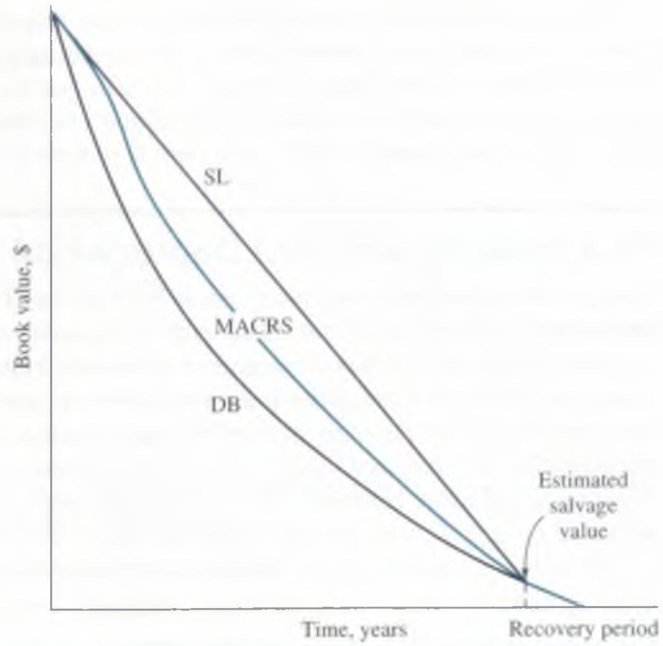


Figure 16–1
General shape of book value curves for different depreciation methods.

As mentioned before, there are several models for depreciating assets. The straight line (SL) method is used historically and internationally. Accelerated models, such as the declining balance (DB) method, decrease the book value to zero (or to the salvage value) more rapidly than the straight line method, as shown by the general book value curves in Figure 16–1.

For each of the methods—straight line, declining balance, MACRS, and sum-of-years-digits—there are spreadsheet functions available to determine annual depreciation. Each function is introduced and illustrated as the method is explained.

As expected, there are many rules and exceptions to the depreciation laws of a country. One that may be of interest to a U.S.-based *small or medium-sized business* performing an economic analysis is the *Section 179 Deduction.* This is an economic incentive that changes over the years and encourages businesses to invest capital in equipment directly used in the company. Up to a specified amount, the entire basis of an asset is treated as a business expense in the year of purchase. This tax treatment reduces federal income taxes, just as depreciation does, but it is allowed in lieu of depreciating the first cost over several years. The limit changes with time; it was \$24,000 in 2002; \$102,000 in 2004; \$125,000 in 2007; and \$250,000 in 2008–2010. The economic stimulus efforts in the United States and around the world during the latter part of the decade made many attempts to put investment capital to work in small and medium-sized businesses. Investments above these limits must be depreciated using MACRS.

In the 1980s the U.S. government standardized accelerated methods for *federal tax depreciation* purposes. In 1981, all classical methods, including straight line, declining balance, and sum-of-years-digits depreciation, were disallowed as tax deductible and replaced by the Accelerated Cost Recovery System (ACRS). In a second round of standardization, MACRS (Modified ACRS) was made the required tax depreciation method in 1986. To this date, the following is the law in the United States.

Tax depreciation must be calculated using MACRS; **book depreciation** may be calculated using any classical method or MACRS.

MACRS has the DB and SL methods, in slightly different forms, embedded in it, but these two methods cannot be used directly if the annual depreciation is to be tax deductible. Many U.S. companies still apply the classical methods for keeping their own books, because these methods are more representative of how the usage patterns of the asset reflect the remaining capital invested in it. Most other countries still recognize the classical methods of straight line and declining balance for tax or book purposes. Because of the continuing importance of the SL and DB methods, they are explained in the next two sections prior to MACRS. Appendix Section 16A.1 discusses two historical methods of depreciation.

Tax law revisions occur often, and depreciation rules are changed from time to time in the United States and other countries. For more depreciation and tax law information, consult the U.S. Department of the Treasury, Internal Revenue Service (IRS), website at www.irs.gov. Pertinent publications can be downloaded. Publication 946, *How to Depreciate Property*, is especially applicable to this chapter. MACRS and most corporate tax depreciation laws are discussed in it.

16.2 Straight Line (SL) Depreciation

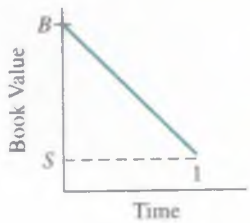
Straight line depreciation derives its name from the fact that the book value decreases **linearly with time.** The depreciation rate is the same $(1/n)$ each year of the recovery period n.

Straight line depreciation is considered the standard against which any depreciation model is compared. For *book depreciation* purposes, it offers an excellent representation of book value for any asset that is used regularly over an estimated number of years. For *tax depreciation*, as mentioned earlier, it is not used directly in the United States, but it is commonly used in most countries for tax purposes. However, the U.S. MACRS method includes a version of SL depreciation with a larger n value than that prescribed by regular MACRS (see Section 16.5).

The annual SL depreciation is determined by multiplying the first cost minus the salvage value by d_t. In equation form,

$$D_t = (B - S)d_t = \frac{B - S}{n} \qquad [16.1]$$

where t = year ($t = 1, 2, \ldots, n$)
D_t = annual depreciation charge
B = first cost or unadjusted basis
S = estimated salvage value
n = recovery period
d_t = depreciation rate = $1/n$



Since the asset is depreciated by the same amount each year, the book value after t years of service, denoted by BV_t, will be equal to the first cost B minus the annual depreciation times t.

$$BV_t = B - tD_t \qquad [16.2]$$

Earlier we defined d_t as a depreciation rate for a specific year t. However, the SL model has the same rate for all years, that is,

$$d = d_t = \frac{1}{n} \qquad [16.3]$$

The format for the spreadsheet function to display the annual depreciation D_t in a single-cell operation is

$$= \text{SLN}(B,S,n) \qquad [16.4]$$

EXAMPLE 16.1

If an asset has a first cost of \$50,000 with a \$10,000 estimated salvage value after 5 years, (*a*) calculate the annual depreciation and (*b*) calculate and plot the book value of the asset after each year, using straight line depreciation.

Solution

(*a*) The depreciation each year for 5 years can be found by Equation [16.1].

$$D_t = \frac{B - S}{n} = \frac{50{,}000 - 10{,}000}{5} = \$8000$$

Enter the function = SLN(50000,10000,5) in any cell to display the D_t of \$8000.

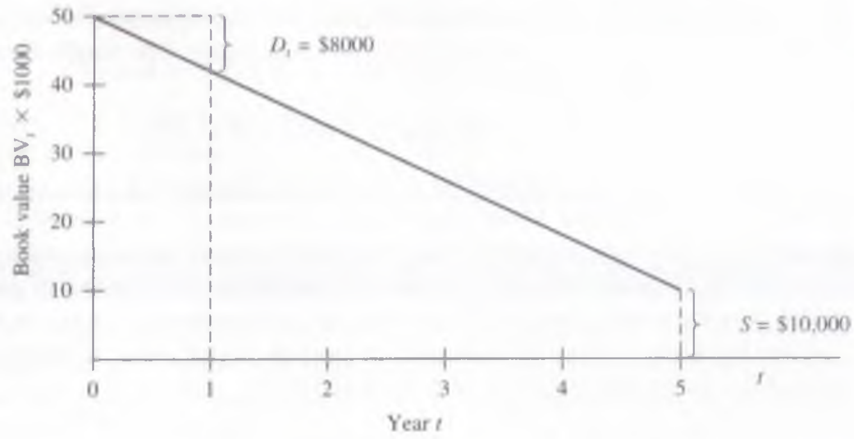


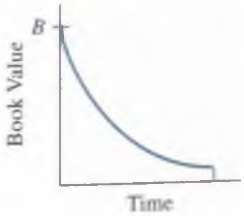
Figure 16–2
Book value of an asset using straight line depreciation, Example 16.1.

(*b*) The book value after each year t is computed using Equation [16.2]. The BV_t values are plotted in Figure 16–2. For years 1 and 5, for example,

$$BV_1 = 50{,}000 - 1(8000) = \$42{,}000$$
$$BV_5 = 50{,}000 - 5(8000) = \$10{,}000 = S$$

16.3 Declining Balance (DB) and Double Declining Balance (DDB) Depreciation ● ● ●

The **declining balance method** is commonly applied as the book depreciation method. Like the SL method, DB is embedded in the MACRS method, but the DB method itself cannot be used to determine the annual tax-deductible depreciation in the United States. This method is used routinely in most other countries for tax and book depreciation purposes.



Declining balance is also known as the fixed percentage or uniform percentage method. DB depreciation accelerates the write-off of asset value because the annual depreciation is determined by multiplying the *book value at the beginning of a year* by a fixed (uniform) percentage d, expressed in decimal form. If $d = 0.1$, then 10% of the book value is removed each year. Therefore, the depreciation amount decreases each year.

The maximum annual depreciation rate for the DB method is twice the straight line rate, that is,

$$d_{max} = 2/n \qquad [16.5]$$

In this case the method is called *double declining balance (DDB)*. If $n = 10$ years, the DDB rate is $2/10 = 0.2$; so 20% of the book value is removed annually. Another commonly used percentage for the DB method is 150% of the SL rate, where $d = 1.5/n$.

The depreciation for year t is the fixed rate d times the book value at the end of the previous year.

$$D_t = (d)BV_{t-1} \qquad [16.6]$$

The actual depreciation rate for each year t, relative to the basis B, is

$$d_t = d(1 - d)^{t-1} \qquad [16.7]$$

If BV_{t-1} is not known, the depreciation in year t can be calculated using B and d.

$$D_t = dB(1 - d)^{t-1} \qquad [16.8]$$

Book value in year t is determined in one of two ways: by using the rate d and basis B or by subtracting the current depreciation charge from the previous book value. The equations are

$$BV_t = B(1 - d)^t \qquad [16.9]$$
$$BV_t = BV_{t-1} - D_t \qquad [16.10]$$

It is important to understand that the book value for the DB method never goes to zero, because the book value is always decreased by a fixed percentage. The **implied salvage value** after n years is the BV_n amount, that is,

$$\text{Implied } S = BV_n = B(1-d)^n \qquad [16.11]$$

If a salvage value is estimated for the asset, this *estimated S value is not used in the DB or DDB method* to calculate annual depreciation. However, if the implied $S <$ estimated S, it is necessary to stop charging further depreciation when the book value is at or below the estimated salvage value. In most cases, the estimated S is in the range of zero to the implied S value. (This guideline is important when the DB method can be used directly for tax depreciation purposes.)

If the fixed percentage d is not stated, it is possible to determine an implied fixed rate using the estimated S value, if $S > 0$. The range for d is $0 < d < 2/n$.

$$\text{Implied } d = 1 - \left(\frac{S}{B}\right)^{1/n} \qquad [16.12]$$

The spreadsheet functions DDB and DB are used to display depreciation amounts for specific years. The function is repeated in consecutive spreadsheet cells because the depreciation amount D_t changes with t. For the double declining balance method, the format is

$$= \mathbf{DDB}(B,S,n,t,d) \qquad [16.13]$$

The entry d is the fixed rate expressed as a number between 1 and 2. If omitted, this optional entry is assumed to be 2 for DDB. An entry of $d = 1.5$ makes the DDB function display 150% declining balance method amounts. The DDB function automatically checks to determine when the book value equals the estimated S value. No further depreciation is charged when this occurs. (To allow *full* depreciation charges to be made, ensure that the S entered is between zero and the implied S from Equation [16.11].) Note that $d = 1$ is the same as the straight line rate $1/n$, but *D_t will not be the SL amount* because declining balance depreciation is determined as a fixed percentage of the previous year's book value, which is completely different from the SL calculation in Equation [16.1].

The format for the DB function is $= \text{DB}(B,S,n,t)$. Caution is needed when using this function. The fixed rate d is not entered in the DB function; d is an embedded calculation using a spreadsheet equivalent of Equation [16.12]. Also, only three significant digits are maintained for d, so the book value may go below the estimated salvage value due to round-off errors. Therefore, if the depreciation rate is known, always use the DDB function to ensure correct results. Examples 16.2 and 16.3 illustrate DB and DDB depreciation and their spreadsheet functions.

EXAMPLE 16.2

Underwater electroacoustic transducers were purchased for use in SONAR applications. The equipment will be DDB depreciated over an expected life of 12 years. There is a first cost of \$25,000 and an estimated salvage of \$2500. (*a*) Calculate the depreciation and book value for years 1 and 4. Write the spreadsheet functions to display depreciation for years 1 and 4. (*b*) Calculate the implied salvage value after 12 years.

Solution

(*a*) The DDB fixed depreciation rate is $d = 2/n = 2/12 = 0.1667$ per year. Use Equations [16.8] and [16.9].

Year 1:

$$D_1 = (0.1667)(25{,}000)(1-0.1667)^{1-1} = \$4167$$
$$BV_1 = 25{,}000(1-0.1667)^1 = \$20{,}833$$

Year 4:

$$D_4 = (0.1667)(25{,}000)(1-0.1667)^{4-1} = \$2411$$
$$BV_4 = 25{,}000(1-0.1667)^4 = \$12{,}054$$

The DDB functions for D_1 and D_4 are, respectively, = DDB(25000,2500,12,1) and = DDB(25000,2500,12,4).

(*b*) From Equation [16.11], the implied salvage value after 12 years is

$$\text{Implied } S = 25{,}000(1 - 0.1667)^{12} = \$2803$$

Since the estimated $S = \$2500$ is less than \$2803, the asset is not fully depreciated when its 12-year expected life is reached.

EXAMPLE 16.3

Freeport-McMoRan Copper and Gold has purchased a new ore grading unit for \$80,000. The unit has an anticipated life of 10 years and a salvage value of \$10,000. Use the DB and DDB methods to compare the schedule of depreciation and book values for each year. Solve by hand and by spreadsheet.

Solution by Hand

An implied DB depreciation rate is determined by Equation [16.12].

$$d = 1 - \left(\frac{10{,}000}{80{,}000}\right)^{1/10} = 0.1877$$

Note that $0.1877 < 2/n = 0.2$, so this DB model does not exceed twice the straight line rate. Table 16–1 presents the D_t values using Equation [16.6] and the BV_t values from Equation [16.10] rounded to the nearest dollar. For example, in year $t = 2$, the DB results are

$$D_2 = d(BV_1) = 0.1877(64{,}984) = \$12{,}197$$
$$BV_2 = 64{,}984 - 12{,}197 = \$52{,}787$$

Because we round off to even dollars, \$2312 is calculated for depreciation in year 10, but \$2318 is deducted to make $BV_{10} = S = \$10{,}000$ exactly. Similar calculations for DDB with $d = 0.2$ result in the depreciation and book value series in Table 16–1.

TABLE 16–1 D_t and BV_t Values for DB and DDB Depreciation, Example 16.3

	Declining Balance, \$		Double Declining Balance, \$	
Year t	D_t	BV_t	D_t	BV_t
0	—	80,000	—	80,000
1	15,016	64,984	16,000	64,000
2	12,197	52,787	12,800	51,200
3	9,908	42,879	10,240	40,960
4	8,048	34,831	8,192	32,768
5	6,538	28,293	6,554	26,214
6	5,311	22,982	5,243	20,972
7	4,314	18,668	4,194	16,777
8	3,504	15,164	3,355	13,422
9	2,846	12,318	2,684	10,737
10	2,318	10,000	737	10,000

Solution by Spreadsheet

The spreadsheet in Figure 16–3 displays the results for the DB and DDB methods. The chart plots book values for each year. Since the fixed rates are close—0.1877 for DB and 0.2 for DDB—the annual depreciation and book value series are approximately the same for the two methods.

The depreciation rate (cell B5) is calculated by Equation [16.12], but note in the cell tags that the DDB function is used in both columns B and D to determine annual depreciation. As mentioned earlier, the DB function automatically calculates the implied rate by Equation [16.12] and maintains it to only three significant digits. Therefore, if the DB function

	A	B	C	D	E
1					
2	First cost, B	$ 80,000			
3	Salvage, S	$ 10,000			
4	Rec'vry pd, n	10			
5	Depr. rate, d	0.1877476			
6		Declining balance (DB), $		Double declining (DDB), $	
7	Year, t	D(t)	DB book value	D(t)	DDB book value
8	0		80,000		80,000
9	1	15,020	64,980	16,000	64,000
10	2	12,200	52,780	12,800	51,200
11	3	9,909	42,871	10,240	40,960
12	4	8,049	34,822	8,192	32,768
13	5	6,538	28,284	6,554	26,214
14	6	5,310	22,974	5,243	20,972
15	7	4,313	18,661	4,194	16,777
16	8	3,503	15,157	3,355	13,422
17	9	2,846	12,311	2,684	10,737
18	10	2,311	10,000	737	10,000

= 1-(B3/B2)^0.1

= DDB(B$2,B$3,B$4,$A18,10*B5)

= DDB(B$2,B$3,B$4,$A18)

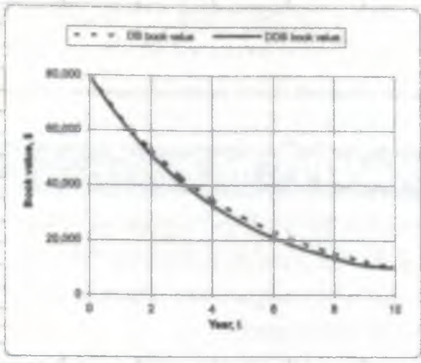


Figure 16–3
Annual depreciation and book value using DB and DDB methods, Example 16.3.

were used in column B (Figure 16–3), the fixed rate applied would be 0.188. The resulting D_t and BV_t values for years 8, 9, and 10 would be as follows:

t	D_t, $	BV_t, $
8	3,501	15,120
9	2,842	12,277
10	2,308	9,969

Also noteworthy is the fact that the DB function uses the implied rate without a check to halt the book value at the estimated salvage value. Thus, BV_{10} will go slightly below $S = \$10{,}000$, as shown above. However, the DDB function uses a relation different from that of the DB function to determine annual depreciation—one that correctly stops depreciating at the estimated salvage value, as shown in Figure 16–3, cells E17–E18.

Online Presentation

16.4 Modified Accelerated Cost Recovery System (MACRS) ● ● ●

In the 1980s, the United States introduced MACRS as the **required tax depreciation method** for all depreciable assets. Through MACRS, the 1986 Tax Reform Act defined statutory depreciation rates that take advantage of the accelerated DB and DDB methods. Corporations are free to apply any of the classical methods for book depreciation. When developed, MACRS and its predecessor ACRS were intended to create economic growth through the investment of new capital and the tax advantages that accelerated depreciation methods offer corporations and businesses.[1]

Many aspects of MACRS deal with the specific depreciation accounting aspects of tax law. This section covers only the elements that materially affect after-tax economic analysis. Additional information on how the DDB, DB, and SL methods are embedded into MACRS and how to derive the MACRS depreciation rates is presented and illustrated in the chapter appendix, Sections 16A.2 and 16A.3.

MACRS determines annual depreciation amounts using the relation

$$D_t = d_t B \qquad [16.14]$$

[1] R. Lundquist, "The Pedagogy of Taxes and Tax Purpose Depreciation," *Proceedings*, ASEE Annual Conference, Austin, TX, June 2009.

where the depreciation rate d_t is provided in tabulated form. As for other methods, the book value in year t is determined by subtracting the depreciation amount from the previous year's book value

$$\mathrm{BV}_t = \mathrm{BV}_{t-1} - D_t \qquad [16.15]$$

or by subtracting the total depreciation from the first cost.

$$\mathbf{BV}_t = \textbf{first cost} - \textbf{sum of accumulated depreciation}$$

$$= B - \sum_{j=1}^{j=t} D_j \qquad [16.16]$$

MACRS has standardized and simplified many of the decisions and calculations of depreciation.

The basis B (or first cost P) is completely depreciated; salvage is always assumed to be zero, or $S = \$0$.

Recovery periods are standardized to specific values:

$n = 3, 5, 7, 10, 15,$ or 20 years for personal property (e.g., equipment or vehicles)
$n = 27.5$ or 39 years for real property (e.g., rental property or structures)

Depreciation rates provide accelerated write-off by incorporating switching between classical methods.

Section 16.5 explains how to determine an allowable MACRS recovery period. The MACRS personal property depreciation rates (d_t values) for $n = 3, 5, 7, 10, 15$, and 20 for use in Equation [16.14] are included in Table 16–2.

MACRS depreciation rates incorporate the DDB method ($d = 2/n$) and switch to SL depreciation during the recovery period as an inherent component for *personal property* depreciation. The MACRS rates start with the DDB rate or the 150% DB rate and switch when the SL method offers faster write-off.

TABLE 16–2 Depreciation Rates d_t Applied to the Basis B for the MACRS Method

	Depreciation Rate (%) for Each MACRS Recovery Period in Years					
Year	$n = 3$	$n = 5$	$n = 7$	$n = 10$	$n = 15$	$n = 20$
1	33.33	20.00	14.29	10.00	5.00	3.75
2	44.45	32.00	24.49	18.00	9.50	7.22
3	14.81	19.20	17.49	14.40	8.55	6.68
4	7.41	11.52	12.49	11.52	7.70	6.18
5		11.52	8.93	9.22	6.93	5.71
6		5.76	8.92	7.37	6.23	5.29
7			8.93	6.55	5.90	4.89
8			4.46	6.55	5.90	4.52
9				6.56	5.91	4.46
10				6.55	5.90	4.46
11				3.28	5.91	4.46
12					5.90	4.46
13					5.91	4.46
14					5.90	4.46
15					5.91	4.46
16					2.95	4.46
17–20						4.46
21						2.23

For *real property*, MACRS utilizes the SL method for $n = 39$ throughout the recovery period. The annual percentage depreciation rate is $d = 1/39 = 0.02564$. However, MACRS forces partial-year recovery in years 1 and 40. The MACRS real property rates in percentage amounts are

Year 1	$100d_1 = 1.391\%$
Years 2–39	$100d_t = 2.564\%$
Year 40	$100d_{40} = 1.177\%$

The real property recovery period of 27.5 years, which applies only to residential rental property, uses the SL method in a similar fashion.

Note that all MACRS depreciation rates in Table 16–2 are presented for 1 year longer than the stated recovery period. Also note that the extra-year rate is one-half of the previous year's rate. This is so because a built-in **half-year convention** is imposed by MACRS. This convention assumes that all property is placed in service at the midpoint of the tax year of installation. Therefore, only 50% of the first-year DB depreciation applies for tax purposes. This removes some of the accelerated depreciation advantage and requires that one-half year of depreciation be taken in year $n + 1$.

No specially designed spreadsheet function is present for MACRS depreciation. However, the variable declining balance (VDB) function, which is used to determine when to switch between classical methods, can be adapted to display MACRS deprecation for each year. (The VDB function is explained in detail in Section 16A.2 of this chapter and Appendix A of the text.) The MACRS depreciation format of the VDB function requires embedded MAX and MIN functions, as follows:

$$= \mathbf{VDB}(B,0,n,\ \mathbf{MAX}(0,t-1.5),\ \mathbf{MIN}(n,t-0.5),d) \qquad [16.17]$$

where B = first cost
0 = salvage value of $S = 0$
n = recovery period

$$d = \begin{cases} 2 & \text{if MACRS } n = 3, 5, 7, \text{ or } 10 \\ 1.5 & \text{if MACRS } n = 15 \text{ or } 20 \end{cases}$$

The MAX and MIN functions ensure that the MACRS half-year conventions are followed; that is, only one-half of the first year's depreciation is charged in year 1, and one-half of the last year's charge is carried over to year $n + 1$.

EXAMPLE 16.4

Chevron Phillips Chemical Company in Baytown, Texas, acquired new equipment for its polyethylene processing line. This chemical is a resin used in plastic pipe, retail bags, blow molding, and injection molding. The equipment has an unadjusted basis of $B = \$400{,}000$, a life of only 3 years, and a salvage value of 5% of B. The chief engineer asked the finance director to provide an analysis of the difference between (1) the DDB method, which is the internal book depreciation and book value method used at the plant, and (2) the required MACRS tax depreciation and its book value. He is especially curious about the differences after 2 years of service for this short-lived, but expensive asset. Use hand and spreadsheet solutions to do the following:

(*a*) Determine which method offers the larger total depreciation after 2 years.
(*b*) Determine the book value for each method after 2 years and at the end of the recovery period.

Solution by Hand

The basis is $B = \$400{,}000$ and the estimated $S = 0.05(400{,}000) = \$20{,}000$. The MACRS rates for $n = 3$ are taken from Table 16–2, and the depreciation rate for DDB is $d_{max} = 2/3 = 0.6667$. Table 16–3 presents the depreciation and book values. Year 3 depreciation for DDB would be $\$44{,}444(0.6667) = \$29{,}629$, except this would make $BV_3 < \$20{,}000$. Only the remaining amount of $24,444 is removed.

TABLE 16–3 Comparing MACRS and DDB Depreciation, Example 16.4

	MACRS			DDB	
Year	**Rate**	**Tax Depreciation, \$**	**Book Value, \$**	**Book Depreciation, \$**	**Book Value, \$**
0			400,000		400,000
1	0.3333	133,320	266,680	266,667	133,333
2	0.4445	177,800	88,880	88,889	44,444
3	0.1481	59,240	29,640	24,444	20,000
4	0.0741	29,640	0		

(*a*) The 2-year accumulated depreciation values from Table 16–3 are

MACRS: $D_1 + D_2 = \$133{,}320 + 177{,}800 = \$311{,}120$

DDB: $D_1 + D_2 = \$266{,}667 + 88{,}889 = \$355{,}556$

The DDB depreciation is larger. (Remember that for tax purposes, the company does not have the choice in the United States of DDB as applied here.)

(*b*) After 2 years the book value for DDB at \$44,444 is 50% of the MACRS book value of \$88,880. At the end of recovery (4 years for MACRS due to the built-in half-year convention, and 3 years for DDB), the MACRS book value is $BV_4 = 0$ and for DDB, $BV_3 = \$20{,}000$. This occurs because MACRS always removes the entire first cost, regardless of the estimated salvage value. This is a tax depreciation advantage of the MACRS method (unless the asset is disposed of for more than the MACRS-depreciated book value, as discussed in Section 17.4).

Solution by Spreadsheet

Figure 16–4 presents the spreadsheet solution using the VDB function (column B) for MACRS depreciation (in lieu of the MACRS rates) and applying the DDB function in column D.

(*a*) The 2-year accumulated depreciation values are

MACRS (add cells B6 + B7): \$133,333 + 177,778 = \$311,111

DDB (add cells D6 + D7): \$266,667 + 88,889 = \$355,556

(*b*) Book values after 2 years are

MACRS (cell C7): \$88,889

DDB (cell E7): \$44,444

The book values are plotted in Figure 16–4. Observe that MACRS goes to zero in year 4, while DDB stops at \$20,000 in year 3.

Comment

It is advisable to set up a *spreadsheet template* for use with depreciation problems in this and future chapters. The format and functions of Figure 16–4 are a good template for MACRS and DDB methods.

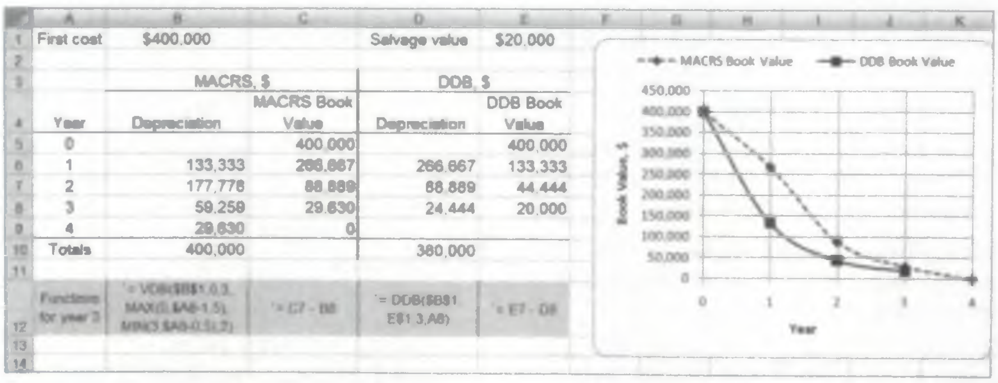


Figure 16–4

Spreadsheet screen shot of MACRS and DDB depreciation and book value, Example 16.4.

MACRS simplifies depreciation computations, but it removes much of the flexibility of method selection for a business or corporation. In general, an economic comparison that includes depreciation may be performed more rapidly and usually without altering the final decision by applying the *classical straight line method in lieu of MACRS.*

16.5 Determining the MACRS Recovery Period

The expected useful life of property is estimated in years and used as the n value in alternative evaluation and in depreciation computations. For book depreciation the n value should be the expected useful life. However, when the depreciation will be claimed as tax deductible, the n value should be lower. There are tables that assist in determining the life and recovery period for tax purposes.

The advantage of a recovery period **shorter** than the anticipated useful life is leveraged by the accelerated depreciation methods that write off more of the basis B in the initial years.

The U.S. government requires that all depreciable property be classified into a **property class** which identifies its MACRS-allowed recovery period. Table 16–4, a summary of material from IRS Publication 946, gives examples of assets and the MACRS n values. Virtually any property considered in an economic analysis has a MACRS n value of 3, 5, 7, 10, 15, or 20 years.

Table 16–4 provides two MACRS n values for each property. The first is the **general depreciation system (GDS)** value, which we use in examples and problems. The depreciation rates in Table 16–2 correspond to the n values for the GDS column and provide the fastest write-off allowed. The rates utilize the DDB method or the 150% DB method with a switch to SL depreciation. Note that any asset not in a stated class is automatically assigned a 7-year recovery period under GDS.

The far right column of Table 16–4 lists the **alternative depreciation system (ADS)** recovery period range. This alternative method allows the use of *SL depreciation over a longer*

TABLE 16–4 Example MACRS Recovery Periods for Various Asset Descriptions

	MACRS n Value, Years	
Asset Description (Personal and Real Property)	**GDS**	**ADS Range**
Special manufacturing and handling devices, tractors, racehorses	3	3–5
Computers and peripherals, oil and gas drilling equipment, construction assets, autos, trucks, buses, cargo containers, some manufacturing equipment	5	6–9.5
Office furniture; some manufacturing equipment; railroad cars, engines, tracks; agricultural machinery; petroleum and natural gas equipment; **all property not in another class**	7	10–15
Equipment for water transportation, petroleum refining, agriculture product processing, durable-goods manufacturing, shipbuilding	10	15–19
Land improvements, docks, roads, drainage, bridges, landscaping, pipelines, nuclear power production equipment, telephone distribution	15	20–24
Municipal sewers, farm buildings, telephone switching buildings, power production equipment (steam and hydraulic), water utilities	20	25–50
Residential rental property (house, mobile home)	27.5	40
Nonresidential real property attached to the land, but not the land itself	39	40

recovery period than the GDS. The half-year convention applies, and any salvage value is neglected, as it is in regular MACRS. The use of ADS is generally a choice left to a company, but it is required for some special asset situations. Since it takes longer to depreciate the asset, and since the SL model is required (thus removing the advantage of accelerated depreciation), ADS is usually not considered an option for the economic analysis. This electable SL option is, however, sometimes chosen by businesses that are young and do not need the tax benefit of accelerated depreciation during the first years of operation and asset ownership. If ADS is selected, tables of d_t rates are available.

16.6 Depletion Methods

Previously, for all assets, facilities, and equipment that can be replaced we have applied depreciation. We now turn to irreplaceable natural resources and the equivalent of depreciation, which is called *depletion.*

Depletion is a book method (noncash) to represent the decreasing value of a **natural resource** as it is recovered, removed, or felled. The two methods of depletion for book or tax purposes are used to write off the first cost, or value of the estimated quantity, of resources in mines, wells, quarries, geothermal deposits, forests, and the like.

The two methods of depletion are *cost* and *percentage depletion,* as described below. Details for U.S. taxes on depletion are found in IRS Publication 535, *Business Expenses*.

Cost depletion Sometimes referred to as factor depletion, cost depletion is based on the level of activity or usage, not time, as in depreciation. Cost depletion may be applied to most types of natural resources and *must* be applied to timber production. The cost depletion factor for year t, denoted by CD_t, is the ratio of the first cost of the resource to the estimated number of units recoverable.

$$\text{CD}_t = \frac{\text{first cost}}{\text{resource capacity}} \qquad [16.18]$$

The annual depletion charge is CD_t times the year's usage or volume. *The total cost depletion cannot exceed the first cost of the resource.* If the capacity of the property is reestimated some year in the future, a new cost depletion factor is determined based upon the undepleted amount and the new capacity estimate.

Percentage depletion This is a special consideration given for natural resources. A constant, stated percentage of the resource's **gross income** may be depleted each year *provided it does not exceed 50% of the company's taxable income.* The depletion amount for year t is calculated as

$$\begin{aligned}\text{Percentage depletion}_t &= \text{percentage depletion rate} \\ &\quad \times \text{gross income from property} \\ &= \text{PD} \times \text{GI}_t \qquad [16.19]\end{aligned}$$

Using percentage depletion, total depletion charges may exceed first cost with no limitation. The U.S. government does not generally allow percentage depletion to be applied to oil and gas wells (except small independent producers).

The annual percentage depletion rates for some common natural deposits are listed below per U.S. tax law.

Deposit	Percentage of Gross Income, PD
Sulfur, uranium, lead, nickel, zinc, and some other ores and minerals	22
Gold, silver, copper, iron ore, and some oil shale	15
Oil and natural gas wells (varies)	15–22
Coal, lignite, sodium chloride	10
Gravel, sand, peat, some stones	5
Most other minerals, metallic ores	14

EXAMPLE 16.5

Temple-Inland Corporation has negotiated the rights to cut timber on privately held forest acreage for $700,000. An estimated 350 million board feet of lumber is harvestable.

(*a*) Determine the depletion amount for the first 2 years if 15 million and 22 million board feet are removed.
(*b*) After 2 years the total recoverable board feet was reestimated upward to be 450 million from the time the rights were purchased. Compute the new cost depletion factor for years 3 and later.

Solution

(*a*) Use Equation [16.18] for CD_t in dollars per million board feet.

$$CD_t = \frac{700,000}{350} = \$2000 \text{ per million board feet}$$

Multiply CD_t by the annual harvest to obtain depletion of $30,000 in year 1 and $44,000 in year 2. Continue until a total of $700,000 is written off.

(*b*) After 2 years, a total of $74,000 has been depleted. A new CD_t value must be calculated based on the remaining $700,000 - 74,000 = \$626,000$ investment. Additionally, with the new estimate of 450 million board feet, a total of $450 - 15 - 22 = 413$ million board feet remains. For years $t = 3, 4, \ldots$, the cost depletion factor is

$$CD_t = \frac{626,000}{413} = \$1516 \text{ per million board feet}$$

EXAMPLE 16.6

A gold mine was purchased for $10 million. It has an anticipated gross income of $5.0 million per year for years 1 to 5 and $3.0 million per year after year 5. Assume that depletion charges do not exceed 50% of taxable income. Compute annual depletion amounts for the mine. How long will it take to recover the initial investment at $i = 0\%$?

Solution

The rate for gold is PD = 0.15. Depletion amounts are

Years 1 to 5:	0.15(5.0 million) = $750,000
Years thereafter:	0.15(3.0 million) = $450,000

A total of $3.75 million is written off in 5 years, and the remaining $6.25 million is written off at $450,000 per year. The total number of years is

$$5 + \frac{\$6.25 \text{ million}}{\$450,000} = 5 + 13.9 = 18.9$$

In 19 years, the initial investment could be fully depleted.

In many of the natural resource depletion situations, the tax law allows the larger of the two depletion amounts to be claimed each year. This is allowed provided the percentage depletion amount does not exceed 50% of taxable income. Therefore, it is wise to calculate both depletion amounts and select the larger. Use the following terminology for year t ($t = 1, 2, \ldots$).

$$\begin{aligned} CDA_t &= \text{cost depletion amount} \\ PDA_t &= \text{percentage depletion amount} \\ TI_t &= \text{taxable income} \end{aligned}$$

The guideline for the tax-allowed depletion amount for year t is

$$\text{Depletion} = \begin{cases} \max[CDA_t, PDA_t] & \text{if } PDA_t \le 50\% \text{ of } TI_t \\ \max[CDA_t, 50\% \text{ of } TI_t] & \text{if } PDA_t > 50\% \text{ of } TI_t \end{cases}$$

For example, assume a medium-sized quarry owner calculates the following for 1 year.

$$TI = \$500,000 \qquad CDA = \$275,000 \qquad PDA = \$280,000$$

Since 50% of TI is \$250,000, the PDA is too large and, therefore, is not allowed. For tax purposes, apply the guideline above and use the cost depletion of \$275,000, since it is larger than 50% of TI.

CHAPTER SUMMARY

Depreciation may be determined for internal company records (book depreciation) or for income tax purposes (tax depreciation). In the United States, the MACRS method is the only one allowed for tax depreciation. In many other countries, straight line and declining balance methods are applied for both tax and book depreciation. Depreciation does not result in cash flow directly. It is a book method by which the capital investment in tangible property is recovered. The annual depreciation amount is tax deductible, which can result in actual cash flow changes.

Some important points about the straight line, declining balance, and MACRS methods are presented below. Common relations for each method are summarized in Table 16–5.

Straight Line (SL)

- It writes off capital investment linearly over n years.
- The estimated salvage value is always considered.
- This is the classical, nonaccelerated depreciation model.

Declining Balance (DB)

- The method accelerates depreciation compared to the straight line method.
- The book value is reduced each year by a fixed percentage.
- The most used rate is twice the SL rate, which is called double declining balance (DDB).
- It has an implied salvage that may be lower than the estimated salvage.
- It is not an approved tax depreciation method in the United States. It is frequently used for book depreciation purposes.

Modified Accelerated Cost Recovery System (MACRS)

- It is the only approved tax depreciation system in the United States.
- It automatically switches from DDB or DB to SL depreciation.
- It always depreciates to zero; that is, it assumes $S = 0$.
- Recovery periods are specified by property classes.
- Depreciation rates are tabulated.
- The actual recovery period is 1 year longer due to the imposed half-year convention.
- MACRS straight line depreciation is an option, but recovery periods are longer than those for regular MACRS.

Cost and *percentage depletion methods* recover investment in natural resources. The annual cost depletion factor is applied to the amount of resource removed. No more than the initial investment can be recovered with cost depletion. Percentage depletion, which can recover more than the initial investment, reduces the investment value by a constant percentage of gross income each year.

TABLE 16–5 Summary of Common Depreciation Method Relations

Method	MACRS	SL	DDB
Fixed depreciation rate d	Not defined	$\frac{1}{n}$	$\frac{2}{n}$
Annual rate d_t	Table 16–2	$\frac{1}{n}$	$d(1-d)^{t-1}$
Annual depreciation D_t	d_tB	$\frac{B-S}{n}$	$d(BV_{t-1})$
Book value BV_t	$BV_{t-1} - D_t$	$B - tD_t$	$B(1-d)^t$

CHAPTER 16 APPENDIX

16A.1 Sum-of-Years-Digits (SYD) and Unit-of-Production (UOP) Depreciation ● ● ●

The **SYD method** is a historical accelerated depreciation technique that removes much of the basis in the first one-third of the recovery period; however, write-off is not as rapid as for DDB or MACRS. This technique may be used in an engineering economy analysis in the book depreciation of multiple-asset accounts (group and composite depreciation).

The mechanics of the method involve the sum of the year's digits from 1 through the recovery period n. The depreciation charge for any given year is obtained by multiplying the basis of the asset, less any salvage value, by the ratio of the number of years remaining in the recovery period to the sum of the year's digits SUM.

$$D_t = \frac{\text{depreciable years remaining}}{\text{sum of years digits}}(\text{basis} - \text{salvage value})$$

$$D_t = \frac{n - t + 1}{\text{SUM}}(B - S) \qquad [16A.1]$$

where SUM is the sum of the digits 1 through n.

$$\text{SUM} = \sum_{j=1}^{j=n} j = \frac{n(n+1)}{2}$$

The book value for any year t is calculated as

$$\text{BV}_t = B - \frac{t(n - t/2 + 0.5)}{\text{SUM}}(B - S) \qquad [16A.2]$$

The rate of depreciation decreases each year and equals the multiplier in Equation [16A.1].

$$d_t = \frac{n - t + 1}{\text{SUM}} \qquad [16A.3]$$

The SYD spreadsheet function displays the depreciation for the year t. The function format is

$$= \text{SYD}(B,S,n,t)$$

EXAMPLE 16A.1

Calculate the SYD depreciation charges for year 2 for electro-optics equipment with $B = \$25{,}000$, $S = \$4000$, and an 8-year recovery period.

Solution

The sum of the year's digits is 36, and the depreciation amount for the second year by Equation [16A.1] is

$$D_2 = \frac{7}{36}(21{,}000) = \$4083$$

The SYD function is = SYD(25000,4000,8,2).

Figure 16A–1 is a plot of the book values for an \$80,000 asset with $S = \$10{,}000$ and $n = 10$ years using the four depreciation methods that we have learned. The MACRS, DDB, and SYD curves track closely except for year 1 and years 9 through 11.

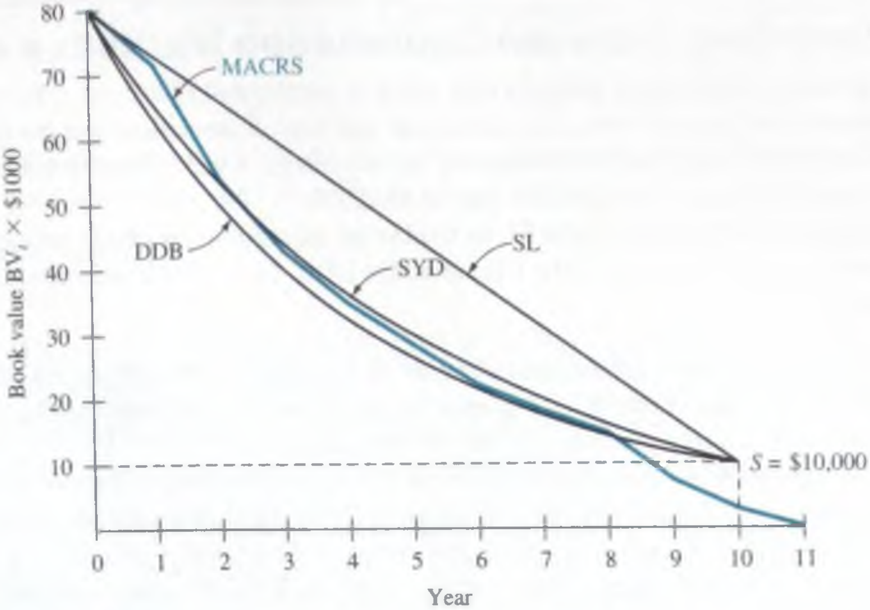


Figure 16A–1
Comparison of book values using SL, SYD, DDB, and MACRS depreciation.

A second depreciation method that is not allowed for tax purposes, but useful in some situations is the **unit-of-production (UOP) method.** When the decreasing value of equipment is **based on usage, not time,** the UOP method is quite applicable. Suppose a highway contractor has a series of state highway department contracts that will last several years and that earth moving equipment is purchased for use on all contracts. If the equipment usage goes up and down significantly over the years, the UOP method is ideal for book depreciation. For year t, UOP deprecation is calculated as

$$D_t = \frac{\text{actual usage for year } t}{\text{total lifetime usage}}(\text{basis} - \text{salvage}) \qquad [16A.4]$$

EXAMPLE 16A.2

Zachry Contractors purchased an \$80,000 mixer for use during the next 10 years for contract work on IH-10 in San Antonio. The mixer will have a negligible salvage value after 10 years, and the total amount of material to process is estimated at 2 million m^3. Use the actual usage per year shown in Table 16A–1 and the unit-of-production method to determine annual depreciation.

Solution

The actual usage each year is placed in the numerator of Equation [16A.4] to determine the annual depreciation based on the estimated total lifetime amount of material, 2 million m^3 in this case. Table 16A–1 shows the annual and cumulative depreciation over the 10 years. If the mixer is continued in service after the 2 million m^3 is processed, no further depreciation is allowed.

TABLE 16A–1 Unit-of-Production Method of Depreciation, Example 16A.2

Year t	Actual Usage, 1000 m^3	Annual Depreciation D_t, \$	Cumulative Depreciation, \$
1	400	16,000	16,000
2–8	200	8,000	72,000
9–10	100	4,000	80,000
Total	2000	80,000	

16A.2 Switching between Depreciation Methods ● ● ●

Switching between depreciation methods may assist in accelerated reduction of the book value. It also maximizes the present value of accumulated and total depreciation over the recovery period. Therefore, switching usually increases the tax advantage in years where the depreciation is larger. The approach below is an inherent part of MACRS.

Switching from a DB method to the SL method is the most common switch because it usually offers a real advantage, especially if the DB method is DDB. General rules of switching are summarized here.

1. Switching is recommended when the depreciation for year t by the currently used method is less than that for a new method. The selected depreciation D_t is the larger amount.
2. Only one switch can take place during the recovery period.
3. Regardless of the (classical) depreciation methods, the book value cannot go below the estimated salvage value. When switching from a DB method, the estimated salvage value, not the DB-implied salvage value, is used to compute the depreciation for the new method; we assume $S = 0$ in all cases. (This does not apply to MACRS, since it already includes switching.)
4. The undepreciated amount, that is, BV_t, is used as the new adjusted basis to select the larger D_t for the next switching decision.

In all situations, the criterion is to **maximize the present worth of the total depreciation PW_D.** The combination of depreciation methods that results in the largest present worth is the best switching strategy.

$$\mathrm{PW}_D = \sum_{t=1}^{t=n} D_t(P/F,i,t) \qquad [16A.5]$$

This logic minimizes tax liability in the early part of an asset's recovery period.

Switching is most advantageous from a rapid write-off method such as DDB to the SL model. This switch is predictably advantageous if the implied salvage value computed by Equation [16.11] exceeds the salvage value estimated at purchase time; that is, switch if

$$\mathrm{BV}_n = B(1-d)^n > \text{estimated } S \qquad [16A.6]$$

Since we assume that S will be zero per rule 3 above, and since *BV_n will be greater than zero,* for a DB method a switch to SL is always advantageous. Depending upon the values of d and n, the switch may be best in the later years or last year of the recovery period, which removes the implied S inherent to the DDB model.

The procedure to switch from DDB to SL depreciation is as follows:

1. For each year t, compute the two depreciation charges.

For DDB: $$D_{\mathrm{DDB}} = d(\mathrm{BV}_{t-1}) \qquad [16A.7]$$

For SL: $$D_{\mathrm{SL}} = \frac{\mathrm{BV}_{t-1}}{n-t+1} \qquad [16A.8]$$

2. Select the larger depreciation value. The depreciation for each year is

$$D_t = \max[D_{\mathrm{DDB}}, D_{\mathrm{SL}}] \qquad [16A.9]$$

3. If needed, determine the present worth of total depreciation, using Equation [16A.5].

It is acceptable, though not usually financially advantageous, to state that a switch will take place in a particular year, for example, a mandated switch from DDB to SL in year 7 of a 10-year recovery period. This approach is usually not taken, but the switching technique will work correctly for all depreciation methods.

To use a spreadsheet for switching, first understand the depreciation model switching rules and practice the switching procedure from declining balance to straight line. Once these are understood, the mechanics of the switching can be speeded up by applying the spreadsheet function

VDB (variable declining balance). This is a quite powerful function that determines the depreciation for 1 year or the total over several years for the DB-to-SL switch. The function format is

$$= \text{VDB}(B, S, n, \text{start\_}t, \text{end\_}t, d, \text{no\_switch}) \qquad [16A.10]$$

Appendix A explains all the fields in detail, but for simple applications, where the DDB and SL annual D_t values are needed, the following are correct entries:

start_t is the year $(t-1)$

end_t is year t

d is optional; 2 for DDB is assumed, the same as in the DDB function

no_switch is an optional logical value:

FALSE or omitted—switch to SL occurs, if advantageous

TRUE—DDB or DB method is applied with no switching to SL depreciation considered.

Entering TRUE for the no_switch option obviously causes the VDB function to display the same depreciation amounts as the DDB function. This is discussed in Example 16A.3*d*. You may notice that the VDB function is the same one used to calculate annual MACRS depreciation.

EXAMPLE 16A.3

The Outback Steakhouse main office has purchased a \$100,000 online document imaging system with an estimated useful life of 8 years and a tax depreciation recovery period of 5 years. Compare the present worth of total depreciation for (*a*) the SL method, (*b*) the DDB method, and (*c*) DDB-to-SL switching. (*d*) Perform the DDB-to-SL switch using a spreadsheet and plot the book values. Use a rate of $i = 15\%$ per year.

Solution by Hand

The MACRS method is not involved in this solution.

(*a*) Equation [16.1] determines the annual SL depreciation.

$$D_t = \frac{100{,}000 - 0}{5} = \$20{,}000$$

Since D_t is the same for all years, the P/A factor replaces P/F to compute PW_D.

$$PW_D = 20{,}000(P/A,15\%,5) = 20{,}000(3.3522) = \$67{,}044$$

(*b*) For DDB, $d = 2/5 = 0.40$. The results are shown in Table 16A–2. The value $PW_D =$ \$69,915 exceeds \$67,044 for SL depreciation. As is predictable, the accelerated depreciation of DDB increases PW_D.

(*c*) Use the DDB-to-SL switching procedure.

1. The DDB values for D_t in Table 16A–2 are repeated in Table 16A–3 for comparison with the D_{SL} values from Equation [16A.8]. The D_{SL} values change each year because BV_{t-1} is different. Only in year 1 is $D_{SL} = \$20{,}000$, the same as computed in part (*a*). For illustration, compute D_{SL} values for years 2 and 4. For $t = 2$, $BV_1 = \$60{,}000$ by the DDB method and

$$D_{SL} = \frac{60{,}000 - 0}{5 - 2 + 1} = \$15{,}000$$

For $t = 4$, $BV_3 = \$21{,}600$ by the DDB method and

$$D_{SL} = \frac{21{,}600 - 0}{5 - 4 + 1} = \$10{,}800$$

2. The column "Larger D_t" indicates a switch in year 4 with $D_4 = \$10{,}800$. The $D_{SL} =$ \$12,960 in year 5 would apply *only* if the switch occurred in year 5. Total depreciation with switching is \$100,000 compared to the DDB amount of \$92,224.
3. With switching, $PW_D = \$73{,}943$, which is an increase over both the SL and DDB methods.

TABLE 16A–2 DDB Model Depreciation and Present Worth Computations, Example 16A.3*b*

Year t	D_t, \$	BV_t, \$	$(P/F,15\%,t)$	Present Worth of D_t, \$
0		100,000		
1	40,000	60,000	0.8696	34,784
2	24,000	36,000	0.7561	18,146
3	14,400	21,600	0.6575	9,468
4	8,640	12,960	0.5718	4,940
5	5,184	7,776	0.4972	2,577
Totals	92,224			69,915

TABLE 16A–3 Depreciation and Present Worth for DDB-to-SL Switching, Example 16A.3*c*

Year	DDB Method, \$		SL Method	Larger	P/F	Present Worth of
t	D_{DDB}	BV_t	D_{SL}, \$	D_t, \$	Factor	D_t, \$
0	—	100,000				
1	40,000	60,000	20,000	40,000	0.8696	34,784
2	24,000	36,000	15,000	24,000	0.7561	18,146
3	14,400	21,600	12,000	14,400	0.6575	9,468
4*	8,640	12,960	10,800	10,800	0.5718	6,175
5	5,184	7,776	12,960	10,800	0.4972	5,370
Totals	92,224			100,000		73,943

*Indicates year of switch from DDB to SL depreciation.

Solution by Spreadsheet

(*d*) In Figure 16A–2, column D entries are the VDB functions to determine that the DDB-to-SL switch should take place in year 4. The entries "2,FALSE" at the end of the VDB function are optional (see the VDB function description). If TRUE were entered, the declining balance model would be maintained throughout the recovery period, and the annual depreciation amounts would be equal to those in column B. The plot in Figure 16A–2 indicates another difference in depreciation methods. The terminal book value in year 5 for the DDB method is BV_5 = \$7776, while the DDB-to-SL switch reduces the book value to zero.

The NPV function determines the PW of depreciation (row 9). The results here are the same as in parts (*b*) and (*c*) above. The DDB-to-SL switch has the larger PW_D value.

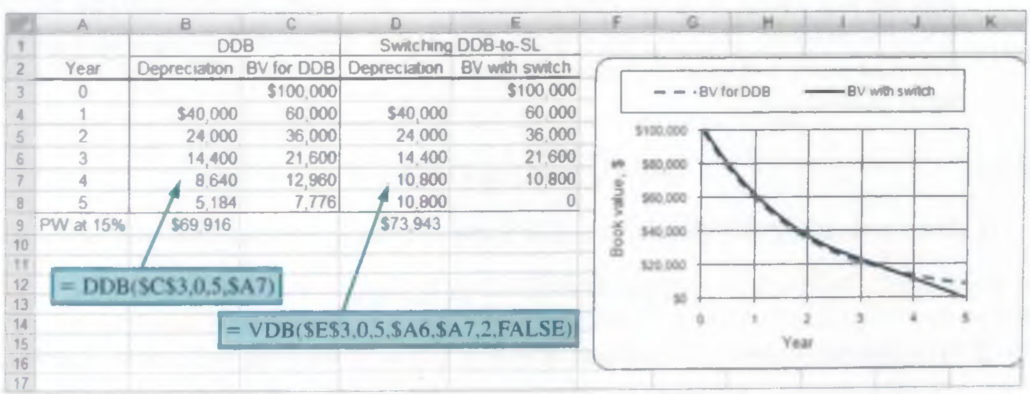


Figure 16A–2
Depreciation for DDB-to-SL switch using the VDB function, Example 16A.3.

In MACRS, recovery periods of 3, 5, 7, and 10 years apply DDB depreciation with half-year convention switching to SL. When the switch to SL takes place, which is usually in the last 1 to 3 years of the recovery period, any remaining basis is charged off in year $n + 1$ so that the book value reaches zero. Usually 50% of the applicable SL amount remains after the switch has occurred. For recovery periods of 15 and 20 years, 150% DB with the half-year convention and the switch to SL apply.

The present worth of depreciation PW_D will always indicate which method is the most advantageous. Only the MACRS rates for the GDS recovery periods (Table 16–4) utilize the DDB-to-SL switch. The MACRS rates for the alternative depreciation system (ADS) have longer recovery periods and impose the SL model for the entire recovery period.

EXAMPLE 16A.4

In Example 16A.3, parts (*c*) and (*d*), the DDB-to-SL switching method was applied to a \$100,000, $n = 5$ years asset resulting in $PW_D = \$73,943$ at $i = 15\%$. Use MACRS to depreciate the same asset for a 5-year recovery period, and compare PW_D values.

Solution

Table 16A–4 summarizes the computations for depreciation (using Table 16–2 rates), book value, and present worth of depreciation. The PW_D values for all four methods are as follows:

DDB-to-SL switching	\$73,943
Double declining balance	\$69,916
MACRS	\$69,016
Straight line	\$67,044

MACRS provides a slightly less accelerated write-off. This is so, in part, because the half-year convention disallows 50% of the first-year DDB depreciation (which amounts to 20% of the basis). Also the MACRS recovery period extends to year 6, further reducing PW_D.

TABLE 16A–4 Depreciation and Book Value Using MACRS, Example 16A.4

t	d_t	D_t, \$	BV_t, \$
0	—	—	100,000
1	0.20	20,000	80,000
2	0.32	32,000	48,000
3	0.192	19,200	28,800
4	0.1152	11,520	17,280
5	0.1152	11,520	5,760
6	0.0576	5,760	0
	1.000	100,000	

$$PW_D = \sum_{t=1}^{t=6} D_t(P/F,15\%,t) = \$69,016$$

16A.3 Determination of MACRS Rates ● ● ●

The depreciation rates for MACRS incorporate the DB-to-SL switching for all GDS recovery periods from 3 to 20 years. In the first year, some adjustments have been made to compute the MACRS rate. The adjustments vary and are not usually considered in detail in economic analyses. The half-year convention is always imposed, and any remaining book value in year n is removed in year $n + 1$. The value $S = 0$ is assumed for all MACRS schedules.

Since different DB depreciation rates apply for different n values, the following summary may be used to determine D_t and BV_t values. The symbols D_{DB} and D_{SL} are used to identify DB and SL depreciation, respectively.

For n = 3, 5, 7, and 10 Use DDB depreciation with the half-year convention, switching to SL depreciation in year t when $D_{SL} \geq D_{DB}$. Use the switching rules of Section 16A.2, and add one-half year when computing D_{SL} to account for the half-year convention. The yearly depreciation rates are

$$d_t = \begin{cases} \dfrac{1}{n} & t = 1 \\[1ex] \dfrac{2}{n} & t = 2, 3, \ldots \end{cases} \qquad [16A.11]$$

Annual depreciation values for each year t applied to the adjusted basis, allowing for the half-year convention, are

$$D_{DB} = d_t(BV_{t-1}) \qquad [16A.12]$$

$$D_{SL} = \begin{cases} \dfrac{1}{2}\left(\dfrac{1}{n}\right)B & t = 1 \\[1ex] \dfrac{BV_{t-1}}{n - t + 1.5} & t = 2, 3, \ldots, n \end{cases} \qquad [16A.13]$$

After the switch to SL depreciation takes place—usually in the last 1 to 3 years of the recovery period—any remaining book value in year n is removed in year $n + 1$.

For n = 15 and 20 Use 150% DB with the half-year convention and the switch to SL when $D_{SL} \geq D_{DB}$. Until SL depreciation is more advantageous, the annual DB depreciation is computed using a form of Equation [16A.7]

$$D_{DB} = d_t(BV_{t-1})$$

where

$$d_t = \begin{cases} \dfrac{0.75}{n} & t = 1 \\[1ex] \dfrac{1.50}{n} & t = 2, 3, \ldots \end{cases} \qquad [16A.14]$$

EXAMPLE 16A.5

A wireless tracking system for shop floor control with a MACRS 5-year recovery period has been purchased for \$10,000. (*a*) Use Equations [16A.11] through [16A.13] to obtain the annual depreciation and book value. (*b*) Determine the resulting annual depreciation rates and compare them with the MACRS rates in Table 16–2 for $n = 5$.

Solution

(*a*) With $n = 5$ and the half-year convention, use the DDB-to-SL switching procedure to obtain the results in Table 16A–5. The switch to SL depreciation, which occurs in year 4 when both depreciation values are equal, is indicated by

$$D_{DB} = 0.4(2880) = \$1152$$

$$D_{SL} = \frac{2880}{5 - 4 + 1.5} = \$1152$$

The SL depreciation of \$1000 in year 1 results from applying the half-year convention included in the first relation of Equation [16A.13]. Also, the SL depreciation of \$576 in year 6 is the result of the half-year convention.

(*b*) The actual rates are computed by dividing the "Larger D_t" column values by the first cost of \$10,000. The rates below are the same as the Table 16–2 rates.

t	1	2	3	4	5	6
d_t	0.20	0.32	0.192	0.1152	0.1152	0.0576

TABLE 16A–5 Depreciation Amounts Used to Determine MACRS Rates for $n = 5$, Example 16A.5

Years	DDB		SL Depreciation	Larger	
t	d_t	D_{DB}, $	D_{SL}, $	D_t, $	BV_t, $
0	—	—	—	—	10,000
1	0.2	2,000	1,000	2,000	8,000
2	0.4	3,200	1,777	3,200	4,800
3	0.4	1,920	1,371	1,920	2,880
4	0.4	1,152	1,152	1,152	1,728
5	0.4	691	1,152	1,152	576
6	—	—	576	576	0
				10,000	

It is clearly easier to use the rates in Table 16–2 or the VDB spreadsheet function than to determine each MACRS rate using the switching logic above. But the logic behind the MACRS rates is described here for those interested. The annual MACRS rates may be derived by using the applicable rate for the DB method. The subscripts DB and SL have been inserted along with the year t. For the first year $t = 1$,

$$d_{DB,1} = \frac{1}{n} \quad \text{or} \quad d_{SL,1} = \frac{1}{2}\left(\frac{1}{n}\right)$$

For summation purposes only, we introduce the subscript i ($i = 1, 2, \ldots, t$) on d. Then the depreciation rates for years $t = 2, 3, \ldots, n$ are

$$d_{DB,t} = d\left(1 - \sum_{i=1}^{i=t-1} d_i\right) \qquad [16A.15]$$

$$d_{SL,t} = \frac{\left(1 - \sum_{i=1}^{i=t-1} d_i\right)}{n - t + 1.5} \qquad [16A.16]$$

Also, for year $n + 1$, the MACRS rate is one-half the SL rate of the previous year n.

$$d_{SL,n+1} = 0.5(d_{SL,n}) \qquad [16A.17]$$

The DB and SL rates are compared each year to determine which is larger and when the switch to SL depreciation should occur.

EXAMPLE 16A.6

Verify the MACRS rates in Table 16–2 for a 3-year recovery period. The rates in percent are 33.33, 44.45, 14.81, and 7.41.

Solution

The fixed rate for DDB with $n = 3$ is $d = 2/3 = 0.6667$. Using the half-year convention in year 1 and Equations [16A.15] through [16A.17], the results are as follows:

d_1: $\quad d_{DB,1} = 0.5d = 0.5(0.6667) = 0.3333$

d_2: Cumulative depreciation rate is 0.3333.

$$d_{DB,2} = 0.6667(1 - 0.3333) = 0.4445 \qquad \text{(larger value)}$$

$$d_{SL,2} = \frac{1 - 0.3333}{3 - 2 + 1.5} = 0.2267$$

d_3: Cumulative depreciation rate is 0.3333 + 0.4445 = 0.7778.

$$d_{DB,3} = 0.6667(1 - 0.7778) = 0.1481$$

$$d_{SL,2} = \frac{1 - 0.7778}{3 - 3 + 1.5} = 0.1481$$

Both values are the same; switch to straight line depreciation.

d_4: This rate is 50% of the last SL rate.

$$d_4 = 0.5(d_{SL,3}) = 0.5(0.1481) = 0.0741$$

PROBLEMS

Fundamentals of Depreciation

16.1 How does depreciation affect a company's cash flow?

16.2 What is the difference between book value and market value?

16.3 State the difference between book depreciation and tax depreciation.

16.4 State the difference between unadjusted and adjusted basis.

16.5 Explain why the recovery period used for tax depreciation purposes may be different from the estimated n value in an engineering economy study.

16.6 Visit the U.S. Internal Revenue Service website at www.irs.gov and answer the following questions about depreciation and MACRS by consulting Publication 946, *How to Depreciate Property*.
(*a*) What is the definition of depreciation according to the IRS?
(*b*) What is the description of the term *salvage value*?
(*c*) What are the two depreciation systems within MACRS, and what are the major differences between them?
(*d*) What are the properties listed that cannot be depreciated under MACRS?
(*e*) When does depreciation begin and end?
(*f*) What is a Section 179 deduction?

16.7 A major energy production company has the following information regarding the acquisition of new-generation equipment.

Purchase price = \$580,000
Transoceanic shipping and delivery cost = \$4300
Installation cost (1 technician at \$1600 per day for 4 days) = \$6400
Tax recovery period = 15 years
Book depreciation recovery period = 10 years
Salvage value = 10% of purchase price
Operating cost (with technician) = \$185,000 per year

The manager of the department asked the newest hire to enter the appropriate data in the tax accounting program. For the MACRS method, what are the values of B, n, and S in depreciating the asset for tax purposes?

16.8 Stahmann Products paid \$350,000 for a numerical controller during the last month of 2007 and had it installed at a cost of \$50,000. The recovery period was 7 years with an estimated salvage value of 10% of the original purchase price. Stahmann sold the system at the end of 2011 for \$45,000.
(*a*) What numerical values are needed to develop a depreciation schedule at purchase time?
(*b*) State the numerical values for the following: remaining life at sale time, market value in 2011, book value at sale time if 65% of the basis had been depreciated.

16.9 An asset with an unadjusted basis of \$50,000 was depreciated over n_{tax} = 10 years for tax depreciation purposes and n_{book} = 5 years for book depreciation purposes. The annual depreciation was $1/n$ using the relevant life value. Use a spreadsheet to plot on one graph the annual book value for both methods of depreciation.

Straight Line Depreciation

16.10 What is the depreciation rate d_t per year for an asset that has an 8-year useful life and is straight line depreciated?

16.11 Pneumatics Engineering purchased a machine that had a first cost of \$40,000, an expected useful life of 8 years, a recovery period of 10 years, and a salvage value of \$10,000. The operating cost of the machine is expected to be \$15,000 per year. The inflation rate is 6% per year and the company's

MARR is 11% per year. Determine (*a*) the depreciation charge for year 3, (*b*) the present worth of the third-year depreciation charge in year 0, the time of asset purchase, and (*c*) the book value for year 3 according to the straight line method.

16.12 An asset that is book-depreciated over a 5-year period by the straight line method has $BV_3 = \$62{,}000$ with a depreciation charge of $26,000 per year. Determine (*a*) the first cost of the asset and (*b*) the assumed salvage value.

16.13 Lee Company of Westbrook, Connecticut, manufactures pressure relief inserts for thermal relief and low-flow hydraulic pressure relief applications where zero leakage is required. A machine purchased 3 years ago has been book-depreciated by the straight line method using a 5-year useful life. If the book value at the end of year 3 is $30,000 and the company assumed that the machine would be worthless at the end of its 5-year useful life, (*a*) what is the book depreciation charge each year and (*b*) what was the first cost of the machine?

16.14 An asset has an unadjusted basis of $200,000, a salvage value of $10,000, and a recovery period of 7 years. Write a single-cell spreadsheet function to display the book value after 5 years of straight line depreciation. Use your function to determine the book value.

16.15 Bristol Myers Squibb purchased a tablet-forming machine in 2008 for $750,000. The company planned to use the machine for 10 years; however, due to rapid obsolescence it will be retired after only 4 years in 2012. Develop a spreadsheet for depreciation and book value amounts necessary to answer the following.

(*a*) What is the amount of capital investment remaining when the asset is prematurely retired?

(*b*) If the asset is sold at the end of 4 years for $175,000, what is the amount of capital investment lost based on straight line depreciation?

(*c*) If the new-technology machine has an estimated cost of $300,000, how many more years should the company retain and depreciate the currently owned machine to make its book value and the first cost of the new machine equal to each other?

16.16 A special-purpose graphics workstation acquired by Busbee Consultants has $B = \$50{,}000$ with a 4-year recovery period. Tabulate the values for SL depreciation, accumulated depreciation, and book value for each year if (*a*) $S = 0$ and (*b*) $S = \$16{,}000$. (*c*) Use a spreadsheet to plot the book value over the 4 years on one chart for both salvage value estimates.

16.17 A company owns the same asset in a U.S. plant (New York) and in a EU plant (Paris). It has $B = \$2{,}000{,}000$ and a salvage value of 20% of B. For tax depreciation purposes, the United States allows a straight line write-off over 5 years, while the EU allows SL write-off over 8 years. The general managers of the two plants want to know the difference in (*a*) the depreciation amount for year 5 and (*b*) the book value after 5 years. Using a spreadsheet, write cell functions in *only* two cells to answer both questions.

Declining Balance Depreciation

16.18 When declining balance (DB) depreciation is applied, there can be three different depreciation rates involved—d, d_{max}, and d_t. Explain the differences between these rates.

16.19 Equipment for immersion cooling of electronic components has an installed value of $182,000 with an estimated trade-in value of $40,000 after 15 years. For years 2 and 10, use DDB book depreciation to determine (*a*) the depreciation charge and (*b*) the book value.

16.20 A cooling-water pumping station at the LCRA plant costs $600,000 to construct, and it is projected to have a 25-year life with an estimated salvage value of 15% of the construction cost. However, the station will be book-depreciated to zero over a recovery period of 30 years. Calculate the annual depreciation charge for years 4, 10, and 25, using (*a*) straight line depreciation and (*b*) DDB depreciation. (*c*) What is the implied salvage value for DDB? (*d*) Use a spreadsheet to build the depreciation and book value schedules for both methods to verify your answers.

16.21 A video recording system was purchased 3 years ago at a cost of $30,000. A 5-year recovery period and DDB depreciation have been used to write off the basis. The system is to be replaced this year with a trade-in value of $5000. What is the difference between the book value and the trade-in value?

16.22 An engineer with Accenture Middle East BV in Dubai was asked by her client to help him understand the difference between 150% DB and DDB depreciation. Answer these questions if $B = \$180{,}000$, $n = 12$ years, and $S = \$30{,}000$.

(*a*) What are the book values after 12 years for both methods?

(*b*) How do the estimated salvage and these book values compare in value after 12 years?

(*c*) Which of the two methods, when calculated correctly considering $S = \$30{,}000$, writes off more of the first cost over 12 years?

16.23 Exactly 10 years ago, Boyditch Professional Associates purchased \$100,000 in depreciable assets with an estimated salvage of \$10,000. For tax depreciation, the SL method with $n = 10$ years was used, but for book depreciation, Boyditch applied the DDB method with $n = 7$ years and neglected the salvage estimate. The company sold the assets today for \$12,500.

(*a*) Compare the sales price today with the book values using the SL and DDB methods.

(*b*) If the salvage of \$12,500 had been estimated exactly 10 years ago, determine the depreciation for each method in year 10.

16.24 Shirley is studying depreciation in her engineering management course. The instructor asked her to graphically compare the total percent of first cost depreciated for an asset costing B dollars over a life of $n = 5$ years for DDB and 125% DB depreciation. Help her by developing the plots of percent of B depreciated versus years. Use a spreadsheet unless otherwise instructed.

MACRS Depreciation

16.25 Explain the difference between an accelerated depreciation method and one that is not accelerated. Give an example of each.

16.26 What was one of the prime reasons that MACRS depreciation was initiated in the mid-1980s?

16.27 A company just purchased an intelligent robot, which has a first cost of \$80,000. Since the robot is unique in its capabilities, the company expects to be able to sell it in 4 years for \$95,000.

(*a*) If the company spends \$10,000 per year in maintenance and operation of the robot, what will the company's MACRS depreciation charge be in year 2? Assume the recovery period for robots is 5 years and the company's MARR is 16% per year when the inflation rate is 9% per year.

(*b*) Determine the book value of the robot at the end of year 2.

16.28 Animatics Corp. of Santa Clara, California, makes small servo systems with built-in controllers, amplifiers, and encoders so that they can control entire machines. The company purchased an asset 2 years ago that has a 5-year recovery period. The depreciation charge by the MACRS method for year 2 is \$24,320.

(*a*) What was the first cost of the asset?

(*b*) How much was the depreciation charge in year 1?

(*c*) Develop the complete MACRS depreciation and book value schedule using the VDB function.

16.29 A plant manager for a large cable company knows that the remaining invested value of certain types of manufacturing equipment is more closely approximated when the equipment is depreciated linearly by the SL method compared to a rapid write-off method such as MACRS. Therefore, he keeps two sets of books, one for tax purposes (MACRS) and one for equipment management purposes (SL). For an asset that has a first cost of \$80,000, a depreciable life of 5 years, and a salvage value equal to 25% of the first cost, determine the difference in the book values shown in the two sets of books at the end of year 4.

16.30 The manager of a Glidden Paint manufacturing plant is aware that MACRS and DDB are both accelerated depreciation methods; however, out of curiosity, she wants to determine which one provides the faster write-off in the first 3 years for a recently purchased mixer that has a first cost of \$300,000, a 5-year recovery period, and a \$60,000 salvage value. Determine which method yields the lower book value and by how much after 3 years. The annual MACRS depreciation rates are 20%, 32%, and 19.2% for years 1, 2, and 3, respectively.

16.31 Railroad cars used to transport coal from Wyoming mines to Texas power plants cost \$1.2 million and have an estimated salvage value of \$300,000. Develop the depreciation and book value schedules for the GDS MACRS method by using two methods on a spreadsheet—the VDB function and the MACRS rates. Are the book value series the same?

16.32 A 120-metric-ton telescoping crane that cost \$320,000 is owned by Upper State Power. Salvage is estimated at \$75,000. (*a*) Compare book values for MACRS and standard SL depreciation over a 7-year recovery period. (*b*) Explain how the estimated salvage is treated using MACRS.

16.33 Youngblood Shipbuilding Yard just purchased \$800,000 in capital equipment for ship repairing functions on dry-docked ships. Estimated salvage is \$150,000 for any year after 5 years of use. Compare the depreciation and book value for year 3 for each of the following depreciation methods.

(*a*) GDS MACRS where a recovery period of 10 years is allowed

(*b*) Double declining balance with a recovery period of 15 years

(*c*) ADS straight line as an alternative to MACRS, with a recovery period of 15 years

16.34 Basketball.com has installed \$100,000 worth of depreciable software and equipment that represents the latest in Internet teaming and basket competition, intended to allow anyone to enjoy the sport on the Web or in the alley. No salvage value is estimated. The company can depreciate using MACRS for a 5-year recovery period or opt for the ADS alternate system over 10 years using the straight line method. The SL rates require the half-year convention; that is, only 50% of the regular annual rate applies for years 1 and 11.

(*a*) Construct the book value curves for both methods on one graph. Show hand or spreadsheet computations as instructed.

(*b*) After 3 years of use, what percentage of the $100,000 basis is removed for each method? Compare the two percentages.

16.35 A company has purchased special-purpose equipment for the manufacture of rubber products (asset class 30.11 in IRS Publication 946) and expects to use it predominately outside the United States. In this case, the ADS alternative to MACRS is required for tax depreciation purposes. The manager wants to understand the difference in yearly recovery rates for classical SL, MACRS, and the ADS alternative to MACRS. Using a recovery period of 3 years, except for the ADS alternative, which requires a 4-year recovery with half-year convention included, prepare a single graph showing the annual recovery rates (in percent) for the three methods.

Depletion

16.36 A coal mine purchased 3 years ago for $7 million was estimated to contain 4,000,000 tons of coal. During the past 3 years the amount of coal removed was 21,000, 18,000, and 20,000 tons, respectively. The gross income obtained in these 3 years was $257,000 for the first year, $320,000 for the second year, and $340,000 for the third year. Determine (*a*) the cost depletion allowance for each year and (*b*) the percentage of the purchase price depleted thus far.

16.37 NA Forest Resources purchased forest acreage for $500,000 from which an estimated 200 million board feet of lumber is recoverable. The company will sell the lumber for $0.10 per board foot. No lumber will be sold for the next 2 years because an environmental impact statement must be completed before harvesting can begin. In years 3 to 10, however, the company expects to remove 20 million board feet per year. The inflation rate is 8%, and the company's MARR is 10%.

(*a*) Determine the depletion amount in year 2 by the cost depletion method.

(*b*) Determine the depletion amount in year 5 by the percentage depletion method.

16.38 A sand and gravel pit purchased for $900,000 is expected to yield 50,000 tons of gravel and 80,000 tons of sand per year. The gravel will sell for $6 per ton and the sand for $9 per ton.

(*a*) Determine the depletion charge according to the percentage depletion method. The percentage depletion rate for sand and gravel is 5%.

(*b*) If taxable income is $100,000 for the year, is this depletion charge allowed? If not, how much is allowed?

16.39 Vesco Mineral Resources purchased mineral rights to land in the foothills of the Santa Cristo mountains. The cost of the purchase was $9 million. Vesco originally thought that it would be able to extract 200,000 tons of lignite from the land, but further exploration revealed that 280,000 tons could be economically removed. If the company sold 20,000 tons in year 1 and 30,000 tons in year 2, what would the depletion charges be each year according to the cost depletion method?

16.40 A company owns gold mining operations in the United States, Australia, and South Africa. The Colorado mine has the taxable income and sales results summarized below. Determine the annual percentage depletion amount for the gold mine.

Year	Taxable Income, $	Sales, Ounces	Average Sales Price, $ per Ounce
1	1,500,000	1800	1115
2	2,000,000	5500	1221
3	800,000	2300	1246

16.41 A highway construction company operates a quarry. During the last 5 years, the amount extracted each year was 60,000, 50,000, 58,000 60,000, and 65,000 tons. The mine is estimated to contain a total of 2.5 million tons of usable stones and gravel. The quarry land had an initial cost of $3.2 million. The company had a per-ton gross income of $30 for the first year, $25 for the second year, $35 for the next 2 years, and $40 for the last year.

(*a*) Determine the depletion charge each year, using the larger of the values for the two depletion methods. Assume all depletion amounts are less than 50% of taxable income.

(*b*) Compute the percent of the initial cost that has been written off in these 5 years, using the depletion charges in part (*a*).

(*c*) If the quarry operation is reevaluated after the first 3 years of operation and estimated to contain a total of 1.5 million tons remaining, rework parts (*a*) and (*b*).

ADDITIONAL PROBLEMS AND FE EXAM REVIEW QUESTIONS

16.42 All of the following types of real property are depreciable except:
(a) Warehouses
(b) Land
(c) Office buildings
(d) Test facilities

16.43 According to information on the IRS website, a taxpayer can take a depreciation deduction as long as the property meets all of the following requirements except:
(a) The taxpayer must own the property.
(b) The taxpayer must use the property in an income-producing activity.
(c) The taxpayer must use the property for personal purposes.
(d) The property must have a determinable useful life of more than 1 year.

16.44 A machine with a 5-year life has a first cost of \$20,000 and a \$2000 salvage value. Its annual operating cost is \$8000 per year. According to the classical straight line method, the depreciation charge in year 2 is nearest to:
(a) \$2800 (b) \$3600
(c) \$4500 (d) \$5300

16.45 A machine with a 10-year life is MACRS-depreciated. The machine has a first cost of \$40,000 with a \$5000 salvage value. Its annual operating cost is \$7000 per year, and d_t for years 1, 2, and 3 is 10.00%, 18.00%, and 14.40%, respectively. The depreciation charge in year 3 is nearest to:
(a) \$5800 (b) \$7200
(c) \$8500 (d) \$9300

16.46 A coal mine purchased for \$5 million has enough coal to operate for 10 years. The annual cost is expected to be \$200,000 per year. The coal is expected to sell for \$150 per ton, with annual production expected to be 10,000 tons. Coal has a depletion percentage rate of 10%. The depletion charge for year 6 according to the percentage depletion method would be closest to:
(a) \$75,000
(b) \$100,000
(c) \$125,000
(d) \$150,000

16.47 The depreciation charge for a 5-year, straight line depreciated vehicle is \$3000 in year 4. If the first cost was \$20,000, the salvage value used in the depreciation calculation was closest to:
(a) \$0 (b) \$2500
(c) \$5000 (d) \$7500

16.48 The book value of an asset that was DDB-depreciated over a 10-year period was \$5832 at the end of year 4. If the first cost of the asset was \$80,000, the salvage value that was used in the depreciation calculation was closest to:
(a) \$0 (b) \$2000
(c) \$5000 (d) \$8000

16.49 For an asset that has $B = \$100{,}000$, $S = \$40{,}000$, and a 10-year depreciable life, the book value at the end of year 4 according to the MACRS method would be closest to (d_t values for years 1, 2, 3, 4, and 5 are 10.00%, 18.00%, 14.40%, 11.52%, and 9.22%, respectively):
(a) \$58,700 (b) \$62,400
(c) \$53,900 (d) \$46,100

16.50 An asset that was depreciated over a 5-year period by the MACRS method had a BV of \$33,025 at the end of year 3. If the MACRS depreciation rates for years 1, 2, and 3, were 0.20, 0.32, and 0.192, respectively, the basis of the asset was closest to:
(a) \$158,000 (b) \$172,000
(c) \$185,000 (d) \$193,000

16.51 A lumber company purchased a tract of land for \$70,000 that contained an estimated 25,000 usable trees. The value of the land was estimated at \$20,000. In the first year of operation, the lumber company cut down 5000 trees. According to the cost depletion method, the depletion deduction for year 1 is closest to:
(a) \$2000 (b) \$7000
(c) \$10,000 (d) \$14,000

16.52 An asset with a first cost of \$50,000 and an estimated salvage value of \$10,000 is depreciated by the MACRS method. If its book value at the end of year 3 is \$21,850 and its market value is \$25,850, the total amount of depreciation charged against the asset up to this time is closest to:
(a) \$18,850 (b) \$21,850
(c) \$25,850 (d) \$28,150

16.53 Under the General Depreciation System (GDS) of asset classification, any asset that is not in a stated class is automatically assigned a recovery period of:
(a) 5 years (b) 7 years
(c) 10 years (d) 15 years

16.54 All of the following statements about the Alternative Depreciation System (ADS) are true except:
(a) The half-year convention applies.
(b) Salvage value is neglected.
(c) The recovery periods are shorter than in GDS.
(d) The straight line method is required.

