Rings and Modules - Definitions

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Contents

1 Preliminaries

Here a couple of preliminariy definitions, before we move on to categories.

Definition 1.1: R-module.
A *left* R-module $(M, +, \cdot)$ is an abelian group $(M, +)$ on which is defined a map

$$\begin{aligned} \cdot\colon R \times M &\to M &(1.1)\\ (r, m) &\mapsto rm. &(1.2)\end{aligned}$$

It is called scalar multiplication and satisfies

1. for any $r \in R$ the induced map

$$\begin{aligned} \dot{r}\colon M &\to M &(1.3)\\ m &\mapsto rm &(1.4)\end{aligned}$$

   is a homomorphism of abelian groups.

2. The map that sends each $r \in R$ to its associated endomorphism (as in the above)

$$\begin{aligned} \phi\colon R &\to \mathrm{End}_{\mathbb{Z}}(M) &(1.5)\\ r &\mapsto \dot{r} &(1.6)\end{aligned}$$

   is a morphism of rings.

If ϕ, instead of being a homomorphism is an antihomomorphism (i.e. it is a homomorphism

$$\phi\colon R^{op} \to \mathrm{End}_{\mathbb{Z}}(M), \quad (1.7)$$

from the opposite ring, in which the operations are computed in the opposite direction), then M is a *right* R-module. We denote *left* R-modules as ${}_RM$, wheareas *right* R-modules as M_R.

Definition 1.2: Bimodule.
Let R, S be rings. An abelian group $(M, +)$ is an R, S-bimodule ${}_RM_S$ iff ${}_RM$ is a left R-module, M_S is a right R-module and

$$r(xs) = (rx)s \quad (1.8)$$

for any $r \in R$, $s \in S$, $x \in M$.

Definition 1.3: Tensor product of modules.
Let S be a ring, $M_S \in \mathsf{Mod}\text{-}S$ and ${}_SN \in S\text{-}\mathsf{Mod}$. A map $\beta\colon M \times N \to G$, to G an abelian group, is called balanced iff it satisfies the following

$$\begin{aligned} \beta(m + m', n) &= \beta(m, n) + \beta(m', n) && \forall\, m, m' \in M \text{ and } \forall\, n \in N &(1.9)\\ \beta(m, n + n') &= \beta(m, n) + \beta(m, n') && \forall\, m \in M \text{ and } \forall\, n, n' \in N &(1.10)\\ \beta(ms, n) &= \beta(m, sn) && \forall\, m \in M,\ \forall\, n \in N \text{ and } \forall\, s \in S. &(1.11)\end{aligned}$$

The tensor product of M and N is the pair $(M \otimes_S N, \tau)$, with $M \otimes_S N$ an abelian group and $\tau\colon M \times N \to M \otimes_S N$ a map s.t. $\forall\, \beta\colon M \times N \to G$ a balanced map, $\exists\,!\, \alpha\colon M \otimes_S N \to G$ an abelian group morphism s.t. the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "\tau"] \arrow[d, "\beta"'] & M \otimes_S N \arrow[dl, "\alpha"] \\ G & \end{tikzcd}, \quad (1.12)$$

i.e. s.t. $\alpha \circ \tau = \beta$. In such a case we say that every balanced map β factors through τ via an abelian group morphism.

Remark 1.4: Construction of the tensor product.
Consider M and N as before. Consider $M \times N$ as a set. Let $\mathbb{Z}^{M\times N}$ be the free abelian group with basis $(m, n) \in M \times N$. Consider $H \triangleleft \mathbb{Z}^{M\times N}$ generated by the elements of the form

$$\begin{aligned} \big\{\, &(m + m', n) - (m, n) - (m', n)\,, (m, n + n') - (m, n) - (m, n')\,, &(1.13)\\ &(ms, n) - (m, sn) \;\big|\; m, m' \in M,\; n, n' \in N,\; s \in S \big\}. &(1.14) \end{aligned}$$

Let $\tau\colon M \times N \to \mathbb{Z}^{M\times N}/H$, defined by $(m, n) \overset{\tau}{\mapsto} (m, n) + H$, then $\big(\mathbb{Z}^{M\times N}, \tau\big)$ is a tensor product of M and N.

Remark 1.5: Tensor product as a module.
Given $N_S \in \mathsf{Mod}\text{-}S$ and ${}_SM_R$ and S-R bimodule, we want to construct a tensor product of the two, which is also a right R-module. (We can choose $S = \mathrm{End}_R(M)$, or $S = \mathbb{Z}$ for example). We define, for any $L_R \in \mathsf{Mod}\text{-}R$, the set

$$\beta \in \mathrm{Bal}(N \times M, L_R), \tag{1.15}$$

consisting of β balanced maps (as defined above) s.t. $\beta(x, yr) = \beta(x, y)r$ for all $x \in N$, $y \in M$ and $r \in R$. Then, in this situation, we define the tensor product $N \otimes_S M$ as the right R-module, with a map τ, s.t. the following diagram commutes

$$\begin{tikzcd} N \times M \arrow[r, "\tau"] \arrow[d, "\beta"'] & N \otimes_S M \arrow[dl, dashed, "\exists\,\alpha"] \\ L_R & \end{tikzcd} \tag{1.16}$$

for all L_R, and $\beta \in \mathrm{Bal}(N \times M, L_R)$. In particular this gives a bijection

$$\mathrm{Bal}(N \times M, L_R) \overset{\varphi}{\longleftrightarrow} \mathrm{Hom}_R\,(N \otimes_S M, L_R)\ . \tag{1.17}$$

2 Category theory

2.1 Categories and morphisms

Definition 2.1: Category.
A category C si determined by the following elements:

- $\mathrm{Ob}(\mathsf{C})$ a *class* of objects,
- $\forall\, X, Y \in \mathrm{Ob}(\mathsf{C})$ the data of a set of *arrows* with *source* X and *target* Y, denoted with $\mathrm{Hom}_{\mathsf{C}}\,(X, Y)$, whose elements are called *morphisms*,
- an operation of composition, that acts as follows

$$\begin{aligned} \circ\colon \mathrm{Hom}_{\mathsf{C}}\,(X, Y) \times \mathrm{Hom}_{\mathsf{C}}\,(Y, Z) &\to \mathrm{Hom}_{\mathsf{C}}\,(X, Z) &(2.1)\\ (f, g) &\mapsto g \circ f, &(2.2) \end{aligned}$$

  for any $X, Y, Z \in \mathrm{Ob}(\mathsf{C})$ and is associative, i.e.

$$h \circ (g \circ f) = (h \circ g) \circ f, \tag{2.3}$$

  whenever defined, i.e. $\forall\, X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$.

Also the set $\mathrm{End}_{\mathsf{C}}(X) := \mathrm{Hom}_{\mathsf{C}}(X, X)$ always contains the element id_X, that is defined to act as: given any $f \in \mathrm{Hom}_C(X, Y)$, $g \in \mathrm{Hom}_{\mathsf{C}}(Z, X)$

$$f \circ id_X = f, \quad id_X \circ g = g. \tag{2.4}$$

Example

- Sets: $\mathrm{Ob}(\mathsf{Sets})$ are sets, and morphisms are set theoretic maps,
- Top: $\mathrm{Ob}(\mathsf{Top})$ are topological spaces, morphisms are continuous maps,
- $\mathsf{Semigroups}$: $\mathrm{Ob}(\mathsf{Semigrousp})$ are sets with an associative operation, morphisms are homomorphisms of semigroups,
- $\mathsf{Monoids}$: $\mathrm{Ob}(\mathsf{Monoids})$ are semigroups with a unit, morphisms are monoid morphisms,
- Clearly one can construct a lot more examples, we'll stop here.

Definition 2.2: Opposite category.
Given a category C, one can define the opposite category C^{op}, characterized by

- $\mathrm{Ob}(\mathsf{C}^{op}) := \mathrm{Ob}(\mathsf{C})$,
- $\mathrm{Hom}_{\mathsf{C}^{op}}(X, Y) := \mathrm{Hom}_{\mathsf{C}}(Y, X)$, with composition given by

$$g^{op} \circ_{\mathsf{C}^{op}} f^{op} := (f \circ_{\mathsf{C}} g)^{op}. \tag{2.5}$$

Definition 2.3: iso-mono-epi morphisms.
Let $X \xrightarrow{f} Y$ be a morphism in a category C, then it is a(n)

monomorphism: iff $\forall\, Z \overset{g_1}{\underset{g_2}{\rightrightarrows}} X$ s.t. $f \circ g_1 = f \circ g_2 \implies g_1 = g_2$. We denote it with $f\colon Y \rightarrowtail Z$. It is said that f is *left* erasable

epimorphism: iff $\forall\, X \overset{h_1}{\underset{h_2}{\rightrightarrows}} Y$ s.t. $h_1 \circ f = h_2 \circ f \implies h_1 = h_2$. We denote it with $f\colon Y \twoheadrightarrow Z$. It is said that f is *right* erasable

isomorphism: iff $\exists\, Y \xrightarrow{g} X$ s.t. $g \circ f = id_X$ and $f \circ g = id_Y$.

Remark 2.4 Note that if $X \xrightarrow{f} Y$ is an *iso*, then it is also *mono* and *epi*, but the converse is not always true.

If, moreover, $X \xrightarrow{f} Y$ is an iso, we say that X and Y are *isomorphic* and we denote it with $X \simeq_{\mathsf{C}} Y$ (especially if we do not want to explicitly cite the isomorphism).

Definition 2.5: Subcategory.
A category C' is a subcategory of C, denoted with $\mathsf{C}' \subset \mathsf{C}$ iff

- $\mathrm{Ob}(\mathsf{C}') \subset \mathrm{Ob}(\mathsf{C})$
- $\forall\, X, Y \in \mathrm{Ob}(\mathsf{C}')$, we have $\mathrm{Hom}_{\mathsf{C}'}(X, Y) \subset \mathrm{Hom}_{\mathsf{C}}(X, Y)$,

and the two categories have the same composition and identities.

Definition 2.6: Full subcategory.
$\mathsf{C}' \subset \mathsf{C}$ is said to be a *full* subcategory iff $\forall\, X, Y \in \mathrm{Ob}(\mathsf{C}')$

$$\mathrm{Hom}_{\mathsf{C}'}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)\,. \tag{2.6}$$

Definition 2.7: Discrete/finite/grupoid.
A category C is said to be

Discrete: iff the only morphisms are the identities. Note that a set can be naturally identified as a *discrete* category.

Finite: iff the family $\mathrm{Mor}(\mathsf{C})$ of all the morphisms in C (and, as a consequence $\mathrm{Ob}(\mathsf{C})$) is a finite set.

Grupoid: iff all the morphisms are isomoprhisms. Note that a group G can be identified with a *grupoid* category C with only one element $X \in \mathrm{Ob}\mathsf{C}$ and

$$\mathrm{Hom}_{\mathsf{C}}(X, X) \coloneqq G. \tag{2.7}$$

Definition 2.8: Product category.
Let C and D be two categories, one can define their product $\mathsf{C} \times \mathsf{D}$ as the category characterized by

- $\mathrm{Ob}(\mathsf{C} \times \mathsf{D}) \coloneqq \mathrm{Ob}(\mathsf{C}) \times \mathrm{Ob}(\mathsf{D})$,
- $\mathrm{Hom}_{\mathsf{C}\times\mathsf{D}}((X, Y), (X', Y')) \coloneqq \mathrm{Hom}_{\mathsf{C}}(X, Y) \times \mathrm{Hom}_{\mathsf{D}}(X', Y')$,
- $(f, g) \circ_{\mathsf{C}\times\mathsf{D}} (f', g') \coloneqq (f \circ_{\mathsf{C}} g, f' \circ_{\mathsf{D}} g')$.

Definition 2.9: Initial/terminal/zero object.
An object $X \in \mathrm{Ob}(\mathsf{C})$ is said to be

Initial: iff $\forall\, Y \in \mathrm{Ob}(\mathsf{C})$ we have $\mathrm{Hom}_{\mathsf{C}}(X, Y) = \{\mathrm{pt}\}$,

Terminal: iff $\forall\, Y \in \mathrm{Ob}(\mathsf{C})$ we have $\mathrm{Hom}_{\mathsf{C}}(Y, X) = \{\mathrm{pt}\}$,

Zero: iff it is both an *initial* and *terminal* object.

In the above list we have denoted with $\{\mathrm{pt}\}$ the singleton, i.e. any set with only one element.

Definition 2.10: Zero morphism.
Let C be category with a zero object 0_{C}. Given $X, Y \in \mathrm{Ob}(\mathsf{C})$ we can define the 0-morphism from X into Y as the unique map

$$X \xrightarrow{\alpha} 0_{\mathsf{C}} \xrightarrow{\beta} Y. \tag{2.8}$$

2.2 Functors

Definition 2.11: Functor.
Given two categories C and D, a functor F between them is defined by:

- a map $F\colon \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$,
- a collection of maps, also denoted by F, given $\forall\, X, Y \in \mathrm{Ob}(\mathsf{C})$

$$F\colon \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(FX, FY)\,, \tag{2.9}$$

  s.t. $F(id_X) = id_Y$ and $\forall\, f, g$ these maps preserve composition, i.e.

$$F(g \circ_{\mathsf{C}} f) = F(g) \circ_{\mathsf{D}} F(f). \tag{2.10}$$

Definition 2.12: Full/faithful/essentially surjective/conservative functors.
Let $\mathsf{C} \xrightarrow{F} \mathsf{D}$ be a functor, then it is said to be

Full iff $\forall X, Y \in \mathrm{Ob}(\mathsf{C})$ the map $\mathrm{Hom}_{\mathsf{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathsf{D}}(FX, FY)$ is surjective,

Faithful iff $\forall X, Y \in \mathrm{Ob}(\mathsf{C})$ the map $\mathrm{Hom}_{\mathsf{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathsf{D}}(FX, FY)$ is injective,

Fully faithful iff $\forall X, Y \in \mathrm{Ob}(\mathsf{C})$ the map $\mathrm{Hom}_{\mathsf{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathsf{D}}(FX, FY)$ is bijective,

Essentially surjective iff $\forall Y \in \mathrm{Ob}(\mathsf{D})\ \exists X \in \mathrm{Ob}(\mathsf{C})$ s.t. $FX \simeq_{\mathsf{D}} Y$,

Conservative iff $X \xrightarrow{f} Y$ is an isomorphism in C as soon as $F(f)$ is an isomorphism in D.

Remark 2.13 A fully faithful functor $F\colon \mathsf{C} \to \mathsf{D}$ is conservative.

Definition 2.14: Concrete category.
A category C is called *concrete* iff it is equipped with a faithful functor to Sets.

Definition 2.15: Contravariant functor.
We define a *contravariant* functor from C to C' to be a functor from C^{op} to C', i.e. it satisfies

$$F(g \circ f) = F(f) \circ F(g). \tag{2.11}$$

We denote with $\mathrm{op}\colon \mathsf{C} \to \mathsf{C}^{op}$ to be the contravariant functor associated with $id_{\mathsf{C}^{op}}$. Sometimes functors are called *covariant* in order to emphasize the fact that they are not *contravariant*.

Remark 2.16 Notice that, given $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{E}$ functors, then

- if both F and G are either covariant or contravariant, then $F \circ G$ is covariant,
- if one of them is covariant and the other is contravariant, then $F \circ G$ is contravariant.

Definition 2.17: Bifunctor.
A *bifunctor* F from (C, D) to E is a functor from the product category, i.e.

$$F\colon \mathsf{C} \times \mathsf{D} \to \mathsf{E}. \tag{2.12}$$

In particular, fixed $X \in \mathsf{C}$ and $Y \in \mathsf{D}$, then $F(X, -)\colon \mathsf{D} \to \mathsf{E}$ and $F(-, Y)\colon \mathsf{C} \to \mathsf{E}$ are functors. Moreover, for any morphism $f\colon X \to X'$ in C and $g\colon Y \to Y'$ in D, then the following diagram commutes:

$$\begin{tikzcd} F(X,Y) \arrow[r, "F(X{,}g)"] \arrow[d, "F(f{,}Y)"'] & F(X,Y') \arrow[d, "F(f{,}Y')"] \\ F(X',Y) \arrow[r, "F(X'{,}g)"'] & F(X',Y') \end{tikzcd} . \tag{2.13}$$

Example Given a category C, there is a natural bifunctor

$$F = \mathrm{Hom}_{\mathsf{C}}(-,-)\colon \mathsf{C}^{op} \times \mathsf{C} \to \mathsf{Sets}. \tag{2.14}$$

It is defined as follows. On objects it acts as

$$\begin{aligned} F\colon \mathsf{C}^{op} \times \mathsf{C} &\to \mathsf{Sets} && (2.15)\\ (C, D) &\mapsto \mathrm{Hom}_{\mathsf{C}}(C, D)\,. && (2.16)\end{aligned}$$

On pairs of morphisms $C' \xrightarrow{f} C$ and $D \xrightarrow{g} D'$, it acts as

$$\begin{tikzcd} (C,D) \arrow[r] \arrow[d, "{(f,g)}"] & \mathrm{Hom}_{\mathsf{C}}(C,D) \arrow[d, "{F(f,g)}"] & \alpha \arrow[d, mapsto] \\ (C',D') \arrow[r] & \mathrm{Hom}_{\mathsf{C}}(C',D') & g \circ \alpha \circ f \end{tikzcd}\,. \tag{2.17}$$

Clearly F is covariant in both variables.

Definition 2.18: Morphism of functors.
Given two functors $F, G\colon \mathsf{C} \to \mathsf{D}$, a *morphism of functors* (sometimes called *natural transformation*) $\theta\colon F \to G$ (sometimes denoted with $F \overset{\theta}{\Rightarrow} G$) is the data, for any $X \in \mathsf{C}$, of a map $\theta(X)\colon FX \to GX$ s.t. $\forall\, f\colon X \to X'$ in C the following diagram commutes

$$\begin{tikzcd} FX \arrow[r, "\theta(X)"] \arrow[d, "F(f)"'] & GX \arrow[d, "G(f)"] \\ FX' \arrow[r, "\theta(X')"'] & GX' \end{tikzcd}\,, \tag{2.18}$$

i.e. $G(f) \circ \theta(X) = \theta(X') \circ F(f)$.

Some authors denote one such transformation with the following diagram

$$\begin{tikzcd} C \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd} \tag{2.19}$$

Definition 2.19: Natural isomorphic functors.
Let C and D be two categories, and $G, F\colon \mathsf{C} \to \mathsf{D}$ be two functors. We say that F is *naturally isomorphic* to G iff one of the following (equivalent) conditions is satisfied:

- there exist two natural transformations $\eta\colon F \to G$ and $\theta\colon G \to F$ s.t.

$$id_G = \eta \circ \theta \quad \text{and} \quad \theta \circ \eta = id_F, \tag{2.20}$$

- there exists a natural transformation $\eta\colon F \to G$ s.t. $\eta_X\colon FX \to GX$ is an isomorphism in D for every $C \in \mathrm{Ob}(\mathsf{C})$.

Definition 2.20: Category of functors.
We denote by $\mathsf{D}^{\mathsf{C}} := \mathsf{Fct}(\mathsf{C}, \mathsf{D})$ the *category of functors* from C to D, whose elements are functors $F\colon \mathsf{C} \to \mathsf{D}$ and whose morphisms are the above mentioned morphisms of functors.

Remark 2.21 In general the category of functors is a *large category*, in the sense that its objects might not be sets. Though, if we start from a *small* category, i.e. if $\mathrm{Ob}(\mathsf{C})$ is a set, then $\mathsf{Fct}(\mathsf{C}, \mathsf{D})$ is a small category.

In such case, fixed F, G functors from C to D, then a natural transformation is

$$\eta = \{\eta_X\}_{X \in \mathrm{Ob}(\mathsf{C})} \in \prod_{X \in \mathrm{Ob}(\mathsf{C})} \mathrm{Hom}_{\mathsf{D}}(FX, GX)\,. \tag{2.21}$$

It is important to notice that the infinite product of sets is still a set, hence

$$\mathrm{Nat}\,(F,G) \subset \prod_{X\in \mathrm{Ob}(\mathsf{C})} \mathrm{Hom}_{\mathsf{C}}\,(FX,GX)\,. \tag{2.22}$$

Example Fix $\mathsf{I} := (I, \leq)$ a poset (a small category) and a category C. An element $F \in \mathsf{Fct}\,(\mathsf{I},\mathsf{C}) = \mathsf{C}^{\mathsf{I}}$ is a functor

$$F\colon \mathsf{I} \to \mathsf{C} \tag{2.23}$$

that associates to each element $i \in I$ an object $F(i) \in \mathrm{Ob}\,(\mathsf{C})$. Moreover, with regards to morphisms it acts as follows: given $i \leq j \leq k$ we have $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ and $\beta\circ\alpha = \gamma\colon i \to k$ and the following commutative diagram

$$\begin{tikzcd} F(i) \arrow[rr, "F(\gamma)"] \arrow[dr, "F(\alpha)"'] & & F(k) \\ & F(j) \arrow[ur, "F(\beta)"'] & \end{tikzcd} . \tag{2.24}$$

In particular, given $\mathsf{C} = \mathsf{Mod}\,(R)$, then $F \in \mathsf{Fct}\,(I, \mathsf{Mod}\,(R))$ is a functor s.t., called $f_{ji} := F(i \to j)$, then

$$f_{ki} = f_{kj} \circ f_{ji}. \tag{2.25}$$

This is called a *direct system of modules*.

Definition 2.22: Preadditive category.
A category C is called *preadditive* iff it is a $\mathbb{Z}$ category, i.e. iff given any pair $X, Y \in \mathrm{Ob}\,(\mathsf{C})$ the set $\mathrm{Hom}_{\mathsf{C}}\,(X,Y)$ is a $\mathbb{Z}$-module (an abelian group) and the composition of morphisms is a bilinear map.

Example R-Mod, the category of left R-modules, and Mod-R, the category of right R-modules, are all preadditive categories (even for R division rings or fields).
Rings and Groups are not preadditive: the Hom sets do not have the structure of abelian group.

Definition 2.23: Additive functors.
Given two preadditive categores C and D, a functor $F\colon \mathsf{C} \to \mathsf{D}$ is called *additive* iff, for any $X, Y \in \mathrm{Ob}\,(\mathsf{C})$, for any $f, g\colon X \to Y$, then

$$F(f+g) = F(f) + F(g). \tag{2.26}$$

Remark 2.24 For a samll preadditive category C and a preadditive category D, then we denote with

$$\underline{\mathrm{Hom}\,(\mathsf{C},\mathsf{D})} \tag{2.27}$$

the category of all additive functors from C to D.

Example Given a ring R, we define the category $\underline{\mathsf{R}}$ with one object, $*$, characterized by

$$\mathrm{Hom}_{\underline{\mathsf{R}}}\,(*,*) := R, \tag{2.28}$$

with the composition acting as the product in R. Clearly it is a preadditive category. Let's consider the category

$$\underline{\mathrm{Hom}\,(\underline{\mathsf{R}},\mathsf{Ab})}. \tag{2.29}$$

Definition 2.25: Category of C-modules.
Given a small preadditive category C, then the category

$$\underline{\mathrm{Hom}\left(\mathsf{C},\mathsf{Ab}\right)} \tag{2.30}$$

of additive covariant (contravariant) functors, is called the category of *left (right)* C-modules.

Definition 2.26: Category isomorphism/equivalence.
Given two categories C and D we say they are

> **isomorphic** , notation $\mathsf{C} \cong \mathsf{D}$, iff there exist $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ s.t. $F \circ G = id_{\mathsf{D}}$ and $G \circ F = id_{\mathsf{C}}$,
>
> **equivalent** , notation $\mathsf{C} \simeq \mathsf{D}$, iff there exist $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ s.t. $F \circ G \simeq id_{\mathsf{D}}$ and $G \circ F \simeq id_{\mathsf{C}}$. In this case we just asked for isomorphism of functors, which makes F and G *quasi-inverses*.

