assumed. These six components are σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}. We also note that, at a given point, *shear cannot take place in one plane only;* an equal shearing stress must be exerted on another plane perpendicular to the first one. For example, considering again the bolt of Fig. 1.27 and a small cube at the center Q of the bolt (Fig. 1.37*a*), we find that shearing stresses of equal magnitude must be exerted on the two horizontal faces of the cube and on the two faces that are perpendicular to the forces **P** and **P**′ (Fig. 1.37*b*).

Before concluding our discussion of stress components, let us consider again the case of a member under axial loading. If we consider a small cube with faces respectively parallel to the faces of the member and recall the results obtained in Sec. 1.11, we find that the conditions of stress in the member may be described as shown in Fig. 1.38*a*; the only stresses are normal stresses σ_x exerted on the faces of the cube which are perpendicular to the x axis. However, if the small cube is rotated by 45° about the z axis so that its new orientation matches the orientation of the sections considered in Fig. 1.29*c* and *d*, we conclude that normal and shearing stresses of equal magnitude are exerted on four faces of the cube (Fig. 1.38*b*). We thus observe that the same loading condition may lead to different interpretations of the stress situation at a given point, depending upon the orientation of the element considered. More will be said about this in Chap 7.

1.13 DESIGN CONSIDERATIONS

In the preceding sections you learned to determine the stresses in rods, bolts, and pins under simple loading conditions. In later chapters you will learn to determine stresses in more complex situations. In engineering applications, however, the determination of stresses is seldom an end in itself. Rather, the knowledge of stresses is used by engineers to assist in their most important task, namely, the design of structures and machines that will safely and economically perform a specified function.

a. Determination of the Ultimate Strength of a Material. An important element to be considered by a designer is how the material that has been selected will behave under a load. For a given material, this is determined by performing specific tests on prepared samples of the material. For example, a test specimen of steel may be prepared and placed in a laboratory testing machine to be subjected to a known centric axial tensile force, as described in Sec. 2.3. As the magnitude of the force is increased, various changes in the specimen are measured, for example, changes in its length and its diameter.

Eventually the largest force which may be applied to the specimen is reached, and the specimen either breaks or begins to carry less load. This largest force is called the *ultimate load* for the test specimen and is denoted by P_U. Since the applied load is centric, we may divide the ultimate load by the original cross-sectional area of the rod to obtain the *ultimate normal stress* of the material used. This stress, also known as the *ultimate strength in tension* of the material, is

$$\sigma_U = \frac{P_U}{A} \tag{1.23}$$

Several test procedures are available to determine the *ultimate shearing stress*, or *ultimate strength in shear*, of a material. The one most commonly used involves the twisting of a circular tube (Sec. 3.5). A more direct, if less accurate, procedure consists in clamping a rectangular or round bar in a shear tool (Fig. 1.39) and applying an increasing load P until the ultimate load P_U for single shear is obtained. If the free end of the specimen rests on both of the hardened dies (Fig. 1.40), the ultimate load for double shear is obtained. In either case, the ultimate shearing stress τ_U is obtained by dividing the ultimate load by the total area over which shear has taken place. We recall that, in the case of single shear, this area is the cross-sectional area A of the specimen, while in double shear it is equal to twice the cross-sectional area.

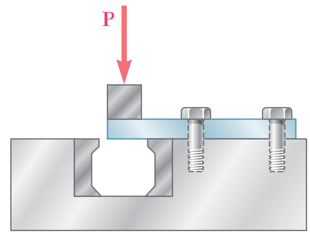


Fig. 1.39 Single shear test.

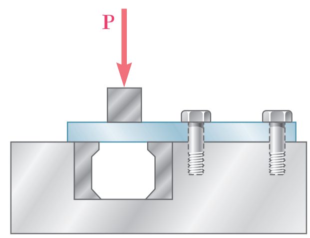


Fig. 1.40 Double shear test.

b. Allowable Load and Allowable Stress; Factor of Safety. The maximum load that a structural member or a machine component will be allowed to carry under normal conditions of utilization is considerably smaller than the ultimate load. This smaller load is referred to as the *allowable load* and, sometimes, as the *working load* or *design load*. Thus, only a fraction of the ultimate-load capacity of the member is utilized when the allowable load is applied. The remaining portion of the load-carrying capacity of the member is kept in reserve to assure its safe performance. The ratio of the ultimate load to the allowable load is used to define the *factor of safety*.† We have

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate load}}{\text{allowable load}} \tag{1.24}$$

An alternative definition of the factor of safety is based on the use of stresses:

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate stress}}{\text{allowable stress}} \tag{1.25}$$

The two expressions given for the factor of safety in Eqs. (1.24) and (1.25) are identical when a linear relationship exists between the load and the stress. In most engineering applications, however, this relationship ceases to be linear as the load approaches its ultimate value, and the factor of safety obtained from Eq. (1.25) does not provide a

†In some fields of engineering, notably aeronautical engineering, the *margin of safety* is used in place of the factor of safety. The margin of safety is defined as the factor of safety minus one; that is, margin of safety = $F.S. - 1.00$.

true assessment of the safety of a given design. Nevertheless, the *allowable-stress method* of design, based on the use of Eq. (1.25), is widely used.

c. Selection of an Appropriate Factor of Safety. The selection of the factor of safety to be used for various applications is one of the most important engineering tasks. On the one hand, if a factor of safety is chosen too small, the possibility of failure becomes unacceptably large; on the other hand, if a factor of safety is chosen unnecessarily large, the result is an uneconomical or nonfunctional design. The choice of the factor of safety that is appropriate for a given design application requires engineering judgment based on many considerations, such as the following:

1. *Variations that may occur in the properties of the member under consideration*. The composition, strength, and dimensions of the member are all subject to small variations during manufacture. In addition, material properties may be altered and residual stresses introduced through heating or deformation that may occur during manufacture, storage, transportation, or construction.
2. *The number of loadings that may be expected during the life of the structure or machine*. For most materials the ultimate stress decreases as the number of load applications is increased. This phenomenon is known as *fatigue* and, if ignored, may result in sudden failure (see Sec. 2.7).
3. *The type of loadings that are planned for in the design, or that may occur in the future*. Very few loadings are known with complete accuracy—most design loadings are engineering estimates. In addition, future alterations or changes in usage may introduce changes in the actual loading. Larger factors of safety are also required for dynamic, cyclic, or impulsive loadings.
4. *The type of failure that may occur*. Brittle materials fail suddenly, usually with no prior indication that collapse is imminent. On the other hand, ductile materials, such as structural steel, normally undergo a substantial deformation called *yielding* before failing, thus providing a warning that overloading exists. However, most buckling or stability failures are sudden, whether the material is brittle or not. When the possibility of sudden failure exists, a larger factor of safety should be used than when failure is preceded by obvious warning signs.
5. *Uncertainty due to methods of analysis*. All design methods are based on certain simplifying assumptions which result in calculated stresses being approximations of actual stresses.
6. *Deterioration that may occur in the future because of poor maintenance or because of unpreventable natural causes*. A larger factor of safety is necessary in locations where conditions such as corrosion and decay are difficult to control or even to discover.
7. *The importance of a given member to the integrity of the whole structure*. Bracing and secondary members may in many cases be designed with a factor of safety lower than that used for primary members.

In addition to the these considerations, there is the additional consideration concerning the risk to life and property that a failure would produce. Where a failure would produce no risk to life and only minimal risk to property, the use of a smaller factor of safety can be considered. Finally, there is the practical consideration that, unless a careful design with a nonexcessive factor of safety is used, a structure or machine might not perform its design function. For example, high factors of safety may have an unacceptable effect on the weight of an aircraft.

For the majority of structural and machine applications, factors of safety are specified by design specifications or building codes written by committees of experienced engineers working with professional societies, with industries, or with federal, state, or city agencies. Examples of such design specifications and building codes are

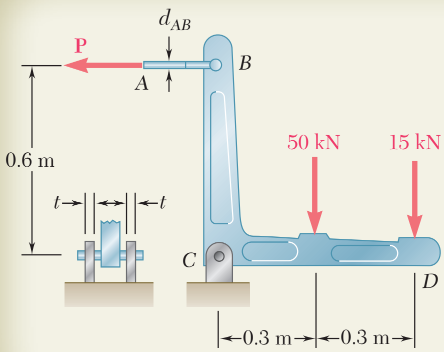
1. *Steel:* American Institute of Steel Construction, Specification for Structural Steel Buildings
2. *Concrete:* American Concrete Institute, Building Code Requirement for Structural Concrete
3. *Timber:* American Forest and Paper Association, National Design Specification for Wood Construction
4. *Highway bridges:* American Association of State Highway Officials, Standard Specifications for Highway Bridges

***d. Load and Resistance Factor Design.** As we saw previously, the allowable-stress method requires that all the uncertainties associated with the design of a structure or machine element be grouped into a single factor of safety. An alternative method of design, which is gaining acceptance chiefly among structural engineers, makes it possible through the use of three different factors to distinguish between the uncertainties associated with the structure itself and those associated with the load it is designed to support. This method, referred to as *Load and Resistance Factor Design (LRFD),* further allows the designer to distinguish between uncertainties associated with the *live load,* P_L, that is, with the load to be supported by the structure, and the *dead load,* P_D, that is, with the weight of the portion of structure contributing to the total load.

When this method of design is used, the *ultimate load,* P_U, of the structure, that is, the load at which the structure ceases to be useful, should first be determined. The proposed design is then acceptable if the following inequality is satisfied:

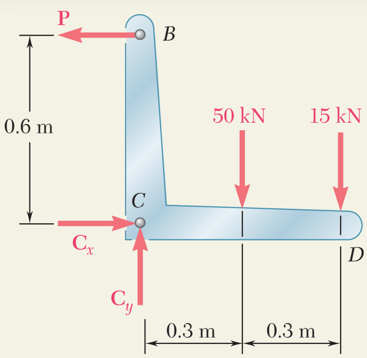
$$\gamma_D P_D + \gamma_L P_L \le \phi P_U \tag{1.26}$$

The coefficient ϕ is referred to as the *resistance factor;* it accounts for the uncertainties associated with the structure itself and will normally be less than 1. The coefficients γ_D and γ_L are referred to as the *load factors;* they account for the uncertainties associated, respectively, with the dead and live load and will normally be greater than 1, with γ_L generally larger than γ_D. While a few examples or assigned problems using LRFD are included in this chapter and in Chaps. 5 and 10, the allowable-stress method of design will be used in this text.



SAMPLE PROBLEM 1.3

Two forces are applied to the bracket *BCD* as shown. (*a*) Knowing that the control rod *AB* is to be made of a steel having an ultimate normal stress of 600 MPa, determine the diameter of the rod for which the factor of safety with respect to failure will be 3.3. (*b*) The pin at *C* is to be made of a steel having an ultimate shearing stress of 350 MPa. Determine the diameter of the pin *C* for which the factor of safety with respect to shear will also be 3.3. (*c*) Determine the required thickness of the bracket supports at *C* knowing that the allowable bearing stress of the steel used is 300 MPa.



SOLUTION

Free Body: Entire Bracket. The reaction at *C* is represented by its components $\mathbf{C}_x$ and $\mathbf{C}_y$.

$+\circlearrowleft \Sigma M_C = 0$: $P(0.6 \text{ m}) - (50 \text{ kN})(0.3 \text{ m}) - (15 \text{ kN})(0.6 \text{ m}) = 0 \quad P = 40 \text{ kN}$

$\Sigma F_x = 0$: $\quad C_x = 40 \text{ k}$

$\Sigma F_y = 0$: $\quad C_y = 65 \text{ kN} \qquad C = \sqrt{C_x^2 + C_y^2} = 76.3 \text{ kN}$

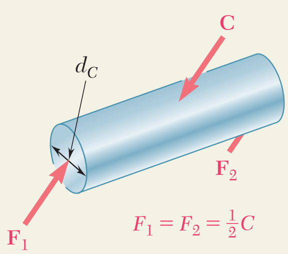
a. Control Rod *AB*. Since the factor of safety is to be 3.3, the allowable stress is

$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{600 \text{ MPa}}{3.3} = 181.8 \text{ MPa}$$

For $P = 40$ kN the cross-sectional area required is

$$A_{\text{req}} = \frac{P}{\sigma_{\text{all}}} = \frac{40 \text{ kN}}{181.8 \text{ MPa}} = 220 \times 10^{-6} \text{ m}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_{AB}^2 = 220 \times 10^{-6} \text{ m}^2 \qquad d_{AB} = 16.74 \text{ mm} \quad ◀$$



b. Shear in Pin *C*. For a factor of safety of 3.3, we have

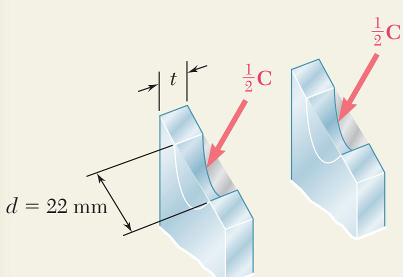
$$\tau_{\text{all}} = \frac{\tau_U}{F.S.} = \frac{350 \text{ MPa}}{3.3} = 106.1 \text{ MPa}$$

Since the pin is in double shear, we write

$$A_{\text{req}} = \frac{C/2}{\tau_{\text{all}}} = \frac{(76.3 \text{ kN})/2}{106.1 \text{ MPa}} = 360 \text{ mm}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_C^2 = 360 \text{ mm}^2 \qquad d_C = 21.4 \text{ mm} \qquad \text{Use: } d_C = 22 \text{ mm} \quad ◀$$

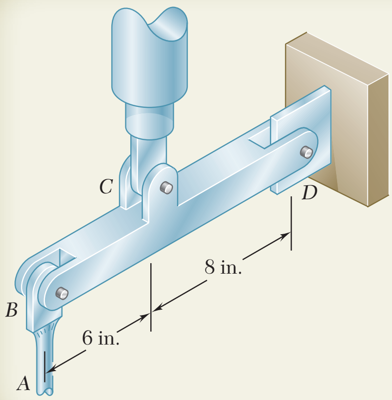
The next larger size pin available is of 22-mm diameter and should be used.



c. Bearing at *C*. Using $d = 22$ mm, the nominal bearing area of each bracket is $22t$. Since the force carried by each bracket is $C/2$ and the allowable bearing stress is 300 MPa, we write

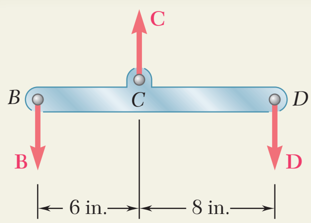
$$A_{\text{req}} = \frac{C/2}{\sigma_{\text{all}}} = \frac{(76.3 \text{ kN})/2}{300 \text{ MPa}} = 127.2 \text{ mm}^2$$

Thus $22t = 127.2 \qquad t = 5.78 \text{ mm} \qquad \text{Use: } t = 6 \text{ mm} \quad ◀$



SAMPLE PROBLEM 1.4

The rigid beam *BCD* is attached by bolts to a control rod at *B*, to a hydraulic cylinder at *C*, and to a fixed support at *D*. The diameters of the bolts used are: $d_B = d_D = \frac{3}{8}$ in., $d_C = \frac{1}{2}$ in. Each bolt acts in double shear and is made from a steel for which the ultimate shearing stress is $\tau_U = 40$ ksi. The control rod *AB* has a diameter $d_A = \frac{7}{16}$ in. and is made of a steel for which the ultimate tensile stress is $\sigma_U = 60$ ksi. If the minimum factor of safety is to be 3.0 for the entire unit, determine the largest upward force which may be applied by the hydraulic cylinder at *C*.



SOLUTION

The factor of safety with respect to failure must be 3.0 or more in each of the three bolts and in the control rod. These four independent criteria will be considered separately.

Free Body: Beam *BCD*. We first determine the force at *C* in terms of the force at *B* and in terms of the force at *D*.

$$+\circlearrowleft \Sigma M_D = 0: \qquad B(14 \text{ in.}) - C(8 \text{ in.}) = 0 \qquad C = 1.750B \qquad (1)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -D(14 \text{ in.}) + C(6 \text{ in.}) = 0 \qquad C = 2.33D \qquad (2)$$

Control Rod. For a factor of safety of 3.0 we have

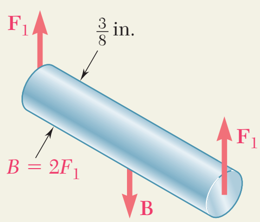
$$\sigma_{all} = \frac{\sigma_U}{F.S.} = \frac{60 \text{ ksi}}{3.0} = 20 \text{ ksi}$$

The allowable force in the control rod is

$$B = \sigma_{all}(A) = (20 \text{ ksi})\tfrac{1}{4}\pi(\tfrac{7}{16} \text{ in.})^2 = 3.01 \text{ kips}$$

Using Eq. (1) we find the largest permitted value of *C*:

$$C = 1.750B = 1.750(3.01 \text{ kips}) \qquad C = 5.27 \text{ kips} \blacktriangleleft$$



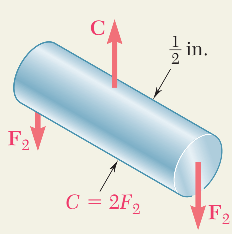
Bolt at *B*. $\tau_{all} = \tau_U/F.S. = (40 \text{ ksi})/3 = 13.33$ ksi. Since the bolt is in double shear, the allowable magnitude of the force **B** exerted on the bolt is

$$B = 2F_1 = 2(\tau_{all}A) = 2(13.33 \text{ ksi})(\tfrac{1}{4}\pi)(\tfrac{3}{8} \text{ in.})^2 = 2.94 \text{ kips}$$

From Eq. (1): $\qquad C = 1.750B = 1.750(2.94 \text{ kips}) \qquad C = 5.15 \text{ kips} \blacktriangleleft$

Bolt at *D*. Since this bolt is the same as bolt *B*, the allowable force is $D = B = 2.94$ kips. From Eq. (2):

$$C = 2.33D = 2.33(2.94 \text{ kips}) \qquad C = 6.85 \text{ kips} \blacktriangleleft$$



Bolt at *C*. We again have $\tau_{all} = 13.33$ ksi and write

$$C = 2F_2 = 2(\tau_{all}A) = 2(13.33 \text{ ksi})(\tfrac{1}{4}\pi)(\tfrac{1}{2} \text{ in.})^2 \qquad C = 5.23 \text{ kips} \blacktriangleleft$$

Summary. We have found separately four maximum allowable values of the force *C*. In order to satisfy all these criteria we must choose the smallest value, namely: $\qquad C = 5.15 \text{ kips} \blacktriangleleft$

PROBLEMS

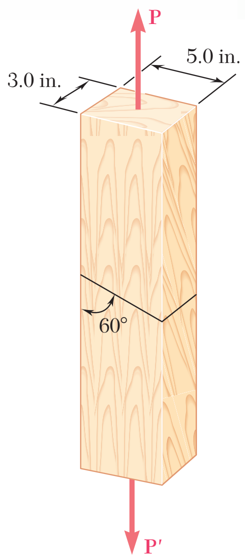


Fig. P1.29 and P1.30

1.29 The 1.4-kip load **P** is supported by two wooden members of uniform cross section that are joined by the simple glued scarf splice shown. Determine the normal and shearing stresses in the glued splice.

1.30 Two wooden members of uniform cross section are joined by the simple scarf splice shown. Knowing that the maximum allowable tensile stress in the glued splice is 75 psi, determine (*a*) the largest load **P** that can be safely supported, (*b*) the corresponding shearing stress in the splice.

1.31 Two wooden members of uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that $P = 11$ kN, determine the normal and shearing stresses in the glued splice.

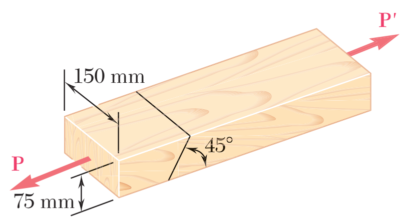


Fig. P1.31 and P1.32

1.32 Two wooden members of uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that the maximum allowable shearing stress in the glued splice is 620 kPa, determine (*a*) the largest load **P** that can be safely applied, (*b*) the corresponding tensile stress in the splice.

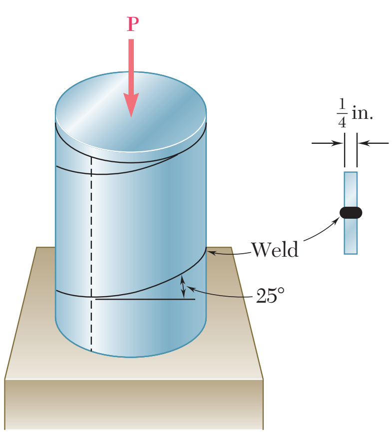


Fig. *P1.33* and *P1.34*

1.33 A steel pipe of 12-in. outer diameter is fabricated from $\frac{1}{4}$-in.-thick plate by welding along a helix that forms an angle of 25° with a plane perpendicular to the axis of the pipe. Knowing that the maximum allowable normal and shearing stresses in the directions respectively normal and tangential to the weld are $\sigma = 12$ ksi and $\tau = 7.2$ ksi, determine the magnitude P of the largest axial force that can be applied to the pipe.

1.34 A steel pipe of 12-in. outer diameter is fabricated from $\frac{1}{4}$-in.-thick plate by welding along a helix that forms an angle of 25° with a plane perpendicular to the axis of the pipe. Knowing that a 66 kip axial force **P** is applied to the pipe, determine the normal and shearing stresses in directions respectively normal and tangential to the weld.

1.35 A 1060-kN load **P** is applied to the granite block shown. Determine the resulting maximum value of (*a*) the normal stress, (*b*) the shearing stress. Specify the orientation of that plane on which each of these maximum values occurs.

1.36 A centric load **P** is applied to the granite block shown. Knowing that the resulting maximum value of the shearing stress in the block is 18 MPa, determine (*a*) the magnitude of **P**, (*b*) the orientation of the surface on which the maximum shearing stress occurs, (*c*) the normal stress exerted on that surface, (*d*) the maximum value of the normal stress in the block.

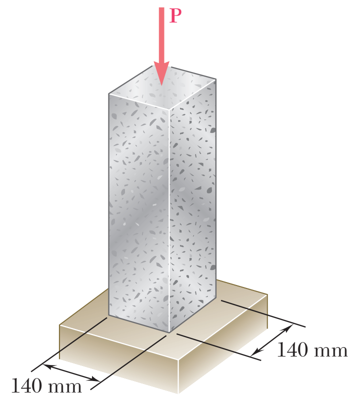


Fig. P1.35 and P1.36

1.37 Link *BC* is 6 mm thick, has a width $w = 25$ mm, and is made of a steel with a 480-MPa ultimate strength in tension. What is the safety factor used if the structure shown was designed to support a 16-kN load **P**?

1.38 Link *BC* is 6 mm thick and is made of a steel with a 450-MPa ultimate strength in tension. What should be its width *w* if the structure shown is being designed to support a 20-kN load **P** with a factor of safety of 3?

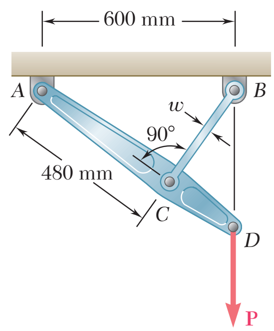


Fig. P1.37 and *P1.38*

1.39 A $\frac{3}{4}$-in.-diameter rod made of the same material as rods *AC* and *AD* in the truss shown was tested to failure and an ultimate load of 29 kips was recorded. Using a factor of safety of 3.0, determine the required diameter (*a*) of rod *AC*, (*b*) of rod *AD*.

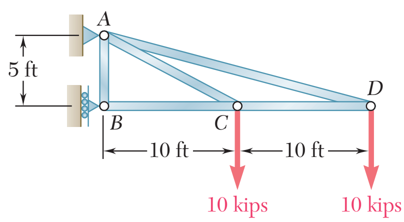


Fig. P1.39 and P1.40

1.40 In the truss shown, members *AC* and *AD* consist of rods made of the same metal alloy. Knowing that *AC* is of 1-in. diameter and that the ultimate load for that rod is 75 kips, determine (*a*) the factor of safety for *AC*, (*b*) the required diameter of *AD* if it is desired that both rods have the same factor of safety.

1.41 Link *AB* is to be made of a steel for which the ultimate normal stress is 450 MPa. Determine the cross-sectional area of *AB* for which the factor of safety will be 3.50. Assume that the link will be adequately reinforced around the pins at *A* and *B*.

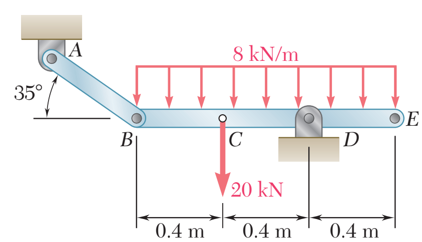


Fig. P1.41

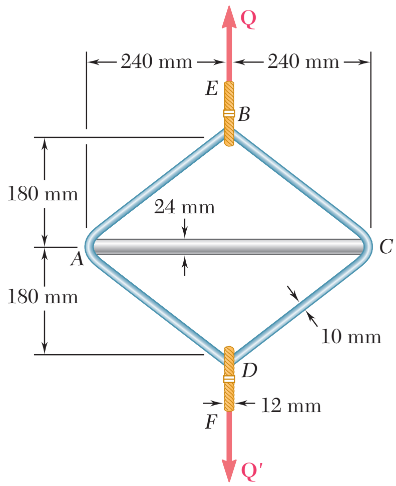


Fig. *P1.42*

1.42 A steel loop *ABCD* of length 1.2 m and of 10-mm diameter is placed as shown around a 24-mm-diameter aluminum rod *AC*. Cables *BE* and *DF*, each of 12-mm diameter, are used to apply the load **Q.** Knowing that the ultimate strength of the steel used for the loop and the cables is 480 MPa and that the ultimate strength of the aluminum used for the rod is 260 MPa, determine the largest load **Q** that can be applied if an overall factor of safety of 3 is desired.

1.43 Two wooden members shown, which support a 3.6-kip load, are joined by plywood splices fully glued on the surfaces in contact. The ultimate shearing stress in the glue is 360 psi and the clearance between the members is $\frac{1}{4}$ in. Determine the required length L of each splice if a factor of safety of 2.75 is to be achieved.

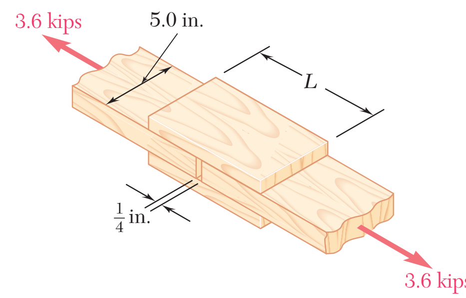


Fig. P1.43

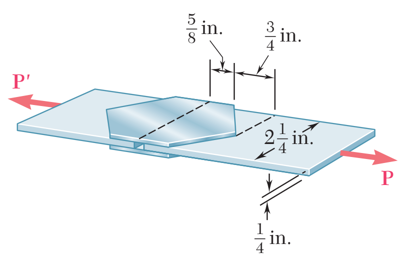


Fig. P1.44

1.44 Two plates, each $\frac{1}{8}$-in. thick, are used to splice a plastic strip as shown. Knowing that the ultimate shearing stress of the bonding between the surfaces is 130 psi, determine the factor of safety with respect to shear when $P = 325$ lb.

1.45 A load **P** is supported as shown by a steel pin that has been inserted in a short wooden member hanging from the ceiling. The ultimate strength of the wood used is 60 MPa in tension and 7.5 MPa in shear, while the ultimate strength of the steel is 145 MPa in shear. Knowing that $b = 40$ mm, $c = 55$ mm, and $d = 12$ mm, determine the load **P** if an overall factor of safety of 3.2 is desired.

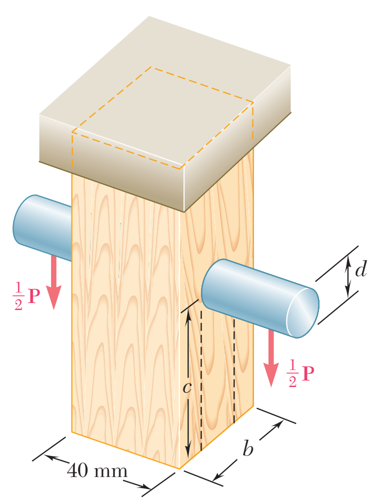


Fig. P1.45

1.46 For the support of Prob. 1.45, knowing that the diameter of the pin is $d = 16$ mm and that the magnitude of the load is $P = 20$ kN, determine (*a*) the factor of safety for the pin, (*b*) the required values of b and c if the factor of safety for the wooden member is the same as that found in part *a* for the pin.

1.47 Three steel bolts are to be used to attach the steel plate shown to a wooden beam. Knowing that the plate will support a 110-kN load, that the ultimate shearing stress for the steel used is 360 MPa, and that a factor of safety of 3.35 is desired, determine the required diameter of the bolts.

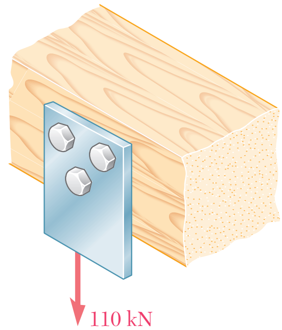


Fig. *P1.47* and P1.48

1.48 Three 18-mm-diameter steel bolts are to be used to attach the steel plate shown to a wooden beam. Knowing that the plate will support a 110-kN load and that the ultimate shearing stress for the steel used is 360 MPa, determine the factor of safety for this design.

1.49 A steel plate $\frac{5}{16}$ in. thick is embedded in a horizontal concrete slab and is used to anchor a high-strength vertical cable as shown. The diameter of the hole in the plate is $\frac{3}{4}$ in., the ultimate strength of the steel used is 36 ksi, and the ultimate bonding stress between plate and concrete is 300 psi. Knowing that a factor of safety of 3.60 is desired when $P = 2.5$ kips, determine (*a*) the required width a of the plate, (*b*) the minimum depth b to which a plate of that width should be embedded in the concrete slab. (Neglect the normal stresses between the concrete and the bottom edge of the plate.)

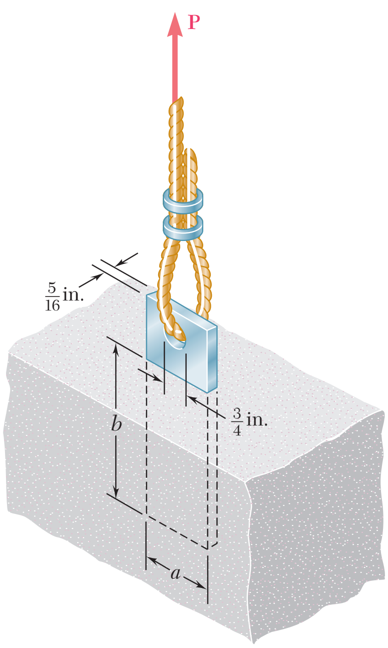


Fig. P1.49

1.50 Determine the factor of safety for the cable anchor in Prob. 1.49 when $P = 3$ kips, knowing that $a = 2$ in. and $b = 7.5$ in.

1.51 In the steel structure shown, a 6-mm-diameter pin is used at C and 10-mm-diameter pins are used at B and D. The ultimate shearing stress is 150 MPa at all connections, and the ultimate normal stress is 400 MPa in link BD. Knowing that a factor of safety of 3.0 is desired, determine the largest load $\mathbf{P}$ that can be applied at A. Note that link BD is not reinforced around the pin holes.

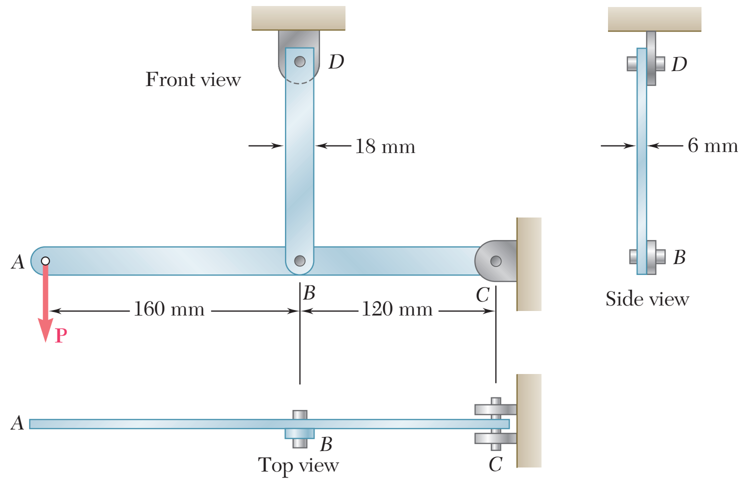


Fig. P1.51

1.52 Solve Prob. 1.51, assuming that the structure has been redesigned to use 12-mm-diameter pins at B and D and no other change has been made.

1.53 Each of the two vertical links CF connecting the two horizontal members AD and EG has a uniform rectangular cross section $\frac{1}{4}$ in. thick and 1 in. wide, and is made of a steel with an ultimate strength in tension of 60 ksi. The pins at C and F each have a $\frac{1}{2}$-in. diameter and are made of a steel with an ultimate strength in shear of 25 ksi. Determine the overall factor of safety for the links CF and the pins connecting them to the horizontal members.

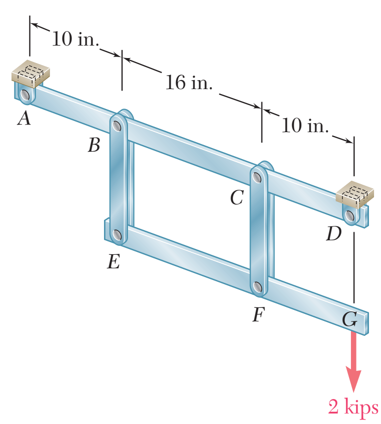


Fig. P1.53

1.54 Solve Prob. 1.53, assuming that the pins at C and F have been replaced by pins with a $\frac{3}{4}$-in. diameter.

1.55 In the structure shown, an 8-mm-diameter pin is used at A, and 12-mm-diameter pins are used at B and D. Knowing that the ultimate shearing stress is 100 MPa at all connections and that the ultimate normal stress is 250 MPa in each of the two links joining B and D, determine the allowable load **P** if an overall factor of safety of 3.0 is desired.

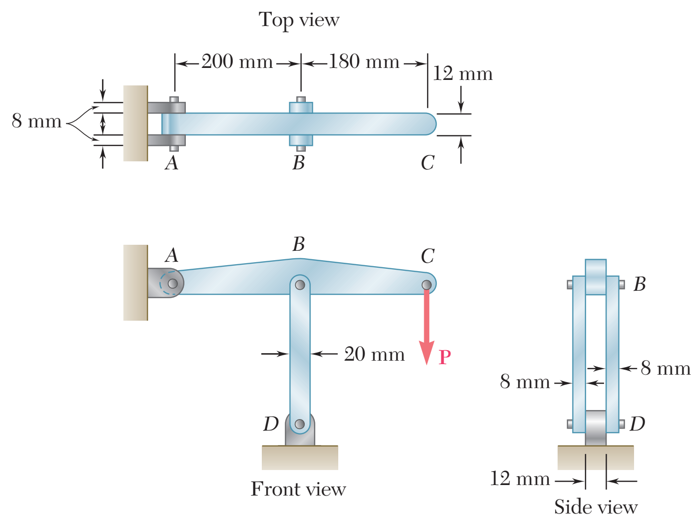


Fig. P1.55

1.56 In an alternative design for the structure of Prob. 1.55, a pin of 10-mm diameter is to be used at A. Assuming that all other specifications remain unchanged, determine the allowable load **P** if an overall factor of safety of 3.0 is desired.

***1.57** The Load and Resistance Factor Design method is to be used to select the two cables that will raise and lower a platform supporting two window washers. The platform weighs 160 lb and each of the window washers is assumed to weigh 195 lb with equipment. Since these workers are free to move on the platform, 75% of their total weight and the weight of their equipment will be used as the design live load of each cable. (*a*) Assuming a resistance factor $\phi = 0.85$ and load factors $\gamma_D = 1.2$ and $\gamma_L = 1.5$, determine the required minimum ultimate load of one cable. (*b*) What is the conventional factor of safety for the selected cables?

Fig. P1.57

***1.58** A 40-kg platform is attached to the end B of a 50-kg wooden beam AB, which is supported as shown by a pin at A and by a slender steel rod BC with a 12-kN ultimate load. (*a*) Using the Load and Resistance Factor Design method with a resistance factor $\phi = 0.90$ and load factors $\gamma_D = 1.25$ and $\gamma_L = 1.6$, determine the largest load that can be safely placed on the platform. (*b*) What is the corresponding conventional factor of safety for rod BC?

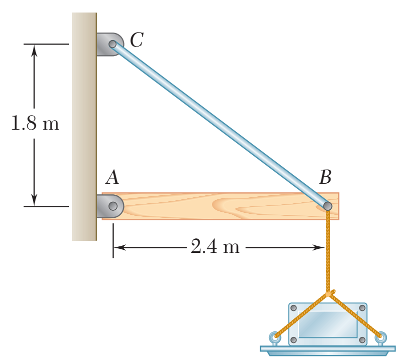


Fig. P1.58

REVIEW AND SUMMARY

This chapter was devoted to the concept of stress and to an introduction to the methods used for the analysis and design of machines and load-bearing structures.

Section 1.2 presented a short review of the methods of statics and of their application to the determination of the reactions exerted by its supports on a simple structure consisting of pin-connected members. Emphasis was placed on the use of a *free-body diagram* to obtain equilibrium equations which were solved for the unknown reactions. Free-body diagrams were also used to find the internal forces in the various members of the structure.

Axial loading. Normal stress

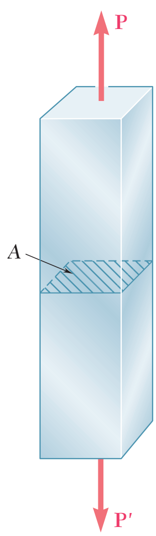


Fig. 1.41

The concept of *stress* was first introduced in Sec. 1.3 by considering a two-force member under an *axial loading*. The *normal stress* in that member was obtained by dividing the magnitude P of the load by the cross-sectional area A of the member (Fig. 1.41). We wrote

$$\sigma = \frac{P}{A} \tag{1.5}$$

Section 1.4 was devoted to a short discussion of the two principal tasks of an engineer, namely, the *analysis* and the *design* of structures and machines.

As noted in Sec. 1.5, the value of σ obtained from Eq. (1.5) represents the *average stress* over the section rather than the stress at a specific point Q of the section. Considering a small area ΔA surrounding Q and the magnitude ΔF of the force exerted on ΔA, we defined the stress at point Q as

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1.6}$$

In general, the value obtained for the stress σ at point Q is different from the value of the average stress given by formula (1.5) and is found to vary across the section. However, this variation is small in any section away from the points of application of the loads. In practice, therefore, the distribution of the normal stresses in an axially loaded member is assumed to be *uniform*, except in the immediate vicinity of the points of application of the loads.

However, for the distribution of stresses to be uniform in a given section, it is necessary that the line of action of the loads **P** and **P**′ pass through the centroid C of the section. Such a loading is called a *centric* axial loading. In the case of an *eccentric* axial loading, the distribution of stresses is *not* uniform. Stresses in members subjected to an eccentric axial loading will be discussed in Chap 4.

Transverse forces. Shearing stress

When equal and opposite *transverse forces* **P** and **P′** of magnitude P are applied to a member AB (Fig. 1.42), *shearing stresses* τ are created over any section located between the points of application of the two forces [Sec 1.6]. These stresses vary greatly across the section and their distribution *cannot* be assumed uniform. However, dividing the magnitude P—referred to as the *shear* in the section—by the cross-sectional area A, we defined the *average shearing stress* over the section:

$$\tau_{\text{ave}} = \frac{P}{A} \tag{1.8}$$

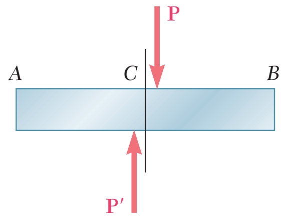


Fig. 1.42

Single and double shear

Shearing stresses are found in bolts, pins, or rivets connecting two structural members or machine components. For example, in the case of bolt CD (Fig. 1.43), which is in *single shear,* we wrote

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A} \tag{1.9}$$

while, in the case of bolts EG and HJ (Fig. 1.44), which are both in *double shear,* we had

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F/2}{A} = \frac{F}{2A} \tag{1.10}$$

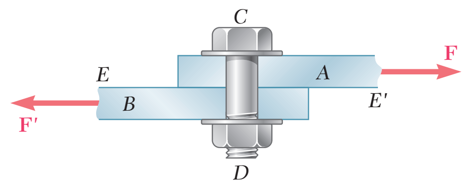


Fig. 1.43

Bearing stress

Bolts, pins, and rivets also create stresses in the members they connect, along the *bearing surface,* or surface of contact [Sec. 1.7]. The bolt CD of Fig. 1.43, for example, creates stresses on the semicylindrical surface of plate A with which it is in contact (Fig. 1.45). Since the distribution of these stresses is quite complicated, one uses in practice an average nominal value σ_b of the stress, called *bearing stress,* obtained by dividing the load P by the area of the rectangle representing the projection of the bolt on the plate section. Denoting by t the thickness of the plate and by d the diameter of the bolt, we wrote

$$\sigma_b = \frac{P}{A} = \frac{P}{td} \tag{1.11}$$

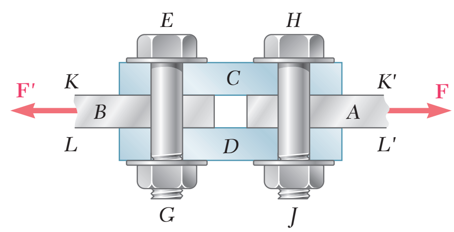


Fig. 1.44

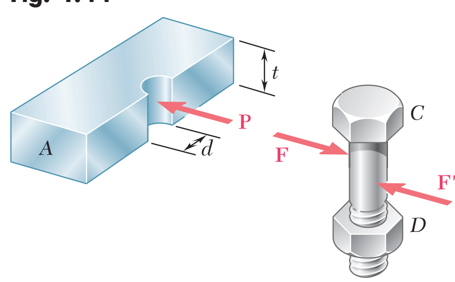


Fig. 1.45

In Sec. 1.8, we applied the concept introduced in the previous sections to the analysis of a simple structure consisting of two pin-connected members supporting a given load. We determined successively the normal stresses in the two members, paying special attention to their narrowest sections, the shearing stresses in the various pins, and the bearing stress at each connection.

Method of Solution

The method you should use in solving a problem in mechanics of materials was described in Sec. 1.9. Your solution should begin with a clear and precise *statement* of the problem. You will then draw one or several *free-body diagrams* that you will use to write *equilibrium equations*. These equations will be solved for *unknown forces,* from which the required *stresses* and *deformations* can be computed. Once the answer has been obtained, it should be *carefully checked.*

The first part of the chapter ended with a discussion of *numerical accuracy* in engineering, which stressed the fact that the accuracy of an answer can never be greater than the accuracy of the given data [Sec. 1.10].

Stresses on an oblique section

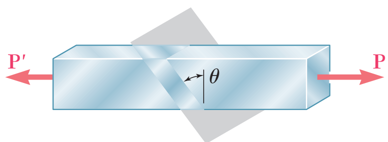


Fig. 1.46

In Sec. 1.11, we considered the stresses created on an *oblique section* in a two-force member under axial loading. We found that both *normal* and *shearing* stresses occurred in such a situation. Denoting by θ the angle formed by the section with a normal plane (Fig. 1.46) and by A_0 the area of a section perpendicular to the axis of the member, we derived the following expressions for the normal stress σ and the shearing stress τ on the oblique section:

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta \tag{1.14}$$

We observed from these formulas that the normal stress is maximum and equal to $\sigma_m = P/A_0$ for $\theta = 0$, while the shearing stress is maximum and equal to $\tau_m = P/2A_0$ for $\theta = 45°$. We also noted that $\tau = 0$ when $\theta = 0$, while $\sigma = P/2A_0$ when $\theta = 45°$.

Stress under general loading

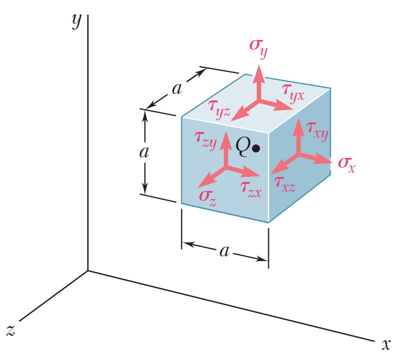


Fig. 1.47

Next, we discussed the state of stress at a point Q in a body under the most general loading condition [Sec. 1.12]. Considering a small cube centered at Q (Fig. 1.47), we denoted by σ_x the normal stress exerted on a face of the cube perpendicular to the x axis, and by τ_{xy} and τ_{xz}, respectively, the y and z components of the shearing stress exerted on the same face of the cube. Repeating this procedure for the other two faces of the cube and observing that $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, and $\tau_{zx} = \tau_{xz}$, we concluded that *six stress components* are required to define the state of stress at a given point Q, namely, σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}.

Section 1.13 was devoted to a discussion of the various concepts used in the design of engineering structures. The *ultimate load* of a given structural member or machine component is the load at which the member or component is expected to fail; it is computed from the *ultimate stress* or *ultimate strength* of the material used, as determined by a laboratory test on a specimen of that material. The ultimate load should be considerably larger than the *allowable load*, i.e., the load that the member or component will be allowed to carry under normal conditions. The ratio of the ultimate load to the allowable load is defined as the *factor of safety:*

Factor of safety

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate load}}{\text{allowable load}} \tag{1.24}$$

The determination of the factor of safety that should be used in the design of a given structure depends upon a number of considerations, some of which were listed in this section.

Load and Resistance Factor Design

Section 1.13 ended with the discussion of an alternative approach to design, known as *Load and Resistance Factor Design,* which allows the engineer to distinguish between the uncertainties associated with the structure and those associated with the load.

REVIEW PROBLEMS

1.59 A strain gage located at C on the surface of bone AB indicates that the average normal stress in the bone is 3.80 MPa when the bone is subjected to two 1200-N forces as shown. Assuming the cross section of the bone at C to be annular and knowing that its outer diameter is 25 mm, determine the inner diameter of the bone's cross section at C.

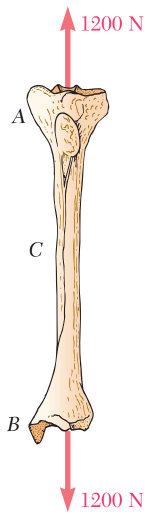


Fig. P1.59

1.60 Two horizontal 5-kip forces are applied to pin B of the assembly shown. Knowing that a pin of 0.8-in. diameter is used at each connection, determine the maximum value of the average normal stress (*a*) in link AB, (*b*) in link BC.

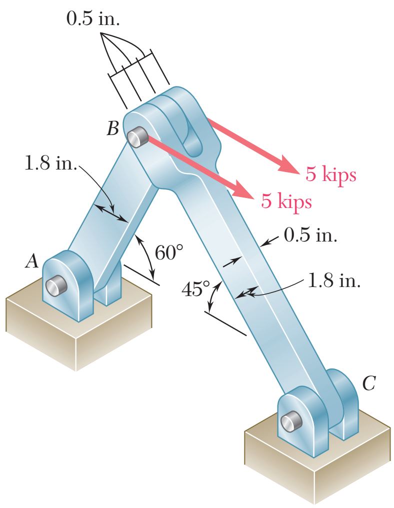


Fig. *P1.60*

1.61 For the assembly and loading of Prob. 1.60, determine (*a*) the average shearing stress in the pin at C, (*b*) the average bearing stress at C in member BC, (*c*) the average bearing stress at B in member BC.

1.62 In the marine crane shown, link CD is known to have a uniform cross section of 50×150 mm. For the loading shown, determine the normal stress in the central portion of that link.

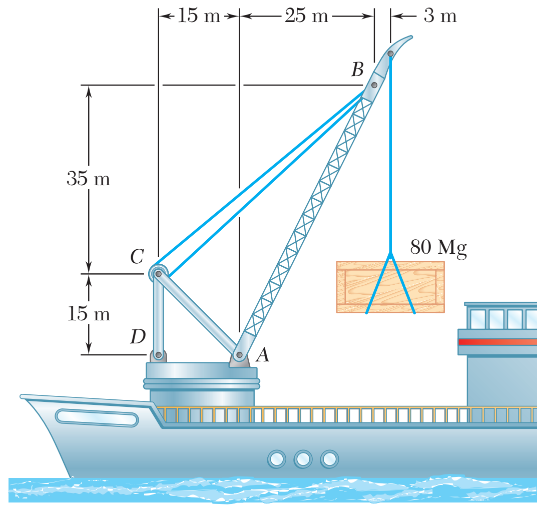


Fig. *P1.62*

1.63 Two wooden planks, each $\frac{1}{2}$ in. thick and 9 in. wide, are joined by the dry mortise joint shown. Knowing that the wood used shears off along its grain when the average shearing stress reaches 1.20 ksi, determine the magnitude P of the axial load that will cause the joint to fail.

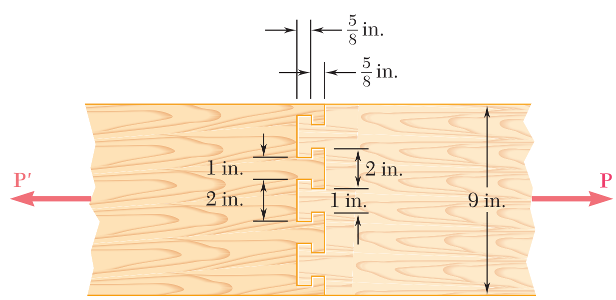


Fig. P1.63

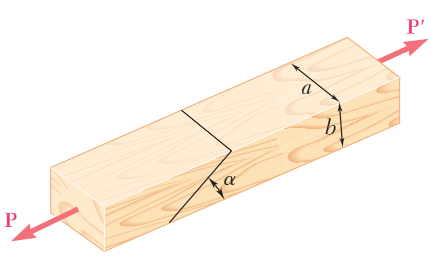


Fig. *P1.64*

1.64 Two wooden members of uniform rectangular cross section of sides a = 100 mm and b = 60 mm are joined by a simple glued joint as shown. Knowing that the ultimate stresses for the joint are σ_U = 1.26 MPa in tension and τ_U = 1.50 MPa in shear and that P = 6 kN, determine the factor of safety for the joint when (*a*) α = 20°, (*b*) α = 35°, (*c*) α = 45°. For each of these values of α, also determine whether the joint will fail in tension or in shear if P is increased until rupture occurs.

1.65 Member *ABC*, which is supported by a pin and bracket at *C* and a cable *BD*, was designed to support the 16-kN load **P** as shown. Knowing that the ultimate load for cable *BD* is 100 kN, determine the factor of safety with respect to cable failure.

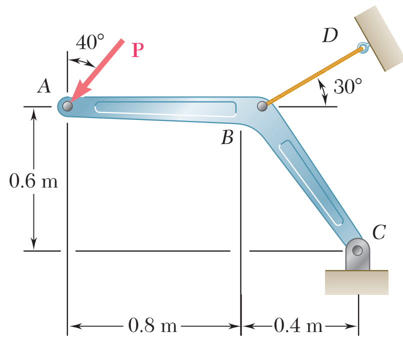


Fig. P1.65

1.66 The 2000-lb load may be moved along the beam *BD* to any position between stops at *E* and *F*. Knowing that $\sigma_{all} = 6$ ksi for the steel used in rods *AB* and *CD*, determine where the stops should be placed if the permitted motion of the load is to be as large as possible.

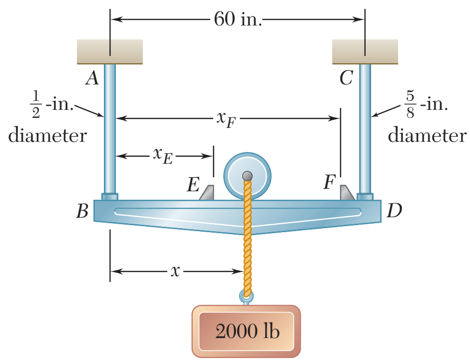


Fig. *P1.66*

1.67 Knowing that a force **P** of magnitude 750 N is applied to the pedal shown, determine (*a*) the diameter of the pin at *C* for which the average shearing stress in the pin is 40 MPa, (*b*) the corresponding bearing stress in the pedal at *C*, (*c*) the corresponding bearing stress in each support bracket at *C*.

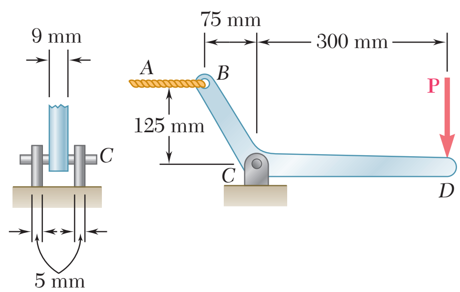


Fig. P1.67

1.68 A force **P** is applied as shown to a steel reinforcing bar that has been embedded in a block of concrete. Determine the smallest length L for which the full allowable normal stress in the bar can be developed. Express the result in terms of the diameter d of the bar, the allowable normal stress σ_{all} in the steel, and the average allowable bond stress τ_{all} between the concrete and the cylindrical surface of the bar. (Neglect the normal stresses between the concrete and the end of the bar.)

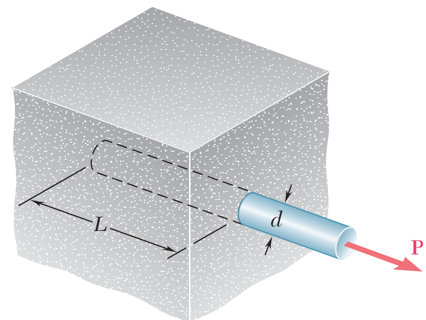


Fig. P1.68

1.69 The two portions of member AB are glued together along a plane forming an angle θ with the horizontal. Knowing that the ultimate stress for the glued joint is 2.5 ksi in tension and 1.3 ksi in shear, determine the range of values of θ for which the factor of safety of the members is at least 3.0.

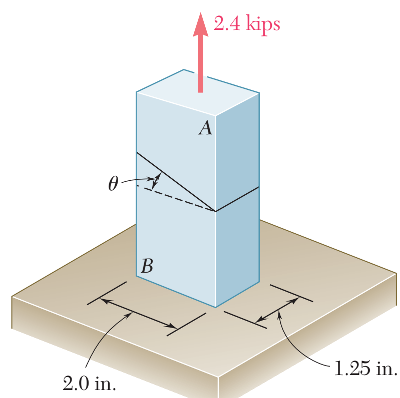


Fig. P1.69 and P1.70

1.70 The two portions of member AB are glued together along a plane forming an angle θ with the horizontal. Knowing that the ultimate stress for the glued joint is 2.5 ksi in tension and 1.3 ksi in shear, determine (*a*) the value of θ for which the factor of safety of the member is maximum, (*b*) the corresponding value of the factor of safety. (*Hint:* Equate the expressions obtained for the factors of safety with respect to normal stress and shear.)

COMPUTER PROBLEMS

The following problems are designed to be solved with a computer.

1.C1 A solid steel rod consisting of n cylindrical elements welded together is subjected to the loading shown. The diameter of element i is denoted by d_i and the load applied to its lower end by $\mathbf{P}_i$, with the magnitude P_i of this load being assumed positive if $\mathbf{P}_i$ is directed downward as shown and negative otherwise. (*a*) Write a computer program that can used with either SI or U.S. customary units to determine the average stress in each element of the rod. (*b*) Use this program to solve Probs. 1.2 and 1.4.

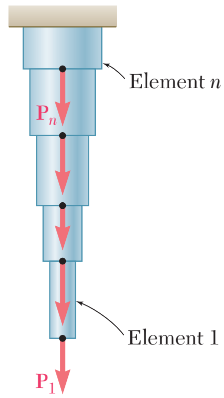


Fig. P1.C1

1.C2 A 20-kN load is applied as shown to the horizontal member ABC. Member ABC has a 10 × 50-mm uniform rectangular cross section and is supported by four vertical links, each of 8 × 36-mm uniform rectangular cross section. Each of the four pins at A, B, C, and D has the same diameter d and is in double shear. (*a*) Write a computer program to calculate for values of d from 10 to 30 mm, using 1-mm increments, (1) the maximum value of the average normal stress in the links connecting pins B and D, (2) the average normal stress in the links connecting pins C and E, (3) the average shearing stress in pin B, (4) the average shearing stress in pin C, (5) the average bearing stress at B in member ABC, (6) the average bearing stress at C in member ABC. (*b*) Check your program by comparing the values obtained for d = 16 mm with the answers given for Probs. 1.7 and 1.27. (*c*) Use this program to find the permissible values of the diameter d of the pins, knowing that the allowable values of the normal, shearing, and bearing stresses for the steel used are, respectively, 150 MPa, 90 MPa, and 230 MPa. (*d*) Solve part *c*, assuming that the thickness of member ABC has been reduced from 10 to 8 mm.

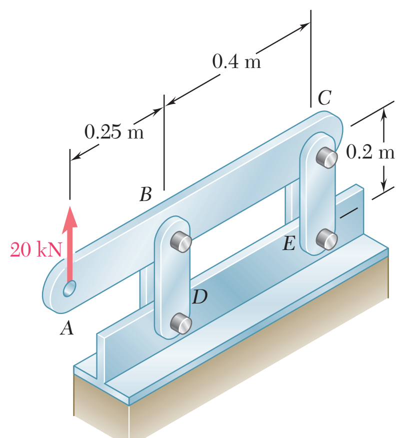


Fig. P1.C2

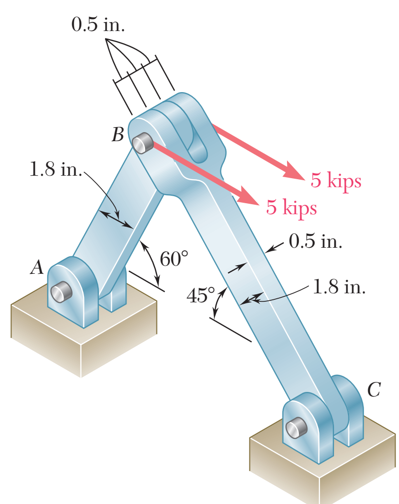


Fig. P1.C3

1.C3 Two horizontal 5-kip forces are applied to pin B of the assembly shown. Each of the three pins at A, B, and C has the same diameter d and is in double shear. (*a*) Write a computer program to calculate for values of d from 0.50 to 1.50 in., using 0.05-in. increments, (1) the maximum value of the average normal stress in member AB, (2) the average normal stress in member BC, (3) the average shearing stress in pin A, (4) the average shearing stress in pin C, (5) the average bearing stress at A in member AB, (6) the average bearing stress at C in member BC, (7) the average bearing stress at B in member BC. (*b*) Check your program by comparing the values obtained for $d = 0.8$ in. with the answers given for Probs. 1.60 and 1.61. (*c*) Use this program to find the permissible values of the diameter d of the pins, knowing that the allowable values of the normal, shearing, and bearing stresses for the steel used are, respectively, 22 ksi, 13 ksi, and 36 ksi. (*d*) Solve part *c*, assuming that a new design is being investigated in which the thickness and width of the two members are changed, respectively, from 0.5 to 0.3 in. and from 1.8 to 2.4 in.

1.C4 A 4-kip force **P** forming an angle α with the vertical is applied as shown to member ABC, which is supported by a pin and bracket at C and by a cable BD forming an angle β with the horizontal. (*a*) Knowing that the ultimate load of the cable is 25 kips, write a computer program to construct a table of the values of the factor of safety of the cable for values of α and β from 0 to 45°, using increments in α and β corresponding to 0.1 increments in $\tan \alpha$ and $\tan \beta$. (*b*) Check that for any given value of α, the maximum value of the factor of safety is obtained for $\beta = 38.66°$ and explain why. (*c*) Determine the smallest possible value of the factor of safety for $\beta = 38.66°$, as well as the corresponding value of α, and explain the result obtained.

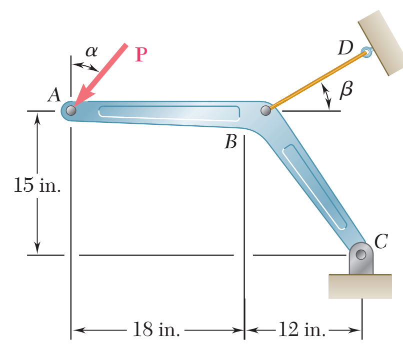


Fig. P1.C4

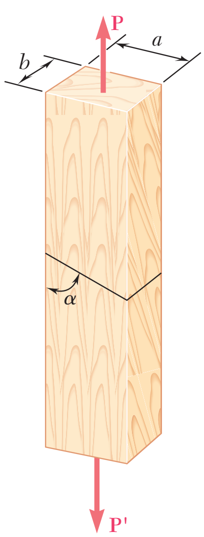


Fig. P1.C5

1.C5 A load **P** is supported as shown by two wooden members of uniform rectangular cross section that are joined by a simple glued scarf splice. (*a*) Denoting by σ_U and τ_U, respectively, the ultimate strength of the joint in tension and in shear, write a computer program which, for given values of *a*, *b*, *P*, σ_U and τ_U, expressed in either SI or U.S. customary units, and for values of a from 5 to 85° at 5° intervals, can calculate (1) the normal stress in the joint, (2) the shearing stress in the joint, (3) the factor of safety relative to failure in tension, (4) the factor of safety relative to failure in shear, (5) the overall factor of safety for the glued joint. (*b*) Apply this program, using the dimensions and loading of the members of Probs. 1.29 and 1.31, knowing that $\sigma_U = 150$ psi and $\tau_U = 214$ psi for the glue used in Prob. 1.29, and that $\sigma_U = 1.26$ MPa and $\tau_U = 1.50$ MPa for the glue used in Prob. 1.31. (*c*) Verify in each of these two cases that the shearing stress is maximum for $\alpha = 45°$.

1.C6 Member ABC is supported by a pin and bracket at A, and by two links that are pin-connected to the member at B and to a fixed support at D. (*a*) Write a computer program to calculate the allowable load P_{all} for any given values of (1) the diameter d_1 of the pin at A, (2) the common diameter d_2 of the pins at B and D, (3) the ultimate normal stress σ_U in each of the two links, (4) the ultimate shearing stress τ_U in each of the three pins, (5) the desired overall factor of safety *F.S.* Your program should also indicate which of the following three stresses is critical: the normal stress in the links, the shearing stress in the pin at A, or the shearing stress in the pins at B and D (*b* and *c*). Check your program by using the data of Probs. 1.55 and 1.56, respectively, and comparing the answers obtained for P_{all} with those given in the text. (*d*) Use your program to determine the allowable load P_{all}, as well as which of the stresses is critical, when $d_1 = d_2 = 15$ mm, $\sigma_U = 110$ MPa for aluminum links, $\tau_U = 100$ MPa for steel pins, and $F.S. = 3.2$.

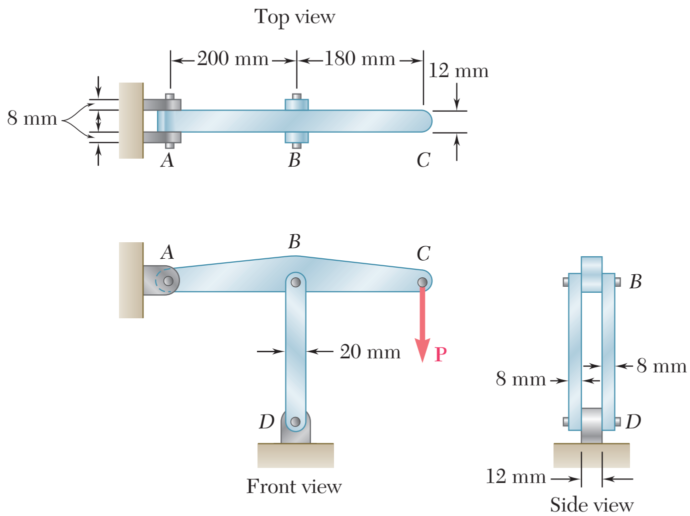
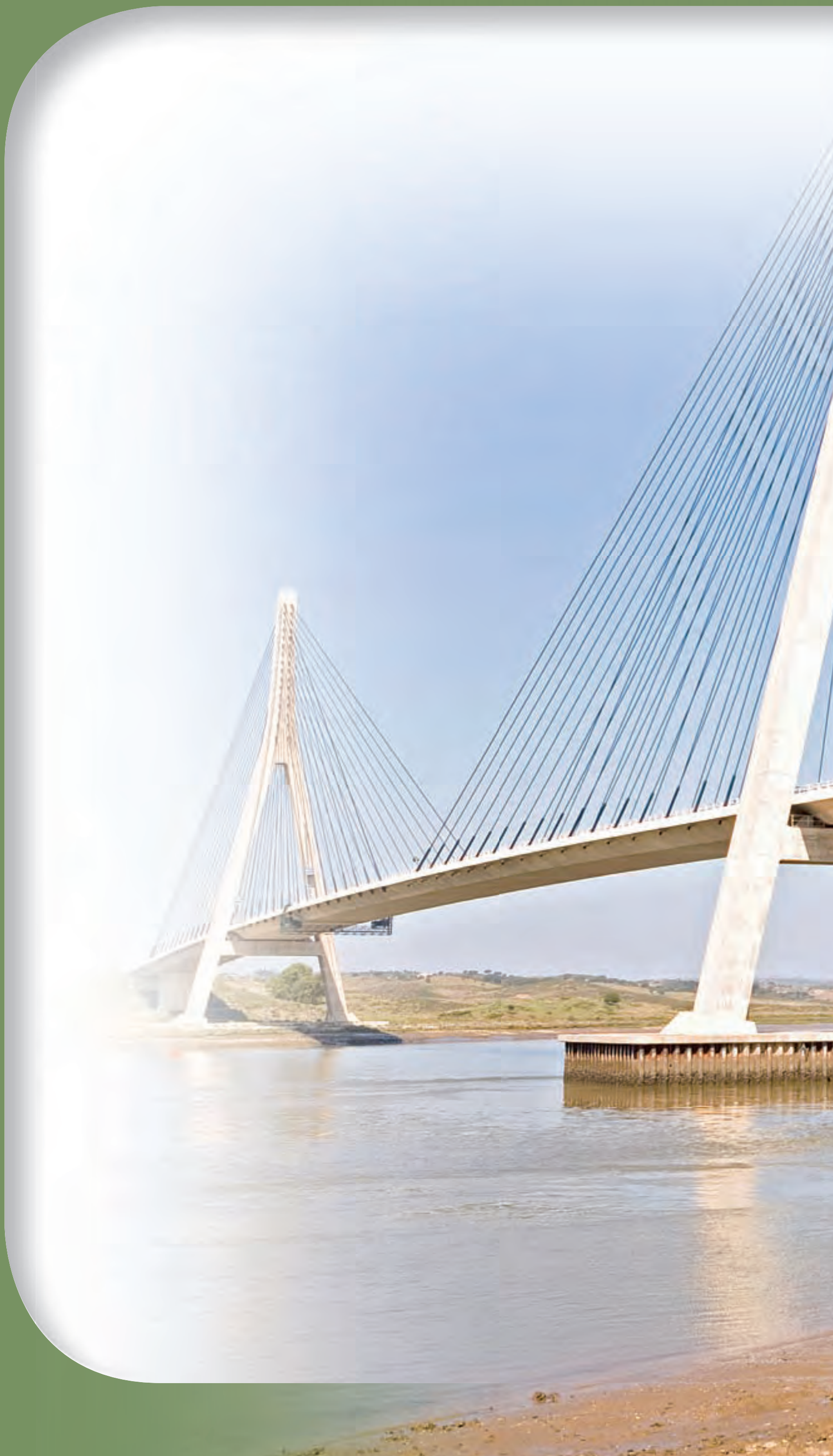


Fig. P1.C6

This chapter is devoted to the study of deformations occurring in structural components subjected to axial loading. The change in length of the diagonal stays was carefully accounted for in the design of this cable-stayed bridge.



CHAPTER 2

Stress and Strain—Axial Loading

Chapter 2 Stress and Strain—Axial Loading

- **2.1** Introduction
- **2.2** Normal Strain Under Axial Loading
- **2.3** Stress-Strain Diagram
- ***2.4** True Stress and True Strain
- **2.5** Hooke's Law; Modulus of Elasticity
- **2.6** Elastic versus Plastic Behavior of a Material
- **2.7** Repeated Loadings; Fatigue
- **2.8** Deformations of Members Under Axial Loading
- **2.9** Statically Indeterminate Problems
- **2.10** Problems Involving Temperature Changes
- **2.11** Poisson's Ratio
- **2.12** Multiaxial Loading; Generalized Hooke's Law
- ***2.13** Dilatation; Bulk Modulus
- **2.14** Shearing Strain
- **2.15** Further Discussions of Deformations Under Axial Loading; Relation Among E, ν, and G
- ***2.16** Stress-Strain Relationships for Fiber-Reinforced Composite Materials
- **2.17** Stress and Strain Distribution Under Axial Loading; Saint-Venant's Principle
- **2.18** Stress Concentrations
- **2.19** Plastic Deformations
- ***2.20** Residual Stresses

2.1 INTRODUCTION

In Chap. 1 we analyzed the stresses created in various members and connections by the loads applied to a structure or machine. We also learned to design simple members and connections so that they would not fail under specified loading conditions. Another important aspect of the analysis and design of structures relates to the *deformations* caused by the loads applied to a structure. Clearly, it is important to avoid deformations so large that they may prevent the structure from fulfilling the purpose for which it was intended. But the analysis of deformations may also help us in the determination of stresses. Indeed, it is not always possible to determine the forces in the members of a structure by applying only the principles of statics. This is because statics is based on the assumption of undeformable, rigid structures. By considering engineering structures as *deformable* and analyzing the deformations in their various members, it will be possible for us to compute forces that are *statically indeterminate,* i.e., indeterminate within the framework of statics. Also, as we indicated in Sec. 1.5, the distribution of stresses in a given member is statically indeterminate, even when the force in that member is known. To determine the actual distribution of stresses within a member, it is thus necessary to analyze the deformations that take place in that member. In this chapter, you will consider the deformations of a structural member such as a rod, bar, or plate under *axial loading*.

First, the *normal strain* ϵ in a member will be defined as the *deformation of the member per unit length*. Plotting the stress σ versus the strain ϵ as the load applied to the member is increased will yield a *stress-strain diagram* for the material used. From such a diagram we can determine some important properties of the material, such as its *modulus of elasticity,* and whether the material is *ductile* or *brittle* (Secs. 2.2 to 2.5). You will also see in Sec. 2.5 that, while the behavior of most materials is independent of the direction in which the load is applied, the response of fiber-reinforced composite materials depends upon the direction of the load.

From the stress-strain diagram, we can also determine whether the strains in the specimen will disappear after the load has been removed—in which case the material is said to behave *elastically*—or whether *a permanent set* or *plastic deformation* will result (Sec. 2.6).

Section 2.7 is devoted to the phenomenon of *fatigue,* which causes structural or machine components to fail after a very large number of repeated loadings, even though the stresses remain in the elastic range.

The first part of the chapter ends with Sec. 2.8, which is devoted to the determination of the deformation of various types of members under various conditions of axial loading.

In Secs. 2.9 and 2.10, *statically indeterminate problems* will be considered, i.e., problems in which the reactions and the internal forces *cannot* be determined from statics alone. The equilibrium equations derived from the free-body diagram of the member under consideration must be complemented by relations involving deformations; these relations will be obtained from the geometry of the problem.

In Secs. 2.11 to 2.15, additional constants associated with isotropic materials—i.e., materials with mechanical characteristics independent of direction—will be introduced. They include *Poisson's ratio,* which relates lateral and axial strain, the *bulk modulus,* which characterizes the change in volume of a material under hydrostatic pressure, and the *modulus of rigidity,* which relates the components of the shearing stress and shearing strain. Stress-strain relationships for an isotropic material under a multiaxial loading will also be derived.

In Sec. 2.16, stress-strain relationships involving several distinct values of the modulus of elasticity, Poisson's ratio, and the modulus of rigidity, will be developed for fiber-reinforced composite materials under a multiaxial loading. While these materials are not isotropic, they usually display special properties, known as *orthotropic* properties, which facilitate their study.

In the text material described so far, stresses are assumed uniformly distributed in any given cross section; they are also assumed to remain within the elastic range. The validity of the first assumption is discussed in Sec. 2.17, while *stress concentrations* near circular holes and fillets in flat bars are considered in Sec. 2.18. Sections 2.19 and 2.20 are devoted to the discussion of stresses and deformations in members made of a ductile material when the yield point of the material is exceeded. As you will see, permanent *plastic deformations* and *residual stresses* result from such loading conditions.

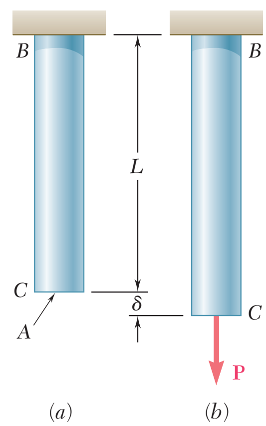


Fig. 2.1 Deformation of axially-loaded rod.

2.2 NORMAL STRAIN UNDER AXIAL LOADING

Let us consider a rod BC, of length L and uniform cross-sectional area A, which is suspended from B (Fig. 2.1*a*). If we apply a load $\mathbf{P}$ to end C, the rod elongates (Fig. 2.1*b*). Plotting the magnitude P of the load against the deformation δ (Greek letter delta), we obtain a certain load-deformation diagram (Fig. 2.2). While this diagram contains information useful to the analysis of the rod under consideration, it cannot be used directly to predict the deformation of a rod of the same material but of different dimensions. Indeed, we observe that, if a deformation δ is produced in rod BC by a load $\mathbf{P}$, a load $2\mathbf{P}$ is required to cause the same deformation in a rod $B'C'$ of the same length L, but of cross-sectional area $2A$ (Fig. 2.3). We note that, in both cases, the value of the stress is the same: $\sigma = P/A$. On the other hand, a load $\mathbf{P}$ applied to a rod $B''C''$, of the same

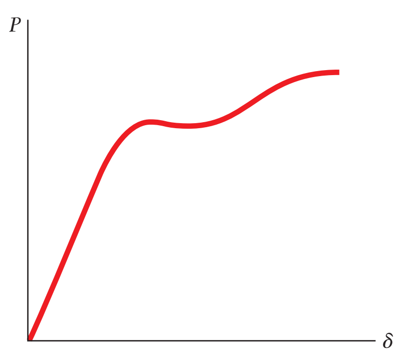


Fig. 2.2 Load-deformation diagram.

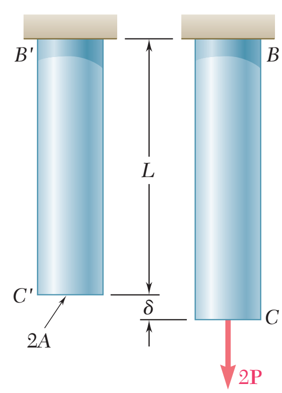


Fig. 2.3

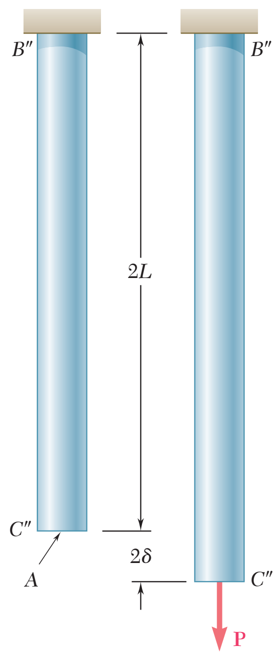


Fig. 2.4

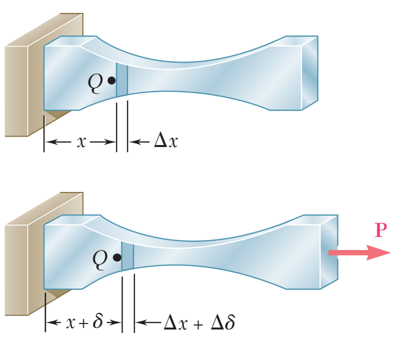


Fig. 2.5 Deformation of axially-loaded member of variable cross-sectional area.

cross-sectional area A, but of length $2L$, causes a deformation 2δ in that rod (Fig. 2.4), i.e., a deformation twice as large as the deformation δ it produces in rod BC. But in both cases the ratio of the deformation over the length of the rod is the same; it is equal to δ/L. This observation brings us to introduce the concept of *strain:* We define the *normal strain* in a rod under axial loading as the *deformation per unit length* of that rod. Denoting the normal strain by ϵ (Greek letter epsilon), we write

$$\epsilon = \frac{\delta}{L} \tag{2.1}$$

Plotting the stress $\sigma = P/A$ against the strain $\epsilon = \delta/L$, we obtain a curve that is characteristic of the properties of the material and does not depend upon the dimensions of the particular specimen used. This curve is called a *stress-strain diagram* and will be discussed in detail in Sec. 2.3.

Since the rod BC considered in the preceding discussion had a uniform cross section of area A, the normal stress σ could be assumed to have a constant value P/A throughout the rod. Thus, it was appropriate to define the strain ϵ as the ratio of the total deformation δ over the total length L of the rod. In the case of a member of variable cross-sectional area A, however, the normal stress $\sigma = P/A$ varies along the member, and it is necessary to define the strain at a given point Q by considering a small element of undeformed length Δx (Fig. 2.5). Denoting by $\Delta\delta$ the deformation of the element under the given loading, we define the *normal strain at point Q* as

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx} \tag{2.2}$$

Since deformation and length are expressed in the same units, the normal strain ϵ obtained by dividing δ by L (or $d\delta$ by dx) is a *dimensionless quantity.* Thus, the same numerical value is obtained for the normal strain in a given member, whether SI metric units or U.S. customary units are used. Consider, for instance, a bar of length $L = 0.600$ m and uniform cross section, which undergoes a deformation $\delta = 150 \times 10^{-6}$ m. The corresponding strain is

$$\epsilon = \frac{\delta}{L} = \frac{150 \times 10^{-6}\ \text{m}}{0.600\ \text{m}} = 250 \times 10^{-6}\ \text{m/m} = 250 \times 10^{-6}$$

Note that the deformation could have been expressed in micrometers: $\delta = 150\ \mu\text{m}$. We would then have written

$$\epsilon = \frac{\delta}{L} = \frac{150\ \mu\text{m}}{0.600\ \text{m}} = 250\ \mu\text{m/m} = 250\ \mu$$

and read the answer as "250 micros." If U.S. customary units are used, the length and deformation of the same bar are, respectively,

$L = 23.6$ in. and $\delta = 5.91 \times 10^{-3}$ in. The corresponding strain is

$$\epsilon = \frac{\delta}{L} = \frac{5.91 \times 10^{-3} \text{ in.}}{23.6 \text{ in.}} = 250 \times 10^{-6} \text{ in./in.}$$

which is the same value that we found using SI units. It is customary, however, when lengths and deformations are expressed in inches or microinches (μin.), to keep the original units in the expression obtained for the strain. Thus, in our example, the strain would be recorded as $\epsilon = 250 \times 10^{-6}$ in./in. or, alternatively, as $\epsilon = 250\ \mu$in./in.

2.3 STRESS-STRAIN DIAGRAM

We saw in Sec. 2.2 that the diagram representing the relation between stress and strain in a given material is an important characteristic of the material. To obtain the stress-strain diagram of a material, one usually conducts a *tensile test* on a specimen of the material. One type of specimen commonly used is shown in Photo 2.1. The cross-sectional area of the cylindrical central portion of the specimen has been accurately determined and two gage marks have been inscribed on that portion at a distance L_0 from each other. The distance L_0 is known as the *gage length* of the specimen.

The test specimen is then placed in a testing machine (Photo 2.2), which is used to apply a centric load **P**. As the load **P** increases, the

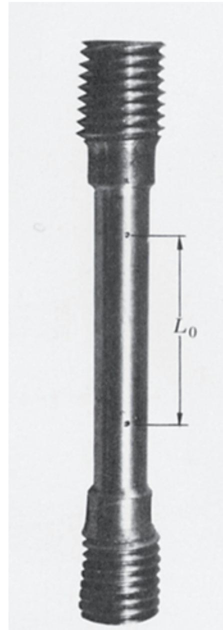


Photo 2.1 Typical tensile-test specimen.

Photo 2.2 This machine is used to test tensile test specimens, such as those shown in this chapter.

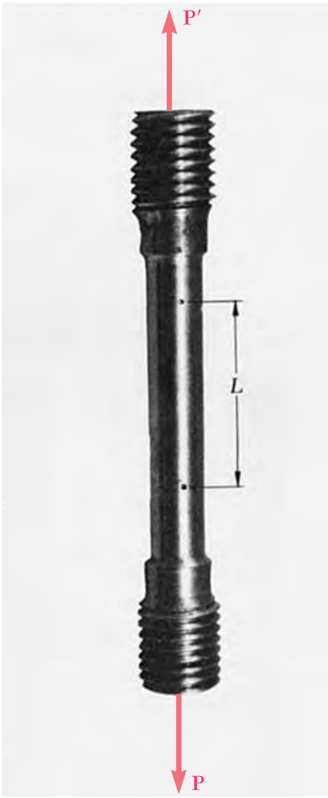


Photo 2.3 Test specimen with tensile load.

distance L between the two gage marks also increases (Photo 2.3). The distance L is measured with a dial gage, and the elongation $\delta = L - L_0$ is recorded for each value of P. A second dial gage is often used simultaneously to measure and record the change in diameter of the specimen. From each pair of readings P and δ, the stress σ is computed by dividing P by the original cross-sectional area A_0 of the specimen, and the strain ϵ by dividing the elongation δ by the original distance L_0 between the two gage marks. The stress-strain diagram may then be obtained by plotting ϵ as an abscissa and σ as an ordinate.

Stress-strain diagrams of various materials vary widely, and different tensile tests conducted on the same material may yield different results, depending upon the temperature of the specimen and the speed of loading. It is possible, however, to distinguish some common characteristics among the stress-strain diagrams of various groups of materials and to divide materials into two broad categories on the basis of these characteristics, namely, the *ductile* materials and the *brittle* materials.

Ductile materials, which comprise structural steel, as well as many alloys of other metals, are characterized by their ability to *yield* at normal temperatures. As the specimen is subjected to an increasing load, its length first increases linearly with the load and at a very slow rate. Thus, the initial portion of the stress-strain diagram is a straight line with a steep slope (Fig. 2.6). However, after a critical value σ_Y of the stress has been reached, the specimen undergoes a large deformation with a relatively small increase in the applied load. This deformation is caused by slippage of the material along oblique surfaces and is due, therefore, primarily to shearing stresses. As we can note from the stress-strain diagrams of two typical ductile materials (Fig. 2.6), the elongation of the specimen after it has started to yield can be 200 times as large as its deformation before yield. After a certain maximum value of the load has been reached, the diameter of a portion of the specimen begins to decrease, because of local instability (Photo 2.4*a*). This phenomenon is known as *necking*. After necking has begun, somewhat lower loads are sufficient to keep the specimen elongating further, until it finally ruptures (Photo 2.4*b*). We note that rupture occurs along a cone-shaped surface that forms an angle of approximately 45° with the original surface of the specimen. This indicates that shear is primarily responsible for the failure of ductile materials, and confirms the fact that, under an axial load,

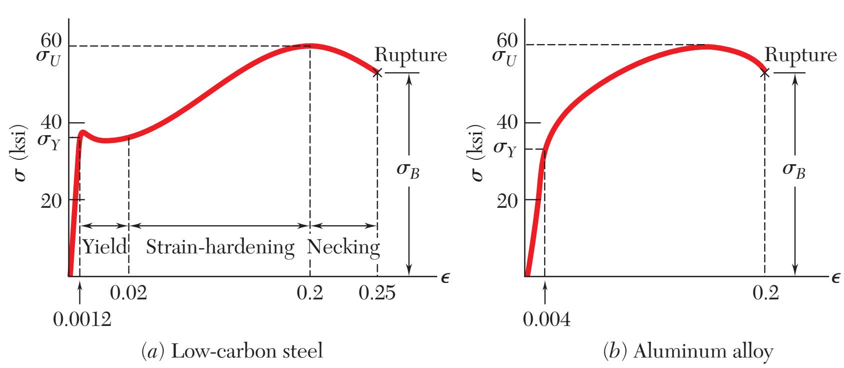


Fig. 2.6 Stress-strain diagrams of two typical ductile materials.

Photo 2.4 Tested specimen of a ductile material.

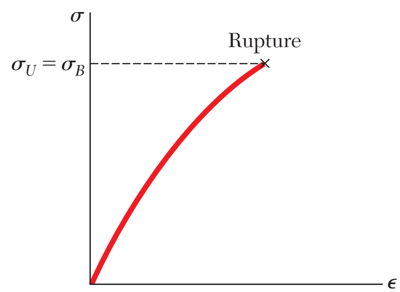


Fig. 2.7 Stress-strain diagram for a typical brittle material.

shearing stresses are largest on surfaces forming an angle of 45° with the load (cf. Sec. 1.11). The stress σ_Y at which yield is initiated is called the *yield strength* of the material, the stress σ_U corresponding to the maximum load applied to the specimen is known as the *ultimate strength,* and the stress σ_B corresponding to rupture is called the *breaking strength.*

Brittle materials, which comprise cast iron, glass, and stone, are characterized by the fact that rupture occurs without any noticeable prior change in the rate of elongation (Fig. 2.7). Thus, for brittle materials, there is no difference between the ultimate strength and the breaking strength. Also, the strain at the time of rupture is much smaller for brittle than for ductile materials. From Photo 2.5, we note the absence of any necking of the specimen in the case of a brittle material, and observe that rupture occurs along a surface perpendicular to the load. We conclude from this observation that normal stresses are primarily responsible for the failure of brittle materials.†

The stress-strain diagrams of Fig. 2.6 show that structural steel and aluminum, while both ductile, have different yield characteristics. In the case of structural steel (Fig. 2.6*a*), the stress remains constant over a large range of values of the strain after the onset of yield. Later the stress must be increased to keep elongating the specimen, until the maximum value σ_U has been reached. This is due to a property of the material known as strain-hardening. The

Photo 2.5 Tested specimen of a brittle material.

†The tensile tests described in this section were assumed to be conducted at normal temperatures. However, a material that is ductile at normal temperatures may display the characteristics of a brittle material at very low temperatures, while a normally brittle material may behave in a ductile fashion at very high temperatures. At temperatures other than normal, therefore, one should refer to *a material in a ductile state* or to *a material in a brittle state,* rather than to a ductile or brittle material.

yield strength of structural steel can be determined during the tensile test by watching the load shown on the display of the testing machine. After increasing steadily, the load is observed to suddenly drop to a slightly lower value, which is maintained for a certain period while the specimen keeps elongating. In a very carefully conducted test, one may be able to distinguish between the *upper yield point,* which corresponds to the load reached just before yield starts, and the *lower yield point,* which corresponds to the load required to maintain yield. Since the upper yield point is transient, the lower yield point should be used to determine the yield strength of the material.

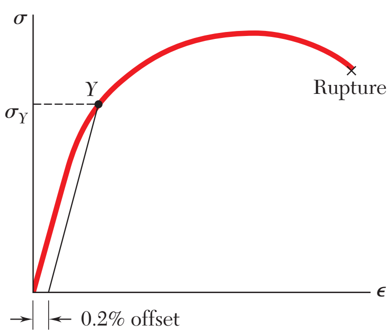


Fig. 2.8 Determination of yield strength by offset method.

In the case of aluminum (Fig. 2.6*b*) and of many other ductile materials, the onset of yield is not characterized by a horizontal portion of the stress-strain curve. Instead, the stress keeps increasing—although not linearly—until the ultimate strength is reached. Necking then begins, leading eventually to rupture. For such materials, the yield strength σ_Y can be defined by the offset method. The yield strength at 0.2% offset, for example, is obtained by drawing through the point of the horizontal axis of abscissa $\epsilon = 0.2\%$ (or $\epsilon = 0.002$), a line parallel to the initial straight-line portion of the stress-strain diagram (Fig. 2.8). The stress σ_Y corresponding to the point Y obtained in this fashion is defined as the yield strength at 0.2% offset.

A standard measure of the ductility of a material is its *percent elongation,* which is defined as

$$\text{Percent elongation} = 100 \frac{L_B - L_0}{L_0}$$

where L_0 and L_B denote, respectively, the initial length of the tensile test specimen and its final length at rupture. The specified minimum elongation for a 2-in. gage length for commonly used steels with yield strengths up to 50 ksi is 21 percent. We note that this means that the average strain at rupture should be at least 0.21 in./in.

Another measure of ductility which is sometimes used is the *percent reduction in area,* defined as

$$\text{Percent reduction in area} = 100 \frac{A_0 - A_B}{A_0}$$

where A_0 and A_B denote, respectively, the initial cross-sectional area of the specimen and its minimum cross-sectional area at rupture. For structural steel, percent reductions in area of 60 to 70 percent are common.

Thus far, we have discussed only tensile tests. If a specimen made of a ductile material were loaded in compression instead of tension, the stress-strain curve obtained would be essentially the same through its initial straight-line portion and through the beginning of the portion corresponding to yield and strain-hardening. Particularly noteworthy is the fact that for a given steel, the yield strength is the same in both tension and compression. For larger values of the strain, the tension and compression stress-strain curves diverge, and it should be noted that necking cannot occur in compression.

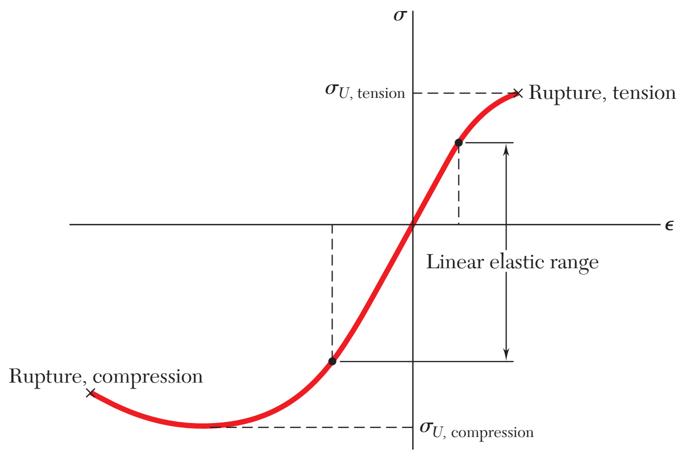


Fig. 2.9 Stress-strain diagram for concrete.

For most brittle materials, one finds that the ultimate strength in compression is much larger than the ultimate strength in tension. This is due to the presence of flaws, such as microscopic cracks or cavities, which tend to weaken the material in tension, while not appreciably affecting its resistance to compressive failure.

An example of brittle material with different properties in tension and compression is provided by *concrete,* whose stress-strain diagram is shown in Fig. 2.9. On the tension side of the diagram, we first observe a linear elastic range in which the strain is proportional to the stress. After the yield point has been reached, the strain increases faster than the stress until rupture occurs. The behavior of the material in compression is different. First, the linear elastic range is significantly larger. Second, rupture does not occur as the stress reaches its maximum value. Instead, the stress decreases in magnitude while the strain keeps increasing until rupture occurs. Note that the modulus of elasticity, which is represented by the slope of the stress-strain curve in its linear portion, is the same in tension and compression. This is true of most brittle materials.

*2.4 TRUE STRESS AND TRUE STRAIN

We recall that the stress plotted in the diagrams of Figs. 2.6 and 2.7 was obtained by dividing the load P by the cross-sectional area A_0 of the specimen measured before any deformation had taken place. Since the cross-sectional area of the specimen decreases as P increases, the stress plotted in our diagrams does not represent the actual stress in the specimen. The difference between the *engineering stress* $\sigma = P/A_0$ that we have computed and the *true stress* $\sigma_t = P/A$ obtained by dividing P by the cross-sectional area A of the deformed specimen becomes apparent in ductile materials after yield has started. While the engineering stress σ, which is directly proportional to the load P, decreases with P during the necking phase, the true stress σ_t, which is proportional to P but also inversely proportional to A, is observed to keep increasing until rupture of the specimen occurs.

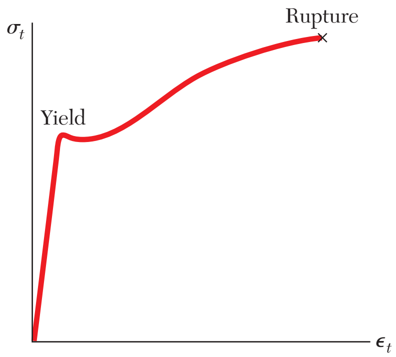


Fig. 2.10 True stress versus true strain for a typical ductile material.

Many scientists also use a definition of strain different from that of the *engineering strain* $\epsilon = \delta/L_0$. Instead of using the total elongation δ and the original value L_0 of the gage length, they use all the successive values of L that they have recorded. Dividing each increment ΔL of the distance between the gage marks, by the corresponding value of L, they obtain the elementary strain $\Delta\epsilon = \Delta L/L$. Adding the successive values of $\Delta\epsilon$, they define the *true strain* ϵ_t:

$$\epsilon_t = \Sigma\Delta\epsilon = \Sigma(\Delta L/L)$$

With the summation replaced by an integral, they can also express the true strain as follows:

$$\epsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln \frac{L}{L_0} \tag{2.3}$$

The diagram obtained by plotting true stress versus true strain (Fig. 2.10) reflects more accurately the behavior of the material. As we have already noted, there is no decrease in true stress during the necking phase. Also, the results obtained from tensile and from compressive tests will yield essentially the same plot when true stress and true strain are used. This is not the case for large values of the strain when the engineering stress is plotted versus the engineering strain. However, engineers, whose responsibility is to determine whether a load P will produce an acceptable stress and an acceptable deformation in a given member, will want to use a diagram based on the engineering stress $\sigma = P/A_0$ and the engineering strain $\epsilon = \delta/L_0$, since these expressions involve data that are available to them, namely the cross-sectional area A_0 and the length L_0 of the member in its undeformed state.

2.5 HOOKE'S LAW; MODULUS OF ELASTICITY

Most engineering structures are designed to undergo relatively small deformations, involving only the straight-line portion of the corresponding stress-strain diagram. For that initial portion of the diagram (Fig. 2.6), the stress σ is directly proportional to the strain ϵ, and we can write

$$\sigma = E\epsilon \tag{2.4}$$

This relation is known as *Hooke's law*, after Robert Hooke (1635–1703), an English scientist and one of the early founders of applied mechanics. The coefficient E is called the *modulus of elasticity* of the material involved, or also *Young's modulus*, after the English scientist Thomas Young (1773–1829). Since the strain ϵ is a dimensionless quantity, the modulus E is expressed in the same units as the stress σ, namely in pascals or one of its multiples if SI units are used, and in psi or ksi if U.S. customary units are used.

The largest value of the stress for which Hooke's law can be used for a given material is known as the *proportional limit* of that material. In the case of ductile materials possessing a well-defined yield point, as in Fig. 2.6*a*, the proportional limit almost coincides with the yield point. For other materials, the proportional limit cannot be defined as easily, since it is difficult to determine with accuracy the value of the stress σ for which the relation between σ and ϵ ceases to be linear. But from this very difficulty we can conclude for such materials that using Hooke's law for values of the stress slightly larger than the actual proportional limit will not result in any significant error.

Some of the physical properties of structural metals, such as strength, ductility, and corrosion resistance, can be greatly affected by alloying, heat treatment, and the manufacturing process used. For example, we note from the stress-strain diagrams of pure iron and of three different grades of steel (Fig. 2.11) that large variations in the yield strength, ultimate strength, and final strain (ductility) exist among these four metals. All of them, however, possess the same modulus of elasticity; in other words, their "stiffness," or ability to resist a deformation within the linear range, is the same. Therefore, if a high-strength steel is substituted for a lower-strength steel in a given structure, and if all dimensions are kept the same, the structure will have an increased load-carrying capacity, but its stiffness will remain unchanged.

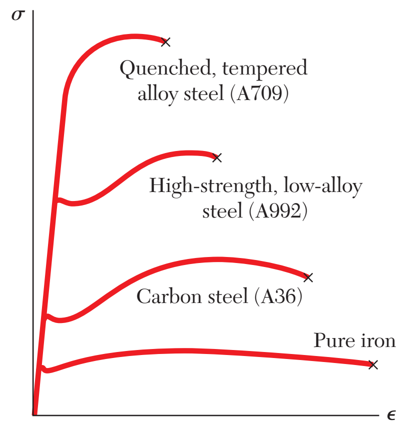


Fig. 2.11 Stress-strain diagrams for iron and different grades of steel.

For each of the materials considered so far, the relation between normal stress and normal strain, $\sigma = E\epsilon$, is independent of the direction of loading. This is because the mechanical properties of each material, including its modulus of elasticity E, are independent of the direction considered. Such materials are said to be *isotropic.* Materials whose properties depend upon the direction considered are said to be *anisotropic.*

An important class of anisotropic materials consists of *fiber-reinforced composite materials.* These composite materials are obtained by embedding fibers of a strong, stiff material into a weaker, softer material, referred to as a *matrix*. Typical materials used as fibers are graphite, glass, and polymers, while various types of resins are used as a matrix. Figure 2.12 shows a layer, or *lamina,* of a composite material consisting of a large number of parallel fibers embedded in a matrix. An axial load applied to the lamina along the x axis, that is, in a direction parallel to the fibers, will create a normal stress σ_x in the lamina and a corresponding normal strain ϵ_x which will satisfy Hooke's law as the load is increased and as long as the elastic limit of the lamina is not exceeded. Similarly, an axial load applied along the y axis, that is, in a direction perpendicular to the lamina, will create a normal stress σ_y and a normal strain ϵ_y satisfying Hooke's law, and an axial load applied along the z axis will create a normal stress σ_z and a normal strain ϵ_z which again satisfy Hooke's law. However, the moduli of elasticity E_x, E_y, and E_z corresponding, respectively, to each of the above loadings will be different. Because the fibers are parallel to the x axis, the lamina will offer a much stronger resistance to a loading directed along the x axis than to a loading directed along the y or z axis, and E_x will be much larger than either E_y or E_z.

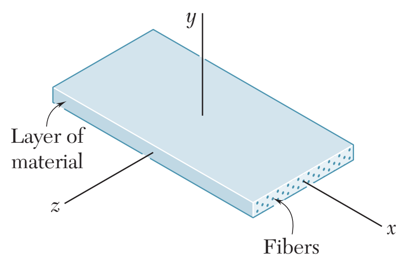


Fig. 2.12 Layer of fiber-reinforced composite material.

A flat *laminate* is obtained by superposing a number of layers or laminas. If the laminate is to be subjected only to an axial load causing tension, the fibers in all layers should have the same orientation as the load in order to obtain the greatest possible strength. But if the laminate may be in compression, the matrix material may not be sufficiently strong to prevent the fibers from kinking or buckling. The lateral stability of the laminate may then be increased by positioning some of the layers so that their fibers will be perpendicular to the load. Positioning some layers so that their fibers are oriented at 30°, 45°, or 60° to the load may also be used to increase the resistance of the laminate to in-plane shear. Fiber-reinforced composite materials will be further discussed in Sec. 2.16, where their behavior under multiaxial loadings will be considered.

2.6 ELASTIC VERSUS PLASTIC BEHAVIOR OF A MATERIAL

If the strains caused in a test specimen by the application of a given load disappear when the load is removed, the material is said to behave *elastically*. The largest value of the stress for which the material behaves elastically is called the *elastic limit* of the material.

If the material has a well-defined yield point as in Fig. 2.6*a*, the elastic limit, the proportional limit (Sec. 2.5), and the yield point are essentially equal. In other words, the material behaves elastically and linearly as long as the stress is kept below the yield point. If the yield point is reached, however, yield takes place as described in Sec. 2.3 and, when the load is removed, the stress and strain decrease in a linear fashion, along a line CD parallel to the straight-line portion AB of the loading curve (Fig. 2.13). The fact that ϵ does not return to zero after the load has been removed indicates that a *permanent set* or *plastic deformation* of the material has taken place. For most materials, the plastic deformation depends not only upon the maximum value reached by the stress, but also upon the time elapsed before the load is removed. The stress-dependent part of the plastic deformation is referred to as *slip*, and the time-dependent part—which is also influenced by the temperature—as *creep*.

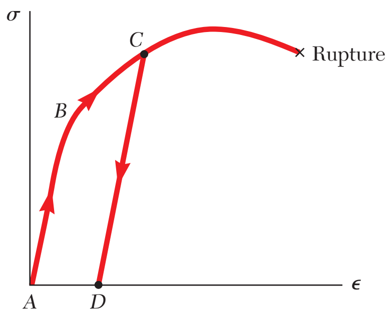


Fig. 2.13 Stress-strain characteristics of ductile material loaded beyond yield and unloaded.

When a material does not possess a well-defined yield point, the elastic limit cannot be determined with precision. However, assuming the elastic limit equal to the yield strength as defined by the offset method (Sec. 2.3) results in only a small error. Indeed, referring to Fig. 2.8, we note that the straight line used to determine point Y also represents the unloading curve after a maximum stress σ_Y has been reached. While the material does not behave truly elastically, the resulting plastic strain is as small as the selected offset.

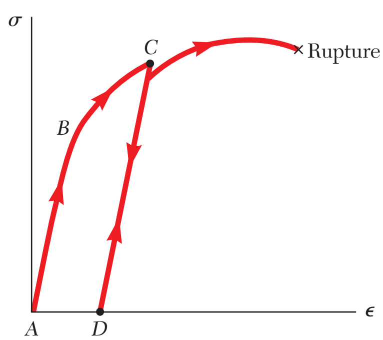


Fig. 2.14 Stress-strain characteristics of ductile material reloaded after prior yielding.

If, after being loaded and unloaded (Fig. 2.14), the test specimen is loaded again, the new loading curve will closely follow the earlier unloading curve until it almost reaches point C; it will then bend to the right and connect with the curved portion of the original stress-strain diagram. We note that the straight-line portion of the new loading curve is longer than the corresponding portion of the initial one. Thus, the proportional limit and the elastic limit have increased as a result of the strain-hardening that occurred during the earlier loading of the specimen. However, since the point of rupture R remains unchanged, the ductility of the specimen, which should now be measured from point D, has decreased.

We have assumed in our discussion that the specimen was loaded twice in the same direction, i.e., that both loads were tensile loads. Let us now consider the case when the second load is applied in a direction opposite to that of the first one. We assume that the

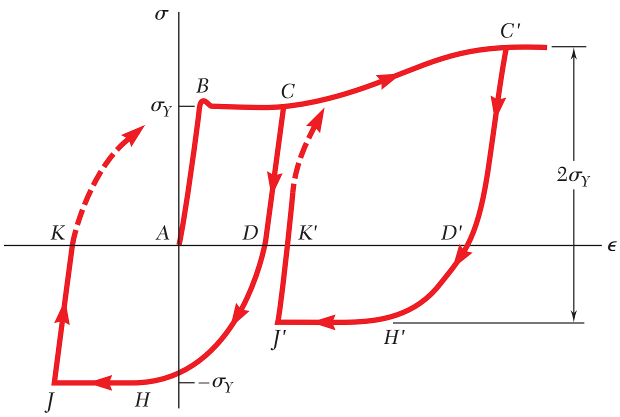


Fig. 2.15 Stress-strain characteristics for mild steel subjected to reverse loading.

material is mild steel, for which the yield strength is the same in tension and in compression. The initial load is tensile and is applied until point C has been reached on the stress-strain diagram (Fig. 2.15). After unloading (point D), a compressive load is applied, causing the material to reach point H, where the stress is equal to $-\sigma_Y$. We note that portion DH of the stress-strain diagram is curved and does not show any clearly defined yield point. This is referred to as the *Bauschinger effect.* As the compressive load is maintained, the material yields along line HJ.

If the load is removed after point J has been reached, the stress returns to zero along line JK, and we note that the slope of JK is equal to the modulus of elasticity E. The resulting permanent set AK may be positive, negative, or zero, depending upon the lengths of the segments BC and HJ. If a tensile load is applied again to the test specimen, the portion of the stress-strain diagram beginning at K (dashed line) will curve up and to the right until the yield stress σ_Y has been reached.

If the initial loading is large enough to cause strain-hardening of the material (point C'), unloading takes place along line $C'D'$. As the reverse load is applied, the stress becomes compressive, reaching its maximum value at H' and maintaining it as the material yields along line $H'J'$. We note that while the maximum value of the compressive stress is less than σ_Y, the total change in stress between C' and H' is still equal to $2\sigma_Y$.

If point K or K' coincides with the origin A of the diagram, the permanent set is equal to zero, and the specimen may appear to have returned to its original condition. However, internal changes will have taken place and, while the same loading sequence may be repeated, the specimen will rupture without any warning after relatively few repetitions. This indicates that the excessive plastic deformations to which the specimen was subjected have caused a radical change in the characteristics of the material. Reverse loadings into the plastic range, therefore, are seldom allowed, and only under carefully controlled conditions. Such situations occur in the straightening of damaged material and in the final alignment of a structure or machine.

2.7 REPEATED LOADINGS; FATIGUE

In the preceding sections we have considered the behavior of a test specimen subjected to an axial loading. We recall that, if the maximum stress in the specimen does not exceed the elastic limit of the material, the specimen returns to its initial condition when the load is removed. You might conclude that a given loading may be repeated many times, provided that the stresses remain in the elastic range. Such a conclusion is correct for loadings repeated a few dozen or even a few hundred times. However, as you will see, it is not correct when loadings are repeated thousands or millions of times. In such cases, rupture will occur at a stress much lower than the static breaking strength; this phenomenon is known as *fatigue*. A fatigue failure is of a brittle nature, even for materials that are normally ductile.

Fatigue must be considered in the design of all structural and machine components that are subjected to repeated or to fluctuating loads. The number of loading cycles that may be expected during the useful life of a component varies greatly. For example, a beam supporting an industrial crane may be loaded as many as two million times in 25 years (about 300 loadings per working day), an automobile crankshaft will be loaded about half a billion times if the automobile is driven 200,000 miles, and an individual turbine blade may be loaded several hundred billion times during its lifetime.

Some loadings are of a fluctuating nature. For example, the passage of traffic over a bridge will cause stress levels that will fluctuate about the stress level due to the weight of the bridge. A more severe condition occurs when a complete reversal of the load occurs during the loading cycle. The stresses in the axle of a railroad car, for example, are completely reversed after each half-revolution of the wheel.

The number of loading cycles required to cause the failure of a specimen through repeated successive loadings and reverse loadings may be determined experimentally for any given maximum stress level. If a series of tests is conducted, using different maximum stress levels, the resulting data may be plotted as a σ-n curve. For each test, the maximum stress σ is plotted as an ordinate and the number of cycles n as an abscissa; because of the large number of cycles required for rupture, the cycles n are plotted on a logarithmic scale.

A typical σ-n curve for steel is shown in Fig. 2.16. We note that, if the applied maximum stress is high, relatively few cycles are required to cause rupture. As the magnitude of the maximum stress is reduced, the number of cycles required to cause rupture increases, until a stress, known as the *endurance limit,* is reached. The endurance limit is the stress for which failure does not occur, even for an indefinitely large number of loading cycles. For a low-carbon steel, such as structural steel, the endurance limit is about one-half of the ultimate strength of the steel.

For nonferrous metals, such as aluminum and copper, a typical σ-n curve (Fig. 2.16) shows that the stress at failure continues to

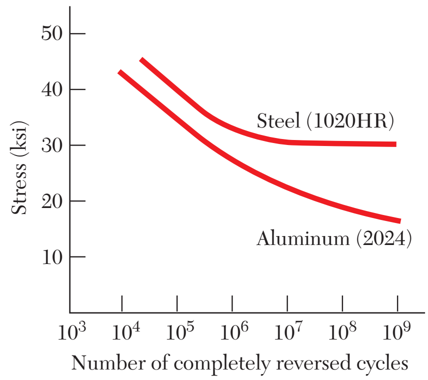


Fig. 2.16 Typical σ-n curves.

decrease as the number of loading cycles is increased. For such metals, one defines the *fatigue limit* as the stress corresponding to failure after a specified number of loading cycles, such as 500 million.

Examination of test specimens, of shafts, of springs, and of other components that have failed in fatigue shows that the failure was initiated at a microscopic crack or at some similar imperfection. At each loading, the crack was very slightly enlarged. During successive loading cycles, the crack propagated through the material until the amount of undamaged material was insufficient to carry the maximum load, and an abrupt, brittle failure occurred. Because fatigue failure may be initiated at any crack or imperfection, the surface condition of a specimen has an important effect on the value of the endurance limit obtained in testing. The endurance limit for machined and polished specimens is higher than for rolled or forged components, or for components that are corroded. In applications in or near seawater, or in other applications where corrosion is expected, a reduction of up to 50% in the endurance limit can be expected.

2.8 DEFORMATIONS OF MEMBERS UNDER AXIAL LOADING

Consider a homogeneous rod BC of length L and uniform cross section of area A subjected to a centric axial load $\mathbf{P}$ (Fig. 2.17). If the resulting axial stress $\sigma = P/A$ does not exceed the proportional limit of the material, we may apply Hooke's law and write

$$\sigma = E\epsilon \tag{2.4}$$

from which it follows that

$$\epsilon = \frac{\sigma}{E} = \frac{P}{AE} \tag{2.5}$$

Recalling that the strain ϵ was defined in Sec. 2.2 as $\epsilon = \delta/L$, we have

$$\delta = \epsilon L \tag{2.6}$$

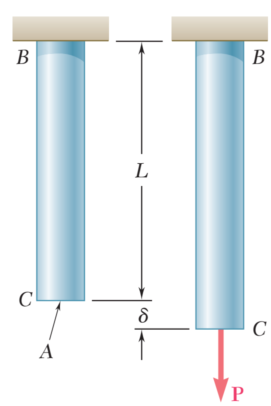


Fig. 2.17 Deformation of axially loaded rod.

and, substituting for ϵ from (2.5) into (2.6):

$$\delta = \frac{PL}{AE} \tag{2.7}$$

Equation (2.7) may be used only if the rod is homogeneous (constant E), has a uniform cross section of area A, and is loaded at its ends. If the rod is loaded at other points, or if it consists of several portions of various cross sections and possibly of different materials, we must divide it into component parts that satisfy individually the required conditions for the application of formula (2.7). Denoting, respectively, by P_i, L_i, A_i, and E_i the internal force, length, cross-sectional area, and modulus of elasticity corresponding to part i, we express the deformation of the entire rod as

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} \tag{2.8}$$

We recall from Sec. 2.2 that, in the case of a member of variable cross section (Fig. 2.18), the strain ϵ depends upon the position of the point Q where it is computed and is defined as $\epsilon = d\delta/dx$. Solving for $d\delta$ and substituting for ϵ from Eq. (2.5), we express the deformation of an element of length dx as

$$d\delta = \epsilon\, dx = \frac{P\,dx}{AE}$$

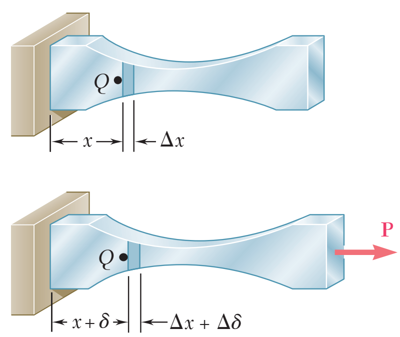


Fig. 2.18 Deformation of axially loaded member of variable cross-sectional area.

The total deformation δ of the member is obtained by integrating this expression over the length L of the member:

$$\delta = \int_0^L \frac{P\,dx}{AE} \tag{2.9}$$

Formula (2.9) should be used in place of (2.7), not only when the cross-sectional area A is a function of x, but also when the internal force P depends upon x, as is the case for a rod hanging under its own weight.

EXAMPLE 2.01

Determine the deformation of the steel rod shown in Fig. 2.19*a* under the given loads ($E = 29 \times 10^6$ psi).

We divide the rod into three component parts shown in Fig. 2.19*b* and write

$$L_1 = L_2 = 12 \text{ in.} \qquad L_3 = 16 \text{ in.}$$
$$A_1 = A_2 = 0.9 \text{ in}^2 \qquad A_3 = 0.3 \text{ in}^2$$

To find the internal forces P_1, P_2, and P_3, we must pass sections through each of the component parts, drawing each time the free-body diagram of the portion of rod located to the right of the section (Fig. 2.19*c*). Expressing that each of the free bodies is in equilibrium, we obtain successively

$$P_1 = 60 \text{ kips} = 60 \times 10^3 \text{ lb}$$
$$P_2 = -15 \text{ kips} = -15 \times 10^3 \text{ lb}$$
$$P_3 = 30 \text{ kips} = 30 \times 10^3 \text{ lb}$$

Carrying the values obtained into Eq. (2.8), we have

$$\delta = \sum_i \frac{P_iL_i}{A_iE_i} = \frac{1}{E}\left(\frac{P_1L_1}{A_1} + \frac{P_2L_2}{A_2} + \frac{P_3L_3}{A_3}\right)$$
$$= \frac{1}{29 \times 10^6}\left[\frac{(60 \times 10^3)(12)}{0.9} + \frac{(-15 \times 10^3)(12)}{0.9} + \frac{(30 \times 10^3)(16)}{0.3}\right]$$
$$\delta = \frac{2.20 \times 10^6}{29 \times 10^6} = 75.9 \times 10^{-3} \text{ in.}$$

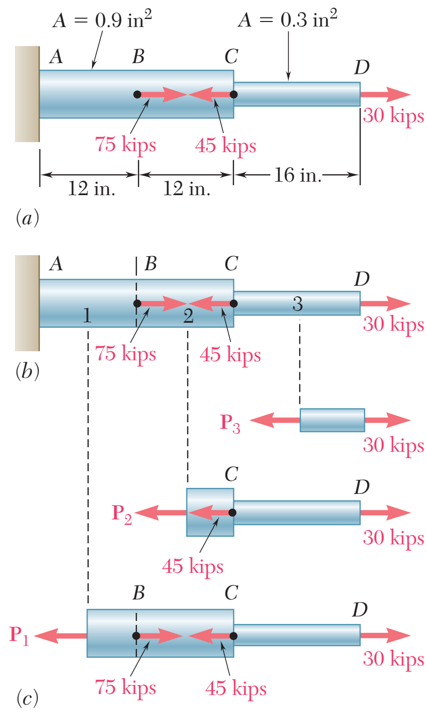


Fig. 2.19

The rod BC of Fig. 2.17, which was used to derive formula (2.7), and the rod AD of Fig. 2.19, which has just been discussed in Example 2.01, both had one end attached to a fixed support. In each case, therefore, the deformation δ of the rod was equal to the displacement of its free end. When both ends of a rod move, however, the deformation of the rod is measured by the *relative displacement* of one end of the rod with respect to the other. Consider, for instance, the assembly shown in Fig. 2.20*a*, which consists of three elastic bars of length L connected by a rigid pin at A. If a load **P** is applied at B (Fig. 2.20*b*), each of the three bars will deform. Since the bars AC and AC' are attached to fixed supports at C and C', their common deformation is measured by the displacement δ_A of point A. On the other hand, since both ends of bar AB move, the deformation of AB is measured by the difference between the displacements δ_A and δ_B of points A and B, i.e., by the relative displacement of B with respect to A. Denoting this relative displacement by $\delta_{B/A}$, we write

$$\delta_{B/A} = \delta_B - \delta_A = \frac{PL}{AE} \tag{2.10}$$

where A is the cross-sectional area of AB and E is its modulus of elasticity.

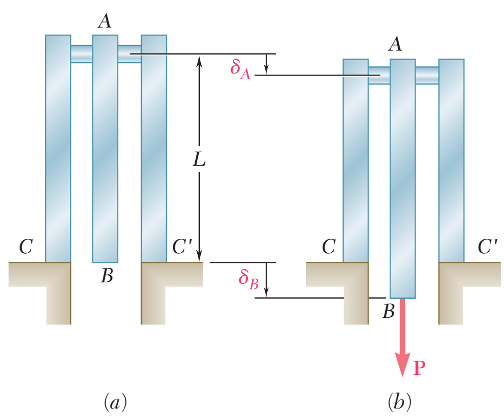
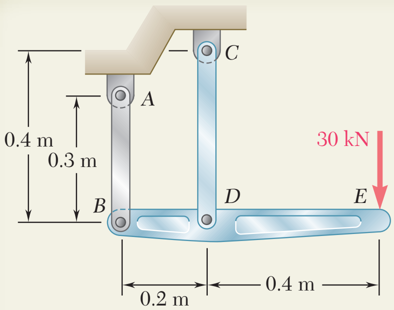
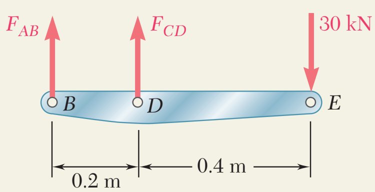


Fig. 2.20 Example of relative end displacement, as exhibited by the middle bar.



SAMPLE PROBLEM 2.1

The rigid bar *BDE* is supported by two links *AB* and *CD*. Link *AB* is made of aluminum ($E = 70$ GPa) and has a cross-sectional area of 500 mm^2; link *CD* is made of steel ($E = 200$ GPa) and has a cross-sectional area of 600 mm^2. For the 30-kN force shown, determine the deflection (*a*) of *B*, (*b*) of *D*, (*c*) of *E*.



SOLUTION

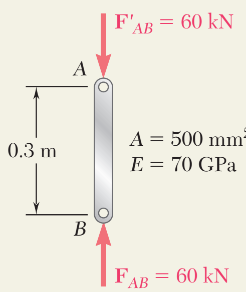
Free Body: Bar *BDE*

$$+\circlearrowleft \Sigma M_B = 0: \quad -(30 \text{ kN})(0.6 \text{ m}) + F_{CD}(0.2 \text{ m}) = 0$$

$$F_{CD} = +90 \text{ kN} \qquad F_{CD} = 90 \text{ kN} \quad \textit{tension}$$

$$+\circlearrowleft \Sigma M_D = 0: \quad -(30 \text{ kN})(0.4 \text{ m}) - F_{AB}(0.2 \text{ m}) = 0$$

$$F_{AB} = -60 \text{ kN} \qquad F_{AB} = 60 \text{ kN} \quad \textit{compression}$$

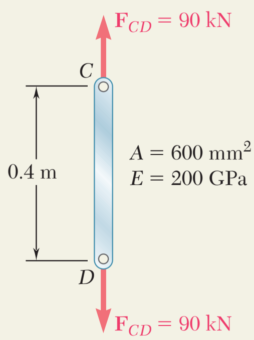


a. Deflection of *B*. Since the internal force in link *AB* is compressive, we have $P = -60$ kN

$$\delta_B = \frac{PL}{AE} = \frac{(-60 \times 10^3 \text{ N})(0.3 \text{ m})}{(500 \times 10^{-6} \text{ m}^2)(70 \times 10^9 \text{ Pa})} = -514 \times 10^{-6} \text{ m}$$

The negative sign indicates a contraction of member *AB*, and, thus, an upward deflection of end *B*:

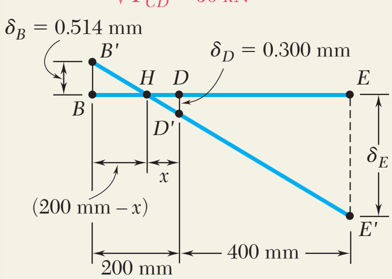
$$\delta_B = 0.514 \text{ mm} \uparrow \quad \blacktriangleleft$$



b. Deflection of *D*. Since in rod *CD*, $P = 90$ kN, we write

$$\delta_D = \frac{PL}{AE} = \frac{(90 \times 10^3 \text{ N})(0.4 \text{ m})}{(600 \times 10^{-6} \text{ m}^2)(200 \times 10^9 \text{ Pa})}$$

$$= 300 \times 10^{-6} \text{ m} \qquad \delta_D = 0.300 \text{ mm} \downarrow \quad \blacktriangleleft$$

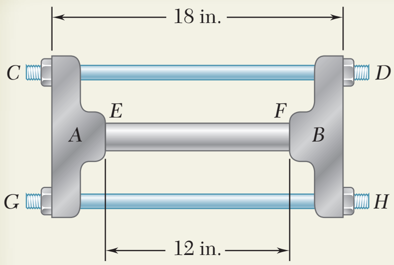


c. Deflection of *E*. We denote by *B′* and *D′* the displaced positions of points *B* and *D*. Since the bar *BDE* is rigid, points *B′*, *D′*, and *E′* lie in a straight line and we write

$$\frac{BB'}{DD'} = \frac{BH}{HD} \qquad \frac{0.514 \text{ mm}}{0.300 \text{ mm}} = \frac{(200 \text{ mm}) - x}{x} \qquad x = 73.7 \text{ mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD} \qquad \frac{\delta_E}{0.300 \text{ mm}} = \frac{(400 \text{ mm}) + (73.7 \text{ mm})}{73.7 \text{ mm}}$$

$$\delta_E = 1.928 \text{ mm} \downarrow \quad \blacktriangleleft$$



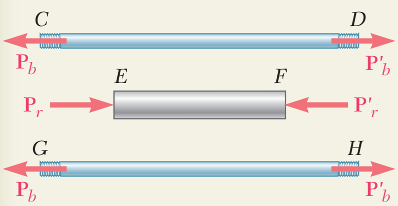
SAMPLE PROBLEM 2.2

The rigid castings A and B are connected by two $\frac{3}{4}$-in.-diameter steel bolts CD and GH and are in contact with the ends of a 1.5-in.-diameter aluminum rod EF. Each bolt is single-threaded with a pitch of 0.1 in., and after being snugly fitted, the nuts at D and H are both tightened one-quarter of a turn. Knowing that E is 29×10^6 psi for steel and 10.6×10^6 psi for aluminum, determine the normal stress in the rod.

SOLUTION

Deformations

Bolts CD and GH. Tightening the nuts causes tension in the bolts. Because of symmetry, both are subjected to the same internal force P_b and undergo the same deformation δ_b. We have



$$\delta_b = +\frac{P_b L_b}{A_b E_b} = +\frac{P_b(18\text{ in.})}{\frac{1}{4}\pi(0.75\text{ in.})^2(29 \times 10^6\text{ psi})} = +1.405 \times 10^{-6}\,P_b \tag{1}$$

Rod EF. The rod is in compression. Denoting by P_r the magnitude of the force in the rod and by δ_r the deformation of the rod, we write

$$\delta_r = -\frac{P_r L_r}{A_r E_r} = -\frac{P_r(12\text{ in.})}{\frac{1}{4}\pi(1.5\text{ in.})^2(10.6 \times 10^6\text{ psi})} = -0.6406 \times 10^{-6}\,P_r \tag{2}$$

Displacement of D Relative to B. Tightening the nuts one-quarter of a turn causes ends D and H of the bolts to undergo a displacement of $\frac{1}{4}(0.1\text{ in.})$ *relative to casting* B. Considering end D, we write

$$\delta_{D/B} = \tfrac{1}{4}(0.1\text{ in.}) = 0.025\text{ in.} \tag{3}$$

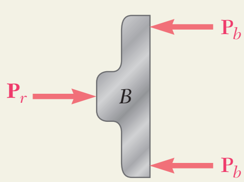
But $\delta_{D/B} = \delta_D - \delta_B$, where δ_D and δ_B represent the displacements of D and B. If we assume that casting A is held in a fixed position while the nuts at D and H are being tightened, these displacements are equal to the deformations of the bolts and of the rod, respectively. We have, therefore,

$$\delta_{D/B} = \delta_b - \delta_r \tag{4}$$

Substituting from (1), (2), and (3) into (4), we obtain

$$0.025\text{ in.} = 1.405 \times 10^{-6}\,P_b + 0.6406 \times 10^{-6}\,P_r \tag{5}$$

Free Body: Casting *B*



$$\overset{+}{\rightarrow}\Sigma F = 0: \qquad P_r - 2P_b = 0 \qquad P_r = 2P_b \tag{6}$$

Forces in Bolts and Rod Substituting for P_r from (6) into (5), we have

$$\begin{aligned} 0.025\text{ in.} &= 1.405 \times 10^{-6}\,P_b + 0.6406 \times 10^{-6}(2P_b) \\ P_b &= 9.307 \times 10^3\text{ lb} = 9.307\text{ kips} \\ P_r &= 2P_b = 2(9.307\text{ kips}) = 18.61\text{ kips} \end{aligned}$$

Stress in Rod

$$\sigma_r = \frac{P_r}{A_r} = \frac{18.61\text{ kips}}{\frac{1}{4}\pi(1.5\text{ in.})^2} \qquad \sigma_r = 10.53\text{ ksi} \blacktriangleleft$$

PROBLEMS

2.1 An 80-m-long wire of 5-mm diameter is made of a steel with $E = 200$ GPa and an ultimate tensile strength of 400 MPa. If a factor of safety of 3.2 is desired, determine (*a*) the largest allowable tension in the wire, (*b*) the corresponding elongation of the wire.

2.2 A steel control rod is 5.5 ft long and must not stretch more than 0.04 in. when a 2-kip tensile load is applied to it. Knowing that $E = 29 \times 10^6$ psi, determine (*a*) the smallest diameter rod that should be used, (*b*) the corresponding normal stress caused by the load.

2.3 Two gage marks are placed exactly 10 in. apart on a $\frac{1}{2}$-in.-diameter aluminum rod with $E = 10.1 \times 10^6$ psi and an ultimate strength of 16 ksi. Knowing that the distance between the gage marks is 10.009 in. after a load is applied, determine (*a*) the stress in the rod, (*b*) the factor of safety.

2.4 An 18-m-long steel wire of 5-mm diameter is to be used in the manufacture of a prestressed concrete beam. It is observed that the wire stretches 45 mm when a tensile force **P** is applied. Knowing that $E = 200$ GPa, determine (*a*) the magnitude of the force **P,** (*b*) the corresponding normal stress in the wire.

2.5 A polystyrene rod of length 12 in. and diameter 0.5 in. is subjected to an 800-lb tensile load. Knowing that $E = 0.45 \times 10^6$ psi, determine (*a*) the elongation of the rod, (*b*) the normal stress in the rod.

2.6 A nylon thread is subjected to a 8.5-N tension force. Knowing that $E = 3.3$ GPa and that the length of the thread increases by 1.1%, determine (*a*) the diameter of the thread, (*b*) the stress in the thread.

2.7 Two gage marks are placed exactly 250 mm apart on a 12-mm-diameter aluminum rod. Knowing that, with an axial load of 6000 N acting on the rod, the distance between the gage marks is 250.18 mm, determine the modulus of elasticity of the aluminum used in the rod.

2.8 An aluminum pipe must not stretch more than 0.05 in. when it is subjected to a tensile load. Knowing that $E = 10.1 \times 10^6$ psi and that the maximum allowable normal stress is 14 ksi, determine (*a*) the maximum allowable length of the pipe, (*b*) the required area of the pipe if the tensile load is 127.5 kips.

2.9 An aluminum control rod must stretch 0.08 in. when a 500-lb tensile load is applied to it. Knowing that $\sigma_{all} = 22$ ksi and $E = 10.1 \times 10^6$ psi, determine the smallest diameter and shortest length that can be selected for the rod.

2.10 A square yellow-brass bar must not stretch more than 2.5 mm when it is subjected to a tensile load. Knowing that $E = 105$ GPa and that the allowable tensile strength is 180 MPa, determine (*a*) the maximum allowable length of the bar, (*b*) the required dimensions of the cross section if the tensile load is 40 kN.

2.11 A 4-m-long steel rod must not stretch more than 3 mm and the normal stress must not exceed 150 MPa when the rod is subjected to a 10-kN axial load. Knowing that $E = 200$ GPa, determine the required diameter of the rod.

2.12 A nylon thread is to be subjected to a 10-N tension. Knowing that $E = 3.2$ GPa, that the maximum allowable normal stress is 40 MPa, and that the length of the thread must not increase by more than 1%, determine the required diameter of the thread.

2.13 The 4-mm-diameter cable BC is made of a steel with $E = 200$ GPa. Knowing that the maximum stress in the cable must not exceed 190 MPa and that the elongation of the cable must not exceed 6 mm, find the maximum load **P** that can be applied as shown.

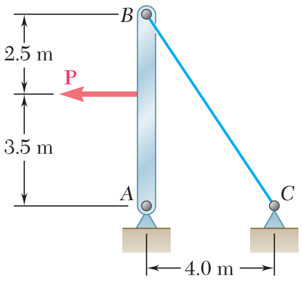


Fig. P2.13

2.14 The aluminum rod ABC ($E = 10.1 \times 10^6$ psi), which consists of two cylindrical portions AB and BC, is to be replaced with a cylindrical steel rod DE ($E = 29 \times 10^6$ psi) of the same overall length. Determine the minimum required diameter d of the steel rod if its vertical deformation is not to exceed the deformation of the aluminum rod under the same load and if the allowable stress in the steel rod is not to exceed 24 ksi.

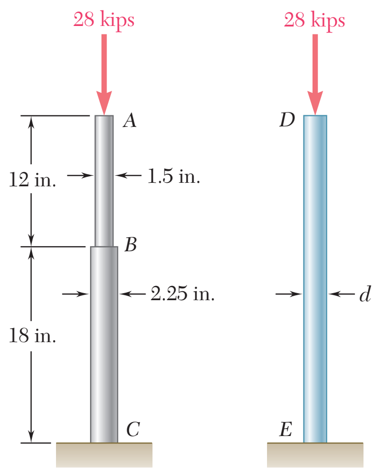


Fig. P2.14

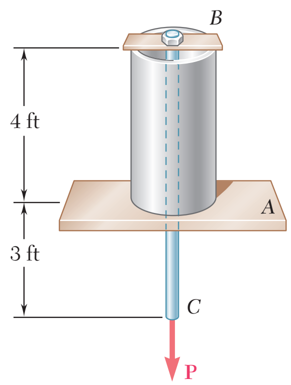


Fig. P2.15

2.15 A 4-ft section of aluminum pipe of cross-sectional area 1.75 in^2 rests on a fixed support at A. The $\frac{5}{8}$-in.-diameter steel rod BC hangs from a rigid bar that rests on the top of the pipe at B. Knowing that the modulus of elasticity is 29×10^6 psi for steel and 10.4×10^6 psi for aluminum, determine the deflection of point C when a 15-kip force is applied at C.

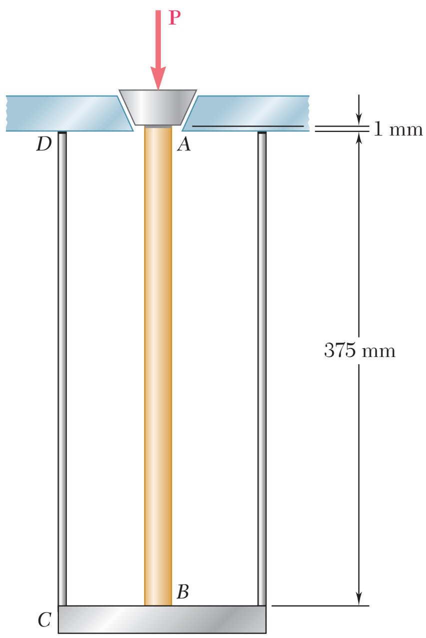


Fig. ***P2.16***

2.16 The brass tube *AB* ($E = 105$ GPa) has a cross-sectional area of 140 mm^2 and is fitted with a plug at *A*. The tube is attached at *B* to a rigid plate that is itself attached at *C* to the bottom of an aluminum cylinder ($E = 72$ GPa) with a cross-sectional area of 250 mm^2. The cylinder is then hung from a support at *D*. In order to close the cylinder, the plug must move down through 1 mm. Determine the force **P** that must be applied to the cylinder.

2.17 A 250-mm-long aluminum tube ($E = 70$ GPa) of 36-mm outer diameter and 28-mm inner diameter can be closed at both ends by means of single-threaded screw-on covers of 1.5-mm pitch. With one cover screwed on tight, a solid brass rod ($E = 105$ GPa) of 25-mm diameter is placed inside the tube and the second cover is screwed on. Since the rod is slightly longer than the tube, it is observed that the cover must be forced against the rod by rotating it one-quarter of a turn before it can be tightly closed. Determine (*a*) the average normal stress in the tube and in the rod, (*b*) the deformations of the tube and of the rod.

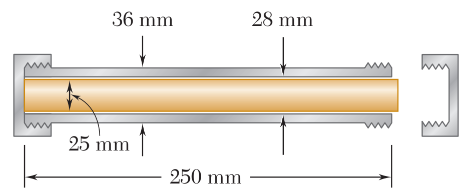


Fig. ***P2.17***

2.18 The specimen shown is made from a 1-in.-diameter cylindrical steel rod with two 1.5-in.-outer-diameter sleeves bonded to the rod as shown. Knowing that $E = 29 \times 10^6$ psi, determine (*a*) the load **P** so that the total deformation is 0.002 in., (*b*) the corresponding deformation of the central portion *BC*.

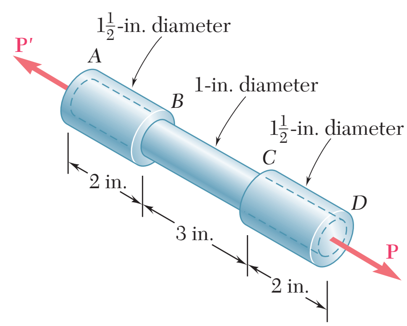


Fig. P2.18

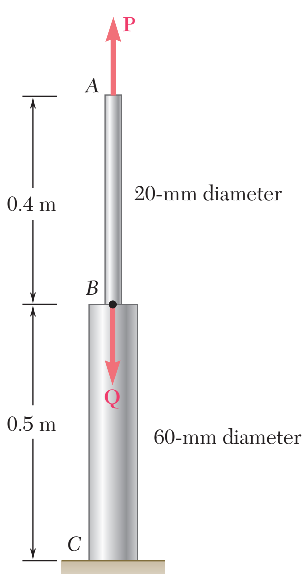


Fig. P2.19 and P2.20

2.19 Both portions of the rod *ABC* are made of an aluminum for which $E = 70$ GPa. Knowing that the magnitude of **P** is 4 kN, determine (*a*) the value of **Q** so that the deflection at *A* is zero, (*b*) the corresponding deflection of *B*.

2.20 The rod *ABC* is made of an aluminum for which $E = 70$ GPa. Knowing that $P = 6$ kN and $Q = 42$ kN, determine the deflection of (*a*) point *A*, (*b*) point *B*.

2.21 Members AB and BC are made of steel ($E = 29 \times 10^6$ psi) with cross-sectional areas of 0.80 in^2 and 0.64 in^2, respectively. For the loading shown, determine the elongation of (*a*) member AB, (*b*) member BC.

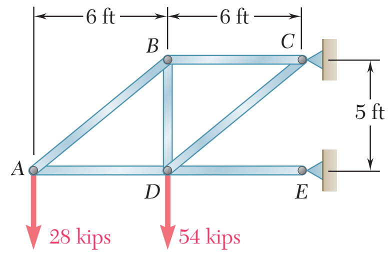


Fig. P2.21

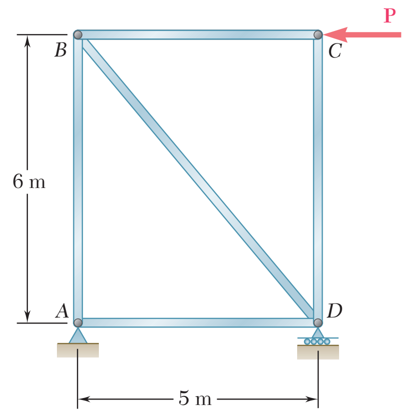


Fig. P2.22

2.22 The steel frame ($E = 200$ GPa) shown has a diagonal brace BD with an area of 1920 mm^2. Determine the largest allowable load **P** if the change in length of member BD is not to exceed 1.6 mm.

2.23 For the steel truss ($E = 200$ GPa) and loading shown, determine the deformations of members AB and AD, knowing that their cross-sectional areas are 2400 mm^2 and 1800 mm^2, respectively.

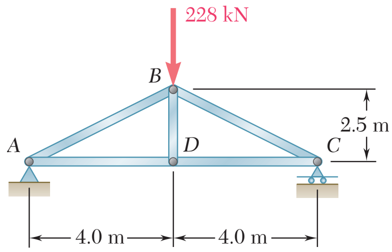


Fig. P2.23

2.24 For the steel truss ($E = 29 \times 10^6$ psi) and loading shown, determine the deformations of members BD and DE, knowing that their cross-sectional areas are 2 in^2 and 3 in^2, respectively.

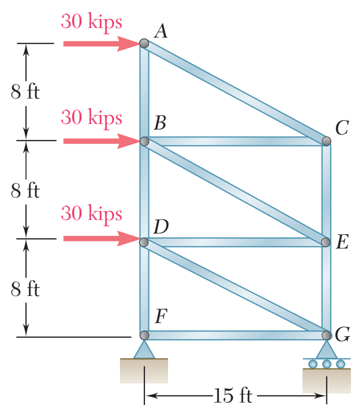


Fig. *P2.24*

2.25 Each of the links AB and CD is made of aluminum ($E = 10.9 \times 10^6$ psi) and has a cross-sectional area of 0.2 in^2. Knowing that they support the rigid member BC, determine the deflection of point E.

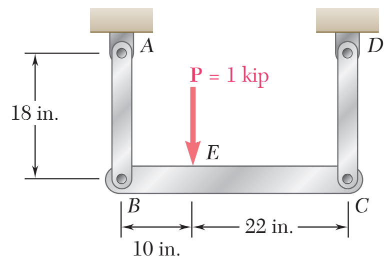


Fig. P2.25

2.26 The length of the $\frac{3}{32}$-in.-diameter steel wire CD has been adjusted so that with no load applied, a gap of $\frac{1}{16}$ in. exists between the end B of the rigid beam ACB and a contact point E. Knowing that $E = 29 \times 10^6$ psi, determine where a 50-lb block should be placed on the beam in order to cause contact between B and E.

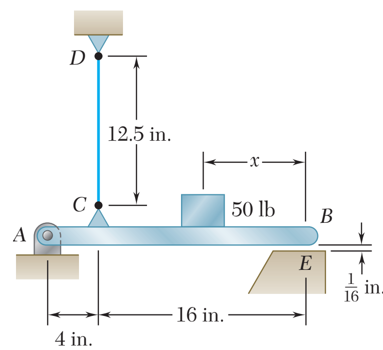


Fig. *P2.26*

2.27 Link BD is made of brass ($E = 105$ GPa) and has a cross-sectional area of 240 mm^2. Link CE is made of aluminum ($E = 72$ GPa) and has a cross-sectional area of 300 mm^2. Knowing that they support rigid member ABC, determine the maximum force **P** that can be applied vertically at point A if the deflection of A is not to exceed 0.35 mm.

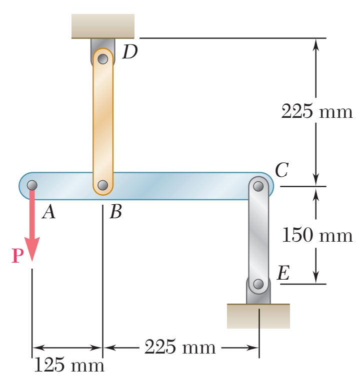


Fig. P2.27

2.28 Each of the four vertical links connecting the two rigid horizontal members is made of aluminum ($E = 70$ GPa) and has a uniform rectangular cross section of 10×40 mm. For the loading shown, determine the deflection of (*a*) point *E*, (*b*) point *F*, (*c*) point *G*.

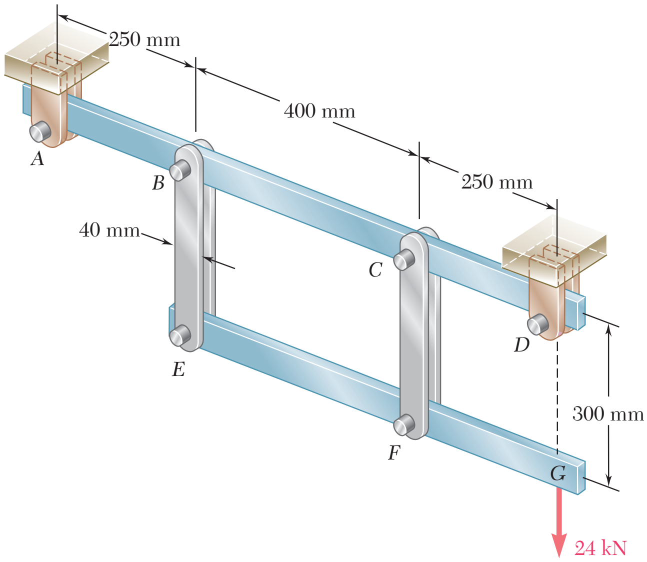


Fig. P2.28

2.29 The vertical load **P** is applied at the center *A* of the upper section of a homogeneous frustum of a circular cone of height *h*, minimum radius *a*, and maximum radius *b*. Denoting by *E* the modulus of elasticity of the material and neglecting the effect of its weight, determine the deflection of point *A*.

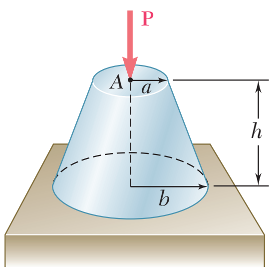


Fig. P2.29

2.30 A homogenous cable of length *L* and uniform cross section is suspended from one end. (*a*) Denoting by ρ the density (mass per unit volume) of the cable and by *E* its modulus of elasticity, determine the elongation of the cable due to its own weight. (*b*) Show that the same elongation would be obtained if the cable were horizontal and if a force equal to half of its weight were applied at each end.

2.31 The volume of a tensile specimen is essentially constant while plastic deformation occurs. If the initial diameter of the specimen is d_1, show that when the diameter is d, the true strain is $\epsilon_t = 2 \ln(d_1/d)$.

2.32 Denoting by ϵ the "engineering strain" in a tensile specimen, show that the true strain is $\epsilon_t = \ln(1 + \epsilon)$.

2.9 STATICALLY INDETERMINATE PROBLEMS

In the problems considered in the preceding section, we could always use free-body diagrams and equilibrium equations to determine the internal forces produced in the various portions of a member under given loading conditions. The values obtained for the internal forces were then entered into Eq. (2.8) or (2.9) to obtain the deformation of the member.

There are many problems, however, in which the internal forces cannot be determined from statics alone. In fact, in most of these problems the reactions themselves—which are external forces—cannot be determined by simply drawing a free-body diagram of the member and writing the corresponding equilibrium equations. The equilibrium equations must be complemented by relations involving deformations obtained by considering the geometry of the problem. Because statics is not sufficient to determine either the reactions or the internal forces, problems of this type are said to be *statically indeterminate*. The following examples will show how to handle this type of problem.

EXAMPLE 2.02

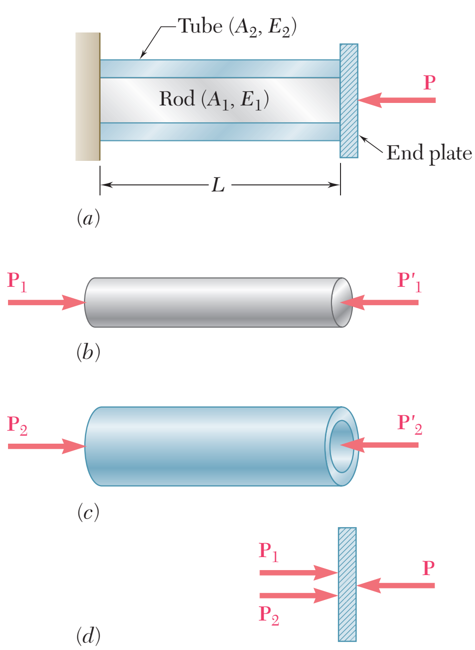


Fig. 2.21

A rod of length L, cross-sectional area A_1, and modulus of elasticity E_1, has been placed inside a tube of the same length L, but of cross-sectional area A_2 and modulus of elasticity E_2 (Fig. 2.21*a*). What is the deformation of the rod and tube when a force **P** is exerted on a rigid end plate as shown?

Denoting by P_1 and P_2, respectively, the axial forces in the rod and in the tube, we draw free-body diagrams of all three elements (Fig. 2.21*b*, *c*, *d*). Only the last of the diagrams yields any significant information, namely:

$$P_1 + P_2 = P \tag{2.11}$$

Clearly, one equation is not sufficient to determine the two unknown internal forces P_1 and P_2. The problem is statically indeterminate.

However, the geometry of the problem shows that the deformations δ_1 and δ_2 of the rod and tube must be equal. Recalling Eq. (2.7), we write

$$\delta_1 = \frac{P_1 L}{A_1 E_1} \qquad \delta_2 = \frac{P_2 L}{A_2 E_2} \tag{2.12}$$

Equating the deformations δ_1 and δ_2, we obtain:

$$\frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2} \tag{2.13}$$

Equations (2.11) and (2.13) can be solved simultaneously for P_1 and P_2:

$$P_1 = \frac{A_1 E_1 P}{A_1 E_1 + A_2 E_2} \qquad P_2 = \frac{A_2 E_2 P}{A_1 E_1 + A_2 E_2}$$

Either of Eqs. (2.12) can then be used to determine the common deformation of the rod and tube.

EXAMPLE 2.03

A bar AB of length L and uniform cross section is attached to rigid supports at A and B before being loaded. What are the stresses in portions AC and BC due to the application of a load P at point C (Fig. 2.22*a*)?

Drawing the free-body diagram of the bar (Fig. 2.22*b*), we obtain the equilibrium equation

$$R_A + R_B = P \tag{2.14}$$

Since this equation is not sufficient to determine the two unknown reactions R_A and R_B, the problem is statically indeterminate.

However, the reactions may be determined if we observe from the geometry that the total elongation δ of the bar must be zero. Denoting by δ_1 and δ_2, respectively, the elongations of the portions AC and BC, we write

$$\delta = \delta_1 + \delta_2 = 0$$

or, expressing δ_1 and δ_2 in terms of the corresponding internal forces P_1 and P_2:

$$\delta = \frac{P_1 L_1}{AE} + \frac{P_2 L_2}{AE} = 0 \tag{2.15}$$

But we note from the free-body diagrams shown respectively in parts *b* and *c* of Fig. 2.23 that $P_1 = R_A$ and $P_2 = -R_B$. Carrying these values into (2.15), we write

$$R_A L_1 - R_B L_2 = 0 \tag{2.16}$$

Equations (2.14) and (2.16) can be solved simultaneously for R_A and R_B; we obtain $R_A = PL_2/L$ and $R_B = PL_1/L$. The desired stresses σ_1 in AC and σ_2 in BC are obtained by dividing, respectively, $P_1 = R_A$ and $P_2 = -R_B$ by the cross-sectional area of the bar:

$$\sigma_1 = \frac{PL_2}{AL} \qquad \sigma_2 = -\frac{PL_1}{AL}$$

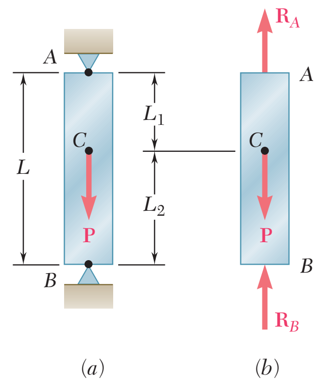


Fig. 2.22

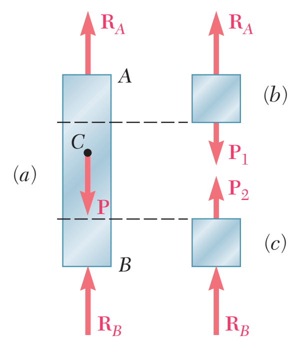


Fig. 2.23

Superposition Method. We observe that a structure is statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium. This results in more unknown reactions than available equilibrium equations. It is often found convenient to designate one of the reactions as *redundant* and to eliminate the corresponding support. Since the stated conditions of the problem cannot be arbitrarily changed, the redundant reaction must be maintained in the solution. But it will be treated as an *unknown load* that, together with the other loads, must produce deformations that are compatible with the original constraints. The actual solution of the problem is carried out by considering separately the deformations caused by the given loads and by the redundant reaction, and by adding—or *superposing*—the results obtained.†

†The general conditions under which the combined effect of several loads can be obtained in this way are discussed in Sec. 2.12.

EXAMPLE 2.04

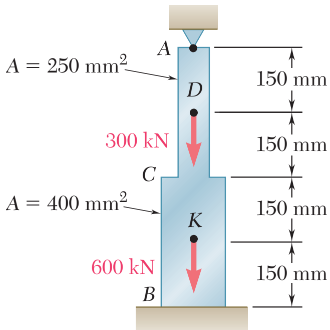


Fig. 2.24

Determine the reactions at A and B for the steel bar and loading shown in Fig. 2.24, assuming a close fit at both supports before the loads are applied.

We consider the reaction at B as redundant and release the bar from that support. The reaction $\mathbf{R}_B$ is now considered as an unknown load (Fig. 2.25*a*) and will be determined from the condition that the deformation δ of the rod must be equal to zero. The solution is carried out by considering separately the deformation δ_L caused by the given loads (Fig. 2.25*b*) and the deformation δ_R due to the redundant reaction $\mathbf{R}_B$ (Fig. 2.25*c*).

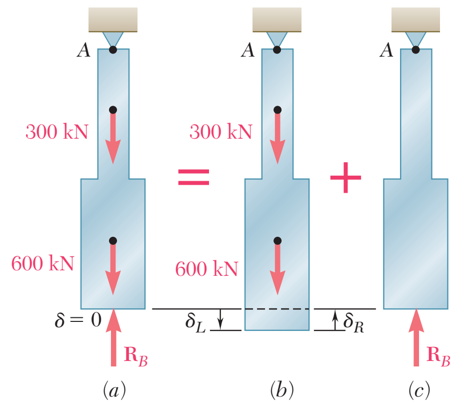


Fig. 2.25

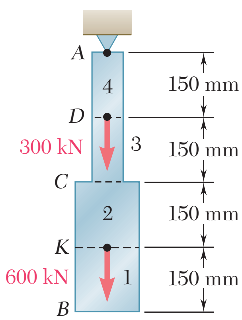


Fig. 2.26

The deformation δ_L is obtained from Eq. (2.8) after the bar has been divided into four portions, as shown in Fig. 2.26. Following the same procedure as in Example 2.01, we write

$$P_1 = 0 \qquad P_2 = P_3 = 600 \times 10^3 \text{ N} \qquad P_4 = 900 \times 10^3 \text{ N}$$
$$A_1 = A_2 = 400 \times 10^{-6} \text{ m}^2 \qquad A_3 = A_4 = 250 \times 10^{-6} \text{ m}^2$$
$$L_1 = L_2 = L_3 = L_4 = 0.150 \text{ m}$$

Substituting these values into Eq. (2.8), we obtain

$$\delta_L = \sum_{i=1}^{4} \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3 \text{ N}}{400 \times 10^{-6} \text{ m}^2} + \frac{600 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2} + \frac{900 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2}\right) \frac{0.150 \text{ m}}{E}$$

$$\delta_L = \frac{1.125 \times 10^9}{E} \tag{2.17}$$

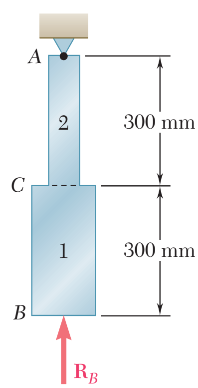


Fig. 2.27

Considering now the deformation δ_R due to the redundant reaction R_B, we divide the bar into two portions, as shown in Fig. 2.27, and write

$$P_1 = P_2 = -R_B$$
$$A_1 = 400 \times 10^{-6} \text{ m}^2 \qquad A_2 = 250 \times 10^{-6} \text{ m}^2$$
$$L_1 = L_2 = 0.300 \text{ m}$$

Substituting these values into Eq. (2.8), we obtain

$$\delta_R = \frac{P_1L_1}{A_1E} + \frac{P_2L_2}{A_2E} = -\frac{(1.95 \times 10^3)R_B}{E} \tag{2.18}$$

Expressing that the total deformation δ of the bar must be zero, we write

$$\delta = \delta_L + \delta_R = 0 \tag{2.19}$$

and, substituting for δ_L and δ_R from (2.17) and (2.18) into (2.19),

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3)R_B}{E} = 0$$

Solving for R_B, we have

$$R_B = 577 \times 10^3 \text{ N} = 577 \text{ kN}$$

The reaction R_A at the upper support is obtained from the free-body diagram of the bar (Fig. 2.28). We write

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300 \text{ kN} - 600 \text{ kN} + R_B = 0$$
$$R_A = 900 \text{ kN} - R_B = 900 \text{ kN} - 577 \text{ kN} = 323 \text{ kN}$$

Once the reactions have been determined, the stresses and strains in the bar can easily be obtained. It should be noted that, while the total deformation of the bar is zero, each of its component parts *does deform* under the given loading and restraining conditions.

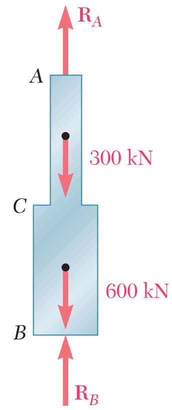


Fig. 2.28

EXAMPLE 2.05

Determine the reactions at A and B for the steel bar and loading of Example 2.04, assuming now that a 4.50-mm clearance exists between the bar and the ground before the loads are applied (Fig. 2.29). Assume E = 200 GPa.

We follow the same procedure as in Example 2.04. Considering the reaction at B as redundant, we compute the deformations δ_L and δ_R caused, respectively, by the given loads and by the redundant reaction $\mathbf{R}_B$. However, in this case the total deformation is not zero, but δ = 4.5 mm. We write therefore

$$\delta = \delta_L + \delta_R = 4.5 \times 10^{-3} \text{ m} \tag{2.20}$$

Substituting for δ_L and δ_R from (2.17) and (2.18) into (2.20), and recalling that $E = 200 \text{ GPa} = 200 \times 10^9$ Pa, we have

$$\delta = \frac{1.125 \times 10^9}{200 \times 10^9} - \frac{(1.95 \times 10^3)R_B}{200 \times 10^9} = 4.5 \times 10^{-3} \text{ m}$$

Solving for R_B, we obtain

$$R_B = 115.4 \times 10^3 \text{ N} = 115.4 \text{ kN}$$

The reaction at A is obtained from the free-body diagram of the bar (Fig. 2.28):

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300 \text{ kN} - 600 \text{ kN} + R_B = 0$$
$$R_A = 900 \text{ kN} - R_B = 900 \text{ kN} - 115.4 \text{ kN} = 785 \text{ kN}$$

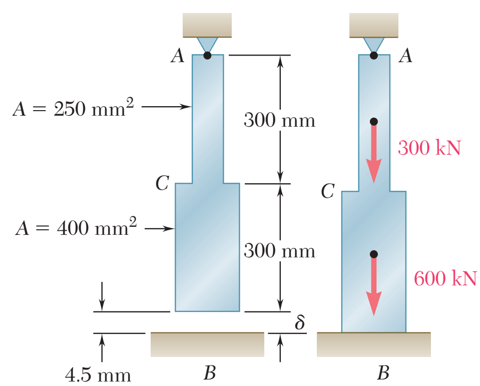


Fig. 2.29

2.10 PROBLEMS INVOLVING TEMPERATURE CHANGES

All of the members and structures that we have considered so far were assumed to remain at the same temperature while they were being loaded. We are now going to consider various situations involving changes in temperature.

Let us first consider a homogeneous rod AB of uniform cross section, which rests freely on a smooth horizontal surface (Fig. 2.30*a*). If the temperature of the rod is raised by ΔT, we observe that the rod elongates by an amount δ_T which is proportional to both the temperature change ΔT and the length L of the rod (Fig. 2.30*b*). We have

$$\delta_T = \alpha(\Delta T)L \tag{2.21}$$

where α is a constant characteristic of the material, called the *coefficient of thermal expansion*. Since δ_T and L are both expressed in units of length, α represents a quantity *per degree C*, or *per degree F*, depending whether the temperature change is expressed in degrees Celsius or in degrees Fahrenheit.

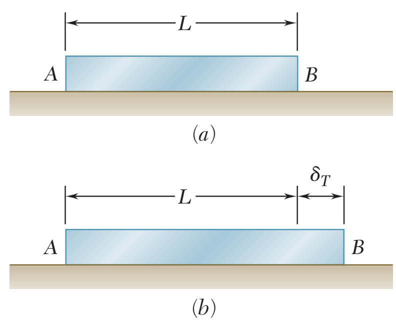


Fig. 2.30 Elongation of rod due to temperature increase.

With the deformation δ_T must be associated a strain $\epsilon_T = \delta_T/L$. Recalling Eq. (2.21), we conclude that

$$\epsilon_T = \alpha\Delta T \tag{2.22}$$

The strain ϵ_T is referred to as a *thermal strain*, since it is caused by the change in temperature of the rod. In the case we are considering here, there is *no stress associated with the strain* ϵ_T.

Let us now assume that the same rod AB of length L is placed between two fixed supports at a distance L from each other (Fig. 2.31*a*). Again, there is neither stress nor strain in this initial condition. If we raise the temperature by ΔT, the rod cannot elongate because of the restraints imposed on its ends; the elongation δ_T of the rod is thus zero. Since the rod is homogeneous and of uniform cross section, the strain ϵ_T at any point is $\epsilon_T = \delta_T/L$ and, thus, also zero. However, the supports will exert equal and opposite forces $\mathbf{P}$ and $\mathbf{P}'$ on the rod after the temperature has been raised, to keep it

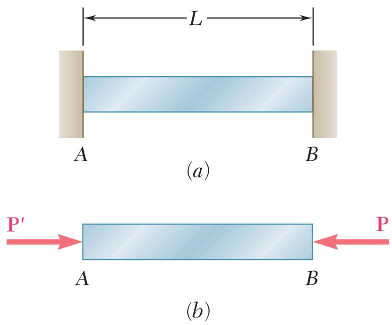


Fig. 2.31 Rod with ends restrained against thermal expansion.

from elongating (Fig. 2.31*b*). It thus follows that a state of stress (with no corresponding strain) is created in the rod.

As we prepare to determine the stress σ created by the temperature change ΔT, we observe that the problem we have to solve is statically indeterminate. Therefore, we should first compute the magnitude P of the reactions at the supports from the condition that the elongation of the rod is zero. Using the superposition method described in Sec. 2.9, we detach the rod from its support B (Fig. 2.32*a*) and let it elongate freely as it undergoes the temperature change ΔT (Fig. 2.32*b*). According to formula (2.21), the corresponding elongation is

$$\delta_T = \alpha(\Delta T)L$$

Applying now to end B the force **P** representing the redundant reaction, and recalling formula (2.7), we obtain a second deformation (Fig. 2.32*c*)

$$\delta_P = \frac{PL}{AE}$$

Expressing that the total deformation δ must be zero, we have

$$\delta = \delta_T + \delta_P = \alpha(\Delta T)L + \frac{PL}{AE} = 0$$

from which we conclude that

$$P = -AE\alpha(\Delta T)$$

and that the stress in the rod due to the temperature change ΔT is

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T) \qquad (2.23)$$

It should be kept in mind that the result we have obtained here and our earlier remark regarding the absence of any strain in the rod *apply only in the case of a homogeneous rod of uniform cross section*. Any other problem involving a restrained structure undergoing a change in temperature must be analyzed on its own merits. However, the same general approach can be used, i.e., we can consider separately the deformation due to the temperature change and the deformation due to the redundant reaction and superpose the solutions obtained.

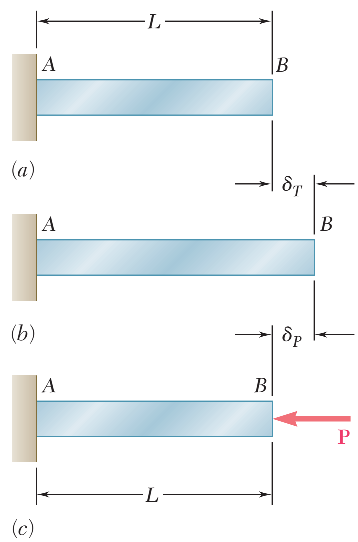


Fig. 2.32 Superposition method applied to rod restrained against thermal expansion.

EXAMPLE 2.06

Determine the values of the stress in portions AC and CB of the steel bar shown (Fig. 2.33) when the temperature of the bar is $-50°F$, knowing that a close fit exists at both of the rigid supports when the temperature is $+75°F$. Use the values $E = 29 \times 10^6$ psi and $\alpha = 6.5 \times 10^6/°F$ for steel.

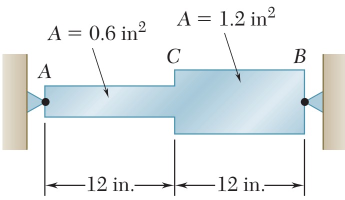


Fig. 2.33

We first determine the reactions at the supports. Since the problem is statically indeterminate, we detach the bar from its support at B and let it undergo the temperature change

$$\Delta T = (-50°F) - (75°F) = -125°F$$

The corresponding deformation (Fig. 2.34*b*) is

$$\begin{aligned}\delta_T &= \alpha(\Delta T)L = (6.5 \times 10^{-6}/°F)(-125°F)(24 \text{ in.})\\ &= -19.50 \times 10^{-3} \text{ in.}\end{aligned}$$

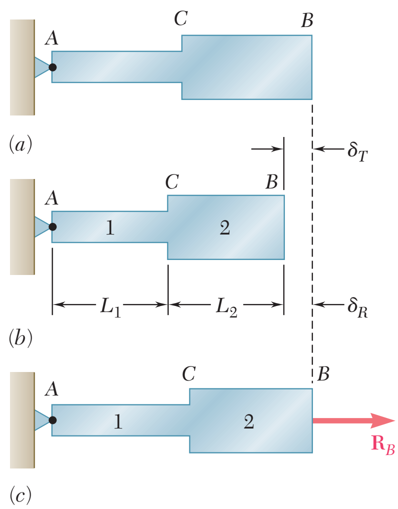


Fig. 2.34

Applying now the unknown force $\mathbf{R}_B$ at end B (Fig. 2.34*c*), we use Eq. (2.8) to express the corresponding deformation δ_R. Substituting

$$L_1 = L_2 = 12 \text{ in.}$$
$$A_1 = 0.6 \text{ in}^2 \qquad A_2 = 1.2 \text{ in}^2$$
$$P_1 = P_2 = R_B \qquad E = 29 \times 10^6 \text{ psi}$$

into Eq. (2.8), we write

$$\begin{aligned}\delta_R &= \frac{P_1L_1}{A_1E} + \frac{P_2L_2}{A_2E}\\ &= \frac{R_B}{29 \times 10^6 \text{ psi}}\left(\frac{12 \text{ in.}}{0.6 \text{ in}^2} + \frac{12 \text{ in.}}{1.2 \text{ in}^2}\right)\\ &= (1.0345 \times 10^{-6} \text{ in./lb})R_B\end{aligned}$$

Expressing that the total deformation of the bar must be zero as a result of the imposed constraints, we write

$$\begin{aligned}\delta &= \delta_T + \delta_R = 0\\ &= -19.50 \times 10^{-3} \text{ in.} + (1.0345 \times 10^{-6} \text{ in./lb})R_B = 0\end{aligned}$$

from which we obtain

$$R_B = 18.85 \times 10^3 \text{ lb} = 18.85 \text{ kips}$$

The reaction at A is equal and opposite.