APPENDIX PROBLEMS

Sum-of-Years-Digits Depreciation

16A.1 A European manufacturing company has new equipment with a first cost of 12,000 euros, an estimated salvage value of 2000 euros, and a recovery period of 8 years. Use the SYD method to tabulate annual depreciation and book value.

16A.2 Earthmoving equipment with a first cost of \$150,000 is expected to have a life of 10 years. The salvage value is expected to be 10% of the first cost. Calculate (*a*) by hand and (*b*) by spreadsheet the depreciation charge and book value for years 2 and 7 using the SYD method.

16A.3 If B = \$12,000, n = 6 years, and S is estimated at 15% of B, use the SYD method to determine (*a*) the book value after 3 years and (*b*) the rate of depreciation and the depreciation amount in year 4.

Unit-of-Production Depreciation

16A.4 A robot used in simulated car crashes cost \$70,000, has no salvage value, and has an expected capacity of tests not to exceed 10,000 according to the manufacturer. Volvo Motors decided to use the unit-of-production depreciation method because the number of test crashes per year in which the robot would be involved was not estimable. Determine the annual depreciation and book value for the first 3 years if the number of tests were 3810, 2720, and 5390 per year.

16A.5 A new hybrid car was purchased by Pedernales Electric Cooperative as a courier vehicle to transport items between its 12 city offices. The car cost \$30,000 and was retained for 5 years. Alternatively, it could have been retained for 100,000 miles. Salvage value is nil. Five-year DDB depreciation was applied. The car pool manager stated that he prefers UOP depreciation on vehicles because it writes off the first cost faster. Use the actual annual miles driven, listed below, to plot the book values for both methods. Determine which method would have removed the \$30,000 faster. Show hand or spreadsheet solution, as instructed.

Year	1	2	3	4	5
Miles Driven, 1000	15	22	16	18	24

Switching Methods

16A.6 An asset has a first cost of \$45,000, a recovery period of 5 years, and a \$3000 salvage value. Use the switching procedure from DDB to SL depreciation, and calculate the present worth of depreciation at i = 18% per year.

16A.7 If B = \$45,000, S = \$3000, and n = 5-year recovery period, use a spreadsheet and i = 18% per year to maximize the present worth of depreciation, using the following methods: DDB-to-SL switching (this was determined in Problem 16A.6) and MACRS. Given that MACRS is the required depreciation system in the United States, comment on the results.

16A.8 Hempstead Industries has a new milling machine with B = \$110,000, n = 10 years, and S = \$10,000. Determine the depreciation schedule and present worth of depreciation at i = 12% per year, using the 175% DB method for the first 5 years and switching to the classical SL method for the last 5 years. Use a spreadsheet to solve this problem.

16A.9 Reliant Electric Company has erected a large portable building with a first cost of \$155,000 and an anticipated salvage of \$50,000 after 25 years. (*a*) Should the switch from DDB to SL depreciation be made? (*b*) For what values of the uniform depreciation rate in the DB method would it be advantageous to switch from DB to SL depreciation at some point in the life of the building?

MACRS Rates

16A.10 Verify the 5-year recovery period rates for MACRS given in Table 16–2. Start with the DDB method in year 1, and switch to SL depreciation when it offers a larger recovery rate.

16A.11 A video recording system was purchased 3 years ago at a cost of \$30,000. A 5-year recovery period and MACRS depreciation have been used to write off the basis. The system is to be prematurely replaced with a trade-in value of \$5000. Determine the MACRS depreciation, using the switching rules to find the difference between the book value and the trade-in value after 3 years.

16A.12 Use the computations in Equations [16A.11] through [16A.13] to determine the MACRS annual depreciation for the following asset data: B = \$50,000 and a recovery period of 7 years.

16A.13 The 3-year MACRS recovery rates are 33.33%, 44.45%, 14.81%, and 7.41%, respectively. (*a*) What are the corresponding rates for the alternative MACRS straight line ADS method with the half-year convention imposed? (*b*) Compare the PW_D values for these two methods if B = \$80,000 and i = 15% per year.

CHAPTER 17

After-Tax Economic Analysis

LEARNING OUTCOMES

Purpose: Perform an after-tax economic evaluation considering the impact of pertinent tax regulations, income taxes, and depreciation.

SECTION	TOPIC	LEARNING OUTCOME
17.1	Terminology and rates	• Know the fundamental terms and relations of after-tax analysis; use a marginal tax rate table.
17.2	CFBT and CFAT	• Determine cash flow series before taxes and after taxes.
17.3	Taxes and depreciation	• Demonstrate the tax advantage of accelerated depreciation and shortened recovery periods.
17.4	Depreciation recapture	• Calculate the tax impact of DR; explain the use of capital gains and capital losses.
17.5	After-tax analysis	• Evaluate one project or multiple alternatives using after-tax PW, AW, and ROR analysis.
17.6	After-tax replacement	• Evaluate a defender and challenger in an after-tax replacement study.
17.7	EVA analysis	• Evaluate an alternative using after-tax economic value-added analysis; compare to CFAT analysis.
17.8	Taxes outside the United States	• Understand the fundamental practices for depreciation and tax rates in international settings.
17.9	VAT	• Demonstrate the use and computation of a value-added tax on manufactured products.

This chapter provides an overview of tax terminology, income tax rates, and tax equations pertinent to an after-tax economic analysis. The change from estimating cash flow before taxes (CFBT) to **cash flow after taxes (CFAT)** involves a consideration of significant tax effects that may alter the final decision, and estimates the magnitude of the effect on cash flow over the life of the alternative that taxes may have.

Mutually exclusive alternative comparisons using after-tax PW, AW, and ROR methods are explained with major tax implications considered. **Replacement studies** are discussed with tax effects that occur at the time that a defender is replaced. Also, the after-tax **economic value added** by an alternative is discussed in the context of annual worth analysis. All these methods use the procedures learned in earlier chapters, except now with tax effects considered.

An after-tax evaluation using any method requires more computations than in previous chapters. Templates for tabulation of cash flow after taxes by hand and by spreadsheet are developed. Additional information on U.S. federal taxes—tax law and annually updated tax rates—is available through Internal Revenue Service publications and, more readily, on the IRS website www.irs.gov. Publications 542, *Corporations*, and 544, *Sales and Other Dispositions of Assets*, are especially applicable to this chapter. Some differences in tax considerations outside the United States are summarized.

17.1 Income Tax Terminology and Basic Relations ● ● ●

The perspective taken in engineering economy when performing an after-tax evaluation is that of the *project* and how relevant tax rules and allowances influence the economic decision. The perspective of a financial study is that of the *corporation* and how the tax structure and laws affect profitability. We take the engineering economy viewpoint in the sections that follow.

There are many types of taxes levied upon corporations and individuals in all countries, including the United States. Some are sales tax, valued-added tax, import tax, income tax, highway tax, gasoline tax, and property (real estate) tax. Federal governments rely on income taxes for a significant portion of their annual revenue. States, provinces, and municipal-level governments rely on sales, value-added, and property taxes to maintain services, schools, etc. for the citizenry. For a starting point on *corporate income taxes* and how they are used when performing an after-tax economic evaluation of a project or multiple alternatives, this section covers basic definitions, terms, and relations.

Income tax is the amount of the payment (taxes) on income or profit that must be delivered to a federal (or lower-level) government unit. Taxes are **real cash flows;** however, for corporations tax computation requires some noncash elements, such as depreciation. Corporate income taxes are usually submitted quarterly, and the last payment of the year is submitted with the annual tax return.

The Internal Revenue Service (IRS), a part of the U.S. Department of the Treasury, collects the taxes and enforces tax laws. The website www.irs.gov provides information on tax laws, rates, publications, etc. that are referenced in this chapter.

Though the formulas are much more complex when applied to a specific situation, two fundamental relations form the basis for income tax computations. The first involves only actual cash flows:

$$\text{Net operating income} = \text{revenue} - \text{operating expenses}$$

The second involves actual cash flows and noncash deductibles, such as depreciation.

$$\text{Taxable income} = \text{revenue} - \text{operating expenses} - \text{depreciation}$$

These terms and relations for corporations are now described. Since each term is calculated for 1 year, there can be a subscript t ($t = 1, 2, \ldots$) added; t is omitted here for simplicity.

Operating revenue *R*, also commonly called **gross income GI,** is the total income realized from all revenue-producing sources. These incomes are listed in the income statement. (See Appendix B on accounting reports.) Other, nonoperating revenues such as sale of assets, license fee income, and royalties are considered separately for tax purposes.

Operating expenses OE include all costs incurred in the transaction of business. These expenses are tax-deductible for corporations. For after-tax economic evaluations, the AOC (annual operating costs) and M&O (maintenance and operating) costs are applicable here. Depreciation is not included here since it is *not an operating expense*.

Net operating income NOI, often called **EBIT** (earnings before interest and income taxes), is the difference between gross income and operating expenses.

$$\text{NOI} = \text{EBIT} = \text{GI} - \text{OE} \tag{17.1}$$

Taxable income TI is the amount of income upon which taxes are based. A corporation is allowed to **remove depreciation,** depletion and amortization, and some other deductibles from net operating income in determining the taxable income for a year. For our evaluations, we define taxable income as

$$\begin{aligned}\text{TI} &= \text{gross income} - \text{operating expenses} - \text{depreciation}\\ &= \text{GI} - \text{OE} - \text{D}\end{aligned} \tag{17.2}$$

Though there may be subtleties and varying interpretations over time, in essence, the differences between NOI and TI are tax-law-allowed deductibles, such as depreciation. (In keeping with the project view of engineering economics, we will primarily use the TI relation when conducting an after-tax evaluation.)

Tax rate T is a percentage, or decimal equivalent, of TI that is owed in taxes. The tax rates in many countries (including the United States) are **graduated** (or **progressive**) by level of TI; that is, higher rates apply as the TI increases. The **marginal tax rate** is the percentage paid on the *last dollar of income*. The average tax rate paid is calculated separately from the highest marginal rate used, as shown later. The general tax computation relation is

$$\begin{aligned}\text{Income taxes} &= \text{applicable tax rate} \times \text{taxable income}\\ &= (T)(\text{TI})\end{aligned} \tag{17.3}$$

Net operating profit after taxes NOPAT is the amount remaining each year after taxes are subtracted from taxable income.

$$\begin{aligned}\text{NOPAT} &= \text{TI} - \text{taxes} = \text{TI} - (T)(\text{TI})\\ &= \text{TI}(1 - T)\end{aligned} \tag{17.4}$$

Basically, NOPAT represents the money remaining in the corporation as a result of the capital invested during the year. It is also called *net profit after taxes* (NPAT).

The graduated tax rate schedule for corporations is presented in Table 17–1 as taken from IRS Publication 542, *Corporations*. These are rates for the entire corporation, not for an individual project, though they are often applied in the after-tax analysis of a single project. The rates can change annually based upon government legislation; however, the corporate tax rate schedule has remained the same for some years. To illustrate the use of the graduated tax rate, assume a company is expected to generate a taxable income of \$500,000 in 1 year. From Table 17–1, the marginal tax rate for the last dollar of TI is 34%, but the graduated rates become progressively larger as TI increases. In this case, for TI = \$500,000,

$$\begin{aligned}\text{Taxes} &= 113{,}900 + 0.34(500{,}000 - 335{,}000)\\ &= 113{,}900 + 56{,}100\\ &= \$170{,}000\end{aligned}$$

Alternatively, the rates for each TI level can be used to calculate taxes the longer way.

$$\begin{aligned}\text{Taxes} &= 0.15(50{,}000) + 0.25(75{,}000 - 50{,}000) + 0.34(100{,}000 - 75{,}000)\\ &\quad + 0.39(335{,}000 - 100{,}000) + 0.34(500{,}000 - 335{,}000)\\ &= 7500 + 6250 + 8500 + 91{,}650 + 56{,}100\\ &= \$170{,}000\end{aligned}$$

TABLE 17–1 U.S. Corporate Income Tax Rate Schedule (2010)

If Taxable Income ($) Is:			
Over	But Not over	Tax Is	Of the Amount over
0	50,000	15%	0
50,000	75,000	7,500 + 25%	50,000
75,000	100,000	13,750 + 34%	75,000
100,000	335,000	22,250 + 39%	100,000
335,000	10,000,000	113,900 + 34%	335,000
10,000,000	15,000,000	3,400,000 + 35%	10,000,000
15,000,000	18,333,333	5,150,000 + 38%	15,000,000
18,333,333	—	35%	0

Smaller businesses (with TI < \$335,000) receive a slight tax advantage compared to large corporations. Once the TI exceeds \$335,000, an effective federal tax rate of 34% applies, and when TI > \$18.33 million, there is a flat tax rate of 35%.

As we move forward with after-tax analysis, it is important to keep the following in mind.

The corporate tax rates apply to a corporation as a whole, not to a specific project, unless the project *is* the company. Tax rates are usually graduated by level of taxable income. Therefore, the last dollar of TI is taxed at a marginal rate. The average federal tax rate actually paid is lower than the highest marginal rate paid because the rates, in general, graduate to higher percentages as TI increases.

Because the marginal tax rates change with TI, it is not possible to quote directly the percent of TI paid in income taxes. Alternatively, a single-value number, the *average tax rate*, is calculated as

$$\text{Average tax rate} = \frac{\text{total taxes paid}}{\text{taxable income}} = \frac{\text{taxes}}{\text{TI}} \qquad [17.5]$$

Referring to Table 17–1, for a small business with TI = \$100,000, the federal income tax burden averages \$22,250/100,000 = 22.25%. If TI = \$15 million, the average tax rate is \$5.15 million/15 million = 34.33%.

As mentioned earlier, there are federal, state, and local taxes imposed. For the sake of simplicity, the tax rate used in an economy study is often a single-figure **effective tax rate** T_e, which accounts for all taxes. Effective tax rates are in the range of 35% to 50%. One reason to use the effective tax rate is that state taxes are deductible for federal tax computation. The effective tax rate and taxes are calculated as

$$T_e = \text{state rate} + (1 - \text{state rate})(\text{federal rate}) \qquad [17.6]$$
$$\text{Taxes} = (T_e)(\text{TI}) \qquad [17.7]$$

EXAMPLE 17.1

REI (Recreational Equipment Incorporated) sells outdoor equipment and sporting goods through retail outlets, the Internet, and catalogs. Assume that for 1 year REI has the following financial results in the state of Kentucky, which has a flat tax rate of 6% on corporate taxable income.

Total revenue	\$19.9 million
Operating expenses	\$8.6 million
Depreciation and other allowed deductions	\$1.8 million

(*a*) Determine the state taxes and federal taxes due using Table 17–1 rates.
(*b*) Find the average federal tax rate paid for the year.
(*c*) Determine a single-value tax rate useful in economic evaluations using the average federal tax rate determined in part (*b*).
(*d*) Estimate federal and state taxes using the single-value rate, and compare their total with the total in part (*a*).

Solution

(*a*) Calculate TI by Equation [17.2] and use Table 17–1 rates for federal taxes due.

$$\begin{aligned} \text{Kentucky state TI} &= \text{GI} - \text{OE} - D = 19.9 \text{ million} - 8.6 \text{ million} - 1.8 \text{ million} \\ &= \$9.5 \text{ million} \\ \text{Kentucky state taxes} &= 0.06(\text{TI}) = 0.06(9{,}500{,}000) = \$570{,}000 \\ \text{Federal TI} &= \text{GI} - \text{OE} - D - \text{state taxes} = 9{,}500{,}000 - 570{,}000 \\ &= \$8{,}930{,}000 \\ \text{Federal taxes} &= 113{,}900 + 0.34(8{,}930{,}000 - 335{,}000) = \$3{,}036{,}200 \\ \text{Total federal and state taxes} &= 3{,}036{,}200 + 570{,}000 = \$3{,}606{,}200 \qquad [17.8] \end{aligned}$$

(*b*) From Equation [17.5], the average tax rate paid is approximately 32% of TI.

$$\text{Average federal tax rate} = 3{,}036{,}200/9{,}500{,}000 = 0.3196$$

(*c*) By Equation [17.6], T_e is slightly over 36% per year for combined state and federal taxes.

$$T_e = 0.06 + (1 - 0.06)(0.3196) = 0.3604 \quad (36.04\%)$$

(*d*) Use the effective tax rate and TI = \$9.5 million from part (*a*) in Equation [17.7] to approximate total taxes.

$$\text{Taxes} = 0.3604(9{,}500{,}000) = \$3{,}423{,}800$$

Compared to Equation [17.8], this approximation is \$182,400 low, a 5.06% underestimate.

It is interesting to understand how corporate tax and individual tax computations differ. Gross income for an individual taxpayer is comparable if revenue is replaced by salaries and wages. However, for an individual's taxable income, most of the expenses for living and working are not tax deductible to the same degree as operating expenses are for corporations. For individual taxpayers,

$$\begin{aligned} \text{GI} &= \text{salaries} + \text{wages} + \text{interest and dividends} + \text{other income} \\ \text{TI} &= \text{GI} - \text{personal exemption} - \text{standard or itemized deductions} \\ \text{Taxes} &= (T)(\text{TI}) \end{aligned}$$

For TI, operating expenses are replaced by individual exemptions and specific deductions. Exemptions are yourself, your spouse, your children, and your other dependents. Each exemption reduces TI by \$3500 to \$4000 per year, depending upon current exemption allowances.

In the United States, the tax rates for individuals, like those for corporations, are graduated by level of TI. In 2010, the marginal rates ranged from 10% to 35%; however, the top marginal rates are increasing for individuals with larger TI amounts. These rates and TI levels are the subject of ongoing political debates at the U.S. national level depending upon the balance of power in the congressional bodies and the economic condition of the country. Once the marginal rates are set, the TI levels are adjusted each year to account for inflation and other factors. This process is called **indexing.** Clearly, tax rates for individuals change much more frequently than the rates for corporations change. Current information is available on the IRS website www.irs.gov through Publication 17, *Your Federal Income Tax.* The current rate schedule is published in the back of this document for four filing status categories:

Unmarried individuals (single)
Married filing jointly
Married filing separately
Head of household

17.2 Calculation of Cash Flow after Taxes ●●●

Early in the text, the term *net cash flow* (*NCF*) was identified as the best estimate of actual cash flow each year. The NCF is calculated as cash inflows minus cash outflows. Since then, the annual NCF amounts have been used many times to perform alternative evaluations via the PW, AW, ROR, and B/C methods. Now that the impact on cash flow of depreciation and related taxes will be considered, it is time to expand our terminology. NCF is replaced by the term **cash flow before taxes (CFBT)**, and we introduce the new term **cash flow after taxes (CFAT)**.

CFBT and CFAT are **actual cash flows;** that is, they represent the estimated actual flow of money into and out of the corporation that will result from the alternative. The remainder of this section explains how to transition from before-tax to after-tax cash flows for solutions by hand and by spreadsheet, using tax regulations described in the next few sections. Once the CFAT estimates are developed, the economic evaluation is performed using the same methods and selection guidelines applied previously. However, the analysis is performed on the CFAT estimates.

We learned that net operating income (NOI) does not include the purchase or sale of capital assets. However, the annual CFBT estimate *must include* the initial capital investment and salvage value for the years in which they occur. Incorporating the definitions of gross income and operating expenses from NOI, CFBT for any year is defined as

$$\begin{aligned}\textbf{CFBT} &= \textbf{gross income} - \textbf{operating expenses} - \textbf{initial investment} + \textbf{salvage value}\\ &= \text{GI} - \text{OE} - P + S \qquad [17.9]\end{aligned}$$

As in previous chapters, P is the initial investment (year 0) and S is the estimated salvage value in year n. Therefore, only in year 0 will the CFBT include P, and only in year n will an S value be present. Once all taxes are estimated, the annual after-tax cash flow is simply

$$\textbf{CFAT} = \textbf{CFBT} - \textbf{taxes} \qquad [17.10]$$

where taxes are estimated using the relation $(T)(\text{TI})$ or $(T_e)(\text{TI})$.

We know from Equation [17.2] that depreciation D is considered when calculating TI. It is very important to understand the different roles of depreciation for income tax computations and in CFAT estimation.

Depreciation is not an operating expense and is a *non*cash flow. Depreciation is tax deductible for determining the amount of income taxes only, but it does not represent a direct, after-tax cash flow. Therefore, the after-tax engineering economy study must be based on actual cash flow estimates, that is, annual CFAT estimates that do not include depreciation as an expense (negative cash flow).

Accordingly, if the CFAT expression is determined using the TI relation, depreciation must not be included outside of the TI component. Equations [17.9] and [17.10] are now combined.

$$\text{CFAT} = \text{GI} - \text{OE} - P + S - (\text{GI} - \text{OE} - D)(T_e) \qquad [17.11]$$

Suggested table column headings for CFBT and CFAT calculations by hand or by spreadsheet are shown in Table 17–2. The equations are shown in column numbers, with the effective tax rate T_e used for income tax estimation. Expenses OE and initial investment P carry negative signs in all tables and spreadsheets.

A **negative TI value** may occur in some years due to a depreciation amount that is larger than (GI – OE). It is possible to account for this in a detailed after-tax analysis using carry-forward and carry-back rules for operating losses. It is the exception that the engineering economy study will consider this level of detail. Rather, the associated negative income tax is considered as a **tax savings for the year.** The assumption is that the negative tax will offset taxes for the same year in other income-producing areas of the corporation.

TABLE 17–2 Suggested Column Headings for Calculation of CFAT

Year	Gross Income GI	Operating Expenses OE	Investment and Salvage P and S	CFBT	Depreciation D	Taxable Income TI	Taxes	CFAT
	(1)	(2)	(3)	(4) = (1) + (2) + (3)	(5)	(6) = (1) + (2) − (5)	(7) = T_e(6)	(8) = (4) − (7)

EXAMPLE 17.2

Wilson Security has received a contract to provide additional security for corporate and government personnel along the Arizona-Mexico border. Wilson plans to purchase listening and detection equipment for use in the 6-year contract. The equipment is expected to cost \$550,000 and have a resale value of \$150,000 after 6 years. Based on the incentive clause in the contract, Wilson estimates that the equipment will increase contract revenue by \$200,000 per year and require an additional M&O expense of \$90,000 per year. MACRS depreciation allows recovery in 5 years, and the effective corporate tax rate is 35% per year. Tabulate and plot the CFBT and CFAT series.

Solution

The spreadsheet in Figure 17–1 presents before-tax and after-tax cash flows using the format of Table 17–2. The functions for year 6 are detailed in row 11. Discussion and sample calculations follow.

CFBT: The operating expenses OE and initial investment P are shown as negative cash flows. The \$150,000 salvage (resale) is a positive cash flow in year 6. CFBT is calculated by Equation [17.9]. In year 6, for example, when the equipment is sold, the function in row 11 indicates that

$$\text{CFBT}_6 = 200{,}000 - 90{,}000 + 150{,}000 = \$260{,}000$$

CFAT: Column F for MACRS depreciation, which is determined using the VDB function over the 6-year period, writes off the entire \$550,000 investment. Taxable income, taxes, and CFAT are calculated, using year 4 as an example.

$$\text{TI}_4 = \text{GI} - \text{OE} - D = 200{,}000 - 90{,}000 - 63{,}360 = \$46{,}640$$

$$\text{Taxes}_4 = (0.35)\,(\text{TI}) = (0.35)\,(46{,}640) = \$16{,}324$$

$$\text{CFAT}_4 = \text{GI} - \text{OE} - \text{taxes} = 200{,}000 - 90{,}000 - 16{,}324 = \$93{,}676$$

In year 2, MACRS depreciation is large enough to cause TI to be negative (\$–66,000). As mentioned above, the negative tax (\$–23,100) is considered a *tax savings* in year 2, thus increasing CFAT.

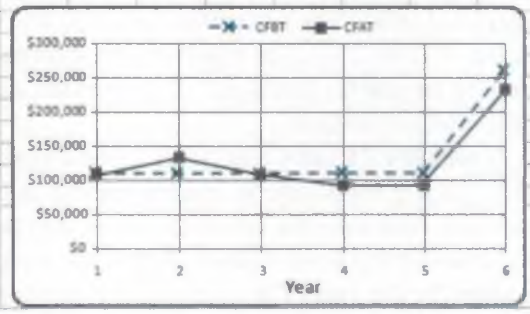
Comment

MACRS depreciates to a salvage value of $S = 0$. Later we will learn about a tax implication due to "recapturing of depreciation" when an asset is sold for an amount larger than zero and MACRS was applied to fully depreciate the asset to zero.

Figure 17–1

Computation of CFBT and CFAT using MACRS depreciation and $T_e = 35\%$, Example 17.2.

	A	B	C	D	E	F	G	H	I
1	Year	GI	OE	P and S	CFBT	D	TI	Taxes	CFAT
2	0			-550,000	-550,000		0	0	-550,000
3	1	200,000	-90,000		110,000	110,000	0	0	110,000
4	2	200,000	-90,000		110,000	176,000	-66,000	-23,100	133,100
5	3	200,000	-90,000		110,000	105,600	4,400	1,540	108,460
6	4	200,000	-90,000		110,000	63,360	46,640	16,324	93,676
7	5	200,000	-90,000		110,000	63,360	46,640	16,324	93,676
8	6	200,000	-90,000	150,000	260,000	31,680	78,320	27,412	232,588
9	Totals					550,000			
10									
11	Functions				= \$B8 +\$C8 +\$D8	= VDB(550000 0 5 MAX(0 \$A8-1 5),MIN(5 \$A8-0 5) 2)	= \$B8 +\$C8 -\$F8	= G8 *0 35	= B8 +C8+D8 -H8



17.3 Effect on Taxes of Different Depreciation Methods and Recovery Periods

It is important to understand why *accelerated depreciation rates* give the corporation a tax advantage relative to the straight line method with the same recovery period. Larger rates in earlier years of the recovery period require less taxes due to the larger reductions in taxable income. The criterion of **minimizing the present worth of taxes** is used to demonstrate the tax effect. For the recovery period n, choose the depreciation rates that result in the **minimum** present worth value for taxes.

$$PW_{tax} = \sum_{t=1}^{t=n} (\text{taxes in year } t)(P/F,i,t) \qquad [17.12]$$

This is equivalent to maximizing the present worth of total depreciation PW_D.

Compare any two depreciation methods. Assume the following: (1) There is a constant single-value tax rate, (2) CFBT exceeds the annual depreciation amount, (3) both methods reduce book value to the same salvage value, and (4) the same recovery period is used. On the basis of these assumptions, the following statements are correct:

The total taxes paid are **equal** for all depreciation methods.
The present worth of taxes is **less** for accelerated depreciation methods.

As we learned in Chapter 16, MACRS is the prescribed tax depreciation method in the United States, and the only alternative is MACRS straight line depreciation with an extended recovery period. The accelerated write-off of MACRS always provides a smaller PW_{tax} compared to less accelerated methods. If the DDB method were still allowed directly, rather than embedded in MACRS, DDB would not fare as well as MACRS. This is so because DDB does not reduce the book value to zero. This is illustrated in Example 17.3.

EXAMPLE 17.3

An after-tax analysis for a new $50,000 machine proposed for a fiber optics manufacturing line is in process. The CFBT for the machine is estimated at $20,000. If a recovery period of 5 years applies, use the present worth of taxes criterion, an effective tax rate of 35%, and a return of 8% per year to compare the following: classical straight line, classical DDB, and required MACRS depreciation. Use a 6-year period for the comparison to accommodate the half-year convention imposed by MACRS.

Solution

Table 17–3 presents a summary of annual depreciation, taxable income, and taxes for each method. For classical straight line depreciation with $n = 5$, $D_t = \$10,000$ for 5 years and $D_6 = 0$ (column 3). The CFBT of $20,000 is fully taxed at 35% in year 6.

The classical DDB percentage of $d = 2/n = 0.40$ is applied for 5 years. The implied salvage value is $50,000 − 46,112 = $3888, so not all $50,000 is tax-deductible. The taxes using classical DDB will be $3888(0.35) = $1361 larger than for the classical SL method.

TABLE 17–3 Comparison of Taxes and Present Worth of Taxes for Different Depreciation Methods

		Classical Straight Line			Classical Double Declining Balance			MACRS		
(1) Year t	(2) CFBT, $	(3) D_t, $	(4) TI, $	(5) = 0.35(4) Taxes, $	(6) D_t, $	(7) TI, $	(8) = 0.35(7) Taxes, $	(9) D_t, $	(10) TI, $	(11) = 0.35(10) Taxes, $
1	+20,000	10,000	10,000	3,500	20,000	0	0	10,000	10,000	3,500
2	+20,000	10,000	10,000	3,500	12,000	8,000	2,800	16,000	4,000	1,400
3	+20,000	10,000	10,000	3,500	7,200	12,800	4,480	9,600	10,400	3,640
4	+20,000	10,000	10,000	3,500	4,320	15,680	5,488	5,760	14,240	4,984
5	+20,000	10,000	10,000	3,500	2,592	17,408	6,093	5,760	14,240	4,984
6	+20,000	0	20,000	7,000	0	20,000	7,000	2,880	17,120	5,992
Totals		50,000		24,500	46,112		25,861*	50,000		24,500
PW_{tax}				18,386			18,549			18,162

*Larger than other values since there is an implied salvage value of $3888 not recovered.

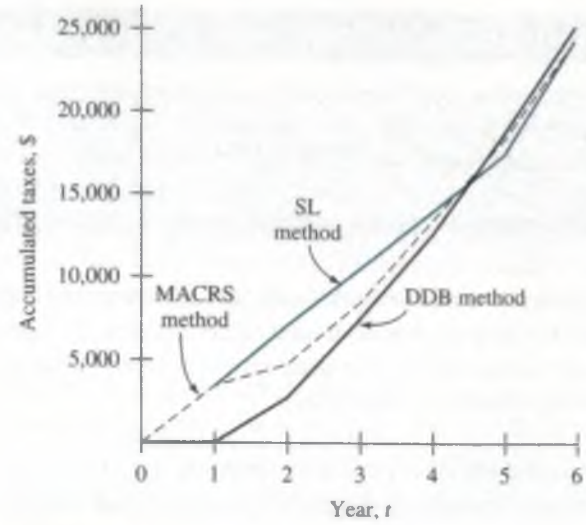


Figure 17–2

Cumulative taxes incurred by different depreciation rates for a 6-year comparison period, Example 17.3.

MACRS writes off $50,000 in 6 years using the rates of Table 16–2. Total taxes are $24,500, the same as for classical SL depreciation.

The annual taxes (columns 5, 8, and 11) are accumulated year by year in Figure 17–2. Note the pattern of the curves, especially the lower total taxes relative to the SL model after year 1 for MACRS and in years 1 through 4 for DDB. These higher tax values for SL cause PW_{tax} for SL depreciation to be larger. The PW_{tax} values at the bottom of Table 17–3 are calculated using Equation [17.12]. The MACRS PW_{tax} value is the smallest at $18,162.

To compare taxes for different *recovery periods*, change only assumption 4 at the beginning of this section to read: The same depreciation method is applied. It can be shown that a shorter recovery period will offer a tax advantage over a longer period using the criterion to minimize PW_{tax}. Comparison will indicate that

The total taxes paid are **equal** for all n values.
The present worth of taxes is **less** for smaller n values.

This is why corporations want to use the shortest MACRS recovery period allowed for income tax purposes. Example 17.4 demonstrates these conclusions for classical straight line depreciation, but the conclusions are correct for MACRS or any other tax depreciation method.

EXAMPLE 17.4

Grupo Grande Maquinaria, a diversified manufacturing corporation based in Mexico, maintains parallel records for depreciable assets in its European operations in Germany. This is common for multinational corporations. One set is for corporate use that reflects the estimated useful life of assets. The second set is for foreign government purposes, such as depreciation and taxes.

The company just purchased an asset for $90,000 with an estimated useful life of 9 years; however, a shorter recovery period of 5 years is allowed by German tax law. Demonstrate the tax advantage for the smaller n if net operating income (NOI) is $30,000 per year, an effective tax rate of 35% applies, invested money is returning 5% per year after taxes, and classical SL depreciation is allowed. Neglect the effect of any salvage value.

Solution

Determine the annual TI and taxes by Equations [17.2] through [17.3] and the present worth of taxes using Equation [17.12] for both n values.

Useful life $n = 9$ years:

$$D = \frac{90{,}000}{9} = 10{,}000$$

$$\text{TI} = 30{,}000 - 10{,}000 = \$20{,}000 \text{ per year}$$

$$\text{Taxes} = (0.35)(20{,}000) = \$7000 \text{ per year}$$

$$\text{PW}_{\text{tax}} = 7000(P/A,5\%,9) = \$49{,}755$$

$$\text{Total taxes} = (7000)(9) = \$63{,}000$$

Recovery period $n = 5$ years:

Use the same comparison period of 9 years, but depreciation occurs only during the first 5 years.

$$D_t = \begin{cases} \dfrac{90{,}000}{5} = \$18{,}000 & t = 1 \text{ to } 5 \\ 0 & t = 6 \text{ to } 9 \end{cases}$$

$$\text{Taxes} = \begin{cases} (0.35)(30{,}000 - 18{,}000) = \$4200 & t = 1 \text{ to } 5 \\ (0.35)(30{,}000) = \$10{,}500 & t = 6 \text{ to } 9 \end{cases}$$

$$\begin{aligned} \text{PW}_{\text{tax}} &= 4200(P/A,5\%,5) + 10{,}500(P/A,5\%,4)(P/F,5\%,5) \\ &= \$47{,}356 \end{aligned}$$

$$\text{Total taxes} = 4200(5) + 10{,}500(4) = \$63{,}000$$

A total of \$63,000 in taxes is paid in both cases. However, the more rapid write-off for $n = 5$ results in a present worth tax savings of nearly \$2400 (49,755 − 47,356).

17.4 Depreciation Recapture and Capital Gains (Losses) ● ● ●

All the tax implications discussed here are the result of disposing of a depreciable asset before, at, or after its recovery period. In an after-tax economic analysis of large investment assets, these tax effects should be considered. The key is the size of the selling price (or salvage or market value) relative to the current book value at disposal time and relative to the first cost, which is called the *unadjusted basis B* in depreciation terminology. There are three relevant terms.

Depreciation recapture DR, also called **ordinary gain,** occurs when a depreciable asset is sold for more than the current book value BV_t. As shown in Figure 17–3,

Depreciation recapture = selling price − book value

$$\text{DR} = \text{SP}_2 - \text{BV}_t \qquad [17.13]$$

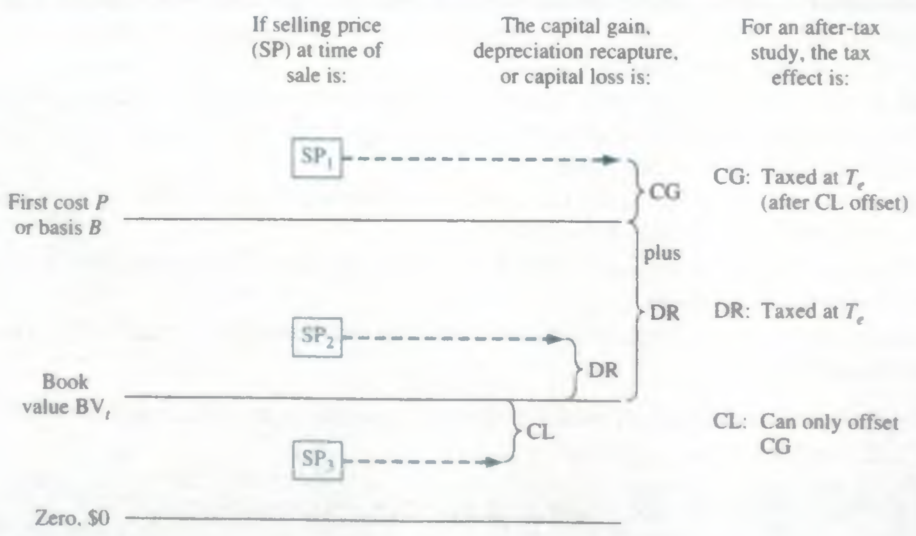


Figure 17–3
Summary of calculations and tax treatment for depreciation recapture and capital gains (losses).

Depreciation recapture is often present in the after-tax study. In the United States, an amount equal to the estimated salvage value can always be anticipated as DR when the asset is disposed of after the MACRS recovery period. This is correct simply because MACRS depreciates every asset to zero in $n + 1$ years. The amount of DR is treated as ordinary taxable income in the year of asset disposal.

Capital gain CG is an amount incurred when the selling price exceeds its (unadjusted) basis B. See Figure 17–3.

$$\text{Capital gain} = \text{selling price} - \text{basis}$$
$$CG = SP_1 - B \qquad [17.14]$$

Since future capital gains are difficult to predict, they are usually not detailed in an after-tax economy study. An exception is for assets that historically increase in value, such as buildings and land.

When the selling price exceeds B, the TI due to the sale is the capital gain *plus* the depreciation recapture, as shown in Figure 17–3. The DR is now the total amount of depreciation taken thus far, that is, $B - BV$.

Capital loss CL occurs when a depreciable asset is disposed of for less than its current book value. In Figure 17–3,

$$\text{Capital loss} = \text{book value} - \text{selling price}$$
$$CL = BV_t - SP_3 \qquad [17.15]$$

An economic analysis does not commonly account for capital loss, simply because it is not estimable for a specific alternative. However, an *after-tax replacement study* should account for any capital loss if the defender must be traded at a "sacrifice" price. For the purposes of the economic study, this provides a tax savings in the year of replacement. Use the effective tax rate to estimate the tax savings. These savings are assumed to be offset elsewhere in the corporation by other income-producing assets that generate taxes.

There are several additional points worth mentioning about capital gains and capital losses for a corporation, apart from their presence in an economic evaluation.

- U.S. tax law defines capital gains as long-term (items retained for more than 1 year) or short-term.
- Capital gains actually take place for property that is not depreciated or amortized. The term *capital gain* correctly applies at sale time to property such as investments (stocks and bonds), art, jewelry, land, and the like. When a depreciable asset's selling price is higher than the original cost (its basis), it is correctly termed an **ordinary gain.** Corporate tax treatment at this time is the same for both: taxed as ordinary income. All said, it is common to classify an expected ordinary gain on a depreciable asset as a capital gain, without altering the economic decision.
- Capital gains are taxed as ordinary taxable income at the corporation's regular tax rates.
- Capital losses do not directly reduce annual income taxes because they can only be *netted* against capital gains to the maximum extent of the capital gains for the year. The terms used then are *net* capital gains (losses).
- When capital losses exceed capital gains, the corporation can take advantage of carry-back and carry-forward tax laws for the excess, something of value to a finance or tax officer, not an engineer doing an economic analysis.
- When an asset is disposed of, the tax treatment is referred to as a *Section 1231 transaction*, which is the IRS rule section of the same number.
- IRS Publication 544, *Sales and Other Dispositions of Assets*, may be helpful if gains and losses are present in a study.
- All these rules apply to corporations. Individual tax rules and rates are different concerning asset disposition.

If the three additional income and tax elements covered here are incorporated into Equation [17.2], *taxable income* is defined as

$$\begin{aligned}\textbf{TI} &= \textbf{gross income} - \textbf{operating expenses} - \textbf{depreciation} \\ &\quad + \textbf{depreciation recapture} + \textbf{net capital gain} - \textbf{net capital loss} \\ &= \textbf{GI} - \textbf{OE} - \textbf{D} + \textbf{DR} + \textbf{CG} - \textbf{CL} \qquad [17.16]\end{aligned}$$

In keeping with our perspective of an engineering economy study rather than that of a financial study, depreciation recapture (i.e., ordinary gain) will be the primary element considered in after-tax evaluations. Only when a capital gain or loss must be included due to the nature of the problem will the calculations involve it.

EXAMPLE 17.5

Biotech, a medical imaging and modeling company, must purchase a bone cell analysis system for use by a team of bioengineers and mechanical engineers studying bone density in athletes. This particular part of a 3-year contract with the NBA will provide additional gross income of $100,000 per year. The effective tax rate is 35%. Estimates for two alternatives are summarized below.

	Analyzer 1	Analyzer 2
Basis *B*, $	150,000	225,000
Operating expenses, $ per year	30,000	10,000
MACRS recovery, years	5	5

Answer the following questions, solving by hand and spreadsheet.

(*a*) The Biotech president, who is very tax conscious, wishes to use a criterion of minimizing total taxes incurred over the 3 years of the contract. Which analyzer should be purchased?

(*b*) Assume that 3 years have now passed, and the company is about to sell the analyzer. Using the same total tax criterion, did either analyzer have an advantage? Assume the selling price is $130,000 for analyzer 1, or $225,000 for analyzer 2.

Solution by Hand

(*a*) Table 17–4 details the tax computations. First, the yearly MACRS depreciation is determined. Equation [17.2], TI = GI – OE – *D*, is used to calculate TI, after which the 35% tax rate is applied each year. Taxes for the 3-year period are summed, with no consideration of the time value of money.

Analyzer 1 tax total: $36,120 Analyzer 2 tax total: $38,430

The two analyzers are very close, but analyzer 1 wins with $2310 less in total taxes.

(*b*) When the analyzer is sold after 3 years of service, there is a depreciation recapture (DR) that is taxed at the 35% rate. This tax is in addition to the third-year tax. For each analyzer, account for the DR by Equation [17.13], SP $-$ BV_3; then determine the TI, using Equation [17.16], TI = GI – OE – *D* + DR. Again, find the total taxes over 3 years, and select the analyzer with the smaller total.

TABLE 17–4 Comparison of Total Taxes for Two Alternatives, Example 17.5a

Year	Gross Income GI, $	Operating Expenses OE, $	Basis *B*, $	MACRS Depreciation *D*, $	Book Value BV, $	Taxable Income TI, $	Taxes at 0.35TI, $
				Analyzer 1			
0			150,000		150,000		
1	100,000	30,000		30,000	120,000	40,000	14,000
2	100,000	30,000		48,000	72,000	22,000	7,700
3	100,000	30,000		28,800	43,200	41,200	14,420
							36,120
				Analyzer 2			
0			225,000		225,000		
1	100,000	10,000		45,000	180,000	45,000	15,750
2	100,000	10,000		72,000	108,000	18,000	6,300
3	100,000	10,000		43,200	64,800	46,800	16,380
							38,430

$$
\begin{aligned}
\text{Analyzer 1:}\quad \text{DR} &= 130{,}000 - 43{,}200 = \$86{,}800 \\
\text{Year 3 TI} &= 100{,}000 - 30{,}000 - 28{,}800 + 86{,}800 = \$128{,}000 \\
\text{Year 3 taxes} &= (0.35)(128{,}000) = \$44{,}800 \\
\text{Total taxes} &= 14{,}000 + 7700 + 44{,}800 = \$66{,}500
\end{aligned}
$$

$$
\begin{aligned}
\text{Analyzer 2:}\quad \text{DR} &= 225{,}000 - 64{,}800 = \$160{,}200 \\
\text{Year 3 TI} &= 100{,}000 - 10{,}000 - 43{,}200 + 160{,}200 = \$207{,}000 \\
\text{Year 3 taxes} &= (0.35)(207{,}000) = \$72{,}450 \\
\text{Total taxes} &= 15{,}750 + 6300 + 72{,}450 = \$94{,}500
\end{aligned}
$$

Now, analyzer 1 has a considerable advantage in total taxes ($94,500 versus $66,500).

Solution by Spreadsheet

(*a*) Rows 5 through 9 of Figure 17–4 perform the same computations as the hand solution for analyzer 1 with total taxes of $36,120. Similar analysis in rows 14 to 18 results in total taxes of $38,430 for analyzer 2, indicating that the company should select analyzer 1, based on taxes only.

(*b*) Revised year 3 entries for analyzer 1 in row 10 show the sales price of $130,000, the updated TI of $128,000, and a 3-year tax total of $66,500. The new TI in year 3 has the depreciation recapture incorporated as DR = selling price − book value = SP − BV_3, which is shown in the cell tag as the last term (D10 − F10). With a similar updating for analyzer 2 (row 19), total taxes of $94,500 now show a significantly larger tax advantage for analyzer 1 over 3 years.

Comment

Note that no time value of money is considered in these analyses, as we have used in previous alternative evaluations. In Section 17.5 below we will rely upon PW, AW, and ROR analyses at an established MARR to make an after-tax decision based upon CFAT values.

	A	B	C	D	E	F	G	H
1								
2								
3	Year	GI	OE	B and S	D	BV	TI	Taxes
4				Analyzer 1				
5	0			-150,000		150,000		
6	1	100,000	-30,000		30,000	120,000	40,000	14,000
7	2	100,000	-30,000		48,000	72,000	22,000	7,700
8	3	100,000	-30,000		28,800	43,200	41,200	14,420
9	Total							36,120
10	Revised 3	100,000	-30,000	130,000	28,800	43,200	128,000	44,800
11	New Total							66,500
12								
13				Analyzer 2				
14	0			-225,000		225,000		
15	1	100,000	-10,000		45,000	180,000	45,000	15,750
16	2	100,000	-10,000		72,000	108,000	18,000	6,300
17	3	100,000	-10,000		43,200	64,800	46,800	16,380
18	Total							38,430
19	Revised 3	100,000	-10,000	225,000	43,200	64,800	207,000	72,450
20	New Total							94,500

= B6+C6−E6

= B10+C10−E10+(D10−F10)
Last term calculates the depreciation recapture

Total tax advantage of analyzer 1 over 2 without sale and with sale in year 3

Figure 17–4
Impact of depreciation recapture on total taxes, Example 17.5.

17.5 After-Tax Evaluation

The required after-tax MARR is established using the market interest rate, the corporation's effective tax rate, and its weighted average cost of capital. The CFAT estimates are used to compute the PW or AW at the after-tax MARR. When *positive and negative CFAT values* are present, a PW or AW < 0 indicates the MARR is not met. For a single project or mutually exclusive alternatives, apply the same logic as in Chapters 5 and 6. The guidelines are:

One project. PW or AW ≥ 0, the project is financially viable because the after-tax MARR is met or exceeded.

Two or more alternatives. Select the alternative with the best (numerically largest) PW or AW value.

ME alternative selection

If *only cost CFAT amounts* are estimated, calculate the after-tax savings generated by the operating expenses and depreciation. Assign a plus sign to each saving and apply the guidelines above.

Equal service

Remember, the **equal-service** assumption requires that the PW analysis be performed over the least common multiple (LCM) of alternative lives. This requirement must be met for every analysis—before or after taxes.

Since the CFAT estimates usually vary from year to year in an after-tax evaluation, the spreadsheet offers a much speedier analysis than solution by hand.

For AW analysis: Use the PMT function with an embedded NPV function *over one life cycle*. The general format is as follows, with the NPV function in italics for the CFAT series.

= −PMT(MARR,n,*NPV(MARR,year_1: year_n) + year_0*) [17.17]

For PW analysis: Obtain the PMT function results first, followed by the PV function taken over the LCM. (There is an LCM function in Excel.) The cell containing the PMT function result is entered as the A value. The general format is

= −PV(MARR,LCM_years,*PMT_result_cell*) [17.18]

EXAMPLE 17.6

Paul is designing the interior walls of an industrial building. In some places, it is important to reduce noise transmission across the wall. Two construction options—stucco on metal lath (S) and bricks (B)—each have about the same transmission loss, approximately 33 decibels. This will reduce noise attenuation costs in adjacent office areas. Paul has estimated the first costs and after-tax savings each year for both designs. (*a*) Use the CFAT values and an after-tax MARR of 7% per year to determine which is economically better. (*b*) Use a spreadsheet to select the alternative and determine the required first cost for the plans to break even.

Plan S		Plan B	
Year	CFAT, $	Year	CFAT, $
0	−28,800	0	−50,000
1–6	5,400	1	14,200
7–10	2,040	2	13,300
10	2,792	3	12,400
		4	11,500
		5	10,600

Solution by Hand

(*a*) In this example, both AW and PW analyses are shown. Develop the AW relations using the CFAT values over each plan's life. Select the larger value.

$$\begin{aligned}AW_S &= [-28{,}800 + 5400(P/A,7\%,6) + 2040(P/A,7\%,4)(P/F,7\%,6)\\&\quad + 2792(P/F,7\%,10)](A/P,7\%,10)\\&= \$422\end{aligned}$$

$$\begin{aligned}AW_B &= [-50{,}000 + 14{,}200(P/F,7\%,1) + \cdots + 10{,}600(P/F,7\%,5)](A/P,7\%,5)\\&= \$327\end{aligned}$$

Both plans are financially viable; select plan S because AW_S is larger.

For the PW analysis, the LCM is 10 years. Use the AW values and the P/A factor for the LCM of 10 years to select stucco on metal lath, plan S.

$$PW_S = AW_S(P/A,7\%,10) = 422(7.0236) = \$2964$$
$$PW_B = AW_B(P/A,7\%,10) = 327(7.0236) = \$2297$$

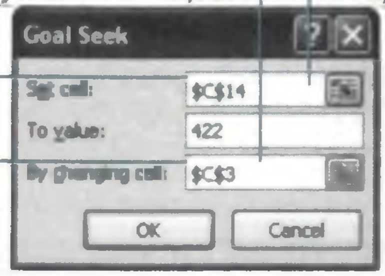
	A	B	C	D
1		CFAT, $		
2	Year	Plan S	Plan B	
3	0	-28,800	-50,000	
4	1	5,400	14,200	
5	2	5,400	13,300	
6	3	5,400	12,400	
7	4	5,400	11,500	
8	5	5,400	10,600	
9	6	5,400		
10	7	2,040		
11	8	2,040		
12	9	2,040		
13	10	4,832		
14	AW @ 7%	422	327	
15	10-yr PW	2,963	2,297	
16				
17				
18				
19				

= −PMT(7%,5,NPV(7%,C4:C8)+C3)

= −PV(7%,10,C14)

(*a*)

	A	B	C
1		CFAT, $	
2	Year	Plan S	Plan B
3	0	-28,800	-49,611
4	1	5,400	14,200
5	2	5,400	13,300
6	3	5,400	12,400
7	4	5,400	11,500
8	5	5,400	10,600
9	6	5,400	
10	7	2,040	
11	8	2,040	
12	9	2,040	
13	10	4,832	
14	AW @ 7%	422	422
15	10-yr PW	2,963	2,964



(*b*)

Figure 17–5

(*a*) After-tax AW and PW analysis, and (*b*) breakeven first cost using Goal Seek, Example 17.6.

Solution by Spreadsheet

(*b*) Figure 17–5*a*, row 14, displays the AW value calculated using the PMT function defined by Equation [17.17], and row 15 shows the 10-year PW that results from the PV function in Equation [17.18]. Plan S is chosen by a relatively small margin.

Figure 17–5*b* shows the Goal Seek template used to equate the AW values and determine the plan B first cost of $−49,611 that causes the plans to break even. This is a small reduction from the $-50,000 first cost initially estimated.

Comment

It is important to remember the minus signs in PMT and PV functions when utilizing them to obtain the corresponding PW and AW values. If the minus is omitted, the AW and PW values have the wrong sign and it appears that the plans are not financially viable in that they do not return at least the after-tax MARR. That would happen in this example.

To utilize the **ROR method,** apply exactly the same procedures as in Chapter 7 (single project) and Chapter 8 (two or more alternatives) to the CFAT series. A PW or AW relation is developed to estimate the rate of return i^* for a project, or Δi^* for the incremental CFAT between two alternatives. Multiple roots may exist in the CFAT series, as they can for any cash flow series. For a single project, set the PW or AW equal to zero and solve for i^*.

Project evaluation

Present worth: $$0 = \sum_{t=1}^{t=n} \text{CFAT}_t(P/F,i^*,t) \qquad [17.19]$$

Annual worth: $$0 = \sum_{t=1}^{t=n} \text{CFAT}_t(P/F,i^*,t)(A/P,i^*,n) \qquad [17.20]$$

If $i^* \geq$ after-tax MARR, the project is economically justified.

Spreadsheet solution for i^* is faster for most CFAT series. It is performed using the IRR function with the general format

$$= \text{IRR(year\_0\_CFAT:year\_n\_CFAT)} \qquad [17.21]$$

If the after-tax ROR is important to the analysis, but the details of an after-tax study are not of interest, the before-tax ROR (or MARR) can be adjusted with the effective tax rate T_e by using the *approximating* relation

$$\textbf{Before-tax ROR} = \frac{\textbf{after-tax ROR}}{1 - T_e} \qquad [17.22]$$

For example, assume a company has an effective tax rate of 40% and normally uses an after-tax MARR of 12% per year for economic analyses that consider taxes explicitly. To *approximate* the effect of taxes without performing the details of an after-tax study, the before-tax MARR can be estimated as

$$\text{Before-tax MARR} = \frac{0.12}{1 - 0.40} = 20\% \text{ per year}$$

If the decision concerns the economic viability of a project and the resulting PW or AW value is close to zero, the details of an after-tax analysis should be developed.

EXAMPLE 17.7

A fiber optics manufacturing company operating in Hong Kong has spent \$50,000 for a 5-year-life machine that has a projected \$20,000 annual NOI and annual depreciation of \$10,000 for years 1 through 5. The company has a T_e of 40%. (*a*) Determine the after-tax rate of return. (*b*) Approximate the before-tax return.

Solution

(*a*) The CFAT in year 0 is \$−50,000. For years 1 through 5 there is no capital purchase or sale, so NOI = CFBT. (See Equations [17.1] and [17.9].) Determine CFAT.

$$\text{TI} = \text{NOI} - \text{D} = 20{,}000 - 10{,}000 = \$10{,}000$$
$$\text{Taxes} = T_e(\text{TI}) = 0.4(10{,}000) = \$4000$$
$$\text{CFAT} = \text{CFBT} - \text{taxes} = 20{,}000 - 4000 = \$16{,}000$$

Since the CFAT for years 1 through 5 has the same value, use the P/A factor in Equation [17.19].

$$0 = -50{,}000 + 16{,}000(P/A,i^*,5)$$
$$(P/A,i^*,5) = 3.125$$

Solution gives $i^* = 18.03\%$ as the after-tax rate of return.

(*b*) Use Equation [17.22] for the before-tax return estimate.

$$\text{Before-tax ROR} = \frac{0.1803}{1 - 0.40} = 0.3005 \quad (30.05\%)$$

The actual before-tax i^* using CFBT = \$20,000 for 5 years is 28.65% from the relation

$$0 = -50{,}000 + 20{,}000(P/A,i^*,5)$$

The tax effect will be slightly overestimated if a MARR of 30.05% is used in a before-tax analysis.

A rate of return evaluation performed by hand on two or more alternatives utilizes a PW or AW relation to determine the incremental return Δi^* of the incremental CFAT series between two alternatives. Solution by spreadsheet is accomplished using the incremental CFAT values and the IRR function. The equations and procedures applied are the same as in Chapter 8 (Sections 8.4 through 8.6) for selection from mutually exclusive alternatives using the ROR method. You should review and understand these sections before proceeding with this section.

From this review, several important facts can be recalled:

Selection guideline: The fundamental rule of incremental ROR evaluation at a stated MARR is as follows:

> Select the one alternative that requires the largest initial investment, provided the extra investment is justified relative to another justified alternative.

Incremental ROR: Incremental analysis must be performed. Overall i^* values cannot be depended upon to select the correct alternative, unlike the PW or AW method at the MARR, which will always indicate the correct alternative.

Equal-service requirement: Incremental ROR analysis requires that the alternatives be evaluated over equal time periods. The LCM of the two alternative lives must be used to find the PW or AW of incremental cash flows. (The only exception, mentioned in Section 8.5, occurs when the AW analysis is performed on *actual cash flows, not the increments;* then one-life-cycle analysis is acceptable over the respective alternative lives.)

Revenue and cost alternatives: Revenue alternatives (positive and negative cash flows) may be treated differently from cost alternatives (cost-only cash flow estimates). For revenue alternatives, the overall i^* may be used to perform an initial screening. Alternatives with $i^* <$ MARR can be removed from further evaluation. An i^* for cost-only alternatives cannot be determined, so incremental analysis is required with all alternatives included.

Breakeven ROR

Once the CFAT series are developed, the **breakeven ROR** can be obtained using a plot of PW versus i^* by solving the PW relation for each alternative over the LCM at several interest rates. For any after-tax MARR greater than the breakeven ROR, the extra investment is not justified.

The next examples solve CFAT problems using incremental ROR analysis and the breakeven ROR plot of PW versus i.

EXAMPLE 17.8

In Example 17.6, Paul estimated the CFAT for interior wall materials to reduce sound transmission; plan S is to construct with stucco on metal lath, and plan B is to construct using brick. Figure 17–5a presented both a PW analysis over 10 years and an AW analysis over the respective lives. Plan S was selected. After reviewing this earlier solution, (*a*) perform an ROR evaluation at the after-tax MARR of 7% per year and (*b*) plot the PW versus Δi graph to determine the breakeven ROR.

Solution by Spreadsheet

(*a*) The LCM is 10 years for the incremental ROR analysis, and plan B requires the extra investment that must be justified. Apply the procedure in Section 8.6 for incremental ROR analysis. Figure 17–6 shows the estimated CFAT for each alternative and the incremental CFAT series. Since these are revenue alternatives, the overall Δi^* is calculated first to ensure that they both make at least the MARR of 7%. Row 14 indicates they do. The IRR function (cell E14) is applied to the incremental CFAT, indicating that $\Delta i^* = 6.35\%$. Since this is lower than the MARR, the extra investment in brick walls is not justified. Plan S is selected, the same as with the PW and AW methods.

(*b*) The NPV function is used to find the PW of the incremental CFAT series at various i values. The graph indicates that the breakeven Δi^* occurs at 6.35%—the same that the IRR function found. Whenever the after-tax MARR is above 6.35%, as is the case here with MARR = 7%, the extra investment in plan B is not justified.

Comment

Note that the incremental CFAT series has three sign changes. The cumulative series also has three sign changes (Norstrom's criterion). Accordingly, there may be multiple Δi^* values. The application of the IRR function using the "guess" option finds no other real number roots in the normal rate of return range.

	Year	Plan S CFAT, \$	Plan B CFAT, \$	B - S Incr CFAT, \$
	0	-28,800	-50,000	-21,200
	1	5,400	14,200	8,800
	2	5,400	13,300	7,900
	3	5,400	12,400	7,000
	4	5,400	11,500	6,100
	5	5,400	-39,400	-44,800
	6	5,400	14,200	8,800
	7	2,040	13,300	11,260
	8	2,040	12,400	10,360
	9	2,040	11,500	9,460
	10	4,832	10,600	5,768
i* and Δi*		9.49%	8.05%	6.35%

Rate, Δi	PW of Incremental CFAT, \$
5%	1,530
6%	377
7%	-666
8%	-1,612
9%	-2,469
10%	-3,248

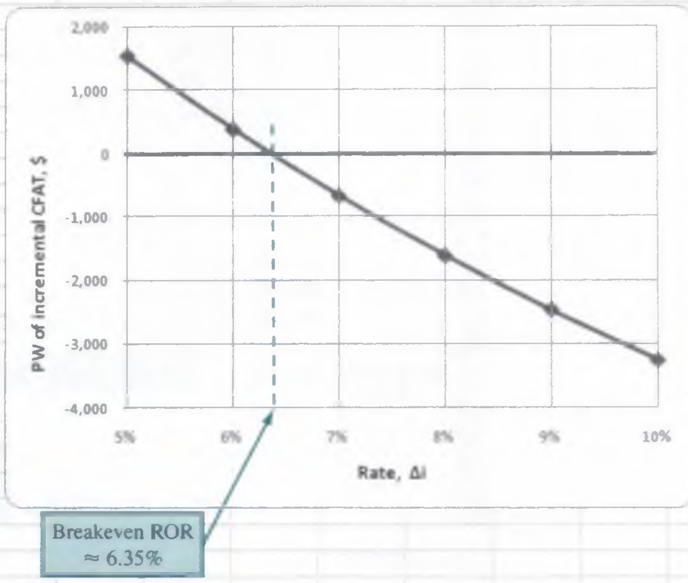


Figure 17–6
Incremental evaluation of CFAT and determination of breakeven ROR, Example 17.8.

EXAMPLE 17.9

In Example 17.5 an after-tax analysis of two bone cell analyzers was initiated due to a new 3-year NBA contract. The criterion used to select analyzer 1 was the total taxes for the 3 years. The complete solution is in Table 17–4 (hand) and Figure 17–4 (spreadsheet).

Continue the spreadsheet analysis by performing an after-tax ROR evaluation, assuming the analyzers are sold after 3 years for the amounts estimated in Example 17.5: \$130,000 for analyzer 1 and \$225,000 for analyzer 2. The after-tax MARR is 10% per year.

Solution

A spreadsheet solution is presented here, but a hand solution is equivalent, just slower. Figure 17–7 is an updated version of the spreadsheet in Figure 17–4 to include the sale of the analyzers in year 3. The CFAT series (column I) are determined by the relation CFAT = CFBT − taxes, with the taxable income determined using Equation [17.16], where DR is included. For example, in year 3 when analyzer 2 is sold for S = \$225,000, the CFAT calculation is

$$\text{CFAT}_3 = \text{CFBT} - (\text{TI})(T_e) = \text{GI} - \text{OE} - P + S - (\text{GI} - \text{OE} - D + \text{DR})(T_e)$$

The depreciation recapture DR is the amount above the year 3 book value received at sale time. Using the book value after 3 years (F14),

$$\text{DR} = \text{selling price} - \text{BV}_3 = 225{,}000 - 64{,}800 = \$160{,}200$$

Now the CFAT in year 3 for analyzer 2 can be determined.

$$\begin{aligned}\text{CFAT}_3 &= 100{,}000 - 10{,}000 + 0 + 225{,}000 \\ &\quad -(100{,}000 - 10{,}000 - 43{,}200 + 160{,}200)(0.35) \\ &= 315{,}000 - 207{,}000(0.35) = \$242{,}550\end{aligned}$$

The cell tags in row 14 of Figure 17–7 follow this same progression. The incremental CFAT is calculated in column J, ready for the after-tax incremental ROR analysis.

These are revenue alternatives, so the overall i^* values indicate that both CFAT series are acceptable. The value Δi^* = 23.6% (cell J17) also exceeds MARR = 10%, so **analyzer 2** *is selected.* This decision applies the ROR method guideline: Select the alternative that requires the largest, *incrementally* justified investment.

	A	B	C	D	E	F	G	H	I	J	K
1	Year	GI	OE	B and S	D	BV	TI	Taxes	CFAT		
2					**Analyzer 1**						
3	0			-150,000		150,000			-150,000		
4	1	100,000	-30,000		30,000	120,000	40,000	14,000	56,000		
5	2	100,000	-30,000		48,000	72,000	22,000	7,700	62,300		
6	3	100,000	-30,000	130,000	28,800	43,200	128,000	44,800	155,200		
7	i*								**30.2%**	Overall i*	
8	PW at 10%								$69,001		
9										Incremental	
10					**Analyzer 2**					CFAT	
11	0			-225,000		225,000			-225,000	-75,000	
12	1	100,000	-10,000		45,000	180,000	45,000	15,750	74,250	18,250	
13	2	100,000	-10,000		72,000	108,000	18,000	6,300	83,700	21,400	
14	3	100,000	-10,000	225,000	43,200	64,800	207,000	72,450	242,550	87,350	
15	i*								**27.9%**		
16	PW at 10%								$93,905		
17	Incr i*									**23.6%**	
18											
19											
20											
21											
22											

= B14+C14−E14+(D14−F14)

This term calculates DR = $160,200

CFAT calculation = B14+C14+D14−H14

Incremental i* = IRR(J11:J14)

Figure 17–7

Incremental ROR analysis of CFAT with depreciation recapture, Example 17.9.

Comment

In Section 8.4, we demonstrated the fallacy of selecting an alternative based solely on the overall i^*, because of the ranking inconsistency problem of the ROR method. The incremental ROR must be used. The same fact is demonstrated in this example. If the larger i^* alternative is chosen, analyzer 1 is incorrectly selected. When Δi^* exceeds the MARR, the larger investment is correctly chosen—analyzer 2 in this case. For verification, the PW at 10% is calculated for each analyzer (column I). Again, analyzer 2 is the winner, based on its larger PW of $93,905.

Online Presentation

17.6 After-Tax Replacement Study ● ● ●

When a currently installed asset (the defender) is challenged with possible replacement, the effect of taxes can have an impact upon the decision of the replacement study. The final decision may not be reversed by taxes, but the difference between before-tax AW values may be significantly different from the after-tax difference. Tax considerations in the year of the replacement are as follows:

Depreciation recapture or tax savings due to a sizable **capital loss** are possible, if it is necessary to trade the defender at a sacrifice price. Additionally, the after-tax replacement study considers tax-deductible **depreciation** and **operating expenses** not accounted for in a before-tax analysis.

The effective tax rate T_e is used to estimate the amount of annual taxes (or tax savings) from TI. The same procedure as the before-tax replacement study in Chapter 11 is applied here, but for CFAT estimates. The procedure should be thoroughly understood before proceeding. Special attention to Sections 11.3 and 11.5 is recommended.

Example 17.10 presents a solution by hand of an after-tax replacement study using a simplifying assumption of classical SL (straight line) depreciation. Example 17.11 solves the same problem by spreadsheet, but includes the detail of MACRS depreciation. This provides an opportunity to observe the difference in the AW values between the two depreciation methods.

EXAMPLE 17.10

Midcontinent Power Authority purchased emission control equipment 3 years ago for $600,000. Management has discovered that it is technologically and legally outdated now. New equipment has been identified. If a market value of $400,000 is offered as the trade-in for the current equipment, perform a replacement study using (*a*) a before-tax MARR of 10% per year and (*b*) a 7% per year after-tax MARR. Assume an effective tax rate of 34%. As a simplifying assumption, use classical straight line depreciation with $S = 0$ for both alternatives.

	Defender	Challenger
Market value, $	400,000	
First cost, $		−1,000,000
Annual cost, $/year	−100,000	−15,000
Recovery period, years	8 (originally)	5

Solution

Assume that an ESL (economic service life) analysis has determined the best life values to be 5 more years for the defender and 5 years total for the challenger.

(*a*) For the *before-tax replacement study*, find the AW values. The defender AW uses the market value as the first cost, $P_D = \$-400{,}000$.

$$AW_D = -400{,}000(A/P,10\%,5) - 100{,}000 = \$-205{,}520$$
$$AW_C = -1{,}000{,}000(A/P,10\%,5) - 15{,}000 = \$-278{,}800$$

Applying step 1 of the replacement study procedure (Section 11.3), we select the better AW value. The defender is retained now with a plan to keep it for the 5 remaining years. The defender has a $73,280 lower equivalent annual cost compared to the challenger. This complete solution is included in Table 17–5 (left half) for comparison with the after-tax study.

(*b*) For the *after-tax replacement study*, there are no tax effects other than income tax for the defender. The annual SL depreciation is $75,000, determined when the equipment was purchased 3 years ago.

$$D_t = 600{,}000/8 = \$75{,}000 \qquad t = 1 \text{ to } 8 \text{ years}$$

Table 17–5 shows the TI and taxes at 34%. The taxes are actually tax savings of $59,500 per year, as indicated by the minus sign. (Remember that for tax savings in an economic

TABLE 17–5 Before-Tax and After-Tax Replacement Analyses, Example 17.10

		Before Taxes			After Taxes			
Defender Age	Year	Expenses OE, $	P and S, $	CFBT, $	Depreciation D, $	Taxable Income TI, $	Taxes* at 0.34TI, $	CFAT, $
				Defender				
3	0		−400,000	−400,000				−400,000
4	1	−100,000		−100,000	75,000	−175,000	−59,500	−40,500
5	2	−100,000		−100,000	75,000	−175,000	−59,500	−40,500
6	3	−100,000		−100,000	75,000	−175,000	−59,500	−40,500
7	4	−100,000		−100,000	75,000	−175,000	−59,500	−40,500
8	5	−100,000	0	−100,000	75,000	−175,000	−59,500	−40,500
AW at 10%				−205,520	AW at 7%			−138,056
				Challenger				
	0		−1,000,000	−1,000,000		+25,000†	8,500	−1,008,500
	1	−15,000		−15,000	200,000	−215,000	−73,100	+58,100
	2	−15,000		−15,000	200,000	−215,000	−73,100	+58,100
	3	−15,000		−15,000	200,000	−215,000	−73,100	+58,100
	4	−15,000		−15,000	200,000	−215,000	−73,100	+58,100
	5	−15,000	0	−15,000	200,000	−215,000‡	−73,100	+58,100
AW at 10%				−278,800	AW at 7%			−187,863

* Minus sign indicates a tax savings for the year.
† Depreciation recapture on defender trade-in.
‡ Assumes challenger's salvage actually realized is $S = 0$; no tax.

analysis it is assumed that there is positive taxable income elsewhere in the corporation to offset the saving.) Since only costs are estimated, the annual CFAT is negative, but the \$59,500 tax savings has reduced it. The CFAT and AW at 7% per year are

$$\text{CFAT} = \text{CFBT} - \text{taxes} = -100{,}000 - (-59{,}500) = \$-40{,}500$$
$$\text{AW}_D = -400{,}000(A/P,7\%,5) - 40{,}500 = \$-138{,}056$$

For the challenger, depreciation recapture on the defender occurs when it is replaced because the trade-in amount of \$400,000 is larger than the current book value. In year 0 for the challenger, Table 17–5 includes the following computations to arrive at a tax of \$8500.

Defender book value, year 3:	$\text{BV}_3 = 600{,}000 - 3(75{,}000) = \$375{,}000$
Depreciation recapture:	$\text{DR}_3 = \text{TI} = 400{,}000 - 375{,}000 = \$25{,}000$
Taxes on trade-in, year 0:	$\text{Taxes} = 0.34(25{,}000) = \$8500$

The SL depreciation is \$1,000,000/5 = \$200,000 per year. This results in tax saving and CFAT as follows:

$$\text{Taxes} = (-15{,}000 - 200{,}000)(0.34) = \$-73{,}100$$
$$\text{CFAT} = \text{CFBT} - \text{taxes} = -15{,}000 - (-73{,}100) = \$+58{,}100$$

In year 5, it is assumed the challenger is sold for \$0; there is no depreciation recapture. The AW for the challenger at the 7% after-tax MARR is

$$\text{AW}_C = -1{,}008{,}500(A/P,7\%,5) + 58{,}100 = \$-187{,}863$$

The defender is again selected; however, the equivalent annual advantage has decreased from \$73,280 before taxes to \$49,807 after taxes.

Conclusion: By either analysis, retain the defender now and plan to keep it for 5 more years. Additionally, plan to evaluate the estimates for both alternatives 1 year hence. If and when cash flow estimates change significantly, perform another replacement analysis.

Comment

If the market value (trade-in) had been less than the current defender book value of \$375,000, a capital loss, rather than depreciation recapture, would occur in year 0. The resulting tax savings would decrease the CFAT (which is to reduce costs if CFAT is negative) of the challenger. For example, a trade-in amount of \$350,000 would result in a TI of \$350,000 − 375,000 = \$−25,000 and a *tax savings* of \$−8500 in year 0. The CFAT is then \$−1,000,000 − (−8500) = \$−991,500.

EXAMPLE 17.11

Repeat the after-tax replacement study of Example 17.10*b* using 7-year MACRS depreciation for the defender and 5-year MACRS depreciation for the challenger. Assume either asset is sold after 5 years for exactly its book value. Determine if the answers are significantly different from those obtained when the simplifying assumption of classical SL depreciation was made.

Solution

Figure 17–8 shows the complete analysis. MACRS requires more computation than SL depreciation, but this effort is easily reduced by the use of a spreadsheet. Again the *defender is selected* for retention, but now by an advantage of \$44,142 annually. This compares to the \$49,807 advantage using classical SL depreciation and the \$73,280 before-tax advantage of the defender. Therefore, taxes and MACRS have reduced the defender's economic advantage, but not enough to reverse the decision to retain it.

Several other differences in the results between SL and MACRS depreciation are worth noting. There is depreciation recapture in year 0 of the challenger due to trade-in of the defender at \$400,000, a value larger than the book value of the 3-year-old defender. This amount,

	A	B	C	D	E	F	G	H	I	J
1	MARR =	7%								
2	Purchase =	$ 600,000		Defender after-tax MACRS analysis						
3	Defender		Basis B &	(Expenses)	MACRS					
4	age	Year	salvage S(1)	CFBT	rates	Depr	TI	Tax savings	CFAT	
5	3	0	-400,000						-400,000	
6	4	1		-100,000	0.1249	74,940	-174,940	-59,480	-40,520	
7	5	2		-100,000	0.0893	53,580	-153,580	-52,217	-47,783	
8	6	3		-100,000	0.0892	53,520	-153,520	-52,197	-47,803	
9	7	4		-100,000	0.0893	53,580	-153,580	-52,217	-47,783	
10	8	5	0	-100,000	0.0446	26,760	-126,760	-43,098	-56,902	
11	Total					262,380				
12	(1)Defender assumed to be sold in year 5 (year 8 of its life) for exactly BV = 0							AW at 7%	-$145,273	
13	All the original B = $600,000 is depreciated over the 8 years; no tax effect									
14										
15	Purchase =	$1,000,000		Challenger after-tax MACRS analysis						
16	Challenger		Basis B &	(Expenses)	MACRS			Taxes or		
17	age	Year	salvage S(1)	CFBT	rates	Depr	TI (2)	Tax savings	CFAT	
18	0	0	-1,000,000				137,620	46,791	-1,046,791	
19	1	1		-15,000	0.2000	200,000	-215,000	-73,100	58,100	
20	2	2		-15,000	0.3200	320,000	-335,000	-113,900	98,900	
21	3	3		-15,000	0.1920	192,000	-207,000	-70,380	55,380	
22	4	4		-15,000	0.1152	115,200	-130,200	-44,268	29,268	
23	5	5	57,600	-15,000	0.1152	115,200	-130,200	-44,268	86,868	
24	Total					942,400				
25	(1)Challenger assumed to be sold in year 5 for exactly							AW at 7%	-$189,416	
26	BV = 1,000,000-942,400 = $57,600.									
27	No tax effect, but CFAT increases in year 5.									
28	(2) TI of $137,620 in year 0 is depreciation recapture from									
29	trade of defender. DR = B - current BV = 400,000 - 262,380									

Depreciation recapture = −C5 − F11

Challenger sales price = B15 − F24

Figure 17–8
After-tax replacement study with MACRS depreciation and depreciation recapture, Example 17.11.

$137,620 (cell G18), is treated as ordinary taxable income. The calculations for the DR and associated tax, by hand, are as follows:

$$
\begin{aligned}
BV_3 &= \text{first cost} - \text{MACRS depreciation for 3 years} \\
&= \text{total MACRS depreciation for years 4 through 8} \\
&= \$262{,}380 \qquad \text{(cell F11)} \\
DR &= TI_0 = \text{trade-in} - BV_3 \\
&= 400{,}000 - 262{,}380 = \$137{,}620 \qquad \text{(cell G18)} \\
\text{Taxes} &= (0.34)(137{,}620) = \$46{,}791 \qquad \text{(cell H18)}
\end{aligned}
$$

See the cell tags and table notes that duplicate this logic.

The assumption that the challenger is sold after 5 years at its book value implies a positive cash flow in year 5. The entry $57,600 (C23) reflects this assumption, since the forgone MACRS depreciation in year 6 would be 1,000,000(0.0576) = $57,600. The spreadsheet relation = B15 − F24 determines this value using the accumulated depreciation in F24. [*Note:* If the salvage $S = 0$ is anticipated after 5 years, then a capital loss of $57,600 will be incurred. This implies an additional tax saving of 57,600(0.34) = $19,584 in year 5. Conversely, if the salvage value exceeds the book value, a depreciation recapture and associated tax should be estimated.]

17.7 After-Tax Value-Added Analysis ● ● ●

When a person or company is willing to pay more for an item, it is likely that some processing has been performed on an earlier version of the item to make it more valuable now to the purchaser. This is value added.

Value added is a term used to indicate that a product or service has **added worth** from the perspective of a consumer, owner, investor, or purchaser. It is common to leverage value-adding activities on a product or service.

Value added

For an example of highly leveraged value-added activities, consider onions that are grown and sold at the farm level for cents per pound. They may be purchased by the shopper in a store at 50 cents to $1.25 per pound. But when onions are cut and coated with a special batter, they may be fried in hot oil and sold as onion rings for several dollars per pound. Thus, from the perspective of the consumer, there has been a large amount of value added by the processing from raw onions in the ground into onion rings sold at a restaurant or fast-food shop.

The value-added measure was briefly introduced in conjunction with AW analysis before taxes. When value-added analysis is performed after taxes, the approach is somewhat different from that of CFAT analysis developed previously in this chapter. However, as shown below,

The decision about an alternative will be the same for both the value-added and CFAT methods, because the AW of economic value-added estimates is the same as the AW of CFAT estimates.

Value-added analysis starts with Equation [17.4], net operating profit after taxes (NOPAT), which includes the depreciation for *year 1* through *year n*. Depreciation D is included in that TI = GI − OE − D. This is different from CFAT, where the depreciation has been specifically removed so that only *actual* cash flow estimates are used for *year 0* through year n.

The term **economic value added (EVA)** indicates the monetary worth added by an alternative to the corporation's bottom line. (The term *EVA* is a trademark of Stern Stewart & Co.) The technique discussed below was first publicized in several articles[1] in the mid-1990s, and it has since become very popular as a means to evaluate the ability of a corporation to increase its economic worth, especially from the shareholders' viewpoint.

The annual EVA is the amount of NOPAT remaining on corporate books after removing the **cost of invested capital** during the year. That is, EVA indicates the project's **contribution to the net profit** of the corporation after taxes.

The **cost of invested capital** is the after-tax rate of return (usually the MARR value) multiplied by the book value of the asset during the year. This is the interest incurred by the current level of capital invested in the asset. (If different tax and book depreciation methods are used, the *book depreciation value is used* here, because it more closely represents the remaining capital invested in the asset from the corporation's perspective.) Computationally,

$$\begin{aligned}\textbf{EVA} &= \textbf{NOPAT} - \textbf{cost of invested capital}\\ &= \textbf{NOPAT} - \textbf{(after-tax interest rate)(book value in year } t-1\textbf{)}\\ &= \textbf{TI}(1 - T_e) - (i)(\textbf{BV}_{t-1}) \qquad [17.23]\end{aligned}$$

Since both TI and the book value consider depreciation, EVA is a measure of worth that mingles actual cash flow with noncash flows to determine the estimated financial worth contribution to the corporation. This financial worth is the amount used in public documents of the corporation (balance sheet, income statement, stock reports, etc.). Because corporations want to present the largest value possible to the stockholders and other owners, the EVA method is often more appealing than the AW method from the financial perspective.