Moreover an equivalence $F\colon \mathsf{C} \to \mathsf{D}^{op}$ is called a *duality*.

Remark 2.27 Fixed a ring R, then

$$\underline{\mathrm{Hom}\left(\underline{\mathsf{R}},\mathsf{Ab}\right)} \cong R\text{-}\mathsf{Mod} \quad \text{and} \quad \underline{\mathrm{Hom}\left(\underline{\mathsf{R}}^{op},\mathsf{Ab}\right)} \cong \mathsf{Mod}\text{-}R. \tag{2.31}$$

Example: duality. Let K be a division ring and K-Vect the category of finite dimensionale left K-Vector Spaces, then

$$D\colon K\text{-}\mathsf{Vect} \to \mathsf{Vect}\text{-}K \tag{2.32}$$
$$V \mapsto V^* \tag{2.33}$$

is a duality.

Proposition 2.28. *A functor $F\colon \mathsf{C} \to \mathsf{D}$ is an equivalence of categories iff it is* fully faithful *and* essentially surjective.

2.3 Yoneda lemma

Definition 2.29 Let C be a category, one defines the following:

$$\mathsf{C}^{\wedge} := \mathsf{Fct}\left(\mathsf{C}^{op},\mathsf{Sets}\right), \quad \mathsf{C}^{\vee} := \mathsf{Fct}\left(\mathsf{C}^{op},\mathsf{Sets}^{op}\right), \tag{2.34}$$

and the functors

$$h_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{C}^{\wedge} \text{ s.t. } X \mapsto \mathrm{Hom}_{\mathsf{C}}\left(-,X\right) \tag{2.35}$$
$$k_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{C}^{\vee} \text{ s.t. } X \mapsto \mathrm{Hom}_{\mathsf{C}}\left(X,-\right). \tag{2.36}$$

Lemma 2.30 (Yoneda)**.** *The functor h_{C} is fully faithful.*

Definition 2.31: Representable functor.

1. A functor $F\colon \mathsf{C}^{op} \to \mathsf{Sets}$ is *representable* iff there exists $X \in \mathsf{C}$ s.t. $F(Y) \simeq \mathrm{Hom}_{\mathsf{C}}\left(Y,X\right)$ functorially in $Y \in \mathsf{C}$. In other words we have $F \simeq h_{\mathsf{C}}(X)$ in $\mathsf{C}^{\wedge}$. Such object X is called a representative of F.
2. A functor $G\colon \mathsf{C} \to \mathsf{Sets}$ is *corepresentable* iff there exists a representative $X \in \mathsf{C}$ s.t. $G(Y) \simeq \mathrm{Hom}_{\mathsf{C}}\left(X,Y\right)$ functorially in $Y \in \mathsf{C}$.

Proposition 2.32. *Let* $F\colon \mathsf{C}^{op} \to \mathsf{Sets}$ *be a* representable *functor, i.e.* $\exists\, X \in \mathrm{Ob}\,(\mathsf{C})$ *s.t.*

$$F \simeq \mathrm{Hom}_{\mathsf{C}}\,(-, X)\,. \tag{2.37}$$

Then X *is unique up to isomorphism.*

Definition 2.33: Adjoint functors.
Let $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ be two functors. One says that (F, G) is an *adjoint* pair, or equivalently that F is a *left adjoint* to G or that G is a *right adjoint* to F, iff there exists an isomorphism of bifunctors:

$$\mathrm{Hom}_{\mathsf{D}}\,(F(-), -) \simeq \mathrm{Hom}_{\mathsf{C}}\,(-, G(-))\,. \tag{2.38}$$

Remark 2.34 Note that, given two categories C and D and a pair (F, G) of *adjoint* functors, one has the following morphism of functors:

$$F \circ G \to id_{\mathsf{D}}, \quad G \circ F \to id_{\mathsf{C}}. \tag{2.39}$$

2.4 Kernel and Cokernel

Definition 2.35: (Co)kernel.
Let C be a preadditive category, with a zero object. Let $A \xrightarrow{f} B$ a morphism in C.

- A *kernel* of f is a pair (K, ϵ), with $K \xrightarrow{\epsilon} A$ satisfying

  K1 $f \circ \epsilon = 0$,

  K2 for any $\epsilon'\colon K' \to A$ s.t. $f \circ \epsilon' = 0$, then $\exists\,!K' \xrightarrow{\alpha} K$ s.t. $\epsilon \circ \alpha = \epsilon'$, i.e. s.t. the following diagram commutes

$$\begin{tikzcd} K \arrow[r, "\epsilon"] & A \arrow[r, "f"] & B \\ & K' \arrow[ul, dashed, "\exists\,!\alpha"] \arrow[u, "\epsilon'"'] \arrow[ur, "0"'] & \end{tikzcd}\,. \tag{2.40}$$

- A *cokernel* of f is a kernel of $B \xrightarrow{f} A$ in C^{op}. In other words it is a pair (C, p), with $B \xrightarrow{p} C$ s.t.

  CK1 $p \circ f = 0$,

  CK2 for any $p'\colon B \to C'$ s.t. $p' \circ f = 0$, then $\exists\,!C \xrightarrow{\gamma} C'$ s.t. $\gamma \circ p = p'$, i.e. s.t. the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "p"] \arrow[d, "p'"] & C \arrow[dl, dashed, "\exists\,!\gamma"] \\ & C' & \end{tikzcd}\,. \tag{2.41}$$

We denote with the uppercase Ker the object K, and with the lowercase ker the morphism $\epsilon\colon K \to A$.
Analogously for the cokernel, we denote with the uppercase Coker the object C, and with the lower case coker the morphism $p\colon B \to C$.

Remark 2.36 Property *K2* grants that Ker satisfies a universal property (U.P.). Objects that satisfy universal properties are unique up to a unique isomorphism.

Definition 2.37: (Co)equalizer.
Let f, g be two parallel morphisms $A \rightrightarrows B$ in a category C.

- An *equalizer* of f and g is a pair (C, e), with $C \xrightarrow{e} A$, satisfying

  eq1 $f \circ e = g \circ e$,

  eq2 for (C', e') with $C' \xrightarrow{e'} A$ s.t. $f \circ e' = g \circ e'$, then $\exists ! \alpha \colon C' \to C$ s.t. $e \circ \alpha = e'$, i.e. the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ & C' \arrow[u, "e'"'] \arrow[ul, dashed, "\exists ! \alpha"] & \end{tikzcd} . \tag{2.42}$$

- A *coequalizer* of f and g is an equalizer of f and g in C^{op}. In other words it is a pair (C, p), with $B \xrightarrow{p} C$ s.t.

  coeq1 $p \circ f = p \circ g$,

  coeq2 for (C', p') with $B \xrightarrow{p'} C'$ s.t. $p' \circ f = p' \circ g$, then $\exists ! \gamma \colon C \to C'$, with $\gamma \circ p = p'$, i.e. s.t. the following diagram commutes

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "P"] \arrow[d, "p'"] & C \arrow[dl, dashed, "\exists ! \gamma"] \\ & C' & \end{tikzcd} . \tag{2.43}$$

Remark 2.38

- The kernel of $A \xrightarrow{f} B$ is just the equalizer of f and 0, if it exists.
- The cokernel of $A \xrightarrow{f} B$ is just the coequalizer of f and 0, if it exists.

Lemma 2.39. *Let C be a preadditive category with 0 object. Let $f \colon A \to B$ in C.*

- *f is a mono (epi) iff $f \circ h = 0 \implies h = 0$ ($h \circ f = 0 \implies h = 0$),*
- *f is a mono (epi) iff $0 \to A$ is a kernel of f ($B \to 0$ is a cokernel of f),*
- *A kernel (cokernel) is mono (epi).*

Definition 2.40: Ker functor.
Let C be a preadditive category admitting zero object. Consider $A \xrightarrow{f} B$ a morphism in C. This induces a natural transformation $f_* \colon h^A \to h^B$, given by the collection of maps

$$f_*(X) \colon h^A(X) = \mathrm{Hom}_{\mathsf{C}}(X, A) \to \mathrm{Hom}_{\mathsf{C}}(X, B) = h^B(X) \tag{2.44}$$
$$\alpha \mapsto f \circ \alpha \tag{2.45}$$

for $X \in \mathrm{Ob}(\mathsf{C})$. For any $X \in \mathrm{Ob}(\mathsf{C})$, $f_*(X)$ is a morphism of abelian groups, hence it admits a kernel.

$$\ker f_*(X) = \left\{ X \xrightarrow{\alpha} A \,\middle|\, f \circ \alpha = 0 \right\} \leq \mathrm{Hom}_{\mathsf{C}}(X, A) . \tag{2.46}$$

We can define the *contravariant* functor

$$F := \ker [f_* \colon \mathrm{Hom}_{\mathsf{C}}(-, A) \to \mathrm{Hom}_{\mathsf{C}}(-, B)] \tag{2.47}$$

That acts on a morphism $X \xrightarrow{h} Y$ as

$$F(h)\colon F(Y) \to F(X) \tag{2.48}$$
$$\beta \mapsto \beta \circ f. \tag{2.49}$$

Proposition 2.41. *A morphism $A \xrightarrow{f} B$ in a preadditive category admitting zero object has a kernel iff the associated functor F is representable. In this case a kernel of f is given by (K, ϵ), as follows: Let $F \simeq_\eta \operatorname{Hom}_{\mathsf{C}}(-, K)$, for $K \in \operatorname{Ob}(\mathsf{C})$ a representative of F. Then ϵ is given by*

$$\operatorname{Hom}_{\mathsf{C}}(K, K) \xrightarrow{\eta_K} F(K) \subset \operatorname{Hom}_{\mathsf{C}}(K, A) \tag{2.50}$$
$$1_K \mapsto \epsilon. \tag{2.51}$$

Definition 2.42: Coker functor.
Let C be a preadditive category admitting zero object. Consider $A \xrightarrow{f} B$ a morphism in C. This induces a natural transformation $f^*\colon h_B \to h_A$, given by the collection of maps

$$f^*(X)\colon h_B(X) = \operatorname{Hom}_{\mathsf{C}}(B, X) \to \operatorname{Hom}_{\mathsf{C}}(A, X) = h_A(X) \tag{2.52}$$
$$\beta \mapsto \beta \circ f \tag{2.53}$$

for $X \in \operatorname{Ob}(\mathsf{C})$. For any $X \in \operatorname{Ob}(\mathsf{C})$, $f^*(X)$ is a morphism of abelian groups, hence it admits a kernel in Ab:

$$\ker f^*(X) = \left\{ B \xrightarrow{\beta} X \,\middle|\, \beta \circ f = 0 \right\}. \tag{2.54}$$

We can define a *covariant* functor

$$F := \ker \left[f^*\colon \operatorname{Hom}_{\mathsf{C}}(B, -) \to \operatorname{Hom}_{\mathsf{C}}(A, -) \right] \tag{2.55}$$

that acts on a morphism $X \xrightarrow{h} Y$ as

$$F(h)\colon F(X) \to F(Y) \tag{2.56}$$
$$\beta \mapsto h \circ \beta. \tag{2.57}$$

Proposition 2.43. *Let C be a preadditive category admitting zero object. The morphism $A \xrightarrow{f} B$ has a cokernel iff F is corepresentable. In other words, iff there exists $C \in \operatorname{Ob}(\mathsf{C})$ and a natural isomorphism*

$$F \simeq_\eta \operatorname{Hom}_{\mathsf{C}}(C, -). \tag{2.58}$$

In this case a cokernel is given by (C, p), with $C \in \operatorname{Ob}(\mathsf{C})$ a representative of F and p given by

$$\operatorname{Hom}_{\mathsf{C}}(C, C) \to F(C) \subset \operatorname{Hom}_{\mathsf{C}}(B, C) \tag{2.59}$$
$$1_C \mapsto p. \tag{2.60}$$

Lemma 2.44. *Let C be a preadditive category with 0 object. Let $A \xrightarrow{f} B$ be a kernel of some other morphism. Then, if $\operatorname{coker} f$ exists, we have*

$$f = \ker(\operatorname{coker} f). \tag{2.61}$$

Lemma 2.45. *Let C be a preadditive category with 0 object. Let $A \xrightarrow{f} B$ be a cokernel of some morphism. Lat f admit a kernel, then*

$$f = \operatorname{coker}\,(\ker f). \tag{2.62}$$

2.5 Product and Coproduct

Definition 2.46: Product.
Let $A, B \in \mathrm{Ob}(\mathsf{C})$ for an arbitrary category C. A *product* of A and B, if it exists, is a triple $(A \prod B, \pi_A, \pi_B)$, where $A \prod B \in \mathrm{Ob}(\mathsf{C})$, and the morphisms π_A and π_B in C, called *projections,*

$$A \prod B \xrightarrow{\pi_A} A \quad \text{and} \quad A \prod B \xrightarrow{\pi_B} B \tag{2.63}$$

satisfy the universal property: Given an arbitrary (X, α, β), with $X \in \mathrm{Ob}(\mathsf{C})$, $X \xrightarrow{\alpha} A$ and $X \xrightarrow{\beta} B$ a pair of morphism, there exists a unique morphism $X \xrightarrow{\exists ! h} A \prod B$ s.t.

$$\begin{tikzcd}
& X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, dashed, "\exists ! h"] & \\
A & & B \\
& A \prod B \arrow[ul, "\pi_A"] \arrow[ur, "\pi_B"'] &
\end{tikzcd} \tag{2.64}$$

the above diagram commutes. In other words, s.t. $\alpha = \pi_A \circ h$ and $\beta = \pi_B \circ h$.

Remark 2.47 If it exists, a product, is unique up to a unique isomorphism. This, as usual, is due to the universal property used to define the product.

Proposition 2.48. *Define the functor*

$$F := \mathrm{Hom}_{\mathsf{C}}(-, A) \times \mathrm{Hom}_{\mathsf{C}}(-, B) : \mathsf{C} \to \mathsf{Sets} \tag{2.65}$$

on objects as $F(X) := \mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B)$*, and on morphisms* $X \xrightarrow{f} Y$*, for a couple of arrows* $Y \xrightarrow{\alpha} A$ *and* $Y \xrightarrow{\beta} B$*, as*

$$F(f) \colon \mathrm{Hom}_{\mathsf{C}}(Y, A) \times \mathrm{Hom}_{\mathsf{C}}(Y, B) \to \mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B) \tag{2.66}$$

$$(\alpha, \beta) \mapsto (\alpha \circ f, \beta \circ f) \,. \tag{2.67}$$

A product $(A \prod B, \pi_A, \pi_B)$ *exists iff the functor* F *is representable. In other words iff* $F \simeq_\eta \mathrm{Hom}_{\mathsf{C}}(-, P)$ *for some* $P \in \mathrm{Ob}(\mathsf{C})$*. In this case* (P, π_A, π_B) *is a product of* A *and* B*, where* (π_A, π_B) *are given by*

$$\eta_P \colon \mathrm{Hom}_{\mathsf{C}}(P, P) \to F(P) = \mathrm{Hom}_{\mathsf{C}}(P, A) \times \mathrm{Hom}_{\mathsf{C}}(P, B) \tag{2.68}$$

$$1_P \mapsto (\pi_A, \pi_B) \,. \tag{2.69}$$

Example

- $\mathsf{C} = \mathsf{Sets}$, then $A \prod B = A \times B$ is the cartesian product of sets, with π_A and π_B the projections.
- $\mathsf{C} = \mathsf{Mod}\text{-}R$, then $A \prod B = A \times B$ is the set theoretic cartesian product, with componentwise operations. The projections are the set-theoretic projections.
- $\mathsf{C} = \mathsf{Rings}$, as above, $A \prod B = A \times B$ is the set theoretic cartesian product, with componentwise operations. The projections are the set-theoretic projections.

Definition 2.49: Coproduct.
Let $A, B \in \mathrm{Ob}(\mathsf{C})$ for an arbitrary category C. A *coproduct* of A and B, if it exists, is a triple $(A \coprod B, \epsilon_A, \epsilon_B)$, where $A \coprod B \in \mathrm{Ob}(\mathsf{C})$ and the morphisms ϵ_A and ϵ_B, called *embeddings*,

$$A \xrightarrow{\epsilon_A} A \coprod B \quad \text{and} \quad B \xrightarrow{\epsilon_B} A \coprod B \tag{2.70}$$

satisfy the universal property: Given an arbitrary (X, α, β), with $X \in \mathrm{Ob}(\mathsf{C})$, $A \xrightarrow{\alpha} X$ and $B \xrightarrow{\beta} X$ a pair of morphism, there exists a unique morphism $A \coprod B \xrightarrow{\exists ! h} X$ s.t.

$$\begin{tikzcd}
 & A \coprod B \arrow[dd, dashed, "\exists ! h"] & \\
A \arrow[ur, "\epsilon_A"] \arrow[dr, "\alpha"'] & & B \arrow[ul, "\epsilon_B"'] \arrow[dl, "\beta"] \\
 & X &
\end{tikzcd} \tag{2.71}$$

the above diagram commutes. In other words, s.t. $h \circ \epsilon_A = \alpha$ and $h \circ \epsilon_B = \beta$.

Remark 2.50 A coproduct is a product in C^{op}. Moreover, if it exists, then it is unique up to a unique isomorphism.

Proposition 2.51. *Define the functor*

$$F := \mathrm{Hom}_{\mathsf{C}}(A, -) \times \mathrm{Hom}_{\mathsf{C}}(B, -) : \mathsf{C} \to \mathsf{Sets} \tag{2.72}$$

on objects as $F(X) := \mathrm{Hom}_{\mathsf{C}}(A, X) \times \mathrm{Hom}_{\mathsf{C}}(B, X)$, *and on morphisms* $X \xrightarrow{f} Y$, *for a couple of arrows* $Y \xrightarrow{\alpha} A$ *and* $Y \xrightarrow{\beta} B$, *as*

$$F(f) \colon \mathrm{Hom}_{\mathsf{C}}(A, X) \times \mathrm{Hom}_{\mathsf{C}}(A, X) \to \mathrm{Hom}_{\mathsf{C}}(A, Y) \times \mathrm{Hom}_{\mathsf{C}}(A, Y) \tag{2.73}$$
$$(\alpha, \beta) \mapsto (f \circ \alpha, f \circ \beta)\,. \tag{2.74}$$

A coproduct $(A \coprod B, \epsilon_A, \epsilon_B)$ *exists iff the functor* F *is corepresentable. In other words iff* $F \simeq_\eta \mathrm{Hom}_{\mathsf{C}}(C, -)$ *for some* $C \in \mathrm{Ob}(\mathsf{C})$. *In this case* $(C, \epsilon_A, \epsilon_B)$ *is a coproduct of* A *and* B, *where* (ϵ_A, ϵ_B) *are given by*

$$\eta_C \colon \mathrm{Hom}_{\mathsf{C}}(C, C) \to F(C) = \mathrm{Hom}_{\mathsf{C}}(A, C) \times \mathrm{Hom}_{\mathsf{C}}(B, C) \tag{2.75}$$
$$1_C \mapsto (\epsilon_A, \epsilon_B)\,. \tag{2.76}$$

Example

- Let $\mathsf{C} = \mathsf{Sets}$, then $A \coprod B = A \sqcup B$, the disjoint union, with embeddings given by the inclusions.
- Let $\mathsf{C} = R$-Mod, then ${}_RM \coprod {}_RN = (M \times N, \epsilon_M, \epsilon_N)$, set-theoretically is the cartesian product, with componentwise operations and inclusions.
- Let $\mathsf{C} = \mathsf{CRings}$ the category of commutative rings. Then $R \coprod S = (R \otimes_{\mathbb{Z}} S, \epsilon_R, \epsilon_S)$ the coproduct of two commutative rings is given by their tensor product over $\mathbb{Z}$.

Definition 2.52: Additive category.
Let C be a preadditive category with 0 object. C is said *additive* iff, given any pair (or finite family) of objects in C, their product exists in C.

Proposition 2.53. *Let* C *be a preadditive category with* 0 *object. If product exist in* C*, then coproduct exist and they are isomorphic. In particular we have the following for embeddings and projections:*

$$\epsilon_A = \begin{bmatrix} 1_A \\ 0 \end{bmatrix}, \quad \pi_A = \begin{bmatrix} 1_A & 0 \end{bmatrix}, \quad \epsilon_B = \begin{bmatrix} 0 \\ 1_B \end{bmatrix}, \quad \pi_B = \begin{bmatrix} 0 & 1_B \end{bmatrix}. \tag{2.77}$$

This implies that these morphisms compose as

$$\pi_A \circ \epsilon_A = 1_A, \quad \pi_A \circ \epsilon_B = 0, \quad \pi_B \circ \epsilon_A = 0, \quad \pi_B \circ \epsilon_B = 1_B. \tag{2.78}$$

Definition 2.54: (Co)product in preadditive categories.
If C is a preadditive category, and the (co)product between $A, B \in \mathrm{Ob}(\mathsf{C})$ exists in C, they are denoted with

$$A \oplus B. \tag{2.79}$$

Proposition 2.55. *Let* C *be an* additive *category with* 0*. Let* $A, B \in \mathrm{Ob}(\mathsf{C})$*. The structure of abelian group of* $\mathrm{Hom}_{\mathsf{C}}(A, B)$ *is determined by* C*.*

2.6 Infinite product and coproduct

Definition 2.56: (Co)product of an arbitrary family of objects.
Let $\{A_i\}_{i \in I} \subset \mathrm{Ob}(\mathsf{C})$ an arbitrary family of objects in the category C.

- A *product* of the A_is is the couple $(\prod_i A_i, (\pi_i)_{i \in I})$, with $\prod_i A_i \in \mathrm{Ob}(\mathsf{C})$, and morphisms $\pi_i \colon \prod_j A_j \to A_i$ for any $i \in I$, satisfying the universal property: Given $X \in \mathrm{Ob}(\mathsf{C})$ and a family of morphisms $X \xrightarrow{\alpha_i} A_i$, then $\exists\,!\,\alpha \colon X \to \prod_i A_i$ s.t. $\pi_i \circ \alpha = \alpha_i$ for all i.
- A *coproduct* of the A_is is the couple $(\coprod_i A_i, (\epsilon_i)_{i \in I})$, with $\coprod_i A_i \in \mathrm{Ob}(\mathsf{C})$, and morphisms $\epsilon_i \colon A_i \to \coprod_j A_j$ for any $i \in I$, satisfying the universal property: Given $X \in \mathrm{Ob}(\mathsf{C})$ and a family of morphisms $A_i \xrightarrow{\alpha_i} X$, then $\exists\,!\,\alpha \colon \coprod_i A_i \to X$ s.t. $\alpha \circ \epsilon_i = \alpha_i$ for all i. In other words it is a product in C^{op}.

Example

- Let $\mathsf{C} = \mathsf{Sets}$ and $\{A_i\}_{i \in I} \subset \mathrm{Ob}(\mathsf{C})$. The set $\prod_{i \in I} A_i$ (the infinite cartesian product), with usual projections, is a product in Sets,
- Analogously, $\sqcup_{i \in I} A_i$ (the disjoint union), with the usual embeddings, is a coproduct in Sets.
- Let $\mathsf{C} = \mathsf{Mod}\text{-}R$ and $\{M_i\}_{i \in I} \subset \mathrm{Ob}(\mathsf{C})$. The set

  $$\prod_{i \in I} M_i \coloneqq \left\{ (x_i)_{i \in I} \mid x_i \in M_i \,\forall\, i \in I \right\}. \tag{2.80}$$

  (the infinite cartesian product), with componentwise operations and usual projections, is a product in $\mathsf{Mod}\text{-}R$. Clearly, given a family of morphisms $\alpha_i \colon X \to M_i$, one defines

  $$\alpha \colon X \to \prod_{i \in I} M_i \tag{2.81}$$

  $$x \mapsto (a_i(x))_{i \in I} \tag{2.82}$$

  and easily checks the universal properties of products.