Noting that the forces in the two portions of the bar are $P_1 = P_2 = 18.85$ kips, we obtain the following values of the stress in portions AC and CB of the bar:

$$\sigma_1 = \frac{P_1}{A_1} = \frac{18.85 \text{ kips}}{0.6 \text{ in}^2} = +31.42 \text{ ksi}$$

$$\sigma_2 = \frac{P_2}{A_2} = \frac{18.85 \text{ kips}}{1.2 \text{ in}^2} = +15.71 \text{ ksi}$$

We cannot emphasize too strongly the fact that, while the *total deformation* of the bar must be zero, the deformations of the portions AC and CB *are not zero*. A solution of the problem based on the assumption that these deformations are zero would therefore be wrong. Neither can the values of the strain in AC or CB be assumed equal to zero. To amplify this point, let us determine the strain ϵ_{AC} in portion AC of the bar. The strain ϵ_{AC} can be divided into two component parts; one is the thermal strain ϵ_T produced in the unrestrained bar by the temperature change ΔT (Fig. 2.34*b*). From Eq. (2.22) we write

$$\begin{aligned} \epsilon_T &= \alpha \, \Delta T = (6.5 \times 10^{-6}/°\text{F})(-125°\text{F}) \\ &= -812.5 \times 10^{-6} \text{ in./in.} \end{aligned}$$

The other component of ϵ_{AC} is associated with the stress σ_1 due to the force $\mathbf{R}_B$ applied to the bar (Fig. 2.34*c*). From Hooke's law, we express this component of the strain as

$$\frac{\sigma_1}{E} = \frac{+31.42 \times 10^3 \text{ psi}}{29 \times 10^6 \text{ psi}} = +1083.4 \times 10^{-6} \text{ in./in.}$$

Adding the two components of the strain in AC, we obtain

$$\begin{aligned} \epsilon_{AC} &= \epsilon_T + \frac{\sigma_1}{E} = -812.5 \times 10^{-6} + 1083.4 \times 10^{-6} \\ &= +271 \times 10^{-6} \text{ in./in.} \end{aligned}$$

A similar computation yields the strain in portion CB of the bar:

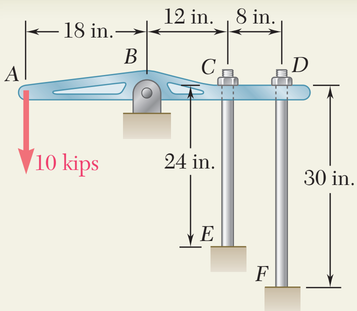
$$\begin{aligned} \epsilon_{CB} &= \epsilon_T + \frac{\sigma_2}{E} = -812.5 \times 10^{-6} + 541.7 \times 10^{-6} \\ &= -271 \times 10^{-6} \text{ in./in.} \end{aligned}$$

The deformations δ_{AC} and δ_{CB} of the two portions of the bar are expressed respectively as

$$\begin{aligned} \delta_{AC} &= \epsilon_{AC}(AC) = (+271 \times 10^{-6})(12 \text{ in.}) \\ &= +3.25 \times 10^{-3} \text{ in.} \\ \delta_{CB} &= \epsilon_{CB}(CB) = (-271 \times 10^{-6})(12 \text{ in.}) \\ &= -3.25 \times 10^{-3} \text{ in.} \end{aligned}$$

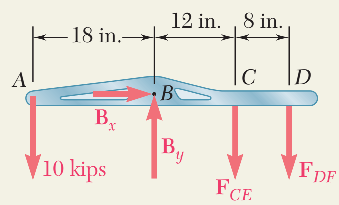
We thus check that, while the sum $\delta = \delta_{AC} + \delta_{CB}$ of the two deformations is zero, neither of the deformations is zero.

SAMPLE PROBLEM 2.3



The $\frac{1}{2}$-in.-diameter rod CE and the $\frac{3}{4}$-in.-diameter rod DF are attached to the rigid bar $ABCD$ as shown. Knowing that the rods are made of aluminum and using $E = 10.6 \times 10^6$ psi, determine (*a*) the force in each rod caused by the loading shown, (*b*) the corresponding deflection of point A.

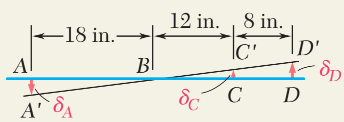
SOLUTION



Statics. Considering the free body of bar $ABCD$, we note that the reaction at B and the forces exerted by the rods are indeterminate. However, using statics, we may write

$$+\circlearrowleft \Sigma M_B = 0: \qquad (10 \text{ kips})(18 \text{ in.}) - F_{CE}(12 \text{ in.}) - F_{DF}(20 \text{ in.}) = 0$$

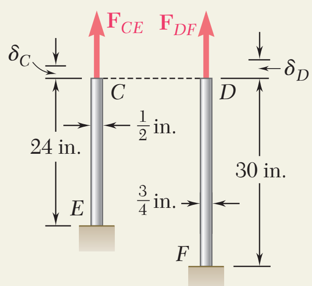
$$12F_{CE} + 20F_{DF} = 180 \tag{1}$$



Geometry. After application of the 10-kip load, the position of the bar is $A'BC'D'$. From the similar triangles BAA', BCC', and BDD' we have

$$\frac{\delta_C}{12 \text{ in.}} = \frac{\delta_D}{20 \text{ in.}} \qquad \delta_C = 0.6\delta_D \tag{2}$$

$$\frac{\delta_A}{18 \text{ in.}} = \frac{\delta_D}{20 \text{ in.}} \qquad \delta_A = 0.9\delta_D \tag{3}$$



Deformations. Using Eq. (2.7), we have

$$\delta_C = \frac{F_{CE}L_{CE}}{A_{CE}E} \qquad \delta_D = \frac{F_{DF}L_{DF}}{A_{DF}E}$$

Substituting for δ_C and δ_D into (2), we write

$$\delta_C = 0.6\delta_D \qquad \frac{F_{CE}L_{CE}}{A_{CE}E} = 0.6\frac{F_{DF}L_{DF}}{A_{DF}E}$$

$$F_{CE} = 0.6\frac{L_{DF}}{L_{CE}}\frac{A_{CE}}{A_{DF}}F_{DF} = 0.6\left(\frac{30 \text{ in.}}{24 \text{ in.}}\right)\left[\frac{\frac{1}{4}\pi(\frac{1}{2}\text{ in.})^2}{\frac{1}{4}\pi(\frac{3}{4}\text{ in.})^2}\right]F_{DF} \qquad F_{CE} = 0.333F_{DF}$$

Force in Each Rod. Substituting for F_{CE} into (1) and recalling that all forces have been expressed in kips, we have

$$12(0.333F_{DF}) + 20F_{DF} = 180 \qquad F_{DF} = 7.50 \text{ kips} \blacktriangleleft$$

$$F_{CE} = 0.333F_{DF} = 0.333(7.50 \text{ kips}) \qquad F_{CE} = 2.50 \text{ kips} \blacktriangleleft$$

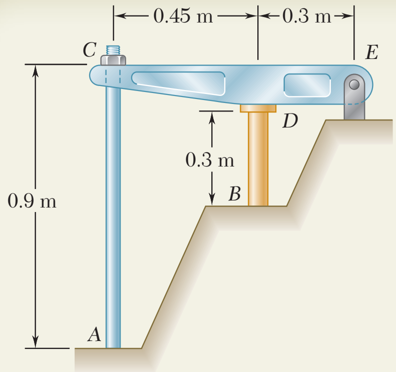
Deflections. The deflection of point D is

$$\delta_D = \frac{F_{DF}L_{DF}}{A_{DF}E} = \frac{(7.50 \times 10^3 \text{ lb})(30 \text{ in.})}{\frac{1}{4}\pi(\frac{3}{4}\text{ in.})^2(10.6 \times 10^6 \text{ psi})} \qquad \delta_D = 48.0 \times 10^{-3} \text{ in.}$$

Using (3), we write

$$\delta_A = 0.9\delta_D = 0.9(48.0 \times 10^{-3} \text{ in.}) \qquad \delta_A = 43.2 \times 10^{-3} \text{ in.} \blacktriangleleft$$

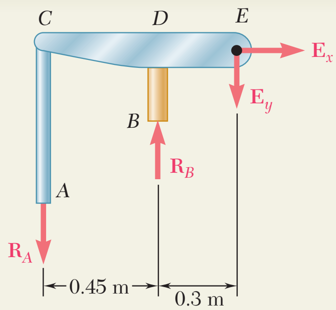
SAMPLE PROBLEM 2.4



The rigid bar *CDE* is attached to a pin support at *E* and rests on the 30-mm-diameter brass cylinder *BD*. A 22-mm-diameter steel rod *AC* passes through a hole in the bar and is secured by a nut which is snugly fitted when the temperature of the entire assembly is 20°C. The temperature of the brass cylinder is then raised to 50°C while the steel rod remains at 20°C. Assuming that no stresses were present before the temperature change, determine the stress in the cylinder.

Rod *AC*: Steel	Cylinder *BD*: Brass
$E = 200$ GPa	$E = 105$ GPa
$\alpha = 11.7 \times 10^{-6}/°\text{C}$	$\alpha = 20.9 \times 10^{-6}/°\text{C}$

SOLUTION



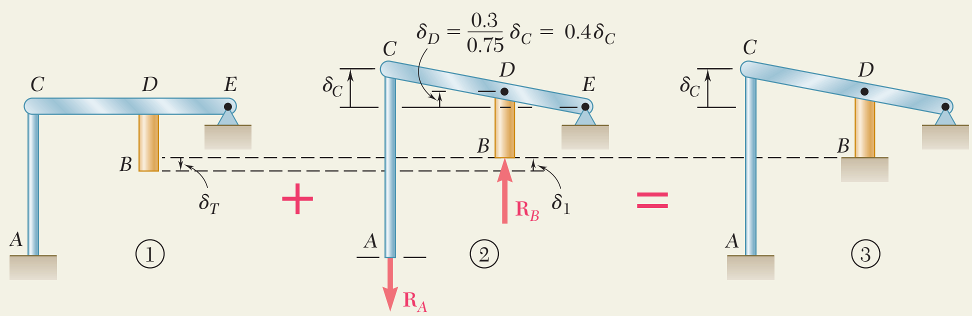
Statics. Considering the free body of the entire assembly, we write

$$+\circlearrowleft \Sigma M_E = 0: \qquad R_A(0.75 \text{ m}) - R_B(0.3 \text{ m}) = 0 \qquad R_A = 0.4R_B \qquad (1)$$

Deformations. We use the method of superposition, considering $\mathbf{R}_B$ as redundant. With the support at *B* removed, the temperature rise of the cylinder causes point *B* to move down through δ_T. The reaction $\mathbf{R}_B$ must cause a deflection δ_1 equal to δ_T so that the final deflection of *B* will be zero (Fig. 3).

Deflection δ_T. Because of a temperature rise of $50° - 20° = 30°\text{C}$, the length of the brass cylinder increases by δ_T.

$$\delta_T = L(\Delta T)\alpha = (0.3 \text{ m})(30°\text{C})(20.9 \times 10^{-6}/°\text{C}) = 188.1 \times 10^{-6} \text{ m} \downarrow$$



Deflection δ_1. We note that $\delta_D = 0.4\delta_C$ and $\delta_1 = \delta_D + \delta_{B/D}$.

$$\delta_C = \frac{R_A L}{AE} = \frac{R_A(0.9 \text{ m})}{\frac{1}{4}\pi(0.022 \text{ m})^2(200 \text{ GPa})} = 11.84 \times 10^{-9} R_A \uparrow$$

$$\delta_D = 0.40\delta_C = 0.4(11.84 \times 10^{-9} R_A) = 4.74 \times 10^{-9} R_A \uparrow$$

$$\delta_{B/D} = \frac{R_B L}{AE} = \frac{R_B(0.3 \text{ m})}{\frac{1}{4}\pi(0.03 \text{ m})^2(105 \text{ GPa})} = 4.04 \times 10^{-9} R_B \uparrow$$

We recall from (1) that $R_A = 0.4R_B$ and write

$$\delta_1 = \delta_D + \delta_{B/D} = [4.74(0.4R_B) + 4.04R_B]10^{-9} = 5.94 \times 10^{-9} R_B \uparrow$$

But $\delta_T = \delta_1$: $\qquad 188.1 \times 10^{-6} \text{ m} = 5.94 \times 10^{-9} R_B \qquad R_B = 31.7 \text{ kN}$

Stress in Cylinder: $\displaystyle \sigma_B = \frac{R_B}{A} = \frac{31.7 \text{ kN}}{\frac{1}{4}\pi(0.03 \text{ m})^2} \qquad \sigma_B = 44.8 \text{ MPa}$ ◀

PROBLEMS

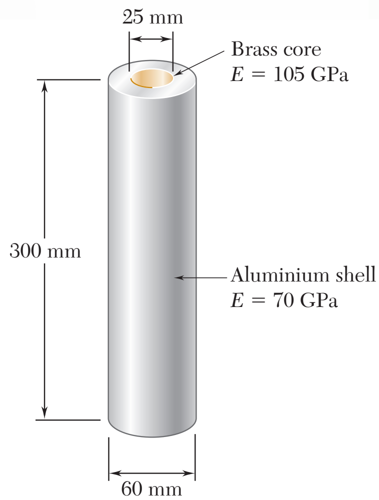


Fig. *P2.33* and *P2.34*

2.33 An axial force of 200 kN is applied to the assembly shown by means of rigid end plates. Determine (*a*) the normal stress in the aluminum shell, (*b*) the corresponding deformation of the assembly.

2.34 The length of the assembly shown decreases by 0.40 mm when an axial force is applied by means of rigid end plates. Determine (*a*) the magnitude of the applied force, (*b*) the corresponding stress in the brass core.

2.35 A 4-ft concrete post is reinforced with four steel bars, each with a $\frac{3}{4}$-in. diameter. Knowing that $E_s = 29 \times 10^6$ psi and $E_c = 3.6 \times 10^6$ psi, determine the normal stresses in the steel and in the concrete when a 150-kip axial centric force **P** is applied to the post.

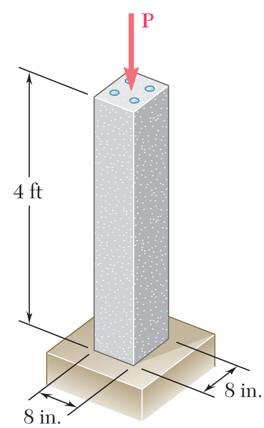


Fig. P2.35

2.36 A 250-mm bar of 150 × 30-mm rectangular cross section consists of two aluminum layers, 5 mm thick, brazed to a center brass layer of the same thickness. If it is subjected to centric forces of magnitude $P = 30$ kN, and knowing that $E_a = 70$ GPa and $E_b = 105$ GPa, determine the normal stress (*a*) in the aluminum layers, (*b*) in the brass layer.

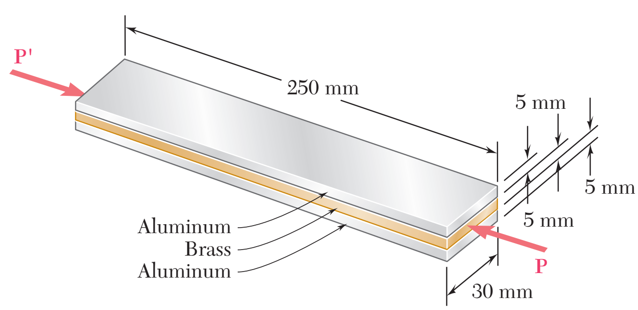


Fig. P2.36

2.37 Determine the deformation of the composite bar of Prob. 2.36 if it is subjected to centric forces of magnitude $P = 45$ kN.

2.38 Compressive centric forces of 40 kips are applied at both ends of the assembly shown by means of rigid end plates. Knowing that $E_s = 29 \times 10^6$ psi and $E_a = 10.1 \times 10^6$ psi, determine (*a*) the normal stresses in the steel core and the aluminum shell, (*b*) the deformation of the assembly.

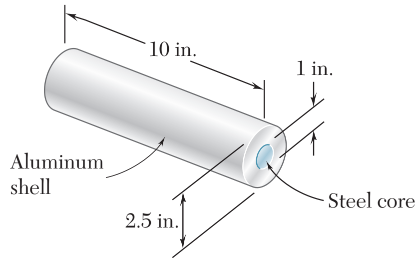


Fig. P2.38

2.39 Three wires are used to suspend the plate shown. Aluminum wires of $\frac{1}{8}$-in. diameter are used at A and B while a steel wire of $\frac{1}{12}$-in. diameter is used at C. Knowing that the allowable stress for aluminum ($E_a = 10.4 \times 10^6$ psi) is 14 ksi and that the allowable stress for steel ($E_s = 29 \times 10^6$ psi) is 18 ksi, determine the maximum load **P** that can be applied.

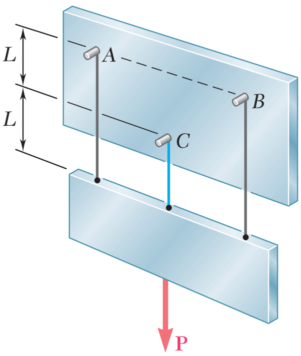


Fig. P2.39

2.40 A polystyrene rod consisting of two cylindrical portions AB and BC is restrained at both ends and supports two 6-kip loads as shown. Knowing that $E = 0.45 \times 10^6$ psi, determine (*a*) the reactions at A and C, (*b*) the normal stress in each portion of the rod.

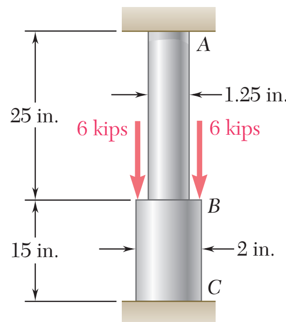


Fig. P2.40

2.41 Two cylindrical rods, one of steel and the other of brass, are joined at C and restrained by rigid supports at A and E. For the loading shown and knowing that $E_s = 200$ GPa and $E_b = 105$ GPa, determine (*a*) the reactions at A and E, (*b*) the deflection of point C.

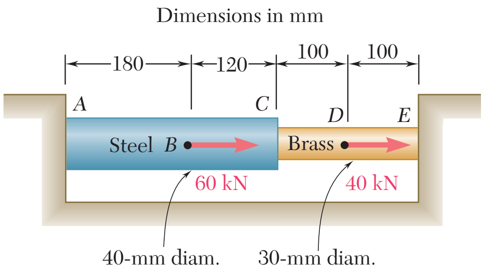


Fig. P2.41

2.42 Solve Prob. 2.41, assuming that rod AC is made of brass and rod CE is made of steel.

2.43 The rigid bar $ABCD$ is suspended from four identical wires. Determine the tension in each wire caused by the load **P** shown.

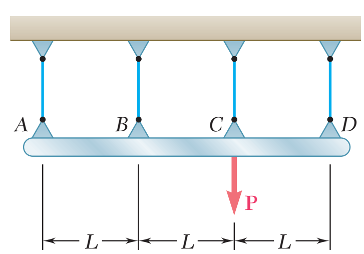


Fig. P2.43

2.44 The rigid bar AD is supported by two steel wires of $\frac{1}{16}$-in. diameter ($E = 29 \times 10^6$ psi) and a pin and bracket at D. Knowing that the wires were initially taut, determine (*a*) the additional tension in each wire when a 120-lb load **P** is applied at B, (*b*) the corresponding deflection of point B.

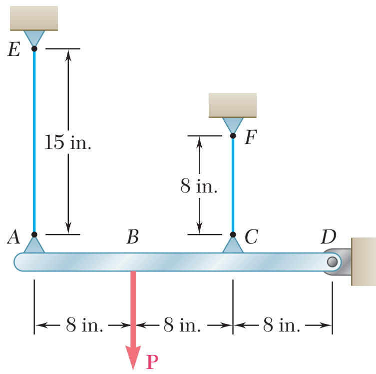


Fig. *P2.44*

2.45 The steel rods BE and CD each have a 16-mm diameter ($E = 200$ GPa); the ends of the rods are single-threaded with a pitch of 2.5 mm. Knowing that after being snugly fitted, the nut at C is tightened one full turn, determine (*a*) the tension in rod CD, (*b*) the deflection of point C of the rigid member ABC.

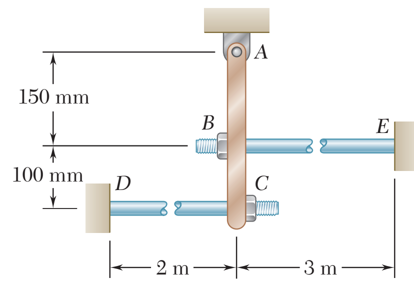


Fig. P2.45

2.46 Links BC and DE are both made of steel ($E = 29 \times 10^6$ psi) and are $\frac{1}{2}$ in. wide and $\frac{1}{4}$ in. thick. Determine (*a*) the force in each link when a 600-lb force **P** is applied to the rigid member AF shown, (*b*) the corresponding deflection of point A.

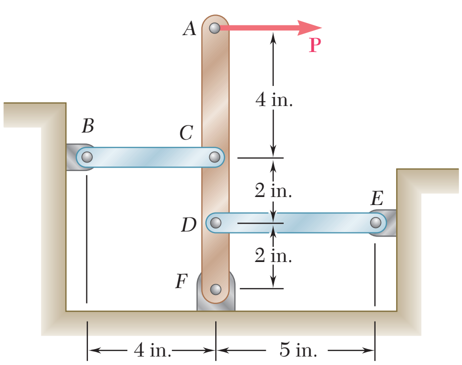


Fig. P2.46

2.47 The concrete post ($E_c = 3.6 \times 10^6$ psi and $\alpha_c = 5.5 \times 10^{-6}/°F$) is reinforced with six steel bars, each of $\frac{7}{8}$-in diameter ($E_s = 29 \times 10^6$ psi and $\alpha_s = 6.5 \times 10^{-6}/°F$). Determine the normal stresses induced in the steel and in the concrete by a temperature rise of 65°F.

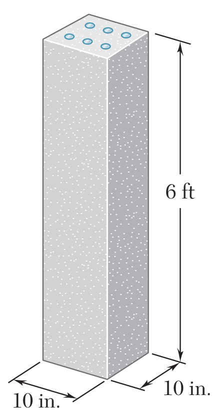


Fig. P2.47

2.48 The assembly shown consists of an aluminum shell ($E_a = 10.6 \times 10^6$ psi, $\alpha_a = 12.9 \times 10^{-6}/°F$) fully bonded to a steel core ($E_s = 29 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}/°F$) and is unstressed. Determine (*a*) the largest allowable change in temperature if the stress in the aluminum shell is not to exceed 6 ksi, (*b*) the corresponding change in length of the assembly.

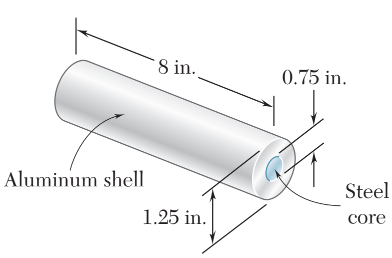


Fig. *P2.48*

2.49 The aluminum shell is fully bonded to the brass core and the assembly is unstressed at a temperature of 15°C. Considering only axial deformations, determine the stress in the aluminum when the temperature reaches 195°C.

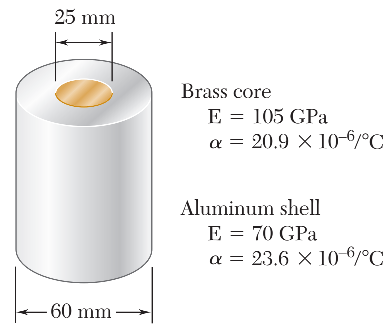


Fig. P2.49

2.50 Solve Prob. 2.49, assuming that the core is made of steel (E_s = 200 GPa, $\alpha_s = 11.7 \times 10^{-6}/°C$) instead of brass.

2.51 A rod consisting of two cylindrical portions *AB* and *BC* is restrained at both ends. Portion *AB* is made of steel (E_s = 200 GPa, $\alpha_s = 11.7 \times 10^{-6}/°C$) and portion *BC* is made of brass (E_b = 105 GPa, $\alpha_b = 20.9 \times 10^{-6}/°C$). Knowing that the rod is initially unstressed, determine the compressive force induced in *ABC* when there is a temperature rise of 50°C.

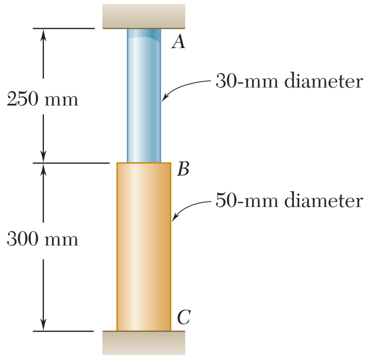


Fig. P2.51

2.52 A steel railroad track (E_s = 200 GPa, $\alpha_s = 11.7 \times 10^{-6}/°C$) was laid out at a temperature of 6°C. Determine the normal stress in the rails when the temperature reaches 48°C, assuming that the rails (*a*) are welded to form a continuous track, (*b*) are 10 m long with 3-mm gaps between them.

2.53 A rod consisting of two cylindrical portions *AB* and *BC* is restrained at both ends. Portion *AB* is made of steel ($E_s = 29 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}/°F$) and portion *BC* is made of aluminum ($E_a = 10.4 \times 10^6$ psi, $\alpha_a = 13.3 \times 10^{-6}/°F$). Knowing that the rod is initially unstressed, determine (*a*) the normal stresses induced in portions *AB* and *BC* by a temperature rise of 70°F, (*b*) the corresponding deflection of point *B*.

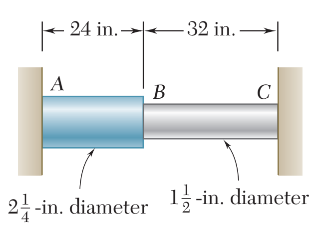


Fig. P2.53

2.54 Solve Prob. 2.53, assuming that portion *AB* of the composite rod is made of aluminum and portion *BC* is made of steel.

2.55 A brass link (E_b = 105 GPa, α_b = 20.9 × 10^{-6}/°C) and a steel rod (E_s = 200 GPa, α_s = 11.7 × 10^{-6}/°C) have the dimensions shown at a temperature of 20°C. The steel rod is cooled until it fits freely into the link. The temperature of the whole assembly is then raised to 45°C. Determine (*a*) the final normal stress in the steel rod, (*b*) the final length of the steel rod.

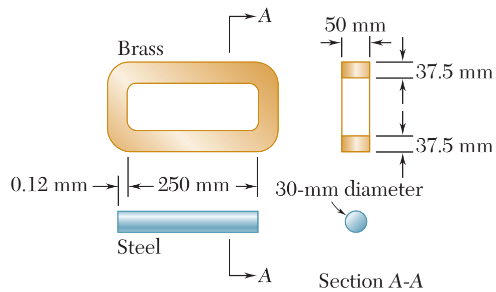


Fig. *P2.55*

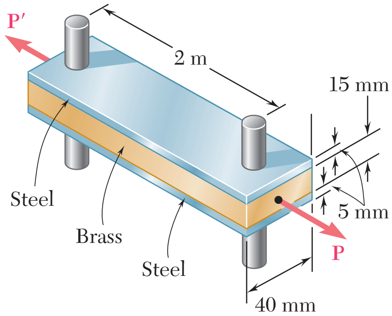
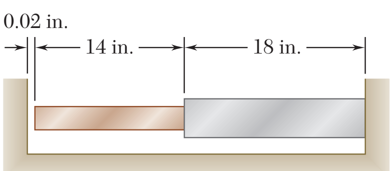


Fig. P2.56

2.56 Two steel bars (E_s = 200 GPa and α_s = 11.7 × 10^{-6}/°C) are used to reinforce a brass bar (E_b = 105 GPa, α_b = 20.9 × 10^{-6}/°C) that is subjected to a load P = 25 kN. When the steel bars were fabricated, the distance between the centers of the holes that were to fit on the pins was made 0.5 mm smaller than the 2 m needed. The steel bars were then placed in an oven to increase their length so that they would just fit on the pins. Following fabrication, the temperature in the steel bars dropped back to room temperature. Determine (*a*) the increase in temperature that was required to fit the steel bars on the pins, (*b*) the stress in the brass bar after the load is applied to it.

2.57 Determine the maximum load P that can be applied to the brass bar of Prob. 2.56 if the allowable stress in the steel bars is 30 MPa and the allowable stress in the brass bar is 25 MPa.



Bronze
A = 2.4 in^2
E = 15 × 10^6 psi
α = 12 × 10^{-6}/°F

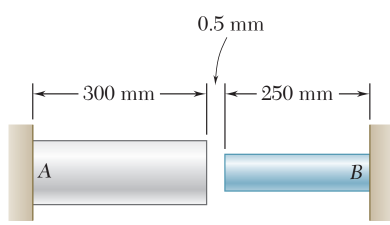
Aluminum
A = 2.8 in^2
E = 10.6 × 10^6 psi
α = 12.9 × 10^{-6}/°F

Fig. P2.58 and P2.59

2.58 Knowing that a 0.02-in. gap exists when the temperature is 75°F, determine (*a*) the temperature at which the normal stress in the aluminum bar will be equal to −11 ksi, (*b*) the corresponding exact length of the aluminum bar.

2.59 Determine (*a*) the compressive force in the bars shown after a temperature rise of 180°F, (*b*) the corresponding change in length of the bronze bar.

2.60 At room temperature (20°C) a 0.5-mm gap exists between the ends of the rods shown. At a later time when the temperature has reached 140°C, determine (*a*) the normal stress in the aluminum rod, (*b*) the change in length of the aluminum rod.



Aluminum
A = 2000 mm^2
E = 75 GPa
α = 23 × 16^{-6}/°C

Stainless steel
A = 800 mm^2
E = 190 GPa
α = 17.3 × 10^{-6}/°C

Fig. *P2.60*

2.11 POISSON'S RATIO

We saw in the earlier part of this chapter that, when a homogeneous slender bar is axially loaded, the resulting stress and strain satisfy Hooke's law, as long as the elastic limit of the material is not exceeded. Assuming that the load **P** is directed along the x axis (Fig. 2.35a), we have $\sigma_x = P/A$, where A is the cross-sectional area of the bar, and, from Hooke's law,

$$\epsilon_x = \sigma_x/E \tag{2.24}$$

where E is the modulus of elasticity of the material.

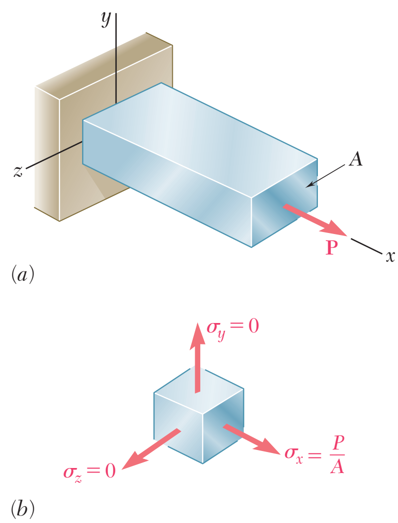


Fig. 2.35 Stresses in an axially-loaded bar.

We also note that the normal stresses on faces respectively perpendicular to the y and z axes are zero: $\sigma_y = \sigma_z = 0$ (Fig. 2.35b). It would be tempting to conclude that the corresponding strains ϵ_y and ϵ_z are also zero. This, however, *is not the case*. In all engineering materials, the elongation produced by an axial tensile force **P** in the direction of the force is accompanied by a contraction in any transverse direction (Fig. 2.36).† In this section and the following sections (Secs. 2.12 through 2.15), all materials considered will be assumed to be both *homogeneous* and *isotropic*, i.e., their mechanical properties will be assumed independent of both *position* and *direction*. It follows that the strain must have the same value for any transverse direction. Therefore, for the loading shown in Fig. 2.35 we must have $\epsilon_y = \epsilon_z$. This common value is referred to as the *lateral strain*. An important constant for a given material is its *Poisson's ratio*, named after the French mathematician Siméon Denis Poisson (1781–1840) and denoted by the Greek letter ν (nu). It is defined as

$$\boxed{\nu = -\frac{\text{lateral strain}}{\text{axial strain}}} \tag{2.25}$$

or

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x} \tag{2.26}$$

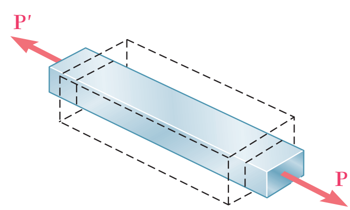


Fig. 2.36 Transverse contraction of bar under axial tensile force.

for the loading condition represented in Fig. 2.35. Note the use of a minus sign in the above equations to obtain a positive value for ν, the axial and lateral strains having opposite signs for all engineering materials.‡ Solving Eq. (2.26) for ϵ_y and ϵ_z, and recalling (2.24), we write the following relations, which fully describe the condition of strain under an axial load applied in a direction parallel to the x axis:

$$\boxed{\epsilon_x = \frac{\sigma_x}{E}} \qquad \boxed{\epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E}} \tag{2.27}$$

†It would also be tempting, but equally wrong, to assume that the volume of the rod remains unchanged as a result of the combined effect of the axial elongation and transverse contraction (see Sec. 2.13).

‡However, some experimental materials, such as polymer foams, expand laterally when stretched. Since the axial and lateral strains have then the same sign, the Poisson's ratio of these materials is negative. (*See* Roderic Lakes, "Foam Structures with a Negative Poisson's Ratio," *Science*, 27 February 1987, Volume 235, pp. 1038–1040.)

EXAMPLE 2.07

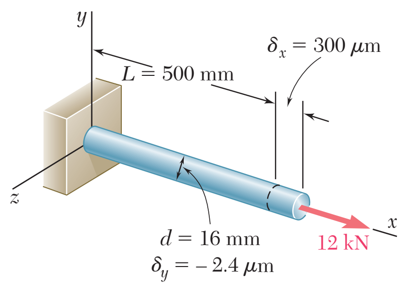


Fig. 2.37

A 500-mm-long, 16-mm-diameter rod made of a homogenous, isotropic material is observed to increase in length by 300 μm, and to decrease in diameter by 2.4 μm when subjected to an axial 12-kN load. Determine the modulus of elasticity and Poisson's ratio of the material.

The cross-sectional area of the rod is

$$A = \pi r^2 = \pi(8 \times 10^{-3}\ \text{m})^2 = 201 \times 10^{-6}\ \text{m}^2$$

Choosing the x axis along the axis of the rod (Fig. 2.37), we write

$$\sigma_x = \frac{P}{A} = \frac{12 \times 10^3\ \text{N}}{201 \times 10^{-6}\ \text{m}^2} = 59.7\ \text{MPa}$$

$$\epsilon_x = \frac{\delta_x}{L} = \frac{300\ \mu\text{m}}{500\ \text{mm}} = 600 \times 10^{-6}$$

$$\epsilon_y = \frac{\delta_y}{d} = \frac{-2.4\ \mu\text{m}}{16\ \text{mm}} = -150 \times 10^{-6}$$

From Hooke's law, $\sigma_x = E\epsilon_x$, we obtain

$$E = \frac{\sigma_x}{\epsilon_x} = \frac{59.7\ \text{MPa}}{600 \times 10^{-6}} = 99.5\ \text{GPa}$$

and, from Eq. (2.26),

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{-150 \times 10^{-6}}{600 \times 10^{-6}} = 0.25$$

2.12 MULTIAXIAL LOADING; GENERALIZED HOOKE'S LAW

All the examples considered so far in this chapter have dealt with slender members subjected to axial loads, i.e., to forces directed along a single axis. Choosing this axis as the x axis, and denoting by P the internal force at a given location, the corresponding stress components were found to be $\sigma_x = P/A$, $\sigma_y = 0$, and $\sigma_z = 0$.

Let us now consider structural elements subjected to loads acting in the directions of the three coordinate axes and producing normal stresses σ_x, σ_y, and σ_z which are all different from zero (Fig. 2.38). This condition is referred to as a *multiaxial loading*. Note that this is not the general stress condition described in Sec. 1.12, since no shearing stresses are included among the stresses shown in Fig. 2.38.

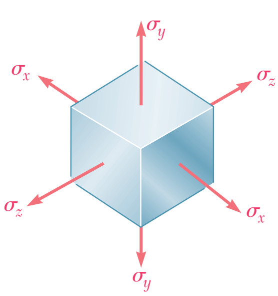


Fig. 2.38 Stress state for multiaxial loading.

Consider an element of an isotropic material in the shape of a cube (Fig. 2.39*a*). We can assume the side of the cube to be equal to unity, since it is always possible to select the side of the cube as a unit of length. Under the given multiaxial loading, the element will deform into a *rectangular parallelepiped* of sides equal, respectively, to $1 + \epsilon_x$, $1 + \epsilon_y$, and $1 + \epsilon_z$, where ϵ_x, ϵ_y, and ϵ_z denote the values of the normal strain in the directions of the three coordinate axes (Fig. 2.39*b*). You should note that, as a result of the deformations of the other elements of the material, the element under consideration could also undergo a translation, but we are concerned here only with the *actual deformation* of the element, and not with any possible superimposed rigid-body displacement.

In order to express the strain components ϵ_x, ϵ_y, ϵ_z in terms of the stress components σ_x, σ_y, σ_z, we will consider separately the effect of each stress component and combine the results obtained. The approach we propose here will be used repeatedly in this text, and is based on the *principle of superposition*. This principle states that the effect of a given combined loading on a structure can be obtained by *determining separately the effects of the various loads and combining the results obtained*, provided that the following conditions are satisfied:

1. Each effect is linearly related to the load that produces it.
2. The deformation resulting from any given load is small and does not affect the conditions of application of the other loads.

In the case of a multiaxial loading, the first condition will be satisfied if the stresses do not exceed the proportional limit of the material, and the second condition will also be satisfied if the stress on any given face does not cause deformations of the other faces that are large enough to affect the computation of the stresses on those faces.

Considering first the effect of the stress component σ_x, we recall from Sec. 2.11 that σ_x causes a strain equal to σ_x/E in the x direction, and strains equal to $-\nu\sigma_x/E$ in each of the y and z directions. Similarly, the stress component σ_y, if applied separately, will cause a strain σ_y/E in the y direction and strains $-\nu\sigma_y/E$ in the other two directions. Finally, the stress component σ_z causes a strain σ_z/E in the z direction and strains $-\nu\sigma_z/E$ in the x and y directions. Combining the results obtained, we conclude that the components of strain corresponding to the given multiaxial loading are

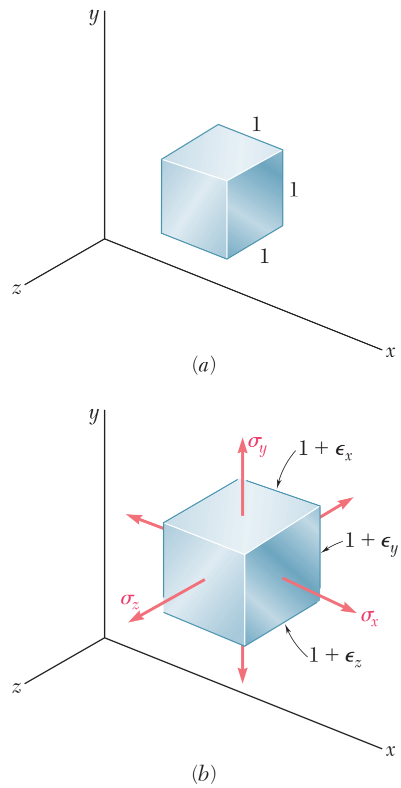


Fig. 2.39 Deformation of cube under multiaxial loading.

$$\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}
\end{aligned} \qquad (2.28)$$

The relations (2.28) are referred to as the *generalized Hooke's law for the multiaxial loading of a homogeneous isotropic material*. As we indicated earlier, the results obtained are valid only as long as the stresses do not exceed the proportional limit, and as long as the deformations involved remain small. We also recall that a positive value for a stress component signifies tension, and a negative value compression. Similarly, a positive value for a strain component indicates expansion in the corresponding direction, and a negative value contraction.

EXAMPLE 2.08

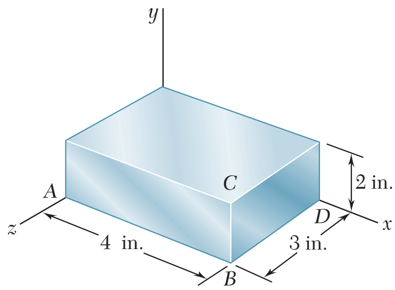


Fig. 2.40

The steel block shown (Fig. 2.40) is subjected to a uniform pressure on all its faces. Knowing that the change in length of edge AB is -1.2×10^{-3} in., determine (*a*) the change in length of the other two edges, (*b*) the pressure p applied to the faces of the block. Assume $E = 29 \times 10^6$ psi and $\nu = 0.29$.

(a) Change in Length of Other Edges. Substituting $\sigma_x = \sigma_y = \sigma_z = -p$ into the relations (2.28), we find that the three strain components have the common value

$$\epsilon_x = \epsilon_y = \epsilon_z = -\frac{p}{E}(1 - 2\nu) \tag{2.29}$$

Since

$$\begin{aligned}\epsilon_x &= \delta_x/AB = (-1.2 \times 10^{-3}\text{ in.})/(4\text{ in.})\\ &= -300 \times 10^{-6}\text{ in./in.}\end{aligned}$$

we obtain

$$\epsilon_y = \epsilon_z = \epsilon_x = -300 \times 10^{-6}\text{ in./in.}$$

from which it follows that

$$\begin{aligned}\delta_y &= \epsilon_y(BC) = (-300 \times 10^{-6})(2\text{ in.}) = -600 \times 10^{-6}\text{ in.}\\ \delta_z &= \epsilon_z(BD) = (-300 \times 10^{-6})(3\text{ in.}) = -900 \times 10^{-6}\text{ in.}\end{aligned}$$

(b) Pressure. Solving Eq. (2.29) for p, we write

$$p = -\frac{E\epsilon_x}{1 - 2\nu} = -\frac{(29 \times 10^6\text{ psi})(-300 \times 10^{-6})}{1 - 0.58}$$

$$p = 20.7\text{ ksi}$$

*2.13 DILATATION; BULK MODULUS

In this section you will examine the effect of the normal stresses σ_x, σ_y, and σ_z on the volume of an element of isotropic material. Consider the element shown in Fig. 2.39. In its unstressed state, it is in the shape of a cube of unit volume; and under the stresses σ_x, σ_y, σ_z, it deforms into a rectangular parallelepiped of volume

$$v = (1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z)$$

Since the strains ϵ_x, ϵ_y, ϵ_z are much smaller than unity, their products will be even smaller and may be omitted in the expansion of the product. We have, therefore,

$$v = 1 + \epsilon_x + \epsilon_y + \epsilon_z$$

Denoting by e the change in volume of our element, we write

$$e = v - 1 = 1 + \epsilon_x + \epsilon_y + \epsilon_z - 1$$

or

$$e = \epsilon_x + \epsilon_y + \epsilon_z \tag{2.30}$$

Since the element had originally a unit volume, the quantity e represents *the change in volume per unit volume;* it is referred to as the *dilatation* of the material. Substituting for ϵ_x, ϵ_y, and ϵ_z from Eqs. (2.28) into (2.30), we write

$$e = \frac{\sigma_x + \sigma_y + \sigma_z}{E} - \frac{2\nu(\sigma_x + \sigma_y + \sigma_z)}{E}$$

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \qquad (2.31)†$$

A case of special interest is that of a body subjected to a uniform hydrostatic pressure p. Each of the stress components is then equal to $-p$ and Eq. (2.31) yields

$$e = -\frac{3(1 - 2\nu)}{E}p \qquad (2.32)$$

Introducing the constant

$$k = \frac{E}{3(1 - 2\nu)} \qquad (2.33)$$

we write Eq. (2.32) in the form

$$e = -\frac{p}{k} \qquad (2.34)$$

The constant k is known as the *bulk modulus* or *modulus of compression* of the material. It is expressed in the same units as the modulus of elasticity E, that is, in pascals or in psi.

Observation and common sense indicate that a stable material subjected to a hydrostatic pressure can only *decrease* in volume; thus the dilatation e in Eq. (2.34) is negative, from which it follows that the bulk modulus k is a positive quantity. Referring to Eq. (2.33), we conclude that $1 - 2\nu > 0$, or $\nu < \frac{1}{2}$. On the other hand, we recall from Sec. 2.11 that ν is positive for all engineering materials. We thus conclude that, for any engineering material,

$$0 < \nu < \tfrac{1}{2} \qquad (2.35)$$

We note that an ideal material having a value of v equal to zero could be stretched in one direction without any lateral contraction. On the other hand, an ideal material for which $\nu = \frac{1}{2}$, and thus $k = \infty$, would be perfectly incompressible ($e = 0$). Referring to Eq. (2.31) we also note that, since $\nu < \frac{1}{2}$ in the elastic range, stretching an engineering material in one direction, for example in the x direction ($\sigma_x > 0$, $\sigma_y = \sigma_z = 0$), will result in an increase of its volume ($e > 0$).‡

†Since the dilatation e represents a change in volume, it must be independent of the orientation of the element considered. It then follows from Eqs. (2.30) and (2.31) that the quantities $\epsilon_x + \epsilon_y + \epsilon_z$ and $\sigma_x + \sigma_y + \sigma_z$ are also independent of the orientation of the element. This property will be verified in Chap. 7.

‡However, in the plastic range, the volume of the material remains nearly constant.

EXAMPLE 2.09

Determine the change in volume ΔV of the steel block shown in Fig. 2.40, when it is subjected to the hydrostatic pressure $p = 180$ MPa. Use $E = 200$ GPa and $\nu = 0.29$.

From Eq. (2.33), we determine the bulk modulus of steel,

$$k = \frac{E}{3(1 - 2\nu)} = \frac{200 \text{ GPa}}{3(1 - 0.58)} = 158.7 \text{ GPa}$$

and, from Eq. (2.34), the dilatation,

$$e = -\frac{p}{k} = -\frac{180 \text{ MPa}}{158.7 \text{ GPa}} = -1.134 \times 10^{-3}$$

Since the volume V of the block in its unstressed state is

$$V = (80 \text{ mm})(40 \text{ mm})(60 \text{ mm}) = 192 \times 10^3 \text{ mm}^3$$

and since e represents the change in volume per unit volume, $e = \Delta V/V$, we have

$$\Delta V = eV = (-1.134 \times 10^{-3})(192 \times 10^3 \text{ mm}^3)$$

$$\Delta V = -218 \text{ mm}^3$$

2.14 SHEARING STRAIN

When we derived in Sec. 2.12 the relations (2.28) between normal stresses and normal strains in a homogeneous isotropic material, we assumed that no shearing stresses were involved. In the more general stress situation represented in Fig. 2.41, shearing stresses τ_{xy}, τ_{yz}, and τ_{zx} will be present (as well, of course, as the corresponding shearing stresses τ_{yx}, τ_{zy}, and τ_{xz}). These stresses have no direct effect on the normal strains and, as long as all the deformations involved remain small, they will not affect the derivation nor the validity of the relations (2.28). The shearing stresses, however, will tend to deform a cubic element of material into an *oblique* parallelepiped.

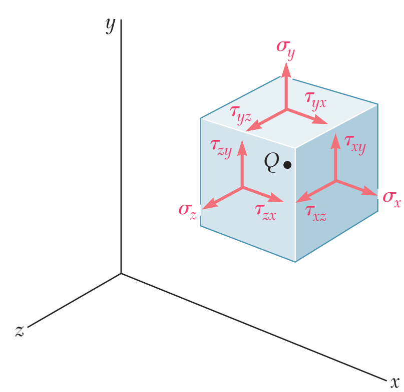


Fig. 2.41 General state of stress.

Consider first a cubic element of side one (Fig. 2.42) subjected to no other stresses than the shearing stresses τ_{xy} and τ_{yx} applied to faces of the element respectively perpendicular to the x and y axes. (We recall from Sec. 1.12 that $\tau_{xy} = \tau_{yx}$.) The element is observed to deform into a rhomboid of sides equal to one (Fig. 2.43). Two of the angles formed by the four faces under stress are reduced from $\frac{\pi}{2}$ to $\frac{\pi}{2} - \gamma_{xy}$, while the other two are increased from $\frac{\pi}{2}$ to $\frac{\pi}{2} + \gamma_{xy}$. The small angle γ_{xy} (expressed in radians) defines the *shearing strain* corresponding to the x and y directions. When the deformation involves a *reduction* of the angle formed by the two faces oriented respectively toward the positive x and y axes (as shown in Fig. 2.43), the shearing strain γ_{xy} is said to be *positive;* otherwise, it is said to be negative.

We should note that, as a result of the deformations of the other elements of the material, the element under consideration can also undergo an overall rotation. However, as was the case in our study of normal strains, we are concerned here only with the *actual deformation* of the element, and not with any possible superimposed rigid-body displacement.†

Plotting successive values of τ_{xy} against the corresponding values of γ_{xy}, we obtain the shearing stress-strain diagram for the material under consideration. This can be accomplished by carrying out a torsion test, as you will see in Chap. 3. The diagram obtained is similar to the normal stress-strain diagram obtained for the same material from the tensile test described earlier in this chapter. However, the values obtained for the yield strength, ultimate strength, etc., of a given material are only about half as large in shear as they are in tension. As was the case for normal stresses and strains, the initial portion of the shearing stress-strain diagram is a straight line. For values of the shearing stress that do not exceed the proportional

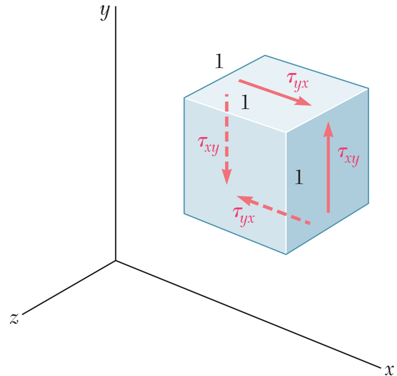


Fig. 2.42 Cubic element subjected to shearing stresses.

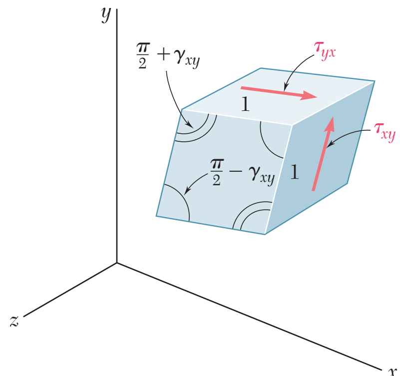


Fig. 2.43 Deformation of cubic element due to shearing stresses.

†In defining the strain γ_{xy}, some authors arbitrarily assume that the actual deformation of the element is accompanied by a rigid-body rotation such that the horizontal faces of the element do not rotate. The strain γ_{xy} is then represented by the angle through which the other two faces have rotated (Fig. 2.44). Others assume a rigid-body rotation such that the horizontal faces rotate through $\frac{1}{2}\gamma_{xy}$ counterclockwise and the vertical faces through $\frac{1}{2}\gamma_{xy}$ clockwise (Fig. 2.45). Since both assumptions are unnecessary and may lead to confusion, we prefer in this text to associate the shearing strain γ_{xy} with the *change in the angle* formed by the two faces, rather than with the *rotation of a given face* under restrictive conditions.

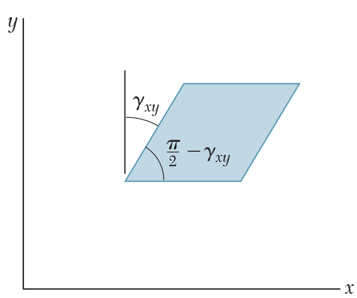


Fig. 2.44

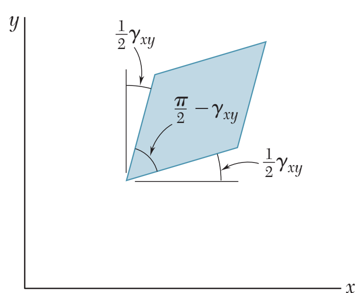


Fig. 2.45

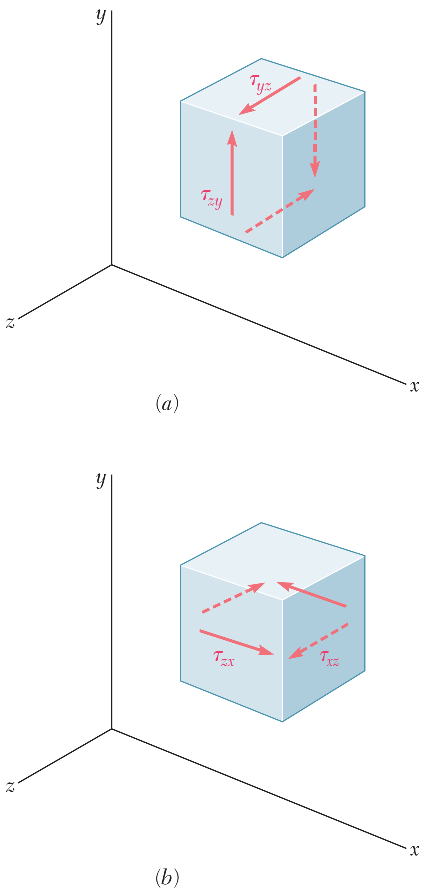


Fig. 2.46

limit in shear, we can therefore write for any homogeneous isotropic material,

$$\tau_{xy} = G\gamma_{xy} \tag{2.36}$$

This relation is known as *Hooke's law for shearing stress and strain*, and the constant G is called the *modulus of rigidity* or *shear modulus* of the material. Since the strain γ_{xy} was defined as an angle in radians, it is dimensionless, and the modulus G is expressed in the same units as τ_{xy}, that is, in pascals or in psi. The modulus of rigidity G of any given material is less than one-half, but more than one-third of the modulus of elasticity E of that material.†

Considering now a small element of material subjected to shearing stresses τ_{yz} and τ_{zy} (Fig. 2.46*a*), we define the shearing strain γ_{yz} as the change in the angle formed by the faces under stress. The shearing strain γ_{zx} is defined in a similar way by considering an element subjected to shearing stresses τ_{zx} and τ_{xz} (Fig. 2.46*b*). For values of the stress that do not exceed the proportional limit, we can write the two additional relations

$$\tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx} \tag{2.37}$$

where the constant G is the same as in Eq. (2.36).

For the general stress condition represented in Fig. 2.41, and as long as none of the stresses involved exceeds the corresponding proportional limit, we can apply the principle of superposition and combine the results obtained in this section and in Sec. 2.12. We obtain the following group of equations representing the generalized Hooke's law for a homogeneous isotropic material under the most general stress condition.

$$\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}
\end{aligned} \tag{2.38}$$

An examination of Eqs. (2.38) might lead us to believe that three distinct constants, E, ν, and G, must first be determined experimentally, if we are to predict the deformations caused in a given material by an arbitrary combination of stresses. Actually, only two of these constants need be determined experimentally for any given material. As you will see in the next section, the third constant can then be obtained through a very simple computation.

†See Prob. 2.91.

EXAMPLE 2.10

A rectangular block of a material with a modulus of rigidity $G = 90$ ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force **P** (Fig. 2.47). Knowing that the upper plate moves through 0.04 in. under the action of the force, determine (*a*) the average shearing strain in the material, (*b*) the force **P** exerted on the upper plate.

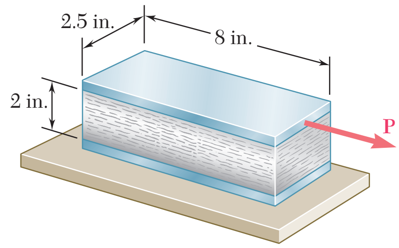


Fig. 2.47

(a) Shearing Strain. We select coordinate axes centered at the midpoint *C* of edge *AB* and directed as shown (Fig. 2.48). According to its definition, the shearing strain γ_{xy} is equal to the angle formed by the vertical and the line *CF* joining the midpoints of edges *AB* and *DE*. Noting that this is a very small angle and recalling that it should be expressed in radians, we write

$$\gamma_{xy} \approx \tan \gamma_{xy} = \frac{0.04 \text{ in.}}{2 \text{ in.}} \qquad \gamma_{xy} = 0.020 \text{ rad}$$

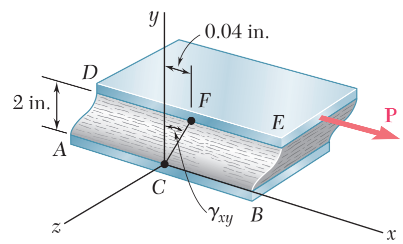


Fig. 2.48

(b) Force Exerted on Upper Plate. We first determine the shearing stress τ_{xy} in the material. Using Hooke's law for shearing stress and strain, we have

$$\tau_{xy} = G\gamma_{xy} = (90 \times 10^3 \text{ psi})(0.020 \text{ rad}) = 1800 \text{ psi}$$

The force exerted on the upper plate is thus

$$P = \tau_{xy} A = (1800 \text{ psi})(8 \text{ in.})(2.5 \text{ in.}) = 36.0 \times 10^3 \text{ lb}$$

$$P = 36.0 \text{ kips}$$

2.15 FURTHER DISCUSSION OF DEFORMATIONS UNDER AXIAL LOADING; RELATION AMONG E, ν, AND G

We saw in Sec. 2.11 that a slender bar subjected to an axial tensile load **P** directed along the x axis will elongate in the x direction and contract in both of the transverse y and z directions. If ϵ_x denotes the axial strain, the lateral strain is expressed as $\epsilon_y = \epsilon_z = -\nu\epsilon_x$, where ν is Poisson's ratio. Thus, an element in the shape of a cube of side equal to one and oriented as shown in Fig. 2.49*a* will deform into a rectangular parallelepiped of sides $1 + \epsilon_x$, $1 - \nu\epsilon_x$, and $1 - \nu\epsilon_x$. (Note that only one face of the element is shown in the figure.) On the other hand, if the element is oriented at 45° to the axis of the load (Fig. 2.49*b*), the face shown in the figure is observed to deform into a rhombus. We conclude that the axial load **P** causes in this element a shearing strain γ' equal to the amount by which each of the angles shown in Fig. 2.49*b* increases or decreases.†

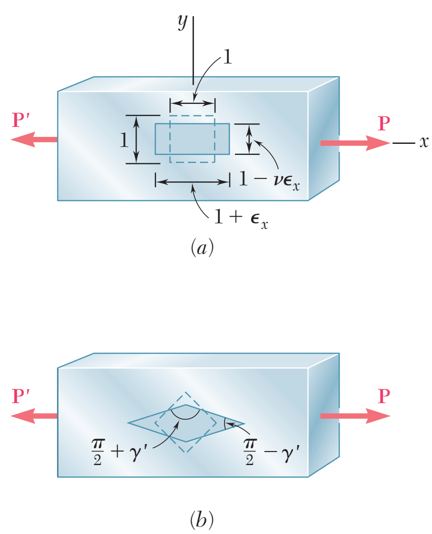


Fig. 2.49 Representations of strain in an axially-loaded bar.

†Note that the load **P** also produces normal strains in the element shown in Fig. 2.49*b* (see Prob. 2.73).

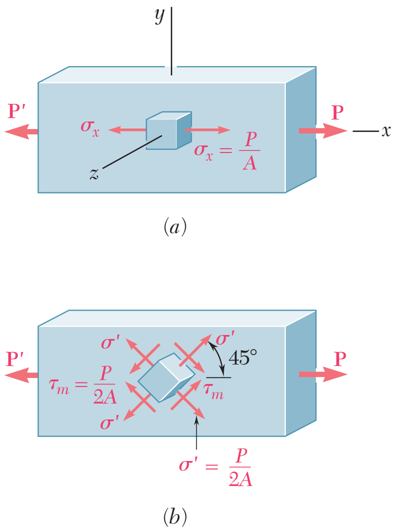


Fig. 1.38 (*repeated*)

The fact that shearing strains, as well as normal strains, result from an axial loading should not come to us as a surprise, since we already observed at the end of Sec. 1.12 that an axial load **P** causes normal and shearing stresses of equal magnitude on four of the faces of an element oriented at 45° to the axis of the member. This was illustrated in Fig. 1.38, which, for convenience, has been repeated here. It was also shown in Sec. 1.11 that the shearing stress is maximum on a plane forming an angle of 45° with the axis of the load. It follows from Hooke's law for shearing stress and strain that the shearing strain γ' associated with the element of Fig. 2.49*b* is also maximum: $\gamma' = \gamma_m$.

While a more detailed study of the transformations of strain will be postponed until Chap. 7, we will derive in this section a relation between the maximum shearing strain $\gamma' = \gamma_m$ associated with the element of Fig. 2.49*b* and the normal strain ϵ_x in the direction of the load. Let us consider for this purpose the prismatic element obtained by intersecting the cubic element of Fig. 2.49*a* by a diagonal plane (Fig. 2.50*a* and *b*). Referring to Fig. 2.49*a*, we conclude that this new element will deform into the element shown in Fig. 2.50*c*, which has horizontal and vertical sides respectively equal to $1 + \epsilon_x$ and $1 - \nu\epsilon_x$. But the angle formed by the oblique and horizontal faces of the element of Fig. 2.50*b* is precisely half of one of the right angles of the cubic element considered in Fig. 2.49*b*. The angle β into which this angle deforms must therefore be equal to half of $\pi/2 - \gamma_m$. We write

$$\beta = \frac{\pi}{4} - \frac{\gamma_m}{2}$$

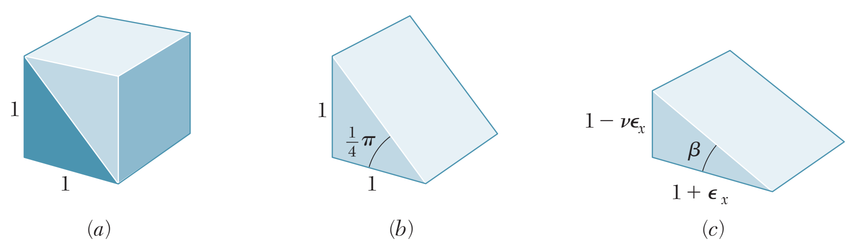


Fig. 2.50

Applying the formula for the tangent of the difference of two angles, we obtain

$$\tan\beta = \frac{\tan\dfrac{\pi}{4} - \tan\dfrac{\gamma_m}{2}}{1 + \tan\dfrac{\pi}{4}\tan\dfrac{\gamma_m}{2}} = \frac{1 - \tan\dfrac{\gamma_m}{2}}{1 + \tan\dfrac{\gamma_m}{2}}$$

or, since $\gamma_m/2$ is a very small angle,

$$\tan\beta = \frac{1 - \dfrac{\gamma_m}{2}}{1 + \dfrac{\gamma_m}{2}} \tag{2.39}$$

But, from Fig. 2.50*c*, we observe that

$$\tan \beta = \frac{1 - \nu\epsilon_x}{1 + \epsilon_x} \tag{2.40}$$

Equating the right-hand members of (2.39) and (2.40), and solving for γ_m, we write

$$\gamma_m = \frac{(1 + \nu)\epsilon_x}{1 + \dfrac{1 - \nu}{2}\epsilon_x}$$

Since $\epsilon_x \ll 1$, the denominator in the expression obtained can be assumed equal to one; we have, therefore,

$$\gamma_m = (1 + \nu)\epsilon_x \tag{2.41}$$

which is the desired relation between the maximum shearing strain γ_m and the axial strain ϵ_x.

To obtain a relation among the constants E, ν, and G, we recall that, by Hooke's law, $\gamma_m = \tau_m/G$, and that, for an axial loading, $\epsilon_x = \sigma_x/E$. Equation (2.41) can therefore be written as

$$\frac{\tau_m}{G} = (1 + \nu)\frac{\sigma_x}{E}$$

or

$$\frac{E}{G} = (1 + \nu)\frac{\sigma_x}{\tau_m} \tag{2.42}$$

We now recall from Fig. 1.38 that $\sigma_x = P/A$ and $\tau_m = P/2A$, where A is the cross-sectional area of the member. It thus follows that $\sigma_x/\tau_m = 2$. Substituting this value into (2.42) and dividing both members by 2, we obtain the relation

$$\boxed{\frac{E}{2G} = 1 + \nu} \tag{2.43}$$

which can be used to determine one of the constants E, ν, or G from the other two. For example, solving Eq. (2.43) for G, we write

$$G = \frac{E}{2(1 + \nu)} \tag{2.43$'$}$$

*2.16 STRESS-STRAIN RELATIONSHIPS FOR FIBER-REINFORCED COMPOSITE MATERIALS

Fiber-reinforced composite materials were briefly discussed in Sec. 2.5. It was shown at that time that these materials are obtained by embedding fibers of a strong, stiff material into a weaker, softer material, referred to as a *matrix*. It was also shown that the relationship between the normal stress and the corresponding normal strain created in a lamina, or layer, of a composite material depends upon the direction in which the load is applied. Different moduli of elasticity, E_x, E_y, and E_z, are therefore required to describe the relationship between normal stress and normal strain, according to whether the

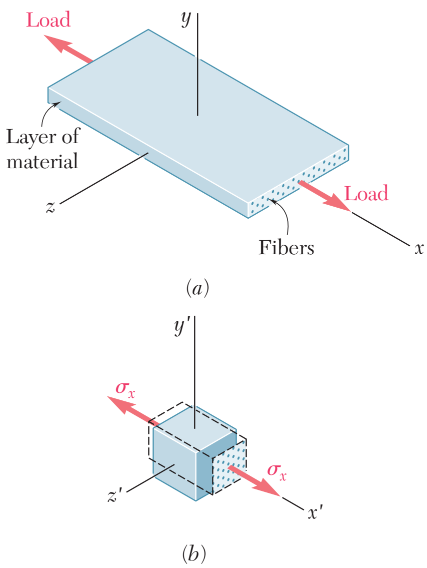


Fig. 2.51 Fiber-reinforced composite material under uniaxial tensile load.

load is applied in a direction parallel to the fibers, in a direction perpendicular to the layer, or in a transverse direction.

Let us consider again the layer of composite material discussed in Sec. 2.5 and let us subject it to a uniaxial tensile load parallel to its fibers, i.e., in the x direction (Fig. 2.51*a*). To simplify our analysis, it will be assumed that the properties of the fibers and of the matrix have been combined, or "smeared," into a fictitious equivalent homogeneous material possessing these combined properties. We now consider a small element of that layer of smeared material (Fig. 2.51*b*). We denote by σ_x the corresponding normal stress and observe that $\sigma_y = \sigma_z = 0$. As indicated earlier in Sec. 2.5, the corresponding normal strain in the x direction is $\epsilon_x = \sigma_x/E_x$, where E_x is the modulus of elasticity of the composite material in the x direction. As we saw for isotropic materials, the elongation of the material in the x direction is accompanied by contractions in the y and z directions. These contractions depend upon the placement of the fibers in the matrix and will generally be different. It follows that the lateral strains ϵ_y and ϵ_z will also be different, and so will the corresponding Poisson's ratios:

$$\nu_{xy} = -\frac{\epsilon_y}{\epsilon_x} \qquad \text{and} \qquad \nu_{xz} = -\frac{\epsilon_z}{\epsilon_x} \tag{2.44}$$

Note that the first subscript in each of the Poisson's ratios ν_{xy} and ν_{xz} in Eqs. (2.44) refers to the direction of the load, and the second to the direction of the contraction.

It follows from the above that, in the case of the *multiaxial loading* of a layer of a composite material, equations similar to Eqs. (2.28) of Sec. 2.12 can be used to describe the stress-strain relationship. In the present case, however, three different values of the modulus of elasticity and six different values of Poisson's ratio will be involved. We write

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E_x} - \frac{\nu_{yx}\sigma_y}{E_y} - \frac{\nu_{zx}\sigma_z}{E_z} \\
\epsilon_y &= -\frac{\nu_{xy}\sigma_x}{E_x} + \frac{\sigma_y}{E_y} - \frac{\nu_{zy}\sigma_z}{E_z} \\
\epsilon_z &= -\frac{\nu_{xz}\sigma_x}{E_x} - \frac{\nu_{yz}\sigma_y}{E_y} + \frac{\sigma_z}{E_z}
\end{aligned} \tag{2.45}$$

Equations (2.45) may be considered as defining the transformation of stress into strain for the given layer. It follows from a general property of such transformations that the coefficients of the stress components are symmetric, i.e., that

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y} \qquad \frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z} \qquad \frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x} \tag{2.46}$$

These equations show that, while different, the Poisson's ratios ν_{xy} and ν_{yx} are not independent; either of them can be obtained from the other if the corresponding values of the modulus of elasticity are known. The same is true of ν_{yz} and ν_{zy}, and of ν_{zx} and ν_{xz}.

Consider now the effect of the presence of shearing stresses on the faces of a small element of smeared layer. As pointed out in

Sec. 2.14 in the case of isotropic materials, these stresses come in pairs of equal and opposite vectors applied to opposite sides of the given element and have no effect on the normal strains. Thus, Eqs. (2.45) remain valid. The shearing stresses, however, will create shearing strains which are defined by equations similar to the last three of the equations (2.38) of Sec. 2.14, except that three different values of the modulus of rigidity, G_{xy}, G_{yz}, and G_{zx}, must now be used. We have

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G_{yz}} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G_{zx}} \tag{2.47}$$

The fact that the three components of strain ϵ_x, ϵ_y, and ϵ_z can be expressed in terms of the normal stresses only and do not depend upon any shearing stresses characterizes *orthotropic materials* and distinguishes them from other anisotropic materials.

As we saw in Sec. 2.5, a flat *laminate* is obtained by superposing a number of layers or laminas. If the fibers in all layers are given the same orientation to better withstand an axial tensile load, the laminate itself will be orthotropic. If the lateral stability of the laminate is increased by positioning some of its layers so that their fibers are at a right angle to the fibers of the other layers, the resulting laminate will also be orthotropic. On the other hand, if any of the layers of a laminate are positioned so that their fibers are neither parallel nor perpendicular to the fibers of other layers, the lamina, generally, will not be orthotropic.†

EXAMPLE 2.11

A 60-mm cube is made from layers of graphite epoxy with fibers aligned in the x direction. The cube is subjected to a compressive load of 140 kN in the x direction. The properties of the composite material are: E_x = 155.0 GPa, E_y = 12.10 GPa, E_z = 12.10 GPa, ν_{xy} = 0.248, ν_{xz} = 0.248, and ν_{yz} = 0.458. Determine the changes in the cube dimensions, knowing that (*a*) the cube is free to expand in the y and z directions (Fig. 2.52); (*b*) the cube is free to expand in the z direction, but is restrained from expanding in the y direction by two fixed frictionless plates (Fig. 2.53).

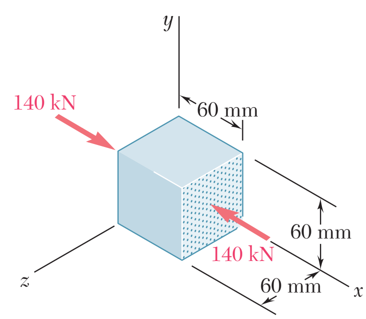


Fig. 2.52

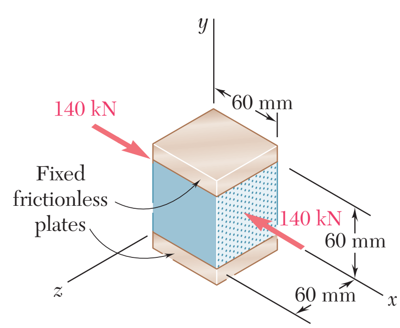


Fig. 2.53

(a) Free in *y* and *z* Directions. We first determine the stress σ_x in the direction of loading. We have

$$\sigma_x = \frac{P}{A} = \frac{-140 \times 10^3 \text{ N}}{(0.060 \text{ m})(0.060 \text{ m})} = -38.89 \text{ MPa}$$

Since the cube is not loaded or restrained in the y and z directions, we have $\sigma_y = \sigma_z = 0$. Thus, the right-hand members of Eqs. (2.45) reduce to their first terms. Substituting the given data into these equations, we write

$$\epsilon_x = \frac{\sigma_x}{E_x} = \frac{-38.89 \text{ MPa}}{155.0 \text{ GPa}} = -250.9 \times 10^{-6}$$

$$\epsilon_y = -\frac{\nu_{xy}\sigma_x}{E_x} = -\frac{(0.248)(-38.89 \text{ MPa})}{155.0 \text{ GPa}} = +62.22 \times 10^{-6}$$

$$\epsilon_z = -\frac{\nu_{xz}\sigma_x}{E_x} = -\frac{(0.248)(-38.69 \text{ MPa})}{155.0 \text{ GPa}} = +62.22 \times 10^{-6}$$

†For more information on fiber-reinforced composite materials, see Hyer, M. W., *Stress Analysis of Fiber-Reinforced Composite Materials,* McGraw-Hill, New York, 1998.

The changes in the cube dimensions are obtained by multiplying the corresponding strains by the length $L = 0.060$ m of the side of the cube:

$$\delta_x = \epsilon_x L = (-250.9 \times 10^{-6})(0.060 \text{ m}) = -15.05 \ \mu\text{m}$$
$$\delta_y = \epsilon_y L = (+62.2 \times 10^{-6})(0.060 \text{ m}) = +3.73 \ \mu\text{m}$$
$$\delta_z = \epsilon_z L = (+62.2 \times 10^{-6})(0.060 \text{ m}) = +3.73 \ \mu\text{m}$$

(b) Free in *z* Direction, Restrained in *y* Direction. The stress in the x direction is the same as in part a, namely, $\sigma_x = -38.89$ MPa. Since the cube is free to expand in the z direction as in part a, we again have $\sigma_z = 0$. But since the cube is now restrained in the y direction, we should expect a stress σ_y different from zero. On the other hand, since the cube cannot expand in the y direction, we must have $\delta_y = 0$ and, thus, $\epsilon_y = \delta_y/L = 0$. Making $\sigma_z = 0$ and $\epsilon_y = 0$ in the second of Eqs. (2.45), solving that equation for σ_y, and substituting the given data, we have

$$\sigma_y = \left(\frac{E_y}{E_x}\right)\nu_{xy}\sigma_x = \left(\frac{12.10}{155.0}\right)(0.248)(-38.89 \text{ MPa})$$
$$= -752.9 \text{ kPa}$$

Now that the three components of stress have been determined, we can use the first and last of Eqs. (2.45) to compute the strain components ϵ_x and ϵ_z. But the first of these equations contains Poisson's ratio ν_{yx} and, as we saw earlier, this ratio *is not equal* to the ratio ν_{xy} which was among the given data. To find ν_{yx} we use the first of Eqs. (2.46) and write

$$\nu_{yx} = \left(\frac{E_y}{E_x}\right)\nu_{xy} = \left(\frac{12.10}{155.0}\right)(0.248) = 0.01936$$

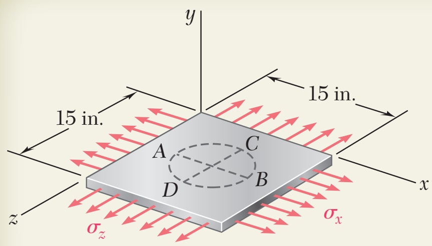
Making $\sigma_z = 0$ in the first and third of Eqs. (2.45) and substituting in these equations the given values of E_x, E_y, ν_{xz}, and ν_{yz}, as well as the values obtained for σ_x, σ_y, and ν_{yx}, we have

$$\epsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_{yx}\sigma_y}{E_y} = \frac{-38.89 \text{ MPa}}{155.0 \text{ GPa}} - \frac{(0.01936)(-752.9 \text{ kPa})}{12.10 \text{ GPa}}$$
$$= -249.7 \times 10^{-6}$$
$$\epsilon_z = -\frac{\nu_{xz}\sigma_x}{E_x} - \frac{\nu_{yz}\sigma_y}{E_y} = -\frac{(0.248)(-38.89 \text{ MPa})}{155.0 \text{ GPa}} - \frac{(0.458)(-752.9 \text{ kPa})}{12.10 \text{ GPa}}$$
$$= +90.72 \times 10^{-6}$$

The changes in the cube dimensions are obtained by multiplying the corresponding strains by the length $L = 0.060$ m of the side of the cube:

$$\delta_x = \epsilon_x L = (-249.7 \times 10^{-6})(0.060 \text{ m}) = -14.98 \ \mu\text{m}$$
$$\delta_y = \epsilon_y L = (0)(0.060 \text{ m}) = 0$$
$$\delta_z = \epsilon_z L = (+90.72 \times 10^{-6})(0.060 \text{ m}) = +5.44 \ \mu\text{m}$$

Comparing the results of parts a and b, we note that the difference between the values obtained for the deformation δ_x in the direction of the fibers is negligible. However, the difference between the values obtained for the lateral deformation δ_z is not negligible. This deformation is clearly larger when the cube is restrained from deforming in the y direction.



SAMPLE PROBLEM 2.5

A circle of diameter $d = 9$ in. is scribed on an unstressed aluminum plate of thickness $t = \frac{3}{4}$ in. Forces acting in the plane of the plate later cause normal stresses $\sigma_x = 12$ ksi and $\sigma_z = 20$ ksi. For $E = 10 \times 10^6$ psi and $\nu = \frac{1}{3}$, determine the change in (*a*) the length of diameter *AB*, (*b*) the length of diameter *CD*, (*c*) the thickness of the plate, (*d*) the volume of the plate.

SOLUTION

Hooke's Law. We note that $\sigma_y = 0$. Using Eqs. (2.28) we find the strain in each of the coordinate directions.

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[(12 \text{ ksi}) - 0 - \frac{1}{3}(20 \text{ ksi})\right] = +0.533 \times 10^{-3} \text{ in./in.}$$

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[-\frac{1}{3}(12 \text{ ksi}) + 0 - \frac{1}{3}(20 \text{ ksi})\right] = -1.067 \times 10^{-3} \text{ in./in.}$$

$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[-\frac{1}{3}(12 \text{ ksi}) - 0 + (20 \text{ ksi})\right] = +1.600 \times 10^{-3} \text{ in./in.}$$

a. Diameter *AB*. The change in length is $\delta_{B/A} = \epsilon_x d$.

$$\delta_{B/A} = \epsilon_x d = (+0.533 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{B/A} = +4.8 \times 10^{-3} \text{ in.} \quad \blacktriangleleft$$

b. Diameter *CD*.

$$\delta_{C/D} = \epsilon_z d = (+1.600 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{C/D} = +14.4 \times 10^{-3} \text{ in.} \quad \blacktriangleleft$$

c. Thickness. Recalling that $t = \frac{3}{4}$ in., we have

$$\delta_t = \epsilon_y t = (-1.067 \times 10^{-3} \text{ in./in.})(\tfrac{3}{4} \text{ in.})$$

$$\delta_t = -0.800 \times 10^{-3} \text{ in.} \quad \blacktriangleleft$$

d. Volume of the Plate. Using Eq. (2.30), we write

$$e = \epsilon_x + \epsilon_y + \epsilon_z = (+0.533 - 1.067 + 1.600)10^{-3} = +1.067 \times 10^{-3}$$

$$\Delta V = eV = +1.067 \times 10^{-3}[(15 \text{ in.})(15 \text{ in.})(\tfrac{3}{4} \text{ in.})] \quad \Delta V = +0.187 \times \text{in}^3 \quad \blacktriangleleft$$

PROBLEMS

2.61 A 600-lb tensile load is applied to a test coupon made from $\frac{1}{16}$-in. flat steel plate ($E = 29 \times 10^6$ psi, $\nu = 0.30$). Determine the resulting change (*a*) in the 2-in. gage length, (*b*) in the width of portion *AB* of the test coupon, (*c*) in the thickness of portion *AB*, (*d*) in the cross-sectional area of portion *AB*.

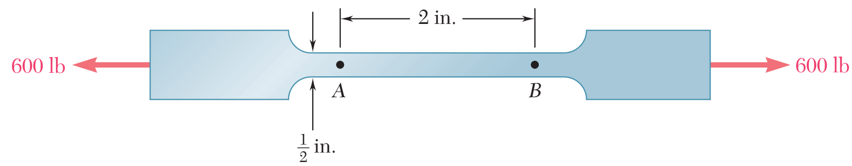


Fig. P2.61

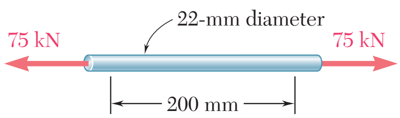


Fig. *P2.62*

2.62 In a standard tensile test a steel rod of 22-mm diameter is subjected to a tension force of 75 kN. Knowing that $\nu = 0.3$ and $E = 200$ GPa, determine (*a*) the elongation of the rod in a 200-mm gage length, (*b*) the change in diameter of the rod.

2.63 A 20-mm-diameter rod made of an experimental plastic is subjected to a tensile force of magnitude $P = 6$ kN. Knowing that an elongation of 14 mm and a decrease in diameter of 0.85 mm are observed in a 150-mm length, determine the modulus of elasticity, the modulus of rigidity, and Poisson's ratio for the material.

2.64 The change in diameter of a large steel bolt is carefully measured as the nut is tightened. Knowing that $E = 29 \times 10^6$ psi and $\nu = 0.30$, determine the internal force in the bolt, if the diameter is observed to decrease by 0.5×10^{-3} in.

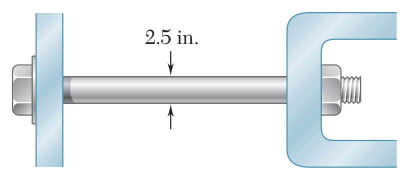


Fig. P2.64

2.65 A 2.5-m length of a steel pipe of 300-mm outer diameter and 15-mm wall thickness is used as a column to carry a 700-kN centric axial load. Knowing that $E = 200$ GPa and $\nu = 0.30$, determine (*a*) the change in length of the pipe, (*b*) the change in its outer diameter, (*c*) the change in its wall thickness.

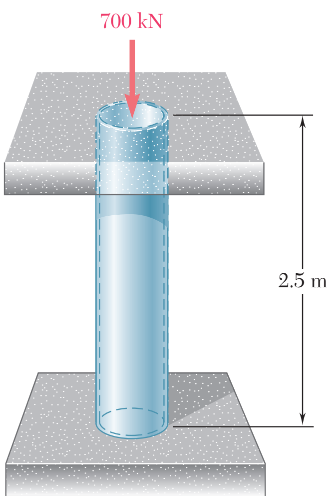


Fig. *P2.65*

2.66 An aluminum plate ($E = 74$ GPa, $\nu = 0.33$) is subjected to a centric axial load that causes a normal stress σ. Knowing that, before loading, a line of slope 2:1 is scribed on the plate, determine the slope of the line when $\sigma = 125$ MPa.

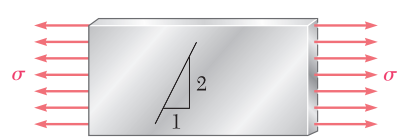


Fig. P2.66

2.67 The block shown is made of a magnesium alloy for which $E = 45$ GPa and $\nu = 0.35$. Knowing that $\sigma_x = -180$ MPa, determine (*a*) the magnitude of σ_y for which the change in the height of the block will be zero, (*b*) the corresponding change in the area of the face *ABCD*, (*c*) the corresponding change in the volume of the block.

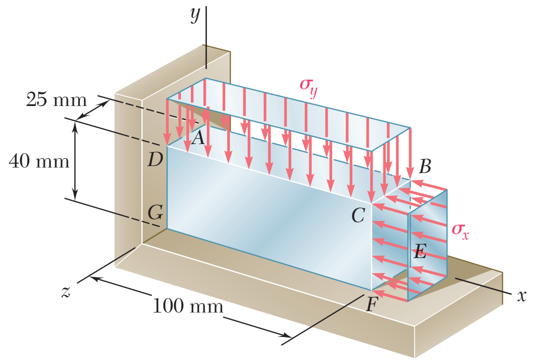


Fig. P2.67

2.68 A 30-mm square was scribed on the side of a large steel pressure vessel. After pressurization the biaxial stress condition at the square is as shown. For $E = 200$ GPa and $\nu = 0.30$, determine the change in length of (*a*) side *AB*, (*b*) side *BC*, (*c*) diagonal *AC*.

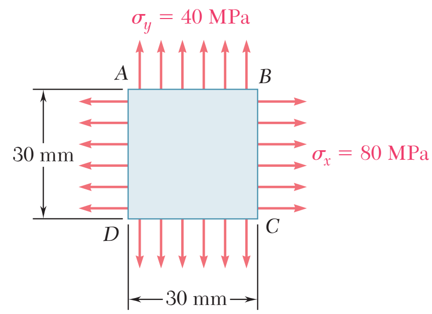


Fig. P2.68

2.69 The aluminum rod *AD* is fitted with a jacket that is used to apply a hydrostatic pressure of 6000 psi to the 12-in. portion *BC* of the rod. Knowing that $E = 10.1 \times 10^6$ psi and $\nu = 0.36$, determine (*a*) the change in the total length *AD*, (*b*) the change in diameter at the middle of the rod.

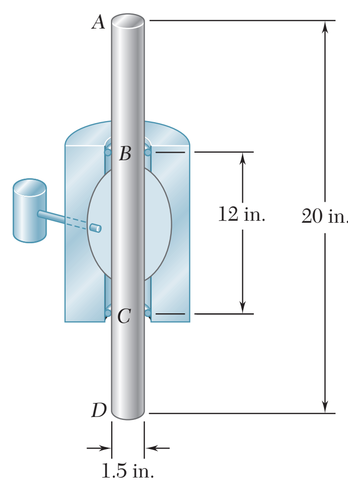


Fig. P2.69

2.70 For the rod of Prob. 2.69, determine the forces that should be applied to the ends *A* and *D* of the rod (*a*) if the axial strain in portion *BC* of the rod is to remain zero as the hydrostatic pressure is applied, (*b*) if the total length *AD* of the rod is to remain unchanged.

2.71 In many situations physical constraints prevent strain from occurring in a given direction. For example, $\epsilon_z = 0$ in the case shown, where longitudinal movement of the long prism is prevented at every point. Plane sections perpendicular to the longitudinal axis remain plane and the same distance apart. Show that for this situation, which is known as *plane strain*, we can express σ_z, ϵ_x, and ϵ_y as follows:

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\epsilon_x = \frac{1}{E}[(1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y]$$

$$\epsilon_y = \frac{1}{E}[(1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x]$$

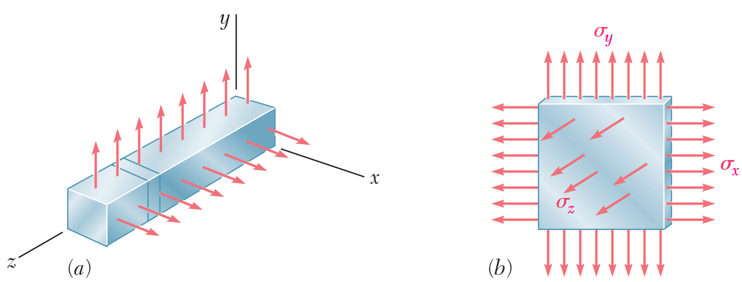


Fig. P2.71

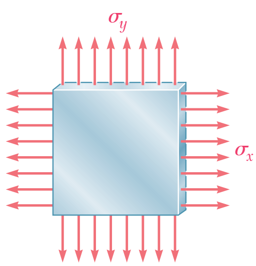


Fig. P2.72

2.72 In many situations it is known that the normal stress in a given direction is zero. For example, $\sigma_z = 0$ in the case of the thin plate shown. For this case, which is known as *plane stress,* show that if the strains ϵ_x and ϵ_y have been determined experimentally, we can express σ_x, σ_y and ϵ_z as follows:

$$\sigma_x = E\frac{\epsilon_x + \nu\epsilon_y}{1 - \nu^2}$$

$$\sigma_y = E\frac{\epsilon_y + \nu\epsilon_x}{1 - \nu^2}$$

$$\epsilon_z = -\frac{\nu}{1 - \nu}(\epsilon_x + \epsilon_y)$$

2.73 For a member under axial loading, express the normal strain ϵ' in a direction forming an angle of 45° with the axis of the load in terms of the axial strain ϵ_x by (*a*) comparing the hypotenuses of the triangles shown in Fig. 2.50, which represent respectively an element before and after deformation, (*b*) using the values of the corresponding stresses σ' and σ_x shown in Fig. 1.38, and the generalized Hooke's law.

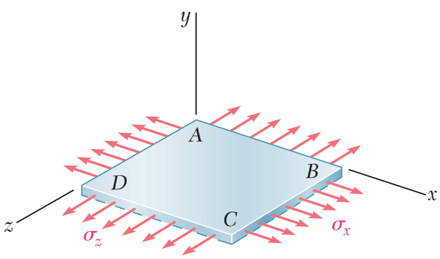


Fig. *P2.74*

2.74 The homogeneous plate *ABCD* is subjected to a biaxial loading as shown. It is known that $\sigma_z = \sigma_0$ and that the change in length of the plate in the x direction must be zero, that is, $\epsilon_x = 0$. Denoting by E the modulus of elasticity and by ν Poisson's ratio, determine (*a*) the required magnitude of σ_x, (*b*) the ratio σ_0/ϵ_z.

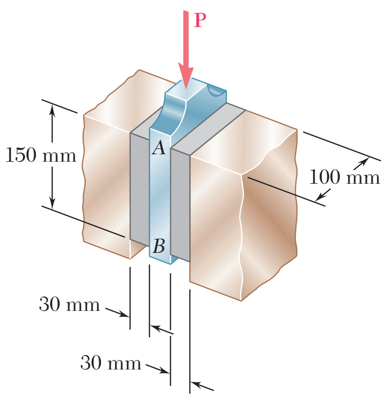


Fig. P2.75 and P2.76

2.75 A vibration isolation unit consists of two blocks of hard rubber bonded to a plate *AB* and to rigid supports as shown. Knowing that a force of magnitude $P = 25$ kN causes a deflection $\delta = 1.5$ mm of plate *AB*, determine the modulus of rigidity of the rubber used.

2.76 A vibration isolation unit consists of two blocks of hard rubber with a modulus of rigidity $G = 19$ MPa bonded to a plate *AB* and to rigid supports as shown. Denoting by P the magnitude of the force applied to the plate and by δ the corresponding deflection, determine the effective spring constant, $k = P/\delta$, of the system.

2.77 The plastic block shown is bonded to a fixed base and to a horizontal rigid plate to which a force **P** is applied. Knowing that for the plastic used $G = 55$ ksi, determine the deflection of the plate when $P = 9$ kips.

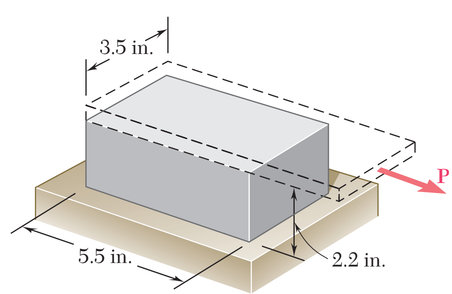


Fig. P2.77

2.78 A vibration isolation unit consists of two blocks of hard rubber bonded to plate AB and to rigid supports as shown. For the type and grade of rubber used $\tau_{all} = 220$ psi and $G = 1800$ psi. Knowing that a centric vertical force of magnitude $P = 3.2$ kips must cause a 0.1-in. vertical deflection of the plate AB, determine the smallest allowable dimensions a and b of the block.

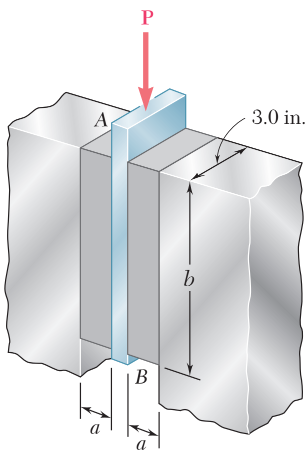


Fig. P2.78

2.79 The plastic block shown is bonded to a rigid support and to a vertical plate to which a 55-kip load **P** is applied. Knowing that for the plastic used $G = 150$ ksi, determine the deflection of the plate.

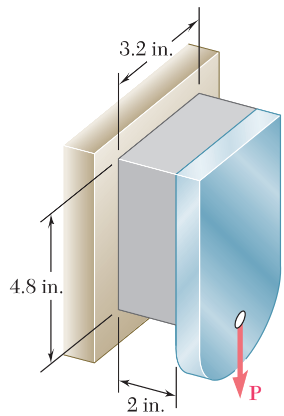
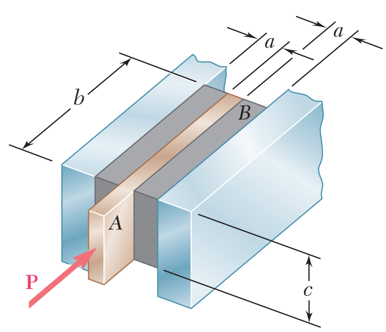


Fig. *P2.79*

2.80 What load **P** should be applied to the plate of Prob. 2.79 to produce a $\frac{1}{16}$-in. deflection?

2.81 Two blocks of rubber with a modulus of rigidity $G = 12$ MPa are bonded to rigid supports and to a plate AB. Knowing that $c = 100$ mm and $P = 45$ kN, determine the smallest allowable dimensions a and b of the blocks if the shearing stress in the rubber is not to exceed 1.4 MPa and the deflection of the plate is to be at least 5 mm.

2.82 Two blocks of rubber with a modulus of rigidity $G = 10$ MPa are bonded to rigid supports and to a plate AB. Knowing that $b = 200$ mm and $c = 125$ mm, determine the largest allowable load P and the smallest allowable thickness a of the blocks if the shearing stress in the rubber is not to exceed 1.5 MPa and the deflection of the plate is to be at least 6 mm.



Figs. P2.81 and P2.82

***2.83** Determine the dilatation e and the change in volume of the 200-mm length of the rod shown if (a) the rod is made of steel with $E = 200$ GPa and $\nu = 0.30$, (b) the rod is made of aluminum with $E = 70$ GPa and $\nu = 0.35$.

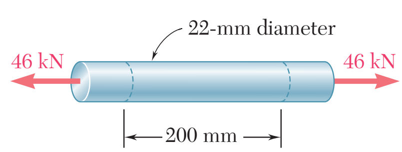


Fig. *P2.83*

***2.84** Determine the change in volume of the 2-in. gage length segment AB in Prob. 2.61 (a) by computing the dilatation of the material, (b) by subtracting the original volume of portion AB from its final volume.

***2.85** A 6-in.-diameter solid steel sphere is lowered into the ocean to a point where the pressure is 7.1 ksi (about 3 miles below the surface). Knowing that $E = 29 \times 10^6$ psi and $\nu = 0.30$, determine (a) the decrease in diameter of the sphere, (b) the decrease in volume of the sphere, (c) the percent increase in the density of the sphere.

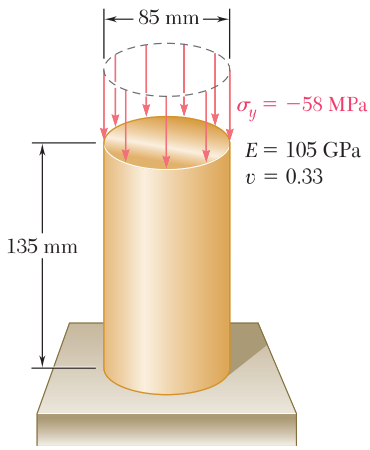


Fig. P2.86

***2.86** (*a*) For the axial loading shown, determine the change in height and the change in volume of the brass cylinder shown. (*b*) Solve part *a*, assuming that the loading is hydrostatic with $\sigma_x = \sigma_y = \sigma_z = -70$ MPa.

***2.87** A vibration isolation support consists of a rod *A* of radius $R_1 = 10$ mm and a tube *B* of inner radius $R_2 = 25$ mm bonded to an 80-mm-long hollow rubber cylinder with a modulus of rigidity $G = 12$ MPa. Determine the largest allowable force **P** that can be applied to rod *A* if its deflection is not to exceed 2.50 mm.

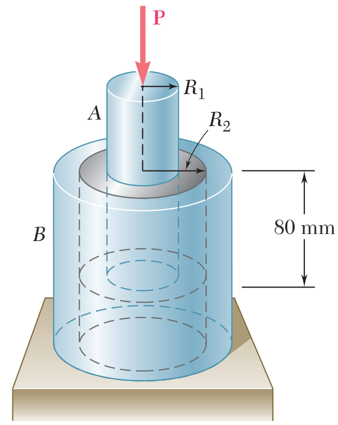


Fig. *P2.87* and P2.88

***2.88** A vibration isolation support consists of a rod *A* of radius R_1 and a tube *B* of inner radius R_2 bonded to an 80-mm-long hollow rubber cylinder with a modulus of rigidity $G = 10.93$ MPa. Determine the required value of the ratio R_2/R_1 if a 10-kN force **P** is to cause a 2-mm deflection of rod *A*.

***2.89** The material constants *E*, *G*, *k*, and ν are related by Eqs. (2.33) and (2.43). Show that any one of the constants may be expressed in terms of any other two constants. For example, show that (*a*) $k = GE/(9G - 3E)$ and (*b*) $\nu = (3k - 2G)/(6k + 2G)$.

***2.90** Show that for any given material, the ratio *G/E* of the modulus of rigidity over the modulus of elasticity is always less than $\frac{1}{2}$ but more than $\frac{1}{3}$. [*Hint:* Refer to Eq. (2.43) and to Sec. 2.13.]

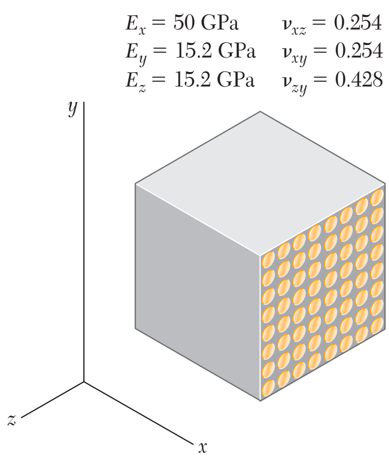


Fig. P2.91

***2.91** A composite cube with 40-mm sides and the properties shown is made with glass polymer fibers aligned in the *x* direction. The cube is constrained against deformations in the *y* and *z* directions and is subjected to a tensile load of 65 kN in the *x* direction. Determine (*a*) the change in the length of the cube in the *x* direction, (*b*) the stresses σ_x, σ_y, and σ_z.

***2.92** The composite cube of Prob. 2.91 is constrained against deformation in the *z* direction and elongated in the *x* direction by 0.035 mm due to a tensile load in the *x* direction. Determine (*a*) the stresses σ_x, σ_y, and σ_z, (*b*) the change in the dimension in the *y* direction.

2.17 STRESS AND STRAIN DISTRIBUTION UNDER AXIAL LOADING; SAINT-VENANT'S PRINCIPLE

We have assumed so far that, in an axially loaded member, the normal stresses are uniformly distributed in any section perpendicular to the axis of the member. As we saw in Sec. 1.5, such an assumption may be quite in error in the immediate vicinity of the points of application of the loads. However, the determination of the actual stresses in a given section of the member requires the solution of a statically indeterminate problem.

In Sec. 2.9, you saw that statically indeterminate problems involving the determination of *forces* can be solved by considering the *deformations* caused by these forces. It is thus reasonable to conclude that the determination of the *stresses* in a member requires the analysis of the *strains* produced by the stresses in the member. This is essentially the approach found in advanced textbooks, where the mathematical theory of elasticity is used to determine the distribution of stresses corresponding to various modes of application of the loads at the ends of the member. Given the more limited mathematical means at our disposal, our analysis of stresses will be restricted to the particular case when two rigid plates are used to transmit the loads to a member made of a homogeneous isotropic material (Fig. 2.54).

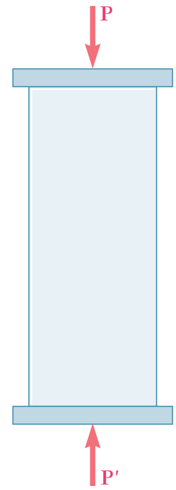


Fig. 2.54 Axial load applied by rigid plates to a member.

If the loads are applied at the center of each plate,† the plates will move toward each other without rotating, causing the member to get shorter, while increasing in width and thickness. It is reasonable to assume that the member will remain straight, that plane sections will remain plane, and that all elements of the member will deform in the same way, since such an assumption is clearly compatible with the given end conditions. This is illustrated in Fig. 2.55,

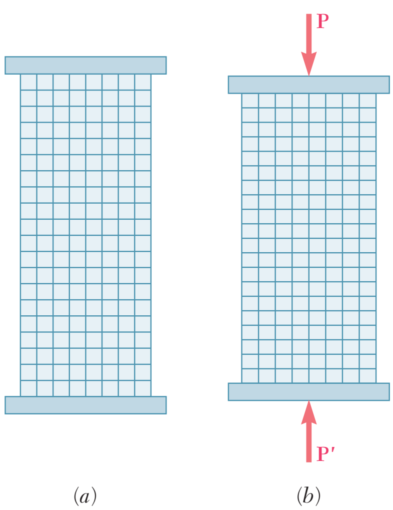


Fig. 2.55 Axial load applied by rigid plates to rubber model.

†More precisely, the common line of action of the loads should pass through the centroid of the cross section (cf. Sec. 1.5).

which shows a rubber model before and after loading.† Now, if all elements deform in the same way, the distribution of strains throughout the member must be uniform. In other words, the axial strain ϵ_y and the lateral strain $\epsilon_x = -\nu\epsilon_y$ are constant. But, if the stresses do not exceed the proportional limit, Hooke's law applies and we may write $\sigma_y = E\epsilon_y$, from which it follows that the normal stress σ_y is also constant. Thus, the distribution of stresses is uniform throughout the member and, at any point,

$$\sigma_y = (\sigma_y)_{\text{ave}} = \frac{P}{A}$$

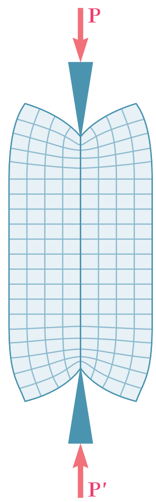


Fig. 2.56 Concentrated axial load applied to rubber model.

On the other hand, if the loads are concentrated, as illustrated in Fig. 2.56, the elements in the immediate vicinity of the points of application of the loads are subjected to very large stresses, while other elements near the ends of the member are unaffected by the loading. This may be verified by observing that strong deformations, and thus large strains and large stresses, occur near the points of application of the loads, while no deformation takes place at the corners. As we consider elements farther and farther from the ends, however, we note a progressive equalization of the deformations involved, and thus a more nearly uniform distribution of the strains and stresses across a section of the member. This is further illustrated in Fig. 2.57, which shows the result of the calculation by advanced mathematical methods of the distribution of stresses across various sections of a thin rectangular plate subjected to concentrated loads. We note that at a distance b from either end, where b is the width of the plate, the stress distribution is nearly uniform across the section, and the value of the stress σ_y at any point of that section can be assumed equal to the average value P/A. Thus, at a distance equal to, or greater than, the width of the member, the distribution of stresses across a given section is the same, whether the member is loaded as shown in Fig. 2.54 or Fig. 2.56. In other words, except in the immediate vicinity of the points of application of the loads, the stress distribution may be assumed independent of the actual mode of application of the loads. This statement, which applies not only to axial loadings, but to practically any type of load, is known as *Saint-Venant's principle,* after the French mathematician and engineer Adhémar Barré de Saint-Venant (1797–1886).

While Saint-Venant's principle makes it possible to replace a given loading by a simpler one for the purpose of computing the stresses in a structural member, you should keep in mind two important points when applying this principle:

1. The actual loading and the loading used to compute the stresses must be *statically equivalent*.
2. Stresses cannot be computed in this manner in the immediate vicinity of the points of application of the loads. Advanced theoretical or experimental methods must be used to determine the distribution of stresses in these areas.

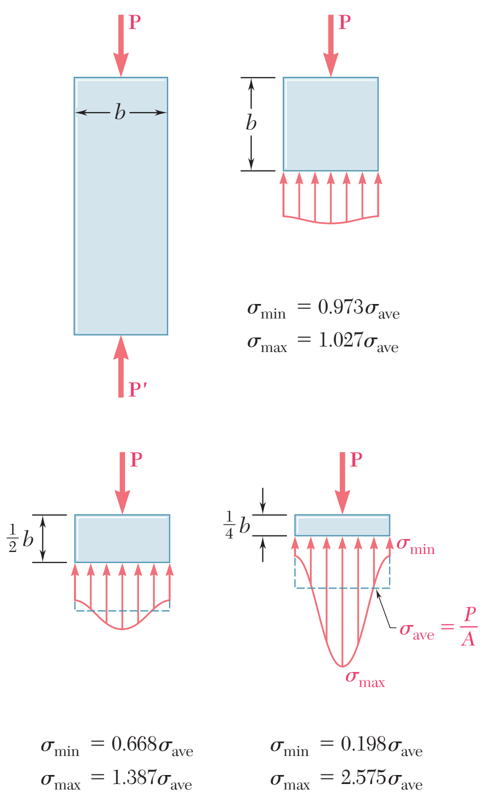


Fig. 2.57 Stress distributions in a plate under concentrated axial loads.

†Note that for long, slender members, another configuration is possible, and indeed will prevail, if the load is sufficiently large; the member *buckles* and assumes a curved shape. This will be discussed in Chap. 10.

You should also observe that the plates used to obtain a uniform stress distribution in the member of Fig. 2.55 must allow the member to freely expand laterally. Thus, the plates cannot be rigidly attached to the member; you must assume them to be just in contact with the member, and smooth enough not to impede the lateral expansion of the member. While such end conditions can actually be achieved for a member in compression, they cannot be physically realized in the case of a member in tension. It does not matter, however, whether or not an actual fixture can be realized and used to load a member so that the distribution of stresses in the member is uniform. The important thing is to *imagine a model* that will allow such a distribution of stresses, and to keep this model in mind so that you may later compare it with the actual loading conditions.

2.18 STRESS CONCENTRATIONS

As you saw in the preceding section, the stresses near the points of application of concentrated loads can reach values much larger than the average value of the stress in the member. When a structural member contains a discontinuity, such as a hole or a sudden change in cross section, high localized stresses can also occur near the discontinuity. Figures 2.58 and 2.59 show the distribution of stresses in critical sections corresponding to two such situations. Figure 2.58 refers to a flat bar with a *circular hole* and shows the stress distribution in a section passing through the center of the hole. Figure 2.59 refers to a flat bar consisting of two portions of different widths connected by *fillets;* it shows the stress distribution in the narrowest part of the connection, where the highest stresses occur.

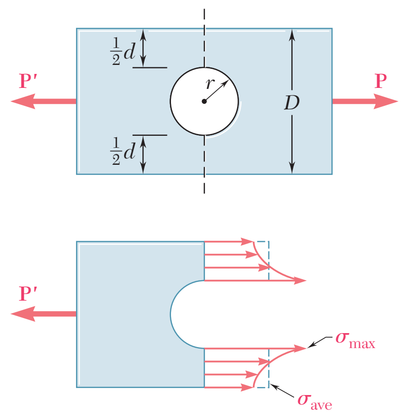


Fig. 2.58 Stress distribution near circular hole in flat bar under axial loading.

These results were obtained experimentally through the use of a photoelastic method. Fortunately for the engineer who has to design a given member and cannot afford to carry out such an analysis, the results obtained are independent of the size of the member and of the material used; they depend only upon the ratios of the geometric parameters involved, i.e., upon the ratio r/d in the case of a circular hole, and upon the ratios r/d and D/d in the case of fillets. Furthermore, the designer is more interested in the *maximum value* of the stress in a given section, than in the actual distribution of stresses in that section, since the main concern is to determine *whether* the allowable stress will be exceeded under a given loading, and not *where* this value will be exceeded. For this reason, one defines the ratio

$$K = \frac{\sigma_{max}}{\sigma_{ave}} \tag{2.48}$$

of the maximum stress over the average stress computed in the critical (narrowest) section of the discontinuity. This ratio is referred to as the *stress-concentration factor* of the given discontinuity. Stress-concentration factors can be computed once and for all in terms of the ratios of the geometric parameters involved, and the results obtained can be expressed in the form of tables or of graphs, as

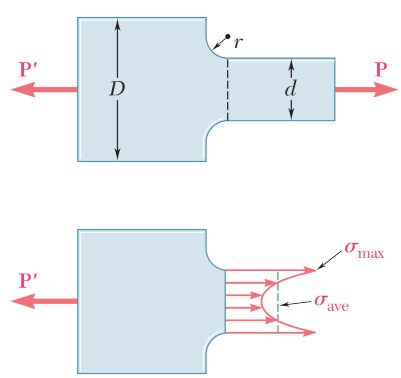
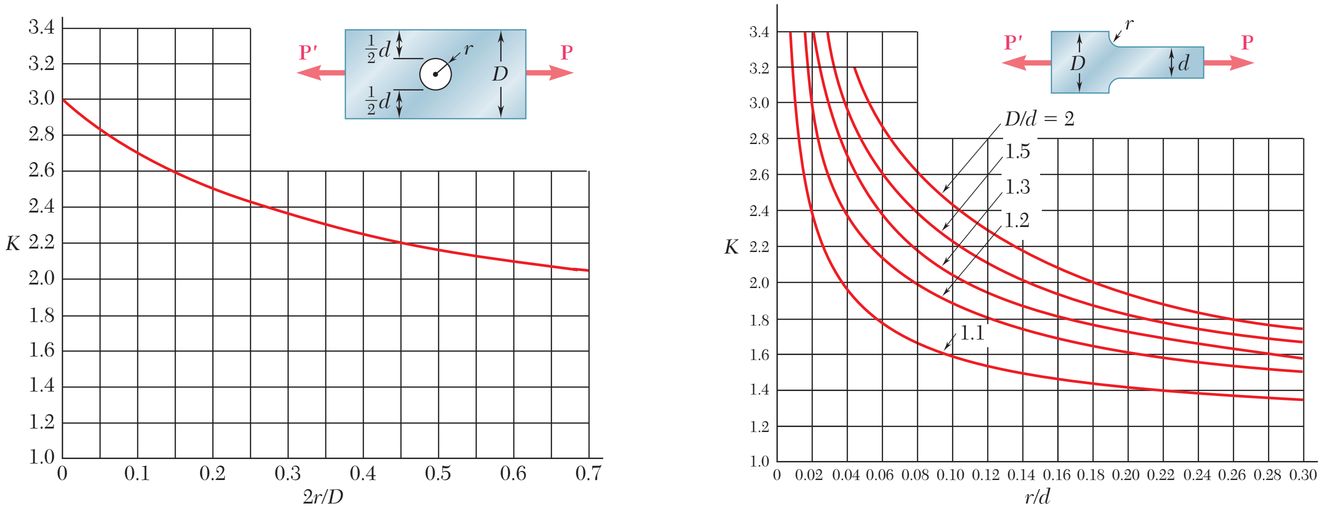


Fig. 2.59 Stress distribution near fillets in flat bar under axial loading.



(*a*) Flat bars with holes

(*b*) Flat bars with fillets

Fig. 2.60 Stress concentration factors for flat bars under axial loading†

Note that the average stress must be computed across the narrowest section: $\sigma_{ave} = P/td$, where t is the thickness of the bar.

shown in Fig. 2.60. To determine the maximum stress occurring near a discontinuity in a given member subjected to a given axial load P, the designer needs only to compute the average stress $\sigma_{ave} = P/A$ in the critical section, and multiply the result obtained by the appropriate value of the stress-concentration factor K. You should note, however, that this procedure is valid only as long as σ_{max} does not exceed the proportional limit of the material, since the values of K plotted in Fig. 2.60 were obtained by assuming a linear relation between stress and strain.

EXAMPLE 2.12

Determine the largest axial load **P** that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick and, respectively, 40 and 60 mm wide, connected by fillets of radius $r = 8$ mm. Assume an allowable normal stress of 165 MPa.

We first compute the ratios

$$\frac{D}{d} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.50 \qquad \frac{r}{d} = \frac{8 \text{ mm}}{40 \text{ mm}} = 0.20$$

Using the curve in Fig. 2.60*b* corresponding to $D/d = 1.50$, we find that the value of the stress-concentration factor corresponding to $r/d = 0.20$ is

$$K = 1.82$$

Carrying this value into Eq. (2.48) and solving for σ_{ave}, we have

$$\sigma_{ave} = \frac{\sigma_{max}}{1.82}$$

But σ_{max} cannot exceed the allowable stress $\sigma_{all} = 165$ MPa. Substituting this value for σ_{max}, we find that the average stress in the narrower portion ($d = 40$ mm) of the bar should not exceed the value

$$\sigma_{ave} = \frac{165 \text{ MPa}}{1.82} = 90.7 \text{ MPa}$$

Recalling that $\sigma_{ave} = P/A$, we have

$$P = A\sigma_{ave} = (40 \text{ mm})(10 \text{ mm})(90.7 \text{ MPa}) = 36.3 \times 10^3 \text{ N}$$

$$P = 36.3 \text{ kN}$$

†W. D. Pilkey, *Peterson's Stress Concentration Factors,* 2nd ed., John Wiley & Sons, New York, 1997.

2.19 PLASTIC DEFORMATIONS

The results obtained in the preceding sections were based on the assumption of a linear stress-strain relationship. In other words, we assumed that the proportional limit of the material was never exceeded. This is a reasonable assumption in the case of brittle materials, which rupture without yielding. In the case of ductile materials, however, this assumption implies that the yield strength of the material is not exceeded. The deformations will then remain within the elastic range and the structural member under consideration will regain its original shape after all loads have been removed. If, on the other hand, the stresses in any part of the member exceed the yield strength of the material, plastic deformations occur and most of the results obtained in earlier sections cease to be valid. A more involved analysis, based on a nonlinear stress-strain relationship, must then be carried out.

While an analysis taking into account the actual stress-strain relationship is beyond the scope of this text, we gain considerable insight into plastic behavior by considering an idealized *elastoplastic material* for which the stress-strain diagram consists of the two straight-line segments shown in Fig. 2.61. We may note that the stress-strain diagram for mild steel in the elastic and plastic ranges is similar to this idealization. As long as the stress σ is less than the yield strength σ_Y, the material behaves elastically and obeys Hooke's law, $\sigma = E\epsilon$. When σ reaches the value σ_Y, the material starts yielding and keeps deforming plastically under a constant load. If the load is removed, unloading takes place along a straight-line segment CD parallel to the initial portion AY of the loading curve. The segment AD of the horizontal axis represents the strain corresponding to the permanent set or plastic deformation resulting from the loading and unloading of the specimen. While no actual material behaves exactly as shown in Fig. 2.61, this stress-strain diagram will prove useful in discussing the plastic deformations of ductile materials such as mild steel.

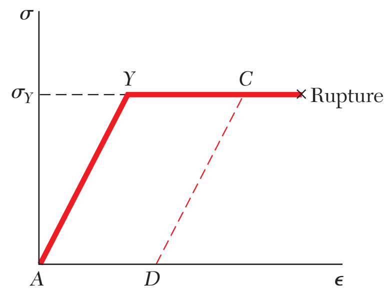


Fig. 2.61 Stress-strain diagram for an idealized elastoplastic material.

EXAMPLE 2.13

A rod of length $L = 500$ mm and cross-sectional area $A = 60\text{ mm}^2$ is made of an elastoplastic material having a modulus of elasticity $E = 200$ GPa in its elastic range and a yield point $\sigma_Y = 300$ MPa. The rod is subjected to an axial load until it is stretched 7 mm and the load is then removed. What is the resulting permanent set?

Referring to the diagram of Fig. 2.61, we find that the maximum strain, represented by the abscissa of point C, is

$$\epsilon_C = \frac{\delta_C}{L} = \frac{7\text{ mm}}{500\text{ mm}} = 14 \times 10^{-3}$$

On the other hand, the yield strain, represented by the abscissa of point Y, is

$$\epsilon_Y = \frac{\sigma_Y}{E} = \frac{300 \times 10^6\text{ Pa}}{200 \times 10^9\text{ Pa}} = 1.5 \times 10^{-3}$$

The strain after unloading is represented by the abscissa ϵ_D of point D. We note from Fig. 2.61 that

$$\begin{aligned}\epsilon_D &= AD = YC = \epsilon_C - \epsilon_Y \\ &= 14 \times 10^{-3} - 1.5 \times 10^{-3} = 12.5 \times 10^{-3}\end{aligned}$$

The permanent set is the deformation δ_D corresponding to the strain ϵ_D. We have

$$\delta_D = \epsilon_D L = (12.5 \times 10^{-3})(500\text{ mm}) = 6.25\text{ mm}$$

EXAMPLE 2.14

A 30-in.-long cylindrical rod of cross-sectional area $A_r = 0.075 \text{ in}^2$ is placed inside a tube of the same length and of cross-sectional area $A_t = 0.100 \text{ in}^2$. The ends of the rod and tube are attached to a rigid support on one side, and to a rigid plate on the other, as shown in the longitudinal section of Fig. 2.62. The rod and tube are both assumed to be elastoplastic, with moduli of elasticity $E_r = 30 \times 10^6$ psi and $E_t = 15 \times 10^6$ psi, and yield strengths $(\sigma_r)_Y = 36$ ksi and $(\sigma_t)_Y = 45$ ksi. Draw the load-deflection diagram of the rod-tube assembly when a load **P** is applied to the plate as shown.

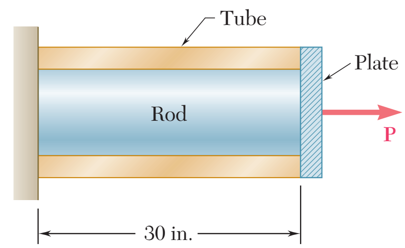


Fig. 2.62

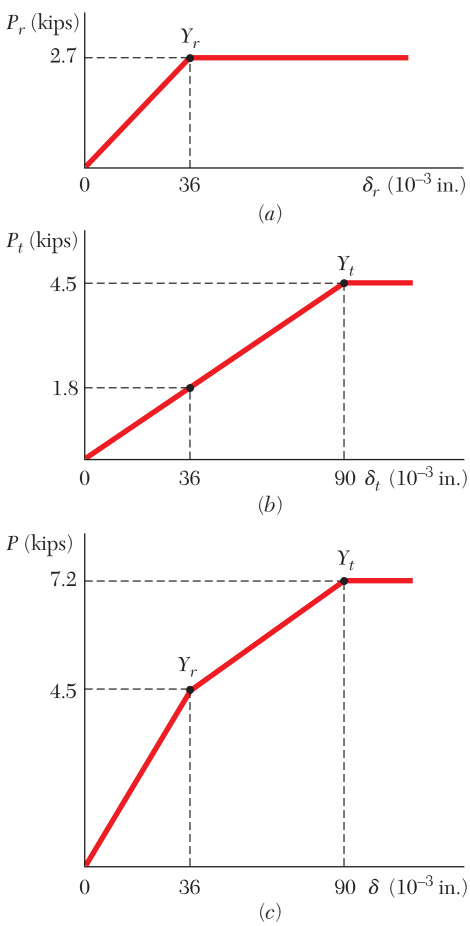


Fig. 2.63

We first determine the internal force and the elongation of the rod as it begins to yield:

$$(P_r)_Y = (\sigma_r)_Y A_r = (36 \text{ ksi})(0.075 \text{ in}^2) = 2.7 \text{ kips}$$

$$(\delta_r)_Y = (\epsilon_r)_Y L = \frac{(\sigma_r)_Y}{E_r} L = \frac{36 \times 10^3 \text{ psi}}{30 \times 10^6 \text{ psi}}(30 \text{ in.})$$
$$= 36 \times 10^{-3} \text{ in.}$$

Since the material is elastoplastic, the force-elongation diagram *of the rod alone* consists of an oblique straight line and of a horizontal straight line, as shown in Fig. 2.63*a*. Following the same procedure for the tube, we have

$$(P_t)_Y = (\sigma_t)_Y A_t = (45 \text{ ksi})(0.100 \text{ in}^2) = 4.5 \text{ kips}$$

$$(\delta_t)_Y = (\epsilon_t)_Y L = \frac{(\sigma_t)_Y}{E_t} L = \frac{45 \times 10^3 \text{ psi}}{15 \times 10^6 \text{ psi}}(30 \text{ in.})$$
$$= 90 \times 10^{-3} \text{ in.}$$

The load-deflection diagram of the tube alone is shown in Fig. 2.63*b*. Observing that the load and deflection of the rod-tube combination are, respectively,

$$P = P_r + P_t \qquad \delta = \delta_r = \delta_t$$

we draw the required load-deflection diagram by adding the ordinates of the diagrams obtained for the rod and for the tube (Fig. 2.63*c*). Points Y_r and Y_t correspond to the onset of yield in the rod and in the tube, respectively.

EXAMPLE 2.15

If the load **P** applied to the rod-tube assembly of Example 2.14 is increased from zero to 5.7 kips and decreased back to zero, determine (*a*) the maximum elongation of the assembly, (*b*) the permanent set after the load has been removed.

(a) Maximum Elongation. Referring to Fig. 2.63*c*, we observe that the load $P_{max} = 5.7$ kips corresponds to a point located on the segment Y_rY_t of the load-deflection diagram of the assembly. Thus, the rod has reached the plastic range, with $P_r = (P_r)_Y = 2.7$ kips and $\sigma_r = (\sigma_r)_Y = 36$ ksi, while the tube is still in the elastic range, with

$$P_t = P - P_r = 5.7 \text{ kips} - 2.7 \text{ kips} = 3.0 \text{ kips}$$

$$\sigma_t = \frac{P_t}{A_t} = \frac{3.0 \text{ kips}}{0.1 \text{ in}^2} = 30 \text{ ksi}$$

$$\delta_t = \epsilon_t L = \frac{\sigma_t}{E_t} L = \frac{30 \times 10^3 \text{ psi}}{15 \times 10^6 \text{ psi}}(30 \text{ in.}) = 60 \times 10^{-3} \text{ in.}$$

The maximum elongation of the assembly, therefore, is

$$\delta_{max} = \delta_t = 60 \times 10^{-3} \text{ in.}$$

(b) Permanent Set. As the load **P** decreases from 5.7 kips to zero, the internal forces P_r and P_t both decrease along a straight line, as shown in Fig. 2.64*a* and *b*, respectively. The force P_r decreases along line *CD* parallel to the initial portion of the loading curve, while the force P_t decreases along the original loading curve, since the yield stress was not exceeded in the tube. Their sum *P*, therefore, will decrease along a line *CE* parallel to the portion $0Y_r$ of the load-deflection curve of the assembly (Fig. 2.64*c*). Referring to Fig. 2.63*c*, we find that the slope of $0Y_r$, and thus of *CE*, is

$$m = \frac{4.5 \text{ kips}}{36 \times 10^{-3} \text{ in.}} = 125 \text{ kips/in.}$$

The segment of line *FE* in Fig. 2.64*c* represents the deformation δ' of the assembly during the unloading phase, and the segment $0E$ the permanent set δ_p after the load **P** has been removed. From triangle *CEF* we have

$$\delta' = -\frac{P_{max}}{m} = -\frac{5.7 \text{ kips}}{125 \text{ kips/in.}} = -45.6 \times 10^{-3} \text{ in.}$$

The permanent set is thus

$$\delta_P = \delta_{max} + \delta' = 60 \times 10^{-3} - 45.6 \times 10^{-3}$$
$$= 14.4 \times 10^{-3} \text{ in.}$$

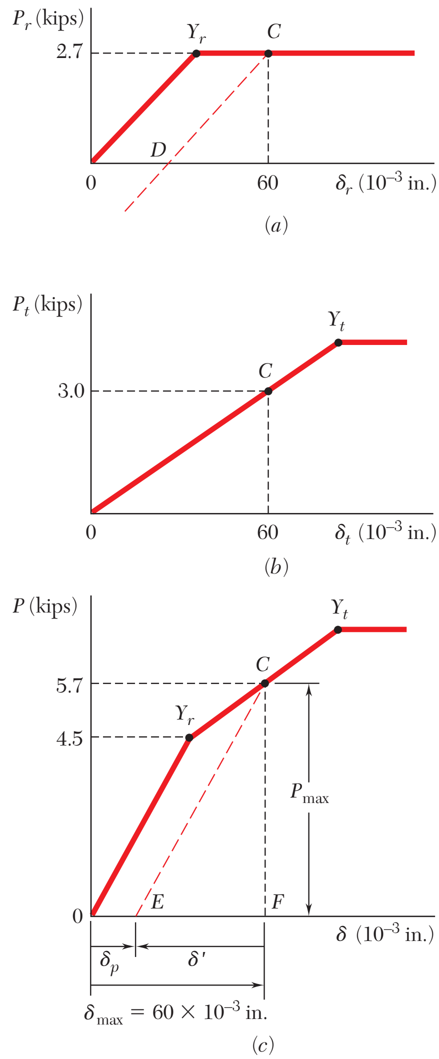


Fig. 2.64

We recall that the discussion of stress concentrations of Sec. 2.18 was carried out under the assumption of a linear stress-strain relationship. The stress distributions shown in Figs. 2.58 and 2.59, and the values of the stress-concentration factors plotted in Fig. 2.60 cannot be used, therefore, when plastic deformations take place, i.e., when the value of σ_{max} obtained from these figures exceeds the yield strength σ_Y.

Let us consider again the flat bar with a circular hole of Fig. 2.58, and let us assume that the material is elastoplastic, i.e., that its stress-strain diagram is as shown in Fig. 2.61. As long as no plastic deformation takes place, the distribution of stresses is as indicated in Sec. 2.18

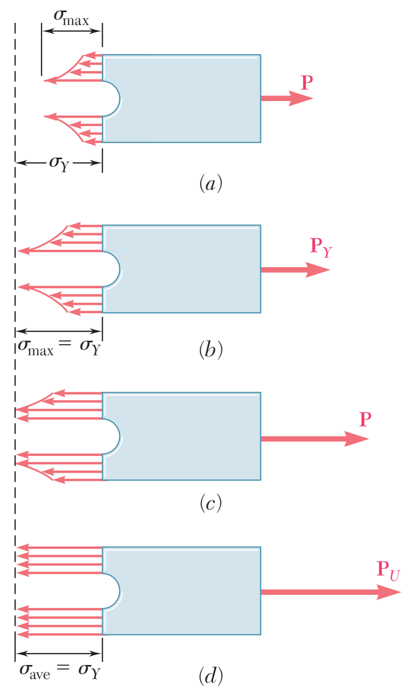


Fig. 2.65 Distribution of stresses in elastoplastic material under increasing load.

(Fig. 2.65*a*). We observe that the area under the stress-distribution curve represents the integral $\int \sigma\, dA$, which is equal to the load P. Thus this area, and the value of σ_{max}, must increase as the load P increases. As long as $\sigma_{max} \leq \sigma_Y$, all the successive stress distributions obtained as P increases will have the shape shown in Fig. 2.58 and repeated in Fig. 2.65*a*. However, as P is increased beyond the value P_Y corresponding to $\sigma_{max} = \sigma_Y$ (Fig. 2.65*b*), the stress-distribution curve must flatten in the vicinity of the hole (Fig. 2.65*c*), since the stress in the material considered cannot exceed the value σ_Y. This indicates that the material is yielding in the vicinity of the hole. As the load P is further increased, the plastic zone where yield takes place keeps expanding, until it reaches the edges of the plate (Fig. 2.65*d*). At that point, the distribution of stresses across the plate is uniform, $\sigma = \sigma_Y$, and the corresponding value $P = P_U$ of the load is the largest that can be applied to the bar without causing rupture.

It is interesting to compare the maximum value P_Y of the load that can be applied if no permanent deformation is to be produced in the bar, with the value P_U that will cause rupture. Recalling the definition of the average stress, $\sigma_{ave} = P/A$, where A is the net cross-sectional area, and the definition of the stress concentration factor, $K = \sigma_{max}/\sigma_{ave}$, we write

$$P = \sigma_{ave} A = \frac{\sigma_{max} A}{K} \tag{2.49}$$

for any value of σ_{max} that does not exceed σ_Y. When $\sigma_{max} = \sigma_Y$ (Fig. 2.65*b*), we have $P = P_Y$, and Eq. (2.49) yields

$$P_Y = \frac{\sigma_Y A}{K} \tag{2.50}$$

On the other hand, when $P = P_U$ (Fig. 2.65*d*), we have $\sigma_{\text{ave}} = \sigma_Y$ and

$$P_U = \sigma_Y A \tag{2.51}$$

Comparing Eqs. (2.50) and (2.51), we conclude that

$$P_Y = \frac{P_U}{K} \tag{2.52}$$

*2.20 RESIDUAL STRESSES

In Example 2.13 of the preceding section, we considered a rod that was stretched beyond the yield point. As the load was removed, the rod did not regain its original length; it had been permanently deformed. However, after the load was removed, all stresses disappeared. You should not assume that this will always be the case. Indeed, when only some of the parts of an indeterminate structure undergo plastic deformations, as in Example 2.15, or when different parts of the structure undergo different plastic deformations, the stresses in the various parts of the structure will not, in general, return to zero after the load has been removed. Stresses, called *residual stresses,* will remain in the various parts of the structure.

While the computation of the residual stresses in an actual structure can be quite involved, the following example will provide you with a general understanding of the method to be used for their determination.

EXAMPLE 2.16

Determine the residual stresses in the rod and tube of Examples 2.14 and 2.15 after the load **P** has been increased from zero to 5.7 kips and decreased back to zero.

We observe from the diagrams of Fig. 2.66 that after the load **P** has returned to zero, the internal forces P_r and P_t are *not* equal to zero. Their values have been indicated by point E in parts *a* and *b*, respectively, of Fig. 2.66. It follows that the corresponding stresses are not equal to zero either after the assembly has been unloaded. To determine these residual stresses, we shall determine the reverse stresses σ'_r and σ'_t caused by the unloading and add them to the maximum stresses $\sigma_r = 36$ ksi and $\sigma_t = 30$ ksi found in part *a* of Example 2.15.

The strain caused by the unloading is the same in the rod and in the tube. It is equal to δ'/L, where δ' is the deformation of the assembly during unloading, which was found in Example 2.15. We have

$$\epsilon' = \frac{\delta'}{L} = \frac{-45.6 \times 10^{-3}\text{ in.}}{30\text{ in.}} = -1.52 \times 10^{-3}\text{ in./in.}$$

The corresponding reverse stresses in the rod and tube are

$$\sigma'_r = \epsilon' E_r = (-1.52 \times 10^{-3})(30 \times 10^6\text{ psi}) = -45.6\text{ ksi}$$
$$\sigma'_t = \epsilon' E_t = (-1.52 \times 10^{-3})(15 \times 10^6\text{ psi}) = -22.8\text{ ksi}$$

The residual stresses are found by superposing the stresses due to loading and the reverse stresses due to unloading. We have

$$(\sigma_r)_{\text{res}} = \sigma_r + \sigma'_r = 36\text{ ksi} - 45.6\text{ ksi} = -9.6\text{ ksi}$$
$$(\sigma_t)_{\text{res}} = \sigma_t + \sigma'_t = 30\text{ ksi} - 22.8\text{ ksi} = +7.2\text{ ksi}$$

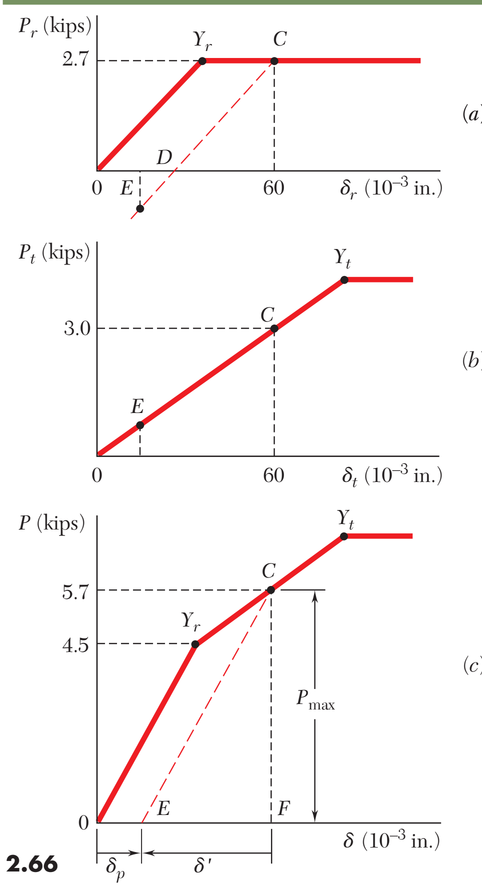


Fig. 2.66

Plastic deformations caused by temperature changes can also result in residual stresses. For example, consider a small plug that is to be welded to a large plate. For discussion purposes the plug will be considered as a small rod AB that is to be welded across a small hole in the plate (Fig. 2.67). During the welding process the temperature of the rod will be raised to over 1000°C, at which temperature its modulus of elasticity, and hence its stiffness and stress, will be almost zero. Since the plate is large, its temperature will not be increased significantly above room temperature (20°C). Thus, when the welding is completed, we have rod AB at $T = 1000°\text{C}$, with no stress, attached to the plate, which is at 20°C.

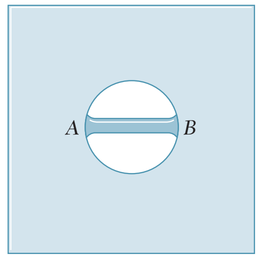


Fig. 2.67 Small rod welded to a large plate.

As the rod cools, its modulus of elasticity increases and, at about 500°C, will approach its normal value of about 200 GPa. As the temperature of the rod decreases further, we have a situation similar to that considered in Sec. 2.10 and illustrated in Fig. 2.31. Solving Eq. (2.23) for ΔT and making σ equal to the yield strength, $\sigma_Y = 300$ MPa, of average steel, and $\alpha = 12 \times 10^{-6}/°\text{C}$, we find the temperature change that will cause the rod to yield:

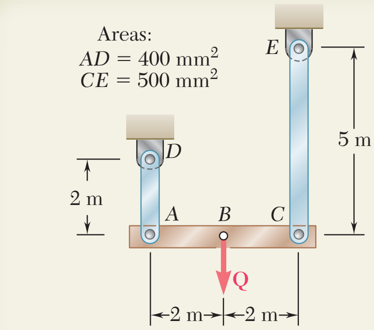
$$\Delta T = -\frac{\sigma}{E\alpha} = -\frac{300 \text{ MPa}}{(200 \text{ GPa})(12 \times 10^{-6}/°\text{C})} = -125°\text{C}$$

This means that the rod will start yielding at about 375°C and will keep yielding at a fairly constant stress level, as it cools down to room temperature. As a result of the welding operation, a residual stress approximately equal to the yield strength of the steel used is thus created in the plug and in the weld.

Residual stresses also occur as a result of the cooling of metals that have been cast or hot rolled. In these cases, the outer layers cool more rapidly than the inner core. This causes the outer layers to reacquire their stiffness (E returns to its normal value) faster than the inner core. When the entire specimen has returned to room temperature, the inner core will have contracted more than the outer layers. The result is residual longitudinal tensile stresses in the inner core and residual compressive stresses in the outer layers.

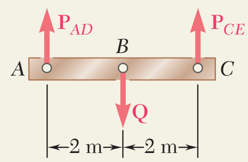
Residual stresses due to welding, casting, and hot rolling can be quite large (of the order of magnitude of the yield strength). These stresses can be removed, when necessary, by reheating the entire specimen to about 600°C, and then allowing it to cool slowly over a period of 12 to 24 hours.

SAMPLE PROBLEM 2.6



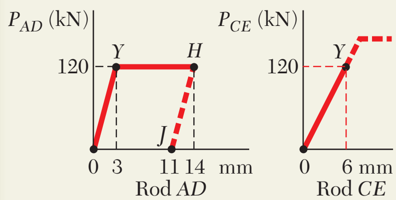
The rigid beam ABC is suspended from two steel rods as shown and is initially horizontal. The midpoint B of the beam is deflected 10 mm downward by the slow application of the force $\mathbf{Q}$, after which the force is slowly removed. Knowing that the steel used for the rods is elastoplastic with $E = 200$ GPa and $\sigma_Y = 300$ MPa, determine (*a*) the required maximum value of Q and the corresponding position of the beam, (*b*) the final position of the beam.

SOLUTION



Statics. Since $\mathbf{Q}$ is applied at the midpoint of the beam, we have

$$P_{AD} = P_{CE} \qquad \text{and} \qquad Q = 2P_{AD}$$



Load-deflection diagrams

Elastic Action. The maximum value of Q and the maximum elastic deflection of point A occur when $\sigma = \sigma_Y$ in rod AD.

$$(P_{AD})_{\max} = \sigma_Y A = (300 \text{ MPa})(400 \text{ mm}^2) = 120 \text{ kN}$$

$$Q_{\max} = 2(P_{AD})_{\max} = 2(120 \text{ kN}) \qquad Q_{\max} = 240 \text{ kN} \blacktriangleleft$$

$$\delta_{A_1} = \epsilon L = \frac{\sigma_Y}{E} L = \left(\frac{300 \text{ MPa}}{200 \text{ GPa}}\right)(2 \text{ m}) = 3 \text{ mm}$$

Since $P_{CE} = P_{AD} = 120$ kN, the stress in rod CE is

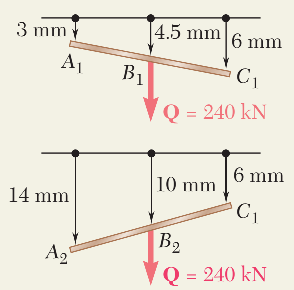
$$\sigma_{CE} = \frac{P_{CE}}{A} = \frac{120 \text{ kN}}{500 \text{ mm}^2} = 240 \text{ MPa}$$

The corresponding deflection of point C is

$$\delta_{C_1} = \epsilon L = \frac{\sigma_{CE}}{E} L = \left(\frac{240 \text{ MPa}}{200 \text{ GPa}}\right)(5 \text{ m}) = 6 \text{ mm}$$

The corresponding deflection of point B is

$$\delta_{B_1} = \tfrac{1}{2}(\delta_{A_1} + \delta_{C_1}) = \tfrac{1}{2}(3 \text{ mm} + 6 \text{ mm}) = 4.5 \text{ mm}$$

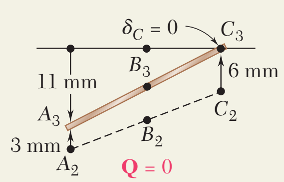


(*a*) Deflections for $\delta_B = 10$ mm $\blacktriangleleft$

Since we must have $\delta_B = 10$ mm, we conclude that plastic deformation will occur.

Plastic Deformation. For $Q = 240$ kN, plastic deformation occurs in rod AD, where $\sigma_{AD} = \sigma_Y = 300$ MPa. Since the stress in rod CE is within the elastic range, δ_C remains equal to 6 mm. The deflection δ_A for which $\delta_B = 10$ mm is obtained by writing

$$\delta_{B_2} = 10 \text{ mm} = \tfrac{1}{2}(\delta_{A_2} + 6 \text{ mm}) \qquad \delta_{A_2} = 14 \text{ mm}$$



(*b*) Final deflections $\blacktriangleleft$

Unloading. As force $\mathbf{Q}$ is slowly removed, the force P_{AD} decreases along line HJ parallel to the initial portion of the load-deflection diagram of rod AD. The final deflection of point A is

$$\delta_{A_3} = 14 \text{ mm} - 3 \text{ mm} = 11 \text{ mm}$$

Since the stress in rod CE remained within the elastic range, we note that the final deflection of point C is zero.

PROBLEMS

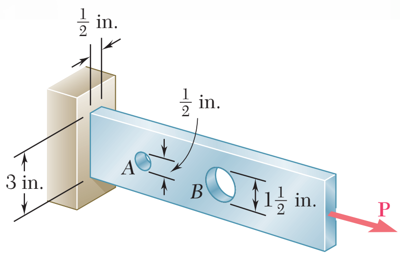


Fig. P2.93 and P2.94

2.93 Two holes have been drilled through a long steel bar that is subjected to a centric axial load as shown. For $P = 6.5$ kips, determine the maximum value of the stress (*a*) at *A*, (*b*) at *B*.

2.94 Knowing that $\sigma_{all} = 16$ ksi, determine the maximum allowable value of the centric axial load **P**.

2.95 Knowing that the hole has a diameter of 9 mm, determine (*a*) the radius r_f of the fillets for which the same maximum stress occurs at the hole *A* and at the fillets, (*b*) the corresponding maximum allowable load **P** if the allowable stress is 100 MPa.

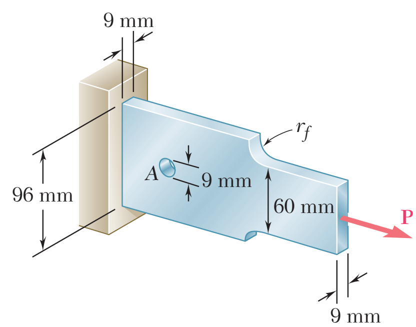


Fig. P2.95

2.96 For $P = 100$ kN, determine the minimum plate thickness t required if the allowable stress is 125 MPa.

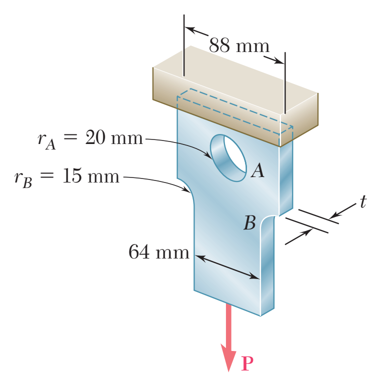
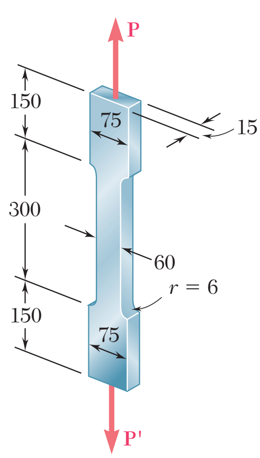


Fig. P2.96

2.97 The aluminum test specimen shown is subjected to two equal and opposite centric axial forces of magnitude *P*. (*a*) Knowing that $E = 70$ GPa and $\sigma_{all} = 200$ MPa, determine the maximum allowable value of *P* and the corresponding total elongation of the specimen. (*b*) Solve part *a*, assuming that the specimen has been replaced by an aluminum bar of the same length and a uniform 60×15-mm rectangular cross section.

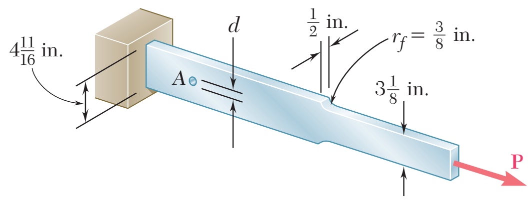


Dimensions in mm

Fig. P2.97

2.98 For the test specimen of Prob. 2.97, determine the maximum value of the normal stress corresponding to a total elongation of 0.75 mm.

2.99 A hole is to be drilled in the plate at A. The diameters of the bits available to drill the hole range from $\frac{1}{2}$ to $1\frac{1}{2}$ in. in $\frac{1}{4}$-in. increments. If the allowable stress in the plate is 21 ksi, determine (*a*) the diameter d of the largest bit that can be used if the allowable load **P** at the hole is to exceed that at the fillets, (*b*) the corresponding allowable load **P**.



Figs. *P2.99* and *P2.100*

2.100 (*a*) For $P = 13$ kips and $d = \frac{1}{2}$ in., determine the maximum stress in the plate shown. (*b*) Solve part *a*, assuming that the hole at A is not drilled.

2.101 Rod ABC consists of two cylindrical portions AB and BC; it is made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. A force **P** is applied to the rod and then removed to give it a permanent set $\delta_p = 2$ mm. Determine the maximum value of the force **P** and the maximum amount δ_m by which the rod should be stretched to give it the desired permanent set.

2.102 Rod ABC consists of two cylindrical portions AB and BC; it is made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. A force **P** is applied to the rod until its end A has moved down by an amount $\delta_m = 5$ mm. Determine the maximum value of the force **P** and the permanent set of the rod after the force has been removed.

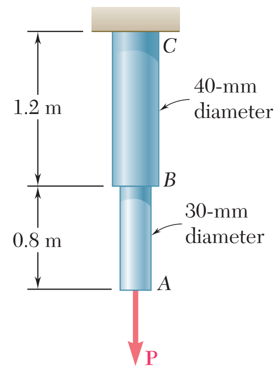


Fig. P2.101 and P2.102

2.103 The 30-mm-square bar AB has a length $L = 2.2$ m; it is made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 345$ MPa. A force **P** is applied to the bar until end A has moved down by an amount δ_m. Determine the maximum value of the force **P** and the permanent set of the bar after the force has been removed, knowing that (*a*) $\delta_m = 4.5$ mm, (*b*) $\delta_m = 8$ mm.

2.104 The 30-mm-square bar AB has a length $L = 2.5$ m; it is made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 345$ MPa. A force **P** is applied to the bar and then removed to give it a permanent set δ_p. Determine the maximum value of the force **P** and the maximum amount δ_m by which the bar should be stretched if the desired value of δ_p is (*a*) 3.5 mm, (*b*) 6.5 mm.

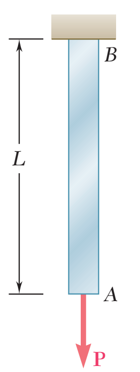


Fig. *P2.103* and *P2.104*

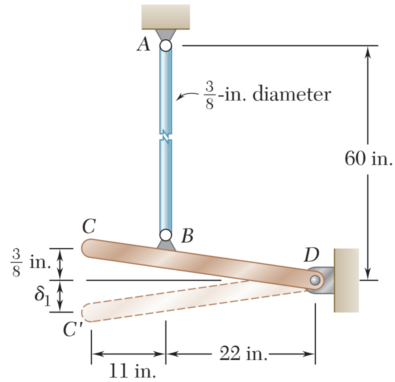


Fig. P2.105

2.105 Rod AB is made of a mild steel that is assumed to be elastoplastic with $E = 29 \times 10^6$ psi and $\sigma_Y = 36$ ksi. After the rod has been attached to the rigid lever CD, it is found that end C is $\frac{3}{8}$ in. too high. A vertical force $\mathbf{Q}$ is then applied at C until this point has moved to position C'. Determine the required magnitude of $\mathbf{Q}$ and the deflection δ_1 if the lever is to *snap* back to a horizontal position after $\mathbf{Q}$ is removed.

2.106 Solve Prob. 2.105, assuming that the yield point of the mild steel is 50 ksi.

2.107 Each cable has a cross-sectional area of 100 mm^2 and is made of an elastoplastic material for which $\sigma_Y = 345$ MPa and $E = 200$ GPa. A force $\mathbf{Q}$ is applied at C to the rigid bar ABC and is gradually increased from 0 to 50 kN and then reduced to zero. Knowing that the cables were initially taut, determine (*a*) the maximum stress that occurs in cable BD, (*b*) the maximum deflection of point C, (*c*) the final displacement of point C. (*Hint:* In part *c*, cable CE is not taut.)

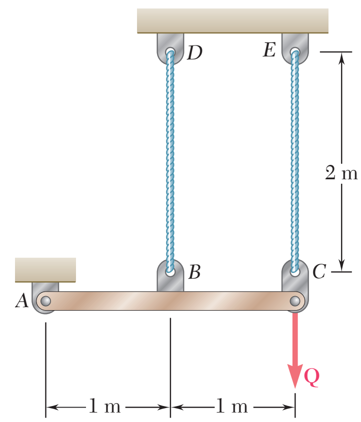


Fig. *P2.107*

2.108 Solve Prob. 2.107, assuming that the cables are replaced by rods of the same cross-sectional area and material. Further assume that the rods are braced so that they can carry compressive forces.

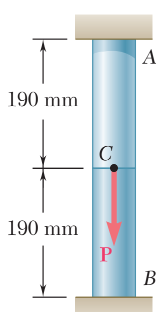


Fig. P2.109

2.109 Rod AB consists of two cylindrical portions AC and BC, each with a cross-sectional area of 1750 mm^2. Portion AC is made of a mild steel with $E = 200$ GPa and $\sigma_Y = 250$ MPa, and portion CB is made of a high-strength steel with $E = 200$ GPa and $\sigma_Y = 345$ MPa. A load $\mathbf{P}$ is applied at C as shown. Assuming both steels to be elastoplastic, determine (*a*) the maximum deflection of C if P is gradually increased from zero to 975 kN and then reduced back to zero, (*b*) the maximum stress in each portion of the rod, (*c*) the permanent deflection of C.

2.110 For the composite rod of Prob. 2.109, if P is gradually increased from zero until the deflection of point C reaches a maximum value of $\delta_m = 0.3$ mm and then decreased back to zero, determine (*a*) the maximum value of P, (*b*) the maximum stress in each portion of the rod, (*c*) the permanent deflection of C after the load is removed.

2.111 Two tempered-steel bars, each $\frac{3}{16}$-in. thick, are bonded to a $\frac{1}{2}$-in. mild-steel bar. This composite bar is subjected as shown to a centric axial load of magnitude P. Both steels are elastoplastic with $E = 29 \times 10^6$ psi and with yield strengths equal to 100 ksi and 50 ksi, respectively, for the tempered and mild steel. The load P is gradually increased from zero until the deformation of the bar reaches a maximum value $\delta_m = 0.04$ in. and then decreased back to zero. Determine (*a*) the maximum value of P, (*b*) the maximum stress in the tempered-steel bars, (*c*) the permanent set after the load is removed.

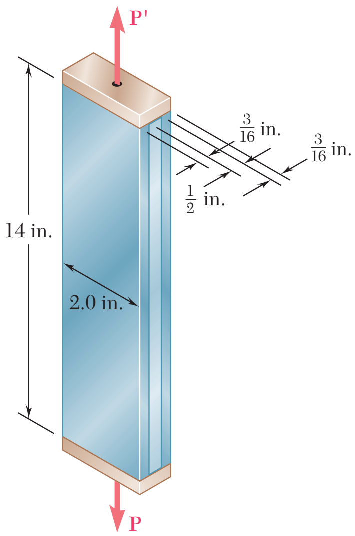


Fig. P2.111

2.112 For the composite bar of Prob. 2.111, if P is gradually increased from zero to 98 kips and then decreased back to zero, determine (*a*) the maximum deformation of the bar, (*b*) the maximum stress in the tempered-steel bars, (*c*) the permanent set after the load is removed.

2.113 The rigid bar ABC is supported by two links, AD and BE, of uniform 37.5 × 6-mm rectangular cross section and made of a mild steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. The magnitude of the force **Q** applied at B is gradually increased from zero to 260 kN. Knowing that $a = 0.640$ m, determine (*a*) the value of the normal stress in each link, (*b*) the maximum deflection of point B.

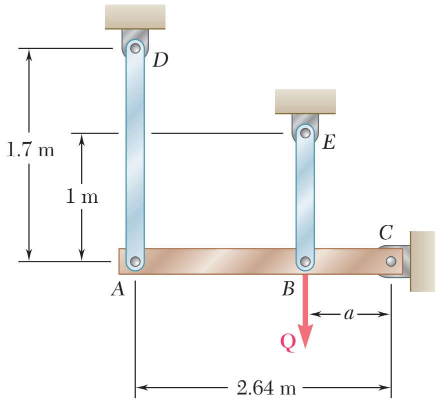


Fig. P2.113

2.114 Solve Prob. 2.113, knowing that $a = 1.76$ m and that the magnitude of the force **Q** applied at B is gradually increased from zero to 135 kN.

***2.115** Solve Prob. 2.113, assuming that the magnitude of the force **Q** applied at B is gradually increased from zero to 260 kN and then decreased back to zero. Knowing that $a = 0.640$ m, determine (*a*) the residual stress in each link, (*b*) the final deflection of point B. Assume that the links are braced so that they can carry compressive forces without buckling.

2.116 A uniform steel rod of cross-sectional area A is attached to rigid supports and is unstressed at a temperature of 45°F. The steel is assumed to be elastoplastic with $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi. Knowing that $\alpha = 6.5 \times 10^{-6}$/°F, determine the stress in the bar (*a*) when the temperature is raised to 320°F, (*b*) after the temperature has returned to 45°F.

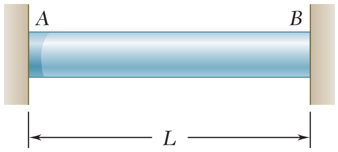


Fig. P2.116

2.117 The steel rod *ABC* is attached to rigid supports and is unstressed at a temperature of 25°C. The steel is assumed elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. The temperature of both portions of the rod is then raised to 150°C. Knowing that $\alpha = 11.7 \times 10^{-6}/°C$, determine (*a*) the stress in both portions of the rod, (*b*) the deflection of point *C*.

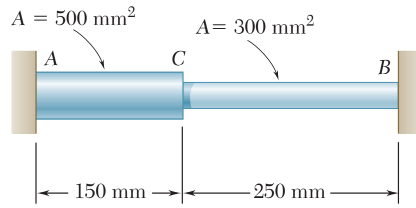


Fig. P2.117

***2.118** Solve Prob. 2.117, assuming that the temperature of the rod is raised to 150°C and then returned to 25°C.

***2.119** For the composite bar of Prob. 2.111, determine the residual stresses in the tempered-steel bars if *P* is gradually increased from zero to 98 kips and then decreased back to zero.

***2.120** For the composite bar in Prob. 2.111, determine the residual stresses in the tempered-steel bars if *P* is gradually increased from zero until the deformation of the bar reaches a maximum value $\delta_m = 0.04$ in. and is then decreased back to zero.

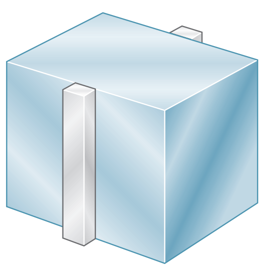


Fig. P2.121

***2.121** Narrow bars of aluminum are bonded to the two sides of a thick steel plate as shown. Initially, at $T_1 = 70°F$, all stresses are zero. Knowing that the temperature will be slowly raised to T_2 and then reduced to T_1, determine (*a*) the highest temperature T_2 that does *not* result in residual stresses, (*b*) the temperature T_2 that will result in a residual stress in the aluminum equal to 58 ksi. Assume $\alpha_a = 12.8 \times 10^{-6}/°F$ for the aluminum and $\alpha_s = 6.5 \times 10^{-6}/°F$ for the steel. Further assume that the aluminum is elastoplastic with $E = 10.9 \times 10^6$ psi and $\sigma_Y = 58$ ksi. (*Hint:* Neglect the small stresses in the plate.)

***2.122** Bar *AB* has a cross-sectional area of 1200 mm² and is made of a steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 250$ MPa. Knowing that the force **F** increases from 0 to 520 kN and then decreases to zero, determine (*a*) the permanent deflection of point *C*, (*b*) the residual stress in the bar.

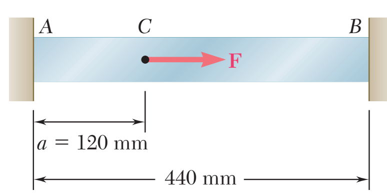


Fig. P2.122

***2.123** Solve Prob. 2.122, assuming that $a = 180$ mm.

REVIEW AND SUMMARY

This chapter was devoted to the introduction of the concept of *strain*, to the discussion of the relationship between stress and strain in various types of materials, and to the determination of the deformations of structural components under axial loading.

Normal strain

Considering a rod of length L and uniform cross section and denoting by δ its deformation under an axial load $\mathbf{P}$ (Fig. 2.68), we defined the *normal strain* ϵ in the rod as the *deformation per unit length* [Sec. 2.2]:

$$\epsilon = \frac{\delta}{L} \tag{2.1}$$

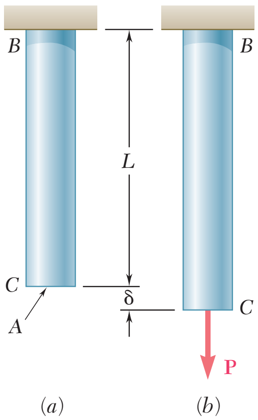


Fig. 2.68

In the case of a rod of variable cross section, the normal strain was defined at any given point Q by considering a small element of rod at Q. Denoting by Δx the length of the element and by $\Delta\delta$ its deformation under the given load, we wrote

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx} \tag{2.2}$$

Stress-strain diagram

Plotting the stress σ versus the strain ϵ as the load increased, we obtained a *stress-strain diagram* for the material used [Sec. 2.3]. From such a diagram, we were able to distinguish between *brittle* and *ductile* materials: A specimen made of a brittle material ruptures without any noticeable prior change in the rate of elongation (Fig. 2.69), while a specimen made of a ductile material *yields* after a critical stress σ_Y, called the *yield strength*, has been reached, i.e., the specimen undergoes a large deformation before rupturing, with a relatively small increase in the applied load (Fig. 2.70). An example of brittle material with different properties in tension and in compression was provided by *concrete*.

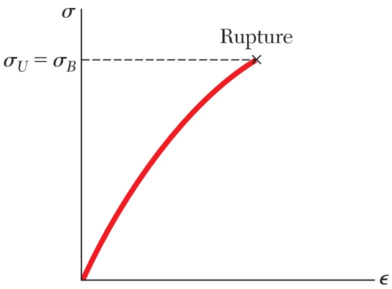


Fig. 2.69

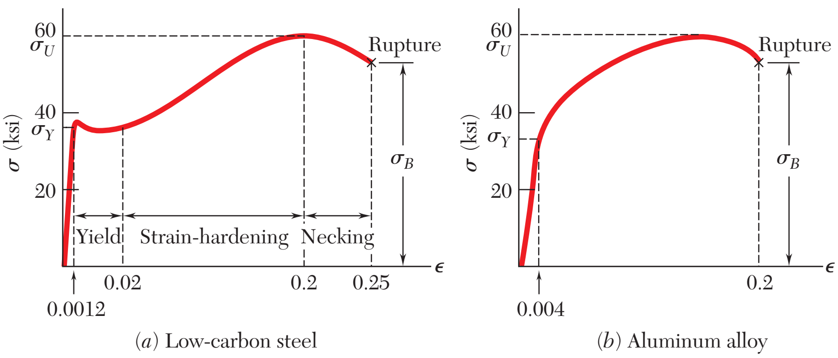


Fig. 2.70

Hooke's law
Modulus of elasticity

We noted in Sec. 2.5 that the initial portion of the stress-strain diagram is a straight line. This means that for small deformations, the stress is directly proportional to the strain:

$$\sigma = E\epsilon \tag{2.4}$$

This relation is known as *Hooke's law* and the coefficient E as the *modulus of elasticity* of the material. The largest stress for which Eq. (2.4) applies is the *proportional limit* of the material.

Materials considered up to this point were *isotropic,* i.e., their properties were independent of direction. In Sec. 2.5 we also considered a class of *anisotropic* materials, i.e., materials whose properties depend upon direction. They were *fiber-reinforced composite materials,* made of fibers of a strong, stiff material embedded in layers of a weaker, softer material (Fig. 2.71). We saw that different moduli of elasticity had to be used, depending upon the direction of loading.

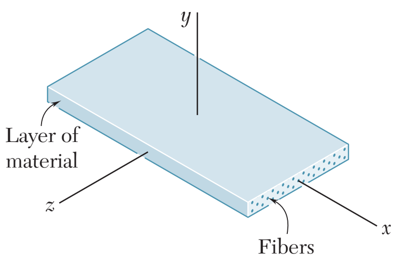


Fig. 2.71

Elastic limit. Plastic deformation

If the strains caused in a test specimen by the application of a given load disappear when the load is removed, the material is said to behave *elastically,* and the largest stress for which this occurs is called the *elastic limit* of the material [Sec. 2.6]. If the elastic limit is exceeded, the stress and strain decrease in a linear fashion when the load is removed and the strain does not return to zero (Fig. 2.72), indicating that a *permanent set* or *plastic deformation* of the material has taken place.

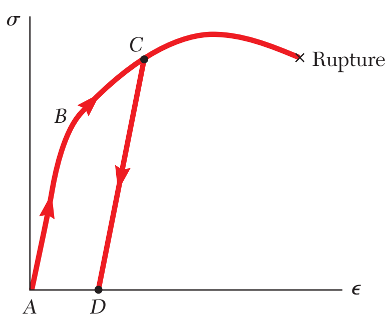


Fig. 2.72

Fatigue. Endurance limit

In Sec. 2.7, we discussed the phenomenon of *fatigue,* which causes the failure of structural or machine components after a very large number of repeated loadings, even though the stresses remain in the elastic range. A standard fatigue test consists in determining the number n of successive loading-and-unloading cycles required to cause the failure of a specimen for any given maximum stress level σ, and plotting the resulting σ-n curve. The value of σ for which failure does not occur, even for an indefinitely large number of cycles, is known as the *endurance limit* of the material used in the test.

Elastic deformation under axial loading

Section 2.8 was devoted to the determination of the elastic deformations of various types of machine and structural components under various conditions of axial loading. We saw that if a rod of length L and uniform cross section of area A is subjected at its end to a centric axial load $\mathbf{P}$ (Fig. 2.73), the corresponding deformation is

$$\delta = \frac{PL}{AE} \tag{2.7}$$

If the rod is loaded at several points or consists of several parts of various cross sections and possibly of different materials, the deformation δ of the rod must be expressed as the sum of the deformations of its component parts [Example 2.01]:

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} \tag{2.8}$$

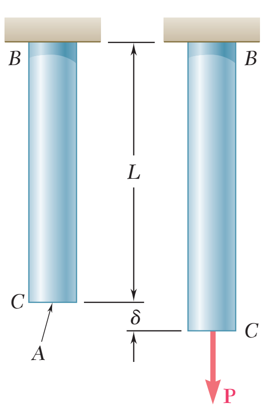


Fig. 2.73

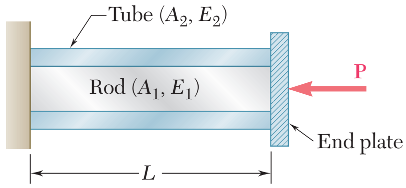


Fig. 2.74

Statically indeterminate problems

Section 2.9 was devoted to the solution of *statically indeterminate problems,* i.e., problems in which the reactions and the internal forces *cannot* be determined from statics alone. The equilibrium equations derived from the free-body diagram of the member under consideration were complemented by relations involving deformations and obtained from the geometry of the problem. The forces in the rod and in the tube of Fig. 2.74, for instance, were determined by observing, on one hand, that their sum is equal to P, and on the other, that they cause equal deformations in the rod and in the tube [Example 2.02]. Similarly, the reactions at the supports of the bar of Fig. 2.75 could not be obtained from the free-body diagram of the bar alone [Example 2.03]; but they could be determined by expressing that the total elongation of the bar must be equal to zero.