The result of an EVA analysis is a series of annual EVA estimates. Two or more alternatives are compared by calculating the AW of EVA estimates and selecting the alternative with the larger AW value. If only one project is evaluated, AW > 0 means the after-tax MARR is exceeded, thus making the project value-adding.

Sullivan and Needy[2] have demonstrated that the AW of EVA and the AW of CFAT are identical in amount. Thus, either method can be used to make a decision. The annual EVA estimates indicate added worth to the corporation generated by the alternative, while the annual CFAT estimates describe how cash will flow. This comparison is made in Example 17.12.

[1] A. Blair, "EVA Fever," *Management Today*, Jan. 1997, pp. 42–45; W. Freedman, "How Do You Add Up?" *Chemical Week*, Oct. 9, 1996, pp. 31–34.

[2] W. G. Sullivan and K. L. Needy, "Determination of Economic Value Added for a Proposed Investment in New Manufacturing," *The Engineering Economist*, vol. 45, no. 2 (2000), pp. 166–181.

EXAMPLE 17.12

Biotechnics Engineering has developed two mutually exclusive plans for investing in new capital equipment with the expectation of increased revenue from its medical diagnostic services to cancer patients. The estimates are summarized below. (*a*) Use classical straight line depreciation, an after-tax MARR of 12%, and an effective tax rate of 40% to perform two annual worth after-tax analyses: EVA and CFAT. (*b*) Explain the fundamental difference between the results of the two analyses.

	Plan A	Plan B
Initial investment, \$	−500,000	−1,200,000
Gross income − expenses, \$	170,000 per year	600,000 in year 1, decreasing by 100,000 per year thereafter
Estimated life, years	4	4
Salvage value	None	None

Solution by Spreadsheet

(*a*) Refer to the spreadsheet and function cells (row 22) in Figure 17–9.

EVA evaluation: All the necessary information for EVA estimation is determined in columns B through G. The net operating profit after taxes (NOPAT) in column H is calculated by Equation [17.4], TI − taxes. The book values (column E) are used to determine the cost of invested capital in column I, using the second term in Equation [17.23], that is, $i(BV_{t-1})$, where i is the 12% after-tax MARR. This represents the amount of interest at 12% per year, after taxes, for the currently invested capital as reflected by the book value at the beginning of the year. The EVA estimate is the sum of columns H and I for years 1 through 4. *Notice there is no EVA estimate for year 0,* since NOPAT and the cost of invested capital are estimated for years 1 through n. Finally, the larger AW of the EVA value is selected, which indicates that plan B is better and that plan A does not make the 12% return.

CFAT evaluation: As shown in function row 22 (plan B for year 3), CFAT estimates (column K) are calculated as (GI − OE) − P − taxes. The AW of CFAT again concludes that plan B is better and that plan A does not return the after-tax MARR of 12% (K10).

(*b*) What is the fundamental difference between the EVA and CFAT series in columns J and K? They are clearly equivalent from the time value of money perspective since the AW values are numerically the same. To answer the question, consider plan A, which has a constant CFAT estimate of \$152,000 per year. To obtain the AW of EVA estimate of \$−12,617 for

	A	B	C	D	E	F	G	H	I	J	K
1						PLAN A					
2									EVA analysis		CFAT analysis
3			Investment P	SL	Book value	Taxable			Cost of		
4	Year	GI - OE	(Basis B)	Depreciation	BV	income, TI	Taxes	NOPAT	inv. capital	EVA	CFAT
5	0		-500,000		500,000						-500,000
6	1	170,000		125,000	375,000	45,000	18,000	27,000	-60,000	-33,000	152,000
7	2	170,000		125,000	250,000	45,000	18,000	27,000	-45,000	-18,000	152,000
8	3	170,000		125,000	125,000	45,000	18,000	27,000	-30,000	-3,000	152,000
9	4	170,000		125,000	0	45,000	18,000	27,000	-15,000	12,000	152,000
10	AW values									-\$12,617	-\$12,617
11						PLAN B					
12									EVA analysis		CFAT analysis
13			Investment P	SL	Book value	Taxable			Cost of		
14	Year	GI - OE	(Basis B)	Depreciation	BV	income, TI	Taxes	NOPAT	inv. capital	EVA	CFAT
15	0		-1,200,000		1,200,000						-1,200,000
16	1	600,000		300,000	900,000	300,000	120,000	180,000	-144,000	36,000	480,000
17	2	500,000		300,000	600,000	200,000	80,000	120,000	-108,000	12,000	420,000
18	3	400,000		300,000	300,000	100,000	40,000	60,000	-72,000	-12,000	360,000
19	4	300,000		300,000	0	0	0	0	-36,000	-36,000	300,000
20	AW values									\$3,388	\$3,388
21											
22	Functions for Plan B, year 3			= -C15/4	= E17-D18	= B18-D18	= F18*0.4	= F18-G18	=-0.12*E17	= H18+I18	= B18+C18-G18

Figure 17–9
Comparison of two plans using EVA and CFAT analyses, Example 17.12.

years 1 through 4, the initial investment of \$500,000 is distributed over the 4-year life using the A/P factor at 12%. That is, an equivalent amount of \$500,000(A/P,12%,4) = \$164,617 is "charged" against the cash inflows in each of years 1 through 4. In effect, the yearly CFAT is reduced by this charge.

$$\begin{aligned} \text{CFAT} - (\text{initial investment})(A/P,12\%,4) &= 152{,}000 - 500{,}000(A/P,12\%,4) \\ 152{,}000 - 164{,}617 &= \$-12{,}617 \\ &= \text{AW of EVA} \end{aligned}$$

This is the AW value for both series, demonstrating that the two methods are economically equivalent. However, the EVA method indicates an alternative's yearly estimated contribution to the *value of the corporation*, whereas the CFAT method estimates the actual cash flows to the corporation. This is why the EVA method is often more popular than the cash flow method with corporate executives.

Comment

The calculation $P(A/P,i,n)$ = \$500,000(A/P,12%,4) is exactly the same as the capital recovery in Equation [6.3], assuming an estimated salvage value of zero. Thus, the cost of invested capital for EVA is the same as the capital recovery discussed in Chapter 6. This further demonstrates why the AW method is economically equivalent to the EVA evaluation.

17.8 After-Tax Analysis for International Projects ● ● ●

Primary questions to be answered prior to performing a corporate-based after-tax analysis for international settings revolve around tax-deductible allowances—depreciation, business expenses, capital asset evaluation—and the effective tax rate needed for Equation [17.7], taxes = (T_e)(TI). As discussed in Chapter 16, most governments of the world recognize and use the straight line (SL) and declining balance (DB) methods of depreciation with some variations to determine the annual tax-deductible allowance. Expense deductions vary widely from country to country. By way of example, some of these are summarized here.

Canada

Depreciation: This is deductible and is normally based on DB calculations, although SL may be used. An equivalent of the half-year convention is applied in the first year of ownership. The annual tax-deductible allowance is termed *capital cost allowance (CCA)*. As in the U.S. system, recovery rates are standardized, so the depreciation amount does not necessarily reflect the useful life of an asset.

Class and CCA rate: Asset classes are defined and annual depreciation rates are specified by class. No specific recovery period (life) is identified, in part because assets of a particular class are grouped together and the annual CCA is determined for the entire class, not individual assets. There are some 44 classes, and CCA rates vary from 4% per year (the equivalent of a 25-year-life asset) for buildings (class 1) to 100% (1-year life) for applications software, chinaware, dies, etc. (class 12). Most rates are in the range of 10% to 30% per year.

Expenses: Business expenses are deductible in calculating TI. Expenses related to capital investments are not deductible, since they are accommodated through the CCA.

Internet: Further details are available on the Revenue Canada website at www.cra.gc.ca in the Forms and Publications section.

China (PRC)

Depreciation: Officially, SL is the primary method of tax-deductible depreciation; however, assets employed in selected industries or types of assets can utilize accelerated DB or SYD (sum-of-years-digits) depreciation, when approved by the government. The selected industries and assets can change over time; currently favored industries serve areas such as technology and oil exploration, and equipment subjected to large vibrations during normal usage is allowed accelerated depreciation.

Recovery period: Standardized recovery periods are published that vary from 3 years (electronic equipment) to 10 years (aircraft, machinery, and other production equipment) to 20 years (buildings). Shortened periods can be approved, but the minimum recovery period cannot be less than 60% for the normal period defined by current tax law.

Expenses: Business expenses are deductible with some limitations and some special incentives. Limitations are placed, for example, on advertising expense deductions (15% of sales for the year). Incentives are generous in some cases; for example, 150% of actual expenses is deductible for new technology and new product R&D activities.

Internet: Summary information for China and several other countries is available at www.worldwide-tax.com.

Mexico

Depreciation: This is a fully deductible allowance for calculating TI. The SL method is applied with an index for inflation considered each year. For some asset types, an immediate deduction of a percentage of the first cost is allowed. (This is a close equivalent to the Section 179 Deduction in the United States.)

Class and rates: Asset types are identified, though not as specifically defined as in some countries. Major classes are identified, and annual recovery rates vary from 5% for buildings (the equivalent of a 20-year life) to 100% for environmental machinery. Most rates range from 10% to 30% per year.

Profit tax: The income tax is levied on profits on income earned from carrying on business in Mexico. Most business expenses are deductible. Corporate income is taxed only once, at the federal level; no state-level taxes are imposed.

Tax on Net Assets (TNA): Under some conditions, a tax of 1.8% of the average value of assets located in Mexico is paid annually in addition to income taxes.

Internet: The best information is via websites for companies that assist international corporations located in Mexico. One example is PriceWaterhouseCoopers at www.pwcglobal.com/mx/eng.

Japan

Depreciation: This is fully deductible and based on classical SL and DB methods. To encourage capital investment for long-term economic growth, in 2007 Japan allowed assets to be depreciated using a *250% DB method;* that is, a significantly increased accelerated rate is allowed compared to historical Japanese allowances. Switching to classical SL depreciation must take place in the year that the accelerated rate amount falls below the corresponding SL amount.

Class and life: A statutory useful life ranging from 4 to 24 years, with a 50-year life for reinforced concrete buildings, is specified.

Expenses: Business expenses are deductible in calculating TI.

Internet: Further details are available on the Japanese Ministry of Finance website at www.mof.go.jp.

The effective tax rate varies considerably among countries. Some countries levy taxes only at the federal level, while others impose taxes at several levels of government (federal, state or provincial, prefecture, county, and city). A summary of international corporate average tax rates is presented in Table 17–6 for a wide range of industrialized countries. These include income taxes at all reported levels of government within each country; however, other types of taxes may be imposed by a particular government. Although these average rates of taxation will vary from year to year, especially as tax reform is enacted, it can be surmised that most corporations face effective rates of about 20% to 40% of taxable income. A close examination of international rates shows that they have decreased significantly over the last decade. In fact, the KPMG report noted in Table 17–6 indicates that the global average corporate tax rate on TI has decreased from 32.7% (1999) to 25.5% (2009). This has encouraged corporate investment and business expansion within country borders and helped soften the massive economic downturn experienced in recent years.

TABLE 17–6 Summary of International Corporate Average Tax Rates

Tax Rate Levied on Taxable Income, %	For These Countries
≥ 40	United States, Japan
35 to < 40	South Africa, Pakistan
32 to < 35	Canada, France, India
28 to < 32	Australia, United Kingdom, New Zealand, Spain, Germany, Mexico, Indonesia
24 to < 28	China, Taiwan, Republic of Korea
20 to < 24	Russia, Turkey, Saudi Arabia
< 20	Singapore, Hong Kong, Chile, Ireland, Iceland, Hungary

Sources: Extracted from KPMG's *Corporate and Indirect Tax Survey 2009* (www.kpmg.com/Global/en/IssuesAndInsights/Pages/default.aspx) and from country websites on corporate taxation.

One of the prime ways that governments have been able to reduce corporate tax rates is by shifting to **indirect taxes** on goods and services for additional tax revenue. These taxes are usually in the form of a **value-added tax (VAT),** goods and services tax (GST), and taxes on products imported from outside its borders. As corporate tax rates have declined, in general the indirect tax rates have increased. This has been especially true during the first decade of the 21st century. However, the worldwide economic slump experienced in recent years has made it necessary for governments around the world to be more cautious of how they tax corporations and maintain a reasonable balance between regular tax rates (as listed in Table 17–6) and indirect tax rates. The VAT system is explained now.

17.9 Value-Added Tax ● ● ●

A value-added tax (VAT) has facetiously been called a sales tax on steroids, because the VAT rates on some items in some countries that impose a VAT can be as high as 90%. It is also called a GST (goods and service tax).

A **value-added tax** is an indirect tax; that is, it is a tax on goods and services rather than on people or corporations. It differs from a sales tax in two ways: (1) when it is charged and (2) who pays (as explained below). A specific percentage, say 10%, is a charge added to the price of the item and paid by the buyer. The seller then sends this 10% VAT to the taxing entity, usually a government unit. This process of 10% VAT continues every time the item is resold—as purchased or in a modified form—thus the term *value-added.*

A value-added tax is commonly used throughout the world. In some countries, VAT is used in lieu of business or individual income taxes. There is no VAT system in the United States yet. In fact, the United States is the only major industrialized country in the world that does not have a VAT system, though other forms of indirect taxation are used liberally. There is mounting evidence, however, that a VAT/GST system will be necessary in the United States in the near future, but not without a great amount of political discord.

A **sales tax** is used by the U.S. government, by nearly all of the states, and by many local entities. A sales tax is charged on goods and services at the time the goods and services reach *the end user or consumer*. That is, businesses do not pay a sales tax on raw material, unfinished goods, or items they purchase that will ultimately be sold to an end user; only the end user pays the sales tax. Businesses *do pay* a sales tax on items for which they are the end user. Total sales tax percentages imposed by multiple government levels can range from 5% to 11%, sometimes larger on specific items. For example, when Home Depot (HD) purchases microwave ovens from Jenn-Air, Inc., Home Depot does not pay a sales tax on the microwave ovens because they will be sold to HD customers who *will* pay the sales tax. On the other hand, if Home Depot purchases a forklift from Caterpillar for loading and unloading merchandise in one of its stores, HD will pay the sales tax on the forklift, since HD is the end user. Thus, a sales tax is **paid only one time,** and that time is when the goods or services are purchased **by the end user.** The sales tax is the responsibility of the merchant to collect and remand to the taxing entity.

A **value-added tax (VAT),** on the other hand, is charged to the buyer at purchase time, whether the buyer is a business or end user. The seller sends the collected VAT to the taxing entity. If the buyer subsequently resells the goods to another buyer (as is or a modification of it), another VAT is collected by the seller. Now, this second seller will send to the taxing entity an amount that is equal to the total tax collected *minus* the amount of VAT already paid.

As an illustration, assume the U.S. government charged a 10% VAT. Here is how the VAT might work.

> Northshore Mining Corporation of Babbitt, Minnesota, sells $100,000 worth of iron ore to Westfall Steel. As part of the price, Northshore collects $110,000, that is, $100,000 + 0.1 × 100,000, from Westfall Steel. Northshore remits the $10,000 VAT to the U.S. Treasury.
>
> Westfall Steel sells all of the steel it made from the iron ore at a price of $300,000 to General Electric (GE). Westfall collects $330,000 from GE and then sends $20,000 to the U.S. Treasury, that is, $30,000 it collected in VAT from GE minus $10,000 it paid in taxes to Northshore Mining.
>
> GE uses the steel to make refrigerators that it sells for $700,000 to retailers, such as Home Depot, Lowe's, and others. GE collects $770,000 and then remits $40,000 to the U.S. Treasury, that is, $70,000 it collected in taxes from retailers minus $30,000 it paid in VAT to Westfall Steel.
>
> If GE purchased machines, tools, or other items to make the refrigerators during this accounting period, and it paid taxes on those items, the taxes paid would also be deducted from the VAT that GE collected before sending the money to the U.S. Treasury. For example, if GE paid $5000 in taxes on motors it purchased for the refrigerators, the amount GE would remit to the U.S. Treasury would be $35,000 (that is, $70,000 it collected from retailers minus $30,000 it paid in taxes to Westfall Steel minus $5000 it paid in taxes on the motors).
>
> The retailers sell the refrigerators for $950,000 and collect $95,000 in taxes from end users—consumers. The retailers remit $25,000 to the U.S. Treasury (that is, $95,000 they collected minus $70,000 they paid previously).

Through this process, the U.S Treasury has received $10,000 from Northshore, $20,000 from Westfall Steel, $35,000 from GE, $5000 from the supplier of the motors, and $25,000 from the retailers, for a total of $95,000. This is 10% of the final sales price of $950,000. The VAT money was deposited into the Treasury at several different times from several different companies.

The taxes that a company *pays* for materials or items it purchases in order to produce goods or services that will subsequently be sold to another business or end user are called *input taxes*, and the company is able to recover them when it collects the VAT from the sale of its products. The taxes that a company *collects* are called *output taxes*, and these are forwarded to the taxing entity, less the amount of input taxes the company paid. Hence, the businesses incur no taxes themselves, the same as with a sales tax.

Several dimensions of a VAT distinguish it from a sales tax or corporate income taxes. Some are as follow:

- Value-added taxes are taxes on consumption, not production or taxable income.
- The end user pays all of the value-added taxes, but VATs are not as obvious as a sales tax that is added to the price of the item at the time of purchase (and displayed on the receipt). Therefore, VAT taxing entities encounter less resistance from consumers.
- Value-added taxes are generally considerably higher than sales taxes, with the average European VAT rate at 20% and the worldwide average at 15.25%.
- The VAT is essentially a "sales tax," but it is charged at each stage of the product development process instead of when the product is sold.
- With VAT, there is less evasion of taxes because it is harder for multiple entities to evade collecting and paying the taxes than it is for one entity to do so.
- VAT rates vary from country to country and from category to category. For example, in some countries, food has a 0% VAT rate, while aviation fuel is taxed at 32%.

EXAMPLE 17.13

Tata Motors is a major player in automobile manufacturing in India. It has three different manufacturing units that specialize in manufacturing different transportation-related products, such as trucks, engines and axles, commercial vehicles, utility vehicles, and passenger cars. The company buys products that fall under different sections of the Indian government VAT tax code, and therefore the products have different VAT rates. In one particular accounting period, Tata had invoices from four different suppliers (vendors A, B, C, and D) in the respective amounts of \$1.5 million, \$3.8 million, \$1.1 million, and \$900,000. The products Tata purchased were subject to VAT rates of 4%, 4%, 12.5%, and 22%, respectively.

(*a*) How much total VAT did Tata pay to its vendors?

(*b*) Assume that Tata's products have a VAT rate of 12.5%. If Tata's sales during the period were \$9.2 million, how much VAT did the Indian Treasury receive from Tata?

Solution

(*a*) Let X equal the product price before the VAT is added. Solve for X and then subtract it from the purchase amount to determine the VAT charged by each vendor. Table 17–7 shows the VAT that Tata paid its four vendors. An example computation for vendor A is as follows:

$$
\begin{aligned}
X + 0.04X &= 1{,}500{,}000 \\
1.04X &= 1{,}500{,}000 \\
X &= \$1{,}442{,}308 \\
\text{VAT}_A &= 1{,}500{,}000 - 1{,}442{,}308 \\
&= \$57{,}692 \\
\text{Total VAT paid} &= 57{,}692 + 146{,}154 + 122{,}222 + 162{,}295 \\
&= \$488{,}363
\end{aligned}
$$

(*b*)
$$
\begin{aligned}
\text{Total from Tata} &= \text{total VAT} - \text{VAT paid by vendors} \\
&= 9{,}200{,}000(0.125) - 488{,}363 \\
&= \$661{,}637
\end{aligned}
$$

TABLE 17–7 VAT Computation, Example 17.13

Vendor	Purchases, \$	VAT Rate, %	Price before VAT, X, \$	VAT, \$
A	1,500,000	4.0	1,442,308	57,692
B	3,800,000	4.0	3,653,846	146,154
C	1,100,000	12.5	977,778	122,222
D	900,000	22.0	737,705	162,295
Total				488,363

CHAPTER SUMMARY

After-tax analysis does not usually change the decision to select one alternative over another; however, it does offer a much clearer estimate of the monetary impact of taxes. After-tax PW, AW, and ROR evaluations of one or more alternatives are performed on the CFAT series using exactly the same procedures as in previous chapters.

Income tax rates for U.S. corporations and individual taxpayers are graduated or progressive—higher taxable incomes pay higher income taxes. A single-value, effective tax rate T_e is usually applied in an after-tax economic analysis. Taxes are reduced because of tax deductible items, such as depreciation and operating expenses. Because depreciation is a noncash flow, it is important to consider depreciation only in TI computations, and not directly in the CFBT and

CFAT calculations. Accordingly, key general cash flow after-tax relations for each year are as follows:

$$\begin{aligned} \text{NOI} &= \text{gross income} - \text{expenses} \\ \text{TI} &= \text{gross income} - \text{operating expenses} - \text{depreciation} + \text{depreciation recapture} \\ \text{CFBT} &= \text{gross income} - \text{operating expenses} - \text{initial investment} + \text{salvage value} \\ \text{CFAT} &= \text{CFBT} - \text{taxes} = \text{CFBT} - (T_e)(\text{TI}) \end{aligned}$$

If an alternative's estimated contribution to corporate financial worth is the economic measure, the economic value added (EVA) should be determined. Unlike CFAT, the EVA includes the effect of depreciation. The equivalent annual worths of CFAT and EVA estimates are the same numerically, because they interpret the annual cost of the capital investment in different, but equivalent manners when the time value of money is taken into account.

In a replacement study, the tax impact of depreciation recapture, which may occur when the defender is traded for the challenger, is accounted for in an after-tax analysis. The replacement study procedure of Chapter 11 is applied. The tax analysis may not reverse the decision to replace or retain the defender, but the effect of taxes will likely reduce (possibly by a significant amount) the economic advantage of one alternative over the other.

International corporate tax rates have steadily decreased, but indirect taxes, such as value-added tax (VAT), have increased. The mechanism of a VAT is explained and compared to a sales tax. The United States currently has no VAT.

PROBLEMS

Basic Tax Computations

17.1 (*a*) Define the following tax terms: graduated tax rates, marginal tax rate, and indexing.
(*b*) Describe a fundamental difference between each of the following terms: net operating income (NOI), taxable income (TI), and net operating profit after taxes (NOPAT).

17.2 Determine the average tax rate for a corporation that has taxable income of (*a*) $150,000 and (*b*) $12,000,000.

17.3 Determine the single-value effective tax rate for a corporation that has a federal tax rate of 35% and a state tax rate of 5%.

17.4 Identify the primary term described by each event below: gross income, depreciation, operating expense, taxable income, income tax, or net operating profit after taxes.
(*a*) A new machine had a first-year write-off of $10,500.
(*b*) A public corporation estimates that it will report a $–750,000 net profit on its annual income statement.
(*c*) An asset with a book value of $8000 was retired and sold for $8450.
(*d*) An over-the-counter software system will generate $420,000 in revenue this quarter.
(*e*) An asset with a MACRS recovery period of 7 years has been owned for 10 years. It was just sold for $2750.
(*f*) The cost of goods sold in the past year was $3,680,200.
(*g*) A convenience store collected $33,550 in lottery ticket sales last month. Based on winners holding these tickets, a rebate of $350 was sent to the manager.
(*h*) An asset with a first cost of $65,000 was utilized on a new product line to increase sales by $150,000.
(*i*) The cost to maintain equipment during the past year was $641,000.

17.5 Two companies have the following values on their annual tax returns.

	Company 1	Company 2
Sales revenue, $	1,500,000	820,000
Interest revenue, $	31,000	25,000
Expenses, $	−754,000	−591,000
Depreciation, $	48,000	18,000

(*a*) Calculate the federal income tax for the year for each company.
(*b*) Determine the percent of sales revenue each company will pay in federal income tax.
(*c*) Estimate the taxes using an effective rate of 34% of the entire TI. Determine the percentage error made relative to the exact taxes in part (*a*).

17.6 Last year, one division of Hagauer.com, a dot-com sports industry service firm that provides real-time analysis of mechanical stress due to athlete injury, had \$250,000 in taxable income. This year, TI is estimated to be \$600,000. Calculate the federal income taxes and answer the following.

(*a*) What was the average federal tax rate paid last year?

(*b*) What is the marginal federal tax rate on the additional TI this year?

(*c*) What will be the average federal tax rate this year?

(*d*) What will be the NOPAT on just the additional \$350,000 in taxable income?

17.7 Yamachi and Nadler of Hawaii has a gross income of \$7.5 million for the year. Depreciation and operating expenses total \$4.3 million. The combined state and local tax rate is 7.2%. If an effective federal rate of 35% applies, estimate the income taxes using the effective tax rate equation.

17.8 Workman Tools reported a TI of \$90,000 last year. If the state income tax rate is 7%, determine the (*a*) average federal tax rate, (*b*) overall effective tax rate, (*c*) total taxes to be paid based on the effective tax rate, and (*d*) total taxes paid to the state and paid to the federal government.

17.9 The taxable income for a motorcycle sales and repair business is estimated to be \$150,000 this year. A single-value tax rate of 39% is used. A new engine diagnostics system will cost \$40,000 and have an average annual depreciation of \$8000. The equipment will increase gross income by an estimated \$9000 and expenses by \$2000 for the year. Compute the expected change in income taxes for the year if the new system is purchased.

17.10 C.F. Jordon Construction Services has operated for the last 26 years in a northern U.S. state where the state income tax on corporate revenue is 6% per year. C.F. Jordon pays an average federal tax of 23% and reports taxable income of \$7 million. Because of pressing labor cost increases, the president wants to move to another state to reduce the total tax burden. The new state may have to be willing to offer tax allowances or an interest-free grant for the first couple of years in order to attract the company. You are an engineer with the company and are asked to do the following.

(*a*) Determine the effective tax rate for C.F. Jordon.

(*b*) Estimate the state tax rate that would be necessary to reduce the overall effective tax rate by 10% per year.

(*c*) Determine what the new state would have to do financially for C.F. Jordon to move there and to reduce its effective tax rate to 22% per year.

CFBT and CFAT

17.11 Identify which of the following items are included in the calculation of cash flow before taxes (CFBT): operating expenses, salvage value, depreciation, initial investment, gross income, tax rate.

17.12 What is the basic difference between cash flow after taxes (CFAT) and net operating profit after taxes (NOPAT)?

17.13 For a year in which there is no initial investment P or salvage value S, derive an equation for CFBT that contains only the following terms: CFAT, CFBT, D, and T_e.

17.14 Determine the cash flow before taxes for Anderson Consultants when the cash flow after taxes was \$600,000, asset depreciation was \$350,000 and the company's effective tax rate was 36%.

17.15 Four years ago Sierra Instruments of Monterey, California spent \$200,000 for equipment to manufacture standard gas flow calibrators. The equipment was depreciated by MACRS using a 3-year recovery period. The gross income for year 4 was \$100,000, with operating expenses of \$50,000. Use an effective tax rate of 40% to determine the CFAT in year 4 if the asset was (*a*) discarded with no salvage value in year 4 and (*b*) sold for \$20,000 at the end of year 4 (neglect any taxes that may be incurred on the sale of the equipment). The MACRS depreciation rate for year 4 is 7.41%.

17.16 Four years ago a division of Harcourt-Banks purchased an asset that was depreciated by the MACRS method using a 3-year recovery period. The total revenue for year 2 was \$48 million, depreciation was \$8.2 million, and operating expenses were \$28 million. Use a federal tax rate of 35% and a state tax rate of 6.5% to determine (*a*) CFAT, (*b*) percentage of total revenue expended on taxes, and (*c*) net profit after taxes for the year.

17.17 Advanced Anatomists, Inc., researchers in medical science, is contemplating a commercial venture concentrating on proteins based on the new X-ray technology of free-electron lasers. To recover the huge investment needed, an annual \$2.5 million CFAT is needed. A favored average federal tax rate of 20% is expected; however, state taxing authorities will levy an 8% tax on TI. Over a 3-year period, the deductible expenses and depreciation are estimated to total \$1.3 million the first year, increasing by \$500,000 per year thereafter. Of this, 50% is expenses and 50% is depreciation. What is the required gross income each year?

The following information is used in Problems 17.18 through 17.21. (Show hand and spreadsheet solutions, as instructed)

After 4 years of use, Procter and Gamble has decided to replace capital equipment used on its Zest bath soap line. The equipment was MACRS-depreciated over a 3-year recovery period. After-tax MARR is 10% per year, and T_e is 35% in the United States. The cash flow data is tabulated in $1000 units.

	Year				
	0	1	2	3	4
Purchase, $	−1900				
Gross income, $		800	950	600	300
Operating expenses, $		−100	−150	−200	−250
Salvage, $					700

17.18 Utilize the CFBT series and AW value to determine whether the equipment investment exceeded the MARR.

17.19 Calculate MACRS depreciation and estimate the CFAT series over 4 years. Neglect any tax impact caused by the $700,000 salvage received in year 4.

17.20 Utilize the CFAT series and AW value to determine whether the investment exceeded the MARR.

17.21 Compare the after-tax ROR values using both methods—CFAT series and approximation from the CFBT values using the before-tax ROR and T_e.

17.22 A Wal-Mart Distribution Center has put into service forklifts and conveyors purchased for $250,000. Use a spreadsheet to tabulate CFBT, CFAT, NOPAT, and i^* before and after taxes for 6 years of ownership. The effective tax rate is 40%, and the estimated cash flow and depreciation amounts are shown. Salvage is expected to be zero.

Year	Gross Income, $	Operating Expenses, $	MACRS Depreciation, $
1	210,000	−120,000	50,000
2	210,000	−120,000	80,000
3	160,000	−122,000	48,000
4	160,000	−124,000	28,800
5	160,000	−126,000	28,800
6	140,000	−128,000	14,400

Taxes and Depreciation

17.23 An asset purchased by Stratasys, Inc. had a first cost of $70,000 with an expected salvage value of $10,000 at the end of its 5-year life. In year 2, the revenue was $490,000 with operating expenses of $140,000. If the company's effective tax rate was 36%, determine the difference in taxes paid in year 2 if the depreciation method had been straight line instead of MACRS.

17.24 Cheryl, an electrical engineering student who is working on a business minor, is studying depreciation and finance in her engineering management course. The assignment is to demonstrate that shorter recovery periods require the same total taxes, but they offer a time value of taxes advantage for depreciable assets. Help her, using asset estimates made for a 6-year study period: P = $65,000, S = $5000, GI = $32,000 per year, AOC is $10,000 per year, SL depreciation, i = 12% per year, T_e = 31%. Make the comparison using recovery periods of 3 and 6 years.

17.25 Complete the last four columns of the table below using an effective tax rate of 40% for an asset that has a first cost of $20,000 and a 3-year recovery period with no salvage value, using (*a*) straight line depreciation and (*b*) MACRS depreciation. All cash flows are in $1000 units.

	Estimates, $						
Year	GI	*P*	OE	*D*	TI	Taxes	CFAT
0	—	−20	—	—	—	—	−20
1	8		−2				
2	15		−4				
3	12		−3				
4	10		−5				

17.26 Use an effective tax rate of 32% to determine the CFAT and NOPAT associated with the asset shown below under two different scenarios: (*a*) with depreciation at $6000 per year and (*b*) with depreciation at $6000, $9600, $5760, and $3456 in years 1 through 4, respectively. All monetary amounts are in $1000.

	Estimates, $							
Year	GI	OE	*P*	*D*	TI	Taxes	CFAT	NOPAT
0	—	—	−30	—	—	—	−30	—
1	8	−2						
2	15	−4						
3	12	−3						
4	10	−5						

17.27 J. B. Hunt, Inc., an overland freight company, has purchased new trailers for $150,000 and expects to realize a net $80,000 in gross income over operating expenses for each of the next 3 years. The trailers have a recovery period of 3 years. Assume an effective tax rate of 35% and an interest rate of 15% per year.

(*a*) Show the advantage of accelerated depreciation by calculating the present worth of taxes for the MACRS method versus the classical

SL method. Since MACRS takes an additional year to fully depreciate the basis, assume no CFBT beyond year 3, but include any negative tax as a tax savings.
(*b*) Show that the total taxes are the same for both methods.

17.28 A bioengineer is evaluating methods used to apply adhesive onto microporous paper tape that is commonly used after surgery. The machinery costs \$200,000, has no salvage value, and the CFBT estimate is \$75,000 per year for up to 10 years. The T_e = 38% and i = 8% per year. The two depreciation methods to consider are: MACRS with n = 5 years and SL with n = 8 years (neglect the half-year convention effect). For a study period of 8 years, (*a*) determine which depreciation method offers the better tax advantage, and (*b*) demonstrate that the same total taxes are paid for MACRS and SL.

17.29 An asset with a first cost of \$9000 is depreciated by MACRS over a 5-year recovery period. The CFBT is estimated at \$10,000 for the first 4 years and \$5000 thereafter as long as the asset is retained. The effective tax rate is 40%, and money is worth 10% per year. In present worth dollars, how much of the cash flow generated by the asset over its recovery period is lost to taxes?

Depreciation Recapture and Capital Gains (Losses)

17.30 Last month, a company specializing in wind power plant design and construction made a capital investment of \$400,000 in physical simulation equipment that will be used for at least 5 years, after which it is expected to be sold for approximately 25% of its first cost. According to tax law, the simulation is MACRS-depreciated using a 3-year recovery period.
(*a*) Explain why there is a predictable tax implication when the simulator is sold.
(*b*) Determine by how much the sale will cause TI and taxes to change in year 5 if T_e = 35%.

The following information is used in Problems 17.31 through 17.34.

Open Access, Inc. is an international provider of computer network communications gear. Different depreciation, recovery period, and tax law practices in the three countries where depreciable assets are located are summarized in the table. Also, information is provided about assets purchased 5 years ago at each location and sold this year. After-tax MARR = 9% per year and T_e = 30% can be used for all countries.

Practice or Estimate	Country 1	Country 2	Country 3
Depreciation method	SL with $n = 5$	MACRS with $n = 3$	DDB with $n = 5$
Depreciation recapture	Not taxed	Taxed as TI	Taxed as TI
First cost, \$	−100,000	−100,000	−100,000
GI − OE, \$ per year	25,000	25,000	25,000
Estimated salvage, \$	0 in year 5	0 in year 5	20,000 in year 5
Life, years	5	5	5
Actual selling price, \$	20,000 in year 5	20,000 in year 5	20,000 in year 5

17.31 For country 1, SL depreciation is \$20,000 per year. Determine the (*a*) CFAT series and (*b*) PW of depreciation, taxes, and CFAT series.

17.32 For country 2, MACRS depreciation for 4 years is \$33,333, \$44,444, \$14,815, and \$7407, respectively. Determine the (*a*) CFAT series and (*b*) PW of depreciation, taxes, and CFAT series.

17.33 For country 3, DDB depreciation for 5 years is \$40,000, \$24,000, \$14,400, \$1600, and 0, respectively. Determine the (*a*) CFAT series and (*b*) PW of depreciation, taxes, and CFAT series.

17.34 If you worked Problems 17.31 through 17.33, develop a table that summarizes, for each country, the total taxes paid and the PW values of the depreciation, taxes, and CFAT series. For each criterion, select the country that provides the best PW value. Explain why the same country is not selected for all three criteria. (*Hint:* The PW should be minimized for some criteria and maximized for other criteria. Review previous material first to be sure you choose correctly.)

17.35 An asset with a first cost of \$350,000 three years ago is sold for \$385,000. The asset was depreciated by the MACRS method and has a book value of \$100,800 at the time of sale. Determine the capital gain and depreciation recapture, if any.

17.36 Sun-Tex Truck Stop is located in the desert southwest and is 5 miles from the nearest municipal water system. In order to have fresh water at the site, the company purchased a turnkey reverse osmosis (RO) system for \$355,000. The company depreciated the RO system using the MACRS method with a 10-year recovery period. Four years after the system was purchased, water lines from a local water system were extended to the truck stop, so Sun-Tex sold the RO system for \$190,000. Determine which of the following apply and the amount, if any, to include in an after-tax analysis of the project: depreciation recapture, capital gain, capital loss. The MACRS depreciation rates are 10%, 18%, 14.4%, and 11.52% for years 1 through 4, respectively.

17.37 Freeman Engineering paid $28,500 for specialized equipment for use with its new GPS/GIS system. The equipment was depreciated over a 3-year recovery period using MACRS depreciation. The company sold the equipment after 2 years for $5000 when it purchased an upgraded system. (*a*) Determine the amount of the depreciation recapture or capital loss involved in selling the asset. (*b*) What tax effect will this amount have?

17.38 Determine any depreciation recapture, capital gain, or capital loss generated by each event described below. Use them to determine the amount of income tax effect, if the effective tax rate is 35%.

(*a*) A MACRS-depreciated asset with a 7-year recovery period has been sold prematurely after 4 years at an amount equal to 40% of its first cost, which was $150,000.

(*b*) A hi-tech machine was sold internationally for $10,000 more than its purchase price just after it was in service 1 year. The asset had P = $100,000, S = $1000, and $n = 3$ years and was depreciated by the MACRS method for the 1 year.

(*c*) Land purchased 4 years ago for $1.8 million was sold at a 10% profit.

(*d*) A 21-year-old asset was removed from service and sold for $500. When purchased, the asset was entered on the books with a basis of P = $180,000, S = $5000, and n = 18 years. Classical straight line depreciation was used for the entire recovery period.

(*e*) A corporate car was depreciated using MACRS over a 3-year recovery period. It was sold in the fourth year of use for $2000.

17.39 Sunnen Products Co. of St. Louis, Missouri, makes actuator hones for gas meter tubes where light-duty metal removal is needed. The company purchased land, a building, and two depreciable assets from MPG Automation Systems Corporation, all of which have recently been disposed of. Use the information shown to determine the presence and amount of any depreciation recapture, capital gain, or capital loss.

Asset	Purchase Price, $	Recovery Period, Years	Current Book Value, $	Sales Price, $
Land	−220,000	—		295,000
Building	−900,000	27.5	300,000	275,000
Asset 1	−50,500	3	15,500	19,500
Asset 2	−20,000	3	5,000	12,500

After-Tax Economic Analysis

17.40 Compute the required before-tax return if an after-tax return of 7% per year is expected and the state and local tax rates total 4.2%. The effective federal tax rate is 34%.

17.41 Estimate the approximate after-tax rate of return (ROR) for a project that has a before-tax ROR of 24%. Assume that the company has an effective tax rate of 35% and it uses MACRS depreciation for an asset that has a $27,000 salvage value.

17.42 Estimate the approximate after-tax rate of return for a project that has a first cost of $500,000, a salvage value of 20% of the first cost after 3 years, and annual gross income less operating expenses of (GI−OE) = $230,000. Assume the company has a 35% effective tax rate.

17.43 An engineer is making an annual return of 12% before taxes on the retirement investments placed through her employer. No taxes are paid on retirement earnings until they are withdrawn; however, she was told by her brother, an accountant, that this is the equivalent of an 8% per year after-tax return. What percent of taxable income is her brother assuming to be taken by income taxes?

17.44 Bart is an economic consultant to the textile industry. In both a small business and a large corporation he performed economic evaluations that have an average 18% per year before-tax return. If the stated MARR in both companies is 12% per year after taxes, determine if management at both companies should accept the projects. The before-tax return is used to approximate the after-tax return. Effective tax rates are 34% for the larger corporation and 28% for the small company.

17.45 Elias wants to perform an after-tax evaluation of equivalent methods A and B to electrostatically remove airborne particulate matter from clean rooms used to package liquid pharmaceutical products. Using the information shown, MACRS depreciation with $n = 3$ years, a 5-year study period, after-tax MARR = 7% per year, and T_e = 34% and a spreadsheet, he obtained the results AW_A = $−2176 and AW_B = $3545. Any tax effects when the equipment is salvaged were neglected. Thus, with MACRS depreciation, method B is the better method. Now, use classical SL depreciation with $n = 5$ years to evaluate the alternatives. Is the decision different from that reached using MACRS?

	Method A	Method B
First cost, $	−100,000	−150,000
Salvage value, $	10,000	20,000
Savings, $ per year	35,000	45,000
AOC, $ per year	−15,000	−6,000
Expected life, years	5	5

17.46 A corporation uses the following: before-tax MARR of 14% per year, after-tax MARR of 7% per year, and T_e of 50%. Two new machines have the following estimates.

	Machine A	Machine B
First cost, $	−15,000	−22,000
Salvage value, $	3,000	5,000
AOC, $ per year	−3,000	−1,500
Life, years	10	10

The machine is retained in use for a total of 10 years, then sold for the estimated salvage value. Select one machine under the following conditions:

(*a*) Before-tax PW analysis.

(*b*) After-tax PW analysis, using classical SL depreciation over the 10-year life.

(*c*) After-tax PW analysis, using MACRS depreciation with a 5-year recovery period.

17.47 Choose between alternatives A and B below if the after-tax MARR is 8% per year, MACRS depreciation is used, and $T_e = 40\%$. The GI−OE estimate is made for only 3 years; it is zero when each asset is sold in year 4.

	Alternative A	Alternative B
First cost, $	−8,000	−13,000
Actual salvage value, $	0	2,000
Gross income − expenses, $	3,500	5,000
Recovery period, years	3	3

17.48 Offshore platform safety equipment, designed for special jobs, will cost $2,500,000, will have no salvage value, and will be kept in service for exactly 5 years, according to company policy. Operating revenue minus expenses is estimated to be $1,500,000 in year 1 and only $300,000 each additional year. The effective tax rate for the multinational oil company is 30%. Show hand and spreadsheet solutions, as instructed, to find the after-tax ROR using (*a*) classical SL depreciation and (*b*) MACRS 5-year depreciation, neglecting any tax effects at the end of 5 years of service.

17.49 Automatic inspection equipment purchased for $78,000 by Stimson Engineering generated an average of $26,080 annually in before-tax cash flow during its 5-year estimated life. This represents a return of 20%. However, the corporate tax expert said the CFAT was only $15,000 per year. If the corporation wants to realize an after-tax return of 10% per year, for how many more years must the equipment remain in service?

After-Tax Replacement

17.50 In a replacement study between a defender and a challenger, there may be a capital gain or loss when the defender asset is sold. (*a*) How is the gain or loss calculated, and (*b*) how does it affect the AW values in the study?

17.51 In an after-tax replacement study of cost alternatives involving one challenger and one defender, how will a capital loss when selling the defender affect the AW of each alternative?

17.52 An asset that was purchased 2 years ago was expected to be kept in service for its projected life of 5 years, but a new version (the challenger) of this asset promises to be more efficient and have lower operating costs. You have been asked to figure out if it would be more economically attractive to replace the defender now or keep it for 3 more years as originally planned. The defender had a first cost of $300,000, but its market value now is only $150,000. It has operating expenses of $120,000 per year and no estimated salvage value after 3 more years. To simplify calculations for this problem only, assume that SL depreciation was charged at $60,000 per year and that it will continue at that rate for the next 3 years.

The challenger will cost $420,000, will have no salvage value after its 3-year life, will have chargeable expenses of $30,000 per year, and will be depreciated at $140,000 per year (again, using SL depreciation for simplicity in this case). Assume the company's effective tax rate is 35%, and its after-tax MARR is 15% per year.

(*a*) Determine the CFAT in year 0 for the challenger and defender.

(*b*) Determine the CFAT in years 1 through 3 for the challenger and defender.

(*c*) Conduct an AW evaluation to determine if the defender should be kept for 3 more years or replaced now.

17.53 The defender in a catalytic oxidizer manufacturing plant has a market value of $130,000 and expected annual operating costs of $70,000 with no salvage value after its remaining life of 3 years. The depreciation charges for the next 3 years will be $69,960, $49,960, and $35,720. Using an effective tax rate of 35%, determine the CFAT for next year only that should be used in a present worth equation to compare this defender against a challenger that also has a 3-year life. Assume the company's after-tax MARR is 12%.

17.54 Perform a PW replacement study (hand and spreadsheet solutions, if instructed) from the information shown using an after-tax MARR = 12% per year, T_e = 35%, and a study period of 4 years.

(Assume that the assets will be salvaged at their original salvage estimates. Since no revenues are estimated, all taxes are negative and considered "savings" to the alternative.) All monetary values are in \$1000 units.

	Defender	Challenger
First cost, \$1000	−45	−24
Estimated S at purchase, \$1000	5	0
Market value now, \$1000	35	—
AOC, \$1000 per year	−7	−8
Depreciation method	SL	MACRS
Recovery period, years	8	3
Useful life, years	8	5
Years owned	3	—

17.55 After 8 years of use, the heavy truck engine overhaul equipment at Pete's Truck Repair was evaluated for replacement. Pete's accountant used an after-tax MARR of 8% per year, $T_e = 30\%$, and a current market value of \$25,000 to determine AW = \$2100. The new equipment costs \$75,000, uses SL depreciation over a 10-year recovery period, and has a \$15,000 salvage estimate. Estimated CFBT is \$15,000 per year. Pete asked his engineer son Ramon to determine if the new equipment should replace what is owned currently. From the accountant, Ramon learned the current equipment cost \$20,000 when purchased and reached a zero book value several years ago. Help Ramon answer his father's question.

17.56 Apple Crisp Foods signed a contract some years ago for maintenance services on its fleet of trucks and cars. The contract is up for renewal now for a period of 1 year or 2 years only. The contract quote is \$300,000 per year if taken for 1 year and \$240,000 per year if taken for 2 years. The finance vice president wants to renew the contract for 2 years without further analysis, but the vice president for engineering believes it is more economical to perform the maintenance in-house. Since much of the fleet is aging and must be replaced in the near future, a fixed 3-year study period has been agreed upon. The estimates for the in-house (challenger) alternative are as follows:

First cost, \$	−800,000
AOC, \$ per year	−120,000
Life, years	4
Estimated salvage	Loses 25% of P annually: End year 1, S = \$600,000 End year 2, S = \$400,000 End year 3, S = \$200,000 End year 4, S = \$0
MACRS depreciation	3-year recovery period

The effective tax rate is 35%, and the after-tax MARR is 10% per year. Perform an after-tax AW analysis, and determine which vice president has the better economic strategy over the next 3 years.

17.57 Nuclear safety devices installed several years ago have been depreciated from a first cost of \$200,000 to zero using MACRS. The devices can be sold on the used equipment market for an estimated \$15,000. Or they can be retained in service for 5 more years with a \$9000 upgrade now and an AOC of \$6000 per year. The upgrade investment will be depreciated over 3 years with no salvage value. The challenger is a replacement with newer technology at a first cost of \$40,000, $n = 5$ years, and $S = 0$. The new units will have operating expenses of \$7000 per year. (*a*) Use a 5-year study period, an effective tax rate of 40%, an after-tax MARR of 12% per year, and an assumption of classical straight line depreciation (no half-year convention) to perform an after-tax replacement study. (*b*) If the challenger is known to be salable after 5 years for an amount between \$2000 and \$4000, will the challenger AW value become more or less costly? Why?

17.58 Three years ago, Silver House Steel purchased a new quenching system for \$550,000. The salvage value after 10 years at that time was estimated to be \$50,000. Currently the expected remaining life is 7 years with an AOC of \$27,000 per year. The new president has recommended early replacement of the system with one that costs \$400,000 and has a 12-year life, a \$35,000 salvage value, and an estimated AOC of \$50,000 per year. The MARR for the corporation is 12% per year. The president wishes to know the replacement value that will make the recommendation to replace now economically advantageous. Use a spreadsheet and Solver to find the minimum trade-in value (*a*) before taxes and (*b*) after taxes, using an effective tax rate of 30%. For solution purposes, use classical SL depreciation for both systems. Comment on the difference in replacement value made by the consideration of taxes.

Economic Value Added

17.59 While an engineering manager may prefer to use CFAT estimates to evaluate the AW of a project, a financial manager may select AW of EVA estimates. Why are these preferences predictable?

17.60 Cardenas and Moreno Engineering is evaluating a very large flood control program for several southern U.S. cities. One component is a 4-year project for a special-purpose transport ship-crane for use in building permanent storm surge protection against hurricanes on the New Orleans coastline. The estimates are $P = \$300{,}000$, $S = 0$, and $n = 3$ years. MACRS depreciation with a 3-year recovery is indicated. Gross income and operating expenses are estimated at \$200,000 and \$80,000, respectively, for each of 4 years. The CFAT is shown below. Calculate the AW values of the CFAT and EVA series. They should have the same value. The after-tax MARR is 9.75% and $T_e = 35\%$.

Year	GI, \$	OE, \$	P and S, \$	D, \$	TI, \$	Taxes, \$	CFAT, \$
0			−300,000				−300,000
1	200,000	−80,000		99,990	20,010	7,003	112,997
2	200,000	−80,000		133,350	−13,350	−4,673	124,673
3	200,000	−80,000	0	44,430	75,570	26,450	93,551
4	200,000	−80,000		22,230	97,770	34,220	85,781

17.61 Triple Play Innovators Corporation (TPIC) plans to offer IPTV (Internet Protocol TV) service to North American customers starting soon. Perform an AW analysis of the EVA series for the two alternative suppliers available for the hardware and software. Let $T_e = 30\%$ and after-tax MARR = 8%; use SL depreciation (neglect half-year convention and MACRS, for simplicity) and a study period of 8 years.

	Vendor	
	Hong Kong	Japan
First cost, \$	4.2 million	3.6 million
Recovery period, years	8	5
Salvage value, \$	0	0
GI − OE, \$ per year	1,500,000 in year 1; increasing by 300,000 per year up to 8 years	

17.62 Sun Microsystems has developed partnerships with several large manufacturing corporations to use Java software in their consumer and industrial products. A new corporation will be formed to manage these applications. One major project involves using Java in commercial and industrial appliances that store and cook food. The gross income and operating expenses are expected to follow the relations shown for the estimated life of 6 years. For $t = 1$ to 6 years,

$$\text{Annual gross income, GI} = 2{,}800{,}000 - 100{,}000t$$
$$\text{Annual operating expenses, OE} = 950{,}000 + 50{,}000t$$

The effective tax rate is 30%, the interest rate is 12% per year, and the depreciation method chosen for the \$3,000,000 capital investment is a 5-year MACRS ADS alternative that allows straight line write-off with the half-year convention in years 1 and 6. Using a spreadsheet, estimate (*a*) the annual economic contribution of the project to the new corporation and (*b*) the equivalent annual worth of these contributions.

Value-Added Tax

17.63 What is the primary difference between a sales tax and a value-added tax?

17.64 In Denmark, VAT is applied at a rate of 25%, with few exceptions. Vendor A sells raw materials to vendor B for \$60,000 plus VAT, vendor B sells a product to vendor C for \$130,000 plus VAT, and vendor C sells an improved, value-added product to an end user for \$250,000 plus VAT. Determine the following.
(*a*) The amount of VAT collected by vendor B.
(*b*) The amount of tax vendor B sends to Denmark's Treasury.
(*c*) The total amount of tax collected by the Treasury department.

The following information is used in Problems 17.65 through 17.70

Ajinkya Electronic Systems, a company in India that manufactures many different electronic products, has to purchase goods and services from a variety of suppliers (wire, diodes, LED displays, plastic components, etc.). The table below shows several suppliers and the VAT rates associated with each. It also shows the purchases in \$1000 units that Ajinkya made (before taxes) from each supplier in the previous accounting period. Ajinkya's sales to end users was \$9.2 million, and Ajinkya's products carry a VAT of 12.5%.

Supplier	VAT Rate, %	Purchases by Ajinkya, \$1000
A	4.0	350
B	12.5	870
C	12.5	620
D	21.3	90
E	32.6	50

17.65 How much VAT did supplier C collect?

17.66 How much tax did Ajinkya keep from the tax it collected based on the purchases it made from supplier A?

17.67 What was the total amount of taxes paid by Ajinkya to the suppliers?

17.68 What was the average VAT rate paid by Ajinkya in purchasing goods and services?

17.69 What was the amount of taxes Ajinkya sent to the Treasury of India?

17.70 What was the total amount of taxes collected by the Treasury of India from Ajinkya and Ajinkya's suppliers?

ADDITIONAL PROBLEMS AND FE EXAM REVIEW QUESTIONS

17.71 A graduated income tax system means:
(*a*) Only taxable incomes above a certain level pay any taxes.
(*b*) A higher flat rate goes with all of the taxable income.
(*c*) Higher tax rates go with higher taxable incomes.
(*d*) Rates are indexed each year to keep up with inflation.

17.72 All of the following are characteristics of a value-added tax system except:
(*a*) Value-added taxes are taxes on consumption.
(*b*) The end user pays value-added taxes.
(*c*) Value-added taxes are charged at each stage of product development.
(*d*) Value-added taxes are charged only on the raw materials for product development.

17.73 A small company has a taxable income that places it in the 35% tax bracket. The amount of taxes that a depreciation charge of \$16,000 would save is closest to:
(*a*) \$0
(*b*) \$3200
(*c*) \$5600
(*d*) \$10,400

17.74 A company that has a 50% effective tax rate had income of \$200 million in each of the last 2 years. In one of those years, the company had deductions of \$100 million. In the other year, the company had deductions of only \$80 million. The difference in income taxes paid by the company in those 2 years was closest to:
(*a*) \$10 million
(*b*) \$20 million
(*c*) \$50 million
(*d*) \$60 million

17.75 Taxable income (TI) is defined as:
(*a*) TI = revenue + operating expenses − depreciation
(*b*) TI = revenue − operating expenses + depreciation
(*c*) TI = revenue − operating expenses − depreciation + amortization
(*d*) TI = revenue − operating expenses − depreciation

17.76 The marginal tax rate is defined as:
(*a*) The percentage paid on the last dollar of income
(*b*) The tax rate that applies to a questionable investment
(*c*) The tax rate that includes federal, state, and local taxes
(*d*) The percentage paid on the first dollar of income

17.77 A subcontractor with an effective tax rate of 25% has gross income of \$55,000, other income of \$4000, operating expenses of \$13,000, and other deductions and exemptions of \$11,000. The income tax due is closest to:
(*a*) \$11,750
(*b*) \$8,750
(*c*) \$10,750
(*d*) \$13,750

17.78 When a depreciable asset is disposed of for less than its current book value, the transaction is known as:
(*a*) An after-tax expense
(*b*) Capital loss
(*c*) Capital gain
(*d*) Depreciation recapture

17.79 The after-tax analysis for a \$60,000 investment with associated gross income minus expenses (GI−OE) is shown below for the first 2 years only. If the effective tax rate is 40%, the values for depreciation (*D*), taxable income (TI), and taxes for year 1 are closest to:

Year	Investment, \$	GI − OE, \$	*D*, \$	TI, \$	Taxes, \$	CFAT, \$
0	−60,000					−60,000
1		30,000				26,000
2		35.000		15,000	6000	29.000

(a) $D = \$5{,}000$, TI = \$25,000, taxes = \$10,000
(b) $D = \$30{,}000$, TI = \$30,000, taxes = \$4,000
(c) $D = \$20{,}000$, TI = \$50,000, taxes = \$20,000
(d) $D = \$20{,}000$, TI = \$10,000, taxes = \$4,000

17.80 If the after-tax rate of return for a cash flow series is 11.9% and the corporate effective tax rate is 34%, the approximated before-tax rate of return is closest to:
(a) 6.8%
(b) 5.4%
(c) 18.0%
(d) 28.7%

17.81 An asset purchased for \$100,000 with $S = \$20{,}000$ after 5 years was depreciated using the 5-year MACRS rates. Expenses averaged \$18,000 per year, and the effective tax rate is 30%. The asset was actually sold after 5 years of service for \$22,000. MACRS rates in years 5 and 6 are 11.53% and 5.76%, respectively. The after-tax cash flow from the sale is closest to:
(a) \$27,760
(b) \$17,130
(c) \$26,870
(d) \$20,585

17.82 When accelerated depreciation methods or shortened recovery periods are applied, there are impacts on the income taxes due. Of the following, the statements that are commonly *incorrect* are:
1. Total taxes paid are the same for all depreciation methods.
2. Present worth of taxes is lower for shorter recovery periods.
3. Accelerated depreciation imposes more taxes in the later years of the recovery period.
4. Present worth of taxes is higher for shorter recovery periods.

(a) 1, 2, and 3
(b) 1 and 4
(c) 2
(d) 4

CASE STUDY

AFTER-TAX ANALYSIS FOR BUSINESS EXPANSION

Background

Charles was always a hands-on type of person. Within a couple of years of graduating from college, he started his own business. After some 20 years, it has grown significantly. He owns and operates Pro-Fence, Inc. in the Metroplex, specializing in custom-made metal and stone fencing for commercial and residential sites. For some time, Charles has thought he should expand into a new geographic region, with the target area being another large metropolitan area about 500 miles north, called Victoria.

Pro-Fence is privately owned by Charles; therefore, the question of how to finance such an expansion has been, and still is, the major challenge. Debt financing would not be a problem in that the Victoria Bank has already offered a loan of up to \$2 million. Taking capital from the retained earnings of Pro-Fence is a second possibility, but taking too much will jeopardize the current business, especially if the expansion were not an economic success and Pro-Fence were stuck with a large loan to repay.

This is where you come in as a long-time friend of Charles. He knows you are quite economically oriented and that you understand the rudiments of debt and equity financing and economic analysis. He wants you to advise him on the balance between using Pro-Fence funds and borrowed funds. You have agreed to help him, as much as you can.

Information

Charles has collected some information that he shares with you. Between his accountant and a small market survey of the business opportunities in Victoria, the following generalized estimates seem reasonable.

Initial capital investment = \$1.5 million

Annual gross income = \$700,000

Annual operating expenses = \$100,000

Effective income tax rate for Pro-Fence = 35%

Five-year MACRS depreciation for all \$1.5 million investment

The terms of the Victoria Bank loan would be 6% per year simple interest based on the initial loan principal. Repayment would be in 5 equal payments of interest and principal. Charles comments that this is not the best loan arrangement he hopes to get, but it is a good worst-case scenario upon which to base the debt portion of the analysis. A range of D-E mixes should be analyzed. Between Charles and yourself, you have developed the following viable options.

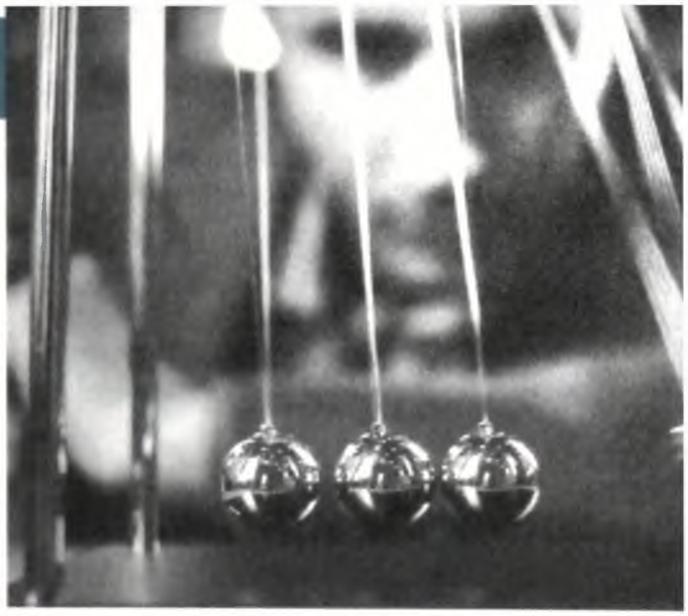
Debt		Equity	
Percentage	Loan Amount, \$	Percentage	Investment Amount, \$
0		100	1,500,000
50	750,000	50	750,000
70	1,050,000	30	450,000
90	1,350,000	10	150,000

Case Study Exercises

1. For each funding option, perform a spreadsheet analysis that shows the total CFAT and its present worth over a 6-year period, the time it will take to realize the full advantage of MACRS depreciation. An after-tax return of 10% is expected. Which funding option is best for Pro-Fence? (*Hint:* For the spreadsheet, sample column headings are: Year, GI – OE, Loan interest, Loan principal, Equity investment, Depreciation rate, Depreciation, Book value, TI, Taxes, and CFAT.)
2. Observe the changes in the total 6-year CFAT as the D-E percentages change. If the time value of money is neglected, what is the constant amount by which this sum changes for every 10% increase in equity funding?
3. Charles noticed that the CFAT total and PW values go in opposite directions as the equity percentage increases. He wants to know why this phenomenon occurs. How should you explain this to Charles?
4. After deciding on the 50-50 split of debt and equity financing, Charles wants to know what additional bottom-line contributions to the economic worth of the company may be added by the new Victoria site. What are the best estimates at this time?

CHAPTER 18

Sensitivity Analysis and Staged Decisions



LEARNING OUTCOMES

Purpose: Perform a sensitivity analysis of parameters; use expected values to evaluate staged funding options.

SECTION	TOPIC	LEARNING OUTCOME
18.1	Sensitivity to variation	• Use a measure of worth to explain sensitivity to variation in one or more parameters.
18.2	Three-estimate variation	• Choose the better alternative using three estimates of variation in selected parameters.
18.3	$E(X)$	• Calculate the expected value of a variable.
18.4	$E(X)$ of cash flows	• Evaluate a project or alternatives using expected values of cash flows.
18.5	Decision trees	• Use a decision tree to evaluate alternatives stage by stage.
18.6	Real options	• Explain a real option in engineering economics and evaluate a staged decision using real options analysis.

This chapter includes several related topics about alternative evaluation. Initially, we expand our capability to perform a **sensitivity analysis** of one or more parameters and of an entire alternative. Then the determination and use of the **expected value** of a cash flow series are treated. The techniques of **decision trees** help make a series of economic decisions for alternatives that have different, but closely connected stages.

Economic decisions that involve **staged funding** are very common in professional and everyday life. The last topic of **real options analysis** introduces a method useful in these circumstances.

18.1 Determining Sensitivity to Parameter Variation

Online Presentation

The term *parameter* is used in this chapter to represent any variable or factor for which an estimate or stated value is necessary. Example parameters are first cost, salvage value, AOC, estimated life, production rate, and materials costs. Estimates such as the loan interest rate and the inflation rate can also be parameters of the analysis.

Economic analysis uses estimates of a parameter's future value to assist decision makers. Since future estimates are always incorrect to some degree, inaccuracy is present in the economic projections. The effect of variation may be determined by using sensitivity analysis.

Sensitivity analysis

> **Sensitivity analysis** determines how a measure of worth—PW, AW, FW, ROR, B/C, or CER—is altered when one or more parameters vary over a **selected range of values.** Usually one parameter at a time is varied, and independence with other parameters is assumed. Though this approach is an oversimplification in real-world situations, since the dependencies are difficult to accurately model, the end results are usually correct.

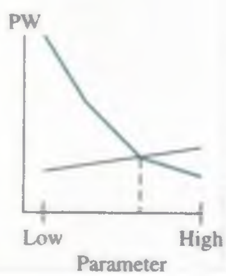
In reality, we have applied this approach (informally) throughout previous chapters to determine the response to variation in a variety of parameters. Variation in a parameter such as MARR will not alter the decision to select an alternative when all compared alternatives return considerably more than the MARR; thus, the decision is relatively insensitive to the MARR. However, variation in the n or AOC value may indicate that the alternative's measure of worth is very sensitive to the estimated life or annual operating costs.

Usually the variations in life, annual costs, and revenues result from variations in selling price, operation at different levels of capacity, inflation, etc. For example, if an operating level of 90% of airline seating capacity for a domestic route is compared with 70% for a proposed international route, the operating cost and revenue per passenger-mile will increase, but anticipated aircraft life will probably decrease only slightly. Usually several important parameters are studied to learn how the uncertainty of estimates affects the economic analysis.

Sensitivity analysis routinely concentrates on the variation expected in estimates of P, AOC, S, n, unit costs, unit revenues, and similar parameters. These parameters are often the result of design questions and their answers, as discussed in Chapter 15. Parameters that are interest rate–based are not treated in the same manner.

> Parameters such as MARR and other interest rates (loan rates, inflation rate) are more stable from project to project. If performed, sensitivity analysis on them is for specific values or over a **narrow range of values.** This point is important to remember if simulation is used for decision making under risk (Chapter 19).

Plotting the sensitivity of PW, AW, or ROR versus the parameter(s) studied is very helpful. Two alternatives can be compared with respect to a given parameter and the breakeven point. This is the value at which the two alternatives are economically equivalent. However, the breakeven chart commonly represents only one parameter per chart. Thus, several charts are constructed, and independence of each parameter is assumed. In previous uses of breakeven analysis, we often computed the measure of worth at only two values of a parameter and connected the points with a straight line. However, if the results are sensitive to the parameter value, several intermediate points should be used to better evaluate the sensitivity, especially if the relationships are not linear.



When several parameters are studied, sensitivity analysis can become quite complex. It may be performed one parameter at a time using a spreadsheet or computations by hand or calculator.

The computer facilitates comparison of multiple parameters and multiple measures of worth, and the software can rapidly plot the results.

Here is a general procedure to follow when conducting a thorough sensitivity analysis.

1. Determine which parameter(s) of interest might vary from the most likely estimated value.
2. Select the probable range and an increment of variation for each parameter.
3. Select the measure of worth.
4. Compute the results for each parameter, using the measure of worth as a basis.
5. To better interpret the sensitivity, graphically display the parameter versus the measure of worth.

This sensitivity analysis procedure should indicate the parameters that warrant closer study or require additional information. When there are two or more alternatives, it is better to use the PW or AW measure of worth in step 3. If ROR is used, it requires the extra efforts of incremental analysis between alternatives. Example 18.1 illustrates sensitivity analysis for one project.

EXAMPLE 18.1

Wild Rice, Inc. expects to purchase a new asset for automated rice handling. Most likely estimates are a first cost of $80,000, zero salvage value, and a cash flow before taxes (CFBT) per year t that follows the relation $27,000 – 2000$t$. The MARR for the company varies over a wide range from 10% to 25% per year for different types of investments. The economic life of similar machinery varies from 8 to 12 years. Evaluate the sensitivity of PW by varying (*a*) MARR, while assuming a constant n value of 10 years, and (*b*) n, while MARR is constant at 15% per year. Perform the analysis by hand and by spreadsheet.

Solution by Hand

(*a*) Follow the procedure above to understand the sensitivity of PW to MARR variation.

1. MARR is the parameter of interest.
2. Select 5% increments to evaluate sensitivity to MARR; the range is 10% to 25%.
3. The measure of worth is PW.
4. Set up the PW relation for 10 years. When MARR = 10%,

$$\begin{aligned} \text{PW} &= -80{,}000 + 25{,}000(P/A,10\%,10) - 2000(P/G,10\%,10) \\ &= \$27{,}830 \end{aligned}$$

The PW for all MARR values at 5% intervals is as follows:

MARR, %	PW, $
10	27,830
15	11,512
20	−962
25	−10,711

5. A plot of MARR versus PW is shown in Figure 18–1. The steep negative slope indicates that the decision to accept the proposal based on PW is quite sensitive to variations in the MARR. If the MARR is established at the upper end of the range, the investment is not attractive.

(*b*) 1. Asset life n is the parameter.
2. Select 2-year increments to evaluate PW sensitivity over the range 8 to 12 years.
3. The measure of worth is PW.
4. Set up the same PW relation as in part (*a*) at i = 15%. The PW results are

n	PW, $
8	7,221
10	11,511
12	13,145

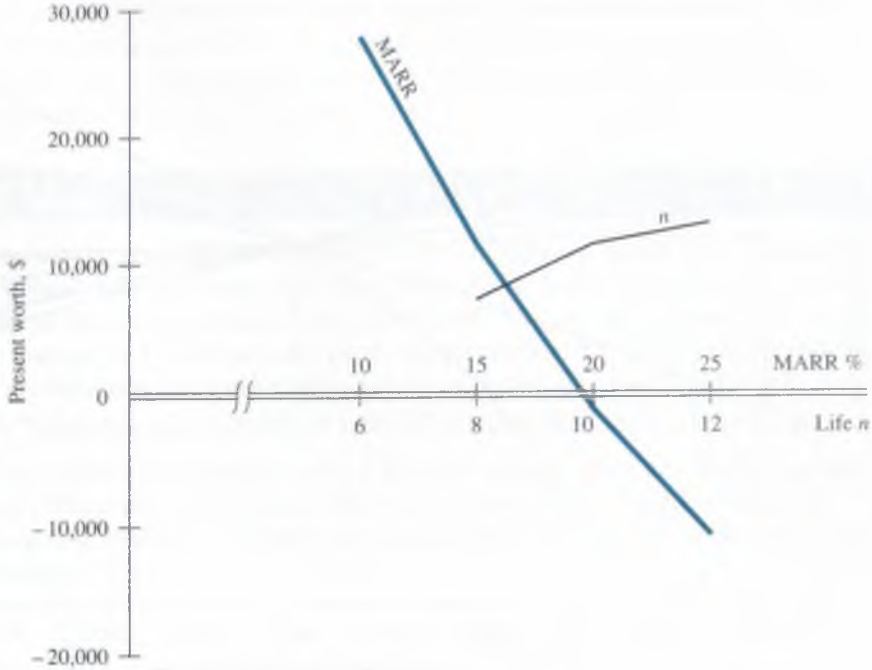


Figure 18–1 Plot of PW versus MARR and n for sensitivity analysis, Example 18.1.

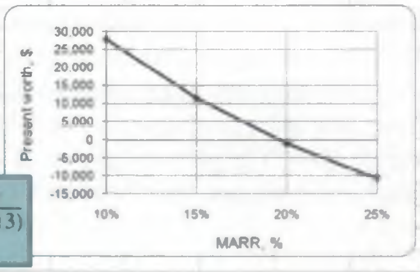
5. Figure 18–1 presents the plot of PW versus n. Since the PW measure is positive for all values of n, the decision to invest is not materially affected by the estimated life. The PW curve levels out above $n = 10$. This insensitivity to changes in cash flow in the distant future is a predictable observation, because the P/F factor gets smaller as n increases.

Solution by Spreadsheet

Figure 18–2 presents two spreadsheets and accompanying plots of PW versus MARR (fixed n) and PW versus n (fixed MARR). The NPV function calculates PW for i values from 10% to 25% and n values from 8 to 12 years. As the solution by hand indicated, so do the charts; PW is sensitive to changes in MARR values, but not very sensitive to variations in n.

Year	Cash flow for n = 10	MARR	PW, $
0	-80,000	10%	27,831
1	25,000	15%	11,510
2	23,000	20%	-962
3	21,000	25%	-10,712
4	19,000		
5	17,000		
6	15,000		
7	13,000		
8	11,000		
9	9,000		
10	7,000		

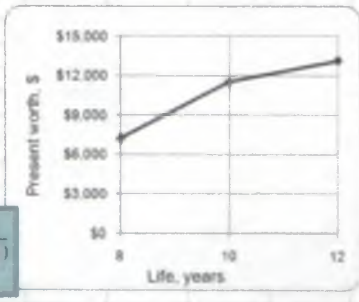
PW computation
= NPV(C6,B$4:B$13)+B$3



(a)

Year	Cash flow for n = 12	Life, n	PW, $
0	-80,000		
1	25,000	8	7,222
2	23,000	10	11,510
3	21,000	12	13,146
4	19,000		
5	17,000		
6	15,000		
7	13,000		
8	11,000		
9	9,000		
10	7,000		
11	5,000		
12	3,000		

PW computation
= NPV(15%,B$4:B$15)+B$3



(b)

Figure 18–2 Sensitivity analysis of PW to variation in (a) MARR values and (b) life estimates, Example 18.1.

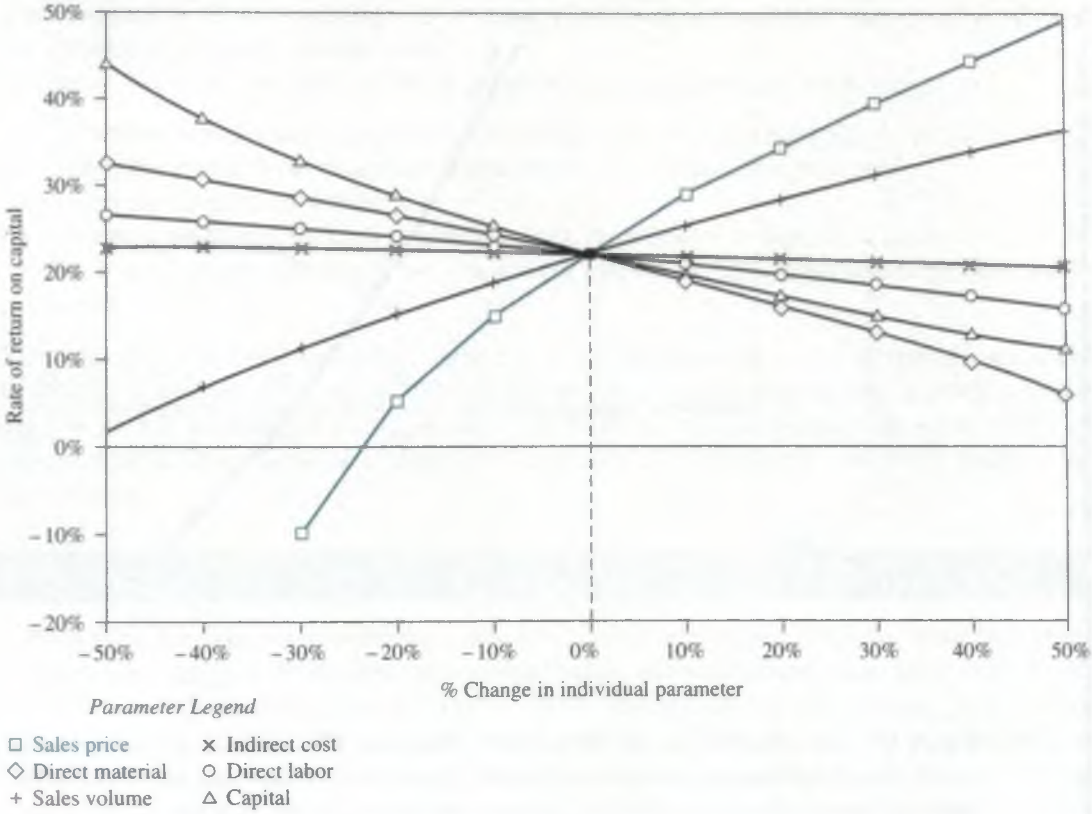


Figure 18–3 Sensitivity analysis graph of percent variation from the most likely estimate.

When the sensitivity of *several parameters* is considered for *one alternative* using a *single measure of worth,* it is helpful to **graph percentage change** for each parameter versus the measure of worth. This is sometimes called a **spider graph.** Figure 18–3 illustrates ROR versus six different parameters for one alternative. The variation in each parameter is indicated as a percentage deviation from the most likely estimate on the horizontal axis. If the ROR response curve is flat and approaches horizontal over the range of total variation graphed for a parameter, there is little sensitivity of ROR to changes in the parameter's value. This is the conclusion for indirect cost in Figure 18–3. On the other hand, ROR is very sensitive to sales price. A reduction of 30% from the expected sales price reduces the ROR from approximately 20% to −10%, whereas a 10% increase in price raises the ROR to about 30%.

If *two alternatives* are compared and the sensitivity to *one parameter* is sought, the graph may show quite nonlinear results. Observe the general shape of the sample sensitivity graphs in Figure 18–4. The plots are shown as linear segments between specific computation points. The graph indicates that the PW of each plan is a nonlinear function of hours of operation. Plan A is

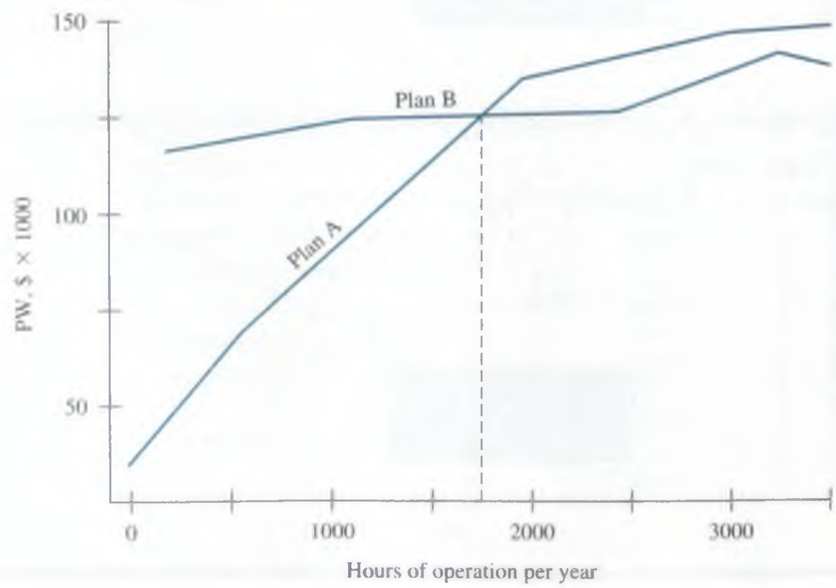


Figure 18–4 Sample PW sensitivity to hours of operation for two alternatives.

very sensitive in the range of 0 to 2000 hours, but it is comparatively insensitive above 2000 hours. Plan B is more attractive due to its relative insensitivity. The breakeven point is at about 1750 hours per year. It may be necessary to plot the measure of worth at intermediate points to better understand the nature of the sensitivity.

EXAMPLE 18.2

Columbus, Ohio needs to resurface a 3-kilometer stretch of highway. Knobel Construction has proposed two methods of resurfacing. The first method is a concrete surface for a cost of \$1.5 million and an annual maintenance cost of \$10,000. The second method is an asphalt covering with a first cost of \$1 million and a yearly maintenance of \$50,000. However, Knobel requests that every third year the asphalt highway be touched up at a cost of \$75,000.

The city uses the interest rate on bonds, 6% on its last bond issue, as the discount rate.

(*a*) Determine the breakeven number of years of the two methods. If the city expects an interstate to replace this stretch of highway in 10 years, which method should be selected?

(*b*) If the touch-up cost increases by \$5000 per kilometer every 3 years, is the decision sensitive to this increase?

Solution

(*a*) Use PW analysis to determine the breakeven n value.

$$\text{PW of concrete} = \text{PW of asphalt}$$

$$-1{,}500{,}000 - 10{,}000(P/A{,}6\%{,}n) = -1{,}000{,}000 - 50{,}000(P/A{,}6\%{,}n) - 75{,}000\left[\sum_j (P/F{,}6\%{,}j)\right]$$

where $j = 3, 6, 9, \ldots, n$. The relation can be rewritten to reflect the incremental cash flows.

$$-500{,}000 + 40{,}000(P/A{,}6\%{,}n) + 75{,}000\left[\sum_j (P/F{,}6\%{,}j)\right] = 0 \qquad [18.1]$$

The breakeven n value can be determined by hand solution by increasing n until Equation [18.1] switches from negative to positive PW values. Alternatively, a spreadsheet solution using the NPV function can find the breakeven n value (Figure 18–5). The NPV functions in column C are the same each year, except that the cash flows are extended 1 year for each

	Part (a)		Part (b)	
Year, n	Incremental cash flow, \$	PW for n years, \$	Incremental cash flow, \$	PW for n years, \$
0	-500,000		-500,000	
1	40,000	-462,264	40,000	-462,264
2	40,000	-426,664	40,000	-426,664
3	115,000	-330,108	115,000	-330,108
4	40,000	-298,424	40,000	-298,424
5	40,000	-268,534	40,000	-268,534
6	115,000	-187,464	130,000	-176,889
7	40,000	-160,861	40,000	-150,287
8	40,000	-135,765	40,000	-125,190
9	115,000	-67,696	145,000	-39,365
10	40,000	-45,361	40,000	**-17,029**
11	40,000	**-24,289**	40,000	**4,042**
12	115,000	**32,862**	160,000	83,557
13	40,000	51,616	40,000	102,311
14	40,000	69,308	40,000	120,003
15	115,000	117,293	175,000	193,024
16	40,000	133,039	40,000	208,770

PW for 12 years
= NPV(6%,\$B\$5:\$B16)+\$B\$4

PW for 11 years
= NPV(6%,\$B\$5:\$B15)+\$B\$4

Year counter advances by 1

Figure 18–5
Sensitivity of the breakeven life between two alternatives, Example 18.2.

present worth calculation. At approximately $n = 11.4$ years, concrete and asphalt resurfacing break even economically. Since the road is needed for 10 more years, the extra cost of concrete is not justified; select the asphalt alternative.