- Analogously the copruduct exists and is defined as follows

$$\coprod_{i\in I} M_i = \left\{(x_i)_{i\in I} \,\middle|\, x_i \in M_i\, \forall\, i\in I \text{ and } x_i = 0 \text{ for almost all } i\right\} \le \prod_{i\in I} M_i. \quad (2.83)$$

with the embeddings

$$\epsilon_i \colon M_i \to \coprod_{i\in I} M_i \quad (2.84)$$

$$x \mapsto (\ldots, 0, x, 0, \ldots)\,, \quad (2.85)$$

with nonzero entry only for the i-th component, is a coproduct in $\mathsf{Mod}\text{-}R$. In fact, given a family of morphisms $\alpha_i \colon M_i \to X$, the unique morphism is defined as

$$\exists!\, \alpha\colon \coprod_{i\in I} M_i \to X \quad (2.86)$$

$$(x_i)_{i\in I} \mapsto \sum_{i\in I} \alpha_i(x_i). \quad (2.87)$$

It is important to remark that the sum makes sense, since $x_i \neq 0$ only for finitely many $i \in I$, hence it is a finite sum.

Proposition 2.57. *Let C be an arbitrary category. let $\{A_i\}_{i\in I} \subset \mathrm{Ob}(\mathsf{C})$ be an arbitrary family of objects. Assume that a product $\left(\prod_{i\in I} A_i, \pi_i\right)$ exists in C, then given $X \in \mathrm{Ob}(\mathsf{C})$, the map*

$$\mathrm{Hom}_{\mathsf{C}}\left(X, \prod_{i\in I} A_i\right) \xrightarrow{\phi_X} \prod_{i\in I} \mathrm{Hom}_{\mathsf{C}}(X, A_i) \quad (2.88)$$

$$f \mapsto (\pi_i \circ f)_{i\in I} \quad (2.89)$$

is an isomorphism in Sets (by U.P.). Moreover the family $\{\phi_X\}_{X\in \mathrm{Ob}(\mathsf{C})}$ gives a natural isomorphism between the functors

$$F \coloneqq \mathrm{Hom}_{\mathsf{C}}\left(-, \prod_{i\in I} A_i\right) \quad \textit{and} \quad G \coloneqq \prod_{i\in I} \mathrm{Hom}_{\mathsf{C}}(-, A_i)\,, \quad (2.90)$$

where G, on morphisms acts as: $G(f) = \prod_{i\in I} \mathrm{Hom}_{\mathsf{C}}(f, A_i)$.

Proposition 2.58. *Let C be an arbitrary category. let $\{A_i\}_{i\in I} \subset \mathrm{Ob}(\mathsf{C})$ be an arbitrary family of objects. Assume that a coproduct $\left(\coprod_{i\in I} A_i, \epsilon_i\right)$ exists in C, then given $X \in \mathrm{Ob}(\mathsf{C})$, the map*

$$\mathrm{Hom}_{\mathsf{C}}\left(\coprod_{i\in I} A_i, X\right) \xrightarrow{\psi_X} \prod_{i\in I} \mathrm{Hom}_{\mathsf{C}}(A_i, X) \quad (2.91)$$

$$f \mapsto (f \circ \epsilon_i)_{i\in I} \quad (2.92)$$

is an isomorphism in Sets (by U.P.). Moreover the family $\{\psi_X\}_{X\in \mathrm{Ob}(\mathsf{C})}$ gives a natural isomorphism between the functors

$$F \coloneqq \mathrm{Hom}_{\mathsf{C}}\left(\coprod_{i\in I} A_i, -\right) \quad \textit{and} \quad G \coloneqq \prod_{i\in I} \mathrm{Hom}_{\mathsf{C}}(A_i, -)\,, \quad (2.93)$$

where G, on morphisms acts as: $G(f) = \prod_{i\in I} \mathrm{Hom}_{\mathsf{C}}(A_i, f)$.

Remark 2.59 Notice that, if C is preadditive with 0, then ϕ_X and ψ_X are both isomorphisms of abelian groups. In particular $\{\phi_X\}_{X\in\mathrm{Ob}(\mathsf{C})}$ and $\{\psi_X\}_{X\in\mathrm{Ob}(\mathsf{C})}$ are both natural isomorphisms of functors with values in Ab.

Proposition 2.60. *Let C be an arbitrary category. Let $\{A_i\}_{i\in I}\subset \mathrm{Ob}(\mathsf{C})$, $\{B_i\}_{i\in I}\subset \mathrm{Ob}(\mathsf{C})$. Let $\{\alpha_i\}_{i\in I}$ a family of morphisms s.t. for each i $\alpha_i\colon A_i\to B_i$. Assume that both products $\left(\prod_{i\in I}A_i,\pi_i\right)$ and $\left(\prod_{i\in I}B_i,p_i\right)$ exist in C. Then*

$$\exists\,!\,\alpha\colon \prod_{i\in I}A_i\to\prod_{i\in I}B_i \tag{2.94}$$

s.t. $p_i\circ\alpha=\alpha_i\circ\pi_i$. Moreover if, for all i, the morphism α_i is a monomorphism, then also α is a monomorphism.

Proposition 2.61. *Let C be an arbitrary category. Let $\{A_i\}_{i\in I}\subset \mathrm{Ob}(\mathsf{C})$, $\{B_i\}_{i\in I}\subset \mathrm{Ob}(\mathsf{C})$. Let $\{\alpha_i\}_{i\in I}$ a family of morphisms s.t. for each i $\alpha_i\colon A_i\to B_i$. Assume that both coproducts $\left(\coprod_{i\in I}A_i,\epsilon_i\right)$ and $\left(\coprod_{i\in I}B_i,\delta_i\right)$ exist in C. Then*

$$\exists\,!\,\alpha\colon \coprod_{i\in I}A_i\to\coprod_{i\in I}B_i \tag{2.95}$$

s.t. $\alpha\circ\epsilon_i=\delta_i\circ\alpha_i$. Moreover if, for all i, the morphism α_i is am epimorphism, then also α is a epimorphism.

Proposition 2.62. *Let C be an arbitrary category. Consider an arbitrary family $\{A_i\}_{i\in I}\subset\mathrm{Ob}(\mathsf{C})$ s.t. the product $\left(\prod_{i\in I}A_i,\pi_i\right)$ (resp. the coproduct $\left(\coprod_{i\in I}A_i,\epsilon_i\right)$) exists in C. Assume, moreover, that $\mathrm{Hom}_{\mathsf{C}}(A_i,A_j)\neq\emptyset$ for $i\neq j\in I$. It follows that π_i (resp. ϵ_i) is an epimorphism (resp. monomorphism) for all $i\in I$.*

Corollary 2.63. *In particular, if C is preadditive with 0 object, then every $\mathrm{Hom}_{\mathsf{C}}(X,Y)\neq\emptyset$. This means that π_i and ϵ_i in the above proposition are always respectively epi and mono. In particular, given $A,B\in\mathrm{Ob}(\mathsf{C})$, then $\pi_A\colon A\prod B\to A$ and $\pi_B\colon A\prod B\to B$ are epi, whereas $\epsilon_A\colon A\to A\prod B$ and $\epsilon_B\colon B\to A\prod B$ are mono.*

3 Abelian categories

Lemma 3.1 (Parallel morphism)**.** *Let C be a preadditive category with 0 object. Assume that every morphism in C admits kernel and cokernel, then*

$$\begin{tikzcd}
\ker f \arrow[r, "\epsilon"] & A \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, dashed, "\beta"] & B \arrow[r, "\pi"] & \operatorname{coker} f \\
& \operatorname{coker}\epsilon \arrow[r, dashed, "\widetilde{f}"'] & \ker\pi \arrow[u, "\mu"'] &
\end{tikzcd} \tag{3.1}$$

$\exists\,!\,\widetilde{f}\colon \operatorname{coker}\epsilon\to\ker\pi$ *s.t.* $\widetilde{f}\circ p=\beta$. $\widetilde{f}$ *is called* parallel morphism *of f.*

Example Let $\mathsf{C}=\mathsf{Mod}\text{-}R$ and $A\xrightarrow{f}B$. Then $\operatorname{coker}(\ker f)=A/\ker f$ ad $\ker(\operatorname{coker} f)\simeq \operatorname{im} f$. By the first isomorphism theorem we have

$$A/\ker f\simeq_{\widetilde{f}}\operatorname{im} f. \tag{3.2}$$

Definition 3.2: Some notation.
We denote the above objects as

$$\operatorname{coim} f := \operatorname{coker}(\ker f) \tag{3.3}$$
$$\operatorname{Im} f := \ker(\operatorname{coker} f)\,. \tag{3.4}$$

Definition 3.3: Abelian category.
A category C is said *abelian* iff it is additive and

1. every morphism has both kernel and cokernel,
2. the parallel morphism $\widetilde{f}$ of f is an isomorphism for any f.

The second condition is equivalent to the following

2'. Every morphism f in C factors as $\nu\beta$ with β a cokernel and ν a kernel.

Lemma 3.4. *Let $f = \nu\beta$ in C a preadditive category with 0 object.*

1. *If ν is a mono, then $\ker f = \ker \beta$, if they exist.*
2. *If β is epi, then $\operatorname{coker} f = \operatorname{coker} \nu$, if they exist.*

Lemma 3.5. *Assume that C is an abelian category. Let $A \xrightarrow{f} B$ be a morphism in C, then*

1. *If f is mono and epi, then f is iso.*
2. *If f is mono, then $f = \ker(\operatorname{coker} f)$.*
3. *If f is epi, then $f = \operatorname{coker}(\ker f)$.*

Example Let $\mathsf{C} = \mathsf{Ab}$. We say that $G \in \mathsf{Ab}$ is torsion free iff $\forall\, 0 \neq x \in G$, for all $n \in \mathbb{Z}$ s.t. $nx = 0$, then $n = 0$. Instead G is torsion iff $\forall\, x \in G$ there exists $0 \neq n \in \mathbb{Z}$ s.t. $nx = 0$. Given $G \in \mathsf{Ab}$, we denote by $t(G) \leq G$ the torsion subgroup of G, i.e.

$$t(G) := \{x \in G \mid \exists\, 0 \neq n \in \mathbb{Z} \text{ s.t. } nx = 0\}\,. \tag{3.5}$$

Clearly, then, $G/t(G)$ is a torsion free group.

Let's now see a few examples of abelian categories:

- Let $\mathsf{C} = \mathsf{Mod}\text{-}R$ the category of abelian groups. C is *abelian*: consider $A_R \xrightarrow{f} B_R$, then

  $$\operatorname{coker}(\ker f) \simeq A/\ker f \quad \text{and} \quad \ker(\operatorname{coker} f) \simeq \operatorname{im} f. \tag{3.6}$$

  From the first isomorphism theorem we obtain an isomorphism of the two. Then one can show that this category is abelian.
- Let $\mathsf{T} \subset \mathsf{Ab}$ the full subcategory of abelian groups consisting of *torsion* abelian groups, then T is abelian. This is the case, since $\ker$ and coker in T correspond to the notions in Ab, which is abelian.

The following, instead, are additive, with kernels and cokernels, but not abelian:

- Let $\mathsf{F} \subset \mathsf{Ab}$ be the full subcategory consisting of the torsion free abelian groups. Clearly F is closed under subgroups. Let $A \xrightarrow{f} B$ a morphism in F. Let $K \xrightarrow{\epsilon} A$ a kernel of f in Ab, clearly $K \hookrightarrow A$, hence $K \in \mathrm{Ob}\,(\mathsf{F})$ and f admits kernel in F. Let (C, π) a cokernel in Ab. It might not be in F. Consider $C/t(C) \in \mathrm{Ob}\,(\mathsf{F})$ and $B \xrightarrow{\pi} C \xrightarrow{q} C/t(C)$, then $q \circ \pi$ is a cokernel of f in F. It follows that f admits also cokernel in F.

  In other words we have just proved that F admits both kernels and cokernels. But F is not abelian. In order to show this we consider

$$\begin{tikzcd} \ker \dot{2} = 0 \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\dot{2}"] \arrow[d, "1_{\mathbb{Z}}"'] & \mathbb{Z} \arrow[r, "0"] & 0 = \operatorname{coker} \dot{2} \\ & \mathbb{Z} \arrow[r, "\widetilde{\dot{2}}"] & \mathbb{Z} \arrow[u, "1_{\mathbb{Z}}"'] & \end{tikzcd} , \tag{3.7}$$

  where $\dot{2} \colon \mathbb{Z} \to \mathbb{Z}$ is the multiplication by 2. In F we have $\operatorname{coker} \dot{2} = 0$, since in Ab $\operatorname{coker} \dot{2} = \mathbb{Z}/2\mathbb{Z}$, which is torsion. Then, in this example, $\widetilde{f} = \widetilde{\dot{2}}$, which is not an isomorphism in F (nor in Ab, and F is a full subcategory of Ab). Also note that $\dot{2}$ is both mono and epi in F, but not an iso.

- Let $G \in \mathrm{Ob}\,(\mathsf{Ab})$ an abelian group. We say that G is *divisible* iff $\forall\, x \in G$ and $\forall\, 0 \neq n \in \mathbb{Z}$, $\exists\, y \in G$ s.t. $ny = x$. Instead an abelian group is called *reduced* iff it has no nonzero divisible subgroups.

  Let $\mathsf{D} \subset \mathsf{Ab}$ the full subcategory consisting of divisible abelian groups. Then D has kernels and cokernels, it is also additive, but not abelian.

I'm actually not sure whether the following definition is correct, but I cannot find it on the internet and I really didn't understand what was part of the definition during the lecture.

Definition 3.6: Torsion pair of full subcategories.
Let C be an abelian category, and $\mathsf{D} \subset \mathsf{C} \supset \mathsf{E}$ be two full subcategories. We say that the pair (D, E) is a *torsion pair* iff given any $D \in \mathrm{Ob}\,(\mathsf{D})$ and $E \in \mathrm{Ob}\,(\mathsf{E})$ we have

$$\mathrm{Hom}_{\mathsf{C}}\,(D, E) = 0. \tag{3.8}$$

Example

- Consider the category Ab of abelian groups and $\mathsf{T} \subset \mathsf{Ab}$ the full subcategory of torsion abelian groups and $\mathsf{F} \subset \mathsf{Ab}$ the full subcategory of torsion-free abelian groups. The pair (T, F) is a torsion pair, in fact, for any $T \in \mathrm{Ob}\,(\mathsf{T})$ and $F \in \mathrm{Ob}\,(\mathsf{F})$, we have

$$\mathrm{Hom}_{\mathsf{Ab}}\,(T, F) = 0. \tag{3.9}$$

- Consider the full subcategories $\mathsf{D} \subset \mathsf{Ab}$ of all divisible groups and $\mathsf{R} \subset \mathsf{Ab}$ of all reduced groups. The pair (D, R) is torsion, in fact, for any $D \in \mathrm{Ob}\,(\mathsf{D})$ and $R \in \mathrm{Ob}\,(\mathsf{R})$, we have

$$\mathrm{Hom}_{\mathsf{Ab}}\,(D, R) = 0. \tag{3.10}$$

3.1 Pullback and Pushout

Definition 3.7: Pullback.
Let C be an arbitrary category. Let $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be morphisms in C. A *pullback* of f and g is a triple (P, p_A, p_B), with $P \in \mathrm{Ob}\,(\mathsf{C})$, $p_A \colon P \to A$ and $p_B \colon P \to B$ s.t. the following conditions are satisfied

PB1 The following square is commutative

$$
\begin{tikzcd}
P \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd} . \tag{3.11}
$$

In other words we ask that $f \circ p_A = g \circ p_B$.

PB2 For any pair of morphisms $X \xrightarrow{\alpha} A$ and $X \xrightarrow{\beta} B$, from a fixed $X \in \mathrm{Ob}(\mathsf{C})$, s.t. $f \circ \alpha = g \circ \beta$ then $\exists\,!\, X \xrightarrow{\gamma} P$ s.t. the following diagram commutes

$$
\begin{tikzcd}
X \arrow[dr, "\exists\,!\,\gamma"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& P \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd} . \tag{3.12}
$$

In other words, s.t. $p_B \circ \gamma = \beta$ and $p_A \circ \gamma = \alpha$.

Remark 3.8 Notice that *PB2* is a universal property. This means that, if a *pullback* of f and g exists, then it is unique up to a unique isomorphism.

Example Let C be a preadditive category with 0 object.

- Consider $A \xrightarrow{f} C$ and $0 \xrightarrow{0} C$. A pullback of f and 0 exists iff $\ker f$ exists in C. In particular (P, p_A) is a kernel of f.
- Consider $A \xrightarrow{0} 0$ and $B \xrightarrow{0} 0$. The pullback of 0 and 0 exists iff the product of A and B exists, then the triple (P, p_A, p_B) is a product of A and B:

$$
\begin{tikzcd}
P \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "0"] \\
B \arrow[r, "0"] & 0
\end{tikzcd} . \tag{3.13}
$$

Proposition 3.9. *Let C be a preadditive category with 0 object. If C admits kernel and finite products, then C has pullbacks. Moreover these are constructed by means of products and kernels.*

Proof. The construction via kernels and products goes as follows: Consider the morphisms $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$. Let $(A \prod B, \pi_A, \pi_B)$ be a product. Let $\mu := f \circ \pi_A - g \circ \pi_B \colon A \prod B \to C$. Finally, consider (K, ϵ) a kernel of μ. Then (K, p_A, p_B), with $p_A := \pi_A \circ \epsilon$ and $p_B = \pi_B \circ \epsilon$, is a pullback of f and g. The corresponding diagram is

$$
\begin{tikzcd}
K \arrow[dr, "\epsilon"] \arrow[drr, bend left, "p_A"] \arrow[ddr, bend right, "p_B"'] & & \\
& B \prod A \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] \arrow[dr, "\mu"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd} . \tag{3.14}
$$

■

Example Consider an abelian category C, for example the category $\mathsf{Mod}\text{-}R$. Take two morphisms $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$, then the pullback of f and g is a submodule $P \leq A \oplus B$, in particular it is

$$
\begin{aligned}
P &= \{(a,b) \in A \oplus B \mid \mu(a,b) = 0\} && (3.15)\\
&= \{(a,b) \in A \oplus B \mid f(a) = g(b)\}\,. && (3.16)
\end{aligned}
$$

Proposition 3.10. *Let C be preadditive with 0 object. Let*

$$
\begin{tikzcd}
P \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}
\tag{3.17}
$$

be a pullback diagram, then:

- *If g (resp. f) is mono, then p_A (resp. p_B) [the parallel arrow] is mono.*
- *If C is abelian and g (resp. f) is epi, then p_A (resp. p_B) is epi.*
- *If g (resp. f) is a kernel of h, then p_A (resp. p_B) is a kernel of $h \circ f$ (resp. $h \circ g$).*

Example: An application of the above result. Let C be an abelian category. Consider the following pullback diagram of the morphisms f and g

$$
\begin{tikzcd}
P \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}\,.
\tag{3.18}
$$

Assume that g is epi. Take (K, ϵ) a kernel of g, then $\exists!\delta\colon K \to P$ s.t. the following diagram commutes

$$
\begin{tikzcd}
K \arrow[r, tail, "\delta"] \arrow[d, equal, "1_K"'] & P \arrow[r, two heads, "p_A"] \arrow[d, "p_B"] & A \arrow[d, "f"] \\
K \arrow[r, tail, "\epsilon"'] & B \arrow[r, two heads, "g"'] & C
\end{tikzcd}\,.
\tag{3.19}
$$

Moreover δ is a kernel of p_A (hence it is a monomorphism).

Definition 3.11: Pushout.
Let C be an arbitrary category. A *pushout* of morphisms $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$ in C is a *pullback* in C^{op}. This means that we can dualize every result for the pullback.

More explicitly, a pushout is a triple (P, ν_A, ν_B), with $P \in \mathrm{Ob}(\mathsf{C})$, $\nu_A\colon A \to P$, and $\nu_B\colon B \to P$ morphisms s.t. the following conditions are satisfied

PO1 The following square is commutative

$$
\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\nu_A"] \\
B \arrow[r, "\nu_B"'] & P
\end{tikzcd}\,.
\tag{3.20}
$$

In other words we ask that $\nu_A \circ f = \nu_B \circ g$.

PO2 For any pair of morphisms $A \xrightarrow{\alpha} X$ and $B \xrightarrow{\beta} X$, into a fixed $X \in \mathrm{Ob}(\mathsf{C})$, s.t. $\alpha \circ f = \beta \circ g$, then $\exists !\, P \xrightarrow{\gamma} X$ s.t. the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\nu_A"] \arrow[ddr, bend left, "\alpha"] & \\ B \arrow[r, "\nu_B"'] \arrow[drr, bend right, "\beta"'] & P \arrow[dr, dashed, "\exists !\gamma"] & \\ & & X \end{tikzcd} . \tag{3.21}$$

In other words, s.t. $\gamma \circ \nu_A = \alpha$ and $\gamma \circ \nu_B = \beta$.

Example Let C be a preadditive category with 0 object.

- Consider $C \xrightarrow{f} A$ and $C \xrightarrow{0} 0$. A pushout of f and 0 exists iff $\operatorname{coker} f$ exists in C. In partcular (P, ν_A) is a cokernel of f.
- Consider $0 \xrightarrow{0} A$ and $0 \xrightarrow{0} B$. The pushout diagram of 0 and 0 exists iff the coproduct of A and B exists. Then the triple (P, ν_A, ν_B) is a coproduct of A and B:

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & A \arrow[d, "\nu_A"] \\ B \arrow[r, "\nu_B"'] & P \end{tikzcd} . \tag{3.22}$$

Proposition 3.12. *Let C be a preadditive category with 0 object. If C admits cokernels and finite coproducts, then C has pushouts. Moreover these are constructed by means of coproducts and cokernels.*

Proof. The construction goes as follows: Consider the morphisms $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$. Let $(A \coprod B, \epsilon_A, \epsilon_B)$ be a coproduct. Let $\delta := \epsilon_A \circ f - \epsilon_B \circ g : C \to A \coprod B$. Finally consider (P, p) a cokernel of δ. Then $(P, p \circ \epsilon_A, p \circ \epsilon_B)$ is a pushout of f and g. The corresponding diagram is

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "\delta"] & A \arrow[d, "\epsilon_A"] \arrow[ddr, bend left] & \\ B \arrow[r, "\epsilon_B"'] \arrow[drr, bend right] & A \coprod B \arrow[dr, "p"] & \\ & & P \end{tikzcd} . \tag{3.23}$$

∎

Proposition 3.13. *Let C be preadditive with 0 object. Let*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\nu_A"] \\ B \arrow[r, "\nu_B"'] & P \end{tikzcd} \tag{3.24}$$

be a pushout diagram, then:

- *If f (resp. g) is epi, then ν_B (resp. ν_A) [the parallel arrow] is epi.*

- *If C is abelian and f (resp. g) is mono, then ν_B (resp. ν_A) is mono.*
- *If f (resp. g) is a cokernel of h, then ν_B (resp. ν_A) is a kernel of $g \circ h$ (resp. $f \circ h$).*

Example Let $\mathsf{C} = \mathsf{Mod}\text{-}R$. Consider the morphisms $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$. Then a pushout $P \simeq \frac{A \oplus B}{H}$, where H is the image of δ as defined above, more explicitly

$$H := \langle (f(c), -g(c)) \mid c \in C \rangle \,. \tag{3.25}$$

More explicitly, the image of C in A and B (resp. via f and g) are glued together in P.