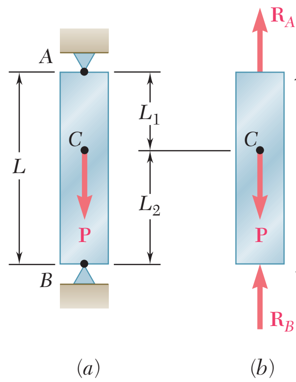


Fig. 2.75

Problems with temperature changes

In Sec. 2.10, we considered problems involving *temperature changes*. We first observed that if the temperature of an *unrestrained rod AB* of length L is increased by ΔT, its elongation is

$$\delta_T = \alpha(\Delta T)L \qquad (2.21)$$

where α is the *coefficient of thermal expansion* of the material. We noted that the corresponding strain, called *thermal strain,* is

$$\epsilon_T = \alpha \Delta T \qquad (2.22)$$

and that *no stress* is associated with this strain. However, if the rod AB is *restrained* by fixed supports (Fig. 2.76), stresses develop in the

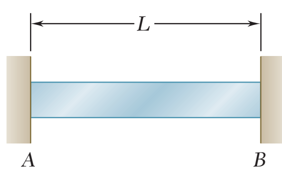


Fig. 2.76

rod as the temperature increases, because of the reactions at the supports. To determine the magnitude P of the reactions, we detached the rod from its support at B (Fig. 2.77) and considered separately the deformation δ_T of the rod as it expands freely because of the temperature change, and the deformation δ_P caused by the force **P** required to bring it back to its original length, so that it may be reattached to the support at B. Writing that the total deformation $\delta = \delta_T + \delta_P$ is equal to zero, we obtained an equation that could be solved for P. While the final strain in rod AB is clearly zero, this will generally not be the case for rods and bars consisting of elements of different cross sections or materials, since the deformations of the various elements will usually *not* be zero [Example 2.06].

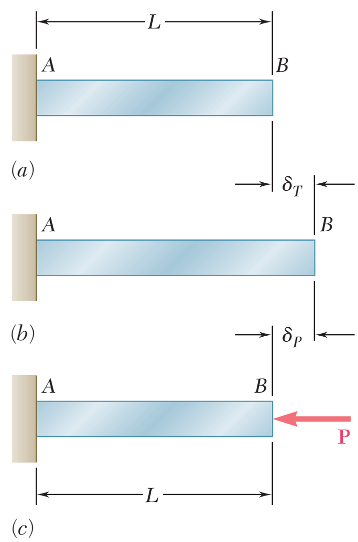


Fig. 2.77

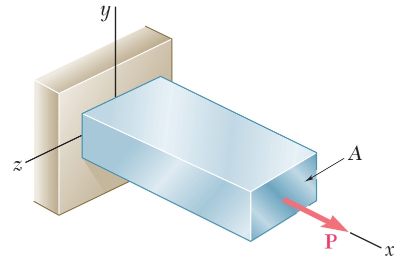


Fig. 2.78

Lateral strain. Poisson's ratio

When an axial load **P** is applied to a homogeneous, slender bar (Fig. 2.78), it causes a strain, not only along the axis of the bar but in any transverse direction as well [Sec. 2.11]. This strain is referred to as the *lateral strain,* and the ratio of the lateral strain over the axial strain is called *Poisson's ratio* and is denoted by ν (Greek letter nu). We wrote

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \tag{2.25}$$

Recalling that the axial strain in the bar is $\epsilon_x = \sigma_x/E$, we expressed as follows the condition of strain under an axial loading in the x direction:

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E} \tag{2.27}$$

Multiaxial loading

This result was extended in Sec. 2.12 to the case of a *multiaxial loading* causing the state of stress shown in Fig. 2.79. The resulting strain condition was described by the following relations, referred to as the *generalized Hooke's law* for a multiaxial loading.

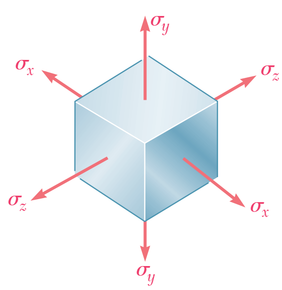


Fig. 2.79

$$\begin{aligned}
\epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\
\epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}
\end{aligned} \tag{2.28}$$

Dilatation

If an element of material is subjected to the stresses σ_x, σ_y, σ_z, it will deform and a certain change of volume will result [Sec. 2.13]. The *change in volume per unit volume* is referred to as the *dilatation* of the material and is denoted by e. We showed that

$$e = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{2.31}$$

When a material is subjected to a hydrostatic pressure p, we have

$$e = -\frac{p}{k} \tag{2.34}$$

Bulk modulus

where k is known as the *bulk modulus* of the material:

$$k = \frac{E}{3(1-2\nu)} \tag{2.33}$$

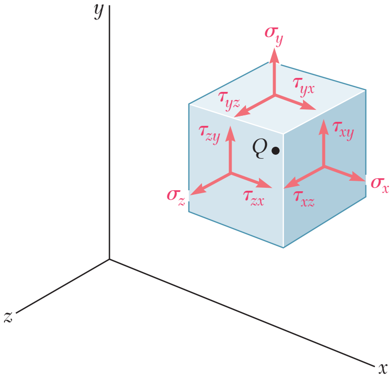


Fig. 2.80

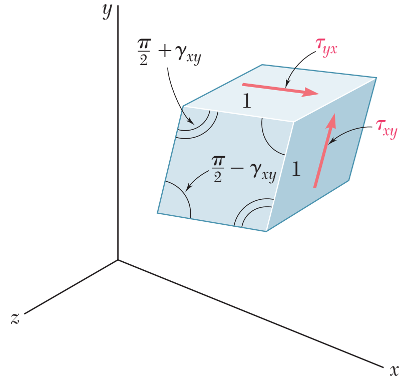


Fig. 2.81

Shearing strain. Modulus of rigidity

As we saw in Chap. 1, the state of stress in a material under the most general loading condition involves shearing stresses, as well as normal stresses (Fig. 2.80). The shearing stresses tend to deform a cubic element of material into an oblique parallelepiped [Sec. 2.14]. Considering, for instance, the stresses τ_{xy} and τ_{yx} shown in Fig. 2.81 (which, we recall, are equal in magnitude), we noted that they cause the angles formed by the faces on which they act to either increase or decrease by a small angle γ_{xy}; this angle, expressed in radians, defines the *shearing strain* corresponding to the x and y directions. Defining in a similar way the shearing strains γ_{yz} and γ_{zx}, we wrote the relations

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx} \tag{2.36, 37}$$

which are valid for any homogeneous isotropic material within its proportional limit in shear. The constant G is called the *modulus of rigidity* of the material and the relations obtained express *Hooke's law for shearing stress and strain*. Together with Eqs. (2.28), they form a group of equations representing the generalized Hooke's law for a homogeneous isotropic material under the most general stress condition.

We observed in Sec. 2.15 that while an axial load exerted on a slender bar produces only normal strains—both axial and transverse—on an element of material oriented along the axis of the bar, it will produce both normal and shearing strains on an element rotated through 45° (Fig. 2.82). We also noted that the three constants E, ν, and G are not independent; they satisfy the relation.

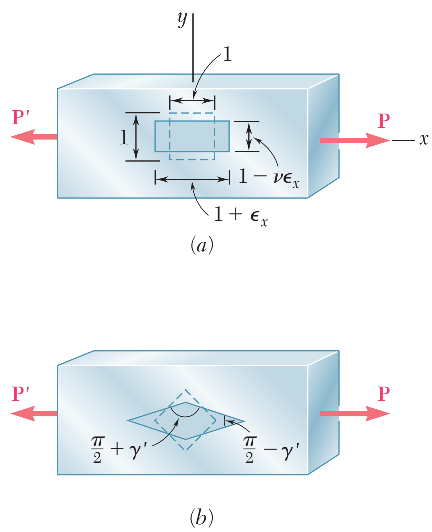


Fig. 2.82

$$\frac{E}{2G} = 1 + \nu \tag{2.43}$$

which may be used to determine any of the three constants in terms of the other two.

Fiber-reinforced composite materials

Stress-strain relationships for fiber-reinforced composite materials were discussed in an optional section (Sec. 2.16). Equations similar to Eqs. (2.28) and (2.36, 37) were derived for these materials, but we noted that direction-dependent moduli of elasticity, Poisson's ratios, and moduli of rigidity had to be used.

Saint-Venant's principle

In Sec. 2.17, we discussed *Saint-Venant's principle,* which states that except in the immediate vicinity of the points of application of the loads, the distribution of stresses in a given member is independent of the actual mode of application of the loads. This principle makes it possible to assume a uniform distribution of stresses in a member subjected to concentrated axial loads, except close to the points of application of the loads, where stress concentrations will occur.

Stress concentrations

Stress concentrations will also occur in structural members near a discontinuity, such as a hole or a sudden change in cross section [Sec. 2.18]. The ratio of the maximum value of the stress occurring near the discontinuity over the average stress computed in the critical section is referred to as the *stress-concentration factor* of the discontinuity and is denoted by K:

$$K = \frac{\sigma_{\max}}{\sigma_{\text{ave}}} \tag{2.48}$$

Values of K for circular holes and fillets in flat bars were given in Fig. 2.64 on p. 108.

Plastic deformotions

In Sec. 2.19, we discussed the *plastic deformations* which occur in structural members made of a ductile material when the stresses in some part of the member exceed the yield strength of the material. Our analysis was carried out for an idealized *elastoplastic material* characterized by the stress-strain diagram shown in Fig. 2.83 [Examples 2.13, 2.14, and 2.15]. Finally, in Sec. 2.20, we observed that when an indeterminate structure undergoes plastic deformations, the stresses do not, in general, return to zero after the load has been removed. The stresses remaining in the various parts of the structure are called *residual stresses* and may be determined by adding the maximum stresses reached during the loading phase and the reverse stresses corresponding to the unloading phase [Example 2.16].

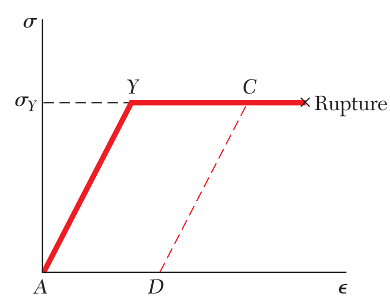


Fig. 2.83

REVIEW PROBLEMS

2.124 Rod BD is made of steel ($E = 29 \times 10^6$ psi) and is used to brace the axially compressed member ABC. The maximum force that can be developed in member BD is $0.02P$. If the stress must not exceed 18 ksi and the maximum change in length of BD must not exceed 0.001 times the length of ABC, determine the smallest-diameter rod that can be used for member BD.

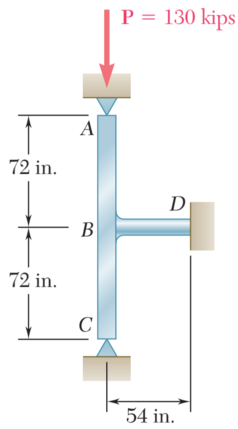


Fig. P2.124

2.125 Two solid cylindrical rods are joined at B and loaded as shown. Rod AB is made of steel ($E = 200$ GPa) and rod BC of brass ($E = 105$ GPa). Determine (*a*) the total deformation of the composite rod ABC, (*b*) the deflection of point B.

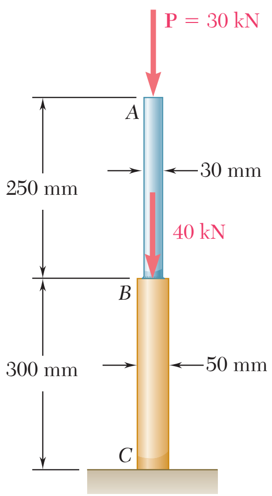


Fig. *P2.125*

2.126 Two solid cylindrical rods are joined at B and loaded as shown. Rod AB is made of steel ($E = 29 \times 10^6$ psi), and rod BC of brass ($E = 15 \times 10^6$ psi). Determine (*a*) the total deformation of the composite rod ABC, (*b*) the deflection of point B.

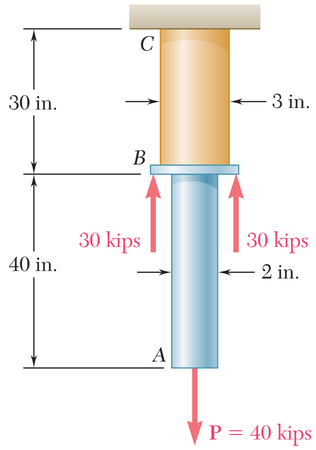


Fig. *P2.126*

2.127 The uniform wire ABC, of unstretched length $2l$, is attached to the supports shown and a vertical load **P** is applied at the midpoint B. Denoting by A the cross-sectional area of the wire and by E the modulus of elasticity, show that, for $\delta \ll l$, the deflection at the midpoint B is

$$\delta = l\sqrt[3]{\frac{P}{AE}}$$

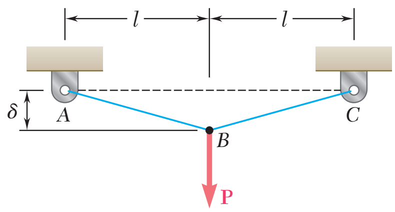


Fig. P2.127

2.128 The brass strip AB has been attached to a fixed support at A and rests on a rough support at B. Knowing that the coefficient of friction is 0.60 between the strip and the support at B, determine the decrease in temperature for which slipping will impend.

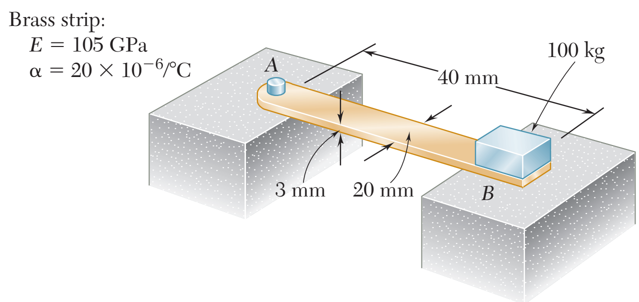


Fig. P2.128

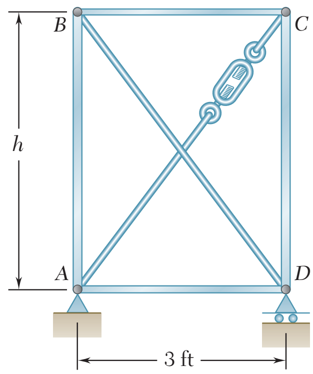


Fig. P2.129

2.129 Members AB and CD are $1\frac{1}{8}$-in.-diameter steel rods, and members BC and AD are $\frac{7}{8}$-in.-diameter steel rods. When the turnbuckle is tightened, the diagonal member AC is put in tension. Knowing that $E = 29 \times 10^6$ psi and $h = 4$ ft, determine the largest allowable tension in AC so that the deformations in members AB and CD do not exceed 0.04 in.

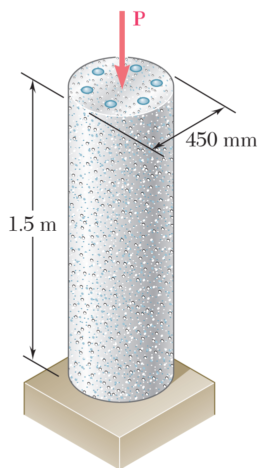


Fig. P2.130

2.130 The 1.5-m concrete post is reinforced with six steel bars, each with a 28-mm diameter. Knowing that $E_s = 200$ GPa and $E_c = 25$ GPa, determine the normal stresses in the steel and in the concrete when a 1550-kN axial centric force $\mathbf{P}$ is applied to the post.

2.131 The brass shell ($\alpha_b = 11.6 \times 10^{-6}/°F$) is fully bonded to the steel core ($\alpha_s = 6.5 \times 10^{-6}/°F$). Determine the largest allowable increase in temperature if the stress in the steel core is not to exceed 8 ksi.

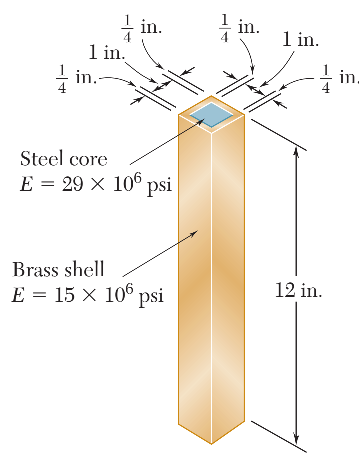


Fig. P2.131

2.132 A fabric used in air-inflated structures is subjected to a biaxial loading that results in normal stresses $\sigma_x = 120$ MPa and $\sigma_z = 160$ MPa. Knowing that the properties of the fabric can be approximated as $E = 87$ GPa and $\nu = 0.34$, determine the change in length of (*a*) side *AB*, (*b*) side *BC*, (*c*) diagonal *AC*.

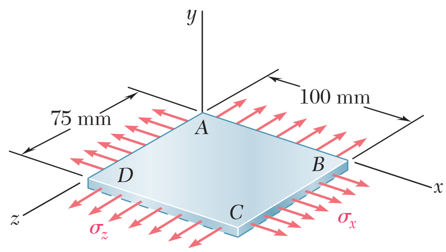


Fig. *P2.132*

2.133 An elastomeric bearing ($G = 0.9$ MPa) is used to support a bridge girder as shown to provide flexibility during earthquakes. The beam must not displace more than 10 mm when a 22-kN lateral load is applied as shown. Knowing that the maximum allowable shearing stress is 420 kPa, determine (*a*) the smallest allowable dimension *b*, (*b*) the smallest required thickness *a*.

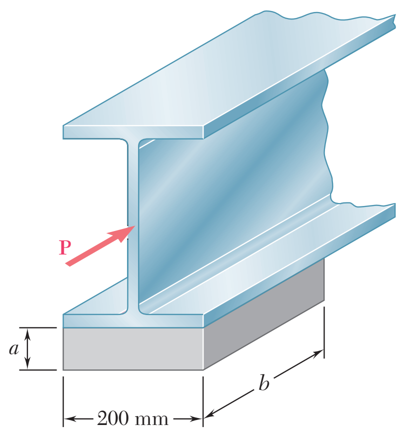


Fig. P2.133

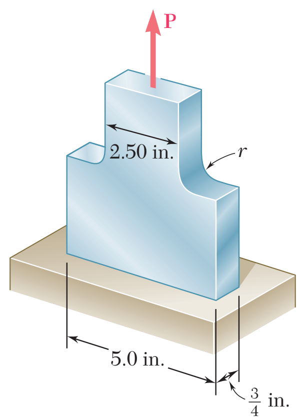


Fig. *P2.134*

2.134 Knowing that $P = 10$ kips, determine the maximum stress when (*a*) $r = 0.50$ in., (*b*) $r = 0.625$ in.

2.135 The uniform rod *BC* has cross-sectional area *A* and is made of a mild steel that can be assumed to be elastoplastic with a modulus of elasticity *E* and a yield strength σ_Y. Using the block-and-spring system shown, it is desired to simulate the deflection of end *C* of the rod as the axial force **P** is gradually applied and removed, that is, the deflection of points *C* and *C'* should be the same for all values of *P*. Denoting by μ the coefficient of friction between the block and the horizontal surface, derive an expression for (*a*) the required mass *m* of the block, (*b*) the required constant *k* of the spring.

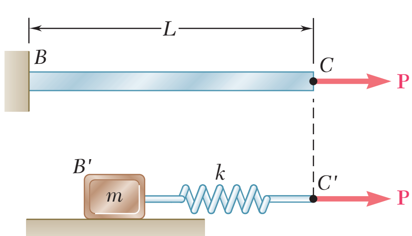


Fig. P2.135

COMPUTER PROBLEMS

The following problems are designed to be solved with a computer. Write each program so that it can be used with either SI or U.S. customary units and in such a way that solid cylindrical elements may be defined by either their diameter or their cross-sectional area.

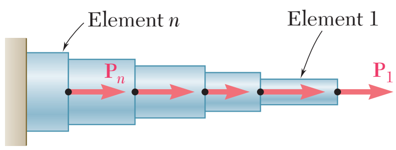


Fig. P2.C1

2.C1 A rod consisting of n elements, each of which is homogeneous and of uniform cross section, is subjected to the loading shown. The length of element i is denoted by L_i, its cross-sectional area by A_i, modulus of elasticity by E_i, and the load applied to its right end by $\mathbf{P}_i$, the magnitude P_i of this load being assumed to be positive if $\mathbf{P}_i$ is directed to the right and negative otherwise. (*a*) Write a computer program that can be used to determine the average normal stress in each element, the deformation of each element, and the total deformation of the rod. (*b*) Use this program to solve Probs. 2.20 and 2.126.

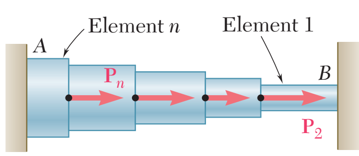


Fig. P2.C2

2.C2 Rod AB is horizontal with both ends fixed; it consists of n elements, each of which is homogeneous and of uniform cross section, and is subjected to the loading shown. The length of element i is denoted by L_i, its cross-sectional area by A_i, its modulus of elasticity by E_i, and the load applied to its right end by $\mathbf{P}_i$, the magnitude P_i of this load being assumed to be positive if $\mathbf{P}_i$ is directed to the right and negative otherwise. (Note that $P_1 = 0$.) (*a*) Write a computer program that can be used to determine the reactions at A and B, the average normal stress in each element, and the deformation of each element. (*b*) Use this program to solve Probs. 2.41 and 2.42.

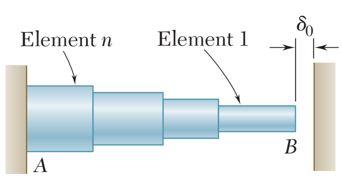


Fig. P2.C3

2.C3 Rod AB consists of n elements, each of which is homogeneous and of uniform cross section. End A is fixed, while initially there is a gap δ_0 between end B and the fixed vertical surface on the right. The length of element i is denoted by L_i, its cross-sectional area by A_i, its modulus of elasticity by E_i, and its coefficient of thermal expansion by α_i. After the temperature of the rod has been increased by ΔT, the gap at B is closed and the vertical surfaces exert equal and opposite forces on the rod. (*a*) Write a computer program that can be used to determine the magnitude of the reactions at A and B, the normal stress in each element, and the deformation of each element. (*b*) Use this program to solve Probs. 2.51, 2.59, and 2.60.

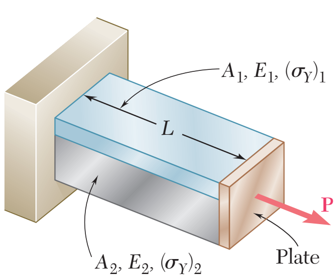


Fig. P2.C4

2.C4 Bar AB has a length L and is made of two different materials of given cross-sectional area, modulus of elasticity, and yield strength. The bar is subjected as shown to a load $\mathbf{P}$ that is gradually increased from zero until the deformation of the bar has reached a maximum value δ_m and then decreased back to zero. (*a*) Write a computer program that, for each of 25 values of δ_m equally spaced over a range extending from 0 to a value equal to 120% of the deformation causing both materials to yield, can be used to determine the maximum value P_m of the load, the maximum normal stress in each material, the permanent deformation δ_p of the bar, and the residual stress in each material. (*b*) Use this program to solve Probs. 2.111 and 2.112.

2.C5 The plate has a hole centered across the width. The stress concentration factor for a flat bar under axial loading with a centric hole is:

$$K = 3.00 - 3.13\left(\frac{2r}{D}\right) + 3.66\left(\frac{2r}{D}\right)^2 - 1.53\left(\frac{2r}{D}\right)^3$$

where r is the radius of the hole and D is the width of the bar. Write a computer program to determine the allowable load **P** for the given values of r, D, the thickness t of the bar, and the allowable stress σ_{all} of the material. Knowing that $t = \frac{1}{4}$ in., $D = 3.0$ in. and $\sigma_{all} = 16$ ksi, determine the allowable load **P** for values of r from 0.125 in. to 0.75 in., using 0.125 in. increments.

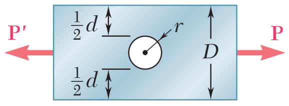


Fig. P2.C5

2.C6 A solid truncated cone is subjected to an axial force **P** as shown. The exact elongation is $(PL)/(2\pi c^2 E)$. By replacing the cone by n circular cylinders of equal thickness, write a computer program that can be used to calculate the elongation of the truncated cone. What is the percentage error in the answer obtained from the program using (*a*) $n = 6$, (*b*) $n = 12$, (*c*) $n = 60$?

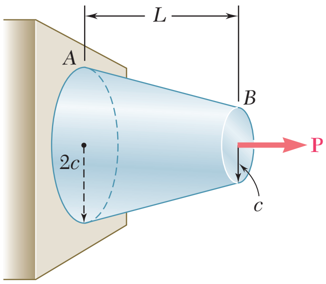


Fig. P2.C6

This chapter is devoted to the study of torsion and of the stresses and deformations it causes. In the jet engine shown here, the central shaft links the components of the engine to develop the thrust that propels the plane.

CHAPTER 3

Torsion

Chapter 3 Torsion

3.1 Introduction
3.2 Preliminary Discussion of the Stresses in a Shaft
3.3 Deformations in a Circular Shaft
3.4 Stresses in the Elastic Range
3.5 Angle of Twist in the Elastic Range
3.6 Statically Indeterminate Shafts
3.7 Design of Transmission Shafts
3.8 Stress Concentrations in Circular Shafts
*3.9 Plastic Deformations in Circular Shafts
*3.10 Circular Shafts Made of an Elastoplastic Material
*3.11 Residual Stresses in Circular Shafts
*3.12 Torsion of Noncircular Members
*3.13 Thin-Walled Hollow Shafts

3.1 INTRODUCTION

In the two preceding chapters you studied how to calculate the stresses and strains in structural members subjected to axial loads, that is, to forces directed along the axis of the member. In this chapter structural members and machine parts that are in *torsion* will be considered. More specifically, you will analyze the stresses and strains in members of circular cross section subjected to twisting couples, or *torques*, **T** and **T′** (Fig. 3.1). These couples have a common magnitude T, and opposite senses. They are vector quantities and can be represented either by curved arrows as in Fig. 3.1*a*, or by couple vectors as in Fig. 3.1*b*.

Members in torsion are encountered in many engineering applications. The most common application is provided by *transmission shafts*, which are used to transmit power from one point to another. For example, the shaft shown in Photo 3.1 is used to transmit power from the engine to the rear wheels of an automobile. These shafts can be either solid, as shown in Fig. 3.1, or hollow.

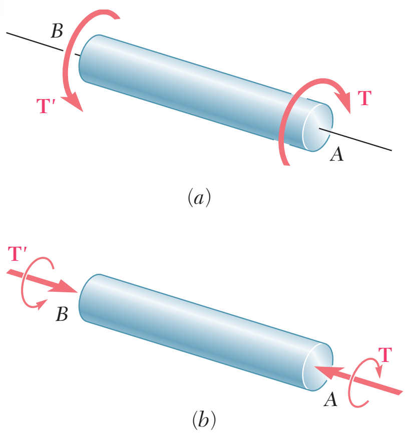


Fig. 3.1 Shaft subject to torsion.

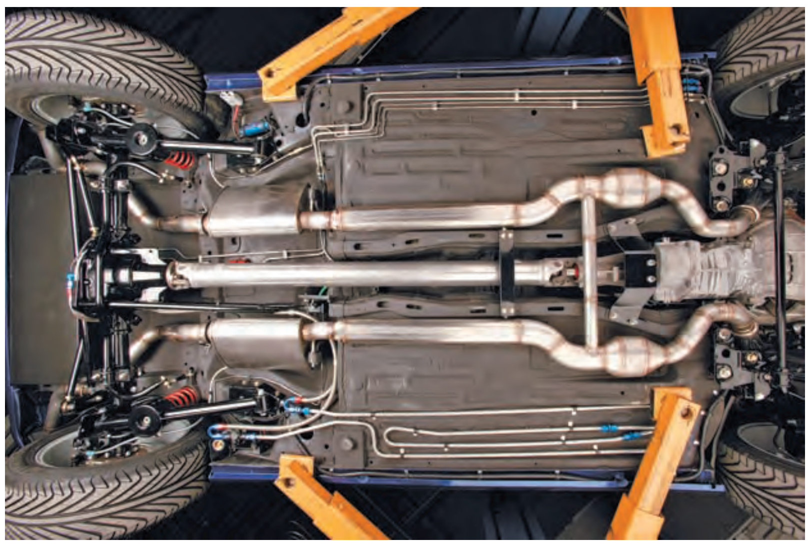


Photo 3.1 In the automotive power train shown, the shaft transmits power from the engine to the rear wheels.

Consider the system shown in Fig. 3.2*a*, which consists of a steam turbine A and an electric generator B connected by a transmission shaft AB. By breaking the system into its three component parts (Fig. 3.2*b*), you can see that the turbine exerts a twisting couple or torque **T** on the shaft and that the shaft exerts an equal torque on the generator. The generator reacts by exerting the equal and opposite torque **T′** on the shaft, and the shaft by exerting the torque **T′** on the turbine.

You will first analyze the stresses and deformations that take place in circular shafts. In Sec. 3.3, an important property of circular shafts is demonstrated: *When a circular shaft is subjected to torsion,*

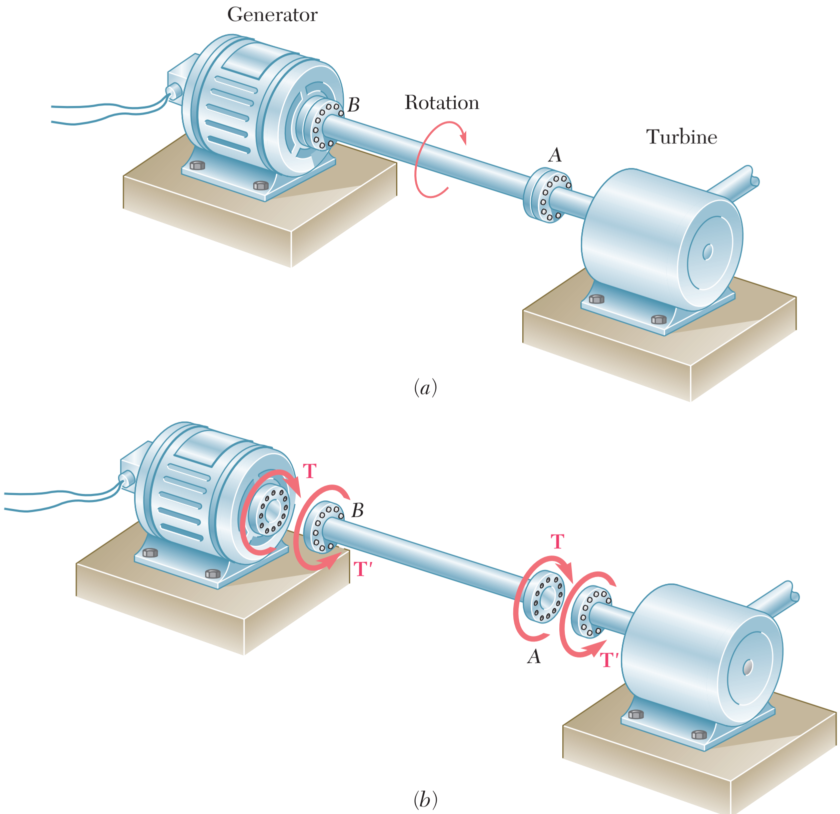


Fig. 3.2 Transmission shaft.

every cross section remains plane and undistorted. In other words, while the various cross sections along the shaft rotate through different angles, each cross section rotates as a solid rigid slab. This property will enable you to determine the *distribution of shearing strains in a circular shaft and to conclude that the shearing strain varies linearly with the distance from the axis of the shaft.*

Considering deformations in the *elastic range* and using Hooke's law for shearing stress and strain, you will determine the *distribution of shearing stresses* in a circular shaft and derive the *elastic torsion formulas* (Sec. 3.4).

In Sec. 3.5, you will learn how to find the *angle of twist* of a circular shaft subjected to a given torque, assuming again elastic deformations. The solution of problems involving *statically indeterminate shafts* is considered in Sec. 3.6.

In Sec. 3.7, you will study the *design of transmission shafts*. In order to accomplish the design, you will learn to determine the required physical characteristics of a shaft in terms of its speed of rotation and the power to be transmitted.

The torsion formulas cannot be used to determine stresses near sections where the loading couples are applied or near a section where an abrupt change in the diameter of the shaft occurs. Moreover, these formulas apply only within the elastic range of the material.

In Sec. 3.8, you will learn how to account for stress concentrations where an abrupt change in diameter of the shaft occurs. In Secs. 3.9 to 3.11, you will consider stresses and deformations in circular shafts made of a ductile material when the yield point of the material is exceeded. You will then learn how to determine the permanent *plastic deformations* and *residual stresses* that remain in a shaft after it has been loaded beyond the yield point of the material.

In the last sections of this chapter, you will study the torsion of noncircular members (Sec. 3.12) and analyze the distribution of stresses in thin-walled hollow noncircular shafts (Sec. 3.13).

3.2 PRELIMINARY DISCUSSION OF THE STRESSES IN A SHAFT

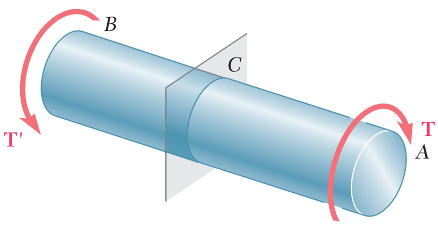


Fig. 3.3 Shaft subject to torques.

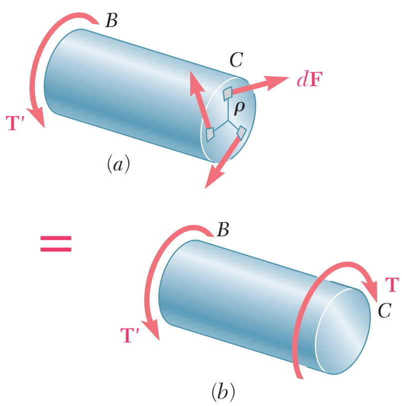


Fig. 3.4

Considering a shaft AB subjected at A and B to equal and opposite torques **T** and **T′**, we pass a section perpendicular to the axis of the shaft through some arbitrary point C (Fig. 3.3). The free-body diagram of the portion BC of the shaft must include the elementary shearing forces $d\mathbf{F}$, perpendicular to the radius of the shaft, that portion AC exerts on BC as the shaft is twisted (Fig. 3.4*a*). But the conditions of equilibrium for BC require that the system of these elementary forces be equivalent to an internal torque **T**, equal and opposite to **T′** (Fig. 3.4*b*). Denoting by ρ the perpendicular distance from the force $d\mathbf{F}$ to the axis of the shaft, and expressing that the sum of the moments of the shearing forces $d\mathbf{F}$ about the axis of the shaft is equal in magnitude to the torque **T**, we write

$$\int \rho\, dF = T$$

or, since $dF = \tau\, dA$, where τ is the shearing stress on the element of area dA,

$$\int \rho(\tau\, dA) = T \tag{3.1}$$

While the relation obtained expresses an important condition that must be satisfied by the shearing stresses in any given cross section of the shaft, it does *not* tell us how these stresses are distributed in the cross section. We thus observe, as we already did in Sec. 1.5, that the actual distribution of stresses under a given load is *statically indeterminate,* i.e., this distribution *cannot be determined by the methods of statics*. However, having assumed in Sec. 1.5 that the normal stresses produced by an axial centric load were uniformly distributed, we found later (Sec. 2.17) that this assumption was justified, except in the neighborhood of concentrated loads. A similar assumption with respect to the distribution of shearing stresses in an elastic shaft *would be wrong*. We must withhold any judgment regarding the distribution of stresses in a shaft until we have analyzed the *deformations* that are produced in the shaft. This will be done in the next section.

One more observation should be made at this point. As was indicated in Sec. 1.12, shear cannot take place in one plane only. Consider the very small element of shaft shown in Fig. 3.5. We know that the torque applied to the shaft produces shearing stresses τ on

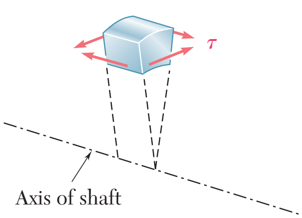


Fig. 3.5 Element in shaft.

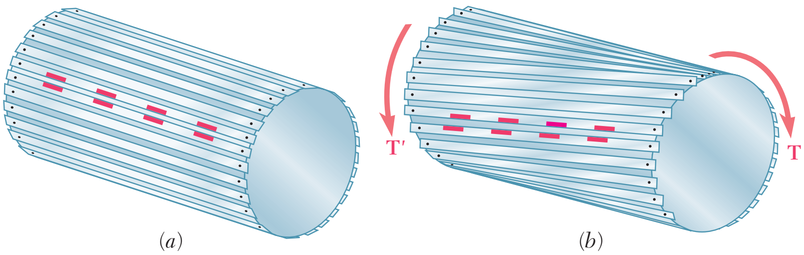


Fig. 3.6 Model of shaft.

the faces perpendicular to the axis of the shaft. But the conditions of equilibrium discussed in Sec. 1.12 require the existence of equal stresses on the faces formed by the two planes containing the axis of the shaft. That such shearing stresses actually occur in torsion can be demonstrated by considering a "shaft" made of separate slats pinned at both ends to disks as shown in Fig. 3.6*a*. If markings have been painted on two adjoining slats, it is observed that the slats slide with respect to each other when equal and opposite torques are applied to the ends of the "shaft" (Fig. 3.6*b*). While sliding will not actually take place in a shaft made of a homogeneous and cohesive material, the tendency for sliding will exist, showing that stresses occur on longitudinal planes as well as on planes perpendicular to the axis of the shaft.†

3.3 DEFORMATIONS IN A CIRCULAR SHAFT

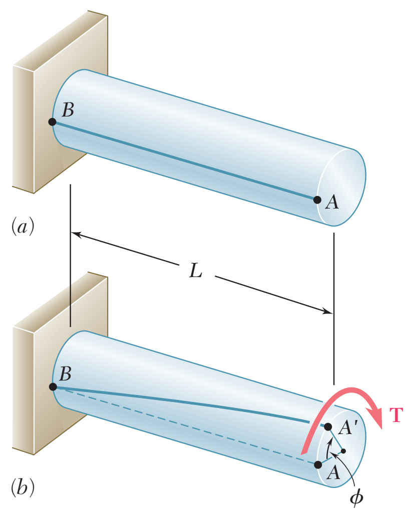


Fig. 3.7 Shaft with fixed support.

Consider a circular shaft that is attached to a fixed support at one end (Fig. 3.7*a*). If a torque **T** is applied to the other end, the shaft will twist, with its free end rotating through an angle ϕ called *the angle of twist* (Fig. 3.7*b*). Observation shows that, within a certain range of values of T, the angle of twist ϕ is proportional to T. It also shows that ϕ is proportional to the length L of the shaft. In other words, the angle of twist for a shaft of the same material and same cross section, but twice as long, will be twice as large under the same torque **T.** One purpose of our analysis will be to find the specific relation existing among ϕ, L, and T; another purpose will be to determine the distribution of shearing stresses in the shaft, which we were unable to obtain in the preceding section on the basis of statics alone.

At this point, an important property of circular shafts should be noted: When a circular shaft is subjected to torsion, *every cross section remains plane and undistorted*. In other words, while the various cross sections along the shaft rotate through different amounts, each cross section rotates as a solid rigid slab. This is illustrated in Fig. 3.8*a*, which shows the deformations in a rubber model subjected to torsion. The property we are discussing is characteristic of circular shafts, whether solid or hollow; it is not enjoyed by members of noncircular cross section. For example, when a bar of square cross section is subjected to torsion, its various cross sections warp and do not remain plane (Fig. 3.8*b*).

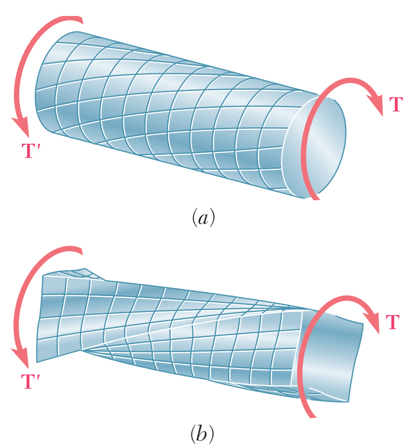


Fig. 3.8 Comparison of deformations in circular and square shafts.

†The twisting of a cardboard tube that has been slit lengthwise provides another demonstration of the existence of shearing stresses on longitudinal planes.

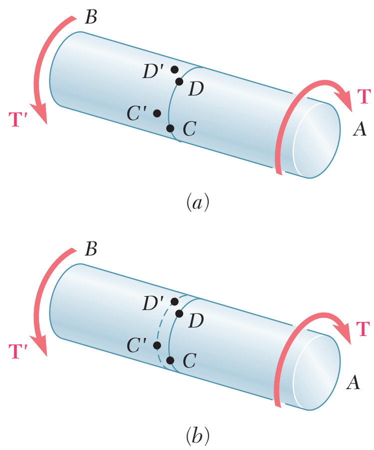


Fig. 3.9 Shaft subject to twisting.

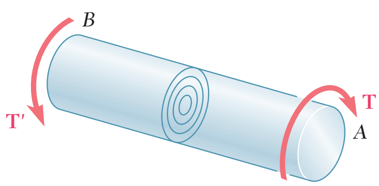


Fig. 3.10 Concentric circles.

The cross sections of a circular shaft remain plane and undistorted because a circular shaft is *axisymmetric,* i.e., its appearance remains the same when it is viewed from a fixed position and rotated about its axis through an arbitrary angle. (Square bars, on the other hand, retain the same appearance only if they are rotated through 90° or 180°.) As we will see presently, the axisymmetry of circular shafts may be used to prove theoretically that their cross sections remain plane and undistorted.

Consider the points C and D located on the circumference of a given cross section of the shaft, and let C' and D' be the positions they will occupy after the shaft has been twisted (Fig. 3.9*a*). The axisymmetry of the shaft and of the loading requires that the rotation which would have brought D into D' should now bring C into C'. Thus C' and D' must lie on the circumference of a circle, and the arc $C'D'$ must be equal to the arc CD (Fig. 3.9*b*). We will now examine whether the circle on which C' and D' lie is different from the original circle. Let us assume that C' and D' do lie on a different circle and that the new circle is located to the left of the original circle, as shown in Fig. 3.9*b*. The same situation will prevail for any other cross section, since all the cross sections of the shaft are subjected to the same internal torque T, and an observer looking at the shaft from its end A will conclude that the loading causes any given circle drawn on the shaft to move *away*. But an observer located at B, to whom the given loading looks the same (a clockwise couple in the foreground and a counterclockwise couple in the background) will reach the opposite conclusion, i.e., that the circle moves *toward* him. This contradiction proves that our assumption is wrong and that C' and D' lie on the same circle as C and D. Thus, as the shaft is twisted, the original circle just rotates in its own plane. Since the same reasoning may be applied to any smaller, concentric circle located in the cross section under consideration, we conclude that the entire cross section remains plane (Fig. 3.10).

The above argument does not preclude the possibility for the various concentric circles of Fig. 3.10 to rotate by different amounts when the shaft is twisted. But if that were so, a given diameter of the cross section would be distorted into a curve which might look as shown in Fig. 3.11*a*. An observer looking at this curve from A would conclude that the outer layers of the shaft get more twisted than the inner ones, while an observer looking from B would reach the opposite conclusion (Fig. 3.11*b*). This inconsistency leads us to conclude that any diameter of a given cross section remains straight (Fig. 3.11*c*) and, therefore, that any given cross section of a circular shaft remains plane and undistorted.

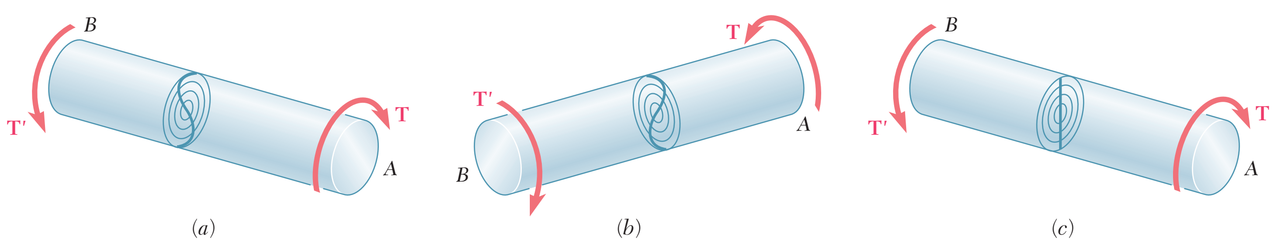


Fig. 3.11 Potential deformations of cross section.

Our discussion so far has ignored the mode of application of the twisting couples **T** and **T**′. If *all* sections of the shaft, from one end to the other, are to remain plane and undistorted, we must make sure that the couples are applied in such a way that the ends of the shaft themselves remain plane and undistorted. This may be accomplished by applying the couples **T** and **T**′ to rigid plates, which are solidly attached to the ends of the shaft (Fig. 3.12a). We can then be sure that all sections will remain plane and undistorted when the loading is applied, and that the resulting deformations will occur in a uniform fashion throughout the entire length of the shaft. All of the equally spaced circles shown in Fig. 3.12*a* will rotate by the same amount relative to their neighbors, and each of the straight lines will be transformed into a curve (helix) intersecting the various circles at the same angle (Fig. 3.12*b*).

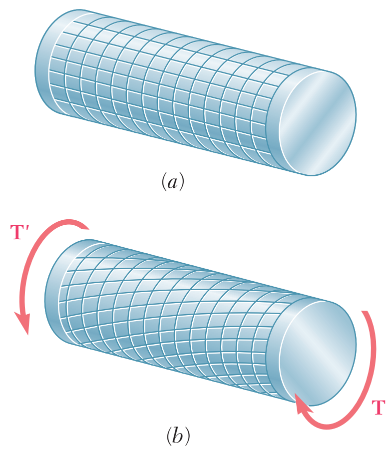


Fig. 3.12 Deformation of shaft subject to twisting couples.

The derivations given in this and the following sections will be based on the assumption of rigid end plates. Loading conditions encountered in practice may differ appreciably from those corresponding to the model of Fig. 3.12. The chief merit of this model is that it helps us define a torsion problem for which we can obtain an exact solution, just as the rigid-end-plates model of Sec. 2.17 made it possible for us to define an axial-load problem which could be easily and accurately solved. By virtue of Saint-Venant's principle, the results obtained for our idealized model may be extended to most engineering applications. However, we should keep these results associated in our mind with the specific model shown in Fig. 3.12.

We will now determine the distribution of *shearing strains* in a circular shaft of length L and radius c that has been twisted through an angle ϕ (Fig. 3.13*a*). Detaching from the shaft a cylinder of radius ρ, we consider the small square element formed by two adjacent circles and two adjacent straight lines traced on the surface of the cylinder before any load is applied (Fig. 3.13*b*). As the shaft is subjected to a torsional load, the element deforms into a rhombus (Fig. 3.13*c*). We now recall from Sec. 2.14 that the shearing strain γ in a given element is measured by the change in the angles formed by the sides of that element. Since the circles defining two of the sides of the element considered here remain unchanged, the shearing strain γ must be equal to the angle between lines AB and $A'B$. (We recall that γ should be expressed in radians.)

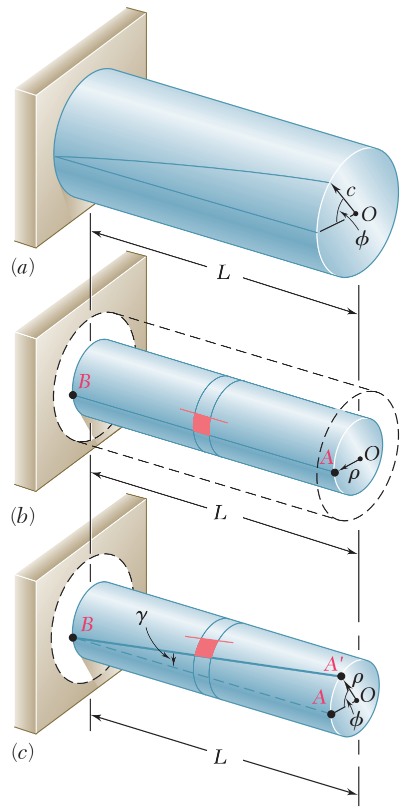


Fig. 3.13 Shearing strain.

We observe from Fig. 3.13*c* that, for small values of γ, we can express the arc length AA' as $AA' = L\gamma$. But, on the other hand, we have $AA' = \rho\phi$. It follows that $L\gamma = \rho\phi$, or

$$\gamma = \frac{\rho\phi}{L} \tag{3.2}$$

where γ and ϕ are both expressed in radians. The equation obtained shows, as we could have anticipated, that the shearing strain γ at a given point of a shaft in torsion is proportional to the angle of twist ϕ. It also shows that γ is proportional to the distance ρ from the axis of the shaft to the point under consideration. Thus, *the shearing strain in a circular shaft varies linearly with the distance from the axis of the shaft*.

It follows from Eq. (3.2) that the shearing strain is maximum on the surface of the shaft, where $\rho = c$. We have

$$\gamma_{\max} = \frac{c\phi}{L} \tag{3.3}$$

Eliminating ϕ from Eqs. (3.2) and (3.3), we can express the shearing strain γ at a distance ρ from the axis of the shaft as

$$\gamma = \frac{\rho}{c}\gamma_{\max} \tag{3.4}$$

3.4 STRESSES IN THE ELASTIC RANGE

No particular stress-strain relationship has been assumed so far in our discussion of circular shafts in torsion. Let us now consider the case when the torque **T** is such that all shearing stresses in the shaft remain below the yield strength τ_Y. We know from Chap. 2 that, for all practical purposes, this means that the stresses in the shaft will remain below the proportional limit and below the elastic limit as well. Thus, Hooke's law will apply and there will be no permanent deformation.

Recalling Hooke's law for shearing stress and strain from Sec. 2.14, we write

$$\tau = G\gamma \tag{3.5}$$

where G is the modulus of rigidity or shear modulus of the material. Multiplying both members of Eq. (3.4) by G, we write

$$G\gamma = \frac{\rho}{c}G\gamma_{\max}$$

or, making use of Eq. (3.5),

$$\tau = \frac{\rho}{c}\tau_{\max} \tag{3.6}$$

The equation obtained shows that, as long as the yield strength (or proportional limit) is not exceeded in any part of a circular shaft, *the shearing stress in the shaft varies linearly with the distance ρ from the axis of the shaft*. Figure 3.14*a* shows the stress distribution in a solid circular shaft of radius c, and Fig. 3.14*b* in a hollow circular shaft of inner radius c_1 and outer radius c_2. From Eq. (3.6), we find that, in the latter case,

$$\tau_{\min} = \frac{c_1}{c_2}\tau_{\max} \tag{3.7}$$

We now recall from Sec. 3.2 that the sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude T of the torque exerted on the shaft:

$$\int \rho(\tau\, dA) = T \tag{3.1}$$

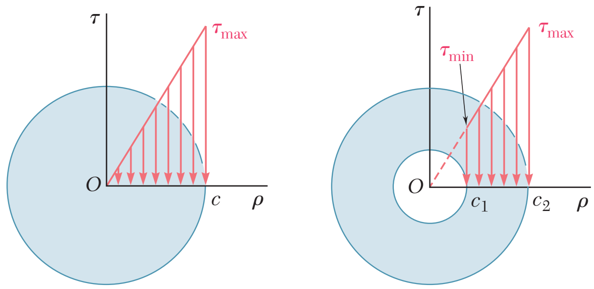


Fig. 3.14 Distribution of shearing stresses.

Substituting for τ from (3.6) into (3.1), we write

$$T = \int \rho\tau \, dA = \frac{\tau_{\max}}{c} \int \rho^2 \, dA$$

But the integral in the last member represents the polar moment of inertia J of the cross section with respect to its center O. We have therefore

$$T = \frac{\tau_{\max} J}{c} \tag{3.8}$$

or, solving for $\tau_{\max}$,

$$\boxed{\tau_{\max} = \frac{Tc}{J}} \tag{3.9}$$

Substituting for $\tau_{\max}$ from (3.9) into (3.6), we express the shearing stress at any distance ρ from the axis of the shaft as

$$\boxed{\tau = \frac{T\rho}{J}} \tag{3.10}$$

Equations (3.9) and (3.10) are known as the *elastic torsion formulas*. We recall from statics that the polar moment of inertia of a circle of radius c is $J = \frac{1}{2}\pi c^4$. In the case of a hollow circular shaft of inner radius c_1 and outer radius c_2, the polar moment of inertia is

$$J = \tfrac{1}{2}\pi c_2^4 - \tfrac{1}{2}\pi c_1^4 = \tfrac{1}{2}\pi(c_2^4 - c_1^4) \tag{3.11}$$

We note that, if SI metric units are used in Eq. (3.9) or (3.10), T will be expressed in $\mathrm{N \cdot m}$, c or ρ in meters, and J in m^4; we check that the resulting shearing stress will be expressed in $\mathrm{N/m^2}$, that is, pascals (Pa). If U.S. customary units are used, T should be expressed in $\mathrm{lb \cdot in.}$, c or ρ in inches, and J in in^4, with the resulting shearing stress expressed in psi.

EXAMPLE 3.01

A hollow cylindrical steel shaft is 1.5 m long and has inner and outer diameters respectively equal to 40 and 60 mm (Fig. 3.15). (*a*) What is the largest torque that can be applied to the shaft if the shearing stress is not to exceed 120 MPa? (*b*) What is the corresponding minimum value of the shearing stress in the shaft?

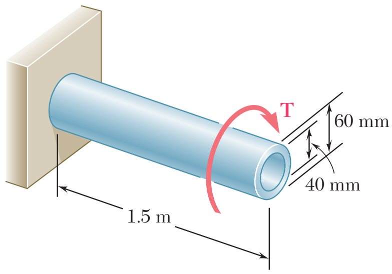


Fig. 3.15

(a) Largest Permissible Torque. The largest torque **T** that can be applied to the shaft is the torque for which $\tau_{max} = 120$ MPa. Since this value is less than the yield strength for steel, we can use Eq. (3.9). Solving this equation for T, we have

$$T = \frac{J\tau_{max}}{c} \tag{3.12}$$

Recalling that the polar moment of inertia J of the cross section is given by Eq. (3.11), where $c_1 = \frac{1}{2}(40 \text{ mm}) = 0.02$ m and $c_2 = \frac{1}{2}(60 \text{ mm}) = 0.03$ m, we write

$$J = \tfrac{1}{2}\pi\,(c_2^4 - c_1^4) = \tfrac{1}{2}\pi(0.03^4 - 0.02^4) = 1.021 \times 10^{-6} \text{ m}^4$$

Substituting for J and τ_{max} into (3.12), and letting $c = c_2 = 0.03$ m, we have

$$T = \frac{J\tau_{max}}{c} = \frac{(1.021 \times 10^{-6} \text{ m}^4)(120 \times 10^6 \text{ Pa})}{0.03 \text{ m}} = 4.08 \text{ kN} \cdot \text{m}$$

(b) Minimum Shearing Stress. The minimum value of the shearing stress occurs on the inner surface of the shaft. It is obtained from Eq. (3.7), which expresses that τ_{min} and τ_{max} are respectively proportional to c_1 and c_2:

$$\tau_{min} = \frac{c_1}{c_2}\tau_{max} = \frac{0.02 \text{ m}}{0.03 \text{ m}}(120 \text{ MPa}) = 80 \text{ MPa}$$

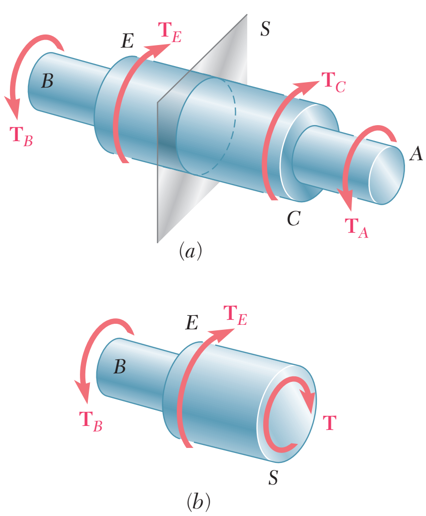


Fig. 3.16 Shaft with variable cross section.

The torsion formulas (3.9) and (3.10) were derived for a shaft of uniform circular cross section subjected to torques at its ends. However, they can also be used for a shaft of variable cross section or for a shaft subjected to torques at locations other than its ends (Fig. 3.16*a*). The distribution of shearing stresses in a given cross section S of the shaft is obtained from Eq. (3.9), where J denotes the polar moment of inertia of that section, and where T represents the *internal torque* in that section. The value of T is obtained by drawing the free-body diagram of the portion of shaft located on one side of the section (Fig. 3.16*b*) and writing that the sum of the torques applied to that portion, including the internal torque **T,** is zero (see Sample Prob. 3.1).

Up to this point, our analysis of stresses in a shaft has been limited to shearing stresses. This is due to the fact that the element we had selected was oriented in such a way that its faces were either parallel or perpendicular to the axis of the shaft (Fig. 3.5). We know from earlier discussions (Secs. 1.11 and 1.12) that normal stresses, shearing stresses, or a combination of both may be found under the same loading condition, depending upon the orientation of the element that has been chosen. Consider the two elements a and b located on the surface of a circular shaft subjected to torsion

(Fig. 3.17). Since the faces of element a are respectively parallel and perpendicular to the axis of the shaft, the only stresses on the element will be the shearing stresses defined by formula (3.9), namely $\tau_{\max} = Tc/J$. On the other hand, the faces of element b, which form arbitrary angles with the axis of the shaft, will be subjected to a combination of normal and shearing stresses.

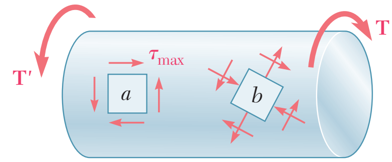


Fig. 3.17 Circular shaft with elements at different orientations.

Let us consider the stresses and resulting forces on faces that are at 45° to the axis of the shaft. In order to determine the stresses on the faces of this element, we consider the two triangular elements shown in Fig. 3.18 and draw their free-body diagrams. In the case of the element of Fig. 3.18a, we know that the stresses exerted on the faces BC and BD are the shearing stresses $\tau_{\max} = Tc/J$. The magnitude of the corresponding shearing forces is thus $\tau_{\max} A_0$, where A_0 denotes the area of the face. Observing that the components along DC of the two shearing forces are equal and opposite, we conclude that the force **F** exerted on DC must be perpendicular to that face. It is a tensile force, and its magnitude is

$$F = 2(\tau_{\max} A_0)\cos 45° = \tau_{\max} A_0 \sqrt{2} \tag{3.13}$$

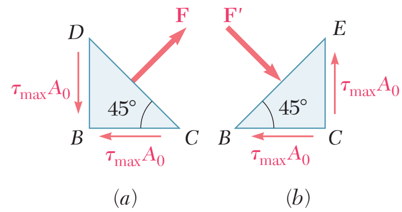


Fig. 3.18 Forces on faces at 45° to shaft axis.

The corresponding stress is obtained by dividing the force F by the area A of face DC. Observing that $A = A_0\sqrt{2}$, we write

$$\sigma = \frac{F}{A} = \frac{\tau_{\max} A_0 \sqrt{2}}{A_0\sqrt{2}} = \tau_{\max} \tag{3.14}$$

A similar analysis of the element of Fig. 3.18b shows that the stress on the face BE is $\sigma = -\tau_{\max}$. We conclude that the stresses exerted on the faces of an element c at 45° to the axis of the shaft (Fig. 3.19) are normal stresses equal to $\pm\tau_{\max}$. Thus, while the element a in Fig. 3.19 is in pure shear, the element c in the same figure is subjected to a tensile stress on two of its faces, and to a compressive stress on the other two. We also note that all the stresses involved have the same magnitude, Tc/J.†

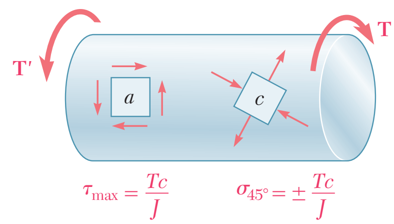
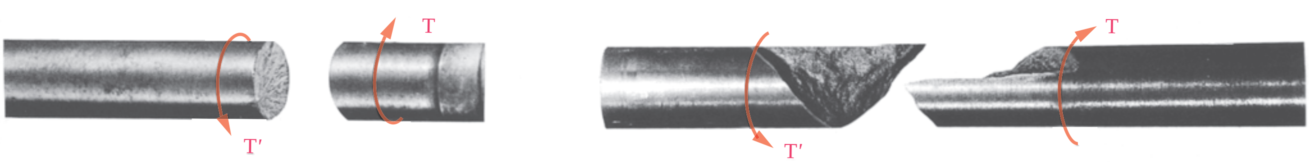


Fig. 3.19 Shaft with elements with only shear stresses or normal stresses.

As you learned in Sec. 2.3, ductile materials generally fail in shear. Therefore, when subjected to torsion, a specimen J made of a ductile material breaks along a plane perpendicular to its longitudinal axis (Photo 3.2a). On the other hand, brittle materials are weaker in tension than in shear. Thus, when subjected to torsion, a specimen made of a brittle material tends to break along surfaces that are perpendicular to the direction in which tension is maximum, i.e., along surfaces forming a 45° angle with the longitudinal axis of the specimen (Photo 3.2b).

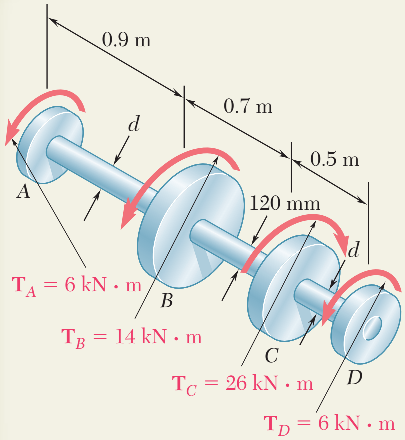


(a) Ductile failure

(b) Brittle failure

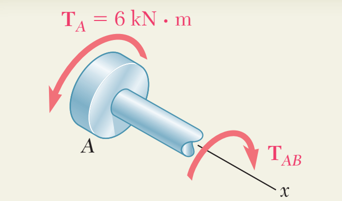
Photo 3.2 Shear failure of shaft subject to torque.

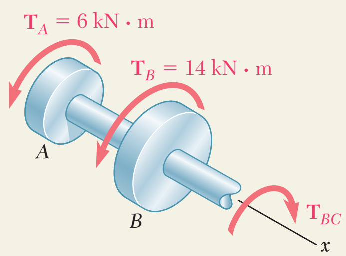
†Stresses on elements of arbitrary orientation, such as element b of Fig. 3.18, will be discussed in Chap. 7.

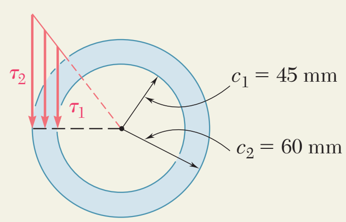


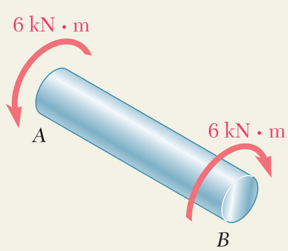
SAMPLE PROBLEM 3.1

Shaft BC is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts AB and CD are solid and of diameter d. For the loading shown, determine (*a*) the maximum and minimum shearing stress in shaft BC, (*b*) the required diameter d of shafts AB and CD if the allowable shearing stress in these shafts is 65 MPa.









SOLUTION

Equations of Statics. Denoting by $\mathbf{T}_{AB}$ the torque in shaft AB, we pass a section through shaft AB and, for the free body shown, we write

$$\Sigma M_x = 0: \qquad (6\text{ kN}\cdot\text{m}) - T_{AB} = 0 \qquad T_{AB} = 6\text{ kN}\cdot\text{m}$$

We now pass a section through shaft BC and, for the free body shown, we have

$$\Sigma M_x = 0: \quad (6\text{ kN}\cdot\text{m}) + (14\text{ kN}\cdot\text{m}) - T_{BC} = 0 \qquad T_{BC} = 20\text{ kN}\cdot\text{m}$$

a. Shaft *BC*. For this hollow shaft we have

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.060)^4 - (0.045)^4] = 13.92 \times 10^{-6}\text{ m}^4$$

Maximum Shearing Stress. On the outer surface, we have

$$\tau_{\max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\text{ kN}\cdot\text{m})(0.060\text{ m})}{13.92 \times 10^{-6}\text{ m}^4} \qquad \tau_{\max} = 86.2\text{ MPa} \blacktriangleleft$$

Minimum Shearing Stress. We write that the stresses are proportional to the distance from the axis of the shaft.

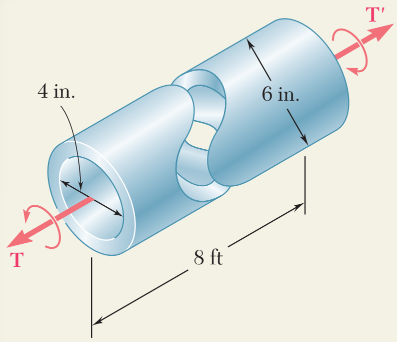
$$\frac{\tau_{\min}}{\tau_{\max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{\min}}{86.2\text{ MPa}} = \frac{45\text{ mm}}{60\text{ mm}} \qquad \tau_{\min} = 64.7\text{ MPa} \blacktriangleleft$$

b. Shafts *AB* and *CD*. We note that in both of these shafts the magnitude of the torque is $T = 6$ kN · m and $\tau_{\text{all}} = 65$ MPa. Denoting by c the radius of the shafts, we write

$$\tau = \frac{Tc}{J} \qquad 65\text{ MPa} = \frac{(6\text{ kN}\cdot\text{m})c}{\frac{\pi}{2}c^4}$$

$$c^3 = 58.8 \times 10^{-6}\text{ m}^3 \qquad c = 38.9 \times 10^{-3}\text{ m}$$

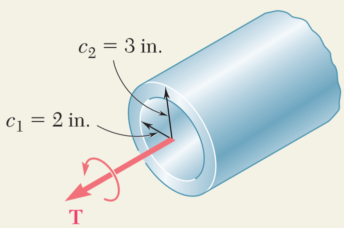
$$d = 2c = 2(38.9\text{ mm}) \qquad d = 77.8\text{ mm} \blacktriangleleft$$



SAMPLE PROBLEM 3.2

The preliminary design of a large shaft connecting a motor to a generator calls for the use of a hollow shaft with inner and outer diameters of 4 in. and 6 in., respectively. Knowing that the allowable shearing stress is 12 ksi, determine the maximum torque that can be transmitted (*a*) by the shaft as designed, (*b*) by a solid shaft of the same weight, (*c*) by a hollow shaft of the same weight and of 8-in. outer diameter.

SOLUTION

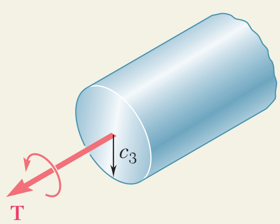


a. Hollow Shaft as Designed. For the hollow shaft we have

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(3\text{ in.})^4 - (2\text{ in.})^4] = 102.1\text{ in}^4$$

Using Eq. (3.9), we write

$$\tau_{\max} = \frac{Tc_2}{J} \qquad 12\text{ ksi} = \frac{T(3\text{ in.})}{102.1\text{ in}^4} \qquad T = 408\text{ kip}\cdot\text{in.} \blacktriangleleft$$



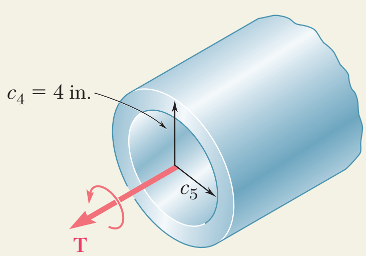
b. Solid Shaft of Equal Weight. For the shaft as designed and this solid shaft to have the same weight and length, their cross-sectional areas must be equal.

$$A_{(a)} = A_{(b)}$$

$$\pi[(3\text{ in.})^2 - (2\text{ in.})^2] = \pi c_3^2 \qquad c_3 = 2.24\text{ in.}$$

Since $\tau_{\text{all}} = 12$ ksi, we write

$$\tau_{\max} = \frac{Tc_3}{J} \qquad 12\text{ ksi} = \frac{T(2.24\text{ in.})}{\frac{\pi}{2}(2.24\text{ in.})^4} \qquad T = 211\text{ kip}\cdot\text{in.} \blacktriangleleft$$



c. Hollow Shaft of 8-in. Diameter. For equal weight, the cross-sectional areas again must be equal. We determine the inside diameter of the shaft by writing

$$A_{(a)} = A_{(c)}$$

$$\pi[(3\text{ in.})^2 - (2\text{ in.})^2] = \pi[(4\text{ in.})^2 - c_5^2] \qquad c_5 = 3.317\text{ in.}$$

For $c_5 = 3.317$ in. and $c_4 = 4$ in.,

$$J = \frac{\pi}{2}[(4\text{ in.})^4 - (3.317\text{ in.})^4] = 212\text{ in}^4$$

With $\tau_{\text{all}} = 12$ ksi and $c_4 = 4$ in.,

$$\tau_{\max} = \frac{Tc_4}{J} \qquad 12\text{ ksi} = \frac{T(4\text{ in.})}{212\text{ in}^4} \qquad T = 636\text{ kip}\cdot\text{in.} \blacktriangleleft$$

PROBLEMS

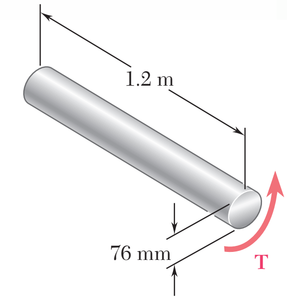


Fig. P3.1

3.1 (*a*) Determine the maximum shearing stress caused by a 4.6-kN · m torque **T** in the 76-mm-diameter solid aluminum shaft shown. (*b*) Solve part *a*, assuming that the solid shaft has been replaced by a hollow shaft of the same outer diameter and of 24-mm inner diameter.

3.2 (*a*) Determine the torque **T** that causes a maximum shearing stress of 45 MPa in the hollow cylindrical steel shaft shown. (*b*) Determine the maximum shearing stress caused by the same torque **T** in a solid cylindrical shaft of the same cross-sectional area.

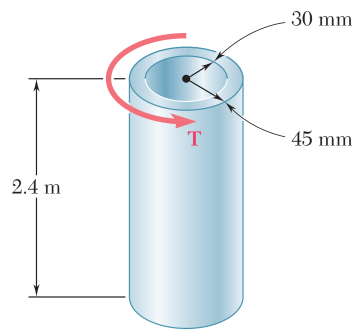


Fig. P3.2

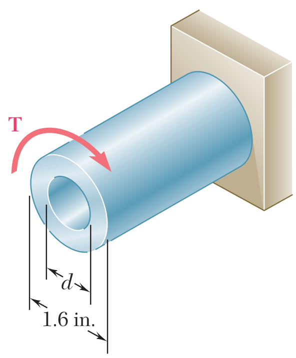


Fig. P3.3 and *P3.4*

3.3 Knowing that $d = 1.2$ in., determine the torque **T** that causes a maximum shearing stress of 7.5 ksi in the hollow shaft shown.

3.4 Knowing that the internal diameter of the hollow shaft shown is $d = 0.9$ in., determine the maximum shearing stress caused by a torque of magnitude $T = 9$ kip · in.

3.5 A torque $T = 3$ kN · m is applied to the solid bronze cylinder shown. Determine (*a*) the maximum shearing stress, (*b*) the shearing stress at point *D*, which lies on a 15-mm-radius circle drawn on the end of the cylinder, (*c*) the percent of the torque carried by the portion of the cylinder within the 15-mm radius.

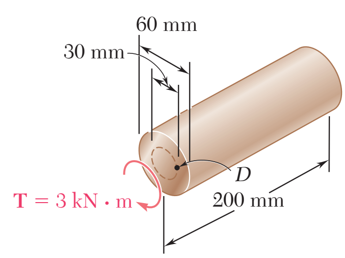


Fig. P3.5

3.6 (*a*) Determine the torque that can be applied to a solid shaft of 20-mm diameter without exceeding an allowable shearing stress of 80 MPa. (*b*) Solve part a, assuming that the solid shaft has been replaced by a hollow shaft of the same cross-sectional area and with an inner diameter equal to half of its outer diameter.

3.7 The solid spindle *AB* has a diameter $d_s = 1.5$ in. and is made of a steel with an allowable shearing stress of 12 ksi, while sleeve *CD* is made of a brass with an allowable shearing stress of 7 ksi. Determine the largest torque **T** that can be applied at *A*.

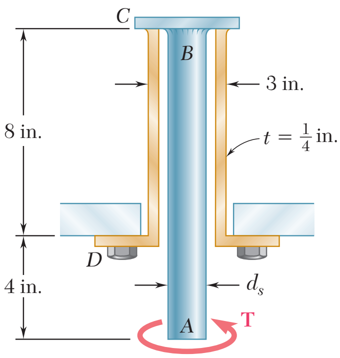


Fig. *P3.7* and P3.8

3.8 The solid spindle *AB* is made of a steel with an allowable shearing stress of 12 ksi, and sleeve *CD* is made of a brass with an allowable shearing stress of 7 ksi. Determine (*a*) the largest torque **T** that can be applied at *A* if the allowable shearing stress is not to be exceeded in sleeve *CD*, (*b*) the corresponding required value of the diameter d_s of spindle *AB*.

3.9 The torques shown are exerted on pulleys *A* and *B*. Knowing that both shafts are solid, determine the maximum shearing stress in (*a*) in shaft *AB*, (*b*) in shaft *BC*.

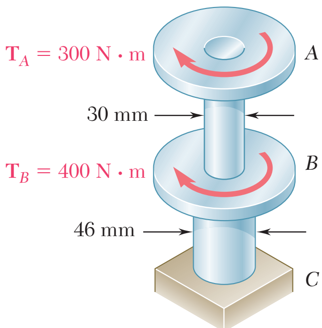


Fig. *P3.9*

3.10 In order to reduce the total mass of the assembly of Prob. 3.9, a new design is being considered in which the diameter of shaft *BC* will be smaller. Determine the smallest diameter of shaft *BC* for which the maximum value of the shearing stress in the assembly will not increase.

3.11 Knowing that each of the shafts *AB*, *BC*, and *CD* consists of a solid circular rod, determine (*a*) the shaft in which the maximum shearing stress occurs, (*b*) the magnitude of that stress.

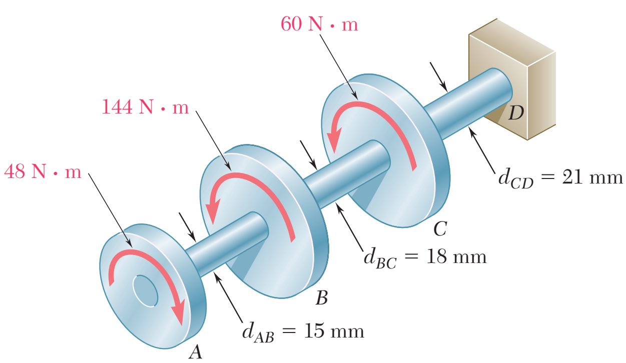


Fig. P3.11 and *P3.12*

3.12 Knowing that an 8-mm-diameter hole has been drilled through each of the shafts *AB*, *BC*, and *CD*, determine (*a*) the shaft in which the maximum shearing stress occurs, (*b*) the magnitude of that stress.

3.13 Under normal operating conditions, the electric motor exerts a 12-kip · in. torque at E. Knowing that each shaft is solid, determine the maximum shearing stress in (*a*) shaft BC, (*b*) shaft CD, (*c*) shaft DE.

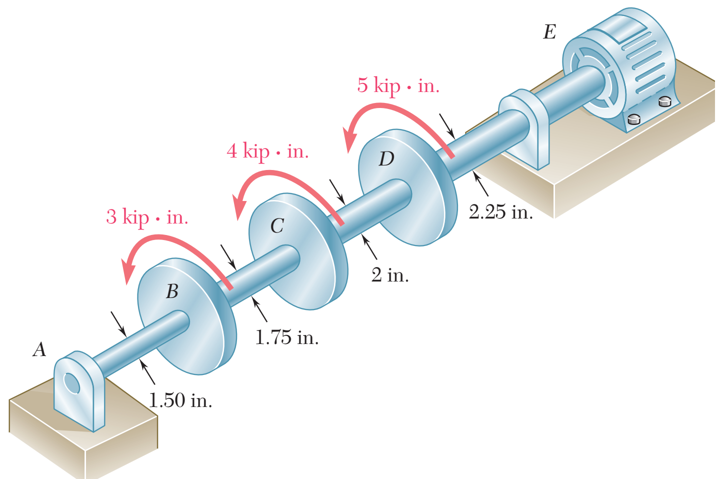


Fig. P3.13

3.14 Solve Prob. 3.13, assuming that a 1-in.-diameter hole has been drilled into each shaft.

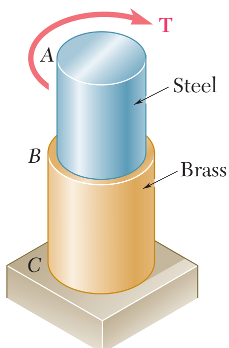


Fig. P3.15 and P3.16

3.15 The allowable shearing stress is 15 ksi in the 1.5-in.-diameter steel rod AB and 8 ksi in the 1.8-in.-diameter brass rod BC. Neglecting the effect of stress concentrations, determine the largest torque that can be applied at A.

3.16 The allowable shearing stress is 15 ksi in the steel rod AB and 8 ksi in the brass rod BC. Knowing that a torque of magnitude $T = 10$ kip · in. is applied at A, determine the required diameter of (*a*) rod AB, (*b*) rod BC.

3.17 The allowable stress is 50 MPa in the brass rod AB and 25 MPa in the aluminum rod BC. Knowing that a torque of magnitude $T = 1250$ N · m is applied at A, determine the required diameter of (*a*) rod AB, (*b*) rod BC.

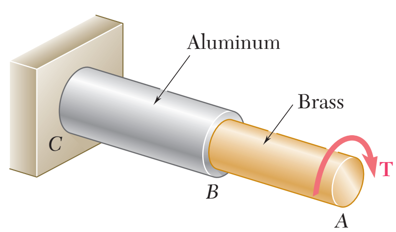


Fig. *P3.17* and *P3.18*

3.18 The *solid* rod BC has a diameter of 30 mm and is made of an aluminum for which the allowable shearing stress is 25 MPa. Rod AB is *hollow* and has an outer diameter of 25 mm; it is made of a brass for which the allowable shearing stress is 50 MPa. Determine (*a*) the largest inner diameter of rod AB for which the factor of safety is the same for each rod, (*b*) the largest torque that can be applied at A.

3.19 The solid rod AB has a diameter $d_{AB} = 60$ mm. The pipe CD has an outer diameter of 90 mm and a wall thickness of 6 mm. Knowing that both the rod and the pipe are made of steel for which the allowable shearing stress is 75 MPa, determine the largest torque **T** that can be applied at A.

3.20 The solid rod AB has a diameter $d_{AB} = 60$ mm and is made of a steel for which the allowable shearing stress is 85 MPa. The pipe CD, which has an outer diameter of 90 mm and a wall thickness of 6 mm, is made of an aluminum for which the allowable shearing stress is 54 MPa. Determine the largest torque **T** that can be applied at A.

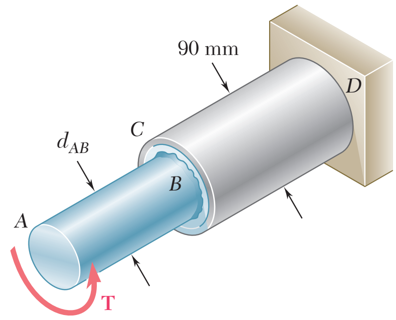


Fig. P3.19 and P3.20

3.21 A torque of magnitude $T = 1000$ N · m is applied at D as shown. Knowing that the diameter of shaft AB is 56 mm and that the diameter of shaft CD is 42 mm, determine the maximum shearing stress in (*a*) shaft AB, (*b*) shaft CD.

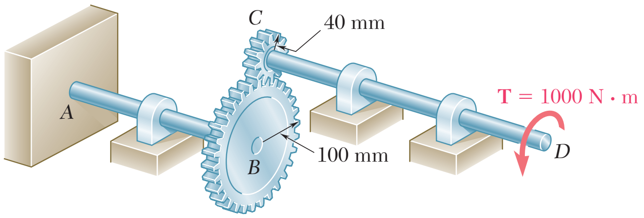


Fig. P3.21 and P3.22

3.22 A torque of magnitude $T = 1000$ N · m is applied at D as shown. Knowing that the allowable shearing stress is 60 MPa in each shaft, determine the required diameter of (*a*) shaft AB, (*b*) shaft CD.

3.23 Under normal operating conditions a motor exerts a torque of magnitude $T_F = 1200$ lb · in. at F. Knowing that $r_D = 8$ in., $r_G = 3$ in., and the allowable shearing stress is 10.5 ksi in each shaft, determine the required diameter of (*a*) shaft CDE, (*b*) shaft FGH.

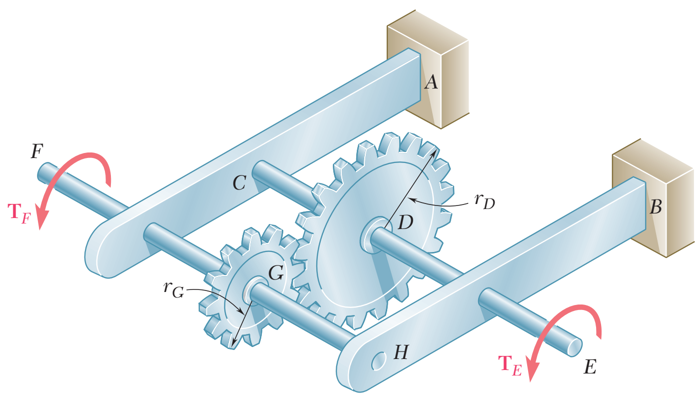


Fig. *P3.23* and P3.24

3.24 Under normal operating conditions a motor exerts a torque of magnitude T_F at F. The shafts are made of a steel for which the allowable shearing stress is 12 ksi and have diameters $d_{CDE} = 0.900$ in. and $d_{FGH} = 0.800$ in. Knowing that $r_D = 6.5$ in. and $r_G = 4.5$ in., determine the largest allowable value of T_F.

3.25 The two solid shafts are connected by gears as shown and are made of a steel for which the allowable shearing stress is 8500 psi. Knowing that a torque of magnitude $T_C = 5$ kip · in. is applied at C and that the assembly is in equilibrium, determine the required diameter of (*a*) shaft BC, (*b*) shaft EF.

3.26 The two solid shafts are connected by gears as shown and are made of a steel for which the allowable shearing stress is 7000 psi. Knowing the diameters of the two shafts are, respectively, $d_{BC} = 1.6$ in. and $d_{EF} = 1.25$ in., determine the largest torque $\mathbf{T}_C$ that can be applied at C.

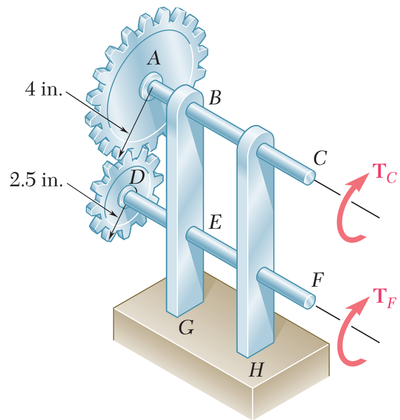


Fig. *P3.25* and P3.26

3.27 A torque of magnitude $T = 100$ N · m is applied to shaft AB of the gear train shown. Knowing that the diameters of the three solid shafts are, respectively, $d_{AB} = 21$ mm, $d_{CD} = 30$ mm, and $d_{EF} = 40$ mm, determine the maximum shearing stress in (*a*) shaft AB, (*b*) shaft CD, (*c*) shaft EF.

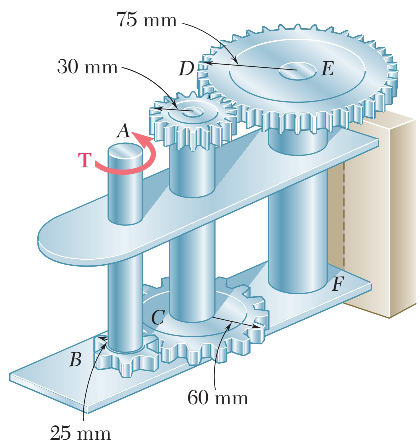


Fig. P3.27 and P3.28

3.28 A torque of magnitude $T = 120$ N · m is applied to shaft AB of the gear train shown. Knowing that the allowable shearing stress is 75 MPa in each of the three solid shafts, determine the required diameter of (*a*) shaft AB, (*b*) shaft CD, (*c*) shaft EF.

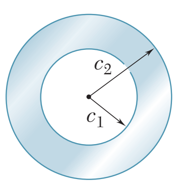


Fig. P3.29

3.29 (*a*) For a given allowable shearing stress, determine the ratio T/w of the maximum allowable torque T and the weight per unit length w for the hollow shaft shown. (*b*) Denoting by $(T/w)_0$ the value of this ratio for a solid shaft of the same radius c_2, express the ratio T/w for the hollow shaft in terms of $(T/w)_0$ and c_1/c_2.

3.30 While the exact distribution of the shearing stresses in a hollow cylindrical shaft is as shown in Fig. P3.30*a*, an approximate value can be obtained for $\tau_{\max}$ by assuming that the stresses are uniformly distributed over the area A of the cross section, as shown in Fig. P3.30*b*, and then further assuming that all of the elementary shearing forces act at a distance from O equal to the mean radius $\frac{1}{2}(c_1 + c_2)$ of the cross section. This approximate value $\tau_0 = T/Ar_m$, where T is the applied torque. Determine the ratio $\tau_{\max}/\tau_0$ of the true value of the maximum shearing stress and its approximate value τ_0 for values of c_1/c_2 respectively equal to 1.00, 0.95, 0.75, 0.50 and 0.

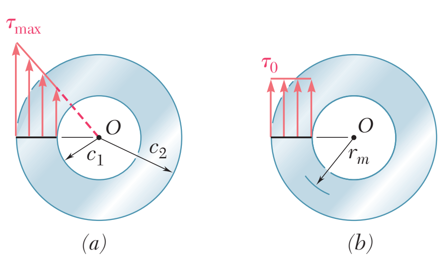


Fig. P3.30

3.5 ANGLE OF TWIST IN THE ELASTIC RANGE

In this section, a relation will be derived between the angle of twist ϕ of a circular shaft and the torque **T** exerted on the shaft. The entire shaft will be assumed to remain elastic. Considering first the case of a shaft of length L and of uniform cross section of radius c subjected to a torque **T** at its free end (Fig. 3.20), we recall from Sec. 3.3 that the angle of twist ϕ and the maximum shearing strain $\gamma_{\max}$ are related as follows:

$$\gamma_{\max} = \frac{c\phi}{L} \tag{3.3}$$

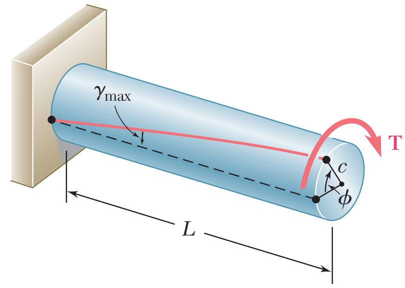


Fig. 3.20 Angle of twist ϕ.

But, in the elastic range, the yield stress is not exceeded anywhere in the shaft, Hooke's law applies, and we have $\gamma_{\max} = \tau_{\max}/G$ or, recalling Eq. (3.9),

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{Tc}{JG} \tag{3.15}$$

Equating the right-hand members of Eqs. (3.3) and (3.15), and solving for ϕ, we write

$$\phi = \frac{TL}{JG} \tag{3.16}$$

where ϕ is expressed in radians. The relation obtained shows that, within the elastic range, *the angle of twist ϕ is proportional to the torque T applied to the shaft*. This is in accordance with the experimental evidence cited at the beginning of Sec. 3.3.

Equation (3.16) provides us with a convenient method for determining the modulus of rigidity of a given material. A specimen of the material, in the form of a cylindrical rod of known diameter and length, is placed in a *torsion testing machine* (Photo 3.3). Torques of increasing magnitude T are applied to the specimen, and the corresponding values of the angle of twist ϕ in a length L of the specimen are recorded. As long as the yield stress of the material is not exceeded, the points obtained by plotting ϕ against T will fall on a straight line. The slope of this line represents the quantity JG/L, from which the modulus of rigidity G may be computed.

Photo 3.3 Torsion testing machine.

EXAMPLE 3.02

What torque should be applied to the end of the shaft of Example 3.01 to produce a twist of 2°? Use the value $G = 77$ GPa for the modulus of rigidity of steel.

Solving Eq. (3.16) for T, we write

$$T = \frac{JG}{L}\phi$$

Substituting the given values

$$G = 77 \times 10^9 \text{ Pa} \qquad L = 1.5 \text{ m}$$
$$\phi = 2°\left(\frac{2\pi \text{ rad}}{360°}\right) = 34.9 \times 10^{-3} \text{ rad}$$

and recalling from Example 3.01 that, for the given cross section,

$$J = 1.021 \times 10^{-6} \text{ m}^4$$

we have

$$T = \frac{JG}{L}\phi = \frac{(1.021 \times 10^{-6} \text{ m}^4)(77 \times 10^9 \text{ Pa})}{1.5 \text{ m}}(34.9 \times 10^{-3} \text{ rad})$$
$$T = 1.829 \times 10^3 \text{ N} \cdot \text{m} = 1.829 \text{ kN} \cdot \text{m}$$

EXAMPLE 3.03

What angle of twist will create a shearing stress of 70 MPa on the inner surface of the hollow steel shaft of Examples 3.01 and 3.02?

The method of attack for solving this problem that first comes to mind is to use Eq. (3.10) to find the torque T corresponding to the given value of τ, and Eq. (3.16) to determine the angle of twist ϕ corresponding to the value of T just found.

A more direct solution, however, may be used. From Hooke's law, we first compute the shearing strain on the inner surface of the shaft:

$$\gamma_{\min} = \frac{\tau_{\min}}{G} = \frac{70 \times 10^6 \text{ Pa}}{77 \times 10^9 \text{ Pa}} = 909 \times 10^{-6}$$

Recalling Eq. (3.2), which was obtained by expressing the length of arc AA' in Fig. 3.13c in terms of both γ and ϕ, we have

$$\phi = \frac{L\gamma_{\min}}{c_1} = \frac{1500 \text{ mm}}{20 \text{ mm}}(909 \times 10^{-6}) = 68.2 \times 10^{-3} \text{ rad}$$

To obtain the angle of twist in degrees, we write

$$\phi = (68.2 \times 10^{-3} \text{ rad})\left(\frac{360°}{2\pi \text{ rad}}\right) = 3.91°$$

Formula (3.16) for the angle of twist can be used only if the shaft is homogeneous (constant G), has a uniform cross section, and is loaded only at its ends. If the shaft is subjected to torques at locations other than its ends, or if it consists of several portions with various cross sections and possibly of different materials, we must divide it into component parts that satisfy individually the required

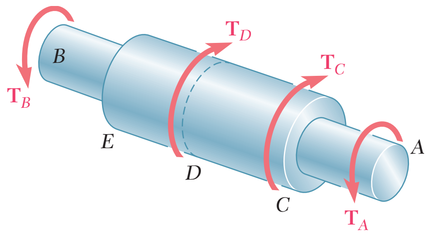


Fig. 3.21 Multiple sections and multiple torques.

conditions for the application of formula (3.16). In the case of the shaft AB shown in Fig. 3.21, for example, four different parts should be considered: AC, CD, DE, and EB. The total angle of twist of the shaft, i.e., the angle through which end A rotates with respect to end B, is obtained by adding *algebraically* the angles of twist of each component part. Denoting, respectively, by T_i, L_i, J_i, and G_i the internal torque, length, cross-sectional polar moment of inertia, and modulus of rigidity corresponding to part i, the total angle of twist of the shaft is expressed as

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i} \tag{3.17}$$

The internal torque T_i in any given part of the shaft is obtained by passing a section through that part and drawing the free-body diagram of the portion of shaft located on one side of the section. This procedure, which has already been explained in Sec. 3.4 and illustrated in Fig. 3.16, is applied in Sample Prob. 3.3.

In the case of a shaft with a variable circular cross section, as shown in Fig. 3.22, formula (3.16) may be applied to a disk of thickness dx. The angle by which one face of the disk rotates with respect to the other is thus

$$d\phi = \frac{T\,dx}{JG}$$

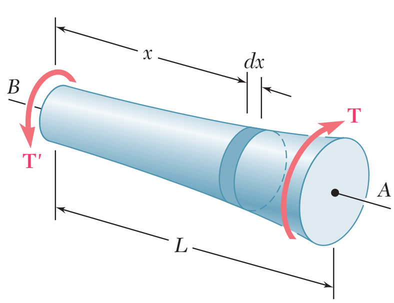


Fig. 3.22 Shaft with variable cross section.

where J is a function of x, which may be determined. Integrating in x from 0 to L, we obtain the total angle of twist of the shaft:

$$\phi = \int_0^L \frac{T\,dx}{JG} \tag{3.18}$$

The shaft shown in Fig. 3.20, which was used to derive formula (3.16), and the shaft of Fig. 3.15, which was discussed in Examples 3.02 and 3.03, both had one end attached to a fixed support. In each case, therefore, the angle of twist ϕ of the shaft was equal to the angle of rotation of its free end. When both ends of a shaft rotate, however, the angle of twist of the shaft is equal to the angle through which one end of the shaft rotates *with respect to the other*. Consider, for instance, the assembly shown in Fig. 3.23*a*, consisting of two elastic shafts AD and BE, each of length L, radius c, and modulus of rigidity G, which are attached to gears meshed at C. If a torque $\mathbf{T}$ is applied at E (Fig. 3.23*b*), both shafts will be twisted. Since the end D of shaft AD is fixed, the angle of twist of AD is measured by the angle of rotation ϕ_A of end A. On the other hand, since both ends of shaft BE rotate, the angle of twist of BE is equal to the difference between the angles of rotation ϕ_B and ϕ_E, i.e., the angle of twist is equal to the angle through which end E rotates with respect to end B. Denoting this relative angle of rotation by $\phi_{E/B}$, we write

$$\phi_{E/B} = \phi_E - \phi_B = \frac{TL}{JG}$$

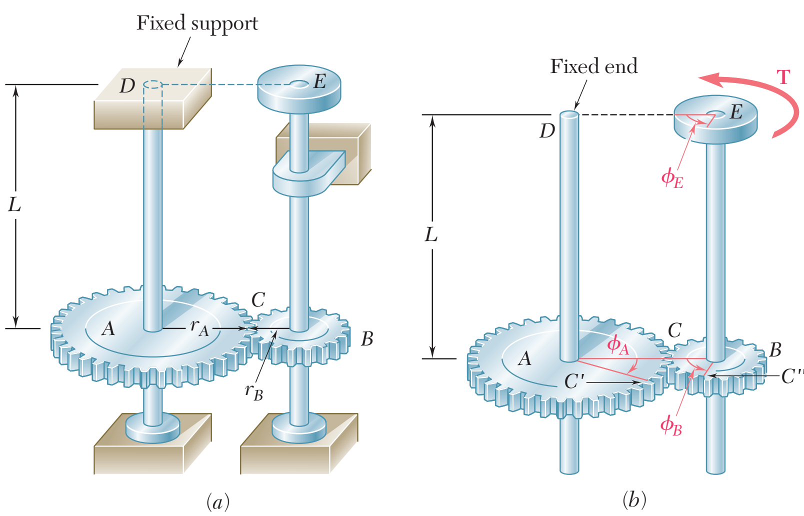


Fig. 3.23 Gear assembly.

EXAMPLE 3.04

For the assembly of Fig. 3.23, knowing that $r_A = 2r_B$, determine the angle of rotation of end E of shaft BE when the torque $\mathbf{T}$ is applied at E.

We first determine the torque $\mathbf{T}_{AD}$ exerted on shaft AD. Observing that equal and opposite forces $\mathbf{F}$ and $\mathbf{F}'$ are applied on the two gears at C (Fig. 3.24), and recalling that $r_A = 2r_B$, we conclude that the torque exerted on shaft AD is twice as large as the torque exerted on shaft BE; thus, $T_{AD} = 2T$.

Since the end D of shaft AD is fixed, the angle of rotation ϕ_A of gear A is equal to the angle of twist of the shaft and is obtained by writing

$$\phi_A = \frac{T_{AD}L}{JG} = \frac{2TL}{JG}$$

Observing that the arcs CC' and CC'' in Fig. 3.26*b* must be equal, we write $r_A\phi_A = r_B\phi_B$ and obtain

$$\phi_B = (r_A/r_B)\phi_A = 2\phi_A$$

We have, therefore,

$$\phi_B = 2\phi_A = \frac{4TL}{JG}$$

Considering now shaft BE, we recall that the angle of twist of the shaft is equal to the angle $\phi_{E/B}$ through which end E rotates with respect to end B. We have

$$\phi_{E/B} = \frac{T_{BE}L}{JG} = \frac{TL}{JG}$$

The angle of rotation of end E is obtained by writing

$$\phi_E = \phi_B + \phi_{E/B}$$

$$= \frac{4TL}{JG} + \frac{TL}{JG} = \frac{5TL}{JG}$$

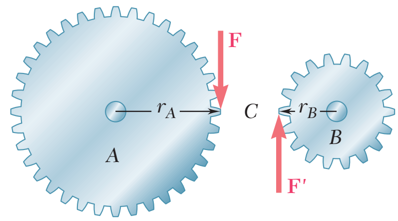


Fig. 3.24

3.6 STATICALLY INDETERMINATE SHAFTS

You saw in Sec. 3.4 that, in order to determine the stresses in a shaft, it was necessary to first calculate the internal torques in the various parts of the shaft. These torques were obtained from statics by drawing the free-body diagram of the portion of shaft located on one side of a given section and writing that the sum of the torques exerted on that portion was zero.

There are situations, however, where the internal torques cannot be determined from statics alone. In fact, in such cases the external torques themselves, i.e., the torques exerted on the shaft by the supports and connections, cannot be determined from the free-body diagram of the entire shaft. The equilibrium equations must be complemented by relations involving the deformations of the shaft and obtained by considering the geometry of the problem. Because statics is not sufficient to determine the external and internal torques, the shafts are said to be *statically indeterminate*. The following example, as well as Sample Prob. 3.5, will show how to analyze statically indeterminate shafts.

EXAMPLE 3.05

A circular shaft AB consists of a 10-in.-long, $\frac{7}{8}$-in.-diameter steel cylinder, in which a 5-in.-long, $\frac{5}{8}$-in.-diameter cavity has been drilled from end B. The shaft is attached to fixed supports at both ends, and a 90 lb · ft torque is applied at its midsection (Fig. 3.25). Determine the torque exerted on the shaft by each of the supports.

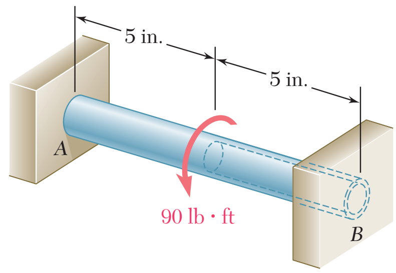


Fig. 3.25

Drawing the free-body diagram of the shaft and denoting by $\mathbf{T}_A$ and $\mathbf{T}_B$ the torques exerted by the supports (Fig. 3.26*a*), we obtain the equilibrium equation

$$T_A + T_B = 90 \text{ lb} \cdot \text{ft}$$

Since this equation is not sufficient to determine the two unknown torques $\mathbf{T}_A$ and $\mathbf{T}_B$, the shaft is statically indeterminate.

However, $\mathbf{T}_A$ and $\mathbf{T}_B$ can be determined if we observe that the total angle of twist of shaft AB must be zero, since both of its ends are restrained. Denoting by ϕ_1 and ϕ_2, respectively, the angles of twist of portions AC and CB, we write

$$\phi = \phi_1 + \phi_2 = 0$$

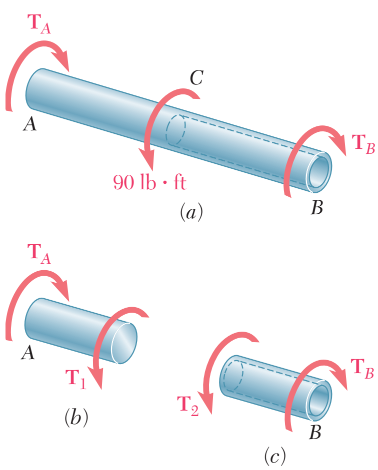


Fig. 3.26

From the free-body diagram of a small portion of shaft including end A (Fig. 3.26*b*), we note that the internal torque T_1 in AC is equal to T_A; from the free-body diagram of a small portion of shaft including end B (Fig. 3.26*c*), we note that the internal torque T_2 in CB is equal to T_B. Recalling Eq. (3.16) and observing that portions AC and CB of the shaft are twisted in opposite senses, we write

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0$$

Solving for T_B, we have

$$T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

Substituting the numerical data gives

$$L_1 = L_2 = 5 \text{ in.}$$
$$J_1 = \tfrac{1}{2}\pi(\tfrac{7}{16} \text{ in.})^4 = 57.6 \times 10^{-3} \text{ in}^4$$
$$J_2 = \tfrac{1}{2}\pi[(\tfrac{7}{16} \text{ in.})^4 - (\tfrac{5}{16} \text{ in.})^4] = 42.6 \times 10^{-3} \text{ in}^4$$

we obtain

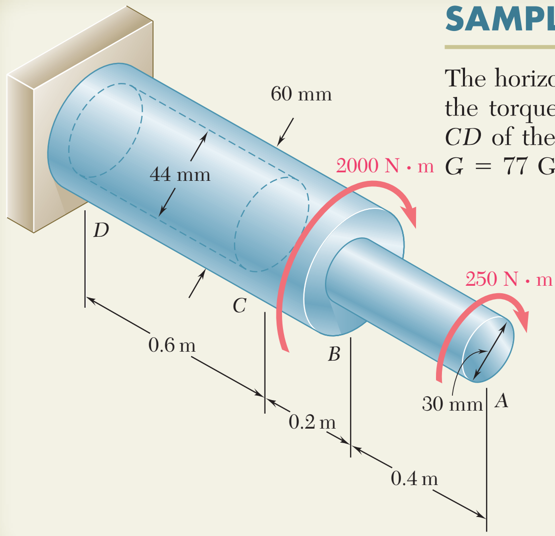
$$T_B = 0.740\ T_A$$

Substituting this expression into the original equilibrium equation, we write

$$1.740\ T_A = 90 \text{ lb} \cdot \text{ft}$$

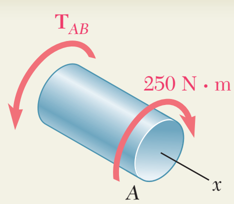
$$T_A = 51.7 \text{ lb} \cdot \text{ft} \qquad T_B = 38.3 \text{ lb} \cdot \text{ft}$$

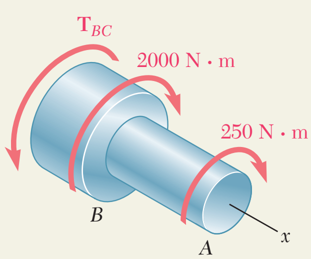
SAMPLE PROBLEM 3.3

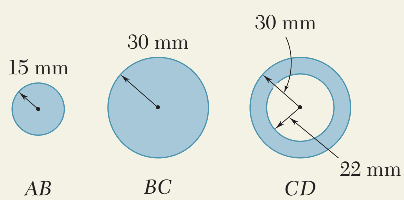


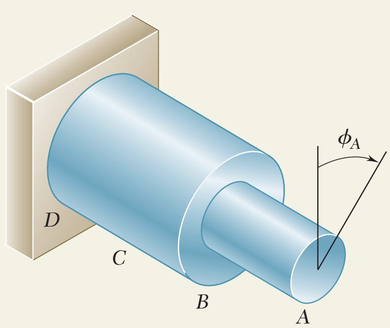
The horizontal shaft AD is attached to a fixed base at D and is subjected to the torques shown. A 44-mm-diameter hole has been drilled into portion CD of the shaft. Knowing that the entire shaft is made of steel for which $G = 77$ GPa, determine the angle of twist at end A.

SOLUTION









Since the shaft consists of three portions AB, BC, and CD, each of uniform cross section and each with a constant internal torque, Eq. (3.17) may be used.

Statics. Passing a section through the shaft between A and B and using the free body shown, we find

$$\Sigma M_x = 0: \qquad (250 \text{ N}\cdot\text{m}) - T_{AB} = 0 \qquad T_{AB} = 250 \text{ N}\cdot\text{m}$$

Passing now a section between B and C, we have

$$\Sigma M_x = 0: (250 \text{ N}\cdot\text{m}) + (2000 \text{ N}\cdot\text{m}) - T_{BC} = 0 \qquad T_{BC} = 2250 \text{ N}\cdot\text{m}$$

Since no torque is applied at C,

$$T_{CD} = T_{BC} = 2250 \text{ N}\cdot\text{m}$$

Polar Moments of Inertia

$$J_{AB} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.015 \text{ m})^4 = 0.0795 \times 10^{-6} \text{ m}^4$$

$$J_{BC} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.030 \text{ m})^4 = 1.272 \times 10^{-6} \text{ m}^4$$

$$J_{CD} = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.030 \text{ m})^4 - (0.022 \text{ m})^4] = 0.904 \times 10^{-6} \text{ m}^4$$

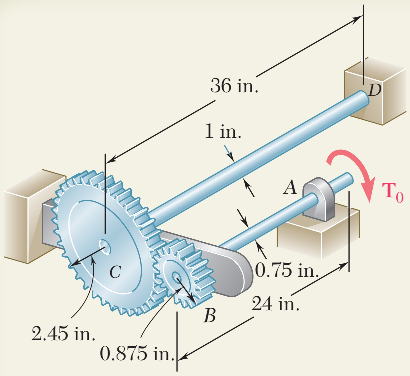
Angle of Twist. Using Eq. (3.17) and recalling that $G = 77$ GPa for the entire shaft, we have

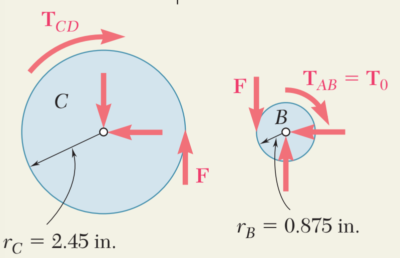
$$\phi_A = \sum_i \frac{T_i L_i}{J_i G} = \frac{1}{G}\left(\frac{T_{AB}L_{AB}}{J_{AB}} + \frac{T_{BC}L_{BC}}{J_{BC}} + \frac{T_{CD}L_{CD}}{J_{CD}}\right)$$

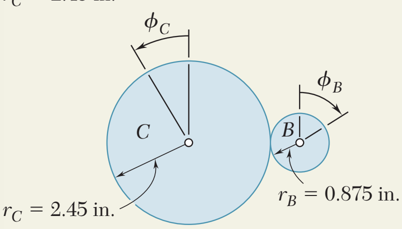
$$\phi_A = \frac{1}{77 \text{ GPa}}\left[\frac{(250 \text{ N}\cdot\text{m})(0.4 \text{ m})}{0.0795 \times 10^{-6} \text{ m}^4} + \frac{(2250)(0.2)}{1.272 \times 10^{-6}} + \frac{(2250)(0.6)}{0.904 \times 10^{-6}}\right]$$

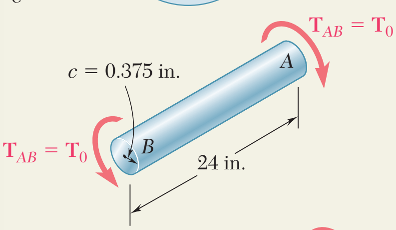
$$= 0.01634 + 0.00459 + 0.01939 = 0.0403 \text{ rad}$$

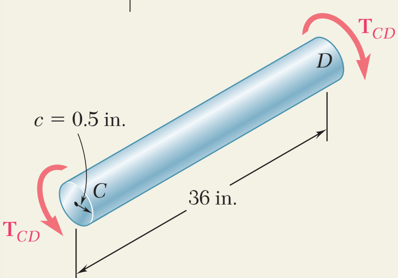
$$\phi_A = (0.0403 \text{ rad})\frac{360°}{2\pi \text{ rad}} \qquad \phi_A = 2.31° \blacktriangleleft$$

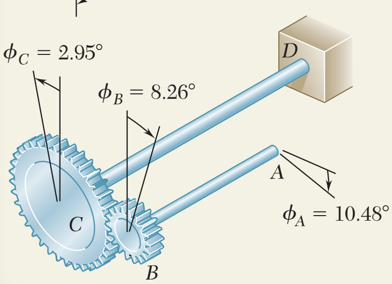












SAMPLE PROBLEM 3.4

Two solid steel shafts are connected by the gears shown. Knowing that for each shaft $G = 11.2 \times 10^6$ psi and that the allowable shearing stress is 8 ksi, determine (*a*) the largest torque $\mathbf{T}_0$ that may be applied to end *A* of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

SOLUTION

Statics. Denoting by F the magnitude of the tangential force between gear teeth, we have

$$\begin{aligned} &\textit{Gear B.} \quad \Sigma M_B = 0: \quad F(0.875 \text{ in.}) - T_0 = 0 \\ &\textit{Gear C.} \quad \Sigma M_C = 0: \quad F(2.45 \text{ in.}) - T_{CD} = 0 \end{aligned} \qquad T_{CD} = 2.8T_0 \qquad (1)$$

Kinematics. Noting that the peripheral motions of the gears are equal, we write

$$r_B\phi_B = r_C\phi_C \qquad \phi_B = \phi_C\frac{r_C}{r_B} = \phi_C\frac{2.45 \text{ in.}}{0.875 \text{ in.}} = 2.8\phi_C \qquad (2)$$

a. Torque T_0

Shaft AB. With $T_{AB} = T_0$ and $c = 0.375$ in., together with a maximum permissible shearing stress of 8000 psi, we write

$$\tau = \frac{T_{AB}c}{J} \qquad 8000 \text{ psi} = \frac{T_0(0.375 \text{ in.})}{\frac{1}{2}\pi(0.375 \text{ in.})^4} \qquad T_0 = 663 \text{ lb} \cdot \text{in.} \quad ◀$$

Shaft CD. From (1) we have $T_{CD} = 2.8T_0$. With $c = 0.5$ in. and $\tau_{\text{all}} = 8000$ psi, we write

$$\tau = \frac{T_{CD}c}{J} \qquad 8000 \text{ psi} = \frac{2.8T_0(0.5 \text{ in.})}{\frac{1}{2}\pi(0.5 \text{ in.})^4} \qquad T_0 = 561 \text{ lb} \cdot \text{in.} \quad ◀$$

Maximum Permissible Torque. We choose the smaller value obtained for T_0

$$T_0 = 561 \text{ lb} \cdot \text{in.} \quad ◀$$

b. Angle of Rotation at End *A*.

We first compute the angle of twist for each shaft.

Shaft AB. For $T_{AB} = T_0 = 561$ lb · in., we have

$$\phi_{A/B} = \frac{T_{AB}L}{JG} = \frac{(561 \text{ lb} \cdot \text{in.})(24 \text{ in.})}{\frac{1}{2}\pi(0.375 \text{ in.})^4(11.2 \times 10^6 \text{ psi})} = 0.0387 \text{ rad} = 2.22°$$

Shaft CD. $T_{CD} = 2.8T_0 = 2.8(561 \text{ lb} \cdot \text{in.})$

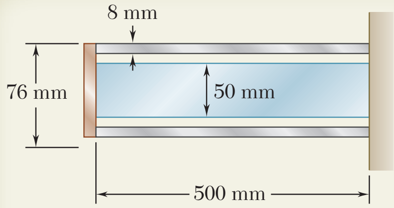
$$\phi_{C/D} = \frac{T_{CD}L}{JG} = \frac{2.8(561 \text{ lb} \cdot \text{in.})(36 \text{ in.})}{\frac{1}{2}\pi(0.5 \text{ in.})^4(11.2 \times 10^6 \text{ psi})} = 0.0514 \text{ rad} = 2.95°$$

Since end *D* of shaft *CD* is fixed, we have $\phi_C = \phi_{C/D} = 2.95°$. Using (2), we find the angle of rotation of gear *B* to be

$$\phi_B = 2.8\phi_C = 2.8(2.95°) = 8.26°$$

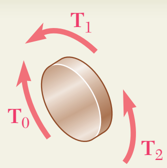
For end *A* of shaft *AB*, we have

$$\phi_A = \phi_B + \phi_{A/B} = 8.26° + 2.22° \qquad \phi_A = 10.48° \quad ◀$$



SAMPLE PROBLEM 3.5

A steel shaft and an aluminum tube are connected to a fixed support and to a rigid disk as shown in the cross section. Knowing that the initial stresses are zero, determine the maximum torque $\mathbf{T}_0$ that can be applied to the disk if the allowable stresses are 120 MPa in the steel shaft and 70 MPa in the aluminum tube. Use $G = 77$ GPa for steel and $G = 27$ GPa for aluminum.



SOLUTION

Statics. ***Free Body of Disk.*** Denoting by $\mathbf{T}_1$ the torque exerted by the tube on the disk and by $\mathbf{T}_2$ the torque exerted by the shaft, we find

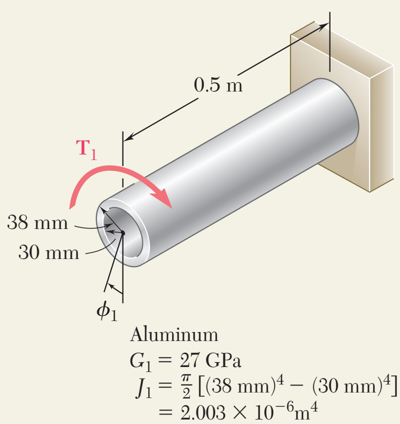
$$T_0 = T_1 + T_2 \tag{1}$$

Deformations. Since both the tube and the shaft are connected to the rigid disk, we have

$$\phi_1 = \phi_2: \qquad \frac{T_1 L_1}{J_1 G_1} = \frac{T_2 L_2}{J_2 G_2}$$

$$\frac{T_1(0.5\text{ m})}{(2.003 \times 10^{-6}\text{ m}^4)(27\text{ GPa})} = \frac{T_2(0.5\text{ m})}{(0.614 \times 10^{-6}\text{ m}^4)(77\text{ GPa})}$$

$$T_2 = 0.874 T_1 \tag{2}$$



Shearing Stresses. We *assume* that the requirement $\tau_{\text{alum}} \le 70$ MPa is critical. For the aluminum tube, we have

$$T_1 = \frac{\tau_{\text{alum}} J_1}{c_1} = \frac{(70\text{ MPa})(2.003 \times 10^{-6}\text{ m}^4)}{0.038\text{ m}} = 3690\text{ N}\cdot\text{m}$$

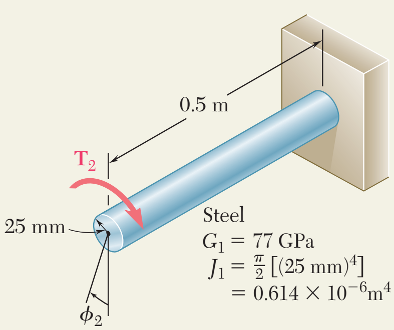
Using Eq. (2), we compute the corresponding value T_2 and then find the maximum shearing stress in the steel shaft.

$$T_2 = 0.874 T_1 = 0.874(3690) = 3225\text{ N}\cdot\text{m}$$

$$\tau_{\text{steel}} = \frac{T_2 c_2}{J_2} = \frac{(3225\text{ N}\cdot\text{m})(0.025\text{ m})}{0.614 \times 10^{-6}\text{ m}^4} = 131.3\text{ MPa}$$

We note that the allowable steel stress of 120 MPa is exceeded; our assumption was *wrong*. Thus the maximum torque $\mathbf{T}_0$ will be obtained by making $\tau_{\text{steel}} = 120$ MPa. We first determine the torque $\mathbf{T}_2$.

$$T_2 = \frac{\tau_{\text{steel}} J_2}{c_2} = \frac{(120\text{ MPa})(0.614 \times 10^{-6}\text{ m}^4)}{0.025\text{ m}} = 2950\text{ N}\cdot\text{m}$$



From Eq. (2), we have

$$2950\text{ N}\cdot\text{m} = 0.874 T_1 \qquad T_1 = 3375\text{ N}\cdot\text{m}$$

Using Eq. (1), we obtain the maximum permissible torque

$$T_0 = T_1 + T_2 = 3375\text{ N}\cdot\text{m} + 2950\text{ N}\cdot\text{m}$$

$$T_0 = 6.325\text{ kN}\cdot\text{m} \blacktriangleleft$$

PROBLEMS

3.31 (*a*) For the solid steel shaft shown (G = 77 GPa), determine the angle of twist at A. (*b*) Solve part *a*, assuming that the steel shaft is hollow with a 30-mm-outer diameter and a 20-mm-inner diameter.

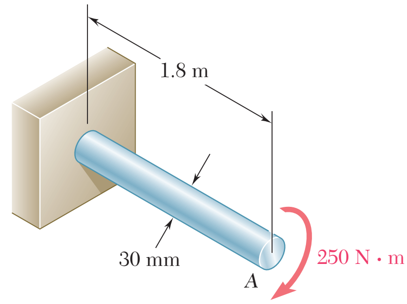


Fig. P3.31

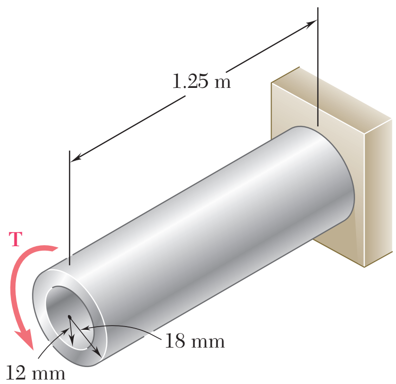


Fig. *P3.32*

3.32 For the aluminum shaft shown (G = 27 GPa), determine (*a*) the torque **T** that causes an angle of twist of 4°, (*b*) the angle of twist caused by the same torque **T** in a solid cylindrical shaft of the same length and cross-sectional area.

3.33 Determine the largest allowable diameter of a 10-ft-long steel rod ($G = 11.2 \times 10^6$ psi) if the rod is to be twisted through 30° without exceeding a shearing stress of 12 ksi.

3.34 While an oil well is being drilled at a depth of 6000 ft, it is observed that the top of the 8-in.-diameter steel drill pipe rotates though two complete revolutions before the drilling bit starts to rotate. Using $G = 11.2 \times 10^6$ psi, determine the maximum shearing stress in the pipe caused by torsion.

3.35 The electric motor exerts a 500 N · m-torque on the aluminum shaft $ABCD$ when it is rotating at a constant speed. Knowing that G = 27 GPa and that the torques exerted on pulleys B and C are as shown, determine the angle of twist between (*a*) B and C, (*b*) B and D.

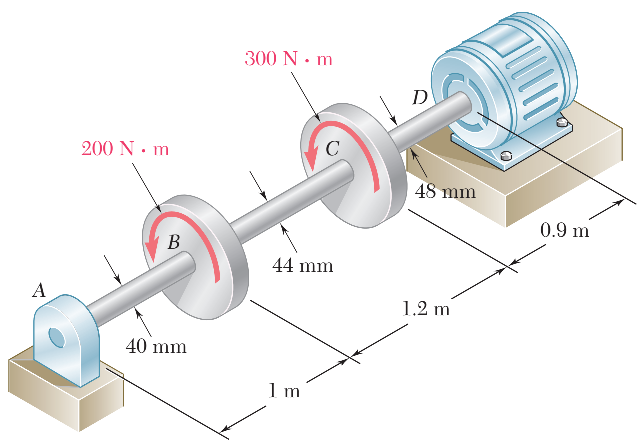


Fig. P3.35

3.36 The torques shown are exerted on pulleys B, C, and D. Knowing that the entire shaft is made of aluminum (G = 27 GPa), determine the angle of twist between (a) C and B, (b) D and B.

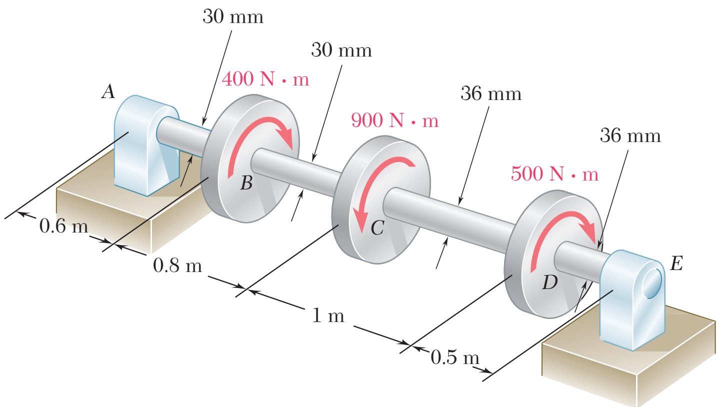


Fig. ***P3.36***

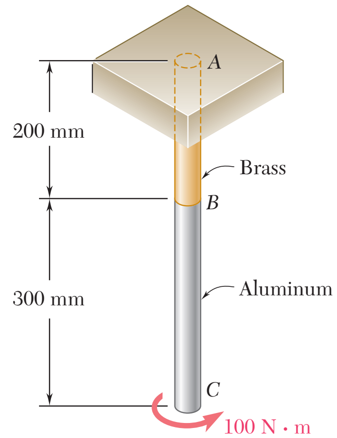


Fig. P3.37

3.37 The aluminum rod BC (G = 26 GPa) is bonded to the brass rod AB (G = 39 GPa). Knowing that each rod is solid and has a diameter of 12 mm, determine the angle of twist (a) at B, (b) at C.

3.38 The aluminum rod AB (G = 27 GPa) is bonded to the brass rod BD (G = 39 GPa). Knowing that portion CD of the brass rod is hollow and has an inner diameter of 40 mm, determine the angle of twist at A.

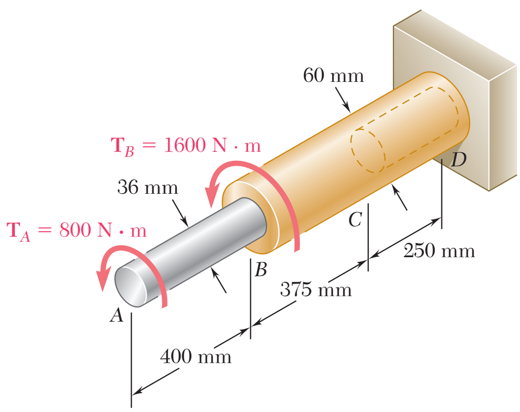


Fig. P3.38

3.39 The solid spindle AB has a diameter d_s = 1.5 in. and is made of a steel with $G = 11.2 \times 10^6$ psi and τ_{all} = 12 ksi, while sleeve CD is made of a brass with $G = 5.6 \times 10^6$ psi and τ_{all} = 7 ksi. Determine the largest angle through which end A can be rotated.

3.40 The solid spindle AB has a diameter d_s = 1.75 in. and is made of a steel with $G = 11.2 \times 10^6$ psi and τ_{all} = 12 ksi, while sleeve CD is made of a brass with $G = 5.6 \times 10^6$ psi and τ_{all} = 7 ksi. Determine (a) the largest torque **T** that can be applied at A if the given allowable stresses are not to be exceeded and if the angle of twist of sleeve CD is not to exceed 0.375°, (b) the corresponding angle through which end A rotates.

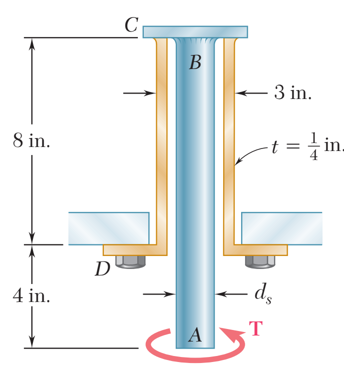


Fig. P3.39 and *P3.40*

3.41 Two shafts, each of $\frac{7}{8}$-in. diameter, are connected by the gears shown. Knowing that $G = 11.2 \times 10^6$ psi and that the shaft at F is fixed, determine the angle through which end A rotates when a 1.2 kip · in. torque is applied at A.

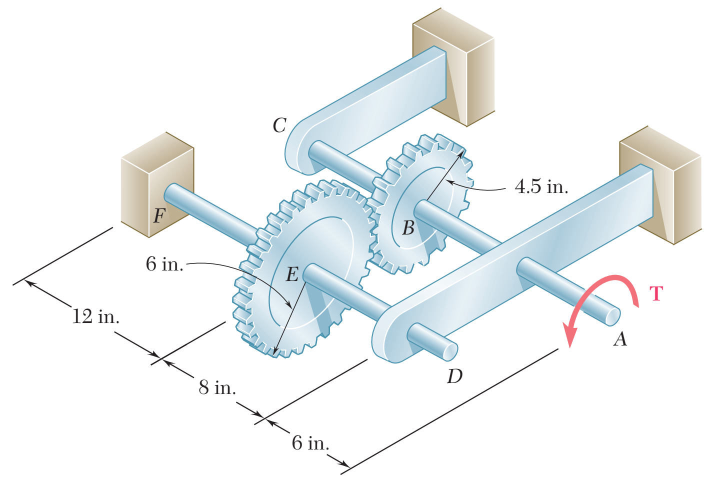


Fig. P3.41

3.42 Two solid shafts are connected by gears as shown. Knowing that $G = 77.2$ GPa for each shaft, determine the angle through which end A rotates when $T_A = 1200$ N · m.

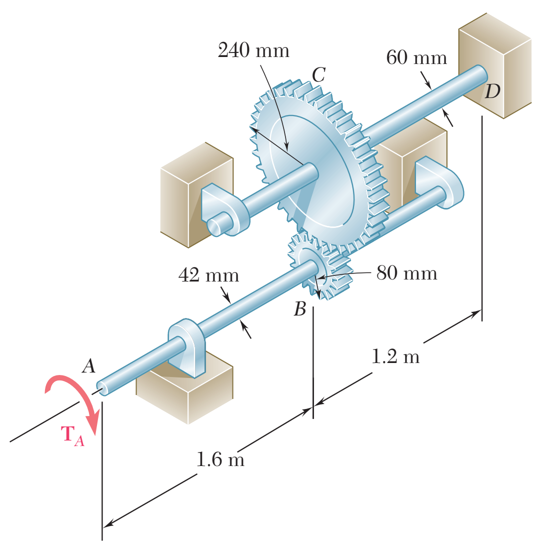


Fig. P3.42

3.43 A coder F, used to record in digital form the rotation of shaft A, is connected to the shaft by means of the gear train shown, which consists of four gears and three solid steel shafts each of diameter d. Two of the gears have a radius r and the other two a radius nr. If the rotation of the coder F is prevented, determine in terms of T, l, G, J, and n the angle through which end A rotates.

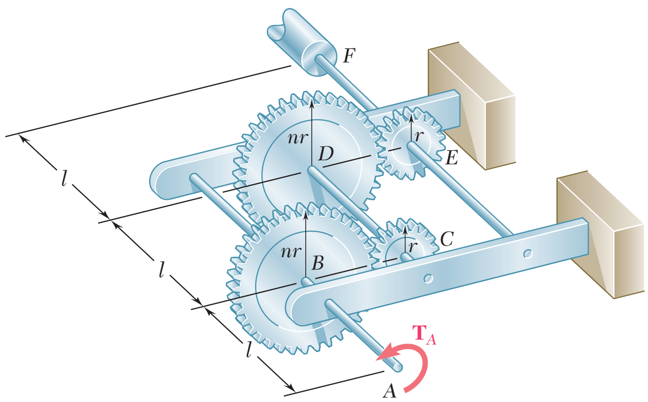


Fig. P3.43

3.44 For the gear train described in Prob. 3.43, determine the angle through which end A rotates when $T = 5$ lb · in., $l = 2.4$ in., $d = \frac{1}{16}$ in., $G = 11.2 \times 10^6$ psi, and $n = 2$.

3.45 The design of the gear-and-shaft system shown requires that steel shafts of the same diameter be used for both AB and CD. It is further required that $\tau_{max} \leq 60$ MPa and that the angle ϕ_D through which end D of shaft CD rotates not exceed 1.5°. Knowing that $G = 77$ GPa, determine the required diameter of the shafts.

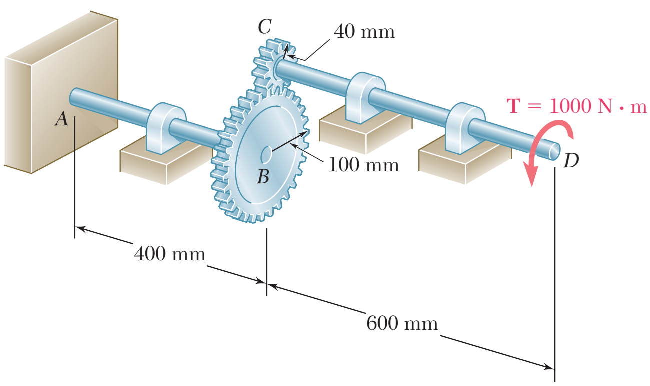


Fig. P3.45

3.46 The electric motor exerts a torque of 800 N · m on the steel shaft *ABCD* when it is rotating at a constant speed. Design specifications require that the diameter of the shaft be uniform from *A* to *D* and that the angle of twist between *A* and *D* not exceed 1.5°. Knowing that $\tau_{max} \leq 60$ MPa and $G = 77$ GPa, determine the minimum diameter shaft that can be used.

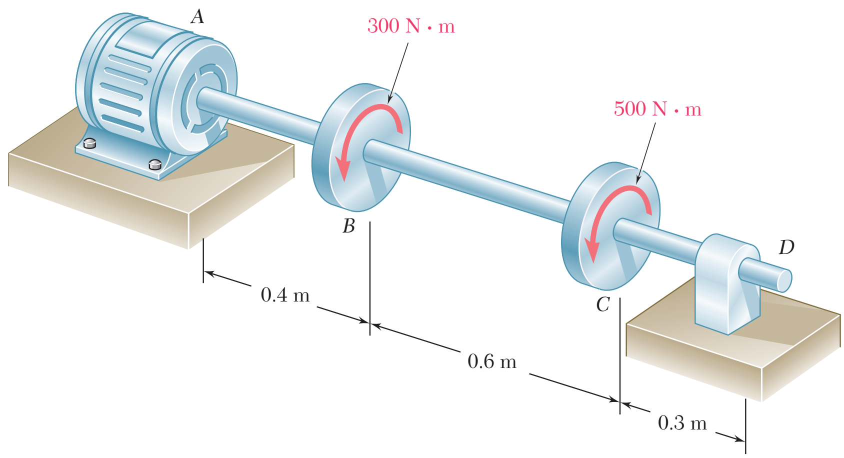


Fig. P3.46

3.47 The design specifications of a 2-m-long solid circular transmission shaft require that the angle of twist of the shaft not exceed 3° when a torque of 9 kN · m is applied. Determine the required diameter of the shaft, knowing that the shaft is made of (*a*) a steel with an allowable shearing stress of 90 MPa and a modulus of rigidity of 77 GPa, (*b*) a bronze with an allowable shearing stress of 35 MPa and a modulus of rigidity of 42 GPa.

3.48 A hole is punched at *A* in a plastic sheet by applying a 600-N force **P** to end *D* of lever *CD*, which is rigidly attached to the solid cylindrical shaft *BC*. Design specifications require that the displacement of *D* should not exceed 15 mm from the time the punch first touches the plastic sheet to the time it actually penetrates it. Determine the required diameter of shaft *BC* if the shaft is made of a steel with $G = 77$ GPa and $\tau_{all} = 80$ MPa.

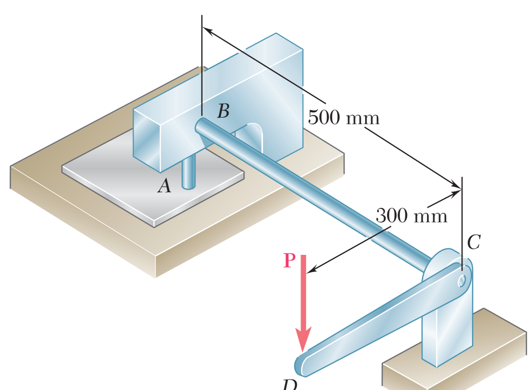


Fig. P3.48

3.49 The design specifications for the gear-and-shaft system shown require that the same diameter be used for both shafts and that the angle through which pulley A will rotate when subjected to a 2-kip · in. torque $\mathbf{T}_A$ while pulley D is held fixed will not exceed 7.5°. Determine the required diameter of the shafts if both shafts are made of a steel with $G = 11.2 \times 10^6$ psi and $\tau_{all} = 12$ ksi.

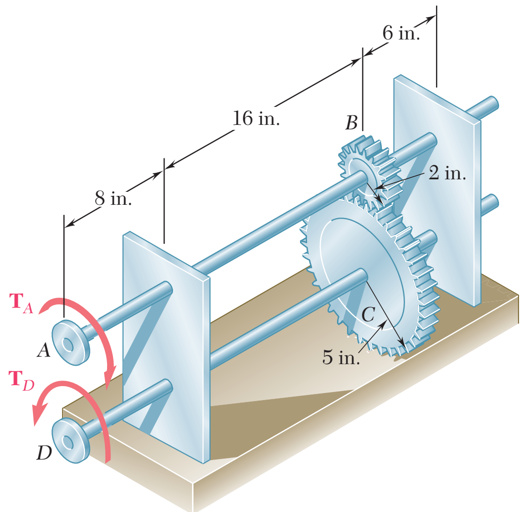


Fig. P3.49

3.50 Solve Prob. 3.49, assuming that both shafts are made of a brass with $G = 5.6 \times 10^6$ psi and $\tau_{all} = 8$ ksi.

3.51 A torque of magnitude $T = 4$ kN · m is applied at end A of the composite shaft shown. Knowing that the modulus of rigidity is 77 GPa for the steel and 27 GPa for the aluminum, determine (*a*) the maximum shearing stress in the steel core, (*b*) the maximum shearing stress in the aluminum jacket, (*c*) the angle of twist at A.

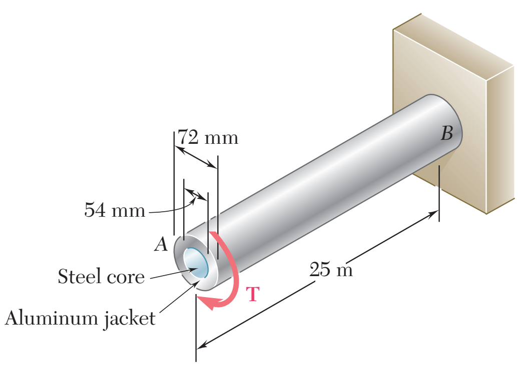


Fig. P3.51 and P3.52

3.52 The composite shaft shown is to be twisted by applying a torque **T** at end A. Knowing that the modulus of rigidity is 77 GPa for the steel and 27 GPa for the aluminum, determine the largest angle through which end A can be rotated if the following allowable stresses are not to be exceeded: $\tau_{steel} = 60$ MPa and $\tau_{aluminum} =$ 45 MPa.

3.53 The solid cylinders AB and BC are bonded together at B and are attached to fixed supports at A and C. Knowing that the modulus of rigidity is 3.7×10^6 psi for aluminum and 5.6×10^6 psi for brass, determine the maximum shearing stress (*a*) in cylinder AB, (*b*) in cylinder BC.

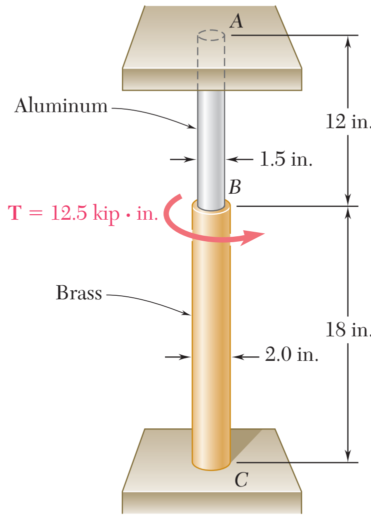


Fig. P3.53

3.54 Solve Prob. 3.53, assuming that cylinder AB is made of steel, for which $G = 11.2 \times 10^6$ psi.

3.55 and 3.56 Two solid steel shafts are fitted with flanges that are then connected by bolts as shown. The bolts are slightly undersized and permit a 1.5° rotation of one flange with respect to the other before the flanges begin to rotate as a single unit. Knowing that $G = 11.2 \times 10^6$ psi, determine the maximum shearing stress in each shaft when a torque of **T** of magnitude 420 kip · ft is applied to the flange indicated.

3.55 The torque **T** is applied to flange B.
3.56 The torque **T** is applied to flange C.

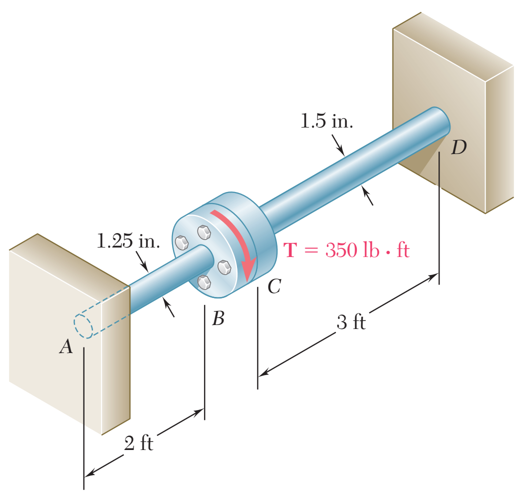


Fig. P3.55 and P3.56

3.57 Ends A and D of the two solid steel shafts AB and CD are fixed, while ends B and C are connected to gears as shown. Knowing that a 4-kN · m torque **T** is applied to gear B, determine the maximum shearing stress (*a*) in shaft AB, (*b*) in shaft CD.

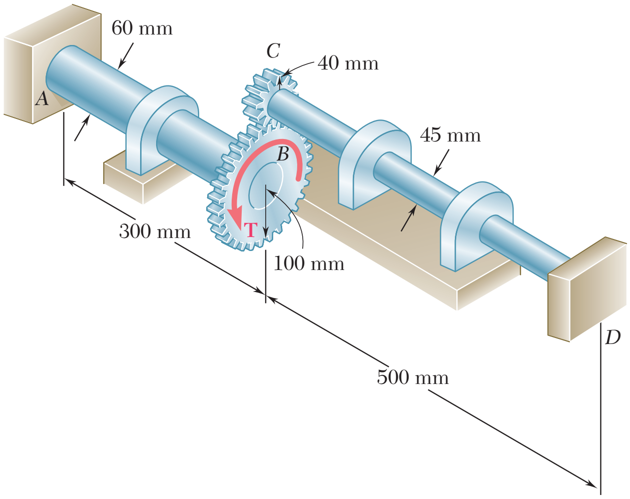


Fig. *P3.57* and *P3.58*

3.58 Ends A and D of the two solid steel shafts AB and CD are fixed, while ends B and C are connected to gears as shown. Knowing that the allowable shearing stress is 50 MPa in each shaft, determine the largest torque **T** that can be applied to gear B.

3.59 The steel jacket CD has been attached to the 40-mm-diameter steel shaft AE by means of *rigid* flanges welded to the jacket and to the rod. The outer diameter of the jacket is 80 mm and its wall thickness is 4 mm. If 500 N · m-torques are applied as shown, determine the maximum shearing stress in the jacket.

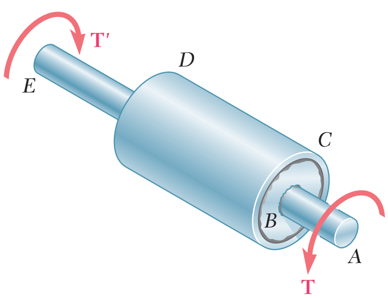


Fig. P3.59

3.60 A solid shaft and a hollow shaft are made of the same material and are of the same weight and length. Denoting by n the ratio c_1/c_2, show that the ratio T_s/T_h of the torque T_s in the solid shaft to the torque T_h in the hollow shaft is (*a*) $\sqrt{(1 - n^2)/(1 + n^2)}$ if the maximum shearing stress is the same in each shaft, (*b*) $(1 - n^2)/(1 + n^2)$ if the angle of twist is the same for each shaft.

3.61 A torque **T** is applied as shown to a solid tapered shaft AB. Show by integration that the angle of twist at A is

$$\phi = \frac{7TL}{12\pi G c^4}$$

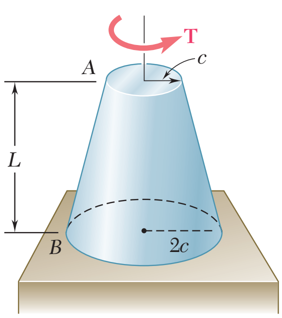


Fig. P3.61

3.62 The mass moment of inertia of a gear is to be determined experimentally by using a torsional pendulum consisting of a 6-ft steel wire. Knowing that $G = 11.2 \times 10^6$ psi, determine the diameter of the wire for which the torsional spring constant will be 4.27 lb · ft/rad.

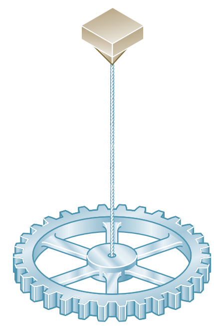


Fig. P3.62

3.63 An annular plate of thickness t and modulus G is used to connect shaft AB of radius r_1 to tube CD of radius r_2. Knowing that a torque **T** is applied to end A of shaft AB and that end D of tube CD is fixed, (a) determine the magnitude and location of the maximum shearing stress in the annular plate, (b) show that the angle through which end B of the shaft rotates with respect to end C of the tube is

$$\phi_{BC} = \frac{T}{4\pi G t}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

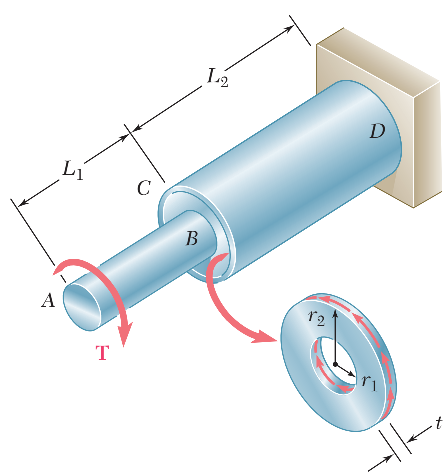


Fig. P3.63

3.7 DESIGN OF TRANSMISSION SHAFTS

The principal specifications to be met in the design of a transmission shaft are the *power* to be transmitted and the *speed of rotation* of the shaft. The role of the designer is to select the material and the dimensions of the cross section of the shaft, so that the maximum shearing stress allowable in the material will not be exceeded when the shaft is transmitting the required power at the specified speed.

To determine the torque exerted on the shaft, we recall from elementary dynamics that the power P associated with the rotation of a rigid body subjected to a torque **T** is

$$P = T\omega \tag{3.19}$$

where ω is the angular velocity of the body expressed in radians per second. But $\omega = 2\pi f$, where f is the frequency of the rotation, i.e., the number of revolutions per second. The unit of frequency is thus 1 s^{-1} and is called a *hertz* (Hz). Substituting for ω into Eq. (3.19), we write

$$\boxed{P = 2\pi f T} \tag{3.20}$$

If SI units are used we verify that, with f expressed in Hz and T in N · m, the power will be expressed in N · m/s, that is, in *watts* (W). Solving Eq. (3.20) for T, we obtain the torque exerted on a shaft transmitting the power P at a frequency of rotation f,

$$\boxed{T = \frac{P}{2\pi f}} \tag{3.21}$$

where P, f, and T are expressed in the units indicated above.

After having determined the torque **T** that will be applied to the shaft and having selected the material to be used, the designer will carry the values of T and of the maximum allowable stress into the elastic torsion formula (3.9). Solving for J/c, we have

$$\frac{J}{c} = \frac{T}{\tau_{\max}} \tag{3.22}$$

and obtain in this way the minimum value allowable for the parameter J/c. We check that, if SI units are used, T will be expressed in N · m, $\tau_{\max}$ in Pa (or N/m^2), and J/c will be obtained in m^3. In the case of a solid circular shaft, $J = \frac{1}{2}\pi c^4$, and $J/c = \frac{1}{2}\pi c^3$; substituting this value for J/c into Eq. (3.22) and solving for c yields the minimum allowable value for the radius of the shaft. In the case of a hollow circular shaft, the critical parameter is J/c_2, where c_2 is the outer radius of the shaft; the value of this parameter may be computed from Eq. (3.11) of Sec. 3.4 to determine whether a given cross section will be acceptable.

When U.S. customary units are used, the frequency is usually expressed in rpm and the power in horsepower (hp). It is then necessary, before applying formula (3.21), to convert the frequency into revolutions per second (i.e., hertzes) and the power into ft · lb/s or in · lb/s through the use of the following relations:

$$1 \text{ rpm} = \frac{1}{60}\,\text{s}^{-1} = \frac{1}{60}\,\text{Hz}$$

$$1 \text{ hp} = 550 \text{ ft} \cdot \text{lb/s} = 6600 \text{ in} \cdot \text{lb/s}$$

If we express the power in in · lb/s, formula (3.21) will yield the value of the torque T in lb · in. Carrying this value of T into Eq. (3.22), and expressing $\tau_{\max}$ in psi, we obtain the value of the parameter J/c in in^3.

EXAMPLE 3.06

What size of shaft should be used for the rotor of a 5-hp motor operating at 3600 rpm if the shearing stress is not to exceed 8500 psi in the shaft?

We first express the power of the motor in in · lb/s and its frequency in cycles per second (or hertzes).

$$P = (5 \text{ hp})\left(\frac{6600 \text{ in} \cdot \text{lb/s}}{1 \text{ hp}}\right) = 33{,}000 \text{ in} \cdot \text{lb/s}$$

$$f = (3600 \text{ rpm})\frac{1 \text{ Hz}}{60 \text{ rpm}} = 60 \text{ Hz} = 60 \text{ s}^{-1}$$

The torque exerted on the shaft is given by Eq. (3.21):

$$T = \frac{P}{2\pi f} = \frac{33{,}000 \text{ in} \cdot \text{lb/s}}{2\pi\,(60 \text{ s}^{-1})} = 87.54 \text{ lb} \cdot \text{in.}$$

Substituting for T and $\tau_{\max}$ into Eq. (3.22), we write

$$\frac{J}{c} = \frac{T}{\tau_{\max}} = \frac{87.54 \text{ lb} \cdot \text{in.}}{8500 \text{ psi}} = 10.30 \times 10^{-3} \text{ in}^3$$

But $J/c = \frac{1}{2}\pi c^3$ for a solid shaft. We have, therefore,

$$\begin{aligned} \tfrac{1}{2}\pi c^3 &= 10.30 \times 10^{-3} \text{ in}^3 \\ c &= 0.1872 \text{ in.} \\ d &= 2c = 0.374 \text{ in.} \end{aligned}$$

A $\frac{3}{8}$-in. shaft should be used.

EXAMPLE 3.07

A shaft consisting of a steel tube of 50-mm outer diameter is to transmit 100 kW of power while rotating at a frequency of 20 Hz. Determine the tube thickness that should be used if the shearing stress is not to exceed 60 MPa.

The torque exerted on the shaft is given by Eq. (3.21):

$$T = \frac{P}{2\pi f} = \frac{100 \times 10^3 \text{ W}}{2\pi\,(20 \text{ Hz})} = 795.8 \text{ N} \cdot \text{m}$$

From Eq. (3.22) we conclude that the parameter J/c_2 must be at least equal to

$$\frac{J}{c_2} = \frac{T}{\tau_{\max}} = \frac{795.8 \text{ N} \cdot \text{m}}{60 \times 10^6 \text{ N/m}^2} = 13.26 \times 10^{-6} \text{ m}^3 \tag{3.23}$$

But, from Eq. (3.10) we have

$$\frac{J}{c_2} = \frac{\pi}{2c_2}(c_2^4 - c_1^4) = \frac{\pi}{0.050}[(0.025)^4 - c_1^4] \tag{3.24}$$

Equating the right-hand members of Eqs. (3.23) and (3.24), we obtain:

$$(0.025)^4 - c_1^4 = \frac{0.050}{\pi}(13.26 \times 10^{-6})$$

$$\begin{aligned} c_1^4 &= 390.6 \times 10^{-9} - 211.0 \times 10^{-9} = 179.6 \times 10^{-9} \text{ m}^4 \\ c_1 &= 20.6 \times 10^{-3} \text{ m} = 20.6 \text{ mm} \end{aligned}$$

The corresponding tube thickness is

$$c_2 - c_1 = 25 \text{ mm} - 20.6 \text{ mm} = 4.4 \text{ mm}$$

A tube thickness of 5 mm should be used.

3.8 STRESS CONCENTRATIONS IN CIRCULAR SHAFTS

The torsion formula $\tau_{\max} = Tc/J$ was derived in Sec. 3.4 for a circular shaft of uniform cross section. Moreover, we had assumed earlier in Sec. 3.3 that the shaft was loaded at its ends through rigid end plates solidly attached to it. In practice, however, the torques are usually applied to the shaft through flange couplings (Fig. 3.27*a*) or through gears connected to the shaft by keys fitted into keyways (Fig. 3.27*b*). In both cases one should expect the distribution of stresses, in and near the section where the torques are applied, to be different from that given by the torsion formula. High concentrations of stresses, for example, will occur in the neighborhood of the keyway shown in Fig. 3.27*b*. The determination of these localized stresses may be carried out by experimental stress analysis methods or, in some cases, through the use of the mathematical theory of elasticity.

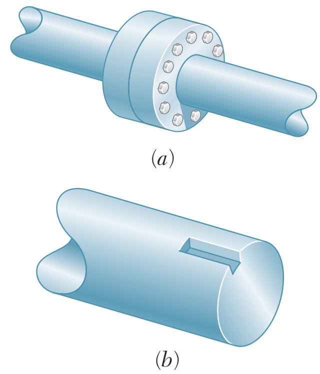


Fig. 3.27 Shaft examples.

As we indicated in Sec. 3.4, the torsion formula can also be used for a shaft of variable circular cross section. In the case of a shaft with an abrupt change in the diameter of its cross section, however, stress concentrations will occur near the discontinuity, with the highest stresses occurring at A (Fig. 3.28). These stresses may

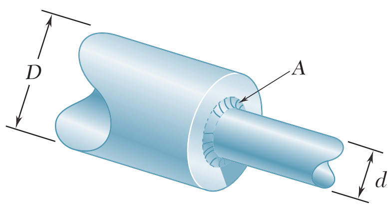


Fig. 3.28 Shaft with change in diameter.

be reduced through the use of a fillet, and the maximum value of the shearing stress at the fillet can be expressed as

$$\tau_{\max} = K\frac{Tc}{J} \tag{3.25}$$

where the stress Tc/J is the stress computed for the smaller-diameter shaft, and where K is a stress-concentration factor. Since the factor K depends only upon the ratio of the two diameters and the ratio of the radius of the fillet to the diameter of the smaller shaft, it may be computed once and for all and recorded in the form of a table or a graph, as shown in Fig. 3.29. We should note, however, that this procedure for determining localized shearing stresses is valid only as long as the value of $\tau_{\max}$ given by Eq. (3.25) does not exceed the proportional limit of the material, since the values of K plotted in Fig. 3.29 were obtained under the assumption of a linear relation between shearing stress and shearing strain. If plastic deformations occur, they will result in values of the maximum stress lower than those indicated by Eq. (3.25).

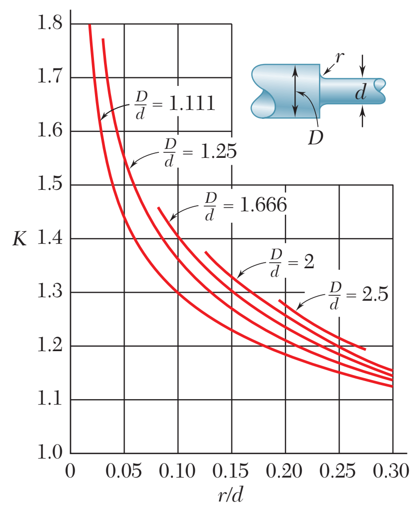
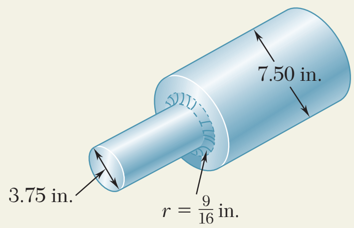


Fig. 3.29 Stress-concentration factors for fillets in circular shafts.†

†W. D. Pilkey, *Peterson's Stress Concentration Factors*, 2nd ed., John Wiley & Sons, New York, 1997.

SAMPLE PROBLEM 3.6



The stepped shaft shown is to rotate at 900 rpm as it transmits power from a turbine to a generator. The grade of steel specified in the design has an allowable shearing stress of 8 ksi. (*a*) For the preliminary design shown, determine the maximum power that can be transmitted. (*b*) If in the final design the radius of the fillet is increased so that $r = \frac{15}{16}$ in., what will be the percent change, relative to the preliminary design, in the power that can be transmitted?

SOLUTION

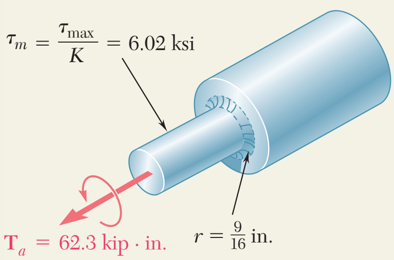
a. Preliminary Design. Using the notation of Fig. 3.32, we have: $D = 7.50$ in., $d = 3.75$ in., $r = \frac{9}{16}$ in. $= 0.5625$ in.

$$\frac{D}{d} = \frac{7.50 \text{ in.}}{3.75 \text{ in.}} = 2 \qquad \frac{r}{d} = \frac{0.5625 \text{ in.}}{3.75 \text{ in.}} = 0.15$$

A stress-concentration factor $K = 1.33$ is found from Fig. 3.29.

Torque. Recalling Eq. (3.25), we write

$$\tau_{\max} = K\frac{Tc}{J} \qquad T = \frac{J}{c}\frac{\tau_{\max}}{K} \tag{1}$$



where J/c refers to the smaller-diameter shaft:

$$J/c = \tfrac{1}{2}\pi c^3 = \tfrac{1}{2}\pi(1.875 \text{ in.})^3 = 10.35 \text{ in}^3$$

and where
$$\frac{\tau_{\max}}{K} = \frac{8 \text{ ksi}}{1.33} = 6.02 \text{ ksi}$$

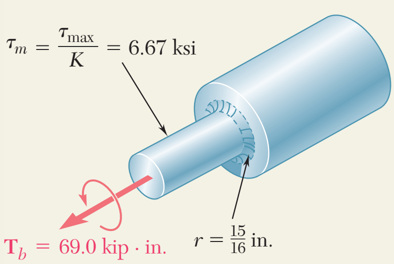
Substituting into Eq. (1), we find $T = (10.35 \text{ in}^3)(6.02 \text{ ksi}) = 62.3 \text{ kip} \cdot \text{in.}$

Power. Since $f = (900 \text{ rpm})\dfrac{1 \text{ Hz}}{60 \text{ rpm}} = 15 \text{ Hz} = 15 \text{ s}^{-1}$, we write

$$P_a = 2\pi fT = 2\pi(15 \text{ s}^{-1})(62.3 \text{ kip} \cdot \text{in.}) = 5.87 \times 10^6 \text{ in.} \cdot \text{lb/s}$$
$$P_a = (5.87 \times 10^6 \text{ in.} \cdot \text{lb/s})(1 \text{ hp}/6600 \text{ in.} \cdot \text{lb/s}) \quad P_a = 890 \text{ hp} \quad ◀$$

b. Final Design. For $r = \frac{15}{16}$ in. $= 0.9375$ in.,

$$\frac{D}{d} = 2 \qquad \frac{r}{d} = \frac{0.9375 \text{ in.}}{3.75 \text{ in.}} = 0.250 \qquad K = 1.20$$



Following the procedure used above, we write

$$\frac{\tau_{\max}}{K} = \frac{8 \text{ ksi}}{1.20} = 6.67 \text{ ksi}$$

$$T = \frac{J}{c}\frac{\tau_{\max}}{K} = (10.35 \text{ in}^3)(6.67 \text{ ksi}) = 69.0 \text{ kip} \cdot \text{in.}$$

$$P_b = 2\pi fT = 2\pi(15 \text{ s}^{-1})(69.0 \text{ kip} \cdot \text{in.}) = 6.50 \times 10^6 \text{ in.} \cdot \text{lb/s}$$
$$P_b = (6.50 \times 10^6 \text{ in.} \cdot \text{lb/s})(1 \text{ hp}/6600 \text{ in.} \cdot \text{lb/s}) = 985 \text{ hp}$$

Percent Change in Power

$$\text{Percent change} = 100\frac{P_b - P_a}{P_a} = 100\frac{985 - 890}{890} = +11\% \quad ◀$$

PROBLEMS

3.64 Determine the maximum shearing stress in a solid shaft of 12-mm diameter as it transmits 2.5 kW at a frequency of (*a*) 25 Hz, (*b*) 50 Hz.

3.65 Determine the maximum shearing stress in a solid shaft of 1.5-in. diameter as it transmits 75 hp at a speed of (*a*) 750 rpm, (*b*) 1500 rpm.

3.66 Design a solid steel shaft to transmit 0.375 kW at a frequency of 29 Hz, if the shearing stress in the shaft is not to exceed 35 MPa.

3.67 Design a solid steel shaft to transmit 100 hp at a speed of 1200 rpm, if the maximum shearing stress is not to exceed 7500 psi.

3.68 Determine the required thickness of the 50-mm tubular shaft of Example 3.07, if it is to transmit the same power while rotating at a frequency of 30 Hz.

3.69 While a steel shaft of the cross section shown rotates at 120 rpm, a stroboscopic measurement indicates that the angle of twist is 2° in a 12-ft length. Using $G = 11.2 \times 10^6$ psi, determine the power being transmitted.

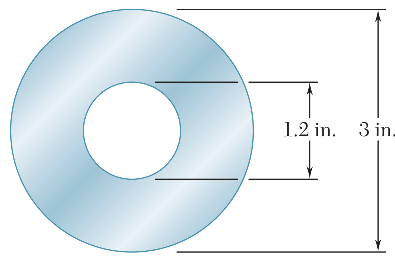


Fig. P3.69

3.70 The hollow steel shaft shown (G = 77.2 GPa, τ_{all} = 50 MPa) rotates at 240 rpm. Determine (*a*) the maximum power that can be transmitted, (*b*) the corresponding angle of twist of the shaft.

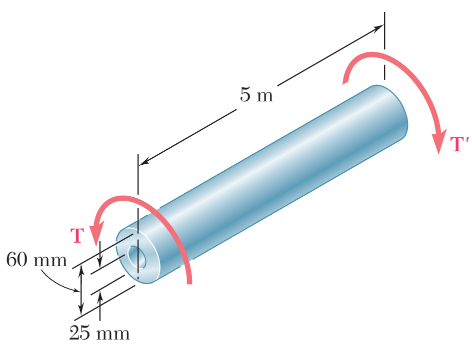


Fig. P3.70 and *P3.71*

3.71 As the hollow steel shaft shown rotates at 180 rpm, a stroboscopic measurement indicates that the angle of twist of the shaft is 3°. Knowing that G = 77.2 GPa, determine (*a*) the power being transmitted, (*b*) the maximum shearing stress in the shaft.

3.72 The design of a machine element calls for a 40-mm-outer-diameter shaft to transmit 45 kW. (*a*) If the speed of rotation is 720 rpm, determine the maximum shearing stress in shaft *a*. (*b*) If the speed of rotation can be increased 50% to 1080 rpm, determine the largest inner diameter of shaft *b* for which the maximum shearing stress will be the same in each shaft.

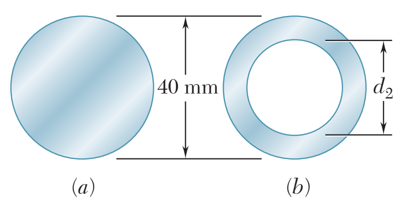


Fig. *P3.72*

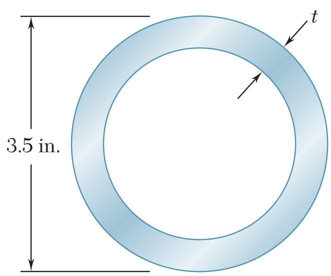


Fig. P3.73

3.73 A steel pipe of 3.5-in. outer diameter is to be used to transmit a torque of 3000 lb · ft without exceeding an allowable shearing stress of 8 ksi. A series of 3.5-in.-outer-diameter pipes is available for use. Knowing that the wall thickness of the available pipes varies from 0.25 in. to 0.50 in. in 0.0625-in. increments, choose the lightest pipe that can be used.

3.74 The two solid shafts and gears shown are used to transmit 16 hp from the motor at *A* operating at a speed of 1260 rpm to a machine tool at *D*. Knowing that the maximum allowable shearing stress is 8 ksi, determine the required diameter (*a*) of shaft *AB*, (*b*) of shaft *CD*.

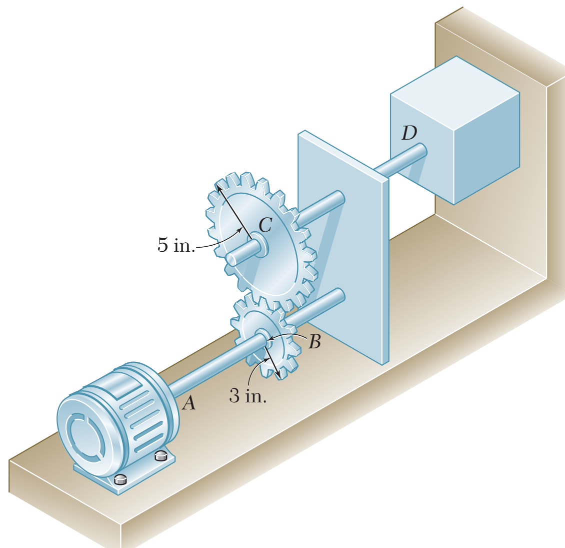


Fig. P3.74 and P3.75

3.75 The two solid shafts and gears shown are used to transmit 16 hp from the motor at *A* operating at a speed of 1260 rpm to a machine tool at *D*. Knowing that each shaft has a diameter of 1 in., determine the maximum shearing stress (*a*) in shaft *AB*, (*b*) in shaft *CD*.