(*b*) The total touch-up cost will increase by \$15,000 every 3 years. Equation [18.1] is now

$$-500{,}000 + 40{,}000(P/A,6\%,n) + \left[75{,}000 + 15{,}000\left(\frac{j-3}{3}\right)\right]\left[\sum_j (P/F,6\%,j)\right] = 0$$

Now the breakeven n value is between 10 and 11 years—10.8 years using linear interpolation (Figure 18–5, column E). The decision has become marginal for asphalt, since the interstate is planned for 10 years hence.

Noneconomic considerations may be used to determine if asphalt is still the better alternative. One conclusion is that the asphalt decision becomes more questionable as the asphalt alternative maintenance costs increase; that is, the PW value is sensitive to increasing touch-up costs.

18.2 Sensitivity Analysis Using Three Estimates

We can thoroughly examine the economic advantages and disadvantages among two or more alternatives by borrowing from the field of project scheduling the concept of making three estimates for each parameter: **a pessimistic, a most likely, and an optimistic estimate.** Depending upon the nature of a parameter, the pessimistic estimate may be the lowest value (alternative life is an example) or the largest value (such as asset first cost).

This approach allows us to study measure of worth and alternative selection sensitivity within a predicted range of variation for each parameter. Usually the most likely estimate is used for all other parameters when the measure of worth is calculated for one particular parameter or one alternative.

EXAMPLE 18.3

An engineer is evaluating three alternatives for new equipment at Emerson Electronics. She has made three estimates for the salvage value, annual operating cost, and life. The estimates are presented on an alternative-by-alternative basis in Table 18–1. For example, alternative B has pessimistic estimates of $S = \$500$, AOC $= \$-4000$, and $n = 2$ years. The first costs are known, so they have the same value. Perform a sensitivity analysis and determine the most economical alternative, using AW analysis at a MARR of 12% per year.

TABLE 18–1 Competing Alternatives with Three Estimates Made for Salvage Value, AOC, and Life Parameters

Strategy		First Cost, \$	Salvage Value S, \$	AOC, \$ per Year	Life *n*, Years
Alternative A					
	P	−20,000	0	−11,000	3
Estimates	ML	−20,000	0	−9,000	5
	O	−20,000	0	−5,000	8
Alternative B					
	P	−15,000	500	−4,000	2
Estimates	ML	−15,000	1,000	−3,500	4
	O	−15,000	2,000	−2,000	7
Alternative C					
	P	−30,000	3,000	−8,000	3
Estimates	ML	−30,000	3,000	−7,000	7
	O	−30,000	3,000	−3,500	9

P = pessimistic; ML = most likely; O = optimistic.

TABLE 18–2 Annual Worth Values, Example 18.3

	Alternative AW Values, $		
Estimates	A	B	C
P	−19,327	−12,640	−19,601
ML	−14,548	−8,229	−13,276
O	−9,026	−5,089	−8,927

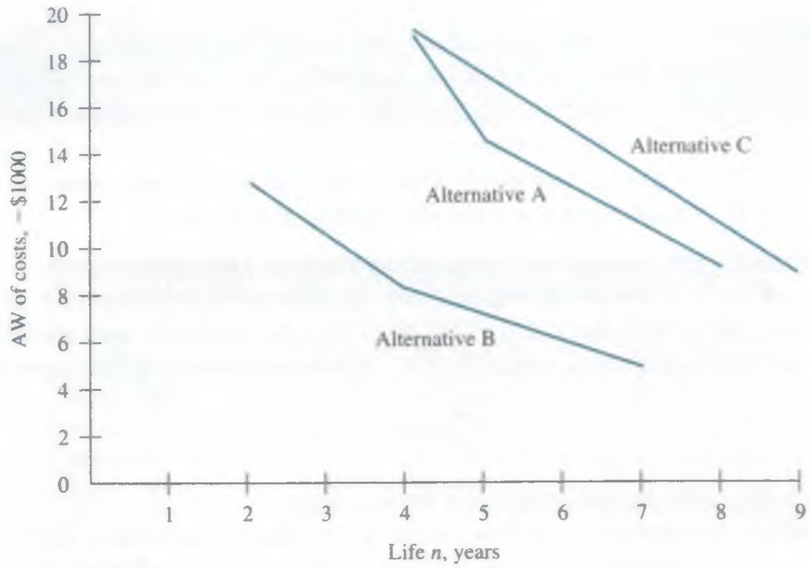


Figure 18–6
Plot of AW of costs for different-life estimates, Example 18.3.

Solution

For each alternative in Table 18–1, calculate the AW value of costs. For example, the AW relation for alternative A, pessimistic estimates, is

$$AW = -20{,}000(A/P,12\%,3) - 11{,}000 = \$-19{,}327$$

Table 18–2 presents all AW values. Figure 18–6 is a plot of AW versus the three estimates of *life* for each alternative. Since the AW calculated using the ML estimates for alternative B ($–8229) is economically better than even the optimistic AW value for alternatives A and C, alternative B is clearly favored.

Comment

While the alternative that should be selected here is quite obvious, this is not normally the case. For example, in Table 18–2, if the pessimistic alternative B equivalent AW were much higher, say, $−21,000 per year (rather than $−12,640), and the optimistic AW values for alternatives A and C were less than that for B ($–5089), the choice of B would not be apparent or correct. In this case, it would be necessary to select one set of estimates (P, ML, or O) upon which to base the decision. Alternatively, the different estimates can be used in an expected value analysis, which is introduced next.

18.3 Estimate Variability and the Expected Value ● ● ●

Online Presentation

Engineers and economic analysts usually deal with estimate variation and risk about an uncertain future by placing appropriate reliance on past data, if any exist. This means that *probability* and *samples* are used. Actually the use of probabilistic analysis is not as common as might be expected. The reason is not that the computations are difficult to perform or understand, but that realistic probabilities associated with cash flow estimates are difficult to assign. Experience and judgment can often be used in conjunction with probabilities and expected values to evaluate the desirability of an alternative.

The **expected value** can be interpreted as a long-run average observable if the project is repeated many times. Since a particular alternative is evaluated or implemented only once, the expected value results in a **point estimate.** However, even for a single occurrence, the expected value is a meaningful number.

The expected value $E(X)$ is computed using the relation

$$E(X) = \sum_{i=1}^{i=m} X_i P(X_i) \qquad [18.2]$$

where X_i = value of the variable X for i from 1 to m different values
$P(X_i)$ = probability that a specific value of X will occur

Probabilities are always correctly stated in decimal form, but they are routinely spoken of in percentages and often referred to as **chance**, such as *the chances are about 10%.* When placing the probability value in Equation [18.2] or any other relation, use the decimal equivalent of 10%, that is, 0.1. In all probability statements the $P(X_i)$ values for a variable X must total to 1.0.

$$\sum_{i=1}^{i=m} P(X_i) = 1.0$$

We may frequently omit the subscript i on X for simplicity.

If X represents the estimated cash flows, some will be positive and others will be negative. If a cash flow sequence includes revenues and costs, and the measure of worth is present worth calculated at the MARR, the result is the expected value of the discounted cash flows $E(\text{PW})$. If the expected value is negative, the overall outcome is expected to be a cash outflow. For example, if $E(\text{PW}) = \$-1500$, this indicates that the proposal is not *expected* to return the MARR.

EXAMPLE 18.4

ANA airlines plans to offer several new electronic services on flights between Tokyo and selected European destinations. The marketing director estimates that for a typical 24-hour period there is a 50% chance of having a net cash flow of \$5000 and a 35% chance of \$10,000. He also estimates there is a small 5% chance of no cash flow and a 10% chance of a loss of \$1000, which is the estimated extra personnel and utility costs to offer the services. Determine the expected net cash flow.

Solution

Let NCF be the net cash flow in dollars, and let $P(\text{NCF})$ represent the associated probabilities. Using Equation [18.2],

$$E(\text{NCF}) = 5000(0.5) + 10{,}000(0.35) + 0(0.05) - 1000(0.1) = \$5900$$

Although the "no cash flow" possibility does not increase or decrease $E(\text{NCF})$, it is included because it makes the probability values sum to 1.0 and it makes the computation complete.

18.4 Expected Value Computations for Alternatives ● ● ●

The expected value computation $E(X)$ is utilized in a variety of ways. Two prime ways are to:

- Prepare information for use in an economic analysis.
- Evaluate the expected viability of a fully formulated alternative.

Example 18.5 illustrates the first situation, and Example 18.6 determines the expected PW when the entire cash flow series and its probabilities are estimated.

EXAMPLE 18.5

There are many government incentives to become more energy-efficient. Installing solar panels on homes, business buildings, and multiple-family dwellings is one of them. The owner pays a portion of the total installation costs, and the government agency pays the rest. Nichole works for the Department of Energy and is responsible for approving solar panel incentive payouts. She has exceeded the annual budgeted amount of \$50 million per year in each of the previous 2 years. Disappointed with this situation, Nichole and her boss decided to collect data to determine what size increase in annual budget the incentive program needs in the future. Over the last 36 months, the amount of average monthly payout and number of months are shown in Table 18–3. She categorized by level the monthly averages according to her experience with the program. Provided the same pattern continues, what is the expected value of the dollar increase in annual budget that is needed to meet the requests?

TABLE 18–3 Solar Panel Incentive Payouts, Example 18.5

Level	Average Payout, \$ Million per Month	Months over Past 3 Years
Very high	6.5	15
High	4.7	10
Moderate	3.2	7
Low	2.9	4

Solution

Use the 36 months of payouts PO_j (j = low, . . . ,very high) to estimate the probability $P(PO_j)$ for each level, and make sure the total is 1.0.

Level, j	Probability of Payout Level, $P(PO_j)$
Very high	$P(PO_1) = 15/36 = 0.417$
High	$P(PO_2) = 10/36 = 0.278$
Moderate	$P(PO_3) = 7/36 = 0.194$
Low	$P(PO_4) = 4/36 = 0.111$
	1.000

The expected monthly payout is calculated using Equation [18.2]. In \$ million units,

$$\begin{aligned} E[PO] &= 6.5(0.417) + 4.7(0.278) + 3.2(0.194) + 2.9(0.111) \\ &= 2.711 + 1.307 + 0.621 + 0.322 \\ &= \$4.961 \quad (\$4{,}961{,}000) \end{aligned}$$

The annual expected budget need is 12×4.961 million = \$59.532 million. The current budget of \$50 million should be increased by an average of \$9.532 million per year.

EXAMPLE 18.6

Lite-Weight Wheelchair Company has a substantial investment in tubular steel bending equipment. A new piece of equipment costs \$5000 and has a life of 3 years. Estimated cash flows (Table 18–4) depend on economic conditions classified as receding, stable, or expanding. A probability is estimated that each of the economic conditions will prevail during the 3-year period. Apply expected value and PW analysis to determine if the equipment should be purchased. Use a MARR of 15% per year.

TABLE 18–4 Equipment Cash Flow and Probabilities, Example 18.6

	Economic Condition		
	Receding (Prob. = 0.4)	Stable (Prob. = 0.4)	Expanding (Prob. = 0.2)
Year	Annual Cash Flow Estimates, $		
0	−5000	−5000	−5000
1	+2500	+2500	+2000
2	+2000	+2500	+3000
3	+1000	+2500	+3500

Solution

First determine the PW of the cash flows in Table 18–4 for each economic condition, and then calculate $E(\text{PW})$ using Equation [18.2]. Define subscripts R for receding economy, S for stable, and E for expanding. The PW values for the three scenarios are

$$\begin{aligned}\text{PW}_R &= -5000 + 2500(P/F,15\%,1) + 2000(P/F,15\%,2) + 1000(P/F,15\%,3)\\ &= -5000 + 4344 = \$-656\end{aligned}$$

$$\text{PW}_S = -5000 + 5708 = \$+708$$

$$\text{PW}_E = -5000 + 6309 = \$+1309$$

Only in a receding economy will the cash flows not return the 15% to justify the investment. The expected present worth is

$$\begin{aligned}E(\text{PW}) &= \sum_{j=R,S,E} \text{PW}_j[P(j)]\\ &= -656(0.4) + 708(0.4) + 1309(0.2)\\ &= \$283\end{aligned}$$

At 15%, $E(\text{PW}) > 0$; the equipment is justified, using an expected value analysis.

Comment

It is also correct to calculate the E(cash flow) for each year and then determine PW of the E(cash flow) series, because the PW computation is a linear function of cash flows. Computing E(cash flow) first may be easier in that it reduces the number of PW computations. In this example, calculate $E(\text{CF}_t)$ for each year, then determine $E(\text{PW})$.

$$\begin{aligned}E(\text{CF}_0) &= \$-5000\\ E(\text{CF}_1) &= 2500(0.4) + 2500(0.4) + 2000(0.2) = \$2400\\ E(\text{CF}_2) &= \$2400\\ E(\text{CF}_3) &= \$2100\end{aligned}$$

$$\begin{aligned}E(\text{PW}) &= -5000 + 2400(P/F,15\%,1) + 2400(P/F,15\%,2) + 2100(P/F,15\%,3)\\ &= \$283\end{aligned}$$

18.5 Staged Evaluation of Alternatives Using a Decision Tree ● ● ●

Alternative evaluation may require a series of decisions in which the outcome from one stage is important to the next stage of decision making. When each alternative is clearly defined and probability estimates can be made to **account for risk,** it is helpful to perform the evaluation using a **decision tree.**

A decision tree includes:

- More than one stage of alternative selection.
- Selection of an alternative at one stage that leads to another stage.
- Expected results from a decision at each stage.
- Probability estimates for each outcome.
- Estimates of economic value (cost or revenue) for each outcome.
- Measure of worth as the selection criterion, such as $E(\text{PW})$.

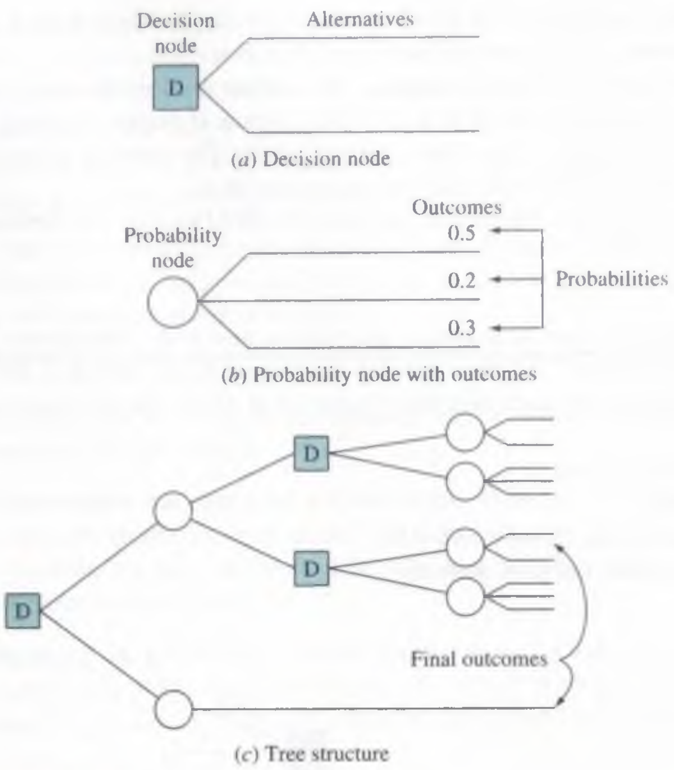


Figure 18–7
Decision and probability nodes used to construct a decision tree.

The decision tree is constructed left to right and includes each possible decision and outcome.

- A square represents a **decision node** with the possible alternatives indicated on the **branches** from the decision node (Figure 18–7*a*).
- A circle represents a **probability node** with the possible **outcomes** and estimated **probabilities** on the branches (Figure 18–7*b*).
- The treelike structure in Figure 18–7*c* results, with outcomes following a decision.

Usually each branch of a decision tree has some estimated economic value (often referred to as *payoff*) in cost, revenue, saving, or benefit. These cash flows are expressed in terms of PW, AW, or FW values and are shown to the right of each final outcome branch. The cash flow and probability estimates on each outcome branch are used in calculating the expected economic value of each decision branch. This process, called **solving the tree** or **rollback**, is explained after Example 18.7, which illustrates the construction of a decision tree.

EXAMPLE 18.7

Jerry Hill is president and CEO of a U.S.-based food processing company, Hill Products and Services. He was recently approached by an international supermarket chain that wants to market in-country its own brand of frozen microwaveable dinners. The offer made to Jerry by the supermarket corporation requires that a series of two decisions be made, now and 2 years hence. The current decision involves two alternatives: (1) *Lease* a facility in the United Arab Emirates (UAE) from the supermarket chain, which has agreed to convert a current processing facility for immediate use by Jerry's company; or (2) *build and own* a processing and packaging facility in the UAE. Possible outcomes of this first decision stage are good market or poor market depending upon the public's response.

The decision choices 2 years hence are dependent upon the lease-or-own decision made now. If Hill *decides to lease,* good market response means that the future decision alternatives are to produce at twice, equal to, or one-half of the original volume. This will be a mutual decision between the supermarket chain and Jerry's company. A poor market response will indicate a one-half level of production, or complete removal from the UAE market. Outcomes for the future decisions are, again, good and poor market responses.

As agreed by the supermarket company, the current decision for Jerry *to own* the facility will allow him to set the production level 2 years hence. If market response is good, the decision alternatives are four or two times original levels. The reaction to poor market response will be production at the same level or no production at all.

Construct the tree of decisions and outcomes for Hill Products and Services.

Solution

This is a two-stage decision tree that has alternatives now and 2 years hence. Identify the decision nodes and branches, and then develop the tree using the branches and the outcomes of good and poor market for each decision. Figure 18–8 details the decision stages and outcome branches.

Stage 1 (decision now):

Label it D1.

Alternatives: lease (L) and own (O).

Outcomes: good and poor markets.

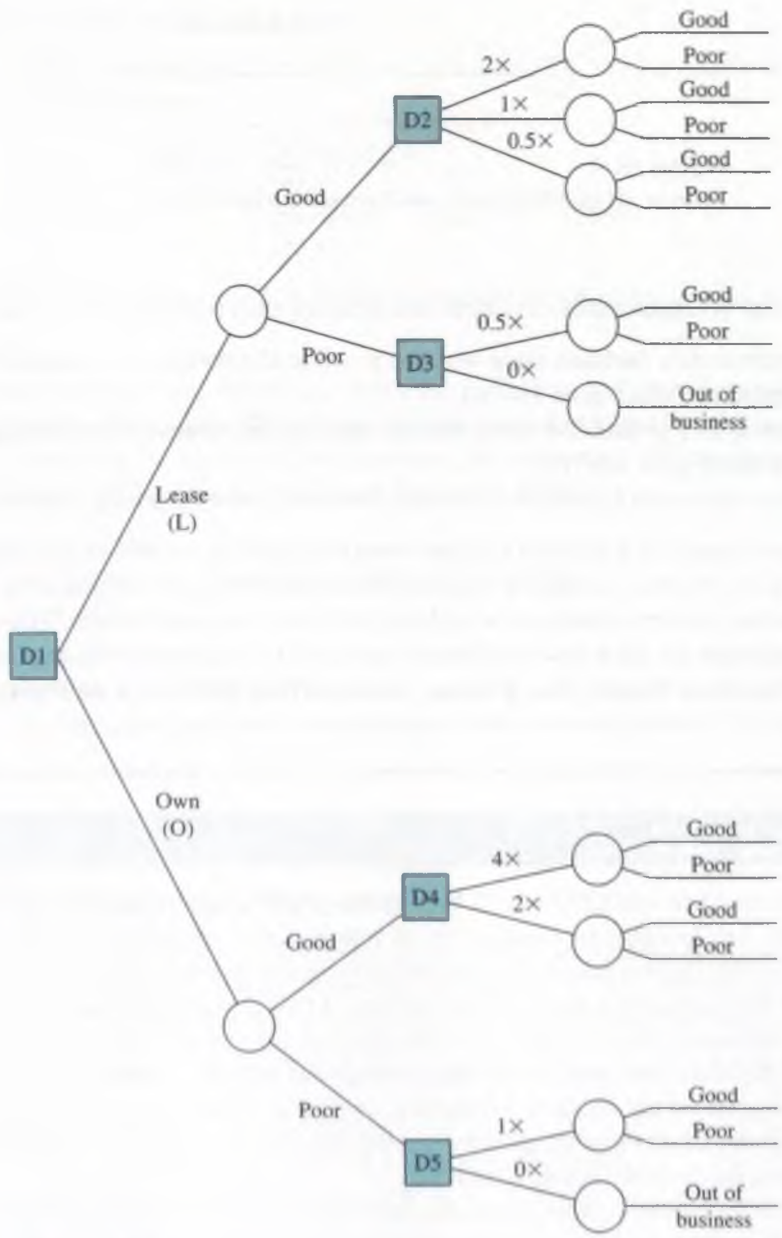


Figure 18–8

A two-stage decision tree identifying alternatives and possible outcomes.

Stage 2 (Decisions 2 years hence):
Label them D2 through D5.
Outcomes: good market, poor market, and out of business.

Choice of production levels for D2 through D5:
Quadruple production (4×); double production (2×); level production (1×); one-half production (0.5×); stop production (0×)

The alternatives for future production levels (D2 through D5) are added to the tree and followed by the market responses of good and poor. If the stop-production (0×) decision is made at D3 or D5, the only outcome is out of business.

To utilize the decision tree for alternative evaluation and selection, the following additional information is necessary for each branch:

- The **estimated probability** that each outcome may occur. These probabilities must sum to 1.0 for each set of outcomes (branches) that result from a decision.
- **Economic information** for each decision alternative and possible outcome, such as initial investment and estimated cash flows.

Decisions are made using the probability estimate and economic value estimate for each outcome branch. Commonly the present worth at the MARR is used in an expected value computation of the type in Equation [18.2]. This is the general procedure to solve the tree using PW analysis:

1. Start at the top right of the tree. Determine the PW value for each outcome branch considering the time value of money.
2. Calculate the expected value for each decision alternative.

$$E(\text{decision}) = \Sigma(\text{outcome estimate})P(\text{outcome}) \qquad [18.3]$$

where the summation is taken over all possible outcomes for each decision alternative.

3. At each decision node, select the best E(decision) value—minimum cost or maximum value (if both costs and revenues are estimated).
4. Continue moving to the left of the tree to the root decision in order to select the best alternative.
5. Trace the best decision path through the tree.

EXAMPLE 18.8

A decision is needed to either market or sell a new invention. If the product is marketed, the next decision is to take it international or national. Assume the details of the outcome branches result in the decision tree of Figure 18–9. The probabilities for each outcome and PW of CFBT (cash flow before taxes) are indicated. These payoffs are in millions of dollars. Determine the best decision at the decision node D1.

Solution

Use the procedure above to determine that the D1 decision alternative to sell the invention should maximize E(PW of CFBT).

1. Present worth of CFBT is supplied.
2. Calculate the expected PW for alternatives from nodes D2 and D3, using Equation [18.3]. In Figure 18–9, to the right of decision node D2, the expected values of 14 and 0.2 in ovals are determined as

$$E(\text{international decision}) = 12(0.5) + 16(0.5) = 14$$
$$E(\text{national decision}) = 4(0.4) - 3(0.4) - 1(0.2) = 0.2$$

The expected PW values of 4.2 and 2 for D3 are calculated in a similar fashion.

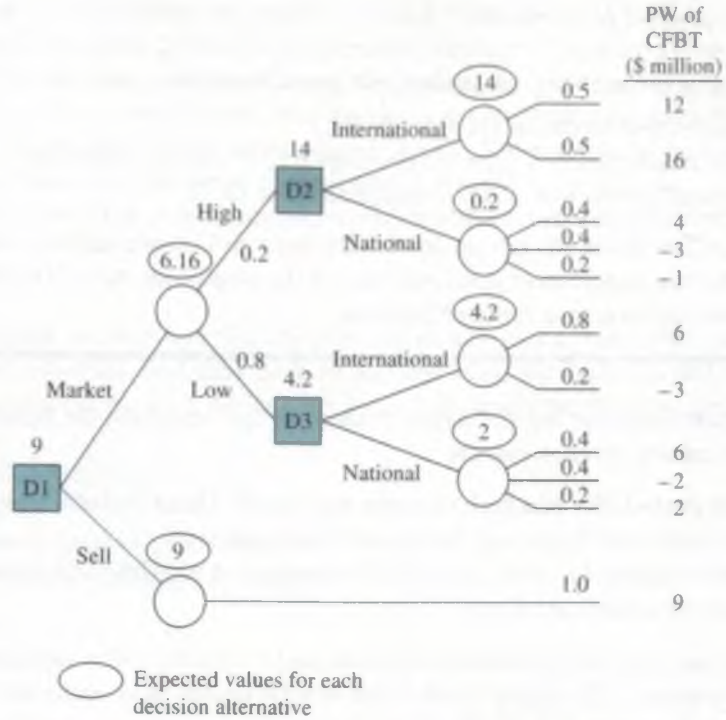


Figure 18–9

Solution of a decision tree with present worth of estimated CFBT values, Example 18.8.

3. Select the larger expected value at each decision node. These are 14 (international) at D2 and 4.2 (international) at D3.
4. Calculate the expected PW for the two D1 branches.

$$E(\text{market decision}) = 14(0.2) + 4.2(0.8) = 6.16$$
$$E(\text{sell decision}) = 9(1.0) = 9$$

The expected value for the sell decision is simple since the one outcome has a payoff of 9. The sell decision yields the larger expected PW of 9.

5. The largest expected PW of CFBT path is to select the sell branch at D1 for a guaranteed $9,000,000.

18.6 Real Options in Engineering Economics ● ● ●

As we learned with decision trees, many of the problems in engineering economy can be viewed as staged decisions. When the decision to invest more or less can be delayed into the future, the problem is called **staged funding.** As an illustration, assume a large company expects to sell an energy-saving, window-mounted residential air conditioning unit at the rate of 100,000 per month by the end of 2 years on the market. Decision makers may opt to (1) build the capacity to supply100,000 per month to market immediately or (2) build capacity to supply 25,000 per month now and test the market's receptivity. If positive, they can stage the increase by 25,000 additional units each 6 months to meet current demand. Of course, if an aggressive competitor enters the scene, or the economy falters, the staged funding decision will change as warranted. These alternatives provide time-based options to the company. Before we go further, some definitions are needed.

An **option** is a purchase or investment that contractually provides the privilege to take a stated action by some stated time in the future, or the right to not accept the offer and forfeit the option.

A **real option,** in engineering economy terms, is the investment (cost) in a project, process, or system. The options usually involve **physical (real)** assets, buildings, equipment, materials, and the like, thus, the word *real*. Options may also be leases, subcontracts, or franchises. The investment alternatives present varying amounts of **risk,** which is estimated by probabilities of occurrence for predictable future events.

Real options analysis is the application of techniques to determine the economic consequences of **delaying** the funding decisions as allowed by the option. The estimated cash flows and other consequences of these delays are analyzed with risk taken into account to the degree possible. A measure of worth, e.g., PW or AW, is the criterion used to make the staged funding decisions. A decision may be to expand, continue as is, contract, abandon, or replicate the alternative at the time the option must be exercised.

An inherent part of real options analysis is the uncertainty of future estimates, as it is for most economic analyses. After some illustrations of real options, we will discuss the probabilistic dimensions. Samples from industry and everyday personal life that can be formulated as real options follow.

Industrial Setting

New markets—Purchase equipment and staff to enter an expanding international market over the next 5 years.

New planes—Purchase commercial airplanes now with an option to buy an additional 5 planes over the next 3 years at the same price as that paid for the current order.

Removing car models—Ford Motor Company can decide to maintain production on an established car model with dwindling sales for the next 3 years or can opt to discontinue the model in stages over a 1- or 2-year period.

Drilling lease—Buy a drilling option contract from landowners to drill for oil and gas at some time in the next 10 years. The drilling may not be justified at this time, but the contract offers the option to drill were it to become economically advantageous based on events such as increased oil prices or improved recovery technology.

Personal Decision Making

Extended car warranty—When purchasing a new car, the option to buy an extended-coverage warranty beyond the manufacturer's warranty is always an option. The price of the option is the cost of the extended warranty. The uncertainties and risks are the future unknown costs for repairs and failed components.

House insurance—When a homeowner has no mortgage to pay, maintaining house insurance is an option. Deductibles are high enough, e.g., 1% to 5% of the fully appraised value, that insurance primarily covers only catastrophic damage to the structure. Self-insurance, where money is set aside for potential damages while accepting the risk that a major event will take place, is an option for the homeowner.

Some of the primary characteristics (with an example) of a real options analysis performed within the context of engineering economics are as follows:

- **Cost** to obtain the option to delay a decision (PW of initial investment, lease cost, or future investment amount).
- Anticipated future **options** and **cash flow** estimates (double production with annual net cash flows estimated).
- **Time period** for follow-on decisions (staged decision time, such as 1 year or a 3-year test period).
- Market and risk-free **interest rates** (expected market MARR of 12% per year and inflation rate estimate of 4% per year).
- Estimates of risk and future **uncertainty** for each option (probability that an estimated cash flow series will actually occur, if a specific option is selected).
- **Economic criterion** used to make a decision (PW, ROR, or other measure of worth).

It is common to use a decision tree to record and understand the options prior to performing a real options analysis with risk included. Example 18.9 demonstrates the use of a decision tree and PW analysis.

EXAMPLE 18.9

A start-up company in the solar energy production business, SolarScale Energy, Inc., has developed and field-tested a modularized, scalar solar thermal electric (STE) generation system that is relatively inexpensive to purchase and has an efficiency considerably better than traditional photovoltaic (PV) panels. The technology is promising enough that Capital Investor Funds (CIF) has provided \$10 million for manufacturing. Additionally, a contract with a consortium of sunbelt states has been offered, but not accepted thus far, for a total of \$1.5 million per year for a 2-year test period. By contract, the units will be marketed through the state energy departments with all revenue going to the state treasuries. The lead engineer at SolarScale, the manager of CIF, and a conservation representative for the state consortium have developed the following staged-funding options, based on the delayed decision to increase manufacturing production level until preliminary results of the 2-year contract are in hand.

Condition	Option	CIF Funding	Consortium Contract
Sales are excellent (> 5000 units/year)	2 × production level	Additional \$10 million in year 2	Additional 8 years; \$4 million in years 3–10
Sales are excellent (3000–5000 units/year)	1 × production level	Nothing; no salvage after 10 years	Additional 8 years; \$1.5 million in years 3–10
Sales are poor (2000–3000 units/year)	Reduce to ½ × production	Nothing; sell for \$2.5 million after 5 years	Additional 3 years; \$1.5 million in years 3–5
Sales are poor (< 2000 units/year)	Stop after 2 years	Nothing; sell for \$5 million after 2 years	Nothing

(*a*) Develop the two-stage decision tree for the options described.

(*b*) The base case is the 1 × production level with the 8-year follow-on contract from the consortium. If the estimates for this option are considered the most likely (expected value) estimates, determine the present worth at a MARR of 10% per year.

(*c*) Determine the PW values for each possible final outcome at 10% per year, and identify the best economic option when the stage 2 funding decision must be made.

Solution

(*a*) Figure 18–10 details the options with the year shown at the bottom. There are 2 outcome branches initially (accept option; decline option) and four final branches for the accept decision at D1, based upon sales level. The decline option has a \$0 outcome. (SolarScale has other ways to pursue revenue that are not represented in this abbreviated example.)

(*b*) Perform a PW evaluation as we have in all previous chapters, assuming the estimates are point estimates over the 10-year life of the project. The resulting $PW_{1\times} < 0$ as shown below indicates the contract is not justified economically. In \$ millions,

$$\begin{aligned} PW_{1\times} &= -10 + 1.5(P/A,10\%,10) \\ &= \$-0.78 \quad (\$-780{,}000) \end{aligned}$$

(*c*) Figure 18–11 is a spreadsheet screen shot that calculates the PW for each option using the NPV function. The i^* values are also shown using the IRR function. Note that the sale of production assets after 5 years for \$2.5 million (½ × level) or after 2 years for \$5 million (stop) is included. Also, the extra \$10 million investment in year 2 for the 2 × level option is included.

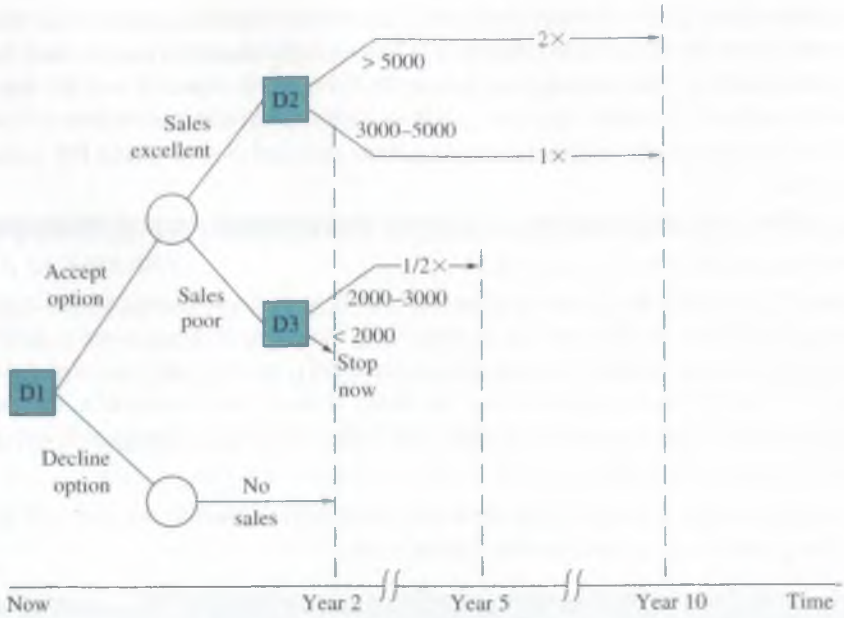


Figure 18–10
Decision tree showing real options over 10-year period, Example 18.9.

	A	B	C	D	E	F
1		Net cash flow estimates, $ million per year				
2	Year	2X level	1X level	1/2X level	Stop	
3	0	-10	-10	-10	-10	
4	1	1.5	1.5	1.5	1.5	
5	2	-8.5	1.5	1.5	6.5	
6	3	4.0	1.5	1.5		
7	4	4.0	1.5	1.5		
8	5	4.0	1.5	4.0		
9	6	4.0	1.5			
10	7	4.0	1.5		Sale of assets in this year	
11	8	4.0	1.5			
12	9	4.0	1.5			
13	10	4.0	1.5			
14	i*	12.6%	8.1%	0.0%	-11.5%	
15	PW @ 10%	1.97	-0.78	-2.76	-3.26	

Figure 18–11
PW analysis of real options without risk considered, Example 18.9.

Only the 2 × production-level option is justified at MARR = 10% per year. If SolarScale and CIF, the financial backers, are not convinced that the sales level will exceed 5000 units per year, the contract option should be declined. Some marketing survey information and risk analysis may be very helpful before this important decision is made.

Of the characteristics listed above for real option situations, the primary one absent in Example 18.9 is that of estimate variation and some measure of **risk.** Decision making under risk is covered more extensively in the next chapter; however, we can use the following definition for this discussion of real options analysis.

When a parameter can take on more than one value and there is any estimate of **chance** or **probability** about the opportunity that each value may be observed, risk is present.

Risk

A coin has two sides. If it is a perfectly balanced coin, a flip of the coin should result in heads 50% of the time and tails 50% of the time. If the coin is intentionally biased in weight, such that 58% of the time it lands heads up, then the long-run probability of heads is $P(\text{heads}) = 0.58$. Since the sum of probabilities across all possible values must add to 1, the biased coin has $P(\text{tails}) = 0.42$.

There are a couple of points worth mentioning about risk and the calculated PW values for real options analysis.

- When the risk is higher and the stakes are larger, the real options analysis is often more valuable. (We shall see this in Example 18.10.)
- Since one of the objectives of real options analysis is to evaluate the economic consequences of delaying a decision, a PW value of the base case that is *moderately positive* means that the project is justified and should be accepted immediately, without the decision delay. On the other hand, if the PW is *largely negative,* the delay is likely not worthwhile, as it would take a very large positive PW to result in $E(\text{PW}) > 0$. Thus, the project should be rejected now, not delayed for a future decision.

We return to the previous example with some risk assessment added to determine if the contract option should be declined, as indicated by the base case.

EXAMPLE 18.10

Before the stage 1 decision is made in Example 18.9 about SolarScale's state-consortium contract offer, some expected sales information was collected. Once the results were reviewed by the three individuals in attendance at the final decision-making meeting—one representative each for SolarScale, CIF, and the consortium—each person recorded her or his estimated probability as a measure of risk that the sales would be excellent or poor, which represents the two outcome possibilities were the option accepted. The results are as follows:

	Probability of Outcome	
	Excellent	**Poor**
SolarScale	0.5	0.5
CIF	0.8	0.2
Consortium	0.6	0.4

Use these probability estimates to determine the expected PW value, provided equal weighting is given to each representative's input.

Solution

For each outcome (excellent and poor), select the best PW value from Figure 18–11, and then find $E(\text{PW})$ for each representative.

Excellent: From 2 × level and 1 × level, select 2 × with PW = \$1.97 million.

Poor: From ½ × level and stop now, select ½ × with PW = \$−2.76 million.

In \$ million, $E(\text{PW})$ for each organization is

$$E(\text{PW for SolarScale}) = 1.97(0.5) - 2.76(0.5) = \$-0.40$$

$$E(\text{PW for CIF}) = 1.97(0.8) - 2.76(0.2) = \$1.02$$

$$E(\text{PW for consortium}) = 1.97(0.6) - 2.76(0.4) = \$0.08$$

With a 1/3 chance assigned to each representative, the overall E(PW of stage 2 decision) is

$$\begin{aligned} E(\text{PW of stage 2 decision}) &= 0.33(-0.40 + 1.02 + 0.08) \\ &= \$0.23 \quad (\$230{,}000) \end{aligned}$$

The base case of 1 × level production in part (*b*) of Example 18.9 resulted in $E(\text{PW}) =$ \$−780,000. When compared with the positive $E(\text{PW})$ result here, we see that with consideration of the different options of production level and probabilities for sales level, the expected PW has increased to a positive value. All other things being equal, the state consortium offer should be accepted; that is, accept the real option of the contract.

There are many other examples and dimensions of real options analysis in engineering economics and in the area of financial analysis, where options analysis got its start some years ago. If you are interested in this new and interesting area of analysis, consult more advanced texts and journal articles on the topic of real options.

CHAPTER SUMMARY

In this chapter the emphasis is on sensitivity to variation in one or more parameters using a specific measure of worth. When two alternatives are compared, compute and graph the measure of worth for different values of the parameter to determine when each alternative is better.

When several parameters are expected to vary over a predictable range, the measure of worth is plotted and calculated using three estimates for a parameter—most likely, pessimistic, and optimistic. This approach can help determine which alternative is best among several. Independence between parameters is assumed in all these analyses.

The combination of parameter and probability estimates results in the expected value relation

$$E(X) = \Sigma XP(X)$$

This expression is also used to calculate E(revenue), E(cost), E(cash flow), and E(PW) for the entire cash flow sequence of an alternative.

Decision trees are used to make a series of alternative selections. This is a way to explicitly take risk into account. It is necessary to make several types of estimates for a decision tree: outcomes for each possible decision, cash flows, and probabilities. Expected value computations are coupled with those for the measure of worth to solve the tree and find the best alternatives stage by stage.

Staged funding over time can be approached using the evolving area of real options. Delaying an investment decision and considering the risks of the future can improve the overall E(PW) of a project, process, or system.

PROBLEMS

Sensitivity to Parameter Variation

18.1 Kahn Instruments is considering an investment of $500,000 in a new product line. The company will make the investment only if it will result in a rate of return of 15% per year or higher. If the revenue is expected to be between $135,000 and $165,000 per year for 5 years, determine if the decision to invest is sensitive to the projected range of income using a present worth analysis.

18.2 A young couple planning ahead for their retirement has decided that $2,600,000 is the amount they will need to retire comfortably 20 years from now. For the past 5 years they have been able to invest one of their salaries ($50,000 per year, which includes employer contributions) while living off the other one. They plan to start a family sometime in the next 10 years, and when they have their first child, one of the parents will quit working, causing the savings to decrease to $15,000 per year thereafter. If they have gotten a rate of return of 10% per year on their investments and expect to continue at this ROR, is reaching their goal of $2.6 million 20 years from now sensitive to when they have their first child (i.e., between now and 10 years from now)? Use an FW analysis.

18.3 A company that manufactures high-speed submersible rotary indexing spindles is considering upgrading the production equipment to reduce costs over a 6-year planning horizon. The company can invest $80,000 now, 1 year from now, or 2 years from now. Depending on when the investment is made, the savings will vary. That is, the savings will be $25,000, $26,000, or $29,000 per year if the investment is made now (year 0), in 1 year, or in 2 years, respectively. Will the timing of the investment affect the request to make at least a 20% per year return? Use future worth analysis.

18.4 Different membrane systems are under consideration for treating 3 million gallons per day (MGD) of cooling tower blowdown water to reduce its volume. Option 1 is a low-pressure seawater reverse osmosis (SWRO) system that will operate at 500 psi with a fixed cost of $465 per day and an operating cost of $0.67 per 1000 gallons. A second option is a higher-pressure SWRO system offered by vendor X that operates at 800 psi and will have a lower fixed cost of $328 per day (because of fewer membranes); however, its operating cost will be $1.35 per 1000 gallons. A third option is also a high-pressure SWRO system from vendor Y, who claims that its system will have a lower operating cost of $1.28 per 1000 gallons and the same fixed cost as that of vendor X. Determine if the selection of a low- or high-pressure system is dependent on the lower operating cost offered by vendor Y.

18.5 A machine that is currently used in manufacturing circuit board card locks has AW = $–63,000 per year. A possible replacement is under consideration with a first cost of $64,000 and an operating cost of $38,000 per year for the next 3 years. Three different engineers have given their opinion, about what the salvage value of the new machine will be 3 years from now: $10,000, $13,000, and $18,000. Is the decision to replace the machine sensitive to the salvage value estimates at the company's MARR of 15% per year?

18.6 An equipment alternative is being economically evaluated separately by three engineers at Raytheon. The first cost will be $77,000, and the life is estimated at 6 years with a salvage value of $10,000. The engineers disagree, however, on the estimated revenue the equipment will generate. Joe has made an estimate of $10,000 per year. Jane states that this is too low and estimates $14,000, while Carlos estimates $18,000 per year. If the before-tax MARR is 8% per year, use PW to determine if these different estimates will change the decision to purchase the equipment.

18.7 The owner of a small construction company is planning to purchase specialized equipment to complete a contract he just received. The first cost of the equipment is $250,000, and it will likely have a salvage value of $90,000 in 3 years, at which time he will not need the equipment anymore. The operating cost is expected to be $75,000 per year. Alternatively, the owner can subcontract the work for $175,000 per year. Because the equipment is specialized, the owner is not sure about the salvage value. He thinks it might be worth as little as $10,000 in 3 years (a scrap value). If his minimum attractive rate of return is 15% per year, determine if the decision to buy the equipment is sensitive to the salvage value.

18.8 A company planning to borrow $10.5 million for a plant expansion is not sure what the interest rate will be when it applies for the loan. The rate could be as low as 10% per year or as high as 12% per year for a 5-year loan. The company will only move forward with the project if the annual worth of the expansion is below $5.7 million. The M&O cost is fixed at $3.1 million per year. The salvage could be $2 million if the interest rate is 10% or $2.5 million if it is 12% per year. Is the decision to move forward with the project sensitive to the interest rate and salvage value estimates?

18.9 A company that manufactures clear PVC pipe is investigating the production options of batch and continuous processes. Estimated cash flows are as follows:

	Batch	Continuous
First cost, $	−80,000	−130,000
Annual cost, $ per year	−55,000	−30,000
Salvage value for any year, $	10,000	40,000
Life, years	3–10	5

The chief operating officer (COO) has asked you to determine if the batch option would ever have a lower annual worth than the continuous flow system, using interest rates over a range of 5% to 15% for the batch option but only 15% for the continuous flow system. (*Note*: The continuous flow process was previously determined to have its lowest cost over a 5-year life cycle; the batch process can be used from 3 to 10 years.)

18.10 An engineer collected average cost and revenue data for Arenson's FC1 handheld financial calculator.

Fixed cost = $300,000 per year

Cost per unit = $40

Revenue per unit = $70

(*a*) What is the range in breakeven quantity if there is possible variation in the fixed cost from $200,000 to $400,000 per year? (Use $50,000 increments.)

(*b*) What is the incremental change in the breakeven quantity for each $50,000 change in fixed cost?

The following information is used for Problems 18.11 through 18.14.

A new online patient diagnostics system for surgeons will cost $200,000 to install, cost $5000 annually to maintain and will have an expected life of 5 years. The added revenue is estimated to be $60,000 per year, and the MARR is 10% per year. Examine the sensitivity of present worth to variation in selected parameter estimates, while others remain constant.

18.11 Sensitivity to first cost variation: $150,000 to $250,000 (−25% to +25%).

18.12 Sensitivity to revenue variation: $45,000 to $75,000 (−25% to +25%).

18.13 Sensitivity to life variation: 4 years to 7 years (−20% to +40%).

18.14 Plot the results on a graph similar to Figure 18–3 and comment on the relative sensitivity of each parameter.

18.15 Charlene plans to place an annual savings amount of A = $27,185 into a retirement program at the end of each year for 20 years starting next year. She expects to retire and start to draw a total of R = $60,000 per year 1 year after the 20th deposit. Assume an effective earning rate of $i = 6\%$ per year on the retirement investments and an infinite life. Determine and comment on the sensitivity of the size of the annual withdrawal R for variations in A and i. Show hand and spreadsheet solutions.

(*a*) Variation of ±5% in the annual deposit A.
(*b*) Variation of ±1% in the effective earning rate i, that is, ranging from 5% to 7% per year.

18.16 Ned Thompson Labs performs tests on super alloys, titanium, aluminum, and most metals. Tests on metal composites that rely upon scanning electron microscope results can be subcontracted, or the labs can purchase new equipment. Evaluate the sensitivity of the economic decision to purchase the equipment over a range of ±20% (in 10% increments) of the estimates for P, AOC, R, n, and MARR (range on MARR is 12% to 16%). Use the AW method and plot the results on a sensitivity graph (like Figure 18–3). For which parameter(s) is the AW most sensitive? Least sensitive?

First cost P = $−220,000
Salvage S = $20,000
Life n = 10 years
Annual operating cost AOC = $−30,000/year
Annual revenue R = $70,000 per year
MARR i = 15% per year

18.17 Titan manufactures and sells gas-powered electricity generators. It can purchase a new line of fuel injectors from either of two companies. Cost and savings estimates are made, but the savings estimate is unreliable at this time. Use an AW analysis at 10% per year to determine if the selection between company A and company B changes when the savings per year may vary as much as 40% from the best estimates made thus far.

	Company A	Company B
First cost, $	−50,000	−37,500
AOC, $ per year	−7,500	−8,000
Savings best estimate, $ per year	15,000	13,000
Salvage, $	5,000	3,700
Life, years	5	5

18.18 (*a*) Graph the sensitivity of what a person should be willing to pay now for a 9%, $10,000 bond due in 10 years if there is a 30% change in (1) face value, (2) dividend rate, or (3) required nominal rate of return, which is expected to be 8% per year, compounded semiannually. The bond pays dividends semiannually.
(*b*) If the investor did purchase the $10,000 face value bond at a premium of 5% (i.e., 5% above face value) and all your other estimates were correct, that is, 0% change, did he pay too much or too little? How much?

Three Estimates

18.19 DVH Technologies purchases several parts for the instruments it makes via a fixed-price contract of $190,000 per year from a local supplier. The company is considering making the parts in-house through the purchase of equipment that will have a first cost of $240,000 with an estimated salvage value of $30,000 after 5 years. The operating cost is difficult to estimate, but company engineers have made optimistic, most likely, and pessimistic estimates of $60,000, $85,000, and $120,000 per year, respectively. Determine if the company should purchase the equipment under any of the operating cost scenarios. The MARR is 20% per year.

18.20 Astor Engineering recently merged with another firm and could lease additional office space or purchase its own building. The $30,000 per year lease agreement will be a net, net, net lease, which means that the lessee (Astor) will pay the real estate taxes on the leased space, the building insurance on the leased space, and the common area maintenance. Since these costs are about the same if Astor owned the building, they do not need to be considered in the analysis. A new building will cost $880,000 to purchase, but there is considerable uncertainty about what it will be worth in 20 years, which is the planning period selected. The individuals involved in the discussion made optimistic, most likely, and pessimistic estimates of $2,400,000, $1,400,000, and $900,000, respectively. Determine if Astor should purchase the building under any of the estimated resale values at i = 10% per year.

18.21 Holly Farms is considering two environmental chambers to accomplish detailed laboratory confirmations of online bacteria tests in chicken meat for the presence of *E. coli* 0157:H7 and *Listeria monocytogenes*. There is some uncertainty about how long the D103 chamber will be useful. A realistic estimate is 3 years, but pessimistic and optimistic estimates of 2 years and 6 years, respectively, are also possible. The estimated salvage value will remain the same. Using an interest rate of 10% per year, determine if any of the D103 estimates would result in a lower cost than that of the 490G chamber for a 6-year planning period.

	Chamber D103	Chamber 490G
Installed cost, \$	−400,000	−250,000
AOC, \$ per year	−4,000	−3,000
Salvage value at 10% of P, \$	40,000	25,000
Life, years	2, 3, or 6	2

18.22 When the country's economy is expanding, AB Investment Company is optimistic and expects a MARR of 15% for new investments. However, in a receding economy the expected return is 8%. Normally a 10% return is required. An expanding economy causes the estimates of asset life to go down about 20%, and a receding economy makes the n values increase about 10%. Calculate and observe or plot the sensitivity of PW values versus (*a*) the MARR and (*b*) the life values for the two plans detailed below, using the most likely estimates for the other factors. (*c*) Considering all the analyses, under which scenario, if any, should plan M or Q be rejected?

	Plan M	Plan Q
Initial investment, \$	−100,000	−110,000
Cash flow, \$ per year	+15,000	+19,000
Life, years	20	20

Expected Value

18.23 Determine the expected net operating income (NOI) from sales of micro turbine components. The probabilities are 20%, 50%, and 30% for revenues of \$800,000, \$1,000,000, and \$1,100,000 per year, respectively, and operating expenses are constant at \$200,000 per year.

18.24 A company that manufactures amplified pressure transducers is trying to decide between a dual-speed and a variable-speed machine. The engineers are not sure about the salvage value of the variable-speed machine, so they have asked several different used-equipment dealers for estimates. The results can be summarized as follows: there is a 32% chance of getting \$20,000, a 45% chance of getting \$28,000, and a 13% chance of getting \$34,000. Also, there is a 10% chance that the company may have to pay \$5000 to dispose of the equipment. Calculate the expected salvage value.

18.25 The average success probability for a wildcat oil well drilled in the Wind River basin 7 miles from the nearest existing production well is estimated to be 13%. If the value of the oil has equal chances of being \$1.5 million, \$1.9 million, and \$2.4 million, what is the expected income from the well?

18.26 Nationwide income from monthly sales data (rounded to the nearest \$100,000) of Stay Flat vacuum hold-down tables for last year is shown below. Determine the expected value of the monthly income, if economic conditions remain the same.

Income, \$ per Month	Number of Months
500,000	4
600,000	2
700,000	1
800,000	2
900,000	3

18.27 Determine the expected maximum rainfall intensity in El Paso, Texas for the month of July using the estimated probabilities shown.

Rainfall Rate, inches per hour	3	4	5	6
Probability	0.4	0.3	0.2	0.1

18.28 There are four estimates made for the anticipated cycle time to produce a subcomponent. The estimates, in seconds, are 10, 20, 30, and 50. (*a*) If equal weight is placed on each estimate, what is the expected cycle time? (*b*) If the largest time is disregarded, what is the percent reduction in the expected time?

18.29 The PW value for an alternative is expected to be one of two values based on bids from two vendors. Your office partner told you that the low bid is \$3200 per year. If she indicates a chance of 70% of accepting the high bid and that her expected PW is \$5875, what is the PW of the high bid?

18.30 A total of 40 different proposals were evaluated by the IRAD (Industrial Research and Development) committee during the past year. Twenty were funded. Their rate of return estimates are summarized with the i^* values rounded to the nearest integer. For the accepted proposals, calculate the expected rate of return $E(i)$.

Proposal ROR, i^*%	Number of Proposals
−8	1
−5	1
0	5
5	5
8	2
10	3
15	3
	20

18.31 Beckman Electronics has performed an economic analysis of proposed service in a new region of the country. The three-estimate approach to sensitivity analysis has been applied. The optimistic and pessimistic values each have an estimated 20% chance of occurring. Use the FW values shown to determine the expected FW.

	Optimistic	Most Likely	Pessimistic
FW value, $	300,000	50,000	−25,000

18.32 A very successful health and recreation club wants to construct a mock mountain for climbing and exercise outside for its customers' use. Because of its location, there is a 30% chance of a 120-day season of good outdoor weather, a 50% chance of a 150-day season, and a 20% chance of a 165-day season. The mountain will be used by an estimated 350 persons each day of the 4-month (120-day) season, but by only 100 per day for each extra day the season lasts. The feature will cost $375,000 to construct and require a $25,000 rework each 4 years; and the annual maintenance and insurance costs will be $56,000. The climbing fee will be $5 per person. If a life of 10 years is anticipated and a 12% per year return is expected, determine if the addition is economically justified.

18.33 The owner of Ace Roofing may invest $200,000 in new equipment. A life of 6 years and a salvage value of 12% of first cost are anticipated. The annual extra revenue will depend upon the state of the housing and construction industry. The extra revenue is expected to be only $20,000 per year if the current slump in the industry continues. Real estate economists estimate a 50% chance of the slump lasting 3 years and they give it a 20% chance of continuing for 3 additional years. However, if the depressed market does improve, during either the first or second 3-year period, the revenue of the investment is expected to increase by a total of $35,000 per year. Can the company expect to make a return of 8% per year on its investment? Use present worth analysis.

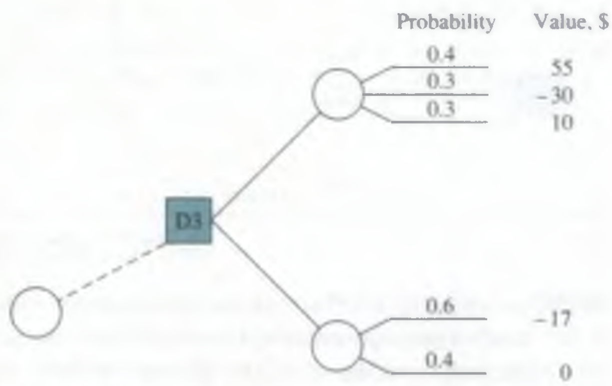
18.34 A flagship hotel in Cedar Falls must construct a retaining wall next to its parking lot due to the widening of the city's main thoroughfare located in front of the hotel. The amount of rainfall experienced in a short time may cause damage in varying amounts, and the wall increases in cost in order to protect against larger and faster rainfalls. The probabilities of a specific amount of rainfall in a 30-minute period and wall cost estimates are as follows:

Rainfall, Inches per 30 Minutes	Probability of Greater Rainfall	First Cost of Wall, $
2.0	0.3	200,000
2.25	0.1	225,000
2.5	0.05	300,000
3.0	0.01	400,000
3.25	0.005	450,000

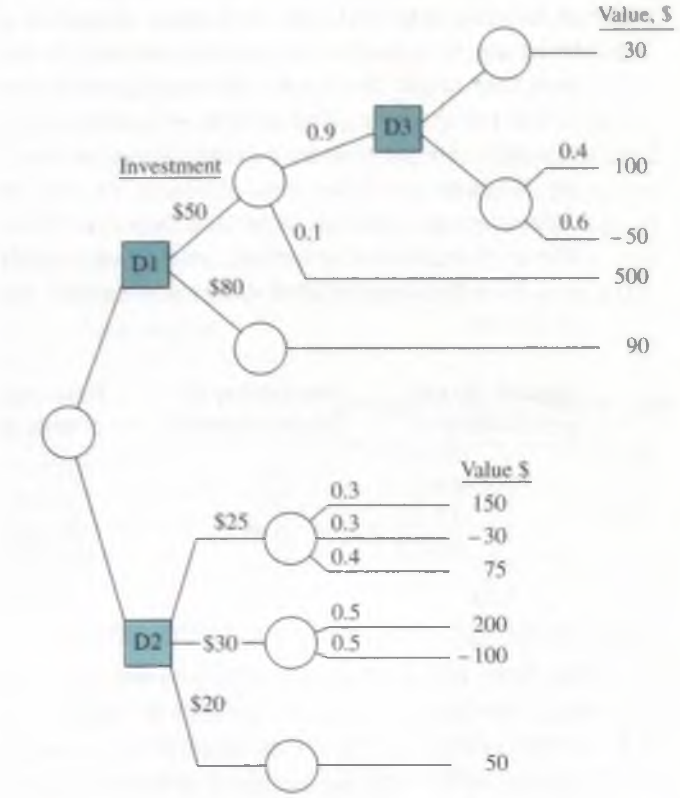
The wall will be financed through a 6% per year loan. The principal and interest will be repaid over a 10-year period. Records indicate an average damage of $50,000 has occurred with heavy rains, due to the relatively poor cohesive properties of the soil along the thoroughfare. A discount rate of 6% per year is applicable. Find the amount of rainfall to protect against by choosing the retaining wall with the smallest AW value over the 10-year period.

Decision Trees

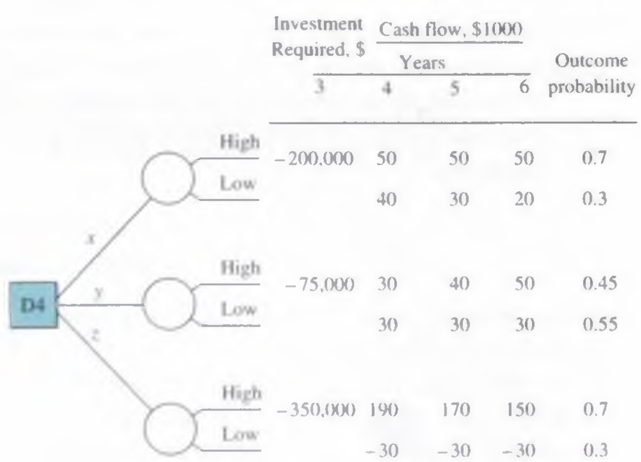
18.35 For the decision tree branch shown, determine the expected values of the two outcomes if decision D3 is already selected and the maximum outcome value is sought. (This decision branch is part of a larger tree.)



18.36 A large decision tree has an outcome branch detailed (next page). If decisions D1, D2, and D3 are all options in a 1-year period, find the decision path that maximizes the outcome value. There are specific investments necessary for decision nodes D1, D2, and D3, as indicated on each branch.



18.37 Decision D4, which has three possible alternatives—x, y, or z—must be made in year 3 of a 6-year study period in order to maximize the expected value of present worth. Using a rate of return of 15% per year, the investment required in year 3 and the estimated cash flows for years 4 through 6, determine which decision should be made in year 3.



18.38 A total of 5000 mechanical subassemblies are needed annually on a final assembly line. The subassemblies can be obtained in one of three ways: (1) *Make them* in one of three plants owned by the company; (2) *buy them off the shelf* from the one and only manufacturer; or (3) *contract to have them made* to specifications by a vendor. The estimated annual equivalent cost for each alternative is dependent upon specific circumstances of the plant, producer, or contractor. The information shown details the circumstance, a probability of occurrence, and the estimated annual cost. Construct and solve a decision tree to determine the least-cost alternative to provide the subassemblies.

Decision Alternative	Outcomes	Probability	Annual Cost for 5000 Units, $ per Year
1. Make	Plant:		
	A	0.3	−250,000
	B	0.5	−400,000
	C	0.2	−350,000
2. Buy off the shelf	Quantity:		
	<5000, pay premium	0.2	−550,000
	5000 available	0.7	−250,000
	>5000, forced to buy	0.1	−290,000
3. Contract	Delivery:		
	Timely delivery	0.5	−175,000
	Late delivery, then buy some off shelf	0.5	−450,000

18.39 The president of ChemTech is trying to decide whether to start a new product line or purchase a small company. It is not financially possible to do both. To make the product for a 3-year period will require an initial investment of $250,000. The expected annual cash flows with probabilities in parentheses are: $75,000 (0.5), $90,000 (0.4), and $150,000 (0.1). To purchase the small company will cost $450,000 now. Market surveys indicate a 55% chance of increased sales for the company and a 45% chance of severe decreases with an annual cash flow of $25,000. If decreases are experienced in the first year, the company will be sold immediately (during year 1) at a price of $200,000. Increased sales could be $100,000 the first 2 years. If this occurs, a decision to expand after 2 years at an additional investment of $100,000 will be considered. This expansion could generate cash flows with indicated probabilities as follows: $120,000 (0.3), $140,000 (0.3), and $175,000 (0.4). If expansion is not chosen, the current size will be maintained with anticipated sales to continue. Assume there are no salvage values on any investments. Use the description given and a 15% per year return to do the following.

(*a*) Construct a decision tree with all values and probabilities shown.

(*b*) Determine the expected PW values at the "expansion/no expansion" decision node after 2 years, provided sales are up.

(*c*) Determine what decision should be made now to offer the greatest return possible for ChemTech.

(*d*) Explain in words what would happen to the expected values at each decision node if the planning horizon were extended beyond 3 years and all cash flow values continued as forecasted in the description.

Real Options

18.40 A privately held company that makes chips that are essential for high-volume data storage is valued at \$3 billion. A computer company that wants to get into cloud computing is considering purchasing the company, but because of the uncertain economy, it would prefer to purchase an option that will allow it to buy the company for up to 1 year from now at a cost of \$3.1 billion. What is the maximum amount the company should be willing to pay for the option, if its MARR is 12% per year?

18.41 A company that is considering adding a new product line has determined that the first cost would be \$80 million. The company is not sure about how the product will be received, so it has projected revenues using optimistic, most likely, and pessimistic estimates of \$35 million, \$25 million, and \$10 million, respectively, with equal probability for each. Instead of expanding now, the company could implement a test program for 1 year in a limited area that will cost \$4 million. (The full-scale project will still cost \$80 million if implemented after the test program is over.) This will provide the company with the option to move forward or cancel the project. The criterion identified to move ahead with full-scale implementation is that revenues must exceed \$900,000. In this case, the pessimistic estimate will be eliminated, and equal probability will be placed on the remaining revenue projections. If the company uses a 5-year planning horizon and a MARR of 12% per year, should the company go ahead with the full-scale project now or take the option to implement the test program for 1 year?

18.42 Dow Chemical is considering licensing a low liquid discharge (LLD) water treatment system from a small company that developed the process. Dow can purchase a 1-year option for \$150,000 that will give it time to pilot-test the LLD process, or Dow can acquire the license now at a cost of \$1.8 million plus 25% of sales. If Dow waits 1 year, the cost will increase to \$1.9 million plus 30% of sales. If Dow projects the sales to be \$1,000,000 per year over the 5-year license period, should the company license the process now or purchase the option to license it after the 1-year test period? Assume the MARR is 15% per year.

18.43 Abby has just negotiated a \$15,000 price on a 2-year-old car and is with the salesman closing the deal. There is a 1-year sales warranty with the purchase; however, an extended warranty is available for \$2500 that will cover the same repairs and component failures as the 1-year warranty for 3 additional years. Abby understands this to be a real options situation with the price of the option (\$2500) paid to avoid future, unknown costs. To help with her decision, the salesman provided three typical sets of historical data on estimated repair costs for used cars. The first-year costs are shown as zero because they will be covered by the sales warranty.

	Year			
	1	2	3	4
Repair cost, \$ per year:				
A	0	−500	−1200	−850
B	0	−1000	−1400	−400
C	0	0	−500	−2000

The salesman said case C is the base case, since it shows that the extended warranty is not needed because the cost of repairs equals the warranty cost. Abby immediately recognized this to be the case only when $i = 0\%$.

(*a*) If Abby assumes that each repair cost scenario has equal probability of occurring with her car, and money is worth 5% per year to her, how much should she be willing to pay for the extended warranty that is offered at \$2500?

(*b*) If the base case actually occurs for her car and she does not purchase the warranty, what is the PW value of the expected future costs at $i = 5\%$ per year?

ADDITIONAL PROBLEMS AND FE EXAM REVIEW QUESTIONS

18.44 In conducting a sensitivity analysis, all of the following could be used as a measure of worth except:

(*a*) Present worth
(*b*) Cost-capacity equations
(*c*) Annual worth
(*d*) Benefit-cost ratio

18.45 When the measure of worth is plotted versus percent change for several parameters, the parameter that is the most sensitive in the economic analysis is the one:

(*a*) That has the steepest curve
(*b*) That has the flattest curve
(*c*) With the largest present worth
(*d*) With the shortest life

18.46 In conducting a formalized sensitivity analysis using three estimates, the three estimates should be:
(a) Strategic, pessimistic, real likely:
(b) Deterministic, realistic, optimistic
(c) Optimistic, pessimistic, most likely
(d) Authentic, realistic, probabilistic

18.47 For annual worth values of $30,000, $40,000, and $50,000 with chances of 20%, 20%, and 60%, respectively, the expected AW is closest to:
(a) $34,000
(b) $40,000
(c) $44,000
(d) $48,000

18.48 The AW of a 3-year-old machine that is used in the manufacture of modular magnetic encoders, which provide data on the speed and positioning of rotating motor shafts, is $-48,000. A challenger will have a first cost of $90,000, an operating cost of $29,000, and a salvage value after 5 years that may vary considerably. For an interest rate of 10% per year and optimistic, most likely, and pessimistic salvage values of $15,000, $10,000, and $2000, respectively, the salvage value(s) for which the challenger AW will be lower than that of the defender are:
(a) All of them
(b) Only the optimistic one
(c) Only the optimistic and pessimistic ones
(d) None of them

18.49 A decision tree includes all of the following except:
(a) Probability estimates for each outcome
(b) Measure of worth as the selection criterion
(c) Expected results from a decision at each stage
(d) The MARR

18.50 A real options analysis is most valuable when:
(a) The risk is low and stakes are high
(b) The stakes are low and risk is high
(c) The stakes are high and risk is high
(d) The stakes are low and risk is low

18.51 A small manufacturing company needs to purchase a machine that will have a first cost of $70,000. The company wants to buy an option that will allow it to purchase the machine for the same price of $70,000 for up to 1 year from now. If the company's MARR is 10% per year, the maximum amount the company should pay for the option is closest to:
(a) $5850
(b) $6365
(c) $6845
(d) $7295

CASE STUDY

SENSITIVITY TO THE ECONOMIC ENVIRONMENT

Background and Information

Berkshire Controllers usually finances its engineering projects with a combination of debt and equity capital. The resulting MARR ranges from a low of 4% per year, if business is slow, to a high of 10% per year. Normally, a 7% per year return is expected. Also the life estimates for assets tend to go down about 20% from normal in a vigorous business environment and up about 10% in a receding economy. The following estimates are the most likely values for two expansion plans currently being evaluated. Plan A will be executed at one location; Plan B will require two locations. All monetary estimates are in $1000 units.

		Plan B	
	Plan A	Location 1	Location 2
First cost, $	−10,000	−30,000	−5,000
AOC, $ per year	−500	−100	−200
Salvage value, $	1,000	5,000	−200
Estimated life, years	40	40	20

Case Study Questions

At the weekly meeting, you were asked to examine the following questions from Berkshire's president.

1. Are the PW values for plans A and B sensitive to changes in the MARR?
2. Are the PW values sensitive to varying life estimates?
3. Is the breakeven point for the first cost of plan A sensitive to the changes in MARR as business goes from vigorous to receding?

CASE STUDY

SENSITIVITY ANALYSIS OF PUBLIC SECTOR PROJECTS—WATER SUPPLY PLANS

Background

One of the most basic services provided by municipal governments is the delivery of a safe, reliable water supply. As cities grow and extend their boundaries to outlying areas, they often inherit water systems that were not constructed according to city codes. The upgrading of these systems is sometimes more expensive than installing one correctly in the first place. To avoid these problems, city officials sometimes install water systems beyond the existing city limits in anticipation of future growth. This case study was extracted from such a countywide water and wastewater management plan and is limited to only some of the water supply alternatives.

From about a dozen suggested plans, five methods were developed by an executive committee as alternative ways of providing water to the study area. These methods were then subjected to a preliminary evaluation to identify the most promising alternatives. Six attributes or factors were used in the initial rating: ability to serve the area, relative cost, engineering feasibility, institutional issues, environmental considerations, and lead time requirement. Each factor carried the same weighting and had values ranging from 1 to 5, with 5 being best. After the top three alternatives were identified, each was subjected to a detailed economic evaluation for selection of the best alternative. These detailed evaluations included an estimate of the capital investment of each alternative amortized over 20 years at 8% per year interest and the annual maintenance and operation (M&O) costs. The annual cost (an AW value) was then divided by the population served to arrive at a monthly cost per household.

Information

Table 18–5 presents the results of the screening using the six factors rated on a scale of 1 to 5. Alternatives 1A, 3, and 4 were determined to be the three best and were chosen for further evaluation.

Detailed Cost Estimates

All amounts are cost estimates.

Alternative 1A

Capital cost	
Land with water rights: 1720 hectares @ \$5000 per hectare	\$8,600,000
Primary treatment plant	2,560,000
Booster station at plant	221,425
Reservoir at booster station	50,325
Site cost	40,260
Transmission line from river	3,020,000
Transmission line right-of-way	23,350
Percolation beds	2,093,500
Percolation bed piping	60,400
Production wells	510,000
Well field gathering system	77,000
Distribution system	1,450,000
Additional distribution system	3,784,800
Reservoirs	250,000
Reservoir site, land, and development	17,000
Subtotal	22,758,060
Engineering and contingencies	5,641,940
Total capital investment	\$28,400,000

TABLE 18–5 Results of Rating Six Factors for Each Alternative, Case Study

		Factors						
Alternative	Description	Ability to Supply Area	Relative Cost	Engineering Feasibility	Institutional Issues	Environmental Considerations	Lead Time Requirement	Total
1A	Receive city water and recharge wells	5	4	3	4	5	3	24
3	Joint city and county plant	5	4	4	3	4	3	23
4	County treatment plant	4	4	3	3	4	3	21
8	Desalt groundwater	1	2	1	1	3	4	12
12	Develop military water	5	5	4	1	3	1	19

Maintenance and operation costs (annual)

Pumping 9,812,610 kWh per year @ $0.08 per kWh	$ 785,009
Fixed operating cost	180,520
Variable operating cost	46,730
Taxes for water rights	48,160
Total annual M&O cost	$1,060,419

$$\begin{aligned}\text{Total annual cost} &= \text{equivalent capital investment} + \text{M\&O cost}\\ &= 28{,}400{,}000(A/P,8\%,20) + 1{,}060{,}419\\ &= 2{,}892{,}540 + 1{,}060{,}419\\ &= \$3{,}952{,}959\end{aligned}$$

Average monthly household cost to serve 95% of 4980 households is

$$\begin{aligned}\text{Household cost} &= (3{,}952{,}959)\left(\frac{1}{12}\right)\left(\frac{1}{4980}\right)\left(\frac{1}{0.95}\right)\\ &= \$69.63 \text{ per month}\end{aligned}$$

Alternative 3

$$\begin{aligned}\text{Total capital investment} &= \$29{,}600{,}000\\ \text{Total annual M\&O cost} &= \$867{,}119\\ \text{Total annual cost} &= 29{,}600{,}000(A/P,8\%,20) + 867{,}119\\ &= 3{,}014{,}760 + 867{,}119\\ &= \$3{,}881{,}879\\ \text{Household cost} &= \$68.38 \text{ per month}\end{aligned}$$

Alternative 4

$$\begin{aligned}\text{Total capital investment} &= \$29{,}000{,}000\\ \text{Total annual M\&O cost} &= \$1{,}063{,}449\\ \text{Total annual cost} &= 29{,}000{,}000(A/P,8\%,20) + 1{,}063{,}449\\ &= 2{,}953{,}650 + 1{,}063{,}449\\ &= \$4{,}017{,}099\\ \text{Household cost} &= \$70.76 \text{ per month}\end{aligned}$$

On the basis of the lowest monthly household cost, alternative 3 (joint city and county plant) is the most economically attractive.

Case Study Exercises

1. If the environmental considerations factor is to have a weighting of twice as much as any of the other five factors, what is its percentage weighting?
2. If the ability to supply area and relative cost factors were each weighted 20% and the other four factors 15% each, which alternatives would be ranked in the top three?
3. By how much would the capital investment of alternative 4 have to decrease to make it more attractive than alternative 3?
4. If alternative 1A served 100% of the households instead of 95%, by how much would the monthly household cost decrease?
5. (*a*) Perform a sensitivity analysis on the two parameters of M&O costs and number of households to determine if alternative 3 remains the best economic choice. Three estimates are made for each parameter in Table 18–6. M&O costs may vary up

TABLE 18–6 Pessimistic, Most Likely, and Optimistic Estimates for Two Parameters

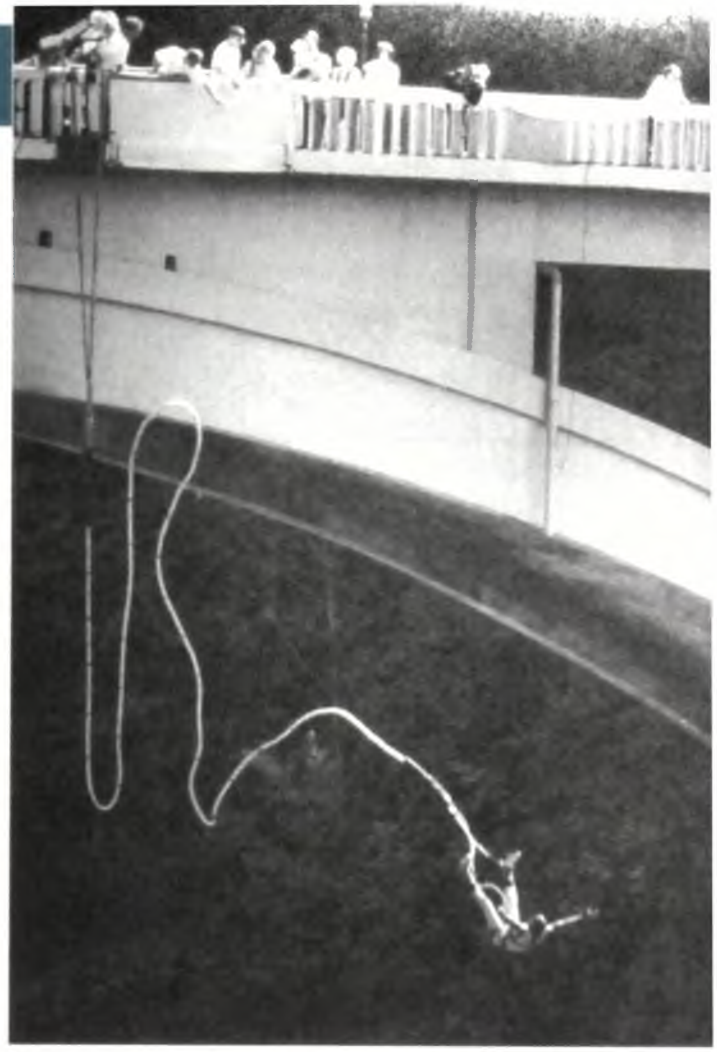
	Annual M&O Costs	Number of Households
Alternative 1A		
Pessimistic	+1%	4980
Most likely	$1,060,419	+2%
Optimistic	−1%	+5%
Alternative 3		
Pessimistic	+5%	4980
Most likely	$867,119	+2%
Optimistic	0%	+5%
Alternative 4		
Pessimistic	+2%	4980
Most likely	$1,063,449	+2%
Optimistic	−10%	+5%

(pessimistic) or down (optimistic) from the most likely estimates presented in the case statement. The estimated number of households (4980) is determined to be the pessimistic estimate. Growth of 2% up to 5% (optimistic) will tend to lower the monthly cost per household.

(*b*) Consider the monthly cost per household for alternative 4, the optimistic estimate. The number of households is 5% above 4980, or 5230. What is the number of households that would have to be available in order for this option to have exactly the same monthly household cost as that for alternative 3 at the optimistic estimate of 5230 households?

CHAPTER 19

More on Variation and Decision Making under Risk



LEARNING OUTCOMES

Purpose: Incorporate decision making under risk into an engineering economy evaluation using probability, sampling, and simulation.

SECTION	TOPIC	LEARNING OUTCOME
19.1	Risk versus certainty	• Understand the approaches to decision making under risk and certainty.
19.2	Probability and distributions	• Construct a probability distribution and cumulative distribution for one variable.
19.3	Random sample	• Obtain a random sample from a cumulative distribution using a random number table.
19.4	μ, σ, and σ^2	• Estimate the population expected value, standard deviation, and variance from a random sample.
19.5	Simulation	• Use Monte Carlo sampling and spreadsheet-based simulation for alternative evaluation.

This chapter further expands our ability to analyze variation in estimates, to consider probability, and to **make decisions under risk**. Fundamentals discussed include variables; probability distributions, especially their graphs and properties of expected value and dispersion; random sampling; and the use of simulation to account for estimate variation in engineering economy studies.

Through coverage of variation and probability, this chapter complements topics in the first sections of Chapter 1: the role of engineering economy in decision making and economic analysis in the problem-solving process. These techniques are more time-consuming than using estimates made with certainty, so they should be used primarily for critical parameters.

19.1 Interpretation of Certainty, Risk, and Uncertainty ● ● ●

All things in the world vary—one from another, over time, and with different environments. We are guaranteed that variation will occur in engineering economy due to its emphasis on decision making for the future. Except for the use of breakeven analysis, sensitivity analysis, and a very brief introduction to expected values, virtually all our estimates have been **certain; that is, no variation** in the amount has entered into the computations of PW, AW, ROR, or any relations used. For example, the estimate that cash flow next year will be $+4500 is one of certainty. Decision making under certainty is, of course, not present in the real world now and surely not in the future. We can observe outcomes with a high degree of certainty, but even this depends upon the accuracy and precision of the scale or measuring instrument.

To allow a parameter of an engineering economy study to vary implies that *risk*, and possibly *uncertainty*, is introduced.