Example: An application of the above result. Let C be an abelian category. Consider the following pushout diagram of the morphisms f and g

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\nu_A"] \\ B \arrow[r, "\nu_B"'] & P \end{tikzcd} . \tag{3.26}$$

Assume that f is mono. Take (D, p) a cokernel of f, then $\exists\,!\, q\colon P \to D$ s.t. the following diagram commutes

$$\begin{tikzcd} C \arrow[r, tail, "f"] \arrow[d, "g"'] & A \arrow[r, two heads, "p"] \arrow[d, "\nu_A"] & D \arrow[d, equal] \\ B \arrow[r, tail, "\nu_B"'] & P \arrow[r, two heads, "q"'] & D \end{tikzcd} . \tag{3.27}$$

Moreover q is a cokernel of ν_B (hence it is an epimorphism).

3.2 Exact categories

Remark 3.14 Let C be an abelian category and $A \xrightarrow{i} B \xrightarrow{d} C$ morphisms in C, s.t. $i = \ker d$ and $d = \operatorname{coker} i$. Then i is mono, d is epi and $\ker d = \operatorname{im} i$. In fact

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[rr, "i"] & & B \arrow[r, "d"] & C \\ & \operatorname{coim} i = A \arrow[u, equal, "1_A"] \arrow[rr, "\sim"] & & \ker d = \operatorname{im} i \arrow[u] & \end{tikzcd} . \tag{3.28}$$

Definition 3.15: Kernel-cokernel pair.
Let C be an additive category. A *kernel cokernel pair* (i, d) in C is a pair of composable morphisms

$$A \xrightarrow{i} B \xrightarrow{d} C \tag{3.29}$$

s.t. i is a kernel of d and d is a cokernel of i.

Definition 3.16: Inflation, deflation, conflation.
Let $\mathcal{E}$ be a fixed class of kernel-cokernel pairs in C. A sequence $E = (i, d) \in \mathcal{E}$

$$A \xrightarrow{i} B \xrightarrow{d} C \tag{3.30}$$

is called a *conflation*. A morphism $i\colon A \rightarrowtail B$ s.t. there exists a morphism d with $(i, d) \in \mathcal{E}$ is called *inflation*. A morphism $d\colon B \twoheadrightarrow C$ s.t. there exists a morphism i with $(i, d) \in \mathcal{E}$ is called *deflation*. Sometimes they are called admissible mono and admissible epi.

Definition 3.17: Exact structure.
Given an additive category C, an *exact structure* on C is a class $\mathcal{E}$ of ker-coker pairs satisfying the following axioms and closed under isomorphisms, i.e. given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[r, "d"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "d'"] & C' \end{tikzcd}, \tag{3.31}$$

in which all the vertical arrows are isomorphisms, and $(i, d) \in \mathcal{E}$, then also $(i', d') \in \mathcal{E}$.

Ex0 1_0 is a deflation,

Ex0^{op} 1_0 is an inflation,

Ex1 the class of deflations is closed under compositions,

Ex1^{op} the class of inflations is closed under compositions,

Ex2 the pullback of a deflation along an arbitrary morphism exists and is a deflation,

Ex2^{op} the pushout of an inflation along an arbitrary morphism exists and is an inflation.

These last 2 axioms correspond to the following diagrams

$$\textbf{Ex2}: \begin{tikzcd} Y' \arrow[r, two heads, "d'"] \arrow[d, "f'"'] & Z' \arrow[d, "f"] \\ Y \arrow[r, two heads, "d"'] & Z \end{tikzcd} \qquad \textbf{Ex2}^{op}: \begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ X' \arrow[r, tail, "i'"] & Y' \end{tikzcd}. \tag{3.32}$$

The interpetation is as follows (for the first diagram): given a deflation $d\colon Y \to Z$ and a morphism $f\colon Z' \to Z$, then, if the pullback of (d, f) exists, let's denote it with (Y', f', d'), also $d'\colon Y' \to Z'$ is a deflation. (These axioms are defined to reflect the properties one can find in abelian categories. In fact these last diagrams are a parallel to the last ones of the previous section).

Definition 3.18: Exact category.
An *exact category* is a pair $(\mathsf{C}, \mathcal{E})$, with C an additive category and $\mathcal{E}$ an exact structure on C. Conflations in $\mathcal{E}$ are called *short exact sequences*.

Remark 3.19 $\mathcal{E}$ is an exact structure in C iff $\mathcal{E}^{op}$ is an exact structure in C^{op}.

Remark 3.20 An abelian category C with $\mathcal{E}$ given by all of its ker-coker pairs is an exact category.

Definition 3.21: Extensions closed subcategory.
Given an abelian category C, a full subcategory $\mathsf{C}' \subset \mathsf{C}$ is *extensions closed* iff, given a ker-coker pair $A \xrightarrow{i} B \xrightarrow{d} C$ with $A, C \in \mathrm{Ob}(\mathsf{C}')$, then $B \in \mathrm{Ob}(\mathsf{C}')$

Remark 3.22 An extensions closed subcategory of an abelian category is an exact category, but need not be abelian. In fact

- $\mathsf{F} \subset \mathsf{Ab}$ the full subcategory of torsion free abelian groups,
- $\mathsf{D} \subset \mathsf{Ab}$ the full subcategory of divisible abelian groups,

are both extensions closed in Ab, but are not abelian. For the first, in fact, given

$$A \xrightarrow{i} B \xrightarrow{d} C, \tag{3.33}$$

with $A, C \in \mathrm{Ob}(\mathsf{F})$, then $B/i(A), i(A) \in \mathrm{Ob}(\mathsf{F})$. From this it can be easily proved that also $B \in \mathrm{Ob}(\mathsf{F})$.

The following proposition can be found in the paper *Chain complexes and stable categories*, by B. Keller. Also in the PhD thesis of T. Bühler *Exact categories* (For more precise references see lecture 8-1, minute 20).

Proposition 3.23 (Keller)**.** *The axioms of exact categories are redundant. The following are enough* Ex0, Ex1, Ex2, Ex2^{op}. *They imply:*

a *given $X, Y \in \mathrm{Ob}(\mathsf{C})$, then the following is a conflation*

$$X \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} X \oplus Z \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}} Z. \tag{3.34}$$

b Ex1^{op}.

c Quillen's obscure axioms*: If a morphism d has a kernel and if $d \circ e$ is a deflation for some morphism e, then also d is a deflation.*

c^{op} Quillen's obscure axioms*: If a morphism i has a cokernel and if $k \circ i$ is an inflation for some morphism k, then also i is an inflation.*

4 (Co)limits

We will concentrate on an arbitrary category C, and on a small category I, i.e. a category where $\mathrm{Ob}(\mathsf{I})$ and $\mathrm{Morph}\,\mathsf{I}$, the family of all morphisms in the category I, are sets. Consider a functor

$$F \colon \mathsf{I} \to \mathsf{C}. \tag{4.1}$$

Then $\forall\, i \in \mathrm{Ob}(\mathsf{I})$, $F(i) \in \mathrm{Ob}(\mathsf{C})$ and, given a morphism $\lambda \colon i \to j$ in I, then $F(\lambda) \colon F(i) \to F(j)$.

Definition 4.1: Compatible family with respect to F.
Consider a family $\{\alpha_i\}_{i \in \mathrm{Ob}(\mathsf{I})}$ of morphisms $\alpha_i \colon X \to F(i)$ for a fixed $X \in \mathrm{Ob}(\mathsf{C})$. It is said to be a *compatible family* with respect to F iff given any morphism $\lambda \colon i \to j$ in I, the following triangle commutes

$$\begin{tikzcd} X \arrow[r, "\alpha_i"] \arrow[dr, "\alpha_j"'] & F(i) \arrow[d, "F(\lambda)"] \\ & F(j) \end{tikzcd} . \tag{4.2}$$

In other words iff $\alpha_j = F(\lambda) \circ \alpha_i$ for every $i, j \in \mathrm{Ob}(\mathsf{I})$ and every $\lambda \colon i \to j$.

Definition 4.2: Projective (inverse) limit.
A (projective/inverse) *limit* of F is an object in C, denoted with $\varprojlim F$, with morphisms $p_i \colon \varprojlim F \to F(i)$ for all $i \in \mathrm{Ob}(\mathsf{I})$ satisfying the following conditions

LIM1 $\{p_i\}_{i\in \mathrm{Ob}(\mathsf{I})}$ is a compatible family of morphisms, i.e.

$$\begin{tikzcd} \varprojlim F \arrow[r, "p_i"] \arrow[dr, "p_j"'] & F(i) \arrow[d, "F(\lambda)"] \\ & F(j) \end{tikzcd} \tag{4.3}$$

the above diagram commutes for all $i, j \in \mathrm{Ob}(\mathsf{I})$ and all $\lambda\colon i \to j$.

LIM2 For any $X \in \mathrm{Ob}(\mathsf{C})$ and any compatible family of morphisms $\{\alpha_i\}_{i\in \mathrm{Ob}(\mathsf{I})}$, with $\alpha_i\colon X \to F(i)$, $\exists\,!\,\alpha\colon X \to \varprojlim F$ s.t. $p_i \circ \alpha = \alpha_i \ \forall\, i \in \mathrm{Ob}(\mathsf{I})$, i.e.

$$\begin{tikzcd} X \arrow[r, "\alpha_i"] \arrow[d, "\alpha"'] & F(i) \\ \varprojlim F \arrow[ur, "p_i"'] & \end{tikzcd} . \tag{4.4}$$

Remark 4.3 As always, since it is defined through a universal property, if $(\varprojlim F, p_i)$ exists, it is unique up to unique isomorphism.

Example Let I be a small discrete category, i.e. the morphisms in I are only the identities. Then, for any functor $F\colon \mathsf{I} \to \mathsf{C}$, $\varprojlim F$ exists iff $\prod_{i\in\mathrm{Ob}(\mathsf{I})} F(i)$ exists and they are isomorphic. In particular the p_i s correspond with the projections of the product.

Example Consider, in an arbitrary category C, the following diagram

$$\begin{tikzcd} & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & A_3 \end{tikzcd} . \tag{4.5}$$

Consider the category I with $\mathrm{Ob}(\mathsf{I}) = \{1, 2, 3\}$ and morphism (other than the identities) $1 \to 3$ and $2 \to 3$. Consider the functor $F\colon \mathsf{I} \to \mathsf{C}$ defined as follows:

$$F(i) = A_i, \quad F(1 \to 3) = f_1, \quad F(2 \to 3) = f_2,. \tag{4.6}$$

Then any $\varprojlim F$ corresponds to a pullback of the above diagram.

Definition 4.4: Complete category.
A category C is called *complete* iff every functor $F\colon \mathsf{I} \to \mathsf{C}$, from a small category I, admits limit in C.

Remark 4.5: Some terminology.
Assume that a preadditive category C has arbitrary products. Consider a functor $F\colon \mathsf{I} \to \mathsf{C}$, from a small category I. For any morphism $\lambda\colon i \to j$ in I, let's define

$$s(\lambda) := i \quad \text{and} \quad t(\lambda) := j, \tag{4.7}$$

where s denotes the source and t the target of the morphism. Consider a product $\left(\prod_{i\in\mathrm{Ob}(\mathsf{I})} F(i), \pi_i\right)$ of $\{F(i)\}_{i\in I}$, and the diagram

$$\begin{tikzcd} \prod_{i\in\mathrm{Ob}(\mathsf{I})} F(i) \arrow[r, "\pi_{s(\lambda)}"] \arrow[dr, "\pi_{t(\lambda)}"'] & F(s(\lambda)) \arrow[d, "F(\lambda)"] \\ & F(t(\lambda)) \end{tikzcd} . \tag{4.8}$$

In general it is not commutative, but we can define the morphism

$$\sigma_\lambda := F(\lambda) \circ \pi_{s(\lambda)} - \pi_{t(\lambda)} \colon \prod_{i \in \mathrm{Ob}(\mathsf{I})} F(i) \to F\left(t(\lambda)\right). \tag{4.9}$$

Let's now consider the product $\left(\prod_{\lambda \in \Lambda} F\left(t(\lambda)\right), q_\lambda\right)$, indexed by $\lambda \in \Lambda := \mathrm{Morph}\,\mathsf{I}$. By the universal property of products, the family $\{\sigma_\lambda\}_{\lambda \in \Lambda}$ induces a unique morphism

$$\sigma \colon \prod_{i \in \mathrm{Ob}(\mathsf{I})} F(i) \to \prod_{\lambda \in \Lambda} F\left(t(\lambda)\right) \tag{4.10}$$

s.t. $q_\lambda \circ \sigma = \sigma_\lambda$.

Proposition 4.6. *If a (preadditive) category* C *admits kernels and (arbitrary) products, then for every functor* $F \colon \mathsf{I} \to \mathsf{C}$*, from a small category* I*,* C *admits* $\varprojlim F$*. Moreover the limit is constructed by means of kernels and products.*

Proof. The proof wants to show that the following construction actually is a limit for F. Consider (K, ϵ) a kernel for the morphism constructed above

$$\sigma \colon \prod_{i \in \mathrm{Ob}(\mathsf{I})} F(i) \to \prod_{\lambda \in \Lambda} F\left(t(\lambda)\right). \tag{4.11}$$

Then (K, p_i), for $p_i := \pi_i \circ \epsilon$ is a projective limit of F. ■

Example Let $\mathsf{C} = \mathsf{Mod}\text{-}R$ and $(\mathsf{I}, \leq)$ a partially ordered set, viewed as a category. Consider a contravariant functor

$$F \colon \mathsf{I}^{op} \to \mathsf{Mod}\text{-}R. \tag{4.12}$$

This is equivalent to the data of $F(i) =: M_i \in \mathsf{Mod}\text{-}R$, and, for all $i \leq j$ of

$$F(i \to j) =: f_{ij} \colon M_j \to M_i. \tag{4.13}$$

Now, given $i \leq j \leq k$, then the following diagram commutes

$$\begin{tikzcd} M_k \arrow[r, "f_{jk}"] \arrow[dr, "f_{ik}"'] & M_j \arrow[d, "f_{ij}"] \\ & M_i \end{tikzcd}, \tag{4.14}$$

in other words $f_{ij} \circ f_{jk} = f_{ik}$. Moreover we require $f_{ii} = id_{M_i}$.

We have, in fact, a correspondence, between contravariant functors from partially ordered sets and inverse systems of modules, which are families $\{M_i, F_{ij}\}_{i \leq j}$ of modules M_i and morphism f_{ij} between them, satisfying the above compatibility conditions.

Then, $\varprojlim F$ is called the *inverse limit* of $\{M_i, f_{ij}\}_{i \leq j}$. The morphisms f_{ij} are called the structural morphisms of the inverse system. Sometimes this is also denoted with $\varprojlim M_i$.

Let's describe $\varprojlim M_i$ explicitly: for every $i \leq j$ we have the (not necessarily commutative) diagram

$$\begin{tikzcd} \prod_{i \in \mathrm{Ob}(\mathsf{I})} M_i \arrow[r, "\pi_j"] \arrow[dr, "\pi_i"'] & M_j \arrow[d, "f_{ij}"] \\ & M_i \end{tikzcd}. \tag{4.15}$$

Let's define, for each $i \leq j$, $\sigma_{ij} := f_{ij} \circ \pi_j - \pi_i$. Then, by universal property of the product, $\exists ! \, \sigma \colon \prod_{i \in \mathrm{Ob}(\mathsf{I})} M_i \to \prod_{i \leq j} M_{ij}$, where $M_{ij} := M_i$ for every $i \leq j$. In the above construction $\pi_i \circ \sigma = \sigma_{ij} \colon \prod_{i \in \mathrm{Ob}(\mathsf{I})} M_i \to M_{ij}$. Then we have

$$\varprojlim M_i \simeq \ker \sigma = \left\{ \mathbf{x} \in \prod_{i \in \mathrm{Ob}(\mathsf{I})} M_i \,\middle|\, \sigma(\mathbf{x}) = 0 \text{ i.e. } \sigma_{ij}(\mathbf{x}) = 0 \, \forall \, i \leq j \right\} \tag{4.16}$$

$$= \left\{ \mathbf{x} = (x_i)_{i \in \mathrm{Ob}(\mathsf{I})} \in \prod_{i \in \mathrm{Ob}(\mathsf{I})} M_i \,\middle|\, f_{ij}(x_j) = x_i, \ \forall \, i \leq j \right\}. \tag{4.17}$$

It is a submodule of the product, in which, chosen x_j, then $\forall \, i \leq j$, x_i is determined by x_j, via the structural morphisms.

Definition 4.7: I-adic topology on a ring.
Given a commutative ring R and an ideal $I \triangleleft R$ of R. We define the **I**-*adic topology on* R as the linear topology determined by $\{I^n\}_{n \in \mathbb{N}}$ as a basis for the neighbourhoods of 0. The open subsets are generated by cosets of these ideals.

Remark 4.8 The **I**-*adic topology* on R is Hausdorff iff $\bigcap_{n \in \mathbb{N}} I^n = 0$.

Example: Completion of a ring in the I-adic topology. Let R be a commutative ring and $I \triangleleft R$ an ideal of R. For $n \leq m$, then $I^m \subset I^n$, hence the canonical projections

$$\pi_{n,m} \colon R/I^m \to R/I^n \tag{4.18}$$

$$x + I^m \mapsto x + I^n \tag{4.19}$$

are well defined. We can check that $\{R/I^n, \pi_{n,m}\}_{n \leq m}$ is a countable inverse system.

$$\varprojlim R/I^n = \left\{ (x_n + I^n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} R/I^n \,\middle|\, \pi_{n,m}(x_m + I^m) = x_n + I^n \, \forall \, n \leq m \right\} \tag{4.20}$$

$$= \left\{ (x_n + I^n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} R/I^n \,\middle|\, x_m - x_n \in I^n \, \forall \, n \leq m \right\}. \tag{4.21}$$

This is the *completion of* R in the I-adic topology. It is called completion since, given $\{x_n\}_{n \in \mathbb{N}}$ it is a *Cauchy* sequence iff $\forall \, V$ neighbourhood of 0, $\exists \, n_0 \in \mathbb{N}$ s.t. $x_n - x_m \in V$ for all $n, m \geq n_0$. Moreover we can define a *neat Cauchy* sequence as a sequence $\{x_n\}_{n \in \mathbb{N}}$ s.t. $\forall \, V_n := I^n$ m then $x_m - x_n \in V_n$ for all $m \geq n$.

In particular an element $(x_n + I^n)_{n \in \mathbb{N}} \in \varprojlim R/I^n$ can be viewed as a limit of the Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$. (This is the reason why it can be seen as the completion in the topology).

Moreover we have a canonical projection

$$\mu \colon R \to \varprojlim R/I^n \tag{4.22}$$

$$x \mapsto (x + I^n)_{n \in \mathbb{N}}. \tag{4.23}$$

Clearly $\ker \mu = \bigcap_{n \in \mathbb{N}} I^n$ (i.e. μ is injective iff R is Hausdorff with the I-adic topology).

Example: p-adic completion of the ring of integers. Let $R := \mathbb{Z}$ and $I := p\mathbb{Z}$.

$$\hat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z} \tag{4.24}$$

is the p-adic completion of the ring of integers. An element $\zeta \in \hat{\mathbb{Z}}$ can be written as

$$\zeta = a_0 + a_1 p + a_2 p^2 + \dots, \tag{4.25}$$

with $0 \leq a_i < p$ for all $i \geq 1$. In fact $x_0 + p\mathbb{Z} = a_0 + p\mathbb{Z}$, with $0 \leq a_0 < p$. Then $x_1 - x_0 \in p\mathbb{Z}$, hence $x_1 = a_0 + a_1 p$. Then, by induction, given $x_n = a_0 + a_1 p + \dots + a_n p^n$ and $x_{n+1} - x_n \in p^{n+1}\mathbb{Z}$, hence

$$x_{n+1} = a_0 + \dots + a_{n+1} p^{n+1}. \tag{4.26}$$

4.1 The functor projective lim

Fix I a small category and let C be a complete category. Let C^{I} be the functor category.

Proposition 4.9.

$$\varprojlim \colon \mathsf{C}^{\mathsf{I}} \to \mathsf{C} \tag{4.27}$$

$$F \mapsto \varprojlim F \tag{4.28}$$

is a functor.

Proof. Given $\eta \colon F \to G$ a natural transformation between the functors $F, G \in \mathsf{C}^{\mathsf{I}}$, the functor associates it a morphism in the natural way

$$\varprojlim \eta \colon \varprojlim F \to \varprojlim G.$$ ■

Let's study a little the category C^{I}, for a small category I.

Proposition 4.10. C^{I} *inherits the properties of* C*. More explicitly if* C *is preadditive/additive/abelian, then also* C^{I} *is preadditive/additive/abelian.*

Moreover construction in C *can be done in* C^{I} *locally, for every* $i \in \mathrm{Ob}(\mathsf{I})$*. For instance:*

- *Given* $\eta, \zeta \in \mathrm{Hom}_{\mathsf{C}^{\mathsf{I}}}(F, G)$*, if* C *is preadditive, then* $(\eta + \zeta)_i = \eta_i + \zeta_i$*, for each object* $i \in \mathrm{Ob}(\mathsf{I})$.
- *If* C *has products, then also* C^{I} *has products. In particular, given two functors* $F, G \in \mathrm{Ob}\left(\mathsf{C}^{\mathsf{I}}\right)$*, we need to define the product* $(F \Pi G, \pi_F, \pi_G)$*, s.t. this is a product of* F *and* G *in* C^{I}*. On objects it is defined as expected*

$$(F \prod G)(i) \coloneqq F(i) \prod G(i). \tag{4.29}$$

Moreover, on morphisms it is defined as follows: given $\lambda \colon i \to j$*, then*

$$\left(F \prod G\right)(\lambda) = \begin{bmatrix} F(\lambda) & 0 \\ 0 & G(\lambda) \end{bmatrix}, \tag{4.30}$$

is our morphism $(F \Pi G)(\lambda) \colon F(i) \Pi G(i) \to F(j) \Pi G(j)$*. Finally we have to define the projections. They are constructed naturally as*

$$(\pi_F)_i \coloneqq \pi_{F(i)} \qquad \text{and} \qquad (\pi_G)_i \coloneqq \pi_{G(i)}. \tag{4.31}$$

- *If* C *has kernels, then also* C^{I} *has. Let* $\eta \colon F \to G$ *be a natural transformation. Let's define* $\ker \eta$ *as an object of* C^{I}*. For every* $i \in \mathrm{Ob}(\mathsf{I})$ *we define* $K(i) \coloneqq \ker \eta_i$ *as an object in* C*. This, for any morphism* $\lambda \colon i \to j$*, gives rise to the commutative diagram*

$$\begin{tikzcd} K(i) \arrow[r, "\epsilon_i"] \arrow[d, dashed, "\exists ! \nu"] & F(i) \arrow[r, "\eta_i"] \arrow[d, "F(\lambda)"] & G(i) \arrow[d, "G(\lambda)"] \\ K(j) \arrow[r, "\epsilon_j"'] & F(j) \arrow[r, "\eta_j"'] & G(j) \end{tikzcd} . \tag{4.32}$$

From this we define $K(\lambda) \coloneqq \nu$. Then, the couple $\left(K, \{\epsilon_i\}_{i\in \mathrm{Ob}(\mathsf{I})}\right)$ is the kernel of $\{\eta_i\}_{i\in \mathrm{Ob}(\mathsf{I})}$.

- *If C has cokernels, then also C^{I} has. Let $\eta\colon F \to G$ be a natural transformation. Let's define* $\operatorname{coker}\eta$ *as an object of C^{I}. For every $i \in \mathrm{Ob}(\mathsf{I})$ we define $C(i) \coloneqq \operatorname{coker}\eta_i$ as an object in C. This, for any morphism $\lambda\colon i \to j$, gives rise to the commutative diagram*

$$\begin{tikzcd} F(i) \arrow[r, "\eta_i"] \arrow[d, "F(\lambda)"] & G(i) \arrow[r, "p_i"] \arrow[d, "G(\lambda)"] & C(i) \arrow[d, dashed, "\exists!\,\nu"] \\ F(j) \arrow[r, "\eta_j"'] & G(j) \arrow[r, "p_j"'] & C(j). \end{tikzcd} \tag{4.33}$$

From this we define $C(\lambda) \coloneqq \nu$. Then, the couple $\left(C, \{p_i\}_{i\in \mathrm{Ob}(\mathsf{I})}\right)$ is the cokernel of $\{\eta_i\}_{i\in \mathrm{Ob}(\mathsf{I})}$.