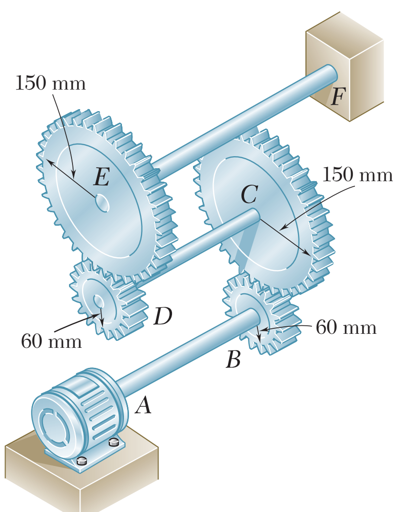


Fig. P3.76 and P3.77

3.76 Three shafts and four gears are used to form a gear train that will transmit 7.5 kW from the motor at *A* to a machine tool at *F*. (Bearings for the shafts are omitted in the sketch.) Knowing that the frequency of the motor is 30 Hz and that the allowable stress for each shaft is 60 MPa, determine the required diameter of each shaft.

3.77 Three shafts and four gears are used to form a gear train that will transmit power from the motor at *A* to a machine tool at *F*. (Bearings for the shafts are omitted in the sketch.) The diameter of each shaft is as follows: $d_{AB} = 16$ mm, $d_{CD} = 20$ mm, $d_{EF} = 28$ mm. Knowing that the frequency of the motor is 24 Hz and that the allowable shearing stress for each shaft is 75 MPa, determine the maximum power that can be transmitted.

3.78 A 1.5-m-long solid steel shaft of 48-mm diameter is to transmit 36 kW between a motor and a machine tool. Determine the lowest speed at which the shaft can rotate, knowing that $G = 77.2$ GPa, that the maximum shearing stress must not exceed 60 MPa, and the angle of twist must not exceed 2.5°.

3.79 A 2.5-m-long steel shaft of 30-mm diameter rotates at a frequency of 30 Hz. Determine the maximum power that the shaft can transmit, knowing that $G = 77.2$ GPa, that the allowable shearing stress is 50 MPa, and that the angle of twist must not exceed 7.5°.

3.80 A steel shaft must transmit 210 hp at a speed of 360 rpm. Knowing that $G = 11.2 \times 10^6$ psi, design a solid shaft so that the maximum shearing stress will not exceed 12 ksi and the angle of twist in an 8.2-ft length will not exceed 3°.

3.81 The shaft-disk-belt arrangement shown is used to transmit 3 hp from point A to point D. (a) Using an allowable shearing stress of 9500 psi, determine the required speed of shaft AB. (b) Solve part a, assuming that the diameters of shafts AB and CD are, respectively, 0.75 in. and 0.625 in.

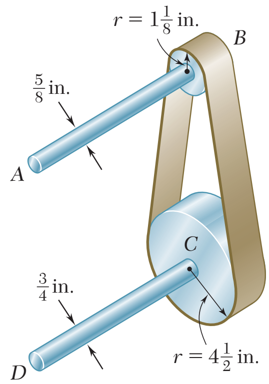


Fig. P3.81

3.82 A 1.6-m-long tubular steel shaft of 42-mm outer diameter d_1 is to be made of a steel for which $\tau_{all} = 75$ MPa and $G = 77.2$ GPa. Knowing that the angle of twist must not exceed 4° when the shaft is subjected to a torque of 900 N · m, determine the largest inner diameter d_2 that can be specified in the design.

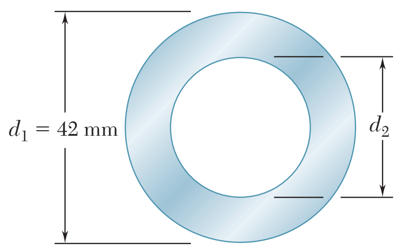


Fig. *P3.82* and P3.83

3.83 A 1.6-m-long tubular steel shaft ($G = 77.2$ GPa) of 42-mm outer diameter d_1 and 30-mm inner diameter d_2 is to transmit 120 kW between a turbine and a generator. Knowing that the allowable shearing stress is 65 MPa and that the angle of twist must not exceed 3°, determine the minimum frequency at which the shaft can rotate.

3.84 Knowing that the stepped shaft shown transmits a torque of magnitude $T = 2.50$ kip · in., determine the maximum shearing stress in the shaft when the radius of the fillet is (a) $r = \frac{1}{8}$ in., (b) $r = \frac{3}{16}$ in.

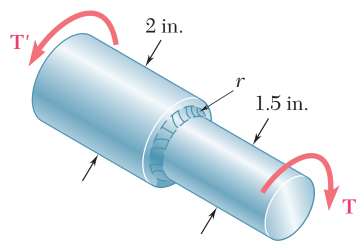


Fig. P3.84 and *P3.85*

3.85 Knowing that the allowable shearing stress is 8 ksi for the stepped shaft shown, determine the magnitude T of the largest torque that can be transmitted by the shaft when the radius of the fillet is (a) $r = \frac{3}{16}$ in., (b) $r = \frac{1}{4}$ in.

3.86 The stepped shaft shown must transmit 40 kW at a speed of 720 rpm. Determine the minimum radius r of the fillet if an allowable stress of 36 MPa is not to be exceeded.

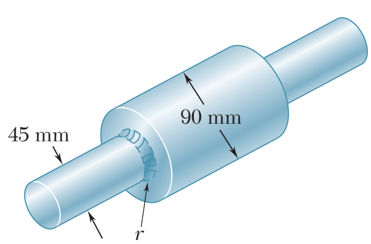


Fig. P3.86

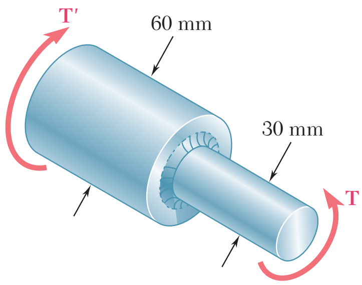
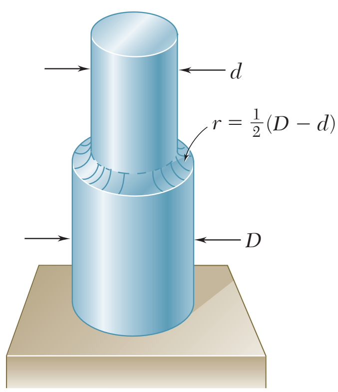


Fig. P3.87 and P3.88



Full quarter-circular fillet extends to edge of larger shaft.
Fig. *P3.89,* P3.90, and P3.91

3.87 The stepped shaft shown must transmit 45 kW. Knowing that the allowable shearing stress in the shaft is 40 MPa and that the radius of the fillet is $r = 6$ mm, determine the smallest permissible speed of the shaft.

3.88 The stepped shaft shown must rotate at a frequency of 50 Hz. Knowing that the radius of the fillet is $r = 8$ mm and the allowable shearing stress is 45 MPa, determine the maximum power that can be transmitted.

3.89 In the stepped shaft shown, which has a full quarter-circular fillet, $D = 1.25$ in. and $d = 1$ in. Knowing that the speed of the shaft is 2400 rpm and that the allowable shearing stress is 7500 psi, determine the maximum power that can be transmitted by the shaft.

3.90 A torque of magnitude $T = 200$ lb · in. is applied to the stepped shaft shown, which has a full quarter-circular fillet. Knowing that $D = 1$ in., determine the maximum shearing stress in the shaft when (*a*) $d = 0.8$ in., (*b*) $d = 0.9$ in.

3.91 In the stepped shaft shown, which has a full quarter-circular fillet, the allowable shearing stress is 80 MPa. Knowing that $D = 30$ mm, determine the largest allowable torque that can be applied to the shaft if (*a*) $d = 26$ mm, (*b*) $d = 24$ mm.

*3.9 PLASTIC DEFORMATIONS IN CIRCULAR SHAFTS

When we derived Eqs. (3.10) and (3.16), which define, respectively, the stress distribution and the angle of twist for a circular shaft subjected to a torque **T,** we assumed that Hooke's law applied throughout the shaft. If the yield strength is exceeded in some portion of the shaft, or if the material involved is a brittle material with a nonlinear shearing-stress-strain diagram, these relations cease to be valid. The purpose of this section is to develop a more general method—which may be used when Hooke's law does not apply—for determining the distribution of stresses in a solid circular shaft, and for computing the torque required to produce a given angle of twist.

We first recall that no specific stress-strain relationship was assumed in Sec. 3.3, when we proved that the shearing strain γ varies linearly with the distance ρ from the axis of the shaft (Fig. 3.30). Thus, we may still use this property in our present analysis and write

$$\gamma = \frac{\rho}{c}\gamma_{\max} \tag{3.4}$$

where c is the radius of the shaft.

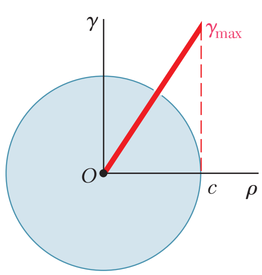


Fig. 3.30 Shearing strain variation.

Assuming that the maximum value τ_{max} of the shearing stress τ has been specified, the plot of τ versus ρ may be obtained as follows. We first determine from the shearing-stress-strain diagram the value of γ_{max} corresponding to τ_{max} (Fig. 3.31), and carry this value into Eq. (3.4). Then, for each value of ρ, we determine the corresponding value of γ from Eq. (3.4) or Fig. 3.30 and obtain from the stress-strain diagram of Fig. 3.31 the shearing stress τ corresponding to this value of γ. Plotting τ against ρ yields the desired distribution of stresses (Fig. 3.32).

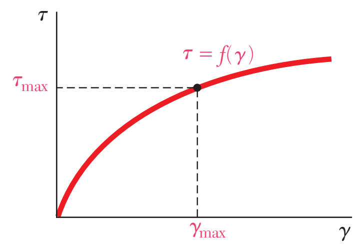


Fig. 3.31 Nonlinear, shear stress-strain diagram.

We now recall that, when we derived Eq. (3.1) in Sec. 3.2, we assumed no particular relation between shearing stress and strain. We may therefore use Eq. (3.1) to determine the torque **T** corresponding to the shearing-stress distribution obtained in Fig. 3.32. Considering an annular element of radius ρ and thickness $d\rho$, we express the element of area in Eq. (3.1) as $dA = 2\pi\rho\, d\rho$ and write

$$T = \int_0^c \rho\tau(2\pi\rho\, d\rho)$$

or

$$T = 2\pi \int_0^c \rho^2 \tau\, d\rho \tag{3.26}$$

where τ is the function of ρ plotted in Fig. 3.32.

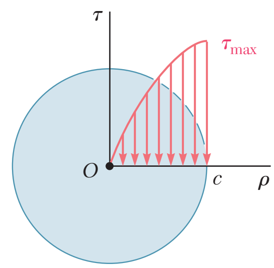


Fig. 3.32 Shearing strain variation for shaft with nonlinear stress-strain diagram.

If τ is a known analytical function of γ, Eq. (3.4) may be used to express τ as a function of ρ, and the integral in (3.26) may be determined analytically. Otherwise, the torque **T** may be obtained through a numerical integration. This computation becomes more meaningful if we note that the integral in Eq. (3.26) represents the second moment, or moment of inertia, with respect to the vertical axis of the area in Fig. 3.32 located above the horizontal axis and bounded by the stress-distribution curve.

An important value of the torque is the ultimate torque T_U which causes failure of the shaft. This value may be determined from the ultimate shearing stress τ_U of the material by choosing $\tau_{max} = \tau_U$ and carrying out the computations indicated earlier. However, it is found more convenient in practice to determine T_U experimentally by twisting a specimen of a given material until it breaks. Assuming a fictitious linear distribution of stresses, Eq. (3.9) is then used to determine the corresponding maximum shearing stress R_T:

$$R_T = \frac{T_U c}{J} \tag{3.27}$$

The fictitious stress R_T is called the *modulus of rupture in torsion* of the given material. It may be used to determine the ultimate torque T_U of a shaft made of the same material, but of different dimensions, by solving Eq. (3.27) for T_U. Since the actual and the fictitious linear stress distributions shown in Fig. 3.33 must yield the same value T_U for the ultimate torque, the areas they define must have the same moment of inertia with respect to the vertical axis. It is thus clear that the modulus of rupture R_T will always be larger than the actual ultimate shearing stress τ_U.

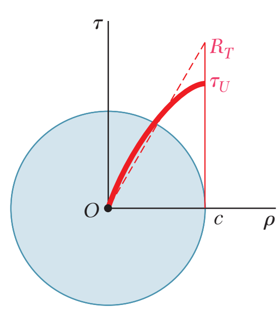


Fig. 3.33 Shaft at failure.

In some cases, we may wish to determine the stress distribution and the torque **T** corresponding to a given angle of twist ϕ. This may be done by recalling the expression obtained in Sec. 3.3 for the shearing strain γ in terms of ϕ, ρ, and the length L of the shaft:

$$\gamma = \frac{\rho\phi}{L} \tag{3.2}$$

With ϕ and L given, we may determine from Eq. (3.2) the value of γ corresponding to any given value of ρ. Using the stress-strain diagram of the material, we may then obtain the corresponding value of the shearing stress τ and plot τ against ρ. Once the shearing-stress distribution has been obtained, the torque **T** may be determined analytically or numerically as explained earlier.

*3.10 CIRCULAR SHAFTS MADE OF AN ELASTOPLASTIC MATERIAL

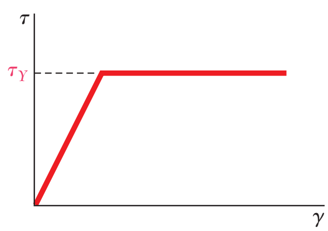


Fig. 3.34 Elastoplastic stress-strain diagram.

Further insight into the plastic behavior of a shaft in torsion is obtained by considering the idealized case of a *solid circular shaft made of an elastoplastic material*. The shearing-stress-strain diagram of such a material is shown in Fig. 3.34. Using this diagram, we can proceed as indicated earlier and find the stress distribution across a section of the shaft for any value of the torque **T.**

As long as the shearing stress τ does not exceed the yield strength τ_Y, Hooke's law applies, and the stress distribution across the section is linear (Fig. 3.35*a*), with τ_{max} given by Eq. (3.9):

$$\tau_{max} = \frac{Tc}{J} \tag{3.9}$$

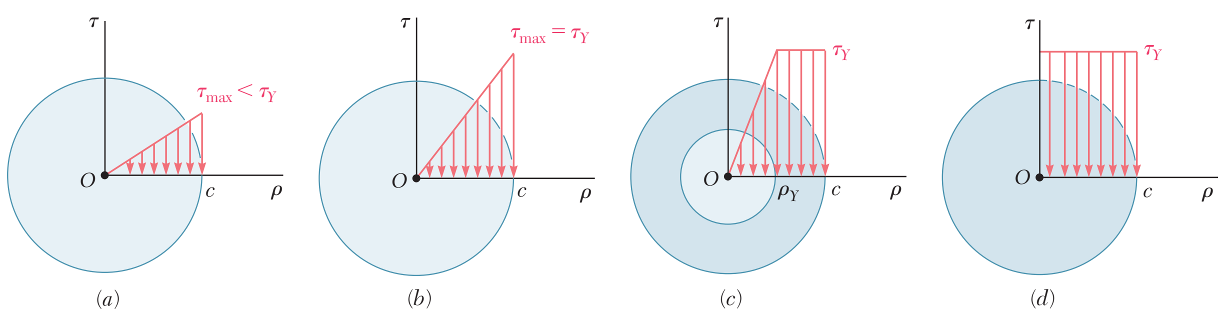


Fig. 3.35 Stress-strain diagrams for shaft made of elastoplastic material.

As the torque increases, τ_{max} eventually reaches the value τ_Y (Fig. 3.35*b*). Substituting this value into Eq. (3.9), and solving for the corresponding value of T, we obtain the value T_Y of the torque at the onset of yield:

$$T_Y = \frac{J}{c}\tau_Y \tag{3.28}$$

The value obtained is referred to as the *maximum elastic torque*, since it is the largest torque for which the deformation remains fully elastic. Recalling that for a solid circular shaft $J/c = \frac{1}{2}\pi c^3$, we have

$$T_Y = \tfrac{1}{2}\pi c^3 \tau_Y \tag{3.29}$$

As the torque is further increased, a plastic region develops in the shaft, around an elastic core of radius ρ_Y (Fig. 3.35*c*). In the plastic region the stress is uniformly equal to τ_Y, while in the elastic core the stress varies linearly with ρ and may be expressed as

$$\tau = \frac{\tau_Y}{\rho_Y}\rho \tag{3.30}$$

As T is increased, the plastic region expands until, at the limit, the deformation is fully plastic (Fig. 3.35*d*).

Equation (3.26) will be used to determine the value of the torque T corresponding to a given radius ρ_Y of the elastic core. Recalling that τ is given by Eq. (3.30) for $0 \leq \rho \leq \rho_Y$, and is equal to τ_Y for $\rho_Y \leq \rho \leq c$, we write

$$\begin{aligned} T &= 2\pi \int_0^{\rho_Y} \rho^2 \left(\frac{\tau_Y}{\rho_Y}\rho\right) d\rho + 2\pi \int_{\rho_Y}^{c} \rho^2 \tau_Y \, d\rho \\ &= \frac{1}{2}\pi \rho_Y^3 \tau_Y + \frac{2}{3}\pi c^3 \tau_Y - \frac{2}{3}\pi \rho_Y^3 \tau_Y \\ T &= \frac{2}{3}\pi c^3 \tau_Y \left(1 - \frac{1}{4}\frac{\rho_Y^3}{c^3}\right) \end{aligned} \tag{3.31}$$

or, in view of Eq. (3.29),

$$T = \frac{4}{3} T_Y \left(1 - \frac{1}{4}\frac{\rho_Y^3}{c^3}\right) \tag{3.32}$$

where T_Y is the maximum elastic torque. We note that, as ρ_Y approaches zero, the torque approaches the limiting value

$$T_p = \frac{4}{3} T_Y \tag{3.33}$$

This value of the torque, which corresponds to a fully plastic deformation (Fig. 3.35*d*), is called the *plastic torque* of the shaft considered. We note that Eq. (3.33) is valid only for a *solid circular shaft made of an elastoplastic material*.

Since the distribution of *strain* across the section remains linear after the onset of yield, Eq. (3.2) remains valid and can be used to express the radius ρ_Y of the elastic core in terms of the angle of twist ϕ. If ϕ is large enough to cause a plastic deformation, the radius ρ_Y of the elastic core is obtained by making γ equal to the yield strain γ_Y in Eq. (3.2) and solving for the corresponding value ρ_Y of the distance ρ. We have

$$\rho_Y = \frac{L\gamma_Y}{\phi} \tag{3.34}$$

Let us denote by ϕ_Y the angle of twist at the onset of yield, i.e., when $\rho_Y = c$. Making $\phi = \phi_Y$ and $\rho_Y = c$ in Eq. (3.34), we have

$$c = \frac{L\gamma_Y}{\phi_Y} \tag{3.35}$$

Dividing (3.34) by (3.35), member by member, we obtain the following relation:†

$$\frac{\rho_Y}{c} = \frac{\phi_Y}{\phi} \tag{3.36}$$

If we carry into Eq. (3.32) the expression obtained for ρ_Y/c, we express the torque T as a function of the angle of twist ϕ,

$$T = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\phi_Y^3}{\phi^3}\right) \tag{3.37}$$

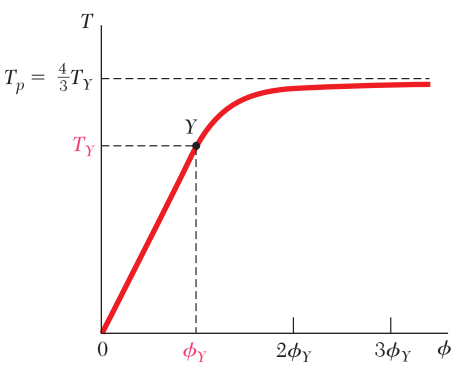


Fig. 3.36 Load displacement relation for elastoplastic material.

where T_Y and ϕ_Y represent, respectively, the torque and the angle of twist at the onset of yield. Note that Eq. (3.37) may be used only for values of ϕ larger than ϕ_Y. For $\phi < \phi_Y$, the relation between T and ϕ is linear and given by Eq. (3.16). Combining both equations, we obtain the plot of T against ϕ represented in Fig. 3.39. We check that, as ϕ increases indefinitely, T approaches the limiting value $T_p = \frac{4}{3}T_Y$ corresponding to the case of a fully developed plastic zone (Fig. 3.35*d*). While the value T_p cannot actually be reached, we note from Eq. (3.37) that it is rapidly approached as ϕ increases. For $\phi = 2\phi_Y$, T is within about 3% of T_p, and for $\phi = 3\phi_Y$ within about 1%.

Since the plot of T against ϕ that we have obtained for an idealized elastoplastic material (Fig. 3.36) differs greatly from the shearing-stress-strain diagram of that material (Fig. 3.34), it is clear that the shearing-stress-strain diagram of an actual material cannot be obtained directly from a torsion test carried out on a solid circular rod made of that material. However, a fairly accurate diagram may be obtained from a torsion test if the specimen used incorporates a portion consisting of a thin circular tube.‡ Indeed, we may assume that the shearing stress will have a constant value τ in that portion. Equation (3.1) thus reduces to

$$T = \rho A \tau$$

where ρ denotes the average radius of the tube and A its cross-sectional area. The shearing stress is thus proportional to the torque, and successive values of τ can be easily computed from the corresponding values of T. On the other hand, the values of the shearing strain γ may be obtained from Eq. (3.2) and from the values of ϕ and L measured on the tubular portion of the specimen.

†Equation (3.36) applies to any ductile material with a well-defined yield point, since its derivation is independent of the shape of the stress-strain diagram beyond the yield point.

‡In order to minimize the possibility of failure by buckling, the specimen should be made so that the length of the tubular portion is no longer than its diameter.

EXAMPLE 3.08

A solid circular shaft, 1.2 m long and 50 mm in diameter, is subjected to a 4.60 kN · m torque at each end (Fig. 3.37). Assuming the shaft to be made of an elastoplastic material with a yield strength in shear of 150 MPa and a modulus of rigidity of 77 GPa, determine (*a*) the radius of the elastic core, (*b*) the angle of twist of the shaft.

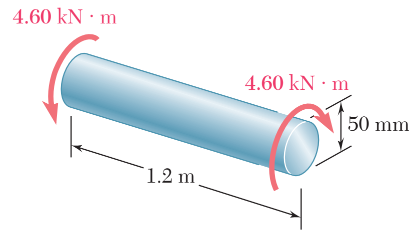


Fig. 3.37

(a) Radius of Elastic Core. We first determine the torque T_Y at the onset of yield. Using Eq. (3.28) with $\tau_Y = 150$ MPa, $c = 25$ mm, and

$$J = \tfrac{1}{2}\pi c^4 = \tfrac{1}{2}\pi(25 \times 10^{-3} \text{ m})^4 = 614 \times 10^{-9} \text{ m}^4$$

we write

$$T_Y = \frac{J\tau_Y}{c} = \frac{(614 \times 10^{-9} \text{ m}^4)(150 \times 10^6 \text{ Pa})}{25 \times 10^{-3} \text{ m}} = 3.68 \text{ kN} \cdot \text{m}$$

Solving Eq. (3.32) for $(\rho_Y/c)^3$ and substituting the values of T and T_Y, we have

$$\left(\frac{\rho_Y}{c}\right)^3 = 4 - \frac{3T}{T_Y} = 4 - \frac{3(4.60 \text{ kN} \cdot \text{m})}{3.68 \text{ kN} \cdot \text{m}} = 0.250$$

$$\frac{\rho_Y}{c} = 0.630 \qquad \rho_Y = 0.630(25 \text{ mm}) = 15.8 \text{ mm}$$

(b) Angle of Twist. We first determine the angle of twist ϕ_Y at the onset of yield from Eq. (3.16):

$$\phi_Y = \frac{T_Y L}{JG} = \frac{(3.68 \times 10^3 \text{ N} \cdot \text{m})(1.2 \text{ m})}{(614 \times 10^{-9} \text{ m}^4)(77 \times 10^9 \text{ Pa})} = 93.4 \times 10^{-3} \text{ rad}$$

Solving Eq. (3.36) for ϕ and substituting the values obtained for ϕ_Y and ρ_Y/c, we write

$$\phi = \frac{\phi_Y}{\rho_Y/c} = \frac{93.4 \times 10^{-3} \text{ rad}}{0.630} = 148.3 \times 10^{-3} \text{ rad}$$

or

$$\phi = (148.3 \times 10^{-3} \text{ rad})\left(\frac{360°}{2\pi \text{ rad}}\right) = 8.50°$$

*3.11 RESIDUAL STRESSES IN CIRCULAR SHAFTS

In the two preceding sections, we saw that a plastic region will develop in a shaft subjected to a large enough torque, and that the shearing stress τ at any given point in the plastic region may be obtained from the shearing-stress-strain diagram of Fig. 3.31. If the torque is removed, the resulting reduction of stress and strain at the point considered will take place along a straight line (Fig. 3.38). As you will see further in this section, the final value of the stress will not, in general, be zero. There will be a residual stress at most points, and that stress may be either positive or negative. We note that, as was the case for the normal stress, the shearing stress will keep decreasing until it has reached a value equal to its maximum value at C minus twice the yield strength of the material.

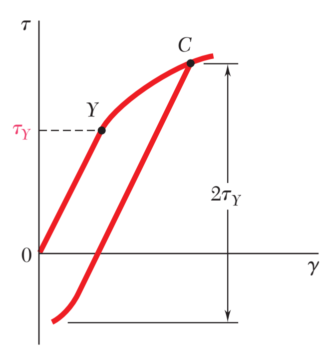


Fig. 3.38 Unloading of shaft with nonlinear stress-strain diagram.

Consider again the idealized case of the elastoplastic material characterized by the shearing-stress-strain diagram of Fig. 3.34.

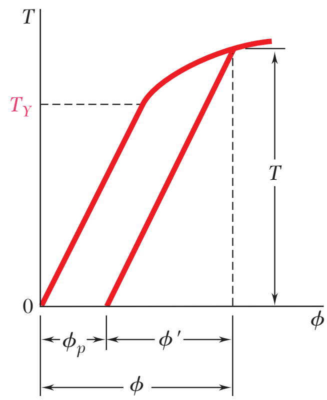


Fig. 3.39 Unloading of shaft with elastoplastic material.

Assuming that the relation between τ and γ at any point of the shaft remains linear as long as the stress does not decrease by more than $2\tau_Y$, we can use Eq. (3.16) to obtain the angle through which the shaft untwists as the torque decreases back to zero. As a result, the unloading of the shaft will be represented by a straight line on the T-ϕ diagram (Fig. 3.39). We note that the angle of twist does not return to zero after the torque has been removed. Indeed, the loading and unloading of the shaft result in a permanent deformation characterized by the angle

$$\phi_p = \phi - \phi' \tag{3.38}$$

where ϕ corresponds to the loading phase and may be obtained from T by solving Eq. (3.38), and where ϕ' corresponds to the unloading phase and may be obtained from Eq. (3.16).

The residual stresses in an elastoplastic material are obtained by applying the principle of superposition in a manner similar to that described in Sec. 2.20 for an axial loading. We consider, on one hand, the stresses due to the application of the given torque **T** and, on the other, the stresses due to the equal and opposite torque which is applied to unload the shaft. The first group of stresses reflects the elastoplastic behavior of the material during the loading phase (Fig. 3.40*a*), and the second group the linear behavior of the same material during the unloading phase (Fig. 3.40*b*). Adding the two groups of stresses, we obtain the distribution of the residual stresses in the shaft (Fig. 3.40*c*).

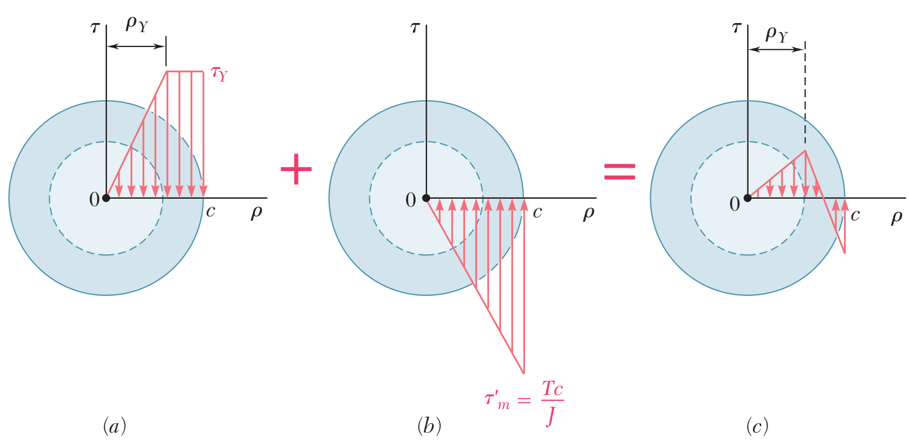


Fig. 3.40 Stress distributions for unloading of shaft with elastoplastic material.

We note from Fig. 3.40*c* that some residual stresses have the same sense as the original stresses, while others have the opposite sense. This was to be expected since, according to Eq. (3.1), the relation

$$\int \rho(\tau\, dA) = 0 \tag{3.39}$$

must be verified after the torque has been removed.

EXAMPLE 3.09

For the shaft of Example 3.08 determine (*a*) the permanent twist, (*b*) the distribution of residual stresses, after the 4.60 kN · m torque has been removed.

(a) Permanent Twist. We recall from Example 3.08 that the angle of twist corresponding to the given torque is $\phi = 8.50°$. The angle ϕ'

through which the shaft untwists as the torque is removed is obtained from Eq. (3.16). Substituting the given data,

$$T = 4.60 \times 10^3 \text{ N} \cdot \text{m}$$
$$L = 1.2 \text{ m}$$
$$G = 77 \times 10^9 \text{ Pa}$$

and the value $J = 614 \times 10^{-9} \text{ m}^4$ obtained in the solution of Example 3.08, we have

$$\phi' = \frac{TL}{JG} = \frac{(4.60 \times 10^3 \text{ N} \cdot \text{m})(1.2 \text{ m})}{(614 \times 10^{-9} \text{ m}^4)(77 \times 10^9 \text{ Pa})}$$
$$= 116.8 \times 10^{-3} \text{ rad}$$

or

$$\phi' = (116.8 \times 10^{-3} \text{ rad})\frac{360°}{2\pi \text{ rad}} = 6.69°$$

The permanent twist is therefore

$$\phi_p = \phi - \phi' = 8.50° - 6.69° = 1.81°$$

(b) Residual Stresses. We recall from Example 3.08 that the yield strength is $\tau_Y = 150$ MPa and that the radius of the elastic core corresponding to the given torque is $\rho_Y = 15.8$ mm. The distribution of the stresses in the loaded shaft is thus as shown in Fig. 3.41*a*.

The distribution of stresses due to the opposite 4.60 kN · m torque required to unload the shaft is linear and as shown in Fig. 3.41*b*. The maximum stress in the distribution of the reverse stresses is obtained from Eq. (3.9):

$$\tau'_{\max} = \frac{Tc}{J} = \frac{(4.60 \times 10^3 \text{ N} \cdot \text{m})(25 \times 10^{-3} \text{ m})}{614 \times 10^{-9} \text{ m}^4}$$
$$= 187.3 \text{ MPa}$$

Superposing the two distributions of stresses, we obtain the residual stresses shown in Fig. 3.41*c*. We check that, even though the reverse stresses exceed the yield strength τ_Y, the assumption of a linear distribution of these stresses is valid, since they do not exceed $2\tau_Y$.

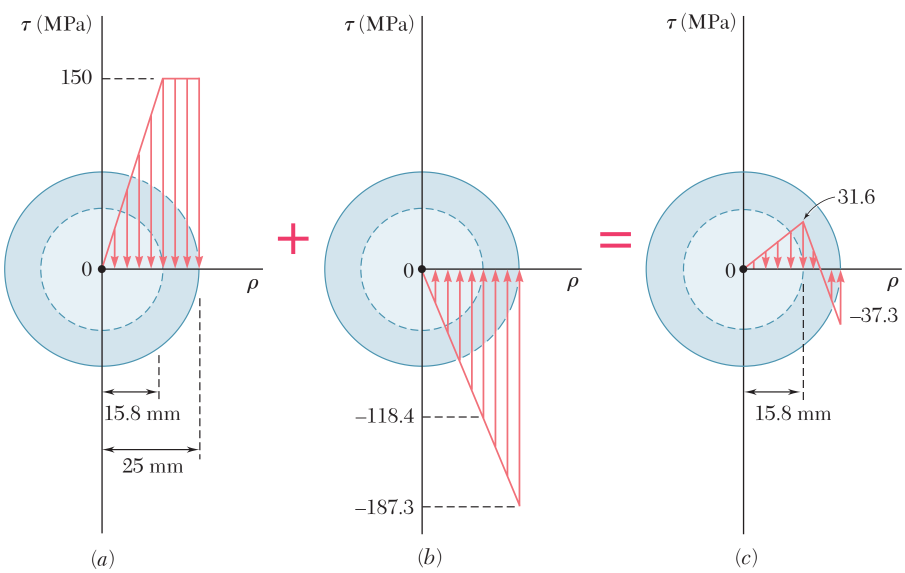
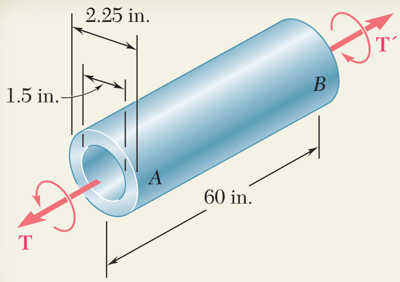
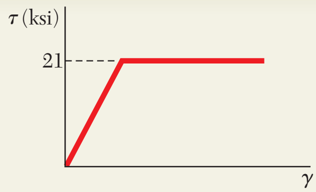


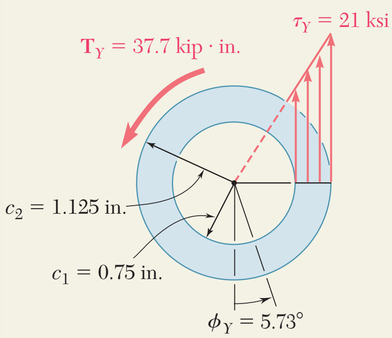
Fig. 3.41

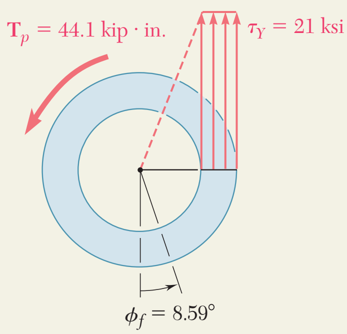


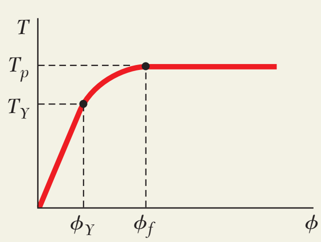
SAMPLE PROBLEM 3.7

Shaft AB is made of a mild steel that is assumed to be elastoplastic with $G = 11.2 \times 10^6$ psi and $\tau_Y = 21$ ksi. A torque $\mathbf{T}$ is applied and gradually increased in magnitude. Determine the magnitude of $\mathbf{T}$ and the corresponding angle of twist (*a*) when yield first occurs, (*b*) when the deformation has become fully plastic.









SOLUTION

Geometric Properties

The geometric properties of the cross section are

$$c_1 = \tfrac{1}{2}(1.5 \text{ in.}) = 0.75 \text{ in.} \qquad c_2 = \tfrac{1}{2}(2.25 \text{ in.}) = 1.125 \text{ in.}$$

$$J = \tfrac{1}{2}\pi(c_2^4 - c_1^4) = \tfrac{1}{2}\pi[(1.125 \text{ in.})^4 - (0.75 \text{ in.})^4] = 2.02 \text{ in}^4$$

a. Onset of Yield. For $\tau_{max} = \tau_Y = 21$ ksi, we find

$$T_Y = \frac{\tau_Y J}{c_2} = \frac{(21 \text{ ksi})(2.02 \text{ in}^4)}{1.125 \text{ in.}}$$

$$T_Y = 37.7 \text{ kip} \cdot \text{in.} \ ◀$$

Making $\rho = c_2$ and $\gamma = \gamma_Y$ in Eq. (3.2) and solving for ϕ, we obtain the value of ϕ_Y:

$$\phi_Y = \frac{\gamma_Y L}{c_2} = \frac{\tau_Y L}{c_2 G} = \frac{(21 \times 10^3 \text{ psi})(60 \text{ in.})}{(1.125 \text{ in.})(11.2 \times 10^6 \text{ psi})} = 0.100 \text{ rad}$$

$$\phi_Y = 5.73° \ ◀$$

b. Fully Plastic Deformation. When the plastic zone reaches the inner surface, the stresses are uniformly distributed as shown. Using Eq. (3.26), we write

$$T_p = 2\pi\tau_Y \int_{c_1}^{c_2} \rho^2 \, d\rho = \tfrac{2}{3}\pi\tau_Y(c_2^3 - c_1^3)$$

$$= \tfrac{2}{3}\pi(21 \text{ ksi})[(1.125 \text{ in.})^3 - (0.75 \text{ in.})^3]$$

$$T_p = 44.1 \text{ kip} \cdot \text{in.} \ ◀$$

When yield first occurs on the inner surface, the deformation is fully plastic; we have from Eq. (3.2):

$$\phi_f = \frac{\gamma_Y L}{c_1} = \frac{\tau_Y L}{c_1 G} = \frac{(21 \times 10^3 \text{ psi})(60 \text{ in.})}{(0.75 \text{ in.})(11.2 \times 10^6 \text{ psi})} = 0.150 \text{ rad}$$

$$\phi_f = 8.59° \ ◀$$

For larger angles of twist, the torque remains constant; the T-ϕ diagram of the shaft is as shown.

SAMPLE PROBLEM 3.8

For the shaft of Sample Prob. 3.7, determine the residual stresses and the permanent angle of twist after the torque $T_p = 44.1$ kip · in. has been removed.

SOLUTION

Referring to Sample Prob. 3.7, we recall that when the plastic zone first reached the inner surface, the applied torque was $T_p = 44.1$ kip · in. and the corresponding angle of twist was $\phi_f = 8.59°$. These values are shown in Fig. 1.

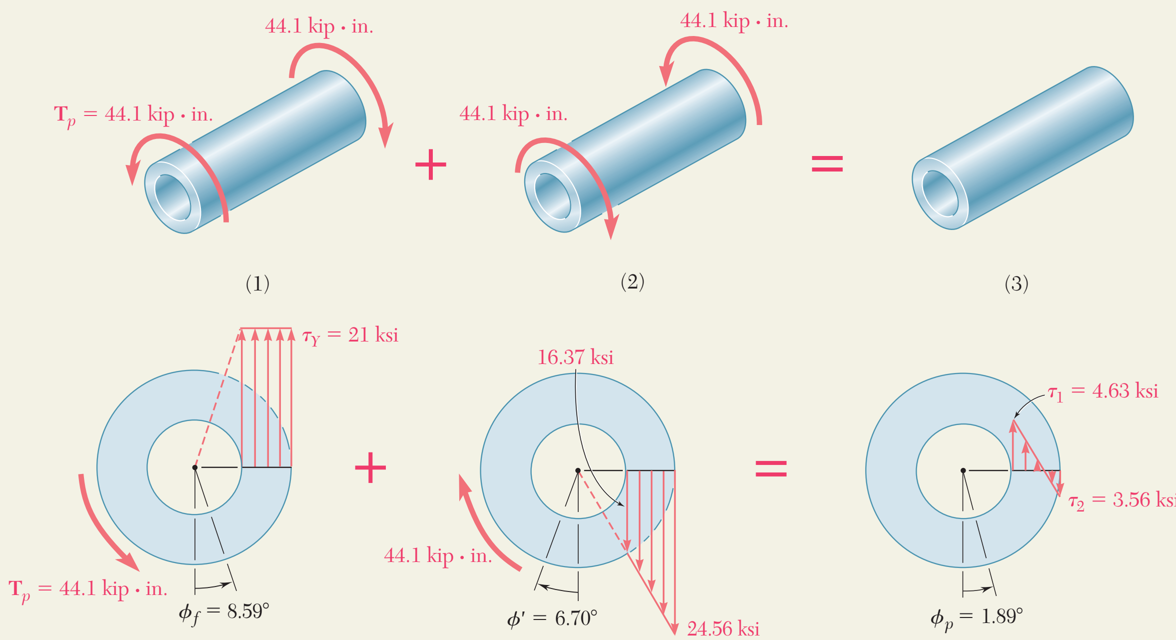
Elastic Unloading. We unload the shaft by applying a 44.1 kip · in. torque in the sense shown in Fig. 2. During this unloading, the behavior of the material is linear. Recalling from Sample Prob. 3.7 the values found for c_1, c_2, and J, we obtain the following stresses and angle of twist:

$$\tau_{\max} = \frac{Tc_2}{J} = \frac{(44.1 \text{ kip} \cdot \text{in.})(1.125 \text{ in.})}{2.02 \text{ in}^4} = 24.56 \text{ ksi}$$

$$\tau_{\min} = \tau_{\max}\frac{c_1}{c_2} = (24.56 \text{ ksi})\frac{0.75 \text{ in.}}{1.125 \text{ in.}} = 16.37 \text{ ksi}$$

$$\phi' = \frac{TL}{JG} = \frac{(44.1 \times 10^3 \text{ psi})(60 \text{ in.})}{(2.02 \text{ in}^4)(11.2 \times 10^6 \text{ psi})} = 0.1170 \text{ rad} = 6.70°$$

Residual Stresses and Permanent Twist. The results of the loading (Fig. 1) and the unloading (Fig. 2) are superposed (Fig. 3) to obtain the residual stresses and the permanent angle of twist ϕ_p.



PROBLEMS

3.92 A 30-mm diameter solid rod is made of an elastoplastic material with $\tau_Y = 3.5$ MPa. Knowing that the elastic core of the rod is 25 mm in diameter, determine the magnitude of the applied torque **T**.

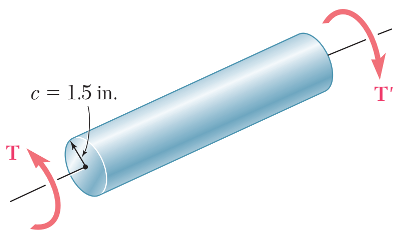


Fig. P3.93

3.93 The solid circular shaft shown is made of a steel that is assumed to be elastoplastic with $\tau_Y = 21$ ksi. Determine the magnitude T of the applied torques when the plastic zone is (*a*) 0.8 in. deep, (*b*) 1.2 in. deep.

3.94 The solid circular shaft shown is made of a steel that is assumed to be elastoplastic with $\tau_Y = 145$ MPa. Determine the magnitude T of the applied torque when the plastic zone is (*a*) 16 mm deep, (*b*) 24 mm deep.

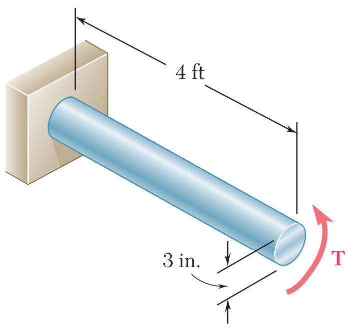


Fig. P3.95

3.95 The solid shaft shown is made of a mild steel that is assumed to be elastoplastic with $G = 11.2 \times 10^6$ psi and $\tau_Y = 21$ ksi. Determine the maximum shearing stress and the radius of the elastic core caused by the application of a torque of magnitude (*a*) $T =$ 100 kip · in., (*b*) $T =$ 140 kip · in.

3.96 It is observed that a straightened paper clip can be twisted through several revolutions by the application of a torque of approximately 60 mN · m. Knowing that the diameter of the wire in the paper clip is 0.9 mm, determine the approximate value of the yield stress of the steel.

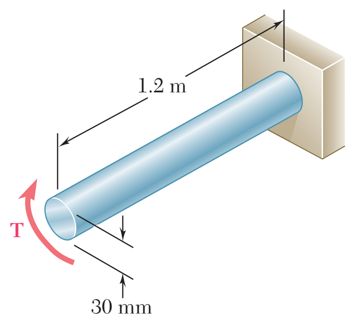


Fig. *P3.97*

3.97 The solid shaft shown is made of a mild steel that is assumed to be elastoplastic with $\tau_Y = 145$ MPa. Determine the radius of the elastic core caused by the application of a torque equal to 1.1 T_Y, where T_Y is the magnitude of the torque at the onset of yield.

3.98 For the solid circular shaft of Prob. 3.95, determine the angle of twist caused by the application of a torque of magnitude (*a*) $T =$ 80 kip · in., (*b*) $T =$ 130 kip · in.

3.99 The solid shaft shown is made of a mild steel that is assumed to be elastoplastic with $G = 77.2$ GPa and $\tau_Y = 145$ MPa. Determine the angle of twist caused by the application of a torque of magnitude (*a*) $T = 600$ N · m, (*b*) $T = 1000$ N · m.

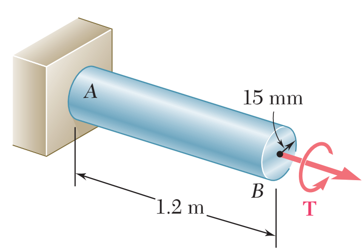


Fig. P3.99

3.100 A 3-ft-long solid shaft has a diameter of 2.5 in. and is made of a mild steel that is assumed to be elastoplastic with $\tau_Y = 21$ ksi and $G = 11.2 \times 10^6$ psi. Determine the torque required to twist the shaft through an angle of (*a*) 2.5°, (*b*) 5°.

3.101 For the solid shaft of Prob. 3.99, determine (*a*) the magnitude of the torque **T** required to twist the shaft through an angle of 15°, (*b*) the radius of the corresponding elastic core.

3.102 The shaft *AB* is made of a material that is elastoplastic with $\tau_Y = 12$ ksi and $G = 4.5 \times 10^6$ psi. For the loading shown, determine (*a*) the radius of the elastic core of the shaft, (*b*) the angle of twist at end *B*.

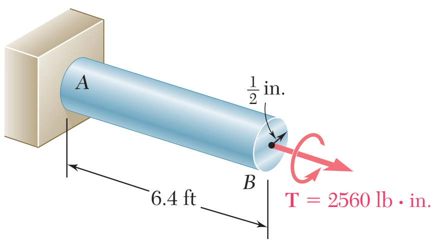


Fig. *P3.102*

3.103 A 1.25-in.-diameter solid circular shaft is made of a material that is assumed to be elastoplastic with $\tau_Y = 18$ ksi and $G = 11.2 \times 10^6$ psi. For an 8-ft length of the shaft, determine the maximum shearing stress and the angle of twist caused by a 7.5-kip · in. torque.

3.104 An 18-mm-diameter solid circular shaft is made of a material that is assumed to be elastoplastic with $\tau_Y = 145$ MPa and $G = 77$ GPa. For an 1.2-m length of the shaft, determine the maximum shearing stress and the angle of twist caused by a 200 N · m-torque.

3.105 A solid circular rod is made of a material that is assumed to be elastoplastic. Denoting by T_Y and ϕ_Y, respectively, the torque and the angle of twist at the onset of yield, determine the angle of twist if the torque is increased to (*a*) $T = 1.1\ T_Y$, (*b*) $T = 1.25\ T_Y$, (*c*) $T = 1.3\ T_Y$.

3.106 The hollow shaft shown is made of steel that is assumed to be elastoplastic with $\tau_Y = 145$ MPa and $G = 77.2$ GPa. Determine the magnitude *T* of the torque and the corresponding angle of twist (*a*) at the onset of yield, (*b*) when the plastic zone is 10 mm deep.

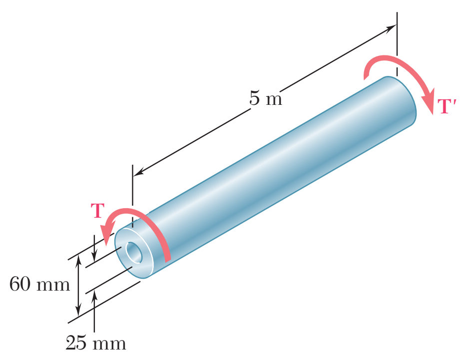


Fig. P3.106

3.107 For the shaft of Prob. 3.106, determine (*a*) angle of twist at which the section first becomes fully plastic, (*b*) the corresponding magnitude *T* of the applied torque. Sketch the *T*-ϕ curve for the shaft.

3.108 A steel rod is machined to the shape shown to form a tapered solid shaft to which torques of magnitude $T = 75$ kip · in. are applied. Assuming the steel to be elastoplastic with $\tau_Y = 21$ ksi and $G = 11.2 \times 10^6$ psi, determine (*a*) the radius of the elastic core in portion *AB* of the shaft, (*b*) the length of portion *CD* that remains fully elastic.

3.109 If the torques applied to the tapered shaft of Prob. 3.108 are slowly increased, determine (*a*) the magnitude *T* of the largest torques that can be applied to the shaft, (*b*) the length of the portion *CD* that remains fully elastic.

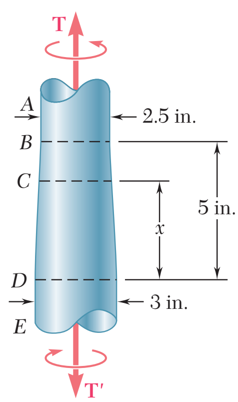


Fig. *P3.108*

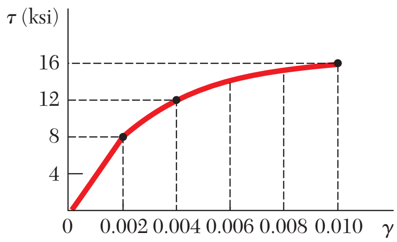


Fig. P3.110 and P3.111

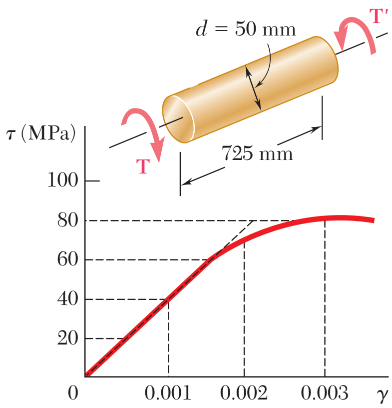


Fig. P3.112

3.110 A hollow shaft of outer and inner diameters respectively equal to 0.6 in. and 0.2 in. is fabricated from an aluminum alloy for which the stress-strain diagram is given in the diagram shown. Determine the torque required to twist a 9-in. length of the shaft through 10°.

3.111 Using the stress-strain diagram shown, determine (*a*) the torque that causes a maximum shearing stress of 15 ksi in a 0.8-in.-diameter solid rod, (*b*) the corresponding angle of twist in a 20-in. length of the rod.

3.112 A 50-mm-diameter cylinder is made of a brass for which the stress-strain diagram is as shown. Knowing that the angle of twist is 5° in a 725-mm length, determine by approximate means the magnitude T of torque applied to the shaft.

3.113 Three points on the nonlinear stress-strain diagram used in Prob. 3.112 are (0, 0), (0.0015, 55 MPa), and (0.003, 80MPa). By fitting the polynomial $T = A + B\gamma + C\gamma^2$ through these points, the following approximate relation has been obtained.

$$T = 46.7 \times 10^9\gamma - 6.67 \times 10^{12}\gamma^2$$

Solve Prob. 3.112 using this relation, Eq. (3.2), and Eq. (3.26).

3.114 The solid circular drill rod AB is made of a steel that is assumed to be elastoplastic with $\tau_Y = 22$ ksi and $G = 11.2 \times 10^6$ psi. Knowing that a torque $T = 75$ kip · in. is applied to the rod and then removed, determine the maximum residual shearing stress in the rod.

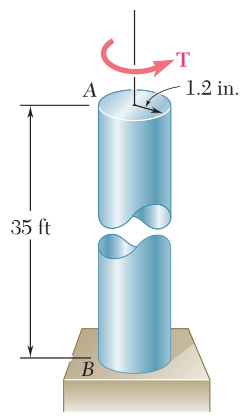


Fig. P3.114

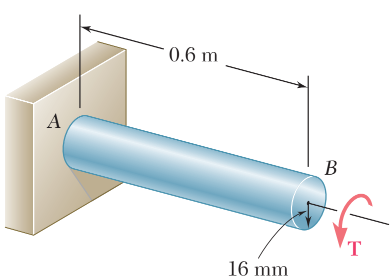


Fig. P3.116

3.115 In Prob. 3.114, determine the permanent angle of twist of the rod.

3.116 The solid shaft shown is made of a steel that is assumed to be elastoplastic with $\tau_Y = 145$ MPa and $G = 77.2$ GPa. The torque is increased in magnitude until the shaft has been twisted through 6°; the torque is then removed. Determine (*a*) the magnitude and location of the maximum residual shearing stress, (*b*) the permanent angle of twist.

3.117 After the solid shaft of Prob. 3.116 has been loaded and unloaded as described in that problem, a torque $\mathbf{T}_1$ of sense opposite to the original torque $\mathbf{T}$ is applied to the shaft. Assuming no change in the value of ϕ_Y, determine the angle of twist ϕ_1 for which yield is initiated in this second loading and compare it with the angle ϕ_Y for which the shaft started to yield in the original loading.

3.118 The hollow shaft shown is made of a steel that is assumed to be elastoplastic with $\tau_Y = 145$ MPa and $G = 77.2$ GPa. The magnitude T of the torques is slowly increased until the plastic zone first reaches the inner surface of the shaft; the torques are then removed. Determine the magnitude and location of the maximum residual shearing stress in the rod.

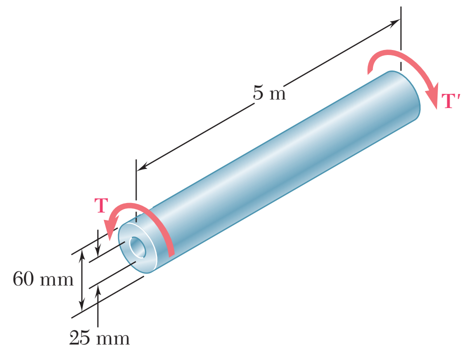


Fig. P3.118

3.119 In Prob. 3.118, determine the permanent angle of twist of the rod.

3.120 A torque $\mathbf{T}$ applied to a solid rod made of an elastoplastic material is increased until the rod is fully plastic and then removed. (*a*) Show that the distribution of residual shearing stresses is as represented in the figure. (*b*) Determine the magnitude of the torque due to the stresses acting on the portion of the rod located within a circle of radius c_0.

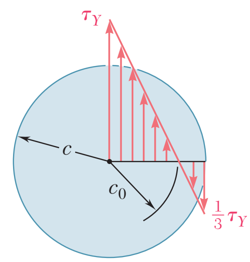


Fig. P3.120

*3.12 TORSION OF NONCIRCULAR MEMBERS

The formulas obtained in Secs. 3.3 and 3.4 for the distributions of strain and stress under a torsional loading apply only to members with a circular cross section. Indeed, their derivation was based on the assumption that the cross section of the member remained plane and undistorted, and we saw in Sec. 3.3 that the validity of this assumption depends upon the *axisymmetry* of the member, i.e., upon the fact that its appearance remains the same when it is viewed from a fixed position and rotated about its axis through an arbitrary angle.

A square bar, on the other hand, retains the same appearance only when it is rotated through 90° or 180°. Following a line of reasoning similar to that used in Sec. 3.3, one could show that the diagonals of the square cross section of the bar and the lines joining the midpoints of the sides of that section remain straight (Fig. 3.42). However, because of the lack of axisymmetry of the bar, any other line drawn in its cross section will deform when the bar is twisted, and the cross section itself will be warped out of its original plane.

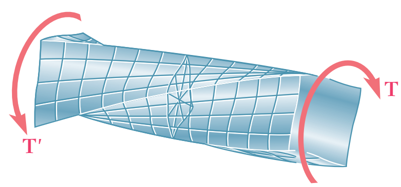


Fig. 3.42 Twisting of shaft with square cross section.

It follows that Eqs. (3.4) and (3.6), which define, respectively, the distributions of strain and stress in an elastic circular shaft, cannot be used for noncircular members. For example, it would be wrong to assume that the shearing stress in the cross section of a square bar varies linearly with the distance from the axis of the bar and is, therefore, largest at the corners of the cross section. As you will see presently, the shearing stress is actually zero at these points.

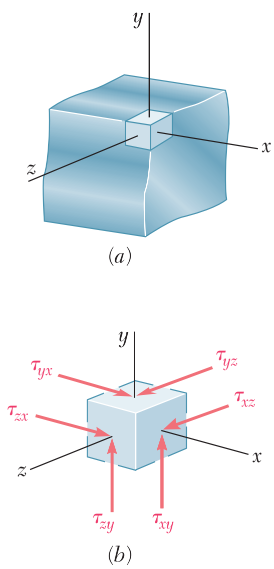


Fig. 3.43 Corner element.

Consider a small cubic element located at a corner of the cross section of a square bar in torsion and select coordinate axes parallel to the edges of the element (Fig. 3.43*a*). Since the face of the element perpendicular to the y axis is part of the free surface of the bar, all stresses on this face must be zero. Referring to Fig. 3.43*b*, we write

$$\tau_{yx} = 0 \qquad \tau_{yz} = 0 \tag{3.40}$$

For the same reason, all stresses on the face of the element perpendicular to the z axis must be zero, and we write

$$\tau_{zx} = 0 \qquad \tau_{zy} = 0 \tag{3.41}$$

It follows from the first of Eqs. (3.40) and the first of Eqs. (3.41) that

$$\tau_{xy} = 0 \qquad \tau_{xz} = 0 \tag{3.42}$$

Thus, both components of the shearing stress on the face of the element perpendicular to the axis of the bar are zero. We conclude that there is no shearing stress at the corners of the cross section of the bar.

By twisting a rubber model of a square bar, one easily verifies that no deformations—and, thus, no stresses—occur along the edges of the bar, while the largest deformations—and, thus, the largest stresses—occur along the center line of each of the faces of the bar (Fig. 3.44).

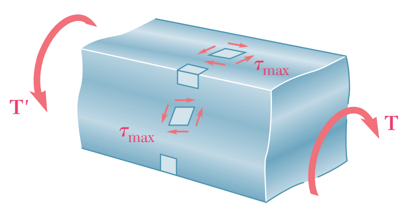


Fig. 3.44 Deformation of square bar.

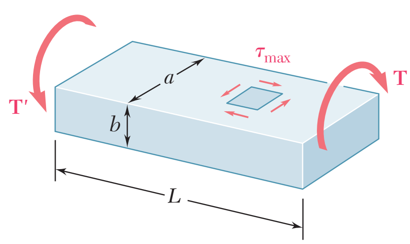


Fig. 3.45 Shaft with rectangular cross section.

The determination of the stresses in noncircular members subjected to a torsional loading is beyond the scope of this text. However, results obtained from the mathematical theory of elasticity for straight bars with a *uniform rectangular cross section* will be indicated here for convenience.† Denoting by L the length of the bar, by a and b, respectively, the wider and narrower side of its cross section, and by T the magnitude of the torques applied to the bar (Fig. 3.45), we find that the maximum shearing stress occurs along the center line of the *wider* face of the bar and is equal to

$$\tau_{\max} = \frac{T}{c_1 ab^2} \tag{3.43}$$

The angle of twist, on the other hand, may be expressed as

$$\phi = \frac{TL}{c_2 ab^3 G} \tag{3.44}$$

†See S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3d ed., McGraw-Hill, New York, 1969, sec. 109.

The coefficients c_1 and c_2 depend only upon the ratio a/b and are given in Table 3.1 for a number of values of that ratio. Note that Eqs. (3.43) and (3.44) are valid only within the elastic range.

We note from Table 3.1 that for $a/b \geq 5$, the coefficients c_1 and c_2 are equal. It may be shown that for such values of a/b, we have

$$c_1 = c_2 = \tfrac{1}{3}(1 - 0.630b/a) \qquad \textit{(for } a/b \geq 5 \textit{ only)} \qquad (3.45)$$

The distribution of shearing stresses in a noncircular member may be visualized more easily by using the *membrane analogy*. A homogeneous elastic membrane attached to a fixed frame and subjected to a uniform pressure on one of its sides happens to constitute an *analog* of the bar in torsion, i.e., the determination of the deformation of the membrane depends upon the solution of the same partial differential equation as the determination of the shearing stresses in the bar.† More specifically, if Q is a point of the cross section of the bar and Q' the corresponding point of the membrane (Fig. 3.46), the shearing stress τ at Q will have the same direction as the horizontal tangent to the membrane at Q', and its magnitude will be proportional to the maximum slope of the membrane at Q'.‡ Furthermore, the applied torque will be proportional to the volume between the membrane and the plane of the fixed frame. In the case of the membrane of Fig. 3.46, which is attached to a rectangular frame, the steepest slope occurs at the midpoint N' of the larger side of the frame. Thus, we verify that the maximum shearing stress in a bar of rectangular cross section will occur at the midpoint N of the larger side of that section.

The membrane analogy may be used just as effectively to visualize the shearing stresses in any straight bar of uniform, noncircular cross section. In particular, let us consider several thin-walled members with the cross sections shown in Fig. 3.47, which are subjected

TABLE 3.1. Coefficients for Rectangular Bars in Torsion

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

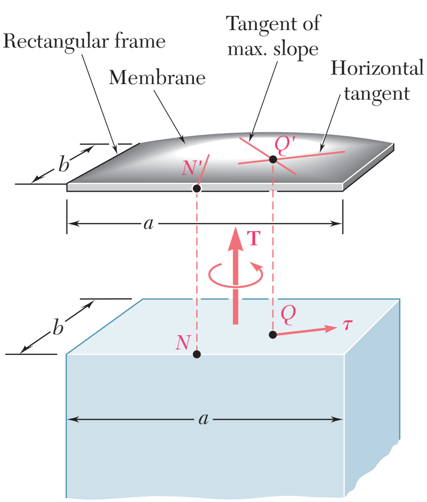


Fig. 3.46 Application of *membrane analogy* to shaft with rectangular cross section.

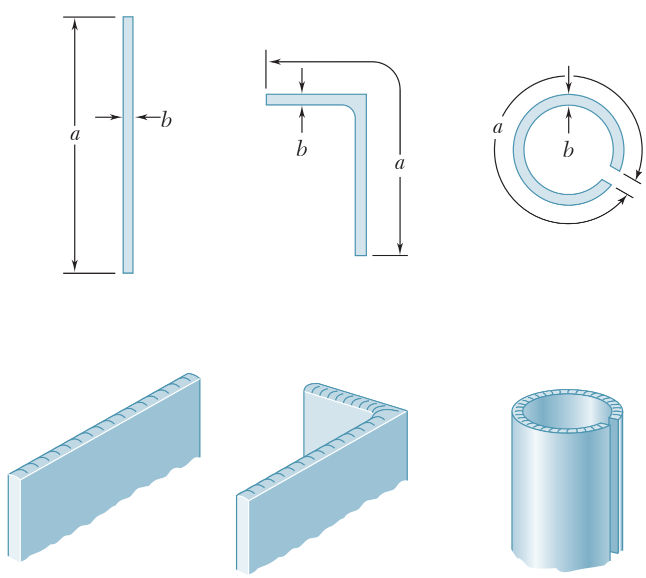


Fig. 3.47 Various thin-walled members.

†See ibid. Sec. 107.

‡This is the slope measured in a direction perpendicular to the horizontal tangent at Q'.

to the same torque. Using the membrane analogy to help us visualize the shearing stresses, we note that, since the same torque is applied to each member, the same volume will be located under each membrane, and the maximum slope will be about the same in each case. Thus, for a thin-walled member of uniform thickness and arbitrary shape, the maximum shearing stress is the same as for a rectangular bar with a very large value of a/b and may be determined from Eq. (3.43) with $c_1 = 0.333$.†

*3.13 THIN-WALLED HOLLOW SHAFTS

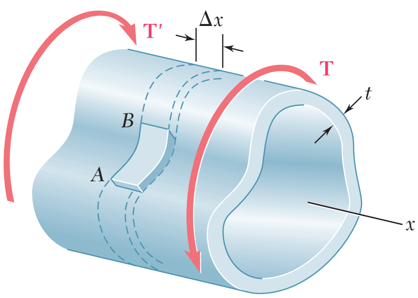


Fig. 3.48 Thin-walled hollow shaft.

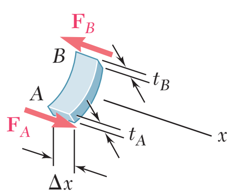


Fig. 3.49 Segment of thin-walled hollow shaft.

In the preceding section we saw that the determination of stresses in noncircular members generally requires the use of advanced mathematical methods. In the case of thin-walled hollow noncircular shafts, however, a good approximation of the distribution of stresses in the shaft can be obtained by a simple computation.

Consider a hollow cylindrical member of *noncircular* section subjected to a torsional loading (Fig. 3.48).‡ While the thickness t of the wall may vary within a transverse section, it will be assumed that it remains small compared to the other dimensions of the member. We now detach from the member the colored portion of wall AB bounded by two transverse planes at a distance Δx from each other, and by two longitudinal planes perpendicular to the wall. Since the portion AB is in equilibrium, the sum of the forces exerted on it in the longitudinal x direction must be zero (Fig. 3.49). But the only forces involved are the shearing forces $\mathbf{F}_A$ and $\mathbf{F}_B$ exerted on the ends of portion AB. We have therefore

$$\Sigma F_x = 0: \qquad F_A - F_B = 0 \tag{3.46}$$

We now express F_A as the product of the longitudinal shearing stress τ_A on the small face at A and of the area $t_A\,\Delta x$ of that face:

$$F_A = \tau_A(t_A\,\Delta x)$$

We note that, while the shearing stress is independent of the x coordinate of the point considered, it may vary across the wall; thus, τ_A represents the average value of the stress computed across the wall. Expressing F_B in a similar way and substituting for F_A and F_B into (3.46), we write

$$\tau_A(t_A\,\Delta x) - \tau_B(t_B\,\Delta x) = 0$$

or

$$\tau_A t_A = \tau_B t_B \tag{3.47}$$

Since A and B were chosen arbitrarily, Eq. (3.47) expresses that the product τt of the longitudinal shearing stress τ and of the wall thickness t is constant throughout the member. Denoting this product by q, we have

$$q = \tau t = \text{constant} \tag{3.48}$$

†It could also be shown that the angle of twist may be determined from Eq. (3.44) with $c_2 = 0.333$.

‡The wall of the member must enclose a single cavity and must not be slit open. In other words, the member should be topologically equivalent to a hollow circular shaft.

We now detach a small element from the wall portion AB (Fig. 3.50). Since the upper and lower faces of this element are part of the free surface of the hollow member, the stresses on these faces are equal to zero. Recalling relations (1.21) and (1.22) of Sec. 1.12, it follows that the stress components indicated on the other faces by dashed arrows are also zero, while those represented by solid arrows are equal. Thus, the shearing stress at any point of a transverse section of the hollow member is parallel to the wall surface (Fig. 3.51) and its average value computed across the wall satisfies Eq. (3.48).

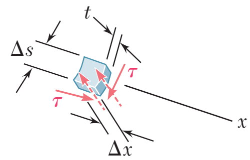


Fig. 3.50 Small element from segment.

At this point we can note an analogy between the distribution of the shearing stresses τ in the transverse section of a thin-walled hollow shaft and the distribution of the velocities v in water flowing through a closed channel of unit depth and variable width. While the velocity v of the water varies from point to point on account of the variation in the width t of the channel, the rate of flow, $q = vt$, remains constant throughout the channel, just as τt in Eq. (3.48). Because of this analogy, the product $q = \tau t$ is referred to as the *shear flow* in the wall of the hollow shaft.

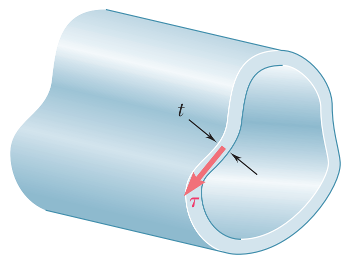


Fig. 3.51 Direction of shearing stress on cross section.

We will now derive a relation between the torque T applied to a hollow member and the shear flow q in its wall. We consider a small element of the wall section, of length ds (Fig. 3.52). The area of the element is $dA = t\,ds$, and the magnitude of the shearing force $d\mathbf{F}$ exerted on the element is

$$dF = \tau\, dA = \tau(t\, ds) = (\tau t)\, ds = q\, ds \tag{3.49}$$

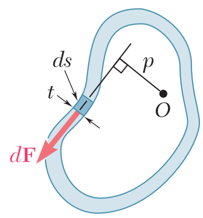


Fig. 3.52

The moment dM_O of this force about an arbitrary point O within the cavity of the member may be obtained by multiplying dF by the perpendicular distance p from O to the line of action of $d\mathbf{F}$. We have

$$dM_O = p\, dF = p(q\, ds) = q(p\, ds) \tag{3.50}$$

But the product $p\, ds$ is equal to twice the area $d\mathfrak{a}$ of the colored triangle in Fig. 3.53. We thus have

$$dM_O = q(2d\mathfrak{a}) \tag{3.51}$$

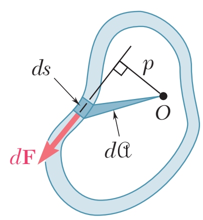


Fig. 3.53

Since the integral around the wall section of the left-hand member of Eq. (3.51) represents the sum of the moments of all the elementary shearing forces exerted on the wall section, and since this sum is equal to the torque T applied to the hollow member, we have

$$T = \oint dM_O = \oint q(2d\mathfrak{a})$$

The shear flow q being a constant, we write

$$T = 2q\mathfrak{a} \tag{3.52}$$

where $\mathfrak{a}$ is the area bounded by the center line of the wall cross section (Fig. 3.54).

The shearing stress τ at any given point of the wall may be expressed in terms of the torque T if we substitute for q from (3.48) into (3.52) and solve for τ the equation obtained. We have

$$\boxed{\tau = \frac{T}{2t\mathfrak{a}}} \tag{3.53}$$

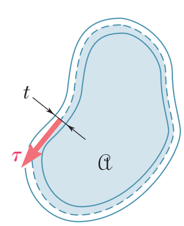


Fig. 3.54 Area for shear flow.

where t is the wall thickness at the point considered and $\mathcal{A}$ the area bounded by the center line. We recall that τ represents the average value of the shearing stress across the wall. However, for elastic deformations the distribution of stresses across the wall may be assumed uniform, and Eq. (3.53) will yield the actual value of the shearing stress at a given point of the wall.

The angle of twist of a thin-walled hollow shaft may be obtained by using the method of energy (Chap. 11). Assuming an elastic deformation, it may be shown† that the angle of twist of a thin-walled shaft of length L and modulus of rigidity G is

$$\phi = \frac{TL}{4\mathcal{A}^2 G} \oint \frac{ds}{t} \tag{3.54}$$

where the integral is computed along the center line of the wall section.

EXAMPLE 3.10

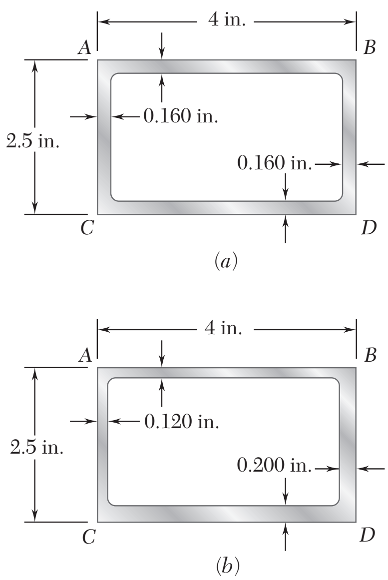


Fig. 3.55

Structural aluminum tubing of 2.5 × 4-in. rectangular cross section was fabricated by extrusion. Determine the shearing stress in each of the four walls of a portion of such tubing when it is subjected to a torque of 24 kip · in., assuming (*a*) a uniform 0.160-in. wall thickness (Fig. 3.55*a*), (*b*) that, as a result of defective fabrication, walls *AB* and *AC* are 0.120-in. thick, and walls *BD* and *CD* are 0.200-in. thick (Fig. 3.55*b*).

(a) Tubing of Uniform Wall Thickness. The area bounded by the center line (Fig. 3.56) is

$$\mathcal{A} = (3.84 \text{ in.})(2.34 \text{ in.}) = 8.986 \text{ in}^2$$

Since the thickness of each of the four walls is $t = 0.160$ in., we find from Eq. (3.53) that the shearing stress in each wall is

$$\tau = \frac{T}{2t\mathcal{A}} = \frac{24 \text{ kip} \cdot \text{in.}}{2(0.160 \text{ in.})(8.986 \text{ in}^2)} = 8.35 \text{ ksi}$$

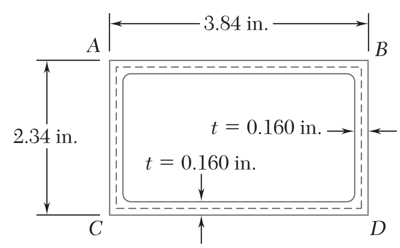


Fig. 3.56

(b) Tubing with Variable Wall Thickness. Observing that the area $\mathcal{A}$ bounded by the center line is the same as in part *a*, and substituting successively $t = 0.120$ in. and $t = 0.200$ in. into Eq. (3.53), we have

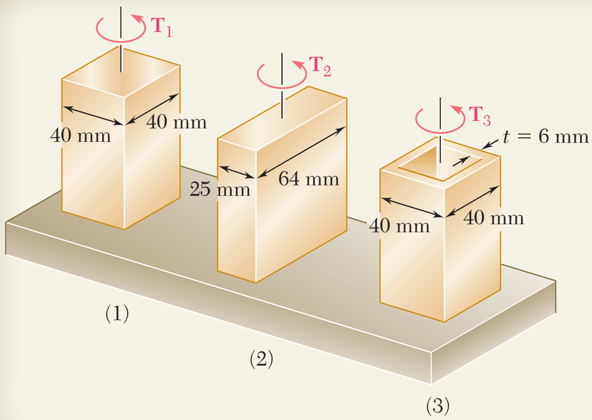
$$\tau_{AB} = \tau_{AC} = \frac{24 \text{ kip} \cdot \text{in.}}{2(0.120 \text{ in.})(8.986 \text{ in}^2)} = 11.13 \text{ ksi}$$

and

$$\tau_{BD} = \tau_{CD} = \frac{24 \text{ kip} \cdot \text{in.}}{2(0.200 \text{ in.})(8.986 \text{ in}^2)} = 6.68 \text{ ksi}$$

We note that the stress in a given wall depends only upon its thickness.

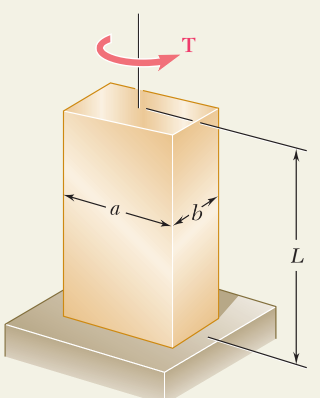
†See Prob. 11.70.



SAMPLE PROBLEM 3.9

Using $\tau_{all} = 40$ MPa, determine the largest torque that may be applied to each of the brass bars and to the brass tube shown. Note that the two solid bars have the same cross-sectional area, and that the square bar and square tube have the same outside dimensions.

SOLUTION



1. Bar with Square Cross Section. For a solid bar of rectangular cross section the maximum shearing stress is given by Eq. (3.43)

$$\tau_{max} = \frac{T}{c_1 ab^2}$$

where the coefficient c_1 is obtained from Table 3.1 in Sec. 3.12. We have

$$a = b = 0.040 \text{ m} \qquad \frac{a}{b} = 1.00 \qquad c_1 = 0.208$$

For $\tau_{max} = \tau_{all} = 40$ MPa, we have

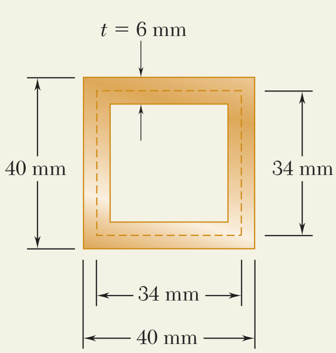
$$\tau_{max} = \frac{T_1}{c_1 ab^2} \qquad 40 \text{ MPa} = \frac{T_1}{0.208(0.040 \text{ m})^3} \qquad T_1 = 532 \text{ N} \cdot \text{m} \quad \blacktriangleleft$$

2. Bar with Rectangular Cross Section. We now have

$$a = 0.064 \text{ m} \qquad b = 0.025 \text{ m} \qquad \frac{a}{b} = 2.56$$

Interpolating in Table 3.1: $c_1 = 0.259$

$$\tau_{max} = \frac{T_2}{c_1 ab^2} \qquad 40 \text{ MPa} = \frac{T_2}{0.259(0.064 \text{ m})(0.025 \text{ m})^2} \qquad T_2 = 414 \text{ N} \cdot \text{m} \quad \blacktriangleleft$$



3. Square Tube. For a tube of thickness t, the shearing stress is given by Eq. (3.53)

$$\tau = \frac{T}{2t\mathcal{A}}$$

where $\mathcal{A}$ is the area bounded by the center line of the cross section. We have

$$\mathcal{A} = (0.034 \text{ m})(0.034 \text{ m}) = 1.156 \times 10^{-3} \text{ m}^2$$

We substitute $\tau = \tau_{all} = 40$ MPa and $t = 0.006$ m and solve for the allowable torque:

$$\tau = \frac{T}{2t\mathcal{A}} \qquad 40 \text{ MPa} = \frac{T_3}{2(0.006 \text{ m})(1.156 \times 10^{-3} \text{ m}^2)} \qquad T_3 = 555 \text{ N} \cdot \text{m} \quad \blacktriangleleft$$

PROBLEMS

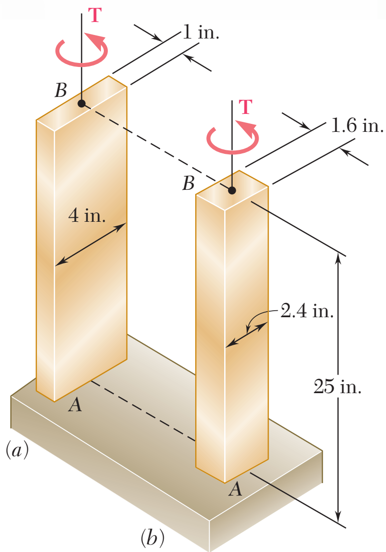


Fig. P3.121 and P3.122

3.121 Determine the largest torque **T** that can be applied to each of the two brass bars shown and the corresponding angle of twist at B, knowing that $\tau_{all} = 12$ ksi and $G = 5.6 \times 10^6$ psi.

3.122 Each of the two brass bars shown is subjected to a torque of magnitude $T = 12.5$ kip · in. Knowing that $G = 5.6 \times 10^6$ psi, determine for each bar the maximum shearing stress and the angle of twist at B.

3.123 Each of the two aluminum bars shown is subjected to a torque of magnitude $T = 1800$ N · m. Knowing that $G = 26$ GPa, determine for each bar the maximum shearing stress and the angle of twist at B.

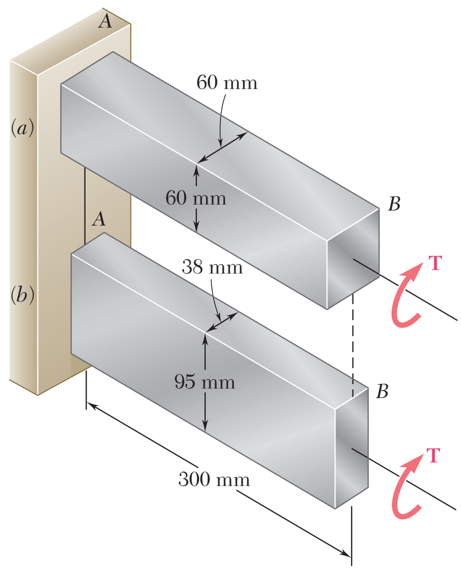


Fig. P3.123 and P3.124

3.124 Determine the largest torque **T** that can be applied to each of the two aluminum bars shown and the corresponding angle of twist at B, knowing that $\tau_{all} = 50$ MPa and $G = 26$ GPa.

3.125 Determine the largest allowable square cross section of a steel shaft of length 20 ft if the maximum shearing stress is not to exceed 10 ksi when the shaft is twisted through one complete revolution. Use $G = 11.2 \times 10^6$ psi.

3.126 Determine the largest allowable length of a stainless steel shaft of $\frac{3}{8} \times \frac{3}{4}$-in. cross section if the shearing stress is not to exceed 15 ksi when the shaft is twisted through 15°. Use $G = 11.2 \times 10^6$ psi.

3.127 The torque **T** causes a rotation of 2° at end B of the stainless steel bar shown. Knowing that $b = 20$ mm and $G = 75$ GPa, determine the maximum shearing stress in the bar.

3.128 The torque **T** causes a rotation of 0.6° at end B of the aluminum bar shown. Knowing that $b = 15$ mm and $G = 26$ GPa, determine the maximum shearing stress in the bar.

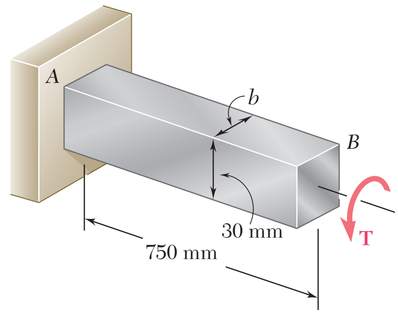


Fig. P3.127 and P3.128

3.129 Two shafts are made of the same material. The cross section of shaft A is a square of side b and that of shaft B is a circle of diameter b. Knowing that the shafts are subjected to the same torque, determine the ratio τ_A/τ_B of maximum shearing stresses occurring in the shafts.

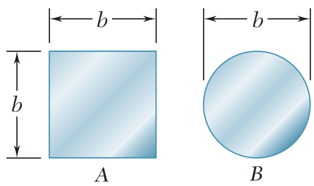


Fig. P3.129

3.130 Shafts A and B are made of the same material and have the same cross-sectional area, but A has a circular cross section and B has a square cross section. Determine the ratio of the maximum shearing stresses occurring in A and B, respectively, when the two shafts are subjected to the same torque ($T_A = T_B$). Assume both deformations to be elastic.

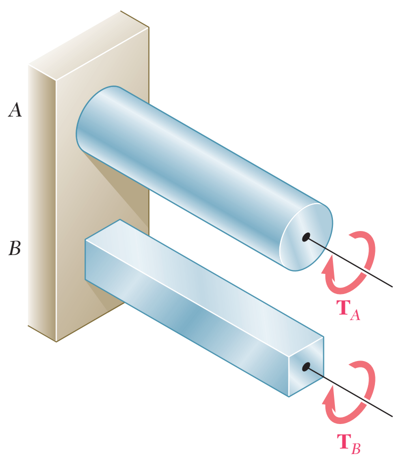


Fig. *P3.130*, P3.131 and P3.132

3.131 Shafts A and B are made of the same material and have the same cross-sectional area, but A has a circular cross section and B has a square cross section. Determine the ratio of the maximum torques T_A and T_B that can be safely applied to A and B, respectively.

3.132 Shafts A and B are made of the same material and have the same length and cross-sectional area, but A has a circular cross section and B has a square cross section. Determine the ratio of the maximum values of the angles ϕ_A and ϕ_B through which shafts A and B, respectively, can be twisted.

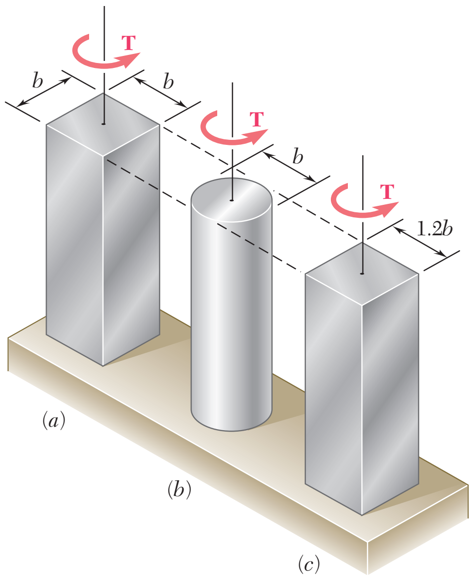


Fig. *P3.133* and P3.134

3.133 Each of the three aluminum bars shown is to be twisted through an angle of 2°. Knowing that $b = 30$ mm, $\tau_{all} = 50$ MPa, and $G = 27$ GPa, determine the shortest allowable length of each bar.

3.134 Each of the three steel bars is subjected to a torque as shown. Knowing that the allowable shearing stress is 8 ksi and that $b = 1.4$ in., determine the maximum torque **T** that can be applied to each bar.

3.135 A 36-kip · in. torque is applied to a 10-ft-long steel angle with an L8 × 8 × 1 cross section. From Appendix C we find that the thickness of the section is 1 in. and that its area is 15 in^2. Knowing that $G = 11.2 \times 10^6$ psi, determine (*a*) the maximum shearing stress along line *a-a*, (*b*) the angle of twist.

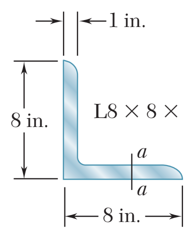


Fig. P3.135

3.136 A 3-m-long steel angle has an L203 × 152 × 12.7 cross section. From Appendix C we find that the thickness of the section is 12.7 mm and that its area is 4350 mm^2. Knowing that $\tau_{all} = 50$ MPa and that $G = 77.2$ GPa, and ignoring the effect of stress concentrations, determine (*a*) the largest torque **T** that can be applied, (*b*) the corresponding angle of twist.

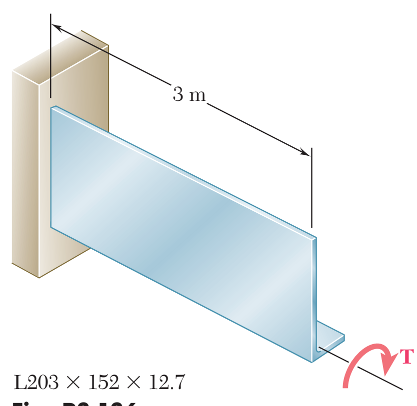


Fig. P3.136

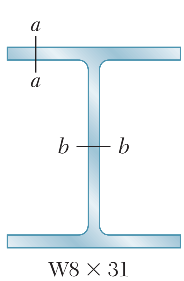


Fig. P3.137

3.137 An 8-ft-long steel member with a W8 × 31 cross section is subjected to a 5-kip · in. torque. The properties of the rolled-steel section are given in Appendix C. Knowing that $G = 11.2 \times 10^6$ psi, determine (*a*) the maximum shearing stress along line *a-a*, (*b*) the maximum shearing stress along line *b-b*, (*c*) the angle of twist. (*Hint*: consider the web and flanges separately and obtain a relation between the torques exerted on the web and a flange, respectively, by expressing that the resulting angles of twist are equal.)

3.138 A 4-m-long steel member has a W310 × 60 cross section. Knowing that $G = 77.2$ GPa and that the allowable shearing stress is 40 MPa, determine (*a*) the largest torque **T** that can be applied, (*b*) the corresponding angle of twist. Refer to Appendix C for the dimensions of the cross section and neglect the effect of stress concentrations. (See hint of Prob. 3.137.)

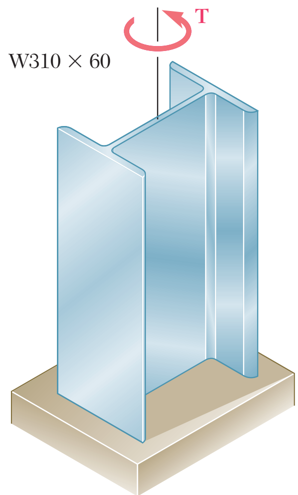


Fig. P3.138

3.139 A torque $T = 750$ kN · m is applied to the hollow shaft shown that has a uniform 8-mm wall thickness. Neglecting the effect of stress concentrations, determine the shearing stress at points *a* and *b*.

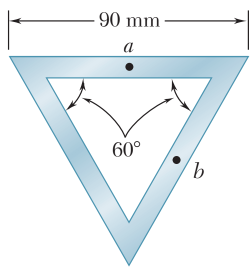


Fig. *P3.139*

3.140 A torque $T = 5$ kN · m is applied to a hollow shaft having the cross section shown. Neglecting the effect of stress concentrations, determine the shearing stress at points *a* and *b*.

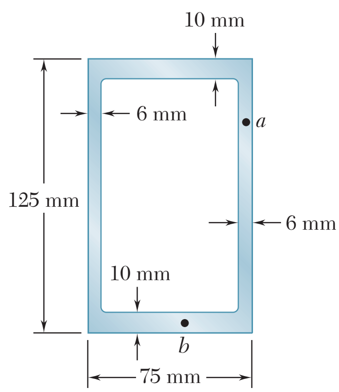


Fig. *P3.140*

3.141 A 90-N · m torque is applied to a hollow shaft having the cross section shown. Neglecting the effect of stress concentrations, determine the shearing stress at points *a* and *b*.

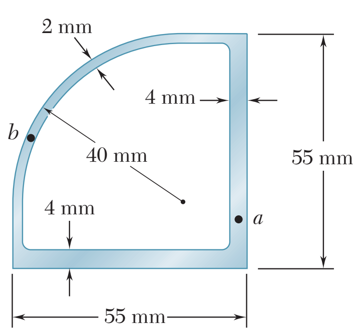


Fig. P3.141

3.142 A 5.6 kN · m-torque is applied to a hollow shaft having the cross section shown. Neglecting the effect of stress concentrations, determine the shearing stress at points *a* and *b*.

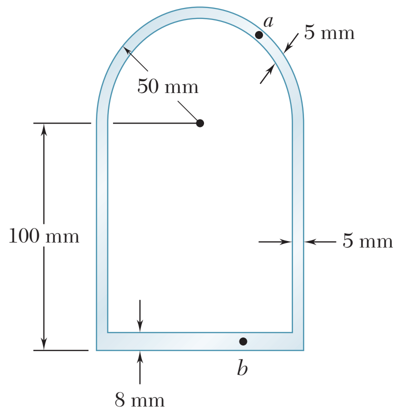


Fig. P3.142

3.143 A hollow member having the cross section shown is formed from sheet metal of 2-mm thickness. Knowing that the shearing stress must not exceed 3 MPa, determine the largest torque that can be applied to the member.

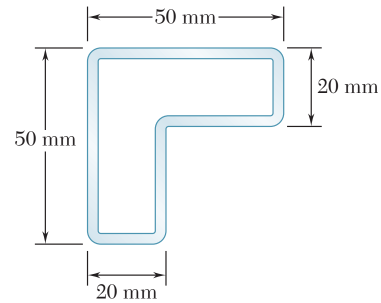


Fig. P3.143

3.144 A hollow brass shaft has the cross section shown. Knowing that the shearing stress must not exceed 12 ksi and neglecting the effect of stress concentrations, determine the largest torque that can be applied to the shaft.

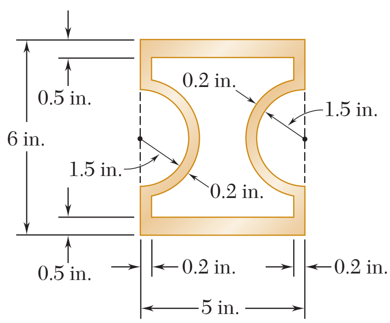


Fig. P3.144

***3.145* and 3.146** A hollow member having the cross section shown is to be formed from sheet metal of 0.06-in. thickness. Knowing that a 1250 lb · in.-torque will be applied to the member, determine the smallest dimension *d* that can be used if the shearing stress is not to exceed 750 psi.

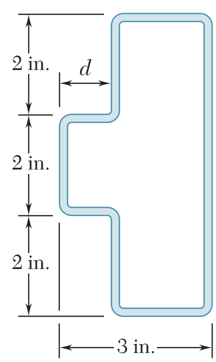


Fig. *P3.145*

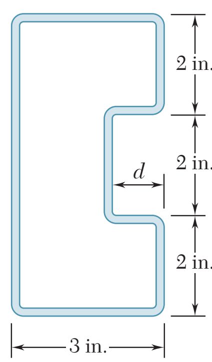


Fig. P3.146

3.147 A hollow cylindrical shaft was designed to have a uniform wall thickness of 0.1 in. Defective fabrication, however, resulted in the shaft having the cross section shown. Knowing that a 15 kip · in.-torque is applied to the shaft, determine the shearing stresses at points *a* and *b*.

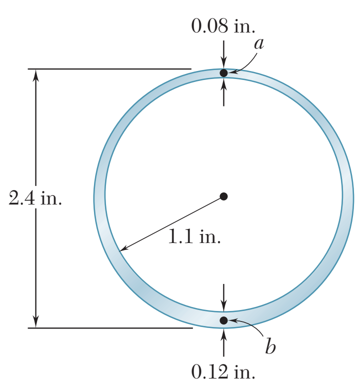


Fig. *P3.147*

3.148 A cooling tube having the cross section shown is formed from a sheet of stainless steel of 3-mm thickness. The radii $c_1 = 150$ mm and $c_2 = 100$ mm are measured to the center line of the sheet metal. Knowing that a torque of magnitude $T = 3$ kN · m is applied to the tube, determine (*a*) the maximum shearing stress in the tube, (*b*) the magnitude of the torque carried by the outer circular shell. Neglect the dimension of the small opening where the outer and inner shells are connected.

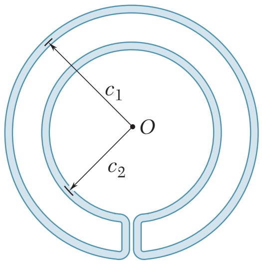


Fig. P3.148

3.149 A hollow cylindrical shaft of length L, mean radius c_m, and uniform thickness t is subjected to a torque of magnitude T. Consider, on the one hand, the values of the average shearing stress τ_{ave} and the angle of twist ϕ obtained from the elastic torsion formulas developed in Secs. 3.4 and 3.5 and, on the other hand, the corresponding values obtained from the formulas developed in Sec. 3.13 for thin-walled shafts. (*a*) Show that the relative error introduced by using the thin-walled-shaft formulas rather than the elastic torsion formulas is the same for τ_{ave} and ϕ and that the relative error is positive and proportional to the ratio t/c_m. (*b*) Compare the percent error corresponding to values of the ratio t/c_m of 0.1, 0.2, and 0.4.

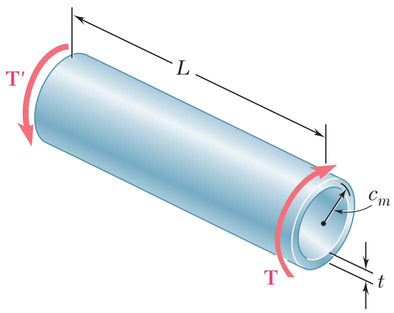
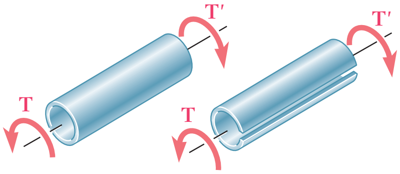


Fig. P3.149

3.150 Equal torques are applied to thin-walled tubes of the same length L, same thickness t, and same radius c. One of the tubes has been slit lengthwise as shown. Determine (*a*) the ratio τ_b/τ_a of the maximum shearing stresses in the tubes, (*b*) the ratio ϕ_b/ϕ_a of the angles of twist of the tubes.



(*a*) (*b*)

Fig. P3.150

REVIEW AND SUMMARY

This chapter was devoted to the analysis and design of *shafts* subjected to twisting couples, or *torques*. Except for the last two sections of the chapter, our discussion was limited to *circular shafts*.

Deformations in circular shafts

In a preliminary discussion [Sec. 3.2], it was pointed out that the distribution of stresses in the cross section of a circular shaft is *statically indeterminate*. The determination of these stresses, therefore, requires a prior analysis of the *deformations* occurring in the shaft [Sec. 3.3]. Having demonstrated that in a circular shaft subjected to torsion, *every cross section remains plane and undistorted,* we derived the following expression for the *shearing strain* in a small element with sides parallel and perpendicular to the axis of the shaft and at a distance ρ from that axis:

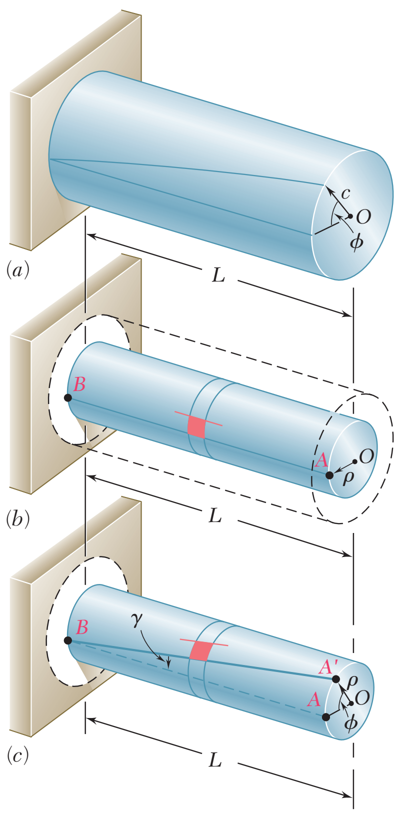


Fig. 3.57

$$\gamma = \frac{\rho\phi}{L} \tag{3.2}$$

where ϕ is the angle of twist for a length L of the shaft (Fig. 3.57). Equation (3.2) shows that the *shearing strain in a circular shaft varies linearly with the distance from the axis of the shaft.* It follows that the strain is maximum at the surface of the shaft, where ρ is equal to the radius c of the shaft. We wrote

$$\gamma_{\max} = \frac{c\phi}{L} \qquad \gamma = \frac{\rho}{c}\gamma_{\max} \tag{3.3, 4}$$

Shearing stresses in elastic range

Considering *shearing stresses* in a circular shaft within the elastic range [Sec. 3.4] and recalling Hooke's law for shearing stress and strain, $\tau = G\gamma$, we derived the relation

$$\tau = \frac{\rho}{c}\tau_{\max} \tag{3.6}$$

which shows that within the elastic range, the *shearing stress τ in a circular shaft also varies linearly with the distance from the axis of the shaft.* Equating the sum of the moments of the elementary forces exerted on any section of the shaft to the magnitude T of the torque applied to the shaft, we derived the *elastic torsion formulas*

$$\tau_{\max} = \frac{Tc}{J} \qquad \tau = \frac{T\rho}{J} \tag{3.9, 10}$$

where c is the radius of the cross section and J its centroidal polar moment of inertia. We noted that $J = \frac{1}{2}\pi c^4$ for a solid shaft and $J = \frac{1}{2}\pi(c_2^4 - c_1^4)$ for a hollow shaft of inner radius c_1 and outer radius c_2.

We noted that while the element *a* in Fig. 3.58 is in pure shear, the element *c* in the same figure is subjected to normal stresses of the same magnitude, Tc/J, two of the normal stresses being tensile and two compressive. This explains why in a torsion test ductile materials, which generally fail in shear, will break along a plane perpendicular to the axis of the specimen, while brittle materials, which are weaker in tension than in shear, will break along surfaces forming a 45° angle with that axis.

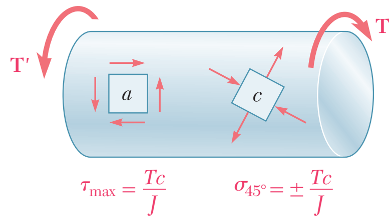


Fig. 3.58

Angle of twist

In Sec. 3.5, we found that within the elastic range, the angle of twist ϕ of a circular shaft is proportional to the torque T applied to it (Fig. 3.59). Expressing ϕ in *radians*, we wrote

$$\phi = \frac{TL}{JG} \tag{3.16}$$

where L = length of shaft
J = polar moment of inertia of cross section
G = modulus of rigidity of material

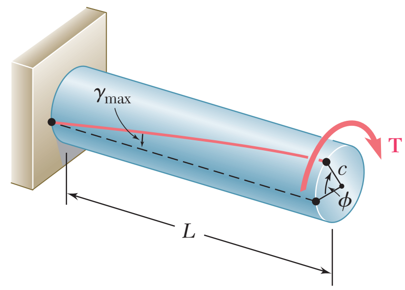


Fig. 3.59

If the shaft is subjected to torques at locations other than its ends or consists of several parts of various cross sections and possibly of different materials, the angle of twist of the shaft must be expressed as the *algebraic sum* of the angles of twist of its component parts [Sample Prob. 3.3]:

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i} \tag{3.17}$$

We observed that when both ends of a shaft *BE* rotate (Fig. 3.60), the angle of twist of the shaft is equal to the *difference* between the angles of rotation ϕ_B and ϕ_E of its ends. We also noted that when two shafts *AD* and *BE* are connected by gears *A* and *B*, the torques applied, respectively, by gear *A* on shaft *AD* and by gear *B* on shaft *BE* are *directly proportional* to the radii r_A and r_B of the two gears—since the forces applied on each other by the gear teeth at *C* are equal and opposite. On the other hand, the angles ϕ_A and ϕ_B through which the two gears rotate are *inversely proportional* to r_A and r_B—since the arcs CC' and CC'' described by the gear teeth are equal [Example 3.04 and Sample Prob. 3.4].

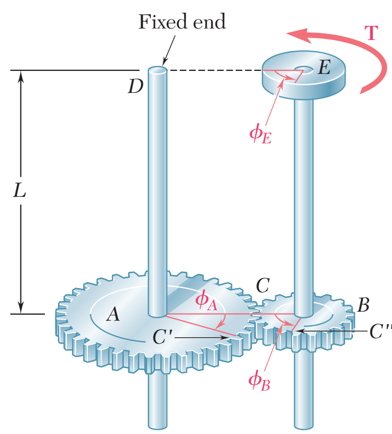


Fig. 3.60

Statically indeterminate shafts

If the reactions at the supports of a shaft or the internal torques cannot be determined from statics alone, the shaft is said to be *statically indeterminate* [Sec. 3.6]. The equilibrium equations obtained from free-body diagrams must then be complemented by relations involving the deformations of the shaft and obtained from the geometry of the problem [Example 3.05, Sample Prob. 3.5].

Transmission shafts

In Sec. 3.7, we discussed the *design of transmission shafts*. We first observed that the power P transmitted by a shaft is

$$P = 2\pi f T \tag{3.20}$$

where T is the torque exerted at each end of the shaft and f the *frequency* or speed of rotation of the shaft. The unit of frequency is the revolution per second (s^{-1}) or *hertz* (Hz). If SI units are used, T is expressed in newton-meters ($N \cdot m$) and P in *watts* (W). If U.S. customary units are used, T is expressed in lb · ft or lb · in., and P in ft · lb/s or in · lb/s; the power may then be converted into *horsepower* (hp) through the use of the relation

$$1 \text{ hp} = 550 \text{ ft} \cdot \text{lb/s} = 6600 \text{ in} \cdot \text{lb/s}$$

To design a shaft to transmit a given power P at a frequency f, you should first solve Eq. (3.20) for T. Carrying this value and the maximum allowable value of τ for the material used into the elastic formula (3.9), you will obtain the corresponding value of the parameter J/c, from which the required diameter of the shaft may be calculated [Examples 3.06 and 3.07].

Stress concentrations

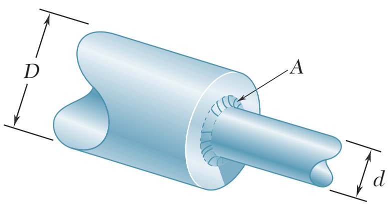


Fig. 3.61

In Sec. 3.8, we discussed *stress concentrations* in circular shafts. We saw that the stress concentrations resulting from an abrupt change in the diameter of a shaft can be reduced through the use of a *fillet* (Fig. 3.61). The maximum value of the shearing stress at the fillet is

$$\tau_{\max} = K\frac{Tc}{J} \tag{3.25}$$

where the stress Tc/J is computed for the smaller-diameter shaft, and where K is a stress-concentration factor. Values of K were plotted in Fig. 3.29 on p. 179 against the ratio r/d, where r is the radius of the fillet, for various values of D/d.

Plastic deformations

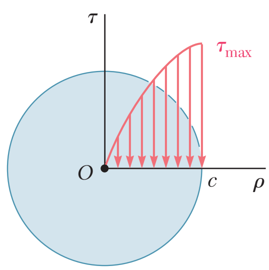


Fig. 3.62

Sections 3.9 through 3.11 were devoted to the discussion of *plastic deformations* and *residual stresses* in circular shafts. We first recalled that even when Hooke's law does not apply, the distribution of *strains* in a circular shaft is always linear [Sec. 3.9]. If the shearing-stress-strain diagram for the material is known, it is then possible to plot the shearing stress τ against the distance ρ from the axis of the shaft for any given value of $\tau_{\max}$ (Fig. 3.62). Summing the contributions to the torque of annular elements of radius ρ and thickness $d\rho$, we expressed the torque T as

$$T = \int_0^c \rho\tau(2\pi\rho\, d\rho) = 2\pi \int_0^c \rho^2 \tau\, d\rho \tag{3.26}$$

where τ is the function of ρ plotted in Fig. 3.62.

Modulus of rupture

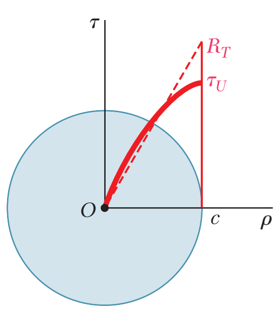


Fig. 3.63

An important value of the torque is the ultimate torque T_U which causes failure of the shaft. This value can be determined, either experimentally, or by carrying out the computations indicated above with $\tau_{\max}$ chosen equal to the ultimate shearing stress τ_U of the material. From T_U, and assuming a linear stress distribution (Fig 3.63), we determined the corresponding fictitious stress $R_T = T_U c/J$, known as the *modulus of rupture in torsion* of the given material.

Considering the idealized case of a *solid circular shaft* made of an *elastoplastic material* [Sec. 3.10], we first noted that, as long as

τ_{max} does not exceed the yield strength τ_Y of the material, the stress distribution across a section of the shaft is linear (Fig. 3.64*a*). The torque T_Y corresponding to $\tau_{max} = \tau_Y$ (Fig. 3.64*b*) is known as the *maximum elastic torque;* for a solid circular shaft of radius c, we have

$$T_Y = \tfrac{1}{2}\pi c^3 \tau_Y \tag{3.29}$$

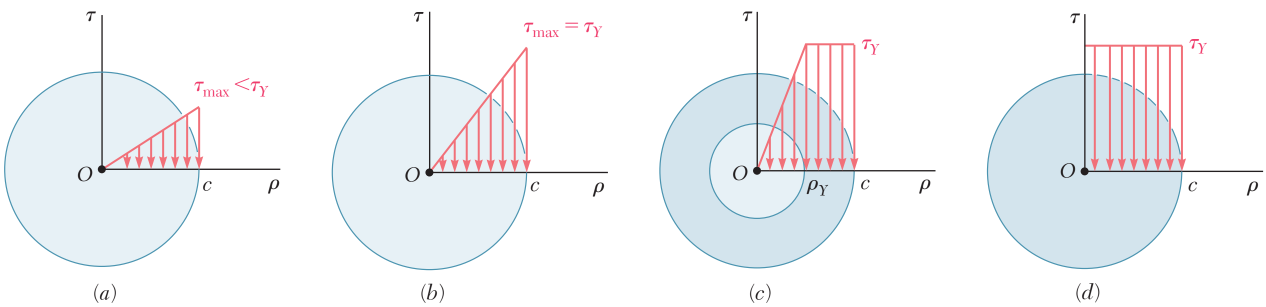


Fig. 3.64

Solid shaft of elastoplastic material

As the torque increases, a plastic region develops in the shaft around an elastic core of radius ρ_Y. The torque T corresponding to a given value of ρ_Y was found to be

$$T = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\rho_Y^3}{c^3}\right) \tag{3.32}$$

We noted that as ρ_Y approaches zero, the torque approaches a limiting value T_p, called the *plastic torque* of the shaft considered:

$$T_p = \frac{4}{3}T_Y \tag{3.33}$$

Plotting the torque T against the angle of twist ϕ of a solid circular shaft (Fig. 3.65), we obtained the segment of straight line $0Y$ defined by Eq. (3.16), followed by a curve approaching the straight line $T = T_p$ and defined by the equation

$$T = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\phi_Y^3}{\phi^3}\right) \tag{3.37}$$

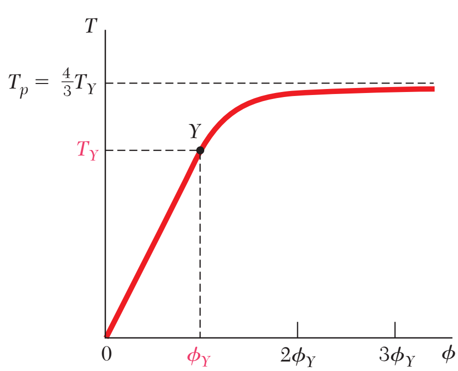


Fig. 3.65

Permanent deformation. Residual stresses

Loading a circular shaft beyond the onset of yield and unloading it [Sec. 3.11] results in a *permanent deformation* characterized by the angle of twist $\phi_p = \phi - \phi'$, where ϕ corresponds to the loading phase described in the previous paragraph, and ϕ' to the unloading phase represented by a straight line in Fig. 3.66. There will also be *residual stresses* in the shaft, which can be determined by adding the maximum stresses reached during the loading phase and the reverse stresses corresponding to the unloading phase [Example 3.09].

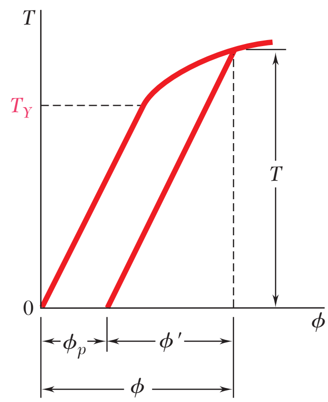


Fig. 3.66

Torsion of noncircular members

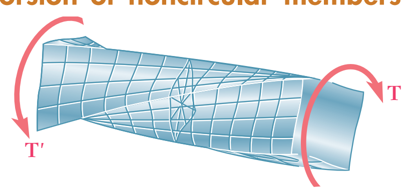


Fig. 3.67

The last two sections of the chapter dealt with the torsion of *noncircular members.* We first recalled that the derivation of the formulas for the distribution of strain and stress in circular shafts was based on the fact that due to the axisymmetry of these members, cross sections remain plane and undistorted. Since this property does not hold for noncircular members, such as the square bar of Fig. 3.67, none of the formulas derived earlier can be used in their analysis [Sec. 3.12].

Bars of rectangular cross section

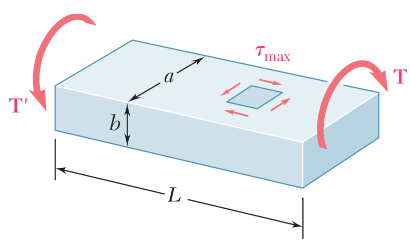


Fig. 3.68

It was indicated in Sec. 3.12 that in the case of straight bars with a *uniform rectangular cross section* (Fig. 3.68), the maximum shearing stress occurs along the center line of the *wider* face of the bar. Formulas for the maximum shearing stress and the angle of twist were given without proof. The *membrane analogy* for visualizing the distribution of stresses in a noncircular member was also discussed.

Thin-walled hollow shafts

We next analyzed the distribution of stresses in *noncircular thin-walled hollow shafts* [Sec. 3.13]. We saw that the shearing stress is parallel to the wall surface and varies both across the wall and along the wall cross section. Denoting by τ the average value of the shearing stress computed across the wall at a given point of the cross section, and by t the thickness of the wall at that point (Fig. 3.69), we showed that the product $q = \tau t$, called the *shear flow,* is constant along the cross section.

Furthermore, denoting by T the torque applied to the hollow shaft and by $\mathcal{a}$ the area bounded by the center line of the wall cross section, we expressed as follows the average shearing stress τ at any given point of the cross section:

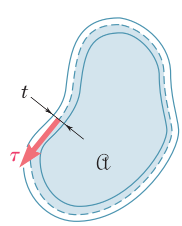


Fig. 3.69

$$\tau = \frac{T}{2t\mathcal{a}} \tag{3.53}$$

REVIEW PROBLEMS

3.151 The ship at A has just started to drill for oil on the ocean floor at a depth of 5000 ft. Knowing that the top of the 8-in.-diameter steel drill pipe ($G = 11.2 \times 10^6$ psi) rotates through two complete revolutions before the drill bit at B starts to operate, determine the maximum shearing stress caused in the pipe by torsion.

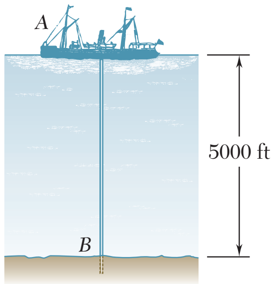


Fig. P3.151

3.152 The shafts of the pulley assembly shown are to be designed. Knowing that the allowable shearing stress in each shaft is 8.5 ksi, determine the smallest allowable diameter of (*a*) shaft AB, (*b*) shaft BC.

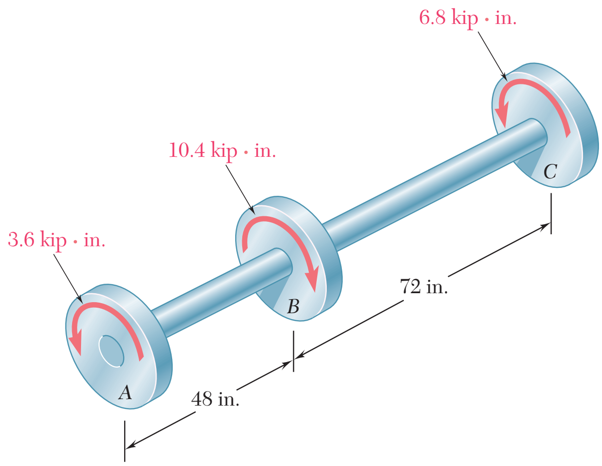


Fig. *P3.152*

3.153 A steel pipe of 12-in. outer diameter is fabricated from $\frac{1}{4}$-in.-thick plate by welding along a helix that forms an angle of 45° with a plane perpendicular to the axis of the pipe. Knowing that the maximum allowable tensile stress in the weld is 12 ksi, determine the largest torque that can be applied to the pipe.

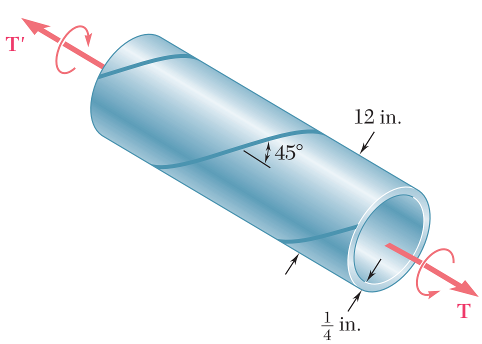


Fig. P3.153

3.154 For the gear train shown, the diameters of the three solid shafts are:

$$d_{AB} = 20 \text{ mm} \qquad d_{CD} = 25 \text{ mm} \qquad d_{EF} = 40 \text{ mm}$$

Knowing that for each shaft the allowable shearing stress is 60 MPa, determine the largest torque **T** that can be applied.

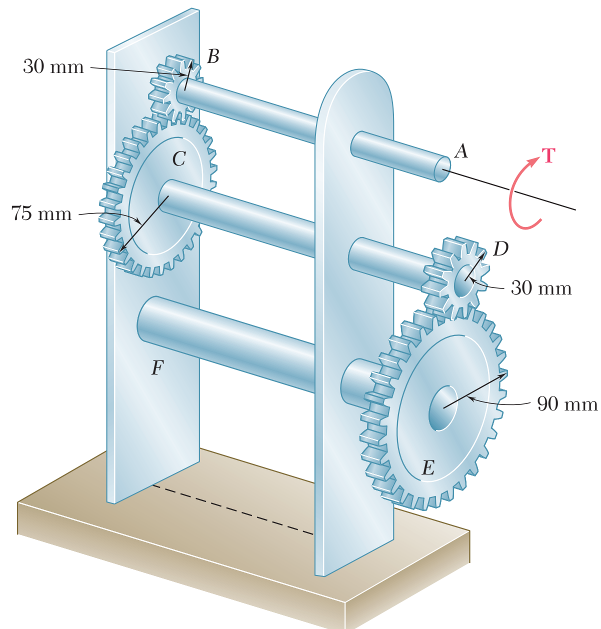


Fig. *P3.154*

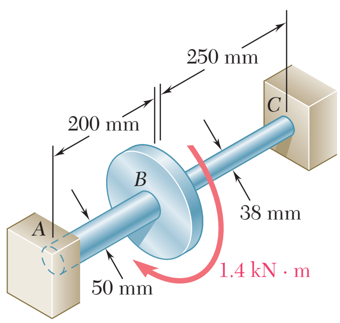


Fig. P3.155

3.155 Two solid steel shafts ($G = 77.2$ GPa) are connected to a coupling disk B and to fixed supports at A and C. For the loading shown, determine (*a*) the reaction at each support, (*b*) the maximum shearing stress in shaft AB, (*c*) the maximum shearing stress in shaft BC.

3.156 In the bevel-gear system shown, $\alpha = 18.43°$. Knowing that the allowable shearing stress is 8 ksi in each shaft and that the system is in equilibrium, determine the largest torque $\mathbf{T}_A$ that can be applied at A.

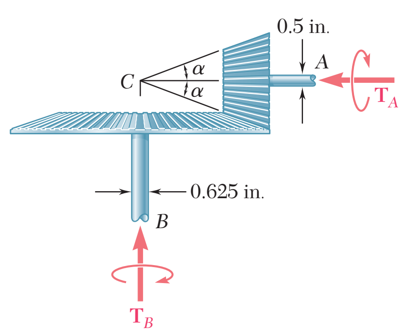


Fig. P3.156

3.157 Three solid shafts, each of $\frac{3}{4}$-in. diameter, are connected by the gears as shown. Knowing that $G = 11.2 \times 10^6$ psi, determine (*a*) the angle through which end *A* of shaft *AB* rotates, (*b*) the angle through which end *E* of shaft *EF* rotates.

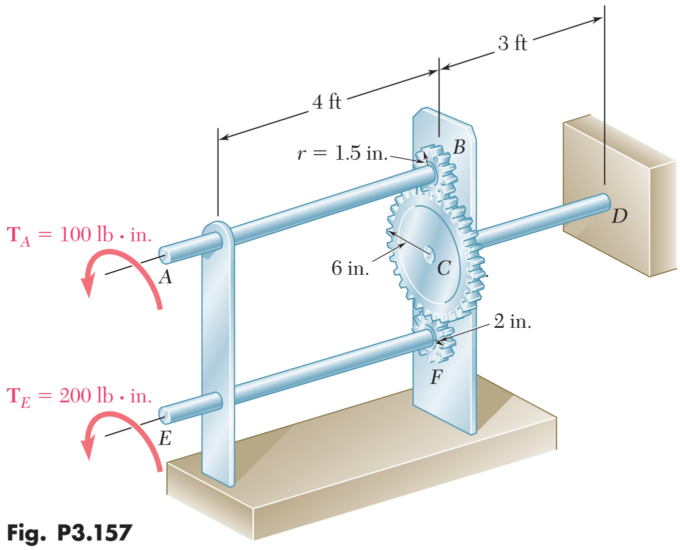


Fig. P3.157

3.158 The design specifications of a 1.2-m-long solid transmission shaft require that the angle of twist of the shaft not exceed 4° when a torque of 750 N · m is applied. Determine the required diameter of the shaft, knowing that the shaft is made of a steel with an allowable shearing stress of 90 MPa and a modulus of rigidity of 77.2 GPa.

3.159 The stepped shaft shown rotates at 450 rpm. Knowing that $r = 0.5$ in., determine the maximum power that can be transmitted without exceeding an allowable shearing stress of 7500 psi.

3.160 A 750-N · m torque is applied to a hollow shaft having the cross section shown and a uniform 6-mm wall thickness. Neglecting the effect of stress concentrations, determine the shearing stress at points *a* and *b*.

3.161 The composite shaft shown is twisted by applying a torque **T** at end *A*. Knowing that the maximum shearing stress in the steel shell is 150 MPa, determine the corresponding maximum shearing stress in the aluminum core. Use $G = 77.2$ GPa for steel and $G = 27$ GPa for aluminum.

3.162 Two solid brass rods *AB* and *CD* are brazed to a brass sleeve *EF*. Determine the ratio d_2/d_1 for which the same maximum shearing stress occurs in the rods and in the sleeve.

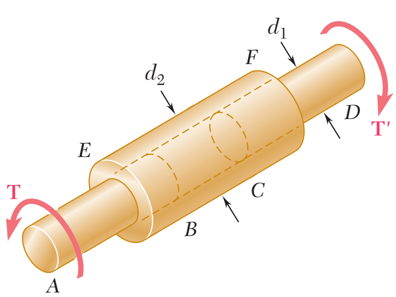


Fig. P3.162

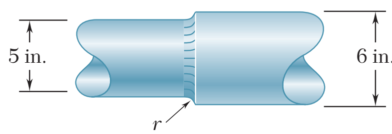


Fig. *P3.159*

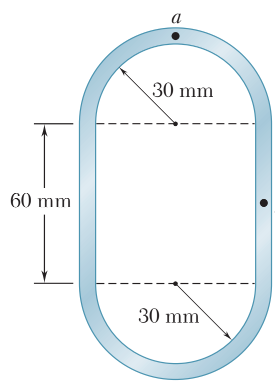


Fig. P3.160

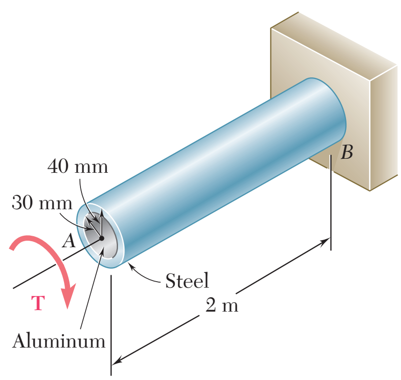


Fig. *P3.161*

COMPUTER PROBLEMS

The following problems are designed to be solved with a computer. Write each program so that it can be used with either SI or U.S. customary units.

3.C1 Shaft AB consists of n homogeneous cylindrical elements, which can be solid or hollow. Its end A is fixed, while its end B is free, and it is subjected to the loading shown. The length of element i is denoted by L_i, its outer diameter by OD_i, its inner diameter by ID_i, its modulus of rigidity by G_i, and the torque applied to its right end by $\mathbf{T}_i$, the magnitude T_i of this torque being assumed to be positive if $\mathbf{T}_i$ is observed as counterclockwise from end B and negative otherwise. (Note that $ID_i = 0$ if the element is solid.) (*a*) Write a computer program that can be used to determine the maximum shearing stress in each element, the angle of twist of each element, and the angle of twist of the entire shaft. (*b*) Use this program to solve Probs. 3.35 and 3.38.

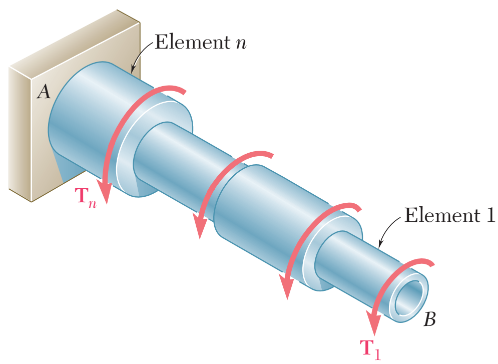


Fig. P3.C1

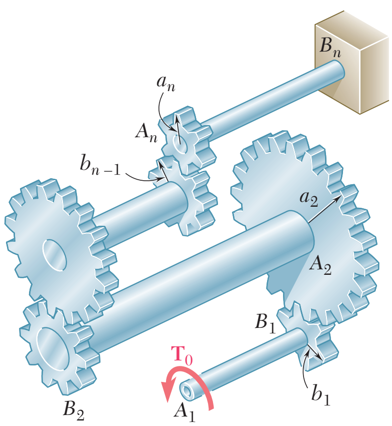


Fig. P3.C2

3.C2 The assembly shown consists of n cylindrical shafts, which can be solid or hollow, connected by gears and supported by brackets (not shown). End A_1 of the first shaft is free and is subjected to a torque $\mathbf{T}_0$, while end B_n of the last shaft is fixed. The length of shaft A_iB_i is denoted by L_i, its outer diameter by OD_i, its inner diameter by ID_i, and its modulus of rigidity by G_i. (Note that $ID_i = 0$ if the element is solid.) The radius of gear A_i is denoted by a_i, and the radius of gear B_i by b_i. (*a*) Write a computer program that can be used to determine the maximum shearing stress in each shaft, the angle of twist of each shaft, and the angle through which end A_i rotates. (*b*) Use this program to solve Probs. 3.41 and 3.44.

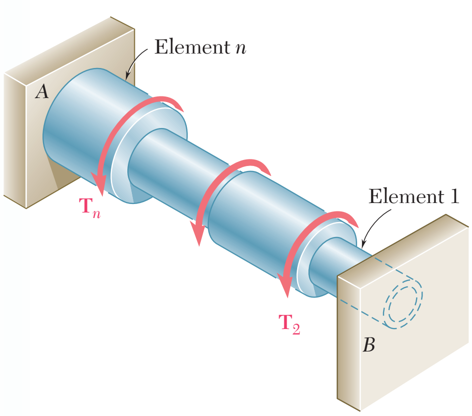


Fig. P3.C3

3.C3 Shaft AB consists of n homogeneous cylindrical elements, which can be solid or hollow. Both of its ends are fixed, and it is subjected to the loading shown. The length of element i is denoted by L_i, its outer diameter by OD_i, its inner diameter by ID_i, its modulus of rigidity by G_i, and the torque applied to its right end by T_i, the magnitude T_i of this torque being assumed to be positive if $\mathbf{T}_i$ is observed as counterclockwise from end B and negative otherwise. Note that $ID_i = 0$ if the element is solid and also that $T_1 = 0$. Write a computer program that can be used to determine the reactions at A and B, the maximum shearing stress in each element, and the angle of twist of each element. Use this program (*a*) to solve Prob. 3.155, (*b*) to determine the maximum shearing stress in the shaft of Example 3.05.

3.C4 The homogeneous, solid cylindrical shaft AB has a length L, a diameter d, a modulus of rigidity G, and a yield strength τ_Y. It is subjected to a torque $\mathbf{T}$ that is gradually increased from zero until the angle of twist of the shaft has reached a maximum value ϕ_m and then decreased back to zero. (a) Write a computer program that, for each of 16 values of ϕ_m equally spaced over a range extending from 0 to a value 3 times as large as the angle of twist at the onset of yield, can be used to determine the maximum value T_m of the torque, the radius of the elastic core, the maximum shearing stress, the permanent twist, and the residual shearing stress both at the surface of the shaft and at the interface of the elastic core and the plastic region. (b) Use this program to obtain approximate answers to Probs. 3.114, 3.115, 3.116.

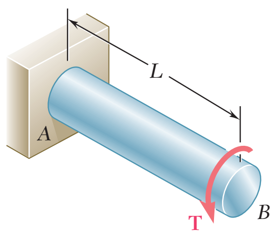


Fig. P3.C4

3.C5 The exact expression is given in Prob. 3.61 for the angle of twist of the solid tapered shaft AB when a torque $\mathbf{T}$ is applied as shown. Derive an approximate expression for the angle of twist by replacing the tapered shaft by n cylindrical shafts of equal length and of radius $r_i = (n + i - \frac{1}{2})(c/n)$, where $i = 1, 2, \ldots, n$. Using for T, L, G, and c values of your choice, determine the percentage error in the approximate expression when (a) $n = 4$, (b) $n = 8$, (c) $n = 20$, (d) $n = 100$.

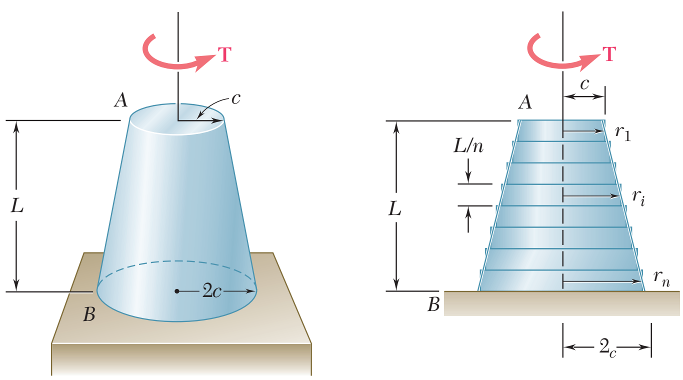


Fig. P3.C5

3.C6 A torque $\mathbf{T}$ is applied as shown to the long, hollow, tapered shaft AB of uniform thickness t. Derive an approximate expression for the angle of twist by replacing the tapered shaft by n cylindrical rings of equal length and of radius $r_i = (n + i - \frac{1}{2})(c/n)$, where $i = 1, 2, \ldots, n$. Using for T, L, G, c, and t values of your choice, determine the percentage error in the approximate expression when (a) $n = 4$, (b) $n = 8$, (c) $n = 20$, (d) $n = 100$.

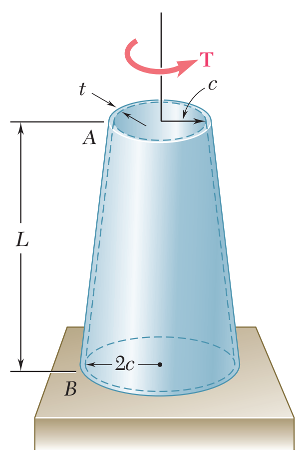


Fig. P3.C6

The athlete shown holds the barbell with his hands placed at equal distances from the weights. This results in pure bending in the center portion of the bar. The normal stresses and the curvature resulting from pure bending will be determined in this chapter.

CHAPTER 4

Pure Bending

Chapter 4 Pure Bending

4.1 Introduction
4.2 Symmetric Member in Pure Bending
4.3 Deformations in a Symmetric Member in Pure Bending
4.4 Stresses and Deformations in the Elastic Range
4.5 Deformations in a Transverse Cross Section
4.6 Bending of Members Made of Several Materials
4.7 Stress Concentrations
*4.8 Plastic Deformations
*4.9 Members Made of Elastoplastic Material
*4.10 Plastic Deformations of Members with a Single Plane of Symmetry
*4.11 Residual Stresses
4.12 Eccentric Axial Loading in a Plane of Symmetry
4.13 Unsymmetric Bending
4.14 General Case of Eccentric Axial Loading
*4.15 Bending of Curved Members

4.1 INTRODUCTION

In the preceding chapters you studied how to determine the stresses in prismatic members subjected to axial loads or to twisting couples. In this chapter and in the following two you will analyze the stresses and strains in prismatic members subjected to *bending*. Bending is a major concept used in the design of many machine and structural components, such as beams and girders.

This chapter will be devoted to the analysis of prismatic members subjected to equal and opposite couples **M** and **M′** acting in the same longitudinal plane. Such members are said to be in *pure bending*. In most of the chapter, the members will be assumed to possess a plane of symmetry and the couples **M** and **M′** to be acting in that plane (Fig. 4.1).

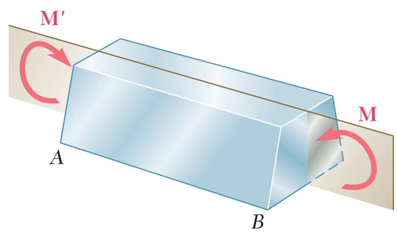


Fig. 4.1 Member in pure bending.

An example of pure bending is provided by the bar of a typical barbell as it is held overhead by a weight lifter as shown in the opening photo for this chapter. The bar carries equal weights at equal distances from the hands of the weight lifter. Because of the symmetry of the free-body diagram of the bar (Fig. 4.2*a*), the reactions at the hands must be equal and opposite to the weights. Therefore, as far as the middle portion *CD* of the bar is concerned, the weights and the reactions can be replaced by two equal and opposite 960-lb · in. couples (Fig. 4.2*b*), showing that the middle portion of the bar is in pure bending. A similar analysis of the axle of a small sport buggy (Photo 4.1) would show that, between the two points where it is attached to the frame, the axle is in pure bending.

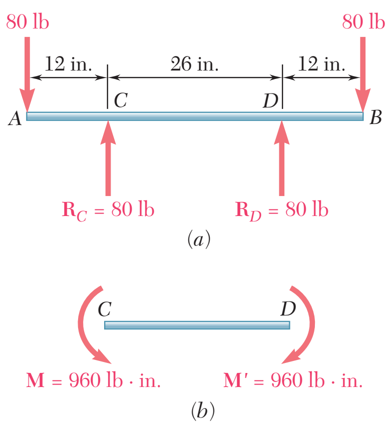


Fig. 4.2 Beam in which portion *CD* is in pure bending.

Photo 4.1 For the sport buggy shown, the center portion of the rear axle is in pure bending.

As interesting as the direct applications of pure bending may be, devoting an entire chapter to its study would not be justified if it were not for the fact that the results obtained will be used in the analysis of other types of loadings as well, such as *eccentric axial loadings* and *transverse loadings.*

Photo 4.2 shows a 12-in. steel bar clamp used to exert 150-lb forces on two pieces of lumber as they are being glued together. Figure 4.3*a* shows the equal and opposite forces exerted by the lumber on the clamp. These forces result in an *eccentric loading* of the straight portion of the clamp. In Fig. 4.3*b* a section CC' has been passed through the clamp and a free-body diagram has been drawn of the upper half of the clamp, from which we conclude that the internal forces in the section are equivalent to a 150-lb axial tensile force **P** and a 750-lb · in. couple **M.** We can thus combine our knowledge of the stresses under a *centric* load and the results of our forthcoming analysis of stresses in pure bending to obtain the distribution of stresses under an *eccentric* load. This will be further discussed in Sec. 4.12.

Photo 4.2 Clamp used to glue lumber pieces together.

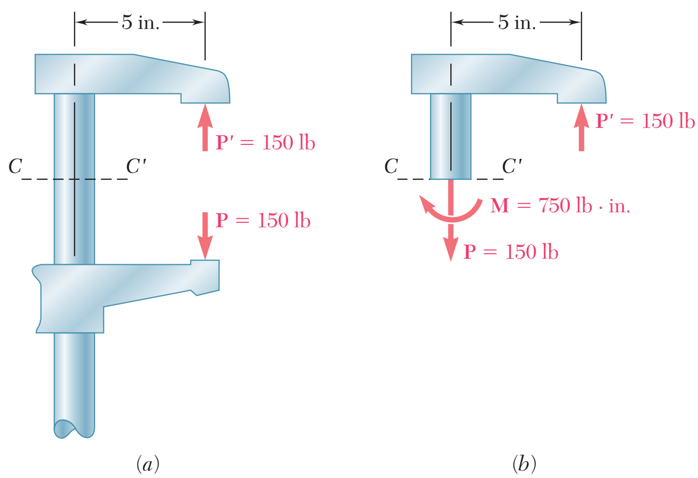


Fig. 4.3 Forces exerted on clamp.

The study of pure bending will also play an essential role in the study of beams, i.e., the study of prismatic members subjected to various types of *transverse loads*. Consider, for instance, a cantilever beam AB supporting a concentrated load **P** at its free end (Fig. 4.4*a*). If we pass a section through C at a distance x from A, we observe from the free-body diagram of AC (Fig. 4.4*b*) that the internal forces in the section consist of a force $\mathbf{P}'$ equal and opposite to **P** and a couple **M** of magnitude $M = Px$. The distribution of normal stresses in the section can be obtained from the couple **M** as if the beam were in pure bending. On the other hand, the shearing stresses in the section depend on the force $\mathbf{P}'$, and you will learn in Chap. 6 how to determine their distribution over a given section.

The first part of the chapter is devoted to the analysis of the stresses and deformations caused by pure bending in a homogeneous member possessing a plane of symmetry and made of a material following Hooke's law. In a preliminary discussion of the stresses due to bending (Sec. 4.2), the methods of statics will be used to derive

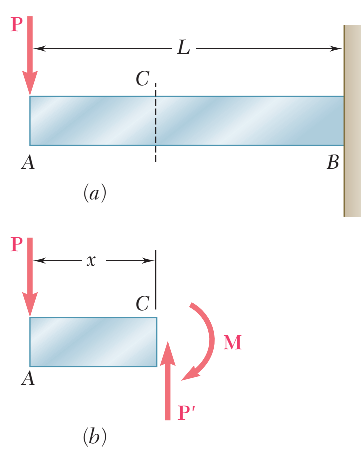


Fig. 4.4 Cantilever beam, not in pure bending.

three fundamental equations which must be satisfied by the normal stresses in any given cross section of the member. In Sec. 4.3, it will be proved that *transverse sections remain plane* in a member subjected to pure bending, while in Sec. 4.4 formulas will be developed that can be used to determine the *normal stresses,* as well as the *radius of curvature* for that member within the elastic range.

In Sec. 4.6, you will study the stresses and deformations in *composite members* made of more than one material, such as reinforced-concrete beams, which utilize the best features of steel and concrete and are extensively used in the construction of buildings and bridges. You will learn to draw a *transformed section* representing the section of a member made of a homogeneous material that undergoes the same deformations as the composite member under the same loading. The transformed section will be used to find the stresses and deformations in the original composite member. Section 4.7 is devoted to the determination of the *stress concentrations* occurring at locations where the cross section of a member undergoes a sudden change.

In the next part of the chapter you will study *plastic deformations* in bending, i.e., the deformations of members which are made of a material which does not follow Hooke's law and are subjected to bending. After a general discussion of the deformations of such members (Sec. 4.8), you will investigate the stresses and deformations in members made of an *elastoplastic material* (Sec. 4.9). Starting with the *maximum elastic moment* M_Y, which corresponds to the onset of yield, you will consider the effects of increasingly larger moments until the *plastic moment* M_p is reached, at which time the member has yielded fully. You will also learn to determine the *permanent deformations* and *residual stresses* that result from such loadings (Sec. 4.11). It should be noted that during the past half-century the elastoplastic property of steel has been widely used to produce designs resulting in both improved safety and economy.

In Sec. 4.12, you will learn to analyze an *eccentric axial loading* in a plane of symmetry, such as the one shown in Fig. 4.4, by superposing the stresses due to pure bending and the stresses due to a centric axial loading.

Your study of the bending of prismatic members will conclude with the analysis of *unsymmetric bending* (Sec. 4.13), and the study of the general case of *eccentric axial loading* (Sec. 4.14). The final section of the chapter will be devoted to the determination of the stresses in *curved members* (Sec. 4.15).

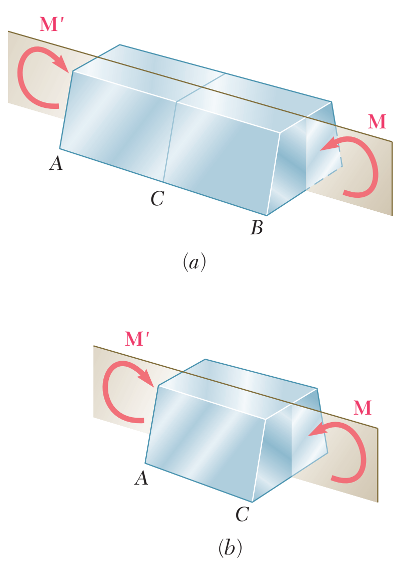


Fig. 4.5 Member in pure bending.

4.2 SYMMETRIC MEMBER IN PURE BENDING

Consider a prismatic member AB possessing a plane of symmetry and subjected to equal and opposite couples **M** and **M′** acting in that plane (Fig. 4.5*a*). We observe that if a section is passed through the member AB at some arbitrary point C, the conditions of equilibrium of the portion AC of the member require that the internal forces in the section be equivalent to the couple **M** (Fig. 4.5*b*). Thus, the internal forces in any cross section of a symmetric member in pure bending are equivalent to a couple. The moment M of that couple is referred to as the *bending moment* in the section. Following the usual convention, a positive sign will be assigned to M when the member is bent as shown in Fig. 4.5a, i.e., when the concavity of the beam faces upward, and a negative sign otherwise.

Denoting by σ_x the normal stress at a given point of the cross section and by τ_{xy} and τ_{xz} the components of the shearing stress, we express that the system of the elementary internal forces exerted on the section is equivalent to the couple **M** (Fig. 4.6).

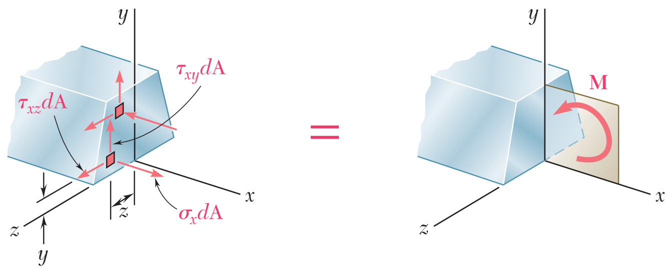


Fig. 4.6

We recall from statics that a couple **M** actually consists of two equal and opposite forces. The sum of the components of these forces in any direction is therefore equal to zero. Moreover, the moment of the couple is the same about *any* axis perpendicular to its plane, and is zero about any axis contained in that plane. Selecting arbitrarily the z axis as shown in Fig. 4.6, we express the equivalence of the elementary internal forces and of the couple **M** by writing that the sums of the components and of the moments of the elementary forces are equal to the corresponding components and moments of the couple **M:**

$$x \text{ components:} \qquad \int \sigma_x \, dA = 0 \tag{4.1}$$

$$\text{moments about } y \text{ axis:} \qquad \int z\sigma_x \, dA = 0 \tag{4.2}$$

$$\text{moments about } z \text{ axis:} \qquad \int (-y\sigma_x \, dA) = M \tag{4.3}$$

Three additional equations could be obtained by setting equal to zero the sums of the y components, z components, and moments about the x axis, but these equations would involve only the components of the shearing stress and, as you will see in the next section, the components of the shearing stress are both equal to zero.

Two remarks should be made at this point: (1) The minus sign in Eq. (4.3) is due to the fact that a tensile stress ($\sigma_x > 0$) leads to a negative moment (clockwise) of the normal force $\sigma_x \, dA$ about the z axis. (2) Equation (4.2) could have been anticipated, since the application of couples in the plane of symmetry of member AB will result in a distribution of normal stresses that is symmetric about the y axis.

Once more, we note that the actual distribution of stresses in a given cross section cannot be determined from statics alone. It is *statically indeterminate* and may be obtained only by analyzing the *deformations* produced in the member.

4.3 DEFORMATIONS IN A SYMMETRIC MEMBER IN PURE BENDING

Let us now analyze the deformations of a prismatic member possessing a plane of symmetry and subjected at its ends to equal and opposite couples **M** and **M′** acting in the plane of symmetry. The member will bend under the action of the couples, but will remain symmetric with respect to that plane (Fig. 4.7). Moreover, since the bending moment M is the same in any cross section, the member will bend uniformly. Thus, the line AB along which the upper face of the member intersects the plane of the couples will have a constant curvature. In other words, the line AB, which was originally a straight line, will be transformed into a circle of center C, and so will the line $A'B'$ (not shown in the figure) along which the lower face of the member intersects the plane of symmetry. We also note that the line AB will decrease in length when the member is bent as shown in the figure, i.e., when $M > 0$, while $A'B'$ will become longer.

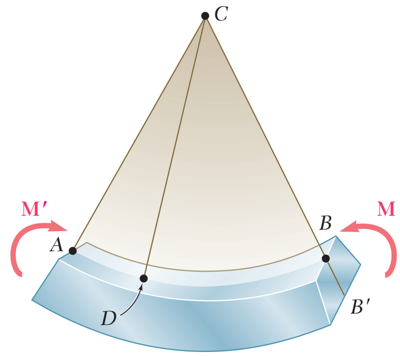


Fig. 4.7 Deformation of member in pure bending.

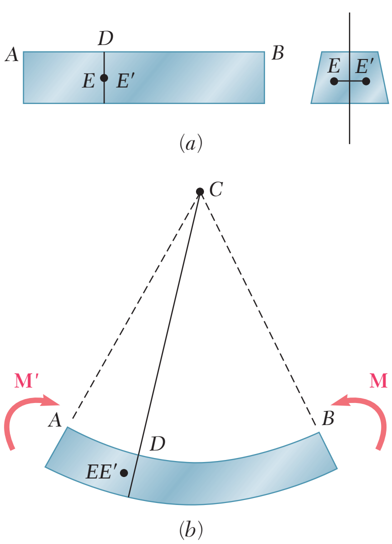
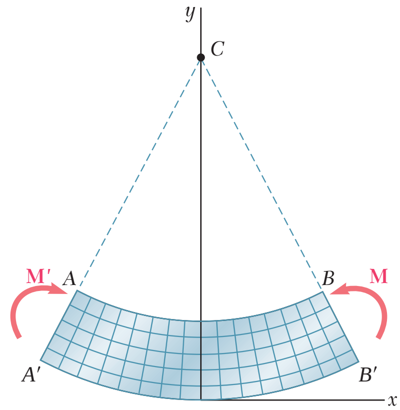


Fig. 4.8

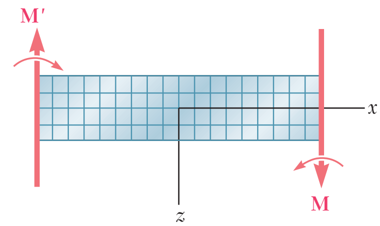
Next we will prove that any cross section perpendicular to the axis of the member remains plane, and that the plane of the section passes through C. If this were not the case, we could find a point E of the original section through D (Fig. 4.8*a*) which, after the member has been bent, would *not* lie in the plane perpendicular to the plane of symmetry that contains line CD (Fig. 4.8*b*). But, because of the symmetry of the member, there would be another point E' that would be transformed exactly in the same way. Let us assume that, after the beam has been bent, both points would be located to the left of the plane defined by CD, as shown in Fig. 4.8*b*. Since the bending moment M is the same throughout the member, a similar situation would prevail in any other cross section, and the points corresponding to E and E' would also move to the left. Thus, an observer at A would conclude that the loading causes the points E and E' in the various cross sections to move forward (toward the observer). But an observer at B, to whom the loading looks the same, and who observes the points E and E' in the same positions (except that they are now inverted) would reach the opposite conclusion. This inconsistency leads us to conclude that E and E' will lie in the plane defined by CD and, therefore, that the section remains plane and passes through C. We should note, however, that this discussion does not rule out the possibility of deformations *within* the plane of the section (see Sec. 4.5).

Suppose that the member is divided into a large number of small cubic elements with faces respectively parallel to the three coordinate planes. The property we have established requires that these elements be transformed as shown in Fig. 4.9 when the member is subjected to the couples **M** and **M′**. Since all the faces represented in the two projections of Fig. 4.9 are at 90° to each other, we conclude that $\gamma_{xy} = \gamma_{zx} = 0$ and, thus, that $\tau_{xy} = \tau_{xz} = 0$. Regarding the three stress components that we have not yet discussed, namely, σ_y, σ_z, and τ_{yz}, we note that they must be zero on the surface of the member. Since, on the other hand, the deformations involved do not require any interaction between the elements of a given transverse cross section, we can assume that these three stress components are equal to zero throughout the member. This assumption is verified, both from experimental evidence and from the theory of elasticity, for slender members undergoing small deformations.† We conclude that the only nonzero stress component exerted on any of the small cubic elements considered here is the normal component σ_x. Thus, at any point of a slender member in pure bending, we have a state of *uniaxial stress*. Recalling that, for $M > 0$, lines AB and $A'B'$ are observed, respectively, to decrease and increase in length, we note that the strain ϵ_x and the stress σ_x are negative in the upper portion of the member (*compression*) and positive in the lower portion (*tension*).

It follows from the above that there must exist a surface parallel to the upper and lower faces of the member, where ϵ_x and σ_x are zero. This surface is called the *neutral surface*. The neutral surface intersects the plane of symmetry along an arc of circle DE (Fig. 4.10*a*), and it intersects a transverse section along a straight line called the *neutral axis* of the section (Fig. 4.10*b*). The origin of coordinates will now be selected on the neutral surface, rather than on the lower face of the member as done earlier, so that the distance from any point to the neutral surface will be measured by its coordinate y.

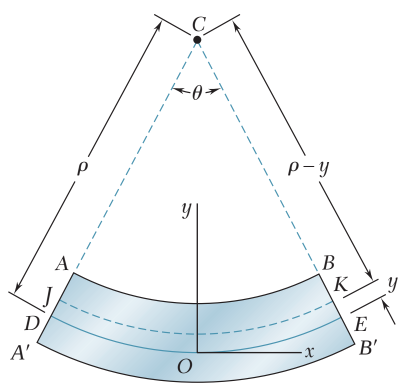


(*a*) Longitudinal, vertical section (plane of symmetry)

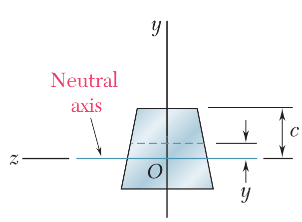


(*b*) Longitudinal, horizontal section

Fig. 4.9 Member subject to pure bending.



(*a*) Longitudinal, vertical section (plane of symmetry)



(*b*) Transverse section

Fig. 4.10 Deformation with respect to neutral axis.

†Also see Prob. 4.32.

Denoting by ρ the radius of arc DE (Fig. 4.10a), by θ the central angle corresponding to DE, and observing that the length of DE is equal to the length L of the undeformed member, we write

$$L = \rho\theta \tag{4.4}$$

Considering now the arc JK located at a distance y above the neutral surface, we note that its length L' is

$$L' = (\rho - y)\theta \tag{4.5}$$

Since the original length of arc JK was equal to L, the deformation of JK is

$$\delta = L' - L \tag{4.6}$$

or, if we substitute from (4.4) and (4.5) into (4.6),

$$\delta = (\rho - y)\theta - \rho\theta = -y\theta \tag{4.7}$$

The longitudinal strain ϵ_x in the elements of JK is obtained by dividing δ by the original length L of JK. We write

$$\epsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta}$$

or

$$\boxed{\epsilon_x = -\frac{y}{\rho}} \tag{4.8}$$

The minus sign is due to the fact that we have assumed the bending moment to be positive and, thus, the beam to be concave upward.

Because of the requirement that transverse sections remain plane, identical deformations will occur in all planes parallel to the plane of symmetry. Thus the value of the strain given by Eq. (4.8) is valid anywhere, and we conclude that the *longitudinal normal strain ϵ_x varies linearly with the distance y from the neutral surface.*

The strain ϵ_x reaches its maximum absolute value when y itself is largest. Denoting by c the largest distance from the neutral surface (which corresponds to either the upper or the lower surface of the member), and by ϵ_m the *maximum absolute value* of the strain, we have

$$\epsilon_m = \frac{c}{\rho} \tag{4.9}$$

Solving (4.9) for ρ and substituting the value obtained into (4.8), we can also write

$$\boxed{\epsilon_x = -\frac{y}{c}\epsilon_m} \tag{4.10}$$

We conclude our analysis of the deformations of a member in pure bending by observing that we are still unable to compute the strain or stress at a given point of the member, since we have not yet located the neutral surface in the member. In order to locate this surface, we must first specify the stress-strain relation of the material used.†

†Let us note, however, that if the member possesses both a vertical and a horizontal plane of symmetry (e.g., a member with a rectangular cross section), and if the stress-strain curve is the same in tension and compression, the neutral surface will coincide with the plane of symmetry (cf. Sec. 4.8).

4.4 STRESSES AND DEFORMATIONS IN THE ELASTIC RANGE

We now consider the case when the bending moment M is such that the normal stresses in the member remain below the yield strength σ_Y. This means that, for all practical purposes, the stresses in the member will remain below the proportional limit and the elastic limit as well. There will be no permanent deformation, and Hooke's law for uniaxial stress applies. Assuming the material to be homogeneous, and denoting by E its modulus of elasticity, we have in the longitudinal x direction

$$\sigma_x = E\epsilon_x \tag{4.11}$$

Recalling Eq. (4.10), and multiplying both members of that equation by E, we write

$$E\epsilon_x = -\frac{y}{c}(E\epsilon_m)$$

or, using (4.11),

$$\boxed{\sigma_x = -\frac{y}{c}\sigma_m} \tag{4.12}$$

where σ_m denotes the *maximum absolute value* of the stress. This result shows that, *in the elastic range, the normal stress varies linearly with the distance from the neutral surface* (Fig. 4.11).

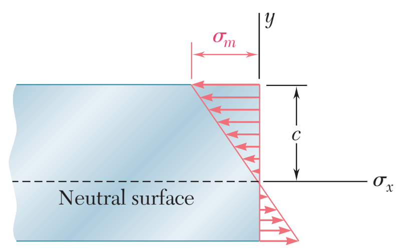


Fig. 4.11 Bending stresses.

It should be noted that, at this point, we do not know the location of the neutral surface, nor the maximum value σ_m of the stress. Both can be found if we recall the relations (4.1) and (4.3) which were obtained earlier from statics. Substituting first for σ_x from (4.12) into (4.1), we write

$$\int \sigma_x \, dA = \int\left(-\frac{y}{c}\sigma_m\right)dA = -\frac{\sigma_m}{c}\int y\,dA = 0$$

from which it follows that

$$\int y\,dA = 0 \tag{4.13}$$

This equation shows that the first moment of the cross section about its neutral axis must be zero.† In other words, for a member subjected to pure bending, and *as long as the stresses remain in the elastic range, the neutral axis passes through the centroid of the section.*

We now recall Eq. (4.3), which was derived in Sec. 4.2 with respect to an *arbitrary* horizontal z axis,

$$\int(-y\sigma_x\,dA) = M \tag{4.3}$$

Specifying that the z axis should coincide with the neutral axis of the cross section, we substitute for σ_x from (4.12) into (4.3) and write

$$\int(-y)\left(-\frac{y}{c}\sigma_m\right)dA = M$$

†See Appendix A for a discussion of the moments of areas.

or

$$\frac{\sigma_m}{c}\int y^2\,dA = M \tag{4.14}$$

Recalling that in the case of pure bending the neutral axis passes through the centroid of the cross section, we note that I is the moment of inertia, or second moment, of the cross section with respect to a centroidal axis perpendicular to the plane of the couple **M.** Solving (4.14) for σ_m, we write therefore†

$$\boxed{\sigma_m = \frac{Mc}{I}} \tag{4.15}$$

Substituting for σ_m from (4.15) into (4.12), we obtain the normal stress σ_x at any distance y from the neutral axis:

$$\boxed{\sigma_x = -\frac{My}{I}} \tag{4.16}$$

Equations (4.15) and (4.16) are called the *elastic flexure formulas*, and the normal stress σ_x caused by the bending or "flexing" of the member is often referred to as the *flexural stress*. We verify that the stress is compressive ($\sigma_x < 0$) above the neutral axis ($y > 0$) when the bending moment M is positive, and tensile ($\sigma_x > 0$) when M is negative.

Returning to Eq. (4.15), we note that the ratio I/c depends only upon the geometry of the cross section. This ratio is called the *elastic section modulus* and is denoted by S. We have

$$\text{Elastic section modulus} = S = \frac{I}{c} \tag{4.17}$$

Substituting S for I/c into Eq. (4.15), we write this equation in the alternative form

$$\boxed{\sigma_m = \frac{M}{S}} \tag{4.18}$$

Since the maximum stress σ_m is inversely proportional to the elastic section modulus S, it is clear that beams should be designed with as large a value of S as practicable. For example, in the case of a wooden beam with a rectangular cross section of width b and depth h, we have

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \frac{1}{6}bh^2 = \frac{1}{6}Ah \tag{4.19}$$

†We recall that the bending moment was assumed to be positive. If the bending moment is negative, M should be replaced in Eq. (4.15) by its absolute value $|M|$.

where A is the cross-sectional area of the beam. This shows that, of two beams with the same cross-sectional area A (Fig. 4.12), the beam with the larger depth h will have the larger section modulus and, thus, will be the more effective in resisting bending.†

In the case of structural steel, American standard beams (S-beams) and wide-flange beams (W-beams), Photo 4.3, are preferred to other shapes because a large portion of their cross section is located far from the neutral axis (Fig. 4.13). Thus, for a given cross-sectional area and a given depth, their design provides large values

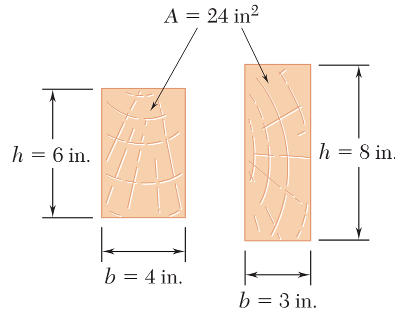


Fig. 4.12 Wood beam cross sections.

Photo 4.3 Wide-flange steel beams form the frame of many buildings.

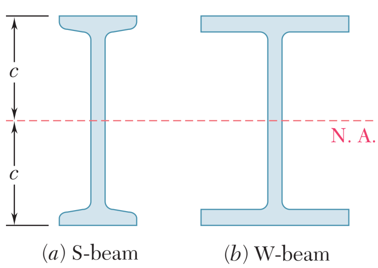


Fig. 4.13 Steel beam cross sections.

†However, large values of the ratio h/b could result in lateral instability of the beam.

of I and, consequently, of S. Values of the elastic section modulus of commonly manufactured beams can be obtained from tables listing the various geometric properties of such beams. To determine the maximum stress σ_m in a given section of a standard beam, the engineer needs only to read the value of the elastic section modulus S in a table, and divide the bending moment M in the section by S.

The deformation of the member caused by the bending moment M is measured by the *curvature* of the neutral surface. The curvature is defined as the reciprocal of the radius of curvature ρ, and can be obtained by solving Eq. (4.9) for $1/\rho$:

$$\frac{1}{\rho} = \frac{\epsilon_m}{c} \tag{4.20}$$

But, in the elastic range, we have $\epsilon_m = \sigma_m/E$. Substituting for ϵ_m into (4.20), and recalling (4.15), we write

$$\frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$

or

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}} \tag{4.21}$$

EXAMPLE 4.01

A steel bar of 0.8×2.5-in. rectangular cross section is subjected to two equal and opposite couples acting in the vertical plane of symmetry of the bar (Fig. 4.14). Determine the value of the bending moment M that causes the bar to yield. Assume $\sigma_Y = 36$ ksi.

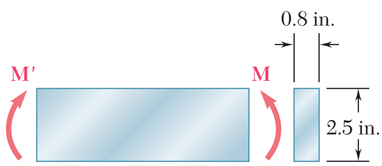


Fig. 4.14

Since the neutral axis must pass through the centroid C of the cross section, we have $c = 1.25$ in. (Fig. 4.15). On the other hand, the centroidal moment of inertia of the rectangular cross section is

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(0.8 \text{ in.})(2.5 \text{ in.})^3 = 1.042 \text{ in}^4$$

Solving Eq. (4.15) for M, and substituting the above data, we have

$$M = \frac{I}{c}\sigma_m = \frac{1.042 \text{ in}^4}{1.25 \text{ in.}}(36 \text{ ksi})$$
$$M = 30 \text{ kip} \cdot \text{in.}$$

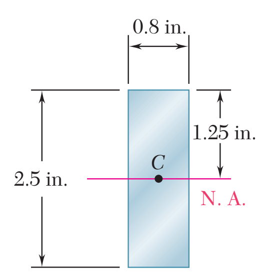


Fig. 4.15

EXAMPLE 4.02

An aluminum rod with a semicircular cross section of radius $r = 12$ mm (Fig. 4.16) is bent into the shape of a circular arc of mean radius $\rho = 2.5$ m. Knowing that the flat face of the rod is turned toward the center of curvature of the arc, determine the maximum tensile and compressive stress in the rod. Use $E = 70$ GPa.

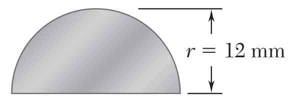


Fig. 4.16

We could use Eq. (4.21) to determine the bending moment M corresponding to the given radius of curvature ρ, and then Eq. (4.15) to determine σ_m. However, it is simpler to use Eq. (4.9) to determine ϵ_m, and Hooke's law to obtain σ_m.

The ordinate $\bar{y}$ of the centroid C of the semicircular cross section is

$$\bar{y} = \frac{4r}{3\pi} = \frac{4(12 \text{ mm})}{3\pi} = 5.093 \text{ mm}$$

The neutral axis passes through C (Fig. 4.17) and the distance c to the point of the cross section farthest away from the neutral axis is

$$c = r - \bar{y} = 12 \text{ mm} - 5.093 \text{ mm} = 6.907 \text{ mm}$$

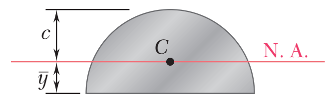


Fig. 4.17

Using Eq. (4.9), we write

$$\epsilon_m = \frac{c}{\rho} = \frac{6.907 \times 10^{-3} \text{ m}}{2.5 \text{ m}} = 2.763 \times 10^{-3}$$

and, applying Hooke's law,

$$\sigma_m = E\epsilon_m = (70 \times 10^9 \text{ Pa})(2.763 \times 10^{-3}) = 193.4 \text{ MPa}$$

Since this side of the rod faces away from the center of curvature, the stress obtained is a tensile stress. The maximum compressive stress occurs on the flat side of the rod. Using the fact that the stress is proportional to the distance from the neutral axis, we write

$$\sigma_{\text{comp}} = -\frac{\bar{y}}{c}\sigma_m = -\frac{5.093 \text{ mm}}{6.907 \text{ mm}}(193.4 \text{ MPa})$$
$$= -142.6 \text{ MPa}$$

4.5 DEFORMATIONS IN A TRANSVERSE CROSS SECTION

When we proved in Sec. 4.3 that the transverse cross section of a member in pure bending remains plane, we did not rule out the possibility of deformations within the plane of the section. That such deformations will exist is evident, if we recall from Sec. 2.11 that elements in a state of uniaxial stress, $\sigma_x \neq 0$, $\sigma_y = \sigma_z = 0$, are deformed in the transverse y and z directions, as well as in the axial x direction. The normal strains ϵ_y and ϵ_z depend upon Poisson's ratio ν for the material used and are expressed as

$$\epsilon_y = -\nu\epsilon_x \qquad \epsilon_z = -\nu\epsilon_x$$

or, recalling Eq. (4.8),

$$\epsilon_y = \frac{\nu y}{\rho} \qquad \epsilon_z = \frac{\nu y}{\rho} \tag{4.22}$$

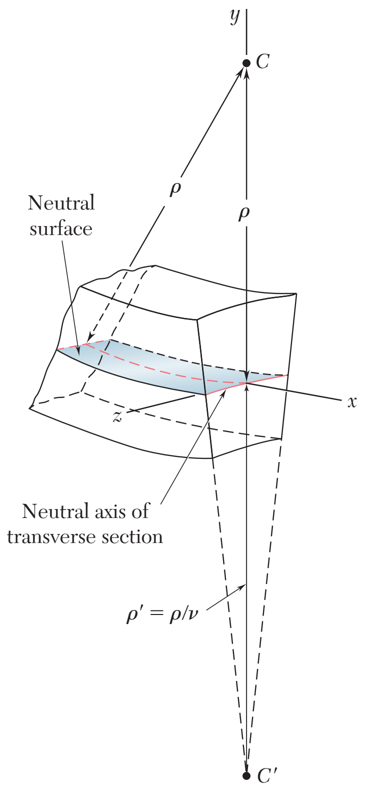


Fig. 4.18 Deformation of transverse cross section.