When there may be two or more observable values for a parameter *and* it is possible to estimate the chance that each value may occur, **risk** is present. Virtually all decision making is performed *under risk.*

Risk

As an illustration, decision making under risk is introduced when an annual cash flow estimate has a 50-50 chance of being either $−1000 or $+500.

Decision making under **uncertainty** means there are two or more values observable, but the chances of their occurring cannot be estimated or no one is willing to assign the chances. The observable values in uncertainty analysis are often referred to as *states of nature.*

For example, consider the states of nature to be the rate of national inflation in a particular country during the next 2 to 4 years: remain low, increase 2% to 6% annually, or increase 6% to 8% annually. If there is absolutely no indication that the three values are equally likely, or that one is more likely than the others, this is a statement that indicates decision making under uncertainty.

Example 19.1 explains how a parameter can be described and graphed to prepare for decision making under risk.

EXAMPLE 19.1

CMS in Fairfield, Virginia received three bids each from vendors for two different pieces of large equipment, A and B. One of each piece of equipment must be purchased. Tom, an engineer at CMS, performed an evaluation of each bid and assigned it a rating between 0 and 100, with 100 points being the best of the three. The total for each piece of equipment is 100%. The bid amounts and ratings are shown at the top of Figure 19–1.

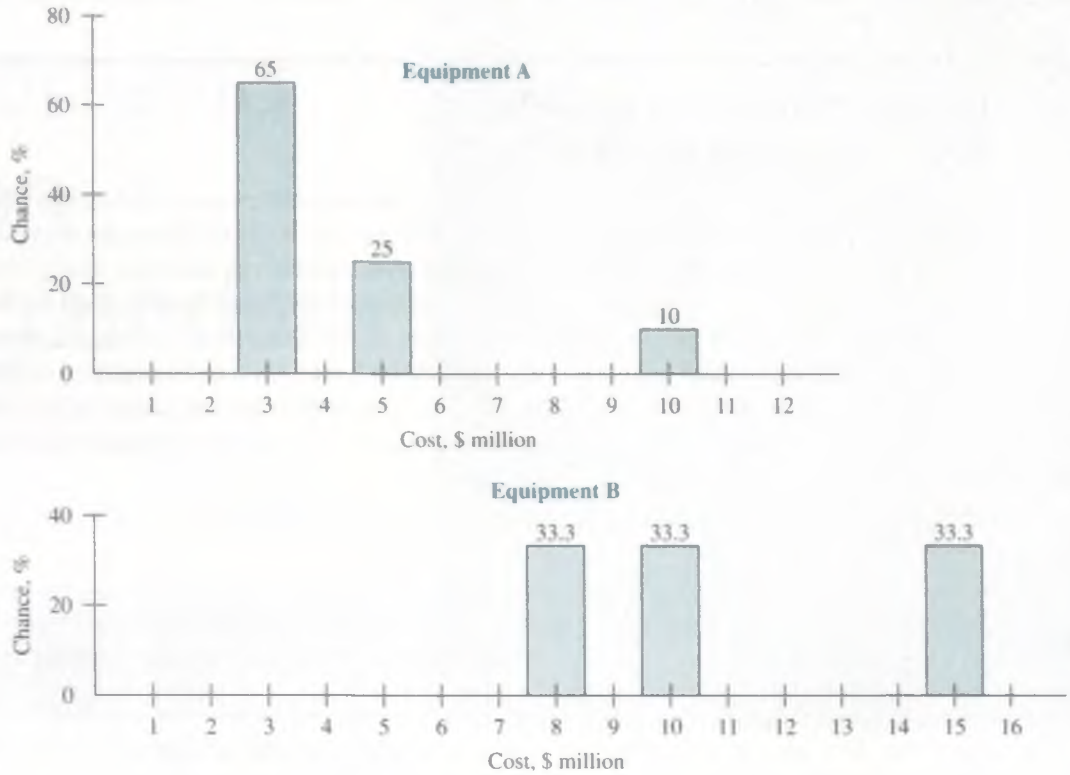
(*a*) Consider the ratings as the chance out of 100 that the bid will be chosen, and plot cost versus chance for each vendor.

(*b*) Since one each of A and B must be purchased, the total cost will vary somewhere between the sum of the lowest bids ($11 million) and the sum of the highest bids ($25 million). Plot this range with an equal chance of 1 in 14 that any amount in between these limits is possible.

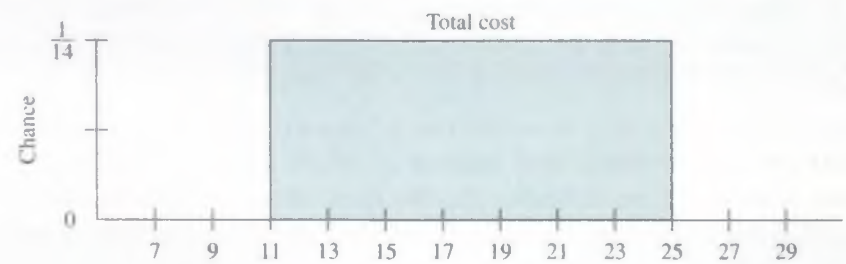
(*c*) Discuss the significant difference between the values of the cost (x axis values) in the graphs in (*a*) and (*b*) above and how the chances are stated (y axis values).

Figure 19–1
Plot of cost estimates versus chance for (*a*) each piece of equipment and (*b*) total cost range, Example 19.1.

Equipment A		Equipment B	
Bid, $1000	**Rating, %**	**Bid, $1000**	**Rating, %**
3,000	65	8,000	33.3
5,000	25	10,000	33.3
10,000	10	15,000	33.3



(*a*) Specific values



(*b*) Continuous range

Solution

(*a*) Figure 19–1*a* plots the specific bids for equipment A and B. The chances (ratings) for A and for B add to 100%. No values between the specific bids have any chance of occurring, according to the single-estimate bids from the three vendors.

(*b*) The range of total cost is between $11 million and $25 million, as shown in Figure 19–1*b*. Tom decided to make his estimate of total cost continuous between these two extremes. This means that the discrete sums of bids ($11 million, $15 million, and $25 million) are no longer used. Rather the entire range from $11 million to $25 million with a chance for every total cost in between is included. Every value has a chance of 1 in 14 of being observed. Now, the sum is a continuous value.

(*c*) In the graph for bid values (Figure 19–1*a*), only specific or discrete estimates are included on the x axis. In the graph for the sum of the cost for equipment A and B (Figure 19–1*b*), the y axis values are continuous over a specific range.

In the next section, the term *variable* is defined and two types of variables are explained—*discrete* and *continuous*—as illustrated here in an elementary form.

Before initiating an engineering economy study, it is important to decide if the analysis will be conducted with certainty for all parameters or if risk will be introduced. A summary of the meaning and use for each type of analysis follows.

Decision Making under Certainty This is what we have done in most analyses thus far. Deterministic estimates are made and entered into measure of worth relations—PW, AW, FW, ROR, B/C—and decision making is based on the results. The values estimated can be considered the most likely to occur with all chance placed on the single-value estimate. A typical example is an asset's first cost estimate made with certainty, say, $P = \$50{,}000$. A plot of P versus chance has the general form of Figure 19–1*a* with one vertical bar at \$50,000 and 100% chance placed on it. The term ***deterministic,*** in lieu of *certainty*, is often used when **single-value** or **single-point estimates** are used exclusively.

In fact, sensitivity analysis using different values of an estimate is simply another form of analysis with certainty, except that the analysis is repeated with different values, *each estimated with certainty*. The resulting measure of worth values are calculated and graphically portrayed to determine the decision's sensitivity to different estimates for one or more parameters.

Decision Making under Risk Now the element of chance is formally taken into account. However, it is more difficult to make a clear decision because the analysis attempts to accommodate **variation.** One or more parameters in an alternative will be allowed to vary. The estimates will be expressed as in Example 19.1 or in slightly more complex forms. Fundamentally, there are two ways to consider risk in an analysis:

Expected value analysis. Use the chance and parameter estimates to calculate expected values E(parameter) via formulas such as Equation [18.2]. Analysis results in E(cash flow), E(AOC), and the like; and the final result is the expected value for a measure of worth, such as E(PW), E(AW), E(ROR), E(B/C). To select the alternative, choose the most favorable expected value of the measure of worth. In an elementary form, this is what we learned about expected values in Chapter 18. The computations may become more elaborate, but the principle is fundamentally the same.

Simulation analysis. Use the chance and parameter estimates to generate repeated computations of the measure of worth relation by randomly sampling from a plot for each varying parameter similar to those in Figure 19–1. When a representative and random sample is complete, an alternative is selected utilizing a table or plot of the results. Usually, graphics are an important part of decision making via simulation analysis. Basically, this is the approach discussed in the rest of this chapter.

Decision Making under Uncertainty When chances are not known for the identified states of nature (or values) of the uncertain parameters, the use of expected value–based decision making under risk as outlined above is *not an option*. In fact, it is difficult to determine what criterion to use to even make the decision. If it is possible to agree that each state is equally likely, then all states have the same chance, and the situation reduces to one of decision making under risk, because expected values can be determined. Because of the relatively inconclusive approaches necessary to incorporate decision making under uncertainty into an engineering economy study, the techniques can be quite useful but are beyond the intended scope of this text.

In an engineering economy study, observed parameter values will vary from the value estimated at the time of the study. However, when performing the analysis, not all parameters should be considered as probabilistic (or at risk). Those that are estimable with a relatively high degree of certainty should be fixed for the study. Accordingly, the methods of sampling, simulation, and statistical data analysis are selectively used on parameters deemed important to the decision-making process. Parameters such as P, AOC, material and unit costs, sales price, revenues, etc., are the targets of decision making under risk. Anticipated variation in interest rates is more commonly addressed by sensitivity analysis.

The remainder of this chapter concentrates on decision making under risk as applied in an engineering economy study. Sections 19.2 to 19.4 provide foundation material necessary to design and correctly conduct a simulation analysis (Section 19.5).

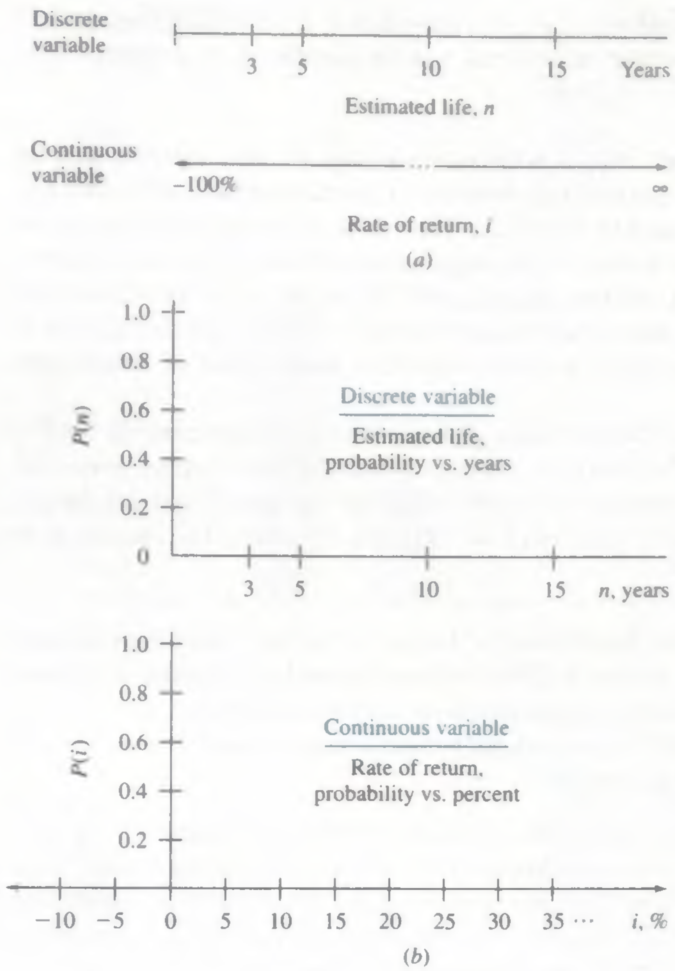


Figure 19–2
(*a*) Discrete and continuous variable scales and (*b*) scales for a variable versus its probability.

19.2 Elements Important to Decision Making under Risk ● ● ●

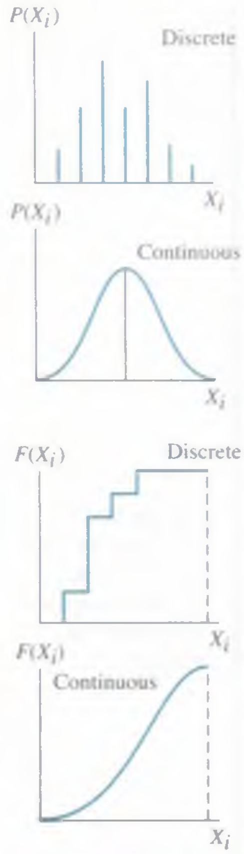
Some basics of probability and statistics are essential to correctly perform decision making under risk via expected value or simulation analysis. They are the *random variable, probability, probability distribution*, and *cumulative distribution*, as defined here. (If you are already familiar with them, this section will provide a review.)

A **random variable** or **variable** is a characteristic or parameter that can take on any one of several values. Variables are classified as *discrete* or *continuous.* Discrete variables have several specific, isolated values, while continuous variables can assume any value between two stated limits, called the *range* of the variable.

The estimated life of an asset is a discrete variable. For example, n may be expected to have values of $n = 3, 5, 10$, or 15 years, and no others. The rate of return is an example of a continuous variable; i can vary from −100% to ∞, that is, $-100\% \leq i < \infty$. The ranges of possible values for n (discrete) and i (continuous) are shown as the x axes in Figure 19–2*a*. (In probability texts, capital letters symbolize a variable, say X, and small letters x identify a specific value of the variable. Though correct, this level of rigor in terminology is not applied in this chapter.)

Probability is a number between 0 and 1.0 that expresses the chance in decimal form that a random variable (discrete or continuous) will take on any value from those identified for it. Probability is simply the amount of chance, divided by 100.

Probabilities are commonly identified by $P(X_i)$ or $P(X = X_i)$, which is read as the probability that the variable X takes on the value X_i. (Actually, for a continuous variable, the probability at a single value is zero, as shown in a later example.) The sum of all $P(X_i)$ for a variable must be 1.0, a requirement already discussed. The probability scale, like the percentage scale for chance in Figure 19–1, is indicated on the ordinate (y axis) of a graph. Figure 19–2b shows the 0 to 1.0 range of probability for the variables n and i.



A **probability distribution** describes how probability is distributed over the different values of a variable. Discrete variable distributions look significantly different from continuous variable distributions, as indicated by the inset at the right.

The individual probability values are stated as

$$P(X_i) = \text{probability that } X \text{ equals } X_i \qquad [19.1]$$

The distribution may be developed in one of two ways: by listing each probability value for each possible variable value (see Example 19.2) or by a mathematical description or expression that states probability in terms of the possible variable values (Example 19.3).

Cumulative distribution, also called the **cumulative probability distribution,** is the accumulation of probability over all values of a variable up to and including a specified value.

Identified by $F(X_i)$, each cumulative value is calculated as

$$\begin{aligned} F(X_i) &= \text{sum of all probabilities through the value } X_i \\ &= P(X \le X_i) \end{aligned} \qquad [19.2]$$

As with a probability distribution, cumulative distributions appear differently for discrete (stair-stepped) and continuous variables (smooth curve). Examples 19.2 and 19.3 illustrate cumulative distributions that correspond to specific probability distributions. These fundamentals about $F(X_i)$ are applied in the next section to develop a random sample.

EXAMPLE 19.2

Alvin is a medical doctor and biomedical engineering graduate who practices at Medical Center Hospital. He is planning to start prescribing an antibiotic that may reduce infection in patients with flesh wounds. Tests indicate the drug has been applied up to 6 times per day without harmful side effects. If no drug is used, there is always a positive probability that the infection will be reduced by a person's own immune system.

Published drug test results provide good probability estimates of positive reaction (i.e., reduction in the infection count) within 48 hours for increased treatments per day. Use the probabilities listed below to construct a probability distribution and a cumulative distribution for the total number of treatments per day.

Number of Added Treatments per Day	Probability of Infection Reduction for Each Added Treatment
0	0.07
1	0.08
2	0.10
3	0.12
4	0.13
5	0.25
6	0.25

Solution

Define the random variable T as the number of added treatments per day. Since T can take on only seven different values, it is a **discrete variable.** The probability of infection reduction is listed for each value in column 2 of Table 19–1. The cumulative probability $F(T_i)$ is determined using Equation [19.2] by adding all $P(T_i)$ values through T_i, as indicated in column 3.

Figure 19–3a and b shows plots of the probability distribution and cumulative distribution, respectively. The summing of probabilities to obtain $F(T_i)$ gives the cumulative distribution the stair-stepped appearance, and in all cases the final $F(T_i) = 1.0$, since the total of all $P(T_i)$ values must equal 1.0.

TABLE 19–1 Probability Distribution and Cumulative Distribution for Example 19.2

(1) Number per Day T_i	(2) Probability $P(T_i)$	(3) Cumulative Probability $F(T_i)$
0	0.07	0.07
1	0.08	0.15
2	0.10	0.25
3	0.12	0.37
4	0.13	0.50
5	0.25	0.75
6	0.25	1.00

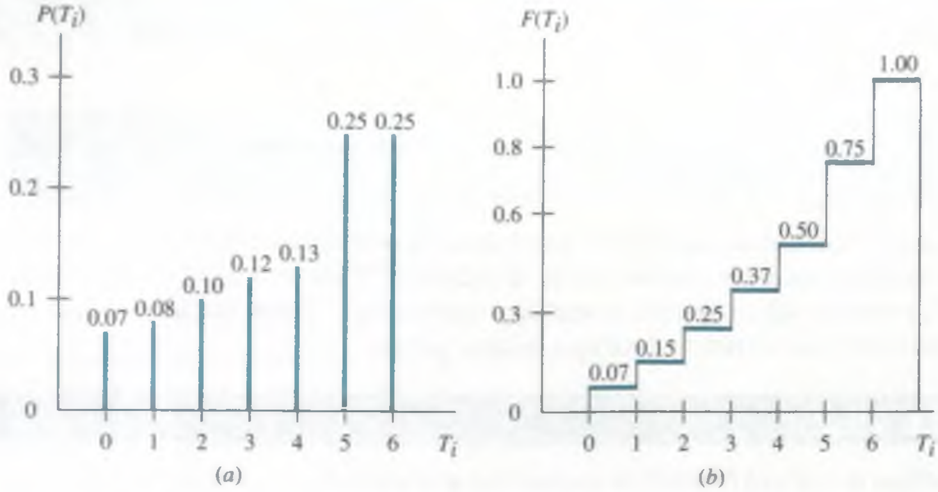


Figure 19–3 (*a*) Probability distribution $P(T_i)$ and (*b*) cumulative distribution $F(T_i)$ for Example 19.2.

Comment

Rather than use a tabular form as in Table 19–1 to state $P(T_i)$ and $F(T_i)$ values, it is possible to express them for each value of the variable.

$$P(T_i) = \begin{cases} 0.07 & T_1 = 0 \\ 0.08 & T_2 = 1 \\ 0.10 & T_3 = 2 \\ 0.12 & T_4 = 3 \\ 0.13 & T_5 = 4 \\ 0.25 & T_6 = 5 \\ 0.25 & T_7 = 6 \end{cases} \qquad F(T_i) = \begin{cases} 0.07 & T_1 = 0 \\ 0.15 & T_2 = 1 \\ 0.25 & T_3 = 2 \\ 0.37 & T_4 = 3 \\ 0.50 & T_5 = 4 \\ 0.75 & T_6 = 5 \\ 1.00 & T_7 = 6 \end{cases}$$

In basic engineering economy situations, the probability distribution for a **continuous variable** is commonly expressed as a mathematical function, such as a *uniform distribution*, a *triangular distribution* (both discussed in Example 19.3 in terms of cash flow), or the more complex, but commonly used, *normal distribution*. For continuous variable distributions, the symbol $f(X)$ is routinely used instead of $P(X_i)$, and $F(X)$ is used instead of $F(X_i)$, simply because the point probability for a continuous variable is zero. Thus, $f(X)$ and $F(X)$ are continuous lines and curves.

EXAMPLE 19.3

As president of a manufacturing systems consultancy, Sallie has observed the monthly cash flows that have occurred over the last 3 years into company accounts from two longstanding clients. Sallie has concluded the following about the distribution of these monthly cash flows:

Client 1

Estimated low cash flow: \$10,000
Estimated high cash flow: \$15,000
Most likely cash flow: same for all values
Distribution of probability: uniform

Client 2

Estimated low cash flow: \$20,000
Estimated high cash flow: \$30,000
Most likely cash flow: \$28,000
Distribution of probability: mode at \$28,000

The *mode* is the most frequently observed value for a variable. Sallie assumes cash flow to be a continuous variable referred to as C. (*a*) Write and graph the two probability distributions and cumulative distributions for monthly cash flow, and (*b*) determine the probability that monthly cash flow is no more than \$12,000 for client 1 and at least \$25,000 for client 2.

Solution

All cash flow values are expressed in \$1000 units.

Client 1: monthly cash flow distribution

(*a*) The distribution of cash flows for client 1, identified by the variable C_1, follows the *uniform distribution.* Probability and cumulative probability take the following general forms.

$$f(C_1) = \frac{1}{\text{high} - \text{low}} \qquad \text{low value} \le C_1 \le \text{high value}$$

$$f(C_1) = \frac{1}{H - L} \qquad L \le C_1 \le H \qquad [19.3]$$

$$F(C_1) = \frac{\text{value} - \text{low}}{\text{high} - \text{low}} \qquad \text{low value} \le C_1 \le \text{high value}$$

$$F(C_1) = \frac{C_1 - L}{H - L} \qquad L \le C_1 \le H \qquad [19.4]$$

For client 1, monthly cash flow is uniformly distributed with $L = \$10$, $H = \$15$, and $\$10 \le C_1 \le \$15$. Figure 19–4 is a plot of $f(C_1)$ and $F(C_1)$ from Equations [19.3] and [19.4].

$$f(C_1) = \frac{1}{5} = 0.2 \qquad \$10 \le C_1 \le \$15$$

$$F(C_1) = \frac{C_1 - 10}{5} \qquad \$10 \le C_1 \le \$15$$

(*b*) The probability that client 1 has a monthly cash flow of no more than \$12 is easily determined from the $F(C_1)$ plot as 0.4, or a 40% chance. If the $F(C_1)$ relation is used directly, the computation is

$$F(\$12) = P(C_1 \le \$12) = \frac{12 - 10}{5} = 0.4$$

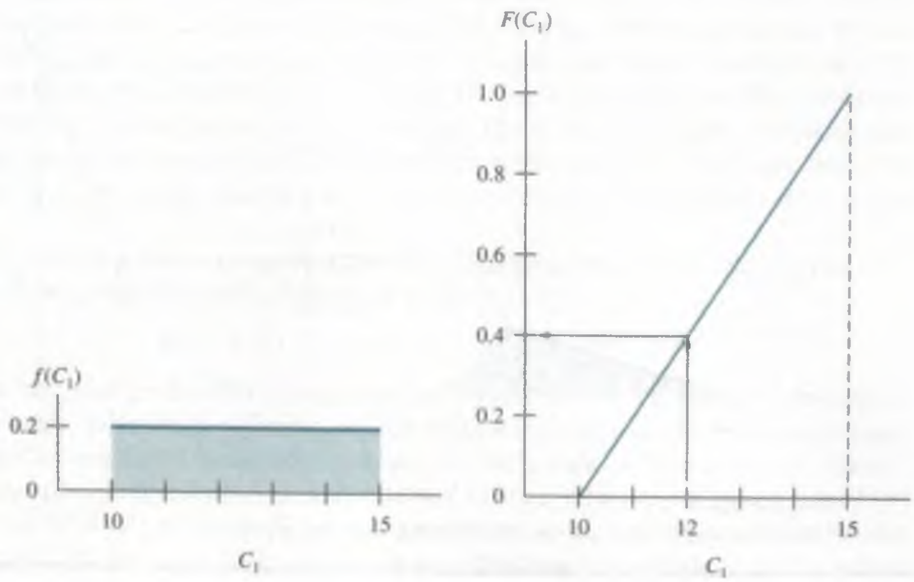


Figure 19–4
Uniform distribution for client 1 monthly cash flow, Example 19.3.

Client 2: monthly cash flow distribution

(a) The distribution of cash flows for client 2, identified by the variable C_2, follows the *triangular distribution.* This probability distribution has the shape of an upward-pointing triangle with the peak at the mode M, and downward-sloping lines joining the x axis on either side at the low (L) and high (H) values. The mode of the triangular distribution has the maximum probability value.

$$f(\text{mode}) = f(M) = \frac{2}{H - L} \qquad [19.5]$$

The cumulative distribution is comprised of two curved line segments from 0 to 1 with a break point at the mode, where

$$F(\text{mode}) = F(M) = \frac{M - L}{H - L} \qquad [19.6]$$

For C_2, the low value is $L = \$20$, the high is $H = \$30$, and the most likely cash flow is the mode $M = \$28$. The probability at M from Equation [19.5] is

$$f(28) = \frac{2}{30 - 20} = \frac{2}{10} = 0.2$$

The break point in the cumulative distribution occurs at $C_2 = 28$. Using Equation [19.6],

$$F(28) = \frac{28 - 20}{30 - 20} = 0.8$$

Figure 19–5 presents the plots for $f(C_2)$ and $F(C_2)$. Note that $f(C_2)$ is skewed, since the mode is not at the midpoint of the range $H - L$, and $F(C_2)$ is a smooth S-shaped curve with an inflection point at the mode.

(b) From the cumulative distribution in Figure 19–5, there is an estimated 31.25% chance that cash flow is \$25 or less. Therefore,

$$F(\$30) - F(\$25) = P(C_2 \geq \$25) = 1 - 0.3125 = 0.6875$$

Comment

The general relations $f(C_2)$ and $F(C_2)$ are not developed here. The variable C_2 is *not* a uniform distribution; it is triangular. Therefore, it requires the use of an integral to find cumulative probability values from the probability distribution $f(C_2)$.

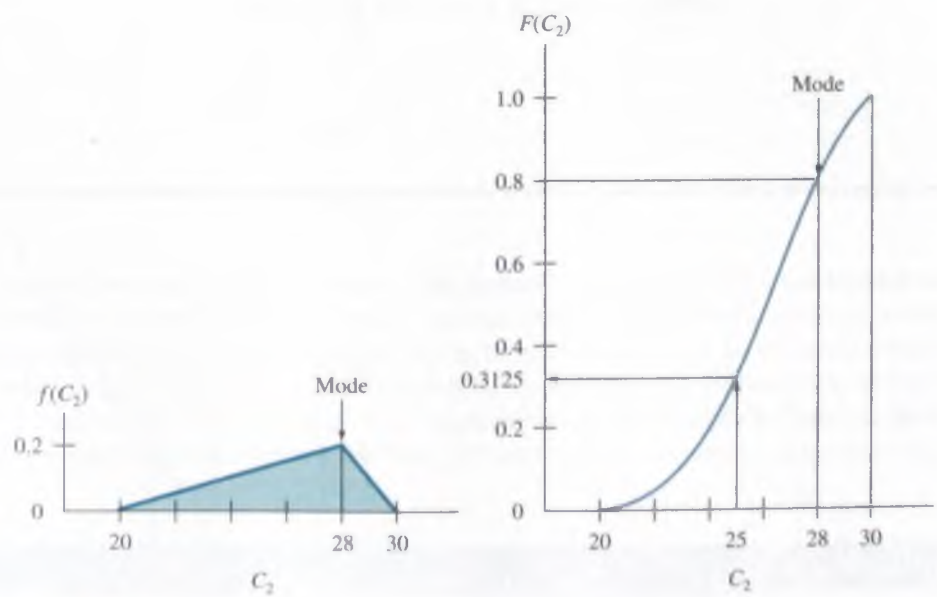


Figure 19–5
Triangular distribution for client 2 monthly cash flow, Example 19.3.

19.3 Random Samples ● ● ●

Estimating a parameter with a single value in previous chapters is the equivalent of taking a *random sample of size 1 from an entire population* of possible values. As an illustration, assume that estimates of first cost, annual operating cost, interest rate, and other parameters are used to compute one PW value in order to accept or reject an alternative. Each estimate is a sample of size 1 from an entire population of possible values for each parameter. Now, if a second estimate is made for each parameter and a second PW value is determined, a sample of size 2 has been taken. If all values in the population were known, the probability distribution and cumulative distribution would be known. Then a sample would not be necessary.

When we perform an engineering economy study and utilize decision making under certainty, we use one estimate for each parameter to calculate a measure of worth (i.e., a sample of size 1 for each parameter). The estimate is the most likely value, that is, one estimate of the expected value. We know that all parameters will vary somewhat; yet some are important enough, or will vary enough, that a probability distribution should be determined or assumed for it and the parameter treated as a random variable. This is using risk, and a sample from the parameter's probability distribution—$P(X)$ for discrete or $f(X)$ for continuous—helps formulate probability statements about the estimates. This approach complicates the analysis somewhat; however, it also provides a sense of confidence (or possibly a lack of confidence in some cases) about the decision made concerning the economic viability of the alternative based on the varying parameter. (We will further discuss this aspect later, after we learn how to correctly take a random sample from any probability distribution.)

A **random sample** of size n is the selection in a random fashion of n values from a population with an assumed or known probability distribution, such that the values of the variable have the **same chance of occurring** in the sample as they are expected to occur in the population.

Suppose Yvon is an engineer with 20 years of experience working for the Aircraft Safety Commission. For a two-crew aircraft, there are three parachutes on board. The safety standard states that 99% of the time, all three chutes must be "fully ready for emergency deployment." Yvon is relatively sure that nationwide the probability distribution of N, the specific number of chutes fully ready, may be described by the probability distribution

$$P(N = N_i) = \begin{cases} 0.005 & N = 0 \text{ chutes ready} \\ 0.015 & N = 1 \text{ chute ready} \\ 0.060 & N = 2 \text{ chutes ready} \\ 0.920 & N = 3 \text{ chutes ready} \end{cases}$$

This means that the safety standard is clearly not met nationwide. Yvon is in the process of sampling 200 (randomly selected) corporate and private aircraft across the nation to determine how many chutes are classified as fully ready. If the sample is truly random and Yvon's probability distribution is a correct representation of actual parachute readiness, the observed N values in the 200 aircraft will approximate the same proportions as the population probabilities, that is, 1 aircraft with 0 chutes ready, etc. Since this is a sample, it is likely that the results won't track the population exactly. However, if the results are relatively close, the study indicates that the sample results may be useful in predicting parachute safety across the nation.

To develop a random sample, use **random numbers (RN)** generated from a uniform probability distribution for the discrete numbers 0 through 9, that is,

$$P(X_i) = 0.1 \qquad \text{for } X_i = 0, 1, 2, \ldots, 9$$

In tabular form, the random digits so generated are commonly clustered in groups of two digits, three digits, or more. Table 19–2 is a sample of 264 random digits clustered into two-digit numbers. This format is very useful because the numbers 00 to 99 conveniently relate to the cumulative distribution values 0.01 to 1.00. This makes it easy to select a two-digit RN and enter $F(X)$ to determine a value of the variable with the same proportions as it occurs in the probability distribution. To apply this logic manually and develop a random sample of size n from a known

TABLE 19–2 Random Digits Clustered into Two-Digit Numbers

51	82	88	18	19	81	03	88	91	46	39	19	28	94	70	76	33	15	64	20	14	52
73	48	28	59	78	38	54	54	93	32	70	60	78	64	92	40	72	71	77	56	39	27
10	42	18	31	23	80	80	26	74	71	03	90	55	61	61	28	41	49	00	79	96	78
45	44	79	29	81	58	66	70	24	82	91	94	42	10	61	60	79	30	01	26	31	42
68	65	26	71	44	37	93	94	93	72	84	39	77	01	97	74	17	19	46	61	49	67
75	52	14	99	67	74	06	50	97	46	27	88	10	10	70	66	22	56	18	32	06	24

discrete probability distribution $P(X)$ or a continuous variable distribution $f(X)$, the following procedure may be used.

1. Develop the cumulative distribution $F(X)$ from the probability distribution. Plot $F(X)$.
2. Assign the RN values from 00 to 99 to the $F(X)$ scale (the y axis) in the same proportion as the probabilities. For the parachute safety example, the probabilities from 0.0 to 0.15 are represented by the random numbers 00 to 14. Indicate the RNs on the graph.
3. To use a table of random numbers, determine the scheme or sequence of selecting RN values—down, up, across, diagonally. Any direction and pattern is acceptable, but the scheme should be used consistently for one entire sample.
4. Select the first number from the RN table, enter the $F(X)$ scale, and observe and record the corresponding variable value. Repeat this step until there are n values of the variable that constitute the random sample.
5. Use the n sample values for analysis and decision making under risk. These may include
   - Plotting the sample probability distribution.
   - Developing probability statements about the parameter.
   - Comparing sample results with the assumed population distribution.
   - Determining sample statistics (Section 19.4).
   - Performing a simulation analysis (Section 19.5).

EXAMPLE 19.4

Develop a random sample of size 10 for the variable N, number of months, as described by the probability distribution

$$P(N = N_i) = \begin{cases} 0.20 & N = 24 \\ 0.50 & N = 30 \\ 0.30 & N = 36 \end{cases} \qquad [19.7]$$

Solution

Apply the procedure above, using the $P(N = N_i)$ values in Equation [19.7].

1. The cumulative distribution, Figure 19–6, is for the discrete variable N, which can assume three different values.
2. Assign 20 numbers (00 through 19) to $N_1 = 24$ months, where $P(N = 24) = 0.2$; 50 numbers to $N_2 = 30$; and 30 numbers to $N_3 = 36$.
3. Initially select any position in Table 19–2, and go across the row to the right and onto the row below toward the left. (Any routine can be developed, and a different sequence for each random sample may be used.)
4. Select the initial number 45 (4th row, 1st column), and enter Figure 19–6 in the RN range of 20 to 69 to obtain $N = 30$ months.
5. Select and record the remaining nine values from Table 19–2 as shown below.

RN	45	44	79	29	81	58	66	70	24	82
N	30	30	36	30	36	30	30	36	30	36

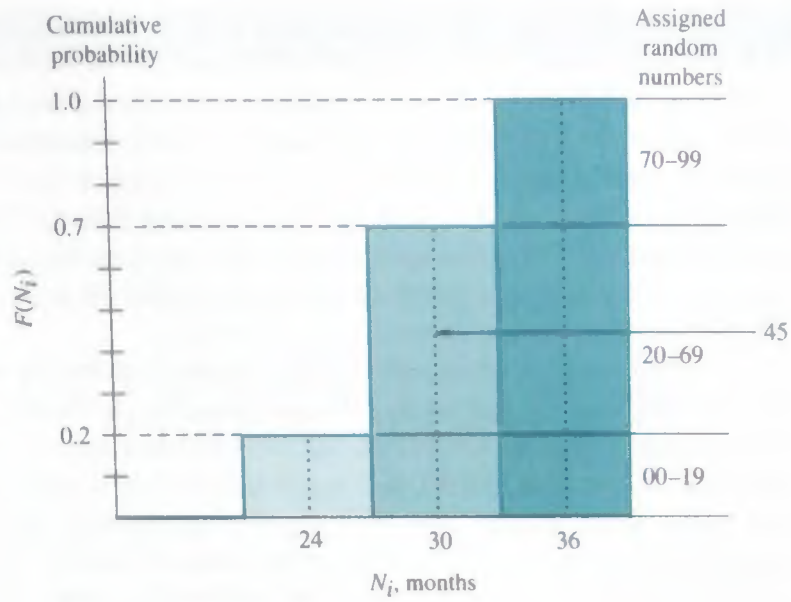


Figure 19–6
Cumulative distribution with random number values assigned in proportion to probabilities, Example 19.4.

Now, using the 10 values, develop the sample probabilities.

Months N	Times in Sample	Sample Probability	Equation [19.7] Probability
24	0	0.00	0.2
30	6	0.60	0.5
36	4	0.40	0.3

With only 10 values, we can expect the sample probability estimates to be different from the values in Equation [19.7]. Only the value $N = 24$ months is significantly different, since no RN of 19 or less occurred. A larger sample will definitely make the probabilities closer to the original data.

To take a *random sample of size n for a continuous variable*, the procedure above is applied, except the random number values are assigned to the cumulative distribution on a continuous scale of 00 to 99 corresponding to the $F(X)$ values. As an illustration, consider Figure 19–4, where C_1 is the *uniformly distributed* cash flow variable for client 1 in Example 19.3. Here $L =$ \$10, $H =$ \$15, and $f(C_1) = 0.2$ for all values between L and H (all values are divided by \$1000). The $F(C_1)$ is repeated as Figure 19–7 with the assigned random number values shown on the right scale. If the two-digit RN of 45 is chosen, the corresponding C_1 is graphically estimated to be \$12.25. It can also be linearly interpolated as $\$12.25 = 10 + (45/100)(15 - 10)$.

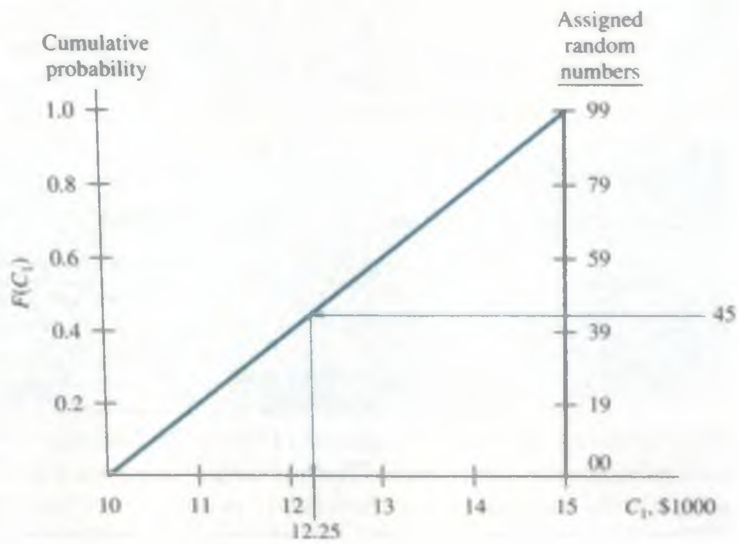


Figure 19–7
Random numbers assigned to the continuous variable of client 1 cash flows in Example 19.3.

For greater accuracy when developing a random sample, especially for a continuous variable, it is possible to use 3-, 4-, or 5-digit RNs. These can be developed from Table 19–2 simply by combining digits in the columns and rows or obtained from tables with RNs printed in larger clusters of digits. In computer-based sampling, most simulation software packages have an RN generator built in that will generate values in the range of 0 to 1 from a continuous variable uniform distribution, usually identified by the symbol $U(0, 1)$. The RN values, usually between 0.00000 and 0.99999, are used to sample directly from the cumulative distribution employing essentially the same procedure explained here. The Excel functions RAND and RANDBETWEEN are described in Appendix A, Section A.3.

An initial question in random sampling usually concerns the **minimum size of *n*** required to ensure confidence in the results. Without detailing the mathematical logic, sampling theory, which is based upon the law of large numbers and the central limit theorem (check a basic statistics book to learn about these), indicates that an n of 30 is sufficient. However, since reality does not follow theory exactly, and since engineering economy often deals with sketchy estimates, samples in the *range of 100 to 200* are the common practice. But samples as small as 10 to 25 provide a much better foundation for decision making under risk than the single-point estimate for a parameter that is known to vary widely.

19.4 Expected Value and Standard Deviation

Two very important measures or properties of a random variable are the expected value and standard deviation. If the entire population for a variable were known, these properties would be calculated directly. Since they are usually not known, random samples are commonly used to estimate them via the sample mean and the sample standard deviation, respectively. The following is a brief introduction to the interpretation and calculation of these properties using a random sample of size n from the population.

The usual symbols are Greek letters for the true population measures and English letters for the sample estimates.

	True Population Measure		Sample Estimate	
	Symbol	**Name**	**Symbol**	**Name**
Expected value	μ or $E(X)$	Mu or true mean	$\overline{X}$	Sample mean
Standard deviation	σ or $\sqrt{\text{Var}(X)}$ or $\sqrt{\sigma^2}$	Sigma or true standard deviation	s or $\sqrt{s^2}$	Sample standard deviation

The **expected value** $E(X)$ is the long-run expected average if the variable is sampled many times.

The population expected value is not known exactly, since the population itself is not known completely, so μ is estimated either by $E(X)$ from a distribution or by $\overline{X}$, the sample mean. Equation [18.2], repeated here as Equation [19.8], is used to compute the $E(X)$ of a probability distribution, and Equation [19.9] is the sample mean, also called the *sample average*.

Population: μ

Probability distribution:
$$E(X) = \Sigma X_i P(X_i) \qquad [19.8]$$

Sample:
$$\overline{X} = \frac{\text{sum of sample values}}{\text{sample size}}$$
$$= \frac{\Sigma X_i}{n} = \frac{\Sigma f_i X_i}{n} \qquad [19.9]$$

The f_i in the second form of Equation [19.9] is the frequency of X_i, that is, the number of times each value occurs in the sample. The resulting $\overline{X}$ is not necessarily an observed value of the variable; it is the long-run average value and can take on any value within the range of the variable. (We omit the subscript i on X and f when there is no confusion introduced.)

EXAMPLE 19.5

Kayeu, an engineer with Pacific NW Utilities, is planning to test several hypotheses about residential electricity bills in North American and Asian countries. The variable of interest is X, the monthly residential bill in U.S. dollars (rounded to the nearest dollar). Two small samples have been collected from different countries of North America and Asia. Estimate the population expected value. Do the samples (from a nonstatistical viewpoint) appear to be drawn from one population of electricity bills or from two different populations?

North American, Sample 1, $	40	66	75	92	107	159	275
Asian, Sample 2, $	84	90	104	187	190		

Solution

Use Equation [19.9] for the sample mean.

Sample 1: $n = 7$ $\quad \Sigma X_i = 814 \quad \overline{X} = \$116.29$
Sample 2: $n = 5$ $\quad \Sigma X_i = 655 \quad \overline{X} = \$131.00$

Based solely on the small sample averages, the approximate \$15 difference, which is only 11% of the larger average bill, does not seem sufficiently large to conclude that the two populations are different. There are several statistical tests available to determine if samples come from the same or different populations. (Check a basic statistics text to learn about them.)

Comment

There are three commonly used measures of central tendency for data. The sample average is the most popular, but the *mode* and the *median* are also good measures. The mode, which is the most frequently observed value, was utilized in Example 19.3 for a triangular distribution. There is no specific mode in Kayeu's two samples, since all values are different. The *median is the middle value* of the sample. It is not biased by extreme sample values, as is the mean. The two medians in the samples are \$92 and \$104. Based solely on the medians, the conclusion is still that the samples do not necessarily come from two different populations of electricity bills.

The **standard deviation** s or $s(X)$ is the dispersion or spread of values **about the expected value** $\boldsymbol{E(X)}$ or sample average $\overline{X}$.

The sample standard deviation s estimates the property σ, which is the population measure of dispersion about the expected value of the variable. A probability distribution for data with strong central tendency is more closely clustered about the center of the data, and has a smaller s, than a wider, more dispersed distribution. In Figure 19–8, the samples with larger s values—s_1 and s_4—have a flatter, wider probability distribution.

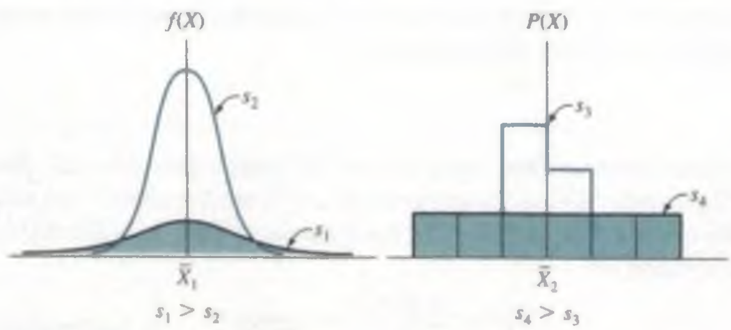


Figure 19–8
Sketches of distributions with different separate lines standard deviation values.

Actually, the variance s^2 is often quoted as the measure of dispersion. The standard deviation is simply the square root of the variance, so either measure can be used. The s value is what we use routinely in making computations about risk and probability. Mathematically, the formulas and symbols for variance and standard deviation of a discrete variable and a random sample of size n are as follows:

Population: $\sigma^2 = \text{Var}(X)$ and $\sigma = \sqrt{\sigma^2} = \sqrt{\text{Var}(X)}$

Probability distribution: $\text{Var}(X) = \Sigma[X_i - E(X)]^2 P(X_i)$ **[19.10]**

Sample:
$$s^2 = \frac{\text{sum of (sample value} - \text{sample average)}^2}{\text{sample size} - 1}$$
$$= \frac{\Sigma(X_i - \overline{X})^2}{n-1} \qquad [19.11]$$
$$s = \sqrt{s^2}$$

Equation [19.11] for sample variance is usually applied in a more computationally convenient form.

$$s^2 = \frac{\Sigma X_i^2}{n-1} - \frac{n}{n-1}\overline{X}^2 = \frac{\Sigma f_i X_i^2}{n-1} - \frac{n}{n-1}\overline{X}^2 \qquad [19.12]$$

The standard deviation uses the sample average as a basis about which to measure the spread or dispersion of data via the calculation $(X - \overline{X})$, which can have a minus or plus sign. To accurately measure the dispersion in both directions from the average, the quantity $(X - \overline{X})$ is squared. To return to the dimension of the variable itself, the square root of Equation [19.11] is extracted. The term $(X - \overline{X})^2$ is called the *mean-squared deviation,* and s has historically also been referred to as the *root-mean-square deviation.* The f_i in the second form of Equation [19.12] uses the frequency of each X_i value to calculate s^2.

One simple way to combine the average and standard deviation is to determine the percentage or fraction of the sample that is within ± 1, ± 2, or ± 3 standard deviations of the average, that is,

$$\overline{X} \pm ts \qquad \text{for } t = 1, 2, \text{ or } 3 \qquad [19.13]$$

In probability terms, this is stated as

$$P(\overline{X} - ts \leq X \leq \overline{X} + ts) \qquad [19.14]$$

Virtually all the sample values will always be within the $\pm 3s$ range of $\overline{X}$, but the percent within $\pm 1s$ will vary depending on how the data points are distributed about $\overline{X}$. Example 19.6 illustrates the calculation of s to estimate σ and incorporates s with the sample average using $\overline{X} \pm ts$.

EXAMPLE 19.6

(*a*) Use the two samples of Example 19.5 to estimate population variance and standard deviation for electricity bills. (*b*) Determine the percentages of each sample that are inside the ranges of 1 and 2 standard deviations from the mean.

Solution

(*a*) For illustration purposes only, apply the two different relations to calculate s for the two samples. For sample 1 (North American) with $n = 7$, use X to identify the values. Table 19–3 presents the computation of $\Sigma(X - \overline{X})^2$ for Equation [19.11], with $\overline{X} = \$116.29$. The resulting s^2 and s values are

$$s^2 = \frac{37{,}743.40}{6} = 6290.57$$
$$s = \$79.31$$

TABLE 19–3 Computation of Standard Deviation Using Equation [19.11] with $\overline{X}$ = \$116.29, Example 19.6

X, \$	$X - \overline{X}$	$(X - \overline{X})^2$
40	−76.29	5,820.16
66	−50.29	2,529.08
75	−41.29	1,704.86
92	−24.29	590.00
107	−9.29	86.30
159	+42.71	1,824.14
275	+158.71	25,188.86
814		37,743.40

TABLE 19–4 Computation of Standard Deviation Using Equation [19.12] with $\overline{Y}$ = \$131, Example 19.6

Y, \$	Y^2
84	7,056
90	8,100
104	10,816
187	34,969
190	36,100
655	97,041

For sample 2 (Asian), use Y to identify the values. With $n = 5$ and $\overline{Y} = 131$, Table 19–4 shows ΣY^2 for Equation [19.12]. Then

$$s^2 = \frac{97{,}041}{4} - \frac{5}{4}(131)^2 = 42{,}260.25 - 1.25(17{,}161) = 2809$$

$$s = \$53$$

The dispersion is smaller for the Asian sample (\$53) than for the North American sample (\$79.31).

(*b*) Equation [19.13] determines the ranges of $\overline{X} \pm 1s$ and $\overline{X} \pm 2s$. Count the number of sample data points between the limits, and calculate the corresponding percentage. See Figure 19–9 for a plot of the data and the standard deviation ranges.

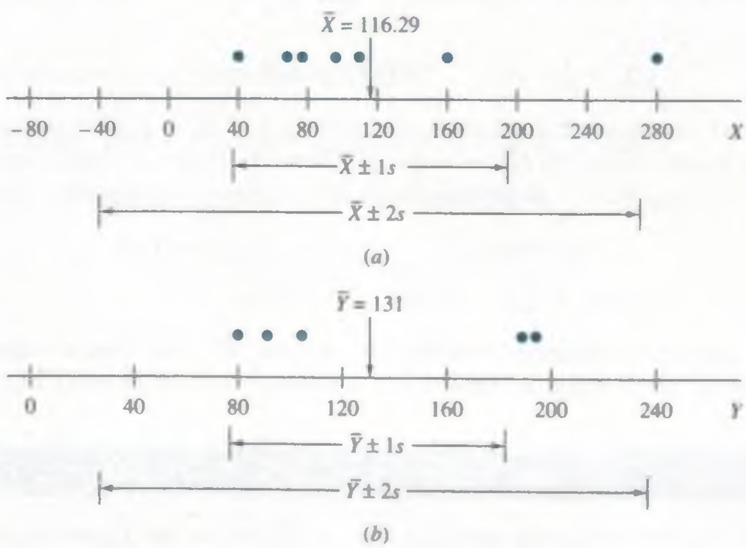


Figure 19–9
Values, averages, and standard deviation ranges for (*a*) North American and (*b*) Asian samples, Example 19.6.

North American sample

$$\overline{X} \pm 1s = 116.29 \pm 79.31 \qquad \text{for a range of \$36.98 to \$195.60}$$

Six out of seven values are within this range, so the percentage is 85.7%.

$$\overline{X} \pm 2s = 116.29 \pm 158.62 \qquad \text{for a range of \$}-42.33 \text{ to \$274.91}$$

There are still six of the seven values within the $\overline{X} \pm 2s$ range. The limit $\$-42.33$ is meaningful only from the probabilistic perspective; from the practical viewpoint, use zero, that is, no amount billed.

Asian sample

$$\overline{Y} \pm 1s = 131 \pm 53 \qquad \text{for a range of \$78 to \$184}$$

There are three of five values, or 60%, within the range.

$$\overline{Y} \pm 2s = 131 \pm 106 \qquad \text{for a range of \$25 to \$237}$$

All five of the values are within the $\overline{Y} \pm 2s$ range.

Comment

A second common measure of dispersion is the *range,* which is simply the largest minus the smallest sample values. In the two samples here, the range estimates are \$235 and \$106.

Only the hand computations for $E(X)$, s, and s^2 have been demonstrated here. Calculators and spreadsheets all have functions to determine these values by simply entering the data.

Before we perform simulation analysis in engineering economy, it may be of use to summarize the expected value and standard deviation relations for a continuous variable, since Equations [19.8] through [19.12] address only discrete variables. The primary differences are that the summation symbol is replaced by the integral over the defined range of the variable, which we identify as R, and that $P(X)$ is replaced by the differential element $f(X)\,dX$. For a stated continuous probability distribution $f(X)$, the formulas are

Expected value: $$E(X) = \int_R Xf(X)\,dX \qquad [19.15]$$

Variance: $$\text{Var}(X) = \int_R X^2f(X)\,dX - [E(X)]^2 \qquad [19.16]$$

For a numerical example, again use the uniform distribution in Example 19.3 (Figure 19–4) over the range R from \$10 to \$15. If we identify the variable as X, rather than C_1, the following are correct.

$$f(X) = \frac{1}{5} = 0.2 \qquad \$10 \leq X \leq \$15$$

$$E(X) = \int_R X(0.2)\,dX = 0.1X^2\Big|_{10}^{15} = 0.1(225 - 100) = \$12.5$$

$$\text{Var}(X) = \int_R X^2(0.2)\,dX - (12.5)^2 = \frac{0.2}{3}X^3\Big|_{10}^{15} - (12.5)^2$$

$$= 0.06667(3375 - 1000) - 156.25 = 2.08$$

$$\sigma = \sqrt{2.08} = \$1.44$$

Therefore, the uniform distribution between $L = \$10$ and $H = \$15$ has an expected value of \$12.5 (the midpoint of the range, as expected) and a standard deviation of \$1.44.

EXAMPLE 19.7

Christy is the regional safety engineer for a chain of franchise-based gasoline and food stores. The home office has had many complaints and several legal actions from employees and customers about slips and falls due to liquids (water, oil, gas, soda, etc.) on concrete surfaces.

Corporate management has authorized each regional engineer to contract locally to apply to all exterior concrete surfaces a newly marketed product that absorbs up to 100 times its own weight in liquid and to charge a home office account for the installation. The authorizing letter to Christy states that, based upon their simulation and random samples that assume a normal population, the cost of the locally arranged installation should be about \$10,000 and almost always is within the range of \$8000 to \$12,000.

You have been asked to write a brief but thorough summary about the normal distribution, explain the \$8000 to \$12,000 range statement, and explain the phrase "random samples that assume a normal population."

Solution

The following summaries about the normal distribution and sampling will help explain the authorization letter.

Normal distribution, probabilities, and random samples

The normal distribution is also referred to as the *bell-shaped curve,* the *Gaussian distribution,* or the *error distribution*. It is, by far, the most commonly used probability distribution in all applications. It places exactly one-half of the probability on either side of the mean or expected value. It is used for continuous variables over the entire range of numbers. The normal distribution is found to accurately predict many types of outcomes, such as IQ values; manufacturing errors about a specified size, volume, weight, etc.; and the distribution of sales revenues, costs, and many other business parameters around a specified mean, which is why it may apply in this situation.

The normal distribution, identified by the symbol $N(\mu, \sigma^2)$, where μ is the expected value or mean and σ^2 is the variance, or measure of spread, can be described as follows:

- The mean μ locates the probability distribution (Figure 19–10*a*), and the spread of the distribution varies with variance (Figure 19–10*b*), growing wider and flatter for larger variance values.
- When a sample is taken, the estimates are identified as sample mean $\overline{X}$ for μ and sample standard deviation s for σ.
- The normal probability distribution $f(X)$ for a variable X is quite complicated, because its formula is

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\left[\frac{(X-\mu)^2}{2\sigma^2}\right]\right\}$$

  where exp represents the number $e = 2.71828$.

Since $f(X)$ is so unwieldy, random samples and probability statements are developed using a transformation, called the *standard normal distribution (SND),* which uses μ and σ (population) or $\overline{X}$ and s (sample) to compute values of the variable Z.

Population : $$Z = \frac{\text{deviation from mean}}{\text{standard deviation}} = \frac{X-\mu}{\sigma} \qquad [19.17]$$

Sample: $$Z = \frac{X-\overline{X}}{s} \qquad [19.18]$$

The SND for Z (Figure 19–10*c*) is the same as for X, except that it always has a mean of 0 and a standard deviation of 1, and it is identified by the symbol $N(0, 1)$. Therefore, the probability values under the SND curve can be stated exactly. It is always possible to transfer back to the original values from sample data by solving Equation [19.17] for X:

$$X = Z\sigma + \mu \qquad [19.19]$$

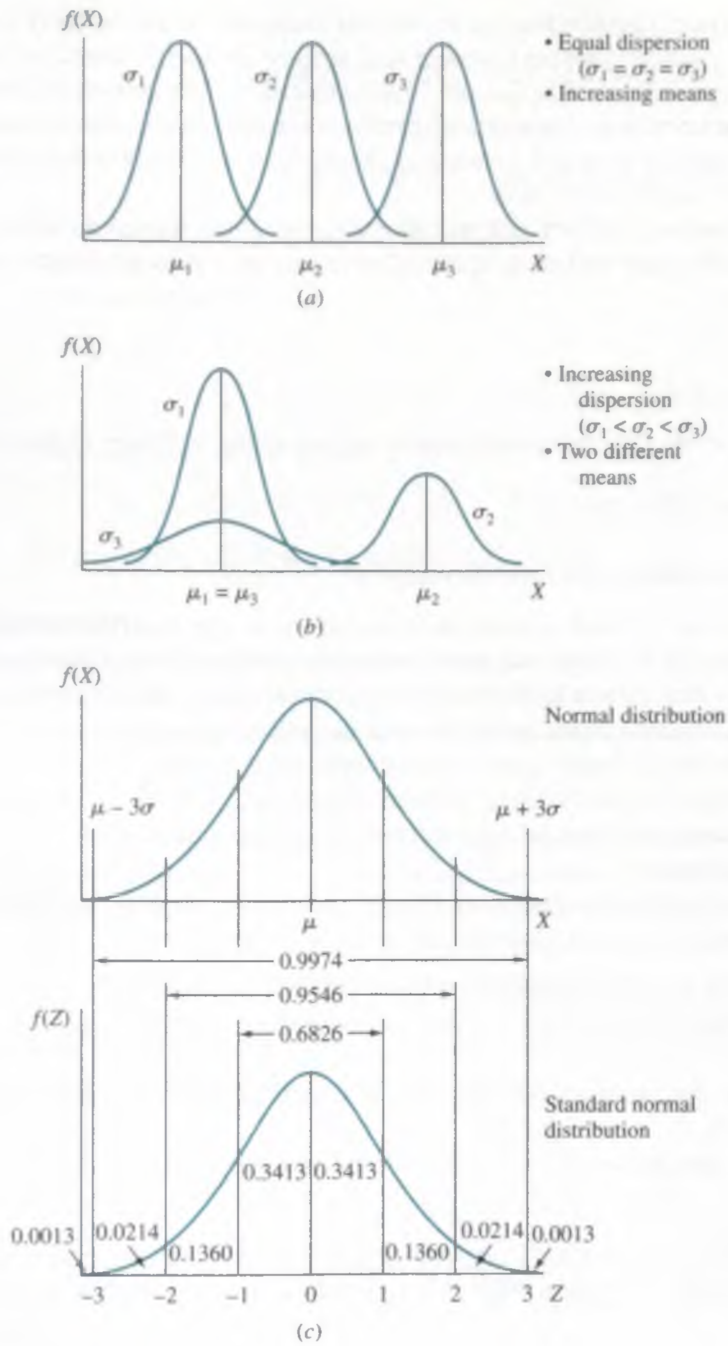


Figure 19–10 Normal distribution showing (a) different mean values μ, (b) different standard deviation values σ, and (c) relation of normal X to standard normal Z.

Several probability statements for Z and X are summarized in the following table and are shown on the distribution curve for Z in Figure 19–10c.

Variable X Range	Probability	Variable Z Range
$\mu + 1\sigma$	0.3413	0 to +1
$\mu \pm 1\sigma$	0.6826	−1 to +1
$\mu + 2\sigma$	0.4773	0 to +2
$\mu \pm 2\sigma$	0.9546	−2 to +2
$\mu + 3\sigma$	0.4987	0 to +3
$\mu \pm 3\sigma$	0.9974	−3 to +3

As an illustration, probability statements from this tabulation and Figure 19–10c for X and Z are as follows:

The probability that X is within 2σ of its mean is 0.9546.

The probability that Z is within 2σ of its mean, which is the same as between the values −2 and +2, is also 0.9546.

In order to take a random sample from a normal $N(\mu, \sigma^2)$ population, a specially prepared table of SND random numbers is used. (Tables of SND values are available in many statistics books.) The numbers are actually values from the Z or $N(0, 1)$ distribution and have values such as -2.10, $+1.24$, etc. Translation from the Z value back to the sample values for X is via Equation [19.19].

Interpretation of the home office memo

The statement that virtually all the local contract amounts should be between \$8000 and \$12,000 may be interpreted as follows: A normal distribution is assumed with a mean of μ = \$10,000 and a standard deviation for σ = \$667, or a variance of $\sigma^2 = (\$667)^2$; that is, an $N[\$10{,}000, (\$667)^2]$ distribution is assumed. The value σ = \$667 is calculated using the fact that virtually all the probability (99.74%) is within 3σ of the mean, as stated above. Therefore,

$$3\sigma = \$2000 \qquad \text{and} \qquad \sigma = \$667 \qquad \text{(rounded off)}$$

As an illustration, if six SND random numbers are selected and used to take a sample of size 6 from the normal distribution $N[\$10{,}000, (\$667)^2]$, the results are as follows:

SND Random Number Z	X Using Equation [19.19] $X = Z\sigma + \mu$
−2.10	$X = (-2.10)(667) + 10{,}000 = \$8599$
+3.12	$X = (+3.12)(667) + 10{,}000 = \$12{,}081$
−0.23	$X = (-0.23)(667) + 10{,}000 = \$9847$
+1.24	$X = (+1.24)(667) + 10{,}000 = \$10{,}827$
−2.61	$X = (-2.61)(667) + 10{,}000 = \$8259$
−0.99	$X = (-0.99)(667) + 10{,}000 = \$9340$

In this sample of six typical concrete surfacing contract amounts for sites in our region, the average is \$9825 and five of six values are within the range of \$8000 to \$12,000, with the sixth being only \$81 above the upper limit.

19.5 Monte Carlo Sampling and Simulation Analysis

Up to this point, all alternative selections have been made using estimates with certainty, possibly followed by some testing of the decision via sensitivity analysis or expected values. In this section, we will use a simulation approach that incorporates the material of the previous sections to facilitate the engineering economy decision about one alternative or between two or more alternatives.

The random sampling technique discussed in Section 19.3 is called **Monte Carlo sampling.** The general procedure outlined below uses Monte Carlo sampling to obtain samples of size n for selected parameters of formulated alternatives. These parameters, expected to vary according to a stated probability distribution, warrant decision making under risk. All other parameters in an alternative are considered certain; that is, they are known, or they can be estimated with enough precision to consider them certain. An important assumption is made, usually without realizing it.

All parameters are **independent;** that is, one variable's distribution does not affect the value of any other variable of the alternative. This is referred to as the property of **independent random variables.**

The simulation approach to engineering economy analysis is summarized in the following basic steps.

Step 1. **Formulate alternative(s).** Set up each alternative in the form to be considered using engineering economic analysis, and select the measure of worth upon which to base the decision. Determine the form of the relation(s) to calculate the measure of worth.

Step 2. **Parameters with variation.** Select the parameters in each alternative to be treated as random variables. Estimate values for all other (certain) parameters for the analysis.

Step 3. Determine probability distributions. Determine whether each variable is discrete or continuous, and describe a probability distribution for each variable in each alternative. Use standard distributions, where possible, to simplify the sampling process and to prepare for computer-based simulation.

Step 4. Random sampling. Incorporate the random sampling procedure of Section 19.3 (the first four steps) into this procedure. This results in the cumulative distribution, assignment of RNs, selection of the RNs, and a sample of size n for each variable.

Step 5. Measure of worth calculation. Compute n values of the selected measure of worth from the relation(s) determined in step 1. Use the estimates made with certainty and the n sample values for the varying parameters. (This is when the property of independent random variables is actually applied.)

Step 6. Measure of worth description. Construct the probability distribution of the measure of worth, using between 10 and 20 cells of data, and calculate measures such as $\overline{X}$, s, $\overline{X} \pm ts$, and relevant probabilities.

Step 7. Conclusions. Draw conclusions about each alternative, and decide which is to be selected. If the alternative(s) has (have) been previously evaluated under the assumption of certainty for all parameters, comparison of results may help with the final decision.

Example 19.8 illustrates this procedure using an abbreviated manual simulation analysis, and Example 19.9 utilizes spreadsheet simulation for the same estimates.

EXAMPLE 19.8

Yvonne Ramos is the CEO of a chain of 50 fitness centers in the United States and Canada. An equipment salesperson has offered Yvonne two long-term opportunities on new aerobic exercise systems, for which the usage is charged to customers on a per-use basis on top of the monthly fees paid by customers. As an enticement, the offer includes a guarantee of annual revenue for one of the systems for the first 5 years.

Since this is an entirely new and risky concept of revenue generation, Yvonne wants to do a careful analysis of each alternative. Details for the two systems follow:

System 1. First cost is P = \$12,000 for a set period of n = 7 years with no salvage value. No guarantee for annual net revenue is offered.

System 2. First cost is P = \$8000, there is no salvage value, and there is a guaranteed annual net revenue of \$1000 for each of the first 5 years, but after this period, there is no guarantee. The equipment with updates may be useful up to 15 years, but the exact number is not known. Cancellation anytime after the initial 5 years is allowed, with no penalty.

For either system, new versions of the equipment will be installed with no added costs. If the MARR is 15% per year, use PW analysis to determine if neither, one, or both of the systems should be installed.

Solution by Hand

Estimates that Yvonne makes to use the simulation analysis procedure are included in the following steps.

Step 1. Formulate alternatives. Using PW analysis, the relations for system 1 and system 2 are developed. The symbol NCF identifies the net cash flows (revenues), and NCF_G is the guaranteed NCF of \$1000 for system 2.

$$\text{PW}_1 = -P_1 + \text{NCF}_1(P/A,15\%,n_1) \qquad [19.20]$$

$$\text{PW}_2 = -P_2 + \text{NCF}_G(P/A,15\%,5) + \text{NCF}_2(P/A,15\%,n_2-5)(P/F,15\%,5) \qquad [19.21]$$

Step 2. Parameters with variation. Yvonne summarizes the parameters estimated with certainty and makes distribution assumptions about three parameters treated as random variables.

System 1

Certainty. $P_1 = \$12{,}000$; $n_1 = 7$ years.

Variable. NCF_1 is a continuous variable, uniformly distributed between $L = \$-4000$ and $H = \$6000$ per year, because this is considered a high-risk venture.

System 2

Certainty. $P_2 = \$8000$; $NCF_G = \$1000$ for first 5 years.

Variable. NCF_2 is a discrete variable, uniformly distributed over the values $L = \$1000$ to $H = \$6000$ only in \$1000 increments, that is, \$1000, \$2000, etc.

Variable. n_2 is a continuous variable that is uniformly distributed between $L = 6$ and $H = 15$ years.

Now, rewrite Equations [19.20] and [19.21] to reflect the estimates made with certainty.

$$\begin{aligned} PW_1 &= -12{,}000 + NCF_1(P/A,15\%,7) \\ &= -12{,}000 + NCF_1(4.1604) \end{aligned} \qquad [19.22]$$

$$\begin{aligned} PW_2 &= -8000 + 1000(P/A,15\%,5) \\ &\quad + NCF_2(P/A,15\%,n_2-5)(P/F,15\%,5) \\ &= -4648 + NCF_2(P/A,15\%,n_2-5)(0.4972) \end{aligned} \qquad [19.23]$$

Step 3. Determine probability distributions. Figure 19–11 (left side) shows the assumed probability distributions for NCF_1, NCF_2, and n_2.

Step 4. Random sampling. Yvonne decides on a sample of size 30 and applies the first four of the random sample steps in Section 19.3. Figure 19–11 (right side) shows the cumulative distributions (step 1) and assigns RNs to each variable (step 2). The RNs for NCF_2 identify the x axis values so that all net cash flows will be in even \$1000 amounts. For the continuous variable n_2, three-digit RN values are used to make the numbers come out evenly, and they are shown in cells only as "indexers" for easy reference when a RN is used to find a variable value. However, we round the number to the next higher value of n_2 because it is likely the contract may be canceled on an anniversary date. Also, now the tabulated compound interest factors for $(n_2 - 5)$ years can be used directly (see Table 19–5).

Once the first RN is selected randomly from Table 19–2, the sequence (step 3) used will be to proceed down the RN table column and then up the column to the left. Table 19–5 shows only the first five RN values selected for each sample and the corresponding variable values taken from the cumulative distributions in Figure 19–11 (step 4).

Step 5. Measure of worth calculation. With the five sample values in Table 19–5, calculate the PW values using Equations [19.22] and [19.23].

1. $PW_1 = -12{,}000 + (-2200)(4.1604) = \$-21{,}153$
2. $PW_1 = -12{,}000 + 2000(4.1604) = \$-3679$
3. $PW_1 = -12{,}000 + (-1100)(4.1604) = \$-16{,}576$
4. $PW_1 = -12{,}000 + (-900)(4.1604) = \$-15{,}744$
5. $PW_1 = -12{,}000 + 3100(4.1604) = \$+897$

1. $PW_2 = -4648 + 1000(P/A,15\%,7)(0.4972) = \$-2579$
2. $PW_2 = -4648 + 1000(P/A,15\%,5)(0.4972) = \$-2981$
3. $PW_2 = -4648 + 5000(P/A,15\%,8)(0.4972) = \$+6507$
4. $PW_2 = -4648 + 3000(P/A,15\%,10)(0.4972) = \$+2838$
5. $PW_2 = -4648 + 4000(P/A,15\%,3)(0.4972) = \$-107$

Now, 25 more RNs are selected for each variable from Table 19–2, and the PW values are calculated.

Step 6. Measure of worth description. Figure 19–12*a* and *b* presents the PW_1 and PW_2 probability distributions for the 30 samples with 14 and 15 cells, respectively, as well as the range of individual PW values and the $\overline{X}$ and s values.

Figure 19-11
Distributions used for random samples, Example 19.8.

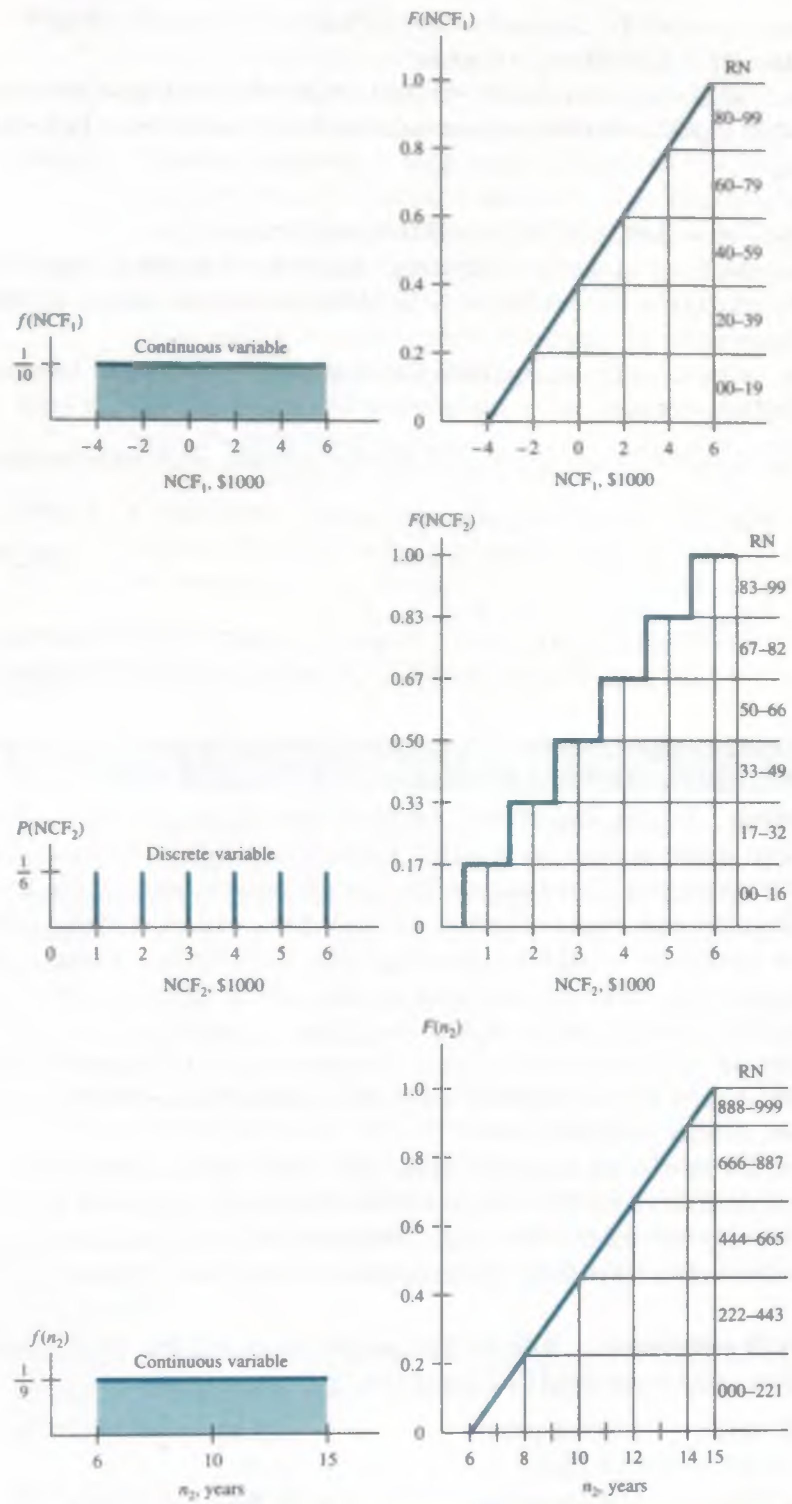


TABLE 19–5 Random Numbers and Variable Values for NCF_1, NCF_2, and n_2, Example 19.8

NCF_1		NCF_2		n_2		
RN*	Value, $	RN†	Value, $	RN‡	Value, Year	Rounded§
18	−2200	10	1000	586	11.3	12
59	+2000	10	1000	379	9.4	10
31	−1100	77	5000	740	12.7	13
29	−900	42	3000	967	14.4	15
71	+3100	55	4000	144	7.3	8

*Randomly start with row 1, column 4 in Table 19–2.
†Start with row 6, column 14.
‡Start with row 4, column 6.
§The n_2 value is rounded up.

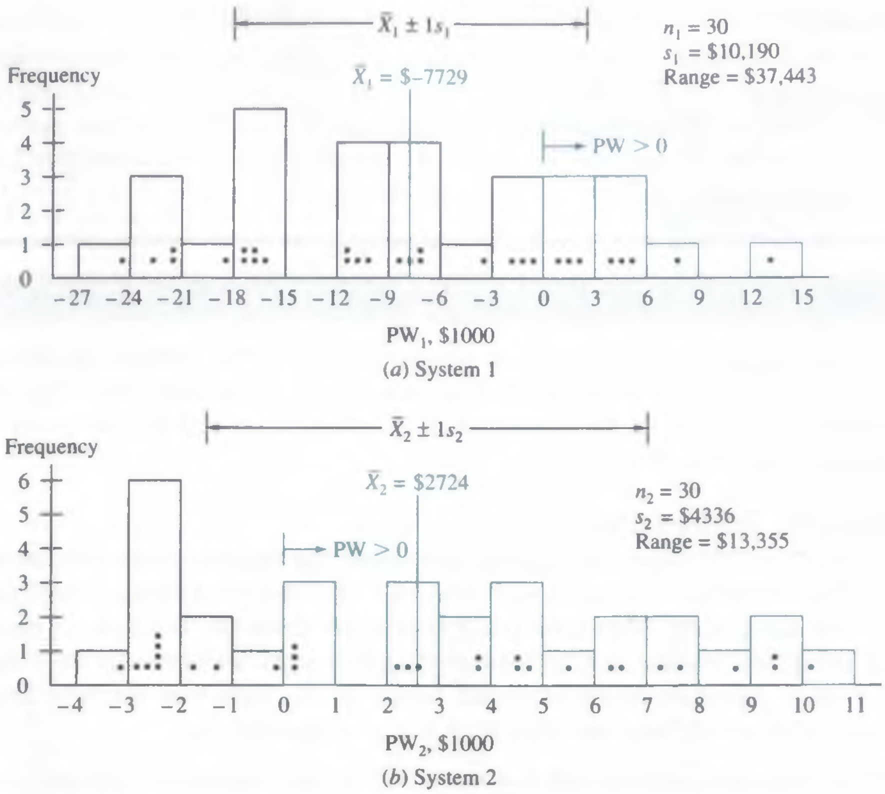


Figure 19–12
Probability distributions of simulated PW values for a sample of size 30, Example 19.8.

PW$_1$. Sample values range from \$–24,481 to \$+12,962. The calculated measures of the 30 values are

$$\bar{X}_1 = \$-7729$$
$$s_1 = \$10,190$$

PW$_2$. Sample values range from \$–3031 to \$+10,324. The sample measures are

$$\bar{X}_2 = \$2724$$
$$s_2 = \$4336$$

Step 7. Conclusions. Additional sample values will surely make the central tendency of the PW distributions more evident and may reduce the s values, which are quite large. Of course, many conclusions are possible once the PW distributions are known, but the following seem clear at this point.

System 1. Based on this small sample of 30 observations, *do not accept* this alternative. The likelihood of making the MARR = 15% is relatively small, since the sample indicates a probability of 0.27 (8 out of 30 values) that the PW will be positive, and $\bar{X}_1$ is a large negative. Though appearing large, the standard deviation may be used to determine that about 20 of the 30 sample PW values (two-thirds) are within the limits $\bar{X} \pm 1s$, which are \$−17,919 and \$2461. A larger sample may alter this analysis somewhat.

System 2. If Yvonne is willing to accept the longer-term commitment that may increase the NCF some years out, the sample of 30 observations indicates to *accept* this alternative. At a MARR of 15%, the simulation approximates the chance for a positive PW as 67% (20 of the 30 PW values in Figure 19–12*b* are positive). However, the probability of observing PW within the $\bar{X} \pm 1s$ limits (\$−1612 and \$7060) is 0.53 (16 of 30 sample values).

Conclusion at this point. Reject system 1; accept system 2; and carefully watch net cash flow, especially after the initial 5-year period.

Comment

The estimates in Example 13.5 are very similar to those here, except all estimates were made with certainty (NCF_1 = \$3000, NCF_2 = \$3000, and n_2 = 14 years). The alternatives were evaluated by the payback period method at MARR = 15%, and the first alternative was selected. However, the subsequent PW analysis in Example 13.5 selected alternative 2 based, in part, upon the anticipated larger cash flow in the later years.

EXAMPLE 19.9

Help Yvonne Ramos set up a spreadsheet simulation for the three random variables and PW analysis in Example 19.8. Does the PW distribution vary appreciably from that developed using manual simulation? Do the decisions to reject the system 1 proposal and accept the system 2 proposal still seem reasonable?

Solution by Spreadsheet

Figures 19–13 and 19–14 are spreadsheet screen shots that accomplish the simulation portion of the analysis described above in steps 3 (determine probability distribution) through 6 (measure of worth description). Most spreadsheet systems are limited in the variety of distributions they can accept for sampling, but common ones such as uniform and normal are available.

Figure 19–13 shows the results of a small sample of 30 values from the three distributions using the RAND and IF functions. (See Section A.3 in Appendix A.)

NCF_1: Continuous uniform from \$–4000 to \$6000. The spreadsheet relation in column B translates RN1 values (column A) into NCF1 amounts.

NCF_2: Discrete uniform in \$1000 increments from \$1000 to \$6000. Column D cells display NCF2 in the \$1000 increments using the logical IF function to translate from the RN2 values.

n_2: Continuous uniform from 6 to 15 years. The results in column F are integer values obtained using the INT function operating on the RN3 values.