4.2 Characterization of projective limit

Let, as before, I be a small category, and C^{I} the category of functors $F\colon \mathsf{I} \to \mathsf{C}$.

Definition 4.11: Constant functor.
Consider a fixed $X \in \mathrm{Ob}(\mathsf{C})$. We define the constant functor

$$\Delta_X\colon \mathsf{I} \to \mathsf{C}. \tag{4.34}$$

On objects as $\Delta_X(i) = X$ for all $i \in \mathrm{Ob}(\mathsf{I})$. On morphism $\Delta_X(\lambda) = id_X$ for all $\lambda\colon i \to j$.

Definition 4.12: Diagonal functor.
We define, in terms of the constant functor, the diagonal functor

$$\Delta\colon \mathsf{C} \to \mathsf{C}^{\mathsf{I}}. \tag{4.35}$$

On objects as $\Delta(X) \coloneqq \Delta_X$. On morphisms $f\colon X \to Y$, then

$$\Delta(f) \coloneqq \bar{f}\colon \Delta_X \to \Delta_Y, \tag{4.36}$$

where $\bar{f}$ is a natural transformation s.t. for every $i \in \mathrm{Ob}(\mathsf{I})$, $\bar{f}_i = f$.

Definition 4.13: Some notation for the following proposition.
Fix a functor $F \in \mathsf{C}^{\mathsf{I}}$. Let $H\colon \mathsf{C}^{op} \to \mathsf{Sets}$ be a contravariant functor from C, defined as follows. On the objects $Y \in \mathrm{Ob}(\mathsf{C})$, $H(Y) \coloneqq \mathrm{Nat}(\Delta_Y, F)$. On the morphisms, for $f\colon X \to Y$,

$$H(f)\colon \mathrm{Nat}(\Delta_Y, F) \to \mathrm{Nat}(\Delta_X, F) \tag{4.37}$$
$$\eta \mapsto \eta \circ \bar{f}. \tag{4.38}$$

Proposition 4.14. *Given a functor $F \in \mathsf{C}^{\mathsf{I}}$, then $\varprojlim F$ exists iff the functor H defined above is representable. In other words iff $\exists\, P \in \mathrm{Ob}(\mathsf{C})$ s.t. the following two functors are naturally isomorphic*

$$\mathrm{Hom}_{\mathsf{C}}(-, P) \simeq_{\varphi} H = \mathrm{Nat}\left(\Delta_{(-)}, F\right). \tag{4.39}$$

In such case $P \simeq \varprojlim F$, and the compatible family is defined as

$$\varphi_P\colon \mathrm{Hom}_{\mathsf{C}}(P, P) \to \mathrm{Nat}(\Delta_P, F) \tag{4.40}$$
$$1_P \mapsto \bar{p} = \{p_i\}_{i\in \mathrm{Ob}(\mathsf{I})}. \tag{4.41}$$

4.3 Colimits

Let's dualize the notion of limit, to obtain the notion of colimit. As usual we consider I a small category, and $F\colon \mathsf{I} \to \mathsf{C}$ a functor.

Before we introduce the notion of colimit let's dualize that of compatible family

Definition 4.15: Compatible family.
Fix $X \in \mathrm{Ob}(\mathsf{C})$ and consider a family $\{\alpha_i\}_{i\in\mathrm{Ob}(\mathsf{I})}$ of morphisms $\alpha_i\colon F(i) \to X$. This is said to be a *compatible family* with respect to F iff, given any morphism $\lambda\colon i \to j$ in I, the following triangle commutes

$$\begin{tikzcd} F(i) \arrow[r, "\alpha_i"] \arrow[dr, "F(\lambda)"'] & X \\ & F(j) \arrow[u, "\alpha_j"'] \end{tikzcd} . \tag{4.42}$$

In other words iff $\alpha_i = \alpha_j \circ F(\lambda)$ for every $i, j \in \mathrm{Ob}(\mathsf{I})$ and every $\lambda\colon i \to j$.

Definition 4.16: Colimit/Injective (inverse) limit.
A *colimit* of F, denoted with $\varinjlim F$ is a limit of F in C^{op}. More explicitly a colimit is an object in C, still denoted with $\varinjlim F$, with morphisms $\mu_i\colon F(i) \to \varinjlim F$ satisfying the following conditions

CoLIM1 $\{\mu_i\}_{i\in\mathrm{Ob}(\mathsf{I})}$ is a compatible family of morphisms, i.e.

$$\begin{tikzcd} F(i) \arrow[r, "\mu_i"] \arrow[dr, "F(\lambda)"'] & \varinjlim F \\ & F(j) \arrow[u, "\mu_j"'] \end{tikzcd} \tag{4.43}$$

the above diagram commutes for all $i, j \in \mathrm{Ob}(\mathsf{I})$ and all $\lambda\colon i \to j$.

CoLIM2 For any $X \in \mathrm{Ob}(\mathsf{C})$ and any compatible family of morphisms $\{\alpha_i\}_{i\in\mathrm{Ob}(\mathsf{I})}$, with $\alpha_i\colon F(i) \to X$, $\exists!\, \alpha\colon \varinjlim F \to X$ s.t. $\alpha \circ \mu_i = \alpha_i \;\; \forall\, i \in \mathrm{Ob}(\mathsf{I})$, i.e.

$$\begin{tikzcd} F(i) \arrow[r, "\alpha_i"] \arrow[d, "\mu_i"'] & X \\ \varinjlim F \arrow[ur, "\alpha"'] & \end{tikzcd} . \tag{4.44}$$

Remark 4.17 As always, since it is defined through a universal property, if $\left(\varinjlim F, \mu_i\right)$ exists, it is unique up to a unique isomorphism.

Example Let I be a small discrete category, i.e. the morphisms in I are only the identities. Then, for any functor $F\colon \mathsf{I} \to \mathsf{C}$, $\varinjlim F$ exists iff $\coprod_{i\in\mathrm{Ob}(\mathsf{I})} F(i)$ exists and they are isomorphic. In particular the μ_i s correspond with the embeddings of the coproduct.

Example Consider, in an arbitrary category C, the following diagram

$$\begin{tikzcd} A_3 \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \\ A_2 & \end{tikzcd} . \tag{4.45}$$

Consider the small category I, with $\mathrm{Ob}(\mathsf{I}) := \{1, 2, 3\}$ and morphisms, other than the identities, $3 \to 1$ and $3 \to 2$. Consider the functor $F : \mathsf{I} \to \mathsf{C}$ defined as follows:

$$F(i) = A_i, \qquad F(3 \to 1) = f_1, \qquad F(3 \to 2) = f_2. \tag{4.46}$$

Then any colimit of F corresponds to a pushout of the above diagram.

Definition 4.18: Cocomplete category.
A category C is called *cocomplete* iff every functor $F : \mathsf{I} \to \mathsf{C}$, from a small category I, admits colimit in C.

Proposition 4.19. *If a (preadditive) category C admits cokernels and (arbitrary) coproducts, then for every functor $F : \mathsf{I} \to \mathsf{C}$, from a small category I, C admits $\varinjlim F$. Moreover the colimit is constructed by means of cokernels and coproducts.*

Proof. As above, the proof wants to show that the following construction actually is a direct limit for F. Consider $\left(\coprod_{i \in \mathrm{Ob}(\mathsf{I})} F(i), \epsilon_i\right)$ the coproduct of $F(i)$. Define

$$\psi_\lambda := \epsilon_{t(\lambda)} \circ F(\lambda) - \epsilon_{s(\lambda)} : F(s(\lambda)) \to \coprod_{i \in \mathrm{Ob}(\mathsf{I})} F(i). \tag{4.47}$$

Then the family ψ_λ induces a unique

$$\psi \colon \coprod_{\lambda \in \Lambda} F(s(\lambda)) \to \coprod_{i \in \mathrm{Ob}(\mathsf{I})} F(i) \tag{4.48}$$

s.t. $\psi \circ \epsilon_{s(\lambda)} = \psi_\lambda$. Moreover, we recall that $\Lambda := \mathrm{Morph}\,\mathsf{I}$. Then, denoted by (C, p) a cokernel of ψ, (C, μ_i), where $\mu_i := p \circ \epsilon_i$, is an injective limit of F. ■

Fix a small category I and let C be a cocomplete category.

Proposition 4.20.

$$\varinjlim \colon \mathsf{C}^{\mathsf{I}} \to \mathsf{C} \tag{4.49}$$
$$F \mapsto \varinjlim F \tag{4.50}$$

is a functor.

Proof. It is clear how the functor acts on objects. Let's define how it acts on $\eta \colon F \to G$ a natural transformation between the functors $F, G \in \mathsf{C}^{\mathsf{I}}$. It associates to η a morphism in the natural way

$$\varinjlim \eta \colon \varinjlim F \to \varinjlim G.$$ ■

Proposition 4.21. *Given a functor $F \in \mathsf{C}^{\mathsf{I}}$, then $\varinjlim F$ exists iff the functor*

$$H \colon \mathsf{C} \to \mathsf{Sets} \tag{4.51}$$
$$Y \mapsto \mathrm{Nat}(F, \Delta_Y) \tag{4.52}$$

is corepresentable. In other words iff $\exists\, C \in \mathrm{Ob}(\mathsf{C})$ s.t. the following two functors are naturally isomorphic

$$\mathrm{Hom}_{\mathsf{C}}(C, -) \simeq_\varphi H = \mathrm{Nat}\left(F, \Delta_{(-)}\right). \tag{4.53}$$

In such case $C \simeq \varinjlim F$, and the compatible family is defined as

$$\varphi_C \colon \mathrm{Hom}_{\mathsf{C}}(C, C) \to \mathrm{Nat}(F, \Delta_C) \tag{4.54}$$
$$1_C \mapsto \bar{\mu} = \{\mu_i\}_{i \in \mathrm{Ob}(\mathsf{I})}. \tag{4.55}$$

Let's now describe a particular case of colimits:

Example Let $\mathsf{C} := \mathsf{Mod}\text{-}R$ and $(\mathsf{I}, \leq)$ a partially ordered set, viewed as a category. Consider a functor

$$F\colon \mathsf{I} \to \mathsf{Mod}\text{-}R. \tag{4.56}$$

This is equivalent to the data of $F(i) =: M_i \in \mathsf{Mod}\text{-}R$ and, for all $i \leq j$, of

$$F(i \to j) =: f_{ji}\colon M_i \to M_j. \tag{4.57}$$

Now, given $i \leq j \leq k$, then the following diagram commutes

$$\begin{tikzcd} M_i \arrow[r, "f_{ji}"] \arrow[dr, "f_{ki}"'] & M_j \arrow[d, "f_{kj}"] \\ & M_k \end{tikzcd}, \tag{4.58}$$

in other words $f_{kj} \circ f_{ji} = f_{ki}$. Moreover we require $f_{ii} = id_{M_i}$.

We have, in fact, a correspondence, between functors from partially ordered sets and direct systems of modules, which are families $\{M_i, f_{ij}\}_{i \leq j}$ of modules M_i and morphism f_{ij} between them, satisfying the above compatibility conditions.

Then, $\varinjlim F$ is called the *direct limit* of $\{M_i, f_{ij}\}_{i \leq j}$. The morphisms f_{ij} are called the structural morphisms of the direct system. Sometimes this is also denoted with $\varinjlim M_i$.

Let's describe $\varinjlim M_i$ explicitly: for every $i \leq j$ we have the (not necessarily commutative) diagram

$$\begin{tikzcd} M_i \arrow[r, "\epsilon_i"] \arrow[d, "f_{ji}"'] & \bigoplus_{i \in \mathrm{Ob}(\mathsf{I})} M_i \\ M_j \arrow[ur, "\epsilon_j"'] & \end{tikzcd}. \tag{4.59}$$

Let's define, for each $i \leq j$, $\psi_{ij} := \epsilon_j \circ f_{ji} - \epsilon_i$. Then, by universal property of the coproduct, $\exists\,!\,\psi\colon \bigoplus_{i \leq j} M_{ij} \to \bigoplus_{i \in \mathrm{Ob}(\mathsf{I})} M_i$, where $M_{ij} := M_i$ for every $i \leq j$. In the above construction $\psi \circ \epsilon_i = \psi_{ij}\colon M_{ij} \to \bigoplus_{i \in \mathrm{Ob}(\mathsf{I})} M_i$. Then we have

$$\varinjlim M_i \simeq \operatorname{coker} \psi = \frac{\bigoplus_{i \in \mathrm{Ob}(\mathsf{I})} M_i}{\operatorname{Im} \psi}, \tag{4.60}$$

where $\operatorname{Im} \psi$ is generated by

$$\{\epsilon_j \circ f_{ji}(x_i) - \epsilon_i(x_i) \mid x_i \in M_i,\ i \leq j\} \subset \bigoplus_{i \in \mathrm{Ob}(\mathsf{I})} M_i. \tag{4.61}$$

In particular the generators of $\operatorname{Im} \psi$ are of the form

$$(\ldots, 0, \ldots, 0, x_i, 0, \ldots, 0, -f_{ji}(x_i), 0, \ldots, 0, \ldots)\,. \tag{4.62}$$

It is a submodule of the product, in which x_i, in position i, and $f_{ji}(x_i)$, in position $j \geq i$, are identified.

Definition 4.22: Directed poset.
A poset $(I, \leq)$ is said *directed* (or *filtered*) iff

$$\forall\, i, j \in I\ \exists\, k \in I \text{ s.t. } i \leq k \text{ and } i \leq k. \tag{4.63}$$

Moreover, if $(M_i, f_{ji})_{i\leq j}$, for I filtered, then $\varinjlim M_i$ is called *directed* (or *filtered*) limit. In general it is easier to describe a colimit on a directed poset.

Before giving an example of one such limit, let's recall a definition:

Definition 4.23: Finitely generated module.
A module M_R is *finitely generated* iff there is an epimorphism $\phi\colon R^N := \bigoplus_{i=1}^N R \to M$, for some $N \in \mathbb{N}$. If we denote by e_i the generators of R^N, then $\phi(e_i) = x_i$ are the generators of M. In other words we are saying that there exists $\{x_1, \dots, x_N\} \subset M$ a finite set of generators such that, for all $x \in R$ there exist $r_1, \dots, r_n \in R$ s.t.

$$x = \sum_{i=1}^{N} x_i r_i. \tag{4.64}$$

Definition 4.24: Finitely presented module.
For a finitely generated module, with epimorphism $\phi\colon R^N \to M$, we denote by $K := \ker \phi$, the module of relations of M:

$$K = \left\{ (r_1, \dots, r_N) \in R^N \;\middle|\; \sum_{i=1}^{N} x_i r_i = 0 \right\}. \tag{4.65}$$

We say that K is the module of relations of M (also known as the first syzygy module). We say that M is finitely presented if, being finitely generated, has also finitely generated first syzygy module.

Example Let $M \in \mathsf{Mod}\text{-}R$. Consider the family

$$\mathcal{F} := \{N \leq M \mid N \text{ is finitely generated }\}. \tag{4.66}$$

Let's label the elements $N \in \mathcal{F}$ as $N = N_i$, for some index $i \in \mathsf{I}$, with I a set of indices. Let's define a partial order on I: $i \leq j$ iff $N_i \subset N_j$. Moreover, if $i, j \in \mathsf{I}$, then $N_i + N_j$ is finitely generated, hence $\exists\, k \in \mathsf{I}$ s.t. $N_i + N_j = N_k$, for some $k \in \mathsf{I}$. This makes $(\mathsf{I}, \leq)$ a filtered poset. We then label the inclusions as $\epsilon_{ji}\colon N_i \to N_j$ and $\epsilon_i\colon N_i \to M$. Clearly this makes $(N_i, \epsilon_{ji})_{i\leq j}$ into a directed system.

Proposition 4.25. *Every R-Mod M is a directed limit of its finitely generated submodules. More explicitly, in the above notation,*

$$(M, \epsilon_i) \simeq \varinjlim N_i. \tag{4.67}$$

Example: Prüfer group. An example of a direct limit construction in $\mathsf{C} = \mathsf{Ab}$. Let $M_n := \mathbb{Z}/p^n\mathbb{Z} = \langle c_n \rangle$, for $n \in \mathbb{N}$ and $p \in \mathbb{N}$ a prime number. Notice that c_n has order p^n, hence $p^n c_n = 0$. We define the structural morphisms of the direct system as

$$f_{n+1,n}\colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z} \tag{4.68}$$

$$c_n \mapsto p \cdot c_{n+1} \tag{4.69}$$

extending it by linearity. Moreover, composing consecutive maps, we obtain

$$f_{m,n}\colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z} \tag{4.70}$$

$$c_n \mapsto p^{m-n} c_m \tag{4.71}$$

(also this extended by linearity). Clearly $\{\mathbb{Z}/p^n\mathbb{Z}, f_{m,n}\}_{n\leq m}$ is a directed system (compatibility follows from the definition of $f_{m,n}$). We can consider the directed limit, denoted as follows, and called *Prüfer* group

$$\varinjlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}(p^\infty) \simeq \bigcup_{n\in\mathbb{N}} \langle c_n \rangle, \tag{4.72}$$

where, in the last union, we consider the map $f_{n+1,n}$ as the inclusion of $\langle c_n \rangle$ in $\langle c_{n+1} \rangle$. Carrying out the construction described in the proposition we obtain that

$$\varinjlim \langle c_n \rangle \simeq \frac{\bigoplus_{n\in\mathbb{N}} \langle c_n \rangle}{\langle (c_n, -p \cdot c_{n+1}) \mid n \in \mathbb{N} \rangle}. \tag{4.73}$$

4.4 Direct limit of modules

Lemma 4.26. *Let $\mathsf{C} = \mathsf{Mod}\text{-}R$ and $(\mathsf{I}, \leq)$ be a filtered poset. Let $\{M_i, f_{ji}\}_{i\leq j}$ be a directed system of modules. In the notation of proposition 4.19, the directed limit $\left(\varinjlim M_i, \mu_i\right)$, has the compatible family of maps*

$$\begin{tikzcd} M_i \arrow[r, "\mu_i"] \arrow[d, "\epsilon_i"] & \varinjlim M_i \\ \bigoplus_{i\in\mathsf{I}} M_i \arrow[ur, "p"'] & \end{tikzcd} . \tag{4.74}$$

Where $\mu_i := p \circ \epsilon_i$ and $p = \operatorname{coker}\psi$. If we denote with $D := \operatorname{Im}\psi$, then every element $x \in \varinjlim M_i = (\bigoplus M_i)/D$ can be written as $\mu_i(x_i)$, for some $i \in \mathsf{I}$ and $x_i \in M_i$.
(Then we can interpret $\varinjlim M_i = \sum_{i\in\mathsf{I}} \mu_i(M_i)$).

Proof. The idea is simply the fact that I is filtered (hence for any finite set of indices we can find an index which is bigger than all of them). Given this one can easily use the relations to express any finite sum in terms of an element from a single M_k. ■

Lemma 4.27. *In the above notation and hypothesis, let $x = x_{i_1} + \ldots + x_{i_n} \in \bigoplus_{i\in\mathsf{I}} M_i$. $x \in D$ iff $\exists\, k \in \mathsf{I}$, $k \geq i_1, \ldots, i_n$ s.t.*

$$f_{k,i_1}(x_{i_1}) + \ldots f_{k,i_n}(x_{i_n}) = 0 \in M_k. \tag{4.75}$$

Lemma 4.28. *In the above notation and hypothesis, let $x_i \in M_i$. Then*

$$\mu_i(x_i) = 0 \in \varinjlim M_i \iff \exists\, j \geq i \in \mathsf{I} \text{ s.t. } f_{ji}(x_i) = 0. \tag{4.76}$$

Proposition 4.29. *Let $M_R \in \mathsf{Mod}\text{-}R$, then M_R is a direct limit of finitely presented modules.*

5 Exactness

5.1 Subobjects and quotients

Definition 5.1: Subobject.
Let A be an object of a category (abelian) C. Consider two monomorphism $f\colon B \to A$ and $g\colon C \to A$. We say that $f \sim g$ iff $\exists\, \alpha\colon B \to C$ an isomorphism s.t. the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "\alpha"'] & A \\ & C \arrow[u, "g"'] \end{tikzcd} \tag{5.1}$$

in other words s.t. $g \circ \alpha = f$. Clearly this is an equivalence relation. An equivalence class of monomorphisms ending in A is called a *subobject* of A. Chosen a representative $f\colon B \to A$ we denote the corresponding subobject by $B \subseteq A$.

Moreover, given B_1 and B_2 subobjects of A, we say that B_1 is a subobject of B_2, denoted by $B_1 \subseteq B_2$, iff $\exists\, \alpha\colon B_1 \to B_2$ a morphism s.t. the following diagram commutes

$$\begin{tikzcd} B_1 \arrow[r, "f_1"] \arrow[dr, "\alpha"'] & A \\ & B_2 \arrow[u, "f_2"'] \end{tikzcd} \tag{5.2}$$

i.e. s.t. $f_2 \circ \alpha = f_1$. Notice that, in this case, α has to be mono.

Remark 5.2 If $B_1 \subseteq B_2 \subseteq A$ and $B_2 \subseteq B_1$, then B_1 and B_2 represent the same subobject of A.

Let's now give the dual definition.

Definition 5.3: Quotient.
Consider $f\colon A \to B$ and $g\colon A \to C$ two epimorphisms. We say that $f \sim g$ iff $\exists\, \alpha\colon B \to C$ an isomorphism s.t. the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[d, "\alpha"] \\ & C \end{tikzcd} \tag{5.3}$$

i.e. s.t. $\alpha \circ f = g$. Given one such morphism $f\colon A \to B$ we call the equivalence class a *quotient* of A.

Remark 5.4: notation.
Assume that $f\colon B \to A$ is a subobject of A (i.e. f is a mono). We write A/B for the quotient object represented by $\operatorname{coker} f$.

Lemma 5.5. *Let* C *be an abelian category. Consider two composable morphisms* $A \xrightarrow{f} B \xrightarrow{g} C$. *Then* $g \circ f = 0$ *iff* $\operatorname{Im} f \subseteq \ker g$*, viewed as subobjects of* B.

Lemma 5.6. *Let* C *be an abelian category. Consider two composable morphisms* $A \xrightarrow{f} B \xrightarrow{g} C$. *Then* $\ker g \subseteq \operatorname{Im} f$*, viewed as subobjects of* B*, iff* $\forall\, h\colon D \to B$ *s.t.* $g \circ h = 0$, $\exists\,!\, h'\colon D \to \operatorname{Im} f$ *s.t.* $\mu \circ h' = h$*, where* $\operatorname{Im} f \xrightarrow{\mu} B$ *is the natural morphism. In other words s.t. the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\beta"'] & B \\ \operatorname{Im} f \arrow[ur, "\mu"'] & D \arrow[u, "h"'] \arrow[l, dashed, "\exists\,!\, h'"] \end{tikzcd} . \tag{5.4}$$

Definition 5.7: Exact sequence.
Let C be an abelian category. Consider a sequence of composable morphisms in C

$$\ldots \to A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} A_{n+2} \xrightarrow{f_{n+2}} \ldots . \tag{5.5}$$

The sequence is *exact* at n iff $\operatorname{Im} f_n = \ker f_{n+1}$ as subobjects of A_{n+1}. It is said to be *exact* iff it is exact at n for every n.

Definition 5.8: Short exact sequence.
An *exact* sequence of the form

$$0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0 \tag{5.6}$$

is called *short exact sequence*. In particular this sequence is exact iff f_1 is a mono, f_2 is an epi, and $\operatorname{Im} f_1 = \ker f_2$.