The relations we have obtained show that the elements located above the neutral surface ($y > 0$) will expand in both the y and z directions, while the elements located below the neutral surface ($y < 0$) will contract. In the case of a member of rectangular cross section, the expansion and contraction of the various elements in the vertical direction will compensate, and no change in the vertical dimension of the cross section will be observed. As far as the deformations in the horizontal transverse z direction are concerned, however, the expansion of the elements located above the neutral surface and the corresponding contraction of the elements located below that surface will result in the various horizontal lines in the section being bent into arcs of circle (Fig. 4.18). The situation observed here is similar to that observed earlier in a longitudinal cross section. Comparing the second of Eqs. (4.22) with Eq. (4.8), we conclude that the neutral axis of the transverse section will be bent into a circle of radius $\rho' = \rho/\nu$. The center C' of this circle is located below the neutral surface (assuming $M > 0$), i.e., on the side opposite to the center of curvature C of the member. The reciprocal of the radius of curvature ρ' represents the curvature of the transverse cross section and is called the *anticlastic curvature.* We have

$$\text{Anticlastic curvature} = \frac{1}{\rho'} = \frac{\nu}{\rho} \tag{4.23}$$

In our discussion of the deformations of a symmetric member in pure bending, in this section and in the preceding ones, we have ignored the manner in which the couples **M** and **M′** were actually applied to the member. If *all* transverse sections of the member, from one end to the other, are to remain plane and free of shearing stresses, we must make sure that the couples are applied in such a way that the ends of the member themselves remain plane and free of shearing stresses. This can be accomplished by applying the couples **M** and **M′** to the member through the use of rigid and smooth plates (Fig. 4.19). The elementary forces exerted by the plates on the member will be normal to the end sections, and these sections, while remaining plane, will be free to deform as described earlier in this section.

We should note that these loading conditions cannot be actually realized, since they require each plate to exert tensile forces on the corresponding end section below its neutral axis, while allowing the section to freely deform in its own plane. The fact that the rigid-end-plates model of Fig. 4.19 cannot be physically realized, however, does not detract from its importance, which is to allow us to *visualize* the loading conditions corresponding to the relations derived in the preceding sections. Actual loading conditions may differ appreciably from this idealized model. By virtue of Saint-Venant's principle, however, the relations obtained can be used to compute stresses in engineering situations, as long as the section considered is not too close to the points where the couples are applied.

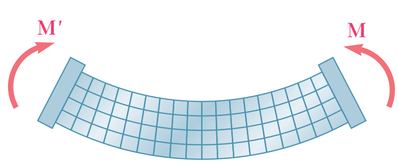
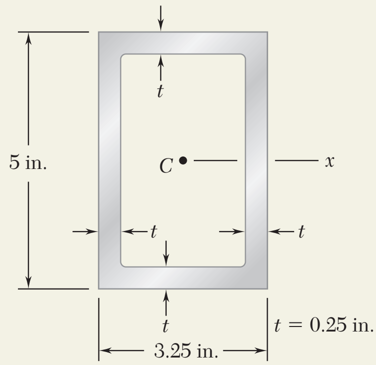
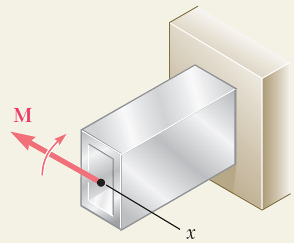


Fig. 4.19 Deformation of longitudinal segment.

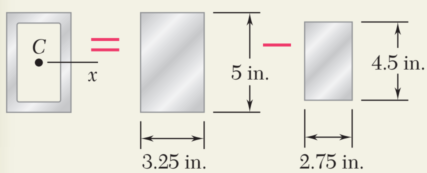
SAMPLE PROBLEM 4.1



The rectangular tube shown is extruded from an aluminum alloy for which $\sigma_Y = 40$ ksi, $\sigma_U = 60$ ksi, and $E = 10.6 \times 10^6$ psi. Neglecting the effect of fillets, determine (*a*) the bending moment M for which the factor of safety will be 3.00, (*b*) the corresponding radius of curvature of the tube.



SOLUTION



Moment of Inertia. Considering the cross-sectional area of the tube as the difference between the two rectangles shown and recalling the formula for the centroidal moment of inertia of a rectangle, we write

$$I = \tfrac{1}{12}(3.25)(5)^3 - \tfrac{1}{12}(2.75)(4.5)^3 \qquad I = 12.97 \text{ in}^4$$

Allowable Stress. For a factor of safety of 3.00 and an ultimate stress of 60 ksi, we have

$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{60 \text{ ksi}}{3.00} = 20 \text{ ksi}$$

Since $\sigma_{\text{all}} < \sigma_Y$, the tube remains in the elastic range and we can apply the results of Sec. 4.4.

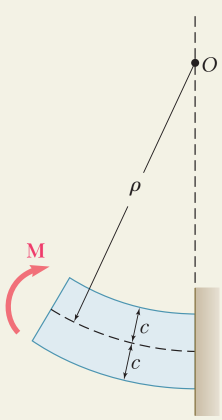
a. Bending Moment. With $c = \frac{1}{2}(5 \text{ in.}) = 2.5$ in., we write

$$\sigma_{\text{all}} = \frac{Mc}{I} \qquad M = \frac{I}{c}\sigma_{\text{all}} = \frac{12.97 \text{ in}^4}{2.5 \text{ in.}}(20 \text{ ksi}) \qquad M = 103.8 \text{ kip} \cdot \text{in.} \ ◀$$

b. Radius of Curvature. Recalling that $E = 10.6 \times 10^6$ psi, we substitute this value and the values obtained for I and M into Eq. (4.21) and find

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{103.8 \times 10^3 \text{ lb} \cdot \text{in.}}{(10.6 \times 10^6 \text{ psi})(12.97 \text{ in}^4)} = 0.755 \times 10^{-3} \text{ in}^{-1}$$

$$\rho = 1325 \text{ in.} \qquad \rho = 110.4 \text{ ft} \ ◀$$



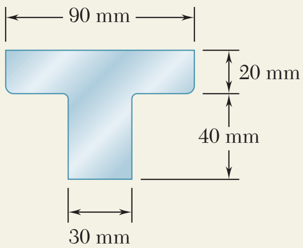
Alternative Solution. Since we know that the maximum stress is $\sigma_{\text{all}} = 20$ ksi, we can determine the maximum strain ϵ_m and then use Eq. (4.9),

$$\epsilon_m = \frac{\sigma_{\text{all}}}{E} = \frac{20 \text{ ksi}}{10.6 \times 10^6 \text{ psi}} = 1.887 \times 10^{-3} \text{ in./in.}$$

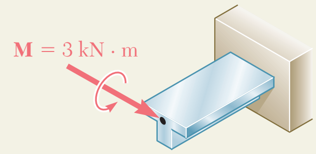
$$\epsilon_m = \frac{c}{\rho} \qquad \rho = \frac{c}{\epsilon_m} = \frac{2.5 \text{ in.}}{1.887 \times 10^{-3} \text{ in./in.}}$$

$$\rho = 1325 \text{ in.} \qquad \rho = 110.4 \text{ ft} \ ◀$$

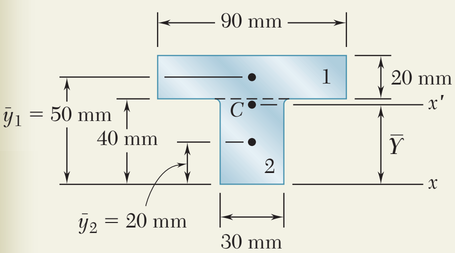
SAMPLE PROBLEM 4.2



A cast-iron machine part is acted upon by the 3 kN · m couple shown. Knowing that $E = 165$ GPa and neglecting the effect of fillets, determine (*a*) the maximum tensile and compressive stresses in the casting, (*b*) the radius of curvature of the casting.



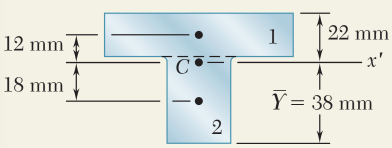
SOLUTION



Centroid. We divide the T-shaped cross section into the two rectangles shown and write

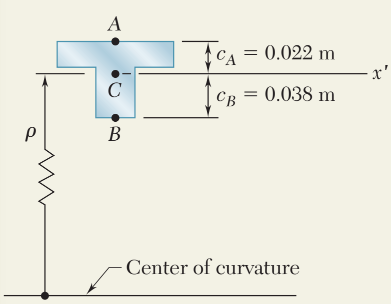
	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$(20)(90) = 1800$	50	90×10^3
2	$(40)(30) = 1200$	20	24×10^3
	$\Sigma A = 3000$		$\Sigma \bar{y}A = 114 \times 10^3$

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$
$$\bar{Y}(3000) = 114 \times 10^6$$
$$\bar{Y} = 38 \text{ mm}$$



Centroidal Moment of Inertia. The parallel-axis theorem is used to determine the moment of inertia of each rectangle with respect to the axis x' that passes through the centroid of the composite section. Adding the moments of inertia of the rectangles, we write

$$\begin{aligned} I_{x'} &= \Sigma(\bar{I} + Ad^2) = \Sigma(\tfrac{1}{12}bh^3 + Ad^2) \\ &= \tfrac{1}{12}(90)(20)^3 + (90 \times 20)(12)^2 + \tfrac{1}{12}(30)(40)^3 + (30 \times 40)(18)^2 \\ &= 868 \times 10^3 \text{ mm}^4 \\ I &= 868 \times 10^{-9} \text{ m}^4 \end{aligned}$$



a. Maximum Tensile Stress. Since the applied couple bends the casting downward, the center of curvature is located below the cross section. The maximum tensile stress occurs at point A, which is farthest from the center of curvature.

$$\sigma_A = \frac{Mc_A}{I} = \frac{(3 \text{ kN} \cdot \text{m})(0.022 \text{ m})}{868 \times 10^{-9} \text{ m}^4} \qquad \sigma_A = +76.0 \text{ MPa} \blacktriangleleft$$

Maximum Compressive Stress. This occurs at point B; we have

$$\sigma_B = -\frac{Mc_B}{I} = -\frac{(3 \text{ kN} \cdot \text{m})(0.038 \text{ m})}{868 \times 10^{-9} \text{ m}^4} \qquad \sigma_B = -131.3 \text{ MPa} \blacktriangleleft$$

b. Radius of Curvature. From Eq. (4.21), we have

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{3 \text{ kN} \cdot \text{m}}{(165 \text{ GPa})(868 \times 10^{-9} \text{ m}^4)}$$
$$= 20.95 \times 10^{-3} \text{ m}^{-1} \qquad \rho = 47.7 \text{ m} \blacktriangleleft$$

PROBLEMS

4.1 and 4.2 Knowing that the couple shown acts in a vertical plane, determine the stress at (*a*) point *A*, (*b*) point *B*.

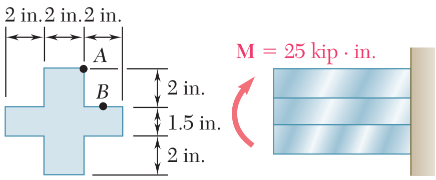


Fig. P4.1

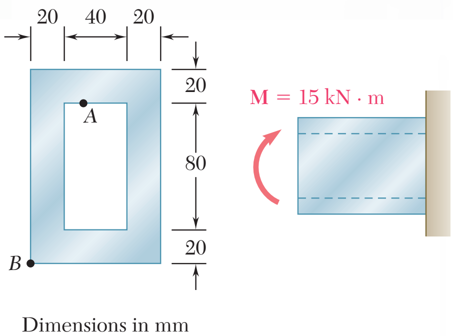


Fig. P4.2

4.3 Using an allowable stress of 16 ksi, determine the largest couple that can be applied to each pipe.

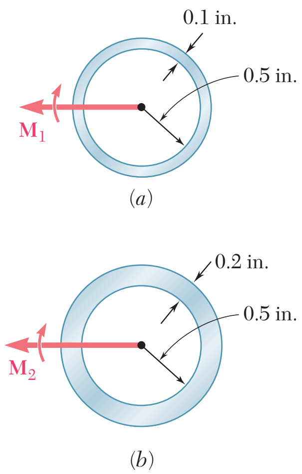


Fig. P4.3

4.4 A nylon spacing bar has the cross section shown. Knowing that the allowable stress for the grade of nylon used is 24 MPa, determine the largest couple $\mathbf{M}_z$ that can be applied to the bar.

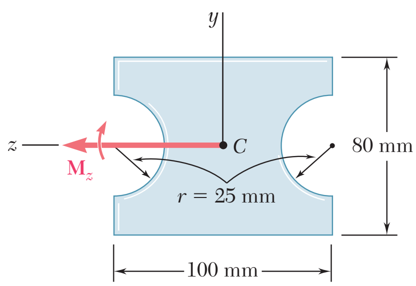


Fig. P4.4

4.5 A beam of the cross section shown is extruded from an aluminum alloy for which $\sigma_Y = 250$ MPa and $\sigma_U = 450$ MPa. Using a factor of safety of 3.00, determine the largest couple that can be applied to the beam when it is bent about the z axis.

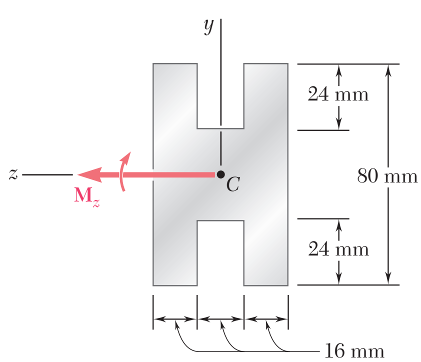


Fig. P4.5

4.6 Solve Prob. 4.5, assuming that the beam is bent about the y axis.

4.7 and 4.8 Two W4 × 13 rolled sections are welded together as shown. Knowing that for the steel alloy used, $\sigma_Y = 36$ ksi and $\sigma_U = 58$ ksi and using a factor of safety of 3.0, determine the largest couple that can be applied when the assembly is bent about the z axis.

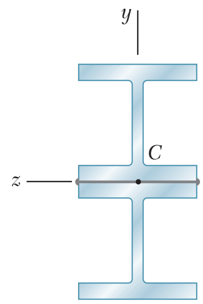


Fig. *P4.7*

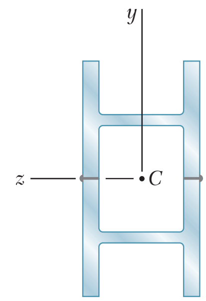


Fig. *P4.8*

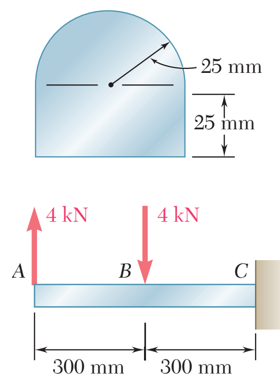


Fig. P4.9

4.9 through 4.11 Two vertical forces are applied to a beam of the cross section shown. Determine the maximum tensile and compressive stresses in portion BC of the beam.

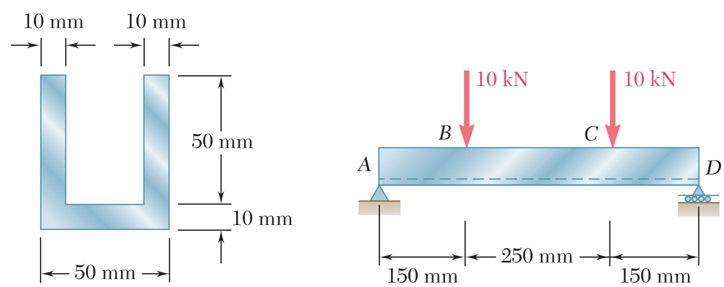


Fig. *P4.10*

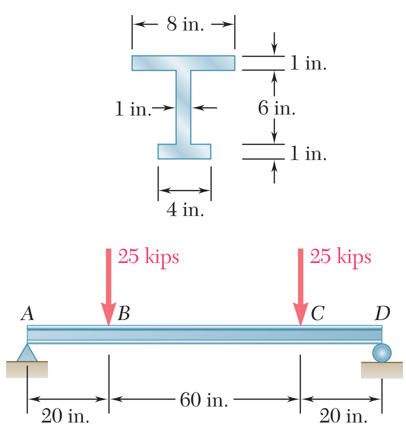


Fig. P4.11

4.12 Knowing that a beam of the cross section shown is bent about a horizontal axis and that the bending moment is 6 kN · m, determine the total force acting on the top flange.

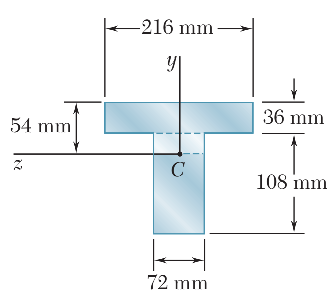


Fig. P4.12 and *P4.13*

4.13 Knowing that a beam of the cross section shown is bent about a horizontal axis and that the bending moment is 6 kN · m, determine the total force acting on the shaded portion of the web.

4.14 Knowing that a beam of the cross section shown is bent about a horizontal axis and that the bending moment is 50 kip · in., determine the total force acting (*a*) on the top flange (*b*) on the shaded portion of the web.

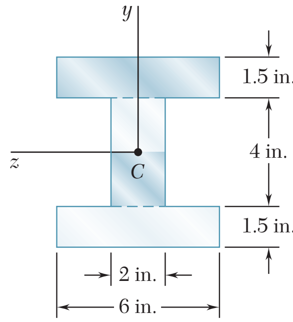


Fig. P4.14

4.15 The beam shown is made of a nylon for which the allowable stress is 24 MPa in tension and 30 MPa in compression. Determine the largest couple **M** that can be applied to the beam.

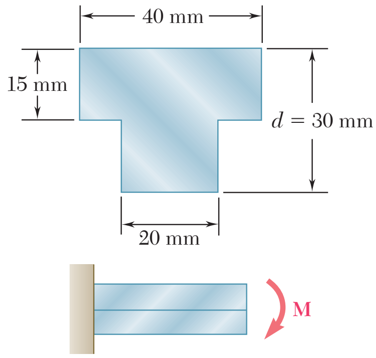


Fig. P4.15

4.16 Solve Prob. 4.15, assuming that $d = 40$ mm.

4.17 Knowing that for the extruded beam shown the allowable stress is 12 ksi in tension and 16 ksi in compression, determine the largest couple **M** that can be applied.

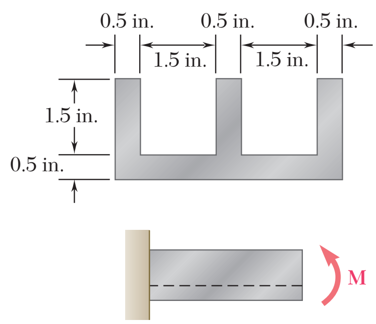


Fig. P4.17

4.18 Knowing that for the casting shown the allowable stress is 5 ksi in tension and 18 ksi in compression, determine the largest couple **M** that can be applied.

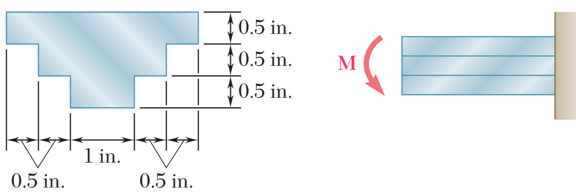


Fig. P4.18

4.19 and *4.20* Knowing that for the extruded beam shown the allowable stress is 120 MPa in tension and 150 MPa in compression, determine the largest couple **M** that can be applied.

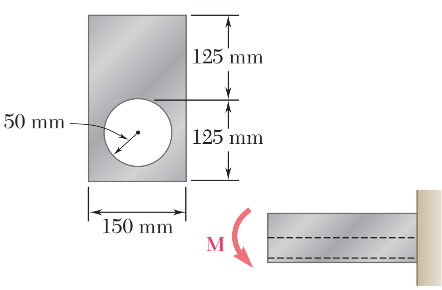


Fig. P4.19

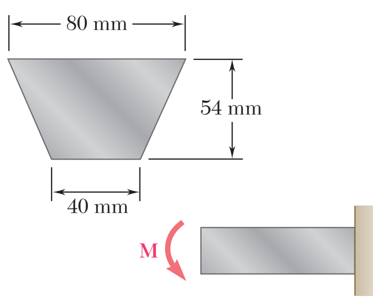


Fig. *P4.20*

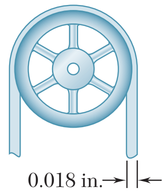


Fig. P4.21

4.21 A steel band saw blade, that was originally straight, passes over 8-in.-diameter pulleys when mounted on a band saw. Determine the maximum stress in the blade, knowing that it is 0.018 in. thick and 0.625 in. wide. Use $E = 29 \times 10^6$ psi.

4.22 Straight rods of 0.30-in. diameter and 200-ft length are sometimes used to clear underground conduits of obstructions or to thread wires through a new conduit. The rods are made of high-strength steel and, for storage and transportation, are wrapped on spools of 5-ft diameter. Assuming that the yield strength is not exceeded, determine (*a*) the maximum stress in a rod, when the rod, which is initially straight, is wrapped on a spool, (*b*) the corresponding bending moment in the rod. Use $E = 29 \times 10^6$ psi.

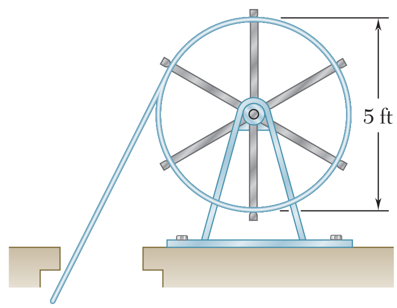


Fig. *P4.22*

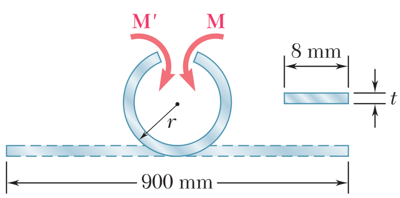


Fig. P4.23

4.23 A 900-mm strip of steel is bent into a full circle by two couples applied as shown. Determine (*a*) the maximum thickness t of the strip if the allowable stress of the steel is 420 MPa, (*b*) the corresponding moment M of the couples. Use $E = 200$ GPa.

4.24 A 60-N · m couple is applied to the steel bar shown. (*a*) Assuming that the couple is applied about the z axis as shown, determine the maximum stress and the radius of curvature of the bar. (*b*) Solve part *a*, assuming that the couple is applied about the y axis. Use $E = 200$ GPa.

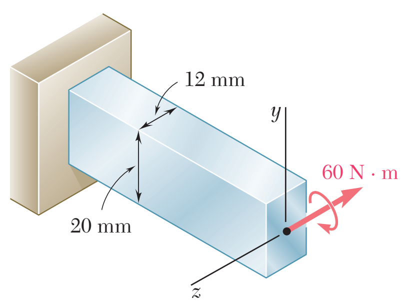


Fig. P4.24

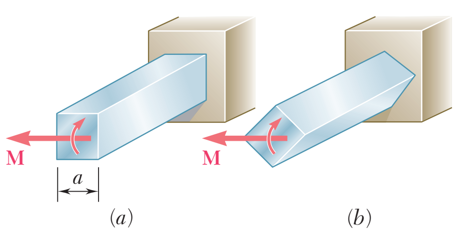


Fig. P4.25

4.25 A couple of magnitude M is applied to a square bar of side a. For each of the orientations shown, determine the maximum stress and the curvature of the bar.

4.26 A portion of a square bar is removed by milling, so that its cross section is as shown. The bar is then bent about its horizontal axis by a couple **M**. Considering the case where $h = 0.9h_0$, express the maximum stress in the bar in the form $\sigma_m = k\sigma_0$ where σ_0 is the maximum stress that would have occurred if the original square bar had been bent by the same couple **M**, and determine the value of k.

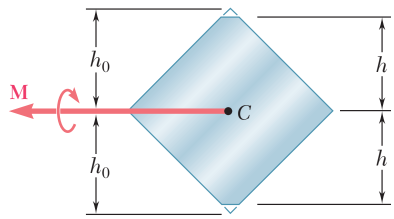


Fig. *P4.26*

4.27 In Prob. 4.26, determine (a) the value of h for which the maximum stress σ_m is as small as possible, (b) the corresponding value of k.

4.28 A couple **M** will be applied to a beam of rectangular cross section that is to be sawed from a log of circular cross section. Determine the ratio d/b for which (a) the maximum stress σ_m will be as small as possible, (b) the radius of curvature of the beam will be maximum.

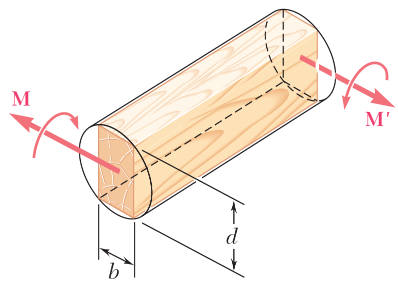


Fig. P4.28

4.29 For the aluminum bar and loading of Sample Prob. 4.1, determine (a) the radius of curvature ρ' of a transverse cross section, (b) the angle between the sides of the bar that were originally vertical. Use $E = 10.6 \times 10^6$ psi and $\nu = 0.33$.

4.30 For the bar and loading of Example 4.01, determine (a) the radius of curvature ρ, (b) the radius of curvature ρ' of a transverse cross section, (c) the angle between the sides of the bar that were originally vertical. Use $E = 29 \times 10^6$ psi and $\nu = 0.29$.

4.31 A W200 $\times$ 31.3 rolled-steel beam is subjected to a couple **M** of moment 45 kN $\cdot$ m. Knowing that $E = 200$ GPa and $\nu = 0.29$, determine (a) the radius of curvature ρ, (b) the radius of curvature ρ' of a transverse cross section.

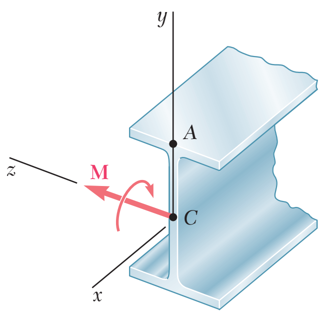


Fig. P4.31

4.32 It was assumed in Sec. 4.3 that the normal stresses σ_y in a member in pure bending are negligible. For an initially straight elastic member of rectangular cross section, (a) derive an approximate expression for σ_y as a function of y, (b) show that $(\sigma_y)_{\max} = -(c/2\rho)(\sigma_x)_{\max}$ and, thus, that σ_y can be neglected in all practical situations. (*Hint*: Consider the free-body diagram of the portion of beam located below the surface of ordinate y and assume that the distribution of the stress σ_x is still linear.)

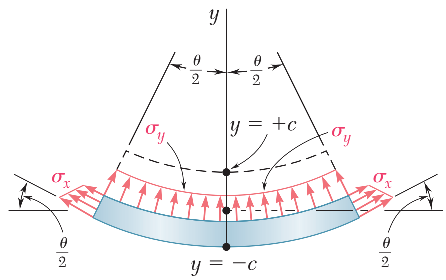


Fig. P4.32

4.6 BENDING OF MEMBERS MADE OF SEVERAL MATERIALS

The derivations given in Sec. 4.4 were based on the assumption of a homogeneous material with a given modulus of elasticity E. If the member subjected to pure bending is made of two or more materials with different moduli of elasticity, our approach to the determination of the stresses in the member must be modified.

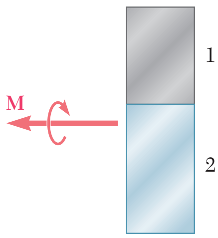


Fig. 4.20 Cross section with two materials.

Consider, for instance, a bar consisting of two portions of different materials bonded together as shown in cross section in Fig. 4.20. This composite bar will deform as described in Sec. 4.3, since its cross section remains the same throughout its entire length, and since no assumption was made in Sec. 4.3 regarding the stress-strain relationship of the material or materials involved. Thus, the normal strain ϵ_x still varies linearly with the distance y from the neutral axis of the section (Fig. 4.21*a* and *b*), and formula (4.8) holds:

$$\epsilon_x = -\frac{y}{\rho} \tag{4.8}$$

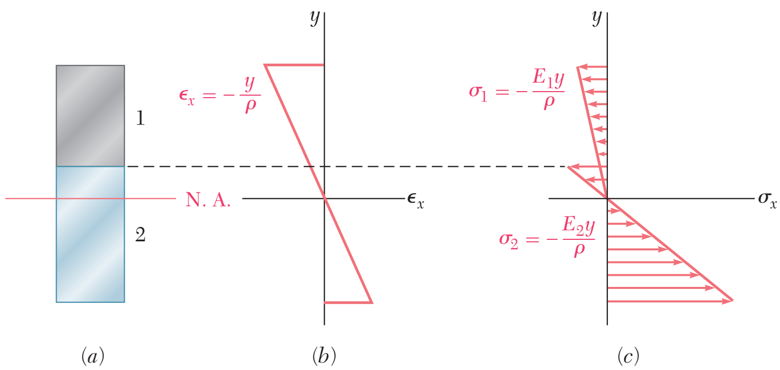


Fig. 4.21 Strain and stress distribution in bar made of two materials.

However, we cannot assume that the neutral axis passes through the centroid of the composite section, and one of the goals of the present analysis will be to determine the location of this axis.

Since the moduli of elasticity E_1 and E_2 of the two materials are different, the expressions obtained for the normal stress in each material will also be different. We write

$$\begin{aligned} \sigma_1 &= E_1\epsilon_x = -\frac{E_1 y}{\rho} \\ \sigma_2 &= E_2\epsilon_x = -\frac{E_2 y}{\rho} \end{aligned} \tag{4.24}$$

and obtain a stress-distribution curve consisting of two segments of straight line (Fig. 4.21*c*). It follows from Eqs. (4.24) that the force dF_1 exerted on an element of area dA of the upper portion of the cross section is

$$dF_1 = \sigma_1\, dA = -\frac{E_1 y}{\rho}\, dA \tag{4.25}$$

while the force dF_2 exerted on an element of the same area dA of the lower portion is

$$dF_2 = \sigma_2 \, dA = -\frac{E_2 y}{\rho} \, dA \tag{4.26}$$

But, denoting by n the ratio E_2/E_1 of the two moduli of elasticity, we can express dF_2 as

$$dF_2 = -\frac{(nE_1)y}{\rho} \, dA = -\frac{E_1 y}{\rho} (n \, dA) \tag{4.27}$$

Comparing Eqs. (4.25) and (4.27), we note that the same force dF_2 would be exerted on an element of area $n \, dA$ of the first material. In other words, the resistance to bending of the bar would remain the same if both portions were made of the first material, provided that the width of each element of the lower portion were multiplied by the factor n. Note that this widening (if $n > 1$), or narrowing (if $n < 1$), must be effected *in a direction parallel to the neutral axis of the section,* since it is essential that the distance y of each element from the neutral axis remain the same. The new cross section obtained in this way is called the *transformed section* of the member (Fig. 4.22).

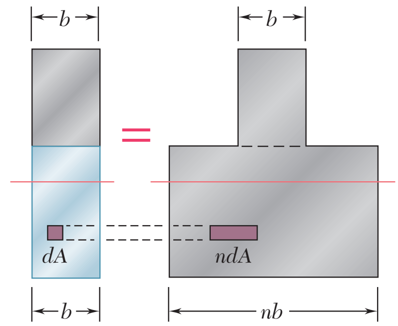


Fig. 4.22 Transformed section for composite bar.

Since the transformed section represents the cross section of a member made of a *homogeneous material* with a modulus of elasticity E_1, the method described in Sec. 4.4 can be used to determine the neutral axis of the section and the normal stress at various points of the section. The neutral axis will be drawn *through the centroid of the transformed section* (Fig. 4.23), and the stress σ_x at any point of the corresponding fictitious homogeneous member will be obtained from Eq. (4.16)

$$\sigma_x = -\frac{My}{I} \tag{4.16}$$

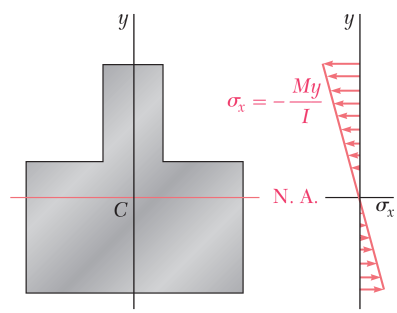


Fig. 4.23 Distribution of stresses in transformed section.

where y is the distance from the neutral surface, and I *the moment of inertia of the transformed section* with respect to its centroidal axis.

To obtain the stress σ_1 at a point located in the upper portion of the cross section of the original composite bar, we simply compute the stress σ_x at the corresponding point of the transformed section. However, to obtain the stress σ_2 at a point in the lower portion of the cross section, we must *multiply by* n the stress σ_x computed at the corresponding point of the transformed section. Indeed, as we saw earlier, the same elementary force dF_2 is applied to an element of area $n \, dA$ of the transformed section and to an element of area dA of the original section. Thus, the stress σ_2 at a point of the original section must be n times larger than the stress at the corresponding point of the transformed section.

The deformations of a composite member can also be determined by using the transformed section. We recall that the transformed section represents the cross section of a member, made of a homogeneous material of modulus E_1, which deforms in the same manner as the composite member. Therefore, using Eq. (4.21), we write that the curvature of the composite member is

$$\frac{1}{\rho} = \frac{M}{E_1 I}$$

where I is the moment of inertia of the transformed section with respect to its neutral axis.

EXAMPLE 4.03

A bar obtained by bonding together pieces of steel ($E_s = 29 \times 10^6$ psi) and brass ($E_b = 15 \times 10^6$ psi) has the cross section shown (Fig. 4.24). Determine the maximum stress in the steel and in the brass when the bar is in pure bending with a bending moment $M = 40$ kip · in.

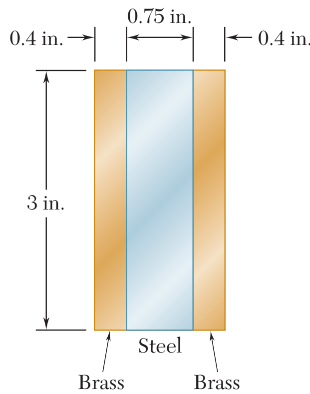


Fig. 4.24

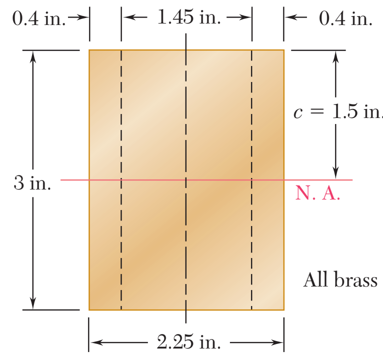


Fig. 4.25

The transformed section corresponding to an equivalent bar made entirely of brass is shown in Fig. 4.25. Since

$$n = \frac{E_s}{E_b} = \frac{29 \times 10^6 \text{ psi}}{15 \times 10^6 \text{ psi}} = 1.933$$

the width of the central portion of brass, which replaces the original steel portion, is obtained by multiplying the original width by 1.933, we have

$$(0.75 \text{ in.})(1.933) = 1.45 \text{ in.}$$

Note that this change in dimension occurs in a direction parallel to the neutral axis. The moment of inertia of the transformed section about its centroidal axis is

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(2.25 \text{ in.})(3 \text{ in.})^3 = 5.063 \text{ in}^4$$

and the maximum distance from the neutral axis is $c = 1.5$ in. Using Eq. (4.15), we find the maximum stress in the transformed section:

$$\sigma_m = \frac{Mc}{I} = \frac{(40 \text{ kip} \cdot \text{in.})(1.5 \text{ in.})}{5.063 \text{ in}^4} = 11.85 \text{ ksi}$$

The value obtained also represents the maximum stress in the brass portion of the original composite bar. The maximum stress in the steel portion, however, will be larger than the value obtained for the transformed section, since the area of the central portion must be reduced by the factor $n = 1.933$ when we return from the transformed section to the original one. We thus conclude that

$$(\sigma_{\text{brass}})_{\max} = 11.85 \text{ ksi}$$
$$(\sigma_{\text{steel}})_{\max} = (1.933)(11.85 \text{ ksi}) = 22.9 \text{ ksi}$$

An important example of structural members made of two different materials is furnished by *reinforced concrete beams* (Photo 4.4). These beams, when subjected to positive bending moments, are reinforced by steel rods placed a short distance above their lower face (Fig. 4.26*a*). Since concrete is very weak in tension, it will crack below the neutral surface and the steel rods will carry the entire tensile load, while the upper part of the concrete beam will carry the compressive load.

To obtain the transformed section of a reinforced concrete beam, we replace the total cross-sectional area A_s of the steel bars by an equivalent area nA_s, where n is the ratio E_s/E_c of the moduli of elasticity of steel and concrete (Fig. 4.26*b*). On the other hand, since the concrete in the beam acts effectively only in compression, only the portion of the cross section located above the neutral axis should be used in the transformed section.

Photo 4.4 Reinforced concrete building.

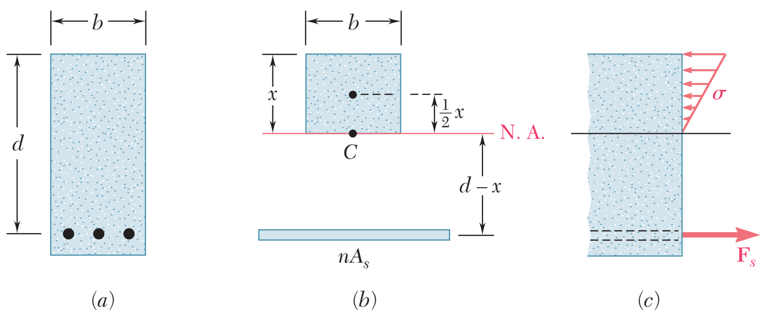


Fig. 4.26 Reinforced concrete beam.

The position of the neutral axis is obtained by determining the distance x from the upper face of the beam to the centroid C of the transformed section. Denoting by b the width of the beam, and by d the distance from the upper face to the center line of the steel rods, we write that the first moment of the transformed section with respect to the neutral axis must be zero. Since the first moment of each of the two portions of the transformed section is obtained by multiplying its area by the distance of its own centroid from the neutral axis, we have

$$(bx)\frac{x}{2} - nA_s(d - x) = 0$$

or

$$\frac{1}{2}bx^2 + nA_s x - nA_s d = 0 \tag{4.28}$$

Solving this quadratic equation for x, we obtain both the position of the neutral axis in the beam, and the portion of the cross section of the concrete beam that is effectively used.

The determination of the stresses in the transformed section is carried out as explained earlier in this section (see Sample Prob. 4.4). The distribution of the compressive stresses in the concrete and the resultant $\mathbf{F}_s$ of the tensile forces in the steel rods are shown in Fig. 4.26*c*.

4.7 STRESS CONCENTRATIONS

The formula $\sigma_m = Mc/I$ was derived in Sec. 4.4 for a member with a plane of symmetry and a uniform cross section, and we saw in Sec. 4.5 that it was accurate throughout the entire length of the member only if the couples **M** and **M**′ were applied through the use of rigid and smooth plates. Under other conditions of application of the loads, stress concentrations will exist near the points where the loads are applied.

Higher stresses will also occur if the cross section of the member undergoes a sudden change. Two particular cases of interest have been studied,† the case of a flat bar with a sudden change in width, and the case of a flat bar with grooves. Since the distribution of stresses in the critical cross sections depends only upon the geometry of the members, stress-concentration factors can be determined for various ratios of the parameters involved and recorded as shown in Figs. 4.27 and 4.28. The value of the maximum stress in the critical cross section can then be expressed as

$$\boxed{\sigma_m = K\frac{Mc}{I}} \tag{4.29}$$

where K is the stress-concentration factor, and where c and I refer to the critical section, i.e., to the section of width d in both of the cases considered here. An examination of Figs. 4.27 and 4.28 clearly shows the importance of using fillets and grooves of radius r as large as practical.

Finally, we should point out that, as was the case for axial loading and torsion, the values of the factors K have been computed under the assumption of a linear relation between stress and strain. In many applications, plastic deformations will occur and result in values of the maximum stress lower than those indicated by Eq. (4.29).

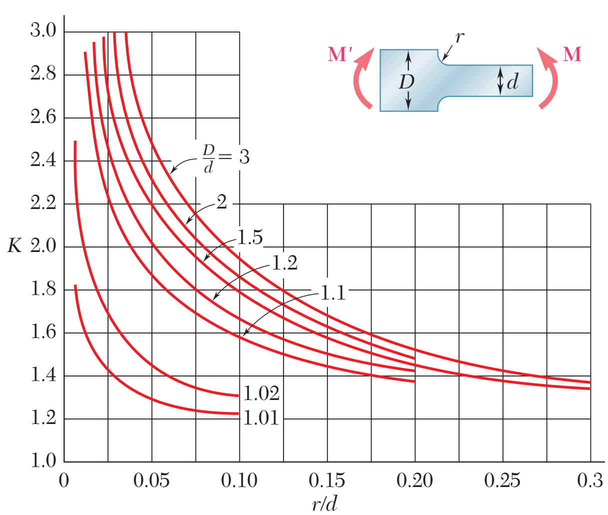


Fig. 4.27 Stress-concentration factors for *flat bars* with fillets under pure bending.†

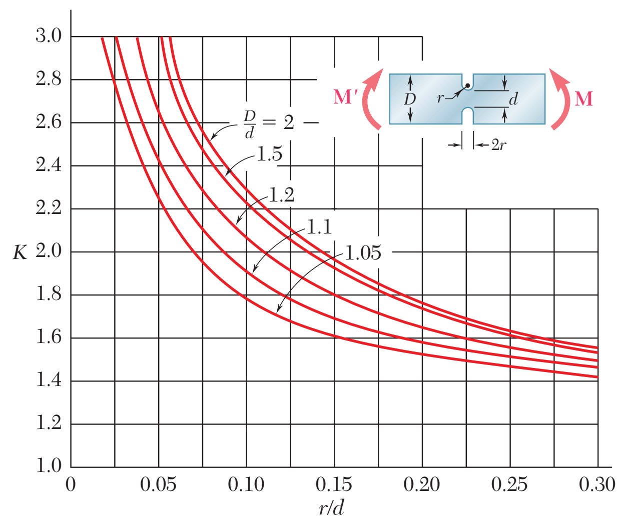


Fig. 4.28 Stress-concentration factors for *flat bars* with grooves under pure bending.†

†W. D. Pilkey, *Peterson's Stress Concentration Factors*, 2d ed., John Wiley & Sons, New York, 1997.

EXAMPLE 4.04

Grooves 10 mm deep are to be cut in a steel bar which is 60 mm wide and 9 mm thick (Fig. 4.29). Determine the smallest allowable width of the grooves if the stress in the bar is not to exceed 150 MPa when the bending moment is equal to 180 N · m.

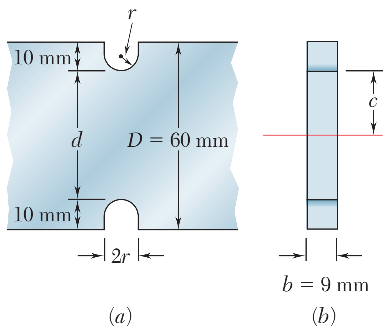


Fig. 4.29

We note from Fig. 4.29*a* that

$$d = 60 \text{ mm} - 2(10 \text{ mm}) = 40 \text{ mm}$$
$$c = \tfrac{1}{2}d = 20 \text{ mm} \qquad b = 9 \text{ mm}$$

The moment of inertia of the critical cross section about its neutral axis is

$$I = \tfrac{1}{12}bd^3 = \tfrac{1}{12}(9 \times 10^{-3} \text{ m})(40 \times 10^{-3} \text{ m})^3$$
$$= 48 \times 10^{-9} \text{ m}^4$$

The value of the stress Mc/I is thus

$$\frac{Mc}{I} = \frac{(180 \text{ N} \cdot \text{m})(20 \times 10^{-3} \text{ m})}{48 \times 10^{-9} \text{ m}^4} = 75 \text{ MPa}$$

Substituting this value for Mc/I into Eq. (4.29) and making $\sigma_m = 150$ MPa, we write

$$150 \text{ MPa} = K(75 \text{ MPa})$$
$$K = 2$$

We have, on the other hand,

$$\frac{D}{d} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.5$$

Using the curve of Fig. 4.32 corresponding to $D/d = 1.5$, we find that the value $K = 2$ corresponds to a value of r/d equal to 0.13. We have, therefore,

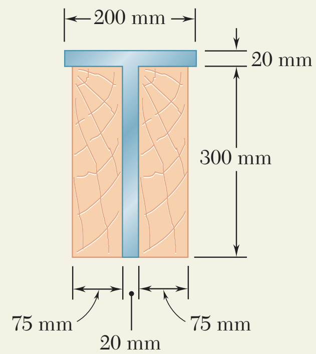
$$\frac{r}{d} = 0.13$$

$$r = 0.13d = 0.13(40 \text{ mm}) = 5.2 \text{ mm}$$

The smallest allowable width of the grooves is thus

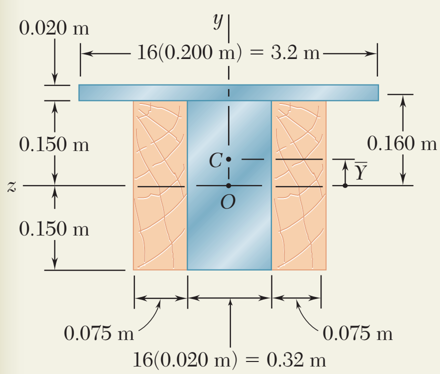
$$2r = 2(5.2 \text{ mm}) = 10.4 \text{ mm}$$

SAMPLE PROBLEM 4.3



Two steel plates have been welded together to form a beam in the shape of a T that has been strengthened by securely bolting to it the two oak timbers shown. The modulus of elasticity is 12.5 GPa for the wood and 200 GPa for the steel. Knowing that a bending moment $M = 50 \text{ kN} \cdot \text{m}$ is applied to the composite beam, determine (*a*) the maximum stress in the wood, (*b*) the stress in the steel along the top edge.

SOLUTION



Transformed Section. We first compute the ratio

$$n = \frac{E_s}{E_w} = \frac{200 \text{ GPa}}{12.5 \text{ GPa}} = 16$$

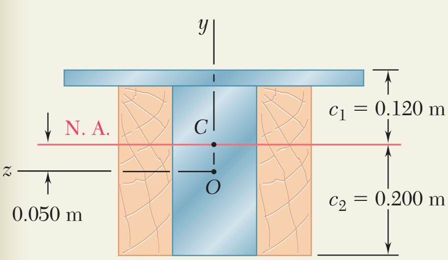
Multiplying the horizontal dimensions of the steel portion of the section by $n = 16$, we obtain a transformed section made entirely of wood.

Neutral Axis. The neutral axis passes through the centroid of the transformed section. Since the section consists of two rectangles, we have

$$\overline{Y} = \frac{\Sigma \overline{y} A}{\Sigma A} = \frac{(0.160 \text{ m})(3.2 \text{ m} \times 0.020 \text{ m}) + 0}{3.2 \text{ m} \times 0.020 \text{ m} + 0.470 \text{ m} \times 0.300 \text{ m}} = 0.050 \text{ m}$$

Centroidal Moment of Inertia. Using the parallel-axis theorem:

$$I = \tfrac{1}{12}(0.470)(0.300)^3 + (0.470 \times 0.300)(0.050)^2$$
$$+ \tfrac{1}{12}(3.2)(0.020)^3 + (3.2 \times 0.020)(0.160 - 0.050)^2$$
$$I = 2.19 \times 10^{-3} \text{ m}^4$$



a. Maximum Stress in Wood. The wood farthest from the neutral axis is located along the bottom edge, where $c_2 = 0.200$ m.

$$\sigma_w = \frac{Mc_2}{I} = \frac{(50 \times 10^3 \text{ N} \cdot \text{m})(0.200 \text{ m})}{2.19 \times 10^{-3} \text{ m}^4}$$

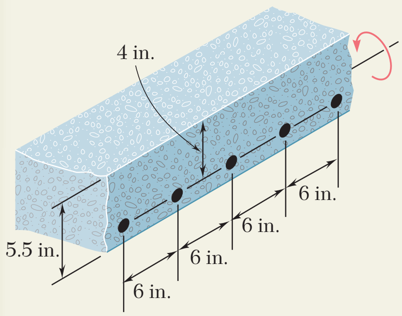
$$\sigma_w = 4.57 \text{ MPa} \quad \blacktriangleleft$$

b. Stress in Steel. Along the top edge $c_1 = 0.120$ m. From the transformed section we obtain an equivalent stress in wood, which must be multiplied by n to obtain the stress in steel.

$$\sigma_s = n\frac{Mc_1}{I} = (16)\frac{(50 \times 10^3 \text{ N} \cdot \text{m})(0.120 \text{ m})}{2.19 \times 10^{-3} \text{ m}^4}$$

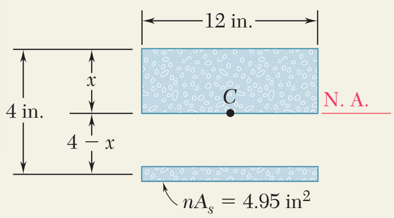
$$\sigma_s = 43.8 \text{ MPa} \quad \blacktriangleleft$$

SAMPLE PROBLEM 4.4



A concrete floor slab is reinforced by $\frac{5}{8}$-in.-diameter steel rods placed 1.5 in. above the lower face of the slab and spaced 6 in. on centers. The modulus of elasticity is 3.6×10^6 psi for the concrete used and 29×10^6 psi for the steel. Knowing that a bending moment of 40 kip · in. is applied to each 1-ft width of the slab, determine (*a*) the maximum stress in the concrete, (*b*) the stress in the steel.

SOLUTION



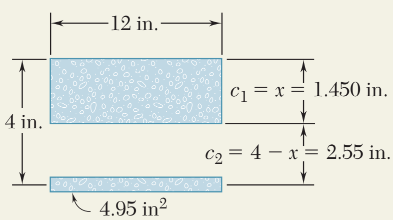
Transformed Section. We consider a portion of the slab 12 in. wide, in which there are two $\frac{5}{8}$-in.-diameter rods having a total cross-sectional area

$$A_s = 2\left[\frac{\pi}{4}\left(\frac{5}{8}\text{ in.}\right)^2\right] = 0.614 \text{ in}^2$$

Since concrete acts only in compression, all the tensile forces are carried by the steel rods, and the transformed section consists of the two areas shown. One is the portion of concrete in compression (located above the neutral axis), and the other is the transformed steel area nA_s. We have

$$n = \frac{E_s}{E_c} = \frac{29 \times 10^6 \text{ psi}}{3.6 \times 10^6 \text{ psi}} = 8.06$$

$$nA_s = 8.06(0.614 \text{ in}^2) = 4.95 \text{ in}^2$$

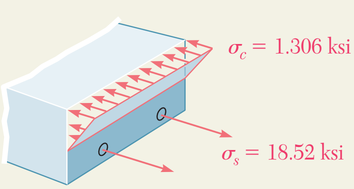


Neutral Axis. The neutral axis of the slab passes through the centroid of the transformed section. Summing moments of the transformed area about the neutral axis, we write

$$12x\left(\frac{x}{2}\right) - 4.95(4 - x) = 0 \qquad x = 1.450 \text{ in.}$$

Moment of Inertia. The centroidal moment of inertia of the transformed area is

$$I = \tfrac{1}{3}(12)(1.450)^3 + 4.95(4 - 1.450)^2 = 44.4 \text{ in}^4$$



a. Maximum Stress in Concrete. At the top of the slab, we have $c_1 = 1.450$ in. and

$$\sigma_c = \frac{Mc_1}{I} = \frac{(40 \text{ kip} \cdot \text{in.})(1.450 \text{ in.})}{44.4 \text{ in}^4} \qquad \sigma_c = 1.306 \text{ ksi} \blacktriangleleft$$

b. Stress in Steel. For the steel, we have $c_2 = 2.55$ in., $n = 8.06$ and

$$\sigma_s = n\frac{Mc_2}{I} = 8.06\frac{(40 \text{ kip} \cdot \text{in.})(2.55 \text{ in.})}{44.4 \text{ in}^4} \qquad \sigma_s = 18.52 \text{ ksi} \blacktriangleleft$$

PROBLEMS

4.33 and 4.34 A bar having the cross section shown has been formed by securely bonding brass and aluminum stock. Using the data given below, determine the largest permissible bending moment when the composite bar is bent about a horizontal axis.

	Aluminum	Brass
Modulus of elasticity	70 GPa	105 GPa
Allowable stress	100 MPa	160 MPa

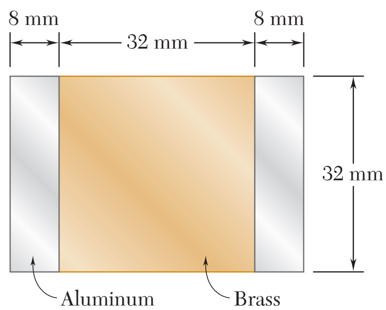


Fig. P4.33

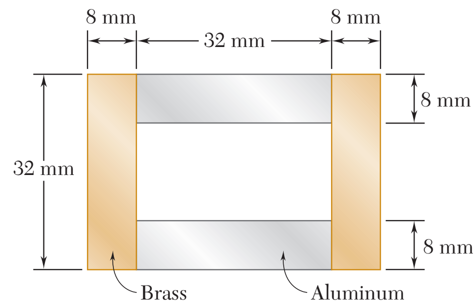


Fig. P4.34

4.35* and *4.36 For the composite bar indicated, determine the largest permissible bending moment when the bar is bent about a vertical axis.

4.35 Bar of Prob. 4.33.
4.36 Bar of Prob. 4.34.

4.37 and 4.38 Wooden beams and steel plates are securely bolted together to form the composite member shown. Using the data given below, determine the largest permissible bending moment when the member is bent about a horizontal axis.

	Wood	Steel
Modulus of elasticity	2×10^6 psi	29×10^6 psi
Allowable stress	2000 psi	22 ksi

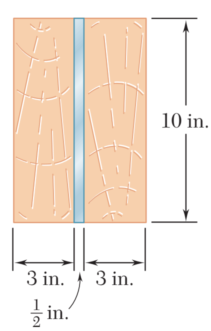


Fig. P4.37

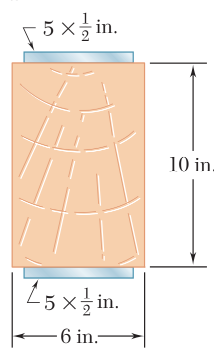


Fig. P4.38

4.39 and 4.40 A steel bar and an aluminum bar are bonded together to form the composite beam shown. The modulus of elasticity for aluminum is 70 GPa and for steel is 200 GPa. Knowing that the beam is bent about a horizontal axis by a couple of moment $M = 1500\ \text{N}\cdot\text{m}$, determine the maximum stress in (*a*) the aluminum, (*b*) the steel.

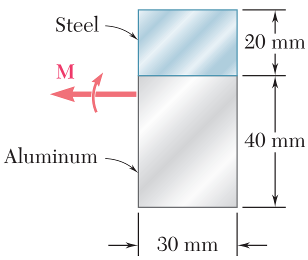


Fig. P4.39

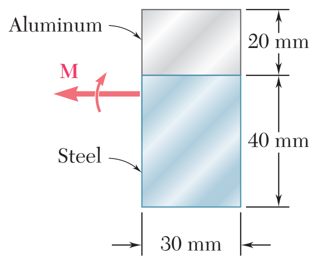


Fig. P4.40

***4.41* and 4.42** The 6 × 12-in. timber beam has been strengthened by bolting to it the steel reinforcement shown. The modulus of elasticity for wood is 1.8×10^6 psi and for steel is 29×10^6 psi. Knowing that the beam is bent about a horizontal axis by a couple of moment $M = 450\ \text{kip}\cdot\text{in.}$, determine the maximum stress in (*a*) the wood, (*b*) the steel.

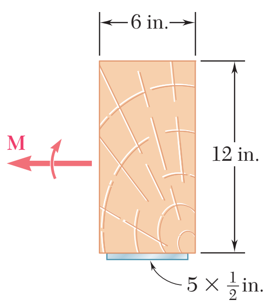


Fig. *P4.41*

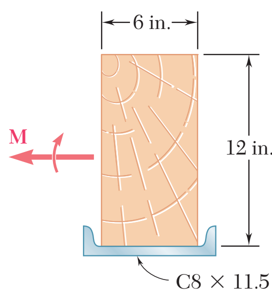


Fig. P4.42

4.43 and 4.44 For the composite beam indicated, determine the radius of curvature caused by the couple of moment $1500\ \text{N}\cdot\text{m}$.

4.43 Beam of Prob. 4.39.

4.44 Beam of Prob. 4.40.

***4.45* and 4.46** For the composite beam indicated, determine the radius of curvature caused by the couple of moment $450\ \text{kip}\cdot\text{in.}$

4.45 Beam of Prob. 4.41.

4.46 Beam of Prob. 4.42.

4.47 The reinforced concrete beam shown is subjected to a positive bending moment of $175\ \text{kN}\cdot\text{m}$. Knowing that the modulus of elasticity is 25 GPa for the concrete and 200 GPa for the steel, determine (*a*) the stress in the steel, (*b*) the maximum stress in the concrete.

4.48 Solve Prob. 4.47, assuming that the 300-mm width is increased to 350 mm.

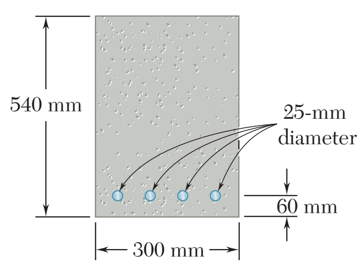


Fig. P4.47

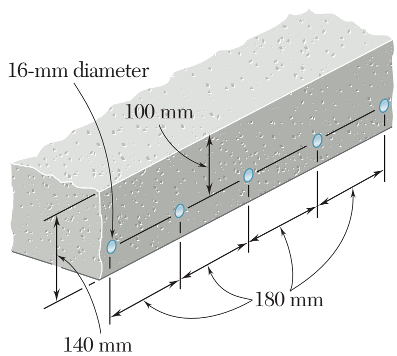


Fig. P4.49

4.49 A concrete slab is reinforced by 16-mm-diameter steel rods placed on 180-mm centers as shown. The modulus of elasticity is 20 GPa for the concrete and 200 GPa for the steel. Using an allowable stress of 9 MPa for the concrete and 120 MPa for the steel, determine the largest bending moment in a portion of slab 1 m wide.

4.50 Solve Prob. 4.49, assuming that the spacing of the 16-mm-diameter rods is increased to 225 mm on centers.

4.51 A concrete beam is reinforced by three steel rods placed as shown. The modulus of elasticity is 3×10^6 psi for the concrete and 29×10^6 psi for the steel. Using an allowable stress of 1350 psi for the concrete and 20 ksi for the steel, determine the largest allowable positive bending moment in the beam.

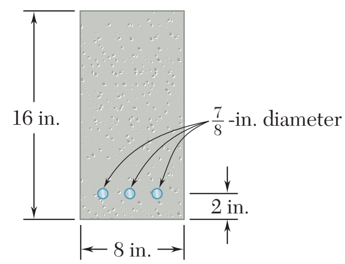


Fig. P4.51

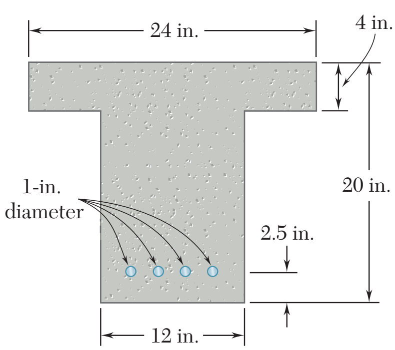


Fig. *P4.52*

4.52 Knowing that the bending moment in the reinforced concrete beam is +100 kip · ft and that the modulus of elasticity is 3.625×10^6 psi for the concrete and 29×10^6 psi for the steel, determine (*a*) the stress in the steel, (*b*) the maximum stress in the concrete.

4.53 The design of a reinforced concrete beam is said to be *balanced* if the maximum stresses in the steel and concrete are equal, respectively, to the allowable stresses σ_s and σ_c. Show that to achieve a balanced design the distance x from the top of the beam to the neutral axis must be

$$x = \frac{d}{1 + \dfrac{\sigma_s E_c}{\sigma_c E_s}}$$

where E_c and E_s are the moduli of elasticity of concrete and steel, respectively, and d is the distance from the top of the beam to the reinforcing steel.

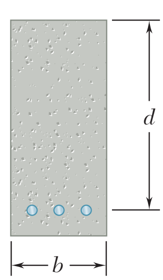


Fig. P4.53 and *P4.54*

4.54 For the concrete beam shown, the modulus of elasticity is 3.5×10^6 psi for the concrete and 29×10^6 psi for the steel. Knowing that $b = 8$ in. and $d = 22$ in., and using an allowable stress of 1800 psi for the concrete and 20 ksi for the steel, determine (*a*) the required area A_s of the steel reinforcement if the beam is to be balanced, (*b*) the largest allowable bending moment. (See Prob. 4.53 for definition of a balanced beam.)

4.55 and 4.56 Five metal strips, each 40 mm wide, are bonded together to form the composite beam shown. The modulus of elasticity is 210 GPa for the steel, 105 GPa for the brass, and 70 GPa for the aluminum. Knowing that the beam is bent about a horizontal axis by a couple of moment 1800 N · m, determine (*a*) the maximum stress in each of the three metals, (*b*) the radius of curvature of the composite beam.

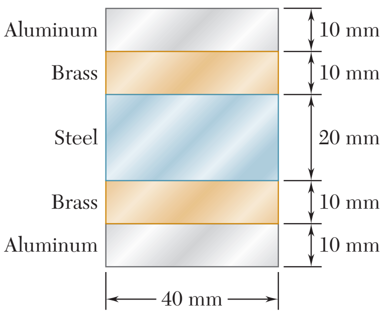


Fig. P4.55

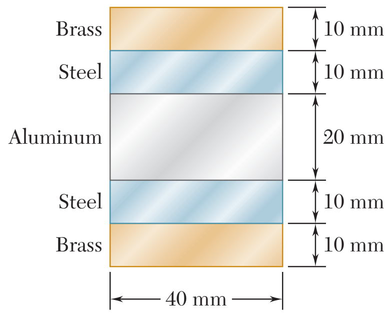


Fig. *P4.56*

4.57 The composite beam shown is formed by bonding together a brass rod and an aluminum rod of semicircular cross sections. The modulus of elasticity is 15×10^6 psi for the brass and 10×10^6 psi for the aluminum. Knowing that the composite beam is bent about a horizontal axis by couples of moment 8 kip · in., determine the maximum stress (*a*) in the brass, (*b*) in the aluminum.

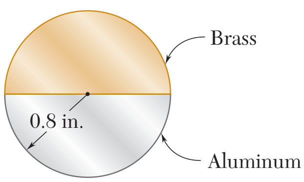


Fig. P4.57

4.58 A steel pipe and an aluminum pipe are securely bonded together to form the composite beam shown. The modulus of elasticity is 200 GPa for the steel and 70 GPa for the aluminum. Knowing that the composite beam is bent by a couple of moment 500 N · m, determine the maximum stress (*a*) in the aluminum, (*b*) in the steel.

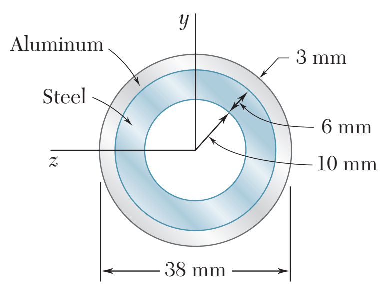


Fig. *P4.58*

4.59 The rectangular beam shown is made of a plastic for which the value of the modulus of elasticity in tension is one-half of its value in compression. For a bending moment $M = 600$ N · m, determine the maximum (*a*) tensile stress, (*b*) compressive stress.

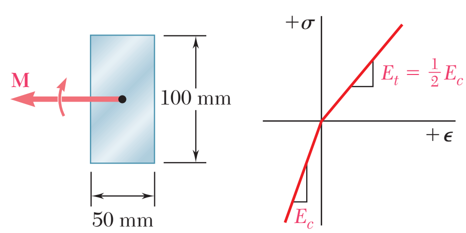


Fig. P4.59

***4.60** A rectangular beam is made of material for which the modulus of elasticity is E_t in tension and E_c in compression. Show that the curvature of the beam in pure bending is

$$\frac{1}{\rho} = \frac{M}{E_r I}$$

where

$$E_r = \frac{4E_t E_c}{(\sqrt{E_t} + \sqrt{E_c})^2}$$

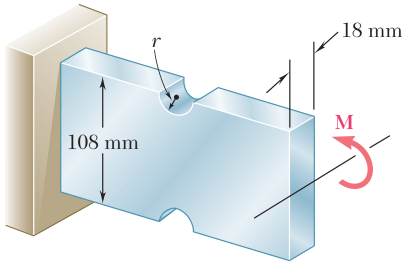


Fig. *P4.61* and *P4.62*

4.61 Semicircular grooves of radius r must be milled as shown in the sides of a steel member. Using an allowable stress of 60 MPa, determine the largest bending moment that can be applied to the member when (*a*) $r = 9$ mm, (*b*) $r = 18$ mm.

4.62 Semicircular grooves of radius r must be milled as shown in the sides of a steel member. Knowing that $M = 450$ N · m, determine the maximum stress in the member when the radius r of the semicircular grooves is (*a*) $r = 9$ mm, (*b*) $r = 18$ mm.

4.63 Knowing that the allowable stress for the beam shown is 90 MPa, determine the allowable bending moment M when the radius r of the fillets is (*a*) 8 mm, (*b*) 12 mm.

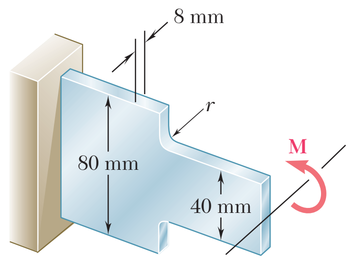


Fig. P4.63 and P4.64

4.64 Knowing that $M = 250$ N · m, determine the maximum stress in the beam shown when the radius r of the fillets is (*a*) 4 mm, (*b*) 8 mm.

4.65 The allowable stress used in the design of a steel bar is 12 ksi. Determine the largest couple **M** that can be applied to the bar (*a*) if the bar is designed with grooves having semicircular portions of radius $r = \frac{3}{4}$ in., as shown in Fig. *a*, (*b*) if the bar is redesigned by removing the material above and below the dashed lines as shown in Fig. *b*.

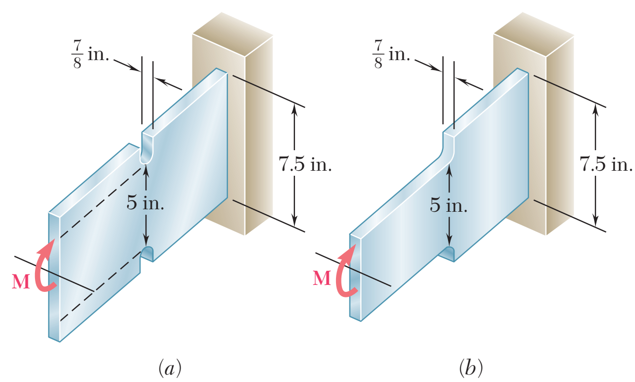


Fig. P4.65 and P4.66

4.66 A couple of moment $M = 20$ kip · in. is to be applied to the end of a steel bar. Determine the maximum stress in the bar (*a*) if the bar is designed with grooves having semicircular portions of radius $r = \frac{1}{2}$ in., as shown in Fig. *a*, (*b*) if the bar is redesigned by removing the material above and below the dashed lines as shown in Fig. *b*.

*4.8 PLASTIC DEFORMATIONS

When we derived the fundamental relation $\sigma_x = -My/I$ in Sec. 4.4, we assumed that Hooke's law applied throughout the member. If the yield strength is exceeded in some portion of the member, or if the material involved is a brittle material with a nonlinear stress-strain diagram, this relation ceases to be valid. The purpose of this section is to develop a more general method for the determination of the distribution of stresses in a member in pure bending, which can be used when Hooke's law does not apply.

We first recall that no specific stress-strain relationship was assumed in Sec. 4.3, when we proved that the normal strain ϵ_x varies linearly with the distance y from the neutral surface. Thus, we can still use this property in our present analysis and write

$$\epsilon_x = -\frac{y}{c}\epsilon_m \tag{4.10}$$

where y represents the distance of the point considered from the neutral surface, and c the maximum value of y.

However, we cannot assume anymore that, in a given section, the neutral axis passes through the centroid of that section, since this property was derived in Sec. 4.4 under the assumption of elastic deformations. In general, the neutral axis must be located by trial and error, until a distribution of stresses has been found, that satisfies Eqs. (4.1) and (4.3) of Sec. 4.2. However, in the particular case of a member possessing both a vertical and a horizontal plane of symmetry, and made of a material characterized by the same stress-strain relation in tension and in compression, the neutral axis will coincide with the horizontal axis of symmetry of the section. Indeed, the properties of the material require that the stresses be symmetric with respect to the neutral axis, i.e., with respect to *some* horizontal axis, and it is clear that this condition will be met, and Eq. (4.1) satisfied at the same time, only if that axis is the horizontal axis of symmetry itself.

Our analysis will first be limited to the special case we have just described. The distance y in Eq. (4.10) is thus measured from the horizontal axis of symmetry z of the cross section, and the distribution of strain ϵ_x is linear and symmetric with respect to that axis (Fig. 4.30). On the other hand, the stress-strain curve is symmetric with respect to the origin of coordinates (Fig. 4.31).

The distribution of stresses in the cross section of the member, i.e., the plot of σ_x versus y, is obtained as follows. Assuming that $\sigma_{\max}$ has been specified, we first determine the corresponding value of ϵ_m from the stress-strain diagram and carry this value into Eq. (4.10). Then, for each value of y, we determine the corresponding value of ϵ_x from Eq. (4.10) or Fig. 4.30, and obtain from the stress-strain diagram of Fig. 4.31 the stress σ_x corresponding to this value of ϵ_x. Plotting σ_x against y yields the desired distribution of stresses (Fig. 4.32).

We now recall that, when we derived Eq. (4.3) in Sec. 4.2, we assumed no particular relation between stress and strain. We can therefore use Eq. (4.3) to determine the bending moment M corresponding to the stress distribution obtained in Fig. 4.32. Considering the particular

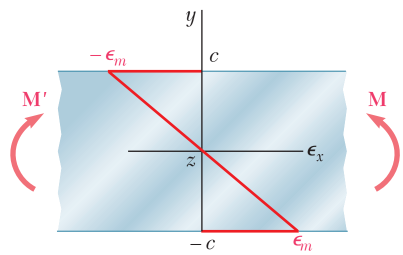


Fig. 4.30 Linear strain distribution in beam.

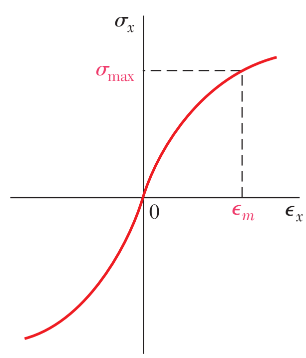


Fig. 4.31 Nonlinear stress-strain material diagram.

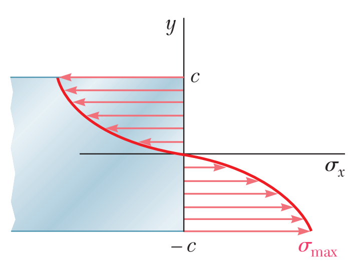


Fig. 4.32 Nonlinear stress distribution in beam.

case of a member with a rectangular cross section of width b, we express the element of area in Eq. (4.3) as $dA = b\,dy$ and write

$$M = -b\int_{-c}^{c} y\sigma_x\,dy \tag{4.30}$$

where σ_x is the function of y plotted in Fig. 4.32. Since σ_x is an odd function of y, we can write Eq. (4.30) in the alternative form

$$M = -2b\int_{0}^{c} y\sigma_x\,dy \tag{4.31}$$

If σ_x is a known analytical function of ϵ_x, Eq. (4.10) can be used to express σ_x as a function of y, and the integral in (4.31) can be determined analytically. Otherwise, the bending moment M can be obtained through a numerical integration. This computation becomes more meaningful if we note that the integral in Eq. (4.31) represents the first moment with respect to the horizontal axis of the area in Fig. 4.32 that is located above the horizontal axis and is bounded by the stress-distribution curve and the vertical axis.

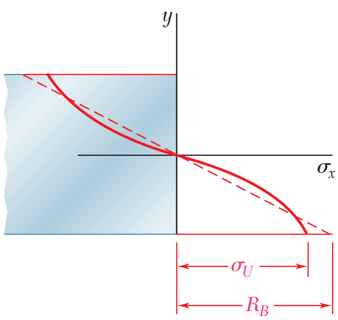


Fig. 4.33 Beam stress distribution at ultimate moment M_U.

An important value of the bending moment is the ultimate bending moment M_U that causes failure of the member. This value can be determined from the ultimate strength σ_U of the material by choosing $\sigma_{max} = \sigma_U$ and carrying out the computations indicated earlier. However, it is found more convenient in practice to determine M_U experimentally for a specimen of a given material. Assuming a fictitious linear distribution of stresses, Eq. (4.15) is then used to determine the corresponding maximum stress R_B:

$$R_B = \frac{M_U c}{I} \tag{4.32}$$

The fictitious stress R_B is called the *modulus of rupture in bending* of the given material. It can be used to determine the ultimate bending moment M_U of a member made of the same material and having a cross section of the same shape, but of different dimensions, by solving Eq. (4.32) for M_U. Since, in the case of a member with a rectangular cross section, the actual and the fictitious linear stress distributions shown in Fig. 4.33 must yield the same value M_U for the ultimate bending moment, the areas they define must have the same first moment with respect to the horizontal axis. It is thus clear that the modulus of rupture R_B will always be larger than the actual ultimate strength σ_U.

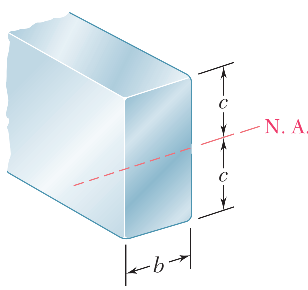


Fig. 4.34 Beam with rectangular cross section.

*4.9 MEMBERS MADE OF AN ELASTOPLASTIC MATERIAL

In order to gain a better insight into the plastic behavior of a member in bending, let us consider the case of a member made of an *elastoplastic material* and first assume the member to have a *rectangular cross section* of width b and depth $2c$ (Fig. 4.34). We recall from Sec. 2.17 that the stress-strain diagram for an idealized elastoplastic material is as shown in Fig. 4.35.

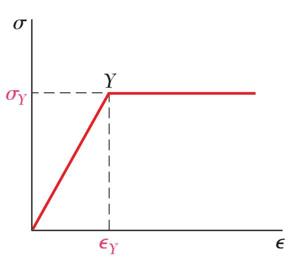


Fig. 4.35 Idealized steel stress-strain diagram.

As long as the normal stress σ_x does not exceed the yield strength σ_Y, Hooke's law applies, and the stress distribution across the section is linear (Fig. 4.36*a*). The maximum value of the stress is

$$\sigma_m = \frac{Mc}{I} \tag{4.15}$$

As the bending moment increases, σ_m eventually reaches the value σ_Y (Fig. 4.36*b*). Substituting this value into Eq. (4.15), and solving for the corresponding value of M, we obtain the value M_Y of the bending moment at the onset of yield:

$$M_Y = \frac{I}{c}\sigma_Y \tag{4.33}$$

The moment M_Y is referred to as the *maximum elastic moment*, since it is the largest moment for which the deformation remains fully elastic. Recalling that, for the rectangular cross section considered here,

$$\frac{I}{c} = \frac{b(2c)^3}{12c} = \frac{2}{3}bc^2 \tag{4.34}$$

we write

$$M_Y = \frac{2}{3}bc^2\sigma_Y \tag{4.35}$$

As the bending moment further increases, plastic zones develop in the member, with the stress uniformly equal to $-\sigma_Y$ in the upper zone, and to $+\sigma_Y$ in the lower zone (Fig. 4.36*c*). Between the plastic zones, an elastic core subsists, in which the stress σ_x varies linearly with y,

$$\sigma_x = -\frac{\sigma_Y}{y_Y}y \tag{4.36}$$

where y_Y represents half the thickness of the elastic core. As M increases, the plastic zones expand until, at the limit, the deformation is fully plastic (Fig. 4.36*d*).

Equation (4.31) will be used to determine the value of the bending moment M corresponding to a given thickness $2y_Y$ of the elastic core. Recalling that σ_x is given by Eq. (4.36) for $0 \le y \le y_Y$, and is equal to $-\sigma_Y$ for $y_Y \le y \le c$, we write

$$\begin{aligned} M &= -2b\int_0^{y_Y} y\left(-\frac{\sigma_Y}{y_Y}y\right)dy - 2b\int_{y_Y}^{c} y(-\sigma_Y)\,dy \\ &= \frac{2}{3}by_Y^2\sigma_Y + bc^2\sigma_Y - by_Y^2\sigma_Y \end{aligned}$$

$$M = bc^2\sigma_Y\left(1 - \frac{1}{3}\frac{y_Y^2}{c^2}\right) \tag{4.37}$$

or, in view of Eq. (4.35),

$$M = \frac{3}{2}M_Y\left(1 - \frac{1}{3}\frac{y_Y^2}{c^2}\right) \tag{4.38}$$

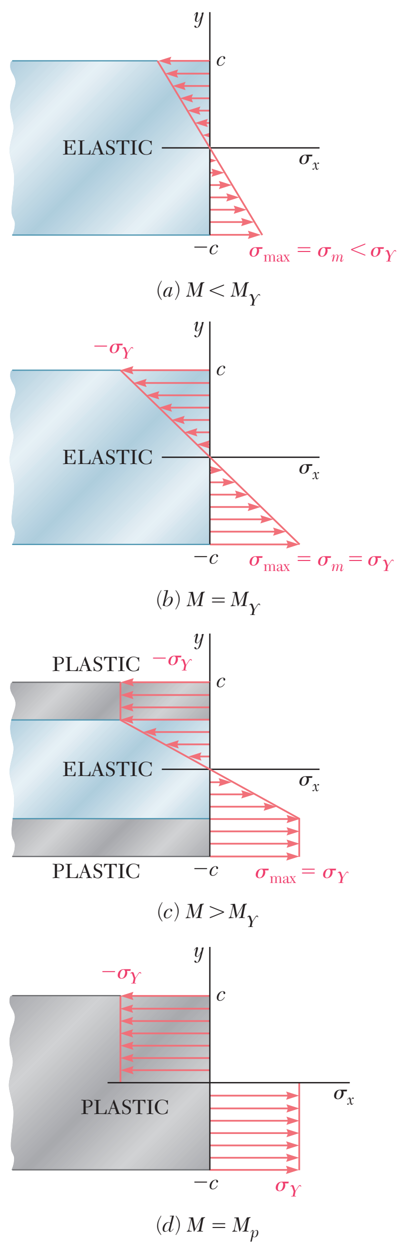


Fig. 4.36 Bending stress distribution in beam for different moments.

where M_Y is the maximum elastic moment. Note that as y_Y approaches zero, the bending moment approaches the limiting value

$$M_p = \frac{3}{2}M_Y \tag{4.39}$$

This value of the bending moment, which corresponds to a fully plastic deformation (Fig. 4.36*d*), is called the *plastic moment* of the member considered. Note that Eq. (4.39) is valid only for a *rectangular member made of an elastoplastic material.*

You should keep in mind that the distribution of *strain* across the section remains linear after the onset of yield. Therefore, Eq. (4.8) of Sec. 4.3 remains valid and can be used to determine the half-thickness y_Y of the elastic core. We have

$$y_Y = \epsilon_Y \rho \tag{4.40}$$

where ϵ_Y is the yield strain and ρ the radius of curvature corresponding to a bending moment $M \geq M_Y$. When the bending moment is equal to M_Y, we have $y_Y = c$ and Eq. (4.40) yields

$$c = \epsilon_Y \rho_Y \tag{4.41}$$

where ρ_Y is the radius of curvature corresponding to the maximum elastic moment M_Y. Dividing (4.40) by (4.41) member by member, we obtain the relation†

$$\frac{y_Y}{c} = \frac{\rho}{\rho_Y} \tag{4.42}$$

Substituting for y_Y/c from (4.42) into Eq. (4.38), we express the bending moment M as a function of the radius of curvature ρ of the neutral surface:

$$M = \frac{3}{2}M_Y\left(1 - \frac{1}{3}\frac{\rho^2}{\rho_Y^2}\right) \tag{4.43}$$

Note that Eq. (4.43) is valid only after the onset of yield, i.e., for values of M larger than M_Y. For $M < M_Y$, Eq. (4.21) of Sec. 4.4 should be used.

We observe from Eq. (4.43) that the bending moment reaches the value $M_p = \frac{3}{2}M_Y$ only when $\rho = 0$. Since we clearly cannot have a zero radius of curvature at every point of the neutral surface, we conclude that a fully plastic deformation cannot develop in pure bending. As you will see in Chap. 5, however, such a situation may occur at one point in the case of a beam under a transverse loading.

The stress distributions in a rectangular member corresponding respectively to the maximum elastic moment M_Y and to the limiting case of the plastic moment M_p have been represented in three dimensions in Fig. 4.37. Since, in both cases, the resultants of the elementary tensile and compressive forces must pass through the centroids of the volumes representing the stress distributions and be equal in magnitude to these volumes, we check that

$$R_Y = \tfrac{1}{2}bc\sigma_Y$$

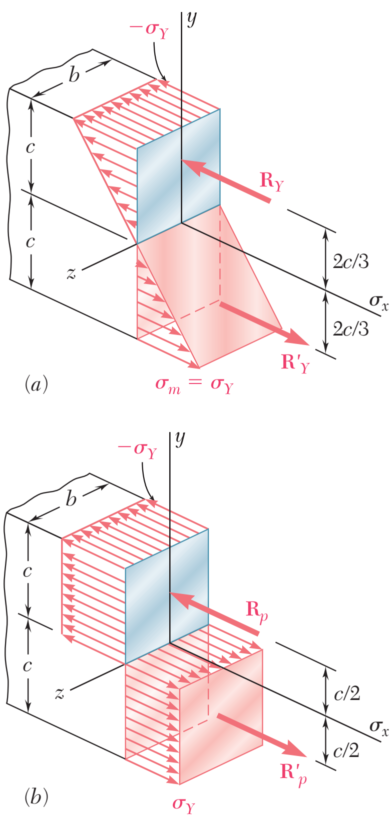


Fig. 4.37 Stress distributions in beam at maximum elastic moment and at plastic moment.

†Equation (4.42) applies to any member made of any ductile material with a well-defined yield point, since its derivation is independent of the shape of the cross section and of the shape of the stress-strain diagram beyond the yield point.

and

$$R_p = bc\sigma_Y$$

and that the moments of the corresponding couples are, respectively,

$$M_Y = (\tfrac{4}{3}c)R_Y = \tfrac{2}{3}bc^2\sigma_Y \tag{4.44}$$

and

$$M_p = cR_p = bc^2\sigma_Y \tag{4.45}$$

We thus verify that, for a rectangular member, $M_p = \frac{3}{2}M_Y$ as required by Eq. (4.39).

For beams of *nonrectangular cross section,* the computation of the maximum elastic moment M_Y and of the plastic moment M_p will usually be simplified if a graphical method of analysis is used, as shown in Sample Prob. 4.5. It will be found in this more general case that the ratio $k = M_p/M_Y$ is generally not equal to $\frac{3}{2}$. For structural shapes such as wide-flange beams, for example, this ratio varies approximately from 1.08 to 1.14. Because it depends only upon the shape of the cross section, the ratio $k = M_p/M_Y$ is referred to as the *shape factor* of the cross section. We note that, if the shape factor k and the maximum elastic moment M_Y of a beam are known, the plastic moment M_p of the beam can be obtained by multiplying M_Y by k:

$$M_p = kM_Y \tag{4.46}$$

The ratio M_p/σ_Y obtained by dividing the plastic moment M_p of a member by the yield strength σ_Y of its material is called the *plastic section modulus* of the member and is denoted by Z. When the plastic section modulus Z and the yield strength σ_Y of a beam are known, the plastic moment M_p of the beam can be obtained by multiplying σ_Y by Z:

$$M_p = Z\sigma_Y \tag{4.47}$$

Recalling from Eq. (4.18) that $M_Y = S\sigma_Y$, and comparing this relation with Eq. (4.47), we note that the shape factor $k = M_p/M_Y$ of a given cross section can be expressed as the ratio of the plastic and elastic section moduli:

$$k = \frac{M_p}{M_Y} = \frac{Z\sigma_Y}{S\sigma_Y} = \frac{Z}{S} \tag{4.48}$$

Considering the particular case of a rectangular beam of width b and depth h, we note from Eqs. (4.45) and (4.47) that the *plastic section modulus* of a rectangular beam is

$$Z = \frac{M_p}{\sigma_Y} = \frac{bc^2\sigma_Y}{\sigma_Y} = bc^2 = \tfrac{1}{4}bh^2$$

On the other hand, we recall from Eq. (4.19) of Sec. 4.4 that the *elastic section modulus* of the same beam is

$$S = \tfrac{1}{6}bh^2$$

Substituting into Eq. (4.48) the values obtained for Z and S, we verify that the shape factor of a rectangular beam is

$$k = \frac{Z}{S} = \frac{\frac{1}{4}bh^2}{\frac{1}{6}bh^2} = \frac{3}{2}$$

EXAMPLE 4.05

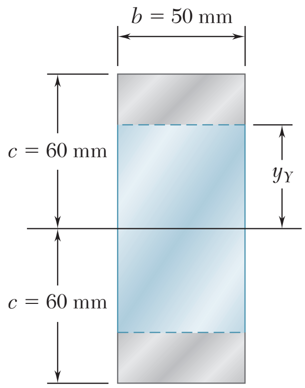


Fig. 4.38

A member of uniform rectangular cross section 50 × 120 mm (Fig. 4.38) is subjected to a bending moment $M = 36.8$ kN · m. Assuming that the member is made of an elastoplastic material with a yield strength of 240 MPa and a modulus of elasticity of 200 GPa, determine (*a*) the thickness of the elastic core, (*b*) the radius of curvature of the neutral surface.

(a) Thickness of Elastic Core. We first determine the maximum elastic moment M_Y. Substituting the given data into Eq. (4.34), we have

$$\frac{I}{c} = \frac{2}{3}bc^2 = \frac{2}{3}(50 \times 10^{-3}\text{ m})(60 \times 10^{-3}\text{ m})^2$$
$$= 120 \times 10^{-6}\text{ m}^3$$

and carrying this value, as well as $\sigma_Y = 240$ MPa, into Eq. (4.33),

$$M_Y = \frac{I}{c}\sigma_Y = (120 \times 10^{-6}\text{ m}^3)(240\text{ MPa}) = 28.8\text{ kN} \cdot \text{m}$$

Substituting the values of M and M_Y into Eq. (4.38), we have

$$36.8\text{ kN} \cdot \text{m} = \frac{3}{2}(28.8\text{ kN} \cdot \text{m})\left(1 - \frac{1}{3}\frac{y_Y^2}{c^2}\right)$$
$$\left(\frac{y_Y}{c}\right)^2 = 0.444 \qquad \frac{y_Y}{c} = 0.666$$

and, since $c = 60$ mm,

$$y_Y = 0.666(60\text{ mm}) = 40\text{ mm}$$

The thickness $2y_Y$ of the elastic core is thus 80 mm.

(b) Radius of Curvature. We note that the yield strain is

$$\epsilon_Y = \frac{\sigma_Y}{E} = \frac{240 \times 10^6\text{ Pa}}{200 \times 10^9\text{ Pa}} = 1.2 \times 10^{-3}$$

Solving Eq. (4.40) for ρ and substituting the values obtained for y_Y and ϵ_Y, we write

$$\rho = \frac{y_Y}{\epsilon_Y} = \frac{40 \times 10^{-3}\text{ m}}{1.2 \times 10^{-3}} = 33.3\text{ m}$$

*4.10 PLASTIC DEFORMATIONS OF MEMBERS WITH A SINGLE PLANE OF SYMMETRY

In our discussion of plastic deformations, we have assumed so far that the member in bending had two planes of symmetry, one containing the couples **M** and **M′**, and the other perpendicular to that plane. Let us now consider the more general case when the member possesses only one plane of symmetry containing the couples **M** and **M′**. However, our analysis will be limited to the situation where the deformation is fully plastic, with the normal stress uniformly equal to $-\sigma_Y$ above the neutral surface, and to $+\sigma_Y$ below that surface (Fig. 4.39*a*).

As indicated in Sec. 4.8, the neutral axis cannot be assumed to coincide with the centroidal axis of the cross section when the cross section is not symmetric with respect to that axis. To locate the neutral axis, we consider the resultant $\mathbf{R}_1$ of the elementary compressive forces exerted on the portion A_1 of the cross section located above the neutral axis, and the resultant $\mathbf{R}_2$ of the tensile forces exerted on the portion A_2 located below the neutral axis (Fig. 4.39*b*). Since the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ form a couple equivalent to the couple applied to the member, they must have the same magnitude. We have therefore $R_1 = R_2$, or $A_1\sigma_Y = A_2\sigma_Y$, from which we conclude that $A_1 = A_2$. In other words, *the neutral axis divides the cross section into portions of equal areas*. Note that the axis obtained in this fashion will *not,* in general, be a centroidal axis of the section.

We also observe that the lines of action of the resultants $\mathbf{R}_1$ and $\mathbf{R}_2$ pass through the centroids C_1 and C_2 of the two portions we have just defined. Denoting by d the distance between C_1 and C_2, and by A the total area of the cross section, we express the plastic moment of the member as

$$M_p = (\tfrac{1}{2}A\sigma_Y)\,d$$

An example of the actual computation of the plastic moment of a member with only one plane of symmetry is given in Sample Prob. 4.6.

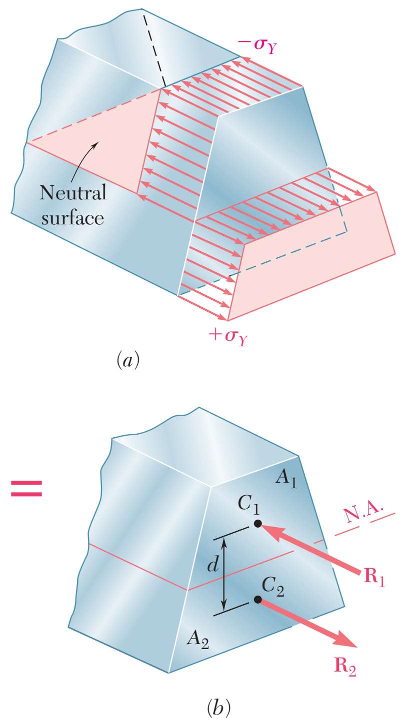


Fig. 4.39 Nonsymmetrical beam subject to plastic moment.

*4.11 RESIDUAL STRESSES

We saw in the preceding sections that plastic zones will develop in a member made of an elastoplastic material if the bending moment is large enough. When the bending moment is decreased back to zero, the corresponding reduction in stress and strain at any given point can be represented by a straight line on the stress-strain diagram, as shown in Fig. 4.40. As you will see presently, the final value of the stress at a point will not, in general, be zero. There will be a residual stress at most points, and that stress may or may not have the same sign as the maximum stress reached at the end of the loading phase.

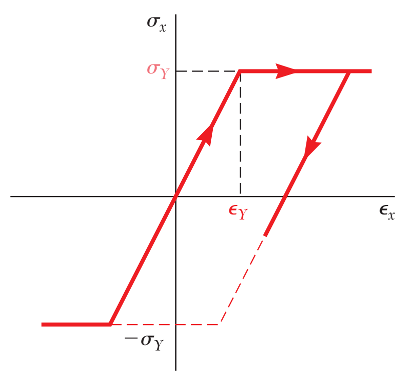


Fig. 4.40 Elastoplastic material stress-strain diagram.

Since the linear relation between σ_x and ϵ_x applies at all points of the member during the unloading phase, Eq. (4.16) can be used to obtain the change in stress at any given point. In other words, the unloading phase can be handled by assuming the member to be fully elastic.

The residual stresses are obtained by applying the principle of superposition in a manner similar to that described in Sec. 2.20 for an axial centric loading and used again in Sec. 3.11 for torsion. We consider, on one hand, the stresses due to the application of the given bending moment M and, on the other, the reverse stresses due to the equal and opposite bending moment $-M$ that is applied to unload the member. The first group of stresses reflect the *elastoplastic* behavior of the material during the loading phase, and the second group the *linear* behavior of the same material during the unloading phase. Adding the two groups of stresses, we obtain the distribution of residual stresses in the member.

EXAMPLE 4.06

For the member of Example 4.05, determine (*a*) the distribution of the residual stresses, (*b*) the radius of curvature, after the bending moment has been decreased from its maximum value of 36.8 kN · m back to zero.

(a) Distribution of Residual Stresses. We recall from Example 4.05 that the yield strength is $\sigma_Y = 240$ MPa and that the thickness of the elastic core is $2y_Y = 80$ mm. The distribution of the stresses in the loaded member is thus as shown in Fig. 4.41*a*.

The distribution of the reverse stresses due to the opposite 36.8 kN · m bending moment required to unload the member is linear and as shown in Fig. 4.41*b*. The maximum stress σ'_m in that distribution is obtained from Eq. (4.15). Recalling from Example 4.05 that $I/c = 120 \times 10^{-6}\ \text{m}^3$, we write

$$\sigma'_m = \frac{Mc}{I} = \frac{36.8\ \text{kN} \cdot \text{m}}{120 \times 10^{-6}\ \text{m}^3} = 306.7\ \text{MPa}$$

Superposing the two distributions of stresses, we obtain the residual stresses shown in Fig. 4.41*c*. We check that, even though the reverse stresses exceed the yield strength σ_Y, the assumption of a linear distribution of the reverse stresses is valid, since they do not exceed $2\sigma_Y$.

(b) Radius of Curvature after Unloading. We can apply Hooke's law at any point of the core $|y| < 40$ mm, since no plastic deformation has occurred in that portion of the member. Thus, the residual strain at the distance $y = 40$ mm is

$$\epsilon_x = \frac{\sigma_x}{E} = \frac{-35.5 \times 10^6\ \text{Pa}}{200 \times 10^9\ \text{Pa}} = -177.5 \times 10^{-6}$$

Solving Eq. (4.8) for ρ and substituting the appropriate values of y and ϵ_x, we write

$$\rho = -\frac{y}{\epsilon_x} = \frac{40 \times 10^{-3}\ \text{m}}{177.5 \times 10^{-6}} = 225\ \text{m}$$

The value obtained for ρ after the load has been removed represents a permanent deformation of the member.

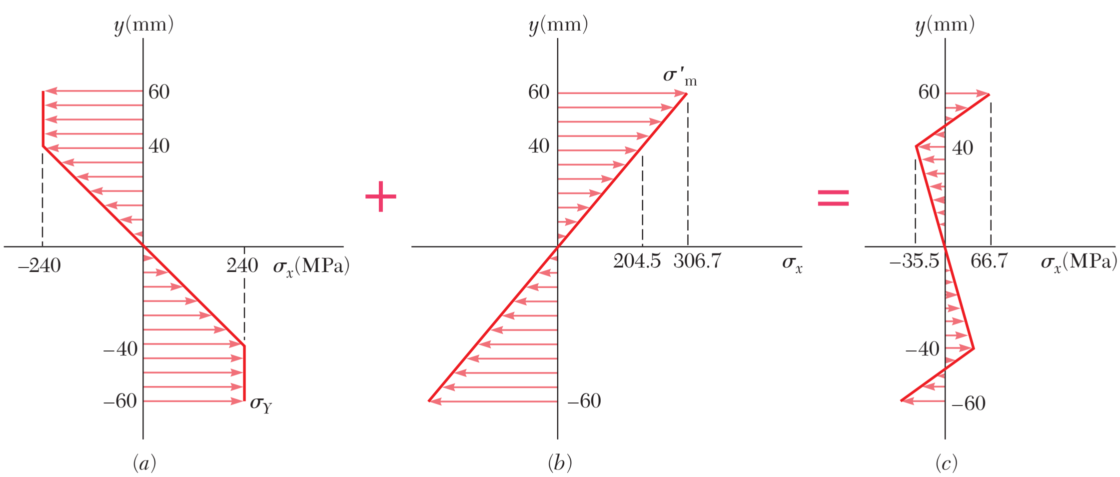
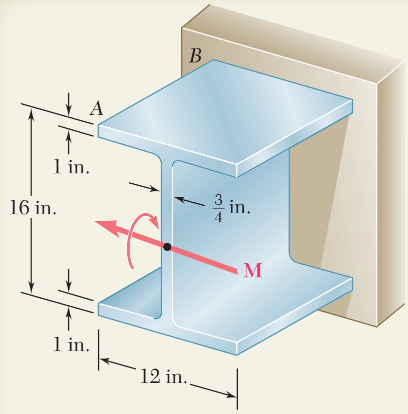
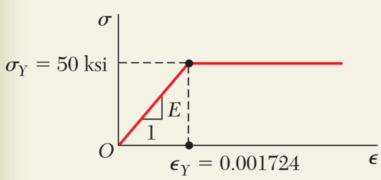


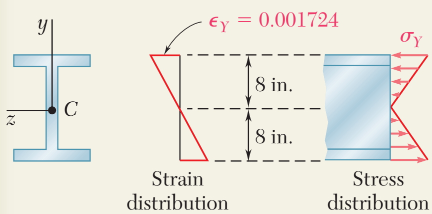
Fig. 4.41

SAMPLE PROBLEM 4.5



Beam AB has been fabricated from a high-strength low-alloy steel that is assumed to be elastoplastic with $E = 29 \times 10^6$ psi and $\sigma_Y = 50$ ksi. Neglecting the effect of fillets, determine the bending moment M and the corresponding radius of curvature (*a*) when yield first occurs, (*b*) when the flanges have just become fully plastic.





SOLUTION

a. Onset of Yield. The centroidal moment of inertia of the section is

$$I = \tfrac{1}{12}(12\text{ in.})(16\text{ in.})^3 - \tfrac{1}{12}(12\text{ in.} - 0.75\text{ in.})(14\text{ in.})^3 = 1524\text{ in}^4$$

Bending Moment. For $\sigma_{max} = \sigma_Y = 50$ ksi and $c = 8$ in., we have

$$M_Y = \frac{\sigma_Y I}{c} = \frac{(50\text{ ksi})(1524\text{ in}^4)}{8\text{ in.}} \qquad M_Y = 9525\text{ kip} \cdot \text{in.} \quad ◀$$

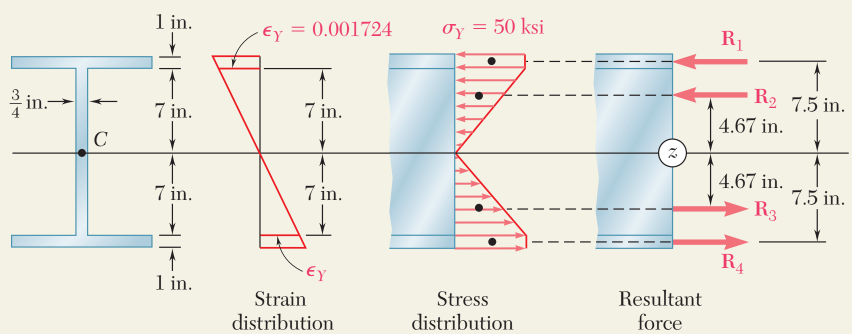
Radius of Curvature. Noting that, at $c = 8$ in., the strain is $\epsilon_Y = \sigma_Y/E = (50\text{ ksi})/(29 \times 10^6\text{ psi}) = 0.001724$, we have from Eq. (4.41)

$$c = \epsilon_Y \rho_Y \qquad 8\text{ in.} = 0.001724\rho_Y \qquad \rho_Y = 4640\text{ in.} \quad ◀$$

b. Flanges Fully Plastic. When the flanges have just become fully plastic, the strains and stresses in the section are as shown in the figure below.

We replace the elementary compressive forces exerted on the top flange and on the top half of the web by their resultants $\mathbf{R}_1$ and $\mathbf{R}_2$, and similarly replace the tensile forces by $\mathbf{R}_3$ and $\mathbf{R}_4$.

$$R_1 = R_4 = (50\text{ ksi})(12\text{ in.})(1\text{ in.}) = 600\text{ kips}$$
$$R_2 = R_3 = \tfrac{1}{2}(50\text{ ksi})(7\text{ in.})(0.75\text{ in.}) = 131.3\text{ kips}$$

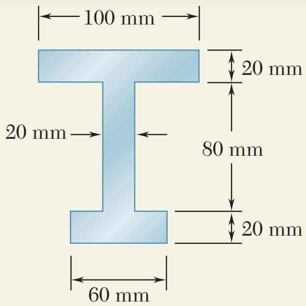


Bending Moment. Summing the moments of $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, and $\mathbf{R}_4$ about the z axis, we write

$$\begin{aligned} M &= 2[R_1(7.5\text{ in.}) + R_2(4.67\text{ in.})] \\ &= 2[(600)(7.5) + (131.3)(4.67)] \qquad M = 10{,}230\text{ kip} \cdot \text{in.} \quad ◀ \end{aligned}$$

Radius of Curvature. Since $y_Y = 7$ in. for this loading, we have from Eq. (4.40)

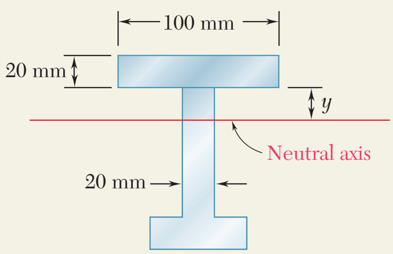
$$y_Y = \epsilon_Y \rho \qquad 7\text{ in.} = (0.001724)\rho \qquad \rho = 4060\text{ in.} = 338\text{ ft} \quad ◀$$



SAMPLE PROBLEM 4.6

Determine the plastic moment M_p of a beam with the cross section shown when the beam is bent about a horizontal axis. Assume that the material is elastoplastic with a yield strength of 240 MPa.

SOLUTION



Neutral Axis. When the deformation is fully plastic, the neutral axis divides the cross section into two portions of equal areas. Since the total area is

$$A = (100)(20) + (80)(20) + (60)(20) = 4800 \text{ mm}^2$$

the area located above the neutral axis must be 2400 mm^2. We write

$$(20)(100) + 20y = 2400 \qquad y = 20 \text{ mm}$$

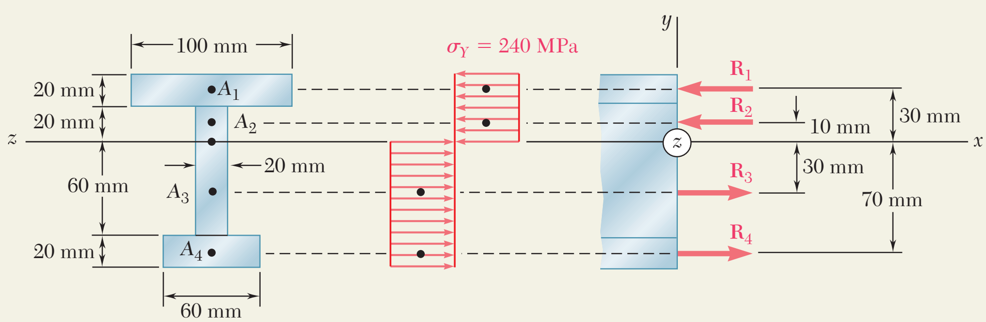
Note that the neutral axis does *not* pass through the centroid of the cross section.

Plastic Moment. The resultant $\mathbf{R}_i$ of the elementary forces exerted on the partial area A_i is equal to

$$R_i = A_i\sigma_Y$$

and passes through the centroid of that area. We have

$$R_1 = A_1\sigma_Y = [(0.100 \text{ m})(0.020 \text{ m})]240 \text{ MPa} = 480 \text{ kN}$$
$$R_2 = A_2\sigma_Y = [(0.020 \text{ m})(0.020 \text{ m})]240 \text{ MPa} = 96 \text{ kN}$$
$$R_3 = A_3\sigma_Y = [(0.020 \text{ m})(0.060 \text{ m})]240 \text{ MPa} = 288 \text{ kN}$$
$$R_4 = A_4\sigma_Y = [(0.060 \text{ m})(0.020 \text{ m})]240 \text{ MPa} = 288 \text{ kN}$$



The plastic moment M_p is obtained by summing the moments of the forces about the z axis.

$$\begin{aligned} M_p &= (0.030 \text{ m})R_1 + (0.010 \text{ m})R_2 + (0.030 \text{ m})R_3 + (0.070 \text{ m})R_4 \\ &= (0.030 \text{ m})(480 \text{ kN}) + (0.010 \text{ m})(96 \text{ kN}) \\ &\quad +(0.030 \text{ m})(288 \text{ kN}) + (0.070 \text{ m})(288 \text{ kN}) \\ &= 44.16 \text{ kN}\cdot\text{m} \end{aligned}$$

$$M_p = 44.2 \text{ kN}\cdot\text{m} \quad ◀$$

Note: Since the cross section is *not* symmetric about the z axis, the sum of the moments of $\mathbf{R}_1$ and $\mathbf{R}_2$ is *not* equal to the sum of the moments of $\mathbf{R}_3$ and $\mathbf{R}_4$.

SAMPLE PROBLEM 4.7

For the beam of Sample Prob. 4.5, determine the residual stresses and the permanent radius of curvature after the 10,230-kip · in. couple **M** has been removed.

SOLUTION

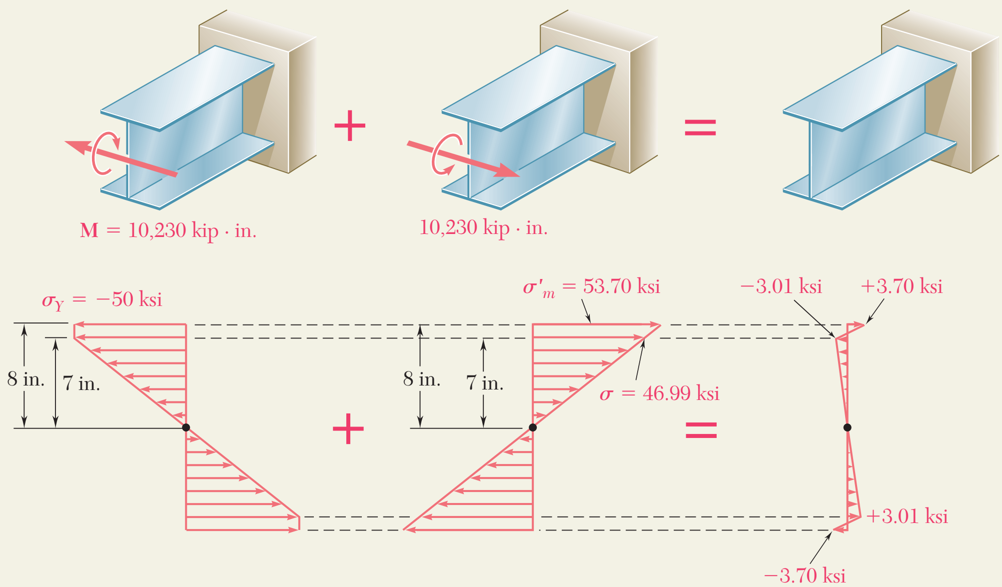
Loading. In Sample Prob. 4.5 a couple of moment $M = 10{,}230$ kip · in. was applied and the stresses shown in Fig. 1 were obtained.

Elastic Unloading. The beam is unloaded by the application of a couple of moment $M = -10{,}230$ kip · in. (which is equal and opposite to the couple originally applied). During this unloading, the action of the beam is fully elastic; recalling from Sample Prob. 4.5 that $I = 1524 \text{ in}^4$, we compute the maximum stress

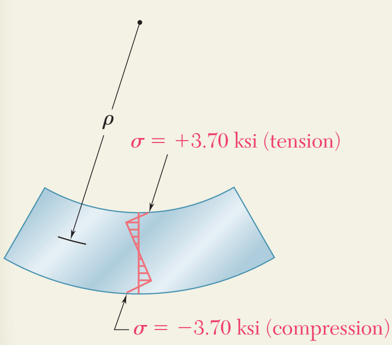
$$\sigma'_m = \frac{Mc}{I} = \frac{(10{,}230 \text{ kip} \cdot \text{in.})(8 \text{ in.})}{1524 \text{ in}^4} = 53.70 \text{ ksi}$$

The stresses caused by the unloading are shown in Fig. 2.

Residual Stresses. We superpose the stresses due to the loading (Fig. 1) and to the unloading (Fig. 2) and obtain the residual stresses in the beam (Fig. 3).



(1) (2) (3)



Permanent Radius of Curvature. At $y = 7$ in. the residual stress is $\sigma = -3.01$ ksi. Since no plastic deformation occurred at this point, Hooke's law can be used and we have $\epsilon_x = \sigma/E$. Recalling Eq. (4.8), we write

$$\rho = -\frac{y}{\epsilon_x} = -\frac{yE}{\sigma} = -\frac{(7 \text{ in.})(29 \times 10^6 \text{ psi})}{-3.01 \text{ ksi}} = +67{,}400 \text{ in.} \qquad \rho = 5620 \text{ ft} \blacktriangleleft$$

We note that the residual stress is tensile on the upper face of the beam and compressive on the lower face, even though the beam is concave upward.

PROBLEMS

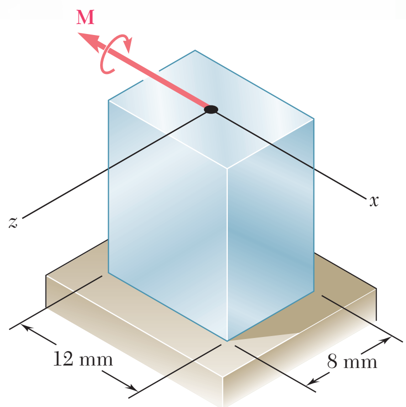


Fig. P4.67

4.67 The prismatic bar shown is made of a steel that is assumed to be elastoplastic with $\sigma_Y = 300$ MPa and is subjected to a couple **M** parallel to the x axis. Determine the moment M of the couple for which (*a*) yield first occurs, (*b*) the elastic core of the bar is 4 mm thick.

4.68 Solve Prob. 4.67, assuming that the couple **M** is parallel to the z axis.

4.69 The prismatic bar shown, made of a steel that is assumed to be elastoplastic with $E = 29 \times 10^6$ psi and $\sigma_Y = 36$ ksi, is subjected to a couple of 1350 lb · in. parallel to the z axis. Determine (*a*) the thickness of the elastic core, (*b*) the radius of curvature of the bar.

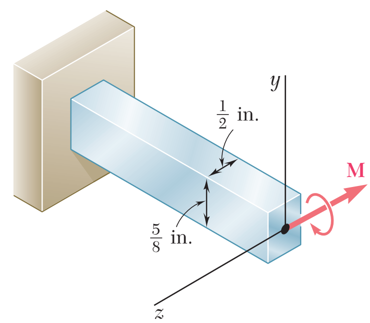


Fig. P4.69

4.70 Solve Prob. 4.69, assuming that the 1350-lb · in. couple is parallel to the y axis.

4.71 A bar of rectangular cross section shown is made of a steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 300$ MPa. Determine the bending moment M for which (*a*) yield first occurs, (*b*) the plastic zones at the top and bottom of the bar are 12 mm thick.

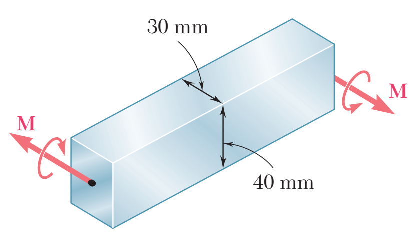


Fig. P4.71 and P4.72

4.72 Bar AB is made of a steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 240$ MPa. Determine the bending moment M for which the radius of curvature of the bar will be (*a*) 18 m, (*b*) 9 m.

4.73 and 4.74 A beam of the cross section shown is made of a steel that is assumed to be elastoplastic with $E = 200$ GPa and $\sigma_Y = 240$ MPa. For bending about the z axis, determine the bending moment at which (*a*) yield first occurs, (*b*) the plastic zones at the top and bottom of the bar are 30 mm thick.

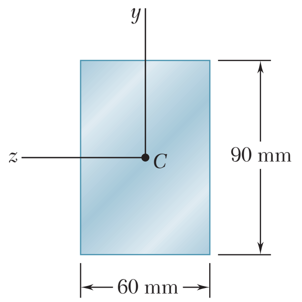


Fig. *P4.73*

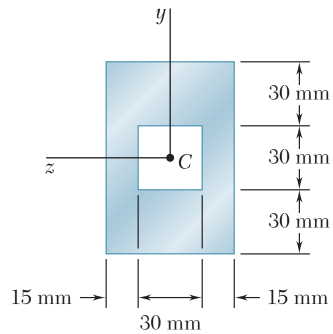


Fig. *P4.74*

4.75 and 4.76 A beam of the cross section shown is made of a steel that is assumed to be elastoplastic with $E = 29 \times 10^6$ psi and $\sigma_Y = 42$ ksi. For bending about the z axis, determine the bending moment at which (*a*) yield first occurs, (*b*) the plastic zones at the top and bottom of the bar are 3 in. thick.

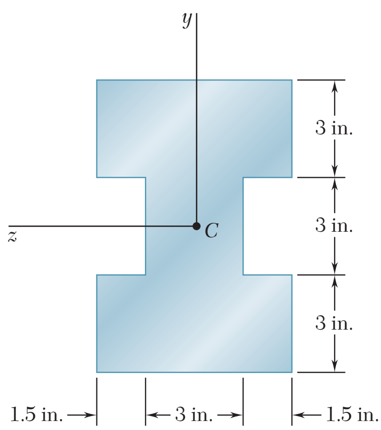


Fig. P4.75

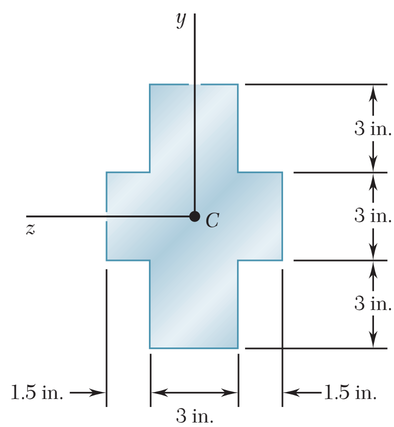


Fig. *P4.76*

4.77 through 4.80 For the beam indicated, determine (*a*) the plastic moment M_p, (*b*) the shape factor of the cross section.

4.77 Beam of Prob. 4.73.
4.78 Beam of Prob. 4.74.
4.79 Beam of Prob. 4.75.
4.80 Beam of Prob. 4.76.

4.81 through 4.84 Determine the plastic moment M_p of a steel beam of the cross section shown, assuming the steel to be elastoplastic with a yield strength of 240 MPa.

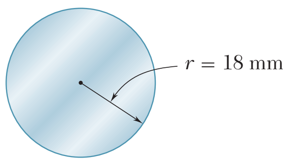


Fig. P4.81

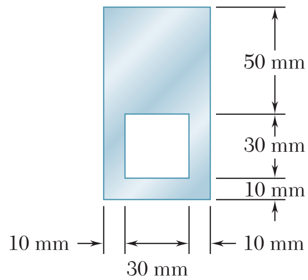


Fig. *P4.82*

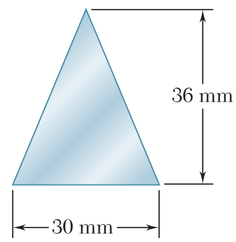


Fig. P4.83

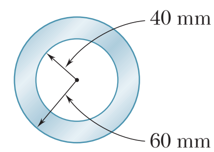


Fig. *P4.84*

4.85 and 4.86 Determine the plastic moment M_p of the cross section shown, assuming the steel to be elastoplastic with a yield strength of 36 ksi.

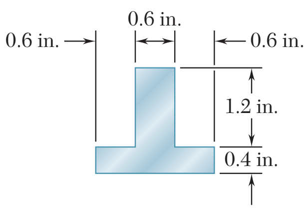


Fig. P4.85

4.87 and 4.88 For the beam indicated, a couple of moment equal to the full plastic moment M_p is applied and then removed. Using a yield strength of 240 MPa, determine the residual stress at $y = 45$ mm.

4.87 Beam of Prob. 4.73.
4.88 Beam of Prob. 4.74.

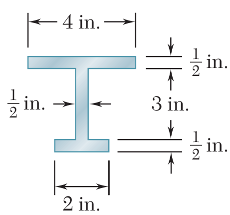


Fig. P4.86

4.89* and *4.90 A bending couple is applied to the bar indicated, causing plastic zones 3 in. thick to develop at the top and bottom of the bar. After the couple has been removed, determine (*a*) the residual stress at $y = 4.5$ in., (*b*) the points where the residual stress is zero, (*c*) the radius of curvature corresponding to the permanent deformation of the bar.

4.89 Beam of Prob. 4.75.
4.90 Beam of Prob. 4.76.

4.91 A bending couple is applied to the beam of Prob. 4.73, causing plastic zones 30 mm thick to develop at the top and bottom of the beam. After the couple has been removed, determine (*a*) the residual stress at $y = 45$ mm, (*b*) the points where the residual stress is zero, (*c*) the radius of curvature corresponding to the permanent deformation of the beam.

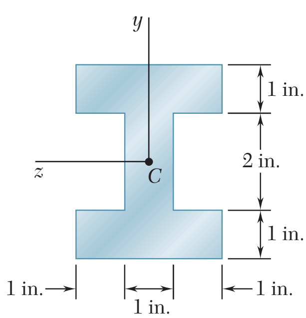


Fig. P4.92

4.92 A beam of the cross section shown is made of a steel that is assumed to be elastoplastic with $E = 29 \times 10^6$ psi and $\sigma_Y = 42$ ksi. A bending couple is applied to the beam about the z axis, causing plastic zones 2 in. thick to develop at the top and bottom of the beam. After the couple has been removed, determine (*a*) the residual stress at $y = 2$ in., (*b*) the points where the residual stress is zero, (*c*) the radius of curvature corresponding to the permanent deformation of the beam.

4.93 A rectangular bar that is straight and unstressed is bent into an arc of circle of radius ρ by two couples of moment M. After the couples are removed, it is observed that the radius of curvature of the bar is ρ_R. Denoting by ρ_Y the radius of curvature of the bar at the onset of yield, show that the radii of curvature satisfy the following relation:

$$\frac{1}{\rho_R} = \frac{1}{\rho}\left\{1 - \frac{3}{2}\frac{\rho}{\rho_Y}\left[1 - \frac{1}{3}\left(\frac{\rho}{\rho_Y}\right)^2\right]\right\}$$

4.94 A solid bar of rectangular cross section is made of a material that is assumed to be elastoplastic. Denoting by M_Y and ρ_Y, respectively, the bending moment and radius of curvature at the onset of yield, determine (*a*) the radius of curvature when a couple of moment $M = 1.25\ M_Y$ is applied to the bar, (*b*) the radius of curvature after the couple is removed. Check the results obtained by using the relation derived in Prob. 4.93.

4.95 The prismatic bar *AB* is made of a steel that is assumed to be elastoplastic and for which $E = 200$ GPa. Knowing that the radius of curvature of the bar is 2.4 m when a couple of moment $M = 350\ \text{N} \cdot \text{m}$ is applied as shown, determine (*a*) the yield strength of the steel, (*b*) the thickness of the elastic core of the bar.

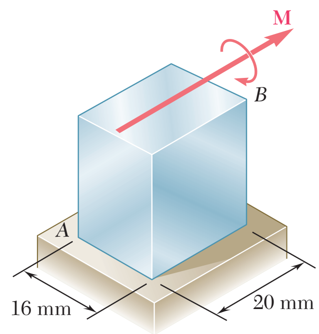


Fig. P4.95

4.96 The prismatic bar *AB* is made of an aluminum alloy for which the tensile stress-strain diagram is as shown. Assuming that the σ-ϵ diagram is the same in compression as in tension, determine (*a*) the radius of curvature of the bar when the maximum stress is 250 MPa, (*b*) the corresponding value of the bending moment. (*Hint*: For part *b*, plot σ versus y and use an approximate method of integration.)

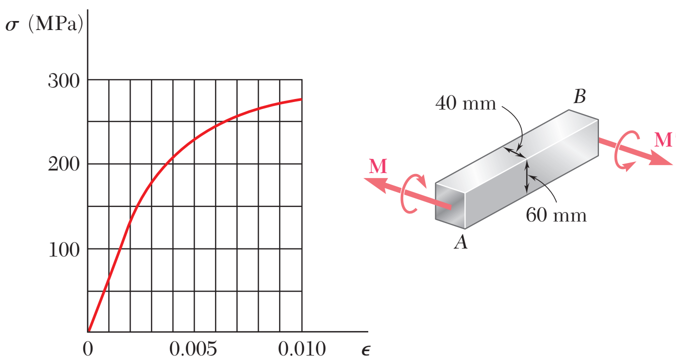


Fig. P4.96

4.97 The prismatic bar *AB* is made of a bronze alloy for which the tensile stress-strain diagram is as shown. Assuming that the σ-ϵ diagram is the same in compression as in tension, determine (*a*) the maximum stress in the bar when the radius of curvature of the bar is 100 in., (*b*) the corresponding value of the bending moment. (See hint given in Prob. 4.96.)

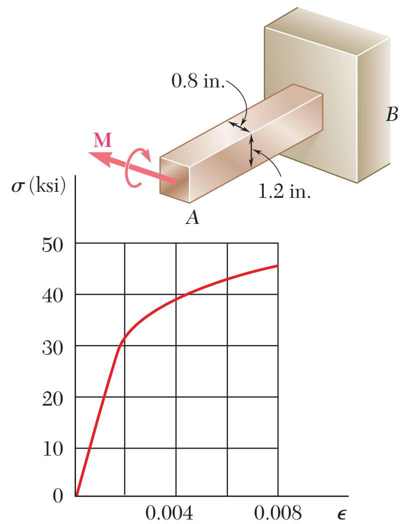


Fig. P4.97

4.98 A prismatic bar of rectangular cross section is made of an alloy for which the stress-strain diagram can be represented by the relation $\epsilon = k\sigma^n$ for $\sigma > 0$ and $\epsilon = -|k\sigma^n|$ for $\sigma < 0$. If a couple **M** is applied to the bar, show that the maximum stress is

$$\sigma_m = \frac{1 + 2n}{3n} \frac{Mc}{I}$$

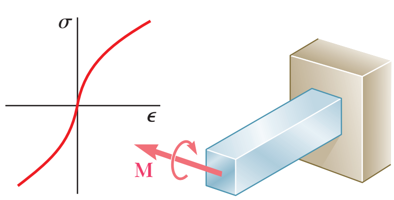


Fig. P4.98

4.12 ECCENTRIC AXIAL LOADING IN A PLANE OF SYMMETRY

We saw in Sec. 1.5 that the distribution of stresses in the cross section of a member under axial loading can be assumed uniform only if the line of action of the loads **P** and **P′** passes through the centroid of the cross section. Such a loading is said to be *centric*. Let us now analyze the distribution of stresses when the line of action of the loads does *not* pass through the centroid of the cross section, i.e., when the loading is *eccentric*.

Two examples of an eccentric loading are shown in Photos 4.5 and 4.6. In the case of the walkway light, the weight of the lamp causes an eccentric loading on the post. Likewise, the vertical forces exerted on the press cause an eccentric loading on the back column of the press.

Photo 4.5

Photo 4.6

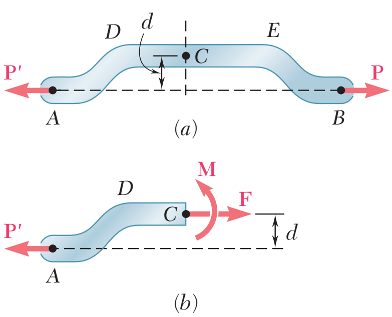


Fig. 4.42 Member with eccentric loading.

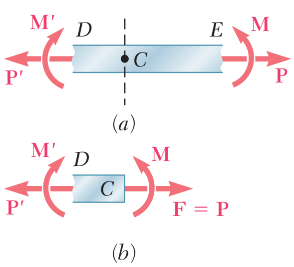


Fig. 4.43 Internal forces in member with eccentric loading.

In this section, our analysis will be limited to members that possess a plane of symmetry, and it will be assumed that the loads are applied in the plane of symmetry of the member (Fig. 4.42*a*). The internal forces acting on a given cross section may then be represented by a force **F** applied at the centroid C of the section and a couple **M** acting in the plane of symmetry of the member (Fig. 4.42*b*). The conditions of equilibrium of the free body AC require that the force **F** be equal and opposite to **P′** and that the moment of the couple **M** be equal and opposite to the moment of **P′** about C. Denoting by d the distance from the centroid C to the line of action AB of the forces **P** and **P′**, we have

$$F = P \qquad \text{and} \qquad M = Pd \tag{4.49}$$

We now observe that the internal forces in the section would have been represented by the same force and couple if the straight portion DE of member AB had been detached from AB and subjected simultaneously to the centric loads **P** and **P′** and to the bending couples **M** and **M′** (Fig. 4.43). Thus, the stress distribution due to the original eccentric loading can be obtained by superposing the uniform stress distribution corresponding to the centric loads **P** and **P′** and the linear distribution corresponding to the bending couples **M** and **M′** (Fig. 4.44). We write

$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$

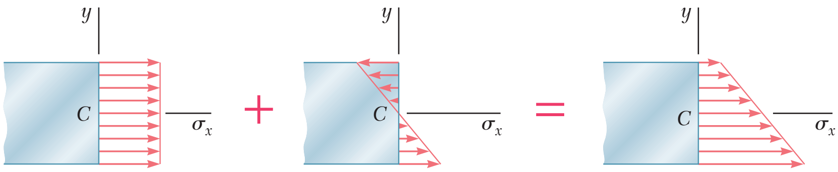


Fig. 4.44 Stress distribution—eccentric loading.

or, recalling Eqs. (1.5) and (4.16):

$$\boxed{\sigma_x = \frac{P}{A} - \frac{My}{I}} \tag{4.50}$$

where A is the area of the cross section and I its centroidal moment of inertia, and where y is measured from the centroidal axis of the cross section. The relation obtained shows that the distribution of stresses across the section is *linear but not uniform*. Depending upon the geometry of the cross section and the eccentricity of the load, the combined stresses may all have the same sign, as shown in Fig. 4.44, or some may be positive and others negative, as shown in Fig. 4.45. In the latter case, there will be a line in the section, along which $\sigma_x = 0$. This line represents the *neutral axis* of the section. We note that the neutral axis does *not* coincide with the centroidal axis of the section, since $\sigma_x \neq 0$ for $y = 0$.

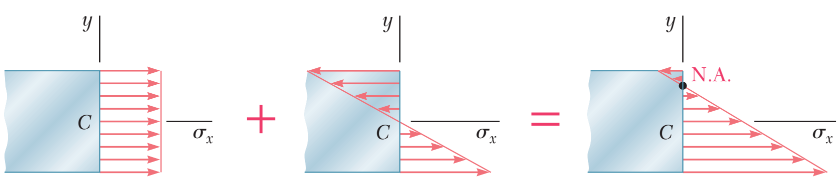


Fig. 4.45 Alternative stress distribution—eccentric loading.

The results obtained are valid only to the extent that the conditions of applicability of the superposition principle (Sec. 2.12) and of Saint-Venant's principle (Sec. 2.17) are met. This means that the stresses involved must not exceed the proportional limit of the material, that the deformations due to bending must not appreciably affect the distance d in Fig. 4.42*a*, and that the cross section where the stresses are computed must not be too close to points D or E in the same figure. The first of these requirements clearly shows that the superposition method cannot be applied to plastic deformations.

EXAMPLE 4.07

An open-link chain is obtained by bending low-carbon steel rods of 0.5-in. diameter into the shape shown (Fig. 4.46). Knowing that the chain carries a load of 160 lb, determine (*a*) the largest tensile and compressive stresses in the straight portion of a link, (*b*) the distance between the centroidal and the neutral axis of a cross section.

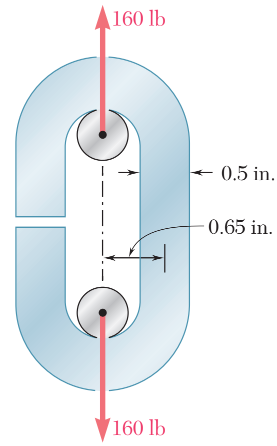


Fig. 4.46

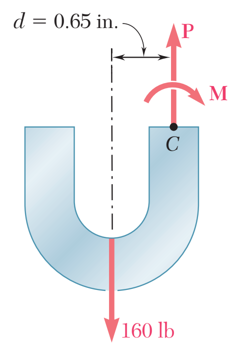


Fig. 4.47

(a) Largest Tensile and Compressive Stresses. The internal forces in the cross section are equivalent to a centric force **P** and a bending couple **M** (Fig. 4.47) of magnitudes

$$P = 160 \text{ lb}$$
$$M = Pd = (160 \text{ lb})(0.65 \text{ in.}) = 104 \text{ lb} \cdot \text{in.}$$

The corresponding stress distributions are shown in parts *a* and *b* of Fig. 4.48. The distribution due to the centric force **P** is uniform and equal to $\sigma_0 = P/A$. We have

$$A = \pi c^2 = \pi(0.25 \text{ in.})^2 = 0.1963 \text{ in}^2$$
$$\sigma_0 = \frac{P}{A} = \frac{160 \text{ lb}}{0.1963 \text{ in}^2} = 815 \text{ psi}$$

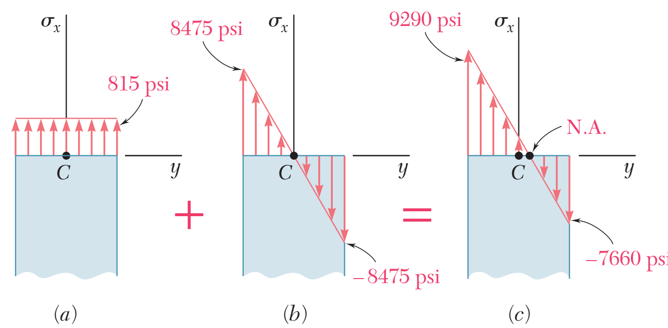


Fig. 4.48

The distribution due to the bending couple **M** is linear with a maximum stress $\sigma_m = Mc/I$. We write

$$I = \tfrac{1}{4}\pi c^4 = \tfrac{1}{4}\pi(0.25 \text{ in.})^4 = 3.068 \times 10^{-3} \text{ in}^4$$
$$\sigma_m = \frac{Mc}{I} = \frac{(104 \text{ lb} \cdot \text{in.})(0.25 \text{ in.})}{3.068 \times 10^{-3} \text{ in}^4} = 8475 \text{ psi}$$

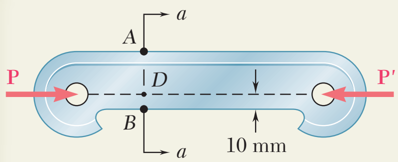
Superposing the two distributions, we obtain the stress distribution corresponding to the given eccentric loading (Fig. 4.48*c*). The largest tensile and compressive stresses in the section are found to be, respectively,

$$\sigma_t = \sigma_0 + \sigma_m = 815 + 8475 = 9290 \text{ psi}$$
$$\sigma_c = \sigma_0 - \sigma_m = 815 - 8475 = -7660 \text{ psi}$$

(b) Distance Between Centroidal and Neutral Axes. The distance y_0 from the centroidal to the neutral axis of the section is obtained by setting $\sigma_x = 0$ in Eq. (4.50) and solving for y_0:

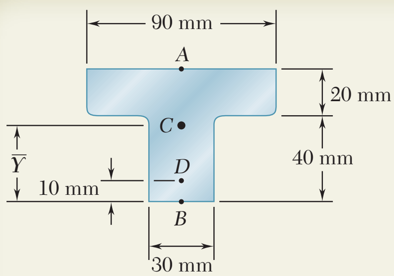
$$0 = \frac{P}{A} - \frac{My_0}{I}$$
$$y_0 = \left(\frac{P}{A}\right)\left(\frac{I}{M}\right) = (815 \text{ psi})\frac{3.068 \times 10^{-3} \text{ in}^4}{104 \text{ lb} \cdot \text{in.}}$$
$$y_0 = 0.0240 \text{ in.}$$

SAMPLE PROBLEM 4.8



Knowing that for the cast iron link shown the allowable stresses are 30 MPa in tension and 120 MPa in compression, determine the largest force **P** which can be applied to the link. (*Note*: The T-shaped cross section of the link has previously been considered in Sample Prob. 4.2.)

SOLUTION



Section a–a

Properties of Cross Section. From Sample Prob. 4.2, we have

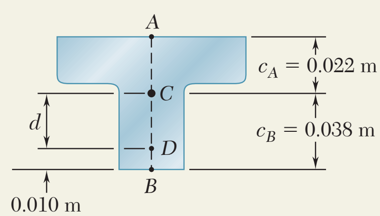
$$A = 3000\ \text{mm}^2 = 3 \times 10^{-3}\ \text{m}^2 \qquad \overline{Y} = 38\ \text{mm} = 0.038\ \text{m}$$

$$I = 868 \times 10^{-9}\ \text{m}^4$$

We now write: $\quad d = (0.038\ \text{m}) - (0.010\ \text{m}) = 0.028\ \text{m}$

Force and Couple at C. We replace **P** by an equivalent force-couple system at the centroid C.

$$P = P \qquad M = P(d) = P(0.028\ \text{m}) = 0.028P$$

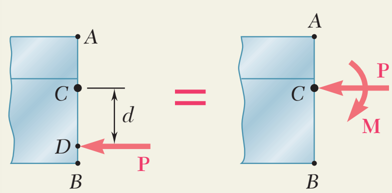


The force **P** acting at the centroid causes a uniform stress distribution (Fig. 1). The bending couple **M** causes a linear stress distribution (Fig. 2).

$$\sigma_0 = \frac{P}{A} = \frac{P}{3 \times 10^{-3}} = 333P \qquad \text{(Compression)}$$

$$\sigma_1 = \frac{Mc_A}{I} = \frac{(0.028P)(0.022)}{868 \times 10^{-9}} = 710P \qquad \text{(Tension)}$$

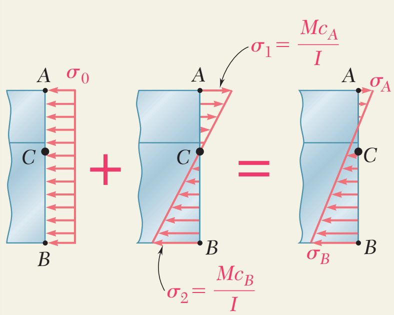
$$\sigma_2 = \frac{Mc_B}{I} = \frac{(0.028P)(0.038)}{868 \times 10^{-9}} = 1226P \qquad \text{(Compression)}$$



Superposition. The total stress distribution (Fig. 3) is found by superposing the stress distributions caused by the centric force **P** and by the couple **M.** Since tension is positive, and compression negative, we have

$$\sigma_A = -\frac{P}{A} + \frac{Mc_A}{I} = -333P + 710P = +377P \qquad \text{(Tension)}$$

$$\sigma_B = -\frac{P}{A} - \frac{Mc_B}{I} = -333P - 1226P = -1559P \qquad \text{(Compression)}$$



(1) (2) (3)

Largest Allowable Force. The magnitude of **P** for which the tensile stress at point A is equal to the allowable tensile stress of 30 MPa is found by writing

$$\sigma_A = 377P = 30\ \text{MPa} \qquad P = 79.6\ \text{kN} \blacktriangleleft$$

We also determine the magnitude of **P** for which the stress at B is equal to the allowable compressive stress of 120 MPa.

$$\sigma_B = -1559P = -120\ \text{MPa} \qquad P = 77.0\ \text{kN} \blacktriangleleft$$

The magnitude of the largest force **P** that can be applied without exceeding either of the allowable stresses is the smaller of the two values we have found.

$$P = 77.0\ \text{kN} \blacktriangleleft$$

PROBLEMS

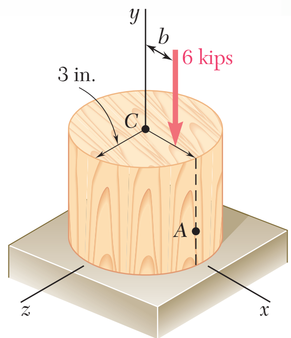


Fig. P4.99

4.99 A short wooden post supports a 6-kip axial load as shown. Determine the stress at point A when (a) $b = 0$, (b) $b = 1.5$ in., (c) $b = 3$ in.

4.100 As many as three axial loads each of magnitude $P = 10$ kips can be applied to the end of a W8 × 21 rolled-steel shape. Determine the stress at point A, (a) for the loading shown, (b) if loads are applied at points 1 and 2 only.

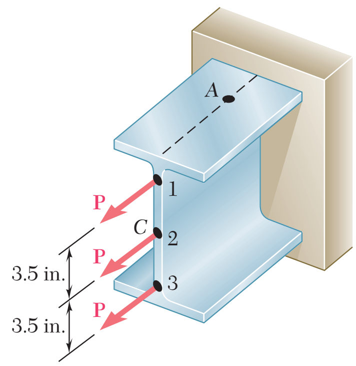


Fig. P4.100

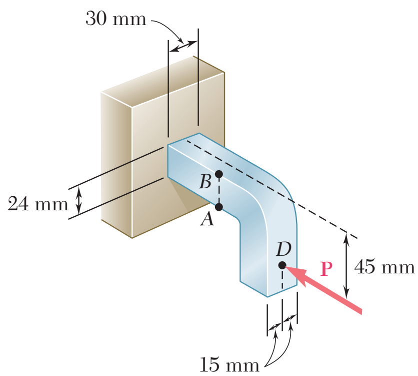


Fig. *P4.101*

4.101 Knowing that the magnitude of the horizontal force **P** is 8 kN, determine the stress at (a) point A, (b) point B.

4.102 The vertical portion of the press shown consists of a rectangular tube of wall thickness $t = 10$ mm. Knowing that the press has been tightened on wooden planks being glued together until $P = 20$ kN, determine the stress at (a) point A, (b) point B.

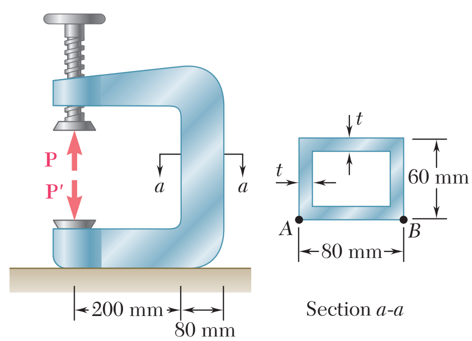


Fig. P4.102

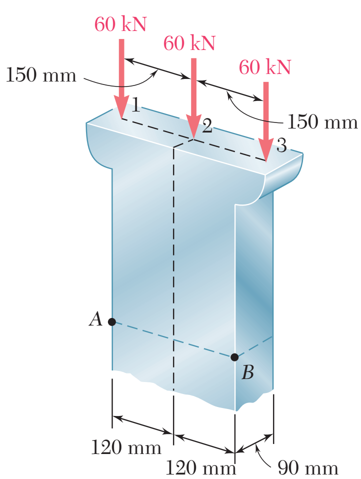


Fig. P4.104

4.103 Solve Prob. 4.102, assuming that $t = 8$ mm.

4.104 Determine the stress at points A and B, (a) for the loading shown, (b) if the 60-kN loads are applied at points 1 and 2 only.

4.105 Knowing that the allowable stress in section ABD is 10 ksi, determine the largest force $\mathbf{P}$ that can be applied to the bracket shown.

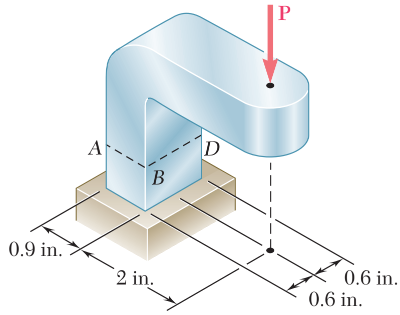


Fig. P4.105

4.106 Portions of a $\frac{1}{2} \times \frac{1}{2}$-in. square bar have been bent to form the two machine components shown. Knowing that the allowable stress is 15 ksi, determine the maximum load that can be applied to each component.

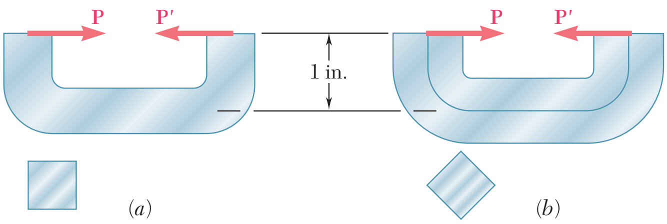


Fig. P4.106

4.107 The four forces shown are applied to a rigid plate supported by a solid steel post of radius a. Knowing that $P = 100$ kN and $a = 40$ mm, determine the maximum stress in the post when (a) the force at D is removed, (b) the forces at C and D are removed.

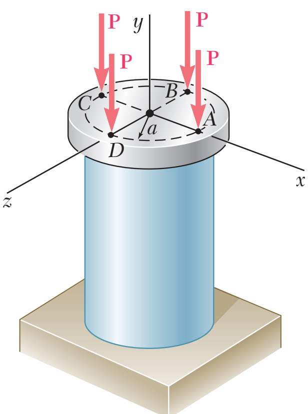


Fig. P4.107

4.108 A milling operation was used to remove a portion of a solid bar of square cross section. Knowing that $a = 30$ mm, $d = 20$ mm, and $\sigma_{all} = 60$ MPa, determine the magnitude P of the largest forces that can be safely applied at the centers of the ends of the bar.

4.109 A milling operation was used to remove a portion of a solid bar of square cross section. Forces of magnitude $P = 18$ kN are applied at the centers of the ends of the bar. Knowing that $a = 30$ mm and $\sigma_{all} = 135$ MPa, determine the smallest allowable depth d of the milled portion of the bar.

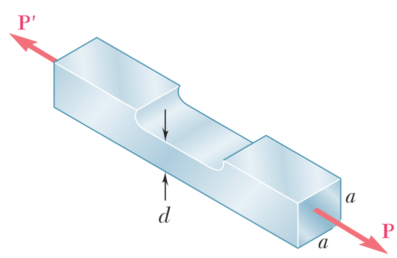


Fig. P4.108 and P4.109

4.110 A short column is made by nailing two 1×4-in. planks to a 2×4-in. timber. Determine the largest compressive stress created in the column by a 12-kip load applied as shown at the center of the top section of the timber if (a) the column is as described, (b) plank 1 is removed, (c) both planks are removed.

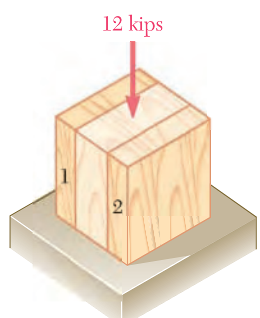


Fig. *P4.110*

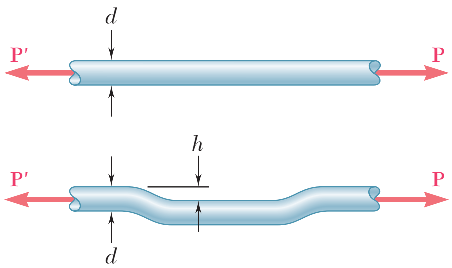


Fig. P4.111 and *P4.112*

4.111 An offset h must be introduced into a solid circular rod of diameter d. Knowing that the maximum stress after the offset is introduced must not exceed 5 times the stress in the rod when it is straight, determine the largest offset that can be used.

4.112 An offset h must be introduced into a metal tube of 0.75-in. outer diameter and 0.08-in. wall thickness. Knowing that the maximum stress after the offset is introduced must not exceed 4 times the stress in the tube when it is straight, determine the largest offset that can be used.

4.113 A steel rod is welded to a steel plate to form the machine element shown. Knowing that the allowable stress is 135 MPa, determine (*a*) the largest force **P** that can be applied to the element, (*b*) the corresponding location of the neutral axis. *Given*: The centroid of the cross section is at C and $I_z = 4195\text{ mm}^4$.

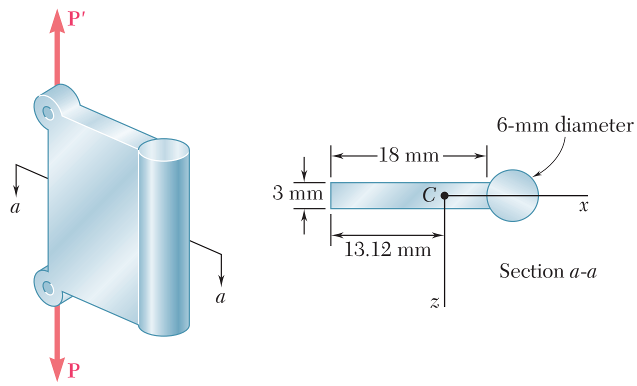


Fig. P4.113

4.114 A vertical rod is attached at point A to the cast iron hanger shown. Knowing that the allowable stresses in the hanger are $\sigma_{all} = +5$ ksi and $\sigma_{all} = -12$ ksi, determine the largest downward force and the largest upward force that can be exerted by the rod.

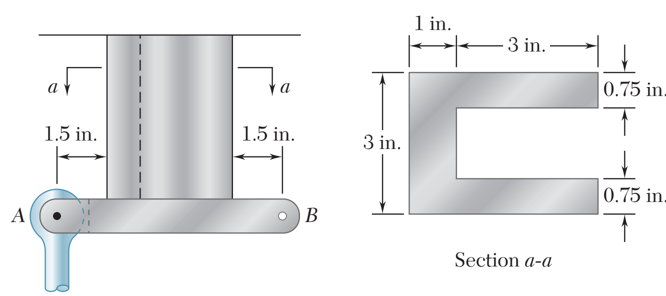


Fig. P4.114

4.115 Solve Prob. 4.114, assuming that the vertical rod is attached at point B instead of point A.

4.116 Three steel plates, each of 25 × 150-mm cross section, are welded together to form a short H-shaped column. Later, for architectural reasons, a 25-mm strip is removed from each side of one of the flanges. Knowing that the load remains centric with respect to the original cross section and that the allowable stress is 100 MPa, determine the largest force **P** (*a*) that could be applied to the original column, (*b*) that can be applied to the modified column.

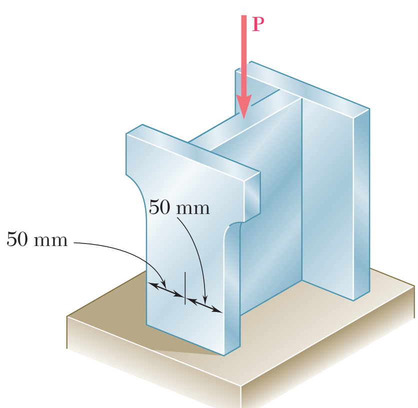


Fig. P4.116

4.117 A vertical force **P** of magnitude 20 kips is applied at point C located on the axis of symmetry of the cross section of a short column. Knowing that $y = 5$ in., determine (*a*) the stress at point A, (*b*) the stress at point B, (*c*) the location of the neutral axis.

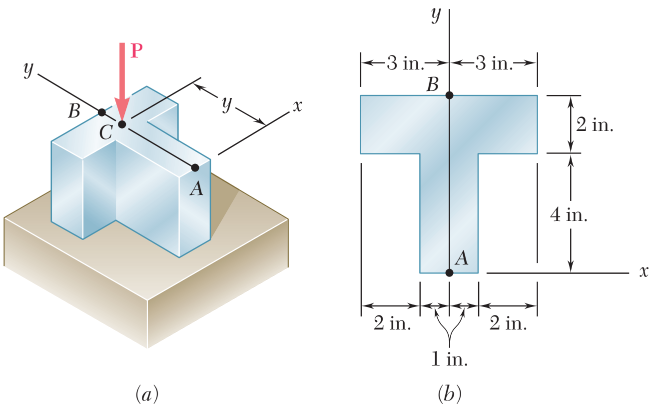


Fig. *P4.117* and P4.118

4.118 A vertical force **P** is applied at point C located on the axis of symmetry of the cross section of a short column. Determine the range of values of y for which tensile stresses do not occur in the column.

4.119 Knowing that the clamp shown has been tightened until $P = 400$ N, determine (*a*) the stress at point A, (*b*) the stress at point B, (*c*) the location of the neutral axis of section *a-a*.

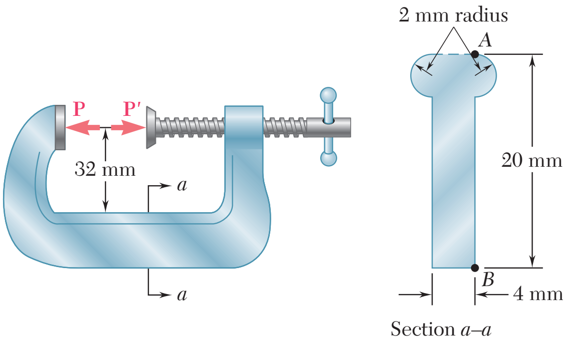


Fig. P4.119

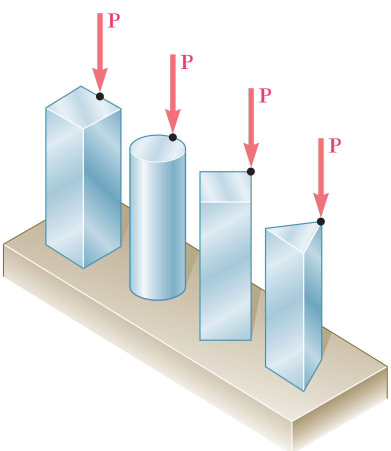


Fig. P4.120

4.120 The four bars shown have the same cross-sectional area. For the given loadings, show that (*a*) the maximum compressive stresses are in the ratio 4:5:7:9, (*b*) the maximum tensile stresses are in the ratio 2:3:5:3. (*Note*: the cross section of the triangular bar is an equilateral triangle.)

4.121 The C-shaped steel bar is used as a dynamometer to determine the magnitude P of the forces shown. Knowing that the cross section of the bar is a square of side 40 mm and that the strain on the inner edge was measured and found to be 450 μ, determine the magnitude P of the forces. Use $E = 200$ GPa.

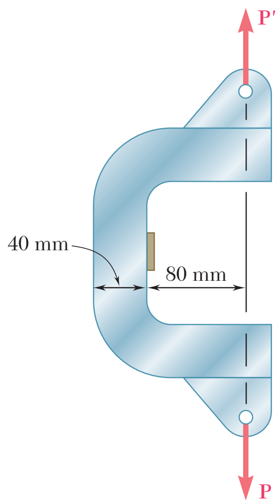


Fig. P4.121

4.122 An eccentric force **P** is applied as shown to a steel bar of 25 × 90-mm cross section. The strains at A and B have been measured and found to be

$$\epsilon_A = +350\ \mu \qquad \epsilon_B = -70\ \mu$$

Knowing that $E = 200$ GPa, determine (*a*) the distance d, (*b*) the magnitude of the force **P.**

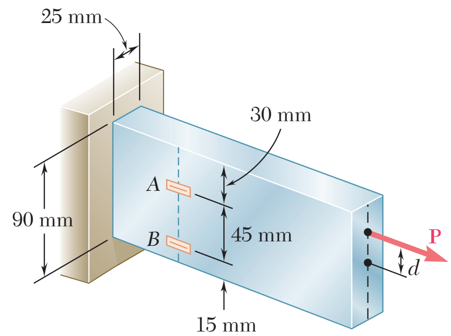


Fig. P4.122

4.123 Solve Prob. 4.122, assuming that the measured strains are

$$\epsilon_A = +600\ \mu \qquad \epsilon_B = +420\ \mu$$

4.124 A short length of a W8 × 31 rolled-steel shape supports a rigid plate on which two loads **P** and **Q** are applied as shown. The strains at two points A and B on the centerline of the outer faces of the flanges have been measured and found to be

$$\epsilon_A = -550 \times 10^{-6} \text{ in./in.} \qquad \epsilon_B = -680 \times 10^{-6} \text{ in./in.}$$

Knowing that $E = 29 \times 10^6$ psi, determine the magnitude of each load.

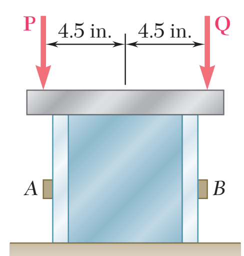


Fig. P4.124

4.125 Solve Prob. 4.124, assuming that the measured strains are

$$\epsilon_A = +35 \times 10^{-6} \text{ in./in.} \qquad \text{and} \qquad \epsilon_B = -450 \times 10^{-6} \text{ in./in.}$$

4.126 The eccentric axial force **P** acts at point D, which must be located 25 mm below the top surface of the steel bar shown. For $P = 60$ kN, determine (*a*) the depth d of the bar for which the tensile stress at point A is maximum, (*b*) the corresponding stress at point A.

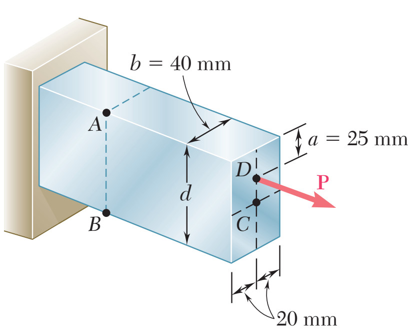


Fig. *P4.126*

4.13 UNSYMMETRIC BENDING

Our analysis of pure bending has been limited so far to members possessing at least one plane of symmetry and subjected to couples acting in that plane. Because of the symmetry of such members and of their loadings, we concluded that the members would remain symmetric with respect to the plane of the couples and thus bend in that plane (Sec. 4.3). This is illustrated in Fig. 4.49; part *a* shows the cross section of a member possessing two planes of symmetry, one vertical and one horizontal, and part *b* the cross section of a member with a single, vertical plane of symmetry. In both cases the couple exerted on the section acts in the vertical plane of symmetry of the member and is represented by the horizontal couple vector **M,** and in both cases the neutral axis of the cross section is found to coincide with the axis of the couple.

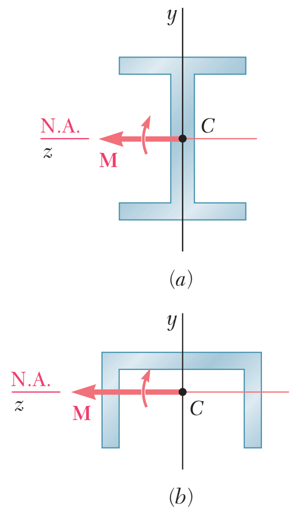


Fig. 4.49 Moment in plane of symmetry.

Let us now consider situations where the bending couples do *not* act in a plane of symmetry of the member, either because they act in a different plane, or because the member does not possess any plane of symmetry. In such situations, we cannot assume that the member will bend in the plane of the couples. This is illustrated in Fig. 4.50. In each part of the figure, the couple exerted on the section has again been assumed to act in a vertical plane and has been represented by a horizontal couple vector **M.** However, since the vertical plane is not a plane of symmetry, *we cannot expect the member to bend in that plane, or the neutral axis of the section to coincide with the axis of the couple.*

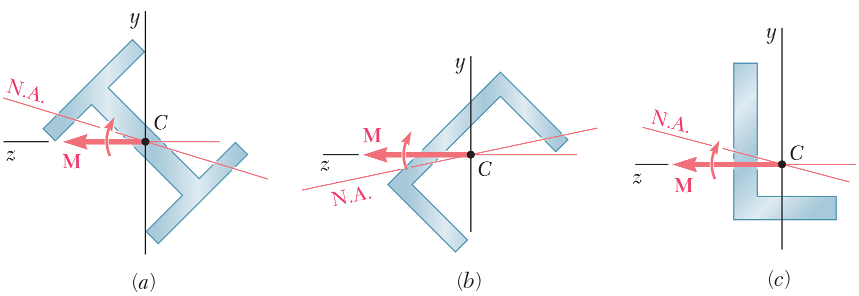


Fig. 4.50 Moment not in plane of symmetry.

We propose to determine the precise conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple **M** representing the forces acting on that section. Such a section is shown in Fig. 4.51, and both the couple vector **M** and the

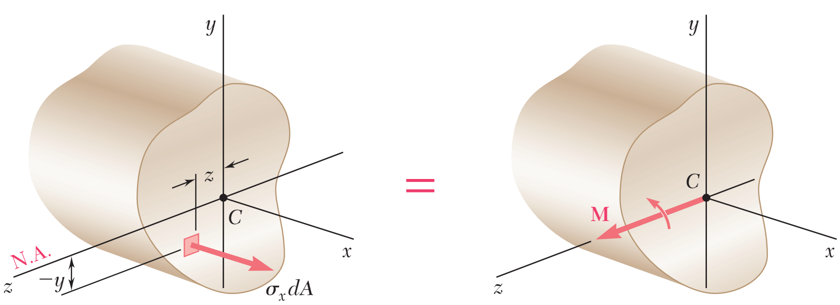


Fig. 4.51 Section with arbitrary shape.

neutral axis have been assumed to be directed along the z axis. We recall from Sec. 4.2 that, if we then express that the elementary internal forces $\sigma_x dA$ form a system equivalent to the couple **M,** we obtain

$$x \text{ components:} \qquad \int \sigma_x dA = 0 \tag{4.1}$$

$$\text{moments about } y \text{ axis:} \qquad \int z\sigma_x dA = 0 \tag{4.2}$$

$$\text{moments about } z \text{ axis:} \qquad \int (-y\sigma_x dA) = M \tag{4.3}$$

As we saw earlier, when all the stresses are within the proportional limit, the first of these equations leads to the requirement that the neutral axis be a centroidal axis, and the last to the fundamental relation $\sigma_x = -My/I$. Since we had assumed in Sec. 4.2 that the cross section was symmetric with respect to the y axis, Eq. (4.2) was dismissed as trivial at that time. Now that we are considering a cross section of arbitrary shape, Eq. (4.2) becomes highly significant. Assuming the stresses to remain within the proportional limit of the material, we can substitute $\sigma_x = -\sigma_m y/c$ into Eq. (4.2) and write

$$\int z\left(-\frac{\sigma_m y}{c}\right) dA = 0 \qquad \text{or} \qquad \int yz\,dA = 0 \tag{4.51}$$

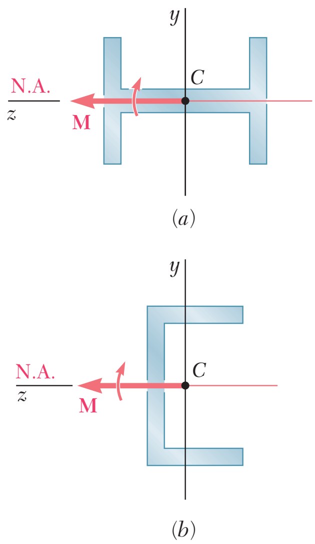


Fig. 4.52 Moment on principal centroidal axis.

The integral $\int yz\,dA$ represents the product of inertia I_{yz} of the cross section with respect to the y and z axes, and will be zero if these axes are the *principal centroidal axes of the cross section*.† We thus conclude that the neutral axis of the cross section will coincide with the axis of the couple **M** representing the forces acting on that section *if, and only if, the couple vector* **M** *is directed along one of the principal centroidal axes of the cross section*.

We note that the cross sections shown in Fig. 4.49 are symmetric with respect to at least one of the coordinate axes. It follows that, in each case, the y and z axes are the principal centroidal axes of the section. Since the couple vector **M** is directed along one of the principal centroidal axes, we verify that the neutral axis will coincide with the axis of the couple. We also note that, if the cross sections are rotated through 90° (Fig. 4.52), the couple vector **M** will still be directed along a principal centroidal axis, and the neutral axis will again coincide with the axis of the couple, even though in case b the couple does *not* act in a plane of symmetry of the member.

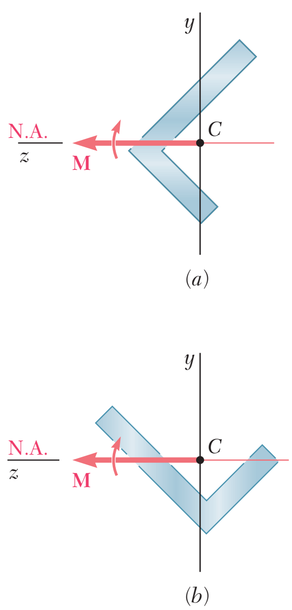


Fig. 4.53 Moment not on principal centroidal axis.

In Fig. 4.50, on the other hand, neither of the coordinate axes is an axis of symmetry for the sections shown, and the coordinate axes are not principal axes. Thus, the couple vector **M** is not directed along a principal centroidal axis, and the neutral axis does not coincide with the axis of the couple. However, any given section possesses principal centroidal axes, even if it is unsymmetric, as the section shown in Fig. 4.50c, and these axes may be determined analytically or by using Mohr's circle.† If the couple vector **M** is directed along one of the principal centroidal axes of the section, the neutral axis will coincide with the axis of the couple (Fig. 4.53) and the equations

†See Ferdinand P. Beer and E. Russell Johnston, Jr., *Mechanics for Engineers,* 5th ed., McGraw-Hill, New York, 2008, or *Vector Mechanics for Engineers,* 9th ed., McGraw-Hill, New York, 2010, Secs. 9.8–9.10.

derived in Secs. 4.3 and 4.4 for symmetric members can be used to determine the stresses in this case as well.

As you will see presently, the principle of superposition can be used to determine stresses in the most general case of unsymmetric bending. Consider first a member with a vertical plane of symmetry, which is subjected to bending couples **M** and **M′** acting in a plane forming an angle θ with the vertical plane (Fig. 4.54). The couple

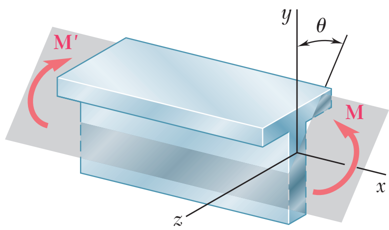


Fig. 4.54 Unsymmetric bending.