	A	B	C	D	E	F
1	Sample of size 30 simulated values					
2	RN1	NCF1, \$	RN2	NCF2, \$	RN3	N2, years
3	12.5625	-2,800	83.6176	6,000	556.2768	12
4	25.0262	-1,500	99.5425	6,000	8.7883	7
5	9.3856	-3,100	26.4693	2,000	507.3598	11
6	38.0199	-200	36.8475	3,000	681.5397	13
7	71.5088	3,100	83.4610	6,000	369.0917	10
8	66.7820	2,600	77.8699	5,000	91.3044	7
9	48.3324	800	8.4308	1,000	457.7487	11
10	39.3886	-100	52.8630	4,000	914.5432	15
11	21.5429	-1,900	57.4819	4,000	698.7624	13
12	44.4996	400	1.9322	1,000	744.2622	13
13	32.9911	-800	70.6307	5,000	190.8139	8
14	96.0249	5,600	61.0023	4,000	714.6685	13
15	99.6875	5,900	55.7741	4,000	648.2268	12
16	13.9560	-2,700	98.9107	6,000	199.9491	8
17	99.8535	5,900	10.7429	1,000	716.5830	13
18	63.2953	2,300	4.6540	1,000	133.4986	8
19	93.0860	5,300	56.7425	4,000	553.2489	11
20	52.6539	1,200	17.1873	2,000	809.5778	14
21	34.1609	-600	46.3758	3,000	810.4792	14
22	86.0288	4,600	99.6589	6,000	950.9657	15
23	46.9626	600	24.9754	2,000	0.9088	7
24	77.4690	3,700	52.2862	4,000	339.4470	10
25	28.5200	-1,200	52.3113	4,000	514.9377	11
26	82.9615	4,200	99.0275	6,000	912.6720	15
27	17.8793	-2,300	50.9493	4,000	800.4352	14
28	89.2411	4,900	6.2988	1,000	118.2531	8
29	9.0495	-3,100	54.6552	4,000	56.4377	7
30	1.4597	-3,900	24.6463	2,000	716.2222	13
31	66.2177	2,600	6.2064	1,000	91.4505	7
32	98.4803	5,800	94.8061	6,000	120.0419	8
33	= RAND()*100	= INT((100*A32-4000)/100)*100	= RAND()*100	= IF(C32<=16,1000, IF(C32<=32,2000, IF(C32<=49,3000, IF(C32<=66,4000, IF(C32<=82,5000, IF(C32<=100,[illegible]000,6000))))))	= RAND()*1000	= INT(0.009*E32+1)+6

Figure 19–13

Random sample of 30 values generated for spreadsheet simulation, Example 19.9.

	A	B	C	D	E	F	G	H
1				Information about Alternatives				
2	System 1, P1	$12,000		System 2	P2	$8,000		
3	n	7	years		NCF	$1,000	5	years
4	MARR	15%			MARR	15%		
5								
6				Results of Analysis				
7	# PW ≥ MARR		10				19	
8	# PW < MARR			20				11
9	PW Average, $	-7,105				1,649		
10	PW Std Dev, $	13,199				3,871		
11								
12				Present Worth Computations				
13	PW1	-23,649	0	1	PW2	7,763	1	0
14	PW1	-18,241	0	1	PW2	202	1	0
15	PW1	-24,897	0	1	PW2	-885	0	1
16	PW1	-12,832	0	1	PW2	2,045	1	0
17	PW1	897	1	0	PW2	5,352	1	0
18	PW1	-1,183	0	1	PW2	-607	0	1
19	PW1	-8,672	0	1	PW2	-2,766	0	1
20	PW1	-12,416	0	1	PW2	5,333	1	0
21	PW1	-19,905	0	1	PW2	4,276	1	0
22	PW1	-10,336	0	1	PW2	-2,417	0	1
23	PW1	-15,328	0	1	PW2	1,028	1	0
24	PW1	11,298	1	0	PW2	4,276	1	0
25	PW1	12,546	1	0	PW2	3,626	1	0
26	PW1	-23,233	0	1	PW2	2,163	1	0
27	PW1	12,546	1	0	PW2	-2,417	0	1
28	PW1	-2,431	0	1	PW2	-3,513	0	1
29	PW1	10,050	1	0	PW2	2,878	1	0
30	PW1	-7,007	0	1	PW2	97	1	0
31	PW1	-14,496	0	1	PW2	2,469	1	0
32	PW1	7,138	1	0	PW2	10,323	1	0
33	PW1	-9,504	0	1	PW2	-3,031	0	1
34	PW1	3,394	1	0	PW2	2,019	1	0
35	PW1	-16,993	0	1	PW2	2,878	1	0
36	PW1	5,474	1	0	PW2	10,323	1	0
37	PW1	-21,569	0	1	PW2	4,841	1	0
38	PW1	8,386	1	0	PW2	-3,513	0	1
39	PW1	-24,897	0	1	PW2	-1,415	0	1
40	PW1	-28,226	0	1	PW2	-186	0	1
41	PW1	-1,183	0	1	PW2	-3,840	0	1
42	PW1	12,130	1	0	PW2	2,163	1	0
43	Functions for last row	'= (PV(B4,B3,-'Random [illegible]	'= IF(B42 >= 0,1,0)	'= IF(B42 < 0,1,0)	'= -F2-PV(F4,G3,F3) +PV(F4,G3,(PV(F4,'Random Numbers'!F32-G3,'Random Numbers'!D32)))		'= IF(F42 >= 0,1,0)	'= IF(F42 < 0,1,0)

=SUM(G13:G42)

=AVERAGE(F13:F42)

=STDEV(F13:F42)

Fig 19-13 Random Numbers | Fig 19-14 PW values | Sheet5 | Sheet8 | Sheet9 | Sheet10 | Sheet11 | Sheet

Figure 19–14
Simulation results for 30 PW values, Example 19.9.

Figure 19–14 presents the two alternatives' estimates in the top section. The PW1 and PW2 computations for the 30 repetitions of NCF1, NCF2, and N2 are the spreadsheet equivalent of Equations [19.22] and [19.23]. The tabular approach used here tallies the number of PW values below zero ($0) and equal to or exceeding zero using the IF operator. For example, cell C17 contains a 1, indicating PW1 > 0 when NCF1 = $3100 (in cell B7 of Figure 19–13), which was used to calculate PW1 = $897 by Equation [19.22]. Cells in rows 7 and 8 show the number of times in the 30 samples that system 1 and system 2 may return at least the MARR = 15% because the corresponding PW ≥ 0. Sample averages and standard deviations are also indicated.

Comparison between the hand and spreadsheet simulations is presented below.

	System 1 PW			System 2 PW		
	$\overline{X}$, $	s, $	No. of PW ≥ 0	$\overline{X}$, $	s, $	No. of PW ≥ 0
Hand	−7,729	10,190	8	2,724	4,336	20
Spreadsheet	−7,105	13,199	10	1,649	3,871	19

For the spreadsheet simulation, 10 (33%) of the PW1 values exceed zero, while the manual simulation included 8 (27%) positive values. These comparative results will change every time this spreadsheet is activated since the RAND function is set up (in this case) to produce a new RN each time. (It is possible to define RAND to keep the same RN values. See the Excel User's Guide.)

The conclusion to reject the system 1 proposal and accept system 2 is still appropriate for the spreadsheet simulation as it was for the hand solution, since there are comparable chances that PW ≥ 0.

CHAPTER SUMMARY

To perform decision making under risk implies that some parameters of an engineering alternative are treated as random variables. Assumptions about the shape of the variable's probability distribution are used to explain how the estimates of parameter values may vary. Additionally, measures such as the expected value and standard deviation describe the characteristic shape of the distribution. In this chapter, we learned several of the simple, but useful, discrete and continuous population distributions used in engineering economy—uniform and triangular—as well as specifying our own distribution or assuming the normal distribution.

Since the population's probability distribution for a parameter is not fully known, a random sample of size n is usually taken, and its sample average and standard deviation are determined. The results are used to make probability statements about the parameter, which help make the final decision with risk considered.

The Monte Carlo sampling method is combined with engineering economy relations for a measure of worth such as PW to implement a simulation approach to risk analysis. The results of such an analysis can then be compared with decisions when parameter estimates are made with certainty.

PROBLEMS

Certainty, Risk, and Uncertainty

19.1 Identify the following variables as either discrete or continuous.

(*a*) The interest rates available in the marketplace for jumbo certificates of deposit
(*b*) Optimistic, most likely, and pessimistic estimates of salvage value
(*c*) The number of cars that are red in the first 100 that pass through a certain intersection
(*d*) The weight of the purse or wallet carried when a person leaves her or his residence
(*e*) The gallons of water that evaporate from Lake Erie in a given day

19.2 For each situation below, determine (1) if the variable is discrete or continuous and (2) if the information involves certainty, risk, and/or uncertainty.

(*a*) The first cost of a new front-end loader is \$34,000 or \$38,000 depending on the size purchased.
(*b*) The raises for engineers and technical staff employees will be 3%, or 5%, with one-half getting 3% and one-half getting 5%.
(*c*) Revenue from a new product line is expected to be between \$350,000 and \$475,000 per year.
(*d*) The salvage value for an old machine will be \$500 (i.e., its asking price) or \$0 (it will be thrown away).
(*e*) Profits are equally likely to be up anywhere from 25% to 60% this year.

19.3 An engineer learned that production output is between 1000 and 2000 units per week 90% of the time, and it may fall below 1000 or go above 2000. He wants to use E(output) in the decision-making process. Identify at least two additional pieces of information that must be obtained or assumed to finalize the output information for this use.

Probability and Distributions

19.4 Royalties received by an investor in an oil well vary according to the price of oil. Data collected from stripper wells in an established oil field were used to develop the probability-royalty relationship shown below.

Royalties, \$ per Year	6200	8500	9600	10,300	12,600	15,500
Probability	0.10	0.21	0.32	0.24	0.09	0.04

(*a*) Calculate the expected value of royalty income (RI) per year.
(*b*) Determine the probability that the royalty income will be at least \$12,600 per year.

19.5 Daily revenue from vending machines placed in various buildings of a major university is as follows:

20, 75, 43, 62, 51, 52, 78, 33, 28, 39, 61, 56, 43, 49, 48, 49, 71, 53, 57, 46, 42, 41, 63, 36, 51, 59, 40, 32, 37, 29, 26

(*a*) Construct a frequency distribution table with a cell size of 12 starting with 19.5 (i.e., first cell is 19.5–31.5, next is 31.5–43.5, etc.).
(*b*) Determine the probability distribution.
(*c*) What is the probability of revenue from a machine being less than \$44?
(*d*) What is the probability that revenue from a machine will equal or exceed \$44?

19.6 A survey of households included a question about the number of operating automobiles N currently owned by people living at the residence and the interest rate i on the lowest-rate loan for the cars. The results for 100 households are shown.

Number of Cars N	Households
0	12
1	56
2	26
3	3
≥4	3

Loan Rate i, %	Households
0.0–2	13
2.01–4	14
4.01–6	19
6.01–8	38
8.01–10	12
10.01–12	4

(*a*) State whether each variable is discrete or continuous.
(*b*) Plot the probability distributions and cumulative distributions for N and i.
(*c*) From the data collected, what is the probability that a household has 1 or 2 cars? Three or more cars?
(*d*) Use the data for i to estimate the chances that the interest rate is between 7% and 11% per year.

19.7 An officer of the state lottery commission sampled lottery ticket purchasers over a 1-week period at one location. The amounts distributed back to the purchasers and the associated probabilities for 5000 tickets are as follows:

Distribution, $	0	2	5	10	100
Probability	0.91	0.045	0.025	0.013	0.007

(*a*) Plot the cumulative distribution of winnings.
(*b*) Calculate the expected value of the distribution of dollars per ticket.
(*c*) If tickets cost $2, what is the expected long-term income to the state per ticket, based upon this sample?

19.8 Bob is working on two separate probability-related projects. The first involves a variable N, which is the number of consecutively manufactured parts that weigh in above the weight specification limit. The variable N is described by the formula $(0.5)^N$ because each unit has a 50-50 chance of being below or above the limit. The second involves a battery life L that varies between 2 and 5 months. The probability distribution is triangular with the mode at 5 months, which is the design life. Some batteries fail early, but 2 months is the smallest life experienced thus far. (*a*) Write out and plot the probability distributions and cumulative distributions for Bob. (*b*) Determine the probability of N being 1, 2, or 3 consecutive units above the weight limit.

19.9 An alternative to buy and an alternative to lease hydraulic lifting equipment have been formulated. Use the parameter estimates and assumed distribution data shown to plot the probability distributions on one graph for each corresponding parameter. Label the parameters carefully.

Purchase Alternative

Parameter	Estimated Value High	Estimated Value Low	Assumed Distribution
First cost, $	25,000	20,000	Uniform; continuous
Salvage value, $	−3,000	2,000	Triangular; mode at $2500
Life, years	8	4	Triangular; mode at 6
AOC, $ per year	9,000	5,000	Uniform; continuous

Lease Alternative

Parameter	Estimated Value High	Estimated Value Low	Assumed Distribution
Lease first cost, $	2000	1800	Uniform; continuous
AOC, $ per year	9000	5000	Triangular; mode at $7000
Lease term, years	3	3	Certainty

19.10 Carla is a statistician with a bank. She has collected debt-to-equity mix data on mature (M) and young (Y) companies. The debt percentages vary from 20% to 80% in her sample. Carla has defined D_M as a variable for the mature companies from 0 to 1, with $D_M = 0$ interpreted as the low of 20% debt and $D_M = 1.0$ as the high of 80% debt. The variable for young corporation debt percentages D_Y is similarly defined. The probability distributions used to describe D_M and D_Y are

$$f(D_M) = 3(1-D_M)^2 \qquad 0 \le D_M \le 1$$
$$f(D_Y) = 2D_Y \qquad 0 \le D_Y \le 1$$

(*a*) Use different values of the debt percentage between 20% and 80% to calculate values for the probability distributions and then plot them. (*b*) What can you comment about the probability that a mature company or a young company will have a low debt percentage? A high debt percentage?

19.11 A discrete variable X can take on integer values of 1 to 10. A sample of size 50 results in the following probability estimates:

X_i	1	2	3	6	9	10
$P(X_i)$	0.2	0.2	0.2	0.1	0.2	0.1

(*a*) Write out and graph the cumulative distribution.
(*b*) Calculate the following probabilities using the cumulative distribution: X is between 6 and 10, and X has the values 4, 5, or 6.
(*c*) Use the cumulative distribution to show that $P(X = 7 \text{ or } 8) = 0.0$. Even though this probability is zero, the statement about X is that it can take on integer values of 1 to 10. How do you explain the apparent contradiction in these two statements?

Random Samples

19.12 A discrete variable X can take on integer values of 1 to 5. A sample of size 100 results in the following probability estimates.

X_i	1	2	3	4	5
$P(X_i)$	0.2	0.3	0.1	03	0.1

(*a*) Use the following random numbers to estimate the probabilities for each value of X.
(*b*) Determine the sample probabilities for $X = 1$ and $X = 5$. Compare the sample results with the probabilities in the problem statement.

RN: 10, 42, 18, 31, 23, 80, 80, 26, 74, 71, 03, 90, 55, 61, 61, 28, 41, 49, 00, 79, 96, 78, 42, 31, 26

19.13 The percent price increase p on a variety of retail food prices over a 1-year period varied from 5% to 10% in all cases. Because of the distribution of p values, the assumed probability distribution for the next year is

$$f(X) = 2X \qquad 0 \leq X \leq 1$$

where

$$X = \begin{cases} 0 & \text{when } p = 5\% \\ 1 & \text{when } p = 10\% \end{cases}$$

For a continuous variable the cumulative distribution $F(X)$ is the integral of $f(X)$ over the same range of the variable. In this case

$$F(X) = X^2 \qquad 0 \leq X \leq 1$$

(*a*) Graphically assign RNs to the cumulative distribution, and take a sample of size 30 for the variable. Transform the X values into interest rates.
(*b*) Calculate the average p value for the sample.

19.14 Develop a discrete probability distribution of your own for the variable G, the expected grade in this course, where G = A, B, C, D, F, or I (incomplete). Assign random numbers to $F(G)$, and take a sample from it. Now plot the probability values from the sample for each G value.

19.15 Use the RAND function in Excel to generate 100 values from a $U(0,1)$ distribution.
(*a*) Calculate the average and compare it to 0.5, the expected value for a random sample between 0 and 1.
(*b*) For the RAND function sample, cluster the results into cells of 0.1 width, that is 0.0–0.1, 0.1–0.2, etc., where the upper-limit value is excluded from each cell. Determine the probability for each grouping from the results. Does your sample come close to having approximately 10% in each cell?

Sample Estimates

19.16 An engineer was asked to determine whether the average air quality in a vehicle assembly plant was within OSHA guidelines. The following air quality readings were collected:

81, 86, 80, 91, 83, 83, 96, 85, 89

(*a*) Determine the sample mean.
(*b*) Calculate the standard deviation.
(*c*) Determine the number of values and percent of values that fall within ±1 standard deviation of the mean.

19.17 Carol sampled the monthly maintenance costs for automated soldering machines a total of 100 times during 1 year. She clustered the costs into $200 cells, for example, $500 to $700, with cell midpoints of $600, $800, $1000, etc. She indicated the number of times (frequency) each cell value was observed. The costs and frequency data are as follows.

Cell Midpoint, $	Frequency
600	6
800	10
1000	7
1200	15
1400	28
1600	15
1800	9
2000	10

(*a*) Estimate the expected value and standard deviation of the maintenance costs the company should anticipate based on Carol's sample.
(*b*) What is the best estimate of the percentage of costs that will fall within 2 standard deviations of the mean?

(c) Develop a probability distribution of the monthly maintenance costs from Carol's sample, and indicate the answers to the previous two questions on it.
(d) Use a spreadsheet to display the sample mean.

19.18 (a) Determine the values of sample average and standard deviation of the data in Problem 19.11. (b) Determine the values 1 and 2 standard deviations from the mean. Of the 50 sample points, how many fall within these two ranges?

19.19 (a) Use the relations in Section 19.4 for continuous variables to determine the expected value and standard deviation for the distribution of $f(D_Y)$ in Problem 19.10. (b) It is possible to calculate the probability of a continuous variable X between two points (a, b) using the following integral:

$$P(a \leq X \leq b) = \int_a^b f(X)\,dx$$

Determine the probability that D_Y is within 2 standard deviations of the expected value.

19.20 (a) Use the relations in Section 19.4 for continuous variables to determine the expected value and variance for the distribution of D_M in Problem 19.10.

$$f(D_M) = 3(1 - D_M)^2 \qquad 0 \leq D_M \leq 1$$

(b) Determine the probability that D_M is within 2 standard deviations of the expected value. Use the relation in Problem 19.19.

19.21 Calculate the expected value for the variable N in Problem 19.8.

19.22 A newsstand manager is tracking Y, the number of weekly magazines left on the shelf when the new edition is delivered. Data collected over a 30-week period are summarized by the following probability distribution. Plot the distribution and the estimates for expected value and 1 standard deviation on either side of $E(Y)$ on the plot.

Y copies	3	7	10	12
P(Y)	1/3	1/4	1/3	1/12

Simulation

19.23 Carl, an engineering colleague, estimated net cash flow after taxes (CFAT) for the project he is working on. The additional CFAT of \$2800 in year 10 is the salvage value of capital assets.

Year	CFAT, \$
0	−28,800
1–6	5,400
7–10	2,040
10	2,800

The PW value at the current MARR of 7% per year is

$$\begin{aligned} PW &= -28{,}800 + 5400(P/A,7\%,6) \\ &\quad + 2040(P/A,7\%,4)(P/F,7\%,6) \\ &\quad + 2800(P/F,7\%,10) \\ &= \$2966 \end{aligned}$$

Carl believes the MARR will vary over a relatively narrow range, as will the CFAT, especially during the out years of 7 through 10. He is willing to accept the other estimates as certain. Use the following probability distribution assumptions for MARR and CFAT to perform a simulation—hand- or spreadsheet-based.

MARR. Uniform distribution over the range 6% to 10%.
CFAT, years 7 through 10. Uniform distribution over the range \$1600 to \$2400 for each year.

Plot the resulting PW distribution. Should the plan be accepted using decision making under certainty? Under risk?

19.24 Repeat Problem 19.23, except use the normal distribution for the CFAT in years 7 through 10 with an expected value of \$2000 and a standard deviation of \$500.

ADDITIONAL PROBLEMS AND FE EXAM REVIEW QUESTIONS

19.25 When there are at least two values for a parameter and it is possible to estimate the chance that each may occur, this situation is known as:
(a) Uncertainty
(b) Risk
(c) Standard deviation
(d) Cost estimating

19.26 A *deterministic* economic analysis is one wherein:
(a) Single-value estimates are used exclusively
(b) Risk is taken into account
(c) A range of values for each parameter is included in the analysis
(d) Taxes and inflation are not considered in the cash flow estimates

19.27 Decision making under risk includes all of the following except:
(*a*) Expected value analysis
(*b*) Simulation
(*c*) Using only single-value estimates
(*d*) Probability

19.28 All of the following are elements in decision making under risk except:
(*a*) Random variable
(*b*) Cost indexes
(*c*) Probability
(*d*) Cumulative distribution

19.29 For the income and probability values shown in the table, the probability that the income in any year will be less than $9600 is closest to:
(*a*) 0.15
(*b*) 0.23
(*c*) 0.32
(*d*) 0.38

Income, $ per Year	6200	8500	9600	10,300	12,600	15,500
Probability	0.15	0.23	0.32	0.24	0.09	0.04

19.30 The symbol that represents the true population mean is:
(*a*) σ
(*b*) s
(*c*) μ
(*d*) $\overline{X}$

19.31 The revenue from an oil dispersant product has averaged $15,000 per month for the past 12 months. If the value of $\Sigma(X_i - \overline{X})^2$ is $1,600,000, the standard deviation is closest to:
(*a*) $381
(*b*) $652
(*c*) $958
(*d*) $1265

19.32 A survey of the types of cars parked at an NFL football stadium revealed that there were equal probabilities of finding cars identified as type A, B, C, and D. Car types were assigned random numbers as follows.

Car Type	Assigned RN
A	0 through 24
B	25 through 49
C	50 through 74
D	75 through 99

The sample probability of a type B car from the 12 random numbers shown is closest to:

RN: 75, 52, 14, 99, 67, 74, 06, 50, 97, 46, 27, 88

(*a*) 0.17
(*b*) 0.25
(*c*) 0.33
(*d*) 0.42

CASE STUDY

USING SIMULATION AND THREE-ESTIMATE SENSITIVITY ANALYSIS

Background

The Knox Brewing company makes specialty-named sodas and flavored drinks for retail grocery chains throughout North and Central America. During the past year, it has become obvious that a new bottle-capping machine is needed to replace the current 10-year-old system. Dr. Knox, the owner and president, knows the business quite well. You just handed him the first cost bids from three vendors for the machine. He looked carefully at the numbers and asked you to sit down. You were quite surprised, as this was the first time you had been in his office, and most other engineers at Knox have a great fear of "the Old Man."

Information

As he examined the three bids on first cost of the machine, he started to write some numbers, which, he explained, were his estimates of the annual operating cost, useful life, and possible salvage value for each of the machines sold by the three vendors.

After a few minutes, he told you to take these numbers and use some of that "new engineering knowledge" you acquired in college to determine which, if any, of these three bids made the best economic sense. He also told you to be innovative and use a computer and some probability to come up with a robust recommendation by tomorrow at 2 P.M.

You have used the estimates from the president to develop Table 19–6 of pessimistic (P), most likely (ML), and optimistic (O) estimates for each vendor's machine. In addition, you developed some possible distributions for the parameters that Dr. Knox estimated, namely, AOC, life, and salvage value. These are summarized in Table 19–7. You plan to use a simple Monte Carlo simulation to help formulate your recommendation for tomorrow.

Case Study Exercises

First, learn to use the RNG (random number generator) in Excel, if you have not already done so. It is necessary to

TABLE 19–6 Parameter Estimates for Bottle-Capping Machine

	First Cost, $	AOC, $ per Year	Salvage, $	Life, Years
Vendor 1				
P	−200,000	−11,000	0	3
ML	−200,000	−10,000	0	5
O	−200,000	−6,000	0	8
Vendor 2				
P	−150,000	−5,000	0	2
ML	−150,000	−3,500	5,000	4
O	−150,000	−2,000	8,000	7
Vendor 3				
P	−300,000	−8,000	5,000	5
ML	−300,000	−6,000	5,000	7
O	−300,000	−4,500	8,000	9

TABLE 19–7 Distribution Assumptions about AOC, Life, and Salvage

Parameter	Vendor 1	Vendor 2	Vendor 3
AOC, $ per year	Normal Mean: 10,000 Std. dev.: 500	Normal Mean: 3500 Std. dev.: 500	Normal Mean: 6000 Std. dev.: 500
Salvage, $	Uniform 0 to 1000	Uniform 0 to 8000	Uniform 5000 to 8000
Life, years	Discrete uniform 3 to 8, equal probability	Discrete uniform 2 to 7, equal probability	Discrete uniform 5 to 9, equal probability

sample from the normal distributions that you have specified in Table 19–7. RNG is part of the Analysis Tool-Pak accessed through the Office Button, Excel Options, Add-Ins path.

1. Prepare the simulation using a spreadsheet; determine which of the vendors offers the best machine from an economic perspective, and take into account the estimates made by Dr. Knox. Use a sample size of at least 50, and base your conclusions on the AW measure of worth.
2. Prepare a short presentation for Dr. Knox (and class) using your analysis.

APPENDIX A

USING SPREADSHEETS AND MICROSOFT EXCEL©

This appendix explains the layout of a spreadsheet and the use of Microsoft Excel (hereafter called Excel) functions in engineering economy. Refer to the Excel help system for your particular computer and version of Excel. Some specific commands and entries refer to Excel 2007 and may differ slightly from your version.

A.1 Introduction to Using Excel

Enter a Formula or Use an Excel Function

The **= sign is required** to perform any formula or function computation in a cell. The formulas and functions on the worksheet can be displayed by simultaneously pressing Ctrl and `. The symbol ` is usually in the upper left of the keyboard with the ~ (tilde) symbol. Pressing Ctrl+` a second time hides the formulas and functions.

1. Run Excel.
2. Move to cell C3. (Move the pointer to C3 and left-click.)
3. Type = PV(5%,12,10) and <Enter>. This function will calculate the present value of 12 payments of $10 at a 5% per year interest rate.

Another example: To calculate the future value of 12 payments of $10 at 6% per year interest, do the following:

1. Move to cell B3, and type INTEREST.
2. Move to cell C3, and type 6% or = 6/100.
3. Move to cell B4, and type PAYMENT.
4. Move to cell C4, and type 10 (to represent the size of each payment).
5. Move to cell B5, and type NUMBER OF PAYMENTS.
6. Move to cell C5, and type 12 (to represent the number of payments).
7. Move to cell B7, and type FUTURE VALUE.
8. Move to cell C7, and type = FV(C3,C5,C4) and hit <Enter>. The answer will appear in cell C7.

To edit the values in cells

1. Move to cell C3 and type =5/100 (the previous value will be replaced).
2. The value in cell C7 will update.

Cell References in Formulas and Functions

If a cell reference is used in lieu of a specific number, it is possible to change the number once and perform sensitivity analysis on any variable that is referenced by the cell number, such as C5. This approach defines the referenced cell as a **global variable** for the worksheet. There are two types of cell references—relative and absolute.

Relative References If a cell reference is entered, for example, A1, into a formula or function that is copied or dragged into another cell, the reference is changed relative to the movement of the original cell. If the formula in C5 is = A1 and it is copied into cell C6, the formula is changed to = A2. This feature is used when dragging a function through several cells, and the source entries must change with the column or row.

Absolute References If adjusting cell references is not desired, place a **$ sign** in front of the part of the cell reference that is not to be adjusted—the column, row, or both. For example,

= A1 will retain the formula when it is moved anywhere on the worksheet. Similarly, = $A1 will retain the column A, but the relative reference on 1 will adjust the row number upon movement around the worksheet.

Absolute references are used in engineering economy for sensitivity analysis of parameters such as MARR, first cost, and annual cash flows. In these cases, a change in the absolute-reference cell entry can help determine the sensitivity of a result, such as PW or AW.

Print the Spreadsheet

First define the portion (or all) of the spreadsheet to be printed.

1. Move the pointer to the top left cell of your spreadsheet.
2. Hold down the left-click button. (Do not release the left-click button.)
3. Drag the mouse to the lower right corner of your spreadsheet or to wherever you want to stop printing.
4. Release the left-click button. (It is ready to print.)
5. Left-click the Office button (see Figure A–1).
6. Move the pointer down to select Print and left-click.
7. In the dialog box, left-click the Print option (or similar command).

Depending on your computer environment, you may have to select a network printer and queue your printout through a server.

Save the Spreadsheet

You can save your spreadsheet at any time during or after completing your work. It is recommended that you save your work regularly.

1. Left-click the Office button.
2. To save the spreadsheet the first time, left-click the Save As . . . option.
3. Type the file name, e.g., Prob 7.9, and left-click the Save button.

To save the spreadsheet after it has been saved the first time, i.e., a file name has been assigned to it, left-click the Office button, move the pointer down, and left-click on Save.

Create an *xy* (Scatter) Chart

This chart is one of the most commonly used in scientific analysis, including engineering economy. It plots pairs of data and can place multiple series of entries on the *Y* axis. The *xy* scatter chart is especially useful for results such as the PW versus *i* graph, where *i* is the *X* axis and the *Y* axis displays the results of the NPV function for several alternatives.

1. Run Excel.
2. Enter the following numbers in columns A, B, and C, respectively.
   Column A, cell A1 through A6: Rate *i*%, 4, 6, 8, 9, 10
   Column B, cell B1 through B6: $ for A, 40, 55, 60, 45, 10
   Column C, cell C1 through C6: $ for B, 100, 70, 65, 50, 30.
3. Move the mouse to A1, left-click, and hold while dragging to cell C6. All cells will be highlighted, including the title cell for each column.
4. If not all the columns for the chart are adjacent to one another, first press and hold the Control key on the keyboard during the entirety of step 3. After dragging over one column of data, momentarily release the left click, then move to the top of the next (nonadjacent) column of the chart. Do not release the Control key until all columns to be plotted have been highlighted.
5. Left-click on the Insert button on the toolbar.
6. Select the Scatter option and choose a subtype of scatter chart. The graph appears with a legend (Figure A–1).

Now a large number of styling effects can be introduced for axis titles, legend, data series, etc. Note that only the bottom row of the title can be highlighted. If titles are not highlighted, the data sets are generically identified as series 1, series 2, etc. on the legend.

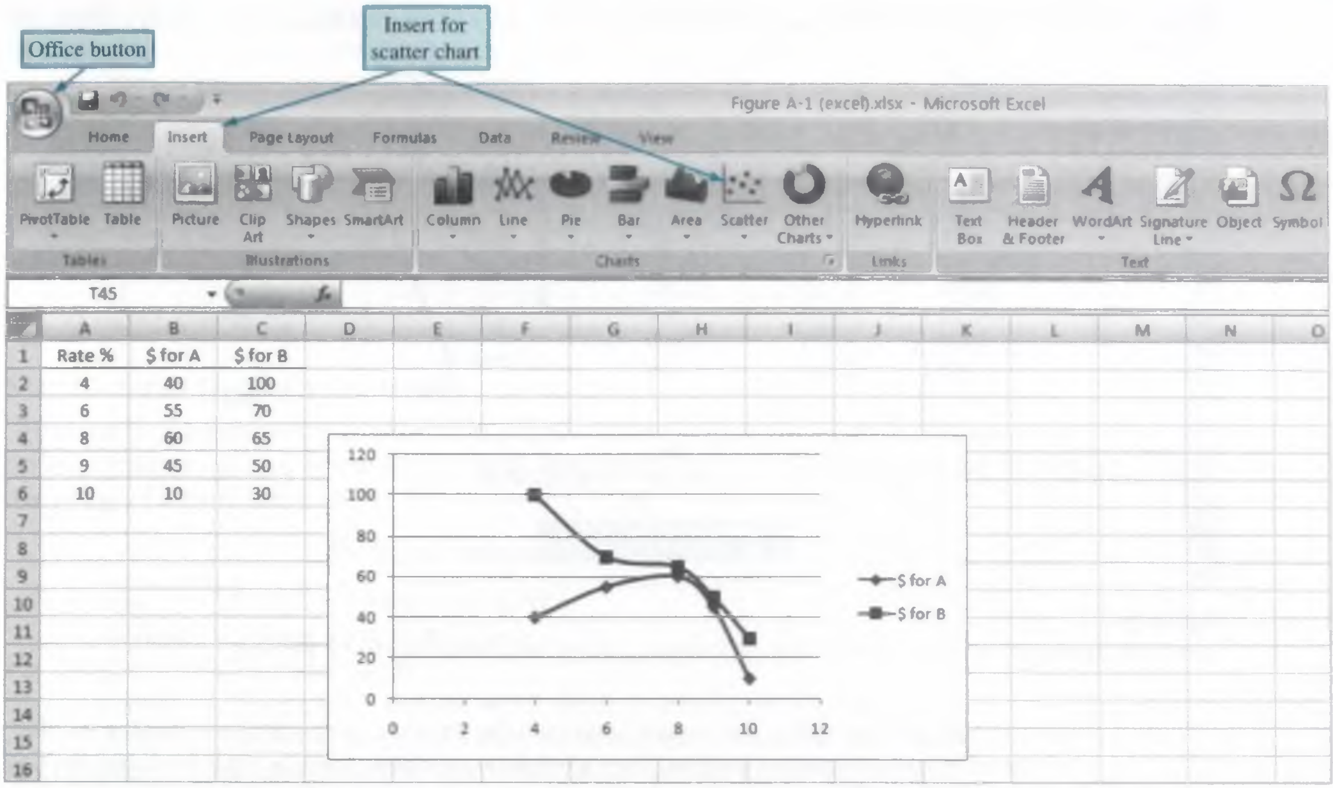


Figure A–1
Scatter chart for data entries and location of commonly used buttons.

Obtain Help While Using Excel

1. To get general help information, left-click on the "?" (upper right).
2. Enter the topic or phrase. For example, if you want to know more about how to save a file, type the word Save.
3. Select the appropriate matching words. You can browse through the options by left-clicking on any item.

A.2 Organization (Layout) of the Spreadsheet

A spreadsheet can be used in several ways to obtain answers to numerical questions. The first is as a rapid solution tool, often with the entry of only a few numbers or one predefined function. For example, to find the future worth in a single-cell operation, move the pointer to any cell and enter = FV(8%,5,-2500). The display of $14,666.50 is the 8% future worth at the end of year 5 of five equal payments of $2500 each.

A second use is more formal; it presents data, solutions, graphs, and tables developed on the spreadsheet and ready for presentation to others. Some fundamental guidelines in spreadsheet organization are presented here. A sample layout is presented in Figure A–2. As solutions become more complex, organization of the spreadsheet becomes increasingly important, especially for presentation to an audience via PowerPoint or similar software.

Cluster the data and the answers. It is advisable to organize the given or estimated data in the top left of the spreadsheet. A very brief label should be used to identify the data, for example, MARR = in cell A1 and the value, 12%, in cell B1. Then B1 can be the referenced cell for all entries requiring the MARR. Additionally, it may be worthwhile to cluster the answers into one area and frame it. Often, the answers are best placed at the bottom or top of the column of entries used in the formula or predefined function.

Enter titles for columns and rows. Each column or row should be labeled so its entries are clear to the reader. It is very easy to select from the wrong column or row when no brief title is present at the head of the data.

Enter income and cost cash flows separately. When there are both income and cost cash flows involved, it is strongly recommended that the cash flow estimates for revenue (usually positive)

	A	B	C	D	E
1	MARR	12%			
2	First cost	$40,000			
3	Annual income	$5,000			
4	Salvage value	$2,000			
5	Life, years	5			
6		Income	Cost		
7	Year	cash flow	cash flow	Net cash flow, $	
8	0	0	-40,000	-40,000	
9	1	5,000	0	5,000	
10	2	5,000	0	5,000	
11	3	5,000	0	5,000	
12	4	5,000	0	5,000	
13	5	5,000	2,000	7,000	
14					
15					
16	Present worth			-$20,841	
17	Future worth			-$36,729	= -FV(B1,5,,D16)
18					
19					
20				Function used to find FW	
21					

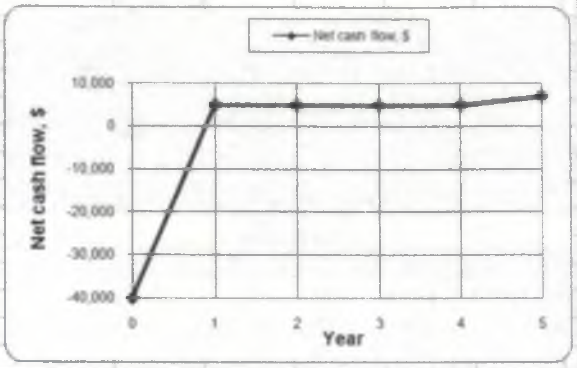


Figure A–2
Spreadsheet layout with cash flow estimates, results of functions, function formula detailed, and a scatter chart.

and first cost, salvage value, and annual costs (usually negative, with salvage a positive number) be entered into two adjacent columns. Then a formula combining them in a third column displays the net cash flow. There are two immediate advantages to this practice: fewer errors are made when performing the summation and subtraction mentally, and changes for sensitivity analysis are more easily made.

Use cell references. The use of absolute and relative cell references is a must when any changes in entries are expected. For example, suppose the MARR is entered in cell B1 and three separate references are made to the MARR in functions on the spreadsheet. The absolute cell reference entry B1 in the three functions allows the MARR to be changed one time, not three.

Obtain a final answer through summing and embedding. When the formulas and functions are kept relatively simple, the final answer can be obtained using the SUM function. For example, if the present worth values (PW) of two columns of cash flows are determined separately, then the total PW is the SUM of the subtotals. This practice is especially useful when the cash flow series are complex.

Prepare for a chart. If a chart (graph) will be developed, plan ahead by leaving sufficient room on the right of the data and answers. Charts can be placed on the same worksheet or on a separate worksheet. Placement on the same worksheet is recommended, especially when the results of sensitivity analysis are plotted.

A.3 Excel Functions Important to Engineering Economy (alphabetical order)

DB (Declining Balance)

Calculates the depreciation amount for an asset for a specified period n using the declining balance method. The depreciation rate d used in the computation is determined from asset values S (salvage value) and B (basis or first cost) as $d = 1 - (S/B)^{1/n}$. This is Equation [16.12]. Three-decimal-place accuracy is used for d.

= DB(cost, salvage, life, period, month)

cost	First cost or basis of the asset.
salvage	Salvage value.
life	Depreciation life (recovery period).
period	The period, year, for which the depreciation is to be calculated.
month	(optional entry) If this entry is omitted, a full year is assumed for the first year.

Example A new machine costs \$100,000 and is expected to last 10 years. At the end of 10 years, the salvage value of the machine is \$50,000. What is the depreciation of the machine in the first year and the fifth year?

Depreciation for the first year: = DB(100000,50000,10,1)

Depreciation for the fifth year: = DB(100000,50000,10,5)

Because of the manner in which the DB function determines the fixed percentage d and the accuracy of the computations, it is **recommended that the DDB function (below) be used** for all declining balance depreciation rates. Simply use the optional factor entry for rates other than $d = 2/n$.

DDB (Double Declining Balance)

Calculates the depreciation of an asset for a specified period n using the double declining balance method. A factor can also be entered for some other declining balance depreciation method by specifying a factor in the function.

= DDB(cost, salvage, life, period, factor)

cost	First cost or basis of the asset.
salvage	Salvage value of the asset.
life	Depreciation life.
period	The period, a year, for which the depreciation is to be calculated.
factor	(optional entry) If this entry is omitted, the function will use a double declining method with 2 times the straight line rate. If, for example, the entry is 1.5, the 150% declining balance method will be used.

Example A new machine costs \$200,000 and is expected to last 10 years. The salvage value is \$10,000. Calculate the depreciation of the machine for the first and the eighth years. Finally, calculate the depreciation for the fifth year using the 175% declining balance method.

Depreciation for the first year: = DDB(200000,10000,10,1)

Depreciation for the eighth year: = DDB(200000,10000,10,8)

Depreciation for the fifth year using 175% DB: = DDB(200000,10000,10,5,1.75)

EFFECT (Effective Interest Rate)

Calculates the effective annual interest rate for a stated nominal annual rate and a given number of compounding periods per year. Excel uses Equation [4.7] to calculate the effective rate.

= EFFECT(nominal, npery)

nominal	Nominal interest rate for the year.
npery	Number of times interest is compounded per year.

Example Claude has applied for a \$10,000 loan. The bank officer told him that the interest rate is 8% per year and that interest is compounded monthly to conveniently match his monthly payments. What effective annual rate will Claude pay?

Effective annual rate: = EFFECT(8%,12)

EFFECT can also be used to find **effective rates other than annually.** Enter the nominal rate for the time period of the required effective rate; npery is the number of times compounding occurs during the time period of the effective rate.

Example Interest is stated as 3.5% per quarter with quarterly compounding. Find the effective semiannual rate.

The 6-month nominal rate is 7%, and compounding is 2 times per 6 months.

Effective semiannual rate: = EFFECT(7%,2)

FV (Future Value)

Calculates the future value (worth) based on periodic payments at a specific interest rate.

= FV(rate, nper, pmt, pv, type)

rate	Interest rate per compounding period.
nper	Number of compounding periods.
pmt	Constant payment amount.
pv	The present value amount. If pv is not specified, the function will assume it to be 0.
type	(optional entry) Either 0 or 1. A 0 represents payments made at the end of the period, and 1 represents payments at the beginning of the period. If omitted, 0 is assumed.

Example Jack wants to start a savings account that can be increased as desired. He will deposit \$12,000 to start the account and plans to add \$500 to the account at the beginning of each month for the next 24 months. The bank pays 0.25% per month. How much will be in Jack's account at the end of 24 months?

Future value in 24 months: = FV(0.25%,24,500,12000,1)

IF (IF Logical Function)

Determines which of two entries is entered into a cell based on the outcome of a logical check on the outcome of another cell. The logical test can be a function or a simple value check, but it must use an equality or inequality sense. If the response is a text string, place it between quote marks (" "). The responses can themselves be IF functions. Up to seven IF functions can be nested for very complex logical tests.

= IF(logical_test,value_if_true,value_if_false)

logical_test	Any worksheet function can be used here, including a mathematical operation.
value_if_true	Result if the logical_test argument is true.
value_if_false	Result if the logical_test argument is false.

Example The entry in cell B4 should be "selected" if the PW value in cell B3 is greater than or equal to zero and "rejected" if PW < 0.

Entry in cell B4: = IF(B3 >=0,"selected","rejected")

Example The entry in cell C5 should be "selected" if the PW value in cell C4 is greater than or equal to zero, "rejected" if PW < 0, and "fantastic" if PW ≥ 200.

Entry in cell C5: = IF(C4<0,"rejected",IF(C4>=200,"fantastic","selected"))

IPMT (Interest Payment)

Calculates the interest accrued for a given period *n* based on constant periodic payments and interest rate.

= IPMT(rate, per, nper, pv, fv, type)

rate	Interest rate per compounding period.
per	Period for which interest is to be calculated.
nper	Number of compounding periods.
pv	Present value. If pv is not specified, the function will assume it to be 0.
fv	Future value. If fv is omitted, the function will assume it to be 0. The fv can also be considered a cash balance after the last payment is made.
type	(optional entry) Either 0 or 1. A 0 represents payments made at the end of the period, and 1 represents payments made at the beginning of the period. If omitted, 0 is assumed.

Example Calculate the interest due in the 10th month for a 48-month, \$20,000 loan. The interest rate is 0.25% per month.

Interest due: = IPMT(0.25%,10,48,20000)

IRR (Internal Rate of Return)

Calculates the internal rate of return between −100% and infinity for a series of cash flows at regular periods.

= IRR(values, guess)

values	A set of numbers in a spreadsheet column (or row) for which the rate of return will be calculated. The set of numbers must consist of at least *one* positive and *one* negative number. Negative numbers denote a payment made or cash outflow, and positive numbers denote income or cash inflow.
guess	(optional entry) To reduce the number of iterations, a *guessed rate of return* can be entered. In most cases, a guess is not required, and a 10% rate of return is initially assumed. If the #NUM! error appears, try using different values for guess. Inputting different guess values makes it possible to determine the multiple roots for the rate of return equation of a nonconventional cash flow series.

Example John wants to start a printing business. He will need \$25,000 in capital and anticipates that the business will generate the following incomes during the first 5 years. Calculate his rate of return after 3 years and after 5 years.

Year 1	\$5,000
Year 2	\$7,500
Year 3	\$8,000
Year 4	\$10,000
Year 5	\$15,000

Set up an array in the spreadsheet.

In cell A1, type −25000 (negative for payment).
In cell A2, type 5000 (positive for income).
In cell A3, type 7500.
In cell A4, type 8000.
In cell A5, type 10000.
In cell A6, type 15000.

Therefore, cells A1 through A6 contain the array of cash flows for the first 5 years, including the capital outlay. *Note that any years with a zero cash flow must have a zero entered* to ensure that the year value is correctly maintained for computation purposes.

To calculate the internal rate of return after 3 years, move to cell A7, and type = IRR(A1:A4).

To calculate the internal rate of return after 5 years and specify a guess value of 5%, move to cell A8, and type = IRR(A1:A6,5%).

MIRR (Modified Internal Rate of Return)

Calculates the modified internal rate of return for a series of cash flows and reinvestment of income and interest at a stated rate.

= MIRR(values, finance_rate, reinvest_rate)

values	Refers to an array of cells in the spreadsheet. Negative numbers represent payments, and positive numbers represent

	income. The series of payments and income must occur at regular periods and must contain at least *one* positive number and *one* negative number.
finance_rate	Interest rate on funds borrowed from external sources (i_b in Equation [7.9]).
reinvest_rate	Interest rate for reinvestment on positive cash flows (i_i in Equation [7.9]). (This is not the same reinvestment rate on the net investments when the cash flow series is nonconventional. See Section 7.5 for comments.)

Example Jane opened a hobby store 4 years ago. When she started the business, Jane borrowed \$50,000 from a bank at 12% per year. Since then, the business has yielded \$10,000 the first year, \$15,000 the second year, \$18,000 the third year, and \$21,000 the fourth year. Jane reinvests her profits, earning 8% per year. What is the modified rate of return after 3 years and after 4 years?

In cell A1, type −50000.

In cell A2, type 10000.

In cell A3, type 15000.

In cell A4, type 18000.

In cell A5, type 21000.

To calculate the modified rate of return after 3 years, move to cell A6, and type
= MIRR(A1:A4,12%,8%).

To calculate the modified rate of return after 4 years, move to cell A7, and type
= MIRR(A1:A5,12%,8%).

NOMINAL (Nominal Interest Rate)

Calculates the nominal **annual** interest rate for a stated effective **annual** rate and a given number of compounding periods per year. *This function is designed to display only nominal annual rates.*

= NOMINAL(effective, npery)

effective	Effective interest rate for the year.
npery	Number of times that interest is compounded per year.

Example Last year, a corporate stock earned an effective return of 12.55% per year. Calculate the nominal annual rate, if interest is compounded quarterly and compounded continuously.

Nominal annual rate, quarterly compounding: = NOMINAL(12.55%,4)

Nominal annual rate, continuous compounding: = NOMINAL(12.55%,100000)

NPER (Number of Periods)

Calculates the number of periods for the present worth of an investment to equal the future value specified, based on uniform regular payments and a stated interest rate.

= NPER(rate, pmt, pv, fv, type)

rate	Interest rate per compounding period.
pmt	Amount paid during each compounding period.
pv	Present value (lump-sum amount).
fv	(optional entry) Future value or cash balance after the last payment. If fv is omitted, the function will assume a value of 0.
type	(optional entry) Enter 0 if payments are due at the end of the compounding period and 1 if payments are due at the beginning of the period. If omitted, 0 is assumed.

Example Sally plans to open a savings account that pays 0.25% per month. Her initial deposit is \$3000, and she plans to deposit \$250 at the beginning of every month. How many payments does she have to make to accumulate \$25,000 to buy a new car?

Number of payments: = NPER(0.25%,−250,−3000,25000,1)

NPV (Net Present Value)

Calculates the net present value of a series of future cash flows at a stated interest rate.

= NPV(rate, series)

rate	Interest rate per compounding period.
series	Series of costs and incomes set up in a range of cells in the spreadsheet.

Example Mark is considering buying a sports store for \$100,000 and expects to receive the following income during the next 6 years of business: \$25,000, \$40,000, \$42,000, \$44,000, \$48,000, \$50,000. The interest rate is 8% per year.

In cells A1 through A7, enter −100,000, followed by the six annual incomes.

Present value: = NPV(8%,A2:A7) + A1

The cell A1 value is already a present value. *Any year with a zero cash flow must have a 0 entered* to ensure a correct result.

PMT (Payments)

Calculates equivalent periodic amounts based on present value and/or future value at a constant interest rate.

= PMT(rate, nper, pv, fv, type)

rate	Interest rate per compounding period.
nper	Total number of periods.
pv	Present value.
fv	Future value.
type	(optional entry) Enter 0 for payments due at the end of the compounding period and 1 if payment is due at the start of the compounding period. If omitted, 0 is assumed.

Example Jim plans to take a \$15,000 loan to buy a new car. The interest rate is 7% per year. He wants to pay the loan off in 5 years (60 months). What are his monthly payments?

Monthly payments: = PMT(7%/12,60,15000)

PPMT (Principal Payment)

Calculates the payment on the principal based on uniform payments at a specified interest rate.

= PPMT(rate, per, nper, pv, fv, type)

rate	Interest rate per compounding period.
per	Period for which the payment on the principal is required.
nper	Total number of periods.
pv	Present value.
fv	Future value.
type	(optional entry) Enter 0 for payments that are due at the end of the compounding period and 1 if payments are due at the start of the compounding period. If omitted, 0 is assumed.

Example Jovita is planning to invest \$10,000 in equipment which is expected to last 10 years with no salvage value. The interest rate is 5%. What is the principal payment at the end of year 4 and year 8?

At the end of year 4: = PPMT(5%,4,10,−10000)

At the end of year 8: = PPMT(5%,8,10,−10000)

PV (Present Value)

Calculates the present value of a future series of equal cash flows and a single lump sum in the last period at a constant interest rate.

= PV(rate, nper, pmt, fv, type)

rate	Interest rate per compounding period.
nper	Total number of periods.
pmt	Cash flow at regular intervals. Negative numbers represent payments (cash outflows), and positive numbers represent income.
fv	Future value or cash balance at the end of the last period.
type	(optional entry) Enter 0 if payments are due at the end of the compounding period and 1 if payments are due at the start of each compounding period. If omitted, 0 is assumed.

There are two primary differences between the PV function and the NPV function: PV allows for end or beginning of period cash flows, and PV requires that all amounts have the same value, whereas they may vary for the NPV function.

Example Jose is considering leasing a car for \$300 a month for 3 years (36 months). After the 36-month lease, he can purchase the car for \$12,000. Using an interest rate of 8% per year, find the present value of this option.

Present value: = PV(8%/12,36,−300,−12000)

Note the minus signs on the pmt and fv amounts.

RAND (Random Number)

Returns an evenly distributed number that is (1) ≥ 0 and < 1; (2) ≥ 0 and < 100; or (3) between two specified numbers.

= RAND()	**for range 0 to 1**
= RAND()*100	**for range 0 to 100**
= RAND()*(b−a)+a	**for range a to b**

a = minimum integer to be generated
b = maximum integer to be generated

The Excel function RANDBETWEEN(a,b) may also be used to obtain a random number between two values.

Example Grace needs random numbers between 5 and 10 with 3 digits after the decimal. What is the Excel function? Here a = 5 and b = 10.

Random number: = RAND()*5 + 5

Example Randi wants to generate random numbers between the limits of −10 and 25. What is the Excel function? The minimum and maximum values are $a = -10$ and $b = 25$, so $b - a = 25 - (-10) = 35$.

Random number: = RAND()*35 − 10

RATE (Interest Rate)

Calculates the interest rate per compounding period for a series of payments or incomes.

= RATE(nper, pmt, pv, fv, type, guess)

nper	Total number of periods.
pmt	Payment amount made each compounding period.
pv	Present value.
fv	Future value (not including the pmt amount).
type	(optional entry) Enter 0 for payments due at the end of the compounding period and 1 if payments are due at the start of each compounding period. If omitted, 0 is assumed.
guess	(optional entry) To minimize computing time, include a guessed interest rate. If a value of guess is not specified, the function will assume a rate of 10%. This function usually converges to a solution if the rate is between 0% and 100%.

Example Alysha wants to start a savings account at a bank. She will make an initial deposit of \$1000 to open the account and plans to deposit \$100 at the beginning of each month. She plans to do this for the next 3 years (36 months). At the end of 3 years, she wants to have at least \$5000. What is the minimum interest required to achieve this result?

Interest rate: = RATE(36,−100,−1000,5000,1)

SLN (Straight Line Depreciation)

Calculates the straight line depreciation of an asset for a given year.

= SLN(cost, salvage, life)

cost	First cost or basis of the asset.
salvage	Salvage value.
life	Depreciation life.

Example Maria purchased a printing machine for \$100,000. The machine has an allowed depreciation life of 8 years and an estimated salvage value of \$15,000. What is the depreciation each year?

Depreciation: = SLN(100000,15000,8)

SYD (Sum-of-Years-Digits Depreciation)

Calculates the sum-of-years-digits depreciation of an asset for a given year.

= SYD(cost, salvage, life, period)

cost	First cost or basis of the asset.
salvage	Salvage value.
life	Depreciation life.
period	The year for which the depreciation is sought.

Example Jack bought equipment for \$100,000 that has a depreciation life of 10 years. The salvage value is \$10,000. What is the depreciation for year 1 and year 9?

Depreciation for year 1: = SYD(100000,10000,10,1)

Depreciation for year 9: = SYD(100000,10000,10,9)

VDB (Variable Declining Balance)

Calculates the depreciation using the declining balance method with a switch to straight line depreciation in the year in which straight line has a larger depreciation amount. This function automatically implements the switch from DB to SL depreciation, unless specifically instructed to not switch.

= VDB (cost, salvage, life, start_period, end_period, factor, no_switch)

cost	First cost of the asset.
salvage	Salvage value.
life	Depreciation life.
start_period	First period for depreciation to be calculated.
end_period	Last period for depreciation to be calculated.
factor	(optional entry) If omitted, the function will use the double declining rate of $2/n$, or twice the straight line rate. Other entries define the declining balance method, for example, 1.5 for 150% declining balance.
no_switch	(optional entry) If omitted or entered as FALSE, the function will switch from declining balance to straight line depreciation when the latter is greater than DB depreciation. If entered as TRUE, the function will not switch to SL depreciation at any time during the depreciation life.

Example Newly purchased equipment with a first cost of $300,000 has a depreciable life of 10 years with no salvage value. Calculate the 175% declining balance depreciation for the first year and the ninth year if switching to SL depreciation is acceptable and if switching is not permitted.

Depreciation for first year, with switching: = VDB(300000,0,10,0,1,1.75)

Depreciation for ninth year, with switching: = VDB(300000,0,10,8,9,1.75)

Depreciation for first year, no switching: = VDB(300000,0,10,0,1,1.75,TRUE)

Depreciation for ninth year, no switching: = VDB(300000,0,10,8,9,1.75,TRUE)

VDB (for MACRS Depreciation)

The VDB function can be adapted to generate the MACRS annual depreciation amount, when the start_period and end_period are replaced with the MAX and MIN functions, respectively. As above, the factor option should be entered if other than DDB rates start the MACRS depreciation. The VDB format is

= VDB(cost,0,life,MAX(0,t−1.5),MIN(life,t−0.5),factor)

Example Determine the MACRS depreciation for year 4 for a $350,000 asset that has a 20% salvage value and a MACRS recovery period of 3 years. D_4 = $25,926 is the display.

Depreciation for year 4: = VDB(350000,0,3,MAX(0,4−1.5),MIN(3,4−0.5),2)

Example Find the MACRS depreciation in year 16 for a $350,000-asset with a recovery period of n = 15 years. The optional factor 1.5 is required here, since MACRS starts with 150% DB for n = 15-year and 20-year recovery periods. D_{16} = $10,334.

Depreciation for year 16: = VDB(350000,0,15,MAX(0,16−1.5),MIN(15,16−0.5),1.5)

Other Functions

There are numerous additional financial functions available on Excel, as well as engineering, mathematics, trigonometry, statistics, data and time, logical, and information functions. These can be viewed by clicking the Formulas tab on the Excel toolbar.

A.4 Goal Seek—A Tool for Breakeven and Sensitivity Analysis ● ● ●

Goal Seek is found on the Excel toolbar labeled Data, followed by What-if Analysis. This tool changes the value in a specific cell based on a numerical value in another (changing) cell as input by the user. It is a good tool for **sensitivity analysis, breakeven analysis,** and **"what if?" questions**

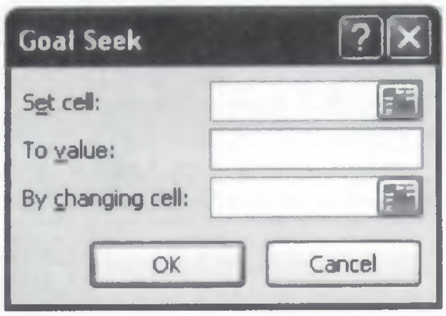


Figure A–3
Goal Seek template used to specify a cell, a value, and the changing cell.

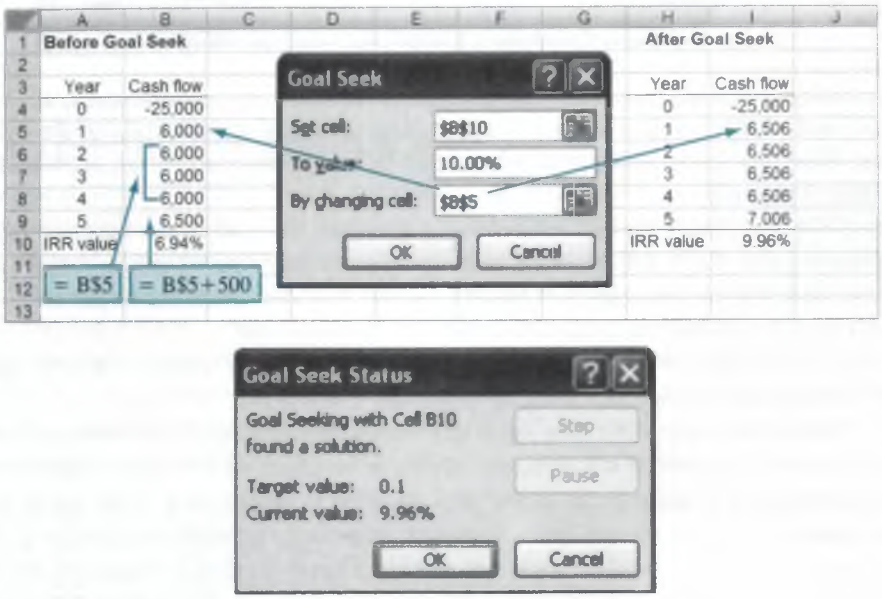


Figure A–4
Use of Goal Seek to determine an annual cash flow to increase the rate of return.

when no constraint relations or inequalities are needed. The initial Goal Seek template is pictured in Figure A–3. One of the cells (set or changing cell) must contain an equation or spreadsheet function that uses the other cell to determine a numeric value. Only a single cell can be identified as the changing cell; however, this limitation can be avoided by using equations rather than specific numerical inputs in any additional cells also to be changed. This is demonstrated below.

Example A new asset will cost \$25,000, generate an annual cash flow of \$6000 over its 5-year life, and have an estimated \$500 salvage value. The rate of return using the IRR function is 6.94%. Determine the annual cash flow necessary to raise the return to 10% per year.

Figure A–4 (top left) shows the cash flows and return displayed using the function = IRR(B4:B9) prior to the use of Goal Seek. Note that the initial \$6000 is input in cell B5, but other years' cash flows are input as equations that refer to B5. The \$500 salvage is added for the last year. This format allows Goal Seek to change only cell B5 while making the other cash flows have the same value. The tool finds the required cash flow of \$6506 to approximate the 10% per year return. The Goal Seek Status inset indicates that a solution is found. Clicking OK saves all changed cells; clicking Cancel returns to the original values.

A.5 Solver—An Optimizing Tool for Capital Budgeting, Breakeven, and Sensitivity Analysis ● ● ●

Solver is a powerful spreadsheet tool to change the value in multiple (one or more) cells based on the value in a specific (target) cell. It is excellent when solving a **capital budgeting problem** to select from independent projects where budget constraints are present. (Section 12.4 details this application.) The initial Solver template is shown in Figure A–5.

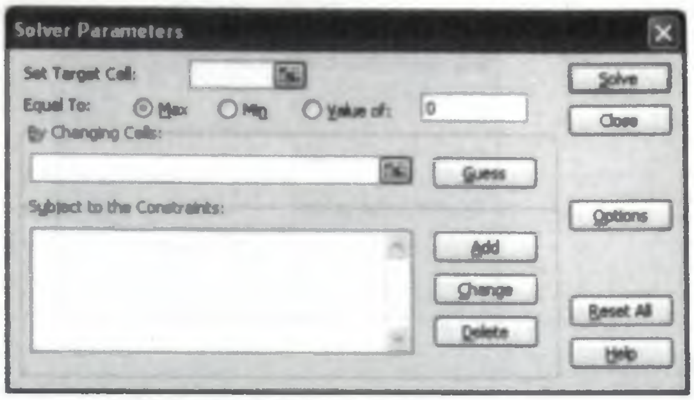


Figure A–5
Solver template used to specify optimization in a target cell, multiple changing cells, and constraint relations.

Set Target Cell box. Enter a cell reference or name. The target cell itself must contain a formula or function. The value in the cell can be maximized (Max), minimized (Min), or restricted to a specified value (Value of).

By Changing Cells box. Enter the cell reference for each cell to be adjusted, using commas between nonadjacent cells. Each cell must be directly or indirectly related to the target cell. Solver proposes a value for the changing cell based on input provided about the target cell. The Guess button will list all possible changing cells related to the target cell.

Subject to the Constraints box. Enter any constraints that may apply, for example, C1 < $50,000. Integer and binary variables are determined in this box.

Options box. Choices here allow the user to specify various parameters of the solution: maximum time and number of iterations allowed, the precision and tolerance of the values determined, and the convergence requirements as the final solution is determined. Also, linear and nonlinear model assumptions can be set here. *If integer or binary variables are involved, the tolerance option must be set to a small number, say, 0.0001.* This is especially important for the binary variables when selecting from independent projects (Chapter 12). If tolerance remains at the default value of 5%, a project may be incorrectly included in the solution set at a very low level.

Solver Results box. This appears after Solve is clicked and a solution appears. It is possible, of course, that no solution can be found for the scenario described. It is possible to update the spreadsheet by clicking Keep Solver Solution, or return to the original entries using Restore Original Values.

A.6 Error Messages ● ● ●

If Excel is unable to complete a formula or function computation, an error message is displayed. Some of the common messages are:

#DIV/0!	Requires division by zero.
#N/A	Refers to a value that is not available.
#NAME?	Uses a name that Excel doesn't recognize.
#NULL!	Specifies an invalid intersection of two areas.
#NUM!	Uses a number incorrectly.
#REF!	Refers to a cell that is not valid.
#VALUE!	Uses an invalid argument or operand.
#####	Produces a result, or includes a constant numeric value, that is too long to fit in the cell. (Widen the column.)

APPENDIX B

BASICS OF ACCOUNTING REPORTS AND BUSINESS RATIOS

This appendix provides a fundamental description of financial statements. The documents discussed here will assist in reviewing or understanding basic financial statements and in gathering information useful in an engineering economy study.

B.1 The Balance Sheet

The fiscal year and the tax year are defined identically for a corporation or an individual—12 months in length. The fiscal year (FY) is commonly not the calendar year (CY) for a corporation. The U.S. government uses October through September as its FY. For example, October 2011 through September 2012 is FY2012. The fiscal or tax year is always the calendar year for an individual citizen.

At the end of each fiscal year, a company publishes a **balance sheet.** A sample balance sheet for JAGBA Corporation is presented in Table B–1. This is a yearly presentation of the state of the firm at a particular time, for example, May 31, 2012; however, a balance sheet is also usually prepared quarterly and monthly. Three main categories are used.

Assets. This section is a summary of all resources owned by or owed to the company. There are two main classes of assets. *Current assets* represent shorter-lived working capital (cash, accounts receivable, etc.), which is more easily converted to cash, usually within 1 year. Longer-lived assets are referred to as *fixed assets* (land, equipment, etc.). Conversion of these holdings to cash in a short time would require a major corporate reorientation.

Liabilities. This section is a summary of all *financial obligations* (debts, mortgages, loans, etc.) of a corporation. Bond indebtedness is included here.

Net worth. Also called *owner's equity,* this section provides a summary of the financial value of ownership, including stocks issued and earnings retained by the corporation.

TABLE B–1 Sample Balance Sheet

JAGBA CORPORATION
Balance Sheet
May 31, 2012

Assets		Liabilities	
Current			
Cash	$10,500	Accounts payable	$19,700
Accounts receivable	18,700	Dividends payable	7,000
Interest accrued receivable	500	Long-term notes payable	16,000
Inventories	52,000	Bonds payable	20,000
Total current assets	$81,700	Total liabilities	$62,700
		Net Worth	
Fixed			
Land	$25,000	Common stock	$275,000
Building and equipment	438,000	Preferred stock	100,000
Less: Depreciation		Retained earnings	25,000
allowance $82,000	356,000		
Total fixed assets	381,000	Total net worth	400,000
Total assets	$462,700	Total liabilities and net worth	$462,700

The balance sheet is constructed using the relation

Assets = liabilities + net worth

In Table B–1 each major category is further divided into standard subcategories. For example, current assets is comprised of cash, accounts receivable, etc. Each subdivision has a specific interpretation, such as accounts receivable, which represents all money owed to the company by its customers.

B.2 Income Statement and Cost of Goods Sold Statement ●●●

A second important financial statement is the **income statement** (Table B–2). The income statement summarizes the profits or losses of the corporation for a stated period of time. Income statements always accompany balance sheets. The major categories of an income statement are

Revenues. This includes all *sales and interest revenue* that the company has received in the past accounting period.

Expenses. This is a summary of *all expenses* (operating and others, including taxes) for the period. Some expense amounts are itemized in other statements, for example, cost of goods sold.

The final result of an income statement is the net profit after taxes (NPAT), or NOPAT (O for operating), the amount used in Chapter 17, Sections 17.1 and 17.7. The income statement, published at the same time as the balance sheet, uses the basic equation

Revenues − expenses = profit (or loss)

The **cost of goods sold** is an important accounting term. It represents the net cost of producing the product marketed by the firm. Cost of goods sold may also be called *factory cost.* A statement of the cost of goods sold, such as that shown in Table B–3, is useful in determining exactly how much it costs to make a particular product over a stated time period, usually a year. The total of the cost of goods sold statement is entered as an expense item on the income statement. This total is determined using the relations

Cost of goods sold = prime cost + indirect cost

Prime cost = direct materials + direct labor [B.1]

TABLE B–2 Sample Income Statement

JAGBA CORPORATION
Income Statement
Year Ended May 31, 2012

Revenues		
Sales	$505,000	
Interest revenue	3,500	
Total revenues		$508,500
Expenses		
Cost of goods sold (from Table B–3)	$290,000	
Selling	28,000	
Administrative	35,000	
Other	12,000	
Total expenses		365,000
Income before taxes		143,500
Taxes for year		64,575
Net profit after taxes (NPAT)		$ 78,925

TABLE B–3 Sample Cost of Goods Sold Statement

JAGBA CORPORATION
Statement of Cost of Goods Sold
Year Ended May 31, 2012

Materials		
Inventory June 1, 2011	$ 54,000	
Purchases during year	174,500	
Total	$228,500	
Less: Inventory May 31, 2012	50,000	
Cost of materials		$178,500
Direct labor		110,000
Prime cost		288,500
Indirect costs		7,000
Factory cost		295,500
Less: Increase in finished goods inventory during year		5,500
Cost of goods sold (into Table B–2)		$290,000

Indirect costs include all indirect and overhead charges made to a product, process, or cost center. Indirect cost allocation methods are discussed in Chapter 15.

B.3 Business Ratios

Accountants, financial analysts, and engineering economists frequently utilize business ratio analysis to evaluate the financial health (status) of a company over time and in relation to industry norms. Because the engineering economist must continually communicate with others, she or he should have a basic understanding of several ratios. For comparison purposes, it is necessary to compute the ratios for several companies in the same industry. Industrywide median ratio values are published annually by firms such as Dun and Bradstreet in *Industry Norms and Key Business Ratios*. The ratios are classified according to their role in measuring the corporation.

Solvency ratios. Assess ability to meet short-term and long-term financial obligations.

Efficiency ratios. Measure management's ability to use and control assets.

Profitability ratios. Evaluate the ability to earn a return for the owners of the corporation.

Numerical data for several important ratios are discussed here and are extracted from the JAGBA balance sheet and income statement, Tables B–1 and B–2.

Current Ratio This ratio is utilized to analyze the company's working capital condition. It is defined as

$$\text{Current ratio} = \frac{\text{current assets}}{\text{current liabilities}}$$

Current liabilities include all short-term debts, such as accounts and dividends payable. Note that only balance sheet data are utilized in the current ratio; that is, no association with revenues or expenses is made. For the balance sheet of Table B–1, current liabilities amount to $19,700 + $7000 = $26,700 and

$$\text{Current ratio} = \frac{81{,}700}{26{,}700} = 3.06$$

Since current liabilities are those debts payable in the next year, the current ratio value of 3.06 means that the current assets would cover short-term debts approximately 3 times. Current ratio values of 2 to 3 are common.

The current ratio assumes that the working capital invested in inventory can be converted to cash quite rapidly. Often, however, a better idea of a company's *immediate* financial position can be obtained by using the acid test ratio.

Acid Test Ratio (Quick Ratio) This ratio is

$$\text{Acid-test ratio} = \frac{\text{quick assets}}{\text{current liabilities}}$$
$$= \frac{\text{current assets} - \text{inventories}}{\text{current liabilities}}$$

It is meaningful for the emergency situation when the firm must cover short-term debts using its readily convertible assets. For JAGBA Corporation,

$$\text{Acid test ratio} = \frac{81{,}700 - 52{,}000}{26{,}700} = 1.11$$

Comparison of this and the current ratio shows that approximately 2 times the current debt of the company is invested in inventories. However, an acid test ratio of approximately 1.0 is generally regarded as a strong current position, regardless of the amount of assets in inventories.

Debt Ratio This ratio is a measure of financial strength since it is defined as

$$\text{Debt ratio} = \frac{\text{total liabilities}}{\text{total assets}}$$

For JAGBA Corporation,

$$\text{Debt ratio} = \frac{62{,}700}{462{,}700} = 0.136$$

JAGBA is 13.6% creditor-owned and 86.4% stockholder-owned. A debt ratio in the range of 20% or less usually indicates a sound financial condition, with little fear of forced reorganization because of unpaid liabilities. However, a company with virtually no debts, that is, one with a very low debt ratio, may not have a promising future, because of its inexperience in dealing with short-term and long-term debt financing. The debt-equity (D-E) mix is another measure of financial strength.

Return on Sales Ratio This often quoted ratio indicates the profit margin for the company. It is defined as

$$\text{Return on sales} = \frac{\text{net profit}}{\text{net sales}}(100\%)$$

Net profit is the after-tax value from the income statement. This ratio measures profit earned per sales dollar and indicates how well the corporation can sustain adverse conditions over time, such as falling prices, rising costs, and declining sales. For JAGBA Corporation,

$$\text{Return on sales} = \frac{78{,}925}{505{,}000}(100\%) = 15.6\%$$

Corporations may point to small return on sales ratios, say, 2.5% to 4.0%, as indications of sagging economic conditions. In truth, for a relatively large-volume, high-turnover business, an income ratio of 3% is quite healthy. Of course, a steadily decreasing ratio indicates rising company expenses, which absorb net profit after taxes.

Return on Assets Ratio This is the key indicator of profitability since it evaluates the ability of the corporation to transfer assets into operating profit. The definition and value for JAGBA are

$$\text{Return on assets} = \frac{\text{net profit}}{\text{total assets}}(100\%)$$
$$= \frac{78{,}925}{462{,}700}(100\%) = 17.1\%$$

Efficient use of assets indicates that the company should earn a high return, while low returns usually accompany lower values of this ratio compared to the industry group ratios.

Inventory Turnover Ratio Two different ratios are used here. They both indicate the number of times the average inventory value passes through the operations of the company. If turnover of inventory to *net sales* is desired, the formula is

$$\text{Net sales to inventory} = \frac{\text{net sales}}{\text{average inventory}}$$

where average inventory is the figure recorded in the balance sheet. For JAGBA Corporation this ratio is

$$\text{Net sales to inventory} = \frac{505{,}000}{52{,}000} = 9.71$$

This means that the average value of the inventory has been sold 9.71 times during the year. Values of this ratio vary greatly from one industry to another.

If inventory turnover is related to *cost of goods sold*, the ratio to use is

$$\text{Cost of goods sold to inventory} = \frac{\text{cost of goods sold}}{\text{average inventory}}$$

Now, average inventory is computed as the average of the beginning and ending inventory values in the statement of cost of goods sold. This ratio is commonly used as a measure of the inventory turnover rate in manufacturing companies. It varies with industries, but management likes to see it remain relatively constant as business increases. For JAGBA, using the values in Table B–3,

$$\text{Cost of goods sold to inventory} = \frac{290{,}000}{\frac{1}{2}(54{,}000 + 50{,}000)} = 5.58$$

There are, of course, many other ratios to use in various circumstances; however, the ones presented here are commonly used by both accountants and economic analysts.

EXAMPLE B.1

Sample values for financial ratios or percentages of four industry sectors are presented below. Compare the corresponding JAGBA Corporation values with these norms, and comment on differences and similarities.

Ratio or Percentage	Motor Vehicles and Parts Manufacturing 336105*	Air Transportation (Medium-Sized) 481000*	Industrial Machinery Manufacturing 333200*	Home Furnishings 442000*
Current ratio	2.4	0.4	2.2	2.6
Quick ratio	1.6	0.3	1.5	1.2
Debt ratio	59.3%	96.8%	49.1%	52.4%
Return on assets	40.9%	8.1%	8.0%	5.1%

*North American Industry Classification System (NAICS) code for this industry sector.
SOURCE: L. Troy, *Almanac of Business and Industrial Financial Ratios*, CCH, Wolters Kluwer, USA.

Solution

It is not correct to compare ratios for one company with indexes in different industries, that is, with indexes for different NAICS codes. So the comparison below is for illustration purposes only. The corresponding values for JAGBA are

$$\text{Current ratio} = 3.06$$
$$\text{Quick ratio} = 1.11$$
$$\text{Debt ratio} = 13.5\%$$
$$\text{Return on assets} = 17.1\%$$

JAGBA has a current ratio larger than all four of these industries, since 3.06 indicates it can cover current liabilities 3 times compared with 2.6 and much less in the case of the "average" air transportation corporation. JAGBA has a significantly lower debt ratio than that of any of the sample industries, so it is likely more financially sound. Return on assets, which is a measure of ability to turn assets into profitability, is not as high at JAGBA as motor vehicles, but JAGBA competes well with the other industry sectors.

To make a fair comparison of JAGBA ratios with other values, it is necessary to have norm values *for its industry type* as well as ratio values for other corporations *in the same NAICS category* and about the same size in total assets. Corporate assets are classified in categories by $100,000 units, such as 100 to 250, 1001 to 5000, over 250,000, etc.

APPENDIX C

CODE OF ETHICS FOR ENGINEERS

Source: National Society of Professional Engineers (www.nspe.org).

National Society of Professional Engineers®

Code of Ethics for Engineers

Preamble

Engineering is an important and learned profession. As members of this profession, engineers are expected to exhibit the highest standards of honesty and integrity. Engineering has a direct and vital impact on the quality of life for all people. Accordingly, the services provided by engineers require honesty, impartiality, fairness, and equity, and must be dedicated to the protection of the public health, safety, and welfare. Engineers must perform under a standard of professional behavior that requires adherence to the highest principles of ethical conduct.

I. Fundamental Canons

Engineers, in the fulfillment of their professional duties, shall:

1. Hold paramount the safety, health, and welfare of the public.
2. Perform services only in areas of their competence.
3. Issue public statements only in an objective and truthful manner.
4. Act for each employer or client as faithful agents or trustees.
5. Avoid deceptive acts.
6. Conduct themselves honorably, responsibly, ethically, and lawfully so as to enhance the honor, reputation, and usefulness of the profession.

II. Rules of Practice

1. Engineers shall hold paramount the safety, health, and welfare of the public.
   a. If engineers' judgment is overruled under circumstances that endanger life or property, they shall notify their employer or client and such other authority as may be appropriate.
   b. Engineers shall approve only those engineering documents that are in conformity with applicable standards.
   c. Engineers shall not reveal facts, data, or information without the prior consent of the client or employer except as authorized or required by law or this Code.
   d. Engineers shall not permit the use of their name or associate in business ventures with any person or firm that they believe is engaged in fraudulent or dishonest enterprise.
   e. Engineers shall not aid or abet the unlawful practice of engineering by a person or firm.
   f. Engineers having knowledge of any alleged violation of this Code shall report thereon to appropriate professional bodies and, when relevant, also to public authorities, and cooperate with the proper authorities in furnishing such information or assistance as may be required.
2. Engineers shall perform services only in the areas of their competence.
   a. Engineers shall undertake assignments only when qualified by education or experience in the specific technical fields involved.
   b. Engineers shall not affix their signatures to any plans or documents dealing with subject matter in which they lack competence, nor to any plan or document not prepared under their direction and control.
   c. Engineers may accept assignments and assume responsibility for coordination of an entire project and sign and seal the engineering documents for the entire project, provided that each technical segment is signed and sealed only by the qualified engineers who prepared the segment.
3. Engineers shall issue public statements only in an objective and truthful manner.
   a. Engineers shall be objective and truthful in professional reports, statements, or testimony. They shall include all relevant and pertinent information in such reports, statements, or testimony, which should bear the date indicating when it was current.
   b. Engineers may express publicly technical opinions that are founded upon knowledge of the facts and competence in the subject matter.
   c. Engineers shall issue no statements, criticisms, or arguments on technical matters that are inspired or paid for by interested parties, unless they have prefaced their comments by explicitly identifying the interested parties on whose behalf they are speaking, and by revealing the existence of any interest the engineers may have in the matters.
4. Engineers shall act for each employer or client as faithful agents or trustees.
   a. Engineers shall disclose all known or potential conflicts of interest that could influence or appear to influence their judgment or the quality of their services.
   b. Engineers shall not accept compensation, financial or otherwise, from more than one party for services on the same project, or for services pertaining to the same project, unless the circumstances are fully disclosed and agreed to by all interested parties.
   c. Engineers shall not solicit or accept financial or other valuable consideration, directly or indirectly, from outside agents in connection with the work for which they are responsible.
   d. Engineers in public service as members, advisors, or employees of a governmental or quasi-governmental body or department shall not participate in decisions with respect to services solicited or provided by them or their organizations in private or public engineering practice.
   e. Engineers shall not solicit or accept a contract from a governmental body on which a principal or officer of their organization serves as a member.
5. Engineers shall avoid deceptive acts.
   a. Engineers shall not falsify their qualifications or permit misrepresentation of their or their associates' qualifications. They shall not misrepresent or exaggerate their responsibility in or for the subject matter of prior assignments. Brochures or other presentations incident to the solicitation of employment shall not misrepresent pertinent facts concerning employers, employees, associates, joint venturers, or past accomplishments.
   b. Engineers shall not offer, give, solicit, or receive, either directly or indirectly, any contribution to influence the award of a contract by public authority, or which may be reasonably construed by the public as having the effect or intent of influencing the awarding of a contract. They shall not offer any gift or other valuable consideration in order to secure work. They shall not pay a commission, percentage, or brokerage fee in order to secure work, except to a bona fide employee or bona fide established commercial or marketing agencies retained by them.