Lemma 5.9. *Consider the following exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C. \tag{5.7}$$

Then $f = \ker g$.

Lemma 5.10. *Consider the following exact sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0. \tag{5.8}$$

Then $g = \operatorname{coker} f$.

Let's combine the above lemmas

Proposition 5.11. *Consider the following sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0. \tag{5.9}$$

This is exact (i.e. a s.e.q.) iff $f = \ker g$ *and* $g = \operatorname{coker} f$.

5.2 Functors

In this section we'll always work with abelian categories C and D.

Definition 5.12: Exact functor.
Let $F\colon \mathsf{C} \to \mathsf{D}$ be an *additive* functor. We say that F is *exact* iff, for every exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \quad \text{in } \mathsf{C}, \tag{5.10}$$

then the image sequence is exact in D

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C). \tag{5.11}$$

Equivalently F is exact if given $\operatorname{Im} f = \ker g$ in C, then $\operatorname{Im} F(f) = \ker F(g)$ in D.

Definition 5.13: Left (resp. right) exact functor.
An additive functor $F\colon \mathsf{C} \to \mathsf{D}$ is *left* (resp. *right*) exact iff, for every exact sequence in C

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \quad (\text{resp. } A \xrightarrow{f} B \xrightarrow{g} C \to 0\,), \tag{5.12}$$

then the image sequence is exact in D

$$0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \quad (\text{resp. } F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0\,). \tag{5.13}$$

Proposition 5.14. *An additive functor* $F\colon \mathsf{C} \to \mathsf{D}$ *between abelian categories, is exact iff it is both left and right exact.*

Definition 5.15: Split exact sequence.
A short exact sequence in C (as usual an abelian category)

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{5.14}$$

is said *split exact* iff $\exists\, \alpha\colon B \to A \oplus C$ s.t. the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "\alpha"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "\epsilon_A"] & A \oplus C \arrow[r, "\pi_C"] & C \arrow[r] & 0 \end{tikzcd} . \tag{5.15}$$

Recall that, in matrix notation, the embedding and projection can be written as

$$\epsilon_A = \begin{bmatrix} 1_A \\ 0 \end{bmatrix} \qquad \text{and} \qquad \pi_C = \begin{bmatrix} 0 & 1_c \end{bmatrix}. \tag{5.16}$$

Proposition 5.16. *Let C be an abelian category. TFAE*

1. *The sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact,*
2. *$\exists\, f'\colon B \to A$ s.t. $f' \circ f = 1_A$,*
3. *$\exists\, g'\colon C \to B$ s.t. $g \circ g' = 1_B$.*

In such a case f' is called a section of f, and g' a retraction of g.

Some examples

Recall that, given a category C, we have the natural bifunctor

$$F = \mathrm{Hom}_{\mathsf{C}}(-,-) : \mathsf{C}^{op} \times \mathsf{C} \to \mathsf{Sets}. \tag{5.17}$$

Clearly, if C is preadditive, F is an additive functor. Moreover

Proposition 5.17. *Let C be an abelian category, then*

$$\mathrm{Hom}_{\mathsf{C}}(-,-) : \mathsf{C}^{op} \times \mathsf{C} \to \mathsf{Ab} \tag{5.18}$$

is left exact in both variables.

Remark 5.18 Recall that a contravariant functor $F\colon \mathsf{C} \to \mathsf{D}$, i.e. a covariant functor $F\colon \mathsf{C}^{op} \to \mathsf{D}$, is left exact iff given any exact sequence in C

$$A \to B \to C \to 0, \tag{5.19}$$

i.e. $0 \to C \to B \to A$ exact in C^{op}, then the image sequence is exact in D

$$0 \to F(A) \to F(B) \to F(C). \tag{5.20}$$

Remark 5.19 Consider $\mathsf{C} = \mathsf{Mod}\text{-}R$ and $(\mathsf{I}, \leq)$ a filtered poset, then the functors $F\colon \mathsf{I} \to \mathsf{Mod}\text{-}R$ are in correspondence with the directed systems of modules $\{M_i, f_{ji}\}_{i\leq j}$. Consider the functors $F, G, L \in \mathsf{C}^{\mathsf{I}}$ and their corresponding directed systems $\{M_i, f_{ji}\}_{i\leq j}$, $\{N_i, g_{ji}\}_{i\leq j}$ and $\{L_i, l_{ji}\}_{i\leq j}$. Then the sequence

$$0 \to F \xrightarrow{\eta} G \xrightarrow{\zeta} L \to 0 \tag{5.21}$$

is exact iff the following diagram is commutative and has exact rows

$$\begin{tikzcd} 0 \arrow[r] & M_i \arrow[r, "\eta_i"] \arrow[d, "f_{ji}"] & N_i \arrow[r, "\zeta_i"] \arrow[d, "g_{ji}"] & L_i \arrow[r] \arrow[d, "l_{ji}"] & 0 \\ 0 \arrow[r] & M_i \arrow[r, "\eta_j"] & N_i \arrow[r, "\zeta_j"] & L_i \arrow[r] & 0 \end{tikzcd} , \tag{5.22}$$

for each $i \leq j$ in I.

Proposition 5.20. *Let* $\mathsf{C} = \mathsf{Mod}\text{-}R$ *and* $(\mathsf{I}, \leq)$ *be a filtered poset. Then the functor* $\varinjlim \colon \mathsf{Mod}\text{-}R^{\mathsf{I}} \to \mathsf{Mod}\text{-}R$ *is exact.*

Remark 5.21 Colimits, in general, are not exact, even in $\mathsf{Mod}\text{-}\mathbb{Z} = \mathsf{Ab}$. Consider, in fact, the category I, characterized by $\mathrm{Ob}(\mathsf{I}) := \{1, 2, 3\}$ and nontrivial arrows $1 \to 2$ and $1 \to 2$. Consider $F, G, H \in \mathsf{Ab}^{\mathsf{I}}$, defined by:

$$F\colon \begin{tikzcd} \mathbb{Z} \arrow[r, "\dot{4}"] \arrow[d, "0"'] & \mathbb{Z} \\ \mathbb{Z} & \end{tikzcd} \qquad G\colon \begin{tikzcd} \mathbb{Z} \arrow[r, "\dot{4}"] \arrow[d, "0"'] & \mathbb{Z} \\ \mathbb{Z} & \end{tikzcd} \qquad H\colon \begin{tikzcd} \mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] \arrow[d, "0"'] & \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & \end{tikzcd} . \tag{5.23}$$

Then $\varinjlim F = \operatorname{coker} \dot{4} = \varinjlim G$ and $\varinjlim H \simeq \mathbb{Z}/2\mathbb{Z}$. (Indeed these are just cokernels of the horizontal maps). Consider the natural transformation $\dot{2}$ and π, that give rise to the sequence

$$0 \longrightarrow F \xrightarrow{\dot{2}} G \xrightarrow{\pi} H \longrightarrow 0 , \tag{5.24}$$

which is exact in $\mathsf{Mod}\text{-}\mathbb{Z}^{\mathsf{I}}$. In fact this corresponds to

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\dot{2}"] \arrow[d, "\dot{4}"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\dot{4}"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\dot{2}"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{5.25}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\dot{2}"] \arrow[d, "0"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "0"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\dot{2}"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd} . \tag{5.26}$$

And both are commutative with exact rows. Taking the image by $\varinjlim$ we obtain

$$0 \longrightarrow \varinjlim F \simeq \mathbb{Z}/4\mathbb{Z} \xrightarrow{\dot{2}} \varinjlim G \simeq \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \varinjlim H \simeq \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \tag{5.27}$$

which is not exact, since $\dot{2} \colon \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ is not injective.

Proposition 5.22. *Let* $C := \mathsf{Mod}\text{-}R$ *and* $(\mathsf{I}, \leq)$ *be a filtered poset. Then the functor* $\varprojlim \colon \mathsf{Mod}\text{-}R^{\mathsf{I}} \to \mathsf{Mod}\text{-}R$ *is left exact.*

Example In general, even in the category $\mathsf{Mod}\text{-}\mathbb{Z} = \mathsf{Ab}$, the functor $\varprojlim$ is not exact. It is enough to construct an epimorphism

$$\{M_i, f_{ji}\}_{i \leq j} \xrightarrow{\zeta} \{N_i, g_{ji}\}_{i \leq j} \to 0 \tag{5.28}$$

s.t. the induced $\varprojlim \zeta \colon \varprojlim M_i \to \varprojlim N_i$ is not epi.

Let $\mathsf{I} = \mathbb{N}$, with the usual order. Let $M_n = \mathbb{Z}$ for every n, with structural morphisms $M_m \xrightarrow{3^{m-n}} M_n$, that acts as $x \mapsto x \cdot 3^{m-n}$, for all $m \leq n$, for all $x \in \mathbb{Z}$. Let $N_n = \mathbb{Z}/2\mathbb{Z}$ for every n, with structural morphisms $\mathrm{id} : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. We define

$$\left\{M_n, 3^{m-n}\right\}_{m \leq n} \xrightarrow{\pi} \left\{\mathbb{Z}/2\mathbb{Z}, id\right\}_{m \leq n}, \tag{5.29}$$

defined for all n as the canonical projection. It clearly is both surjective for all n, and (as can be easily checked) it is a natural transformation, hence it is an epi in the category of functors. Notice that, given $(x_n)_{n\in\mathbb{N}} \in \varprojlim M_n$, then $x_1 = 3 \cdot x_2 = \ldots = 3^n x_{n+1}$, hence $x_1 \in \bigcap_{n\in\mathbb{N}} 3^n\mathbb{Z} = \emptyset$. In other words $\varprojlim M_n = \emptyset$. Instead, clearly, $\varprojlim N_n = \mathbb{Z}/2\mathbb{Z}$. Then, obviously, $\varprojlim \pi$ cannot be surjective.

6 Injective and projective objects

Let C be an *arbitrary* category.

Definition 6.1: Projective object.
Let $P \in \mathrm{Ob}(\mathsf{C})$. P is *projective* iff given any $\varphi \colon B \to C$ epimorphism in C, and any morphism $f \colon P \to C$, then there exists $g \colon P \to B$ s.t. $\varphi \circ g = f$, i.e. s.t. the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "\varphi"] & C \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[ul, dashed, "\exists\, g"] & \end{tikzcd} . \tag{6.1}$$

In such case g is called a *lift* of f.
Equivalently: P is projective iff

$$\mathrm{Hom}_{\mathsf{C}}(P, B) \xrightarrow{\mathrm{Hom}_{\mathsf{C}}(P,\varphi)} \mathrm{Hom}_{\mathsf{C}}(P, C) \tag{6.2}$$

is an epimorphism (a surjection in Sets) for every φ epi.

Remark 6.2 If, moreover, C is abelian, then P is *projective* iff $\mathrm{Hom}_{\mathsf{C}}(P, -)$ is exact. Hence P is projective iff $\mathrm{Hom}_{\mathsf{C}}(P, -)$ is also *right* exact.

Definition 6.3: Injective object.
Let $I \in \mathrm{Ob}(\mathsf{C})$. I is *injective* iff I is projective in C^{op}. More explicitly, iff given any $\mu \colon A \to B$ mono in C, and any morphism $f \colon A \to I$, then there exists $g \colon B \to I$ s.t. $g \circ \mu = f$, i.e. s.t. the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\mu"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists\, g"] \\ & I & \end{tikzcd} . \tag{6.3}$$

In such case g is called an *extension* of f.
Equivalently: I is injective iff

$$\mathrm{Hom}_{\mathsf{C}}(B, I) \xrightarrow{\mathrm{Hom}_{\mathsf{C}}(\mu,I)} \mathrm{Hom}_{\mathsf{C}}(A, I) \tag{6.4}$$

is an epimorphism (a surjection in Sets) for every μ mono.

Remark 6.4 If, moreover, C is abelian, then I is *injective* iff $\mathrm{Hom}_{\mathsf{C}}(-, I)$ is exact. Hence I is injective iff $\mathrm{Hom}_{\mathsf{C}}(-, I)$ is also *right* exact.

Proposition 6.5. *Consider $\{P_i\}_{i\in I}$ a family of objects in C an arbitrary category. Assume that $\coprod_{i\in I} P_i$ exists. Then $\coprod_{i\in I} P_i$ is a projective object in C iff P_i is projective $\forall\, i \in I$.*

Dually we have the following

Proposition 6.6. *Consider $\{I_\lambda\}_{\lambda\in\Lambda}$ a family of objects in C an arbitrary category. Assume that $\prod_{\lambda\in\Lambda} I_\lambda$ exists. Then $\prod_{\lambda\in\Lambda} I_\lambda$ is an injective object in C iff I_λ is injective $\forall\, \lambda \in \Lambda$.*

Proposition 6.7 (Baer's criterion for injectivity)**.** *Let $\mathsf{C} = \mathsf{Mod}\text{-}R$. $E_R \in \mathsf{Mod}\text{-}R$ is an injective module iff for any ideal $I_R \triangleleft R$ and every $f\colon I \to E$, there exists $g\colon R \to E$ s.t. the following diagram commutes*

$$\begin{tikzcd} I_R \arrow[r, "\mu"] \arrow[d, "f"'] & R \arrow[dl, dashed, "\exists\, g"] \\ E & \end{tikzcd}, \tag{6.5}$$

where $\mu\colon I_R \to R$ is the inclusion. In other words we ask $g \circ \mu = f$.

Definition 6.8: Enough projectives/injectives.
Consider a category C.

- We say that C has *enough projectives* iff, given any $C \in \mathrm{Ob}(\mathsf{C})$, there exists a projective object $P \in \mathrm{Ob}(\mathsf{C})$ and an epimorphism $\varphi\colon P \to C$.
- We say that C has *enough injectives* iff, given any object $C \in \mathrm{Ob}(\mathsf{C})$, there exists an injective object $E \in \mathrm{Ob}(\mathsf{C})$ and a monomorphism $\mu\colon C \to E$.

Definition 6.9: Free module.
Let $\mathsf{C} := \mathsf{Mod}\text{-}R$. $M_R \in \mathsf{Mod}\text{-}R$ is *free* iff it has a free set of generators $\{x_i\}_{i\in I}$, with $x_i \in M$ for all i, s.t. $\forall\, x \in M$ it can be written in a unique way as a linear combination of the generators. More explicitly

$$x = \sum_{i\in I} x_i r_i \qquad \text{with } r_i \text{ almost all zero.} \tag{6.6}$$

Clearly M is free iff $M = \bigoplus_{i\in I} R_i$ (clearly interpreting the direct sum as a coproduct in the infinite case), with $R_i \simeq R$ for all i. In such case it has $\{e_i\}_{i\in I}$ as a basis. Another notation for $\bigoplus_{i\in I} R_i$ is $R^{(I)}$.

Remark 6.10 It is easy to show that any free module is projective.

Proposition 6.11. *Let $\mathsf{C} = \mathsf{Mod}\text{-}R$. $P_R \in \mathsf{Mod}\text{-}R$ is projective iff it is a direct summand of a free module.*

Remark 6.12 Projective modules are easy to describe. For injective ones we are able to do so only for a specific class of rings, for example for PIDs.

For this purpose, recall that a module M_R is *divisible* iff

$$\forall\, x \in M,\ \forall\, 0 \neq r \in R, \quad \exists\, y \in M \text{ s.t. } x = yr. \tag{6.7}$$

Proposition 6.13. *Let R be a PID, consider the category $\mathsf{C} := \mathsf{Mod}\text{-}R$. $E_R \in \mathsf{Mod}\text{-}R$ is injective iff it is divisible.*

Proof. It seems to me that any injective module is also divisible as soon as $xR \simeq R$ for any $x \in R$, i.e. I guess for integral domains (Baer's lemma still holds and we can check it on every principal ideal). The converse, however, requires that all ideals are principal. ■

Example: category with no nonzero projective objects. Let $\mathsf{C} := \mathsf{T}$ the full subcategory of Ab of torsion abelian groups. Then T has enough injectives, but no nonzero projective objects.

- Notice that $\mathsf{T} \subset \mathsf{Ab} = \mathsf{Mod}\text{-}\mathbb{Z}$, and $\mathbb{Z}$ is a PID. Then a torsion group is injective iff it is divisible.

  Consider an arbitrary $T \in \mathrm{Ob}(\mathsf{T})$. Then T has a set of generators $\{x_i\}_{i\in I}$, each of order $o(x_i) = n_i \in \mathbb{N}$. Then we have an epimorphism

  $$\varphi\colon \bigoplus_{i\in I} \mathbb{Z}/n_i\mathbb{Z} \twoheadrightarrow T. \tag{6.8}$$

  Then an injective element $I \in \mathsf{T}$ containing T is

  $$\bigoplus_{i\in I} \mathbb{Q}/n_i\mathbb{Z}, \tag{6.9}$$

  which is divisible, hence injective, and contains

  $$\bigoplus_{i\in I} \mathbb{Z}/n_i\mathbb{Z}. \tag{6.10}$$

  Finally we have an injection, given by the inclusion, which states that T has enough injectives:

  $$\frac{\bigoplus_{i\in I} \mathbb{Z}/n_i\mathbb{Z}}{\ker\varphi} \hookrightarrow \frac{\bigoplus_{i\in I} \mathbb{Q}/n_i\mathbb{Z}}{\ker\varphi}. \tag{6.11}$$

- There is a well-known fact saying that a subgroup of a direct sum of cyclic abelian groups is a direct sum of cyclic abelian groups.

  Consider $0 \neq T \in \mathrm{Ob}(\mathsf{T})$, and assume it is projective. Then, for $\{x_i\}_{i\in I}$ the generators of T, as above,

  $$\begin{tikzcd} \bigoplus_{i\in I} \mathbb{Z}/n_i\mathbb{Z} \arrow[r, "\varphi"] & T \arrow[r] & 0 \\ & T \arrow[u, equal, "1_T"'] \arrow[ul, dashed, "\psi"] & \end{tikzcd} . \tag{6.12}$$

  Then T is a subgroup of a direct sum of cyclic groups (1_T is injective, hence so has to be ψ). By the above remark

  $$T \simeq \bigoplus_{j\in J} \mathbb{Z}/m_j\mathbb{Z}. \tag{6.13}$$

  Since $T \neq 0$, then there exists m_0 s.t. $\mathbb{Z}/m_0\mathbb{Z} \neq 0$, and it is a projective object, since it is a direct summand of a projective object. Let's now consider the epimorphism

  $$\mathbb{Z}/m_0^2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m_0\mathbb{Z}. \tag{6.14}$$

  Reasoning as before we obtain that $\mathbb{Z}/m_0\mathbb{Z}$ is a direct summand of $\mathbb{Z}/m_0^2\mathbb{Z}$, which is a contradiction.

6.1 Functor categories

Remark 6.14 Let I be a small and preadditive category. Let C be an abelian category. Define $\underline{\mathrm{Hom}\left(\mathsf{I},\mathsf{C}\right)} \subset \mathsf{C}^{\mathsf{I}}$ the subcategory of all additive functors $F\colon \mathsf{I} \to \mathsf{C}$. In this situation $\underline{\mathrm{Hom}\left(\mathsf{I},\mathsf{C}\right)}$ is abelian.

Lemma 6.15 (Yoneda)**.** *Let I be as above. Let $\mathsf{C} := \mathsf{Ab}$. Fix $X \in \mathrm{Ob}\left(\mathsf{I}\right)$ and $F \in \underline{\mathrm{Hom}\left(\mathsf{I}^{op},\mathsf{Ab}\right)}$. There is an isomorphism*

$$\mathrm{Nat}\left(h^X, F\right) \xrightarrow{\theta_{X,F}} F(X), \tag{6.15}$$

natural in X and in F. Recall that $h^X := \mathrm{Hom}_{\mathsf{I}}\left(-, X\right)$.

Remark 6.16: An application of Yoneda lemma.
Consider $X, X' \in \mathrm{Ob}\left(\mathsf{I}\right)$. Let $F := h^{X'}$, then *Yoneda lemma* implies

$$\mathrm{Nat}\left(h^X, h^{X'}\right) \simeq h^{X'}(X) = \mathrm{Hom}_{\mathsf{I}}\left(X, X'\right). \tag{6.16}$$

Definition 6.17: Yoneda embedding.
Consider I small and preadditive, $\mathsf{C} = \mathsf{Ab}$. We define the *Yoneda embedding* as the functor $Y\colon \mathsf{I} \to \underline{\mathrm{Hom}\left(\mathsf{I}^{op},\mathsf{Ab}\right)}$, defined on objects as

$$Y\colon \mathsf{I} \to \underline{\mathrm{Hom}\left(\mathsf{I}^{op},\mathsf{Ab}\right)} \tag{6.17}$$

$$X \mapsto h^X \tag{6.18}$$

and on morphisms, given $f\colon X \to X'$, by

$$Y(f) := \mathrm{Hom}_{\mathsf{I}}\left(-, f\right) : h^X \to h^{X'}. \tag{6.19}$$

Proposition 6.18. *The Yoneda embedding Y is* fully faithful. *Moreover it sends distinct objects of I to distinct objects of $\underline{\mathrm{Hom}\left(\mathsf{I}^{op},\mathsf{Ab}\right)}$.*

Corollary 6.19. *Consider a small preadditive category I. Then I is equivalent to the full subcategory of $\underline{\mathrm{Hom}\left(\mathsf{I}^{op},\mathsf{Ab}\right)}$ consisting of the representable functors.*

Proposition 6.20. *For I as before (small and preadditive) and $X \in \mathrm{Ob}\left(\mathsf{I}\right)$, then h^X is a projective object of $\underline{\mathrm{Hom}\left(\mathsf{I}^{op},\mathsf{Ab}\right)}$.*

Definition 6.21: Generator of a category.
Let C be a category. An object $G \in \mathrm{Ob}\left(\mathsf{C}\right)$ is a *generator* of C iff $\mathrm{Hom}_{\mathsf{C}}\left(G, -\right) : \mathsf{C} \to \mathsf{Sets}$ is faithful. In other words iff the maps of sets

$$\mathrm{Hom}_{\mathsf{C}}\left(C, D\right) \to \mathrm{Hom}_{\mathsf{Sets}}\left(\mathrm{Hom}_{\mathsf{C}}\left(G, C\right), \mathrm{Hom}_{\mathsf{C}}\left(G, D\right)\right) \tag{6.20}$$

is injective for every $C, D \in \mathrm{Ob}\left(\mathsf{C}\right)$.

Remark 6.22: Equivalent definition.
G is a generator, iff for every pair $f, g\colon C \to D$ s.t. $\mathrm{Hom}_{\mathsf{C}}\left(G, f\right) = \mathrm{Hom}_{\mathsf{C}}\left(G, g\right)$, i.e. $g \circ \alpha = f \circ \alpha$ for all $\alpha\colon G \to C$, then $f = g$.

In the case of a preadditive category C, then G is a generator iff for all morphisms f in C s.t. $\mathrm{Hom}_{\mathsf{C}}\left(G, f\right) = 0$, i.e. s.t. $f \circ \alpha = 0$ (whenever admissible), then $f = 0$.

Definition 6.23: Alternative notation for (co)products.
Fix $X \in \mathrm{Ob}\left(\mathsf{C}\right)$ and I a set.