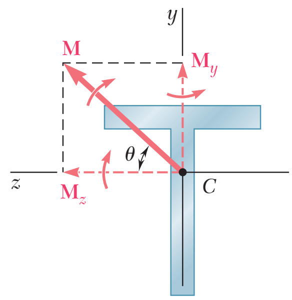


Fig. 4.55

vector **M** representing the forces acting on a given cross section will form the same angle θ with the horizontal z axis (Fig. 4.55). Resolving the vector **M** into component vectors $\mathbf{M}_z$ and $\mathbf{M}_y$ along the z and y axes, respectively, we write

$$M_z = M \cos \theta \qquad M_y = M \sin \theta \tag{4.52}$$

Since the y and z axes are the principal centroidal axes of the cross section, we can use Eq. (4.16) to determine the stresses resulting from the application of either of the couples represented by $\mathbf{M}_z$ and $\mathbf{M}_y$. The couple $\mathbf{M}_z$ acts in a vertical plane and bends the member in that plane (Fig. 4.56). The resulting stresses are

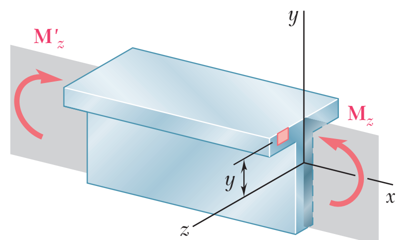


Fig. 4.56

$$\sigma_x = -\frac{M_z y}{I_z} \tag{4.53}$$

where I_z is the moment of inertia of the section about the principal centroidal z axis. The negative sign is due to the fact that we have compression above the xz plane $(y > 0)$ and tension below $(y < 0)$. On the other hand, the couple $\mathbf{M}_y$ acts in a horizontal plane and bends the member in that plane (Fig. 4.57). The resulting stresses are found to be

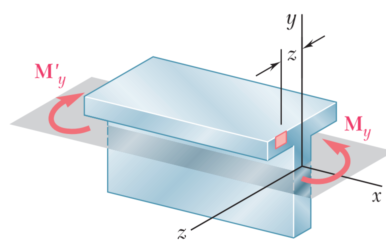


Fig. 4.57

$$\sigma_x = +\frac{M_y z}{I_y} \tag{4.54}$$

where I_y is the moment of inertia of the section about the principal centroidal y axis, and where the positive sign is due to the fact that we have tension to the left of the vertical xy plane $(z > 0)$ and compression to its right $(z < 0)$. The distribution of the stresses caused by the original couple **M** is obtained by superposing the stress distributions defined by Eqs. (4.53) and (4.54), respectively. We have

$$\boxed{\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}} \tag{4.55}$$

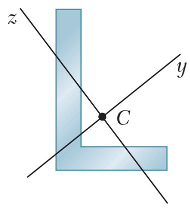


Fig. 4.58 Unsymmetric cross section.

We note that the expression obtained can also be used to compute the stresses in an unsymmetric section, such as the one shown in Fig. 4.58, once the principal centroidal y and z axes have been determined. On the other hand, Eq. (4.55) is valid only if the conditions of applicability of the principle of superposition are met. In other words, it should not be used if the combined stresses exceed the proportional limit of the material, or if the deformations caused by one of the component couples appreciably affect the distribution of the stresses due to the other.

Equation (4.55) shows that the distribution of stresses caused by unsymmetric bending is linear. However, as we have indicated earlier in this section, the neutral axis of the cross section will not, in general, coincide with the axis of the bending couple. Since the normal stress is zero at any point of the neutral axis, the equation defining that axis can be obtained by setting $\sigma_x = 0$ in Eq. (4.55). We write

$$-\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0$$

or, solving for y and substituting for M_z and M_y from Eqs. (4.52),

$$y = \left(\frac{I_z}{I_y}\tan\theta\right)z \qquad (4.56)$$

The equation obtained is that of a straight line of slope $m = (I_z/I_y)$ $\tan\theta$. Thus, the angle ϕ that the neutral axis forms with the z axis (Fig. 4.59) is defined by the relation

$$\boxed{\tan\phi = \frac{I_z}{I_y}\tan\theta} \qquad (4.57)$$

where θ is the angle that the couple vector **M** forms with the same axis. Since I_z and I_y are both positive, ϕ and θ have the same sign. Furthermore, we note that $\phi > \theta$ when $I_z > I_y$, and $\phi < \theta$ when $I_z < I_y$. Thus, the neutral axis is always located between the couple vector **M** and the principal axis corresponding to the minimum moment of inertia.

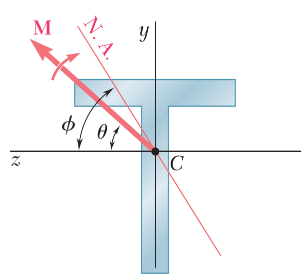


Fig. 4.59

EXAMPLE 4.08

A 1600-lb · in. couple is applied to a wooden beam, of rectangular cross section 1.5 by 3.5 in., in a plane forming an angle of 30° with the vertical (Fig. 4.60). Determine (*a*) the maximum stress in the beam, (*b*) the angle that the neutral surface forms with the horizontal plane.

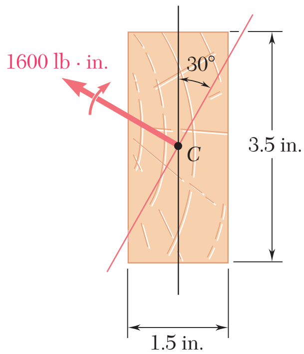


Fig. 4.60

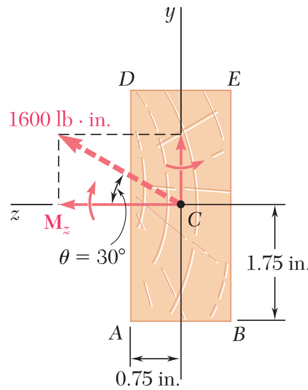


Fig. 4.61

(a) Maximum Stress. The components $\mathbf{M}_z$ and $\mathbf{M}_y$ of the couple vector are first determined (Fig. 4.61):

$$M_z = (1600 \text{ lb} \cdot \text{in.}) \cos 30° = 1386 \text{ lb} \cdot \text{in.}$$
$$M_y = (1600 \text{ lb} \cdot \text{in.}) \sin 30° = 800 \text{ lb} \cdot \text{in.}$$

We also compute the moments of inertia of the cross section with respect to the z and y axes:

$$I_z = \tfrac{1}{12}(1.5 \text{ in.})(3.5 \text{ in.})^3 = 5.359 \text{ in}^4$$
$$I_y = \tfrac{1}{12}(3.5 \text{ in.})(1.5 \text{ in.})^3 = 0.9844 \text{ in}^4$$

The largest tensile stress due to $\mathbf{M}_z$ occurs along AB and is

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386 \text{ lb} \cdot \text{in.})(1.75 \text{ in.})}{5.359 \text{ in}^4} = 452.6 \text{ psi}$$

The largest tensile stress due to $\mathbf{M}_y$ occurs along AD and is

$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(800 \text{ lb} \cdot \text{in.})(0.75 \text{ in.})}{0.9844 \text{ in}^4} = 609.5 \text{ psi}$$

The largest tensile stress due to the combined loading, therefore, occurs at A and is

$$\sigma_{\max} = \sigma_1 + \sigma_2 = 452.6 + 609.5 = 1062 \text{ psi}$$

The largest compressive stress has the same magnitude and occurs at E.

(b) Angle of Neutral Surface with Horizontal Plane. The angle ϕ that the neutral surface forms with the horizontal plane (Fig. 4.62) is obtained from Eq. (4.57):

$$\tan \phi = \frac{I_z}{I_y} \tan \theta = \frac{5.359 \text{ in}^4}{0.9844 \text{ in}^4} \tan 30° = 3.143$$
$$\phi = 72.4°$$

The distribution of the stresses across the section is shown in Fig. 4.63.

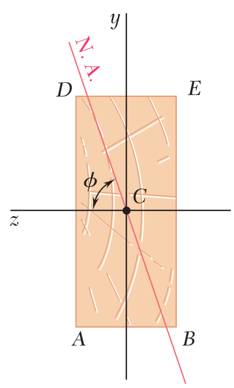


Fig. 4.62

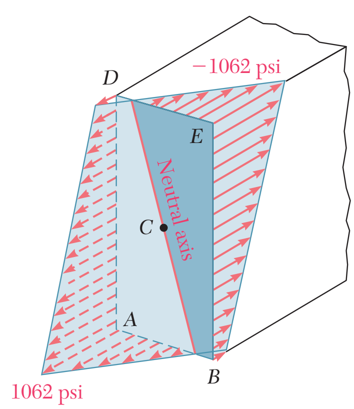


Fig. 4.63

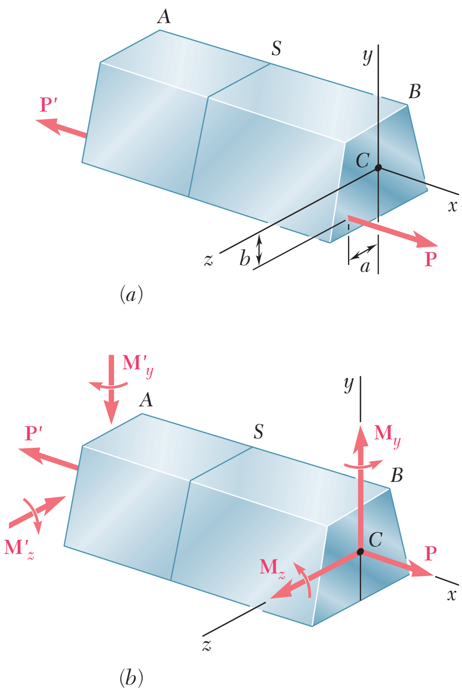


Fig. 4.64 Eccentric axial loading.

4.14 GENERAL CASE OF ECCENTRIC AXIAL LOADING

In Sec. 4.12 you analyzed the stresses produced in a member by an eccentric axial load applied in a plane of symmetry of the member. You will now study the more general case when the axial load is not applied in a plane of symmetry.

Consider a straight member *AB* subjected to equal and opposite eccentric axial forces **P** and **P′** (Fig. 4.64*a*), and let a and b denote the distances from the line of action of the forces to the principal centroidal axes of the cross section of the member. The eccentric force **P** is statically equivalent to the system consisting of a centric force **P** and of the two couples $\mathbf{M}_y$ and $\mathbf{M}_z$ of moments $M_y = Pa$ and $M_z = Pb$ represented in Fig. 4.64*b*. Similarly, the eccentric force **P′** is equivalent to the centric force **P′** and the couples $\mathbf{M}'_y$ and $\mathbf{M}'_z$.

By virtue of Saint-Venant's principle (Sec. 2.17), we can replace the original loading of Fig. 4.64*a* by the statically equivalent loading of Fig. 4.64*b* in order to determine the distribution of stresses in a section *S* of the member, as long as that section is not too close to either end of the member. Furthermore, the stresses due to the loading of Fig. 4.64*b* can be obtained by superposing the stresses corresponding to the centric axial load **P** and to the bending couples $\mathbf{M}_y$ and $\mathbf{M}_z$, as long as the conditions of applicability of the principle of superposition are satisfied (Sec. 2.12). The stresses due to the centric load **P** are given by Eq. (1.5), and the stresses due to the bending couples by Eq. (4.55), since the corresponding couple vectors are directed along the principal centroidal axes of the section. We write, therefore,

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{4.58}$$

where y and z are measured from the principal centroidal axes of the section. The relation obtained shows that the distribution of stresses across the section is *linear*.

In computing the combined stress σ_x from Eq. (4.58), care should be taken to correctly determine the sign of each of the three terms in the right-hand member, since each of these terms can be positive or negative, depending upon the sense of the loads **P** and **P′** and the location of their line of action with respect to the principal centroidal axes of the cross section. Depending upon the geometry of the cross section and the location of the line of action of **P** and **P′**, the combined stresses σ_x obtained from Eq. (4.58) at various points of the section may all have the same sign, or some may be positive and others negative. In the latter case, there will be a line in the section, along which the stresses are zero. Setting $\sigma_x = 0$ in Eq. (4.58), we obtain the equation of a straight line, which represents the *neutral axis* of the section:

$$\frac{M_z}{I_z} y - \frac{M_y}{I_y} z = \frac{P}{A}$$

EXAMPLE 4.09

A vertical 4.80-kN load is applied as shown on a wooden post of rectangular cross section, 80 by 120 mm (Fig. 4.65). (*a*) Determine the stress at points A, B, C, and D. (*b*) Locate the neutral axis of the cross section.

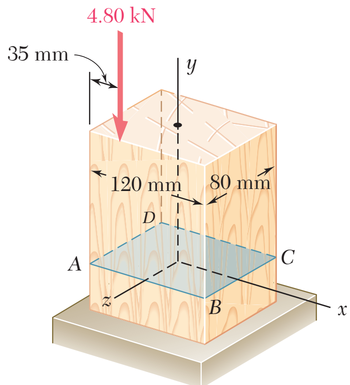


Fig. 4.65

(a) Stresses. The given eccentric load is replaced by an equivalent system consisting of a centric load **P** and two couples $\mathbf{M}_x$ and $\mathbf{M}_z$ represented by vectors directed along the principal centroidal axes of the section (Fig. 4.66). We have

$$M_x = (4.80 \text{ kN})(40 \text{ mm}) = 192 \text{ N} \cdot \text{m}$$
$$M_z = (4.80 \text{ kN})(60 \text{ mm} - 35 \text{ mm}) = 120 \text{ N} \cdot \text{m}$$

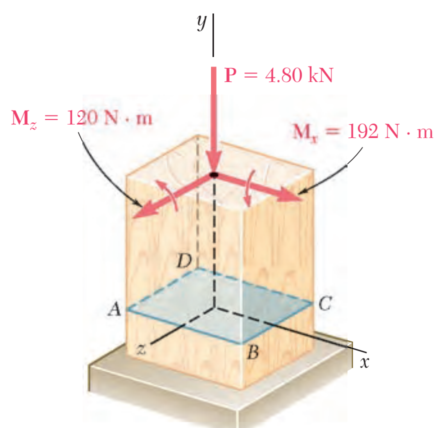


Fig. 4.66

We also compute the area and the centroidal moments of inertia of the cross section:

$$A = (0.080 \text{ m})(0.120 \text{ m}) = 9.60 \times 10^{-3} \text{ m}^2$$
$$I_x = \tfrac{1}{12}(0.120 \text{ m})(0.080 \text{ m})^3 = 5.12 \times 10^{-6} \text{ m}^4$$
$$I_z = \tfrac{1}{12}(0.080 \text{ m})(0.120 \text{ m})^3 = 11.52 \times 10^{-6} \text{ m}^4$$

The stress σ_0 due to the centric load **P** is negative and uniform across the section. We have

$$\sigma_0 = \frac{P}{A} = \frac{-4.80 \text{ kN}}{9.60 \times 10^{-3} \text{ m}^2} = -0.5 \text{ MPa}$$

The stresses due to the bending couples $\mathbf{M}_x$ and $\mathbf{M}_z$ are linearly distributed across the section, with maximum values equal, respectively, to

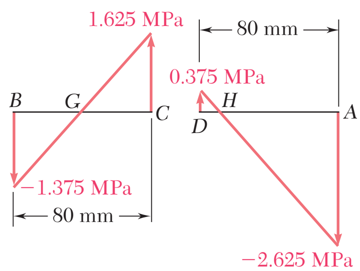
$$\sigma_1 = \frac{M_x z_{\max}}{I_x} = \frac{(192 \text{ N} \cdot \text{m})(40 \text{ mm})}{5.12 \times 10^{-6} \text{ m}^4} = 1.5 \text{ MPa}$$
$$\sigma_2 = \frac{M_z x_{\max}}{I_z} = \frac{(120 \text{ N} \cdot \text{m})(60 \text{ mm})}{11.52 \times 10^{-6} \text{ m}^4} = 0.625 \text{ MPa}$$

The stresses at the corners of the section are

$$\sigma_y = \sigma_0 \pm \sigma_1 \pm \sigma_2$$

where the signs must be determined from Fig. 4.66. Noting that the stresses due to $\mathbf{M}_x$ are positive at C and D, and negative at A and B, and that the stresses due to $\mathbf{M}_z$ are positive at B and C, and negative at A and D, we obtain

$$\sigma_A = -0.5 - 1.5 - 0.625 = -2.625 \text{ MPa}$$
$$\sigma_B = -0.5 - 1.5 + 0.625 = -1.375 \text{ MPa}$$
$$\sigma_C = -0.5 + 1.5 + 0.625 = +1.625 \text{ MPa}$$
$$\sigma_D = -0.5 + 1.5 - 0.625 = +0.375 \text{ MPa}$$



(*a*) (*b*)

Fig. 4.67

(b) Neutral Axis. We note that the stress will be zero at a point G between B and C, and at a point H between D and A (Fig. 4.67). Since the stress distribution is linear, we write

$$\frac{BG}{80 \text{ mm}} = \frac{1.375}{1.625 + 1.375} \qquad BG = 36.7 \text{ mm}$$
$$\frac{HA}{80 \text{ mm}} = \frac{2.625}{2.625 + 0.375} \qquad HA = 70 \text{ mm}$$

The neutral axis can be drawn through points G and H (Fig. 4.68).

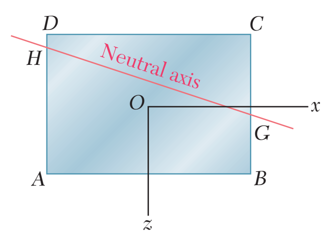


Fig. 4.68

The distribution of the stresses across the section is shown in Fig. 4.69.

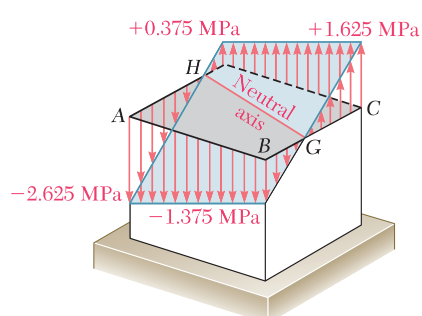
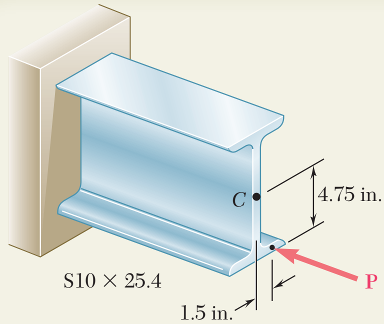


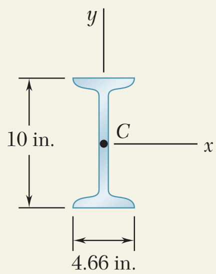
Fig. 4.69



SAMPLE PROBLEM 4.9

A horizontal load **P** is applied as shown to a short section of an S10 × 25.4 rolled-steel member. Knowing that the compressive stress in the member is not to exceed 12 ksi, determine the largest permissible load **P.**

SOLUTION



Properties of Cross Section. The following data are taken from Appendix C.

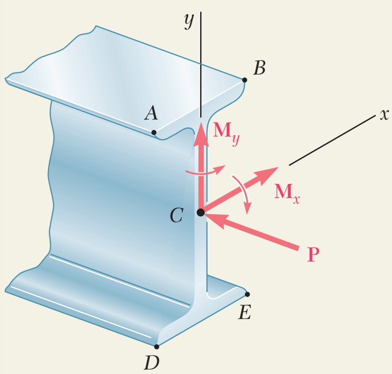
Area: $A = 7.46 \text{ in}^2$

Section moduli: $S_x = 24.7 \text{ in}^3 \qquad S_y = 2.91 \text{ in}^3$

Force and Couple at C. We replace **P** by an equivalent force-couple system at the centroid C of the cross section.

$$M_x = (4.75 \text{ in.})P \qquad M_y = (1.5 \text{ in.})P$$

Note that the couple vectors $\mathbf{M}_x$ and $\mathbf{M}_y$ are directed along the principal axes of the cross section.



Normal Stresses. The absolute values of the stresses at points A, B, D, and E due, respectively, to the centric load **P** and to the couples $\mathbf{M}_x$ and $\mathbf{M}_y$ are

$$\sigma_1 = \frac{P}{A} = \frac{P}{7.46 \text{ in}^2} = 0.1340P$$

$$\sigma_2 = \frac{M_x}{S_x} = \frac{4.75P}{24.7 \text{ in}^3} = 0.1923P$$

$$\sigma_3 = \frac{M_y}{S_y} = \frac{1.5P}{2.91 \text{ in}^3} = 0.5155P$$

Superposition. The total stress at each point is found by superposing the stresses due to **P,** $\mathbf{M}_x$, and $\mathbf{M}_y$. We determine the sign of each stress by carefully examining the sketch of the force-couple system.

$$\sigma_A = -\sigma_1 + \sigma_2 + \sigma_3 = -0.1340P + 0.1923P + 0.5155P = +0.574P$$

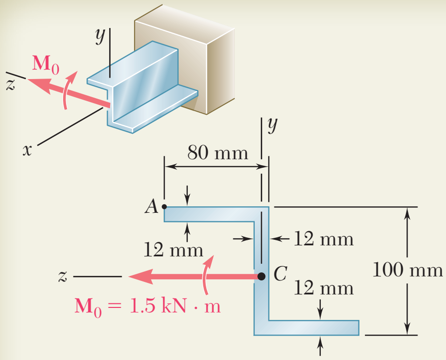
$$\sigma_B = -\sigma_1 + \sigma_2 - \sigma_3 = -0.1340P + 0.1923P - 0.5155P = -0.457P$$

$$\sigma_D = -\sigma_1 - \sigma_2 + \sigma_3 = -0.1340P - 0.1923P + 0.5155P = +0.189P$$

$$\sigma_E = -\sigma_1 - \sigma_2 - \sigma_3 = -0.1340P - 0.1923P - 0.5155P = -0.842P$$

Largest Permissible Load. The maximum compressive stress occurs at point E. Recalling that $\sigma_{\text{all}} = -12$ ksi, we write

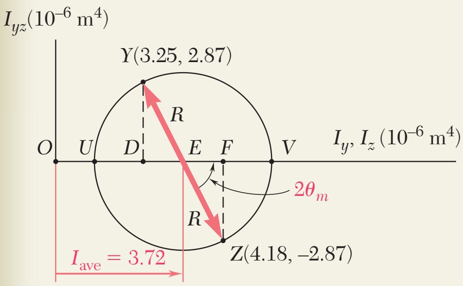
$$\sigma_{\text{all}} = \sigma_E \qquad -12 \text{ ksi} = -0.842P \qquad P = 14.3 \text{ kips} \quad ◀$$



*SAMPLE PROBLEM 4.10

A couple of magnitude $M_0 = 1.5$ kN · m acting in a vertical plane is applied to a beam having the Z-shaped cross section shown. Determine (*a*) the stress at point *A*, (*b*) the angle that the neutral axis forms with the horizontal plane. The moments and product of inertia of the section with respect to the *y* and *z* axes have been computed and are as follows:

$$I_y = 3.25 \times 10^{-6}\ \text{m}^4$$
$$I_z = 4.18 \times 10^{-6}\ \text{m}^4$$
$$I_{yz} = 2.87 \times 10^{-6}\ \text{m}^4$$



SOLUTION

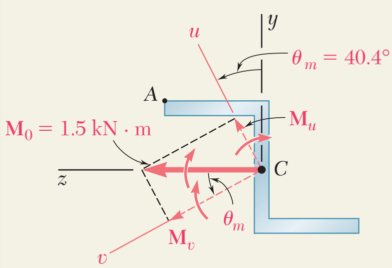
Principal Axes. We draw Mohr's circle and determine the orientation of the principal axes and the corresponding principal moments of inertia.†

$$\tan 2\theta_m = \frac{FZ}{EF} = \frac{2.87}{0.465} \qquad 2\theta_m = 80.8° \qquad \theta_m = 40.4°$$

$$R^2 = (EF)^2 + (FZ)^2 = (0.465)^2 + (2.87)^2 \qquad R = 2.91 \times 10^{-6}\ \text{m}^4$$

$$I_u = I_{\min} = OU = I_{\text{ave}} - R = 3.72 - 2.91 = 0.810 \times 10^{-6}\ \text{m}^4$$

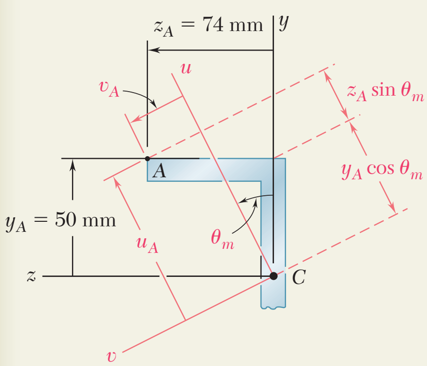
$$I_v = I_{\max} = OV = I_{\text{ave}} + R = 3.72 + 2.91 = 6.63 \times 10^{-6}\ \text{m}^4$$



Loading. The applied couple $\mathbf{M}_0$ is resolved into components parallel to the principal axes.

$$M_u = M_0 \sin \theta_m = 1500 \sin 40.4° = 972\ \text{N} \cdot \text{m}$$

$$M_v = M_0 \cos \theta_m = 1500 \cos 40.4° = 1142\ \text{N} \cdot \text{m}$$



a. Stress at *A*. The perpendicular distances from each principal axis to point *A* are

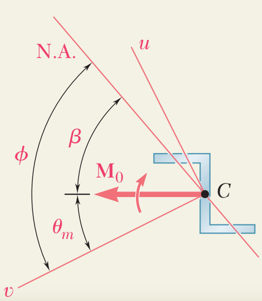
$$u_A = y_A \cos \theta_m + z_A \sin \theta_m = 50 \cos 40.4° + 74 \sin 40.4° = 86.0\ \text{mm}$$

$$v_A = -y_A \sin \theta_m + z_A \cos \theta_m = -50 \sin 40.4° + 74 \cos 40.4° = 23.9\ \text{mm}$$

Considering separately the bending about each principal axis, we note that $\mathbf{M}_u$ produces a tensile stress at point *A* while $\mathbf{M}_v$ produces a compressive stress at the same point.

$$\sigma_A = +\frac{M_u v_A}{I_u} - \frac{M_v u_A}{I_v} = +\frac{(972\ \text{N} \cdot \text{m})(0.0239\ \text{m})}{0.810 \times 10^{-6}\ \text{m}^4} - \frac{(1142\ \text{N} \cdot \text{m})(0.0860\ \text{m})}{6.63 \times 10^{-6}\ \text{m}^4}$$

$$= +(28.68\ \text{MPa}) - (14.81\ \text{MPa}) \qquad \sigma_A = +13.87\ \text{MPa} \ ◀$$



b. Neutral Axis. Using Eq. (4.57), we find the angle ϕ that the neutral axis forms with the *v* axis.

$$\tan \phi = \frac{I_v}{I_u} \tan \theta_m = \frac{6.63}{0.810} \tan 40.4° \qquad \phi = 81.8°$$

The angle β formed by the neutral axis and the horizontal is

$$\beta = \phi - \theta_m = 81.8° - 40.4° = 41.4° \qquad \beta = 41.4° \ ◀$$

†See Ferdinand F. Beer and E. Russell Johnston, Jr., *Mechanics for Engineers,* 5th ed., McGraw-Hill, New York, 2008, or *Vector Mechanics for Engineers*–9th ed., McGraw-Hill, New York, 2010, Secs. 9.8–9.10.

PROBLEMS

4.127 through 4.134 The couple **M** is applied to a beam of the cross section shown in a plane forming an angle β with the vertical. Determine the stress at (*a*) point *A*, (*b*) point *B*, (*c*) point *D*.

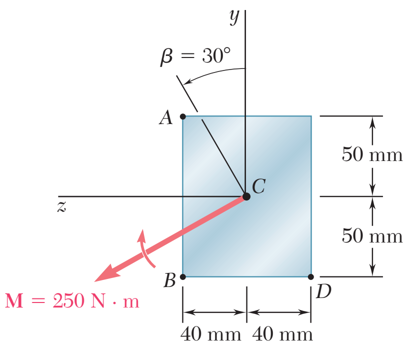


Fig. P4.127

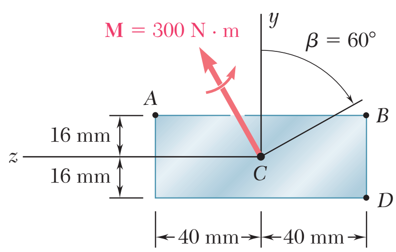


Fig. P4.128

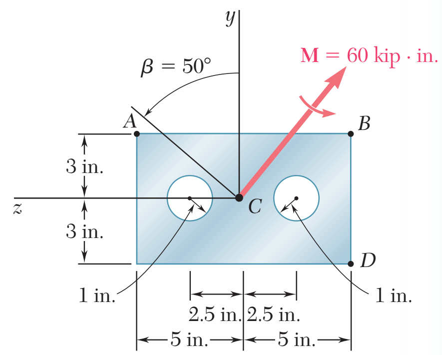


Fig. P4.129

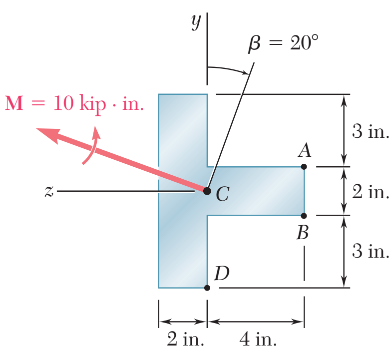


Fig. P4.130

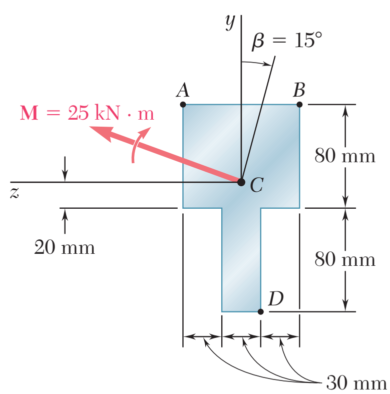


Fig. P4.131

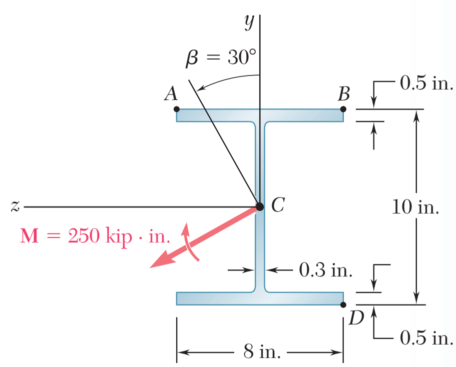


Fig. *P4.132*

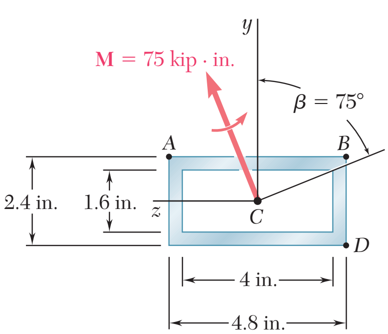


Fig. *P4.133*

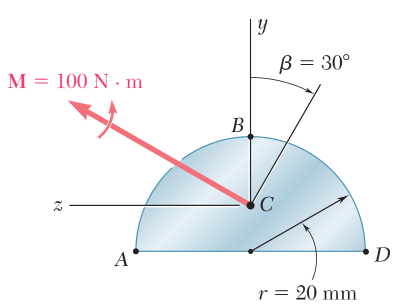


Fig. P4.134

4.135 through 4.140 The couple **M** acts in a vertical plane and is applied to a beam oriented as shown. Determine (*a*) the angle that the neutral axis forms with the horizontal, (*b*) the maximum tensile stress in the beam.

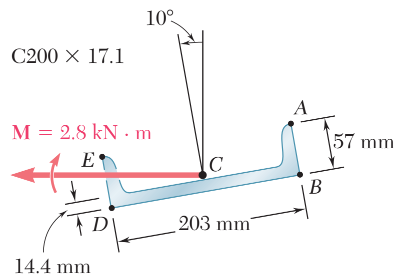


Fig. P4.135

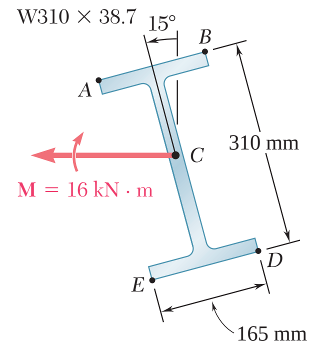
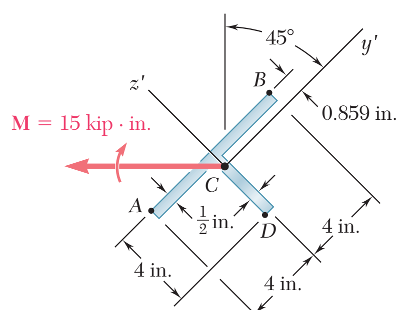
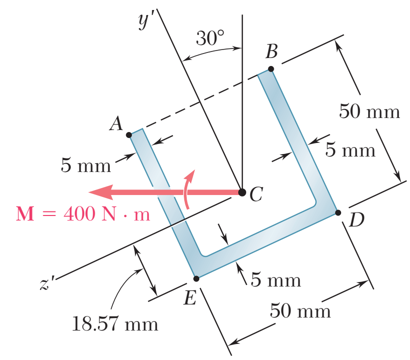


Fig. *P4.136*



$I_{y'} = 6.74 \text{ in}^4$
$I_{z'} = 21.4 \text{ in}^4$

Fig. P4.137



$I_{y'} = 281 \times 10^3 \text{ mm}^4$
$I_{z'} = 176.9 \times 10^3 \text{ mm}^4$

Fig. P4.138

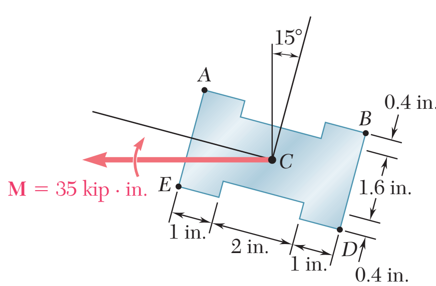
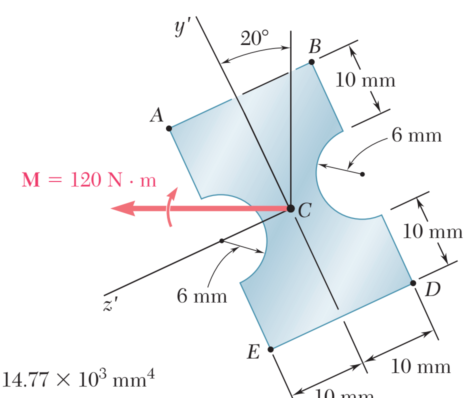


Fig. P4.139



$I_{y'} = 14.77 \times 10^3 \text{ mm}^4$
$I_{z'} = 53.6 \times 10^3 \text{ mm}^4$

Fig. *P4.140*

***4.141 through *4.143** The couple **M** acts in a vertical plane and is applied to a beam oriented as shown. Determine the stress at point *A*.

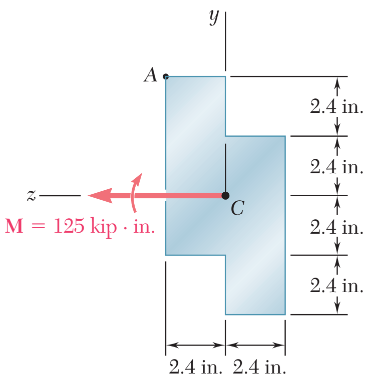


Fig. P4.141

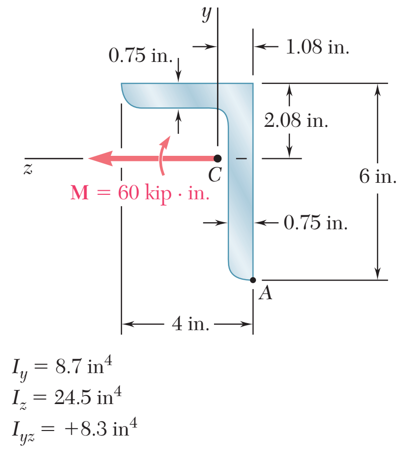


Fig. *P4.142*

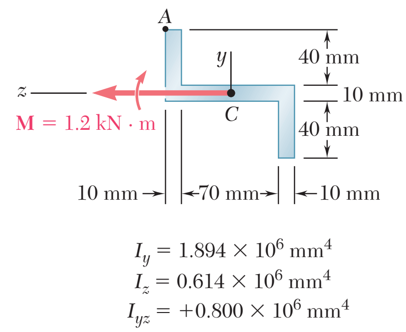


Fig. P4.143

4.144 The tube shown has a uniform wall thickness of 12 mm. For the loading given, determine (*a*) the stress at points *A* and *B*, (*b*) the point where the neutral axis intersects line *ABD*.

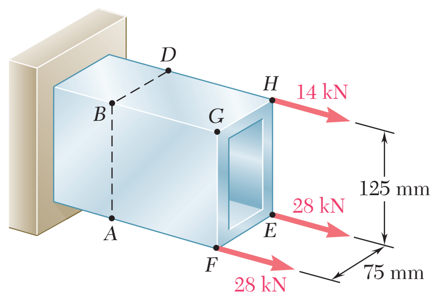


Fig. P4.144

4.145 Solve Prob. 4.144, assuming that the 28-kN force at point *E* is removed.

4.146 A rigid circular plate of 125-mm radius is attached to a solid 150 × 200-mm rectangular post, with the center of the plate directly above the center of the post. If a 4-kN force **P** is applied at *E* with θ = 30°, determine (*a*) the stress at point *A*, (*b*) the stress at point *B*, (*c*) the point where the neutral axis intersects line *ABD*.

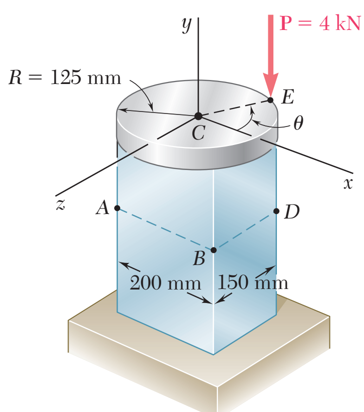


Fig. *P4.146*

4.147 In Prob. 4.146, determine (*a*) the value of θ for which the stress at *D* reaches its largest value, (*b*) the corresponding values of the stress at *A*, *B*, *C*, and *D*.

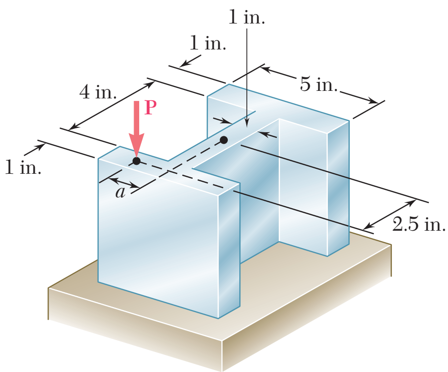


Fig. P4.148 and P4.149

4.148 Knowing that $P = 90$ kips, determine the largest distance a for which the maximum compressive stress does not exceed 18 ksi.

4.149 Knowing that $a = 1.25$ in., determine the largest value of $\mathbf{P}$ that can be applied without exceeding either of the following allowable stresses:

$$\sigma_{\text{ten}} = 10 \text{ ksi} \qquad \sigma_{\text{comp}} = 18 \text{ ksi}$$

4.150 The Z section shown is subjected to a couple $\mathbf{M}_0$ acting in a vertical plane. Determine the largest permissible value of the moment M_0 of the couple if the maximum stress is not to exceed 80 MPa. *Given*: $I_{max} = 2.28 \times 10^{-6}$ m^4, $I_{min} = 0.23 \times 10^{-6}$ m^4, principal axes 25.7° $\angle$ and 64.3° $\angle$.

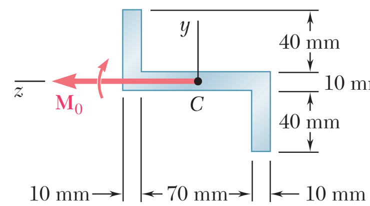


Fig. P4.150

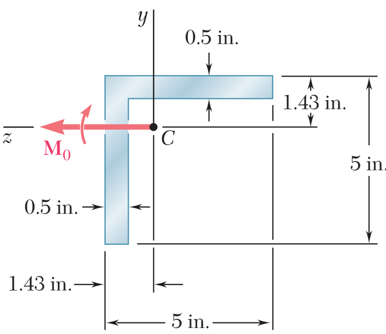


Fig. P4.152

4.151 Solve Prob. 4.150, assuming that the couple $\mathbf{M}_0$ acts in a horizontal plane.

4.152 A beam having the cross section shown is subjected to a couple $\mathbf{M}_0$ that acts in a vertical plane. Determine the largest permissible value of the moment M_0 of the couple if the maximum stress in the beam is not to exceed 12 ksi. *Given*: $I_y = I_z = 11.3$ in^4, $A = 4.75$ in^2, $k_{min} = 0.983$ in. (*Hint*: By reason of symmetry, the principal axes form an angle of 45° with the coordinate axes. Use the relations $I_{min} = Ak^2_{min}$ and $I_{min} + I_{max} = I_y + I_z$.)

4.153 Solve Prob. 4.152, assuming that the couple $\mathbf{M}_0$ acts in a horizontal plane.

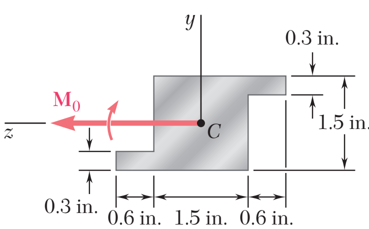


Fig. *P4.154*

4.154 An extruded aluminum member having the cross section shown is subjected to a couple acting in a vertical plane. Determine the largest permissible value of the moment M_0 of the couple if the maximum stress is not to exceed 12 ksi. *Given*: $I_{max} = 0.957$ in^4, $I_{min} = 0.427$ in^4, principal axes 29.4° $\angle$ and 60.6° $\angle$.

4.155 A couple $\mathbf{M}_0$ acting in a vertical plane is applied to a W12 × 16 rolled-steel beam, whose web forms an angle θ with the vertical. Denoting by σ_0 the maximum stress in the beam when $\theta = 0$, determine the angle of inclination θ of the beam for which the maximum stress is $2\sigma_0$.

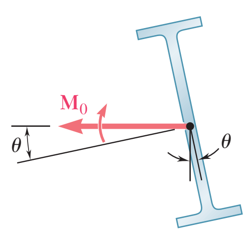


Fig. *P4.155*

4.156 Show that, if a solid rectangular beam is bent by a couple applied in a plane containing one diagonal of a rectangular cross section, the neutral axis will lie along the other diagonal.

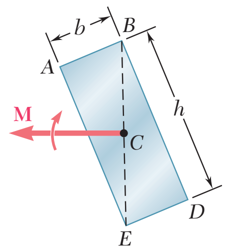


Fig. P4.156

4.157 A beam of unsymmetric cross section is subjected to a couple $\mathbf{M}_0$ acting in the horizontal plane xz. Show that the stress at point A, of coordinates y and z, is

$$\sigma_A = \frac{zI_z - yI_{yz}}{I_yI_z - I_{yz}^2}M_y$$

where I_y, I_z, and I_{yz} denote the moments and product of inertia of the cross section with respect to the coordinate axes, and M_y the moment of the couple.

4.158 A beam of unsymmetric cross section is subjected to a couple $\mathbf{M}_0$ acting in the vertical plane xy. Show that the stress at point A, of coordinates y and z, is

$$\sigma_A = -\frac{yI_y - zI_{yz}}{I_yI_z - I_{yz}^2}M_z$$

where I_y, I_z, and I_{yz} denote the moments and product of inertia of the cross section with respect to the coordinate axes, and M_z the moment of the couple.

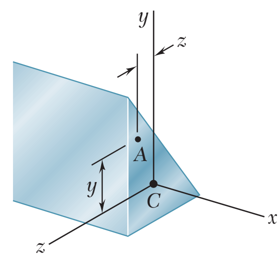


Fig. P4.157 and P4.158

4.159 (a) Show that, if a vertical force $\mathbf{P}$ is applied at point A of the section shown, the equation of the neutral axis BD is

$$\left(\frac{x_A}{r_z^2}\right)x + \left(\frac{z_A}{r_x^2}\right)z = -1$$

where r_z and r_x denote the radius of gyration of the cross section with respect to the z axis and the x axis, respectively. (b) Further show that, if a vertical force $\mathbf{Q}$ is applied at any point located on line BD, the stress at point A will be zero.

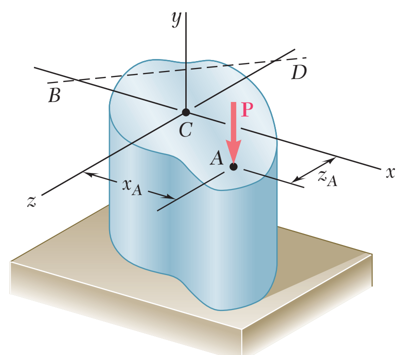


Fig. P4.159

4.160 (a) Show that the stress at corner A of the prismatic member shown in Fig. P4.160a will be zero if the vertical force $\mathbf{P}$ is applied at a point located on the line

$$\frac{x}{b/6} + \frac{z}{h/6} = 1$$

(b) Further show that, if no tensile stress is to occur in the member, the force $\mathbf{P}$ must be applied at a point located within the area bounded by the line found in part a and three similar lines corresponding to the condition of zero stress at B, C, and D, respectively. This area, shown in Fig. P4.160b, is known as the *kern* of the cross section.

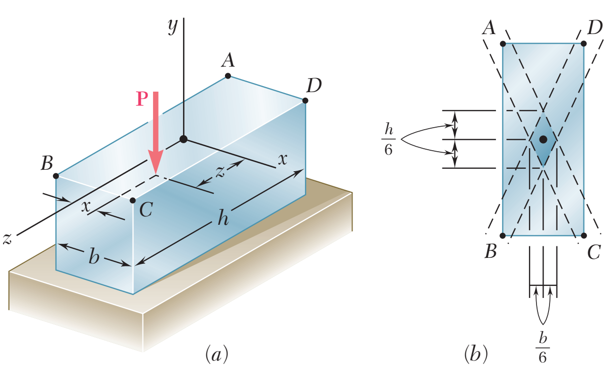


Fig. P4.160

*4.15 BENDING OF CURVED MEMBERS

Our analysis of stresses due to bending has been restricted so far to straight members. In this section we will consider the stresses caused by the application of equal and opposite couples to members that are initially curved. Our discussion will be limited to curved members of uniform cross section possessing a plane of symmetry in which the bending couples are applied, and it will be assumed that all stresses remain below the proportional limit.

If the initial curvature of the member is small, i.e., if its radius of curvature is large compared to the depth of its cross section, a good approximation can be obtained for the distribution of stresses by assuming the member to be straight and using the formulas derived in Secs. 4.3 and 4.4.† However, when the radius of curvature and the dimensions of the cross section of the member are of the same order of magnitude, we must use a different method of analysis, which was first introduced by the German engineer E. Winkler (1835–1888).

Consider the curved member of uniform cross section shown in Fig. 4.70. Its transverse section is symmetric with respect to the y axis (Fig. 4.70b) and, in its unstressed state, its upper and lower surfaces intersect the vertical xy plane along arcs of circle AB and FG centered at C (Fig. 4.70a). We now apply two equal and opposite couples **M**

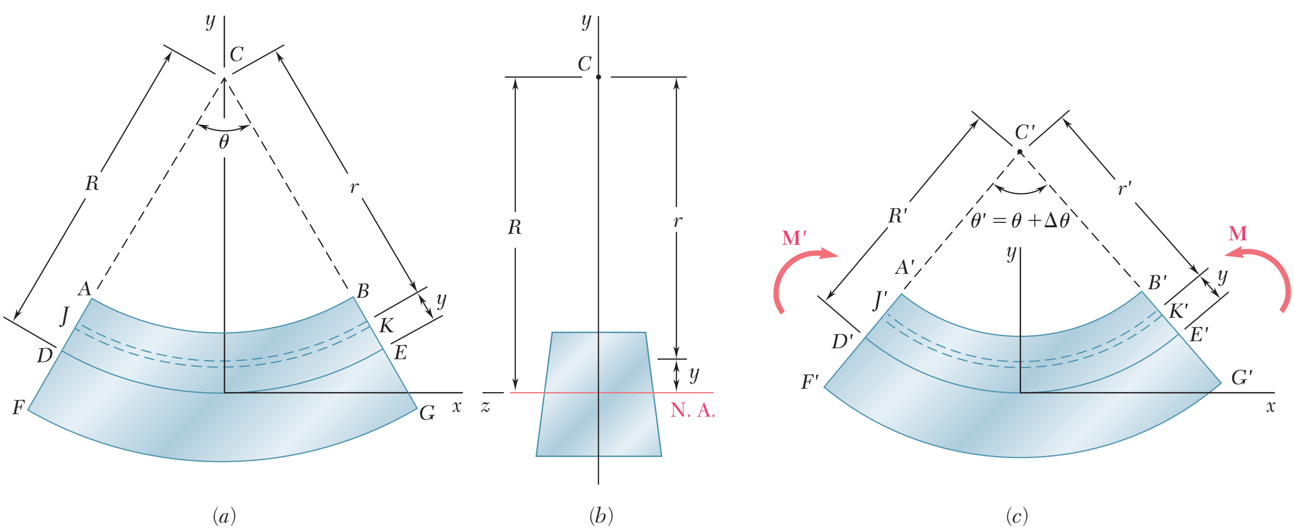


Fig. 4.70 Curved member in pure bending.

and **M**′ in the plane of symmetry of the member (Fig. 4.70c). A reasoning similar to that of Sec. 4.3 would show that any transverse plane section containing C will remain plane, and that the various arcs of circle indicated in Fig. 4.70a will be transformed into circular and concentric arcs with a center C' different from C. More specifically, if the couples **M** and **M**′ are directed as shown, the curvature of the various arcs of circle will increase; that is $A'C' < AC$. We also note that the couples **M** and **M**′ will cause the length of the upper surface

†See Prob. 4.166.

of the member to decrease ($A'B' < AB$) and the length of the lower surface to increase ($F'G' > FG$). We conclude that a *neutral surface* must exist in the member, the length of which remains constant. The intersection of the neutral surface with the xy plane has been represented in Fig. 4.70*a* by the arc DE of radius R, and in Fig. 4.70*c* by the arc $D'E'$ of radius R'. Denoting by θ and θ' the central angles corresponding respectively to DE and $D'E'$, we express the fact that the length of the neutral surface remains constant by writing

$$R\theta = R'\theta' \tag{4.59}$$

Considering now the arc of circle JK located at a distance y above the neutral surface, and denoting respectively by r and r' the radius of this arc before and after the bending couples have been applied, we express the deformation of JK as

$$\delta = r'\theta' - r\theta \tag{4.60}$$

Observing from Fig. 4.70 that

$$r = R - y \qquad r' = R' - y \tag{4.61}$$

and substituting these expressions into Eq. (4.60), we write

$$\delta = (R' - y)\theta' - (R - y)\theta$$

or, recalling Eq. (4.59) and setting $\theta' - \theta = \Delta\theta$,

$$\delta = -y\,\Delta\theta \tag{4.62}$$

The normal strain ϵ_x in the elements of JK is obtained by dividing the deformation δ by the original length $r\theta$ of arc JK. We write

$$\epsilon_x = \frac{\delta}{r\theta} = -\frac{y\,\Delta\theta}{r\theta}$$

or, recalling the first of the relations (4.61),

$$\epsilon_x = -\frac{\Delta\theta}{\theta}\frac{y}{R - y} \tag{4.63}$$

The relation obtained shows that, while each transverse section remains plane, the normal strain ϵ_x *does not vary linearly* with the distance y from the neutral surface.

The normal stress σ_x can now be obtained from Hooke's law, $\sigma_x = E\epsilon_x$, by substituting for ϵ_x from Eq. (4.63). We have

$$\sigma_x = -\frac{E\,\Delta\theta}{\theta}\frac{y}{R - y} \tag{4.64}$$

or, alternatively, recalling the first of Eqs. (4.61),

$$\sigma_x = -\frac{E\,\Delta\theta}{\theta}\frac{R - r}{r} \tag{4.65}$$

Equation (4.64) shows that, like ϵ_x, the normal stress σ_x *does not vary linearly* with the distance y from the neutral surface. Plotting σ_x versus y, we obtain an arc of hyperbola (Fig. 4.71).

In order to determine the location of the neutral surface in the member and the value of the coefficient $E\,\Delta\theta/\theta$ used in Eqs. (4.64)

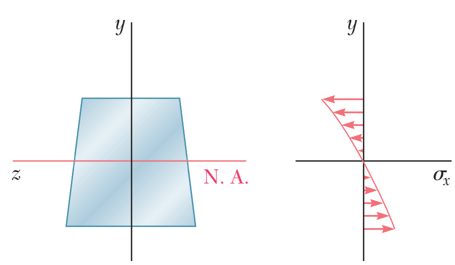


Fig. 4.71

and (4.65), we now recall that the elementary forces acting on any transverse section must be statically equivalent to the bending couple **M.** Expressing, as we did in Sec. 4.2 for a straight member, that the sum of the elementary forces acting on the section must be zero, and that the sum of their moments about the transverse z axis must be equal to the bending moment M, we write the equations

$$\int \sigma_x \, dA = 0 \tag{4.1}$$

and

$$\int (-y\sigma_x \, dA) = M \tag{4.3}$$

Substituting for σ_x from (4.65) into Eq. (4.1), we write

$$-\int \frac{E\,\Delta\theta}{\theta} \frac{R - r}{r} dA = 0$$

$$\int \frac{R - r}{r} dA = 0$$

$$R \int \frac{dA}{r} - \int dA = 0$$

from which it follows that the distance R from the center of curvature C to the neutral surface is defined by the relation

$$R = \frac{A}{\displaystyle\int \frac{dA}{r}} \tag{4.66}$$

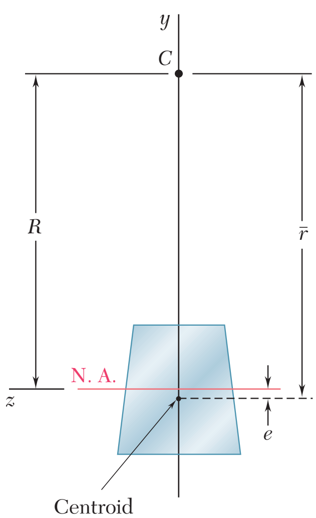


Fig. 4.72

We note that the value obtained for R is not equal to the distance $\bar{r}$ from C to the centroid of the cross section, since $\bar{r}$ is defined by a different relation, namely,

$$\bar{r} = \frac{1}{A} \int r \, dA \tag{4.67}$$

We thus conclude that, in a curved member, *the neutral axis of a transverse section does not pass through the centroid of that section* (Fig. 4.72).† Expressions for the radius R of the neutral surface will be derived for some specific cross-sectional shapes in Example 4.10 and in Probs. 4.188 through 4.190. For convenience, these expressions are shown in Fig. 4.73.

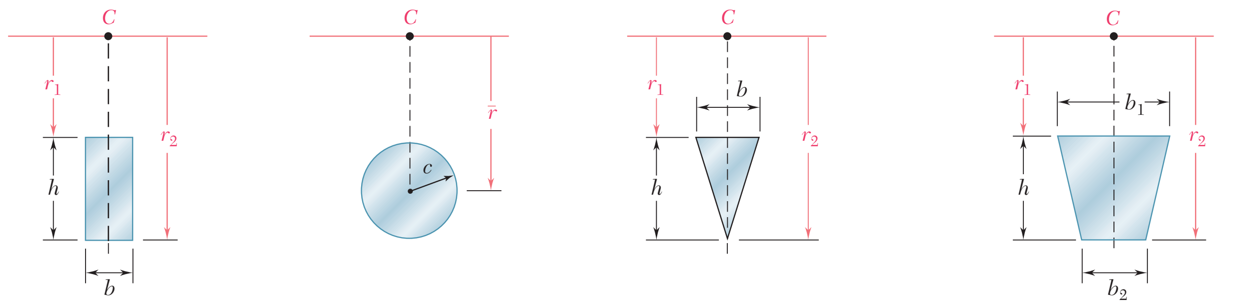
Substituting now for σ_x from (4.65) into Eq. (4.3), we write

$$\int \frac{E\,\Delta\theta}{\theta} \frac{R - r}{r} y \, dA = M$$

†However, an interesting property of the neutral surface can be noted if we write Eq. (4.66) in the alternative form

$$\frac{1}{R} = \frac{1}{A} \int \frac{1}{r} dA \tag{4.66$'$}$$

Equation (4.66′) shows that, if the member is divided into a large number of fibers of cross-sectional area dA, the curvature $1/R$ of the neutral surface will be equal to the average value of the curvature $1/r$ of the various fibers.



Rectangle: $R = \dfrac{h}{\ln \dfrac{r_2}{r_1}}$

Circle: $R = \frac{1}{2}\left(\bar{r} + \sqrt{\bar{r}^2 - c^2}\right)$

Triangle: $R = \dfrac{\frac{1}{2}h}{\dfrac{r_2}{h}\ln\dfrac{r_2}{r_1} - 1}$

Trapezoid: $R = \dfrac{\frac{1}{2}h^2(b_1 + b_2)}{(b_1 r_2 - b_2 r_1)\ln\dfrac{r_2}{r_1} - h(b_1 - b_2)}$

Fig. 4.73 Radius of neutral surface for various cross-sectional shapes.

or, since $y = R - r$,

$$\frac{E\,\Delta\theta}{\theta}\int\frac{(R-r)^2}{r}dA = M$$

Expanding the square in the integrand, we obtain after reductions

$$\frac{E\,\Delta\theta}{\theta}\left[R^2\int\frac{dA}{r} - 2RA + \int r\,dA\right] = M$$

Recalling Eqs. (4.66) and (4.67), we note that the first term in the brackets is equal to RA, while the last term is equal to $\bar{r}A$. We have, therefore,

$$\frac{E\,\Delta\theta}{\theta}(RA - 2RA + \bar{r}A) = M$$

and, solving for $E\,\Delta\theta/\theta$,

$$\frac{E\,\Delta\theta}{\theta} = \frac{M}{A(\bar{r} - R)} \tag{4.68}$$

Referring to Fig. 4.70, we note that $\Delta\theta > 0$ for $M > 0$. It follows that $\bar{r} - R > 0$, or $R < \bar{r}$, regardless of the shape of the section. Thus, the neutral axis of a transverse section is always located between the centroid of the section and the center of curvature of the member (Fig. 4.72). Setting $\bar{r} - R = e$, we write Eq. (4.68) in the form

$$\frac{E\,\Delta\theta}{\theta} = \frac{M}{Ae} \tag{4.69}$$

Substituting now for $E\,\Delta\theta/\theta$ from (4.69) into Eqs. (4.64) and (4.65), we obtain the following alternative expressions for the normal stress σ_x in a curved beam:

$$\sigma_x = -\frac{My}{Ae(R - y)} \tag{4.70}$$

and

$$\sigma_x = \frac{M(r - R)}{Aer} \tag{4.71}$$

We should note that the parameter e in the previous equations is a small quantity obtained by subtracting two lengths of comparable size, R and $\bar{r}$. In order to determine σ_x with a reasonable degree of accuracy, it is therefore necessary to compute R and $\bar{r}$ very accurately, particularly when both of these quantities are large, i.e., when the curvature of the member is small. However, as we indicated earlier, it is possible in such a case to obtain a good approximation for σ_x by using the formula $\sigma_x = -My/I$ developed for straight members.

Let us now determine the change in curvature of the neutral surface caused by the bending moment M. Solving Eq. (4.59) for the curvature $1/R'$ of the neutral surface in the deformed member, we write

$$\frac{1}{R'} = \frac{1}{R}\frac{\theta'}{\theta}$$

or, setting $\theta' = \theta + \Delta\theta$ and recalling Eq. (4.69),

$$\frac{1}{R'} = \frac{1}{R}\left(1 + \frac{\Delta\theta}{\theta}\right) = \frac{1}{R}\left(1 + \frac{M}{EAe}\right)$$

from which it follows that the change in curvature of the neutral surface is

$$\frac{1}{R'} - \frac{1}{R} = \frac{M}{EAeR} \tag{4.72}$$

EXAMPLE 4.10

A curved rectangular bar has a mean radius $\bar{r} = 6$ in. and a cross section of width $b = 2.5$ in. and depth $h = 1.5$ in. (Fig. 4.74). Determine the distance e between the centroid and the neutral axis of the cross section.

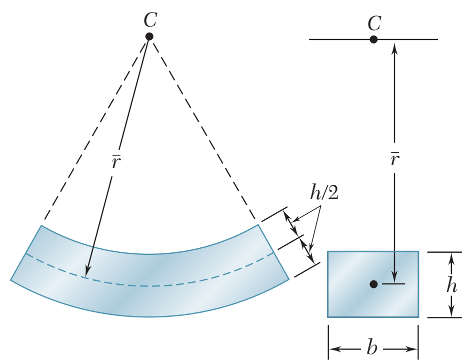


Fig. 4.74

We first derive the expression for the radius R of the neutral surface. Denoting by r_1 and r_2, respectively, the inner and outer radius of the bar (Fig. 4.75), we use Eq. (4.66) and write

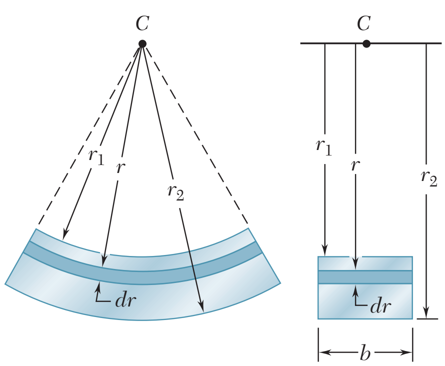


Fig. 4.75

$$R = \frac{A}{\displaystyle\int_{r_1}^{r_2}\frac{dA}{r}} = \frac{bh}{\displaystyle\int_{r_1}^{r_2}\frac{b\,dr}{r}} = \frac{h}{\displaystyle\int_{r_1}^{r_2}\frac{dr}{r}}$$

$$R = \frac{h}{\ln\dfrac{r_2}{r_1}} \tag{4.73}$$

For the given data, we have

$$r_1 = \bar{r} - \tfrac{1}{2}h = 6 - 0.75 = 5.25 \text{ in.}$$
$$r_2 = \bar{r} + \tfrac{1}{2}h = 6 + 0.75 = 6.75 \text{ in.}$$

Substituting for h, r_1, and r_2 into Eq. (4.73), we have

$$R = \frac{h}{\ln \dfrac{r_2}{r_1}} = \frac{1.5 \text{ in.}}{\ln \dfrac{6.75}{5.25}} = 5.9686 \text{ in.}$$

The distance between the centroid and the neutral axis of the cross section (Fig. 4.76) is thus

$$e = \bar{r} - R = 6 - 5.9686 = 0.0314 \text{ in.}$$

We note that it was necessary to calculate R with five significant figures in order to obtain e with the usual degree of accuracy.

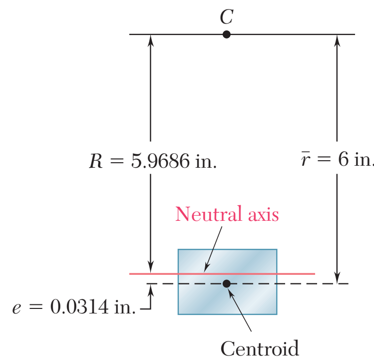


Fig. 4.76

EXAMPLE 4.11

For the bar of Example 4.10, determine the largest tensile and compressive stresses, knowing that the bending moment in the bar is $M = 8$ kip · in.

We use Eq. (4.71) with the given data,

$$M = 8 \text{ kip} \cdot \text{in.} \qquad A = bh = (2.5 \text{ in.})(1.5 \text{ in.}) = 3.75 \text{ in}^2$$

and the values obtained in Example 4.10 for R and e,

$$R = 5.969 \qquad e = 0.0314 \text{ in.}$$

Making first $r = r_2 = 6.75$ in. in Eq. (4.71), we write

$$\sigma_{\max} = \frac{M(r_2 - R)}{Aer_2}$$
$$= \frac{(8 \text{ kip} \cdot \text{in.})(6.75 \text{ in.} - 5.969 \text{ in.})}{(3.75 \text{ in}^2)(0.0314 \text{ in.})(6.75 \text{ in.})}$$
$$\sigma_{\max} = 7.86 \text{ ksi}$$

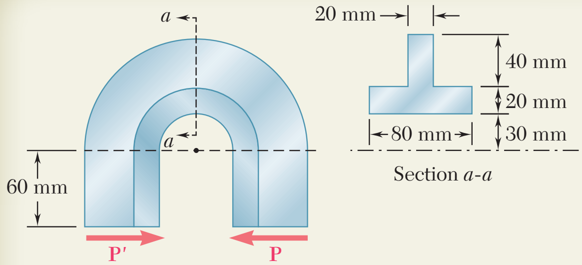
Making now $r = r_1 = 5.25$ in. in Eq. (4.71), we have

$$\sigma_{\min} = \frac{M(r_1 - R)}{Aer_1}$$
$$= \frac{(8 \text{ kip} \cdot \text{in.})(5.25 \text{ in.} - 5.969 \text{ in.})}{(3.75 \text{ in}^2)(0.0314 \text{ in.})(5.25 \text{ in.})}$$
$$\sigma_{\min} = -9.30 \text{ ksi}$$

Remark. Let us compare the values obtained for $\sigma_{\max}$ and $\sigma_{\min}$ with the result we would get for a straight bar. Using Eq. (4.15) of Sec. 4.4, we write

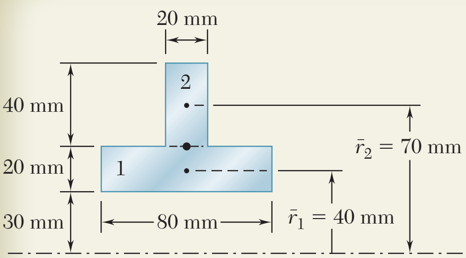
$$\sigma_{\max,\min} = \pm\frac{Mc}{I}$$
$$= \pm\frac{(8 \text{ kip} \cdot \text{in.})(0.75 \text{ in.})}{\frac{1}{12}(2.5 \text{ in.})(1.5 \text{ in.})^3} = \pm 8.53 \text{ ksi}$$

SAMPLE PROBLEM 4.11



A machine component has a T-shaped cross section and is loaded as shown. Knowing that the allowable compressive stress is 50 MPa, determine the largest force **P** that can be applied to the component.

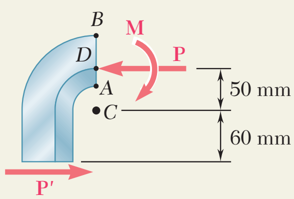
SOLUTION



Centroid of the Cross Section. We locate the centroid D of the cross section

	A_i, mm²	$\bar{r}_i$, mm	$\bar{r}_i A_i$, mm³
1	$(20)(80) = 1600$	40	64×10^3
2	$(40)(20) = 800$	70	56×10^3
	$\Sigma A_i = 2400$		$\Sigma \bar{r}_i A_i = 120 \times 10^3$

$$\bar{r}\Sigma A_i = \Sigma \bar{r}_i A_i$$
$$\bar{r}(2400) = 120 \times 10^3$$
$$\bar{r} = 50 \text{ mm} = 0.050 \text{ m}$$

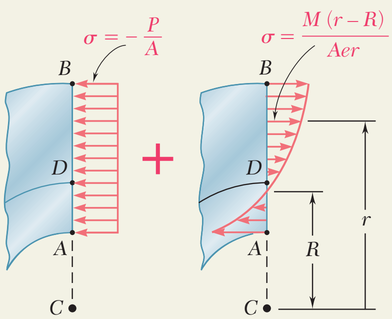


Force and Couple at *D*. The internal forces in section *a-a* are equivalent to a force **P** acting at D and a couple **M** of moment

$$M = P(50 \text{ mm} + 60 \text{ mm}) = (0.110 \text{ m})P$$

Superposition. The centric force **P** causes a uniform compressive stress on section *a-a*. The bending couple **M** causes a varying stress distribution [Eq. (4.71)]. We note that the couple **M** tends to increase the curvature of the member and is therefore positive (cf. Fig. 4.70). The total stress at a point of section *a-a* located at distance r from the center of curvature C is

$$\sigma = -\frac{P}{A} + \frac{M(r - R)}{Aer} \qquad (1)$$



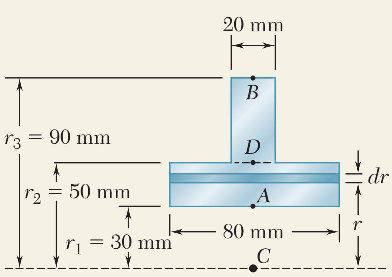
Radius of Neutral Surface. We now determine the radius R of the neutral surface by using Eq. (4.66).

$$R = \frac{A}{\displaystyle\int \frac{dA}{r}} = \frac{2400 \text{ mm}^2}{\displaystyle\int_{r_1}^{r_2} \frac{(80 \text{ mm})\,dr}{r} + \int_{r_2}^{r_3} \frac{(20 \text{ mm})\,dr}{r}}$$

$$= \frac{2400}{80 \ln \frac{50}{30} + 20 \ln \frac{90}{50}} = \frac{2400}{40.866 + 11.756} = 45.61 \text{ mm}$$

$$= 0.04561 \text{ m}$$

We also compute: $e = \bar{r} - R = 0.05000 \text{ m} - 0.04561 \text{ m} = 0.00439 \text{ m}$



Allowable Load. We observe that the largest compressive stress will occur at point A where $r = 0.030$ m. Recalling that $\sigma_{all} = 50$ MPa and using Eq. (1), we write

$$-50 \times 10^6 \text{ Pa} = -\frac{P}{2.4 \times 10^{-3} \text{ m}^2} + \frac{(0.110\text{ P})(0.030 \text{ m} - 0.04561 \text{ m})}{(2.4 \times 10^{-3} \text{ m}^2)(0.00439 \text{ m})(0.030 \text{ m})}$$

$$-50 \times 10^6 = -417P - 5432P$$

$$P = 8.55 \text{ kN}$$

PROBLEMS

4.161 For the machine component and loading shown, determine the stress at point A when (*a*) $h = 2$ in., (*b*) $h = 2.6$ in.

4.162 For the machine component and loading shown, determine the stress at points A and B when $h = 2.5$ in.

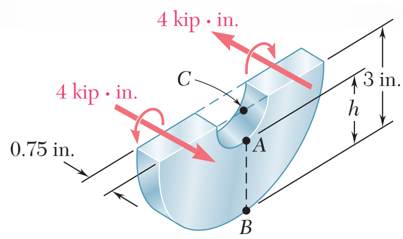


Fig. P4.161 and P4.162

4.163 The curved portion of the bar shown has an inner radius of 20 mm. Knowing that the allowable stress in the bar is 150 MPa, determine the largest permissible distance a from the line of action of the 3-kN force to the vertical plane containing the center of curvature of the bar.

4.164 The curved portion of the bar shown has an inner radius of 20 mm. Knowing that the line of action of the 3-kN force is located at a distance $a = 60$ mm from the vertical plane containing the center of curvature of the bar, determine the largest compressive stress in the bar.

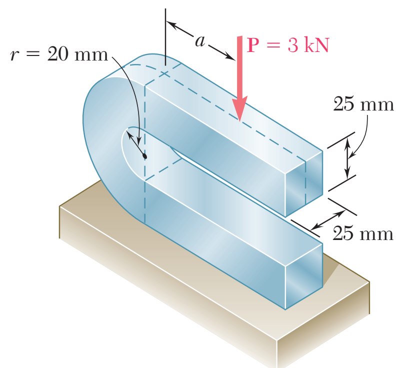


Fig. P4.163 and P4.164

4.165 The curved bar shown has a cross section of 40×60 mm and an inner radius $r_1 = 15$ mm. For the loading shown determine the largest tensile and compressive stresses.

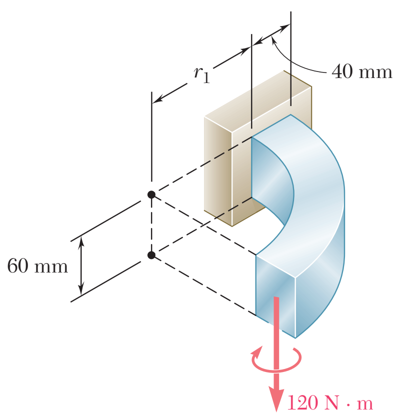


Fig. *P4.165* and *P4.166*

4.166 For the curved bar and loading shown, determine the percent error introduced in the computation of the maximum stress by assuming that the bar is straight. Consider the case when (*a*) $r_1 = 20$ mm, (*b*) $r_1 = 200$ mm, (*c*) $r_1 = 2$ m.

4.167 The curved bar shown has a cross section of 30×30 mm. Knowing that $a = 60$ mm, determine the stress at (*a*) point A, (*b*) point B.

4.168 The curved bar shown has a cross section of 30×30 mm. Knowing that the allowable compressive stress is 175 MPa, determine the largest allowable distance a.

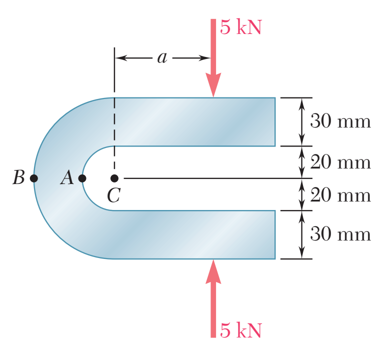


Fig. P4.167 and P4.168

4.169 Steel links having the cross section shown are available with different central angles β. Knowing that the allowable stress is 12 ksi, determine the largest force **P** that can be applied to a link for which $\beta = 90°$.

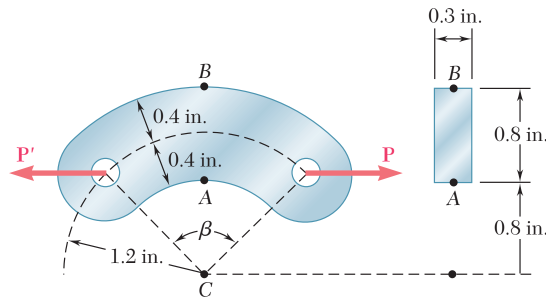


Fig. *P4.169*

4.170 Solve Prob. 4.169, assuming that $\beta = 60°$.

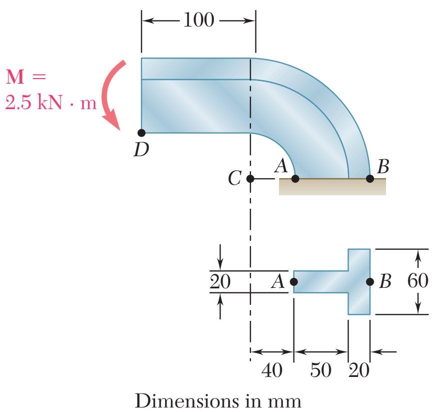


Fig. P4.171 and P4.172

4.171 A machine component has a T-shaped cross section that is oriented as shown. Knowing that $M = 2.5$ kN · m, determine the stress at (*a*) point *A*, (*b*) point *B*.

4.172 Assuming that the couple shown is replaced by a vertical 10-kN force attached at point *D* and acting downward, determine the stress at (*a*) point *A*, (*b*) point *B*.

4.173 Three plates are welded together to form the curved beam shown. For the given loading, determine the distance *e* between the neutral axis and the centroid of the cross section.

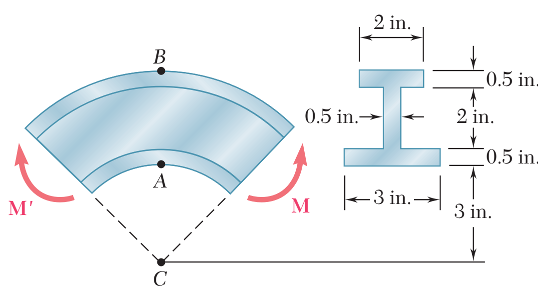


Fig. *P4.173* and P4.174

4.174 Three plates are welded together to form the curved beam shown. For $M = 8$ kip · in., determine the stress at (*a*) point *A*, (*b*) point *B*, (*c*) the centroid of the cross section.

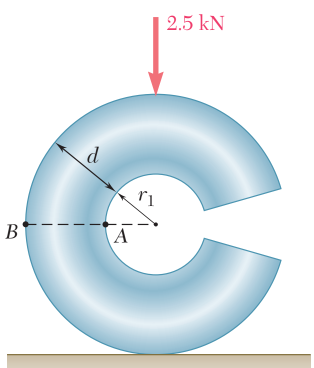


Fig. P4.175 and P4.176

4.175 The split ring shown has an inner radius $r_1 = 20$ mm and a circular cross section of diameter $d = 32$ mm. For the loading shown, determine the stress at (*a*) point *A*, (*b*) point *B*.

4.176 The split ring shown has an inner radius $r_1 = 16$ mm and a circular cross section of diameter $d = 32$ mm. For the loading shown, determine the stress at (*a*) point *A*, (*b*) point *B*.

4.177 The curved bar shown has a circular cross section of 32-mm diameter. Determine the largest couple **M** that can be applied to the bar about a horizontal axis if the maximum stress is not to exceed 60 MPa.

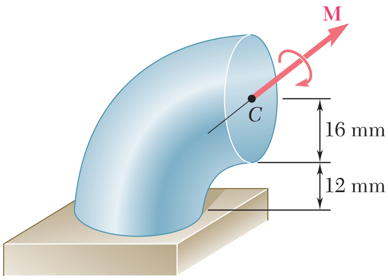


Fig. P4.177

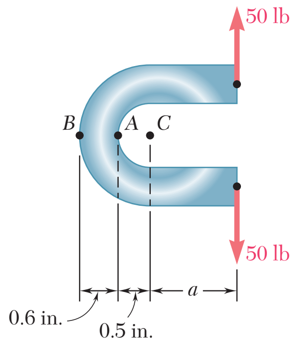


Fig. P4.178 and P4.179

4.178 The bar shown has a circular cross section of 0.6 in.-diameter. Knowing that $a = 1.2$ in., determine the stress at (*a*) point *A*, (*b*) point *B*.

4.179 The bar shown has a circular cross section of 0.6-in. diameter. Knowing that the allowable stress is 8 ksi, determine the largest permissible distance *a* from the line of action of the 50-lb forces to the plane containing the center of curvature of the bar.

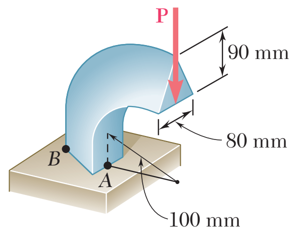


Fig. P4.180

4.180 Knowing that $P = 10$ kN, determine the stress at (*a*) point *A*, (*b*) point *B*.

4.181* and *4.182 Knowing that $M = 5$ kip · in., determine the stress at (*a*) point *A*, (*b*) point *B*.

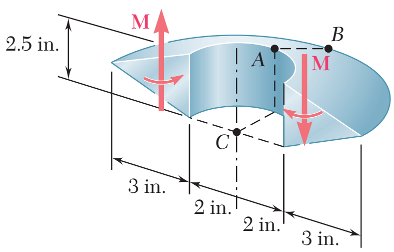


Fig. *P4.181*

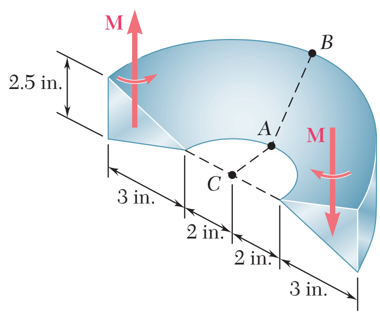


Fig. *P4.182*

4.183 For the curved beam and loading shown, determine the stress at (*a*) point *A*, (*b*) point *B*.

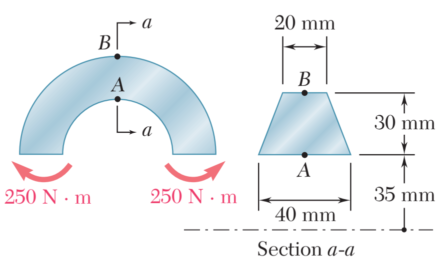


Fig. P4.183

4.184 For the crane hook shown, determine the largest tensile stress in section *a-a*.

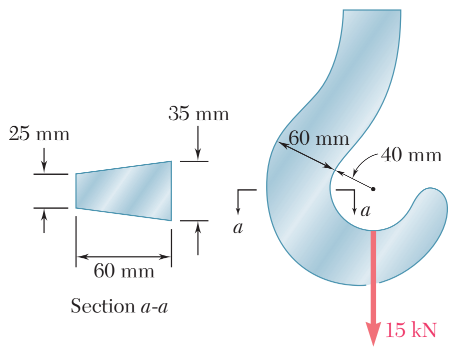


Fig. *P4.184*

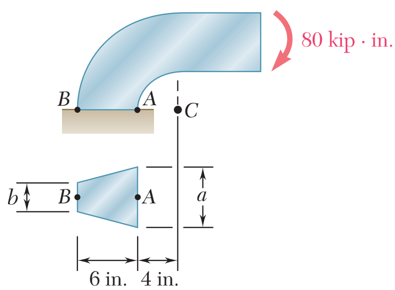


Fig. P4.185 and P4.186

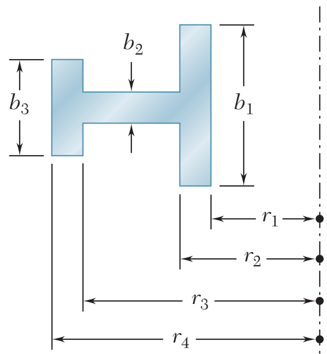


Fig. P4.187

4.185 Knowing that the machine component shown has a trapezoidal cross section with $a = 3.5$ in. and $b = 2.5$ in., determine the stress at (*a*) point *A*, (*b*) point *B*.

4.186 Knowing that the machine component shown has a trapezoidal cross section with $a = 2.5$ in. and $b = 3.5$ in., determine the stress at (*a*) point *A*, (*b*) point *B*.

4.187 Show that if the cross section of a curved beam consists of two or more rectangles, the radius *R* of the neutral surface can be expressed as

$$R = \frac{A}{\ln\left[\left(\frac{r_2}{r_1}\right)^{b_1}\left(\frac{r_3}{r_2}\right)^{b_2}\left(\frac{r_4}{r_3}\right)^{b_3}\right]}$$

where *A* is the total area of the cross section.

4.188 through 4.190 Using Eq. (4.66), derive the expression for *R* given in Fig. 4.73 for

***4.188** A circular cross section.
4.189 A trapezoidal cross section.
4.190 A triangular cross section.

***4.191** For a curved bar of rectangular cross section subjected to a bending couple **M,** show that the radial stress at the neutral surface is

$$\sigma_r = \frac{M}{Ae}\left(1 - \frac{r_1}{R} - \ln\frac{R}{r_1}\right)$$

and compute the value of σ_r for the curved bar of Examples 4.10 and 4.11.

(*Hint:* consider the free-body diagram of the portion of the beam located above the neutral surface.)

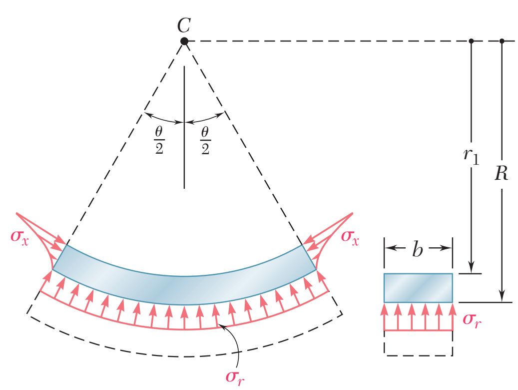


Fig. P4.191

REVIEW AND SUMMARY

This chapter was devoted to the analysis of members in *pure bending.* That is, we considered the stresses and deformation in members subjected to equal and opposite couples **M** and **M**′ acting in the same longitudinal plane (Fig. 4.77).

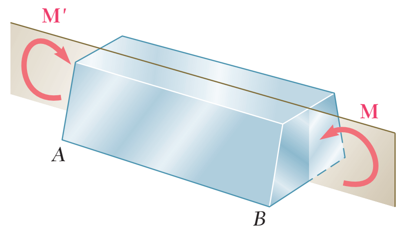


Fig. 4.77

Normal strain in bending

We first studied members possessing a plane of symmetry and subjected to couples acting in that plane. Considering possible deformations of the member, we proved that *transverse sections remain plane* as a member is deformed [Sec. 4.3]. We then noted that a member in pure bending has a *neutral surface* along which normal strains and stresses are zero and that the longitudinal *normal strain* ϵ_x *varies linearly* with the distance y from the neutral surface:

$$\epsilon_x = -\frac{y}{\rho} \tag{4.8}$$

where ρ is the *radius of curvature* of the neutral surface (Fig. 4.78). The intersection of the neutral surface with a transverse section is known as the *neutral axis* of the section.

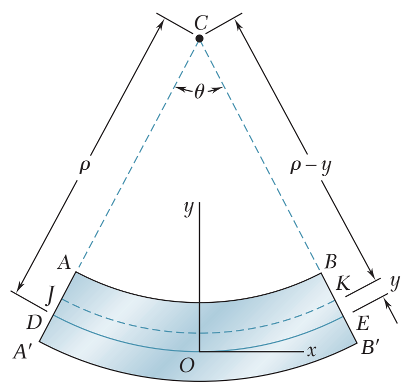


Fig. 4.78

Normal stress in elastic range

For members made of a material that follows Hooke's law [Sec. 4.4], we found that the *normal stress* σ_x *varies linearly* with the distance from the neutral axis (Fig. 4.79). Denoting by σ_m the maximum stress we wrote

$$\sigma_x = -\frac{y}{c}\sigma_m \tag{4.12}$$

where c is the largest distance from the neutral axis to a point in the section.

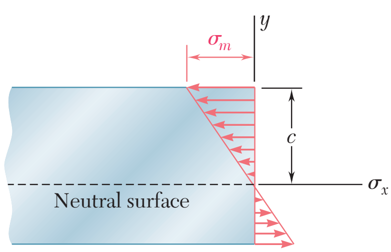


Fig. 4.79

Elastic flexure formula

By setting the sum of the elementary forces, $\sigma_x\, dA$, equal to zero, we proved that the *neutral axis passes through the centroid* of the cross section of a member in pure bending. Then by setting the sum of the moments of the elementary forces equal to the bending moment, we derived the *elastic flexure formula* for the maximum normal stress

$$\sigma_m = \frac{Mc}{I} \tag{4.15}$$

where I is the moment of inertia of the cross section with respect to the neutral axis. We also obtained the normal stress at any distance y from the neutral axis:

$$\sigma_x = -\frac{My}{I} \tag{4.16}$$

Noting that I and c depend only on the geometry of the cross section, we introduced the *elastic section modulus*

Elastic section modulus

$$S = \frac{I}{c} \tag{4.17}$$

and then used the section modulus to write an alternative expression for the maximum normal stress:

$$\sigma_m = \frac{M}{S} \tag{4.18}$$

Curvature of member

Recalling that the curvature of a member is the reciprocal of its radius of curvature, we expressed the *curvature* of the member as

$$\frac{1}{\rho} = \frac{M}{EI} \tag{4.21}$$

Anticlastic curvature

In Sec. 4.5, we completed our study of the bending of homogeneous members possessing a plane of symmetry by noting that deformations occur in the plane of a transverse cross section and result in *anticlastic curvature* of the members.

Members made of several materials

Next we considered the bending of members made of several materials with *different moduli of elasticity* [Sec. 4.6]. While transverse sections remain plane, we found that, in general, the *neutral axis does not pass through the centroid* of the composite cross section (Fig. 4.80). Using the ratio of the moduli of elasticity of the materials,

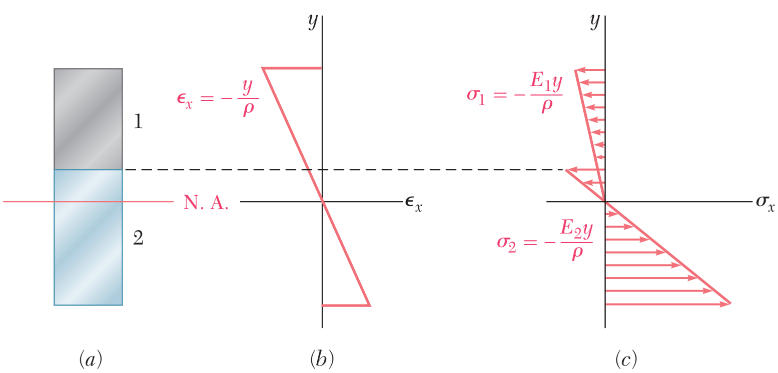


Fig. 4.80

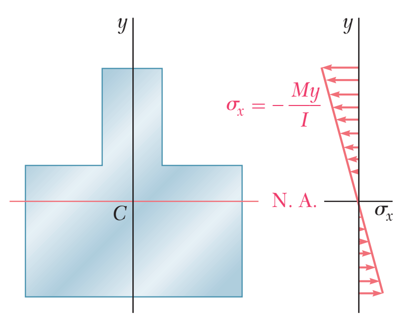


Fig. 4.81

we obtained a *transformed section* corresponding to an equivalent member made entirely of one material. We then used the methods previously developed to determine the stresses in this equivalent homogeneous member (Fig. 4.81) and then again used the ratio of the moduli of elasticity to determine the stresses in the composite beam [Sample Probs. 4.3 and 4.4].

Stress concentrations

In Sec. 4.7, *stress concentrations* that occur in members in pure bending were discussed and charts giving stress-concentration factors for flat bars with fillets and grooves were presented in Figs. 4.27 and 4.28.

Plastic deformations

We next investigated members made of materials that do not follow Hooke's law [Sec. 4.8]. A rectangular beam made of an *elastoplastic material* (Fig. 4.82) was analyzed as the magnitude of the bending moment was increased. The *maximum elastic moment* M_Y occurred when yielding was initiated in the beam (Fig. 4.83). As the bending moment was further increased, plastic zones developed and the size of the elastic core of the member decreased [Sec. 4.9]. Finally the beam became fully plastic and we obtained the maximum or *plastic moment* M_p. In Sec. 4.11, we found that *permanent deformations* and *residual stresses* remain in a member after the loads that caused yielding have been removed.

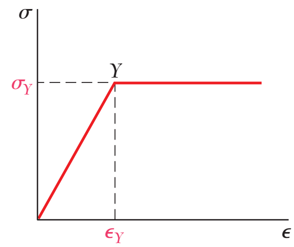


Fig. 4.82

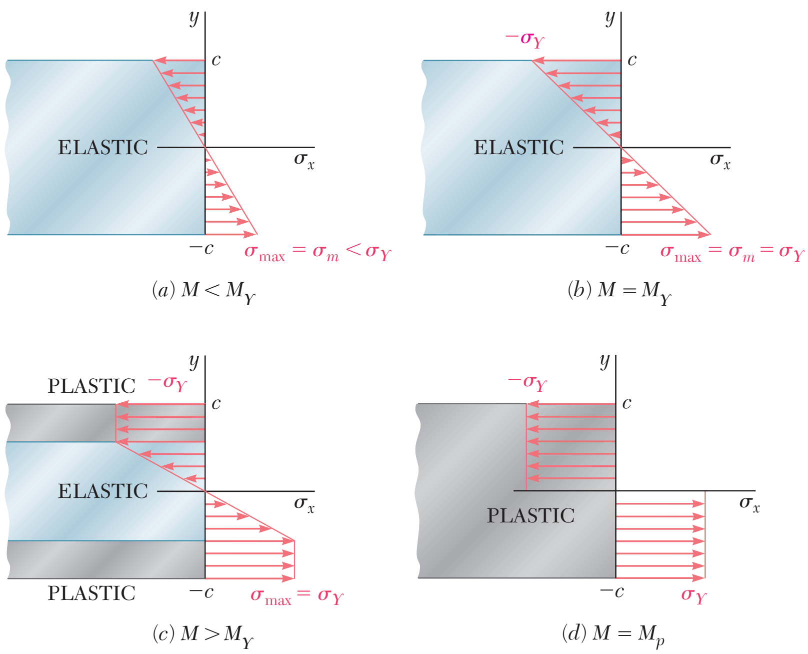


Fig. 4.83

Eccentric axial loading

In Sec. 4.12, we studied the stresses in members loaded *eccentrically in a plane of symmetry*. Our analysis made use of methods developed earlier. We replaced the *eccentric load* by a force-couple system located at the centroid of the cross section (Fig. 4.84) and then superposed stresses due to the centric load and the bending couple (Fig. 4.85):

$$\sigma_x = \frac{P}{A} - \frac{My}{I} \tag{4.50}$$

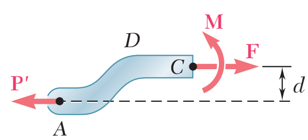


Fig. 4.84

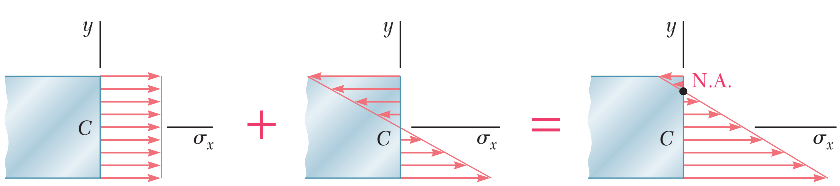


Fig. 4.85

Unsymmetric bending

The bending of members of *unsymmetric cross section* was considered next [Sec. 4.13]. We found that the flexure formula may be used, provided that the couple vector **M** is directed along one of the principal centroidal axes of the cross section. When necessary we

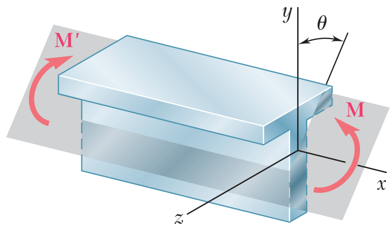


Fig. 4.86

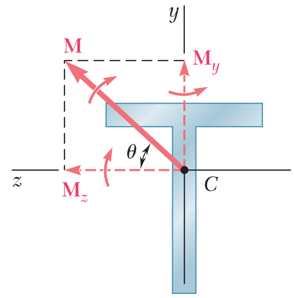


Fig. 4.87

resolved **M** into components along the principal axes and superposed the stresses due to the component couples (Figs. 4.86 and 4.87).

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{4.55}$$

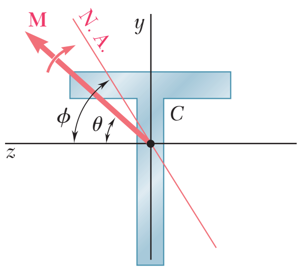


Fig. 4.88

For the couple **M** shown in Fig. 4.88, we determined the orientation of the neutral axis by writing

$$\tan \phi = \frac{I_z}{I_y} \tan \theta \tag{4.57}$$

General eccentric axial loading

The general case of *eccentric axial loading* was considered in Sec. 4.14, where we again replaced the load by a force-couple system located at the centroid. We then superposed the stresses due to the centric load and two component couples directed along the principal axes:

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{4.58}$$

Curved members

The chapter concluded with the analysis of stresses in *curved members* (Fig. 4.89). While transverse sections remain plane when the member is subjected to bending, we found that the *stresses do not vary linearly* and the neutral surface does not pass through the centroid of the section. The distance R from the center of curvature of the member to the neutral surface was found to be

$$R = \frac{A}{\displaystyle\int \frac{dA}{r}} \tag{4.66}$$

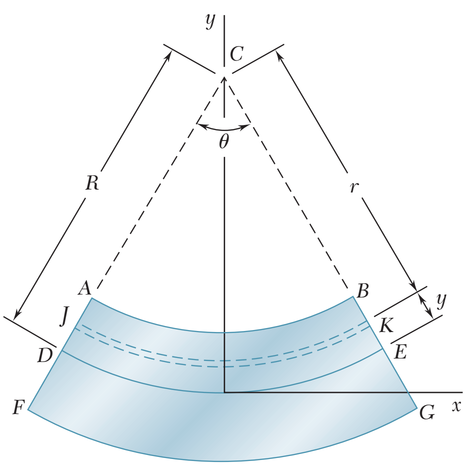


Fig. 4.89

where A is the area of the cross section. The normal stress at a distance y from the neutral surface was expressed as

$$\sigma_x = -\frac{My}{Ae(R - y)} \tag{4.70}$$

where M is the bending moment and e the distance from the centroid of the section to the neutral surface.

REVIEW PROBLEMS

4.192 Two vertical forces are applied to a beam of the cross section shown. Determine the maximum tensile and compressive stresses in portion BC of the beam.

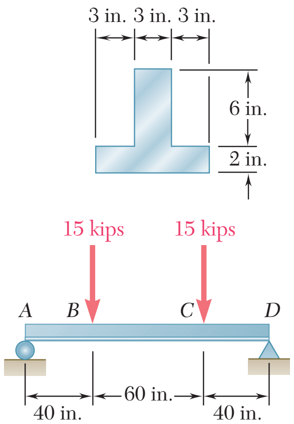


Fig. P4.192

4.193 Straight rods of 6-mm diameter and 30-m length are stored by coiling the rods inside a drum of 1.25-m inside diameter. Assuming that the yield strength is not exceeded, determine (*a*) the maximum stress in a coiled rod, (*b*) the corresponding bending moment in the rod. Use $E = 200$ GPa.

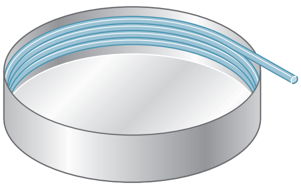


Fig. *P4.193*

4.194 Knowing that for the beam shown the allowable stress is 12 ksi in tension and 16 ksi in compression, determine the largest couple **M** that can be applied.

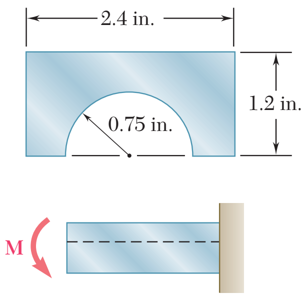


Fig. P4.194

4.195 In order to increase corrosion resistance, a 2-mm-thick cladding of aluminum has been added to a steel bar as shown. The modulus of elasticity is 200 GPa for steel and 70 GPa for aluminum. For a bending moment of 300 N · m, determine (*a*) the maximum stress in the steel, (*b*) the maximum stress in the aluminum, (*c*) the radius of curvature of the bar.

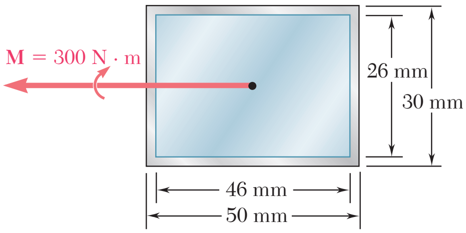


Fig. P4.195

4.196 A single vertical force **P** is applied to a short steel post as shown. Gages located at A, B, and C indicate the following strains:

$$\epsilon_A = -500\ \mu \qquad \epsilon_B = -1000\ \mu \qquad \epsilon_C = -200\ \mu$$

Knowing that $E = 29 \times 10^6$ psi, determine (*a*) the magnitude of **P**, (*b*) the line of action of **P**, (*c*) the corresponding strain at the hidden edge of the post, where $x = -2.5$ in. and $z = -1.5$ in.

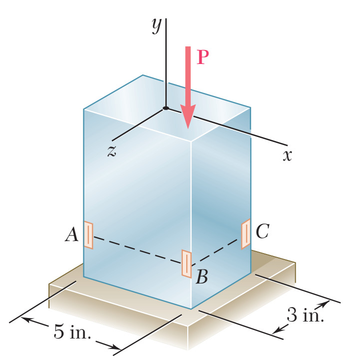


Fig. *P4.196*

4.197 For the split ring shown, determine the stress at (*a*) point *A*, (*b*) point *B*.

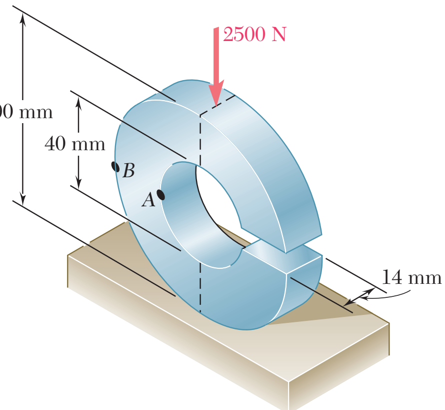


Fig. P4.197

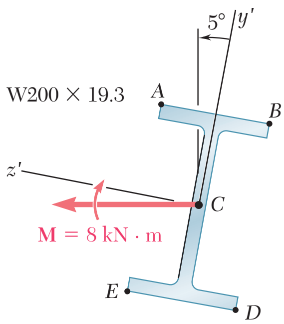


Fig. *P4.198*

4.198 A couple **M** of moment 8 kN · m acting in a vertical plane is applied to a W200 × 19.3 rolled-steel beam as shown. Determine (*a*) the angle that the neutral axis forms with the horizontal plane, (*b*) the maximum stress in the beam.

4.199 Determine the maximum stress in each of the two machine elements shown.

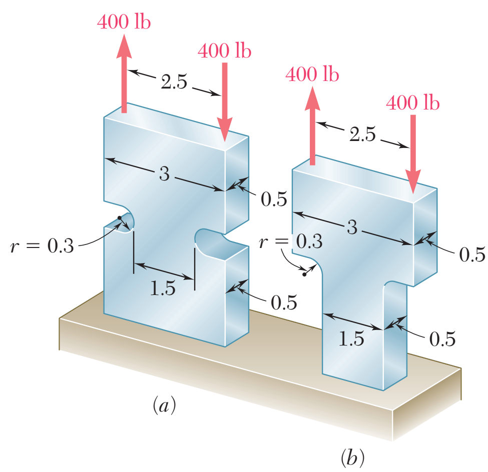


Fig. P4.199 All dimensions given in inches.

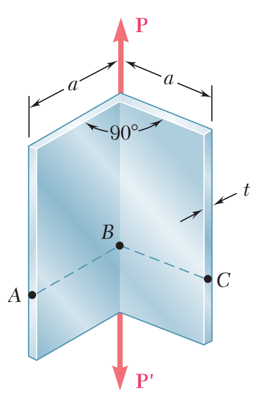


Fig. P4.200

4.200 The shape shown was formed by bending a thin steel plate. Assuming that the thickness *t* is small compared to the length *a* of a side of the shape, determine the stress (*a*) at *A*, (*b*) at *B*, (*c*) at *C*.

4.201 Three 120 × 10-mm steel plates have been welded together to form the beam shown. Assuming that the steel is elastoplastic with $E = 200$ GPa and $\sigma_Y = 300$ MPa, determine (*a*) the bending moment for which the plastic zones at the top and bottom of the beam are 40 mm thick, (*b*) the corresponding radius of curvature of the beam.

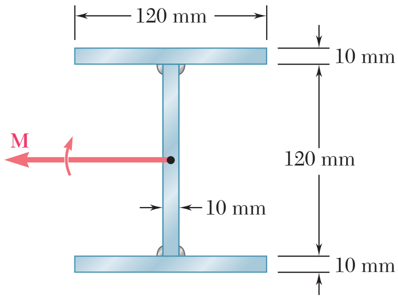


Fig. *P4.201*

4.202 A short column is made by nailing four 1 × 4-in. planks to a 4 × 4-in. timber. Determine the largest compressive stress created in the column by a 16-kip load applied as shown in the center of the top section of the timber if (*a*) the column is as described, (*b*) plank 1 is removed, (*c*) planks 1 and 2 are removed, (*d*) planks 1, 2, and 3 are removed, (*e*) all planks are removed.

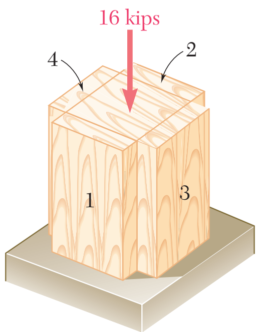


Fig. P4.202

4.203 Two thin strips of the same material and same cross section are bent by couples of the same magnitude and glued together. After the two surfaces of contact have been securely bonded, the couples are removed. Denoting by σ_1 the maximum stress and by ρ_1 the radius of curvature of each strip while the couples were applied, determine (*a*) the final stresses at points *A*, *B*, *C*, and *D*, (*b*) the final radius of curvature.

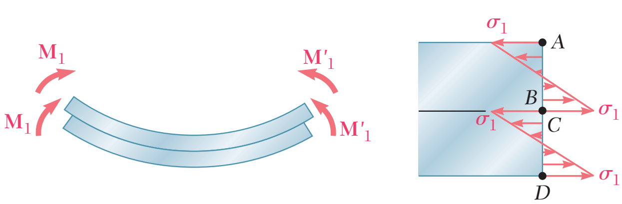


Fig. P4.203

COMPUTER PROBLEMS

The following problems are designed to be solved with a computer.

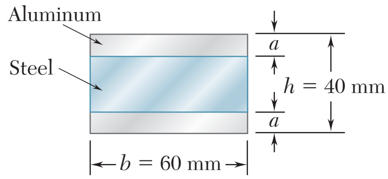


Fig. P4.C1

4.C1 Two aluminum strips and a steel strip are to be bonded together to form a composite member of width $b = 60$ mm and depth $h = 40$ mm. The modulus of elasticity is 200 GPa for the steel and 75 GPa for the aluminum. Knowing that $M = 1500$ N · m, write a computer program to calculate the maximum stress in the aluminum and in the steel for values of a from 0 to 20 mm using 2-mm increments. Using appropriate smaller increments, determine (*a*) the largest stress that can occur in the steel, (*b*) the corresponding value of a.

4.C2 A beam of the cross section shown, made of a steel that is assumed to be elastoplastic with a yield strength σ_Y and a modulus of elasticity E, is bent about the x axis. (*a*) Denoting by y_Y the half thickness of the elastic core, write a computer program to calculate the bending moment M and the radius of curvature ρ for values of y_Y from $\frac{1}{2}d$ to $\frac{1}{6}d$ using decrements equal to $\frac{1}{2}t_f$. Neglect the effect of fillets. (*b*) Use this program to solve Prob. 4.201.

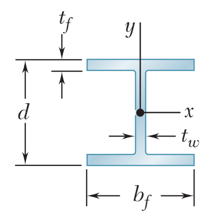


Fig. P4.C2

4.C3 An 8-kip · in. couple **M** is applied to a beam of the cross section shown in a plane forming an angle β with the vertical. Noting that the centroid of the cross section is located at C and that the y and z axes are principal axes, write a computer program to calculate the stress at A, B, C, and D for values of β from 0 to 180° using 10° increments. (*Given:* $I_y = 6.23$ in^4 and $I_z = 1.481$ in^4.)

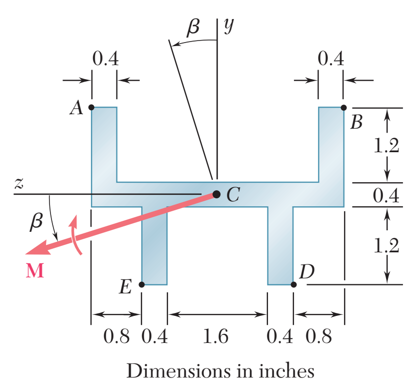


Fig. P4.C3

4.C4 Couples of moment $M = 2$ kN · m are applied as shown to a curved bar having a rectangular cross section with $h = 100$ mm and $b = 25$ mm. Write a computer program and use it to calculate the stresses at points A and B for values of the ratio r_1/h from 10 to 1 using decrements of 1, and from 1 to 0.1 using decrements of 0.1. Using appropriate smaller increments, determine the ratio r_1/h for which the maximum stress in the curved bar is 50% larger than the maximum stress in a straight bar of the same cross section.

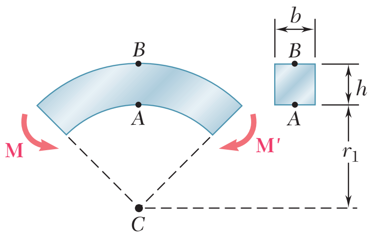


Fig. P4.C4

4.C5 The couple **M** is applied to a beam of the cross section shown. (*a*) Write a computer program that, for loads expressed in either SI or U.S. customary units, can be used to calculate the maximum tensile and compressive stresses in the beam. (*b*) Use this program to solve Probs. 4.10, 4.11, and 4.192.

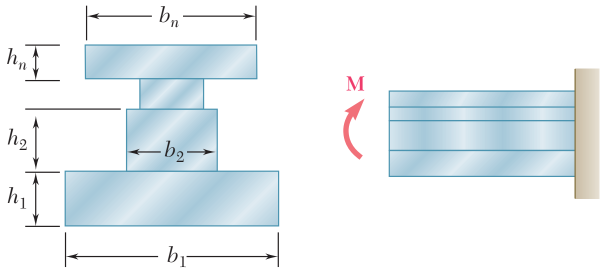


Fig. P4.C5

4.C6 A solid rod of radius $c = 1.2$ in. is made of a steel that is assumed to be elastoplastic with $E = 29{,}000$ ksi and $\sigma_Y = 42$ ksi. The rod is subjected to a couple of moment M that increases from zero to the maximum elastic moment M_Y and then to the plastic moment M_p. Denoting by y_Y the half thickness of the elastic core, write a computer program and use it to calculate the bending moment M and the radius of curvature ρ for values of y_Y from 1.2 in. to 0 using 0.2-in. decrements. (*Hint:* Divide the cross section into 80 horizontal elements of 0.03-in. height.)

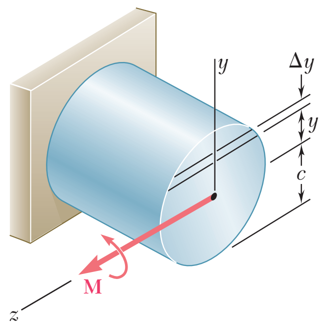


Fig. P4.C6

4.C7 The machine element of Prob. 4.182 is to be redesigned by removing part of the triangular cross section. It is believed that the removal of a small triangular area of width a will lower the maximum stress in the element. In order to verify this design concept, write a computer program to calculate the maximum stress in the element for values of a from 0 to 1 in. using 0.1-in. increments. Using appropriate smaller increments, determine the distance a for which the maximum stress is as small as possible and the corresponding value of the maximum stress.

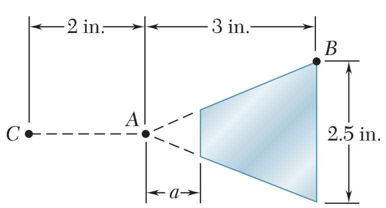
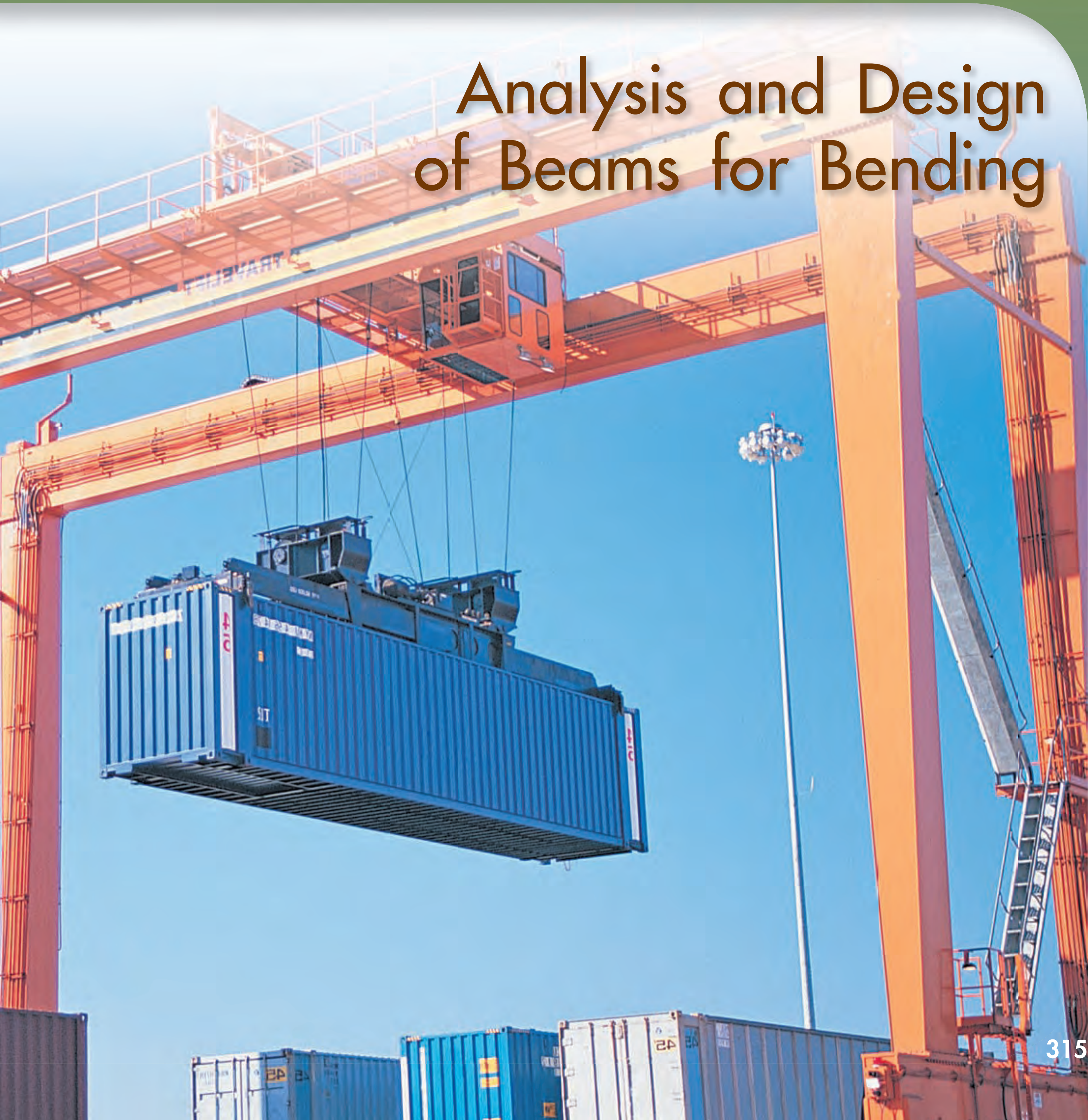


Fig. P4.C7

The beams supporting the multiple overhead cranes system shown in this picture are subjected to transverse loads causing the beams to bend. The normal stresses resulting from such loadings will be determined in this chapter.

CHAPTER 5

Analysis and Design of Beams for Bending



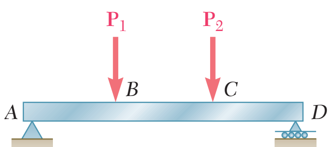
Chapter 5 Analysis and Design of Beams for Bending

5.1 Introduction
5.2 Shear and Bending-Moment Diagrams
5.3 Relations Among Load, Shear, and Bending Moment
5.4 Design of Prismatic Beams for Bending
*5.5 Using Singularity Functions to Determine Shear and Bending Moment in a Beam
*5.6 Nonprismatic Beams

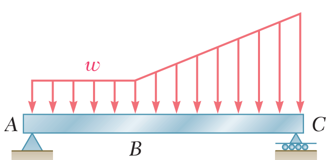
5.1 INTRODUCTION

This chapter and most of the next one will be devoted to the analysis and the design of *beams*, i.e., structural members supporting loads applied at various points along the member. Beams are usually long, straight prismatic members, as shown in the photo on the previous page. Steel and aluminum beams play an important part in both structural and mechanical engineering. Timber beams are widely used in home construction (Photo 5.1). In most cases, the loads are perpendicular to the axis of the beam. Such a *transverse loading* causes only bending and shear in the beam. When the loads are not at a right angle to the beam, they also produce axial forces in the beam.

Photo 5.1 Timber beams used in residential dwelling.



(*a*) Concentrated loads



(*b*) Distributed load

Fig. 5.1 Transversely loaded beams.

The transverse loading of a beam may consist of *concentrated loads* $\mathbf{P}_1$, $\mathbf{P}_2$, . . . , expressed in newtons, pounds, or their multiples, kilonewtons and kips (Fig. 5.1*a*), of a *distributed load* w, expressed in N/m, kN/m, lb/ft, or kips/ft (Fig. 5.1*b*), or of a combination of both. When the load w per unit length has a constant value over part of the beam (as between A and B in Fig. 5.1*b*), the load is said to be *uniformly distributed* over that part of the beam.

Beams are classified according to the way in which they are supported. Several types of beams frequently used are shown in Fig. 5.2. The distance L shown in the various parts of the figure is

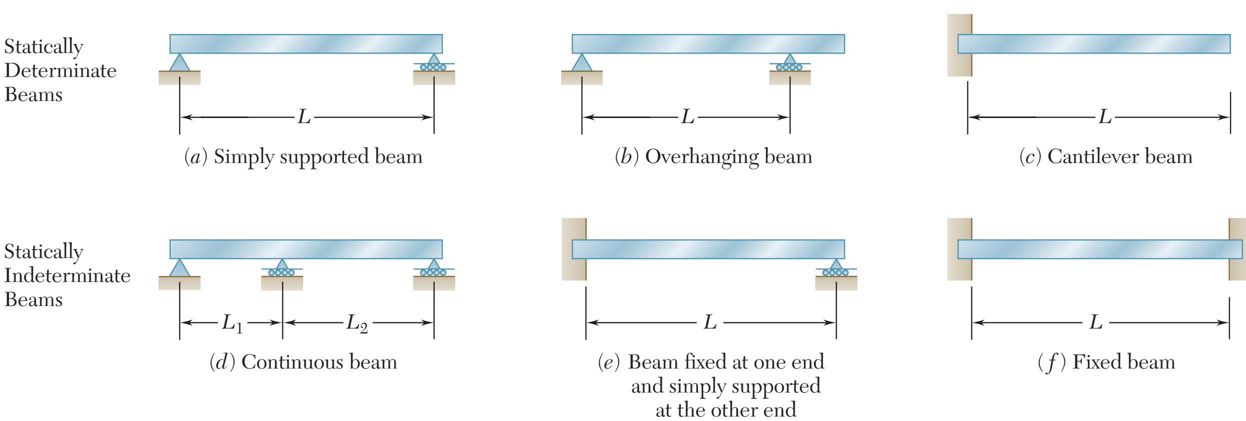


Fig. 5.2 Common beam support configurations.

called the *span*. Note that the reactions at the supports of the beams in parts a, b, and c of the figure involve a total of only three unknowns and, therefore, can be determined by the methods of statics. Such beams are said to be *statically determinate* and will be discussed in this chapter and the next. On the other hand, the reactions at the supports of the beams in parts d, e, and f of Fig. 5.2 involve more than three unknowns and cannot be determined by the methods of statics alone. The properties of the beams with regard to their resistance to deformations must be taken into consideration. Such beams are said to be *statically indeterminate* and their analysis will be postponed until Chap. 9, where deformations of beams will be discussed.

Sometimes two or more beams are connected by hinges to form a single continuous structure. Two examples of beams hinged at a point H are shown in Fig. 5.3. It will be noted that the reactions at the supports involve four unknowns and cannot be determined from the free-body diagram of the two-beam system. They can be determined, however, by recognizing that the internal moment at the hinge is zero. Then, after considering the free-body diagram of each beam separately, six unknowns are involved (including two force components at the hinge), and six equations are available.

When a beam is subjected to transverse loads, the internal forces in any section of the beam will generally consist of a shear force **V** and a bending couple **M.** Consider, for example, a simply supported beam AB carrying two concentrated loads and a uniformly distributed load (Fig. 5.4*a*). To determine the internal forces in a section through point C we first draw the free-body diagram of the entire beam to obtain the reactions at the supports (Fig. 5.4*b*). Passing a section through C, we then draw the free-body diagram of AC (Fig. 5.4*c*), from which we determine the shear force **V** and the bending couple **M.**

The bending couple **M** creates *normal stresses* in the cross section, while the shear force **V** creates *shearing stresses* in that section. In most cases the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam. The determination of the normal stresses in a beam will be the subject of this chapter, while shearing stresses will be discussed in Chap. 6.

Since the distribution of the normal stresses in a given section depends only upon the value of the bending moment M in that section and the geometry of the section,† the elastic flexure formulas derived in Sec. 4.4 can be used to determine the maximum stress, as well as the stress at any given point, in the section. We write‡

$$\sigma_m = \frac{|M|c}{I} \qquad \sigma_x = -\frac{My}{I} \qquad (5.1,\ 5.2)$$

†It is assumed that the distribution of the normal stresses in a given cross section is not affected by the deformations caused by the shearing stresses. This assumption will be verified in Sec. 6.5.

‡We recall from Sec. 4.2 that M can be positive or negative, depending upon whether the concavity of the beam at the point considered faces upward or downward. Thus, in the case considered here of a transverse loading, the sign of M can vary along the beam. On the other hand, since σ_m is a positive quantity, the absolute value of M is used in Eq. (5.1).

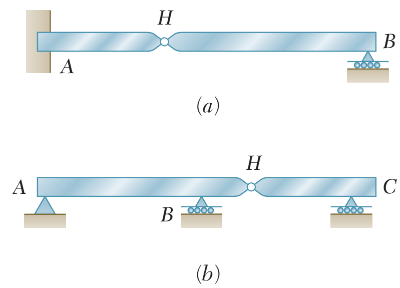


Fig. 5.3 Beams connected by hinges.

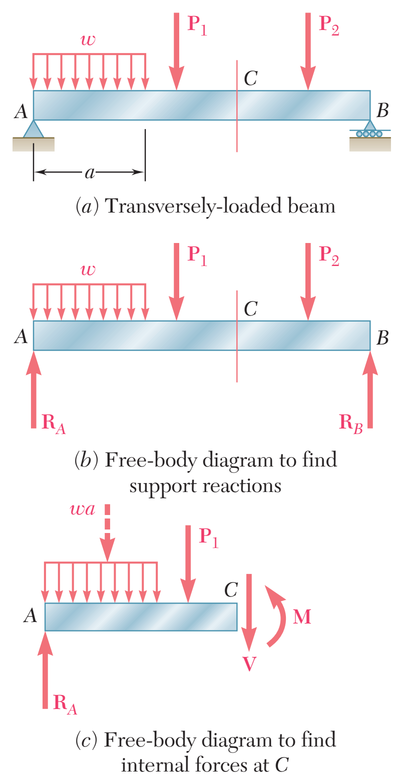


Fig. 5.4 Analysis of a simply supported beam.

where I is the moment of inertia of the cross section with respect to a centroidal axis perpendicular to the plane of the couple, y is the distance from the neutral surface, and c is the maximum value of that distance (Fig. 4.11). We also recall from Sec. 4.4 that, introducing the elastic section modulus $S = I/c$ of the beam, the maximum value σ_m of the normal stress in the section can be expressed as

$$\sigma_m = \frac{|M|}{S} \tag{5.3}$$

The fact that σ_m is inversely proportional to S underlines the importance of selecting beams with a large section modulus. Section moduli of various rolled-steel shapes are given in Appendix C, while the section modulus of a rectangular shape can be expressed, as shown in Sec. 4.4, as

$$S = \tfrac{1}{6}bh^2 \tag{5.4}$$

where b and h are, respectively, the width and the depth of the cross section.

Equation (5.3) also shows that, for a beam of uniform cross section, σ_m is proportional to $|M|$: Thus, the maximum value of the normal stress in the beam occurs in the section where $|M|$ is largest. It follows that one of the most important parts of the design of a beam for a given loading condition is the determination of the location and magnitude of the largest bending moment.

This task is made easier if a *bending-moment diagram* is drawn, i.e., if the value of the bending moment M is determined at various points of the beam and plotted against the distance x measured from one end of the beam. It is further facilitated if a *shear diagram* is drawn at the same time by plotting the shear V against x.

The sign convention to be used to record the values of the shear and bending moment will be discussed in Sec. 5.2. The values of V and M will then be obtained at various points of the beam by drawing free-body diagrams of successive portions of the beam. In Sec. 5.3 relations among load, shear, and bending moment will be derived and used to obtain the shear and bending-moment diagrams. This approach facilitates the determination of the largest absolute value of the bending moment and, thus, the determination of the maximum normal stress in the beam.

In Sec. 5.4 you will learn to design a beam for bending, i.e., so that the maximum normal stress in the beam will not exceed its allowable value. As indicated earlier, this is the dominant criterion in the design of a beam.

Another method for the determination of the maximum values of the shear and bending moment, based on expressing V and M in terms of *singularity functions,* will be discussed in Sec. 5.5. This approach lends itself well to the use of computers and will be expanded in Chap. 9 to facilitate the determination of the slope and deflection of beams.

Finally, the design of *nonprismatic beams,* i.e., beams with a variable cross section, will be discussed in Sec. 5.6. By selecting the shape and size of the variable cross section so that its elastic section modulus $S = I/c$ varies along the length of the beam in the same way as $|M|$, it is possible to design beams for which the maximum normal stress in each section is equal to the allowable stress of the material. Such beams are said to be of *constant strength*.

5.2 SHEAR AND BENDING-MOMENT DIAGRAMS

As indicated in Sec. 5.1, the determination of the maximum absolute values of the shear and of the bending moment in a beam are greatly facilitated if V and M are plotted against the distance x measured from one end of the beam. Besides, as you will see in Chap. 9, the knowledge of M as a function of x is essential to the determination of the deflection of a beam.

In the examples and sample problems of this section, the shear and bending-moment diagrams will be obtained by determining the values of V and M at selected points of the beam. These values will be found in the usual way, i.e., by passing a section through the point where they are to be determined (Fig. 5.5*a*) and considering the equilibrium of the portion of beam located on either side of the section (Fig. 5.5*b*). Since the shear forces $\mathbf{V}$ and $\mathbf{V}'$ have opposite senses, recording the shear at point C with an up or down arrow would be meaningless, unless we indicated at the same time which of the free bodies AC and CB we are considering. For this reason, the shear V will be recorded with a sign: a *plus sign* if the shearing forces are directed as shown in Fig. 5.5*b*, and a *minus sign* otherwise. A similar convention will apply for the bending moment M. It will be considered as positive if the bending couples are directed as shown in that figure, and negative otherwise.† Summarizing the sign conventions we have presented, we state:

The shear V and the bending moment M at a given point of a beam are said to be positive when the internal forces and couples acting on each portion of the beam are directed as shown in Fig. 5.6a.

These conventions can be more easily remembered if we note that

1. *The shear at any given point of a beam is positive when the* **external** *forces (loads and reactions) acting on the beam tend to shear off the beam at that point as indicated in Fig. 5.6b.*

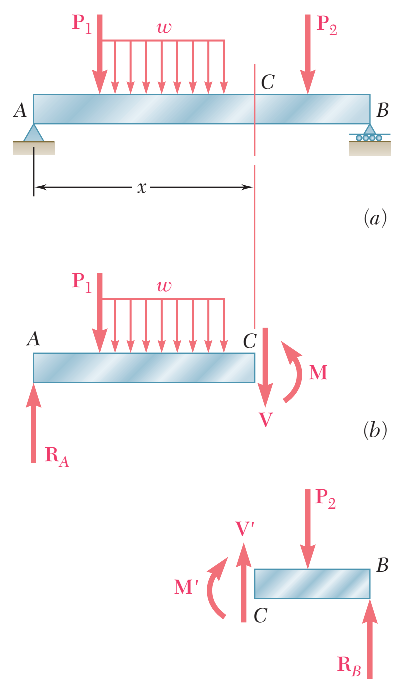


Fig. 5.5 Determination of V and M.

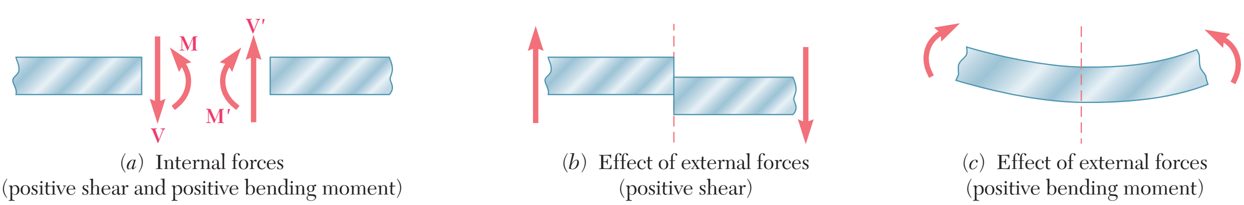


Fig. 5.6 Sign convention for shear and bending moment.

†Note that this convention is the same that we used earlier in Sec. 4.2

2. *The bending moment at any given point of a beam is positive when the* **external** *forces acting on the beam tend to bend the beam at that point as indicated in Fig. 5.6c.*

It is also of help to note that the situation described in Fig. 5.6, in which the values of the shear and of the bending moment are positive, is precisely the situation that occurs in the left half of a simply supported beam carrying a single concentrated load at its midpoint. This particular case is fully discussed in the next example.

EXAMPLE 5.01

Draw the shear and bending-moment diagrams for a simply supported beam AB of span L subjected to a single concentrated load **P** at its midpoint C (Fig. 5.7).

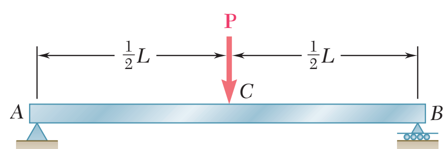


Fig. 5.7

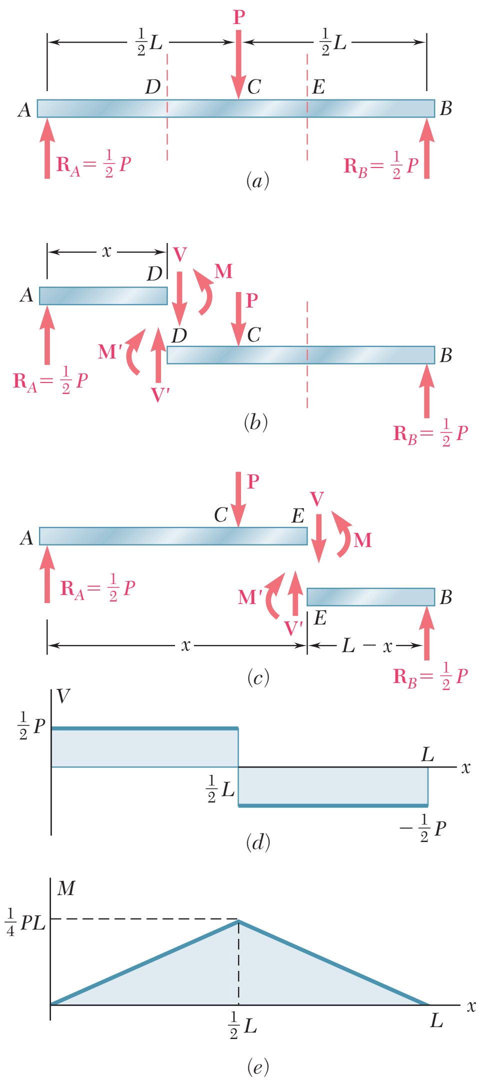


Fig. 5.8

We first determine the reactions at the supports from the free-body diagram of the entire beam (Fig. 5.8*a*); we find that the magnitude of each reaction is equal to $P/2$.

Next we cut the beam at a point D between A and C and draw the free-body diagrams of AD and DB (Fig. 5.8*b*). *Assuming that shear and bending moment are positive,* we direct the internal forces **V** and **V′** and the internal couples **M** and **M′** as indicated in Fig. 5.6*a*. Considering the free body AD and writing that the sum of the vertical components and the sum of the moments about D of the forces acting on the free body are zero, we find $V = +P/2$ and $M = +Px/2$. Both the shear and the bending moment are therefore positive; this may be checked by observing that the reaction at A tends to shear off and to bend the beam at D as indicated in Figs. 5.6*b* and *c*. We now plot V and M between A and C (Figs. 5.8*d* and *e*); the shear has a constant value $V = P/2$, while the bending moment increases linearly from $M = 0$ at $x = 0$ to $M = PL/4$ at $x = L/2$.

Cutting, now, the beam at a point E between C and B and considering the free body EB (Fig. 5.8*c*), we write that the sum of the vertical components and the sum of the moments about E of the forces acting on the free body are zero. We obtain $V = -P/2$ and $M = P(L - x)/2$. The shear is therefore negative and the bending moment positive; this can be checked by observing that the reaction at B bends the beam at E as indicated in Fig. 5.6*c* but tends to shear it off in a manner opposite to that shown in Fig. 5.6*b*. We can complete, now, the shear and bending-moment diagrams of Figs. 5.8*d* and *e*; the shear has a constant value $V = -P/2$ between C and B, while the bending moment decreases linearly from $M = PL/4$ at $x = L/2$ to $M = 0$ at $x = L$.

We note from the foregoing example that, when a beam is subjected only to concentrated loads, the shear is constant between loads and the bending moment varies linearly between loads. In such situations, therefore, the shear and bending-moment diagrams can easily be drawn, once the values of V and M have been obtained at sections selected just to the left and just to the right of the points where the loads and reactions are applied (see Sample Prob. 5.1).

EXAMPLE 5.02

Draw the shear and bending-moment diagrams for a cantilever beam AB of span L supporting a uniformly distributed load w (Fig. 5.9).

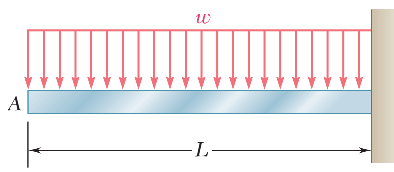


Fig. 5.9

We cut the beam at a point C between A and B and draw the free-body diagram of AC (Fig. 5.10a), directing $\mathbf{V}$ and $\mathbf{M}$ as indicated in Fig. 5.6a. Denoting by x the distance from A to C and replacing the distributed load over AC by its resultant wx applied at the midpoint of AC, we write

$$+\uparrow \Sigma F_y = 0: \qquad -wx - V = 0 \qquad V = -wx$$

$$+\circlearrowleft \Sigma M_C = 0: \qquad wx\left(\frac{x}{2}\right) + M = 0 \qquad M = -\frac{1}{2}wx^2$$

We note that the shear diagram is represented by an oblique straight line (Fig. 5.10b) and the bending-moment diagram by a parabola (Fig. 5.10c). The maximum values of V and M both occur at B, where we have

$$V_B = -wL \qquad M_B = -\tfrac{1}{2}wL^2$$

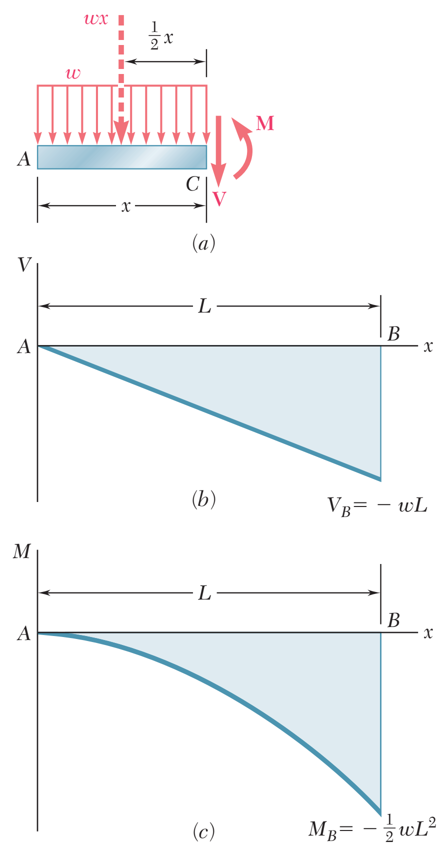
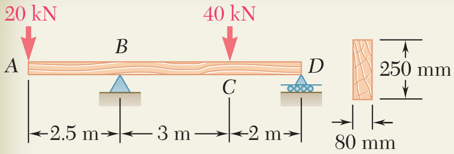
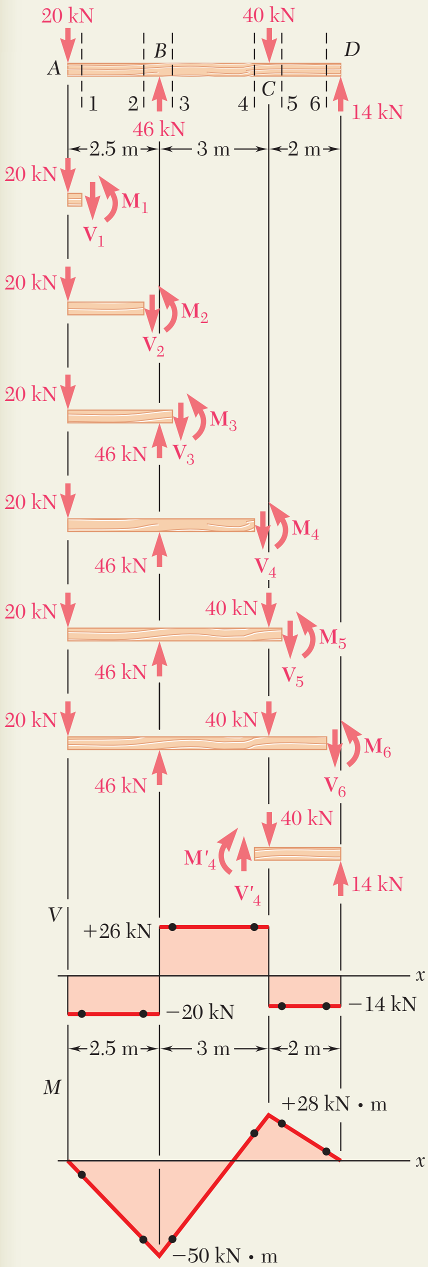


Fig. 5.10



SAMPLE PROBLEM 5.1

For the timber beam and loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.



SOLUTION

Reactions. Considering the entire beam as a free body, we find

$$\mathbf{R}_B = 40 \text{ kN} \uparrow \qquad \mathbf{R}_D = 14 \text{ kN} \uparrow$$

Shear and Bending-Moment Diagrams. We first determine the internal forces just to the right of the 20-kN load at A. Considering the stub of beam to the left of section *1* as a free body and assuming V and M to be positive (according to the standard convention), we write

$$+\uparrow \Sigma F_y = 0: \qquad -20 \text{ kN} - V_1 = 0 \qquad V_1 = -20 \text{ kN}$$
$$+\circlearrowleft \Sigma M_1 = 0: \qquad (20 \text{ kN})(0 \text{ m}) + M_1 = 0 \qquad M_1 = 0$$

We next consider as a free body the portion of beam to the left of section *2* and write

$$+\uparrow \Sigma F_y = 0: \qquad -20 \text{ kN} - V_2 = 0 \qquad V_2 = -20 \text{ kN}$$
$$+\circlearrowleft \Sigma M_2 = 0: \qquad (20 \text{ kN})(2.5 \text{ m}) + M_2 = 0 \qquad M_2 = -50 \text{ kN} \cdot \text{m}$$

The shear and bending moment at sections *3*, *4*, *5*, and *6* are determined in a similar way from the free-body diagrams shown. We obtain

$$\begin{aligned} V_3 &= +26 \text{ kN} & M_3 &= -50 \text{ kN} \cdot \text{m} \\ V_4 &= +26 \text{ kN} & M_4 &= +28 \text{ kN} \cdot \text{m} \\ V_5 &= -14 \text{ kN} & M_5 &= +28 \text{ kN} \cdot \text{m} \\ V_6 &= -14 \text{ kN} & M_6 &= 0 \end{aligned}$$

For several of the latter sections, the results may be more easily obtained by considering as a free body the portion of the beam to the right of the section. For example, for the portion of the beam to the right of section *4*, we have

$$+\uparrow \Sigma F_y = 0: \qquad V_4 - 40 \text{ kN} + 14 \text{ kN} = 0 \qquad V_4 = +26 \text{ kN}$$
$$+\circlearrowleft \Sigma M_4 = 0: \qquad -M_4 + (14 \text{ kN})(2 \text{ m}) = 0 \qquad M_4 = +28 \text{ kN} \cdot \text{m}$$

We can now plot the six points shown on the shear and bending-moment diagrams. As indicated earlier in this section, the shear is of constant value between concentrated loads, and the bending moment varies linearly; we obtain therefore the shear and bending-moment diagrams shown.

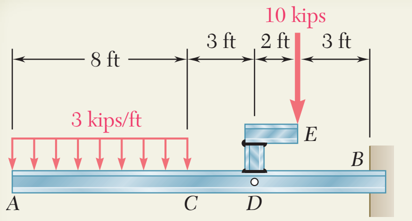
Maximum Normal Stress. It occurs at B, where $|M|$ is largest. We use Eq. (5.4) to determine the section modulus of the beam:

$$S = \tfrac{1}{6}bh^2 = \tfrac{1}{6}(0.080 \text{ m})(0.250 \text{ m})^2 = 833.33 \times 10^{-6} \text{ m}^3$$

Substituting this value and $|M| = |M_B| = 50 \times 10^3 \text{ N} \cdot \text{m}$ into Eq. (5.3) gives

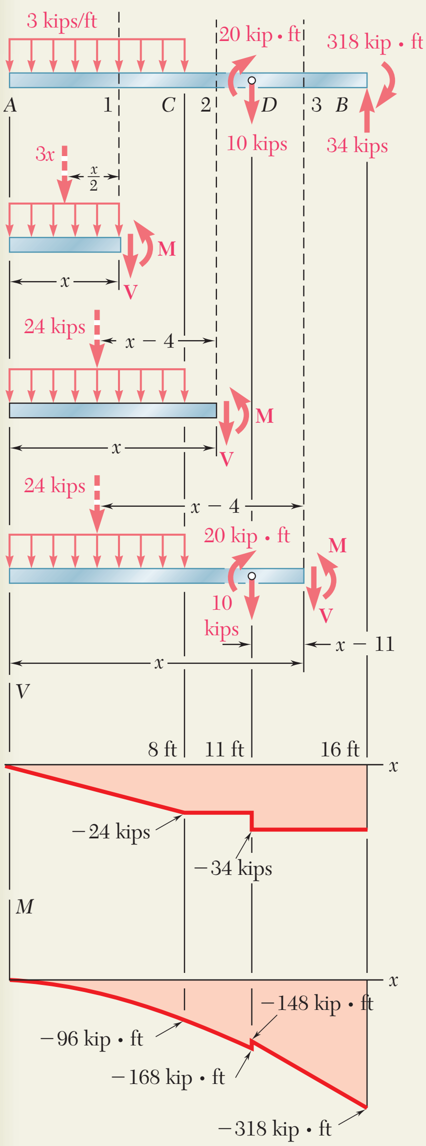
$$\sigma_m = \frac{|M_B|}{S} = \frac{(50 \times 10^3 \text{ N} \cdot \text{m})}{833.33 \times 10^{-6}} = 60.00 \times 10^6 \text{ Pa}$$

Maximum normal stress in the beam = 60.0 MPa ◀



SAMPLE PROBLEM 5.2

The structure shown consists of a W10 × 112 rolled-steel beam AB and of two short members welded together and to the beam. (a) Draw the shear and bending-moment diagrams for the beam and the given loading. (b) Determine the maximum normal stress in sections just to the left and just to the right of point D.



SOLUTION

Equivalent Loading of Beam. The 10-kip load is replaced by an equivalent force-couple system at D. The reaction at B is determined by considering the beam as a free body.

a. Shear and Bending-Moment Diagrams

From A to C. We determine the internal forces at a distance x from point A by considering the portion of beam to the left of section *1*. That part of the distributed load acting on the free body is replaced by its resultant, and we write

$$+\uparrow \Sigma F_y = 0: \qquad -3x - V = 0 \qquad V = -3x \text{ kips}$$
$$+\circlearrowleft \Sigma M_1 = 0: \qquad 3x(\tfrac{1}{2}x) + M = 0 \qquad M = -1.5x^2 \text{ kip} \cdot \text{ft}$$

Since the free-body diagram shown can be used for all values of x smaller than 8 ft, the expressions obtained for V and M are valid in the region $0 < x < 8$ ft.

From C to D. Considering the portion of beam to the left of section 2 and again replacing the distributed load by its resultant, we obtain

$$+\uparrow \Sigma F_y = 0: \qquad -24 - V = 0 \qquad V = -24 \text{ kips}$$
$$+\circlearrowleft \Sigma M_2 = 0: \qquad 24(x - 4) + M = 0 \qquad M = 96 - 24x \quad \text{kip} \cdot \text{ft}$$

These expressions are valid in the region 8 ft $< x <$ 11 ft.

From D to B. Using the position of beam to the left of section 3, we obtain for the region 11 ft $< x <$ 16 ft

$$V = -34 \text{ kips} \qquad M = 226 - 34x \quad \text{kip} \cdot \text{ft}$$

The shear and bending-moment diagrams for the entire beam can now be plotted. We note that the couple of moment 20 kip · ft applied at point D introduces a discontinuity into the bending-moment diagram.

b. Maximum Normal Stress to the Left and Right of Point *D*.

From Appendix C we find that for the W10 × 112 rolled-steel shape, $S = 126 \text{ in}^3$ about the X-X axis.

To the left of D: We have $|M| = 168$ kip · ft $= 2016$ kip · in. Substituting for $|M|$ and S into Eq. (5.3), we write

$$\sigma_m = \frac{|M|}{S} = \frac{2016 \text{ kip} \cdot \text{in.}}{126 \text{ in}^3} = 16.00 \text{ ksi} \qquad \sigma_m = 16.00 \text{ ksi} \blacktriangleleft$$

To the right of D: We have $|M| = 148$ kip · ft $= 1776$ kip · in. Substituting for $|M|$ and S into Eq. (5.3), we write

$$\sigma_m = \frac{|M|}{S} = \frac{1776 \text{ kip} \cdot \text{in.}}{126 \text{ in}^3} = 14.10 \text{ ksi} \qquad \sigma_m = 14.10 \text{ ksi} \blacktriangleleft$$

PROBLEMS

5.1 through 5.6 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the equations of the shear and bending-moment curves.

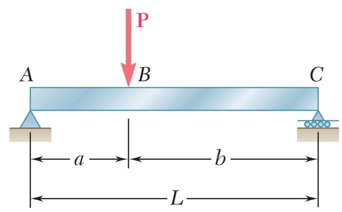


Fig. P5.1

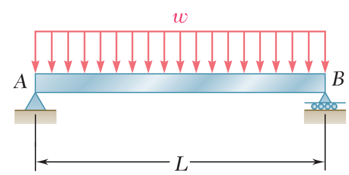


Fig. P5.2

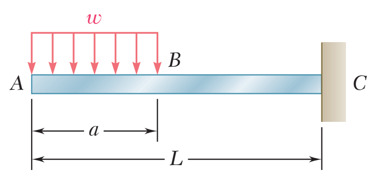


Fig. P5.3

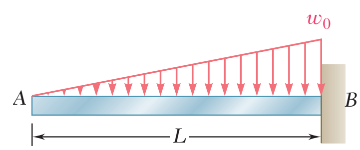


Fig. P5.4

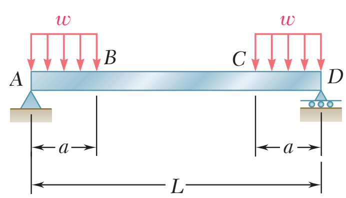


Fig. P5.5

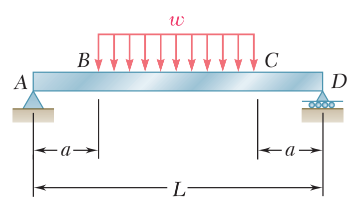


Fig. P5.6

5.7 and 5.8 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

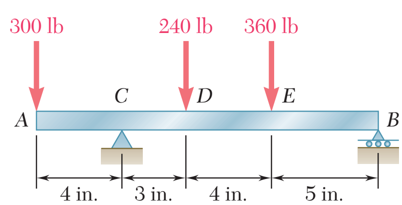


Fig. P5.7

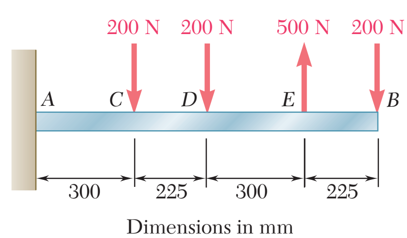


Fig. P5.8

5.9 and *5.10* Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

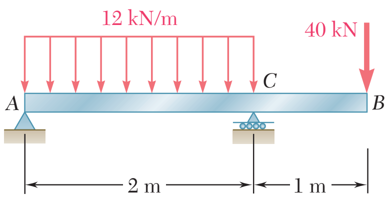


Fig. P5.9

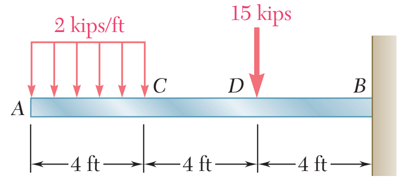


Fig. *P5.10*

5.11 and 5.12 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

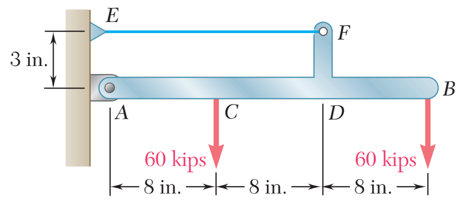


Fig. P5.11

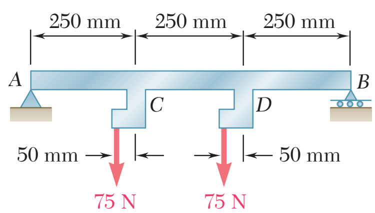


Fig. P5.12

***5.13* and 5.14** Assuming that the reaction of the ground is uniformly distributed, draw the shear and bending-moment diagrams for the beam *AB* and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

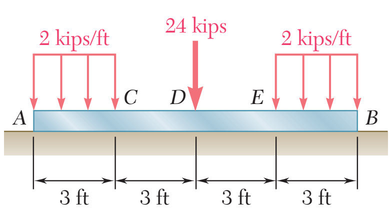


Fig. *P5.13*

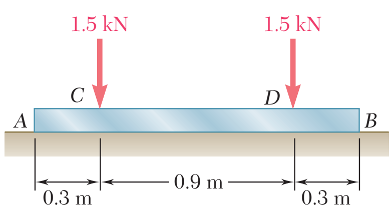


Fig. P5.14

5.15 and 5.16 For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at *C*.

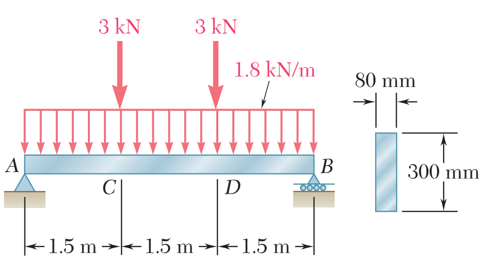


Fig. P5.15

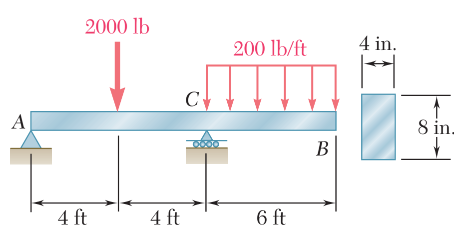


Fig. P5.16

5.17 For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at C.

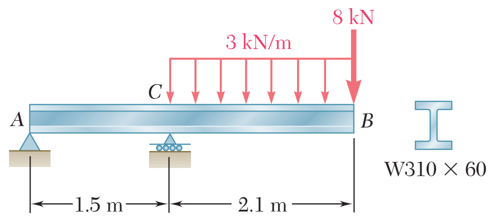


Fig. *P5.17*

5.18 For the beam and loading shown, determine the maximum normal stress due to bending on section *a-a*.

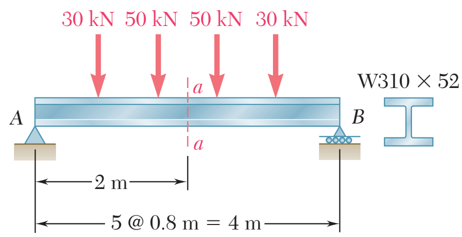


Fig. P5.18

5.19 and *5.20* For the beam and loading shown, determine the maximum normal stress due to bending on a transverse section at C.

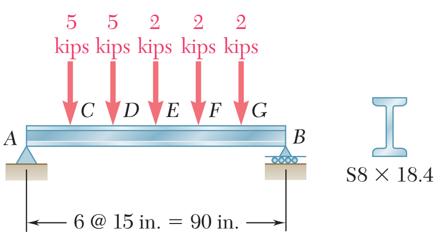


Fig. P5.19

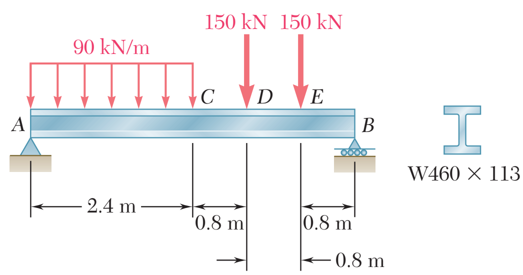


Fig. *P5.20*

5.21 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

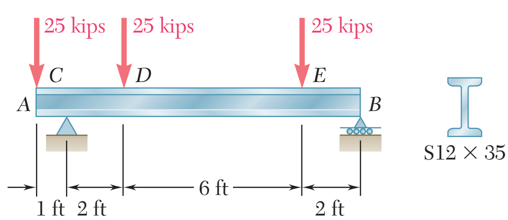


Fig. P5.21

5.22 and *5.23* Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

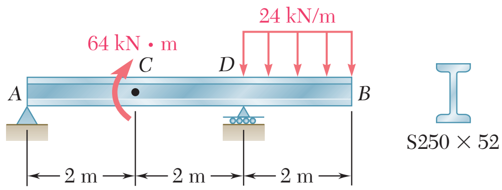


Fig. P5.22

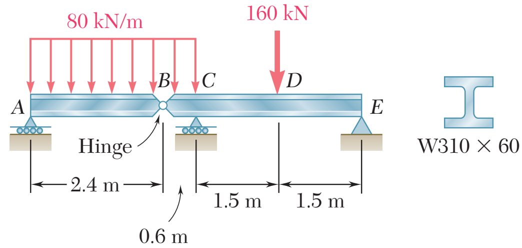


Fig. *P5.23*

***5.24* and 5.25** Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

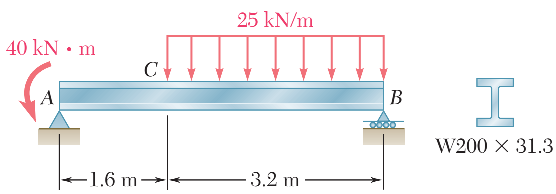


Fig. *P5.24*

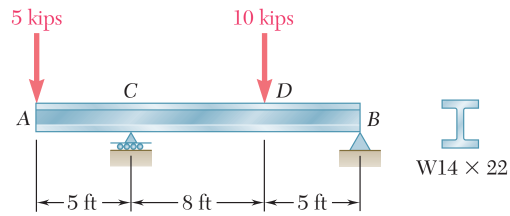
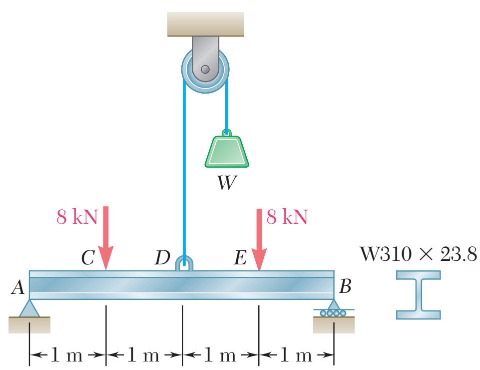


Fig. P5.25

5.26 Knowing that $W = 12$ kN, draw the shear and bending-moment diagrams for beam AB and determine the maximum normal stress due to bending.

5.27 Determine (*a*) the magnitude of the counterweight W for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding maximum normal stress due to bending. (*Hint:* Draw the bending-moment diagram and equate the absolute values of the largest positive and negative bending moments obtained.)



Figs. P5.26 and P5.27

5.28 Determine (*a*) the distance a for which the absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding maximum normal stress due to bending. (See hint of Prob. 5.27.)

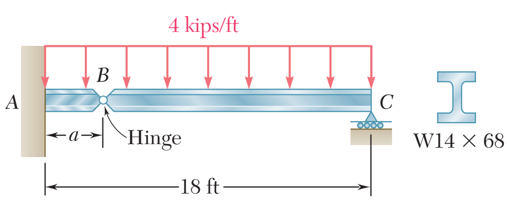


Fig. P5.28

5.29 Determine (a) the distance a for which the absolute value of the bending moment in the beam is as small as possible, (b) the corresponding maximum normal stress due to bending. (See hint of Prob. 5.27.)

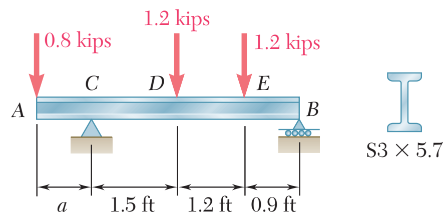


Fig. P5.29

5.30 Knowing that $P = Q = 480$ N, determine (a) the distance a for which the absolute value of the bending moment in the beam is as small as possible, (b) the corresponding maximum normal stress due to bending. (See hint of Prob. 5.27.)

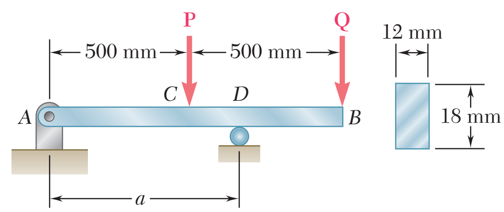


Fig. P5.30

5.31 Solve Prob. 5.30, assuming that $P = 480$ N and $Q = 320$ N.

5.32 A solid steel bar has a square cross section of side b and is supported as shown. Knowing that for steel $\rho = 7860$ kg/m^3, determine the dimension b for which the maximum normal stress due to bending is (a) 10 MPa, (b) 50 MPa.

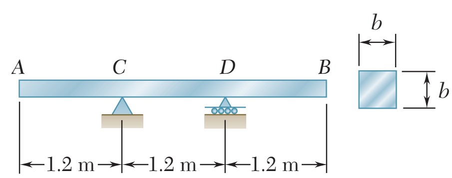


Fig. P5.32

5.33 A solid steel rod of diameter d is supported as shown. Knowing that for steel $\gamma = 490$ lb/ft^3, determine the smallest diameter d that can be used if the normal stress due to bending is not to exceed 4 ksi.

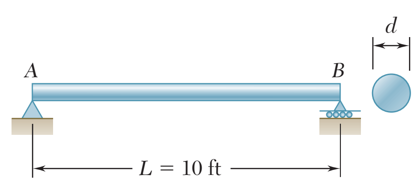


Fig. P5.33

5.3 RELATIONS AMONG LOAD, SHEAR, AND BENDING MOMENT

When a beam carries more than two or three concentrated loads, or when it carries distributed loads, the method outlined in Sec. 5.2 for plotting shear and bending moment can prove quite cumbersome. The construction of the shear diagram and, especially, of the bending-moment diagram will be greatly facilitated if certain relations existing among load, shear, and bending moment are taken into consideration.

Let us consider a simply supported beam AB carrying a distributed load w per unit length (Fig. 5.11*a*), and let C and C' be two points of the beam at a distance Δx from each other. The shear and bending moment at C will be denoted by V and M, respectively, and will be assumed positive; the shear and bending moment at C' will be denoted by $V + \Delta V$ and $M + \Delta M$.

We now detach the portion of beam CC' and draw its free-body diagram (Fig. 5.11*b*). The forces exerted on the free body include a load of magnitude $w\,\Delta x$ and internal forces and couples at C and C'. Since shear and bending moment have been assumed positive, the forces and couples will be directed as shown in the figure.

Relations between Load and Shear. Writing that the sum of the vertical components of the forces acting on the free body CC' is zero, we have

$$+\uparrow\Sigma F_y = 0: \qquad V - (V + \Delta V) - w\,\Delta x = 0$$
$$\Delta V = -w\,\Delta x$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\boxed{\frac{dV}{dx} = -w} \tag{5.5}$$

Equation (5.5) indicates that, for a beam loaded as shown in Fig. 5.11*a*, the slope dV/dx of the shear curve is negative; the numerical value

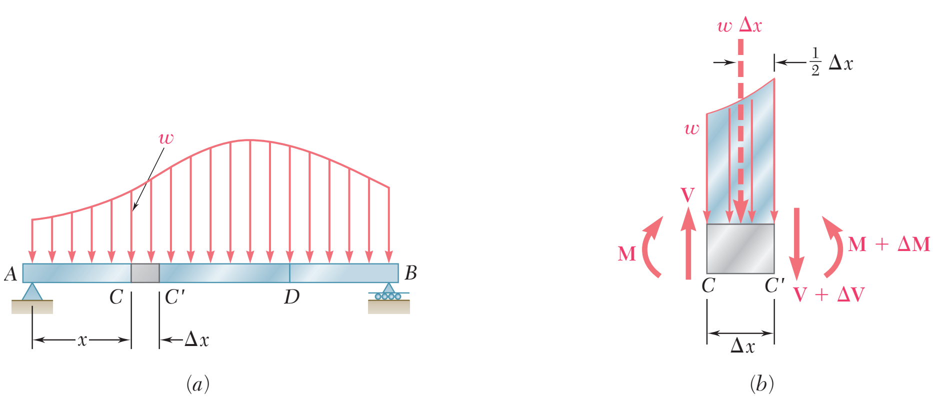


Fig. 5.11 Simply supported beam subjected to a distributed load.

of the slope at any point is equal to the load per unit length at that point.

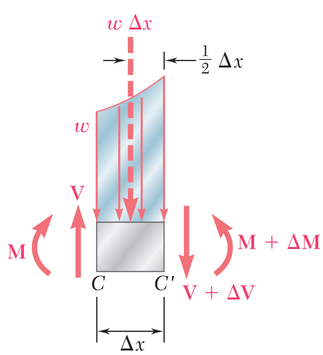
Integrating (5.5) between points C and D, we write

$$V_D - V_C = -\int_{x_C}^{x_D} w\,dx \tag{5.6}$$

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \tag{5.6$'$}$$

Note that this result could also have been obtained by considering the equilibrium of the portion of beam CD, since the area under the load curve represents the total load applied between C and D.

It should be observed that Eq. (5.5) is not valid at a point where a concentrated load is applied; the shear curve is discontinuous at such a point, as seen in Sec. 5.2. Similarly, Eqs. (5.6) and (5.6′) cease to be valid when concentrated loads are applied between C and D, since they do not take into account the sudden change in shear caused by a concentrated load. Equations (5.6) and (5.6′), therefore, should be applied only between successive concentrated loads.