III. Professional Obligations

1. Engineers shall be guided in all their relations by the highest standards of honesty and integrity.
   a. Engineers shall acknowledge their errors and shall not distort or alter the facts.
   b. Engineers shall advise their clients or employers when they believe a project will not be successful.
   c. Engineers shall not accept outside employment to the detriment of their regular work or interest. Before accepting any outside engineering employment, they will notify their employers.
   d. Engineers shall not attempt to attract an engineer from another employer by false or misleading pretenses.
   e. Engineers shall not promote their own interest at the expense of the dignity and integrity of the profession.
2. Engineers shall at all times strive to serve the public interest.
   a. Engineers are encouraged to participate in civic affairs; career guidance for youths; and work for the advancement of the safety, health, and well-being of their community.
   b. Engineers shall not complete, sign, or seal plans and/or specifications that are not in conformity with applicable engineering standards. If the client or employer insists on such unprofessional conduct, they shall notify the proper authorities and withdraw from further service on the project.
   c. Engineers are encouraged to extend public knowledge and appreciation of engineering and its achievements.
   d. Engineers are encouraged to adhere to the principles of sustainable development[1] in order to protect the environment for future generations.

3. Engineers shall avoid all conduct or practice that deceives the public.
   a. Engineers shall avoid the use of statements containing a material misrepresentation of fact or omitting a material fact.
   b. Consistent with the foregoing, engineers may advertise for recruitment of personnel.
   c. Consistent with the foregoing, engineers may prepare articles for the lay or technical press, but such articles shall not imply credit to the author for work performed by others.
4. Engineers shall not disclose, without consent, confidential information concerning the business affairs or technical processes of any present or former client or employer, or public body on which they serve.
   a. Engineers shall not, without the consent of all interested parties, promote or arrange for new employment or practice in connection with a specific project for which the engineer has gained particular and specialized knowledge.
   b. Engineers shall not, without the consent of all interested parties, participate in or represent an adversary interest in connection with a specific project or proceeding in which the engineer has gained particular specialized knowledge on behalf of a former client or employer.
5. Engineers shall not be influenced in their professional duties by conflicting interests.
   a. Engineers shall not accept financial or other considerations, including free engineering designs, from material or equipment suppliers for specifying their product.
   b. Engineers shall not accept commissions or allowances, directly or indirectly, from contractors or other parties dealing with clients or employers of the engineer in connection with work for which the engineer is responsible.
6. Engineers shall not attempt to obtain employment or advancement or professional engagements by untruthfully criticizing other engineers, or by other improper or questionable methods.
   a. Engineers shall not request, propose, or accept a commission on a contingent basis under circumstances in which their judgment may be compromised.
   b. Engineers in salaried positions shall accept part-time engineering work only to the extent consistent with policies of the employer and in accordance with ethical considerations.
   c. Engineers shall not, without consent, use equipment, supplies, laboratory, or office facilities of an employer to carry on outside private practice.
7. Engineers shall not attempt to injure, maliciously or falsely, directly or indirectly, the professional reputation, prospects, practice, or employment of other engineers. Engineers who believe others are guilty of unethical or illegal practice shall present such information to the proper authority for action.
   a. Engineers in private practice shall not review the work of another engineer for the same client, except with the knowledge of such engineer, or unless the connection of such engineer with the work has been terminated.
   b. Engineers in governmental, industrial, or educational employ are entitled to review and evaluate the work of other engineers when so required by their employment duties.
   c. Engineers in sales or industrial employ are entitled to make engineering comparisons of represented products with products of other suppliers.
8. Engineers shall accept personal responsibility for their professional activities, provided, however, that engineers may seek indemnification for services arising out of their practice for other than gross negligence, where the engineer's interests cannot otherwise be protected.
   a. Engineers shall conform with state registration laws in the practice of engineering.
   b. Engineers shall not use association with a nonengineer, a corporation, or partnership as a "cloak" for unethical acts.
9. Engineers shall give credit for engineering work to those to whom credit is due, and will recognize the proprietary interests of others.
   a. Engineers shall, whenever possible, name the person or persons who may be individually responsible for designs, inventions, writings, or other accomplishments.
   b. Engineers using designs supplied by a client recognize that the designs remain the property of the client and may not be duplicated by the engineer for others without express permission.
   c. Engineers, before undertaking work for others in connection with which the engineer may make improvements, plans, designs, inventions, or other records that may justify copyrights or patents, should enter into a positive agreement regarding ownership.
   d. Engineers' designs, data, records, and notes referring exclusively to an employer's work are the employer's property. The employer should indemnify the engineer for use of the information for any purpose other than the original purpose.
   e. Engineers shall continue their professional development throughout their careers and should keep current in their specialty fields by engaging in professional practice, participating in continuing education courses, reading in the technical literature, and attending professional meetings and seminars.

Footnote 1 "Sustainable development" is the challenge of meeting human needs for natural resources, industrial products, energy, food, transportation, shelter, and effective waste management while conserving and protecting environmental quality and the natural resource base essential for future development.

As Revised July 2007

"By order of the United States District Court for the District of Columbia, former Section 11(c) of the NSPE Code of Ethics prohibiting competitive bidding, and all policy statements, opinions, rulings or other guidelines interpreting its scope, have been rescinded as unlawfully interfering with the legal right of engineers, protected under the antitrust laws, to provide price information to prospective clients; accordingly, nothing contained in the NSPE Code of Ethics, policy statements, opinions, rulings or other guidelines prohibits the submission of price quotations or competitive bids for engineering services at any time or in any amount."

Statement by NSPE Executive Committee

In order to correct misunderstandings which have been indicated in some instances since the issuance of the Supreme Court decision and the entry of the Final Judgment, it is noted that in its decision of April 25, 1978, the Supreme Court of the United States declared: "The Sherman Act does not require competitive bidding."

It is further noted that as made clear in the Supreme Court decision:

1. Engineers and firms may individually refuse to bid for engineering services.
2. Clients are not required to seek bids for engineering services.
3. Federal, state, and local laws governing procedures to procure engineering services are not affected, and remain in full force and effect.
4. State societies and local chapters are free to actively and aggressively seek legislation for professional selection and negotiation procedures by public agencies.
5. State registration board rules of professional conduct, including rules prohibiting competitive bidding for engineering services, are not affected and remain in full force and effect. State registration boards with authority to adopt rules of professional conduct may adopt rules governing procedures to obtain engineering services.
6. As noted by the Supreme Court, "nothing in the judgment prevents NSPE and its members from attempting to influence governmental action . . ."

Note: In regard to the question of application of the Code to corporations vis-a-vis real persons, business form or type should not negate nor influence conformance of individuals to the Code. The Code deals with professional services, which services must be performed by real persons. Real persons in turn establish and implement policies within business structures. The Code is clearly written to apply to the Engineer, and it is incumbent on members of NSPE to endeavor to live up to its provisions. This applies to all pertinent sections of the Code.

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APPENDIX D

ALTERNATE METHODS FOR EQUIVALENCE CALCULATIONS

Throughout the text, engineering economy factor formulas, tabulated factor values, or built-in spreadsheet functions have been used to obtain a value of P, F, A, i, or n. Because of advances in programmable and scientific calculators, many of the equivalence computations can be performed without the use of tables or spreadsheets, but rather with a handheld calculator. An overview of the possibilities is presented here.

Alternatively, the recognition that all equivalence calculations involve geometric series can likewise remove the need for tabulated values or spreadsheet functions. From the summation of the series, it is possible to perform calculator-based computations to obtain P, F, or A values. A brief introduction to this technique is presented in Section D.2.

D.1 Using Programmable Calculators

A basic way to calculate one parameter, given the other four, is to use a calculator that allows a present worth relation to be encoded. The software can then solve for any one of the parameters, when the remaining four are entered. For example, consider a PW relation in which all five parameters are included.

$$A(P/A,i,n) + F(P/F,i,n) + P = 0$$

The A, P, and F values can be positive (cash inflow) or negative (cash outflow) or zero, as long as there is at least one value with each sign. The interest rate i can be coded for entry as a percent or decimal. When the unknown variable is identified, the calculator's software can solve the equation for zero, thus providing the answer.

This is the approach taken by relatively simple scientific calculators, such as the Hewlett-Packard (HP) scientific series, for example, HP 33s. By substituting the formulas for the factors, the actual relation entered into the calculator is

$$A\left[\frac{1-(1+i/100)^{-n}}{i/100}\right] + F[1+(i/100)]^{-n} + P = 0$$

The HP calculator uses a slightly different symbol set than we have used thus far. The initial investment is called B rather than P, and the equal uniform amount is termed P rather than A. Once entered, the relation can be solved for any one variable, given values for the other four.

Another example that offers freedom from the spreadsheet and tables is an engineering calculator that has the same functions built in as those on a spreadsheet to determine P, F, A, i, or n. An example of this higher level is Texas Instrument's TI-Nspire series. The functions are basically the same as those on a spreadsheet. They are clustered under the heading of tvm (time value of money) functions. For example, the tvmPV function format is

tvmPV(n,i,Pmt,FV,PpY,CpY,PmtAt)

where
- n = number of periods
- i = annual interest rate as a percent
- Pmt = equal uniform periodic amount A
- FV = future amount F
- PpY = payments per year (optional; default is 1)
- CpY = compounding periods per year (optional; default is 1)
- PmtAt = beginning- or end-of-period payments (optional; default is 0 = end)

It is easy to understand why it is possible to do a lot on a calculator with relatively well-behaved cash flow series. As the series become more complex, it is necessary to move to a spreadsheet for speed and versatility. However, the use of tables or factor formulas is not necessary.

D.2 Using the Summation of a Geometric Series

A geometric progression is a series of n terms with a common ratio or base r. If c is a constant for each term, the series is written in the form

$$cr^a + cr^{a+1} + \cdots + cr^n = c\sum_{j=a}^{j=n} r^j$$

The sum S of a geometric series adds the terms using the closed-end form

$$S = \frac{r^{n+1} - r^a}{r - 1} \qquad \text{[D.1]}$$

Ristroph[1] and others have explained how the recognition that equivalence computations are simple applications of geometric series can be used to determine F, P, and A using Equation [D.1] and a simple handheld calculator with exponentiation capability.

Before explaining how to apply this approach, we define the base r as follows when a future worth F or present worth P is sought.

To find F: $r = 1 + i$

To find P: $r = (1 + i)^{-1}$

A very familiar application of geometric series is the determination of the equivalent future worth F in year n for a single present worth amount P in year 0. This is the same as using a geometric series of only one term. As shown in Figure D–1, if P = \$100, n = 10 years, and i = 10% per year, when $r = 1 + i$,

$$\begin{aligned} F = P(1+i)^n = P(r)^n &= 100(1.1)^{10} \\ &= 100(2.5937) \\ &= \$259.37 \end{aligned}$$

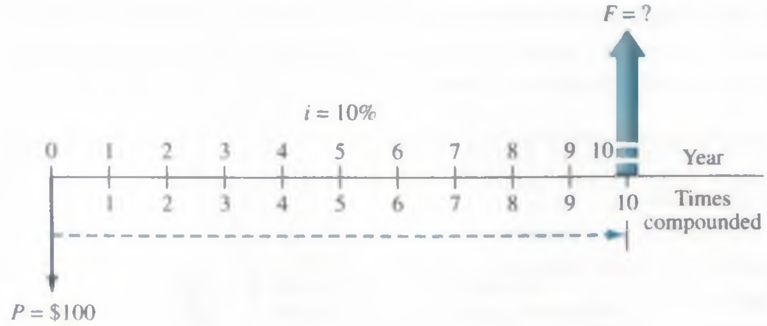
This is identical to using the tabulated value (or formula) for the F/P factor.

$$\begin{aligned} F &= P(F/P,i,n) = 100(F/P,10\%,10) = 100(2.5937) \\ &= \$259.37 \end{aligned}$$

Figure D–2 shows a uniform annual series A = \$100 for 10 years. To calculate F, we can move each A value forward to year 10. Place a subscript j on each A value to indicate the year of occurrence, and determine F for each A_j value.

$$F = A_1(1+i)^9 + A_2(1+i)^8 + \cdots + A_9(1+i)^1 + A_{10}(1+i)^0$$

Figure D–1
Future worth of a single amount in year 0.



[1] J. H. Ristroph, "Engineering Economics: Time for New Directions?" *Proceedings*, ASEE Annual Conference, Austin, TX, June 2009.

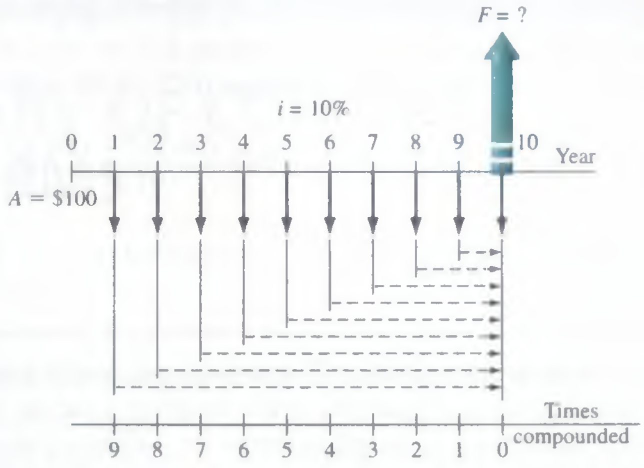


Figure D–2
Future worth of an A series from year 1 through year 10.

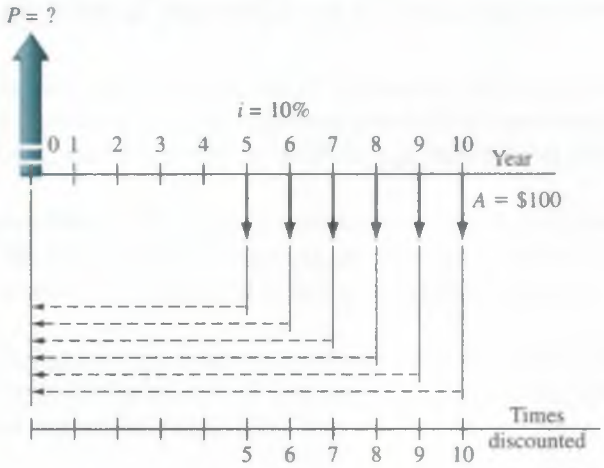


Figure D–3
Present worth of a shifted uniform series.

Note that the first value A_1 is compounded 9 times, not 10. With $r = 1 + i$, the geometric series and its sum S (from Equation [D.1]) can be developed. Removing the subscript on A,

$$F = A[r^9 + r^8 + \cdots + r^1 + r^0] = A\sum_{j=0}^{j=9} r^j$$

$$S = \frac{r^{10} - r^0}{r - 1} = \frac{(1.1)^{10} - 1}{0.1} = 15.9374$$

The F value is the same whether using the geometric series sum or the tabulated F/A factor.

$$F = 100(S) = 100(15.9374) = \$1593.74$$

$$F = 100(F/A,10\%,10) = 100(15.9374) = \$1593.74$$

As a final demonstration of the geometric series approach to equivalence computations, consider the shifted cash flow series in Figure D–3. The A series is present from years 5 through 10, and the P value is sought. If the tables are used, the solution is

$$P = A(P/A,10\%,6)(P/F,10\%,4) = 100(4.3553)(0.6830)$$
$$= \$297.47$$

As shown in the figure, this is a geometric series with the first A value discounted 5 years, therefore, $a = 5$. The last A term is discounted 10 years, making $n = 10$. Since P is sought, the geometric series base is $r = (1 + i)^{-1}$. Again using Equation [D.1] for the summation, we have

$$P = A\left[\sum_{j=5}^{j=10} r^j\right] = 100\left[\frac{r^{11} - r^5}{r - 1}\right] = 100\left[\frac{(1/1.1)^{11} - (1/1.1)^5}{(1/1.1) - 1}\right]$$

$$= 100\left[\frac{(0.9091)^{11} - (0.9091)^5}{0.9091 - 1}\right] = 100(2.9747)$$

$$= \$297.47$$

It is possible to develop similar relations to handle arithmetic and geometric series, conversions from P to A, and vice versa. Proponents of this approach point out the use of standard mathematical notation; the removal of a need to derive any factors; no need to remember the placement of P, F, and A values based on factor formula development; and the easy use of a calculator to determine the equivalence relations. As with the use of programmable calculators, the technique is excellent for well-behaved and reasonably complex series. When the series become quite involved, or when sensitivity analysis is required to reach an economic decision, it may be beneficial to use a spreadsheet. This often helps in performing sidebar and ancillary calculations that assist in the understanding of the problem, not just the math computations necessary to obtain an answer. However, once again, the use of tables and factors is not necessary.

APPENDIX E

GLOSSARY OF CONCEPTS AND TERMS

E.1 Important Concepts and Guidelines

Title

The following elements of engineering economy are identified throughout the text in the margin by this checkmark and a title below it. The numbers in parentheses indicate chapters where the concept or guideline is introduced or essential to obtaining a correct solution.

Time Value of Money It is a fact that money *makes* money. This concept explains the change in the amount of money *over time* for both owned and borrowed funds. (1)

Economic Equivalence A combination of time value of money and interest rate that makes different sums of money at different times have *equal economic value*. (1)

Cash Flow The flow of money into and out of a company, project, or activity. *Revenues are cash inflows* and carry a positive (+) sign; *expenses are outflows* and carry a negative (−) sign. If only costs are involved, the − sign may be omitted, e.g., benefit/cost (B/C) analysis. (1, 9)

End-of-Period Convention To simplify calculations, cash flows (revenues and costs) are assumed to occur at the *end of a time period*. An interest period or fiscal period is commonly 1 *year*. A half-year convention is often used in depreciation calculations. (1)

Cost of Capital The interest rate incurred to obtain capital investment funds. COC is usually a *weighted average* that involves the cost of debt capital (loans, bonds, and mortgages) and equity capital (stocks and retained earnings). (1, 10)

Minimum Attractive Rate of Return (MARR) A reasonable rate of return established for the evaluation of an economic alternative. Also called the *hurdle rate*, MARR is based on cost of capital, market trend, risk, etc. The inequality ROR $\geq$ MARR $>$ COC is correct for an economically viable project. (1, 10)

Opportunity Cost A forgone opportunity caused by the inability to pursue a project. Numerically, it is the *largest rate of return* of all the projects not funded due to the lack of capital funds. Stated differently, it is the ROR of the first project rejected because of unavailability of funds. (1, 10)

Nominal or Effective Interest Rate (r or i) A nominal interest rate *does not include any compounding;* for example, 1% per month is the same as nominal 12% per year. Effective interest rate is the actual rate over a period of time because *compounding is imputed;* for example, 1% per month, compounded monthly, is an effective 12.683% per year. Inflation or deflation is not considered. (4)

Placement of Present Worth (P; PW) In applying the $(P/A,i\%,n)$ factor, P or PW is always located *one interest period (year) prior to the first A amount*. The A or AW is a series of equal, end-of-period cash flows for n consecutive periods, expressed as money per time (say, \$/year; €/year). (2, 3)

Placement of Future Worth (F; FW) In applying the $(F/A,i\%,n)$ factor, F or FW is always located at the *end of the last interest period (year) of the A series*. (2, 3)

Placement of Gradient Present Worth (P_G; P_g) The $(P/G,i\%,n)$ factor for an *arithmetic gradient* finds the P_G of only the gradient series *2 years prior* to the first appearance of the constant gradient G. The base amount A is treated separately from the gradient series.

The $(P/A,g,i,n)$ factor for a *geometric gradient* determines P_g for the gradient and initial amount A_1 *two years prior* to the appearance of the first gradient amount. The initial amount A_1 *is* included in the value of P_g. (2, 3)

Equal-Service Requirement Identical capacity of all alternatives operating over the *same amount of time* is mandated by the equal-service requirement. Estimated costs and revenues for equal service must be evaluated. PW analysis requires evaluation over the same number of years (periods) using the LCM (least common multiple) of lives; AW analysis is performed over one life cycle. Further, equal service assumes that all costs and revenues rise and fall in accordance with the overall rate of inflation or deflation over the total time period of the evaluation. (5, 6, 8)

LCM or Study Period To select from mutually exclusive alternatives under the equal-service requirement for PW computations, use the *LCM of lives with repurchase(s)* as necessary. For a stated study period (planning horizon), evaluate cash flows *only over this period* , neglecting any beyond this time; estimated market values at termination of the study period are the salvage values. (5, 6, 11)

Salvage/Market Value Expected trade-in, market, or scrap value at the *end of the estimated life* or the *study period*. In a replacement study, the defender's estimated market value at the end of a year is considered its "first cost" at the beginning of the next year. MACRS depreciation always reduces the book value to a salvage of zero. (6, 11)

Do Nothing The DN alternative is always an option, unless one of the defined alternatives *must* be selected. DN is status quo; it generates *no new costs, revenues, or savings*. (5)

Revenue or Cost Alternative Revenue alternatives have *costs and revenues* estimated; savings are considered negative costs and carry a + sign. Incremental evaluation requires comparison with DN for revenue alternatives. Cost (or service) alternatives have *only costs* estimated; revenues and savings are assumed equal between alternatives. (5)

Rate of Return An interest rate that equates a PW or AW relation to *zero*. Also defined as the rate on the unpaid balance of borrowed money, or rate earned on the unrecovered balance of an investment such that the *last cash flow brings the balance exactly to zero*. (7, 8)

Project Evaluation *For a specified* MARR, determine a measure of worth for net cash flow series over the life or study period. Guidelines for a *single project* to be economically justified at the MARR (or discount rate) follow. (5, 6, 7, 9, 17)

Present worth: If PW ≥ 0	**Annual worth:** If AW ≥ 0
Future worth: If FW ≥ 0	**Rate of return:** If $i^* \geq$ MARR
Benefit/cost: If B/C ≥ 1.0	**Profitability index:** If PI ≥ 1.0

ME Alternative Selection For mutually exclusive (select only one) alternatives, compare *two alternatives* at a time by determining a measure of worth for the incremental (Δ) cash flow series over the life or study period, adhering to the equal-service requirement. (5, 6, 8, 9, 10, 17)

Present worth or annual worth: Find PW or AW values at MARR; *select numerically largest* (least negative or most positive).

Rate of return: Order by *initial cost,* perform pairwise Δi^* comparison; if $\Delta i^* \geq$ MARR, select *larger cost* alternative; continue until one remains.

Benefit/cost: Order by *total equivalent cost,* perform pairwise ΔB/C comparison; if ΔB/C ≥ 1.0, select *larger cost* alternative; continue until one remains.

Cost-effectiveness ratio: For service sector alternatives; order by *effectiveness measure;* perform pairwise ΔC/E comparison using *dominance;* select from nondominated alternatives without exceeding budget.

Independent Project Selection No comparison between projects; only against DN. Calculate a measure of worth and select using the guidelines below. (5, 6, 8, 9, 12)

Present worth or annual worth: Find PW or AW at MARR; select all projects with PW or AW ≥ 0.

Rate of return: No incremental comparison; select all projects with overall $i^* \geq$ MARR.

Benefit/cost: No incremental comparison; select all projects with overall B/C ≥ 1.0.

Cost-effectiveness ratio: For service sector projects; no incremental comparison; order by CER and select projects to not exceed budget.

When a capital budget limit is defined, independent projects are selected using the *capital budgeting process* based on PW values. The Solver spreadsheet tool is useful here.

Capital Recovery CR is the equivalent annual amount an asset or system must earn to *recover the initial investment plus a stated rate of return.* Numerically, it is the AW value of the initial investment at a stated rate of return. The salvage value is considered in CR calculations. (6)

Economic Service Life The ESL is the number of years n at which the *total AW of costs,* including salvage and AOC, is at its *minimum,* considering all the years the asset may provide service. (11)

Sunk Cost Capital (money) that is lost and cannot be recovered. Sunk costs are not included when making decisions about the future. They should be handled using tax laws and write-off allowances, not the economic study. (11)

Inflation Expressed as a percentage per time (% per year), it is an *increase* in the amount of money required to purchase the *same amount* of goods or services *over time.* Inflation occurs when the value of a currency decreases. Economic evaluations are performed using either a market (inflation-adjusted) interest rate or an inflation-free rate (constant-value terms). (1, 14)

Breakeven For a single project, the value of a parameter that makes *two elements equal,* e.g., sales necessary to equate revenues and costs. For two alternatives, breakeven is the value of a common variable at which the two are equally acceptable. Breakeven analysis is fundamental to make-buy decisions, replacement studies, payback analysis, sensitivity analysis, breakeven ROR analysis, and many others. The Goal Seek spreadsheet tool is useful in breakeven analysis. (8, 13)

Payback Period Amount of time n before *recovery of the initial capital investment* is expected. Payback with $i > 0$ or simple payback at $i = 0$ is useful for preliminary or screening analysis to determine if a full PW, AW, or ROR analysis is needed. (13)

Direct/Indirect Costs Direct costs are primarily human labor, machines, and materials associated with a product, process, system, or service. Indirect costs, which include support functions, utilities, management, legal, taxes, and the like, are more difficult to associate with a specific product or process. (15)

Value Added Activities have added worth to a product or service from the perspective of a consumer, owner, or investor who is willing to pay more for an enhanced value. (17)

Sensitivity Analysis Determination of how a measure of worth is affected by changes in estimated values of a parameter over a stated range. Parameters may be any cost factor, revenue, life, salvage value, inflation rate, etc. (18)

Risk Variation from an expected, desirable, or predicted value that may be detrimental to the product, process, or system. Risk represents an *absence of or deviation from certainty*. Probability estimates of variation (values) help evaluate risk and uncertainty using statistics and simulation. (10, 18, 19, 20)

E.2 Symbols and Terms

This section identifies and defines the common terms and their symbols used throughout the text. The numbers in parentheses indicate sections where the term is introduced and used in various applications.

Term	Symbol	Description
Annual amount or worth	A or AW	Equivalent uniform annual worth of all cash inflows and outflows over estimated life (1.5, 6.1).
Annual operating cost	AOC	Estimated annual costs to maintain and support an alternative (1.3).
Benefit/cost ratio	B/C	Ratio of a project's benefits to costs expressed in PW, AW, or FW terms (9.2).
Book value	BV	Remaining capital investment in an asset after depreciation is accounted for (16.1).
Breakeven point	Q_{BE}	Quantity at which revenues and costs are equal, or two alternatives are equivalent (13.1).
Capital budget	b	Amount of money available for capital investment projects (12.1).
Capital recovery	CR or A	Equivalent annual cost of owning an asset plus the required return on the initial investment (6.2).
Capitalized cost	CC or P	Present worth of an alternative that will last forever (or a long time) (5.5).
Cash flow	CF	Actual cash amounts that are receipts (inflow) and disbursements (outflow) (1.6).
Cash flow before or after taxes	CFBT or CFAT	Cash flow amount before relevant taxes or after taxes are applied (17.2).
Compounding frequency	m	Number of times interest is compounded per period (year) (4.1).
Cost-effectiveness ratio	CER	Ratio of equivalent cost to effectiveness measure to evaluate service sector projects (9.5).
Cost estimating relationships	C_2 or C_T	Relations that use design variables and changing costs over time to estimate current and future costs (15.3–4).
Cost of capital	WACC	Interest rate paid for the use of capital funds; includes both debt and equity funds. For debt and equity considered, it is weighted average cost of capital (1.9, 10.2).
Debt-equity mix	D-E	Percentages of debt and equity investment capital used by a corporation (10.2).
Depreciation	D	Reduction in the value of assets using specific models and rules; there are book and tax depreciation methods (16.1).
Depreciation rate	d_t	Annual rate for reducing the value of assets using different depreciation methods (16.1).
Economic service life	ESL or n	Number of years at which the AW of costs is a minimum (11.2).
Effectiveness measure	E	A nonmonetary measure used in the cost-effectiveness ratio for service sector projects (9.5).

Term	Symbol	Description
Expected value (average)	$\bar{X}$, μ, or $E(X)$	Long-run expected average if a random variable is sampled many times (18.3, 19.4).
Expenses, operating	OE	All corporate costs incurred in transacting business (17.1).
First cost	P	Total initial cost—purchase, construction, setup, etc. (1.3, 16.1).
Future amount or worth	F or FW	Amount at some future date considering time value of money (1.5, 5.4).
Gradient, arithmetic	G	Uniform change (+ or −) in cash flow each time period (2.5).
Gradient, geometric	g	Constant rate of change (+ or −) each time period (2.6).
Gross income	GI	Income from all sources for corporations or individuals (17.1).
Inflation rate	f	Rate that reflects changes in the value of a currency over time (14.1).
Interest	I	Amount earned or paid over time based on an initial amount and interest rate (1.4).
Interest rate	i or r	Interest expressed as a percentage of the original amount per time period; nominal (r) and effective (i) rates (1.4, 4.1).
Interest rate, inflation-adjusted	i_f	Interest rate adjusted to take inflation into account (14.1).
Life (estimated)	n	Number of years or periods over which an alternative or asset will be used; the evaluation time (1.5).
Life-cycle cost	LCC	Evaluation of costs for a system over all stages: feasibility to design to phaseout (6.5).
Measure of worth	Varies	Value, such as PW, AW, i^*, used to judge economic viability (1.1).
Minimum attractive rate of return	MARR	Minimum value of the rate of return for an alternative to be financially viable (1.9, 10.1).
Modified ROR	i' or MIRR	Unique ROR when a reinvestment rate i_r and external borrowing rate i_b are applied to multiple-rate cash flows (7.5).
Net cash flow	NCF	Resulting, actual amount of cash that flows in or out during a time period (1.6).
Net operating income	NOI	Difference between gross income and operating expenses (17.1).
Net operating profit after taxes	NOPAT or NPAT	Amount remaining after taxes are removed from taxable income (17.1).
Net present value	NPV	Another name for the present worth, PW.
Payback period	n_p	Number of years to recover the initial investment and a stated rate of return (13.3).
Present amount or worth	P or PW	Amount of money at the current time or a time denoted as *present* (1.5, 5.2).
Probability distribution	$P(X)$	Distribution of probability over different values of a variable (19.2).
Profitability index	PI	Ratio of PW of net cash flows to initial investment used for revenue projects; rewritten modified B/C ratio (9.2, 12.5).
Random variable	X	Parameter or characteristic that can take on any one of several values; discrete and continuous (19.2).
Rate of return	i^* or ROR	Compound interest rate on unpaid or unrecovered balances such that the final amount results in a zero balance (7.1).

Term	Symbol	Description
Recovery period	n	Number of years to completely depreciate an asset (16.1).
Return on invested capital	i'' or ROIC	Unique ROR when a reinvestment rate i_r is applied to multiple-rate cash flows (7.5).
Salvage/market value	S or MV	Expected trade-in or market value when an asset is traded or disposed of (6.2, 11.1, 16.1).
Standard deviation	s or σ	Measure of dispersion or spread about the expected value or average (19.4).
Study period	n	Specified number of years over which an evaluation takes place (5.3, 11.5).
Taxable income	TI	Amount upon which income taxes are based (17.1).
Tax rate	T	Decimal rate, usually graduated, used to calculate corporate or individual taxes (17.1).
Tax rate, effective	T_e	Single-figure tax rate incorporating several rates and bases (17.1).
Time	t	Indicator for a time period (1.7).
Unadjusted basis	B	Depreciable amount of first cost, delivery, and installation costs of an asset (18.1).
Value added	EVA	Economic value added reflects net profit after taxes (NPAT) after removing cost of invested capital during the year (17.7).
Value-added tax	VAT	An indirect consumption tax collected at each stage of production/distribution process; different from a sales tax paid by end user at purchase time (17.9).

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0.25% **TABLE 1 Discrete Cash Flow: Compound Interest Factors** 0.25%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0025	0.9975	1.00000	1.0000	1.00250	0.9975		
2	1.0050	0.9950	0.49938	2.0025	0.50188	1.9925	0.9950	0.4994
3	1.0075	0.9925	0.33250	3.0075	0.33500	2.9851	2.9801	0.9983
4	1.0100	0.9901	0.24906	4.0150	0.25156	3.9751	5.9503	1.4969
5	1.0126	0.9876	0.19900	5.0251	0.20150	4.9627	9.9007	1.9950
6	1.0151	0.9851	0.16563	6.0376	0.16813	5.9478	14.8263	2.4927
7	1.0176	0.9827	0.14179	7.0527	0.14429	6.9305	20.7223	2.9900
8	1.0202	0.9802	0.12391	8.0704	0.12641	7.9107	27.5839	3.4869
9	1.0227	0.9778	0.11000	9.0905	0.11250	8.8885	35.4061	3.9834
10	1.0253	0.9753	0.09888	10.1133	0.10138	9.8639	44.1842	4.4794
11	1.0278	0.9729	0.08978	11.1385	0.09228	10.8368	53.9133	4.9750
12	1.0304	0.9705	0.08219	12.1664	0.08469	11.8073	64.5886	5.4702
13	1.0330	0.9681	0.07578	13.1968	0.07828	12.7753	76.2053	5.9650
14	1.0356	0.9656	0.07028	14.2298	0.07278	13.7410	88.7587	6.4594
15	1.0382	0.9632	0.06551	15.2654	0.06801	14.7042	102.2441	6.9534
16	1.0408	0.9608	0.06134	16.3035	0.06384	15.6650	116.6567	7.4469
17	1.0434	0.9584	0.05766	17.3443	0.06016	16.6235	131.9917	7.9401
18	1.0460	0.9561	0.05438	18.3876	0.05688	17.5795	148.2446	8.4328
19	1.0486	0.9537	0.05146	19.4336	0.05396	18.5332	165.4106	8.9251
20	1.0512	0.9513	0.04882	20.4822	0.05132	19.4845	183.4851	9.4170
21	1.0538	0.9489	0.04644	21.5334	0.04894	20.4334	202.4634	9.9085
22	1.0565	0.9466	0.04427	22.5872	0.04677	21.3800	222.3410	10.3995
23	1.0591	0.9442	0.04229	23.6437	0.04479	22.3241	243.1131	10.8901
24	1.0618	0.9418	0.04048	24.7028	0.04298	23.2660	264.7753	11.3804
25	1.0644	0.9395	0.03881	25.7646	0.04131	24.2055	287.3230	11.8702
26	1.0671	0.9371	0.03727	26.8290	0.03977	25.1426	310.7516	12.3596
27	1.0697	0.9348	0.03585	27.8961	0.03835	26.0774	335.0566	12.8485
28	1.0724	0.9325	0.03452	28.9658	0.03702	27.0099	360.2334	13.3371
29	1.0751	0.9301	0.03329	30.0382	0.03579	27.9400	386.2776	13.8252
30	1.0778	0.9278	0.03214	31.1133	0.03464	28.8679	413.1847	14.3130
36	1.0941	0.9140	0.02658	37.6206	0.02908	34.3865	592.4988	17.2306
40	1.1050	0.9050	0.02380	42.0132	0.02630	38.0199	728.7399	19.1673
48	1.1273	0.8871	0.01963	50.9312	0.02213	45.1787	1040.06	23.0209
50	1.1330	0.8826	0.01880	53.1887	0.02130	46.9462	1125.78	23.9802
52	1.1386	0.8782	0.01803	55.4575	0.02053	48.7048	1214.59	24.9377
55	1.1472	0.8717	0.01698	58.8819	0.01948	51.3264	1353.53	26.3710
60	1.1616	0.8609	0.01547	64.6467	0.01797	55.6524	1600.08	28.7514
72	1.1969	0.8355	0.01269	78.7794	0.01519	65.8169	2265.56	34.4221
75	1.2059	0.8292	0.01214	82.3792	0.01464	68.3108	2447.61	35.8305
84	1.2334	0.8108	0.01071	93.3419	0.01321	75.6813	3029.76	40.0331
90	1.2520	0.7987	0.00992	100.7885	0.01242	80.5038	3446.87	42.8162
96	1.2709	0.7869	0.00923	108.3474	0.01173	85.2546	3886.28	45.5844
100	1.2836	0.7790	0.00881	113.4500	0.01131	88.3825	4191.24	47.4216
108	1.3095	0.7636	0.00808	123.8093	0.01058	94.5453	4829.01	51.0762
120	1.3494	0.7411	0.00716	139.7414	0.00966	103.5618	5852.11	56.5084
132	1.3904	0.7192	0.00640	156.1582	0.00890	112.3121	6950.01	61.8813
144	1.4327	0.6980	0.00578	173.0743	0.00828	120.8041	8117.41	67.1949
240	1.8208	0.5492	0.00305	328.3020	0.00555	180.3109	19399	107.5863
360	2.4568	0.4070	0.00172	582.7369	0.00422	237.1894	36264	152.8902
480	3.3151	0.3016	0.00108	926.0595	0.00358	279.3418	53821	192.6699

0.5%	TABLE 2 Discrete Cash Flow: Compound Interest Factors							0.5%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0050	0.9950	1.00000	1.0000	1.00500	0.9950		
2	1.0100	0.9901	0.49875	2.0050	0.50375	1.9851	0.9901	0.4988
3	1.0151	0.9851	0.33167	3.0150	0.33667	2.9702	2.9604	0.9967
4	1.0202	0.9802	0.24813	4.0301	0.25313	3.9505	5.9011	1.4938
5	1.0253	0.9754	0.19801	5.0503	0.20301	4.9259	9.8026	1.9900
6	1.0304	0.9705	0.16460	6.0755	0.16960	5.8964	14.6552	2.4855
7	1.0355	0.9657	0.14073	7.1059	0.14573	6.8621	20.4493	2.9801
8	1.0407	0.9609	0.12283	8.1414	0.12783	7.8230	27.1755	3.4738
9	1.0459	0.9561	0.10891	9.1821	0.11391	8.7791	34.8244	3.9668
10	1.0511	0.9513	0.09777	10.2280	0.10277	9.7304	43.3865	4.4589
11	1.0564	0.9466	0.08866	11.2792	0.09366	10.6770	52.8526	4.9501
12	1.0617	0.9419	0.08107	12.3356	0.08607	11.6189	63.2136	5.4406
13	1.0670	0.9372	0.07464	13.3972	0.07964	12.5562	74.4602	5.9302
14	1.0723	0.9326	0.06914	14.4642	0.07414	13.4887	86.5835	6.4190
15	1.0777	0.9279	0.06436	15.5365	0.06936	14.4166	99.5743	6.9069
16	1.0831	0.9233	0.06019	16.6142	0.06519	15.3399	113.4238	7.3940
17	1.0885	0.9187	0.05651	17.6973	0.06151	16.2586	128.1231	7.8803
18	1.0939	0.9141	0.05323	18.7858	0.05823	17.1728	143.6634	8.3658
19	1.0994	0.9096	0.05030	19.8797	0.05530	18.0824	160.0360	8.8504
20	1.1049	0.9051	0.04767	20.9791	0.05267	18.9874	177.2322	9.3342
21	1.1104	0.9006	0.04528	22.0840	0.05028	19.8880	195.2434	9.8172
22	1.1160	0.8961	0.04311	23.1944	0.04811	20.7841	214.0611	10.2993
23	1.1216	0.8916	0.04113	24.3104	0.04613	21.6757	233.6768	10.7806
24	1.1272	0.8872	0.03932	25.4320	0.04432	22.5629	254.0820	11.2611
25	1.1328	0.8828	0.03765	26.5591	0.04265	23.4456	275.2686	11.7407
26	1.1385	0.8784	0.03611	27.6919	0.04111	24.3240	297.2281	12.2195
27	1.1442	0.8740	0.03469	28.8304	0.03969	25.1980	319.9523	12.6975
28	1.1499	0.8697	0.03336	29.9745	0.03836	26.0677	343.4332	13.1747
29	1.1556	0.8653	0.03213	31.1244	0.03713	26.9330	367.6625	13.6510
30	1.1614	0.8610	0.03098	32.2800	0.03598	27.7941	392.6324	14.1265
36	1.1967	0.8356	0.02542	39.3361	0.03042	32.8710	557.5598	16.9621
40	1.2208	0.8191	0.02265	44.1588	0.02765	36.1722	681.3347	18.8359
48	1.2705	0.7871	0.01849	54.0978	0.02349	42.5803	959.9188	22.5437
50	1.2832	0.7793	0.01765	56.6452	0.02265	44.1428	1035.70	23.4624
52	1.2961	0.7716	0.01689	59.2180	0.02189	45.6897	1113.82	24.3778
55	1.3156	0.7601	0.01584	63.1258	0.02084	47.9814	1235.27	25.7447
60	1.3489	0.7414	0.01433	69.7700	0.01933	51.7256	1448.65	28.0064
72	1.4320	0.6983	0.01157	86.4089	0.01657	60.3395	2012.35	33.3504
75	1.4536	0.6879	0.01102	90.7265	0.01602	62.4136	2163.75	34.6679
84	1.5204	0.6577	0.00961	104.0739	0.01461	68.4530	2640.66	38.5763
90	1.5666	0.6383	0.00883	113.3109	0.01383	72.3313	2976.08	41.1451
96	1.6141	0.6195	0.00814	122.8285	0.01314	76.0952	3324.18	43.6845
100	1.6467	0.6073	0.00773	129.3337	0.01273	78.5426	3562.79	45.3613
108	1.7137	0.5835	0.00701	142.7399	0.01201	83.2934	4054.37	48.6758
120	1.8194	0.5496	0.00610	163.8793	0.01110	90.0735	4823.51	53.5508
132	1.9316	0.5177	0.00537	186.3226	0.01037	96.4596	5624.59	58.3103
144	2.0508	0.4876	0.00476	210.1502	0.00976	102.4747	6451.31	62.9551
240	3.3102	0.3021	0.00216	462.0409	0.00716	139.5808	13416	96.1131
360	6.0226	0.1660	0.00100	1004.52	0.00600	166.7916	21403	128.3236
480	10.9575	0.0913	0.00050	1991.49	0.00550	181.7476	27588	151.7949

0.75%	TABLE 3 Discrete Cash Flow: Compound Interest Factors							0.75%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0075	0.9926	1.00000	1.0000	1.00750	0.9926		
2	1.0151	0.9852	0.49813	2.0075	0.50563	1.9777	0.9852	0.4981
3	1.0227	0.9778	0.33085	3.0226	0.33835	2.9556	2.9408	0.9950
4	1.0303	0.9706	0.24721	4.0452	0.25471	3.9261	5.8525	1.4907
5	1.0381	0.9633	0.19702	5.0756	0.20452	4.8894	9.7058	1.9851
6	1.0459	0.9562	0.16357	6.1136	0.17107	5.8456	14.4866	2.4782
7	1.0537	0.9490	0.13967	7.1595	0.14717	6.7946	20.1808	2.9701
8	1.0616	0.9420	0.12176	8.2132	0.12926	7.7366	26.7747	3.4608
9	1.0696	0.9350	0.10782	9.2748	0.11532	8.6716	34.2544	3.9502
10	1.0776	0.9280	0.09667	10.3443	0.10417	9.5996	42.6064	4.4384
11	1.0857	0.9211	0.08755	11.4219	0.09505	10.5207	51.8174	4.9253
12	1.0938	0.9142	0.07995	12.5076	0.08745	11.4349	61.8740	5.4110
13	1.1020	0.9074	0.07352	13.6014	0.08102	12.3423	72.7632	5.8954
14	1.1103	0.9007	0.06801	14.7034	0.07551	13.2430	84.4720	6.3786
15	1.1186	0.8940	0.06324	15.8137	0.07074	14.1370	96.9876	6.8606
16	1.1270	0.8873	0.05906	16.9323	0.06656	15.0243	110.2973	7.3413
17	1.1354	0.8807	0.05537	18.0593	0.06287	15.9050	124.3887	7.8207
18	1.1440	0.8742	0.05210	19.1947	0.05960	16.7792	139.2494	8.2989
19	1.1525	0.8676	0.04917	20.3387	0.05667	17.6468	154.8671	8.7759
20	1.1612	0.8612	0.04653	21.4912	0.05403	18.5080	171.2297	9.2516
21	1.1699	0.8548	0.04415	22.6524	0.05165	19.3628	188.3253	9.7261
22	1.1787	0.8484	0.04198	23.8223	0.04948	20.2112	206.1420	10.1994
23	1.1875	0.8421	0.04000	25.0010	0.04750	21.0533	224.6682	10.6714
24	1.1964	0.8358	0.03818	26.1885	0.04568	21.8891	243.8923	11.1422
25	1.2054	0.8296	0.03652	27.3849	0.04402	22.7188	263.8029	11.6117
26	1.2144	0.8234	0.03498	28.5903	0.04248	23.5422	284.3888	12.0800
27	1.2235	0.8173	0.03355	29.8047	0.04105	24.3595	305.6387	12.5470
28	1.2327	0.8112	0.03223	31.0282	0.03973	25.1707	327.5416	13.0128
29	1.2420	0.8052	0.03100	32.2609	0.03850	25.9759	350.0867	13.4774
30	1.2513	0.7992	0.02985	33.5029	0.03735	26.7751	373.2631	13.9407
36	1.3086	0.7641	0.02430	41.1527	0.03180	31.4468	524.9924	16.6946
40	1.3483	0.7416	0.02153	46.4465	0.02903	34.4469	637.4693	18.5058
48	1.4314	0.6986	0.01739	57.5207	0.02489	40.1848	886.8404	22.0691
50	1.4530	0.6883	0.01656	60.3943	0.02406	41.5664	953.8486	22.9476
52	1.4748	0.6780	0.01580	63.3111	0.02330	42.9276	1022.59	23.8211
55	1.5083	0.6630	0.01476	67.7688	0.02226	44.9316	1128.79	25.1223
60	1.5657	0.6387	0.01326	75.4241	0.02076	48.1734	1313.52	27.2665
72	1.7126	0.5839	0.01053	95.0070	0.01803	55.4768	1791.25	32.2882
75	1.7514	0.5710	0.00998	100.1833	0.01748	57.2027	1917.22	33.5163
84	1.8732	0.5338	0.00859	116.4269	0.01609	62.1540	2308.13	37.1357
90	1.9591	0.5104	0.00782	127.8790	0.01532	65.2746	2578.00	39.4946
96	2.0489	0.4881	0.00715	139.8562	0.01465	68.2584	2853.94	41.8107
100	2.1111	0.4737	0.00675	148.1445	0.01425	70.1746	3040.75	43.3311
108	2.2411	0.4462	0.00604	165.4832	0.01354	73.8394	3419.90	46.3154
120	2.4514	0.4079	0.00517	193.5143	0.01267	78.9417	3998.56	50.6521
132	2.6813	0.3730	0.00446	224.1748	0.01196	83.6064	4583.57	54.8232
144	2.9328	0.3410	0.00388	257.7116	0.01138	87.8711	5169.58	58.8314
240	6.0092	0.1664	0.00150	667.8869	0.00900	111.1450	9494.12	85.4210
360	14.7306	0.0679	0.00055	1830.74	0.00805	124.2819	13312	107.1145
480	36.1099	0.0277	0.00021	4681.32	0.00771	129.6409	15513	119.6620

1%	TABLE 4 Discrete Cash Flow: Compound Interest Factors						1%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0100	0.9901	1.00000	1.0000	1.01000	0.9901		
2	1.0201	0.9803	0.49751	2.0100	0.50751	1.9704	0.9803	0.4975
3	1.0303	0.9706	0.33002	3.0301	0.34002	2.9410	2.9215	0.9934
4	1.0406	0.9610	0.24628	4.0604	0.25628	3.9020	5.8044	1.4876
5	1.0510	0.9515	0.19604	5.1010	0.20604	4.8534	9.6103	1.9801
6	1.0615	0.9420	0.16255	6.1520	0.17255	5.7955	14.3205	2.4710
7	1.0721	0.9327	0.13863	7.2135	0.14863	6.7282	19.9168	2.9602
8	1.0829	0.9235	0.12069	8.2857	0.13069	7.6517	26.3812	3.4478
9	1.0937	0.9143	0.10674	9.3685	0.11674	8.5660	33.6959	3.9337
10	1.1046	0.9053	0.09558	10.4622	0.10558	9.4713	41.8435	4.4179
11	1.1157	0.8963	0.08645	11.5668	0.09645	10.3676	50.8067	4.9005
12	1.1268	0.8874	0.07885	12.6825	0.08885	11.2551	60.5687	5.3815
13	1.1381	0.8787	0.07241	13.8093	0.08241	12.1337	71.1126	5.8607
14	1.1495	0.8700	0.06690	14.9474	0.07690	13.0037	82.4221	6.3384
15	1.1610	0.8613	0.06212	16.0969	0.07212	13.8651	94.4810	6.8143
16	1.1726	0.8528	0.05794	17.2579	0.06794	14.7179	107.2734	7.2886
17	1.1843	0.8444	0.05426	18.4304	0.06426	15.5623	120.7834	7.7613
18	1.1961	0.8360	0.05098	19.6147	0.06098	16.3983	134.9957	8.2323
19	1.2081	0.8277	0.04805	20.8109	0.05805	17.2260	149.8950	8.7017
20	1.2202	0.8195	0.04542	22.0190	0.05542	18.0456	165.4664	9.1694
21	1.2324	0.8114	0.04303	23.2392	0.05303	18.8570	181.6950	9.6354
22	1.2447	0.8034	0.04086	24.4716	0.05086	19.6604	198.5663	10.0998
23	1.2572	0.7954	0.03889	25.7163	0.04889	20.4558	216.0660	10.5626
24	1.2697	0.7876	0.03707	26.9735	0.04707	21.2434	234.1800	11.0237
25	1.2824	0.7798	0.03541	28.2432	0.04541	22.0232	252.8945	11.4831
26	1.2953	0.7720	0.03387	29.5256	0.04387	22.7952	272.1957	11.9409
27	1.3082	0.7644	0.03245	30.8209	0.04245	23.5596	292.0702	12.3971
28	1.3213	0.7568	0.03112	32.1291	0.04112	24.3164	312.5047	12.8516
29	1.3345	0.7493	0.02990	33.4504	0.03990	25.0658	333.4863	13.3044
30	1.3478	0.7419	0.02875	34.7849	0.03875	25.8077	355.0021	13.7557
36	1.4308	0.6989	0.02321	43.0769	0.03321	30.1075	494.6207	16.4285
40	1.4889	0.6717	0.02046	48.8864	0.03046	32.8347	596.8561	18.1776
48	1.6122	0.6203	0.01633	61.2226	0.02633	37.9740	820.1460	21.5976
50	1.6446	0.6080	0.01551	64.4632	0.02551	39.1961	879.4176	22.4363
52	1.6777	0.5961	0.01476	67.7689	0.02476	40.3942	939.9175	23.2686
55	1.7285	0.5785	0.01373	72.8525	0.02373	42.1472	1032.81	24.5049
60	1.8167	0.5504	0.01224	81.6697	0.02224	44.9550	1192.81	26.5333
72	2.0471	0.4885	0.00955	104.7099	0.01955	51.1504	1597.87	31.2386
75	2.1091	0.4741	0.00902	110.9128	0.01902	52.5871	1702.73	32.3793
84	2.3067	0.4335	0.00765	130.6723	0.01765	56.6485	2023.32	35.7170
90	2.4486	0.4084	0.00690	144.8633	0.01690	59.1609	2240.57	37.8724
96	2.5993	0.3847	0.00625	159.9273	0.01625	61.5277	2459.43	39.9727
100	2.7048	0.3697	0.00587	170.4814	0.01587	63.0289	2605.78	41.3426
108	2.9289	0.3414	0.00518	192.8926	0.01518	65.8578	2898.42	44.0103
120	3.3004	0.3030	0.00435	230.0387	0.01435	69.7005	3334.11	47.8349
132	3.7190	0.2689	0.00368	271.8959	0.01368	73.1108	3761.69	51.4520
144	4.1906	0.2386	0.00313	319.0616	0.01313	76.1372	4177.47	54.8676
240	10.8926	0.0918	0.00101	989.2554	0.01101	90.8194	6878.60	75.7393
360	35.9496	0.0278	0.00029	3494.96	0.01029	97.2183	8720.43	89.6995
480	118.6477	0.0084	0.00008	11765	0.01008	99.1572	9511.16	95.9200

1.25% TABLE 5 Discrete Cash Flow: Compound Interest Factors 1.25%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0125	0.9877	1.00000	1.0000	1.01250	0.9877		
2	1.0252	0.9755	0.49680	2.0125	0.50939	1.9631	0.9755	0.4969
3	1.0380	0.9634	0.32920	3.0377	0.34170	2.9265	2.9023	0.9917
4	1.0509	0.9515	0.24536	4.0756	0.25786	3.8781	5.7569	1.4845
5	1.0641	0.9398	0.19506	5.1266	0.20756	4.8178	9.5160	1.9752
6	1.0774	0.9282	0.16153	6.1907	0.17403	5.7460	14.1569	2.4638
7	1.0909	0.9167	0.13759	7.2680	0.15009	6.6627	19.6571	2.9503
8	1.1045	0.9054	0.11963	8.3589	0.13213	7.5681	25.9949	3.4348
9	1.1183	0.8942	0.10567	9.4634	0.11817	8.4623	33.1487	3.9172
10	1.1323	0.8832	0.09450	10.5817	0.10700	9.3455	41.0973	4.3975
11	1.1464	0.8723	0.08537	11.7139	0.09787	10.2178	49.8201	4.8758
12	1.1608	0.8615	0.07776	12.8604	0.09026	11.0793	59.2967	5.3520
13	1.1753	0.8509	0.07132	14.0211	0.08382	11.9302	69.5072	5.8262
14	1.1900	0.8404	0.06581	15.1964	0.07831	12.7706	80.4320	6.2982
15	1.2048	0.8300	0.06103	16.3863	0.07353	13.6005	92.0519	6.7682
16	1.2199	0.8197	0.05685	17.5912	0.06935	14.4203	104.3481	7.2362
17	1.2351	0.8096	0.05316	18.8111	0.06566	15.2299	117.3021	7.7021
18	1.2506	0.7996	0.04988	20.0462	0.06238	16.0295	130.8958	8.1659
19	1.2662	0.7898	0.04696	21.2968	0.05946	16.8193	145.1115	8.6277
20	1.2820	0.7800	0.04432	22.5630	0.05682	17.5993	159.9316	9.0874
21	1.2981	0.7704	0.04194	23.8450	0.05444	18.3697	175.3392	9.5450
22	1.3143	0.7609	0.03977	25.1431	0.05227	19.1306	191.3174	10.0006
23	1.3307	0.7515	0.03780	26.4574	0.05030	19.8820	207.8499	10.4542
24	1.3474	0.7422	0.03599	27.7881	0.04849	20.6242	224.9204	10.9056
25	1.3642	0.7330	0.03432	29.1354	0.04682	21.3573	242.5132	11.3551
26	1.3812	0.7240	0.03279	30.4996	0.04529	22.0813	260.6128	11.8024
27	1.3985	0.7150	0.03137	31.8809	0.04387	22.7963	279.2040	12.2478
28	1.4160	0.7062	0.03005	33.2794	0.04255	23.5025	298.2719	12.6911
29	1.4337	0.6975	0.02882	34.6954	0.04132	24.2000	317.8019	13.1323
30	1.4516	0.6889	0.02768	36.1291	0.04018	24.8889	337.7797	13.5715
36	1.5639	0.6394	0.02217	45.1155	0.03467	28.8473	466.2830	16.1639
40	1.6436	0.6084	0.01942	51.4896	0.03192	31.3269	559.2320	17.8515
48	1.8154	0.5509	0.01533	65.2284	0.02783	35.9315	759.2296	21.1299
50	1.8610	0.5373	0.01452	68.8818	0.02702	37.0129	811.6738	21.9295
52	1.9078	0.5242	0.01377	72.6271	0.02627	38.0677	864.9409	22.7211
55	1.9803	0.5050	0.01275	78.4225	0.02525	39.6017	946.2277	23.8936
60	2.1072	0.4746	0.01129	88.5745	0.02379	42.0346	1084.84	25.8083
72	2.4459	0.4088	0.00865	115.6736	0.02115	47.2925	1428.46	30.2047
75	2.5388	0.3939	0.00812	123.1035	0.02062	48.4890	1515.79	31.2605
84	2.8391	0.3522	0.00680	147.1290	0.01930	51.8222	1778.84	34.3258
90	3.0588	0.3269	0.00607	164.7050	0.01857	53.8461	1953.83	36.2855
96	3.2955	0.3034	0.00545	183.6411	0.01795	55.7246	2127.52	38.1793
100	3.4634	0.2887	0.00507	197.0723	0.01757	56.9013	2242.24	39.4058
108	3.8253	0.2614	0.00442	226.0226	0.01692	59.0865	2468.26	41.7737
120	4.4402	0.2252	0.00363	275.2171	0.01613	61.9828	2796.57	45.1184
132	5.1540	0.1940	0.00301	332.3198	0.01551	64.4781	3109.35	48.2234
144	5.9825	0.1672	0.00251	398.6021	0.01501	66.6277	3404.61	51.0990
240	19.7155	0.0507	0.00067	1497.24	0.01317	75.9423	5101.53	67.1764
360	87.5410	0.0114	0.00014	6923.28	0.01264	79.0861	5997.90	75.8401
480	388.7007	0.0026	0.00003	31016	0.01253	79.7942	6284.74	78.7619

1.5%	TABLE 6 Discrete Cash Flow: Compound Interest Factors							1.5%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0150	0.9852	1.00000	1.0000	1.01500	0.9852		
2	1.0302	0.9707	0.49628	2.0150	0.51128	1.9559	0.9707	0.4963
3	1.0457	0.9563	0.32838	3.0452	0.34338	2.9122	2.8833	0.9901
4	1.0614	0.9422	0.24444	4.0909	0.25944	3.8544	5.7098	1.4814
5	1.0773	0.9283	0.19409	5.1523	0.20909	4.7826	9.4229	1.9702
6	1.0934	0.9145	0.16053	6.2296	0.17553	5.6972	13.9956	2.4566
7	1.1098	0.9010	0.13656	7.3230	0.15156	6.5982	19.4018	2.9405
8	1.1265	0.8877	0.11858	8.4328	0.13358	7.4859	25.6157	3.4219
9	1.1434	0.8746	0.10461	9.5593	0.11961	8.3605	32.6125	3.9008
10	1.1605	0.8617	0.09343	10.7027	0.10843	9.2222	40.3675	4.3772
11	1.1779	0.8489	0.08429	11.8633	0.09929	10.0711	48.8568	4.8512
12	1.1956	0.8364	0.07668	13.0412	0.09168	10.9075	58.0571	5.3227
13	1.2136	0.8240	0.07024	14.2368	0.08524	11.7315	67.9454	5.7917
14	1.2318	0.8118	0.06472	15.4504	0.07972	12.5434	78.4994	6.2582
15	1.2502	0.7999	0.05994	16.6821	0.07494	13.3432	89.6974	6.7223
16	1.2690	0.7880	0.05577	17.9324	0.07077	14.1313	101.5178	7.1839
17	1.2880	0.7764	0.05208	19.2014	0.06708	14.9076	113.9400	7.6431
18	1.3073	0.7649	0.04881	20.4894	0.06381	15.6726	126.9435	8.0997
19	1.3270	0.7536	0.04588	21.7967	0.06088	16.4262	140.5084	8.5539
20	1.3469	0.7425	0.04325	23.1237	0.05825	17.1686	154.6154	9.0057
21	1.3671	0.7315	0.04087	24.4705	0.05587	17.9001	169.2453	9.4550
22	1.3876	0.7207	0.03870	25.8376	0.05370	18.6208	184.3798	9.9018
23	1.4084	0.7100	0.03673	27.2251	0.05173	19.3309	200.0006	10.3462
24	1.4295	0.6995	0.03492	28.6335	0.04992	20.0304	216.0901	10.7881
25	1.4509	0.6892	0.03326	30.0630	0.04826	20.7196	232.6310	11.2276
26	1.4727	0.6790	0.03173	31.5140	0.04673	21.3986	249.6065	11.6646
27	1.4948	0.6690	0.03032	32.9867	0.04532	22.0676	267.0002	12.0992
28	1.5172	0.6591	0.02900	34.4815	0.04400	22.7267	284.7958	12.5313
29	1.5400	0.6494	0.02778	35.9987	0.04278	23.3761	302.9779	12.9610
30	1.5631	0.6398	0.02664	37.5387	0.04164	24.0158	321.5310	13.3883
36	1.7091	0.5851	0.02115	47.2760	0.03615	27.6607	439.8303	15.9009
40	1.8140	0.5513	0.01843	54.2679	0.03343	29.9158	524.3568	17.5277
48	2.0435	0.4894	0.01437	69.5652	0.02937	34.0426	703.5462	20.6667
50	2.1052	0.4750	0.01357	73.6828	0.02857	34.9997	749.9636	21.4277
52	2.1689	0.4611	0.01283	77.9249	0.02783	35.9287	796.8774	22.1794
55	2.2679	0.4409	0.01183	84.5296	0.02683	37.2715	868.0285	23.2894
60	2.4432	0.4093	0.01039	96.2147	0.02539	39.3803	988.1674	25.0930
72	2.9212	0.3423	0.00781	128.0772	0.02281	43.8447	1279.79	29.1893
75	3.0546	0.3274	0.00730	136.9728	0.02230	44.8416	1352.56	30.1631
84	3.4926	0.2863	0.00602	166.1726	0.02102	47.5786	1568.51	32.9668
90	3.8189	0.2619	0.00532	187.9299	0.02032	49.2099	1709.54	34.7399
96	4.1758	0.2395	0.00472	211.7202	0.01972	50.7017	1847.47	36.4381
100	4.4320	0.2256	0.00437	228.8030	0.01937	51.6247	1937.45	37.5295
108	4.9927	0.2003	0.00376	266.1778	0.01876	53.3137	2112.13	39.6171
120	5.9693	0.1675	0.00302	331.2882	0.01802	55.4985	2359.71	42.5185
132	7.1370	0.1401	0.00244	409.1354	0.01744	57.3257	2588.71	45.1579
144	8.5332	0.1172	0.00199	502.2109	0.01699	58.8540	2798.58	47.5512
240	35.6328	0.0281	0.00043	2308.85	0.01543	64.7957	3870.69	59.7368
360	212.7038	0.0047	0.00007	14114	0.01507	66.3532	4310.72	64.9662
480	1269.70	0.0008	0.00001	84580	0.01501	66.6142	4415.74	66.2883

2% TABLE 7 Discrete Cash Flow: Compound Interest Factors 2%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0200	0.9804	1.00000	1.0000	1.02000	0.9804		
2	1.0404	0.9612	0.49505	2.0200	0.51505	1.9416	0.9612	0.4950
3	1.0612	0.9423	0.32675	3.0604	0.34675	2.8839	2.8458	0.9868
4	1.0824	0.9238	0.24262	4.1216	0.26262	3.8077	5.6173	1.4752
5	1.1041	0.9057	0.19216	5.2040	0.21216	4.7135	9.2403	1.9604
6	1.1262	0.8880	0.15853	6.3081	0.17853	5.6014	13.6801	2.4423
7	1.1487	0.8706	0.13451	7.4343	0.15451	6.4720	18.9035	2.9208
8	1.1717	0.8535	0.11651	8.5830	0.13651	7.3255	24.8779	3.3961
9	1.1951	0.8368	0.10252	9.7546	0.12252	8.1622	31.5720	3.8681
10	1.2190	0.8203	0.09133	10.9497	0.11133	8.9826	38.9551	4.3367
11	1.2434	0.8043	0.08218	12.1687	0.10218	9.7868	46.9977	4.8021
12	1.2682	0.7885	0.07456	13.4121	0.09456	10.5753	55.6712	5.2642
13	1.2936	0.7730	0.06812	14.6803	0.08812	11.3484	64.9475	5.7231
14	1.3195	0.7579	0.06260	15.9739	0.08260	12.1062	74.7999	6.1786
15	1.3459	0.7430	0.05783	17.2934	0.07783	12.8493	85.2021	6.6309
16	1.3728	0.7284	0.05365	18.6393	0.07365	13.5777	96.1288	7.0799
17	1.4002	0.7142	0.04997	20.0121	0.06997	14.2919	107.5554	7.5256
18	1.4282	0.7002	0.04670	21.4123	0.06670	14.9920	119.4581	7.9681
19	1.4568	0.6864	0.04378	22.8406	0.06378	15.6785	131.8139	8.4073
20	1.4859	0.6730	0.04116	24.2974	0.06116	16.3514	144.6003	8.8433
21	1.5157	0.6598	0.03878	25.7833	0.05878	17.0112	157.7959	9.2760
22	1.5460	0.6468	0.03663	27.2990	0.05663	17.6580	171.3795	9.7055
23	1.5769	0.6342	0.03467	28.8450	0.05467	18.2922	185.3309	10.1317
24	1.6084	0.6217	0.03287	30.4219	0.05287	18.9139	199.6305	10.5547
25	1.6406	0.6095	0.03122	32.0303	0.05122	19.5235	214.2592	10.9745
26	1.6734	0.5976	0.02970	33.6709	0.04970	20.1210	229.1987	11.3910
27	1.7069	0.5859	0.02829	35.3443	0.04829	20.7069	244.4311	11.8043
28	1.7410	0.5744	0.02699	37.0512	0.04699	21.2813	259.9392	12.2145
29	1.7758	0.5631	0.02578	38.7922	0.04578	21.8444	275.7064	12.6214
30	1.8114	0.5521	0.02465	40.5681	0.04465	22.3965	291.7164	13.0251
36	2.0399	0.4902	0.01923	51.9944	0.03923	25.4888	392.0405	15.3809
40	2.2080	0.4529	0.01656	60.4020	0.03656	27.3555	461.9931	16.8885
48	2.5871	0.3865	0.01260	79.3535	0.03260	30.6731	605.9657	19.7556
50	2.6916	0.3715	0.01182	84.5794	0.03182	31.4236	642.3606	20.4420
52	2.8003	0.3571	0.01111	90.0164	0.03111	32.1449	678.7849	21.1164
55	2.9717	0.3365	0.01014	98.5865	0.03014	33.1748	733.3527	22.1057
60	3.2810	0.3048	0.00877	114.0515	0.02877	34.7609	823.6975	23.6961
72	4.1611	0.2403	0.00633	158.0570	0.02633	37.9841	1034.06	27.2234
75	4.4158	0.2265	0.00586	170.7918	0.02586	38.6771	1084.64	28.0434
84	5.2773	0.1895	0.00468	213.8666	0.02468	40.5255	1230.42	30.3616
90	5.9431	0.1683	0.00405	247.1567	0.02405	41.5869	1322.17	31.7929
96	6.6929	0.1494	0.00351	284.6467	0.02351	42.5294	1409.30	33.1370
100	7.2446	0.1380	0.00320	312.2323	0.02320	43.0984	1464.75	33.9863
108	8.4883	0.1178	0.00267	374.4129	0.02267	44.1095	1569.30	35.5774
120	10.7652	0.0929	0.00205	488.2582	0.02205	45.3554	1710.42	37.7114
132	13.6528	0.0732	0.00158	632.6415	0.02158	46.3378	1833.47	39.5676
144	17.3151	0.0578	0.00123	815.7545	0.02123	47.1123	1939.79	41.1738
240	115.8887	0.0086	0.00017	5744.44	0.02017	49.5686	2374.88	47.9110
360	1247.56	0.0008	0.00002	62328	0.02002	49.9599	2482.57	49.7112
480	13430	0.0001			0.02000	49.9963	2498.03	49.9643

3% TABLE 8 Discrete Cash Flow: Compound Interest Factors 3%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0300	0.9709	1.00000	1.0000	1.03000	0.9709		
2	1.0609	0.9426	0.49261	2.0300	0.52261	1.9135	0.9426	0.4926
3	1.0927	0.9151	0.32353	3.0909	0.35353	2.8286	2.7729	0.9803
4	1.1255	0.8885	0.23903	4.1836	0.26903	3.7171	5.4383	1.4631
5	1.1593	0.8626	0.18835	5.3091	0.21835	4.5797	8.8888	1.9409
6	1.1941	0.8375	0.15460	6.4684	0.18460	5.4172	13.0762	2.4138
7	1.2299	0.8131	0.13051	7.6625	0.16051	6.2303	17.9547	2.8819
8	1.2668	0.7894	0.11246	8.8923	0.14246	7.0197	23.4806	3.3450
9	1.3048	0.7664	0.09843	10.1591	0.12843	7.7861	29.6119	3.8032
10	1.3439	0.7441	0.08723	11.4639	0.11723	8.5302	36.3088	4.2565
11	1.3842	0.7224	0.07808	12.8078	0.10808	9.2526	43.5330	4.7049
12	1.4258	0.7014	0.07046	14.1920	0.10046	9.9540	51.2482	5.1485
13	1.4685	0.6810	0.06403	15.6178	0.09403	10.6350	59.4196	5.5872
14	1.5126	0.6611	0.05853	17.0863	0.08853	11.2961	68.0141	6.0210
15	1.5580	0.6419	0.05377	18.5989	0.08377	11.9379	77.0002	6.4500
16	1.6047	0.6232	0.04961	20.1569	0.07961	12.5611	86.3477	6.8742
17	1.6528	0.6050	0.04595	21.7616	0.07595	13.1661	96.0280	7.2936
18	1.7024	0.5874	0.04271	23.4144	0.07271	13.7535	106.0137	7.7081
19	1.7535	0.5703	0.03981	25.1169	0.06981	14.3238	116.2788	8.1179
20	1.8061	0.5537	0.03722	26.8704	0.06722	14.8775	126.7987	8.5229
21	1.8603	0.5375	0.03487	28.6765	0.06487	15.4150	137.5496	8.9231
22	1.9161	0.5219	0.03275	30.5368	0.06275	15.9369	148.5094	9.3186
23	1.9736	0.5067	0.03081	32.4529	0.06081	16.4436	159.6566	9.7093
24	2.0328	0.4919	0.02905	34.4265	0.05905	16.9355	170.9711	10.0954
25	2.0938	0.4776	0.02743	36.4593	0.05743	17.4131	182.4336	10.4768
26	2.1566	0.4637	0.02594	38.5530	0.05594	17.8768	194.0260	10.8535
27	2.2213	0.4502	0.02456	40.7096	0.05456	18.3270	205.7309	11.2255
28	2.2879	0.4371	0.02329	42.9309	0.05329	18.7641	217.5320	11.5930
29	2.3566	0.4243	0.02211	45.2189	0.05211	19.1885	229.4137	11.9558
30	2.4273	0.4120	0.02102	47.5754	0.05102	19.6004	241.3613	12.3141
31	2.5001	0.4000	0.02000	50.0027	0.05000	20.0004	253.3609	12.6678
32	2.5751	0.3883	0.01905	52.5028	0.04905	20.3888	265.3993	13.0169
33	2.6523	0.3770	0.01816	55.0778	0.04816	20.7658	277.4642	13.3616
34	2.7319	0.3660	0.01732	57.7302	0.04732	21.1318	289.5437	13.7018
35	2.8139	0.3554	0.01654	60.4621	0.04654	21.4872	301.6267	14.0375
40	3.2620	0.3066	0.01326	75.4013	0.04326	23.1148	361.7499	15.6502
45	3.7816	0.2644	0.01079	92.7199	0.04079	24.5187	420.6325	17.1556
50	4.3839	0.2281	0.00887	112.7969	0.03887	25.7298	477.4803	18.5575
55	5.0821	0.1968	0.00735	136.0716	0.03735	26.7744	531.7411	19.8600
60	5.8916	0.1697	0.00613	163.0534	0.03613	27.6756	583.0526	21.0674
65	6.8300	0.1464	0.00515	194.3328	0.03515	28.4529	631.2010	22.1841
70	7.9178	0.1263	0.00434	230.5941	0.03434	29.1234	676.0869	23.2145
75	9.1789	0.1089	0.00367	272.6309	0.03367	29.7018	717.6978	24.1634
80	10.6409	0.0940	0.00311	321.3630	0.03311	30.2008	756.0865	25.0353
84	11.9764	0.0835	0.00273	365.8805	0.03273	30.5501	784.5434	25.6806
85	12.3357	0.0811	0.00265	377.8570	0.03265	30.6312	791.3529	25.8349
90	14.3005	0.0699	0.00226	443.3489	0.03226	31.0024	823.6302	26.5667
96	17.0755	0.0586	0.00187	535.8502	0.03187	31.3812	858.6377	27.3615
108	24.3456	0.0411	0.00129	778.1863	0.03129	31.9642	917.6013	28.7072
120	34.7110	0.0288	0.00089	1123.70	0.03089	32.3730	963.8635	29.7737

4%	TABLE 9 Discrete Cash Flow: Compound Interest Factors							4%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0400	0.9615	1.00000	1.0000	1.04000	0.9615		
2	1.0816	0.9246	0.49020	2.0400	0.53020	1.8861	0.9246	0.4902
3	1.1249	0.8890	0.32035	3.1216	0.36035	2.7751	2.7025	0.9739
4	1.1699	0.8548	0.23549	4.2465	0.27549	3.6299	5.2670	1.4510
5	1.2167	0.8219	0.18463	5.4163	0.22463	4.4518	8.5547	1.9216
6	1.2653	0.7903	0.15076	6.6330	0.19076	5.2421	12.5062	2.3857
7	1.3159	0.7599	0.12661	7.8983	0.16661	6.0021	17.0657	2.8433
8	1.3686	0.7307	0.10853	9.2142	0.14853	6.7327	22.1806	3.2944
9	1.4233	0.7026	0.09449	10.5828	0.13449	7.4353	27.8013	3.7391
10	1.4802	0.6756	0.08329	12.0061	0.12329	8.1109	33.8814	4.1773
11	1.5395	0.6496	0.07415	13.4864	0.11415	8.7605	40.3772	4.6090
12	1.6010	0.6246	0.06655	15.0258	0.10655	9.3851	47.2477	5.0343
13	1.6651	0.6006	0.06014	16.6268	0.10014	9.9856	54.4546	5.4533
14	1.7317	0.5775	0.05467	18.2919	0.09467	10.5631	61.9618	5.8659
15	1.8009	0.5553	0.04994	20.0236	0.08994	11.1184	69.7355	6.2721
16	1.8730	0.5339	0.04582	21.8245	0.08582	11.6523	77.7441	6.6720
17	1.9479	0.5134	0.04220	23.6975	0.08220	12.1657	85.9581	7.0656
18	2.0258	0.4936	0.03899	25.6454	0.07899	12.6593	94.3498	7.4530
19	2.1068	0.4746	0.03614	27.6712	0.07614	13.1339	102.8933	7.8342
20	2.1911	0.4564	0.03358	29.7781	0.07358	13.5903	111.5647	8.2091
21	2.2788	0.4388	0.03128	31.9692	0.07128	14.0292	120.3414	8.5779
22	2.3699	0.4220	0.02920	34.2480	0.06920	14.4511	129.2024	8.9407
23	2.4647	0.4057	0.02731	36.6179	0.06731	14.8568	138.1284	9.2973
24	2.5633	0.3901	0.02559	39.0826	0.06559	15.2470	147.1012	9.6479
25	2.6658	0.3751	0.02401	41.6459	0.06401	15.6221	156.1040	9.9925
26	2.7725	0.3607	0.02257	44.3117	0.06257	15.9828	165.1212	10.3312
27	2.8834	0.3468	0.02124	47.0842	0.06124	16.3296	174.1385	10.6640
28	2.9987	0.3335	0.02001	49.9676	0.06001	16.6631	183.1424	10.9909
29	3.1187	0.3207	0.01888	52.9663	0.05888	16.9837	192.1206	11.3120
30	3.2434	0.3083	0.01783	56.0849	0.05783	17.2920	201.0618	11.6274
31	3.3731	0.2965	0.01686	59.3283	0.05686	17.5885	209.9556	11.9371
32	3.5081	0.2851	0.01595	62.7015	0.05595	17.8736	218.7924	12.2411
33	3.6484	0.2741	0.01510	66.2095	0.05510	18.1476	227.5634	12.5396
34	3.7943	0.2636	0.01431	69.8579	0.05431	18.4112	236.2607	12.8324
35	3.9461	0.2534	0.01358	73.6522	0.05358	18.6646	244.8768	13.1198
40	4.8010	0.2083	0.01052	95.0255	0.05052	19.7928	286.5303	14.4765
45	5.8412	0.1712	0.00826	121.0294	0.04826	20.7200	325.4028	15.7047
50	7.1067	0.1407	0.00655	152.6671	0.04655	21.4822	361.1638	16.8122
55	8.6464	0.1157	0.00523	191.1592	0.04523	22.1086	393.6890	17.8070
60	10.5196	0.0951	0.00420	237.9907	0.04420	22.6235	422.9966	18.6972
65	12.7987	0.0781	0.00339	294.9684	0.04339	23.0467	449.2014	19.4909
70	15.5716	0.0642	0.00275	364.2905	0.04275	23.3945	472.4789	20.1961
75	18.9453	0.0528	0.00223	448.6314	0.04223	23.6804	493.0408	20.8206
80	23.0498	0.0434	0.00181	551.2450	0.04181	23.9154	511.1161	21.3718
85	28.0436	0.0357	0.00148	676.0901	0.04148	24.1085	526.9384	21.8569
90	34.1193	0.0293	0.00121	827.9833	0.04121	24.2673	540.7369	22.2826
96	43.1718	0.0232	0.00095	1054.30	0.04095	24.4209	554.9312	22.7236
108	69.1195	0.0145	0.00059	1702.99	0.04059	24.6383	576.8949	23.4146
120	110.6626	0.0090	0.00036	2741.56	0.04036	24.7741	592.2428	23.9057
144	283.6618	0.0035	0.00014	7066.55	0.04014	24.9119	610.1055	24.4906