- If $\prod_{i \in I} X_i$, with $X_i \coloneqq X$ for all $i \in I$, exists we define the notation

$$X^I \coloneqq \prod_{i \in I} X_i. \tag{6.21}$$

- If $\coprod_{i \in I} X_i$, with $X_i \coloneqq X$ for all $i \in I$, exists we define the notation

$$X^{(I)} \coloneqq \coprod_{i \in I} X_i. \tag{6.22}$$

Proposition 6.24. *Assume that* C *has arbitrary coproducts. TFAE*

1. G *is a generator of* C*,*
2. $\forall X \in \mathrm{Ob}(\mathsf{C})$*, there is an epimorphism* $G^{(I)} \twoheadrightarrow X$*, for some set* I*.*

Definition 6.25: Cogenerator of a category.
Let C be a category. An object $C \in \mathrm{Ob}(\mathsf{C})$ is a *cogenerator* of C iff C is a generator in C^{op}, i.e. iff $\mathrm{Hom}_{\mathsf{C}}(-, C) : \mathsf{C}^{op} \to \mathsf{Sets}$ is faithful. In other words iff the maps of sets

$$\mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Hom}_{\mathsf{C}}(B, C), \mathrm{Hom}_{\mathsf{C}}(A, C)) \tag{6.23}$$

is injective for every $A, B \in \mathrm{Ob}(\mathsf{C})$.

Remark 6.26: Equivalent definition.
C is a cogenerator, iff for every pair $f, g \colon A \to B$ s.t. $\mathrm{Hom}_{\mathsf{C}}(f, C) = \mathrm{Hom}_{\mathsf{C}}(g, C)$, i.e. $\alpha \circ f = \alpha \circ g$ for all $\alpha \colon B \to C$, then $f = g$.

In the case of a preadditive category C, then C is a generator iff for all morphisms f in C s.t. $\mathrm{Hom}_{\mathsf{C}}(f, C) = 0$, i.e. s.t. $\alpha \circ f = 0$ (whenever admissible), then $f = 0$.

Proposition 6.27. *Assume that* C *has arbitrary products. TFAE*

1. C *is a cogenerator of* C*,*
2. $\forall X \in \mathrm{Ob}(\mathsf{C})$*, there is a monomorphism* $\mu \colon X \rightarrowtail C^I$*, for some set* I*.*

Example Let $\mathsf{C} \coloneqq \mathsf{Mod}\text{-}R$. R is a generator of $\mathsf{Mod}\text{-}R$: given a module M_R, and $\{x_i\}_{i \in I}$ a set of generators for M, then

$$R^{(I)} = \bigoplus_{i \in I} R_i \xrightarrow{\phi} M \to 0, \tag{6.24}$$

in which $\phi(e_i) \coloneqq x_i$. Moreover R is projective, hence it is a projective generator.

Remark 6.28: A not-so-easy-to-prove fact about modules.
Let $\mathsf{C} \coloneqq \mathsf{Mod}\text{-}R$. Every module M can be embedded in an injective module (i.e. $\mathsf{Mod}\text{-}R$ has enough injectives). Moreover every module M admits an injective envelope, denoted $E(M)$, where the envelope is a minimal injective module containing M.

Example Let $\mathsf{C} \coloneqq \mathsf{Mod}\text{-}R$ as before. Let $\mathcal{S}$ be the set of simple modules $S \in \mathsf{Mod}\text{-}R$ (i.e. modules with no proper submodules). Recall that $S \in \mathcal{S}$ iff $S \simeq R/\mathfrak{m}_R$, for some maximal ideal $\mathfrak{m}_R \lhd R$. Given $S \in \mathcal{S}$, consider its injective envelope $E(S)$, and finally let's define

$$C \coloneqq \prod_{S \in \mathcal{S}} E(S) \simeq \prod_{\mathfrak{m}_R \in \mathrm{Max}\, R} E(R/\mathfrak{m}_R) \in \mathrm{Ob}(\mathsf{C}). \tag{6.25}$$

Then C is an injective cogenerator of $\mathsf{Mod}\text{-}R$. In fact, consider $0 \neq X_R \in \mathsf{Mod}\text{-}R$, and $0 \neq x \in X_R$. Then $\langle x \rangle \simeq R/I$, for $I = \{r \in R \mid xr = 0\} \lhd R$. Consider any maximal ideal $\mathfrak{m}_R \lhd R$ s.t. $I \subset \mathfrak{m}_R$, then, since $E(R/\mathfrak{m}_R)$ is injective, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \langle x \rangle \arrow[r, hook] \arrow[d, "\pi"'] & X \arrow[ddl, "\exists\, f_x \neq 0"] \\
& R/\mathfrak{m}_R \arrow[d, hook] & \\
& E(R/\mathfrak{m}_R) &
\end{tikzcd} . \tag{6.26}$$

Then, for every $0 \neq x \in X$, we have the map

$$0 \neq f_x \colon X \to E\left(R/\mathfrak{m}_R\right) \hookrightarrow \prod_{\mathfrak{m}_R \in \operatorname{Max} R} E\left(R/\mathfrak{m}_R\right) =: C. \tag{6.27}$$

Then, by the universal property of products, viewing X as a set,

$$\exists\,!\, f \colon X \hookrightarrow C^X \tag{6.28}$$

induced by the various f_x. Moreover this f is mono, since $f_x(x) \neq 0$ for any $0 \neq x$.

Remark 6.29 Notice that, if C has a projective generator, then C has enough projectives. Analogously, if C has an injective cogenerator, then C has enough injectives.

6.2 Grothendieck categories

Definition 6.30: Grothendieck category.
An abelian category C is a *Grothendieck* category iff it is cocomplete, it has a generator, and filtered direct limits are exact in C.

Remark 6.31: Important fact.
A Grothendieck category has injective envelopes, in particular injective cogenerators. Though it might have no nonzero projective objects.

Example

- $\mathsf{Mod}\text{-}R$ and $R\text{-}\mathsf{Mod}$ are both Grothendieck categories.
- The category of coherent sheaves is Grothendieck, but has no nonzero projective objects.
- It can be shown that also the category of torsion abelian groups is Grothendieck.

7 Adjoint functors

Let's introduce this topic with an example

Example Let $\mathbb{K}$ be a field, and $\mathsf{C} := \mathsf{Vect}\text{-}\mathbb{K}$ the category of $\mathbb{K}$-Vector Spaces. Clearly we can define the forgetful functor, which acts on objects as

$$\mathrm{For} \colon \mathsf{Vect}\text{-}\mathbb{K} \to \mathsf{Sets} \tag{7.1}$$
$$V_K \mapsto V, \tag{7.2}$$

forgetting about the structure of Vector Space. For a set X, moreover, we can construct the Vector Space $\langle X \rangle$, which is the Vector Space for which X is a basis. This induces a functor

$$\begin{aligned} \mathsf{Sets} &\to \mathsf{Vect}\text{-}\mathbb{K} && (7.3)\\ X &\mapsto \langle X \rangle =: V. && (7.4) \end{aligned}$$

Recall that, fixed $X \in \mathrm{Ob}\,(\mathsf{Sets})$, and $W \in \mathrm{Ob}\,(\mathsf{Vect} - \mathbb{K})$, for every map $\alpha\colon X \to \mathrm{for}\, W$, we can construct a unique linear map $f\colon \langle X \rangle \to W$ s.t. the diagram commutes

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, hook, "i"] & W \\ \langle X \rangle \arrow[ur, "\exists!\, f"'] & \end{tikzcd} \tag{7.5}$$

i.e. s.t. $f(x) = \alpha(x)\ \forall\, x \in X$. In particular we have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}\,(X, \mathrm{For}\, W) \longleftrightarrow \mathrm{Hom}_{\mathbb{K}}\,(\langle X \rangle_{\mathbb{K}}, W_{\mathbb{K}})\ . \tag{7.6}$$

Definition 7.1: Adjoint pair of functors.
Let C and D be two categories. Consider two functors $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$. The pair (L, R) is called an *adjoint* pair iff there is

$$\mathrm{Hom}_{\mathsf{D}}\,(L(C), D) \xrightarrow{\varphi(C,D)} \mathrm{Hom}_{\mathsf{C}}\,(C, R(D)) \tag{7.7}$$

a bijection natural in C and D. In particular L is the left adjoint of R and R is the right adjoint of L. An adjoint pair is sometimes referred to as an adjunction, and denoted by

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D} \qquad \text{or} \qquad L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R. \tag{7.8}$$

Remark 7.2 In the above remark, the pair $(\langle - \rangle, \mathrm{For}\,)$ is an adjoint pair.

Remark 7.3 Naturality of φ in C and D, more explicitly, means that, for all $f\colon C \to C'$ and all $g\colon D \to D'$, the following diagrams commute

$$\begin{tikzcd} C \arrow[d, "f"] & \mathrm{Hom}_{\mathsf{D}}\,(L(C), D) \arrow[r, "\varphi(C,D)"] & \mathrm{Hom}_{\mathsf{C}}\,(C, R(D)) \\ C' & \mathrm{Hom}_{\mathsf{D}}\,(L(C'), D) \arrow[u, "\mathrm{Hom}_{\mathsf{D}}(L(f),D)"] \arrow[r, "\varphi(C',D)"'] & \mathrm{Hom}_{\mathsf{C}}\,(C', R(D)) \arrow[u, "\mathrm{Hom}_{\mathsf{C}}(f,R(D))"'] \end{tikzcd} \ . \tag{7.9}$$

$$\begin{tikzcd} D \arrow[d, "g"] & \mathrm{Hom}_{\mathsf{D}}\,(L(C), D) \arrow[r, "\varphi(C,D)"] \arrow[d, "\mathrm{Hom}_{\mathsf{D}}(L(C),g)"'] & \mathrm{Hom}_{\mathsf{C}}\,(C, R(D)) \arrow[d, "\mathrm{Hom}_{\mathsf{C}}(C,R(g))"] \\ D' & \mathrm{Hom}_{\mathsf{D}}\,(L(C), D') \arrow[r, "\varphi(C,D')"'] & \mathrm{Hom}_{\mathsf{C}}\,(C, R(D')) \end{tikzcd} \ . \tag{7.10}$$

Definition 7.4: (Co)unit of an adjunction.
Let C and D be two categories. Let $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$ be functors s.t. (L, R) is an adjoint pair. We define

- The *unit* of the adjunction, the natural transformation

$$\eta \colon id_{\mathsf{C}} \to R \circ L \tag{7.11}$$

defined, for every $C \in \mathrm{Ob}\,(\mathsf{C})$, by

$$\eta_C \coloneqq \varphi_{(C,L(C))}\,(1_{LC}) \in \mathrm{Hom}_{\mathsf{C}}\,(C, RL(C))\,. \tag{7.12}$$

- The *counit* of the adjunction, the natural transformation

$$\zeta \colon L \circ R \to id_{\mathsf{D}} \tag{7.13}$$

defined, for every $D \in \mathrm{Ob}\,(\mathsf{D})$, by

$$\zeta_D \coloneqq \varphi^{-1}_{(R(D),D)}\,(1_{RD}) \in \mathrm{Hom}_{\mathsf{C}}\,(LR(D), D)\,. \tag{7.14}$$

Remark 7.5 It is not obvious from the definition that the family of morphisms given by the unit and counit are natural transformation, but they are. Moreover I find it useful to visualize the following diagram to remember how to construct unit and counit of an adjunction (to be read from left to right):

$$\begin{tikzcd} \mathsf{C} \arrow[dr, "L"] \arrow[dd, "\mathrm{id}_{\mathsf{C}}"'] & \\ & \mathsf{D} \arrow[dl, "R"'] \arrow[dd, "\mathrm{id}_{\mathsf{D}}"] \\ \mathsf{C} \arrow[dr, "L"'] & \\ & \mathsf{D}. \end{tikzcd} \tag{7.15}$$

Proposition 7.6. *Given two right adjoints, R and R', of the same functor L, then R and R' are naturally isomorphic. Analogously, given two left adjoints, L and L', of the same functor R, then L and L' are naturally isomorphic.*

Proposition 7.7. *Let $F \colon \mathsf{C} \to \mathsf{D}$ and $R \colon \mathsf{D} \to \mathsf{C}$ be a pair of functors. TFAE*

- *(L, R) is an adjoint pair,*
- *there exist natural transformations*

$$\eta \colon id_{\mathsf{C}} \to R \circ L \qquad \textit{and} \qquad \zeta \colon L \circ R \to id_{\mathsf{D}} \tag{7.16}$$

such that

$$\zeta_{L(C)} \circ L(\eta_C) = id_{L(C)} \qquad \forall\, C \in \mathrm{Ob}\,(\mathsf{C}) \tag{7.17}$$

$$R(\zeta_D) \circ \eta_{R(D)} = id_{R(D)} \qquad \forall\, D \in \mathrm{Ob}\,(\mathsf{D})\,. \tag{7.18}$$

In such case η is the unit, and ζ the counit, of the adjunction.

Remark 7.8 Let (L, R), with $L \colon \mathsf{C} \to \mathsf{D}$ and $R \colon \mathsf{D} \to \mathsf{C}$, be an adjoint pair. Given an arbitrary morphism $\beta \colon C \to RD$, with $C \in \mathrm{Ob}\,(\mathsf{C})$ and $D \in \mathrm{Ob}\,(\mathsf{D})$. Let $\alpha \colon L(C) \to D$ the morphism s.t. $\varphi(C, D) = \beta$. Then there exists a commutative triangle, i.e. that $\beta = R(\alpha) \circ \eta_C$

$$\begin{tikzcd} C \arrow[r, "\beta"] \arrow[d, "\eta_C"'] & R(D) \\ RL(C) \arrow[ur, "R(\alpha)"'] & \end{tikzcd} . \tag{7.19}$$

Remark 7.9 Let (L, R), with $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$, be an adjoint pair. TFAE:

1. R is faithful,
2. R reflects epimorphisms, i.e. if Rf is an epi in C, then f is epi in D,
3. given $\beta\colon C \to R(D)$ epi, then $\alpha := \varphi^{-1}(C, D)(\beta)$ is epi,
4. $\zeta_D\colon LR(D) \to D$ is epi for every $D \in \mathsf{D}$.

Remark 7.10 Given the definitions in the preliminaries, fix two rings S and R, and an R-S bimodule ${}_SM_R$, we can construct the functors (acting on objects as)

$$- \otimes_S M\colon \mathsf{Mod}\text{-}S \to \mathsf{Mod}\text{-}R \tag{7.20}$$
$$N_S \mapsto [N \otimes_S M_R]_R\,, \tag{7.21}$$

$$\mathrm{Hom}_R\left(M_R, -\right)\colon \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S \tag{7.22}$$
$$L_R \mapsto [\mathrm{Hom}_R\left({}_SM_R, L_R\right)]_S\,. \tag{7.23}$$

Proposition 7.11. *The pair $(- \otimes_S M_R, \mathrm{Hom}_R\left(M_R, -\right))$ is an adjoint pair. Moreover also the pair $(M_R \otimes_R -, \mathrm{Hom}_S\left({}_SM, -\right))$ is an adjoint pair. Notice that, if the above functors are between categories of right modules, these are between categories of left modules.*

Example Let $\phi\colon R \to S$ be a ring homomorphism. Then any ${}_SN \in S\text{-}\mathsf{Mod}$ becomes also a left R-module via

$$r \cdot x := \phi(r) \cdot x \qquad \forall\, x \in N,\ \forall\, r \in R. \tag{7.24}$$

And analogously for any right module $N_S \in \mathsf{Mod}\text{-}S$. In particular S becomes both a left and right R-module via ϕ. We can then define the following functors:

$$- \otimes_R S\colon \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S \tag{7.25}$$
$$M_R \mapsto M \otimes_R S \tag{7.26}$$

called *extension of scalars*. And also the *restriction functor*

$$\phi_*\colon \mathsf{Mod}\text{-}S \to \mathsf{Mod}\text{-}S \tag{7.27}$$
$$N_S \mapsto N_R. \tag{7.28}$$

Then the pair $(- \otimes_R S, \phi_*)$ is an adjoint pair.

Analogously we can define the functor

$$\mathrm{Hom}_R\left(S_R, -\right)\colon \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S \tag{7.29}$$
$$M_R \mapsto [\mathrm{Hom}_R\left({}_SS_R, M_R\right)]_S\,, \tag{7.30}$$

and the pair $(\phi_*, \mathrm{Hom}_R\left(S_R, -\right))$ is an adjoint pair.

Proposition 7.12. *Let C and D be arbitrary categories. Let (L, R) be a pair of adjoint functors, $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$. Then*

1. *L preserves colimts, and in particular coproducts, pushouts and cokernels, when they exist,*
2. *R preserves limts, and in particular products, pullbacks and kernels, when they exist.*

Example If $\mathsf{C} \coloneqq R$-Mod and $\mathsf{D} \coloneqq S$-Mod, then $(M \otimes_R -, \mathrm{Hom}_S(M, -))$, for ${}_S M_R$, is an adjoint pair. Then $M \otimes_R -$ preserves colimits. In fact, given a direct system $\{N_i, f_{ji}\}_{i,j \in \mathrm{Ob}(\mathsf{I})}$, for some small category I, then

$$M \otimes_R \varinjlim_i N_i \simeq \varinjlim_i (M \otimes_R N_i) \,. \tag{7.31}$$

Analogously $\mathrm{Hom}_S({}_S M, -)$ preserves limits. Then, given an inverse system $\{L_i, f_{ij}\}_{i,j \in \mathrm{Ob}(\mathsf{I})}$ for some small category I, then

$$\mathrm{Hom}_S\big({}_S M, \varprojlim_i L_i\big) \simeq \varprojlim_i \mathrm{Hom}_S({}_S M, L_i) \,. \tag{7.32}$$

Remark 7.13: Application of the proposition.
Let C and D be abelian categories. Let (L, R) be an adjoint pair, $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$. Then L is *right* exact, and L is *left* exact.

Proposition 7.14. *Let I be a small category, and C be a cocomplete category. Then the colimit functor*

$$\varinjlim \colon \mathsf{C}^{\mathsf{I}} \to \mathsf{C} \tag{7.33}$$

is a left adjoint. If, moreover, C is abelian, $\varinjlim$ is also right exact.

Dually, if C is complete, then $\varprojlim$ is a right adjoint. Again, if C is abelian, then $\varprojlim$ is also left exaxct.

More explicitly, denoting by $\Delta\colon \mathsf{C} \to \mathsf{C}^{\mathsf{I}}$ the diagonal functors, we have the following adjoint pairs:

$$\left(\varinjlim, \Delta\right) \qquad \textit{and} \qquad \left(\Delta, \varprojlim\right) .$$

8 Chain and cochain complexes

Let, in the following, A be a *preadditive* category with 0.

Definition 8.1: Chain complex over A.
We define $\mathrm{Ch}(\mathsf{A})$ the category of *chain complexes* over A as the category whose objects are sequences

$$\ldots \to X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \to \ldots \tag{8.1}$$

s.t. $X_i \in \mathrm{Ob}(\mathsf{A})$, $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. The morphisms d_i are called *differentials* and the sequence is called *complex*, denoted by $\left(X_\bullet, d^X\right)$, with $\left(d^X\right)^2 = 0$.

Morphisms in $\mathrm{Ch}(\mathsf{A})$, denoted by $f\colon \left(X_\bullet, d^X\right) \to \left(Y_\bullet, d^Y\right)$, are a family of morphisms $\{f_n\}_{n \in \mathbb{Z}}$, where $f_n \in \mathrm{Hom}_{\mathsf{A}}(X_n, Y_n)$, making the following diagram commute

$$\begin{tikzcd}
\ldots \arrow[r] & X_n \arrow[r, "d_n^X"] \arrow[d, "f_n"] & X_{n-1} \arrow[r, "d_{n-1}^X"] \arrow[d, "f_{n-1}"] & X_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \ldots \\
\ldots \arrow[r] & Y_n \arrow[r, "d_n^Y"] & Y_{n-1} \arrow[r, "d_{n-1}^Y"] & Y_{n-2} \arrow[r] & \ldots
\end{tikzcd} , \tag{8.2}$$

i.e. such that $d_n^Y \circ f_N = f_{n-1} \circ d_n^X$ for all $n \in \mathbb{Z}$ (more compactly $d^Y \circ f = f \circ d^X$).

Definition 8.2: Cochain complex over A.
We define $\mathrm{Cch}(\mathsf{A})$ the category of *cochain complexes* over A as the category whose objects are sequences

$$\ldots \to X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \to \ldots \tag{8.3}$$

s.t. $X^i \in \mathrm{Ob}(\mathsf{A})$, $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. The morphisms d^i are called *differentials* and the sequence is called *complex*, denoted by $(X^\bullet, d_X)$, with $(d_X)^2 = 0$.

Morphisms in $\mathrm{Cch}(\mathsf{A})$, denoted by $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ are a family of morphisms $\{f^n\}_{n\in\mathbb{Z}}$, where $f^n \in \mathrm{Hom}_{\mathsf{A}}(X^n, Y^n)$, making the following diagram commute

$$\begin{tikzcd} \dots \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r, "d_X^{n+1}"] \arrow[d, "f^{n+1}"] & X^{n+2} \arrow[r] \arrow[d, "f^{n+2}"] & \dots \\ \dots \arrow[r] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r, "d_Y^{n+1}"] & Y^{n+2} \arrow[r] & \dots \end{tikzcd}, \tag{8.4}$$

i.e. such that $f^n \circ d_X^{n-1} = d_Y^{n-1} \circ f^{n-1}$ for all $n \in \mathbb{Z}$ (more compactly $f \circ d_X = d_Y \circ f$).

Remark 8.3: Additive categories.
If A is additive, then also $\mathrm{Ch}(\mathsf{A})$ and $\mathrm{Cch}(\mathsf{A})$ are. In particular, given $(X^\bullet, d_X)$ and $(Y^\bullet, d_Y)$ two cochain complexes, then their coproduct $(X^\bullet \oplus Y^\bullet, d_X \oplus d_Y)$ is given, degree wise, by

$$[X^\bullet \oplus Y^\bullet]^n \coloneqq X^n \oplus Y^n. \tag{8.5}$$

Analogously, degree wise, its differentials are defined by

$$d^n_{X^\bullet \oplus Y^\bullet} \coloneqq d^n_X \oplus d^n_Y = \begin{bmatrix} d^n_X & 0 \\ 0 & d^n_Y \end{bmatrix}. \tag{8.6}$$

Definition 8.4: Bounded (co)chain complex.
A (co)chain complex $(X^\bullet, d_X)$ is *bounded* iff $\exists\, b \in \mathbb{N}$ s.t. $X^n = 0$ for all $|n| > b$. It is bounded *below*, resp. *above*, iff $\exists\, b \in \mathbb{Z}$ s.t. $X^n = 0$ for all $n < b$, resp. $n > b$. (Even though we used the notation for cochain complexes the definitions apply without modification to chain complexes).

We denote respectively with $\mathrm{Ch}(\mathsf{A})^b$, $\mathrm{Ch}(\mathsf{A})^+$ and $\mathrm{Ch}(\mathsf{A})^-$ the full subcategory of bounded, resp. above or below, chain complexes.

Definition 8.5: Canonical functor.
There is a canonical embedding

$$\mathrm{can}\colon \mathsf{A} \to \mathrm{Ch}(\mathsf{A}) \tag{8.7}$$
$$A \mapsto A^\bullet \coloneqq \left[\dots \to 0 \to (A^0 \coloneqq A) \to 0 \to \dots\right]. \tag{8.8}$$

$A^\bullet$ is called complex concentrated in degree 0. Clearly can is fully faithful, hence it is an embedding of A into $\mathrm{Ch}(\mathsf{A})$.

Definition 8.6: Shift functor.
Choose $p \in \mathbb{Z}$, then we can define the functor

$$[p] : \mathrm{Ch}(\mathsf{A}) \to \mathrm{Ch}(\mathsf{A}) \tag{8.9}$$
$$(X^\bullet, d_X) \mapsto (X^\bullet[p], d_X[p]), \tag{8.10}$$

in which we define

$$(X^\bullet[p])^n \coloneqq X^{n+p} \qquad \text{and} \qquad d^n_{X^\bullet[p]} \coloneqq (-1)^p d_X^{n+p}. \tag{8.11}$$

More explicitly this functor shifts the objects in the (co)chain, by p to the left. Analogously it acts on a morphism of complexes $f\colon \left(X_\bullet, d^X\right) \to \left(Y_\bullet, d^Y\right)$ by shifting the morphisms of the family by p to the left. More explicitly

$$([p]f)^n \coloneqq f^{n+p}. \tag{8.12}$$

Moreover we introduce the notation $f[p] \coloneqq [p]f$.