(b)

Fig. 5.11 (repeated)

Relations between Shear and Bending Moment. Returning to the free-body diagram of Fig. 5.11*b*, and writing now that the sum of the moments about C' is zero, we have

$$+\circlearrowleft \Sigma M_{C'} = 0: \qquad (M + \Delta M) - M - V\,\Delta x + w\,\Delta x \frac{\Delta x}{2} = 0$$

$$\Delta M = V\,\Delta x - \frac{1}{2} w\,(\Delta x)^2$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\frac{dM}{dx} = V \tag{5.7}$$

Equation (5.7) indicates that the slope dM/dx of the bending-moment curve is equal to the value of the shear. This is true at any point where the shear has a well-defined value, i.e., at any point where no concentrated load is applied. Equation (5.7) also shows that $V = 0$ at points where M is maximum. This property facilitates the determination of the points where the beam is likely to fail under bending.

Integrating (5.7) between points C and D, we write

$$M_D - M_C = \int_{x_C}^{x_D} V\,dx \tag{5.8}$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \tag{5.8$'$}$$

Note that the area under the shear curve should be considered positive where the shear is positive and negative where the shear is negative. Equations (5.8) and (5.8′) are valid even when concentrated loads are applied between C and D, as long as the shear curve has been correctly drawn. The equations cease to be valid, however, if a couple is applied at a point between C and D, since they do not take into account the sudden change in bending moment caused by a couple (see Sample Prob. 5.6).

EXAMPLE 5.03

Draw the shear and bending-moment diagrams for the simply supported beam shown in Fig. 5.12 and determine the maximum value of the bending moment.

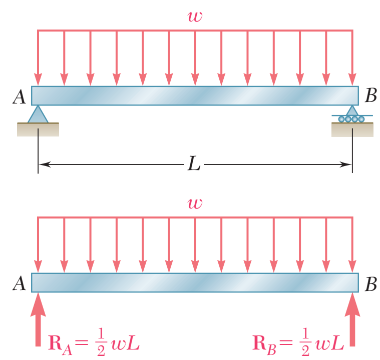


Fig. 5.12

From the free-body diagram of the entire beam, we determine the magnitude of the reactions at the supports.

$$R_A = R_B = \tfrac{1}{2}wL$$

Next, we draw the shear diagram. Close to the end A of the beam, the shear is equal to R_A, that is, to $\frac{1}{2}wL$, as we can check by considering as a free body a very small portion of the beam. Using Eq. (5.6), we then determine the shear V at any distance x from A; we write

$$V - V_A = -\int_0^x w\,dx = -wx$$

$$V = V_A - wx = \tfrac{1}{2}wL - wx = w(\tfrac{1}{2}L - x)$$

The shear curve is thus an oblique straight line which crosses the x axis at $x = L/2$ (Fig. 5.13a). Considering, now, the bending moment, we first observe that $M_A = 0$. The value M of the bending moment at any distance x from A may then be obtained from Eq. (5.8); we have

$$M - M_A = \int_0^x V\,dx$$

$$M = \int_0^x w(\tfrac{1}{2}L - x)dx = \tfrac{1}{2}w(Lx - x^2)$$

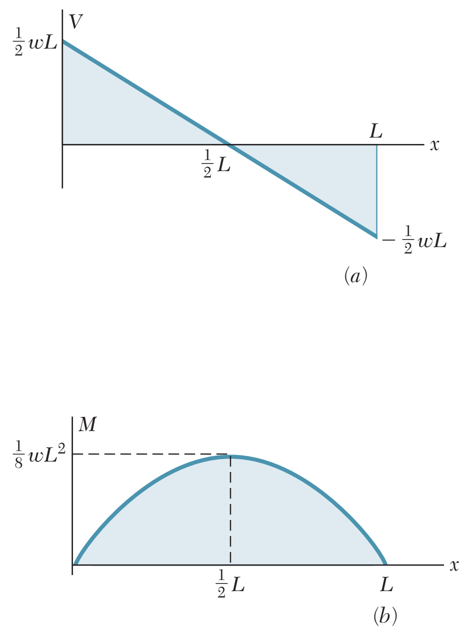


Fig. 5.13

The bending-moment curve is a parabola. The maximum value of the bending moment occurs when $x = L/2$, since V (and thus dM/dx) is zero for that value of x. Substituting $x = L/2$ in the last equation, we obtain $M_{\max} = wL^2/8$ (Fig. 5.13b).

In most engineering applications, one needs to know the value of the bending moment only at a few specific points. Once the shear diagram has been drawn, and after M has been determined at one of the ends of the beam, the value of the bending moment can then be obtained at any given point by computing the area under the shear curve and using Eq. (5.8′). For instance, since $M_A = 0$ for the beam of Example 5.03, the maximum value of the bending moment for that beam can be obtained simply by measuring the area of the shaded triangle in the shear diagram of Fig. 5.13*a*. We have

$$M_{\max} = \frac{1}{2}\frac{L}{2}\frac{wL}{2} = \frac{wL^2}{8}$$

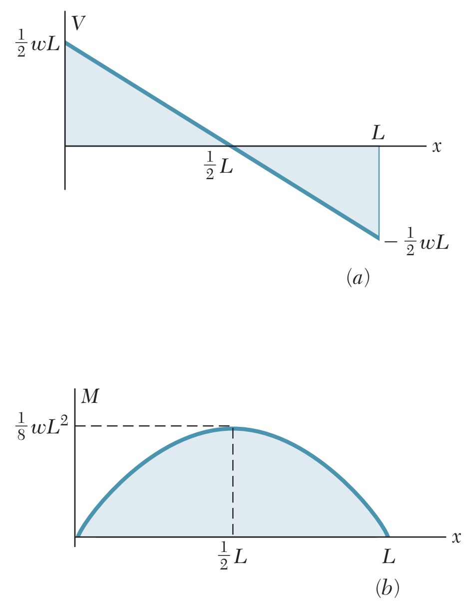
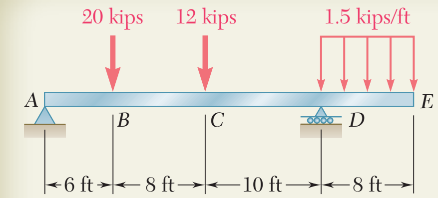


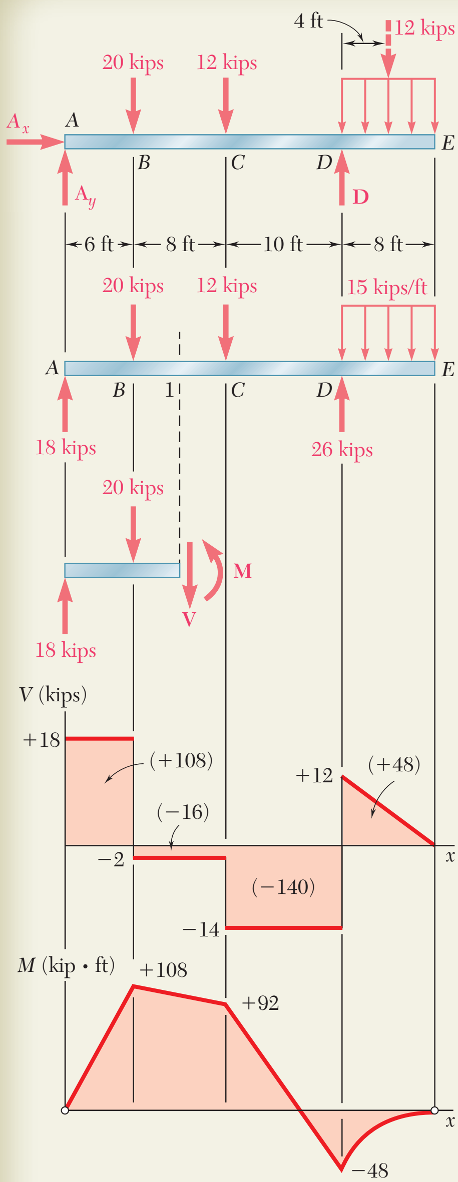
Fig. 5.13

We note that, in this example, the load curve is a horizontal straight line, the shear curve an oblique straight line, and the bending-moment curve a parabola. If the load curve had been an oblique straight line (first degree), the shear curve would have been a parabola (second degree) and the bending-moment curve a cubic (third degree). The shear and bending-moment curves will always be, respectively, one and two degrees higher than the load curve. With this in mind, we should be able to sketch the shear and bending-moment diagrams without actually determining the functions $V(x)$ and $M(x)$, once a few values of the shear and bending moment have been computed. The sketches obtained will be more accurate if we make use of the fact that, at any point where the curves are continuous, the slope of the shear curve is equal to $-w$ and the slope of the bending-moment curve is equal to V.



SAMPLE PROBLEM 5.3

Draw the shear and bending-moment diagrams for the beam and loading shown.



SOLUTION

Reactions. Considering the entire beam as a free body, we write

$+\circlearrowleft \Sigma M_A = 0$:

$$D(24\text{ ft}) - (20\text{ kips})(6\text{ ft}) - (12\text{ kips})(14\text{ ft}) - (12\text{ kips})(28\text{ ft}) = 0$$

$$D = +26\text{ kips} \qquad \mathbf{D} = 26\text{ kips}\uparrow$$

$+\uparrow \Sigma F_y = 0$: $\quad A_y - 20\text{ kips} - 12\text{ kips} + 26\text{ kips} - 12\text{ kips} = 0$

$$A_y = +18\text{ kips} \qquad \mathbf{A}_y = 18\text{ kips}\uparrow$$

$\xrightarrow{+} \Sigma F_x = 0$: $\quad A_x = 0 \qquad \mathbf{A}_x = 0$

We also note that at both A and E the bending moment is zero; thus, two points (indicated by dots) are obtained on the bending-moment diagram.

Shear Diagram. Since $dV/dx = -w$, we find that between concentrated loads and reactions the slope of the shear diagram is zero (i.e., the shear is constant). The shear at any point is determined by dividing the beam into two parts and considering either part as a free body. For example, using the portion of beam to the left of section *1*, we obtain the shear between B and C:

$+\uparrow \Sigma F_y = 0$: $\quad +18\text{ kips} - 20\text{ kips} - V = 0 \qquad V = -2\text{ kips}$

We also find that the shear is $+12$ kips just to the right of D and zero at end E. Since the slope $dV/dx = -w$ is constant between D and E, the shear diagram between these two points is a straight line.

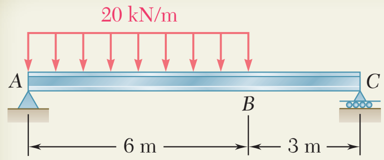
Bending-Moment Diagram. We recall that the area under the shear curve between two points is equal to the change in bending moment between the same two points. For convenience, the area of each portion of the shear diagram is computed and is indicated in parentheses on the diagram. Since the bending moment M_A at the left end is known to be zero, we write

$$\begin{aligned} M_B - M_A &= +108 & M_B &= +108\text{ kip}\cdot\text{ft} \\ M_C - M_B &= -16 & M_C &= +92\text{ kip}\cdot\text{ft} \\ M_D - M_C &= -140 & M_D &= -48\text{ kip}\cdot\text{ft} \\ M_E - M_D &= +48 & M_E &= 0 \end{aligned}$$

Since M_E is known to be zero, a check of the computations is obtained.

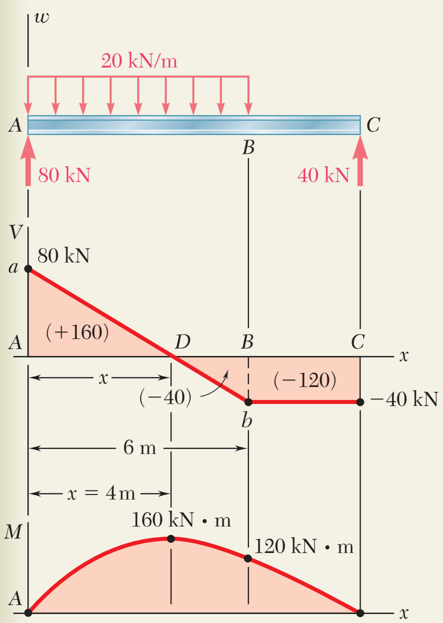
Between the concentrated loads and reactions, the shear is constant; thus, the slope dM/dx is constant, and the bending-moment diagram is drawn by connecting the known points with straight lines. Between D and E where the shear diagram is an oblique straight line, the bending-moment diagram is a parabola.

From the V and M diagrams we note that $V_{max} = 18$ kips and $M_{max} = 108$ kip · ft.



SAMPLE PROBLEM 5.4

The W360 × 79 rolled-steel beam *AC* is simply supported and carries the uniformly distributed load shown. Draw the shear and bending-moment diagrams for the beam and determine the location and magnitude of the maximum normal stress due to bending.



SOLUTION

Reactions. Considering the entire beam as a free body, we find

$$\mathbf{R}_A = 80 \text{ kN} \uparrow \qquad \mathbf{R}_C = 40 \text{ kN} \uparrow$$

Shear Diagram. The shear just to the right of *A* is $V_A = +80$ kN. Since the change in shear between two points is equal to *minus* the area under the load curve between the same two points, we obtain V_B by writing

$$V_B - V_A = -(20 \text{ kN/m})(6 \text{ m}) = -120 \text{ kN}$$
$$V_B = -120 + V_A = -120 + 80 = -40 \text{ kN}$$

The slope $dV/dx = -w$ being constant between *A* and *B*, the shear diagram between these two points is represented by a straight line. Between *B* and *C*, the area under the load curve is zero; therefore,

$$V_C - V_B = 0 \qquad V_C = V_B = -40 \text{ kN}$$

and the shear is constant between *B* and *C*.

Bending-Moment Diagram. We note that the bending moment at each end of the beam is zero. In order to determine the maximum bending moment, we locate the section *D* of the beam where $V = 0$. We write

$$V_D - V_A = -wx$$
$$0 - 80 \text{ kN} = -(20 \text{ kN/m})x$$

and, solving for x we find: $x = 4$ m ◀

The maximum bending moment occurs at point *D*, where we have $dM/dx = V = 0$. The areas of the various portions of the shear diagram are computed and are given (in parentheses) on the diagram. Since the area of the shear diagram between two points is equal to the change in bending moment between the same two points, we write

$$M_D - M_A = +160 \text{ kN} \cdot \text{m} \qquad M_D = +160 \text{ kN} \cdot \text{m}$$
$$M_B - M_D = -\ 40 \text{ kN} \cdot \text{m} \qquad M_B = +120 \text{ kN} \cdot \text{m}$$
$$M_C - M_B = -120 \text{ kN} \cdot \text{m} \qquad M_C = 0$$

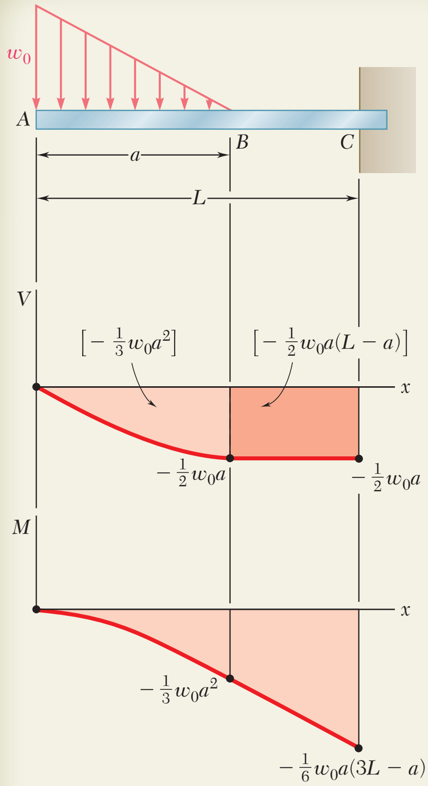
The bending-moment diagram consists of an arc of parabola followed by a segment of straight line; the slope of the parabola at *A* is equal to the value of *V* at that point.

Maximum Normal Stress. It occurs at *D*, where $|M|$ is largest. From Appendix C we find that for a W360 × 79 rolled-steel shape, $S = 1270 \text{ mm}^3$ about a horizontal axis. Substituting this value and $|M| = |M_D| = 160 \times 10^3$ N · m into Eq. (5.3), we write

$$\sigma_m = \frac{|M_D|}{S} = \frac{160 \times 10^3 \text{ N} \cdot \text{m}}{1270 \times 10^{-6} \text{ m}^3} = 126.0 \times 10^6 \text{ Pa}$$

Maximum normal stress in the beam = 126.0 MPa ◀

SAMPLE PROBLEM 5.5



Sketch the shear and bending-moment diagrams for the cantilever beam shown.

SOLUTION

Shear Diagram. At the free end of the beam, we find $V_A = 0$. Between A and B, the area under the load curve is $\frac{1}{2}w_0a$; we find V_B by writing

$$V_B - V_A = -\tfrac{1}{2}w_0a \qquad V_B = -\tfrac{1}{2}w_0a$$

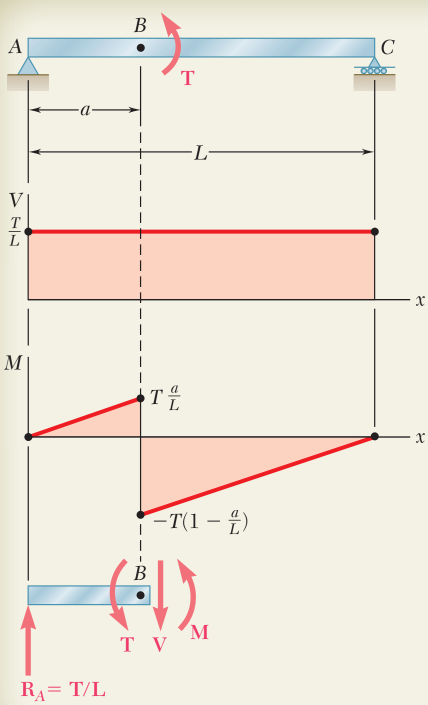
Between B and C, the beam is not loaded; thus $V_C = V_B$. At A, we have $w = w_0$ and, according to Eq. (5.5), the slope of the shear curve is $dV/dx = -w_0$, while at B the slope is $dV/dx = 0$. Between A and B, the loading decreases linearly, and the shear diagram is parabolic. Between B and C, $w = 0$, and the shear diagram is a horizontal line.

Bending-Moment Diagram. The bending moment M_A at the free end of the beam is zero. We compute the area under the shear curve and write

$$M_B - M_A = -\tfrac{1}{3}w_0a^2 \qquad M_B = -\tfrac{1}{3}w_0a^2$$
$$M_C - M_B = -\tfrac{1}{2}w_0a(L-a)$$
$$M_C = -\tfrac{1}{6}w_0a(3L-a)$$

The sketch of the bending-moment diagram is completed by recalling that $dM/dx = V$. We find that between A and B the diagram is represented by a cubic curve with zero slope at A, and between B and C by a straight line.

SAMPLE PROBLEM 5.6



The simple beam AC is loaded by a couple of moment T applied at point B. Draw the shear and bending-moment diagrams of the beam.

SOLUTION

The entire beam is taken as a free body, and we obtain

$$\mathbf{R}_A = \frac{T}{L}\uparrow \qquad \mathbf{R}_C = \frac{T}{L}\downarrow$$

The shear at any section is constant and equal to T/L. Since a couple is applied at B, the bending-moment diagram is discontinuous at B; it is represented by two oblique straight lines and decreases suddenly at B by an amount equal to T. The character of this discontinuity can also be verified by equilibrium analysis. For example, considering the free body of the portion of the beam from A to just beyond the right of B as shown, we find the value of M by

$$+\circlearrowleft\Sigma M_B = 0: \quad -\frac{T}{L}a + T + M = 0 \qquad M = -T\left(1 - \frac{a}{L}\right)$$

PROBLEMS

5.34 Using the method of Sec. 5.3, solve Prob. 5.1*a*.

5.35 Using the method of Sec. 5.3, solve Prob. 5.2*a*.

5.36 Using the method of Sec. 5.3, solve Prob. 5.3*a*.

5.37 Using the method of Sec. 5.3, solve Prob. 5.4*a*.

5.38 Using the method of Sec. 5.3, solve Prob. 5.5*a*.

5.39 Using the method of Sec. 5.3, solve Prob. 5.6*a*.

5.40 Using the method of Sec. 5.3, solve Prob. 5.7.

5.41 Using the method of Sec. 5.3, solve Prob. 5.8.

5.42 Using the method of Sec. 5.3, solve Prob. 5.9.

5.43 Using the method of Sec. 5.3, solve Prob. 5.10.

5.44* and *5.45 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

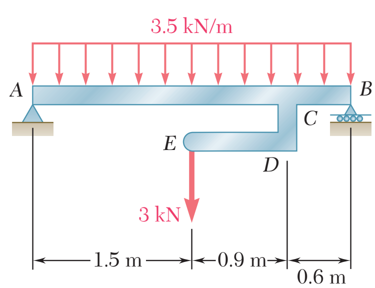


Fig. *P5.44*

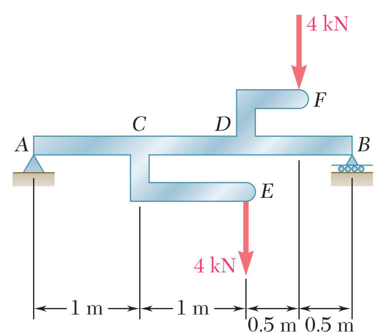


Fig. *P5.45*

5.46 Using the method of Sec. 5.3, solve Prob. 5.15.

5.47 Using the method of Sec. 5.3, solve Prob. 5.16.

5.48 Using the method of Sec. 5.3, solve Prob. 5.18.

5.49 Using the method of Sec. 5.3, solve Prob. 5.19.

5.50 For the beam and loading shown, determine the equations of the shear and bending-moment curves and the maximum absolute value of the bending moment in the beam, knowing that (*a*) $k = 1$, (*b*) $k = 0.5$.

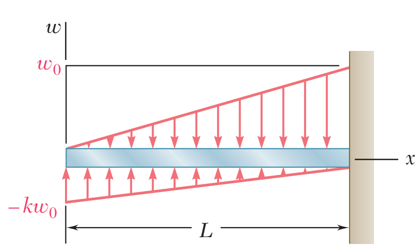


Fig. *P5.50*

5.51 and 5.52 Determine (*a*) the equations of the shear and bending-moment curves for the beam and loading shown, (*b*) the maximum absolute value of the bending moment in the beam.

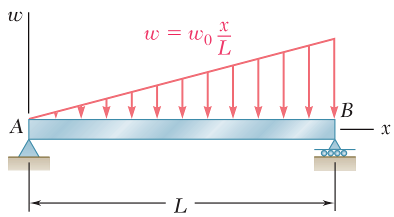


Fig. P5.51

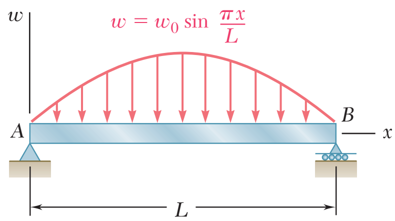


Fig. P5.52

5.53 Determine (*a*) the equations of the shear and bending-moment curves for the beam and loading shown, (*b*) the maximum absolute value of the bending moment in the beam.

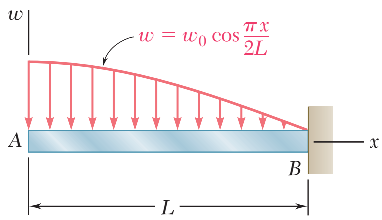


Fig. P5.53

5.54 and 5.55 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

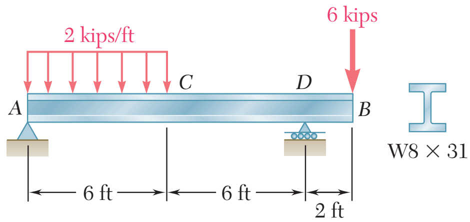


Fig. P5.54

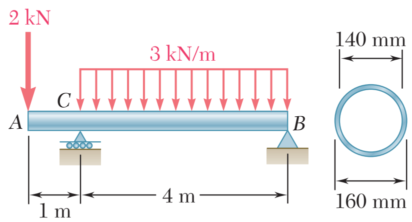


Fig. P5.55

5.56 and 5.57 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

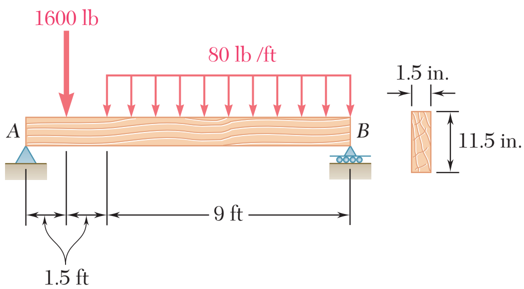


Fig. P5.56

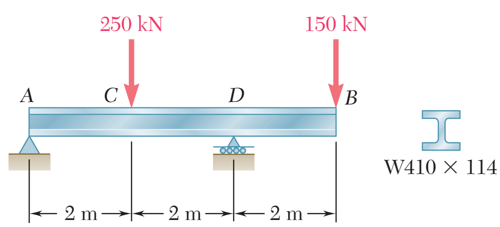


Fig. P5.57

5.58 and *5.59* Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

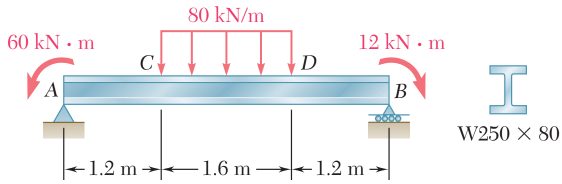


Fig. P5.58

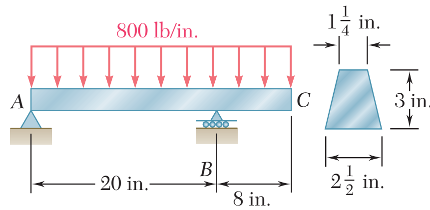


Fig. *P5.59*

5.60 Beam AB, of length L and square cross section of side a, is supported by a pivot at C and loaded as shown. (a) Check that the beam is in equilibrium. (b) Show that the maximum stress due to bending occurs at C and is equal to $w_0L^2/(1.5a)^3$.

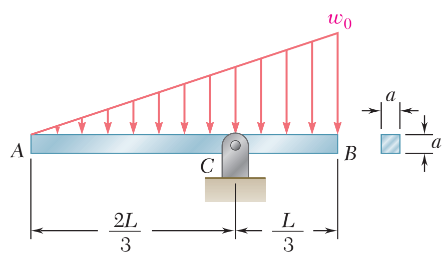


Fig. P5.60

5.61 Knowing that beam AB is in equilibrium under the loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.

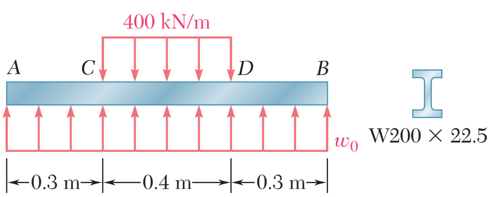


Fig. P5.61

***5.62** The beam AB supports a uniformly distributed load of 480 lb/ft and two concentrated loads **P** and **Q.** The normal stress due to bending on the bottom edge of the lower flange is +14.85 ksi at D and +10.65 ksi at E. (a) Draw the shear and bending-moment diagrams for the beam. (b) Determine the maximum normal stress due to bending that occurs in the beam.

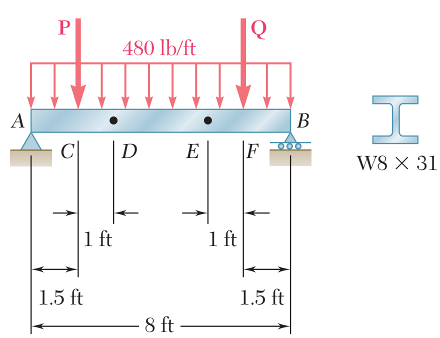


Fig. P5.62

***5.63** Beam AB supports a uniformly distributed load of 2 kN/m and two concentrated loads **P** and **Q.** It has been experimentally determined that the normal stress due to bending in the bottom edge of the beam is -56.9 MPa at A and -29.9 MPa at C. Draw the shear and bending-moment diagrams for the beam and determine the magnitudes of the loads **P** and **Q.**

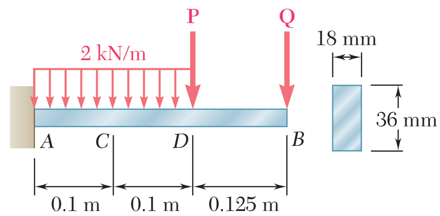


Fig. P5.63

***5.64** The beam AB supports two concentrated loads **P** and **Q.** The normal stress due to bending on the bottom edge of the beam is $+55$ MPa at D and $+37.5$ MPa at F. (*a*) Draw the shear and bending-moment diagrams for the beam. (*b*) Determine the maximum normal stress due to bending that occurs in the beam.

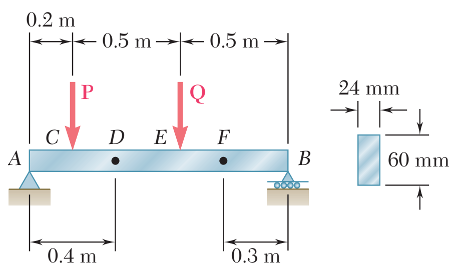


Fig. *P5.64*

5.4 DESIGN OF PRISMATIC BEAMS FOR BENDING

As indicated in Sec. 5.1, the design of a beam is usually controlled by the maximum absolute value $|M|_{max}$ of the bending moment that will occur in the beam. The largest normal stress σ_m in the beam is found at the surface of the beam in the critical section where $|M|_{max}$ occurs and can be obtained by substituting $|M|_{max}$ for $|M|$ in Eq. (5.1) or Eq. (5.3).† We write

$$\sigma_m = \frac{|M|_{max}c}{I} \qquad \sigma_m = \frac{|M|_{max}}{S} \tag{5.1', 5.3'}$$

A safe design requires that $\sigma_m \leq \sigma_{all}$, where σ_{all} is the allowable stress for the material used. Substituting σ_{all} for σ_m in (5.3′) and solving for S yields the minimum allowable value of the section modulus for the beam being designed:

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} \tag{5.9}$$

The design of common types of beams, such as timber beams of rectangular cross section and rolled-steel beams of various cross-sectional shapes, will be considered in this section. A proper procedure should lead to the most economical design. This means that, among beams of the same type and the same material, and other

†For beams that are not symmetrical with respect to their neutral surface, the largest of the distances from the neutral surface to the surfaces of the beam should be used for c in Eq. (5.1) and in the computation of the section modulus $S = I/c$.

things being equal, the beam with the smallest weight per unit length—and, thus, the smallest cross-sectional area—should be selected, since this beam will be the least expensive.

The design procedure will include the following steps†:

1. First determine the value of σ_{all} for the material selected from a table of properties of materials or from design specifications. You can also compute this value by dividing the ultimate strength σ_U of the material by an appropriate factor of safety (Sec. 1.13). Assuming for the time being that the value of σ_{all} is the same in tension and in compression, proceed as follows.
2. Draw the shear and bending-moment diagrams corresponding to the specified loading conditions, and determine the maximum absolute value $|M|_{\text{max}}$ of the bending moment in the beam.
3. Determine from Eq. (5.9) the minimum allowable value S_{min} of the section modulus of the beam.
4. For a timber beam, the depth h of the beam, its width b, or the ratio h/b characterizing the shape of its cross section will probably have been specified. The unknown dimensions may then be selected by recalling from Eq. (4.19) of Sec. 4.4 that b and h must satisfy the relation $\frac{1}{6}bh^2 = S \geq S_{\text{min}}$.
5. For a rolled-steel beam, consult the appropriate table in Appendix C. Of the available beam sections, consider only those with a section modulus $S \geq S_{\text{min}}$ and select from this group the section with the smallest weight per unit length. This is the most economical of the sections for which $S \geq S_{\text{min}}$. Note that this is not necessarily the section with the smallest value of S (see Example 5.04). In some cases, the selection of a section may be limited by other considerations, such as the allowable depth of the cross section, or the allowable deflection of the beam (cf. Chap. 9).

The foregoing discussion was limited to materials for which σ_{all} is the same in tension and in compression. If σ_{all} is different in tension and in compression, you should make sure to select the beam section in such a way that $\sigma_m \leq \sigma_{\text{all}}$ for both tensile and compressive stresses. If the cross section is not symmetric about its neutral axis, the largest tensile and the largest compressive stresses will not necessarily occur in the section where $|M|$ is maximum. One may occur where M is maximum and the other where M is minimum. Thus, step 2 should include the determination of both M_{max} and M_{min}, and step 3 should be modified to take into account both tensile and compressive stresses.

Finally, keep in mind that the design procedure described in this section takes into account only the normal stresses occurring on the surface of the beam. Short beams, especially those made of timber, may fail in shear under a transverse loading. The determination of shearing stresses in beams will be discussed in Chap. 6. Also, in the case of rolled-steel beams, normal stresses larger than those considered here may occur at the junction of the web with the flanges. This will be discussed in Chap. 8.

†We assume that all beams considered in this chapter are adequately braced to prevent lateral buckling, and that bearing plates are provided under concentrated loads applied to rolled-steel beams to prevent local buckling (crippling) of the web.

EXAMPLE 5.04

Select a wide-flange beam to support the 15-kip load as shown in Fig. 5.14. The allowable normal stress for the steel used is 24 ksi.

1. The allowable normal stress is given: $\sigma_{all} = 24$ ksi.
2. The shear is constant and equal to 15 kips. The bending moment is maximum at B. We have

$$|M|_{max} = (15 \text{ kips})(8 \text{ ft}) = 120 \text{ kip} \cdot \text{ft} = 1440 \text{ kip} \cdot \text{in.}$$

3. The minimum allowable section modulus is

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} = \frac{1440 \text{ kip} \cdot \text{in.}}{24 \text{ ksi}} = 60.0 \text{ in}^3$$

4. Referring to the table of *Properties of Rolled-Steel Shapes* in Appendix C, we note that the shapes are arranged in groups of the same depth and that in each group they are listed in order of decreasing weight. We choose in each group the lightest beam having a section modulus $S = I/c$ at least as large as S_{min} and record the results in the following table.

Shape	S, in^3
W21 × 44	81.6
W18 × 50	88.9
W16 × 40	64.7
W14 × 43	62.6
W12 × 50	64.2
W10 × 54	60.0

The most economical is the W16 × 40 shape since it weighs only 40 lb/ft, even though it has a larger section modulus than two of the other shapes. We also note that the total weight of the beam will be (8 ft) × (40 lb) = 320 lb. This weight is small compared to the 15,000-1b load and can be neglected in our analysis.

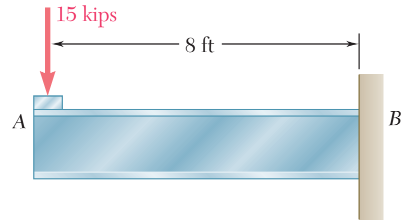
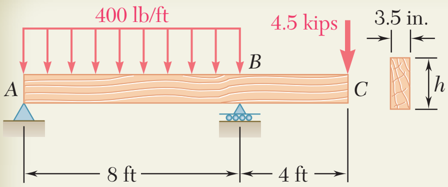


Fig. 5.14

***Load and Resistance Factor Design.** This alternative method of design was briefly described in Sec. 1.13 and applied to members under axial loading. It can readily be applied to the design of beams in bending. Replacing in Eq. (1.26) the loads P_D, P_L, and P_U, respectively, by the bending moments M_D, M_L, and M_U, we write

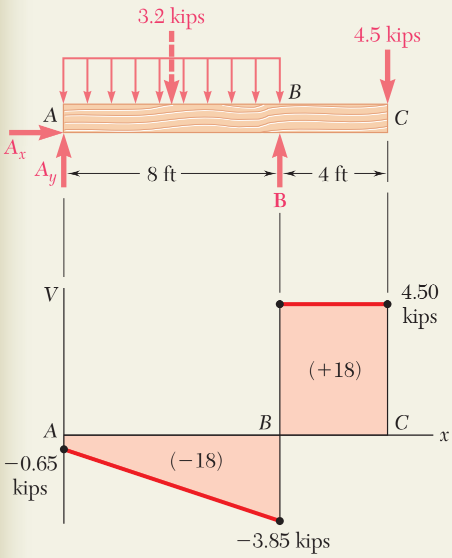
$$\gamma_D M_D + \gamma_L M_L \leq \phi M_U \tag{5.10}$$

The coefficients γ_D and γ_L are referred to as the *load factors* and the coefficient ϕ as the *resistance factor*. The moments M_D and M_L are the bending moments due, respectively, to the dead and the live loads, while M_U is equal to the product of the ultimate strength σ_U of the material and the section modulus S of the beam: $M_U = S\sigma_U$.



SAMPLE PROBLEM 5.7

A 12-ft-long overhanging timber beam *AC* with an 8-ft span *AB* is to be designed to support the distributed and concentrated loads shown. Knowing that timber of 4-in. nominal width (3.5-in. actual width) with a 1.75-ksi allowable stress is to be used, determine the minimum required depth *h* of the beam.



SOLUTION

Reactions. Considering the entire beam as a free body, we write

$$+\circlearrowleft \Sigma M_A = 0: B(8\text{ ft}) - (3.2\text{ kips})(4\text{ ft}) - (4.5\text{ kips})(12\text{ ft}) = 0$$

$$B = 8.35\text{ kips} \qquad \mathbf{B} = 8.35\text{ kips}\uparrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \qquad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: A_y + 8.35\text{ kips} - 3.2\text{ kips} - 4.5\text{ kips} = 0$$

$$A_y = -0.65\text{ kips} \qquad \mathbf{A} = 0.65\text{ kips}\downarrow$$

Shear Diagram. The shear just to the right of *A* is $V_A = A_y = -0.65$ kips. Since the change in shear between *A* and *B* is equal to *minus* the area under the load curve between these two points, we obtain V_B by writing

$$V_B - V_A = -(400\text{ lb/ft})(8\text{ ft}) = -3200\text{ lb} = -3.20\text{ kips}$$

$$V_B = V_A - 3.20\text{ kips} = -0.65\text{ kips} - 3.20\text{ kips} = -3.85\text{ kips}.$$

The reaction at *B* produces a sudden increase of 8.35 kips in *V*, resulting in a value of the shear equal to 4.50 kips to the right of *B*. Since no load is applied between *B* and *C*, the shear remains constant between these two points.

Determination of $|M|_{max}$. We first observe that the bending moment is equal to zero at both ends of the beam: $M_A = M_C = 0$. Between *A* and *B* the bending moment decreases by an amount equal to the area under the shear curve, and between *B* and *C* it increases by a corresponding amount. Thus, the maximum absolute value of the bending moment is $|M|_{max} = 18.00$ kip · ft.

Minimum Allowable Section Modulus. Substituting into Eq. (5.9) the given value of σ_{all} and the value of $|M|_{max}$ that we have found, we write

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} = \frac{(18\text{ kip}\cdot\text{ft})(12\text{ in./ft})}{1.75\text{ ksi}} = 123.43\text{ in}^3$$

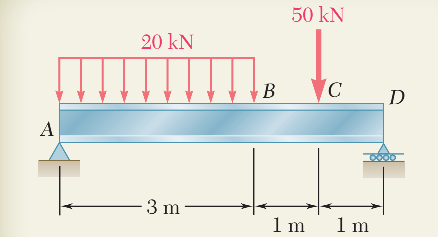
Minimum Required Depth of Beam. Recalling the formula developed in part 4 of the design procedure described in Sec. 5.4 and substituting the values of *b* and S_{min}, we have

$$\tfrac{1}{6}bh^2 \geq S_{min} \qquad \tfrac{1}{6}(3.5\text{ in.})h^2 \geq 123.43\text{ in}^3 \qquad h \geq 14.546\text{ in.}$$

The minimum required depth of the beam is $h = 14.55$ in. ◀

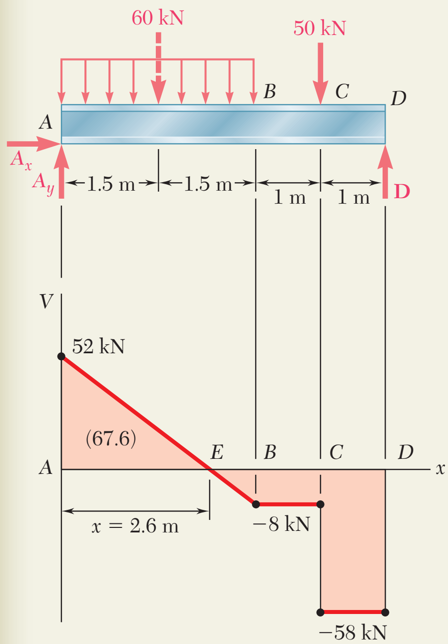
Note: In practice, standard wood shapes are specified by nominal dimensions that are slightly larger than actual. In this case, we would specify a 4-in. × 16-in. member, whose actual dimensions are 3.5 in. × 15.25 in.

SAMPLE PROBLEM 5.8



A 5-m-long, simply supported steel beam AD is to carry the distributed and concentrated loads shown. Knowing that the allowable normal stress for the grade of steel to be used is 160 MPa, select the wide-flange shape that should be used.

SOLUTION



Reactions. Considering the entire beam as a free body, we write

$$+\circlearrowleft\Sigma M_A = 0\colon D(5\text{ m}) - (60\text{ kN})(1.5\text{ m}) - (50\text{ kN})(4\text{ m}) = 0$$

$$D = 58.0\text{ kN} \qquad \mathbf{D} = 58.0\text{ kN}\uparrow$$

$$\xrightarrow{+}\Sigma F_x = 0\colon \qquad A_x = 0$$

$$+\uparrow\Sigma F_y = 0\colon A_y + 58.0\text{ kN} - 60\text{ kN} - 50\text{ kN} = 0$$

$$A_y = 52.0\text{ kN} \qquad \mathbf{A} = 52.0\text{ kN}\uparrow$$

Shear Diagram. The shear just to the right of A is $V_A = A_y = +52.0$ kN. Since the change in shear between A and B is equal to *minus* the area under the load curve between these two points, we have

$$V_B = 52.0\text{ kN} - 60\text{ kN} = -8\text{ kN}$$

The shear remains constant between B and C, where it drops to −58 kN, and keeps this value between C and D. We locate the section E of the beam where $V = 0$ by writing

$$V_E - V_A = -wx$$

$$0 - 52.0\text{ kN} = -(20\text{ kN/m})x$$

Solving for x we find $x = 2.60$ m.

Determination of $|M|_{max}$. The bending moment is maximum at E, where $V = 0$. Since M is zero at the support A, its maximum value at E is equal to the area under the shear curve between A and E. We have, therefore, $|M|_{max} = M_E = 67.6$ kN · m.

Minimum Allowable Section Modulus. Substituting into Eq. (5.9) the given value of σ_{all} and the value of $|M|_{max}$ that we have found, we write

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} = \frac{67.6\text{ kN}\cdot\text{m}}{160\text{ MPa}} = 422.5\times10^{-6}\text{ m}^3 = 422.5\times10^3\text{ mm}^3$$

Selection of Wide-Flange Shape. From Appendix C we compile a list of shapes that have a section modulus larger than S_{min} and are also the lightest shape in a given depth group.

Shape	S, mm^3
W410 × 38.8	629
W360 × 32.9	475
W310 × 38.7	547
W250 × 44.8	531
W200 × 46.1	451

We select the lightest shape available, namely **W360 × 32.9** ◀

PROBLEMS

5.65 and *5.66* For the beam and loading shown, design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 12 MPa.

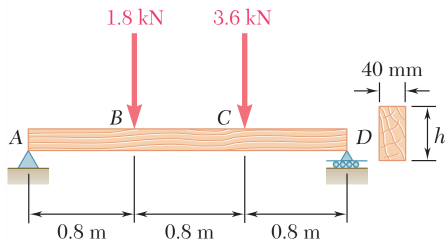


Fig. P5.65

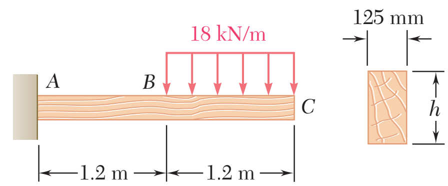


Fig. *P5.66*

5.67 and *5.68* For the beam and loading shown, design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 1750 psi.

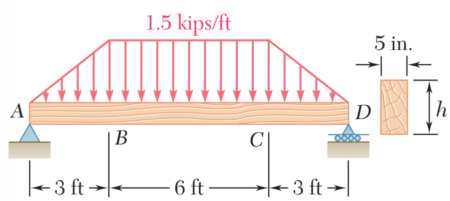


Fig. P5.67

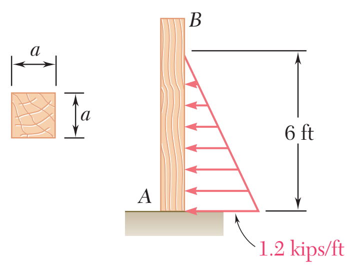


Fig. *P5.68*

5.69 and 5.70 For the beam and loading shown, design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 12 MPa.

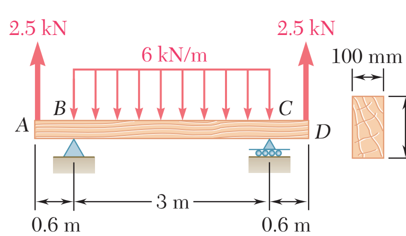


Fig. P5.69

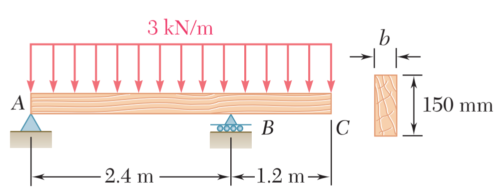


Fig. P5.70

5.71 and 5.72 Knowing that the allowable stress for the steel used is 24 ksi, select the most economical wide-flange beam to support the loading shown.

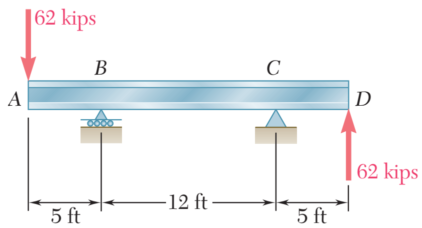


Fig. P5.71

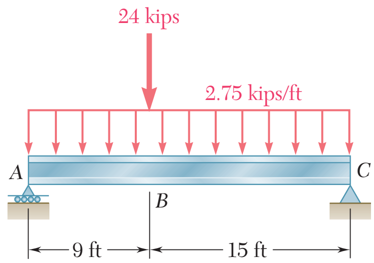


Fig. P5.72

5.73 and 5.74 Knowing that the allowable stress for the steel used is 160 MPa, select the most economical wide-flange beam to support the loading shown.

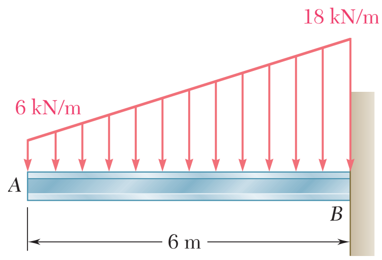


Fig. P5.73

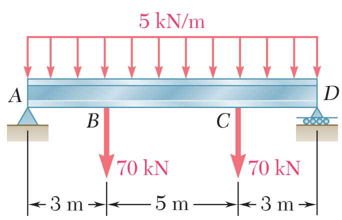


Fig. P5.74

5.75 **and 5.76** Knowing that the allowable stress for the steel used is 160 MPa, select the most economical S-shape beam to support the loading shown.

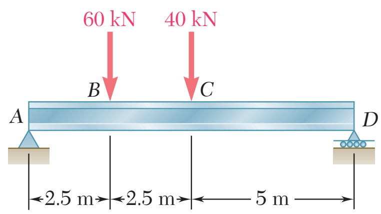


Fig. *P5.75*

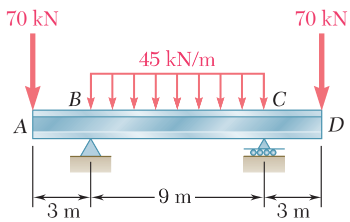


Fig. P5.76

5.77 and *5.78* Knowing that the allowable stress for the steel used is 24 ksi, select the most economical S-shape beam to support the loading shown.

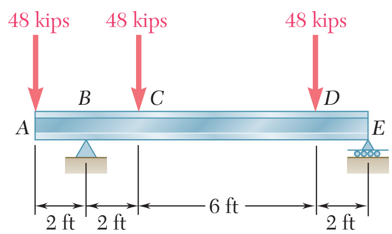


Fig. P5.77

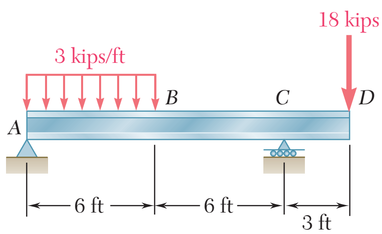


Fig. *P5.78*

5.79 Two L102 × 76 rolled-steel angles are bolted together and used to support the loading shown. Knowing that the allowable normal stress for the steel used is 140 MPa, determine the minimum angle thickness that can be used.

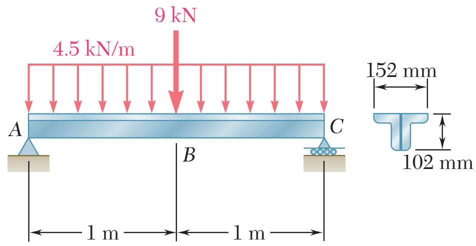


Fig. P5.79

5.80 Two rolled-steel channels are to be welded back to back and used to support the loading shown. Knowing that the allowable normal stress for the steel used is 30 ksi, determine the most economical channels that can be used.

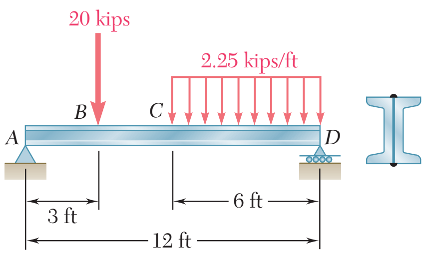


Fig. P5.80

5.81 Three steel plates are welded together to form the beam shown. Knowing that the allowable normal stress for the steel used is 22 ksi, determine the minimum flange width b that can be used.

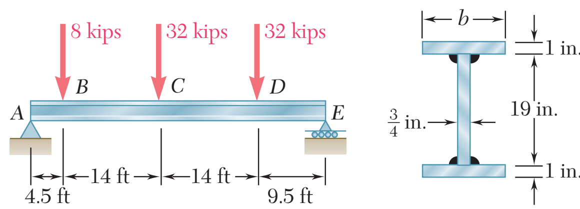


Fig. P5.81

5.82 A steel pipe of 100-mm diameter is to support the loading shown. Knowing that the stock of pipes available has thicknesses varying from 6 mm to 24 mm in 3-mm increments, and that the allowable normal stress for the steel used is 150 MPa, determine the minimum wall thickness t that can be used.

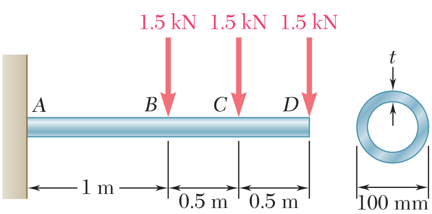


Fig. P5.82

5.83 Assuming the upward reaction of the ground to be uniformly distributed and knowing that the allowable normal stress for the steel used is 24 ksi, select the most economical wide-flange beam to support the loading shown.

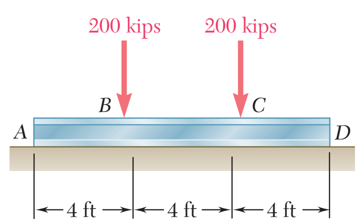


Fig. P5.83

5.84 Assuming the upward reaction of the ground to be uniformly distributed and knowing that the allowable normal stress for the steel used is 170 MPa, select the most economical wide-flange beam to support the loading shown.

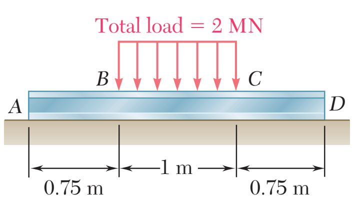


Fig. P5.84

5.85 and 5.86 Determine the largest permissible value of **P** for the beam and loading shown, knowing that the allowable normal stress is +6 ksi in tension and −18 ksi in compression.

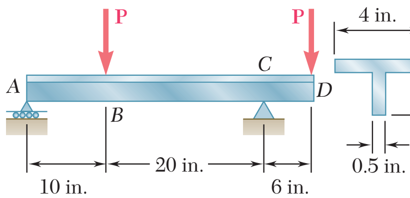


Fig. *P5.85*

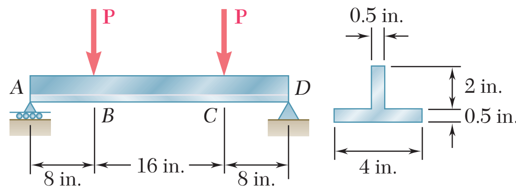


Fig. *P5.86*

5.87 Determine the largest permissible distributed load w for the beam shown, knowing that the allowable normal stress is +80 MPa in tension and −130 MPa in compression.

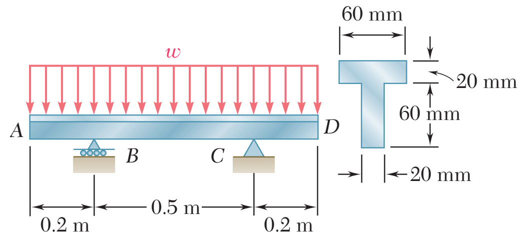


Fig. P5.87

5.88 Solve Prob. 5.87, assuming that the cross section of the beam is reversed, with the flange of the beam resting on the supports at B and C.

5.89 A 54-kip load is to be supported at the center of the 16-ft span shown. Knowing that the allowable normal stress for the steel used is 24 ksi, determine (*a*) the smallest allowable length l of beam CD if the W12 × 50 beam AB is not to be overstressed, (*b*) the most economical W shape that can be used for beam CD. Neglect the weight of both beams.

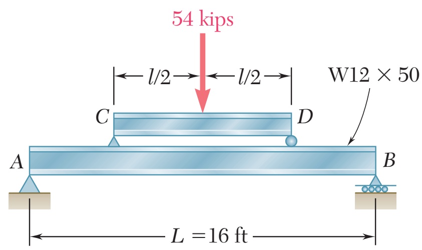


Fig. P5.89

5.90 A uniformly distributed load of 66 kN/m is to be supported over the 6-m span shown. Knowing that the allowable normal stress for the steel used is 140 MPa, determine (*a*) the smallest allowable length l of beam CD if the W460 × 74 beam AB is not to be overstressed, (*b*) the most economical W shape that can be used for beam CD. Neglect the weight of both beams.

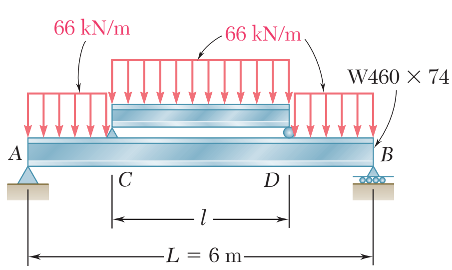


Fig. *P5.90*

5.91 Each of the three rolled-steel beams shown (numbered 1, 2, and 3) is to carry a 64-kip load uniformly distributed over the beam. Each of these beams has a 12-ft span and is to be supported by the two 24-ft rolled-steel girders AC and BD. Knowing that the allowable normal stress for the steel used is 24 ksi, select (*a*) the most economical S shape for the three beams, (*b*) the most economical W shape for the two girders.

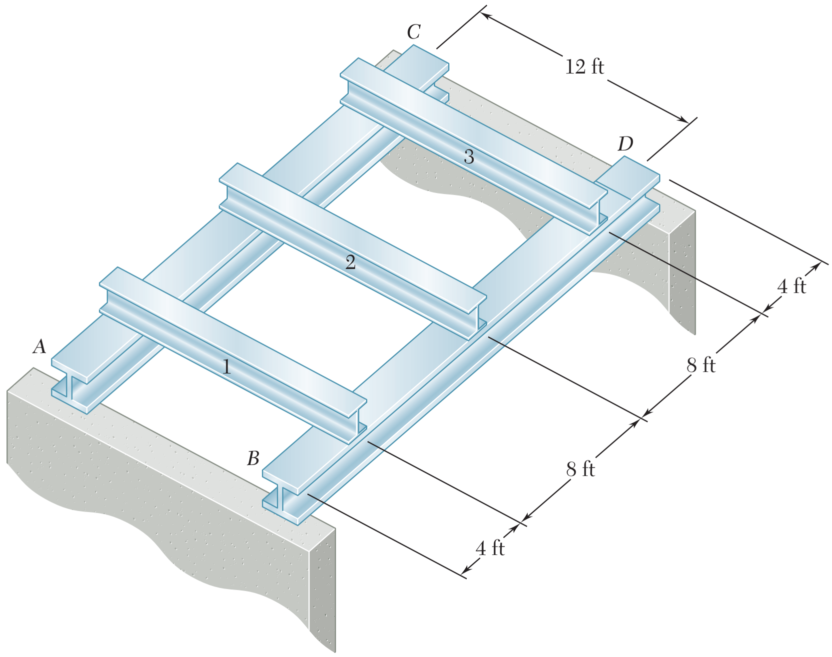


Fig. P5.91

5.92 Beams AB, BC, and CD have the cross section shown and are pin-connected at B and C. Knowing that the allowable normal stress is +110 MPa in tension and −150 MPa in compression, determine (*a*) the largest permissible value of w if beam BC is not to be overstressed, (*b*) the corresponding maximum distance a for which the cantilever beams AB and CD are not overstressed.

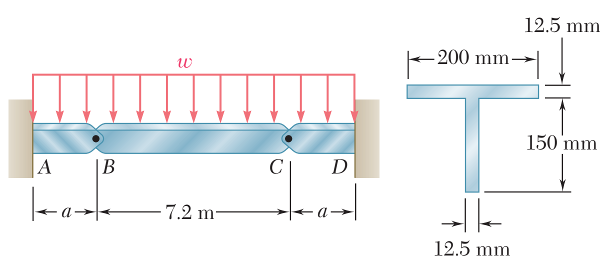


Fig. P5.92

5.93 Beams AB, BC, and CD have the cross section shown and are pin-connected at B and C. Knowing that the allowable normal stress is $+110$ MPa in tension and -150 MPa in compression, determine (*a*) the largest permissible value of **P** if beam BC is not to be overstressed, (*b*) the corresponding maximum distance a for which the cantilever beams AB and CD are not overstressed.

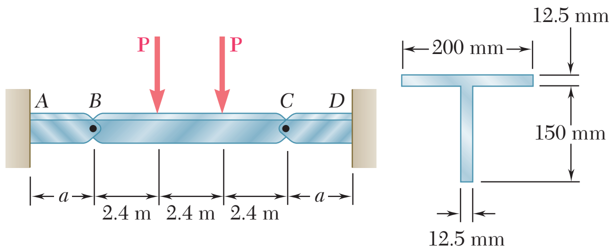


Fig. ***P5.93***

***5.94** A bridge of length $L = 48$ ft is to be built on a secondary road whose access to trucks is limited to two-axle vehicles of medium weight. It will consist of a concrete slab and of simply supported steel beams with an ultimate strength $\sigma_U = 60$ ksi. The combined weight of the slab and beams can be approximated by a uniformly distributed load $w = 0.75$ kips/ft on each beam. For the purpose of the design, it is assumed that a truck with axles located at a distance $a = 14$ ft from each other will be driven across the bridge and that the resulting concentrated loads $\mathbf{P}_1$ and $\mathbf{P}_2$ exerted on each beam could be as large as 24 kips and 6 kips, respectively. Determine the most economical wide-flange shape for the beams, using LRFD with the load factors $\gamma_D = 1.25$, $\gamma_L = 1.75$ and the resistance factor $\phi = 0.9$. [*Hint:* It can be shown that the maximum value of $|M_L|$ occurs under the larger load when that load is located to the left of the center of the beam at a distance equal to $aP_2/2(P_1 + P_2)$.]

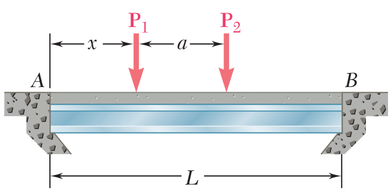


Fig. P5.94

***5.95** Assuming that the front and rear axle loads remain in the same ratio as for the truck of Prob. 5.94, determine how much heavier a truck could safely cross the bridge designed in that problem.

***5.96** A roof structure consists of plywood and roofing material supported by several timber beams of length $L = 16$ m. The dead load carried by each beam, including the estimated weight of the beam, can be represented by a uniformly distributed load $w_D = 350$ N/m. The live load consists of a snow load, represented by a uniformly distributed load $w_L = 600$ N/m, and a 6-kN concentrated load **P** applied at the midpoint C of each beam. Knowing that the ultimate strength for the timber used is $\sigma_U = 50$ MPa and that the width of each beam is $b = 75$ mm, determine the minimum allowable depth h of the beams, using LRFD with the load factors $\gamma_D = 1.2$, $\gamma_L = 1.6$ and the resistance factor $\phi = 0.9$.

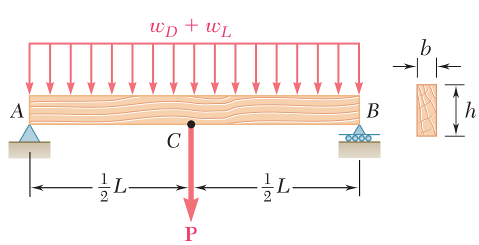


Fig. P5.96

***5.97** Solve Prob. 5.96, assuming that the 6-kN concentrated load **P** applied to each beam is replaced by 3-kN concentrated loads $\mathbf{P}_1$ and $\mathbf{P}_2$ applied at a distance of 4 m from each end of the beams.

*5.5 USING SINGULARITY FUNCTIONS TO DETERMINE SHEAR AND BENDING MOMENT IN A BEAM

Reviewing the work done in the preceding sections, we note that the shear and bending moment could only rarely be described by single analytical functions. In the case of the cantilever beam of Example 5.02 (Fig. 5.9), which supported a uniformly distributed load w, the shear and bending moment *could* be represented by single analytical functions, namely, $V = -wx$ and $M = -\frac{1}{2}wx^2$; this was due to the fact that *no discontinuity* existed in the loading of the beam. On the other hand, in the case of the simply supported beam of Example 5.01, which was loaded only at its midpoint C, the load $\mathbf{P}$ applied at C represented a *singularity* in the beam loading. This singularity resulted in discontinuities in the shear and bending moment and required the use of different analytical functions to represent V and M in the portions of beam located, respectively, to the left and to the right of point C. In Sample Prob. 5.2, the beam had to be divided into three portions, in each of which different functions were used to represent the shear and the bending moment. This situation led us to rely on the graphical representation of the functions V and M provided by the shear and bending-moment diagrams and, later in Sec. 5.3, on a graphical method of integration to determine V and M from the distributed load w.

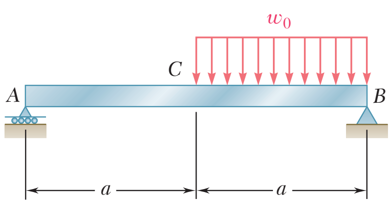


Fig. 5.15

The purpose of this section is to show how the use of *singularity functions* makes it possible to represent the shear V and the bending moment M by single mathematical expressions.

Consider the simply supported beam AB, of length $2a$, which carries a uniformly distributed load w_0 extending from its midpoint C to its right-hand support B (Fig. 5.15). We first draw the free-body diagram of the entire beam (Fig. 5.16*a*); replacing the distributed load by an equivalent concentrated load and, summing moments about B, we write

$$+\circlearrowleft \Sigma M_B = 0: \qquad (w_0 a)(\tfrac{1}{2}a) - R_A(2a) = 0 \qquad R_A = \tfrac{1}{4}w_0 a$$

Next we cut the beam at a point D between A and C. From the free-body diagram of AD (Fig. 5.16*b*) we conclude that, over the interval $0 < x < a$, the shear and bending moment are expressed, respectively, by the functions

$$V_1(x) = \tfrac{1}{4}w_0 a \qquad \text{and} \qquad M_1(x) = \tfrac{1}{4}w_0 ax$$

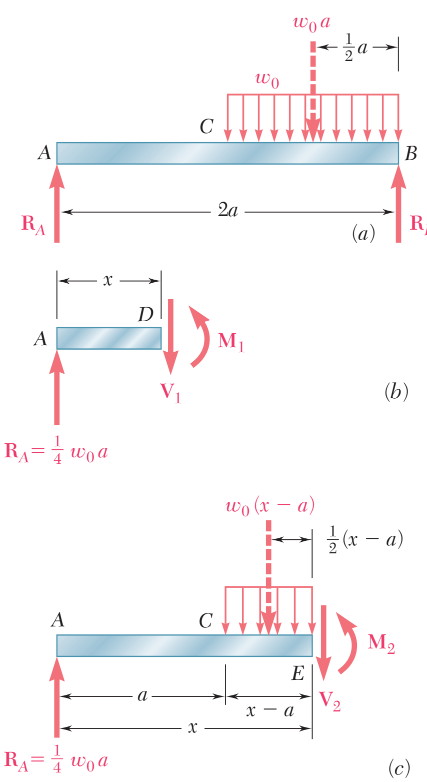


Fig. 5.16

Cutting, now, the beam at a point E between C and B, we draw the free-body diagram of portion AE (Fig. 5.16*c*). Replacing the distributed load by an equivalent concentrated load, we write

$$+\uparrow \Sigma F_y = 0: \qquad \tfrac{1}{4}w_0 a - w_0(x-a) - V_2 = 0$$

$$+\circlearrowleft \Sigma M_E = 0: \qquad -\tfrac{1}{4}w_0 ax + w_0(x-a)[\tfrac{1}{2}(x-a)] + M_2 = 0$$

and conclude that, over the interval $a < x < 2a$, the shear and bending moment are expressed, respectively, by the functions

$$V_2(x) = \tfrac{1}{4}w_0 a - w_0(x-a) \qquad \text{and} \qquad M_2(x) = \tfrac{1}{4}w_0 ax - \tfrac{1}{2}w_0(x-a)^2$$

As we pointed out earlier in this section, the fact that the shear and bending moment are represented by different functions of x, depending upon whether x is smaller or larger than a, is due to the discontinuity in the loading of the beam. However, the functions $V_1(x)$ and $V_2(x)$ can be represented by the single expression

$$V(x) = \tfrac{1}{4}w_0 a - w_0\langle x - a\rangle \tag{5.11}$$

if we specify that the second term should be included in our computations when $x \geq a$ and ignored when $x < a$. In other words, *the brackets* $\langle\ \rangle$ *should be replaced by ordinary parentheses* ( ) *when* $x \geq a$ *and by zero when* $x < a$. With the same convention, the bending moment can be represented at any point of the beam by the single expression

$$M(x) = \tfrac{1}{4}w_0 a x - \tfrac{1}{2}w_0\langle x - a\rangle^2 \tag{5.12}$$

From the convention we have adopted, it follows that brackets $\langle\ \rangle$ can be differentiated or integrated as ordinary parentheses. Instead of calculating the bending moment from free-body diagrams, we could have used the method indicated in Sec. 5.3 and integrated the expression obtained for $V(x)$:

$$M(x) - M(0) = \int_0^x V(x)\,dx = \int_0^x \tfrac{1}{4}w_0 a\,dx - \int_0^x w_0\langle x - a\rangle dx$$

After integration, and observing that $M(0) = 0$, we obtain as before

$$M(x) = \tfrac{1}{4}w_0 a x - \tfrac{1}{2}w_0\langle x - a\rangle^2$$

Furthermore, using the same convention again, we note that the distributed load at any point of the beam can be expressed as

$$w(x) = w_0\langle x - a\rangle^0 \tag{5.13}$$

Indeed, the brackets should be replaced by zero for $x < a$ and by parentheses for $x \geq a$; we thus check that $w(x) = 0$ for $x < a$ and, defining the zero power of any number as unity, that $\langle x - a\rangle^0 = (x - a)^0 = 1$ and $w(x) = w_0$ for $x \geq a$. From Sec. 5.3 we recall that the shear could have been obtained by integrating the function $-w(x)$. Observing that $V = \frac{1}{4}w_0 a$ for $x = 0$, we write

$$V(x) - V(0) = -\int_0^x w(x)\,dx = -\int_0^x w_0\langle x - a\rangle^0\,dx$$

$$V(x) - \tfrac{1}{4}w_0 a = -w_0\langle x - a\rangle^1$$

Solving for $V(x)$ and dropping the exponent 1, we obtain again

$$V(x) = \tfrac{1}{4}w_0 a - w_0\langle x - a\rangle$$

The expressions $\langle x - a\rangle^0$, $\langle x - a\rangle$, $\langle x - a\rangle^2$ are called *singularity functions*. By definition, we have, for $n \geq 0$,

$$\langle x - a\rangle^n = \begin{cases} (x - a)^n & \text{when } x \geq a \\ 0 & \text{when } x < a \end{cases} \tag{5.14}$$

We also note that whenever the quantity between brackets is positive or zero, the brackets should be replaced by ordinary parentheses, and whenever that quantity is negative, the bracket itself is equal to zero.

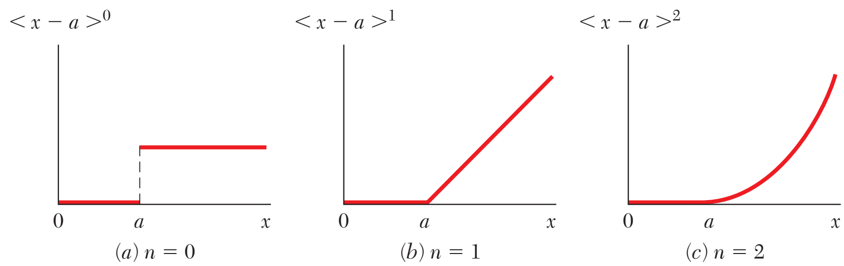


Fig. 5.17 Singularity functions.

The three singularity functions corresponding respectively to $n = 0$, $n = 1$, and $n = 2$ have been plotted in Fig. 5.17. We note that the function $\langle x - a \rangle^0$ is discontinuous at $x = a$ and is in the shape of a "step." For that reason it is referred to as the *step function*. According to (5.14), and with the zero power of any number defined as unity, we have†

$$\langle x - a \rangle^0 = \begin{cases} 1 & \text{when } x \geq a \\ 0 & \text{when } x < a \end{cases} \tag{5.15}$$

It follows from the definition of singularity functions that

$$\int \langle x - a \rangle^n \, dx = \frac{1}{n+1} \langle x - a \rangle^{n+1} \qquad \text{for } n \geq 0 \tag{5.16}$$

and

$$\frac{d}{dx} \langle x - a \rangle^n = n \langle x - a \rangle^{n-1} \qquad \text{for } n \geq 1 \tag{5.17}$$

Most of the beam loadings encountered in engineering practice can be broken down into the basic loadings shown in Fig. 5.18. Whenever applicable, the corresponding functions $w(x)$, $V(x)$, and $M(x)$ have been expressed in terms of singularity functions and plotted against a color background. A heavier color background was used to indicate for each loading the expression that is most easily derived or remembered and from which the other functions can be obtained by integration.

†Since $(x - a)^0$ is discontinuous at $x - a$, it can be argued that this function should be left undefined for $x = a$ or that it should be assigned both of the values 0 and 1 for $x = a$. However, defining $(x - a)^0$ as equal to 1 when $x = a$, as stated in (Eq. 5.15), has the advantage of being unambiguous and, thus, readily applicable to computer programming (cf. page 348).

After a given beam loading has been broken down into the basic loadings of Fig. 5.18, the functions $V(x)$ and $M(x)$ representing the shear and bending moment at any point of the beam can be obtained by adding the corresponding functions associated with each of the basic loadings and reactions. Since all the distributed loadings shown in Fig. 5.18 are open-ended to the right, a distributed loading

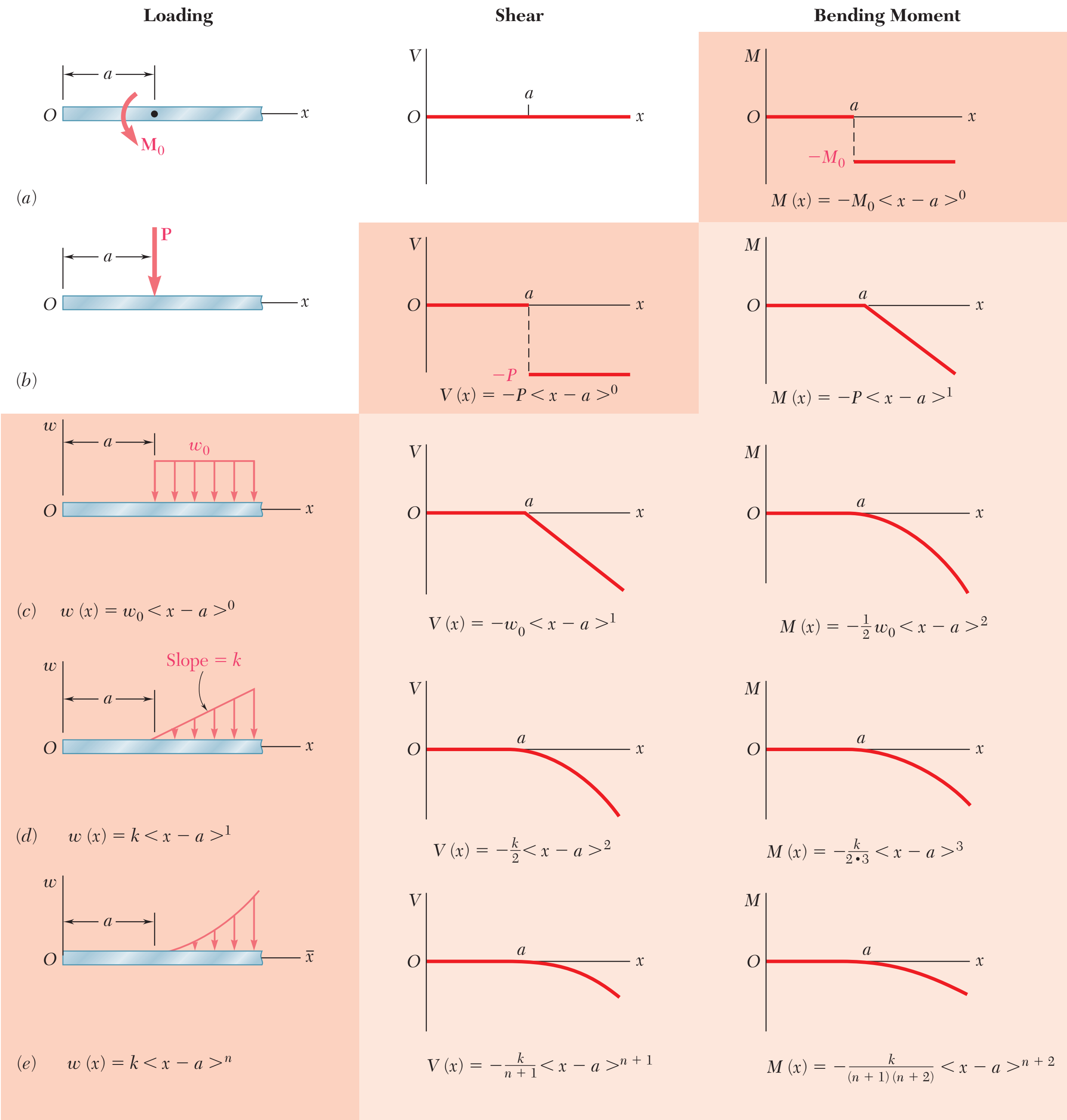
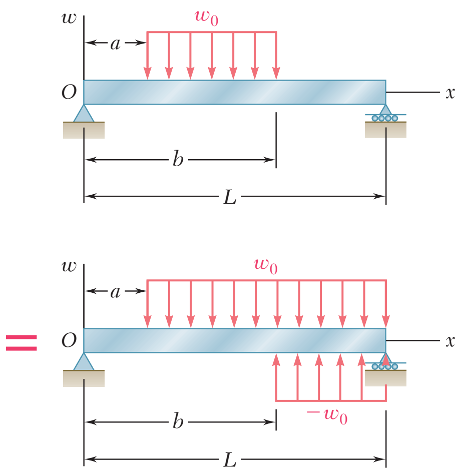


Fig. 5.18 Basic loadings and corresponding shears and bending moments expressed in terms of singularity functions.



$w(x) = w_0 < x - a >^0 - w_0 < x - b >^0$

Fig. 5.19 Use of open-ended loadings to create a closed-ended loading.

that does not extend to the right end of the beam or that is discontinuous should be replaced as shown in Fig. 5.19 by an equivalent combination of open-ended loadings. (See also Example 5.05 and Sample Prob. 5.9.)

As you will see in Sec. 9.6, the use of singularity functions also greatly simplifies the determination of beam deflections. It was in connection with that problem that the approach used in this section was first suggested in 1862 by the German mathematician A. Clebsch (1833–1872). However, the British mathematician and engineer W. H. Macaulay (1853–1936) is usually given credit for introducing the singularity functions in the form used here, and the brackets $\langle\ \rangle$ are generally referred to as *Macaulay's brackets*.†

†W. H. Macaulay, "Note on the Deflection of Beams," *Messenger of Mathematics*, vol. 48, pp. 129–130, 1919.

EXAMPLE 5.05

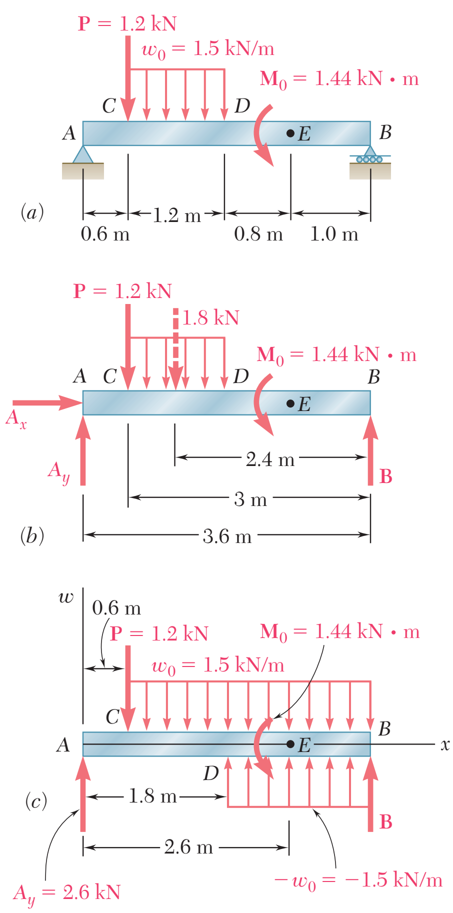


Fig. 5.20

For the beam and loading shown (Fig. 5.20*a*) and using singularity functions, express the shear and bending moment as functions of the distance x from the support at A.

We first determine the reaction at A by drawing the free-body diagram of the beam (Fig. 5.20*b*) and writing

$$\xrightarrow{+} \Sigma F_x = 0: \qquad A_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -A_y(3.6 \text{ m}) + (1.2 \text{ kN})(3 \text{ m})$$
$$+(1.8 \text{ kN})(2.4 \text{ m}) + 1.44 \text{ kN} \cdot \text{m} = 0$$
$$A_y = 2.60 \text{ kN}$$

Next, we replace the given distributed loading by two equivalent open-ended loadings (Fig. 5.20*c*) and express the distributed load $w(x)$ as the sum of the corresponding step functions:

$$w(x) = +w_0\langle x - 0.6\rangle^0 - w_0\langle x - 1.8\rangle^0$$

The function $V(x)$ is obtained by integrating $w(x)$, reversing the + and − signs, and adding to the result the constants A_y and $-P\langle x - 0.6\rangle^0$ representing the respective contributions to the shear of the reaction at A and of the concentrated load. (No other constant of integration is required.) Since the concentrated couple does not directly affect the shear, it should be ignored in this computation. We write

$$V(x) = -w_0\langle x - 0.6\rangle^1 + w_0\langle x - 1.8\rangle^1 + A_y - P\langle x - 0.6\rangle^0$$

In a similar way, the function $M(x)$ is obtained by integrating $V(x)$ and adding to the result the constant $-M_0\langle x - 2.6\rangle^0$ representing the contribution of the concentrated couple to the bending moment. We have

$$M(x) = -\tfrac{1}{2}w_0\langle x - 0.6\rangle^2 + \tfrac{1}{2}w_0\langle x - 1.8\rangle^2 + A_y x - P\langle x - 0.6\rangle^1 - M_0\langle x - 2.6\rangle^0$$

Substituting the numerical values of the reaction and loads into the expressions obtained for $V(x)$ and $M(x)$ and being careful *not* to compute any product or expand any square involving a bracket, we obtain the following expressions for the shear and bending moment at any point of the beam:

$$V(x) = -1.5\langle x - 0.6\rangle^1 + 1.5\langle x - 1.8\rangle^1 + 2.6 - 1.2\langle x - 0.6\rangle^0$$

$$M(x) = -0.75\langle x - 0.6\rangle^2 + 0.75\langle x - 1.8\rangle^2 + 2.6x - 1.2\langle x - 0.6\rangle^1 - 1.44\langle x - 2.6\rangle^0$$

EXAMPLE 5.06

For the beam and loading of Example 5.05, determine the numerical values of the shear and bending moment at the midpoint D.

Making $x = 1.8$ m in the expressions found for $V(x)$ and $M(x)$ in Example 5.05, we obtain

$$V(1.8) = -1.5\langle 1.2\rangle^1 + 1.5\langle 0\rangle^1 + 2.6 - 1.2\langle 1.2\rangle^0$$

$$M(1.8) = -0.75\langle 1.2\rangle^2 + 0.75\langle 0\rangle^2 + 2.6(1.8) - 1.2\langle 1.2\rangle^1 - 1.44\langle -0.8\rangle^0$$

Recalling that whenever a quantity between brackets is positive or zero, the brackets should be replaced by ordinary parentheses, and whenever the quantity is negative, the bracket itself is equal to zero, we write

$$\begin{aligned} V(1.8) &= -1.5(1.2)^1 + 1.5(0)^1 + 2.6 - 1.2(1.2)^0 \\ &= -1.5(1.2) + 1.5(0) + 2.6 - 1.2(1) \\ &= -1.8 + 0 + 2.6 - 1.2 \end{aligned}$$

$$V(1.8) = -0.4 \text{ kN}$$

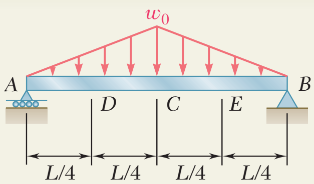
and

$$\begin{aligned} M(1.8) &= -0.75(1.2)^2 + 0.75(0)^2 + 2.6(1.8) - 1.2(1.2)^1 - 1.44(0) \\ &= -1.08 + 0 + 4.68 - 1.44 - 0 \end{aligned}$$

$$M(1.8) = +2.16 \text{ kN} \cdot \text{m}$$

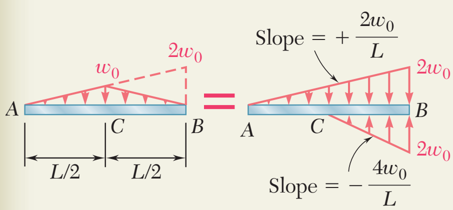
Application to Computer Programming. Singularity functions are particularly well suited to the use of computers. First we note that the step function $\langle x - a\rangle^0$, which will be represented by the symbol STP, can be defined by an IF/THEN/ELSE statement as being equal to 1 for X $\geq$ A and to 0 otherwise. Any other singularity function $\langle x - a\rangle^n$, with $n \geq 1$, can then be expressed as the product of the ordinary algebraic function $(x - a)^n$ and the step function $\langle x - a\rangle^0$.

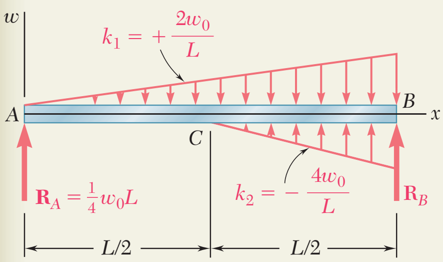
When k different singularity functions are involved, such as $\langle x - a_i\rangle^n$, where $i = 1, 2, \ldots, k$, then the corresponding step functions STP(I), where I = 1, 2, . . ., K, can be defined by a loop containing a single IF/THEN/ELSE statement.

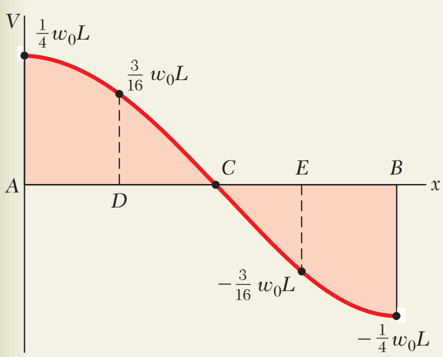


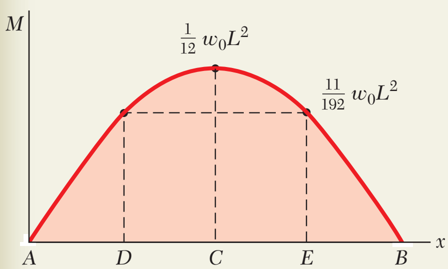
SAMPLE PROBLEM 5.9

For the beam and loading shown, determine (*a*) the equations defining the shear and bending moment at any point, (*b*) the shear and bending moment at points *C*, *D*, and *E*.









SOLUTION

Reactions. The total load is $\frac{1}{2}w_0L$; because of symmetry, each reaction is equal to half that value, namely, $\frac{1}{4}w_0L$.

Distributed Load. The given distributed loading is replaced by two equivalent open-ended loadings as shown. Using a singularity function to express the second loading, we write

$$w(x) = k_1x + k_2\langle x - \tfrac{1}{2}L\rangle = \frac{2w_0}{L}x - \frac{4w_0}{L}\langle x - \tfrac{1}{2}L\rangle \qquad (1)$$

a. Equations for Shear and Bending Moment. We obtain $V(x)$ by integrating (1), changing the signs, and adding a constant equal to R_A:

$$V(x) = -\frac{w_0}{L}x^2 + \frac{2w_0}{L}\langle x - \tfrac{1}{2}L\rangle^2 + \tfrac{1}{4}w_0L \qquad (2)$$

We obtain $M(x)$ by integrating (2); since there is no concentrated couple, no constant of integration is needed:

$$M(x) = -\frac{w_0}{3L}x^3 + \frac{2w_0}{3L}\langle x - \tfrac{1}{2}L\rangle^3 + \tfrac{1}{4}w_0Lx \qquad (3)$$

b. Shear and Bending Moment at *C*, *D*, and *E*

At Point C: Making $x = \frac{1}{2}L$ in Eqs. (2) and (3) and recalling that whenever a quantity between brackets is positive or zero, the brackets may be replaced by parentheses, we have

$$V_C = -\frac{w_0}{L}(\tfrac{1}{2}L)^2 + \frac{2w_0}{L}\langle 0\rangle^2 + \tfrac{1}{4}w_0L \qquad V_C = 0$$

$$M_C = -\frac{w_0}{3L}(\tfrac{1}{2}L)^3 + \frac{2w_0}{3L}\langle 0\rangle^3 + \tfrac{1}{4}w_0L(\tfrac{1}{2}L) \qquad M_C = \frac{1}{12}w_0L^2$$

At Point D: Making $x = \frac{1}{4}L$ in Eqs. (2) and (3) and recalling that a bracket containing a negative quantity is equal to zero, we write

$$V_D = -\frac{w_0}{L}(\tfrac{1}{4}L)^2 + \frac{2w_0}{L}\langle -\tfrac{1}{4}L\rangle^2 + \tfrac{1}{4}w_0L \qquad V_D = \frac{3}{16}w_0L$$

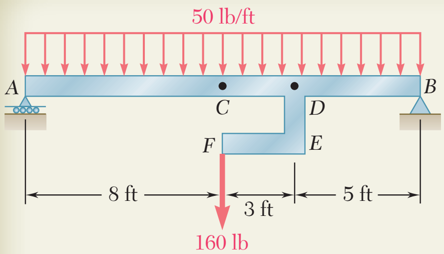
$$M_D = -\frac{w_0}{3L}(\tfrac{1}{4}L)^3 + \frac{2w_0}{3L}\langle -\tfrac{1}{4}L\rangle^3 + \tfrac{1}{4}w_0L(\tfrac{1}{4}L) \qquad M_D = \frac{11}{192}w_0L^2$$

At Point E: Making $x = \frac{3}{4}L$ in Eqs. (2) and (3), we have

$$V_E = -\frac{w_0}{L}(\tfrac{3}{4}L)^2 + \frac{2w_0}{L}\langle \tfrac{1}{4}L\rangle^2 + \tfrac{1}{4}w_0L \qquad V_E = -\frac{3}{16}w_0L$$

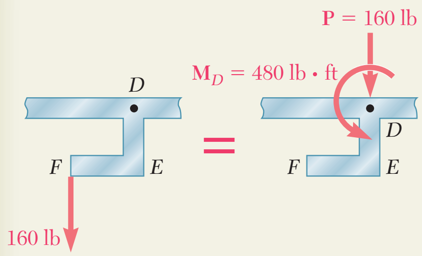
$$M_E = -\frac{w_0}{3L}(\tfrac{3}{4}L)^3 + \frac{2w_0}{3L}\langle \tfrac{1}{4}L\rangle^3 + \tfrac{1}{4}w_0L(\tfrac{3}{4}L) \qquad M_E = \frac{11}{192}w_0L^2$$

SAMPLE PROBLEM 5.10



The rigid bar *DEF* is welded at point *D* to the steel beam *AB*. For the loading shown, determine (*a*) the equations defining the shear and bending moment at any point of the beam, (*b*) the location and magnitude of the largest bending moment.

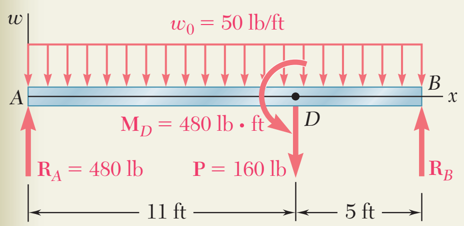
SOLUTION



Reactions. We consider the beam and bar as a free body and observe that the total load is 960 lb. Because of symmetry, each reaction is equal to 480 lb.

Modified Loading Diagram. We replace the 160-lb load applied at *F* by an equivalent force-couple system at *D*. We thus obtain a loading diagram consisting of a concentrated couple, three concentrated loads (including the two reactions), and a uniformly distributed load

$$w(x) = 50 \text{ lb/ft} \tag{1}$$



a. Equations for Shear and Bending Moment. We obtain $V(x)$ by integrating (1), changing the sign, and adding constants representing the respective contributions of $\mathbf{R}_A$ and $\mathbf{P}$ to the shear. Since $\mathbf{P}$ affects $V(x)$ only for values of x larger than 11 ft, we use a step function to express its contribution.

$$V(x) = -50x + 480 - 160\langle x - 11\rangle^0 \tag{2}$$ ◀

We obtain $M(x)$ by integrating (2) and using a step function to represent the contribution of the concentrated couple $\mathbf{M}_D$:

$$M(x) = -25x^2 + 480x - 160\langle x - 11\rangle^1 - 480\langle x - 11\rangle^0 \tag{3}$$ ◀

b. Largest Bending Moment. Since M is maximum or minimum when $V = 0$, we set $V = 0$ in (2) and solve that equation for x to find the location of the largest bending moment. Considering first values of x less than 11 ft and noting that for such values the bracket is equal to zero, we write

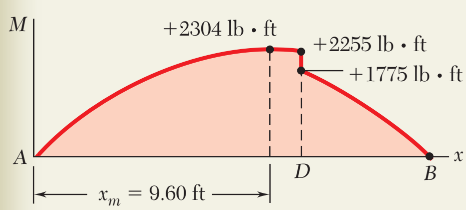
$$-50x + 480 = 0 \qquad x = 9.60 \text{ ft}$$

Considering now values of x larger than 11 ft, for which the bracket is equal to 1, we have

$$-50x + 480 - 160 = 0 \qquad x = 6.40 \text{ ft}$$

Since this value is *not* larger than 11 ft, it must be rejected. Thus, the value of x corresponding to the largest bending moment is

$$x_m = 9.60 \text{ ft}$$ ◀



Substituting this value for x into Eq. (3), we obtain

$$M_{max} = -25(9.60)^2 + 480(9.60) - 160\langle -1.40\rangle^1 - 480\langle -1.40\rangle^0$$

and, recalling that brackets containing a negative quantity are equal to zero,

$$M_{max} = -25(9.60)^2 + 480(9.60) \qquad M_{max} = 2304 \text{ lb} \cdot \text{ft}$$ ◀

The bending-moment diagram has been plotted. Note the discontinuity at point *D* due to the concentrated couple applied at that point. The values of M just to the left and just to the right of *D* were obtained by making $x = 11$ in Eq. (3) and replacing the step function $\langle x - 11\rangle^0$ by 0 and 1, respectively.

PROBLEMS

5.98 through *5.100* (*a*) Using singularity functions, write the equations defining the shear and bending moment for the beam and loading shown. (*b*) Use the equation obtained for M to determine the bending moment at point C and check your answer by drawing the free-body diagram of the entire beam.

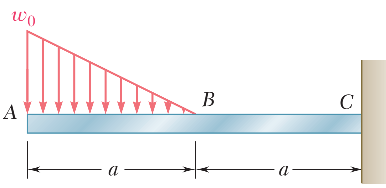


Fig. P5.98

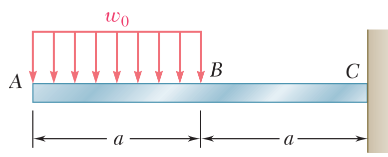


Fig. P5.99

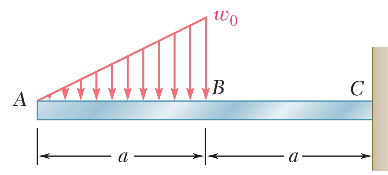


Fig. *P5.100*

5.101 through *5.103* (*a*) Using singularity functions, write the equations defining the shear and bending moment for the beam and loading shown. (*b*) Use the equation obtained for M to determine the bending moment at point E and check your answer by drawing the free-body diagram of the portion of the beam to the right of E.

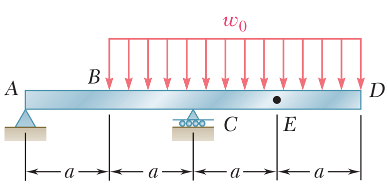


Fig. P5.101

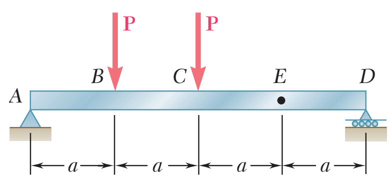


Fig. P5.102

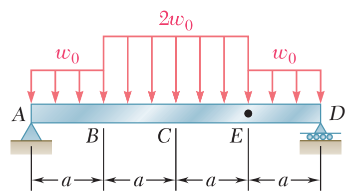


Fig. *P5.103*

5.104 (*a*) Using singularity functions, write the equations for the shear and bending moment for beam ABC under the loading shown. (*b*) Use the equation obtained for M to determine the bending moment just to the right of point B.

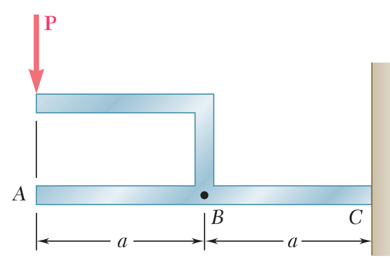


Fig. P5.104

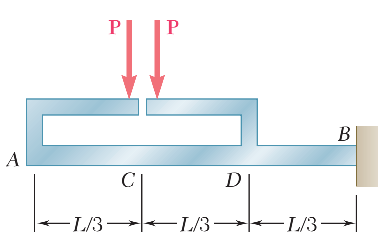


Fig. P5.105

5.105 (*a*) Using singularity functions, write the equations for the shear and bending moment for beam ABC under the loading shown. (*b*) Use the equation obtained for M to determine the bending moment just to the right of point D.

5.106 through 5.109 (*a*) Using singularity functions, write the equations for the shear and bending moment for the beam and loading shown. (*b*) Determine the maximum value of the bending moment in the beam.

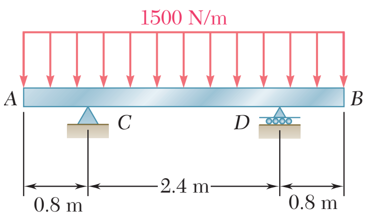


Fig. P5.106

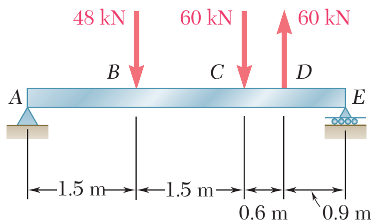


Fig. P5.107

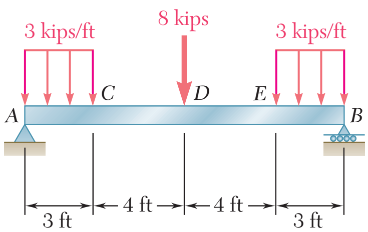


Fig. P5.108

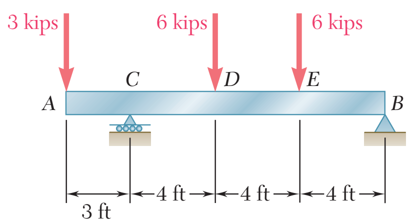


Fig. P5.109

5.110 and 5.111 (*a*) Using singularity functions, write the equations for the shear and bending moment for the beam and loading shown. (*b*) Determine the maximum normal stress due to bending.

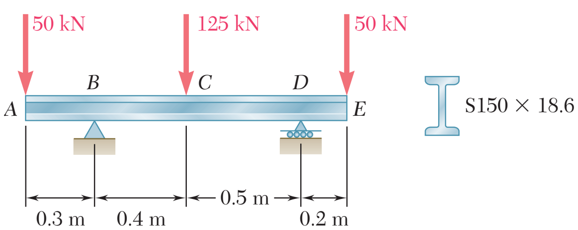


Fig. P5.110

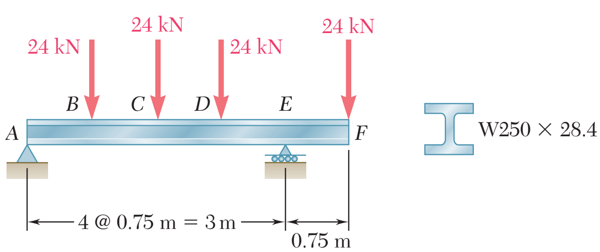


Fig. P5.111

5.112 and 5.113 (*a*) Using singularity functions, find the magnitude and location of the maximum bending moment for the beam and loading shown. (*b*) Determine the maximum normal stress due to bending.

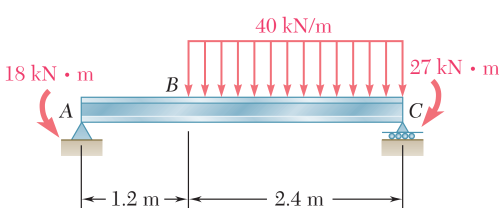


Fig. P5.112

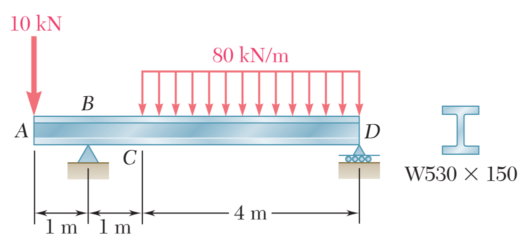


Fig. P5.113

5.114 and 5.115 A beam is being designed to be supported and loaded as shown. (*a*) Using singularity functions, find the magnitude and location of the maximum bending moment in the beam. (*b*) Knowing that the allowable normal stress for the steel to be used is 24 ksi, find the most economical wide-flange shape that can be used.

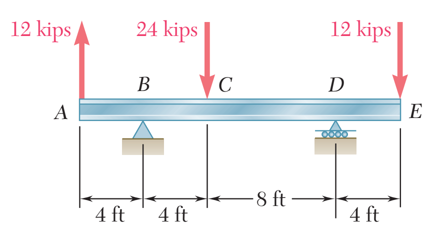


Fig. P5.114

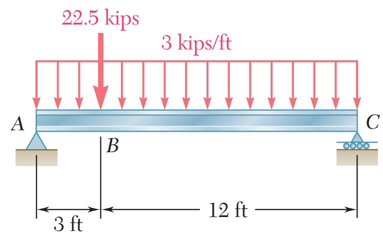


Fig. P5.115

5.116 and 5.117 A timber beam is being designed with supports and loads as shown. (*a*) Using singularity functions, find the magnitude and location of the maximum bending moment in the beam. (*b*) Knowing that the available stock consists of beams with an allowable stress of 12 MPa and a rectangular cross section of 30-mm width and depth h varying from 80 mm to 160 mm in 10-mm increments, determine the most economical cross section that can be used.

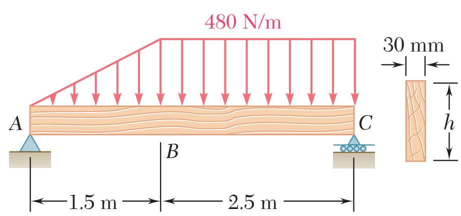


Fig. P5.116

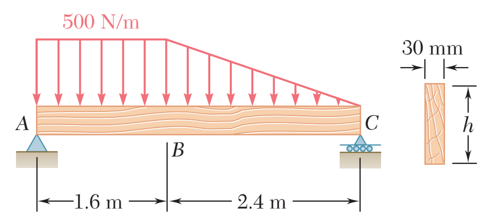


Fig. P5.117

5.118 through 5.121 Using a computer and step functions, calculate the shear and bending moment for the beam and loading shown. Use the specified increment ΔL, starting at point A and ending at the right-hand support.

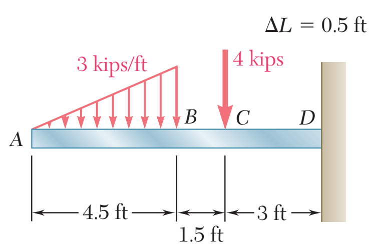


Fig. P5.118

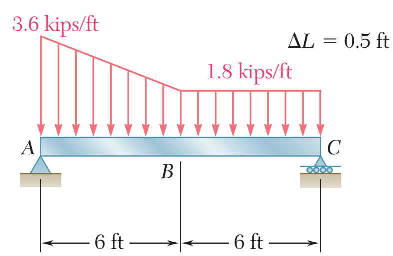


Fig. P5.119

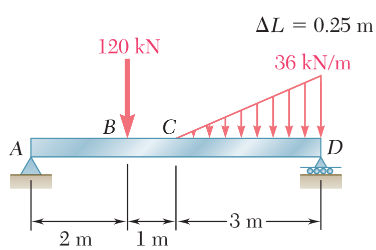


Fig. P5.120

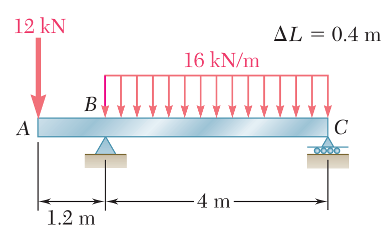


Fig. P5.121

5.122 and 5.123 For the beam and loading shown, and using a computer and step functions, (*a*) tabulate the shear, bending moment, and maximum normal stress in sections of the beam from $x = 0$ to $x = L$, using the increments ΔL indicated, (*b*) using smaller increments if necessary, determine with a 2% accuracy the maximum normal stress in the beam. Place the origin of the x axis at end A of the beam.

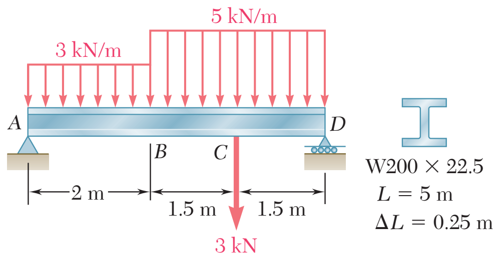


Fig. P5.122

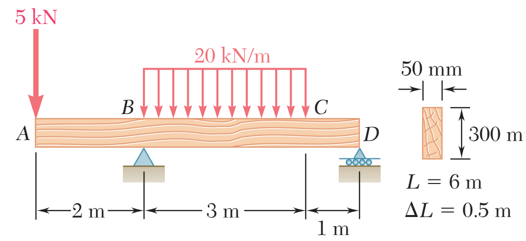


Fig. P5.123

5.124 and *5.125* For the beam and loading shown, and using a computer and step functions, (*a*) tabulate the shear, bending moment, and maximum normal stress in sections of the beam from $x = 0$ to $x = L$, using the increments ΔL indicated, (*b*) using smaller increments if necessary, determine with a 2% accuracy the maximum normal stress in the beam. Place the origin of the x axis at end A of the beam.

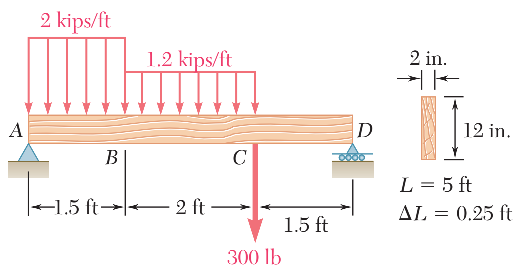


Fig. P5.124

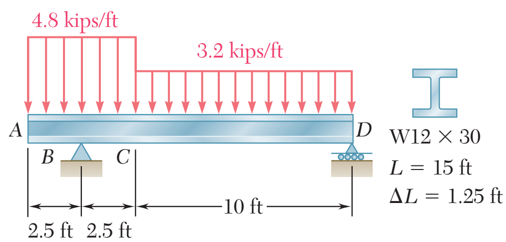


Fig. *P5.125*

*5.6 NONPRISMATIC BEAMS

Our analysis has been limited so far to prismatic beams, i.e., to beams of uniform cross section. As we saw in Sec. 5.4, prismatic beams are designed so that the normal stresses in their critical sections are at most equal to the allowable value of the normal stress for the material being used. It follows that, in all other sections, the normal stresses will be smaller, possibly much smaller, than their allowable value. A prismatic beam, therefore, is almost always overdesigned, and considerable savings of material can be realized by using nonprismatic beams, i.e., beams of variable cross section. The cantilever beams shown in the bridge during construction in Photo 5.2 are examples of nonprismatic beams.

Photo 5.2 Nonprismatic cantilever beams of bridge during construction.

Since the maximum normal stresses σ_m usually control the design of a beam, the design of a nonprismatic beam will be optimum if the section modulus $S = I/c$ of every cross section satisfies Eq. (5.3) of Sec. 5.1. Solving that equation for S, we write

$$S = \frac{|M|}{\sigma_{\text{all}}} \tag{5.18}$$

A beam designed in this manner is referred to as a *beam of constant strength*.

For a forged or cast structural or machine component, it is possible to vary the cross section of the component along its length and to eliminate most of the unnecessary material (see Example 5.07). For a timber beam or a rolled-steel beam, however, it is not possible to vary the cross section of the beam. But considerable savings of material can be achieved by gluing wooden planks of appropriate lengths to a timber beam (see Sample Prob. 5.11) and using cover plates in portions of a rolled-steel beam where the bending moment is large (see Sample Prob. 5.12).

EXAMPLE 5.07

A cast-aluminum plate of uniform thickness b is to support a uniformly distributed load w as shown in Fig. 5.21. (a) Determine the shape of the plate that will yield the most economical design. (b) Knowing that the allowable normal stress for the aluminum used is 72 MPa and that $b = 40$ mm, $L = 800$ mm, and $w = 135$ kN/m, determine the maximum depth h_0 of the plate.

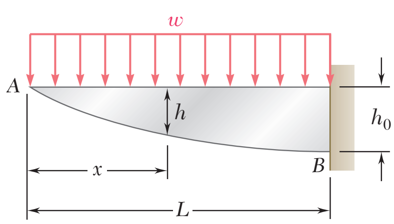


Fig. 5.21

Bending Moment. Measuring the distance x from A and observing that $V_A = M_A = 0$, we use Eqs. (5.6) and (5.8) of Sec. 5.3 and write

$$V(x) = -\int_0^x w\,dx = -wx$$

$$M(x) = \int_0^x V(x)\,dx = -\int_0^x wx\,dx = -\tfrac{1}{2}wx^2$$

(a) Shape of Plate. We recall from Sec. 5.4 that the modulus S of a rectangular cross section of width b and depth h is $S = \frac{1}{6}bh^2$. Carrying this value into Eq. (5.18) and solving for h^2, we have

$$h^2 = \frac{6|M|}{b\sigma_{\text{all}}} \tag{5.19}$$

and, after substituting $|M| = \frac{1}{2}wx^2$,

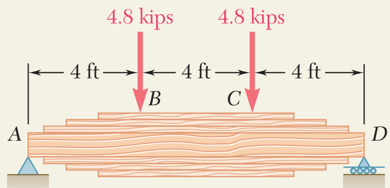
$$h^2 = \frac{3wx^2}{b\sigma_{\text{all}}} \quad \text{or} \quad h = \left(\frac{3w}{b\sigma_{\text{all}}}\right)^{1/2} x \tag{5.20}$$

Since the relation between h and x is linear, the lower edge of the plate is a straight line. Thus, the plate providing the most economical design is of *triangular shape.*

(b) Maximum Depth h_0. Making $x = L$ in Eq. (5.20) and substituting the given data, we obtain

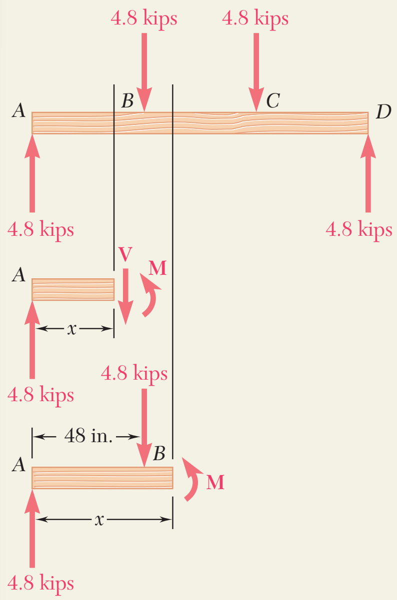
$$h_0 = \left[\frac{3(135\text{ kN/m})}{(0.040\text{ m})(72\text{ MPa})}\right]^{1/2} (800\text{ mm}) = 300\text{ mm}$$

SAMPLE PROBLEM 5.11



A 12-ft-long beam made of a timber with an allowable normal stress of 2.40 ksi and an allowable shearing stress of 0.40 ksi is to carry two 4.8-kip loads located at its third points. As shown in Chap. 6, a beam of uniform rectangular cross section, 4 in. wide and 4.5 in. deep, would satisfy the allowable shearing stress requirement. Since such a beam would not satisfy the allowable normal stress requirement, it will be reinforced by gluing planks of the same timber, 4 in. wide and 1.25 in. thick, to the top and bottom of the beam in a symmetric manner. Determine (*a*) the required number of pairs of planks, (*b*) the length of the planks in each pair that will yield the most economical design.

SOLUTION



Bending Moment. We draw the free-body diagram of the beam and find the following expressions for the bending moment:

From A to B $(0 \le x \le 48 \text{ in.})$: $\quad M = (4.80 \text{ kips})x$

From B to C $(48 \text{ in.} \le x \le 96 \text{ in.})$:

$$M = (4.80 \text{ kips})\,x - (4.80 \text{ kips})(x - 48 \text{ in.}) = 230.4 \text{ kip} \cdot \text{in.}$$

a. Number of Pairs of Planks. We first determine the required total depth of the reinforced beam between B and C. We recall from Sec. 5.4 that $S = \frac{1}{6}bh^2$ for a beam with a rectangular cross section of width b and depth h. Substituting this value into Eq. (5.17) and solving for h^2, we have

$$h^2 = \frac{6|M|}{b\sigma_{\text{all}}} \tag{1}$$

Substituting the value obtained for M from B to C and the given values of b and σ_{all}, we write

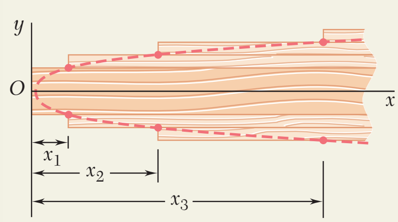
$$h^2 = \frac{6(230.4 \text{ kip} \cdot \text{in.})}{(4 \text{ in.})(2.40 \text{ ksi})} = 144 \text{ in.}^2 \qquad h = 12.00 \text{ in.}$$

Since the original beam has a depth of 4.50 in., the planks must provide an additional depth of 7.50 in. Recalling that each pair of planks is 2.50 in. thick, we write:

Required number of pairs of planks = 3 ◀

b. Length of Planks. The bending moment was found to be $M = (4.80 \text{ kips})\,x$ in the portion AB of the beam. Substituting this expression and the given values of b and σ_{all}, into Eq. (1) and solving for x, we have

$$x = \frac{(4 \text{ in.})(2.40 \text{ ksi})}{6\,(4.80 \text{ kips})}h^2 \qquad x = \frac{h^2}{3 \text{ in.}} \tag{2}$$

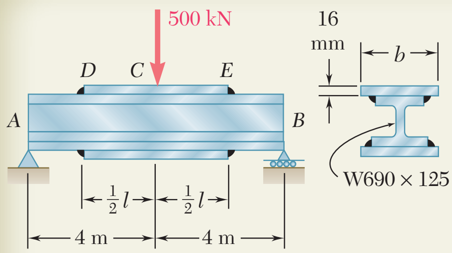


Equation (2) defines the maximum distance x from end A at which a given depth h of the cross section is acceptable. Making $h = 4.50$ in., we find the distance x_1 from A at which the original prismatic beam is safe: $x_1 = 6.75$ in. From that point on, the original beam should be reinforced by the first pair of planks. Making $h = 4.50 \text{ in.} + 2.50 \text{ in.} = 7.00$ in. yields the distance $x_2 = 16.33$ in. from which the second pair of planks should be used, and making $h = 9.50$ in. yields the distance $x_3 = 30.08$ in. from which the third pair of planks should be used. The length l_i of the planks of the pair i, where $i = 1, 2, 3$, is obtained by subtracting $2x_i$ from the 144-in. length of the beam. We find

$l_1 = 130.5$ in., $l_2 = 111.3$ in., $l_3 = 83.8$ in. ◀

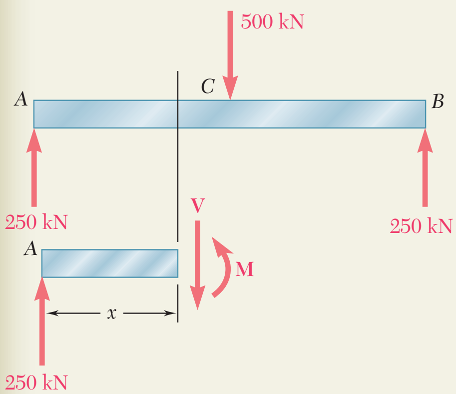
The corners of the various planks lie on the parabola defined by Eq. (2).

SAMPLE PROBLEM 5.12



Two steel plates, each 16 mm thick, are welded as shown to a W690 × 125 beam to reinforce it. Knowing that σ_{all} = 160 MPa for both the beam and the plates, determine the required value of (*a*) the length of the plates, (*b*) the width of the plates.

SOLUTION



Bending Moment. We first find the reactions. From the free body of a portion of beam of length $x \leq 4$ m, we obtain M between A and C:

$$M = (250 \text{ kN})x \tag{1}$$

a. Required Length of Plates. We first determine the maximum allowable length x_m of the portion AD of the unreinforced beam. From Appendix C we find that the section modulus of a W690 × 125 beam is $S = 3490 \times 10^6 \text{ mm}^3$, or $S = 3.49 \times 10^{-3} \text{ m}^3$. Substituting for S and σ_{all} into Eq. (5.17) and solving for M, we write

$$M = S\sigma_{all} = (3.49 \times 10^{-3} \text{ m}^3)(160 \times 10^3 \text{ kN/m}^2) = 558.4 \text{ kN} \cdot \text{m}$$

Substituting for M in Eq. (1), we have

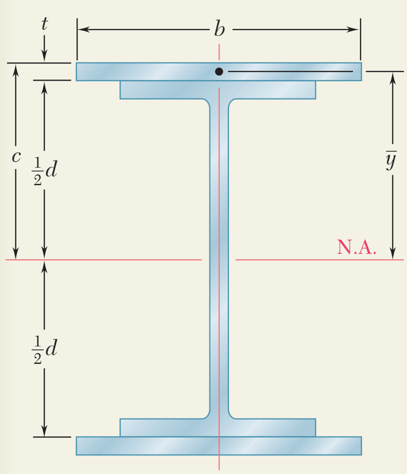
$$558.4 \text{ kN} \cdot \text{m} = (250 \text{ kN})x_m \qquad x_m = 2.234 \text{ m}$$

The required length l of the plates is obtained by subtracting $2x_m$ from the length of the beam:

$$l = 8 \text{ m} - 2(2.234 \text{ m}) = 3.532 \text{ m} \qquad l = 3.53 \text{ m} \blacktriangleleft$$

b. Required Width of Plates. The maximum bending moment occurs in the midsection C of the beam. Making $x = 4$ m in Eq. (1), we obtain the bending moment in that section:

$$M = (250 \text{ kN})(4 \text{ m}) = 1000 \text{ kN} \cdot \text{m}$$



In order to use Eq. (5.1) of Sec. 5.1, we now determine the moment of inertia of the cross section of the reinforced beam with respect to a centroidal axis and the distance c from that axis to the outer surfaces of the plates. From Appendix C we find that the moment of inertia of a W690 × 125 beam is $I_b = 1190 \times 10^6 \text{ mm}^4$ and its depth is $d = 678$ mm. On the other hand, denoting by t the thickness of one plate, by b its width, and by $\bar{y}$ the distance of its centroid from the neutral axis, we express the moment of inertia I_p of the two plates with respect to the neutral axis:

$$I_p = 2(\tfrac{1}{12}bt^3 + A\bar{y}^2) = (\tfrac{1}{6}t^3)b + 2\,bt(\tfrac{1}{2}d + \tfrac{1}{2}t)^2$$

Substituting $t = 16$ mm and $d = 678$ mm, we obtain $I_p = (3.854 \times 10^6 \text{ mm}^3)\,b$. The moment of inertia I of the beam and plates is

$$I = I_b + I_p = 1190 \times 10^6 \text{ mm}^4 + (3.854 \times 10^6 \text{ mm}^3)b \tag{2}$$

and the distance from the neutral axis to the surface is $c = \frac{1}{2}d + t = 355$ mm. Solving Eq. (5.1) for I and substituting the values of M, σ_{all}, and c, we write

$$I = \frac{|M|c}{\sigma_{all}} = \frac{(1000 \text{ kN} \cdot \text{m})(355 \text{ mm})}{160 \text{ MPa}} = 2.219 \times 10^{-3} \text{ m}^4 = 2219 \times 10^6 \text{ mm}^4$$

Replacing I by this value in Eq. (2) and solving for b, we have

$$2219 \times 10^6 \text{ mm}^4 = 1190 \times 10^6 \text{ mm}^4 + (3.854 \times 10^6 \text{ mm}^3)b$$

$$b = 267 \text{ mm} \blacktriangleleft$$

PROBLEMS

5.126 and *5.127* The beam AB, consisting of an aluminum plate of uniform thickness b and length L, is to support the load shown. (a) Knowing that the beam is to be of constant strength, express h in terms of x, L, and h_0 for portion AC of the beam. (b) Determine the maximum allowable load if L = 800 mm, h_0 = 200 mm, b = 25 mm, and σ_{all} = 72 MPa.

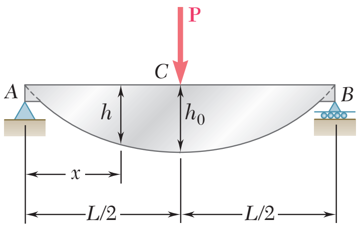


Fig. P5.126

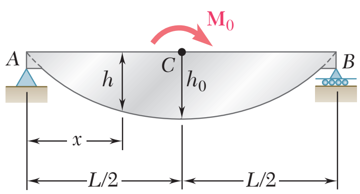


Fig. *P5.127*

5.128 and 5.129 The beam AB, consisting of a cast-iron plate of uniform thickness b and length L, is to support the load shown. (a) Knowing that the beam is to be of constant strength, express h in terms of x, L, and h_0. (b) Determine the maximum allowable load if L = 36 in., h_0 = 12 in., b = 1.25 in., and σ_{all} = 24 ksi.

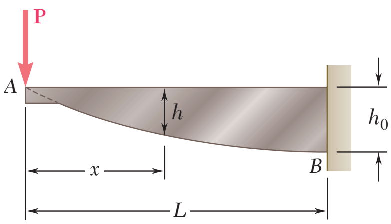


Fig. P5.128

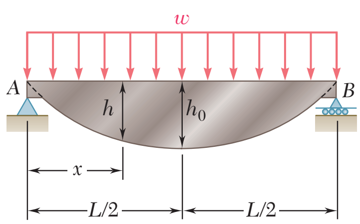


Fig. P5.129

5.130 and *5.131* The beam AB, consisting of a cast-iron plate of uniform thickness b and length L, is to support the distributed load $w(x)$ shown. (a) Knowing that the beam is to be of constant strength, express h in terms of x, L, and h_0. (b) Determine the smallest value of h_0 if L = 750 mm, b = 30 mm, w_0 = 300 kN/m, and σ_{all} = 200 MPa.

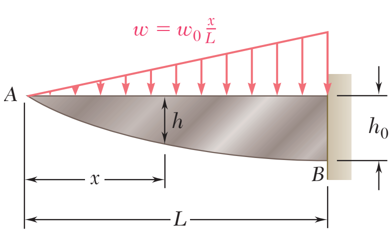


Fig. P5.130

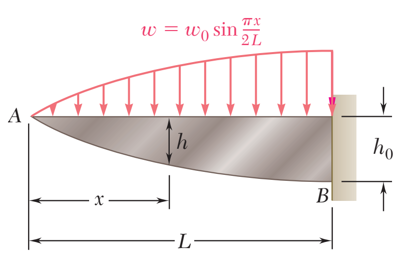


Fig. *P5.131*

5.132 and 5.133 A preliminary design on the use of a simply supported prismatic timber beam indicated that a beam with a rectangular cross section 50 mm wide and 200 mm deep would be required to safely support the load shown in part *a* of the figure. It was then decided to replace that beam with a built-up beam obtained by gluing together, as shown in part *b* of the figure, four pieces of the same timber as the original beam and of 50 × 50-mm cross section. Determine the length l of the two outer pieces of timber that will yield the same factor of safety as the original design.

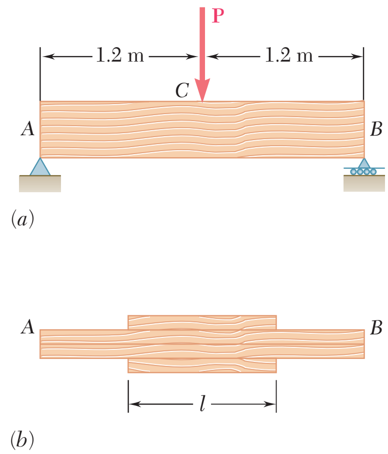


Fig. P5.132

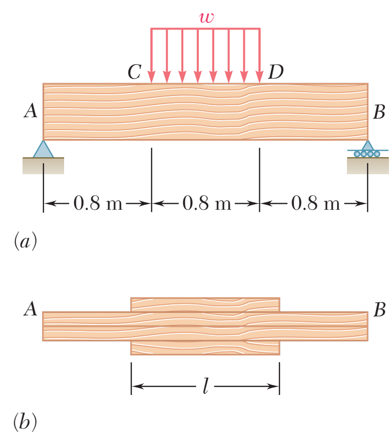


Fig. P5.133

5.134 and 5.135 A preliminary design on the use of a cantilever prismatic timber beam indicated that a beam with a rectangular cross section 2 in. wide and 10 in. deep would be required to safely support the load shown in part *a* of the figure. It was then decided to replace that beam with a built-up beam obtained by gluing together, as shown in part *b* of the figure, five pieces of the same timber as the original beam and of 2 × 2-in. cross section. Determine the respective lengths l_1 and l_2 of the two inner and outer pieces of timber that will yield the same factor of safety as the original design.

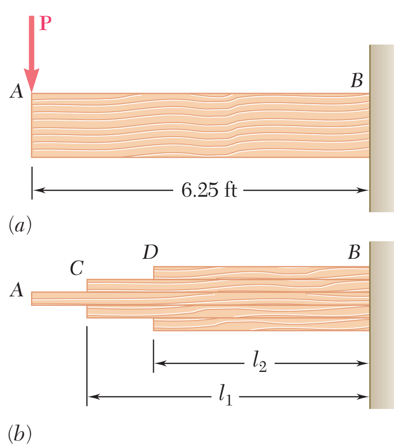


Fig. P5.134

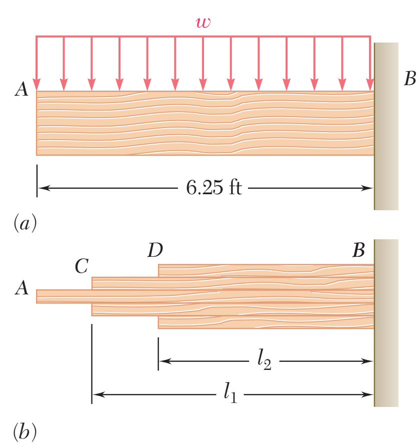


Fig. *P5.135*

5.136 and 5.137 A machine element of cast aluminum and in the shape of a solid of revolution of variable diameter d is being designed to support the load shown. Knowing that the machine element is to be of constant strength, express d in terms of x, L, and d_0.

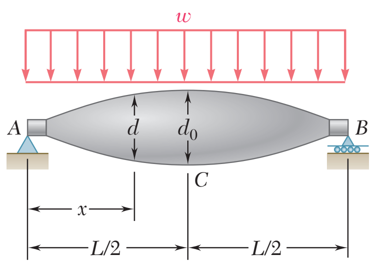


Fig. P5.136

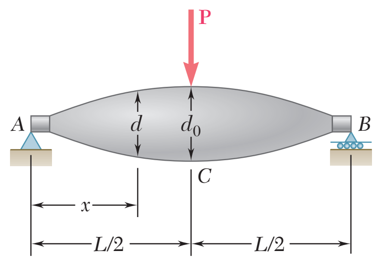


Fig. P5.137

5.138 A cantilever beam AB consisting of a steel plate of uniform depth h and variable width b is to support the distributed load w along its centerline AB. (*a*) Knowing that the beam is to be of constant strength, express b in terms of x, L, and b_0. (*b*) Determine the maximum allowable value of w if $L = 15$ in., $b_0 = 8$ in., $h = 0.75$ in., and $\sigma_{all} = 24$ ksi.

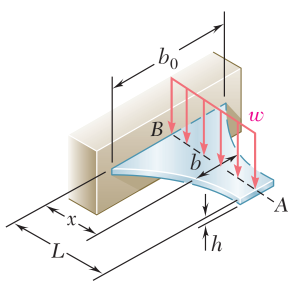


Fig. P5.138

5.139 A cantilever beam AB consisting of a steel plate of uniform depth h and variable width b is to support the concentrated load $\mathbf{P}$ at point A. (*a*) Knowing that the beam is to be of constant strength, express b in terms of x, L, and b_0. (*b*) Determine the smallest allowable value of h if $L = 300$ mm, $b_0 = 375$ mm, $P = 14.4$ kN, and $\sigma_{all} = 160$ MPa.

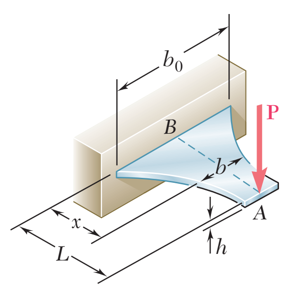


Fig. P5.139

5.140 Assuming that the length and width of the cover plates used with the beam of Sample Prob. 5.12 are, respectively, $l = 4$ m and $b = 285$ mm, and recalling that the thickness of each plate is 16 mm, determine the maximum normal stress on a transverse section (*a*) through the center of the beam, (*b*) just to the left of D.

5.141 Knowing that $\sigma_{all} = 150$ MPa, determine the largest concentrated load **P** that can be applied at end E of the beam shown.

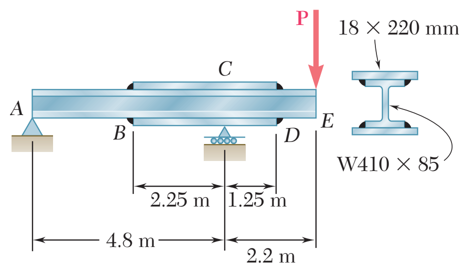


Fig. P5.141

5.142 Two cover plates, each $\frac{5}{8}$ in. thick, are welded to a W30 × 99 beam as shown. Knowing that $l = 9$ ft and $b = 12$ in., determine the maximum normal stress on a transverse section (*a*) through the center of the beam, (*b*) just to the left of D.

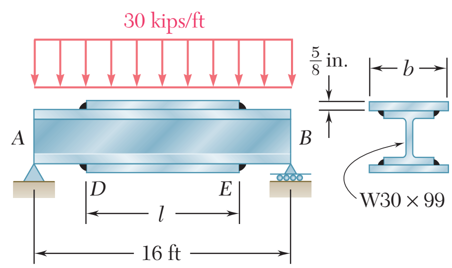


Fig. *P5.142* and P5.143

5.143 Two cover plates, each $\frac{5}{8}$ in. thick, are welded to a W30 × 99 beam as shown. Knowing that $\sigma_{all} = 22$ ksi for both the beam and the plates, determine the required value of (*a*) the length of the plates, (*b*) the width of the plates.

5.144 Two cover plates, each 7.5 mm thick, are welded to a W460 × 74 beam as shown. Knowing that $l = 5$ m and $b = 200$ mm, determine the maximum normal stress on a transverse section (*a*) through the center of the beam, (*b*) just to the left of D.

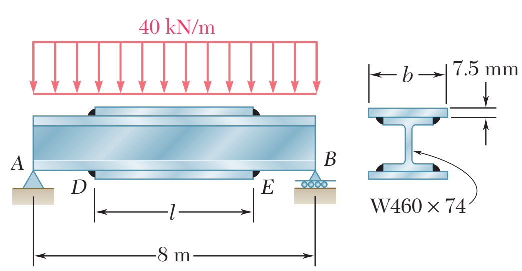


Fig. P5.144 and P5.145

5.145 Two cover plates, each 7.5 mm thick, are welded to a W460 × 74 beam as shown. Knowing that $\sigma_{all} = 150$ MPa for both the beam and the plates, determine the required value of (*a*) the length of the plates, (*b*) the width of the plates.

5.146 Two cover plates, each $\frac{1}{2}$ in. thick, are welded to a W27 × 84 beam as shown. Knowing that l = 10 ft and b = 10.5 in., determine the maximum normal stress on a transverse section (*a*) through the center of the beam, (*b*) just to the left of *D*.

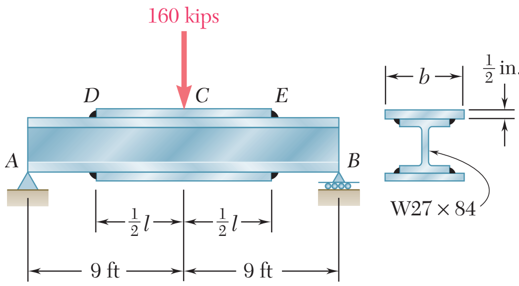


Fig. P5.146 and *P5.147*

5.147 Two cover plates, each $\frac{1}{2}$ in. thick, are welded to a W27 × 84 beam as shown. Knowing that σ_{all} = 24 ksi for both the beam and the plates, determine the required value of (*a*) the length of the plates, (*b*) the width of the plates.

5.148 For the tapered beam shown, determine (*a*) the transverse section in which the maximum normal stress occurs, (*b*) the largest distributed load w that can be applied, knowing that σ_{all} = 140 MPa.

5.149 For the tapered beam shown, knowing that w = 160 kN/m, determine (*a*) the transverse section in which the maximum normal stress occurs, (*b*) the corresponding value of the normal stress.

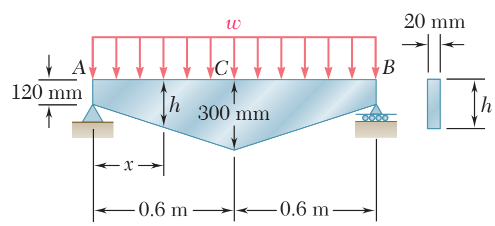


Fig. *P5.148* and P5.149

5.150 For the tapered beam shown, determine (*a*) the transverse section in which the maximum normal stress occurs, (*b*) the largest distributed load w that can be applied, knowing that σ_{all} = 24 ksi.

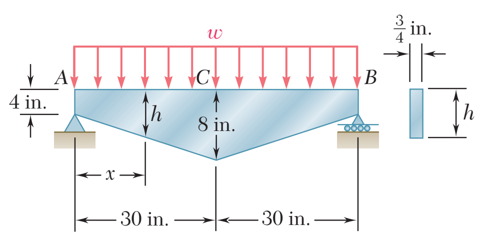


Fig. P5.150

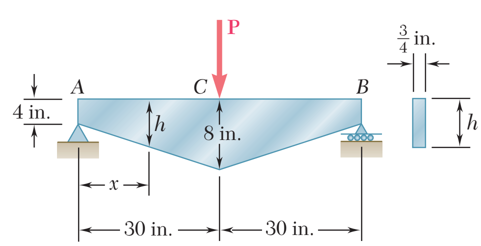


Fig. P5.151

5.151 For the tapered beam shown, determine (*a*) the transverse section in which the maximum normal stress occurs, (*b*) the largest concentrated load **P** that can be applied, knowing that σ_{all} = 24 ksi.

REVIEW AND SUMMARY

Considerations for the design of prismatic beams

This chapter was devoted to the analysis and design of beams under transverse loadings. Such loadings can consist of concentrated loads or distributed loads and the beams themselves are classified according to the way they are supported (Fig. 5.22). Only *statically determinate* beams were considered in this chapter, where all support reactions can be determined by statics. The analysis of statically indeterminate beams is postponed until Chap. 9.

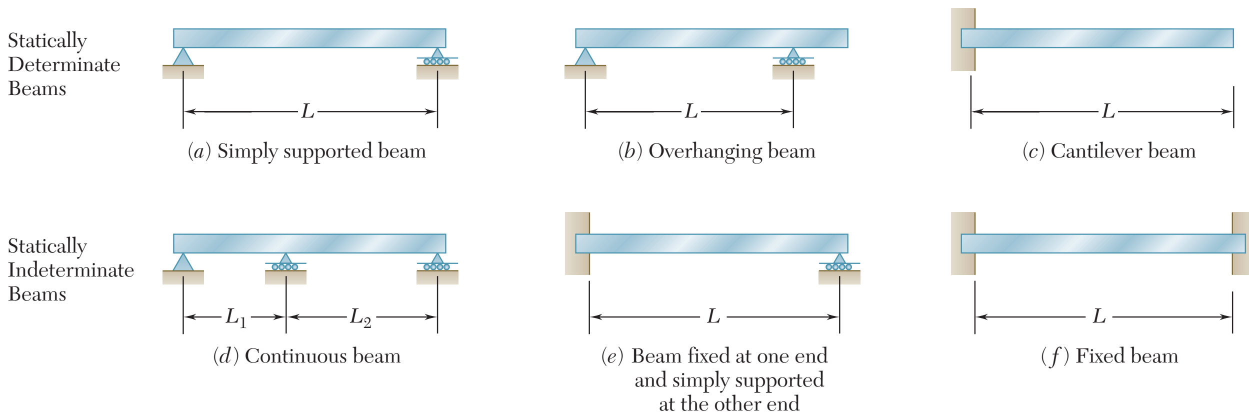


Fig. 5.22

Normal stresses due to bending

While transverse loadings cause both bending and shear in a beam, the normal stresses caused by bending are the dominant criterion in the design of a beam for strength [Sec. 5.1]. Therefore, this chapter dealt only with the determination of the normal stresses in a beam, the effect of shearing stresses being examined in the next one.

We recalled from Sec. 4.4 the flexure formula for the determination of the maximum value σ_m of the normal stress in a given section of the beam,

$$\sigma_m = \frac{|M|c}{I} \tag{5.1}$$

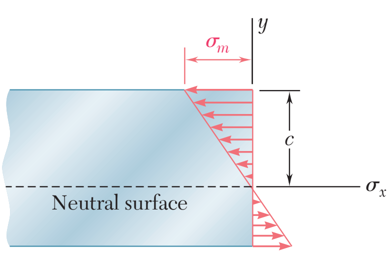


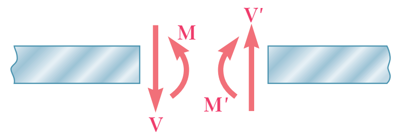
Fig. 5.23

where I is the moment of inertia of the cross section with respect to a centroidal axis perpendicular to the plane of the bending couple $\mathbf{M}$ and c is the maximum distance from the neutral surface (Fig. 5.23). We also recalled from Sec. 4.4 that, introducing the elastic section modulus $S = I/c$ of the beam, the maximum value σ_m of the normal stress in the section can be expressed as

$$\sigma_m = \frac{|M|}{S} \tag{5.3}$$

Shear and bending-moment diagrams

It follows from Eq. (5.1) that the maximum normal stress occurs in the section where $|M|$ is largest, at the point farthest from the neutral axis. The determination of the maximum value of $|M|$ and of the critical section of the beam in which it occurs is greatly simplified if we draw a *shear diagram* and a *bending-moment diagram*. These diagrams represent, respectively, the variation of the shear and of the bending moment along the beam and were obtained by determining the values of V and M at selected points of the beam [Sec. 5.2]. These values were found by passing a section through the point where they were to be determined and drawing the free-body diagram of either of the portions of beam obtained in this fashion. To avoid any confusion regarding the sense of the shearing force **V** and of the bending couple **M** (which act in opposite sense on the two portions of the beam), we followed the sign convention adopted earlier in the text as illustrated in Fig. 5.24 [Examples 5.01 and 5.02, Sample Probs. 5.1 and 5.2].



(*a*) Internal forces
(positive shear and positive bending moment)

Fig. 5.24

Relations among load, shear, and bending moment

The construction of the shear and bending-moment diagrams is facilitated if the following relations are taken into account [Sec. 5.3]. Denoting by w the distributed load per unit length (assumed positive if directed downward), we wrote

$$\frac{dV}{dx} = -w \qquad \frac{dM}{dx} = V \qquad (5.5,\ 5.7)$$

or, in integrated form,

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \qquad (5.6')$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \qquad (5.8')$$

Equation (5.6′) makes it possible to draw the shear diagram of a beam from the curve representing the distributed load on that beam and the value of V at one end of the beam. Similarly, Eq. (5.8′) makes it possible to draw the bending-moment diagram from the shear diagram and the value of M at one end of the beam. However, concentrated loads introduce discontinuities in the shear diagram and concentrated couples in the bending-moment diagram, none of which is accounted for in these equations [Sample Probs. 5.3 and 5.6]. Finally, we noted from Eq. (5.7) that the points of the beam where the bending moment is maximum or minimum are also the points where the shear is zero [Sample Prob. 5.4].

Design of prismatic beams

A proper procedure for the design of a prismatic beam was described in Sec. 5.4 and is summarized here:

Having determined σ_{all} for the material used and assuming that the design of the beam is controlled by the maximum normal stress in the beam, compute the minimum allowable value of the section modulus:

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} \qquad (5.9)$$

For a timber beam of rectangular cross section, $S = \frac{1}{6}bh^2$, where b is the width of the beam and h its depth. The dimensions of the section, therefore, must be selected so that $\frac{1}{6}bh^2 \geq S_{min}$.

For a rolled-steel beam, consult the appropriate table in Appendix C. Of the available beam sections, consider only those with a section modulus $S \geq S_{min}$ and select from this group the section with the smallest weight per unit length. This is the most economical of the sections for which $S \geq S_{min}$.

Singularity functions

In Sec. 5.5, we discussed an alternative method for the determination of the maximum values of the shear and bending moment based on the use of the *singularity functions* $\langle x - a \rangle^n$. By definition, and for $n \geq 0$, we had

$$\langle x - a \rangle^n = \begin{cases} (x-a)^n & \text{when } x \geq a \\ 0 & \text{when } x < a \end{cases} \tag{5.14}$$

Step function

We noted that whenever the quantity between brackets is positive or zero, the brackets should be replaced by ordinary parentheses, and whenever that quantity is negative, the bracket itself is equal to zero. We also noted that singularity functions can be integrated and differentiated as ordinary binomials. Finally, we observed that the singularity function corresponding to $n = 0$ is discontinuous at $x = a$ (Fig. 5.25). This function is called the *step function*. We wrote

$$\langle x - a \rangle^0 = \begin{cases} 1 & \text{when } x \geq a \\ 0 & \text{when } x < a \end{cases} \tag{5.15}$$

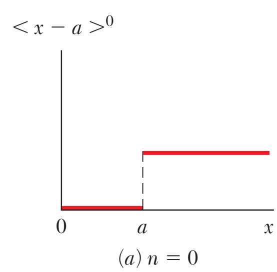


Fig. 5.25

Using singularity functions to express shear and bending moment

The use of singularity functions makes it possible to represent the shear or the bending moment in a beam by a single expression, valid at any point of the beam. For example, the contribution to the shear of the concentrated load **P** applied at the midpoint C of a simply supported beam (Fig. 5.26) can be represented by $-P\langle x - \frac{1}{2}L \rangle^0$, since

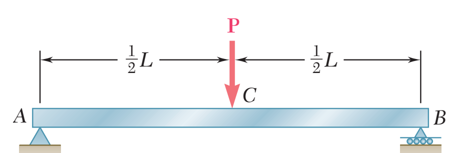


Fig. 5.26

this expression is equal to zero to the left of C, and to $-P$ to the right of C. Adding the contribution of the reaction $R_A = \frac{1}{2}P$ at A, we express the shear at any point of the beam as

$$V(x) = \tfrac{1}{2}P - P\langle x - \tfrac{1}{2}L\rangle^0$$

The bending moment is obtained by integrating this expression:

$$M(x) = \tfrac{1}{2}Px - P\langle x - \tfrac{1}{2}L\rangle^1$$

Equivalent open-ended loadings

The singularity functions representing, respectively, the load, shear, and bending moment corresponding to various basic loadings were given in Fig. 5.18 on page 353. We noted that a distributed loading that does not extend to the right end of the beam, or which is discontinuous, should be replaced by an equivalent combination of open-ended loadings. For instance, a uniformly distributed load extending from $x = a$ to $x = b$ (Fig. 5.27) should be expressed as

$$w(x) = w_0\langle x - a\rangle^0 - w_0\langle x - b\rangle^0$$

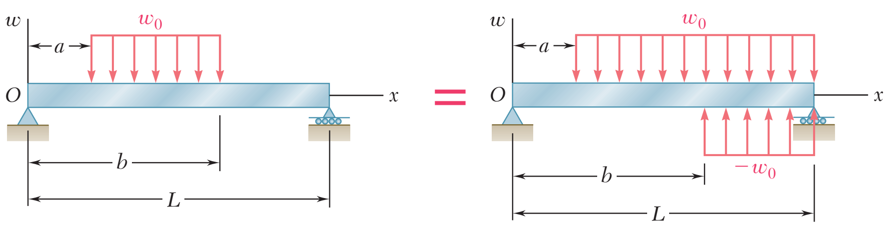


Fig. 5.27

The contribution of this load to the shear and to the bending moment can be obtained through two successive integrations. Care should be taken, however, to also include in the expression for $V(x)$ the contribution of concentrated loads and reactions, and to include in the expression for $M(x)$ the contribution of concentrated couples [Examples 5.05 and 5.06, Sample Probs. 5.9 and 5.10]. We also observed that singularity functions are particularly well suited to the use of computers.

Nonprismatic beams

Beams of constant strength

We were concerned so far only with prismatic beams, i.e., beams of uniform cross section. Considering in Sec. 5.6 the design of nonprismatic beams, i.e., beams of variable cross section, we saw that by selecting the shape and size of the cross section so that its elastic section modulus $S = I/c$ varied along the beam in the same way as the bending moment M, we were able to design beams for which σ_m at each section was equal to σ_{all}. Such beams, called *beams of constant strength,* clearly provide a more effective use of the material than prismatic beams. Their section modulus at any section along the beam was defined by the relation

$$S = \frac{M}{\sigma_{all}} \tag{5.18}$$

REVIEW PROBLEMS

5.152 Draw the shear and bending-moment diagrams for the beam and loading shown, and determine the maximum absolute value (*a*) of the shear, (*b*) of the bending moment.

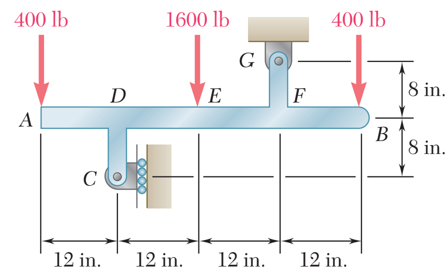


Fig. P5.152

5.153 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

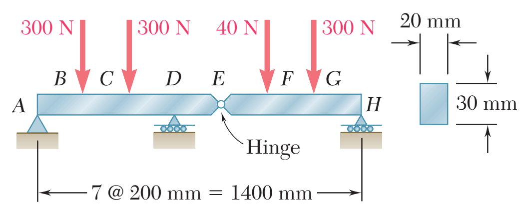


Fig. P5.153

5.154 Determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding maximum normal stress due to bending. (See hint of Prob. 5.27.)

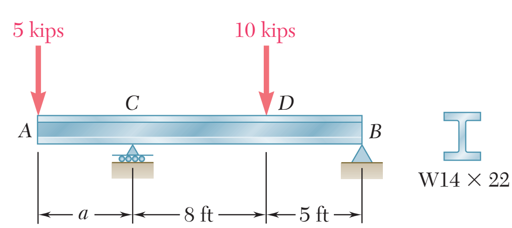


Fig. *P5.154*

5.155 Determine (*a*) the equations of the shear and bending-moment curves for the beam and loading shown, (*b*) the maximum absolute value of the bending moment in the beam.

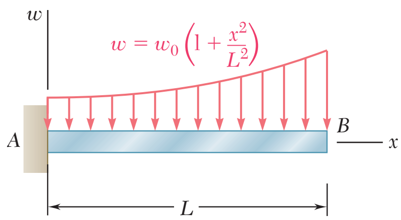


Fig. *P5.155*

5.156 Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

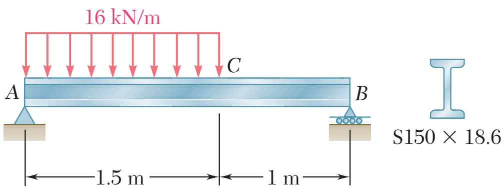


Fig. P5.156

5.157 Knowing that beam *AB* is in equilibrium under the loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.

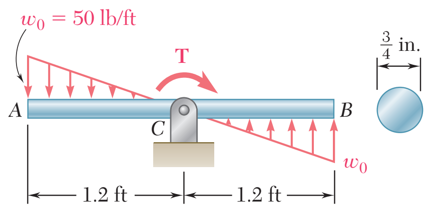


Fig. P5.157

5.158 For the beam and loading shown, design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 1750 psi.

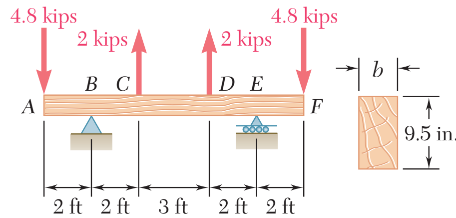


Fig. P5.158

5.159 Knowing that the allowable stress for the steel used is 160 MPa, select the most economical wide-flange beam to support the loading shown.

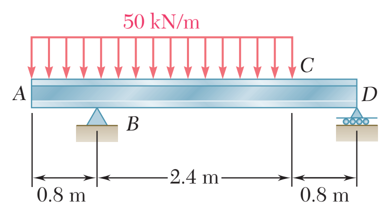


Fig. P5.159

5.160 Determine the largest permissible value of **P** for the beam and loading shown, knowing that the allowable normal stress is +8 ksi in tension and −18 ksi in compression.

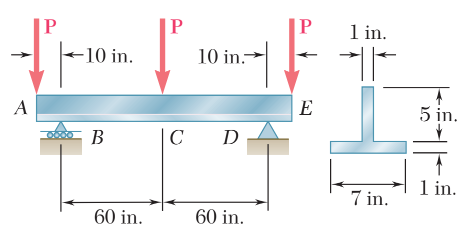


Fig. P5.160

5.161 (*a*) Using singularity functions, find the magnitude and location of the maximum bending moment for the beam and loading shown. (*b*) Determine the maximum normal stress due to bending.

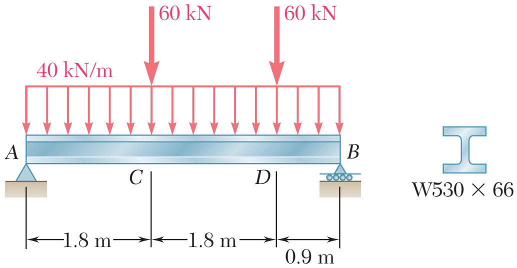


Fig. *P5.161*

5.162 The beam AB, consisting of an aluminum plate of uniform thickness b and length L, is to support the load shown. (*a*) Knowing that the beam is to be of constant strength, express h in terms of x, L, and h_0 for portion AC of the beam. (*b*) Determine the maximum allowable load if $L = 800$ mm, $h_0 = 200$ mm, $b = 25$ mm, and $\sigma_{all} = 72$ MPa.

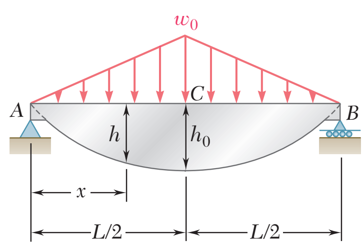


Fig. *P5.162*

5.163 A transverse force **P** is applied as shown at end A of the conical taper AB. Denoting by d_0 the diameter of the taper at A, show that the maximum normal stress occurs at point H, which is contained in a transverse section of diameter $d = 1.5\ d_0$.

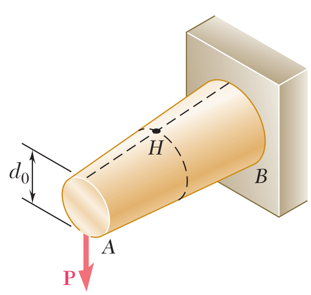


Fig. P5.163

COMPUTER PROBLEMS

The following problems are designed to be solved with a computer.

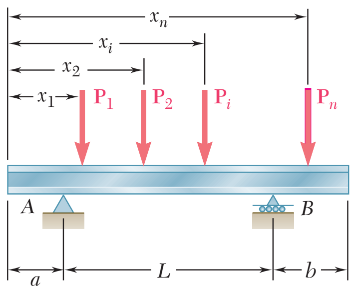


Fig. P5.C1

5.C1 Several concentrated loads P_i ($i = 1, 2, \ldots, n$) can be applied to a beam as shown. Write a computer program that can be used to calculate the shear, bending moment, and normal stress at any point of the beam for a given loading of the beam and a given value of its section modulus. Use this program to solve Probs. 5.18, 5.21, and 5.25. (*Hint:* Maximum values will occur at a support or under a load.)

5.C2 A timber beam is to be designed to support a distributed load and up to two concentrated loads as shown. One of the dimensions of its uniform rectangular cross section has been specified and the other is to be determined so that the maximum normal stress in the beam will not exceed a given allowable value σ_{all}. Write a computer program that can be used to calculate at given intervals ΔL the shear, the bending moment, and the smallest acceptable value of the unknown dimension. Apply this program to solve the following problems, using the intervals ΔL indicated: (*a*) Prob. 5.65 ($\Delta L = 0.1$ m), (*b*) Prob. 5.69 ($\Delta L = 0.3$ m), (*c*) Prob. 5.70 ($\Delta L = 0.2$ m).

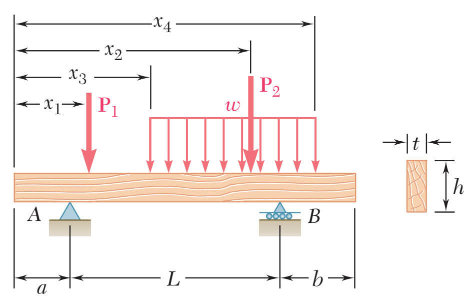


Fig. P5.C2

5.C3 Two cover plates, each of thickness t, are to be welded to a wide-flange beam of length L that is to support a uniformly distributed load w. Denoting by σ_{all} the allowable normal stress in the beam and in the plates, by d the depth of the beam, and by I_b and S_b, respectively, the moment of inertia and the section modulus of the cross section of the unreinforced beam about a horizontal centroidal axis, write a computer program that can be used to calculate the required value of (*a*) the length a of the plates, (*b*) the width b of the plates. Use this program to solve Prob. 5.145.

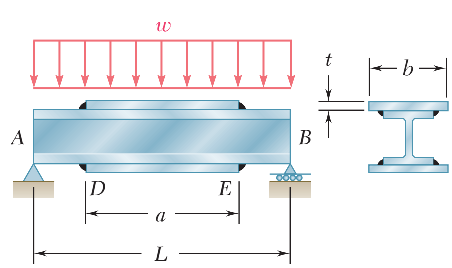


Fig. P5.C3

5.C4 Two 25-kip loads are maintained 6 ft apart as they are moved slowly across the 18-ft beam *AB*. Write a computer program and use it to calculate the bending moment under each load and at the midpoint *C* of the beam for values of *x* from 0 to 24 ft at intervals $\Delta x = 1.5$ ft.

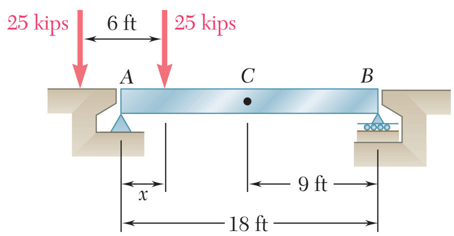


Fig. P5.C4

5.C5 Write a computer program that can be used to plot the shear and bending-moment diagrams for the beam and loading shown. Apply this program with a plotting interval $\Delta L = 0.2$ ft to the beam and loading of (*a*) Prob. 5.72, (*b*) Prob. 5.115.

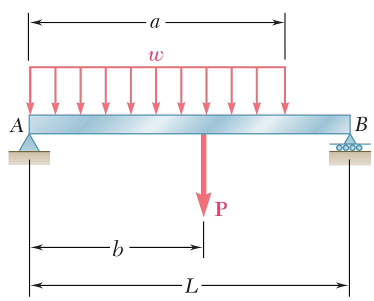


Fig. P5.C5

5.C6 Write a computer program that can be used to plot the shear and bending-moment diagrams for the beam and loading shown. Apply this program with a plotting interval $\Delta L = 0.025$ m to the beam and loading of Prob. 5.112.

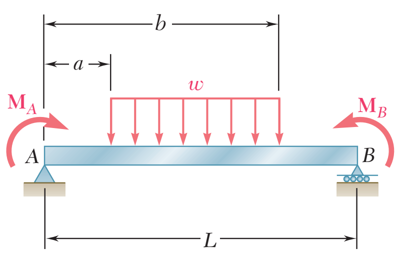
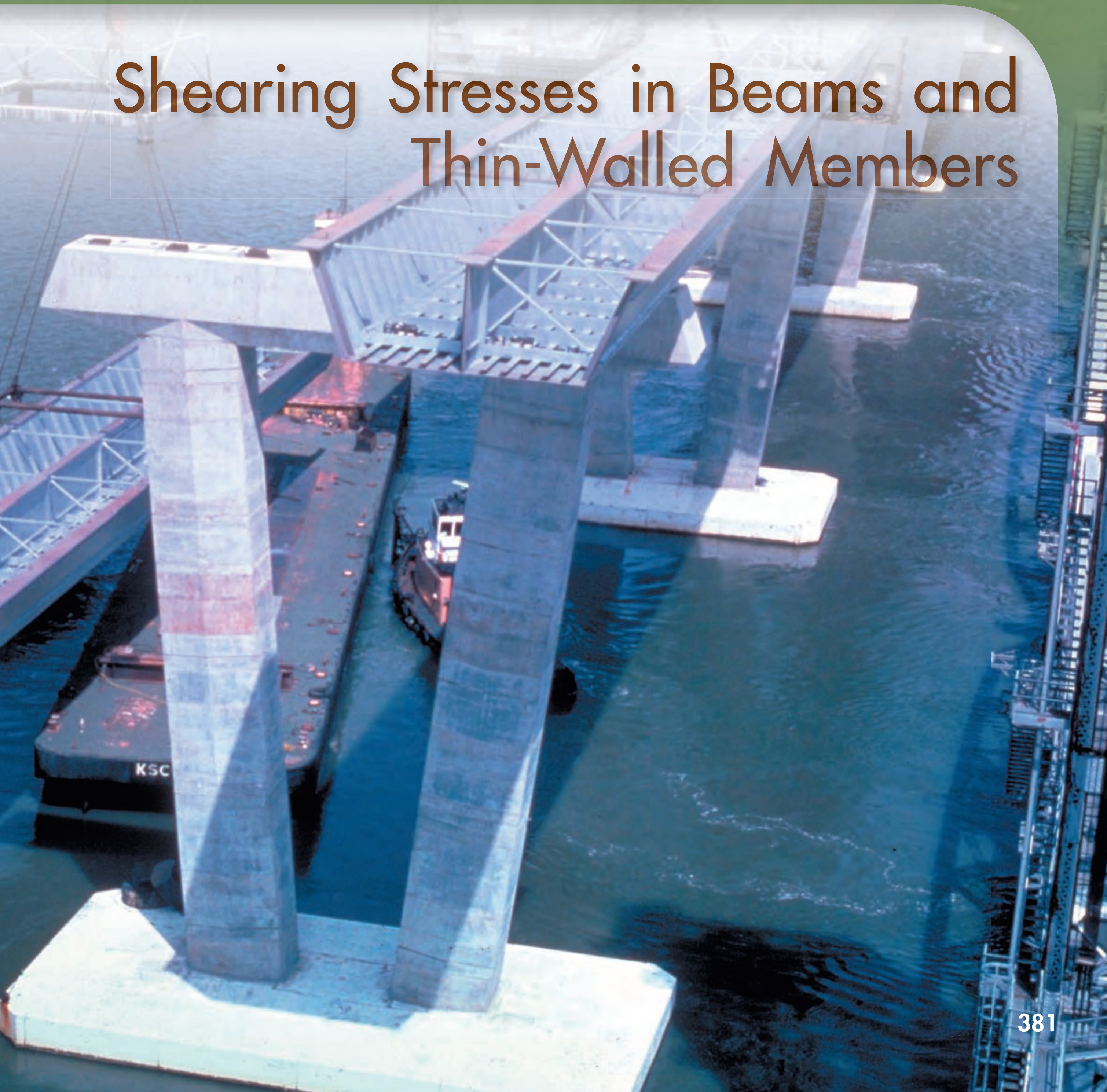


Fig. P5.C6

A reinforced concrete deck will be attached to each of the steel sections shown to form a composite box girder bridge. In this chapter the shearing stresses will be determined in various types of beams and girders.

CHAPTER 6

Shearing Stresses in Beams and Thin-Walled Members



Chapter 6 Shearing Stresses in Beams and Thin-Walled Members

6.1 Introduction
6.2 Shear on the Horizontal Face of a Beam Element
6.3 Determination of the Shearing Stresses in a Beam
6.4 Shearing Stresses τ_{xy} in Common Types of Beams
*6.5 Further Discussion of the Distribution of Stresses in a Narrow Rectangular Beam
6.6 Longitudinal Shear on a Beam Element of Arbitrary Shape
6.7 Shearing Stresses in Thin-Walled Members
*6.8 Plastic Deformations
*6.9 Unsymmetric Loading of Thin-Walled Members; Shear Center

6.1 INTRODUCTION

You saw in Sec. 5.1 that a transverse loading applied to a beam will result in normal and shearing stresses in any given transverse section of the beam. The normal stresses are created by the bending couple **M** in that section and the shearing stresses by the shear **V.** Since the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam, our analysis was limited in Chap. 5 to the determination of the normal stresses. Shearing stresses, however, can be important, particularly in the design of short, stubby beams, and their analysis will be the subject of the first part of this chapter.

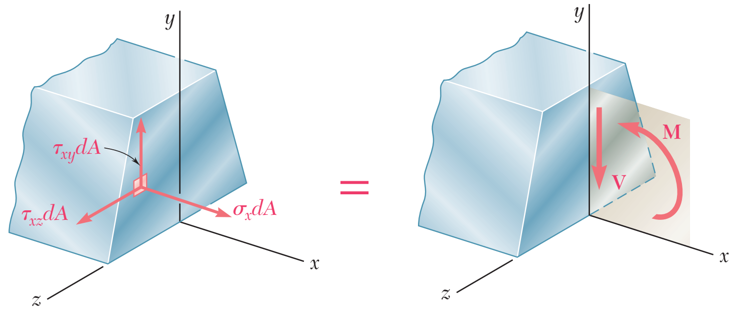


Fig. 6.1 Beam cross section.

Figure 6.1 expresses graphically that the elementary normal and shearing forces exerted on a given transverse section of a prismatic beam with a vertical plane of symmetry are equivalent to the bending couple **M** and the shearing force **V.** Six equations can be written to express that fact. Three of these equations involve only the normal forces $\sigma_x\,dA$ and have already been discussed in Sec. 4.2; they are Eqs. (4.1), (4.2), and (4.3), which express that the sum of the normal forces is zero and that the sums of their moments about the y and z axes are equal to zero and M, respectively. Three more equations involving the shearing forces $\tau_{xy}\,dA$ and $\tau_{xz}\,dA$ can now be written. One of them expresses that the sum of the moments of the shearing forces about the x axis is zero and can be dismissed as trivial in view of the symmetry of the beam with respect to the xy plane. The other two involve the y and z components of the elementary forces and are

$$y \text{ components:} \qquad \int \tau_{xy}\,dA = -V \tag{6.1}$$

$$z \text{ components:} \qquad \int \tau_{xz}\,dA = 0 \tag{6.2}$$

The first of these equations shows that vertical shearing stresses must exist in a transverse section of a beam under transverse loading. The second equation indicates that the average horizontal shearing stress in any section is zero. However, this does not mean that the shearing stress τ_{xz} is zero everywhere.