5%	TABLE 10 Discrete Cash Flow: Compound Interest Factors							5%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0500	0.9524	1.00000	1.0000	1.05000	0.9524		
2	1.1025	0.9070	0.48780	2.0500	0.53780	1.8594	0.9070	0.4878
3	1.1576	0.8638	0.31721	3.1525	0.36721	2.7232	2.6347	0.9675
4	1.2155	0.8227	0.23201	4.3101	0.28201	3.5460	5.1028	1.4391
5	1.2763	0.7835	0.18097	5.5256	0.23097	4.3295	8.2369	1.9025
6	1.3401	0.7462	0.14702	6.8019	0.19702	5.0757	11.9680	2.3579
7	1.4071	0.7107	0.12282	8.1420	0.17282	5.7864	16.2321	2.8052
8	1.4775	0.6768	0.10472	9.5491	0.15472	6.4632	20.9700	3.2445
9	1.5513	0.6446	0.09069	11.0266	0.14069	7.1078	26.1268	3.6758
10	1.6289	0.6139	0.07950	12.5779	0.12950	7.7217	31.6520	4.0991
11	1.7103	0.5847	0.07039	14.2068	0.12039	8.3064	37.4988	4.5144
12	1.7959	0.5568	0.06283	15.9171	0.11283	8.8633	43.6241	4.9219
13	1.8856	0.5303	0.05646	17.7130	0.10646	9.3936	49.9879	5.3215
14	1.9799	0.5051	0.05102	19.5986	0.10102	9.8986	56.5538	5.7133
15	2.0789	0.4810	0.04634	21.5786	0.09634	10.3797	63.2880	6.0973
16	2.1829	0.4581	0.04227	23.6575	0.09227	10.8378	70.1597	6.4736
17	2.2920	0.4363	0.03870	25.8404	0.08870	11.2741	77.1405	6.8423
18	2.4066	0.4155	0.03555	28.1324	0.08555	11.6896	84.2043	7.2034
19	2.5270	0.3957	0.03275	30.5390	0.08275	12.0853	91.3275	7.5569
20	2.6533	0.3769	0.03024	33.0660	0.08024	12.4622	98.4884	7.9030
21	2.7860	0.3589	0.02800	35.7193	0.07800	12.8212	105.6673	8.2416
22	2.9253	0.3418	0.02597	38.5052	0.07597	13.1630	112.8461	8.5730
23	3.0715	0.3256	0.02414	41.4305	0.07414	13.4886	120.0087	8.8971
24	3.2251	0.3101	0.02247	44.5020	0.07247	13.7986	127.1402	9.2140
25	3.3864	0.2953	0.02095	47.7271	0.07095	14.0939	134.2275	9.5238
26	3.5557	0.2812	0.01956	51.1135	0.06956	14.3752	141.2585	9.8266
27	3.7335	0.2678	0.01829	54.6691	0.06829	14.6430	148.2226	10.1224
28	3.9201	0.2551	0.01712	58.4026	0.06712	14.8981	155.1101	10.4114
29	4.1161	0.2429	0.01605	62.3227	0.06605	15.1411	161.9126	10.6936
30	4.3219	0.2314	0.01505	66.4388	0.06505	15.3725	168.6226	10.9691
31	4.5380	0.2204	0.01413	70.7608	0.06413	15.5928	175.2333	11.2381
32	4.7649	0.2099	0.01328	75.2988	0.06328	15.8027	181.7392	11.5005
33	5.0032	0.1999	0.01249	80.0638	0.06249	16.0025	188.1351	11.7566
34	5.2533	0.1904	0.01176	85.0670	0.06176	16.1929	194.4168	12.0063
35	5.5160	0.1813	0.01107	90.3203	0.06107	16.3742	200.5807	12.2498
40	7.0400	0.1420	0.00828	120.7998	0.05828	17.1591	229.5452	13.3775
45	8.9850	0.1113	0.00626	159.7002	0.05626	17.7741	255.3145	14.3644
50	11.4674	0.0872	0.00478	209.3480	0.05478	18.2559	277.9148	15.2233
55	14.6356	0.0683	0.00367	272.7126	0.05367	18.6335	297.5104	15.9664
60	18.6792	0.0535	0.00283	353.5837	0.05283	18.9293	314.3432	16.6062
65	23.8399	0.0419	0.00219	456.7980	0.05219	19.1611	328.6910	17.1541
70	30.4264	0.0329	0.00170	588.5285	0.05170	19.3427	340.8409	17.6212
75	38.8327	0.0258	0.00132	756.6537	0.05132	19.4850	351.0721	18.0176
80	49.5614	0.0202	0.00103	971.2288	0.05103	19.5965	359.6460	18.3526
85	63.2544	0.0158	0.00080	1245.09	0.05080	19.6838	366.8007	18.6346
90	80.7304	0.0124	0.00063	1594.61	0.05063	19.7523	372.7488	18.8712
95	103.0347	0.0097	0.00049	2040.69	0.05049	19.8059	377.6774	19.0689
96	108.1864	0.0092	0.00047	2143.73	0.05047	19.8151	378.5555	19.1044
98	119.2755	0.0084	0.00042	2365.51	0.05042	19.8323	380.2139	19.1714
100	131.5013	0.0076	0.00038	2610.03	0.05038	19.8479	381.7492	19.2337

6%	TABLE 11 Discrete Cash Flow: Compound Interest Factors							6%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0600	0.9434	1.00000	1.0000	1.06000	0.9434		
2	1.1236	0.8900	0.48544	2.0600	0.54544	1.8334	0.8900	0.4854
3	1.1910	0.8396	0.31411	3.1836	0.37411	2.6730	2.5692	0.9612
4	1.2625	0.7921	0.22859	4.3746	0.28859	3.4651	4.9455	1.4272
5	1.3382	0.7473	0.17740	5.6371	0.23740	4.2124	7.9345	1.8836
6	1.4185	0.7050	0.14336	6.9753	0.20336	4.9173	11.4594	2.3304
7	1.5036	0.6651	0.11914	8.3938	0.17914	5.5824	15.4497	2.7676
8	1.5938	0.6274	0.10104	9.8975	0.16104	6.2098	19.8416	3.1952
9	1.6895	0.5919	0.08702	11.4913	0.14702	6.8017	24.5768	3.6133
10	1.7908	0.5584	0.07587	13.1808	0.13587	7.3601	29.6023	4.0220
11	1.8983	0.5268	0.06679	14.9716	0.12679	7.8869	34.8702	4.4213
12	2.0122	0.4970	0.05928	16.8699	0.11928	8.3838	40.3369	4.8113
13	2.1329	0.4688	0.05296	18.8821	0.11296	8.8527	45.9629	5.1920
14	2.2609	0.4423	0.04758	21.0151	0.10758	9.2950	51.7128	5.5635
15	2.3966	0.4173	0.04296	23.2760	0.10296	9.7122	57.5546	5.9260
16	2.5404	0.3936	0.03895	25.6725	0.09895	10.1059	63.4592	6.2794
17	2.6928	0.3714	0.03544	28.2129	0.09544	10.4773	69.4011	6.6240
18	2.8543	0.3503	0.03236	30.9057	0.09236	10.8276	75.3569	6.9597
19	3.0256	0.3305	0.02962	33.7600	0.08962	11.1581	81.3062	7.2867
20	3.2071	0.3118	0.02718	36.7856	0.08718	11.4699	87.2304	7.6051
21	3.3996	0.2942	0.02500	39.9927	0.08500	11.7641	93.1136	7.9151
22	3.6035	0.2775	0.02305	43.3923	0.08305	12.0416	98.9412	8.2166
23	3.8197	0.2618	0.02128	46.9958	0.08128	12.3034	104.7007	8.5099
24	4.0489	0.2470	0.01968	50.8156	0.07968	12.5504	110.3812	8.7951
25	4.2919	0.2330	0.01823	54.8645	0.07823	12.7834	115.9732	9.0722
26	4.5494	0.2198	0.01690	59.1564	0.07690	13.0032	121.4684	9.3414
27	4.8223	0.2074	0.01570	63.7058	0.07570	13.2105	126.8600	9.6029
28	5.1117	0.1956	0.01459	68.5281	0.07459	13.4062	132.1420	9.8568
29	5.4184	0.1846	0.01358	73.6398	0.07358	13.5907	137.3096	10.1032
30	5.7435	0.1741	0.01265	79.0582	0.07265	13.7648	142.3588	10.3422
31	6.0881	0.1643	0.01179	84.8017	0.07179	13.9291	147.2864	10.5740
32	6.4534	0.1550	0.01100	90.8898	0.07100	14.0840	152.0901	10.7988
33	6.8406	0.1462	0.01027	97.3432	0.07027	14.2302	156.7681	11.0166
34	7.2510	0.1379	0.00960	104.1838	0.06960	14.3681	161.3192	11.2276
35	7.6861	0.1301	0.00897	111.4348	0.06897	14.4982	165.7427	11.4319
40	10.2857	0.0972	0.00646	154.7620	0.06646	15.0463	185.9568	12.3590
45	13.7646	0.0727	0.00470	212.7435	0.06470	15.4558	203.1096	13.1413
50	18.4202	0.0543	0.00344	290.3359	0.06344	15.7619	217.4574	13.7964
55	24.6503	0.0406	0.00254	394.1720	0.06254	15.9905	229.3222	14.3411
60	32.9877	0.0303	0.00188	533.1282	0.06188	16.1614	239.0428	14.7909
65	44.1450	0.0227	0.00139	719.0829	0.06139	16.2891	246.9450	15.1601
70	59.0759	0.0169	0.00103	967.9322	0.06103	16.3845	253.3271	15.4613
75	79.0569	0.0126	0.00077	1300.95	0.06077	16.4558	258.4527	15.7058
80	105.7960	0.0095	0.00057	1746.60	0.06057	16.5091	262.5493	15.9033
85	141.5789	0.0071	0.00043	2342.98	0.06043	16.5489	265.8096	16.0620
90	189.4645	0.0053	0.00032	3141.08	0.06032	16.5787	268.3946	16.1891
95	253.5463	0.0039	0.00024	4209.10	0.06024	16.6009	270.4375	16.2905
96	268.7590	0.0037	0.00022	4462.65	0.06022	16.6047	270.7909	16.3081
98	301.9776	0.0033	0.00020	5016.29	0.06020	16.6115	271.4491	16.3411
100	339.3021	0.0029	0.00018	5638.37	0.06018	16.6175	272.0471	16.3711

7% TABLE 12 Discrete Cash Flow: Compound Interest Factors 7%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0700	0.9346	1.00000	1.0000	1.07000	0.9346		
2	1.1449	0.8734	0.48309	2.0700	0.55309	1.8080	0.8734	0.4831
3	1.2250	0.8163	0.31105	3.2149	0.38105	2.6243	2.5060	0.9549
4	1.3108	0.7629	0.22523	4.4399	0.29523	3.3872	4.7947	1.4155
5	1.4026	0.7130	0.17389	5.7507	0.24389	4.1002	7.6467	1.8650
6	1.5007	0.6663	0.13980	7.1533	0.20980	4.7665	10.9784	2.3032
7	1.6058	0.6227	0.11555	8.6540	0.18555	5.3893	14.7149	2.7304
8	1.7182	0.5820	0.09747	10.2598	0.16747	5.9713	18.7889	3.1465
9	1.8385	0.5439	0.08349	11.9780	0.15349	6.5152	23.1404	3.5517
10	1.9672	0.5083	0.07238	13.8164	0.14238	7.0236	27.7156	3.9461
11	2.1049	0.4751	0.06336	15.7836	0.13336	7.4987	32.4665	4.3296
12	2.2522	0.4440	0.05590	17.8885	0.12590	7.9427	37.3506	4.7025
13	2.4098	0.4150	0.04965	20.1406	0.11965	8.3577	42.3302	5.0648
14	2.5785	0.3878	0.04434	22.5505	0.11434	8.7455	47.3718	5.4167
15	2.7590	0.3624	0.03979	25.1290	0.10979	9.1079	52.4461	5.7583
16	2.9522	0.3387	0.03586	27.8881	0.10586	9.4466	57.5271	6.0897
17	3.1588	0.3166	0.03243	30.8402	0.10243	9.7632	62.5923	6.4110
18	3.3799	0.2959	0.02941	33.9990	0.09941	10.0591	67.6219	6.7225
19	3.6165	0.2765	0.02675	37.3790	0.09675	10.3356	72.5991	7.0242
20	3.8697	0.2584	0.02439	40.9955	0.09439	10.5940	77.5091	7.3163
21	4.1406	0.2415	0.02229	44.8652	0.09229	10.8355	82.3393	7.5990
22	4.4304	0.2257	0.02041	49.0057	0.09041	11.0612	87.0793	7.8725
23	4.7405	0.2109	0.01871	53.4361	0.08871	11.2722	91.7201	8.1369
24	5.0724	0.1971	0.01719	58.1767	0.08719	11.4693	96.2545	8.3923
25	5.4274	0.1842	0.01581	63.2490	0.08581	11.6536	100.6765	8.6391
26	5.8074	0.1722	0.01456	68.6765	0.08456	11.8258	104.9814	8.8773
27	6.2139	0.1609	0.01343	74.4838	0.08343	11.9867	109.1656	9.1072
28	6.6488	0.1504	0.01239	80.6977	0.08239	12.1371	113.2264	9.3289
29	7.1143	0.1406	0.01145	87.3465	0.08145	12.2777	117.1622	9.5427
30	7.6123	0.1314	0.01059	94.4608	0.08059	12.4090	120.9718	9.7487
31	8.1451	0.1228	0.00980	102.0730	0.07980	12.5318	124.6550	9.9471
32	8.7153	0.1147	0.00907	110.2182	0.07907	12.6466	128.2120	10.1381
33	9.3253	0.1072	0.00841	118.9334	0.07841	12.7538	131.6435	10.3219
34	9.9781	0.1002	0.00780	128.2588	0.07780	12.8540	134.9507	10.4987
35	10.6766	0.0937	0.00723	138.2369	0.07723	12.9477	138.1353	10.6687
40	14.9745	0.0668	0.00501	199.6351	0.07501	13.3317	152.2928	11.4233
45	21.0025	0.0476	0.00350	285.7493	0.07350	13.6055	163.7559	12.0360
50	29.4570	0.0339	0.00246	406.5289	0.07246	13.8007	172.9051	12.5287
55	41.3150	0.0242	0.00174	575.9286	0.07174	13.9399	180.1243	12.9215
60	57.9464	0.0173	0.00123	813.5204	0.07123	14.0392	185.7677	13.2321
65	81.2729	0.0123	0.00087	1146.76	0.07087	14.1099	190.1452	13.4760
70	113.9894	0.0088	0.00062	1614.13	0.07062	14.1604	193.5185	13.6662
75	159.8760	0.0063	0.00044	2269.66	0.07044	14.1964	196.1035	13.8136
80	224.2344	0.0045	0.00031	3189.06	0.07031	14.2220	198.0748	13.9273
85	314.5003	0.0032	0.00022	4478.58	0.07022	14.2403	199.5717	14.0146
90	441.1030	0.0023	0.00016	6287.19	0.07016	14.2533	200.7042	14.0812
95	618.6697	0.0016	0.00011	8823.85	0.07011	14.2626	201.5581	14.1319
96	661.9766	0.0015	0.00011	9442.52	0.07011	14.2641	201.7016	14.1405
98	757.8970	0.0013	0.00009	10813	0.07009	14.2669	201.9651	14.1562
100	867.7163	0.0012	0.00008	12382	0.07008	14.2693	202.2001	14.1703

8% TABLE 13 Discrete Cash Flow: Compound Interest Factors 8%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0800	0.9259	1.00000	1.0000	1.08000	0.9259		
2	1.1664	0.8573	0.48077	2.0800	0.56077	1.7833	0.8573	0.4808
3	1.2597	0.7938	0.30803	3.2464	0.38803	2.5771	2.4450	0.9487
4	1.3605	0.7350	0.22192	4.5061	0.30192	3.3121	4.6501	1.4040
5	1.4693	0.6806	0.17046	5.8666	0.25046	3.9927	7.3724	1.8465
6	1.5869	0.6302	0.13632	7.3359	0.21632	4.6229	10.5233	2.2763
7	1.7138	0.5835	0.11207	8.9228	0.19207	5.2064	14.0242	2.6937
8	1.8509	0.5403	0.09401	10.6366	0.17401	5.7466	17.8061	3.0985
9	1.9990	0.5002	0.08008	12.4876	0.16008	6.2469	21.8081	3.4910
10	2.1589	0.4632	0.06903	14.4866	0.14903	6.7101	25.9768	3.8713
11	2.3316	0.4289	0.06008	16.6455	0.14008	7.1390	30.2657	4.2395
12	2.5182	0.3971	0.05270	18.9771	0.13270	7.5361	34.6339	4.5957
13	2.7196	0.3677	0.04652	21.4953	0.12652	7.9038	39.0463	4.9402
14	2.9372	0.3405	0.04130	24.2149	0.12130	8.2442	43.4723	5.2731
15	3.1722	0.3152	0.03683	27.1521	0.11683	8.5595	47.8857	5.5945
16	3.4259	0.2919	0.03298	30.3243	0.11298	8.8514	52.2640	5.9046
17	3.7000	0.2703	0.02963	33.7502	0.10963	9.1216	56.5883	6.2037
18	3.9960	0.2502	0.02670	37.4502	0.10670	9.3719	60.8426	6.4920
19	4.3157	0.2317	0.02413	41.4463	0.10413	9.6036	65.0134	6.7697
20	4.6610	0.2145	0.02185	45.7620	0.10185	9.8181	69.0898	7.0369
21	5.0338	0.1987	0.01983	50.4229	0.09983	10.0168	73.0629	7.2940
22	5.4365	0.1839	0.01803	55.4568	0.09803	10.2007	76.9257	7.5412
23	5.8715	0.1703	0.01642	60.8933	0.09642	10.3711	80.6726	7.7786
24	6.3412	0.1577	0.01498	66.7648	0.09498	10.5288	84.2997	8.0066
25	6.8485	0.1460	0.01368	73.1059	0.09368	10.6748	87.8041	8.2254
26	7.3964	0.1352	0.01251	79.9544	0.09251	10.8100	91.1842	8.4352
27	7.9881	0.1252	0.01145	87.3508	0.09145	10.9352	94.4390	8.6363
28	8.6271	0.1159	0.01049	95.3388	0.09049	11.0511	97.5687	8.8289
29	9.3173	0.1073	0.00962	103.9659	0.08962	11.1584	100.5738	9.0133
30	10.0627	0.0994	0.00883	113.2832	0.08883	11.2578	103.4558	9.1897
31	10.8677	0.0920	0.00811	123.3459	0.08811	11.3498	106.2163	9.3584
32	11.7371	0.0852	0.00745	134.2135	0.08745	11.4350	108.8575	9.5197
33	12.6760	0.0789	0.00685	145.9506	0.08685	11.5139	111.3819	9.6737
34	13.6901	0.0730	0.00630	158.6267	0.08630	11.5869	113.7924	9.8208
35	14.7853	0.0676	0.00580	172.3168	0.08580	11.6546	116.0920	9.9611
40	21.7245	0.0460	0.00386	259.0565	0.08386	11.9246	126.0422	10.5699
45	31.9204	0.0313	0.00259	386.5056	0.08259	12.1084	133.7331	11.0447
50	46.9016	0.0213	0.00174	573.7702	0.08174	12.2335	139.5928	11.4107
55	68.9139	0.0145	0.00118	848.9232	0.08118	12.3186	144.0065	11.6902
60	101.2571	0.0099	0.00080	1253.21	0.08080	12.3766	147.3000	11.9015
65	148.7798	0.0067	0.00054	1847.25	0.08054	12.4160	149.7387	12.0602
70	218.6064	0.0046	0.00037	2720.08	0.08037	12.4428	151.5326	12.1783
75	321.2045	0.0031	0.00025	4002.56	0.08025	12.4611	152.8448	12.2658
80	471.9548	0.0021	0.00017	5886.94	0.08017	12.4735	153.8001	12.3301
85	693.4565	0.0014	0.00012	8655.71	0.08012	12.4820	154.4925	12.3772
90	1018.92	0.0010	0.00008	12724	0.08008	12.4877	154.9925	12.4116
95	1497.12	0.0007	0.00005	18702	0.08005	12.4917	155.3524	12.4365
96	1616.89	0.0006	0.00005	20199	0.08005	12.4923	155.4112	12.4406
98	1885.94	0.0005	0.00004	23562	0.08004	12.4934	155.5176	12.4480
100	2199.76	0.0005	0.00004	27485	0.08004	12.4943	155.6107	12.4545

9%	TABLE 14 Discrete Cash Flow: Compound Interest Factors							9%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.0900	0.9174	1.00000	1.0000	1.09000	0.9174		
2	1.1881	0.8417	0.47847	2.0900	0.56847	1.7591	0.8417	0.4785
3	1.2950	0.7722	0.30505	3.2781	0.39505	2.5313	2.3860	0.9426
4	1.4116	0.7084	0.21867	4.5731	0.30867	3.2397	4.5113	1.3925
5	1.5386	0.6499	0.16709	5.9847	0.25709	3.8897	7.1110	1.8282
6	1.6771	0.5963	0.13292	7.5233	0.22292	4.4859	10.0924	2.2498
7	1.8280	0.5470	0.10869	9.2004	0.19869	5.0330	13.3746	2.6574
8	1.9926	0.5019	0.09067	11.0285	0.18067	5.5348	16.8877	3.0512
9	2.1719	0.4604	0.07680	13.0210	0.16680	5.9952	20.5711	3.4312
10	2.3674	0.4224	0.06582	15.1929	0.15582	6.4177	24.3728	3.7978
11	2.5804	0.3875	0.05695	17.5603	0.14695	6.8052	28.2481	4.1510
12	2.8127	0.3555	0.04965	20.1407	0.13965	7.1607	32.1590	4.4910
13	3.0658	0.3262	0.04357	22.9534	0.13357	7.4869	36.0731	4.8182
14	3.3417	0.2992	0.03843	26.0192	0.12843	7.7862	39.9633	5.1326
15	3.6425	0.2745	0.03406	29.3609	0.12406	8.0607	43.8069	5.4346
16	3.9703	0.2519	0.03030	33.0034	0.12030	8.3126	47.5849	5.7245
17	4.3276	0.2311	0.02705	36.9737	0.11705	8.5436	51.2821	6.0024
18	4.7171	0.2120	0.02421	41.3013	0.11421	8.7556	54.8860	6.2687
19	5.1417	0.1945	0.02173	46.0185	0.11173	8.9501	58.3868	6.5236
20	5.6044	0.1784	0.01955	51.1601	0.10955	9.1285	61.7770	6.7674
21	6.1088	0.1637	0.01762	56.7645	0.10762	9.2922	65.0509	7.0006
22	6.6586	0.1502	0.01590	62.8733	0.10590	9.4424	68.2048	7.2232
23	7.2579	0.1378	0.01438	69.5319	0.10438	9.5802	71.2359	7.4357
24	7.9111	0.1264	0.01302	76.7898	0.10302	9.7066	74.1433	7.6384
25	8.6231	0.1160	0.01181	84.7009	0.10181	9.8226	76.9265	7.8316
26	9.3992	0.1064	0.01072	93.3240	0.10072	9.9290	79.5863	8.0156
27	10.2451	0.0976	0.00973	102.7231	0.09973	10.0266	82.1241	8.1906
28	11.1671	0.0895	0.00885	112.9682	0.09885	10.1161	84.5419	8.3571
29	12.1722	0.0822	0.00806	124.1354	0.09806	10.1983	86.8422	8.5154
30	13.2677	0.0754	0.00734	136.3075	0.09734	10.2737	89.0280	8.6657
31	14.4618	0.0691	0.00669	149.5752	0.09669	10.3428	91.1024	8.8083
32	15.7633	0.0634	0.00610	164.0370	0.09610	10.4062	93.0690	8.9436
33	17.1820	0.0582	0.00556	179.8003	0.09556	10.4644	94.9314	9.0718
34	18.7284	0.0534	0.00508	196.9823	0.09508	10.5178	96.6935	9.1933
35	20.4140	0.0490	0.00464	215.7108	0.09464	10.5668	98.3590	9.3083
40	31.4094	0.0318	0.00296	337.8824	0.09296	10.7574	105.3762	9.7957
45	48.3273	0.0207	0.00190	525.8587	0.09190	10.8812	110.5561	10.1603
50	74.3575	0.0134	0.00123	815.0836	0.09123	10.9617	114.3251	10.4295
55	114.4083	0.0087	0.00079	1260.09	0.09079	11.0140	117.0362	10.6261
60	176.0313	0.0057	0.00051	1944.79	0.09051	11.0480	118.9683	10.7683
65	270.8460	0.0037	0.00033	2998.29	0.09033	11.0701	120.3344	10.8702
70	416.7301	0.0024	0.00022	4619.22	0.09022	11.0844	121.2942	10.9427
75	641.1909	0.0016	0.00014	7113.23	0.09014	11.0938	121.9646	10.9940
80	986.5517	0.0010	0.00009	10951	0.09009	11.0998	122.4306	11.0299
85	1517.93	0.0007	0.00006	16855	0.09006	11.1038	122.7533	11.0551
90	2335.53	0.0004	0.00004	25939	0.09004	11.1064	122.9758	11.0726
95	3593.50	0.0003	0.00003	39917	0.09003	11.1080	123.1287	11.0847
96	3916.91	0.0003	0.00002	43510	0.09002	11.1083	123.1529	11.0866
98	4653.68	0.0002	0.00002	51696	0.09002	11.1087	123.1963	11.0900
100	5529.04	0.0002	0.00002	61423	0.09002	11.1091	123.2335	11.0930

10% TABLE 15 Discrete Cash Flow: Compound Interest Factors 10%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.1000	0.9091	1.00000	1.0000	1.10000	0.9091		
2	1.2100	0.8264	0.47619	2.1000	0.57619	1.7355	0.8264	0.4762
3	1.3310	0.7513	0.30211	3.3100	0.40211	2.4869	2.3291	0.9366
4	1.4641	0.6830	0.21547	4.6410	0.31547	3.1699	4.3781	1.3812
5	1.6105	0.6209	0.16380	6.1051	0.26380	3.7908	6.8618	1.8101
6	1.7716	0.5645	0.12961	7.7156	0.22961	4.3553	9.6842	2.2236
7	1.9487	0.5132	0.10541	9.4872	0.20541	4.8684	12.7631	2.6216
8	2.1436	0.4665	0.08744	11.4359	0.18744	5.3349	16.0287	3.0045
9	2.3579	0.4241	0.07364	13.5795	0.17364	5.7590	19.4215	3.3724
10	2.5937	0.3855	0.06275	15.9374	0.16275	6.1446	22.8913	3.7255
11	2.8531	0.3505	0.05396	18.5312	0.15396	6.4951	26.3963	4.0641
12	3.1384	0.3186	0.04676	21.3843	0.14676	6.8137	29.9012	4.3884
13	3.4523	0.2897	0.04078	24.5227	0.14078	7.1034	33.3772	4.6988
14	3.7975	0.2633	0.03575	27.9750	0.13575	7.3667	36.8005	4.9955
15	4.1772	0.2394	0.03147	31.7725	0.13147	7.6061	40.1520	5.2789
16	4.5950	0.2176	0.02782	35.9497	0.12782	7.8237	43.4164	5.5493
17	5.0545	0.1978	0.02466	40.5447	0.12466	8.0216	46.5819	5.8071
18	5.5599	0.1799	0.02193	45.5992	0.12193	8.2014	49.6395	6.0526
19	6.1159	0.1635	0.01955	51.1591	0.11955	8.3649	52.5827	6.2861
20	6.7275	0.1486	0.01746	57.2750	0.11746	8.5136	55.4069	6.5081
21	7.4002	0.1351	0.01562	64.0025	0.11562	8.6487	58.1095	6.7189
22	8.1403	0.1228	0.01401	71.4027	0.11401	8.7715	60.6893	6.9189
23	8.9543	0.1117	0.01257	79.5430	0.11257	8.8832	63.1462	7.1085
24	9.8497	0.1015	0.01130	88.4973	0.11130	8.9847	65.4813	7.2881
25	10.8347	0.0923	0.01017	98.3471	0.11017	9.0770	67.6964	7.4580
26	11.9182	0.0839	0.00916	109.1818	0.10916	9.1609	69.7940	7.6186
27	13.1100	0.0763	0.00826	121.0999	0.10826	9.2372	71.7773	7.7704
28	14.4210	0.0693	0.00745	134.2099	0.10745	9.3066	73.6495	7.9137
29	15.8631	0.0630	0.00673	148.6309	0.10673	9.3696	75.4146	8.0489
30	17.4494	0.0573	0.00608	164.4940	0.10608	9.4269	77.0766	8.1762
31	19.1943	0.0521	0.00550	181.9434	0.10550	9.4790	78.6395	8.2962
32	21.1138	0.0474	0.00497	201.1378	0.10497	9.5264	80.1078	8.4091
33	23.2252	0.0431	0.00450	222.2515	0.10450	9.5694	81.4856	8.5152
34	25.5477	0.0391	0.00407	245.4767	0.10407	9.6086	82.7773	8.6149
35	28.1024	0.0356	0.00369	271.0244	0.10369	9.6442	83.9872	8.7086
40	45.2593	0.0221	0.00226	442.5926	0.10226	9.7791	88.9525	9.0962
45	72.8905	0.0137	0.00139	718.9048	0.10139	9.8628	92.4544	9.3740
50	117.3909	0.0085	0.00086	1163.91	0.10086	9.9148	94.8889	9.5704
55	189.0591	0.0053	0.00053	1880.59	0.10053	9.9471	96.5619	9.7075
60	304.4816	0.0033	0.00033	3034.82	0.10033	9.9672	97.7010	9.8023
65	490.3707	0.0020	0.00020	4893.71	0.10020	9.9796	98.4705	9.8672
70	789.7470	0.0013	0.00013	7887.47	0.10013	9.9873	98.9870	9.9113
75	1271.90	0.0008	0.00008	12709	0.10008	9.9921	99.3317	9.9410
80	2048.40	0.0005	0.00005	20474	0.10005	9.9951	99.5606	9.9609
85	3298.97	0.0003	0.00003	32980	0.10003	9.9970	99.7120	9.9742
90	5313.02	0.0002	0.00002	53120	0.10002	9.9981	99.8118	9.9831
95	8556.68	0.0001	0.00001	85557	0.10001	9.9988	99.8773	9.9889
96	9412.34	0.0001	0.00001	94113	0.10001	9.9989	99.8874	9.9898
98	11389	0.0001	0.00001		0.10001	9.9991	99.9052	9.9914
100	13781	0.0001	0.00001		0.10001	9.9993	99.9202	9.9927

11% TABLE 16 Discrete Cash Flow: Compound Interest Factors 11%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.1100	0.9009	1.00000	1.0000	1.11000	0.9009		
2	1.2321	0.8116	0.47393	2.1100	0.58393	1.7125	0.8116	0.4739
3	1.3676	0.7312	0.29921	3.3421	0.40921	2.4437	2.2740	0.9306
4	1.5181	0.6587	0.21233	4.7097	0.32233	3.1024	4.2502	1.3700
5	1.6851	0.5935	0.16057	6.2278	0.27057	3.6959	6.6240	1.7923
6	1.8704	0.5346	0.12638	7.9129	0.23638	4.2305	9.2972	2.1976
7	2.0762	0.4817	0.10222	9.7833	0.21222	4.7122	12.1872	2.5863
8	2.3045	0.4339	0.08432	11.8594	0.19432	5.1461	15.2246	2.9585
9	2.5580	0.3909	0.07060	14.1640	0.18060	5.5370	18.3520	3.3144
10	2.8394	0.3522	0.05980	16.7220	0.16980	5.8892	21.5217	3.6544
11	3.1518	0.3173	0.05112	19.5614	0.16112	6.2065	24.6945	3.9788
12	3.4985	0.2858	0.04403	22.7132	0.15403	6.4924	27.8388	4.2879
13	3.8833	0.2575	0.03815	26.2116	0.14815	6.7499	30.9290	4.5822
14	4.3104	0.2320	0.03323	30.0949	0.14323	6.9819	33.9449	4.8619
15	4.7846	0.2090	0.02907	34.4054	0.13907	7.1909	36.8709	5.1275
16	5.3109	0.1883	0.02552	39.1899	0.13552	7.3792	39.6953	5.3794
17	5.8951	0.1696	0.02247	44.5008	0.13247	7.5488	42.4095	5.6180
18	6.5436	0.1528	0.01984	50.3959	0.12984	7.7016	45.0074	5.8439
19	7.2633	0.1377	0.01756	56.9395	0.12756	7.8393	47.4856	6.0574
20	8.0623	0.1240	0.01558	64.2028	0.12558	7.9633	49.8423	6.2590
21	8.9492	0.1117	0.01384	72.2651	0.12384	8.0751	52.0771	6.4491
22	9.9336	0.1007	0.01231	81.2143	0.12231	8.1757	54.1912	6.6283
23	11.0263	0.0907	0.01097	91.1479	0.12097	8.2664	56.1864	6.7969
24	12.2392	0.0817	0.00979	102.1742	0.11979	8.3481	58.0656	6.9555
25	13.5855	0.0736	0.00874	114.4133	0.11874	8.4217	59.8322	7.1045
26	15.0799	0.0663	0.00781	127.9988	0.11781	8.4881	61.4900	7.2443
27	16.7386	0.0597	0.00699	143.0786	0.11699	8.5478	63.0433	7.3754
28	18.5799	0.0538	0.00626	159.8173	0.11626	8.6016	64.4965	7.4982
29	20.6237	0.0485	0.00561	178.3972	0.11561	8.6501	65.8542	7.6131
30	22.8923	0.0437	0.00502	199.0209	0.11502	8.6938	67.1210	7.7206
31	25.4104	0.0394	0.00451	221.9132	0.11451	8.7331	68.3016	7.8210
32	28.2056	0.0355	0.00404	247.3236	0.11404	8.7686	69.4007	7.9147
33	31.3082	0.0319	0.00363	275.5292	0.11363	8.8005	70.4228	8.0021
34	34.7521	0.0288	0.00326	306.8374	0.11326	8.8293	71.3724	8.0836
35	38.5749	0.0259	0.00293	341.5896	0.11293	8.8552	72.2538	8.1594
40	65.0009	0.0154	0.00172	581.8261	0.11172	8.9511	75.7789	8.4659
45	109.5302	0.0091	0.00101	986.6386	0.11101	9.0079	78.1551	8.6763
50	184.5648	0.0054	0.00060	1668.77	0.11060	9.0417	79.7341	8.8185
55	311.0025	0.0032	0.00035	2818.20	0.11035	9.0617	80.7712	8.9135
60	524.0572	0.0019	0.00021	4755.07	0.11021	9.0736	81.4461	8.9762
65	883.0669	0.0011	0.00012	8018.79	0.11012	9.0806	81.8819	9.0172
70	1488.02	0.0007	0.00007	13518	0.11007	9.0848	82.1614	9.0438
75	2507.40	0.0004	0.00004	22785	0.11004	9.0873	82.3397	9.0610
80	4225.11	0.0002	0.00003	38401	0.11003	9.0888	82.4529	9.0720
85	7119.56	0.0001	0.00002	64714	0.11002	9.0896	82.5245	9.0790

12%	TABLE 17 Discrete Cash Flow: Compound Interest Factors							12%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.1200	0.8929	1.00000	1.0000	1.12000	0.8929		
2	1.2544	0.7972	0.47170	2.1200	0.59170	1.6901	0.7972	0.4717
3	1.4049	0.7118	0.29635	3.3744	0.41635	2.4018	2.2208	0.9246
4	1.5735	0.6355	0.20923	4.7793	0.32923	3.0373	4.1273	1.3589
5	1.7623	0.5674	0.15741	6.3528	0.27741	3.6048	6.3970	1.7746
6	1.9738	0.5066	0.12323	8.1152	0.24323	4.1114	8.9302	2.1720
7	2.2107	0.4523	0.09912	10.0890	0.21912	4.5638	11.6443	2.5512
8	2.4760	0.4039	0.08130	12.2997	0.20130	4.9676	14.4714	2.9131
9	2.7731	0.3606	0.06768	14.7757	0.18768	5.3282	17.3563	3.2574
10	3.1058	0.3220	0.05698	17.5487	0.17698	5.6502	20.2541	3.5847
11	3.4785	0.2875	0.04842	20.6546	0.16842	5.9377	23.1288	3.8953
12	3.8960	0.2567	0.04144	24.1331	0.16144	6.1944	25.9523	4.1897
13	4.3635	0.2292	0.03568	28.0291	0.15568	6.4235	28.7024	4.4683
14	4.8871	0.2046	0.03087	32.3926	0.15087	6.6282	31.3624	4.7317
15	5.4736	0.1827	0.02682	37.2797	0.14682	6.8109	33.9202	4.9803
16	6.1304	0.1631	0.02339	42.7533	0.14339	6.9740	36.3670	5.2147
17	6.8660	0.1456	0.02046	48.8837	0.14046	7.1196	38.6973	5.4353
18	7.6900	0.1300	0.01794	55.7497	0.13794	7.2497	40.9080	5.6427
19	8.6128	0.1161	0.01576	63.4397	0.13576	7.3658	42.9979	5.8375
20	9.6463	0.1037	0.01388	72.0524	0.13388	7.4694	44.9676	6.0202
21	10.8038	0.0926	0.01224	81.6987	0.13224	7.5620	46.8188	6.1913
22	12.1003	0.0826	0.01081	92.5026	0.13081	7.6446	48.5543	6.3514
23	13.5523	0.0738	0.00956	104.6029	0.12956	7.7184	50.1776	6.5010
24	15.1786	0.0659	0.00846	118.1552	0.12846	7.7843	51.6929	6.6406
25	17.0001	0.0588	0.00750	133.3339	0.12750	7.8431	53.1046	6.7708
26	19.0401	0.0525	0.00665	150.3339	0.12665	7.8957	54.4177	6.8921
27	21.3249	0.0469	0.00590	169.3740	0.12590	7.9426	55.6369	7.0049
28	23.8839	0.0419	0.00524	190.6989	0.12524	7.9844	56.7674	7.1098
29	26.7499	0.0374	0.00466	214.5828	0.12466	8.0218	57.8141	7.2071
30	29.9599	0.0334	0.00414	241.3327	0.12414	8.0552	58.7821	7.2974
31	33.5551	0.0298	0.00369	271.2926	0.12369	8.0850	59.6761	7.3811
32	37.5817	0.0266	0.00328	304.8477	0.12328	8.1116	60.5010	7.4586
33	42.0915	0.0238	0.00292	342.4294	0.12292	8.1354	61.2612	7.5302
34	47.1425	0.0212	0.00260	384.5210	0.12260	8.1566	61.9612	7.5965
35	52.7996	0.0189	0.00232	431.6635	0.12232	8.1755	62.6052	7.6577
40	93.0510	0.0107	0.00130	767.0914	0.12130	8.2438	65.1159	7.8988
45	163.9876	0.0061	0.0074	1358.23	0.12074	8.2825	66.7342	8.0572
50	289.0022	0.0035	0.00042	2400.02	0.12042	8.3045	67.7624	8.1597
55	509.3206	0.0020	0.00024	4236.01	0.12024	8.3170	68.4082	8.2251
60	897.5969	0.0011	0.00013	7471.64	0.12013	8.3240	68.8100	8.2664
65	1581.87	0.0006	0.00008	13174	0.12008	8.3281	69.0581	8.2922
70	2787.80	0.0004	0.00004	23223	0.12004	8.3303	69.2103	8.3082
75	4913.06	0.0002	0.00002	40934	0.12002	8.3316	69.3031	8.3181
80	8658.48	0.0001	0.00001	72146	0.12001	8.3324	69.3594	8.3241
85	15259	0.0001	0.00001		0.12001	8.3328	69.3935	8.3278

14% TABLE 18 Discrete Cash Flow: Compound Interest Factors 14%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.1400	0.8772	1.00000	1.0000	1.14000	0.8772		
2	1.2996	0.7695	0.46729	2.1400	0.60729	1.6467	0.7695	0.4673
3	1.4815	0.6750	0.29073	3.4396	0.43073	2.3216	2.1194	0.9129
4	1.6890	0.5921	0.20320	4.9211	0.34320	2.9137	3.8957	1.3370
5	1.9254	0.5194	0.15128	6.6101	0.29128	3.4331	5.9731	1.7399
6	2.1950	0.4556	0.11716	8.5355	0.25716	3.8887	8.2511	2.1218
7	2.5023	0.3996	0.09319	10.7305	0.23319	4.2883	10.6489	2.4832
8	2.8526	0.3506	0.07557	13.2328	0.21557	4.6389	13.1028	2.8246
9	3.2519	0.3075	0.06217	16.0853	0.20217	4.9464	15.5629	3.1463
10	3.7072	0.2697	0.05171	19.3373	0.19171	5.2161	17.9906	3.4490
11	4.2262	0.2366	0.04339	23.0445	0.18339	5.4527	20.3567	3.7333
12	4.8179	0.2076	0.03667	27.2707	0.17667	5.6603	22.6399	3.9998
13	5.4924	0.1821	0.03116	32.0887	0.17116	5.8424	24.8247	4.2491
14	6.2613	0.1597	0.02661	37.5811	0.16661	6.0021	26.9009	4.4819
15	7.1379	0.1401	0.02281	43.8424	0.16281	6.1422	28.8623	4.6990
16	8.1372	0.1229	0.01962	50.9804	0.15962	6.2651	30.7057	4.9011
17	9.2765	0.1078	0.01692	59.1176	0.15692	6.3729	32.4305	5.0888
18	10.5752	0.0946	0.01462	68.3941	0.15462	6.4674	34.0380	5.2630
19	12.0557	0.0829	0.01266	78.9692	0.15266	6.5504	35.5311	5.4243
20	13.7435	0.0728	0.01099	91.0249	0.15099	6.6231	36.9135	5.5734
21	15.6676	0.0638	0.00954	104.7684	0.14954	6.6870	38.1901	5.7111
22	17.8610	0.0560	0.00830	120.4360	0.14830	6.7429	39.3658	5.8381
23	20.3616	0.0491	0.00723	138.2970	0.14723	6.7921	40.4463	5.9549
24	23.2122	0.0431	0.00630	158.6586	0.14630	6.8351	41.4371	6.0624
25	26.4619	0.0378	0.00550	181.8708	0.14550	6.8729	42.3441	6.1610
26	30.1666	0.0331	0.00480	208.3327	0.14480	6.9061	43.1728	6.2514
27	34.3899	0.0291	0.00419	238.4993	0.14419	6.9352	43.9289	6.3342
28	39.2045	0.0255	0.00366	272.8892	0.14366	6.9607	44.6176	6.4100
29	44.6931	0.0224	0.00320	312.0937	0.14320	6.9830	45.2441	6.4791
30	50.9502	0.0196	0.00280	356.7868	0.14280	7.0027	45.8132	6.5423
31	58.0832	0.0172	0.00245	407.7370	0.14245	7.0199	46.3297	6.5998
32	66.2148	0.0151	0.00215	465.8202	0.14215	7.0350	46.7979	6.6522
33	75.4849	0.0132	0.00188	532.0350	0.14188	7.0482	47.2218	6.6998
34	86.0528	0.0116	0.00165	607.5199	0.14165	7.0599	47.6053	6.7431
35	98.1002	0.0102	0.00144	693.5727	0.14144	7.0700	47.9519	6.7824
40	188.8835	0.0053	0.00075	1342.03	0.14075	7.1050	49.2376	6.9300
45	363.6791	0.0027	0.00039	2590.56	0.14039	7.1232	49.9963	7.0188
50	700.2330	0.0014	0.00020	4994.52	0.14020	7.1327	50.4375	7.0714
55	1348.24	0.0007	0.00010	9623.13	0.14010	7.1376	50.6912	7.1020
60	2595.92	0.0004	0.00005	18535	0.14005	7.1401	50.8357	7.1197
65	4998.22	0.0002	0.00003	35694	0.14003	7.1414	50.9173	7.1298
70	9623.64	0.0001	0.00001	68733	0.14001	7.1421	50.9632	7.1356
75	18530	0.0001	0.00001		0.14001	7.1425	50.9887	7.1388
80	35677				0.14000	7.1427	51.0030	7.1406
85	68693				0.14000	7.1428	51.0108	7.1416

15% TABLE 19 Discrete Cash Flow: Compound Interest Factors 15%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.1500	0.8696	1.00000	1.0000	1.15000	0.8696		
2	1.3225	0.7561	0.46512	2.1500	0.61512	1.6257	0.7561	0.4651
3	1.5209	0.6575	0.28798	3.4725	0.43798	2.2832	2.0712	0.9071
4	1.7490	0.5718	0.20027	4.9934	0.35027	2.8550	3.7864	1.3263
5	2.0114	0.4972	0.14832	6.7424	0.29832	3.3522	5.7751	1.7228
6	2.3131	0.4323	0.11424	8.7537	0.26424	3.7845	7.9368	2.0972
7	2.6600	0.3759	0.09036	11.0668	0.24036	4.1604	10.1924	2.4498
8	3.0590	0.3269	0.07285	13.7268	0.22285	4.4873	12.4807	2.7813
9	3.5179	0.2843	0.05957	16.7858	0.20957	4.7716	14.7548	3.0922
10	4.0456	0.2472	0.04925	20.3037	0.19925	5.0188	16.9795	3.3832
11	4.6524	0.2149	0.04107	24.3493	0.19107	5.2337	19.1289	3.6549
12	5.3503	0.1869	0.03448	29.0017	0.18448	5.4206	21.1849	3.9082
13	6.1528	0.1625	0.02911	34.3519	0.17911	5.5831	23.1352	4.1438
14	7.0757	0.1413	0.02469	40.5047	0.17469	5.7245	24.9725	4.3624
15	8.1371	0.1229	0.02102	47.5804	0.17102	5.8474	26.6930	4.5650
16	9.3576	0.1069	0.01795	55.7175	0.16795	5.9542	28.2960	4.7522
17	10.7613	0.0929	0.01537	65.0751	0.16537	6.0472	29.7828	4.9251
18	12.3755	0.0808	0.01319	75.8364	0.16319	6.1280	31.1565	5.0843
19	14.2318	0.0703	0.01134	88.2118	0.16134	6.1982	32.4213	5.2307
20	16.3665	0.0611	0.00976	102.4436	0.15976	6.2593	33.5822	5.3651
21	18.8215	0.0531	0.00842	118.8101	0.15842	6.3125	34.6448	5.4883
22	21.6447	0.0462	0.00727	137.6316	0.15727	6.3587	35.6150	5.6010
23	24.8915	0.0402	0.00628	159.2764	0.15628	6.3988	36.4988	5.7040
24	28.6252	0.0349	0.00543	184.1678	0.15543	6.4338	37.3023	5.7979
25	32.9190	0.0304	0.00470	212.7930	0.15470	6.4641	38.0314	5.8834
26	37.8568	0.0264	0.00407	245.7120	0.15407	6.4906	38.6918	5.9612
27	43.5353	0.0230	0.00353	283.5688	0.15353	6.5135	39.2890	6.0319
28	50.0656	0.0200	0.00306	327.1041	0.15306	6.5335	39.8283	6.0960
29	57.5755	0.0174	0.00265	377.1697	0.15265	6.5509	40.3146	6.1541
30	66.2118	0.0151	0.00230	434.7451	0.15230	6.5660	40.7526	6.2066
31	76.1435	0.0131	0.00200	500.9569	0.15200	6.5791	41.1466	6.2541
32	87.5651	0.0114	0.00173	577.1005	0.15173	6.5905	41.5006	6.2970
33	100.6998	0.0099	0.00150	664.6655	0.15150	6.6005	41.8184	6.3357
34	115.8048	0.0086	0.00131	765.3654	0.15131	6.6091	42.1033	6.3705
35	133.1755	0.0075	0.00113	881.1702	0.15113	6.6166	42.3586	6.4019
40	267.8635	0.0037	0.00056	1779.09	0.15056	6.6418	43.2830	6.5168
45	538.7693	0.0019	0.00028	3585.13	0.15028	6.6543	43.8051	6.5830
50	1083.66	0.0009	0.00014	7217.72	0.15014	6.6605	44.0958	6.6205
55	2179.62	0.0005	0.00007	14524	0.15007	6.6636	44.2558	6.6414
60	4384.00	0.0002	0.00003	29220	0.15003	6.6651	44.3431	6.6530
65	8817.79	0.0001	0.00002	58779	0.15002	6.6659	44.3903	6.6593
70	17736	0.0001	0.00001		0.15001	6.6663	44.4156	6.6627
75	35673				0.15000	6.6665	44.4292	6.6646
80	71751				0.15000	6.6666	44.4364	6.6656
85					0.15000	6.6666	44.4402	6.6661

16%	TABLE 20 Discrete Cash Flow: Compound Interest Factors							16%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/P	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.1600	0.8621	1.00000	1.0000	1.16000	0.8621		
2	1.3456	0.7432	0.46296	2.1600	0.62296	1.6052	0.7432	0.4630
3	1.5609	0.6407	0.28526	3.5056	0.44526	2.2459	2.0245	0.9014
4	1.8106	0.5523	0.19738	5.0665	0.35738	2.7982	3.6814	1.3156
5	2.1003	0.4761	0.14541	6.8771	0.30541	3.2743	5.5858	1.7060
6	2.4364	0.4104	0.11139	8.9775	0.27139	3.6847	7.6380	2.0729
7	2.8262	0.3538	0.08761	11.4139	0.24761	4.0386	9.7610	2.4169
8	3.2784	0.3050	0.07022	14.2401	0.23022	4.3436	11.8962	2.7388
9	3.8030	0.2630	0.05708	17.5185	0.21708	4.6065	13.9998	3.0391
10	4.4114	0.2267	0.04690	21.3215	0.20690	4.8332	16.0399	3.3187
11	5.1173	0.1954	0.03886	25.7329	0.19886	5.0286	17.9941	3.5783
12	5.9360	0.1685	0.03241	30.8502	0.19241	5.1971	19.8472	3.8189
13	6.8858	0.1452	0.02718	36.7862	0.18718	5.3423	21.5899	4.0413
14	7.9875	0.1252	0.02290	43.6720	0.18290	5.4675	23.2175	4.2464
15	9.2655	0.1079	0.01936	51.6595	0.17936	5.5755	24.7284	4.4352
16	10.7480	0.0930	0.01641	60.9250	0.17641	5.6685	26.1241	4.6086
17	12.4677	0.0802	0.01395	71.6730	0.17395	5.7487	27.4074	4.7676
18	14.4625	0.0691	0.01188	84.1407	0.17188	5.8178	28.5828	4.9130
19	16.7765	0.0596	0.01014	98.6032	0.17014	5.8775	29.6557	5.0457
20	19.4608	0.0514	0.00867	115.3797	0.16867	5.9288	30.6321	5.1666
22	26.1864	0.0382	0.00635	157.4150	0.16635	6.0113	32.3200	5.3765
24	35.2364	0.0284	0.00467	213.9776	0.16467	6.0726	33.6970	5.5490
26	47.4141	0.0211	0.00345	290.0883	0.16345	6.1182	34.8114	5.6898
28	63.8004	0.0157	0.00255	392.5028	0.16255	6.1520	35.7073	5.8041
30	85.8499	0.0116	0.00189	530.3117	0.16189	6.1772	36.4234	5.8964
32	115.5196	0.0087	0.00140	715.7475	0.16140	6.1959	36.9930	5.9706
34	155.4432	0.0064	0.00104	965.2698	0.16104	6.2098	37.4441	6.0299
35	180.3141	0.0055	0.00089	1120.71	0.16089	6.2153	37.6327	6.0548
36	209.1643	0.0048	0.00077	1301.03	0.16077	6.2201	37.8000	6.0771
38	281.4515	0.0036	0.00057	1752.82	0.16057	6.2278	38.0799	6.1145
40	378.7212	0.0026	0.00042	2360.76	0.16042	6.2335	38.2992	6.1441
45	795.4438	0.0013	0.00020	4965.27	0.16020	6.2421	38.6598	6.1934
50	1670.70	0.0006	0.00010	10436	0.16010	6.2463	38.8521	6.2201
55	3509.05	0.0003	0.00005	21925	0.16005	6.2482	38.9534	6.2343
60	7370.20	0.0001	0.00002	46058	0.16002	6.2492	39.0063	6.2419

18% TABLE 21 Discrete Cash Flow: Compound Interest Factors 18%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.1800	0.8475	1.00000	1.0000	1.18000	0.8475		
2	1.3924	0.7182	0.45872	2.1800	0.63872	1.5656	0.7182	0.4587
3	1.6430	0.6086	0.27992	3.5724	0.45992	2.1743	1.9354	0.8902
4	1.9388	0.5158	0.19174	5.2154	0.37174	2.6901	3.4828	1.2947
5	2.2878	0.4371	0.13978	7.1542	0.31978	3.1272	5.2312	1.6728
6	2.6996	0.3704	0.10591	9.4420	0.28591	3.4976	7.0834	2.0252
7	3.1855	0.3139	0.08236	12.1415	0.26236	3.8115	8.9670	2.3526
8	3.7589	0.2660	0.06524	15.3270	0.24524	4.0776	10.8292	2.6558
9	4.4355	0.2255	0.05239	19.0859	0.23239	4.3030	12.6329	2.9358
10	5.2338	0.1911	0.04251	23.5213	0.22251	4.4941	14.3525	3.1936
11	6.1759	0.1619	0.03478	28.7551	0.21478	4.6560	15.9716	3.4303
12	7.2876	0.1372	0.02863	34.9311	0.20863	4.7932	17.4811	3.6470
13	8.5994	0.1163	0.02369	42.2187	0.20369	4.9095	18.8765	3.8449
14	10.1472	0.0985	0.01968	50.8180	0.19968	5.0081	20.1576	4.0250
15	11.9737	0.0835	0.01640	60.9653	0.19640	5.0916	21.3269	4.1887
16	14.1290	0.0708	0.01371	72.9390	0.19371	5.1624	22.3885	4.3369
17	16.6722	0.0600	0.01149	87.0680	0.19149	5.2223	23.3482	4.4708
18	19.6733	0.0508	0.00964	103.7403	0.18964	5.2732	24.2123	4.5916
19	23.2144	0.0431	0.00810	123.4135	0.18810	5.3162	24.9877	4.7003
20	27.3930	0.0365	0.00682	146.6280	0.18682	5.3527	25.6813	4.7978
22	38.1421	0.0262	0.00485	206.3448	0.18485	5.4099	26.8506	4.9632
24	53.1090	0.0188	0.00345	289.4945	0.18345	5.4509	27.7725	5.0950
26	73.9490	0.0135	0.00247	405.2721	0.18247	5.4804	28.4935	5.1991
28	102.9666	0.0097	0.00177	566.4809	0.18177	5.5016	29.0537	5.2810
30	143.3706	0.0070	0.00126	790.9480	0.18126	5.5168	29.4864	5.3448
32	199.6293	0.0050	0.00091	1103.50	0.18091	5.5277	29.8191	5.3945
34	277.9638	0.0036	0.00065	1538.69	0.18065	5.5356	30.0736	5.4328
35	327.9973	0.0030	0.00055	1816.65	0.18055	5.5386	30.1773	5.4485
36	387.0368	0.0026	0.00047	2144.65	0.18047	5.5412	30.2677	5.4623
38	538.9100	0.0019	0.00033	2988.39	0.18033	5.5452	30.4152	5.4849
40	750.3783	0.0013	0.00024	4163.21	0.18024	5.5482	30.5269	5.5022
45	1716.68	0.0006	0.00010	9531.58	0.18010	5.5523	30.7006	5.5293
50	3927.36	0.0003	0.00005	21813	0.18005	5.5541	30.7856	5.5428
55	8984.84	0.0001	0.00002	49910	0.18002	5.5549	30.8268	5.5494
60	20555			114190	0.18001	5.5553	30.8465	5.5526

20% TABLE 22 Discrete Cash Flow: Compound Interest Factors 20%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.2000	0.8333	1.00000	1.0000	1.20000	0.8333		
2	1.4400	0.6944	0.45455	2.2000	0.65455	1.5278	0.6944	0.4545
3	1.7280	0.5787	0.27473	3.6400	0.47473	2.1065	1.8519	0.8791
4	2.0736	0.4823	0.18629	5.3680	0.38629	2.5887	3.2986	1.2742
5	2.4883	0.4019	0.13438	7.4416	0.33438	2.9906	4.9061	1.6405
6	2.9860	0.3349	0.10071	9.9299	0.30071	3.3255	6.5806	1.9788
7	3.5832	0.2791	0.07742	12.9159	0.27742	3.6046	8.2551	2.2902
8	4.2998	0.2326	0.06061	16.4991	0.26061	3.8372	9.8831	2.5756
9	5.1598	0.1938	0.04808	20.7989	0.24808	4.0310	11.4335	2.8364
10	6.1917	0.1615	0.03852	25.9587	0.23852	4.1925	12.8871	3.0739
11	7.4301	0.1346	0.03110	32.1504	0.23110	4.3271	14.2330	3.2893
12	8.9161	0.1122	0.02526	39.5805	0.22526	4.4392	15.4667	3.4841
13	10.6993	0.0935	0.02062	48.4966	0.22062	4.5327	16.5883	3.6597
14	12.8392	0.0779	0.01689	59.1959	0.21689	4.6106	17.6008	3.8175
15	15.4070	0.0649	0.01388	72.0351	0.21388	4.6755	18.5095	3.9588
16	18.4884	0.0541	0.01144	87.4421	0.21144	4.7296	19.3208	4.0851
17	22.1861	0.0451	0.00944	105.9306	0.20944	4.7746	20.0419	4.1976
18	26.6233	0.0376	0.00781	128.1167	0.20781	4.8122	20.6805	4.2975
19	31.9480	0.0313	0.00646	154.7400	0.20646	4.8435	21.2439	4.3861
20	38.3376	0.0261	0.00536	186.6880	0.20536	4.8696	21.7395	4.4643
22	55.2061	0.0181	0.00369	271.0307	0.20369	4.9094	22.5546	4.5941
24	79.4968	0.0126	0.00255	392.4842	0.20255	4.9371	23.1760	4.6943
26	114.4755	0.0087	0.00176	567.3773	0.20176	4.9563	23.6460	4.7709
28	164.8447	0.0061	0.00122	819.2233	0.20122	4.9697	23.9991	4.8291
30	237.3763	0.0042	0.00085	1181.88	0.20085	4.9789	24.2628	4.8731
32	341.8219	0.0029	0.00059	1704.11	0.20059	4.9854	24.4588	4.9061
34	492.2235	0.0020	0.00041	2456.12	0.20041	4.9898	24.6038	4.9308
35	590.6682	0.0017	0.00034	2948.34	0.20034	4.9915	24.6614	4.9406
36	708.8019	0.0014	0.00028	3539.01	0.20028	4.9929	24.7108	4.9491
38	1020.67	0.0010	0.00020	5098.37	0.20020	4.9951	24.7894	4.9627
40	1469.77	0.0007	0.00014	7343.86	0.20014	4.9966	24.8469	4.9728
45	3657.26	0.0003	0.00005	18281	0.20005	4.9986	24.9316	4.9877
50	9100.44	0.0001	0.00002	45497	0.20002	4.9995	24.9698	4.9945
55	22645		0.00001		0.20001	4.9998	24.9868	4.9976

22%	TABLE 23 Discrete Cash Flow: Compound Interest Factors							22%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.2200	0.8197	1.00000	1.0000	1.22000	0.8197		
2	1.4884	0.6719	0.45045	2.2200	0.67045	1.4915	0.6719	0.4505
3	1.8158	0.5507	0.26966	3.7084	0.48966	2.0422	1.7733	0.8683
4	2.2153	0.4514	0.18102	5.5242	0.40102	2.4936	3.1275	1.2542
5	2.7027	0.3700	0.12921	7.7396	0.34921	2.8636	4.6075	1.6090
6	3.2973	0.3033	0.09576	10.4423	0.31576	3.1669	6.1239	1.9337
7	4.0227	0.2486	0.07278	13.7396	0.29278	3.4155	7.6154	2.2297
8	4.9077	0.2038	0.05630	17.7623	0.27630	3.6193	9.0417	2.4982
9	5.9874	0.1670	0.04411	22.6700	0.26411	3.7863	10.3779	2.7409
10	7.3046	0.1369	0.03489	28.6574	0.25489	3.9232	11.6100	2.9593
11	8.9117	0.1122	0.02781	35.9620	0.24781	4.0354	12.7321	3.1551
12	10.8722	0.0920	0.02228	44.8737	0.24228	4.1274	13.7438	3.3299
13	13.2641	0.0754	0.01794	55.7459	0.23794	4.2028	14.6485	3.4855
14	16.1822	0.0618	0.01449	69.0100	0.23449	4.2646	15.4519	3.6233
15	19.7423	0.0507	0.01174	85.1922	0.23174	4.3152	16.1610	3.7451
16	24.0856	0.0415	0.00953	104.9345	0.22953	4.3567	16.7838	3.8524
17	29.3844	0.0340	0.00775	129.0201	0.22775	4.3908	17.3283	3.9465
18	35.8490	0.0279	0.00631	158.4045	0.22631	4.4187	17.8025	4.0289
19	43.7358	0.0229	0.00515	194.2535	0.22515	4.4415	18.2141	4.1009
20	53.3576	0.0187	0.00420	237.9893	0.22420	4.4603	18.5702	4.1635
22	79.4175	0.0126	0.00281	356.4432	0.22281	4.4882	19.1418	4.2649
24	118.2050	0.0085	0.00188	532.7501	0.22188	4.5070	19.5635	4.3407
26	175.9364	0.0057	0.00126	795.1653	0.22126	4.5196	19.8720	4.3968
28	261.8637	0.0038	0.00084	1185.74	0.22084	4.5281	20.0962	4.4381
30	389.7579	0.0026	0.00057	1767.08	0.22057	4.5338	20.2583	4.4683
32	580.1156	0.0017	0.00038	2632.34	0.22038	4.5376	20.3748	4.4902
34	863.4441	0.0012	0.00026	3920.20	0.22026	4.5402	20.4582	4.5060
35	1053.40	0.0009	0.00021	4783.64	0.22021	4.5411	20.4905	4.5122
36	1285.15	0.0008	0.00017	5837.05	0.22017	4.5419	20.5178	4.5174
38	1912.82	0.0005	0.00012	8690.08	0.22012	4.5431	20.5601	4.5256
40	2847.04	0.0004	0.00008	12937	0.22008	4.5439	20.5900	4.5314
45	7694.71	0.0001	0.00003	34971	0.22003	4.5449	20.6319	4.5396
50	20797		0.00001	94525	0.22001	4.5452	20.6492	4.5431
55	56207				0.22000	4.5454	20.6563	4.5445

24% TABLE 24 Discrete Cash Flow: Compound Interest Factors 24%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.2400	0.8065	1.00000	1.0000	1.24000	0.8065		
2	1.5376	0.6504	0.44643	2.2400	0.68643	1.4568	0.6504	0.4464
3	1.9066	0.5245	0.26472	3.7776	0.50472	1.9813	1.6993	0.8577
4	2.3642	0.4230	0.17593	5.6842	0.41593	2.4043	2.9683	1.2346
5	2.9316	0.3411	0.12425	8.0484	0.36425	2.7454	4.3327	1.5782
6	3.6352	0.2751	0.09107	10.9801	0.33107	3.0205	5.7081	1.8898
7	4.5077	0.2218	0.06842	14.6153	0.30842	3.2423	7.0392	2.1710
8	5.5895	0.1789	0.05229	19.1229	0.29229	3.4212	8.2915	2.4236
9	6.9310	0.1443	0.04047	24.7125	0.28047	3.5655	9.4458	2.6492
10	8.5944	0.1164	0.03160	31.6434	0.27160	3.6819	10.4930	2.8499
11	10.6571	0.0938	0.02485	40.2379	0.26485	3.7757	11.4313	3.0276
12	13.2148	0.0757	0.01965	50.8950	0.25965	3.8514	12.2637	3.1843
13	16.3863	0.0610	0.01560	64.1097	0.25560	3.9124	12.9960	3.3218
14	20.3191	0.0492	0.01242	80.4961	0.25242	3.9616	13.6358	3.4420
15	25.1956	0.0397	0.00992	100.8151	0.24992	4.0013	14.1915	3.5467
16	31.2426	0.0320	0.00794	126.0108	0.24794	4.0333	14.6716	3.6376
17	38.7408	0.0258	0.00636	157.2534	0.24636	4.0591	15.0846	3.7162
18	48.0386	0.0208	0.00510	195.9942	0.24510	4.0799	15.4385	3.7840
19	59.5679	0.0168	0.00410	244.0328	0.24410	4.0967	15.7406	3.8423
20	73.8641	0.0135	0.00329	303.6006	0.24329	4.1103	15.9979	3.8922
22	113.5735	0.0088	0.00213	469.0563	0.24213	4.1300	16.4011	3.9712
24	174.6306	0.0057	0.00138	723.4610	0.24138	4.1428	16.6891	4.0284
26	268.5121	0.0037	0.00090	1114.63	0.24090	4.1511	16.8930	4.0695
28	412.8642	0.0024	0.00058	1716.10	0.24058	4.1566	17.0365	4.0987
30	634.8199	0.0016	0.00038	2640.92	0.24038	4.1601	17.1369	4.1193
32	976.0991	0.0010	0.00025	4062.91	0.24025	4.1624	17.2067	4.1338
34	1500.85	0.0007	0.00016	6249.38	0.24016	4.1639	17.2552	4.1440
35	1861.05	0.0005	0.00013	7750.23	0.24013	4.1664	17.2734	4.1479
36	2307.71	0.0004	0.00010	9611.28	0.24010	4.1649	17.2886	4.1511
38	3548.33	0.0003	0.00007	14781	0.24007	4.1655	17.3116	4.1560
40	5455.91	0.0002	0.00004	22729	0.24004	4.1659	17.3274	4.1593
45	15995	0.0001	0.00002	66640	0.24002	4.1664	17.3483	4.1639
50	46890		0.00001		0.24001	4.1666	17.3563	4.1653
55					0.24000	4.1666	17.3593	4.1663

25%	TABLE 25 Discrete Cash Flow: Compound Interest Factors							25%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.2500	0.8000	1.00000	1.0000	1.25000	0.8000		
2	1.5625	0.6400	0.44444	2.2500	0.69444	1.4400	0.6400	0.4444
3	1.9531	0.5120	0.26230	3.8125	0.51230	1.9520	1.6640	0.8525
4	2.4414	0.4096	0.17344	5.7656	0.42344	2.3616	2.8928	1.2249
5	3.0518	0.3277	0.12185	8.2070	0.37185	2.6893	4.2035	1.5631
6	3.8147	0.2621	0.08882	11.2588	0.33882	2.9514	5.5142	1.8683
7	4.7684	0.2097	0.06634	15.0735	0.31634	3.1611	6.7725	2.1424
8	5.9605	0.1678	0.05040	19.8419	0.30040	3.3289	7.9469	2.3872
9	7.4506	0.1342	0.03876	25.8023	0.28876	3.4631	9.0207	2.6048
10	9.3132	0.1074	0.03007	33.2529	0.28007	3.5705	9.9870	2.7971
11	11.6415	0.0859	0.02349	42.5661	0.27349	3.6564	10.8460	2.9663
12	14.5519	0.0687	0.01845	54.2077	0.26845	3.7251	11.6020	3.1145
13	18.1899	0.0550	0.01454	68.7596	0.26454	3.7801	12.2617	3.2437
14	22.7374	0.0440	0.01150	86.9495	0.26150	3.8241	12.8334	3.3559
15	28.4217	0.0352	0.00912	109.6868	0.25912	3.8593	13.3260	3.4530
16	35.5271	0.0281	0.00724	138.1085	0.25724	3.8874	13.7482	3.5366
17	44.4089	0.0225	0.00576	173.6357	0.25576	3.9099	14.1085	3.6084
18	55.5112	0.0180	0.00459	218.0446	0.25459	3.9279	14.4147	3.6698
19	69.3889	0.0144	0.00366	273.5558	0.25366	3.9424	14.6741	3.7222
20	86.7362	0.0115	0.00292	342.9447	0.25292	3.9539	14.8932	3.7667
22	135.5253	0.0074	0.00186	538.1011	0.25186	3.9705	15.2326	3.8365
24	211.7582	0.0047	0.00119	843.0329	0.25119	3.9811	15.4711	3.8861
26	330.8722	0.0030	0.00076	1319.49	0.25076	3.9879	15.6373	3.9212
28	516.9879	0.0019	0.00048	2063.95	0.25048	3.9923	15.7524	3.9457
30	807.7936	0.0012	0.00031	3227.17	0.25031	3.9950	15.8316	3.9628
32	1262.18	0.0008	0.00020	5044.71	0.25020	3.9968	15.8859	3.9746
34	1972.15	0.0005	0.00013	7884.61	0.25013	3.9980	15.9229	3.9828
35	2465.19	0.0004	0.00010	9856.76	.025010	3.9984	15.9367	3.9858
36	3081.49	0.0003	0.00008	12322	0.25008	3.9987	15.9481	3.9883
38	4814.82	0.0002	0.00005	19255	0.25005	3.9992	15.9651	3.9921
40	7523.16	0.0001	0.00003	30089	0.25003	3.9995	15.9766	3.9947
45	22959		0.00001	91831	0.25001	3.9998	15.9915	3.9980
50	70065				0.25000	3.9999	15.9969	3.9993
55					0.25000	4.0000	15.9989	3.9997

30%	TABLE 26 Discrete Cash Flow: Compound Interest Factors							30%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.3000	0.7692	1.00000	1.0000	1.30000	0.7692		
2	1.6900	0.5917	0.43478	2.3000	0.73478	1.3609	0.5917	0.4348
3	2.1970	0.4552	0.25063	3.9900	0.55063	1.8161	1.5020	0.8271
4	2.8561	0.3501	0.16163	6.1870	0.46163	2.1662	2.5524	1.1783
5	3.7129	0.2693	0.11058	9.0431	0.41058	2.4356	3.6297	1.4903
6	4.8268	0.2072	0.07839	12.7560	0.37839	2.6427	4.6656	1.7654
7	6.2749	0.1594	0.05687	17.5828	0.35687	2.8021	5.6218	2.0063
8	8.1573	0.1226	0.04192	23.8577	0.34192	2.9247	6.4800	2.2156
9	10.6045	0.0943	0.03124	32.0150	0.33124	3.0190	7.2343	2.3963
10	13.7858	0.0725	0.02346	42.6195	0.32346	3.0915	7.8872	2.5512
11	17.9216	0.0558	0.01773	56.4053	0.31773	3.1473	8.4452	2.6833
12	23.2981	0.0429	0.01345	74.3270	0.31345	3.1903	8.9173	2.7952
13	30.2875	0.0330	0.01024	97.6250	0.31024	3.2233	9.3135	2.8895
14	39.3738	0.0254	0.00782	127.9125	0.30782	3.2487	9.6437	2.9685
15	51.1859	0.0195	0.00598	167.2863	0.30598	3.2682	9.9172	3.0344
16	66.5417	0.0150	0.00458	218.4722	0.30458	3.2832	10.1426	3.0892
17	86.5042	0.0116	0.00351	285.0139	0.30351	3.2948	10.3276	3.1345
18	112.4554	0.0089	0.00269	371.5180	0.30269	3.3037	10.4788	3.1718
19	146.1920	0.0068	0.00207	483.9734	0.30207	3.3105	10.6019	3.2025
20	190.0496	0.0053	0.00159	630.1655	0.30159	3.3158	10.7019	3.2275
22	321.1839	0.0031	0.00094	1067.28	0.30094	3.3230	10.8482	3.2646
24	542.8008	0.0018	0.00055	1806.00	0.30055	3.3272	10.9433	3.2890
25	705.6410	0.0014	0.00043	2348.80	0.30043	3.3286	10.9773	3.2979
26	917.3333	0.0011	0.00033	3054.44	0.30033	3.3297	11.0045	3.3050
28	1550.29	0.0006	0.00019	5164.31	0.30019	3.3312	11.0437	3.3153
30	2620.00	0.0004	0.00011	8729.99	0.30011	3.3321	11.0687	3.3219
32	4427.79	0.0002	0.00007	14756	0.30007	3.3326	11.0845	3.3261
34	7482.97	0.0001	0.00004	24940	0.30004	3.3329	11.0945	3.3288
35	9727.86	0.0001	0.00003	32423	0.30003	3.3330	11.0980	3.3297

35% TABLE 27 Discrete Cash Flow: Compound Interest Factors 35%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.3500	0.7407	1.00000	1.0000	1.35000	0.7407		
2	1.8225	0.5487	0.42553	2.3500	0.77553	1.2894	0.5487	0.4255
3	2.4604	0.4064	0.23966	4.1725	0.58966	1.6959	1.3616	0.8029
4	3.3215	0.3011	0.15076	6.6329	0.50076	1.9969	2.2648	1.1341
5	4.4840	0.2230	0.10046	9.9544	0.45046	2.2200	3.1568	1.4220
6	6.0534	0.1652	0.06926	14.4384	0.41926	2.3852	3.9828	1.6698
7	8.1722	0.1224	0.04880	20.4919	0.39880	2.5075	4.7170	1.8811
8	11.0324	0.0906	0.03489	28.6640	0.38489	2.5982	5.3515	2.0597
9	14.8937	0.0671	0.02519	39.6964	0.37519	2.6653	5.8886	2.2094
10	20.1066	0.0497	0.01832	54.5902	0.36832	2.7150	6.3363	2.3338
11	27.1439	0.0368	0.01339	74.6967	0.36339	2.7519	6.7047	2.4364
12	36.6442	0.0273	0.00982	101.8406	0.35982	2.7792	7.0049	2.5205
13	49.4697	0.0202	0.00722	138.4848	0.35722	2.7994	7.2474	2.5889
14	66.7841	0.0150	0.00532	187.9544	0.35532	2.8144	7.4421	2.6443
15	90.1585	0.0111	0.00393	254.7385	0.35393	2.8255	7.5974	2.6889
16	121.7139	0.0082	0.00290	344.8970	0.35290	2.8337	7.7206	2.7246
17	164.3138	0.0061	0.00214	466.6109	0.35214	2.8398	7.8180	2.7530
18	221.8236	0.0045	0.00158	630.9247	0.35158	2.8443	7.8946	2.7756
19	299.4619	0.0033	0.00117	852.7483	0.35117	2.8476	7.9547	2.7935
20	404.2736	0.0025	0.00087	1152.21	0.35087	2.8501	8.0017	2.8075
22	736.7886	0.0014	0.00048	2102.25	0.35048	2.8533	8.0669	2.8272
24	1342.80	0.0007	0.00026	3833.71	0.35026	2.8550	8.1061	2.8393
25	1812.78	0.0006	0.00019	5176.50	0.35019	2.8556	8.1194	2.8433
26	2447.25	0.0004	0.00014	6989.28	0.35014	2.8560	8.1296	2.8465
28	4460.11	0.0002	0.00008	12740	0.35008	2.8565	8.1435	2.8509
30	8128.55	0.0001	0.00004	23222	0.35004	2.8568	8.1517	2.8535
32	14814	0.0001	0.00002	42324	0.35002	2.8569	8.1565	2.8550
34	26999		0.00001	77137	0.35001	2.8570	8.1594	2.8559
35	36449		0.00001		0.35001	2.8571	8.1603	2.8562

40%	TABLE 28 Discrete Cash Flow: Compound Interest Factors							40%
	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.4000	0.7143	1.00000	1.0000	1.40000	0.7143		
2	1.9600	0.5102	0.41667	2.4000	0.81667	1.2245	0.5102	0.4167
3	2.7440	0.3644	0.22936	4.3600	0.62936	1.5889	1.2391	0.7798
4	3.8416	0.2603	0.14077	7.1040	0.54077	1.8492	2.0200	1.0923
5	5.3782	0.1859	0.09136	10.9456	0.49136	2.0352	2.7637	1.3580
6	7.5295	0.1328	0.06126	16.3238	0.46126	2.1680	3.4278	1.5811
7	10.5414	0.0949	0.04192	23.8534	0.44192	2.2628	3.9970	1.7664
8	14.7579	0.0678	0.02907	34.3947	0.42907	2.3306	4.4713	1.9185
9	20.6610	0.0484	0.02034	49.1526	0.42034	2.3790	4.8585	2.0422
10	28.9255	0.0346	0.01432	69.8137	0.41432	2.4136	5.1696	2.1419
11	40.4957	0.0247	0.01013	98.7391	0.41013	2.4383	5.4166	2.2215
12	56.6939	0.0176	0.00718	139.2348	0.40718	2.4559	5.6106	2.2845
13	79.3715	0.0126	0.00510	195.9287	0.40510	2.4685	5.7618	2.3341
14	111.1201	0.0090	0.00363	275.3002	0.40363	2.4775	5.8788	2.3729
15	155.5681	0.0064	0.00259	386.4202	0.40259	2.4839	5.9688	2.4030
16	217.7953	0.0046	0.00185	541.9883	0.40185	2.4885	6.0376	2.4262
17	304.9135	0.0033	0.00132	759.7837	0.40132	2.4918	6.0901	2.4441
18	426.8789	0.0023	0.00094	1064.70	0.40094	2.4941	6.1299	2.4577
19	597.6304	0.0017	0.00067	1491.58	0.40067	2.4958	6.1601	2.4682
20	836.6826	0.0012	0.00048	2089.21	0.40048	2.4970	6.1828	2.4761
22	1639.90	0.0006	0.00024	4097.24	0.40024	2.4985	6.2127	2.4866
24	3214.20	0.0003	0.00012	8033.00	0.40012	2.4992	6.2294	2.4925
25	4499.88	0.0002	0.00009	11247	0.40009	2.4994	6.2347	2.4944
26	6299.83	0.0002	0.00006	15747	0.40006	2.4996	6.2387	2.4959
28	12348	0.0001	0.00003	30867	0.40003	2.4998	6.2438	2.4977
30	24201		0.00002	60501	0.40002	2.4999	6.2466	2.4988
32	47435		0.00001		0.40001	2.4999	6.2482	2.4993
34	92972				0.40000	2.5000	6.2490	2.4996
35					0.40000	2.5000	6.2493	2.4997

50% TABLE 29 Discrete Cash Flow: Compound Interest Factors 50%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.5000	0.6667	1.00000	1.0000	1.50000	0.6667		
2	2.2500	0.4444	0.40000	2.5000	0.90000	1.1111	0.4444	0.4000
3	3.3750	0.2963	0.21053	4.7500	0.71053	1.4074	1.0370	0.7368
4	5.0625	0.1975	0.12308	8.1250	0.62308	1.6049	1.6296	1.0154
5	7.5938	0.1317	0.07583	13.1875	0.57583	1.7366	2.1564	1.2417
6	11.3906	0.0878	0.04812	20.7813	0.54812	1.8244	2.5953	1.4226
7	17.0859	0.0585	0.03108	32.1719	0.53108	1.8829	2.9465	1.5648
8	25.6289	0.0390	0.02030	49.2578	0.52030	1.9220	3.2196	1.6752
9	38.4434	0.0260	0.01335	74.8867	0.51335	1.9480	3.4277	1.7596
10	57.6650	0.0173	0.00882	113.3301	0.50882	1.9653	3.5838	1.8235
11	86.4976	0.0116	0.00585	170.9951	0.50585	1.9769	3.6994	1.8713
12	129.7463	0.0077	0.00388	257.4927	0.50388	1.9846	3.7842	1.9068
13	194.6195	0.0051	0.00258	387.2390	0.50258	1.9897	3.8459	1.9329
14	291.9293	0.0034	0.00172	581.8585	0.50172	1.9931	3.8904	1.9519
15	437.8939	0.0023	0.00114	873.7878	0.50114	1.9954	3.9224	1.9657
16	656.8408	0.0015	0.00076	1311.68	0.50076	1.9970	3.9452	1.9756
17	985.2613	0.0010	0.00051	1968.52	0.50051	1.9980	3.9614	1.9827
18	1477.89	0.0007	0.00034	2953.78	0.50034	1.9986	3.9729	1.9878
19	2216.84	0.0005	0.00023	4431.68	0.50023	1.9991	3.9811	1.9914
20	3325.26	0.0003	0.00015	6648.51	0.50015	1.9994	3.9868	1.9940
22	7481.83	0.0001	0.00007	14962	0.50007	1.9997	3.9936	1.9971
24	16834	0.0001	0.00003	33666	0.50003	1.9999	3.9969	1.9986
25	25251		0.00002	50500	0.50002	1.9999	3.9979	1.9990
26	37877		0.00001	75752	0.50001	1.9999	3.9985	1.9993
28	85223		0.00001		0.50001	2.0000	3.9993	1.9997
30					0.50000	2.0000	3.9997	1.9998
32					0.50000	2.0000	3.9998	1.9999
34					0.50000	2.0000	3.9999	2.0000
35					0.50000	2.0000	3.9999	2.0000

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INDEX

A

B

D

G

H

I

L

M

N

O

P

R

S