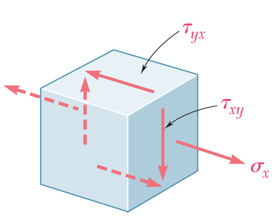


Fig. 6.2 Element from beam.

Let us now consider a small cubic element located in the vertical plane of symmetry of the beam (where we know that τ_{xz} must be zero) and examine the stresses exerted on its faces (Fig. 6.2). As we have just seen, a normal stress σ_x and a shearing stress τ_{xy} are exerted on each of the two faces perpendicular to the x axis. But we know from Chap. 1 that, when shearing stresses τ_{xy} are exerted on the vertical faces of an element, equal stresses must be exerted on the horizontal faces of the same element. We thus conclude that longitudinal shearing stresses must exist in any member subjected to a transverse loading. This can be verified by considering a cantilever beam made of separate planks clamped together at one end (Fig. 6.3*a*). When a transverse load **P** is applied to the free end of this composite beam, the planks are observed to slide with respect to each other (Fig. 6.3*b*). In contrast, if a couple **M** is applied to the free end of the same composite beam (Fig. 6.3*c*), the various planks will bend into concentric arcs of circle and will not slide with respect to each other, thus verifying the fact that shear does not occur in a beam subjected to pure bending (cf. Sec. 4.3).

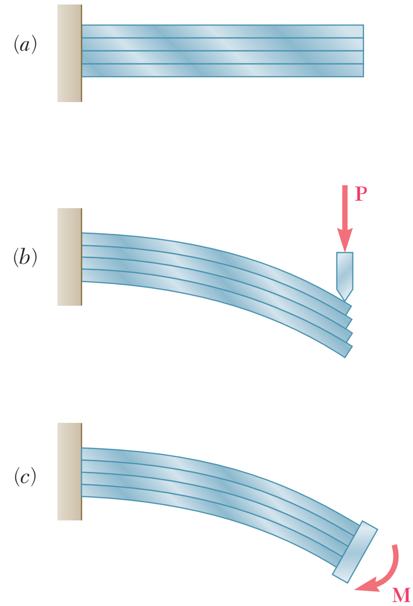


Fig. 6.3 Beam made from planks.

While sliding does not actually take place when a transverse load **P** is applied to a beam made of a homogeneous and cohesive material such as steel, the tendency to slide does exist, showing that stresses occur on horizontal longitudinal planes as well as on vertical transverse planes. In the case of timber beams, whose resistance to shear is weaker between fibers, failure due to shear will occur along a longitudinal plane rather than a transverse plane (Photo 6.1).

In Sec. 6.2, a beam element of length Δx bounded by two transverse planes and a horizontal one will be considered and the shearing force $\Delta\mathbf{H}$ exerted on its horizontal face will be determined, as well as the shear per unit length, q, also known as *shear flow*. A formula for the shearing stress in a beam with a vertical plane of symmetry will be derived in Sec. 6.3 and used in Sec. 6.4 to determine the shearing stresses in common types of beams. The distribution of stresses in a narrow rectangular beam will be further discussed in Sec. 6.5.

The derivation given in Sec. 6.2 will be extended in Sec. 6.6 to cover the case of a beam element bounded by two transverse planes and a curved surface. This will allow us in Sec. 6.7 to determine the shearing stresses at any point of a symmetric thin-walled member, such as the flanges of wide-flange beams and box beams. The effect of plastic deformations on the magnitude and distribution of shearing stresses will be discussed in Sec. 6.8.

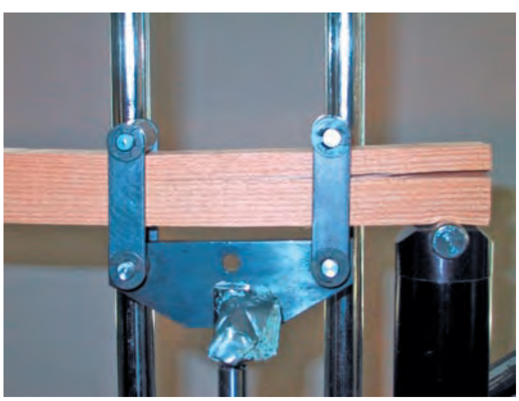


Photo 6.1 Longitudinal shear failure in timber beam.

In the last section of the chapter (Sec. 6.9), the unsymmetric loading of thin-walled members will be considered and the concept of *shear center* will be introduced. You will then learn to determine the distribution of shearing stresses in such members.

6.2 SHEAR ON THE HORIZONTAL FACE OF A BEAM ELEMENT

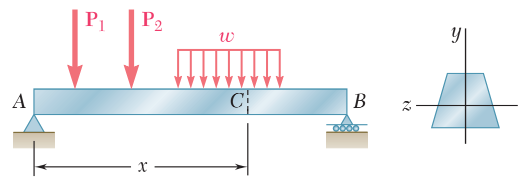


Fig. 6.4 Beam example.

Consider a prismatic beam AB with a vertical plane of symmetry that supports various concentrated and distributed loads (Fig. 6.4). At a distance x from end A we detach from the beam an element $CDD'C'$ of length Δx extending across the width of the beam from the upper surface of the beam to a horizontal plane located at a distance y_1 from the neutral axis (Fig. 6.5). The forces exerted on this element

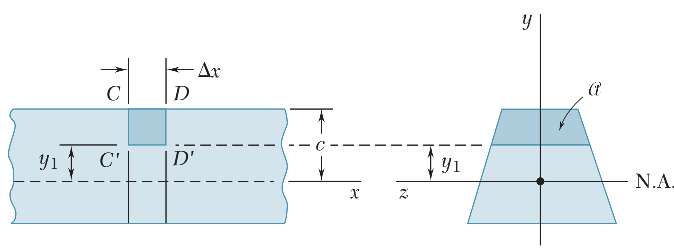


Fig. 6.5 Short segment of beam example.

consist of vertical shearing forces $\mathbf{V}'_C$ and $\mathbf{V}'_D$, a horizontal shearing force $\Delta\mathbf{H}$ exerted on the lower face of the element, elementary horizontal normal forces $\sigma_C\, dA$ and $\sigma_D\, dA$, and possibly a load $w\,\Delta x$ (Fig. 6.6). We write the equilibrium equation

$$\xrightarrow{+}\Sigma F_x = 0: \qquad \Delta H + \int_{\mathcal{A}} (\sigma_C - \sigma_D)\, dA = 0$$

where the integral extends over the shaded area $\mathcal{A}$ of the section located above the line $y = y_1$. Solving this equation for ΔH and using Eq. (5.2) of Sec. 5.1, $\sigma = My/I$, to express the normal stresses in terms of the bending moments at C and D, we have

$$\Delta H = \frac{M_D - M_C}{I} \int_{\mathcal{A}} y\, dA \tag{6.3}$$

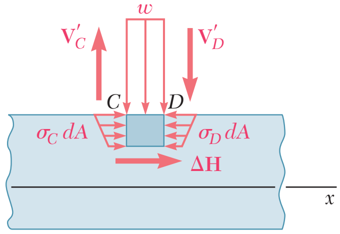


Fig. 6.6 Forces exerted on element.

The integral in (6.3) represents the *first moment* with respect to the neutral axis of the portion $\mathcal{A}$ of the cross section of the beam that is located above the line $y = y_1$ and will be denoted by Q. On the other hand, recalling Eq. (5.7) of Sec. 5.3, we can express the increment $M_D - M_C$ of the bending moment as

$$M_D - M_C = \Delta M = (dM/dx)\,\Delta x = V\,\Delta x$$

Substituting into (6.3), we obtain the following expression for the horizontal shear exerted on the beam element

$$\Delta H = \frac{VQ}{I}\Delta x \tag{6.4}$$

The same result would have been obtained if we had used as a free body the lower element $C'D'D''C''$, rather than the upper element $CDD'C'$ (Fig. 6.7), since the shearing forces $\mathbf{\Delta H}$ and $\mathbf{\Delta H'}$

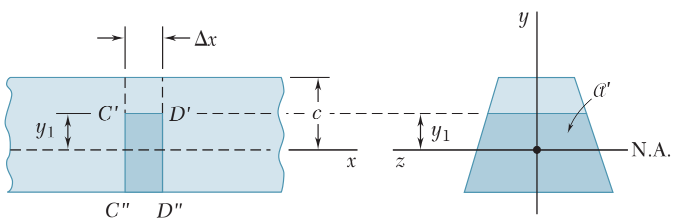


Fig. 6.7 Short segment of beam example.

exerted by the two elements on each other are equal and opposite. This leads us to observe that the first moment Q of the portion $\mathcal{A}'$ of the cross section located below the line $y = y_1$ (Fig. 6.7) is equal in magnitude and opposite in sign to the first moment of the portion $\mathcal{A}$ located above that line (Fig. 6.5). Indeed, the sum of these two moments is equal to the moment of the area of the entire cross section with respect to its centroidal axis and, thus, must be zero. This property can sometimes be used to simplify the computation of Q. We also note that Q is maximum for $y_1 = 0$, since the elements of the cross section located above the neutral axis contribute positively to the integral in (6.3) that defines Q, while the elements located below that axis contribute negatively.

The *horizontal shear per unit length,* which will be denoted by the letter q, is obtained by dividing both members of Eq. (6.4) by Δx:

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} \tag{6.5}$$

We recall that Q is the first moment with respect to the neutral axis of the portion of the cross section located either above or below the point at which q is being computed, and that I is the centroidal moment of inertia of the *entire* cross-sectional area. For a reason that will become apparent later (Sec. 6.7), the horizontal shear per unit length q is also referred to as the *shear flow.*

EXAMPLE 6.01

A beam is made of three planks, 20 by 100 mm in cross section, nailed together (Fig. 6.8). Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is $V = 500$ N, determine the shearing force in each nail.

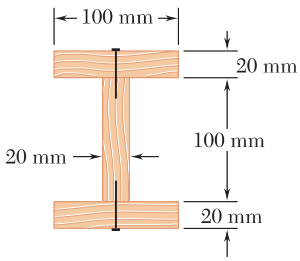


Fig. 6.8

We first determine the horizontal force per unit length, q, exerted on the lower face of the upper plank. We use Eq. (6.5), where Q represents the first moment with respect to the neutral axis of the shaded area A shown in Fig. 6.9*a*, and where I is the moment of inertia about the same axis of the entire cross-sectional area (Fig. 6.9*b*). Recalling that the first moment of an area with respect to a given axis is equal to the product of the area and of the distance from its centroid to the axis,† we have

$$\begin{aligned} Q &= A\bar{y} = (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m}) \\ &= 120 \times 10^{-6} \text{ m}^3 \\ I &= \tfrac{1}{12}(0.020 \text{ m})(0.100 \text{ m})^3 \\ &\quad + 2[\tfrac{1}{12}(0.100 \text{ m})(0.020 \text{ m})^3 \\ &\quad + (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m})^2] \\ &= 1.667 \times 10^{-6} + 2(0.0667 + 7.2)10^{-6} \\ &= 16.20 \times 10^{-6} \text{ m}^4 \end{aligned}$$

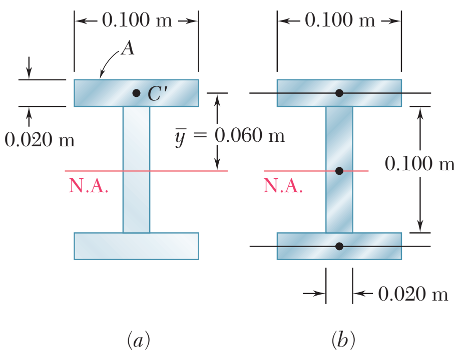


Fig. 6.9

Substituting into Eq. (6.5), we write

$$q = \frac{VQ}{I} = \frac{(500 \text{ N})(120 \times 10^{-6} \text{ m}^3)}{16.20 \times 10^{-6} \text{ m}^4} = 3704 \text{ N/m}$$

Since the spacing between the nails is 25 mm, the shearing force in each nail is

$$F = (0.025 \text{ m})q = (0.025 \text{ m})(3704 \text{ N/m}) = 92.6 \text{ N}$$

6.3 DETERMINATION OF THE SHEARING STRESSES IN A BEAM

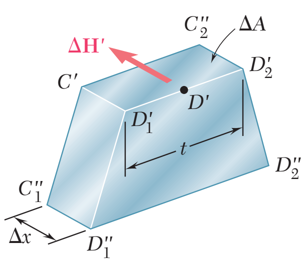


Fig. 6.10 Beam element.

Consider again a beam with a vertical plane of symmetry, subjected to various concentrated or distributed loads applied in that plane. We saw in the preceding section that if, through two vertical cuts and one horizontal cut, we detach from the beam an element of length Δx (Fig. 6.10), the magnitude ΔH of the shearing force exerted on the horizontal face of the element can be obtained from Eq. (6.4). The *average shearing stress* τ_{ave} on that face of the element is obtained by dividing ΔH by the area ΔA of the face. Observing that $\Delta A = t\,\Delta x$, where t is the width of the element at the cut, we write

$$\tau_{\text{ave}} = \frac{\Delta H}{\Delta A} = \frac{VQ}{I}\,\frac{\Delta x}{t\,\Delta x}$$

or

$$\boxed{\tau_{\text{ave}} = \frac{VQ}{It}} \qquad (6.6)$$

†See Appendix A.

We note that, since the shearing stresses τ_{xy} and τ_{yx} exerted respectively on a transverse and a horizontal plane through D' are equal, the expression obtained also represents the average value of τ_{xy} along the line $D'_1D'_2$ (Fig. 6.11).

We observe that $\tau_{yx} = 0$ on the upper and lower faces of the beam, since no forces are exerted on these faces. It follows that $\tau_{xy} = 0$ along the upper and lower edges of the transverse section (Fig. 6.12). We also note that, while Q is maximum for $y = 0$ (see Sec. 6.2), we cannot conclude that τ_{ave} will be maximum along the neutral axis, since τ_{ave} depends upon the width t of the section as well as upon Q.

As long as the width of the beam cross section remains small compared to its depth, the shearing stress varies only slightly along the line $D'_1D'_2$ (Fig. 6.11) and Eq. (6.6) can be used to compute τ_{xy} at any point along $D'_1D'_2$. Actually, τ_{xy} is larger at points D'_1 and D'_2 than at D', but the theory of elasticity shows† that, for a beam of rectangular section of width b and depth h, and as long as $b \leq h/4$, the value of the shearing stress at points C_1 and C_2 (Fig. 6.13) does not exceed by more than 0.8% the average value of the stress computed along the neutral axis.‡

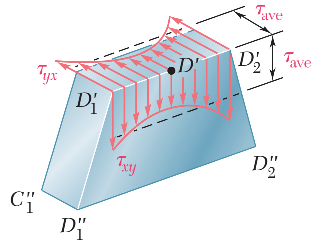


Fig. 6.11 Beam segment.

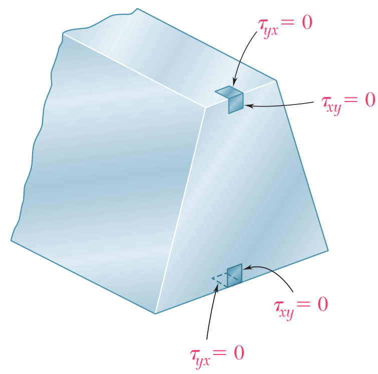


Fig. 6.12 Beam cross section.

6.4 SHEARING STRESSES τ_{xy} IN COMMON TYPES OF BEAMS

We saw in the preceding section that, for a *narrow rectangular beam*, i.e., for a beam of rectangular section of width b and depth h with $b \leq \frac{1}{4}h$, the variation of the shearing stress τ_{xy} across the width of the beam is less than 0.8% of τ_{ave}. We can, therefore, use Eq. (6.6) in practical applications to determine the shearing stress at any point of the cross section of a narrow rectangular beam and write

$$\tau_{xy} = \frac{VQ}{It} \tag{6.7}$$

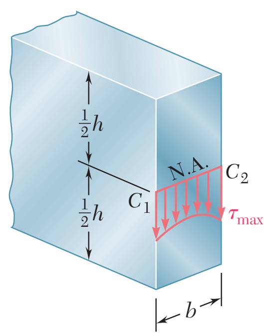


Fig. 6.13 Rectangular beam cross section.

where t is equal to the width b of the beam, and where Q is the first moment with respect to the neutral axis of the shaded area A (Fig. 6.14).

Observing that the distance from the neutral axis to the centroid C' of A is $\bar{y} = \frac{1}{2}(c + y)$, and recalling that $Q = A\bar{y}$, we write

$$Q = A\bar{y} = b(c - y)\tfrac{1}{2}(c + y) = \tfrac{1}{2}b(c^2 - y^2) \tag{6.8}$$

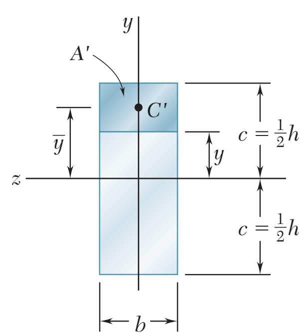


Fig. 6.14 Beam cross section.

†See S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, 3d ed., 1970, sec. 124.

‡On the other hand, for large values of b/h, the value τ_{max} of the stress at C_1 and C_2 may be many times larger then the average value τ_{ave} computed along the neutral axis, as we may see from the following table:

b/h	0.25	0.5	1	2	4	6	10	20	50
τ_{max}/τ_{ave}	1.008	1.033	1.126	1.396	1.988	2.582	3.770	6.740	15.65
τ_{min}/τ_{ave}	0.996	0.983	0.940	0.856	0.805	0.800	0.800	0.800	0.800

Recalling, on the other hand, that $I = bh^3/12 = \frac{2}{3}bc^3$, we have

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{3}{4}\frac{c^2 - y^2}{bc^3}V$$

or, noting that the cross-sectional area of the beam is $A = 2bc$,

$$\tau_{xy} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right) \qquad (6.9)$$

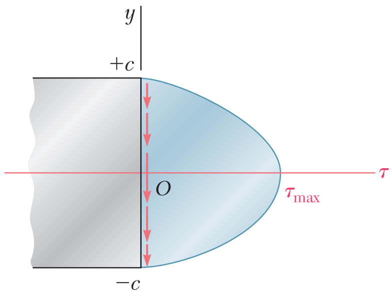


Fig. 6.15 Shear stress distribution on transverse section of rectangular beam.

Equation (6.9) shows that the distribution of shearing stresses in a transverse section of a rectangular beam is *parabolic* (Fig. 6.15). As we have already observed in the preceding section, the shearing stresses are zero at the top and bottom of the cross section ($y = \pm c$). Making $y = 0$ in Eq. (6.9), we obtain the value of the maximum shearing stress in a given section of a *narrow rectangular beam:*

$$\tau_{max} = \frac{3}{2}\frac{V}{A} \qquad (6.10)$$

The relation obtained shows that the maximum value of the shearing stress in a beam of rectangular cross section is 50% larger than the value V/A that would be obtained by wrongly assuming a uniform stress distribution across the entire cross section.

In the case of an *American standard beam* (S-beam) or a *wide-flange beam* (W-beam), Eq. (6.6) can be used to determine the average value of the shearing stress τ_{xy} over a section aa' or bb' of the transverse cross section of the beam (Figs. 6.16*a* and *b*). We write

$$\tau_{ave} = \frac{VQ}{It} \qquad (6.6)$$

where V is the vertical shear, t the width of the section at the elevation considered, Q the first moment of the shaded area with respect to the neutral axis cc', and I the moment of inertia of the entire cross-sectional area about cc'. Plotting τ_{ave} against the vertical distance y, we obtain the curve shown in Fig. 6.16*c*. We note the discontinuities existing in this curve, which reflect the difference between the values of t corresponding respectively to the flanges $ABGD$ and $A'B'G'D'$ and to the web $EFF'E'$.

In the case of the web, the shearing stress τ_{xy} varies only very slightly across the section bb', and can be assumed equal to its average

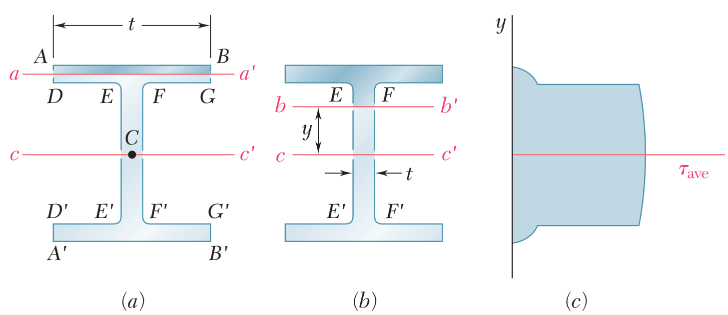


Fig. 6.16 Shear stress distribution on transverse section of wide-flange beam.

value τ_{ave}. This is not true, however, for the flanges. For example, considering the horizontal line $DEFG$, we note that τ_{xy} is zero between D and E and between F and G, since these two segments are part of the free surface of the beam. On the other hand the value of τ_{xy} between E and F can be obtained by making $t = EF$ in Eq. (6.6). In practice, one usually assumes that the entire shear load is carried by the web, and that a good approximation of the maximum value of the shearing stress in the cross section can be obtained by dividing V by the cross-sectional area of the web.

$$\tau_{max} = \frac{V}{A_{web}} \tag{6.11}$$

We should note, however, that while the vertical component τ_{xy} of the shearing stress in the flanges can be neglected, its horizontal component τ_{xz} has a significant value that will be determined in Sec. 6.7.

EXAMPLE 6.02

Knowing that the allowable shearing stress for the timber beam of Sample Prob. 5.7 is $\tau_{all} = 0.250$ ksi, check that the design obtained in that sample problem is acceptable from the point of view of the shearing stresses.

We recall from the shear diagram of Sample Prob. 5.7 that $V_{max} =$ 4.50 kips. The actual width of the beam was given as $b = 3.5$ in., and the value obtained for its depth was $h = 14.55$ in. Using Eq. (6.10) for the maximum shearing stress in a narrow rectangular beam, we write

$$\tau_{max} = \frac{3}{2}\frac{V}{A} = \frac{3}{2}\frac{V}{bh} = \frac{3(4.50 \text{ kips})}{2(3.5 \text{ in.})(14.55 \text{ in.})} = 0.1325 \text{ ksi}$$

Since $\tau_{max} < \tau_{all}$, the design obtained in Sample Prob. 5.7 is acceptable.

EXAMPLE 6.03

Knowing that the allowable shearing stress for the steel beam of Sample Prob. 5.8 is $\tau_{all} = 90$ MPa, check that the W360 × 32.9 shape obtained in that sample problem is acceptable from the point of view of the shearing stresses.

We recall from the shear diagram of Sample Prob. 5.8 that the maximum absolute value of the shear in the beam is $|V|_{max} = 58$ kN. As we saw in Sec. 6.4, it may be assumed in practice that the entire shear load is carried by the web and that the maximum value of the shearing stress in the beam can be obtained from Eq. (6.11). From Appendix C we find that for a W360 × 32.9 shape the depth of the beam and the thickness of its web are, respectively, $d = 349$ mm and $t_w = 5.8$ mm. We thus have

$$A_{web} = d\,t_w = (349 \text{ mm})(5.8 \text{ mm}) = 2024 \text{ mm}^2$$

Substituting the values of $|V|_{max}$ and A_{web} into Eq. (6.11), we obtain

$$\tau_{max} = \frac{|V|_{max}}{A_{web}} = \frac{58 \text{ kN}}{2024 \text{ mm}^2} = 28.7 \text{ MPa}$$

Since $\tau_{max} < \tau_{all}$, the design obtained in Sample Prob. 5.8 is acceptable.

*6.5 FURTHER DISCUSSION OF THE DISTRIBUTION OF STRESSES IN A NARROW RECTANGULAR BEAM

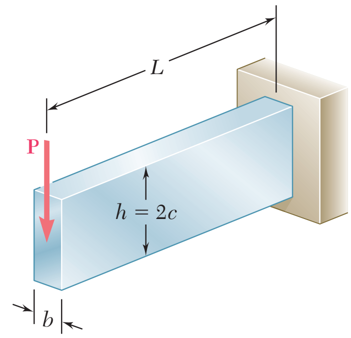


Fig. 6.17 Cantilever beam.

Consider a narrow cantilever beam of rectangular cross section of width b and depth h subjected to a load **P** at its free end (Fig. 6.17). Since the shear V in the beam is constant and equal in magnitude to the load **P,** Eq. (6.9) yields

$$\tau_{xy} = \frac{3}{2}\frac{P}{A}\left(1 - \frac{y^2}{c^2}\right) \tag{6.12}$$

We note from Eq. (6.12) that the shearing stresses depend only upon the distance y from the neutral surface. They are independent, therefore, of the distance from the point of application of the load; it follows that all elements located at the same distance from the neutral surface undergo the same shear deformation (Fig. 6.18). While plane sections do *not* remain plane, the distance between two corresponding points D and D' located in different sections remains the same. This indicates that the normal strains ϵ_x, and thus the normal stresses σ_x, are unaffected by the shearing stresses, and that the assumption made in Sec. 5.1 is justified for the loading condition of Fig. 6.17.

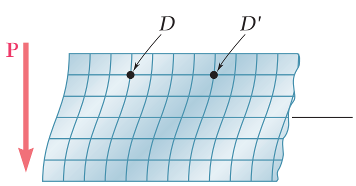


Fig. 6.18 Deformation of segment of cantilever beam.

We conclude that our analysis of the stresses in a cantilever beam of rectangular cross section, subjected to a concentrated load **P** at its free end, is valid. The correct values of the shearing stresses in the beam are given by Eq. (6.12), and the normal stresses at a distance x from the free end are obtained by making $M = -Px$ in Eq. (5.2) of Sec. 5.1. We have

$$\sigma_x = +\frac{Pxy}{I} \tag{6.13}$$

The validity of the above statement, however, depends upon the end conditions. If Eq. (6.12) is to apply everywhere, then the load P must be distributed parabolically over the free-end section. Moreover, the fixed-end support must be of such a nature that it will allow the type of shear deformation indicated in Fig. 6.18. The resulting model (Fig. 6.19) is highly unlikely to be encountered in practice. However, it follows from Saint-Venant's principle that, for other modes of application of the load and for other types of fixed-end supports, Eqs. (6.12) and (6.13) still provide us with the correct distribution of stresses, except close to either end of the beam.

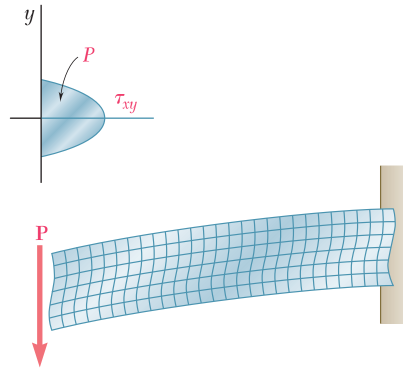


Fig. 6.19 Deformation of cantilever beam with concentrated load.

When a beam of rectangular cross section is subjected to several concentrated loads (Fig. 6.20), the principle of superposition can be used to determine the normal and shearing stresses in sections located between the points of application of the loads. However, since the loads $\mathbf{P}_2$, $\mathbf{P}_3$, etc., are applied on the surface of the beam and cannot be assumed to be distributed parabolically throughout the cross section, the results obtained cease to be valid in the immediate vicinity of the points of application of the loads.

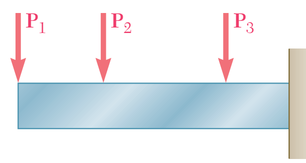


Fig. 6.20 Cantilever beam.

When the beam is subjected to a distributed load (Fig. 6.21), the shear varies with the distance from the end of the beam, and so does the shearing stress at a given elevation y. The resulting shear deformations are such that the distance between two corresponding points of different cross sections, such as D_1 and D_1', or D_2 and D_2', will depend upon their elevation. This indicates that the assumption that plane sections remain plane, under which Eqs. (6.12) and (6.13) were derived, must be rejected for the loading condition of Fig. 6.21. The error involved, however, is small for the values of the span-depth ratio encountered in practice.

We should also note that, in portions of the beam located under a concentrated or distributed load, normal stresses σ_y will be exerted on the horizontal faces of a cubic element of material, in addition to the stresses τ_{xy} shown in Fig. 6.2.

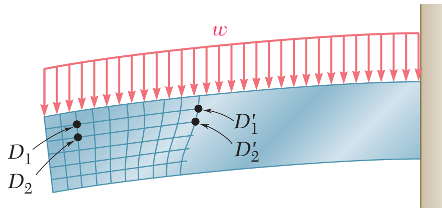
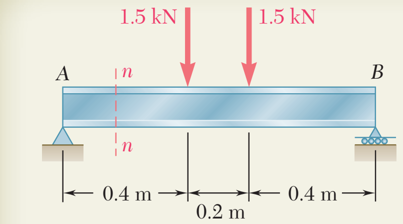
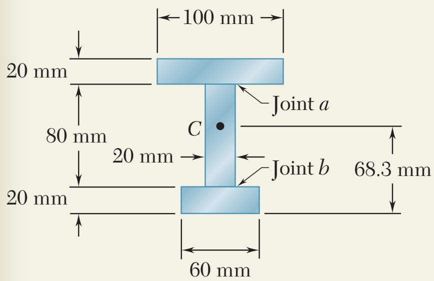


Fig. 6.21 Deformation of cantilever beam with distributed load.

SAMPLE PROBLEM 6.1

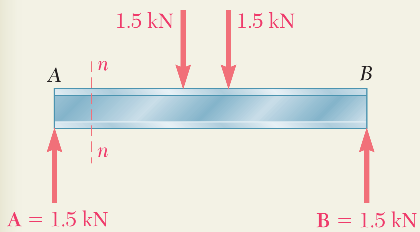


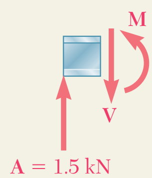
Beam AB is made of three planks glued together and is subjected, in its plane of symmetry, to the loading shown. Knowing that the width of each glued joint is 20 mm, determine the average shearing stress in each joint at section n-n of the beam. The location of the centroid of the section is given in the sketch and the centroidal moment of inertia is known to be $I = 8.63 \times 10^{-6}\ \text{m}^4$.



SOLUTION

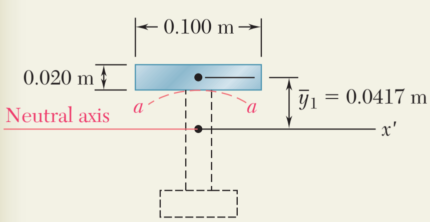
Vertical Shear at Section *n-n*. Since the beam and loading are both symmetric with respect to the center of the beam, we have $\mathbf{A} = \mathbf{B} = 1.5\ \text{kN}\uparrow$.





Considering the portion of the beam to the left of section n-n as a free body, we write

$$+\uparrow \sum F_y = 0: \qquad 1.5\ \text{kN} - V = 0 \qquad V = 1.5\ \text{kN}$$

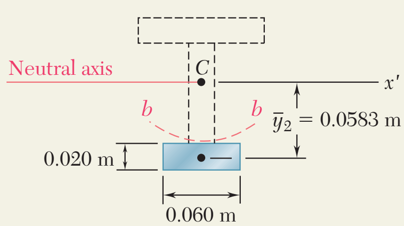


Shearing Stress in Joint *a*. We pass the section a-a through the glued joint and separate the cross-sectional area into two parts. We choose to determine Q by computing the first moment with respect to the neutral axis of the area above section a-a.

$$Q = A\bar{y}_1 = [(0.100\ \text{m})(0.020\ \text{m})](0.0417\ \text{m}) = 83.4 \times 10^{-6}\ \text{m}^3$$

Recalling that the width of the glued joint is $t = 0.020$ m, we use Eq. (6.7) to determine the average shearing stress in the joint.

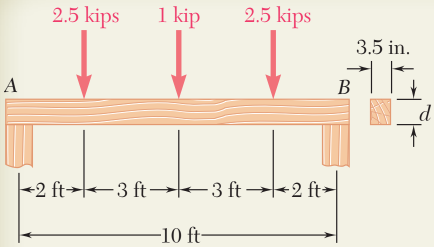
$$\tau_{\text{ave}} = \frac{VQ}{It} = \frac{(1500\ \text{N})(83.4 \times 10^{-6}\ \text{m}^3)}{(8.63 \times 10^{-6}\ \text{m}^4)(0.020\ \text{m})} \qquad \tau_{\text{ave}} = 725\ \text{kPa} \blacktriangleleft$$



Shearing Stress in Joint *b*. We now pass section b-b and compute Q by using the area below the section.

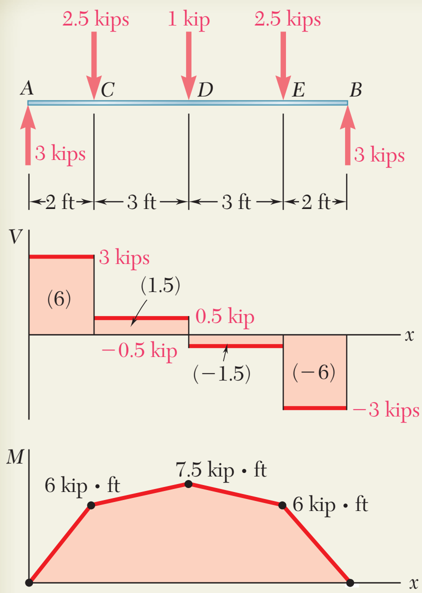
$$Q = A\bar{y}_2 = [(0.060\ \text{m})(0.020\ \text{m})](0.0583\ \text{m}) = 70.0 \times 10^{-6}\ \text{m}^3$$

$$\tau_{\text{ave}} = \frac{VQ}{It} = \frac{(1500\ \text{N})(70.0 \times 10^{-6}\ \text{m}^3)}{(8.63 \times 10^{-6}\ \text{m}^4)(0.020\ \text{m})} \qquad \tau_{\text{ave}} = 608\ \text{kPa} \blacktriangleleft$$



SAMPLE PROBLEM 6.2

A timber beam AB of span 10 ft and nominal width 4 in. (actual width = 3.5 in.) is to support the three concentrated loads shown. Knowing that for the grade of timber used $\sigma_{\text{all}} = 1800$ psi and $\tau_{\text{all}} = 120$ psi, determine the minimum required depth d of the beam.



SOLUTION

Maximum Shear and Bending Moment. After drawing the shear and bending-moment diagrams, we note that

$$M_{\max} = 7.5 \text{ kip} \cdot \text{ft} = 90 \text{ kip} \cdot \text{in.}$$
$$V_{\max} = 3 \text{ kips}$$

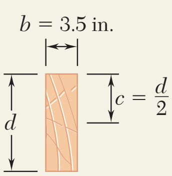
Design Based on Allowable Normal Stress. We first express the elastic section modulus S in terms of the depth d. We have

$$I = \frac{1}{12}bd^3 \qquad S = \frac{1}{c} = \frac{1}{6}bd^2 = \frac{1}{6}(3.5)d^2 = 0.5833d^2$$

For $M_{\max} = 90$ kip · in. and $\sigma_{\text{all}} = 1800$ psi, we write

$$S = \frac{M_{\max}}{\sigma_{\text{all}}} \qquad 0.5833d^2 = \frac{90 \times 10^3 \text{ lb} \cdot \text{in.}}{1800 \text{ psi}}$$
$$d^2 = 85.7 \qquad d = 9.26 \text{ in.}$$

We have satisfied the requirement that $\sigma_m \leq 1800$ psi.



Check Shearing Stress. For $V_{\max} = 3$ kips and $d = 9.26$ in., we find

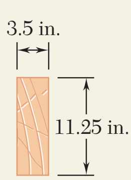
$$\tau_m = \frac{3}{2}\frac{V_{\max}}{A} = \frac{3}{2}\frac{3000 \text{ lb}}{(3.5 \text{ in.})(9.26 \text{ in.})} \qquad \tau_m = 138.8 \text{ psi}$$

Since $\tau_{\text{all}} = 120$ psi, the depth $d = 9.26$ in. is *not* acceptable and we must redesign the beam on the basis of the requirement that $\tau_m \leq 120$ psi.

Design Based on Allowable Shearing Stress. Since we now know that the allowable shearing stress controls the design, we write

$$\tau_m = \tau_{\text{all}} = \frac{3}{2}\frac{V_{\max}}{A} \qquad 120 \text{ psi} = \frac{3}{2}\frac{3000 \text{ lb}}{(3.5 \text{ in.})d}$$

$$d = 10.71 \text{ in.} \quad \blacktriangleleft$$



4 in. × 12 in.
Nominal size

The normal stress is, of course, less than $\sigma_{\text{all}} = 1800$ psi, and the depth of 10.71 in. is fully acceptable.

Comment. Since timber is normally available in depth increments of 2 in., a 4 × 12-in. nominal size timber should be used. The actual cross section would then be 3.5 × 11.25 in.

PROBLEMS

6.1 Three boards, each of 1.5×3.5-in. rectangular cross section, are nailed together to form a beam that is subjected to a vertical shear of 250 lb. Knowing that the spacing between each pair of nails is 2.5 in., determine the shearing force in each nail.

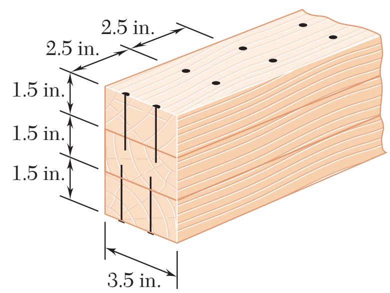


Fig. P6.1

6.2 Three boards, each 2 in. thick, are nailed together to form a beam that is subjected to a vertical shear. Knowing that the allowable shearing force in each nail is 150 lb, determine the allowable shear if the spacing s between the nails is 3 in.

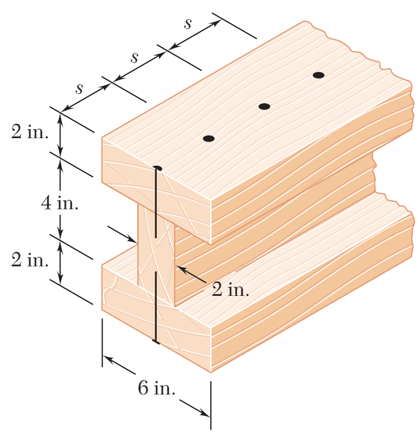


Fig. P6.2

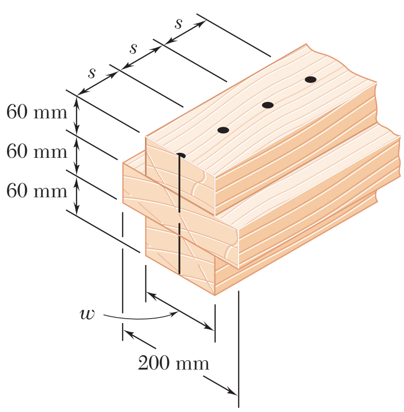


Fig. P6.3

6.3 Three boards are nailed together to form a beam shown, which is subjected to a vertical shear. Knowing that the spacing between the nails is $s = 75$ mm and that the allowable shearing force in each nail is 400 N, determine the allowable shear when $w = 120$ mm.

6.4 Solve Prob. 6.3, assuming that the width of the top and bottom boards is changed to $w = 100$ mm.

6.5 The American Standard rolled-steel beam shown has been reinforced by attaching to it two 16 × 200-mm plates, using 18-mm-diameter bolts spaced longitudinally every 120 mm. Knowing that the average allowable shearing stress in the bolts is 90 MPa, determine the largest permissible vertical shearing force.

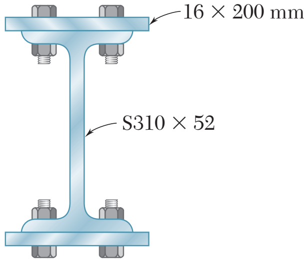


Fig. P6.5

6.6 Solve Prob. 6.5, assuming that the reinforcing plates are only 12 mm thick.

6.7 A column is fabricated by connecting the rolled-steel members shown by bolts of $\frac{3}{4}$-in. diameter spaced longitudinally every 5 in. Determine the average shearing stress in the bolts caused by a shearing force of 30 kips parallel to the y axis.

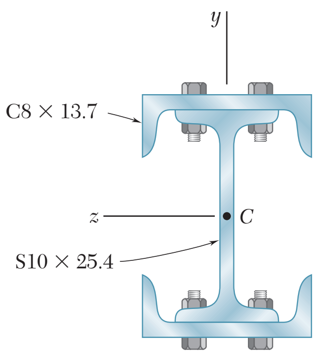


Fig. *P6.7*

6.8 The composite beam shown is fabricated by connecting two W6 × 20 rolled-steel members, using bolts of $\frac{5}{8}$-in. diameter spaced longitudinally every 6 in. Knowing that the average allowable shearing stress in the bolts is 10.5 ksi, determine the largest allowable vertical shear in the beam.

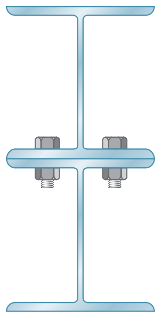


Fig. *P6.8*

6.9 through 6.12 For the beam and loading shown, consider section *n-n* and determine (*a*) the largest shearing stress in that section, (*b*) the shearing stress at point *a*.

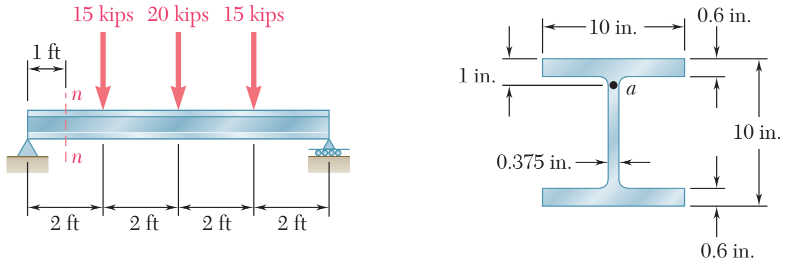


Fig. P6.9

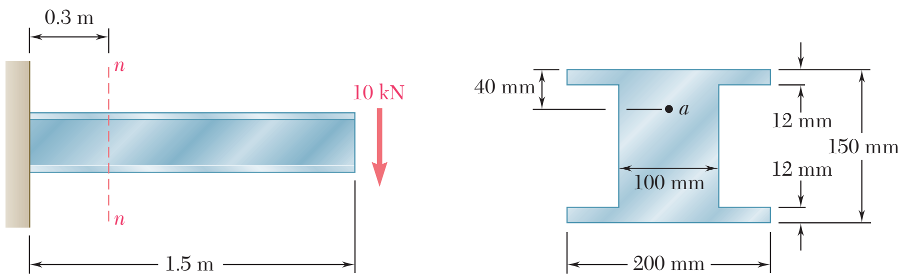
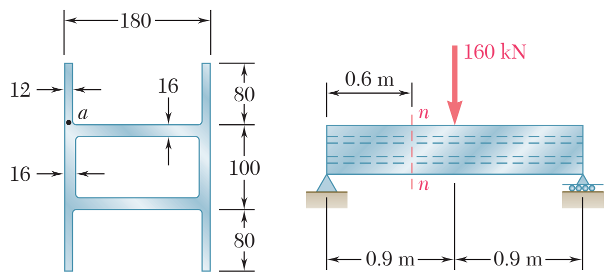


Fig. P6.10



Dimensions in mm

Fig. *P6.11*

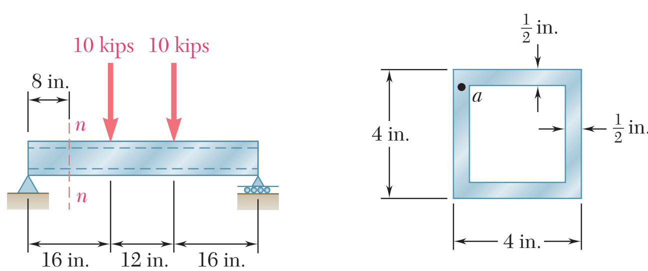


Fig. P6.12

6.13 and *6.14* For a beam having the cross section shown, determine the largest allowable vertical shear if the shearing stress is not to exceed 60 MPa.

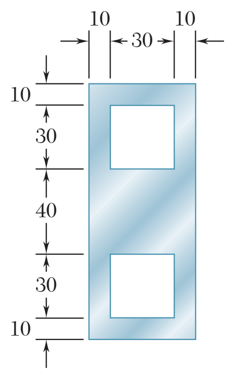


Fig. P6.13

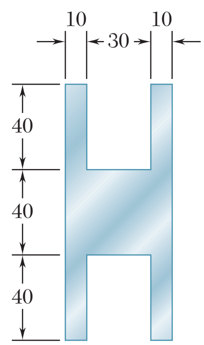


Fig. *P6.14*

6.15 For the beam and loading shown, determine the minimum required depth h, knowing that for the grade of timber used, $\sigma_{all} = 1750$ psi and $\tau_{all} = 130$ psi.

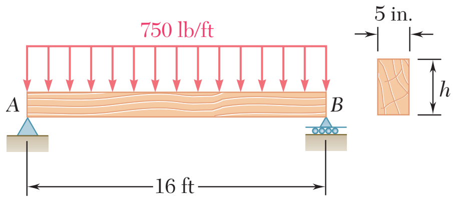


Fig. P6.15

6.16 For the beam and loading shown, determine the minimum required width b, knowing that for the grade of timber used, $\sigma_{all} = 12$ MPa and $\tau_{all} = 825$ kPa.

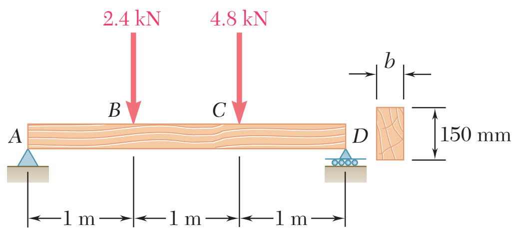


Fig. P6.16

6.17 A timber beam AB of length L and rectangular cross section carries a uniformly distributed load w and is supported as shown. (a) Show that the ratio τ_m/σ_m of the maximum values of the shearing and normal stresses in the beam is equal to $2h/L$, where h and L are, respectively, the depth and the length of the beam. (b) Determine the depth h and the width b of the beam, knowing that $L = 5$ m, $w = 8$ kN/m, $\tau_m = 1.08$ MPa, and $\sigma_m = 12$ MPa.

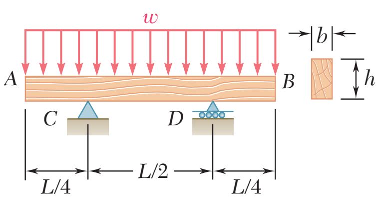


Fig. *P6.17*

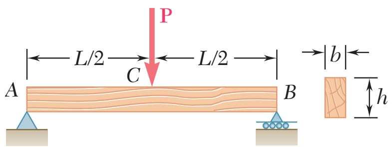


Fig. P6.18

6.18 A timber beam AB of length L and rectangular cross section carries a single concentrated load $\mathbf{P}$ at its midpoint C. (a) Show that the ratio τ_m/σ_m of the maximum values of the shearing and normal stresses in the beam is equal to $h/2L$, where h and L are, respectively, the depth and the length of the beam. (b) Determine the depth h and the width b of the beam, knowing that $L = 2$ m, $P = 40$ kN, $\tau_m = 960$ kPa, and $\sigma_m = 12$ MPa.

6.19 For the wide-flange beam with the loading shown, determine the largest $\mathbf{P}$ that can be applied, knowing that the maximum normal stress is 24 ksi and the largest shearing stress using the approximation $\tau_m = V/A_{\text{web}}$ is 14.5 ksi.

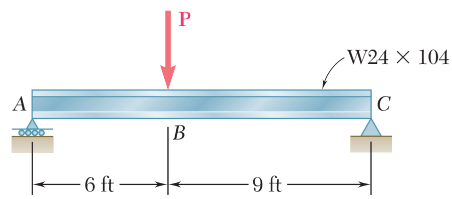


Fig. P6.19

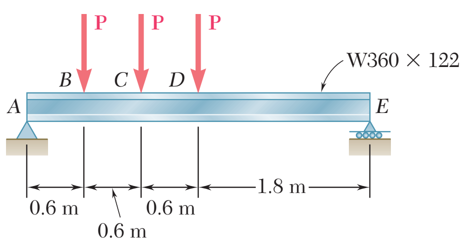


Fig. P6.20

6.20 For the wide-flange beam with the loading shown, determine the largest load $\mathbf{P}$ that can be applied, knowing that the maximum normal stress is 160 MPa and the largest shearing stress using the approximation $\tau_m = V/A_{\text{web}}$ is 100 MPa.

6.21 and 6.22 For the beam and loading shown, consider section n-n and determine the shearing stress at (a) point a, (b) point b.

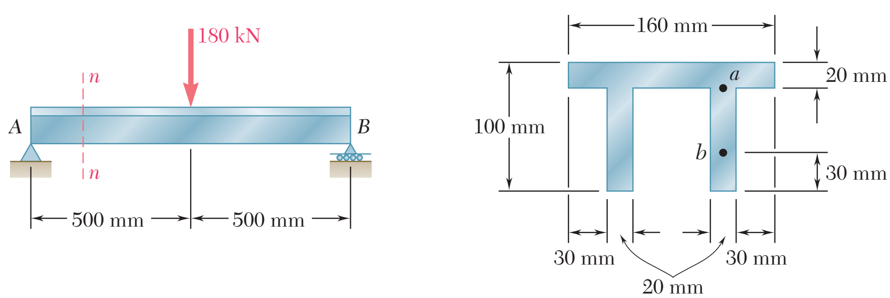


Fig. P6.21 and P6.23

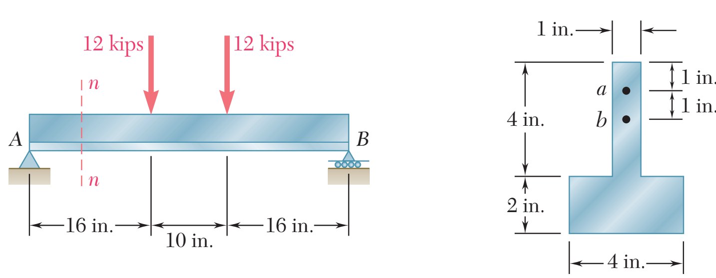


Fig. P6.22 and P6.24

6.23 and 6.24 For the beam and loading shown, determine the largest shearing stress in section n-n.

6.25 through 6.28 A beam having the cross section shown is subjected to a vertical shear **V**. Determine (*a*) the horizontal line along which the shearing stress is maximum, (*b*) the constant k in the following expression for the maximum shearing stress

$$\tau_{\max} = k\frac{V}{A}$$

where A is the cross-sectional area of the beam.

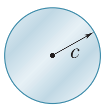


Fig. *P6.25*

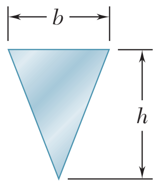


Fig. P6.26

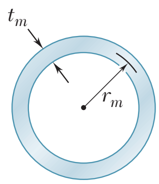


Fig. *P6.27*

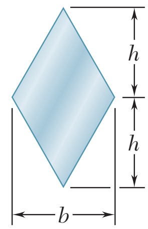


Fig. P6.28

6.6 LONGITUDINAL SHEAR ON A BEAM ELEMENT OF ARBITRARY SHAPE

Consider a box beam obtained by nailing together four planks, as shown in Fig. 6.22*a*. You learned in Sec. 6.2 how to determine the shear per unit length, q, on the horizontal surfaces along which the planks are joined. But could you determine q if the planks had been joined along *vertical* surfaces, as shown in Fig. 6.22*b*? We examined in Sec. 6.4 the distribution of the vertical components τ_{xy} of the stresses on a transverse section of a W-beam or an S-beam and found that these stresses had a fairly constant value in the web of the beam and were negligible in its flanges. But what about the *horizontal* components τ_{xz} of the stresses in the flanges?

To answer these questions we must extend the procedure developed in Sec. 6.2 for the determination of the shear per unit length, q, so that it will apply to the cases just described.

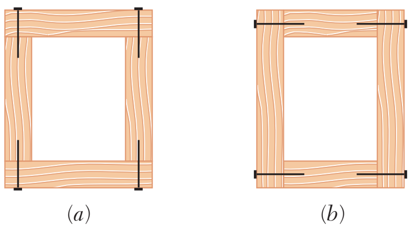


Fig. 6.22 Box beam cross sections.

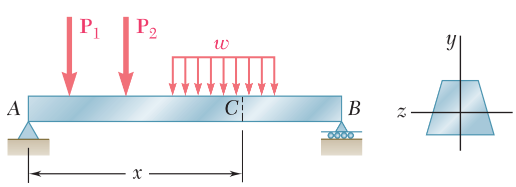


Fig. 6.4 (*repeated*) Beam example.

Consider the prismatic beam AB of Fig. 6.4, which has a vertical plane of symmetry and supports the loads shown. At a distance x from end A we detach again an element $CDD'C'$ of length Δx. This element, however, will now extend from two sides of the beam

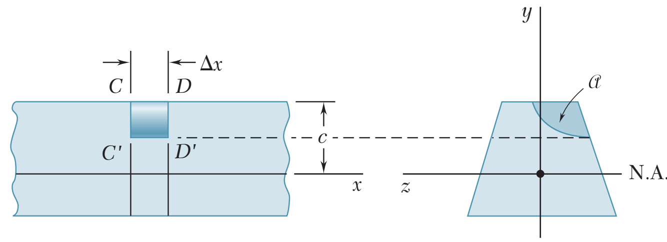


Fig. 6.23 Short segment of beam example.

to an arbitrary curved surface (Fig. 6.23). The forces exerted on the element include vertical shearing forces $\mathbf{V}'_C$ and $\mathbf{V}'_D$, elementary horizontal normal forces $\sigma_C dA$ and $\sigma_D dA$, possibly a load $w\ \Delta x$, and a longitudinal shearing force $\Delta\mathbf{H}$ representing the resultant of the elementary longitudinal shearing forces exerted on the curved surface (Fig. 6.24). We write the equilibrium equation

$$\xrightarrow{+} \sum F_x = 0: \qquad \Delta H + \int_{\mathcal{A}} (\sigma_C - \sigma_D)\, dA = 0$$

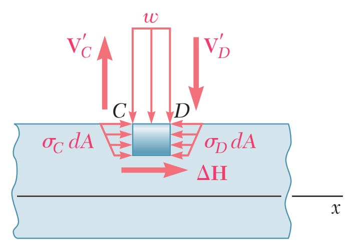


Fig. 6.24 Forces exerted on element.

where the integral is to be computed over the shaded area $\mathcal{A}$ of the section. We observe that the equation obtained is the same as the one we obtained in Sec. 6.2, but that the shaded area $\mathcal{A}$ over which the integral is to be computed now extends to the curved surface.

The remainder of the derivation is the same as in Sec. 6.2. We find that the longitudinal shear exerted on the beam element is

$$\Delta H = \frac{VQ}{I} \Delta x \tag{6.4}$$

where I is the centroidal moment of inertia of the entire section, Q the first moment of the shaded area $\mathcal{A}$ with respect to the neutral axis, and V the vertical shear in the section. Dividing both members of Eq. (6.4) by Δx, we obtain the horizontal shear per unit length, or shear flow:

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} \tag{6.5}$$

EXAMPLE 6.04

A square box beam is made of two 0.75 × 3-in. planks and two 0.75 × 4.5-in. planks, nailed together as shown (Fig. 6.25). Knowing that the spacing between nails is 1.75 in. and that the beam is subjected to a vertical shear of magnitude $V = 600$ lb, determine the shearing force in each nail.

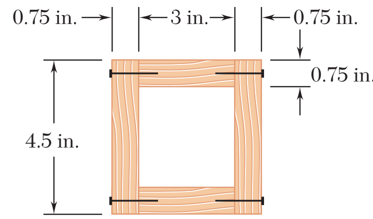


Fig. 6.25

We isolate the upper plank and consider the total force per unit length, q, exerted on its two edges. We use Eq. (6.5), where Q represents the first moment with respect to the neutral axis of the shaded area A' shown in Fig. 6.26*a*, and where I is the moment of inertia about the same axis of the entire cross-sectional area of the box beam (Fig. 6.26*b*). We have

$$Q = A'\bar{y} = (0.75 \text{ in.})(3 \text{ in.})(1.875 \text{ in.}) = 4.22 \text{ in}^3$$

Recalling that the moment of inertia of a square of side a about a centroidal axis is $I = \frac{1}{12}a^4$, we write

$$I = \tfrac{1}{12}(4.5 \text{ in.})^4 - \tfrac{1}{12}(3 \text{ in.})^4 = 27.42 \text{ in}^4$$

Substituting into Eq. (6.5), we obtain

$$q = \frac{VQ}{I} = \frac{(600 \text{ lb})(4.22 \text{ in}^3)}{27.42 \text{ in}^4} = 92.3 \text{ lb/in.}$$

Because both the beam and the upper plank are symmetric with respect to the vertical plane of loading, equal forces are exerted on both edges of the plank. The force per unit length on each of these edges is thus $\frac{1}{2}q = \frac{1}{2}(92.3) = 46.15$ lb/in. Since the spacing between nails is 1.75 in., the shearing force in each nail is

$$F = (1.75 \text{ in.})(46.15 \text{ lb/in.}) = 80.8 \text{ lb}$$

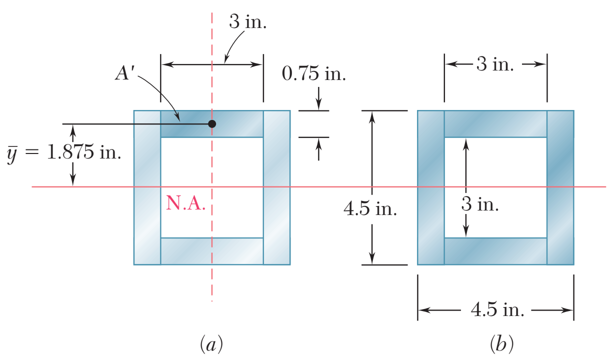


Fig. 6.26

6.7 SHEARING STRESSES IN THIN-WALLED MEMBERS

We saw in the preceding section that Eq. (6.4) may be used to determine the longitudinal shear $\Delta\mathbf{H}$ exerted on the walls of a beam element of arbitrary shape and Eq. (6.5) to determine the corresponding shear flow q. These equations will be used in this section to calculate both the shear flow and the average shearing stress in thin-walled members such as the flanges of wide-flange beams (Photo 6.2) and box beams, or the walls of structural tubes (Photo 6.3).

Photo 6.2 Wide-flange beams.

Photo 6.3 Box beams and tubes.

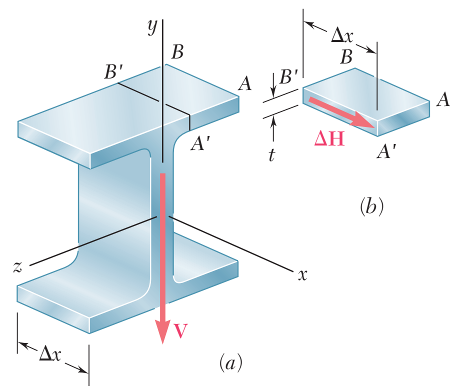


Fig. 6.27 Wide-flange beam segment.

Consider, for instance, a segment of length Δx of a wide-flange beam (Fig. 6.27*a*) and let $\mathbf{V}$ be the vertical shear in the transverse section shown. Let us detach an element $ABB'A'$ of the upper flange (Fig. 6.27*b*). The longitudinal shear $\Delta\mathbf{H}$ exerted on that element can be obtained from Eq. (6.4):

$$\Delta H = \frac{VQ}{I}\,\Delta x \tag{6.4}$$

Dividing ΔH by the area $\Delta A = t\,\Delta x$ of the cut, we obtain for the average shearing stress exerted on the element the same expression that we had obtained in Sec. 6.3 in the case of a horizontal cut:

$$\tau_{\text{ave}} = \frac{VQ}{It} \tag{6.6}$$

Note that τ_{ave} now represents the average value of the shearing stress τ_{zx} over a vertical cut, but since the thickness t of the flange is small, there is very little variation of τ_{zx} across the cut. Recalling that $\tau_{xz} = \tau_{zx}$ (Fig. 6.28), we conclude that the horizontal component τ_{xz} of the

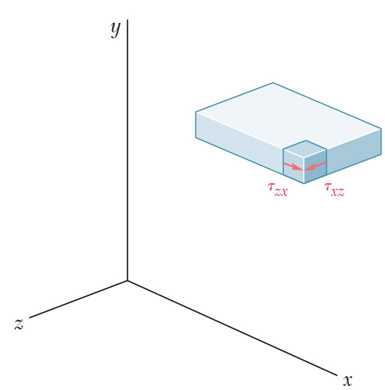


Fig. 6.28 Segment of beam flange.

shearing stress at any point of a transverse section of the flange can be obtained from Eq. (6.6), where Q is the first moment of the shaded area about the neutral axis (Fig. 6.29a). We recall that a similar result was obtained in Sec. 6.4 for the vertical component τ_{xy} of the shearing stress in the web (Fig. 6.29b). Equation (6.6) can be used to determine shearing stresses in box beams (Fig. 6.30), half pipes (Fig. 6.31), and other thin-walled members, as long as the loads are applied in a plane of symmetry of the member. In each case, the cut must be perpendicular to the surface of the member, and Eq. (6.6) will yield the component of the shearing stress in the direction of the tangent to that surface. (The other component may be assumed equal to zero, in view of the proximity of the two free surfaces.)

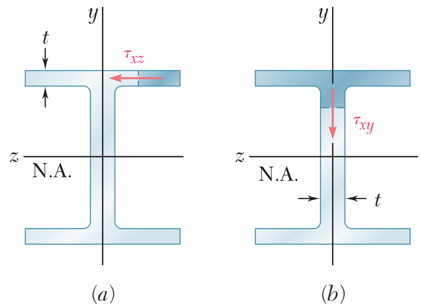


Fig. 6.29 Wide-flange beam.

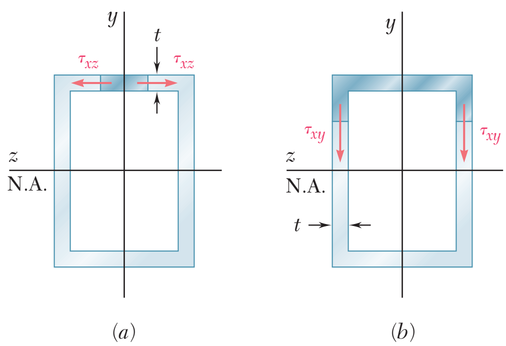


Fig. 6.30 Box beam.

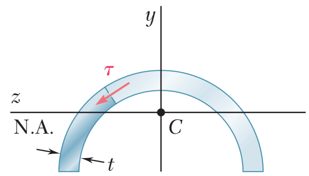


Fig. 6.31 Half pipe beam.

Comparing Eqs. (6.5) and (6.6), we note that the product of the shearing stress τ at a given point of the section and of the thickness t of the section at that point is equal to q. Since V and I are constant in any given section, q depends only upon the first moment Q and, thus, can easily be sketched on the section. In the case of a box beam, for example (Fig. 6.32), we note that q grows smoothly from zero at A to a maximum value at C and C' on the neutral axis, and then decreases back to zero as E is reached. We also note that there is no sudden variation in the magnitude of q as we pass a corner at B, D, B', or D', and that the sense of q in the horizontal portions of the section may be easily obtained from its sense in the vertical portions (which is the same as the sense of the shear $\mathbf{V}$). In the case of a wide-flange section (Fig. 6.33), the values of q in portions AB and $A'B$ of the upper flange are distributed symmetrically. As we turn at B into the web, the values of q corresponding to the two halves of the flange must be combined to obtain the value of q at the top of the web. After reaching a maximum value at C on the neutral axis, q decreases, and at D splits into two equal parts corresponding to the two halves of the lower flange. The name of *shear flow* commonly used to refer to the shear per unit length, q, reflects the similarity between the properties of q that we have just described and some of the characteristics of a fluid flow through an open channel or pipe.†

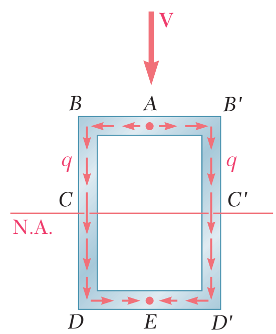


Fig. 6.32 Shear flow q in box beam section.

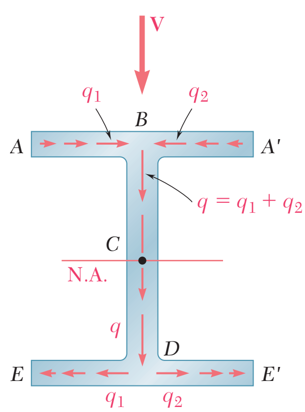


Fig. 6.33 Shear flow q in wide-flange beam section.

†We recall that the concept of shear flow was used to analyze the distribution of shearing stresses in thin-walled hollow shafts (Sec. 3.13). However, while the shear flow in a hollow shaft is constant, the shear flow in a member under a transverse loading is not.

So far we have assumed that all the loads were applied in a plane of symmetry of the member. In the case of members possessing two planes of symmetry, such as the wide-flange beam of Fig. 6.29 or the box beam of Fig. 6.30, any load applied through the centroid of a given cross section can be resolved into components along the two axes of symmetry of the section. Each component will cause the member to bend in a plane of symmetry, and the corresponding shearing stresses can be obtained from Eq. (6.6). The principle of superposition can then be used to determine the resulting stresses.

However, if the member considered possesses no plane of symmetry, or if it possesses a single plane of symmetry and is subjected to a load that is not contained in that plane, the member is observed to *bend and twist* at the same time, except when the load is applied at a specific point, called the *shear center*. Note that the shear center generally does *not* coincide with the centroid of the cross section. The determination of the shear center of various thin-walled shapes is discussed in Sec. 6.9.

*6.8 PLASTIC DEFORMATIONS

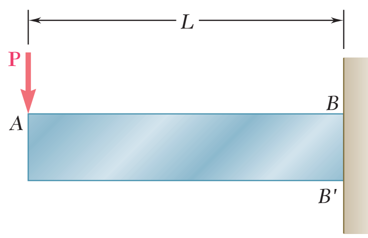


Fig. 6.34 ($PL \leq M_Y$)

Consider a cantilever beam AB of length L and rectangular cross section, subjected at its free end A to a concentrated load $\mathbf{P}$ (Fig. 6.34). The largest value of the bending moment occurs at the fixed end B and is equal to $M = PL$. As long as this value does not exceed the maximum elastic moment M_Y, that is, as long as $PL \leq M_Y$, the normal stress σ_x will not exceed the yield strength σ_Y anywhere in the beam. However, as P is increased beyond the value M_Y/L, yield is initiated at points B and B' and spreads toward the free end of the beam. Assuming the material to be elastoplastic, and considering a cross section CC' located at a distance x from the free end A of the beam (Fig. 6.35), we obtain the half-thickness y_Y of the elastic core in that section by making $M = Px$ in Eq. (4.38) of Sec. 4.9. We have

$$Px = \frac{3}{2}M_Y\left(1 - \frac{1}{3}\frac{y_Y^2}{c^2}\right) \tag{6.14}$$

where c is the half-depth of the beam. Plotting y_Y against x, we obtain the boundary between the elastic and plastic zones.

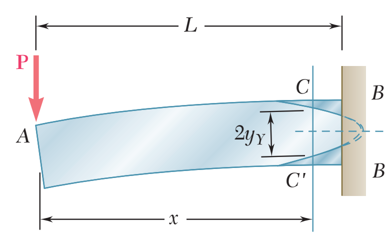


Fig. 6.35 ($PL > M_Y$)

As long as $PL < \frac{3}{2}M_Y$, the parabola defined by Eq. (6.14) intersects the line BB', as shown in Fig. 6.38. However, when PL reaches the value $\frac{3}{2}M_Y$, that is, when $PL = M_p$, where M_p is the plastic moment defined in Sec. 4.9, Eq. (6.14) yields $y_Y = 0$ for $x = L$, which shows that the vertex of the parabola is now located in section BB', and that this section has become fully plastic (Fig. 6.36). Recalling Eq. (4.40) of Sec. 4.9, we also note that the radius of curvature ρ of the neutral surface at that point is equal to zero, indicating the presence of a sharp bend in the beam at its fixed end. We say that a *plastic hinge* has developed at that point. The load $P = M_p/L$ is the largest load that can be supported by the beam.

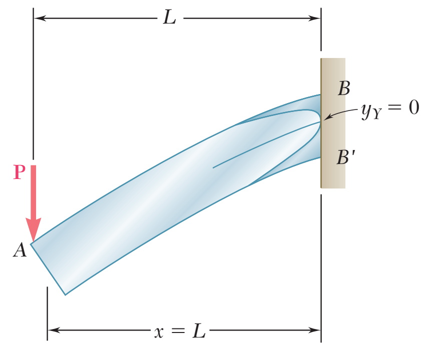


Fig. 6.36 $(PL = M_P = \frac{3}{2}M_Y)$

The above discussion was based only on the analysis of the normal stresses in the beam. Let us now examine the distribution of the shearing stresses in a section that has become partly plastic. Consider the portion of beam $CC''D''D$ located between the transverse sections CC' and DD', and above the horizontal plane $D''C''$ (Fig. 6.37*a*). If this portion is located entirely in the plastic zone, the normal stresses exerted on the faces CC'' and DD'' will be uniformly distributed and equal to the yield strength σ_Y (Fig. 6.40*b*). The equilibrium of the free body $CC''D''D$ thus requires that the horizontal shearing force $\Delta\mathbf{H}$ exerted on its lower face be equal to zero. It follows that the average value of the horizontal shearing stress τ_{yx} across the beam at C'' is zero, as well as the average value of the vertical shearing stress τ_{xy}. We thus conclude that the vertical shear $V = P$ in section CC' must be distributed entirely over the portion EE' of that section that is located within the elastic zone (Fig. 6.38). It can be shown† that the distribution of the shearing stresses over EE' is the same as in an elastic rectangular beam of the same width b as beam AB, and of depth equal to the thickness $2y_Y$ of the elastic zone. Denoting by A' the area $2by_Y$ of the elastic portion of the cross section, we have

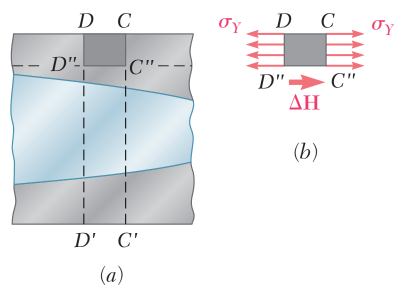


Fig. 6.37 Beam segment.

$$\tau_{xy} = \frac{3}{2}\frac{P}{A'}\left(1 - \frac{y^2}{y_Y^2}\right) \tag{6.15}$$

The maximum value of the shearing stress occurs for $y = 0$ and is

$$\tau_{max} = \frac{3}{2}\frac{P}{A'} \tag{6.16}$$

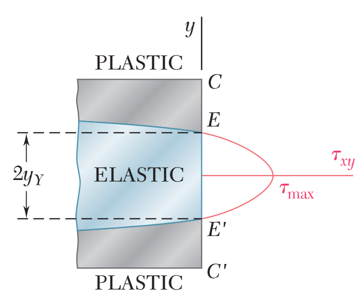
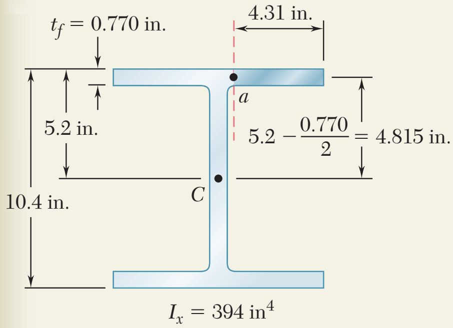


Fig. 6.38

As the area A' of the elastic portion of the section decreases, τ_{max} increases and eventually reaches the yield strength in shear τ_Y. Thus, shear contributes to the ultimate failure of the beam. A more exact analysis of this mode of failure should take into account the combined effect of the normal and shearing stresses.

†See Prob. 6.60.

SAMPLE PROBLEM 6.3



Knowing that the vertical shear is 50 kips in a W10 × 68 rolled-steel beam, determine the horizontal shearing stress in the top flange at a point *a* located 4.31 in. from the edge of the beam. The dimensions and other geometric data of the rolled-steel section are given in Appendix C.

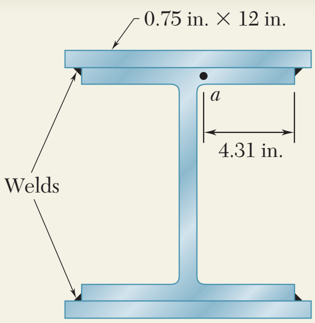
SOLUTION

We isolate the shaded portion of the flange by cutting along the dashed line that passes through point *a*.

$$Q = (4.31\text{ in.})(0.770\text{ in.})(4.815\text{ in.}) = 15.98\text{ in}^3$$

$$\tau = \frac{VQ}{It} = \frac{(50\text{ kips})(15.98\text{ in}^3)}{(394\text{ in}^4)(0.770\text{ in.})} \qquad \tau = 2.63\text{ ksi} \blacktriangleleft$$

SAMPLE PROBLEM 6.4



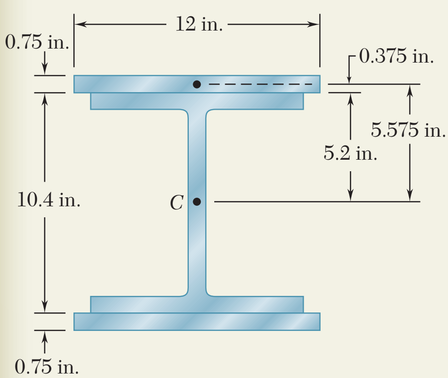
Solve Sample Prob. 6.3, assuming that 0.75 × 12-in. plates have been attached to the flanges of the W10 × 68 beam by continuous fillet welds as shown.

SOLUTION

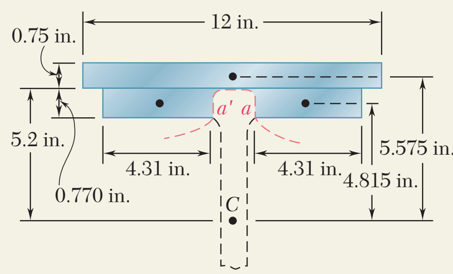
For the composite beam the centroidal moment of inertia is

$$I = 394\text{ in}^4 + 2[\tfrac{1}{12}(12\text{ in.})(0.75\text{ in.})^3 + (12\text{ in.})(0.75\text{ in.})(5.575\text{ in.})^2]$$

$$I = 954\text{ in}^4$$



Since the top plate and the flange are connected only at the welds, we find the shearing stress at *a* by passing a section through the flange at *a*, *between* the plate and the flange, and again through the flange at the symmetric point *a*′.



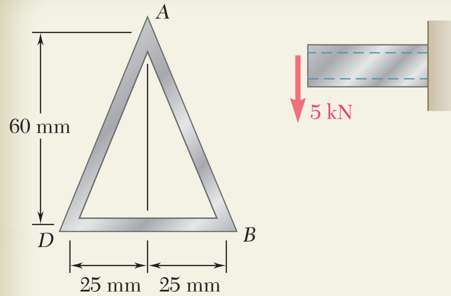
For the shaded area that we have isolated, we have

$$t = 2t_f = 2(0.770\text{ in.}) = 1.540\text{ in.}$$

$$Q = 2[(4.31\text{ in.})(0.770\text{ in.})(4.815\text{ in.})] + (12\text{ in.})(0.75\text{ in.})(5.575\text{ in.})$$

$$Q = 82.1\text{ in}^3$$

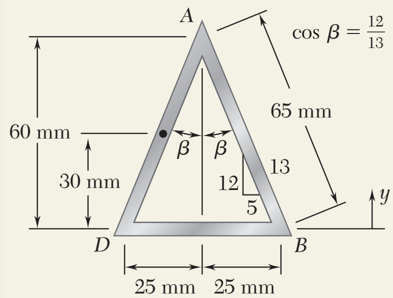
$$\tau = \frac{VQ}{It} = \frac{(50\text{ kips})(82.1\text{ in}^3)}{(954\text{ in}^4)(1.540\text{ in.})} \qquad \tau = 2.79\text{ ksi} \blacktriangleleft$$



SAMPLE PROBLEM 6.5

The thin-walled extruded beam shown is made of aluminum and has a uniform 3-mm wall thickness. Knowing that the shear in the beam is 5 kN, determine (*a*) the shearing stress at point *A*, (*b*) the maximum shearing stress in the beam. *Note:* The dimensions given are to lines midway between the outer and inner surfaces of the beam.

SOLUTION

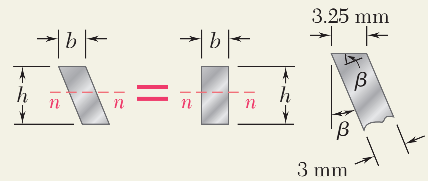


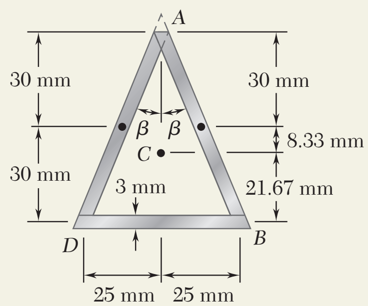
Centroid. We note that $AB = AD = 65$ mm.

$$\bar{Y} = \frac{\Sigma\, \bar{y} A}{\Sigma\, A} = \frac{2[(65 \text{ mm})(3 \text{ mm})(30 \text{ mm})]}{2[(65 \text{ mm})(3 \text{ mm})] + (50 \text{ mm})(3 \text{ mm})}$$

$$\bar{Y} = 21.67 \text{ mm}$$

Centroidal Moment of Inertia. Each side of the thin-walled beam can be considered as a parallelogram, and we recall that for the case shown $I_{nn} = bh^3/12$ where b is measured parallel to the axis nn.



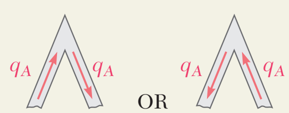


$$b = (3 \text{ mm})/\cos\beta = (3 \text{ mm})/(12/13) = 3.25 \text{ mm}$$

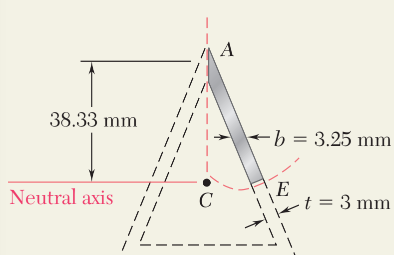
$$I = \Sigma(\bar{I} + Ad^2) = 2[\tfrac{1}{12}(3.25 \text{ mm})(60 \text{ mm})^3 + (3.25 \text{ mm})(60 \text{ mm})(8.33 \text{ mm})^2] + [\tfrac{1}{12}(50 \text{ mm})(3 \text{ mm})^3 + (50 \text{ mm})(3 \text{ mm})(21.67 \text{ mm})^2]$$

$$I = 214.6 \times 10^3 \text{ mm}^4 \qquad I = 0.2146 \times 10^{-6} \text{ m}^4$$

a. Shearing Stress at *A*. If a shearing stress τ_A occurs at A, the shear flow will be $q_A = \tau_A t$ and must be directed in one of the two ways shown. But the cross section and the loading are symmetric about a vertical line through A, and thus the shear flow must also be symmetric. Since neither of the possible shear flows is symmetric, we conclude that $\tau_A = 0$ ◀



b. Maximum Shearing Stress. Since the wall thickness is constant, the maximum shearing stress occurs at the neutral axis, where Q is maximum. Since we know that the shearing stress at A is zero, we cut the section along the dashed line shown and isolate the shaded portion of the beam. In order to obtain the largest shearing stress, the cut at the neutral axis is made perpendicular to the sides, and is of length $t = 3$ mm.



$$Q = [(3.25 \text{ mm})(38.33 \text{ mm})]\left(\frac{38.33 \text{ mm}}{2}\right) = 2387 \text{ mm}^3$$

$$Q = 2.387 \times 10^{-6} \text{ m}^3$$

$$\tau_E = \frac{VQ}{It} = \frac{(5 \text{ kN})(2.387 \times 10^{-6} \text{ m}^3)}{(0.2146 \times 10^{-6} \text{ m}^4)(0.003 \text{ m})} \qquad \tau_{max} = \tau_E = 18.54 \text{ MPa} \; ◀$$

PROBLEMS

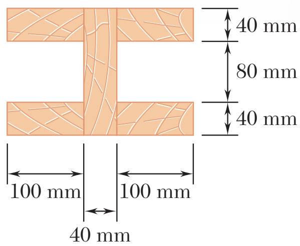


Fig. P6.29

6.29 The built-up beam shown is made by gluing together five planks. Knowing that in the glued joints the average allowable shearing stress is 350 kPa, determine the largest permissible vertical shear in the beam.

6.30 For the beam of Prob. 6.29, determine the largest permissible horizontal shear.

6.31 Several wooden planks are glued together to form the box beam shown. Knowing that the beam is subjected to a vertical shear of 3 kN, determine the average shearing stress in the glued joint (*a*) at *A*, (*b*) at *B*.

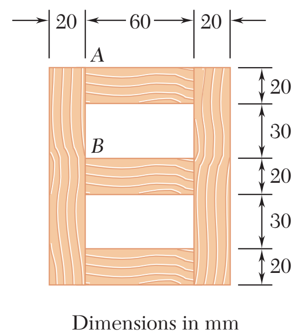


Fig. *P6.31*

6.32 The built-up timber beam is subjected to a 1500-lb vertical shear. Knowing that the longitudinal spacing of the nails is $s = 2.5$ in. and that each nail is 3.5 in. long, determine the shearing force in each nail.

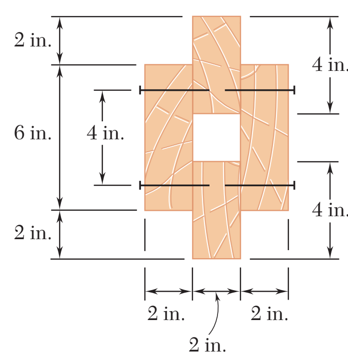


Fig. P6.32

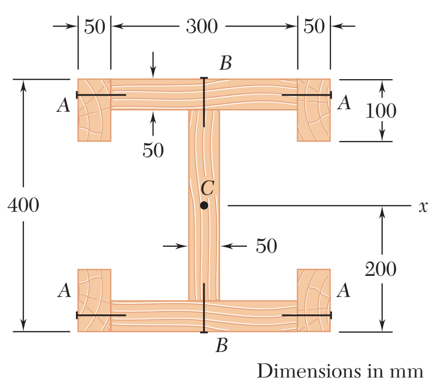


Fig. *P6.33*

6.33 The built-up wooden beam shown is subjected to a vertical shear of 8 kN. Knowing that the nails are spaced longitudinally every 60 mm at *A* and every 25 mm at *B*, determine the shearing force in the nails (*a*) at *A*, (*b*) at *B*. (*Given:* $I_x = 1.504 \times 10^9\ \text{mm}^4$.)

6.34 Knowing that a vertical shear **V** of 50 kips is exerted on a W14 × 82 rolled-steel beam, determine the shearing stress (*a*) at point *a*, (*b*) at the centroid *C*.

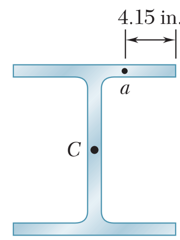


Fig. P6.34

6.35 An extruded aluminum beam has the cross section shown. Knowing that the vertical shear in the beam is 150 kN, determine the shearing stress at (*a*) point *a*, (*b*) point *b*.

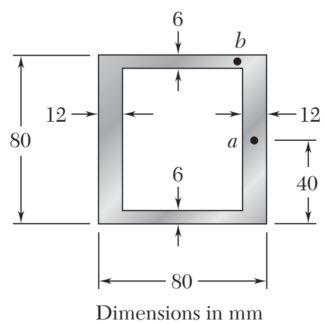


Fig. P6.35

6.36 Knowing that a given vertical shear **V** causes a maximum shearing stress of 75 MPa in the hat-shaped extrusion shown, determine the corresponding shearing stress at (*a*) point *a*, (*b*) point *b*.

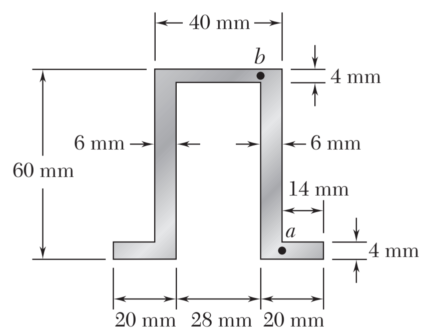


Fig. P6.36

6.37 Knowing that a given vertical shear **V** causes a maximum shearing stress of 75 MPa in an extruded beam having the cross section shown, determine the shearing stress at the three points indicated.

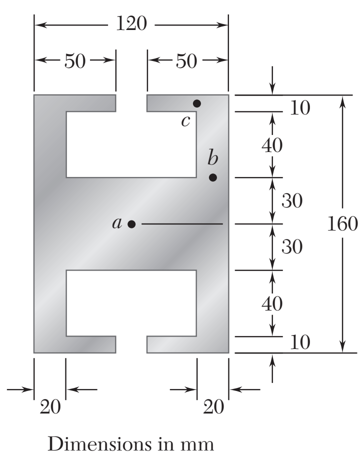


Fig. P6.37

6.38 An extruded beam has the cross section shown and a uniform wall thickness of 0.20 in. Knowing that a given vertical shear **V** causes a maximum shearing stress $\tau = 9$ ksi, determine the shearing stress at the four points indicated.

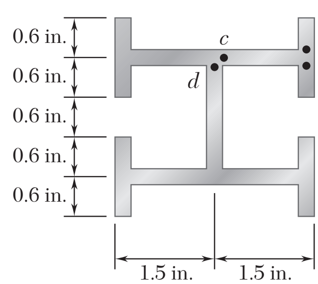


Fig. P6.38

6.39 Solve Prob. 6.38 assuming that the beam is subjected to a horizontal shear **V.**

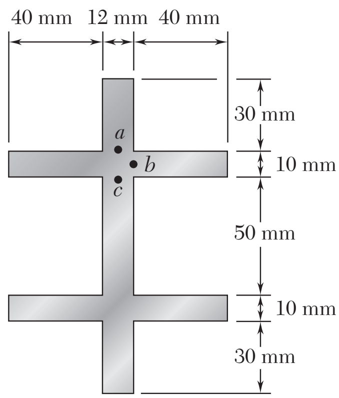


Fig. P6.40

6.40 Knowing that a given vertical shear **V** causes a maximum shearing stress of 50 MPa in a thin-walled member having the cross section shown, determine the corresponding shearing stress at (*a*) point *a*, (*b*) point *b*, (*c*) point *c*.

6.41 and *6.42* The extruded aluminum beam has a uniform wall thickness of $\frac{1}{8}$ in. Knowing that the vertical shear in the beam is 2 kips, determine the corresponding shearing stress at each of the five points indicated.

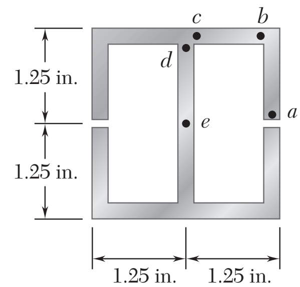


Fig. P6.41

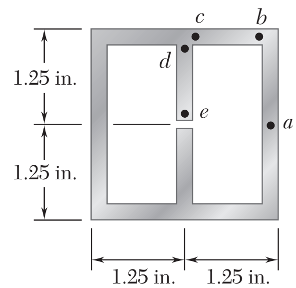


Fig. *P6.42*

6.43 Three 1 × 18-in. steel plates are bolted to four L6 × 6 × 1 angles to form a beam with the cross section shown. The bolts have a $\frac{7}{8}$-in. diameter and are spaced longitudinally every 5 in. Knowing that the allowable average shearing stress in the bolts is 12 ksi, determine the largest permissible vertical shear in the beam. (*Given:* $I_x = 6123\ \text{in}^4$.)

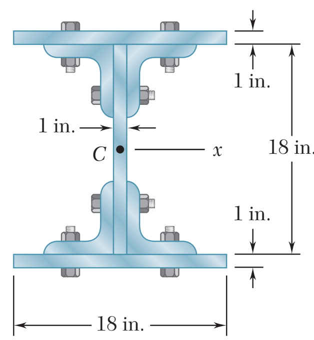


Fig. P6.43

6.44 Three planks are connected as shown by bolts of 14-mm diameter spaced every 150 mm along the longitudinal axis of the beam. For a vertical shear of 10 kN, determine the average shearing stress in the bolts.

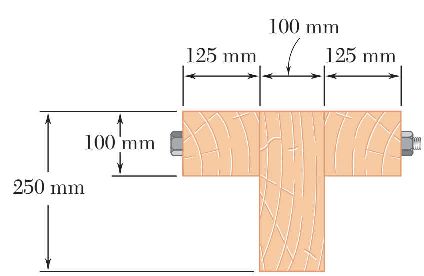


Fig. P6.44

6.45 A beam consists of three planks connected as shown by steel bolts with a longitudinal spacing of 225 mm. Knowing that the shear in the beam is vertical and equal to 6 kN and that the allowable average shearing stress in each bolt is 60 MPa, determine the smallest permissible bolt diameter that can be used.

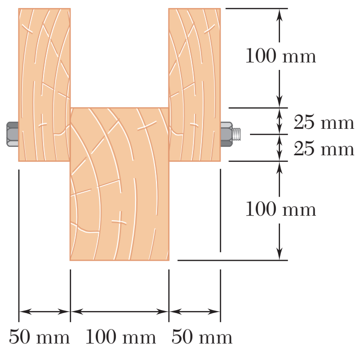


Fig. P6.45

6.46 A beam consists of five planks of 1.5 × 6-in. cross section connected by steel bolts with a longitudinal spacing of 9 in. Knowing that the shear in the beam is vertical and equal to 2000 lb and that the allowable average shearing stress in each bolt is 7500 psi, determine the smallest permissible bolt diameter that can be used.

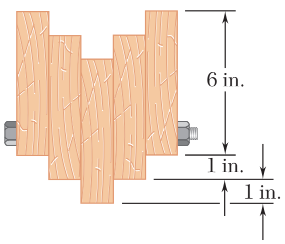


Fig. P6.46

6.47 A plate of $\frac{1}{4}$-in. thickness is corrugated as shown and then used as a beam. For a vertical shear of 1.2 kips, determine (*a*) the maximum shearing stress in the section, (*b*) the shearing stress at point *B*. Also sketch the shear flow in the cross section.

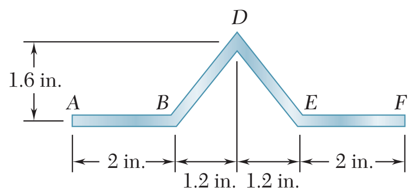
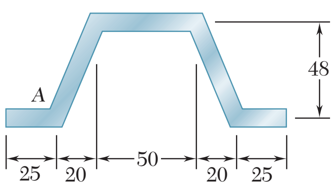


Fig. *P6.47*

6.48 A plate of 4-mm thickness is bent as shown and then used as a beam. For a vertical shear of 12 kN, determine (*a*) the shearing stress at point *A*, (*b*) the maximum shearing stress in the beam. Also sketch the shear flow in the cross section.



Dimensions in mm

Fig. P6.48

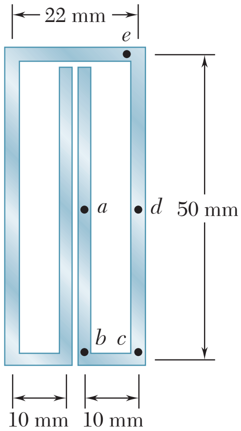


Fig. P6.49

6.49 A plate of 2-mm thickness is bent as shown and then used as a beam. For a vertical shear of 5 kN, determine the shearing stress at the five points indicated and sketch the shear flow in the cross section.

6.50 A plate of thickness t is bent as shown and then used as a beam. For a vertical shear of 600 lb, determine (*a*) the thickness t for which the maximum shearing stress is 300 psi, (*b*) the corresponding shearing stress at point E. Also sketch the shear flow in the cross section.

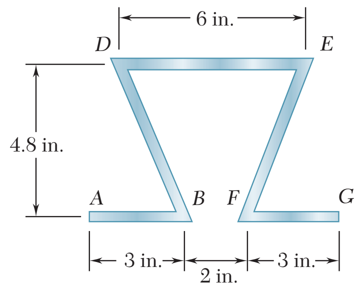


Fig. *P6.50*

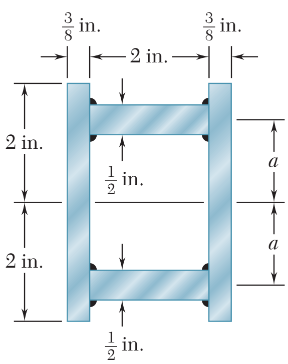


Fig. P6.51

6.51 The design of a beam calls for connecting two vertical rectangular $\frac{3}{8} \times 4$-in. plates by welding them to two horizontal $\frac{1}{2} \times 2$-in. plates as shown. For a vertical shear **V**, determine the dimension a for which the shear flow through the welded surfaces is maximum.

6.52 and 6.53 An extruded beam has a uniform wall thickness t. Denoting by **V** the vertical shear and by A the cross-sectional area of the beam, express the maximum shearing stress as $\tau_{max} = k(V/A)$ and determine the constant k for each of the two orientations shown.

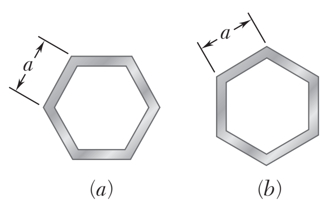


Fig. P6.52

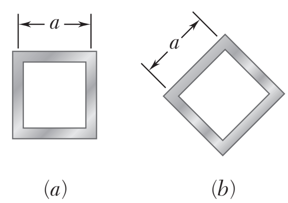


Fig. P6.53

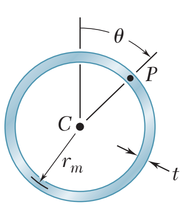


Fig. P6.54

6.54 (*a*) Determine the shearing stress at point P of a thin-walled pipe of the cross section shown caused by a vertical shear **V.** (*b*) Show that the maximum shearing stress occurs for $\theta = 90°$ and is equal to $2V/A$, where A is the cross-sectional area of the pipe.

6.55 For a beam made of two or more materials with different moduli of elasticity, show that Eq. (6.6)

$$\tau_{\text{ave}} = \frac{VQ}{It}$$

remains valid provided that both Q and I are computed by using the transformed section of the beam (see Sec. 4.6) and provided further that t is the actual width of the beam where τ_{ave} is computed.

6.56 and 6.57 A steel bar and an aluminum bar are bonded together as shown to form a composite beam. Knowing that the vertical shear in the beam is 4 kips and that the modulus of elasticity is 29×10^6 psi for the steel and 10.6×10^6 psi for the aluminum, determine (*a*) the average stress at the bonded surface, (*b*) the maximum shearing stress in the beam. (*Hint:* Use the method indicated in Prob. 6.55.)

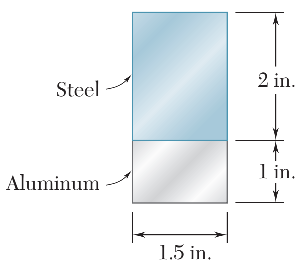


Fig. *P6.56*

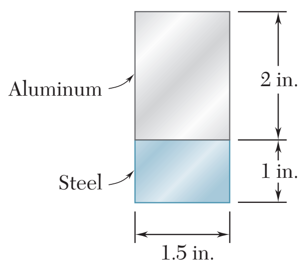


Fig. P6.57

6.58 and 6.59 A composite beam is made by attaching the timber and steel portions shown with bolts of 12-mm diameter spaced longitudinally every 200 mm. The modulus of elasticity is 10 GPa for the wood and 200 GPa for the steel. For a vertical shear of 4 kN, determine (*a*) the average shearing stress in the bolts, (*b*) the shearing stress at the center of the cross section. (*Hint:* Use the method indicated in Prob. 6.55.)

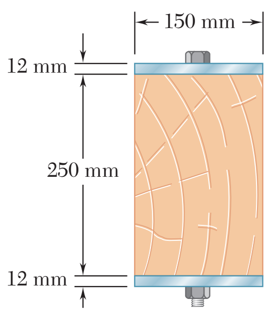


Fig. *P6.58*

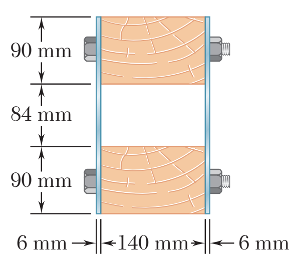


Fig. P6.59

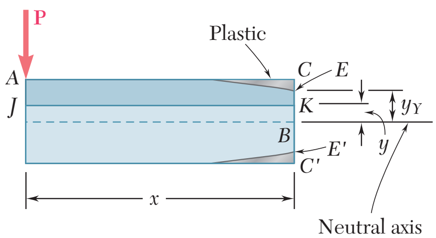


Fig. P6.60

6.60 Consider the cantilever beam AB discussed in Sec. 6.8 and the portion $ACKJ$ of the beam that is located to the left of the transverse section CC' and above the horizontal plane JK, where K is a point at a distance $y < y_Y$ above the neutral axis (Fig. P6.60). (*a*) Recalling that $\sigma_x = \sigma_Y$ between C and E and $\sigma_x = (\sigma_Y/y_Y)y$ between E and K, show that the magnitude of the horizontal shearing force $\mathbf{H}$ exerted on the lower face of the portion of beam $ACKJ$ is

$$H = \frac{1}{2}b\sigma_Y\left(2c - y_Y - \frac{y^2}{y_Y}\right)$$

(*b*) Observing that the shearing stress at K is

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta H}{\Delta A} = \lim_{\Delta x \to 0} \frac{1}{b}\frac{\Delta H}{\Delta x} = \frac{1}{b}\frac{\partial H}{\partial x}$$

and recalling that y_Y is a function of x defined by Eq. (6.14), derive Eq. (6.15).

*6.9 UNSYMMETRIC LOADING OF THIN-WALLED MEMBERS; SHEAR CENTER

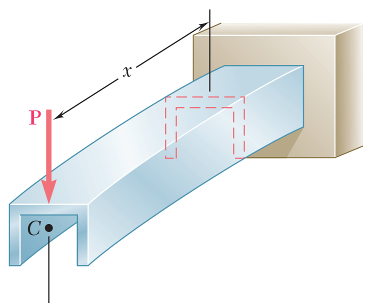


Fig. 6.39 Channel beam.

Our analysis of the effects of transverse loadings in Chap. 5 and in the preceding sections of this chapter was limited to members possessing a vertical plane of symmetry and to loads applied in that plane. The members were observed to bend in the plane of loading (Fig. 6.39) and, in any given cross section, the bending couple $\mathbf{M}$ and the shear $\mathbf{V}$ (Fig. 6.40) were found to result in normal and shearing stresses defined, respectively, by the formulas

$$\sigma_x = -\frac{My}{I} \tag{4.16}$$

and

$$\tau_{\text{ave}} = \frac{VQ}{It} \tag{6.6}$$

In this section, the effects of transverse loadings on *thin-walled members that do not possess a vertical plane of symmetry* will be

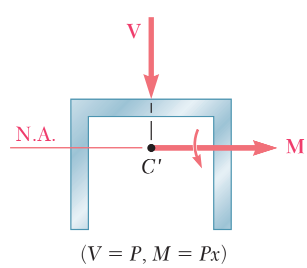


Fig. 6.40 Loaded in vertical plane of symmetry.

examined. Let us assume, for example, that the channel member of Fig. 6.39 has been rotated through 90° and that the line of action of **P** still passes through the centroid of the end section. The couple vector **M** representing the bending moment in a given cross section is still directed along a principal axis of the section (Fig. 6.41), and the neutral axis will coincide with that axis (cf. Sec. 4.13). Equation (4.16), therefore, is applicable and can be used to compute the normal stresses in the section. However, Eq. (6.6) cannot be used to determine the shearing stresses in the section, since this equation was derived for a member possessing a vertical plane of symmetry (cf. Sec. 6.7). Actually, the member will be observed to *bend and twist* under the applied load (Fig. 6.42), and the resulting distribution of shearing stresses will be quite different from that defined by Eq. (6.6).

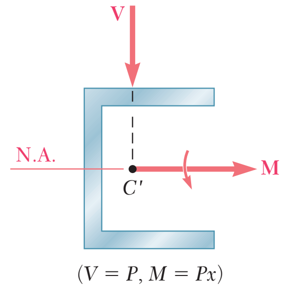


Fig. 6.41 Load perpendicular to vertical plane of symmetry.

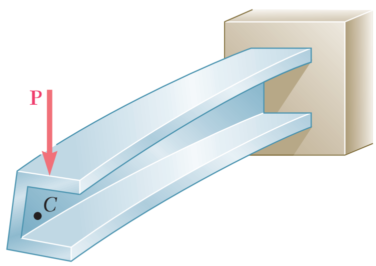


Fig. 6.42 Deformation of channel beam when not loaded in vertical plane of symmetry.

The following question now arises: Is it possible to apply the vertical load **P** in such a way that the channel member of Fig. 6.42 will *bend without twisting* and, if so, where should the load **P** be applied? If the member bends without twisting, then the shearing stress at any point of a given cross section can be obtained from Eq. (6.6), where Q is the first moment of the shaded area with respect to the neutral axis (Fig. 6.43a), and the distribution of stresses will look as shown in Fig. 6.43b, with $\tau = 0$ at both A and E. We note that the shearing force exerted on a small element of cross-sectional area $dA = t\,ds$ is $dF = \tau\,dA = \tau t\,ds$, or $dF = q\,ds$ (Fig.6.44a),

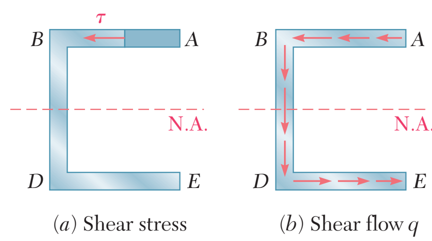


Fig. 6.43 Stresses applied to cross section as a result of load shown in Fig. 6.42.

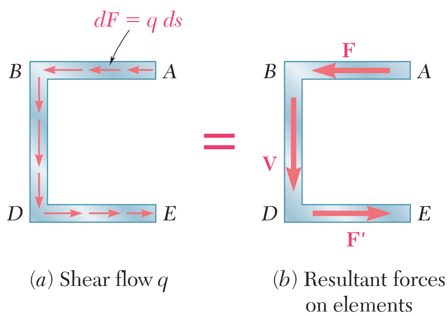


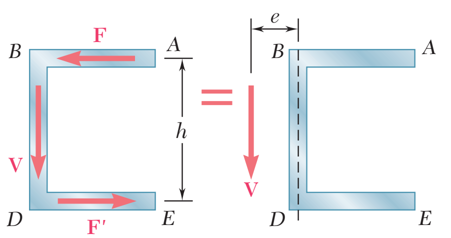
Fig. 6.44

where q is the shear flow $q = \tau t = VQ/I$ at the point considered. The resultant of the shearing forces exerted on the elements of the upper flange AB of the channel is found to be a horizontal force $\mathbf{F}$ (Fig. 6.44*b*) of magnitude

$$F = \int_A^B q\, ds \tag{6.17}$$

Because of the symmetry of the channel section about its neutral axis, the resultant of the shearing forces exerted on the lower flange DE is a force $\mathbf{F}'$ of the same magnitude as $\mathbf{F}$ but of opposite sense. We conclude that the resultant of the shearing forces exerted on the web BD must be equal to the vertical shear $\mathbf{V}$ in the section:

$$V = \int_B^D q\, ds \tag{6.18}$$



(*a*) Resultant forces on elements

(*b*) Placement of **V** to eliminate twisting

Fig. 6.45

We now observe that the forces $\mathbf{F}$ and $\mathbf{F}'$ form a couple of moment Fh, where h is the distance between the center lines of the flanges AB and DE (Fig. 6.45*a*). This couple can be eliminated if the vertical shear $\mathbf{V}$ is moved to the left through a distance e such that the moment of $\mathbf{V}$ about B is equal to Fh (Fig. 6.45*b*). We write $Ve = Fh$ or

$$e = \frac{Fh}{V} \tag{6.19}$$

and conclude that, when the force $\mathbf{P}$ is applied at a distance e to the left of the center line of the web BD, the member bends in a vertical plane without twisting (Fig. 6.46).

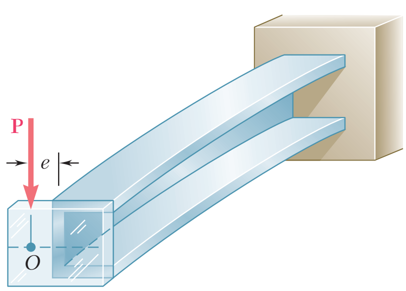
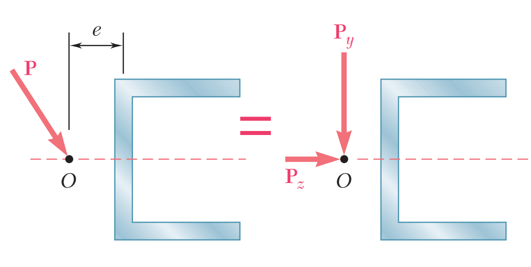


Fig. 6.46 Placement of load to eliminate twisting.

The point O where the line of action of $\mathbf{P}$ intersects the axis of symmetry of the end section is called the *shear center* of that section. We note that, in the case of an oblique load $\mathbf{P}$ (Fig. 6.47*a*), the member will also be free of any twist if the load $\mathbf{P}$ is applied at the shear center of the section. Indeed, the load $\mathbf{P}$ can then be resolved into two components $\mathbf{P}_z$ and $\mathbf{P}_y$ (Fig. 6.47*b*) corresponding respectively to the loading conditions of Figs. 6.39 and 6.46, neither of which causes the member to twist.



(*a*) (*b*)

Fig. 6.47 Beam with oblique load.

EXAMPLE 6.05

Determine the shear center O of a channel section of uniform thickness (Fig. 6.48), knowing that $b = 4$ in., $h = 6$ in., and $t = 0.15$ in.

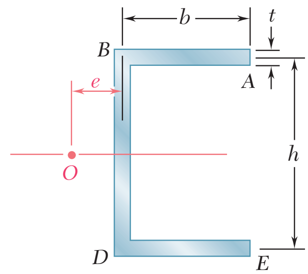


Fig. 6.48

Assuming that the member does not twist, we first determine the shear flow q in flange AB at a distance s from A (Fig. 6.49). Recalling Eq. (6.5) and observing that the first moment Q of the shaded area with respect to the neutral axis is $Q = (st)(h/2)$, we write

$$q = \frac{VQ}{I} = \frac{Vsth}{2I} \tag{6.20}$$

where V is the vertical shear and I the moment of inertia of the section with respect to the neutral axis.

Recalling Eq. (6.17), we determine the magnitude of the shearing force $\mathbf{F}$ exerted on flange AB by integrating the shear flow q from A to B:

$$F = \int_0^b q\,ds = \int_0^b \frac{Vsth}{2I}\,ds = \frac{Vth}{2I}\int_0^b s\,ds$$

$$F = \frac{Vthb^2}{4I} \tag{6.21}$$

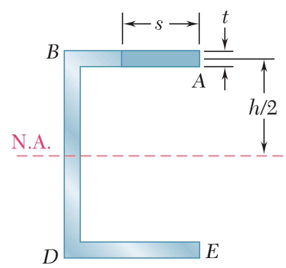


Fig. 6.49

The distance e from the center line of the web BD to the shear center O can now be obtained from Eq. (6.19):

$$e = \frac{Fh}{V} = \frac{Vthb^2}{4I}\frac{h}{V} = \frac{th^2b^2}{4I} \tag{6.22}$$

The moment of inertia I of the channel section can be expressed as follows:

$$\begin{aligned} I &= I_{\text{web}} + 2I_{\text{flange}} \\ &= \frac{1}{12}th^3 + 2\left[\frac{1}{12}bt^3 + bt\left(\frac{h}{2}\right)^2\right] \end{aligned}$$

Neglecting the term containing t^3, which is very small, we have

$$I = \tfrac{1}{12}th^3 + \tfrac{1}{2}tbh^2 = \tfrac{1}{12}th^2(6b + h) \tag{6.23}$$

Substituting this expression into (6.22), we write

$$e = \frac{3b^2}{6b + h} = \frac{b}{2 + \dfrac{h}{3b}} \tag{6.24}$$

We note that the distance e does not depend upon t and can vary from 0 to $b/2$, depending upon the value of the ratio $h/3b$. For the given channel section, we have

$$\frac{h}{3b} = \frac{6 \text{ in.}}{3(4 \text{ in.})} = 0.5$$

and

$$e = \frac{4 \text{ in.}}{2 + 0.5} = 1.6 \text{ in.}$$

EXAMPLE 6.06

For the channel section of Example 6.05 determine the distribution of the shearing stresses caused by a 2.5-kip vertical shear **V** applied at the shear center O (Fig. 6.50).

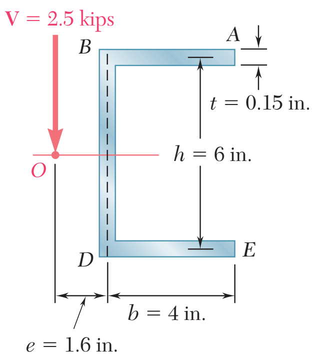


Fig. 6.50

Shearing stresses in flanges. Since **V** is applied at the shear center, there is no torsion, and the stresses in flange AB are obtained from Eq. (6.20) of Example 6.05. We have

$$\tau = \frac{q}{t} = \frac{VQ}{It} = \frac{Vh}{2I}s \tag{6.25}$$

which shows that the stress distribution in flange AB is linear. Letting $s = b$ and substituting for I from Eq. (6.23), we obtain the value of the shearing stress at B:

$$\tau_B = \frac{Vhb}{2(\frac{1}{12}th^2)(6b + h)} = \frac{6Vb}{th(6b + h)} \tag{6.26}$$

Letting $V = 2.5$ kips, and using the given dimensions, we have

$$\tau_B = \frac{6(2.5 \text{ kips})(4 \text{ in.})}{(0.15 \text{ in.})(6 \text{ in.})(6 \times 4 \text{ in.} + 6 \text{ in.})}$$
$$= 2.22 \text{ ksi}$$

Shearing stresses in web. The distribution of the shearing stresses in the web BD is parabolic, as in the case of a W-beam, and the maximum stress occurs at the neutral axis. Computing the first moment of the upper half of the cross section with respect to the neutral axis (Fig. 6.51), we write

$$Q = bt(\tfrac{1}{2}h) + \tfrac{1}{2}ht(\tfrac{1}{4}h) = \tfrac{1}{8}ht(4b + h) \tag{6.27}$$

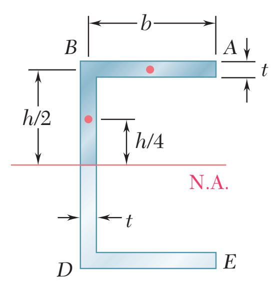


Fig. 6.51

Substituting for I and Q from (6.23) and (6.27), respectively, into the expression for the shearing stress, we have

$$\tau_{\max} = \frac{VQ}{It} = \frac{V(\frac{1}{8}ht)(4b + h)}{\frac{1}{12}th^2(6b + h)t} = \frac{3V(4b + h)}{2th(6b + h)}$$

or, with the given data,

$$\tau_{\max} = \frac{3(2.5 \text{ kips})(4 \times 4 \text{ in.} + 6 \text{ in.})}{2(0.15 \text{ in.})(6 \text{ in.})(6 \times 4 \text{ in.} + 6 \text{ in.})}$$
$$= 3.06 \text{ ksi}$$

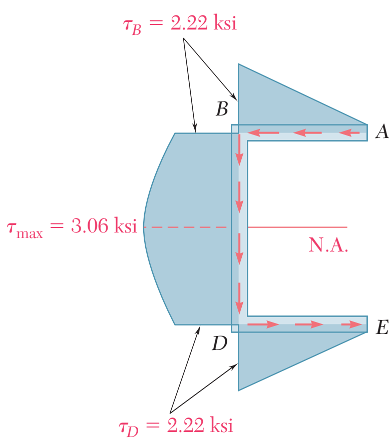


Fig. 6.52

Distribution of stresses over the section. The distribution of the shearing stresses over the entire channel section has been plotted in Fig. 6.52.

EXAMPLE 6.07

For the channel section of Example 6.05, and neglecting stress concentrations, determine the maximum shearing stress caused by a 2.5-kip vertical shear **V** applied at the centroid C of the section, which is located 1.143 in. to the right of the center line of the web BD (Fig. 6.53).

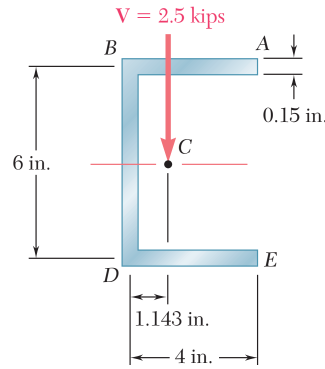


Fig. 6.53

Equivalent force-couple system at shear center. The shear center O of the cross section was determined in Example 6.05 and found to be at a distance e = 1.6 in. to the left of the center line of the web BD. We replace the shear **V** (Fig. 6.54*a*) by an equivalent force-couple system at the shear center O (Fig. 6.54*b*). This system consists of a 2.5-kip force **V** and of a torque **T** of magnitude

$$T = V(OC) = (2.5 \text{ kips})(1.6 \text{ in.} + 1.143 \text{ in.})$$
$$= 6.86 \text{ kip} \cdot \text{in.}$$

Stresses due to bending. The 2.5-kip force **V** causes the member to bend, and the corresponding distribution of shearing stresses in the section (Fig. 6.54*c*) was determined in Example 6.06. We recall that the maximum value of the stress due to this force was found to be

$$(\tau_{\max})_{\text{bending}} = 3.06 \text{ ksi}$$

Stresses due to twisting. The torque **T** causes the member to twist, and the corresponding distribution of stresses is shown in Fig. 6.54*d*. We recall from Sec. 3.12 that the membrane analogy shows that, in a thin-walled member of uniform thickness, the stress caused by a torque **T** is maximum along the edge of the section. Using Eqs. (3.45) and (3.43) with

$$a = 4 \text{ in.} + 6 \text{ in.} + 4 \text{ in.} = 14 \text{ in.}$$
$$b = t = 0.15 \text{ in.} \qquad b/a = 0.0107$$

we have

$$c_1 = \tfrac{1}{3}(1 - 0.630b/a) = \tfrac{1}{3}(1 - 0.630 \times 0.0107) = 0.331$$
$$(\tau_{\max})_{\text{twisting}} = \frac{T}{c_1 ab^2} = \frac{6.86 \text{ kip} \cdot \text{in.}}{(0.331)(14 \text{ in.})(0.15 \text{ in.})^2} = 65.8 \text{ ksi}$$

Combined stresses. The maximum stress due to the combined bending and twisting occurs at the neutral axis, on the inside surface of the web, and is

$$\tau_{\max} = 3.06 \text{ ksi} + 65.8 \text{ ksi} = 68.9 \text{ ksi}$$

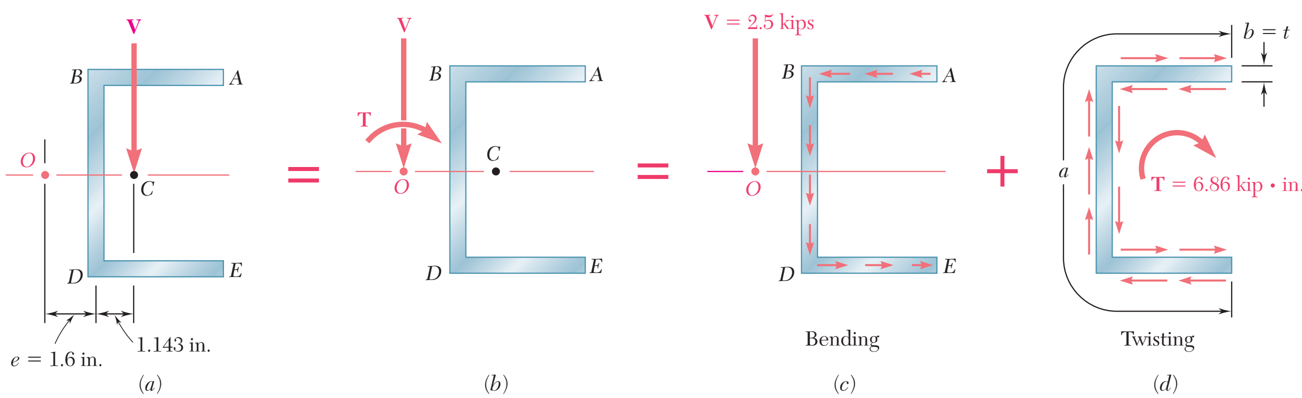


Fig. 6.54

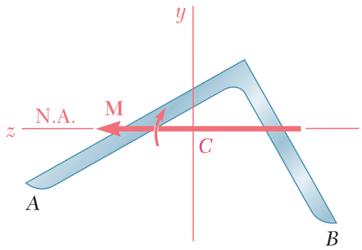
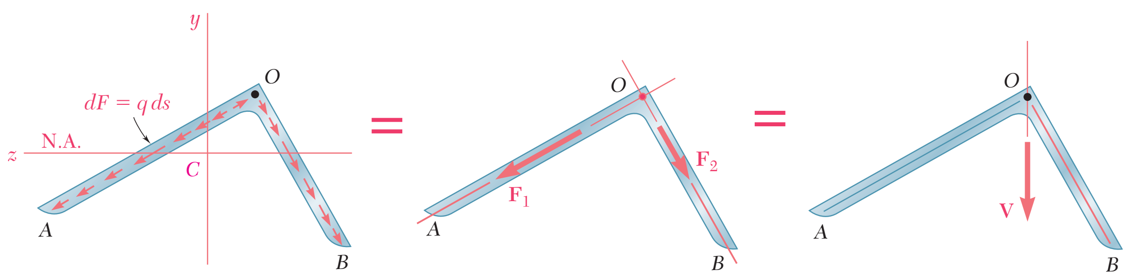


Fig. 6.55 Beam without plane of symmetry.

Turning our attention to thin-walled members possessing no plane of symmetry, we now consider the case of an angle shape subjected to a vertical load **P.** If the member is oriented in such a way that the load **P** is perpendicular to one of the principal centroidal axes Cz of the cross section, the couple vector **M** representing the bending moment in a given section will be directed along Cz (Fig. 6.55), and the neutral axis will coincide with that axis (cf. Sec. 4.13). Equation (4.16), therefore, is applicable and can be used to compute the normal stresses in the section. We now propose to determine where the load **P** should be applied if Eq. (6.6) is to define the shearing stresses in the section, i.e., if the member is to *bend without twisting*.

Let us *assume* that the shearing stresses in the section are defined by Eq. (6.6). As in the case of the channel member considered earlier, the elementary shearing forces exerted on the section can be expressed as $dF = q\,ds$, with $q = VQ/I$, where Q represents a first moment with respect to the neutral axis (Fig. 6.56*a*). We note that the resultant of the shearing forces exerted on portion OA of the cross section is a force $\mathbf{F}_1$ directed along OA, and that the resultant of the shearing forces exerted on portion OB is a force $\mathbf{F}_2$ along OB (Fig. 6.56*b*). Since both $\mathbf{F}_1$ and $\mathbf{F}_2$ pass through point O at the corner of the angle, it follows that their own resultant, which is the shear **V** in the section, must also pass through O (Fig. 6.56*c*). We conclude that the member will not be twisted if the line of action of the load **P** passes through the corner O of the section in which it is applied.



(*a*) Shear stresses (*b*) Resultant forces on elements (*c*) Placement of **V** to eliminate twisting

Fig. 6.56

The same reasoning can be applied when the load **P** is perpendicular to the other principal centroidal axis Cy of the angle section. And, since any load **P** applied at the corner O of a cross section can be resolved into components perpendicular to the principal axes, it follows that the member will not be twisted if each load is applied at the corner O of a cross section. We thus conclude that O is the shear center of the section.

Angle shapes with one vertical and one horizontal leg are encountered in many structures. It follows from the preceding discussion that such members will not be twisted if vertical loads are applied along the center line of their vertical leg. We note from Fig. 6.57 that the resultant of the elementary shearing forces exerted on the vertical portion OA of a given section will be equal to the

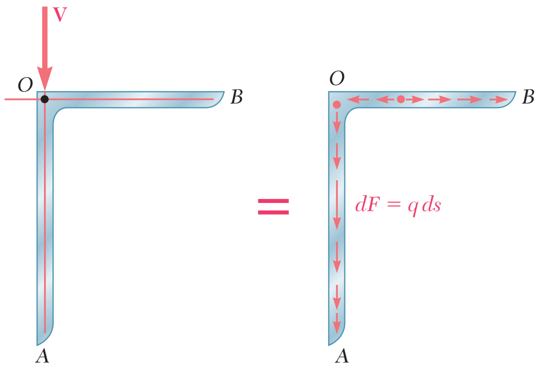


Fig. 6.57 Angle section.

shear **V**, while the resultant of the shearing forces on the horizontal portion OB will be zero:

$$\int_O^A q\,ds = V \qquad \int_O^B q\,ds = 0$$

This does *not* mean, however, that there will be no shearing stress in the horizontal leg of the member. By resolving the shear **V** into components perpendicular to the principal centroidal axes of the section and computing the shearing stress at every point, we would verify that τ is zero at only one point between O and B (see Sample Prob. 6.6).

Another type of thin-walled member frequently encountered in practice is the Z shape. While the cross section of a Z shape does not possess any axis of symmetry, it does possess a *center of symmetry* O (Fig. 6.58). This means that, to any point H of the cross section corresponds another point H' such that the segment of straight line HH' is bisected by O. Clearly, the center of symmetry O coincides with the centroid of the cross section. As you will see presently, point O is also the shear center of the cross section.

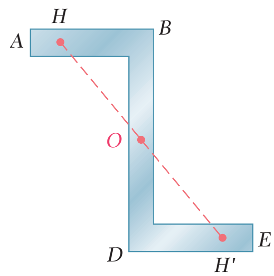


Fig. 6.58 *Z* section.

As we did earlier in the case of an angle shape, we assume that the loads are applied in a plane perpendicular to one of the principal axes of the section, so that this axis is also the neutral axis of the section (Fig. 6.59). We further assume that the shearing stresses in the section are defined by Eq. (6.6), i.e., that the member is bent without being twisted. Denoting by Q the first moment about the neutral axis of portion AH of the cross section, and by Q' the first moment of portion EH', we note that $Q' = -Q$. Thus the shearing stresses at H and H' have the same magnitude and the same direction, and the shearing forces exerted on small elements of area dA located respectively at H and H' are equal forces that have equal and opposite moments about O (Fig. 6.60). Since this is true for any pair of symmetric elements, it follows that the resultant of the shearing forces exerted on the section has a zero moment about O. This means that the shear **V** in the section is directed along a line that passes through O. Since this analysis can be repeated when the loads are applied in a plane perpendicular to the other principal axis, we conclude that point O is the shear center of the section.

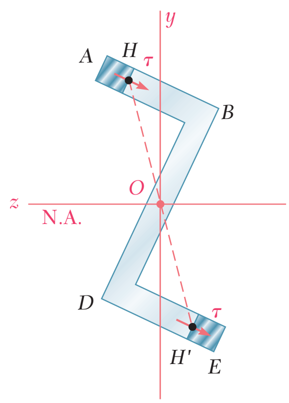


Fig. 6.59

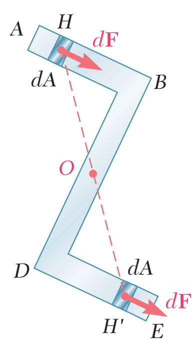
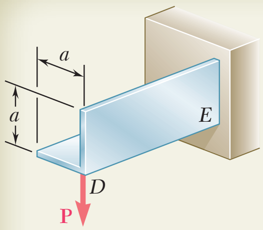


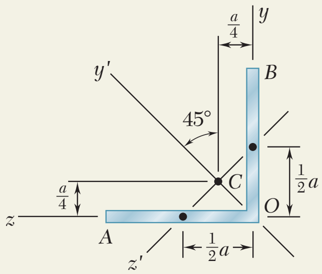
Fig. 6.60

SAMPLE PROBLEM 6.6

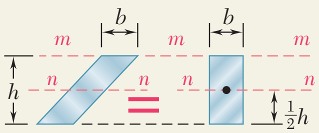


Determine the distribution of shearing stresses in the thin-walled angle shape DE of uniform thickness t for the loading shown.

SOLUTION



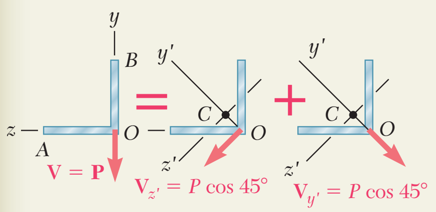
Shear Center. We recall from Sec. 6.9 that the shear center of the cross section of a thin-walled angle shape is located at its corner. Since the load **P** is applied at D, it causes bending but no twisting of the shape.



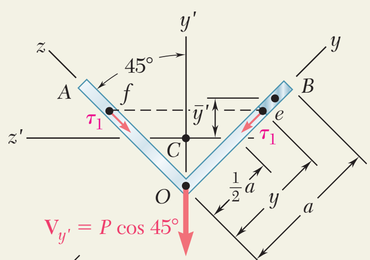
Principal Axes. We locate the centroid C of a given cross section AOB. Since the y' axis is an axis of symmetry, the y' and z' axes are the principal centroidal axes of the section. We recall that for the parallelogram shown $I_{nn} = \frac{1}{12}bh^3$ and $I_{mm} = \frac{1}{3}bh^3$. Considering each leg of the section as a parallelogram, we now determine the centroidal moments of inertia $I_{y'}$ and $I_{z'}$:

$$I_{y'} = 2\left[\frac{1}{3}\left(\frac{t}{\cos 45°}\right)(a \cos 45°)^3\right] = \frac{1}{3}ta^3$$

$$I_{z'} = 2\left[\frac{1}{12}\left(\frac{t}{\cos 45°}\right)(a \cos 45°)^3\right] = \frac{1}{12}ta^3$$



Superposition. The shear **V** in the section is equal to the load **P.** We resolve it into components parallel to the principal axes.



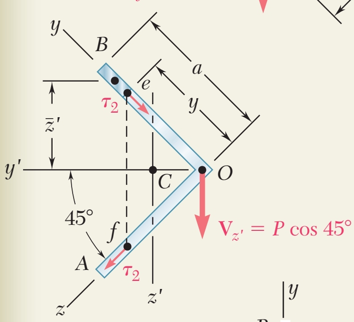
Shearing Stresses Due to $\mathbf{V}_{y'}$. We determine the shearing stress at point e of coordinate y:

$$\bar{y}' = \tfrac{1}{2}(a + y)\cos 45° - \tfrac{1}{2}a \cos 45° = \tfrac{1}{2}y \cos 45°$$

$$Q = t(a - y)\bar{y}' = \tfrac{1}{2}t(a - y)y \cos 45°$$

$$\tau_1 = \frac{V_{y'}Q}{I_{z'}t} = \frac{(P \cos 45°)[\frac{1}{2}t(a - y)y \cos 45°]}{(\frac{1}{12}ta^3)t} = \frac{3P(a - y)y}{ta^3}$$

The shearing stress at point f is represented by a similar function of z.



Shearing Stresses Due to $\mathbf{V}_{z'}$. We again consider point e:

$$\bar{z}' = \tfrac{1}{2}(a + y)\cos 45°$$

$$Q = (a - y)t\bar{z}' = \tfrac{1}{2}(a^2 - y^2)t \cos 45°$$

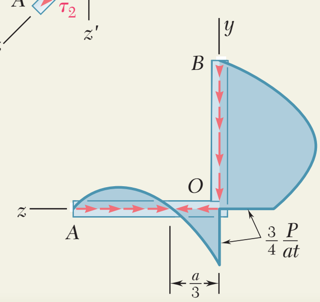
$$\tau_2 = \frac{V_{z'}Q}{I_{y'}t} = \frac{(P \cos 45°)[\frac{1}{2}(a^2 - y^2)t \cos 45°]}{(\frac{1}{3}ta^3)t} = \frac{3P(a^2 - y^2)}{4ta^3}$$

The shearing stress at point f is represented by a similar function of z.

Combined Stresses. ***Along the Vertical Leg.*** The shearing stress at point e is

$$\tau_e = \tau_2 + \tau_1 = \frac{3P(a^2 - y^2)}{4ta^3} + \frac{3P(a - y)y}{ta^3} = \frac{3P(a - y)}{4ta^3}[(a + y) + 4y]$$

$$\tau_e = \frac{3P(a - y)(a + 5y)}{4ta^3} \blacktriangleleft$$



Along the Horizontal Leg. The shearing stress at point f is

$$\tau_f = \tau_2 - \tau_1 = \frac{3P(a^2 - z^2)}{4ta^3} - \frac{3P(a - z)z}{ta^3} = \frac{3P(a - z)}{4ta^3}[(a + z) - 4z]$$

$$\tau_f = \frac{3P(a - z)(a - 3z)}{4ta^3} \blacktriangleleft$$

PROBLEMS

6.61 and 6.62 Determine the location of the shear center *O* of a thin-walled beam of uniform thickness having the cross section shown.

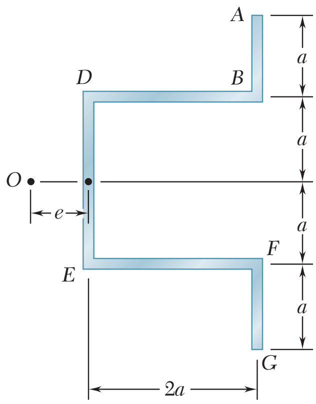


Fig. P6.61

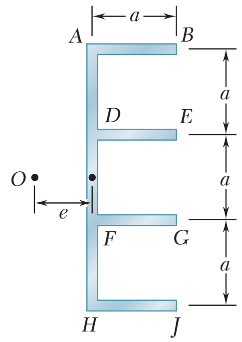


Fig. P6.62

6.63 through 6.66 An extruded beam has the cross section shown. Determine (*a*) the location of the shear center *O*, (*b*) the distribution of the shearing stresses caused by the vertical shearing force **V** shown applied at *O*.

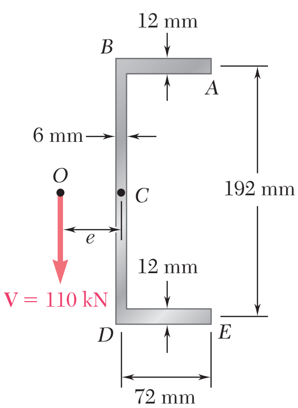


Fig. P6.63

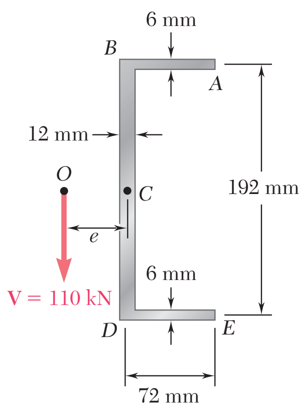


Fig. P6.64

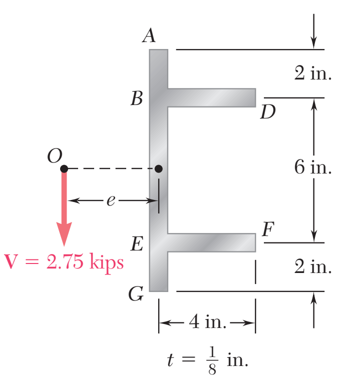


Fig. *P6.65*

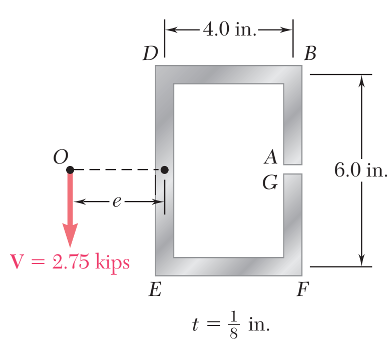
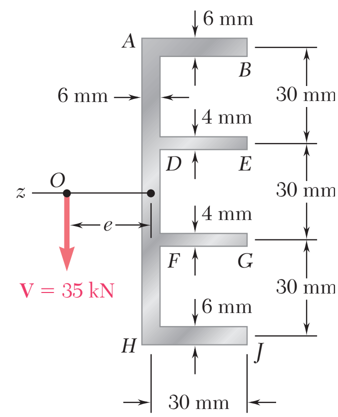


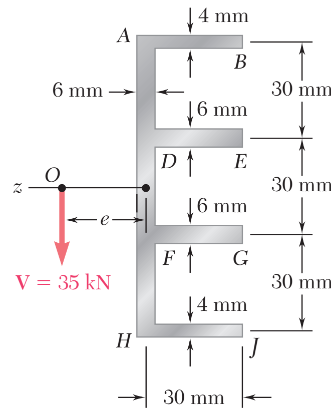
Fig. *P6.66*

6.67 through 6.68 An extruded beam has the cross section shown. Determine (*a*) the location of the shear center *O*, (*b*) the distribution of the shearing stresses caused by the vertical shearing force **V** shown applied at *O*.



$I_z = 1.149 \times 10^6$ mm^4

Fig. P6.67



$I_z = 0.933 \times 10^6$ mm^4

Fig. P6.68

6.69 through *6.74* Determine the location of the shear center *O* of a thin-walled beam of uniform thickness having the cross section shown.

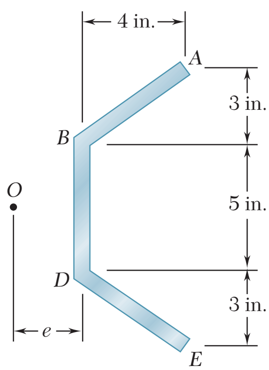


Fig. P6.69

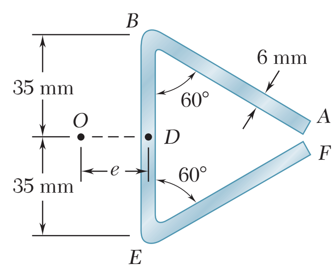


Fig. P6.70

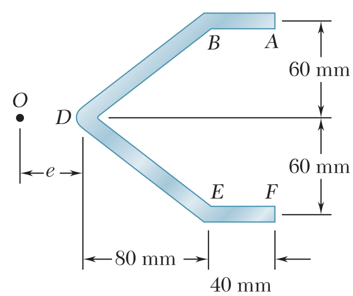


Fig. P6.71

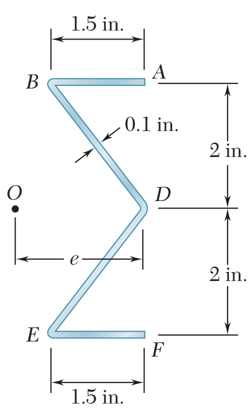


Fig. P6.72

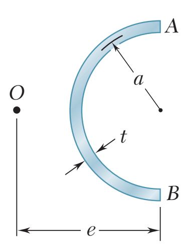


Fig. *P6.73*

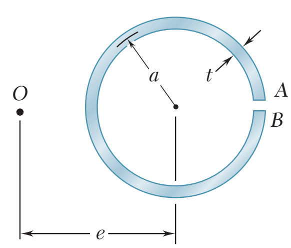


Fig. *P6.74*

6.75 and 6.76 A thin-walled beam has the cross section shown. Determine the location of the shear center O of the cross section.

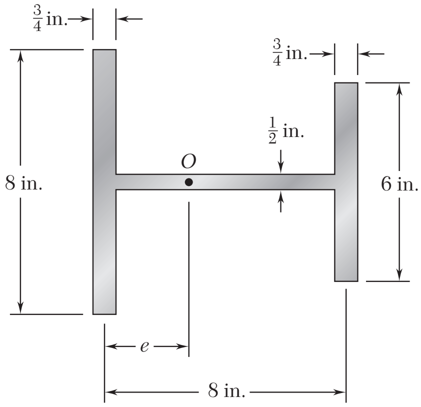


Fig. P6.75

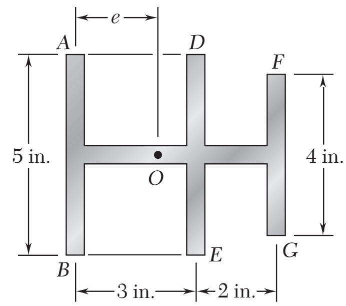


Fig. P6.76

6.77 and 6.78 A thin-walled beam of uniform thickness has the cross section shown. Determine the dimension b for which the shear center O of the cross section is located at the point indicated.

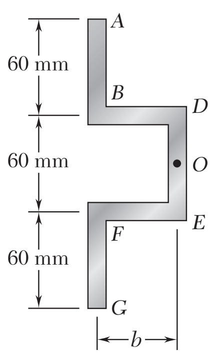


Fig. P6.77

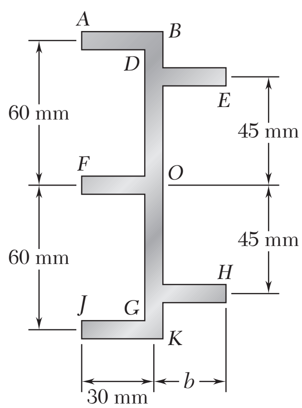


Fig. P6.78

6.79 For the angle shape and loading of Sample Prob. 6.6, check that $\int q\, dz = 0$ along the horizontal leg of the angle and $\int q\, dy = P$ along its vertical leg.

6.80 For the angle shape and loading of Sample Prob. 6.6, (*a*) determine the points where the shearing stress is maximum and the corresponding values of the stress, (*b*) verify that the points obtained are located on the neutral axis corresponding to the given loading.

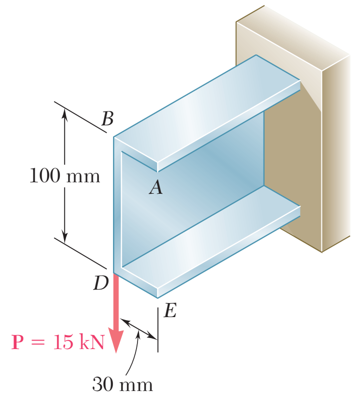


Fig. P6.81

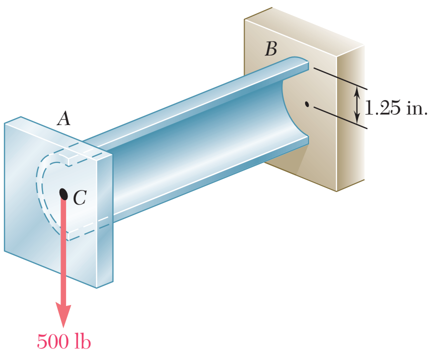


Fig. P6.83

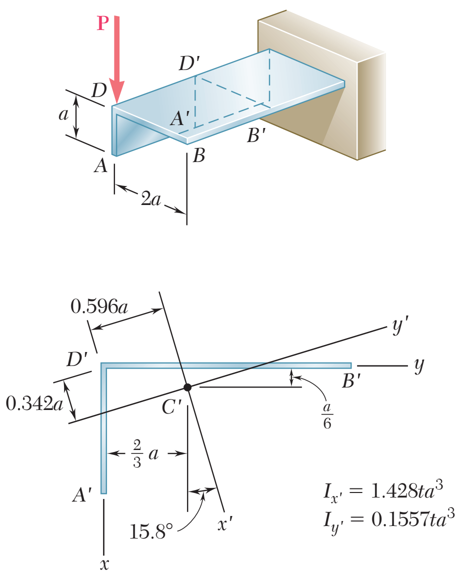


Fig. P6.87

***6.81** A steel plate, 160 mm wide and 8 mm thick, is bent to form the channel shown. Knowing that the vertical load **P** acts at a point in the midplane of the web of the channel, determine (*a*) the torque **T** that would cause the channel to twist in the same way that it does under the load **P,** (*b*) the maximum shearing stress in the channel caused by the load **P.**

***6.82** Solve Prob. 6.81, assuming that a 6-mm-thick plate is bent to form the channel shown.

***6.83** The cantilever beam *AB*, consisting of half of a thin-walled pipe of 1.25-in. mean radius and $\frac{3}{8}$-in. wall thickness, is subjected to a 500-lb vertical load. Knowing that the line of action of the load passes through the centroid *C* of the cross section of the beam, determine (*a*) the equivalent force-couple system at the shear center of the cross section, (*b*) the maximum shearing stress in the beam. (*Hint*: The shear center *O* of this cross section was shown in Prob. 6.73 to be located twice as far from its vertical diameter as its centroid *C*.)

***6.84** Solve Prob. 6.83, assuming that the thickness of the beam is reduced to $\frac{1}{4}$ in.

***6.85** The cantilever beam shown consists of a Z shape of $\frac{1}{4}$-in. thickness. For the given loading, determine the distribution of the shearing stresses along line $A'B'$ in the upper horizontal leg of the Z shape. The x' and y' axes are the principal centroidal axes of the cross section and the corresponding moments of inertia are $I_{x'} = 166.3\ \text{in}^4$ and $I_{y'} = 13.61\ \text{in}^4$.

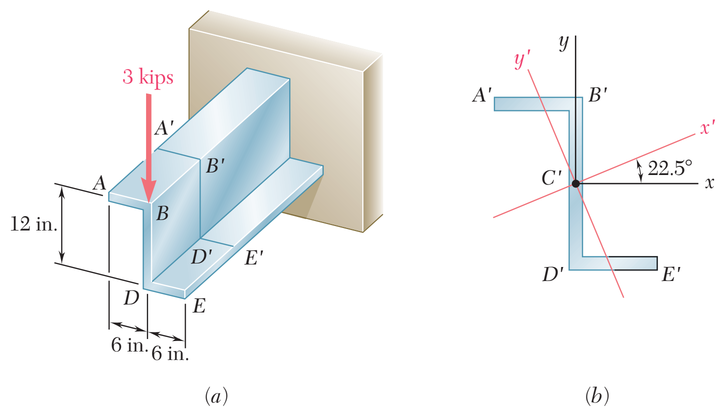


Fig. P6.85

***6.86** For the cantilever beam and loading of Prob. 6.85, determine the distribution of the shearing stress along line $B'D'$ in the vertical web of the Z shape.

***6.87** Determine the distribution of the shearing stresses along line $D'B'$ in the horizontal leg of the angle shape for the loading shown. The x' and y' axes are the principal centroidal axes of the cross section.

***6.88** For the angle shape and loading of Prob. 6.87, determine the distribution of the shearing stresses along line $D'A'$ in the vertical leg.

REVIEW AND SUMMARY

This chapter was devoted to the analysis of beams and thin-walled members under transverse loadings.

Stresses on a beam element

In Sec. 6.1 we considered a small element located in the vertical plane of symmetry of a beam under a transverse loading (Fig. 6.61) and found that normal stresses σ_x and shearing stresses τ_{xy} were exerted on the transverse faces of that element, while shearing stresses τ_{yx}, equal in magnitude to τ_{xy}, were exerted on its horizontal faces.

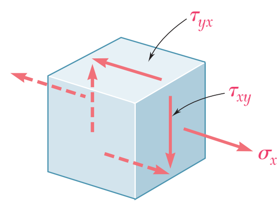


Fig. 6.61

In Sec. 6.2 we considered a prismatic beam AB with a vertical plane of symmetry supporting various concentrated and distributed loads (Fig. 6.62). At a distance x from end A we detached from the beam an element $CDD'C'$ of length Δx extending across the width of the beam from the upper surface of the beam to a horizontal plane located at a distance y_1 from the neutral axis (Fig. 6.63). We found

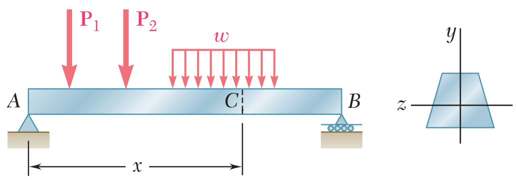


Fig. 6.62

Horizontal shear in a beam

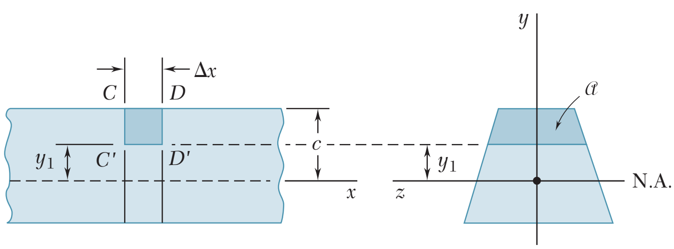


Fig. 6.63

that the magnitude of the shearing force $\Delta\mathbf{H}$ exerted on the lower face of the beam element was

$$\Delta H = \frac{VQ}{I}\Delta x \tag{6.4}$$

where V = vertical shear in the given transverse section
Q = first moment with respect to the neutral axis of the shaded portion $\mathcal{a}$ of the section
I = centroidal moment of inertia of the entire cross-sectional area

Shear flow

The *horizontal shear per unit length,* or *shear flow,* which was denoted by the letter q, was obtained by dividing both members of Eq. (6.4) by Δx:

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} \tag{6.5}$$

Shearing stresses in a beam

Dividing both members of Eq. (6.4) by the area ΔA of the horizontal face of the element and observing that $\Delta A = t\ \Delta x$, where t is the width of the element at the cut, we obtained in Sec. 6.3 the following expression for the *average shearing stress* on the horizontal face of the element

$$\tau_{\text{ave}} = \frac{VQ}{It} \tag{6.6}$$

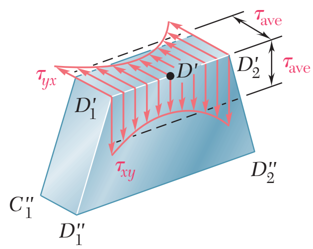


Fig. 6.64

We further noted that, since the shearing stresses τ_{xy} and τ_{yx} exerted, respectively, on a transverse and a horizontal plane through D' are equal, the expression in (6.6) also represents the average value of τ_{xy} along the line $D'_1D'_2$ (Fig. 6.64).

Shearing stresses in a beam of rectangular cross section

In Secs. 6.4 and 6.5 we analyzed the shearing stresses in a beam of rectangular cross section. We found that the distribution of stresses is parabolic and that the maximum stress, which occurs at the center of the section, is

$$\tau_{\max} = \frac{3}{2}\frac{V}{A} \tag{6.10}$$

where A is the area of the rectangular section. For wide-flange beams, we found that a good approximation of the maximum shearing stress can be obtained by dividing the shear V by the cross-sectional area of the web.

Longitudinal shear on curved surface

In Sec. 6.6 we showed that Eqs. (6.4) and (6.5) could still be used to determine, respectively, the longitudinal shearing force $\Delta \mathbf{H}$ and the shear flow q exerted on a beam element if the element was bounded by an arbitrary curved surface instead of a horizontal plane (Fig. 6.65).

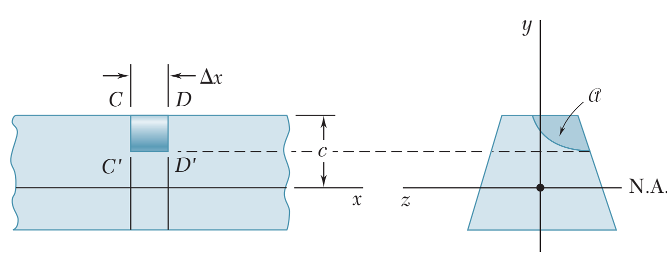


Fig. 6.65

This made it possible for us in Sec. 6.7 to extend the use of Eq. (6.6) to the determination of the average shearing stress in thin-walled members such as wide-flange beams and box beams, in the flanges of such members, and in their webs (Fig. 6.66).

Shearing stresses in thin-walled members

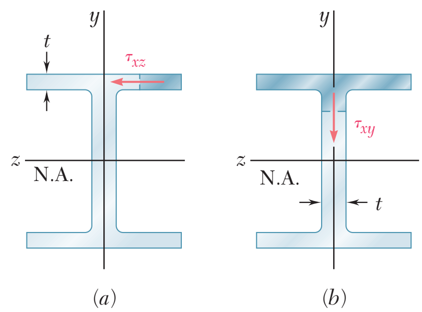


Fig. 6.66

Plastic deformations

In Sec. 6.8 we considered the effect of plastic deformations on the magnitude and distribution of shearing stresses. From Chap. 4 we recalled that once plastic deformation has been initiated, additional loading causes plastic zones to penetrate into the elastic core of a beam. After demonstrating that shearing stresses can occur only in the elastic core of a beam, we noted that both an increase in loading and the resulting decrease in the size of the elastic core contribute to an increase in shearing stresses.

Unsymmetric loading shear center

In Sec. 6.9 we considered prismatic members that are *not* loaded in their plane of symmetry and observed that, in general, both bending and twisting will occur. You learned to locate the point O of the cross section, known as the *shear center,* where the loads should be applied if the member is to bend without twisting (Fig. 6.67) and found that if the loads are applied at that point, the following equations remain valid:

$$\sigma_x = -\frac{My}{I} \qquad \tau_{\text{ave}} = \frac{VQ}{It} \qquad (4.16,\ 6.6)$$

Using the principle of superposition, you also learned to determine the stresses in unsymmetric thin-walled members such as channels, angles, and extruded beams [Example 6.07 and Sample Prob. 6.6]

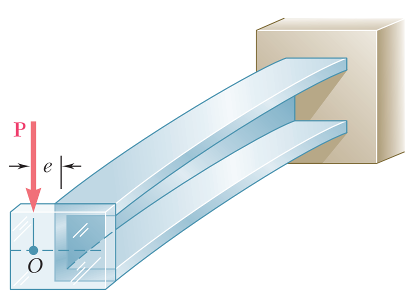


Fig. 6.67

REVIEW PROBLEMS

6.89 A square box beam is made of two 20 × 80-mm planks and two 20 × 120-mm planks nailed together as shown. Knowing that the spacing between the nails is $s = 30$ mm and that the vertical shear in the beam is $V = 1200$ N, determine (*a*) the shearing force in each nail, (*b*) the maximum shearing stress in the beam.

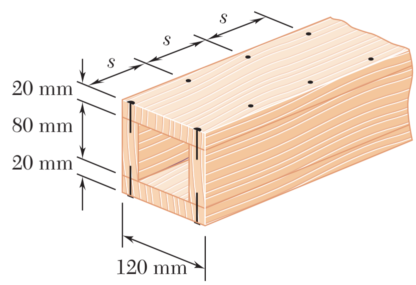


Fig. P6.89

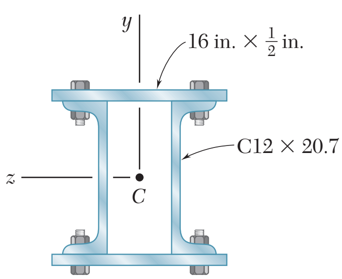
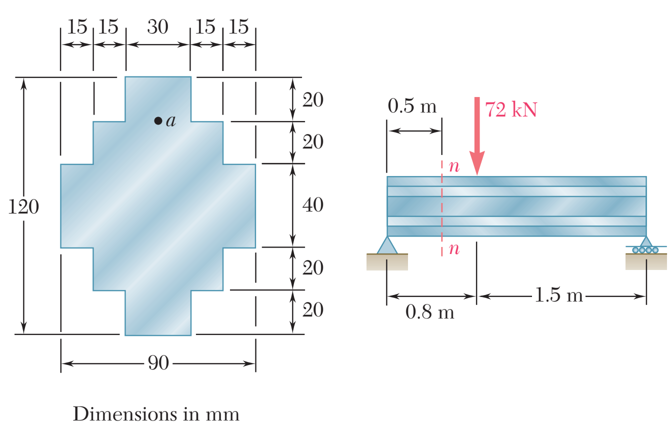


Fig. P6.90

6.90 The beam shown is fabricated by connecting two channel shapes and two plates, using bolts of $\frac{3}{4}$-in. diameter spaced longitudinally every 7.5 in. Determine the average shearing stress in the bolts caused by a shearing force of 25 kips parallel to the y axis.

6.91 For the beam and loading shown, consider section n-n and determine (*a*) the largest shearing stress in that section, (*b*) the shearing stress at point a.



Dimensions in mm

Fig. *P6.91*

6.92 For the beam and loading shown, determine the minimum required width b, knowing that for the grade of timber used, $\sigma_{all} = 12$ MPa and $\tau_{all} = 825$ kPa.

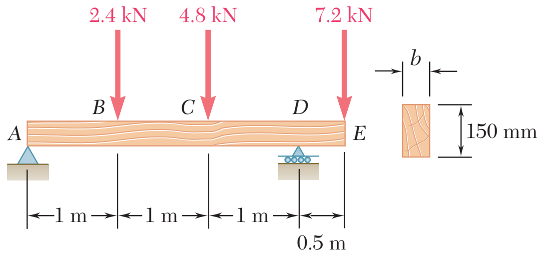


Fig. P6.92

6.93 For the beam and loading shown, consider section n-n and determine the shearing stress at (*a*) point a, (*b*) point b.

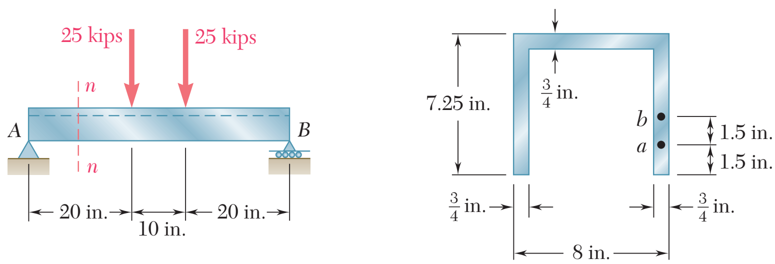


Fig. P6.93 and *P6.94*

6.94 For the beam and loading shown, determine the largest shearing stress in section n-n.

6.95 The composite beam shown is made by welding C200 × 17.1 rolled-steel channels to the flanges of a W250 × 80 wide-flange rolled-steel shape. Knowing that the beam is subjected to a vertical shear of 200 kN, determine (*a*) the horizontal shearing force per meter at each weld, (*b*) the shearing stress at point a of the flange of the wide-flange shape.

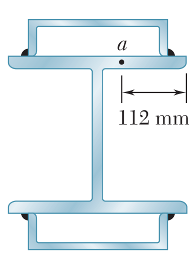


Fig. P6.95

6.96 An extruded beam has the cross section shown and a uniform wall thickness of 3 mm. For a vertical shear of 10 kN, determine (*a*) the shearing stress at point *A*, (*b*) the maximum shearing stress in the beam. Also sketch the shear flow in the cross section.

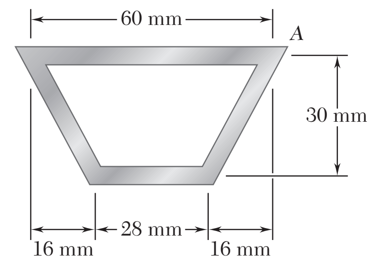


Fig. P6.96

6.97 The design of a beam requires welding four horizontal plates to a vertical 0.5×5-in. plate as shown. For a vertical shear **V**, determine the dimension h for which the shear flow through the welded surfaces is maximum.

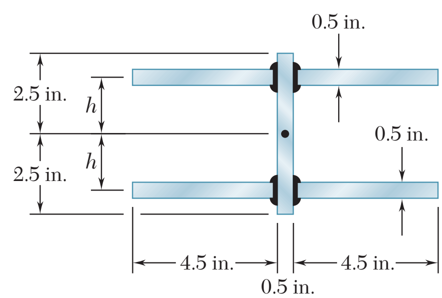


Fig. *P6.97*

6.98 Determine the location of the shear center *O* of a thin-walled beam of uniform thickness having the cross section shown.

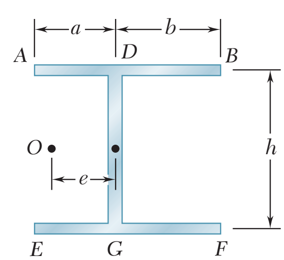


Fig. P6.98

6.99 Determine the location of the shear center O of a thin-walled beam of uniform thickness having the cross section shown.

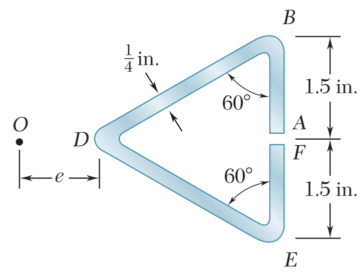


Fig. P6.99

6.100 A thin-walled beam of uniform thickness has the cross section shown. Determine the dimension b for which the shear center O of the cross section is located at the point indicated.

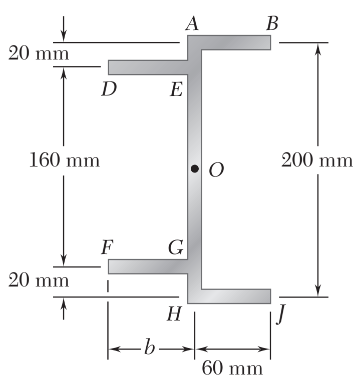


Fig. *P6.100*

COMPUTER PROBLEMS

The following problems are designed to be solved with a computer.

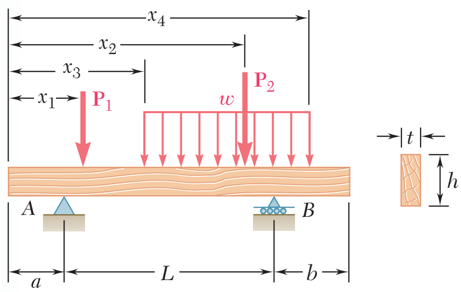


Fig. P6.C1

6.C1 A timber beam is to be designed to support a distributed load and up to two concentrated loads as shown. One of the dimensions of its uniform rectangular cross section has been specified and the other is to be determined so that the maximum normal stress and the maximum shearing stress in the beam will not exceed given allowable values σ_{all} and τ_{all}. Measuring x from end A and using either SI or U.S. customary units, write a computer program to calculate for successive cross sections, from $x = 0$ to $x = L$ and using given increments Δx, the shear, the bending moment, and the smallest value of the unknown dimension that satisfies in that section (1) the allowable normal stress requirement, (2) the allowable shearing stress requirement. Use this program to solve Prob. 5.65 assuming $\sigma_{\text{all}} = 12$ MPa and $\tau_{\text{all}} = 825$ kPa, using $\Delta x = 0.1$ m.

6.C2 A cantilever timber beam AB of length L and of uniform rectangular section shown supports a concentrated load **P** at its free end and a uniformly distributed load w along its entire length. Write a computer program to determine the length L and the width b of the beam for which both the maximum normal stress and the maximum shearing stress in the beam reach their largest allowable values. Assuming $\sigma_{\text{all}} = 1.8$ ksi and $\tau_{\text{all}} = 120$ psi, use this program to determine the dimensions L and b when (a) $P = 1000$ lb and $w = 0$, (b) $P = 0$ and $w = 12.5$ lb/in., (c) $P = 500$ lb and $w = 12.5$ lb/in.

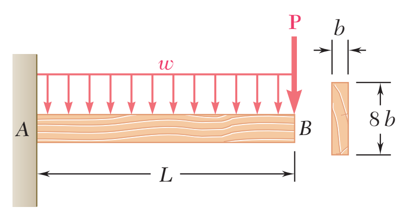


Fig. P6.C2

6.C3 A beam having the cross section shown is subjected to a vertical shear **V.** Write a computer program that, for loads and dimensions expressed in either SI or U.S. customary units, can be used to calculate the shearing stress along the line between any two adjacent rectangular areas forming the cross section. Use this program to solve (a) Prob. 6.10, (b) Prob. 6.12, (c) Prob. 6.21.

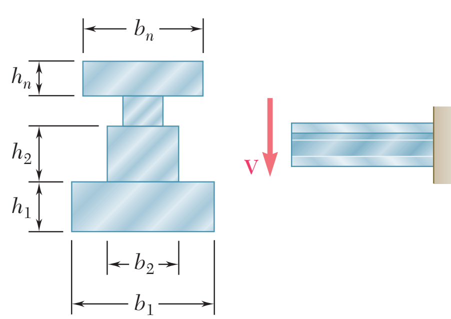


Fig. P6.C3

6.C4 A plate of uniform thickness t is bent as shown into a shape with a vertical plane of symmetry and is then used as a beam. Write a computer program that, for loads and dimensions expressed in either SI or U.S. customary units, can be used to determine the distribution of shearing stresses caused by a vertical shear **V.** Use this program (*a*) to solve Prob. 6.47, (*b*) to find the shearing stress at a point E for the shape and load of Prob. 6.50, assuming a thickness $t = \frac{1}{4}$ in.

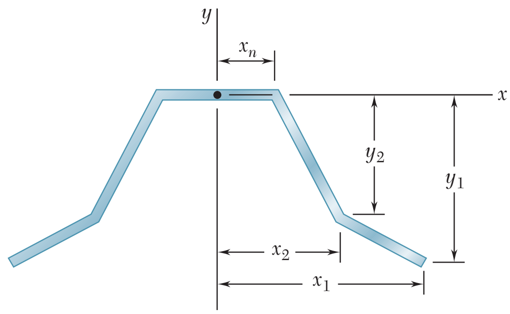


Fig. P6.C4

6.C5 The cross section of an extruded beam is symmetric with respect to the x axis and consists of several straight segments as shown. Write a computer program that, for loads and dimensions expressed in either SI or U.S. customary units, can be used to determine (*a*) the location of the shear center O, (*b*) the distribution of shearing stresses caused by a vertical force applied at O. Use this program to solve Probs. 6.66 and 6.70.

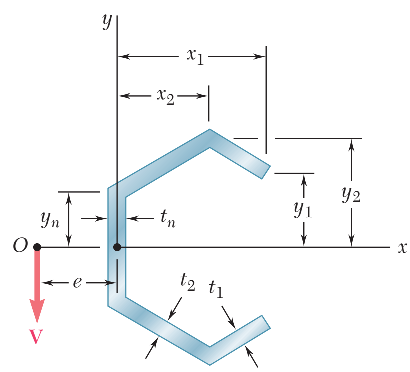


Fig. P6.C5

6.C6 A thin-walled beam has the cross section shown. Write a computer program that, for loads and dimensions expressed in either SI or U.S. customary units, can be used to determine the location of the shear center O of the cross section. Use the program to solve Prob. 6.75.

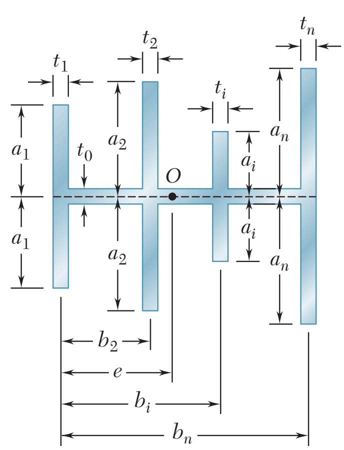


Fig. P6.C6