Remark 8.7: Shift functor.
The above is called the *shift functor* if $p = 1$:

$$[1]\colon \mathrm{Ch}(\mathsf{A}) \to \mathrm{Ch}(\mathsf{A}). \tag{8.13}$$

Remark 8.8 The functor $[p]\colon \mathrm{Ch}(\mathsf{A}) \to \mathrm{Ch}(\mathsf{A})$ is an automorphism of categories. In fact $[p] \circ [-p] = id_{\mathrm{Ch}(\mathsf{A})} = [-p] \circ [p]$.

Remark 8.9: Motivational remark.
From algebraic topology. We define Δ_n the standard n-simplex. Given a topological space X one wants to partition it into finitely many n-simplices. One can construct a chain (the simplicial chain complex) by considering X_k, for every $k \in \mathbb{N}$, the set of k-dimensional simplices appearing in the partition of X. Then one can create for each degree k the free abelian group generated by X_k, we denote it by $(C_\bullet)_k$. One also defines a differential $d_k\colon C_k \to C_{k-1}$, which gives rise to a chain complex.

Proposition 8.10. *Given an abelian category* A*, then* $\mathrm{Ch}(\mathsf{A})$ *is abelian, i.e. it admits kernels, cokernles and* Coim *is canonically isomorphic to* Im*.*

Example Let's, for example, define the kernel of a morphism

$$f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y). \tag{8.14}$$

Then, we denote by $K^\bullet := \ker f$ the cochain s.t. $K^n := \ker f^n$ and with differential defined by:

$$\begin{tikzcd}
 & X^n \arrow[rr, "d_X^n"] \arrow[dd, "f^n"'] & & X^{n+1} \arrow[dd, "f^{n+1}"] \\
\ker f^n \arrow[ur, hook, "\epsilon^n"] \arrow[rr, red, "\exists\,!d^n"] & & \ker f^{n+1} \arrow[ur, hook, "\epsilon^{n+1}"'] & \\
 & Y^n \arrow[rr, "d_Y^n"] & & Y^{n+1}
\end{tikzcd}. \tag{8.15}$$

By the commutativity of the diagram we obtain

$$f^{n+1} \circ d_X^n \circ \epsilon^n = d_Y^n \circ f^n \circ \epsilon^n = 0. \tag{8.16}$$

Then, by the second condition on kernels, we obtain $\exists\,!\,d^n\colon \ker f^n \to \ker f^{n+1}$ s.t. $d_X^n \circ \epsilon^n = \epsilon^{n+1} \circ d^n$.

Definition 8.11: Cohomology.
Let A be an abelian category, and $(X^\bullet, d_X) \in \mathrm{Ch}(\mathsf{A})$. Then, since $d_X^n \circ d_X^{n-1} = 0$, as subobjects we have $\mathrm{Im}\, d_X^{n-1} \subset \ker d_X^n$. Hence we can define, for all $n \in \mathbb{Z}$, the following quotient object

$$H^n(X) := \frac{\ker d_X^n}{\mathrm{Im}\, d_X^{n-1}} \in \mathrm{Ob}\,(\mathsf{A}), \tag{8.17}$$

called the n-th cohomology of the cochain complex $(X^\bullet, d_X)$.

Example Let $\mathsf{A} = \mathsf{Ab}$ the category of abelian groups. Consider the following cochain

$$\ldots \to 0 \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\dot{2}} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\dot{2}} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\dot{2}} \ldots =: (X^\bullet, d_X). \tag{8.18}$$

Then $H^0(X) = 2\mathbb{Z}/4\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$, whereas $H^n(X) = 0$ for all $n \neq 0$. If, instead, we considered the following object

$$\ldots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\dot{2}} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\dot{2}} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\dot{2}} \ldots =: (X^\bullet, d_X). \tag{8.19}$$

Then $H^n(X) = 0$ for all $n \in \mathbb{Z}$ and we say that $(X^\bullet, d_X)$ is *acyclic*.

Proposition 8.12. *Let* A *be an abelian category, then, for every* $n \in \mathbb{Z}$*, we can define*

$$H^n : \mathrm{Ch}(\mathsf{A}) \to \mathsf{A} \tag{8.20}$$
$$(X^\bullet, d_X) \mapsto H^n(X). \tag{8.21}$$

In particular this is an additive functor.

Proof. We need to construct, starting from a cochain map $f : X^\bullet \to Y^\bullet$, the associated cohomology morphism

$$H^n(f) : H^n(X) \to H^n(Y). \tag{8.22}$$

■

Remark 8.13 If $\mathsf{A} := \mathsf{Mod}\text{-}R$, then every part of the above result can be checked by diagram chasing. In fact $z \in \ker d_X^n \iff d_X^n(z) = 0$, then

$$d_Y^n \circ f^n(z) = f^{n+1} \circ d_X^n(z) = 0, \tag{8.23}$$

hence $f^n(\ker d_X^n) \subset \ker d_Y^n$. Moreover, given $x \in \operatorname{Im} d_X^{n-1}$, then $x = d_X^{n-1}(z)$, for some $z \in X^{n-1}$. Then

$$f^n(x) = f^n \circ d_X^{n-1}(z) = d_Y^{n-1} \circ f^{n-1}(z) \in \operatorname{Im} d_Y^{n-1}. \tag{8.24}$$

Hence $f^n(\operatorname{Im} d_X^{n-1}) \subset \operatorname{Im} d_Y^{n-1}$. The f induces a map on the quotient

$$\widetilde{f} : \frac{\ker d_X^n}{\operatorname{Im} d_X^{n-1}} \to \frac{\ker d_Y^n}{\operatorname{Im} d_Y^{n-1}}. \tag{8.25}$$

And now, a very important result!

Theorem 8.14 (Freyd-Mitchell embedding)**.** *Let* A *be a small, abelian category. Then there is a ring* R *and a fully faithful exact functor*

$$F : \mathsf{A} \to \mathsf{Mod}\text{-}R. \tag{8.26}$$

Remark 8.15 The above theorem essentially states that we can consider objects of A as if they were modules. In particular any result in $\mathsf{Mod}\text{-}R$ involving only finitely many objects and morphisms (such as exactness, existence and vanishing of morphisms) holds in any abelian category C. This is true, since we can always construct a small full subcategory A_0 of C, containing only the objects and morphism involved in the result (and, by a remark which will follow, an exact and fully faithful functor reflects exactness).

Notice, however, that results for arbitrary family of objects do not translate so easily. For example the product of an arbitrary family of exact sequences in $\mathsf{Mod}\text{-}R$ is still exact in $\mathsf{Mod}\text{-}R$, but not in an arbitrary abelian category.

Sketch of proof (Freyd-Mitchell). Let $\underline{\mathrm{Hom}\,(\mathsf{A}^{op}, \mathsf{Ab})}$ be the category of the additive functors from A^{op} to Ab. Then, by Yoneda lemma, the Yoneda embedding

$$Y : \mathsf{A} \to \underline{\mathrm{Hom}\,(\mathsf{A}^{op}, \mathsf{Ab})} \tag{8.27}$$
$$A \mapsto h^A = \mathrm{Hom}_{\mathsf{A}}(-, A) \tag{8.28}$$

is fully faithful. Moreover it is left exact, since, for every A the functor h^A is left exact. In fact

$$Y : \mathsf{A} \to \mathsf{L} := \mathrm{Lex}\,(\mathsf{A}^{op}, \mathsf{Ab}) \subset \underline{\mathrm{Hom}\,(\mathsf{A}^{op}, \mathsf{Ab})} \tag{8.29}$$

takes values in the category $\mathrm{Lex}\,(\mathsf{A}^{op}, \mathsf{Ab})$ of left exact functors from A^{op} to Ab. We need some facts about L (which are not trivial to show):

1. L is an abelian category. In particular its kernels coincide with the ones in $\underline{\mathrm{Hom}}\,(\mathsf{A}^{op}, \mathsf{Ab})$, whereas cokernels differ. This implies that the inclusion functor $\mathsf{L} \hookrightarrow \underline{\mathrm{Hom}}\,(\mathsf{A}^{op}, \mathsf{Ab})$ is only left exact.

2. The Yoneda embedding $Y\colon \mathsf{A} \to \mathsf{L}$ is fully faithful and exact.

3. L has arbitrary coproducts, i.e. L is cocomplete, and has a projective generator, which is faithful as a functor, namely
$$P := \coprod_{A \in \mathrm{Ob}(\mathsf{A})} h^A. \tag{8.30}$$
Recall that we can take this coproduct since A is a small category, hence $\mathrm{Ob}\,(\mathsf{A})$ is a set.

Summarizing: A is a small abelian full subcategory of L, which is a cocomplete abelian category with a projective generator. Then Freyd-Mitchell follows from the following theorem. ■

Theorem 8.16. *Let C be a cocomplete abelian category with a projective generator. Then, for every small full abelian category $\mathsf{A} \subset \mathsf{C}$, there is a ring R and a fully faithful exact functor*

$$F\colon \mathsf{A} \to \mathsf{Mod}\text{-}R, \tag{8.31}$$

so that A is equivalent to a full subcategory of $\mathsf{Mod}\text{-}R$.

Definition 8.17: Functor reflecting exactness.
Let C and D be abelian categories, and $F\colon \mathsf{C} \to \mathsf{D}$ be an *additive functor*. We say that F *reflects exactness* iff
$$A \to B \to C \tag{8.32}$$
is exact in C, as soon as
$$F(A) \to F(B) \to F(C) \tag{8.33}$$
is exact in D.

Lemma 8.18. *If F is an exact and fully faithful functor, then F reflects exactness. (you can simplify things if you prove it using Freyd-Mitchell)*

Proposition 8.19. *Let A be a small abelian category. The Yoneda embedding*

$$Y\colon \mathsf{A} \to \underline{\mathrm{Hom}\,(\mathsf{A}^{op}, \mathsf{Ab})} \tag{8.34}$$

reflects exactness.

Definition 8.20: Acyclic complex.
A (co)chain complex $(X^\bullet, d_X)$ is *acyclic* iff $H^n(X) = 0$ for all $n \in \mathbb{Z}$, i.e. as a sequence it is exact

$$\ldots \to X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} X^{n+2} \to \ldots. \tag{8.35}$$

8.1 Homotopy category

Let A be an additive category, and $X^\bullet, Y^\bullet \in \mathrm{Ch}(\mathsf{A})$.

Definition 8.21: Nullhomotopic morphism.
A morphism $f \in \mathrm{Hom}_{\mathrm{Ch}(\mathsf{A})}(X^\bullet, Y^\bullet)$ is *nullhomotopic*, or *homotopic to zero*, iff there exists a family of morphism $\{s^n\}_{n\in\mathbb{N}}$, with $s^n : X^n \to Y^{n-1}$, in pictures

$$\begin{tikzcd} \dots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"'] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"'] \arrow[dl, "s^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] \arrow[dl, "s^{n+1}"'] & \dots \\ \dots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n \arrow[r, "d_Y^n"'] & Y^{n+1} \arrow[r] & \dots \end{tikzcd} \tag{8.36}$$

such that $f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$ for all $n \in \mathbb{Z}$. More compactly we write $f = s \circ d_X + d_Y \circ s$. The morphisms s^n are called *homotopies* or *cochain contractions*. Moreover, if f is nullhomotopic, we write $f \sim 0$.

Definition 8.22: Homotopic morphisms.
Two cochain maps $f, g\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ are called *homotopic*, denoted by $f \sim g$, iff $f - g$ is nullhomotopic.

Remark 8.23 The relation $\sim$ is an equivalence relation.

Definition 8.24: Homotopy category.
Given, as before, an additive category A, we define the homotopy category $K(\mathsf{A})$ as follows. Its objects are exactly the objects in $\mathrm{Ch}(\mathsf{A})$. Its morphisms, instead, are equivalence classes of (co)chain maps, under the homotopy relation $\sim$ we just defined. More explicitly

$$\mathrm{Hom}_{K(\mathsf{A})}(X^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{\mathrm{Ch}(\mathsf{A})}(X^\bullet, Y^\bullet) / \sim \tag{8.37}$$

$$g \mapsto [g]_\sim \,. \tag{8.38}$$

Remark 8.25 The homotopy relation $\sim$ is compatible with addition, hence it is a congruence. In particular, denoted with $\mathrm{Hom}_t(X^\bullet, Y^\bullet) \subset \mathrm{Hom}_{\mathrm{Ch}(\mathsf{A})}(X^\bullet, Y^\bullet)$ the subgroup of nullhomotopic (co)chain maps, then

$$\mathrm{Hom}_{K(\mathsf{A})}(X^\bullet, Y^\bullet) = \frac{\mathrm{Hom}_{\mathrm{Ch}(\mathsf{A})}(X^\bullet, Y^\bullet)}{\mathrm{Hom}_t(X^\bullet, Y^\bullet)}. \tag{8.39}$$

Moreover, let $f, g\colon X^\bullet \to Y^\bullet$ be homotopic cochain maps. Let $\alpha\colon Z^\bullet \to X^\bullet$ and $\beta\colon Y^\bullet \to W^\bullet$ be cochain maps, then, by linearity of composition, we obtain $\beta \circ f \circ \alpha \sim \beta \circ g \circ \alpha$.

Proposition 8.26. *$K(\mathsf{A})$ is an additive category, and the quotient functor, defined*

$$q\colon \mathrm{Ch}(\mathsf{A}) \to K(\mathsf{A}) \tag{8.40}$$

$$X^\bullet \mapsto X^\bullet \tag{8.41}$$

$$f \mapsto [f]_\sim \tag{8.42}$$

is an additive functor.

Definition 8.27: Homotopy equivalence.
A cochain map $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ is said to be a *homotopy equivalence* iff $\exists\, g\colon (Y^\bullet, d_Y) \to (X^\bullet, d_X)$ s.t. $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$. In other words a homotopy equivalence is an isomorphism in $K(\mathsf{A})$.

Proposition 8.28. *Let* A *be an abelian category and* $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ *be a nullhomotopic cochain map. Then the induced cohomology map*

$$H^n(f) =: \overline{f^n}\colon H^n(X) \to H^n(Y) \tag{8.43}$$

is the zero map for every $n \in \mathbb{Z}$.

Proof. We can use Freyd-Mitchell (this proposition deals with a finite number of objects and morphisms). Then, by definition

$$H^n(f)\left(x + \operatorname{im} d_X^{n-1}\right) = f^n(x) + \operatorname{im} d_Y^{n-1}. \tag{8.44}$$

But $f^n(x) = d_Y^{n-1} \circ s^n(x) + s^{n+1} \circ d_X^n(x)$, then

$$H^n(f)(x) = d_Y^{n-1} \circ s^n(x) + \operatorname{im} d_Y^{n-1} = 0 + \operatorname{im} d_Y^{n-1}. \qquad \blacksquare$$

Corollary 8.29. *Let* f *and* g *be homotopic maps, then*

$$H^n(f) = H^n(g) \qquad \forall\, n \in \mathbb{Z}. \tag{8.45}$$

Proof. H^n is an additive functor for each $n \in \mathbb{Z}$. $\blacksquare$

Remark 8.30 In general A abelian implies $\mathrm{Ch}(\mathsf{A})$ abelian, but not $K(\mathsf{A})$ abelian.

Definition 8.31: Semisimple ring.
A ring R is called *semisimple* iff every R-module is projective. Equivalently iff every short exact sequence splits.

Example Any field $\mathbb{K}$ is semisimple, but $\mathbb{Z}$ is not. As a consequence of the following proposition, we get that $K(\mathsf{Mod}\text{-}\mathbb{Z})$ is not abelian.

Proposition 8.32. *The following statement (and more importantly the proof) should be incorrect. Here what should be the correct one (I'm not going to copy the proof again, though). Let* $\mathsf{A} := \mathsf{Mod}\text{-}R$. *If* $K(\mathsf{A})$ *is abelian, then* R *is semisimple.*

Proposition 8.33. *Let* $\mathsf{A} := \mathsf{Mod}\text{-}R$. *If* R *is not semisimple, then* $K(\mathsf{A})$ *is not abelian.*

Proof. Assume that R is not semisimple, but $K(\mathsf{Mod}\text{-}R)$ is abelian. Since R is not semisimple, then there exists a short exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{\pi} Z \to 0 \qquad \text{in } \mathsf{Mod}\text{-}R \tag{8.46}$$

which does not split. Consider now $X^\bullet, Y^\bullet, Z^\bullet$ as complexes concentrated in degree 0. Since $K(\mathsf{Mod}\text{-}R)$ is abelian, then $f := q(f)$ has a cokernel And, by uniqueness up to isomorphism of the cokernel, we can assume that π is a cokernel of f in $K(\mathsf{Mod}\text{-}R)$.

Consider the complex $\operatorname{Cone} f$:

$$0 \to X \xrightarrow{f} Y \to 0, \tag{8.47}$$

where Y is in degree 0. Let's define the cochain map $\alpha\colon Y^\bullet \to \operatorname{Cone} f$ defined by

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & Y \arrow[r] \arrow[d, "1_Y"] & 0 \arrow[r] \arrow[d, "0"] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & X \arrow[r, "f"'] & Y \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd} \; . \tag{8.48}$$

Then we claim that there exist γ, δ s.t. $\alpha = \gamma \circ \pi$ and $\delta \circ \alpha = \pi$, i.e. s.t. the following diagram commutes.

$$\begin{tikzcd}
X^\bullet \arrow[r, "f"] & Y^\bullet \arrow[r, "\pi"] \arrow[dr, "\alpha"'] & Z^\bullet \arrow[d, dashed, shift right, "\gamma"'] \\
& & \mathrm{Cone}\, f \arrow[u, dashed, shift right, "\delta"']
\end{tikzcd} . \tag{8.49}$$

At first we notice that $\alpha \circ f = 0$ in $K(\mathsf{Mod}\text{-}R)$, in fact:

$$\begin{tikzcd}
& 0 \arrow[r] & X \arrow[r] \arrow[d, "f"] \arrow[ddl, dashed, "1_X"'] & 0 \arrow[ddl, dashed, "0"] \\
& 0 \arrow[r] & Y \arrow[r] \arrow[d, "1_Y"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"'] & Y \arrow[r] & 0
\end{tikzcd} . \tag{8.50}$$

Since π is a cokernel of f, then $\exists\,!\,\gamma\colon Z^\bullet \to \mathrm{Cone}\, f$ s.t. $\gamma \circ \pi = \alpha$. With regard to $\delta\colon \mathrm{Cone}\, f \to Z^\bullet$, instead, we define $(0, \pi)$, i.e. the family of maps which all correspond to zero, apart from degree 0, in which it is π. Then $\delta \circ \alpha = \pi$, as described by the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "0"] & Y \arrow[r] \arrow[d, "1_Y"] & 0 \arrow[d, "0"] \\
X \arrow[r, "f"] \arrow[d, "0"] & Y \arrow[r] \arrow[d, "\pi"] & 0 \arrow[d, "0"] \\
0 \arrow[r] & Z \arrow[r] & 0
\end{tikzcd} . \tag{8.51}$$

Then we have $\pi = \delta \circ \alpha = \delta \circ \gamma \circ \pi$. Since π is epi (it is a cokernel), we obtain that $\delta \circ \gamma = id_Z$ in $K(\mathsf{Mod}\text{-}R)$. But then, if we denote by $\gamma_0\colon Z \to Y$ the morphism in degree 0 of γ, we obtain that $\pi \circ \gamma_0 = 1_Z$, hence we have found a retraction of π in (8.46). This is a contradiction, since we assumed it did not split. ■

8.2 Snake lemma and applications

Lemma 8.34. *Let* A *be an abelian category, and let*

$$\begin{tikzcd}
& A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C'
\end{tikzcd} \tag{8.52}$$

be a commutative diagram with exact rows. Then there is an exact sequence:

$$\ker f \xrightarrow{\underline{\alpha}} \ker g \xrightarrow{\underline{\beta}} \ker h \xrightarrow{\partial} \mathrm{coker}\, f \xrightarrow{\overline{\alpha'}} \mathrm{coker}\, g \xrightarrow{\overline{\beta'}} \mathrm{coker}\, h, \tag{8.53}$$

in which ∂ is called the connecting morphism*. Moreover α mono implies $\underline{\alpha}$ is mono, whereas β' epi implies $\overline{\beta'}$ is epi.*

Remark 8.35: Short exact sequences in the category of complexes.
Since the abelian structure of $\mathrm{Ch}(\mathsf{A})$ is defined degree wise, we have that a sequence in $\mathrm{Ch}(\mathsf{A})$

$$0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0 \tag{8.54}$$

is exact in $\mathrm{Ch}(\mathsf{A})$ iff, for every $n \in \mathbb{Z}$, the corresponding

$$0 \to X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \to 0 \tag{8.55}$$

is exact in A.

Theorem 8.36 (Fundamental theorem in (co)homology)**.** *Consider a short exact sequence in $\mathrm{Ch}(\mathsf{A})$, for an abelian category A,*

$$0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} W^\bullet \to 0. \tag{8.56}$$

Then we can associate to it a long exact sequence in A, called the long exact sequence in (co)homology*, given as follows:*

$$\ldots \to H^n(X^\bullet) \xrightarrow{H^n(f)} H^n(Y^\bullet) \xrightarrow{H^n(g)} H^n(W^\bullet) \xrightarrow{\partial} H^{n+1}(X^\bullet) \to H^{n+1}(Y^\bullet) \to \ldots, \tag{8.57}$$

Proof. The proof is essentially an application of the snake lemma. In particular we obtain that $\partial\colon H^n(W^\bullet) \to H^{n+1}(X^\bullet)$ acts as

$$\partial\colon H^n(W^\bullet) \to H^{n+1}(X^\bullet) \tag{8.58}$$

$$[z^n] \mapsto \left[(f^{n+1})^{-1}\left(d_Y^n((g^n)^{-1}(z^n))\right)\right]. \tag{8.59}$$

More visually it is defined by the following diagram chase:

$$\begin{tikzcd}
 & Y^n \arrow[r, "g^n"] \arrow[d, "d_Y^n"'] & Z^n \arrow[d, "0"] \\
X^{n+1} \arrow[r, "f^{n+1}"] \arrow[d, "d_X^{n+1}"'] & Y^{n+1} \arrow[r, "g^{n+1}"] \arrow[d, "d_Y^{n+1}"] & 0 \\
0 = X^{n+2} \arrow[r, "f^{n+2}"] & 0 &
\end{tikzcd} . \tag{8.60}$$

■

Remark 8.37: Notation.
We denote by $Z^n(X^\bullet) := \ker d_X^n$, the n-cycles, and by $B^n(X^\bullet) := \operatorname{im} d_X^{n-1}$, the n-boundaries. Both clearly are subobjects of X^n.

Definition 8.38: Long/short exact sequence category.
Let A be an abelian category.

- We define S, the category of short exact sequences in $\mathrm{Ch}(\mathsf{A})$, as the category whose objects are short exact sequences with objects in $\mathrm{Ob}\,(\mathrm{Ch}(\mathsf{A}))$ and whose morphisms, called morphisms of short exact sequences, are triples (f, g, h) of cochain maps such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r, "\alpha"] \arrow[d, "f"'] & B^\bullet \arrow[r, "\beta"] \arrow[d, "g"'] & C^\bullet \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & X^\bullet \arrow[r, "\alpha'"'] & Y^\bullet \arrow[r, "\beta'"'] & W^\bullet \arrow[r] & 0
\end{tikzcd} . \tag{8.61}$$

- We define L, the category of long exact sequences in A, as the category whose objects are exact sequences in $\mathrm{Ob}\,(\mathrm{Ch}\mathsf{A})$ and whose morphisms are morphisms of complexes, i.e. families of maps $\{f^n\}_{n\in\mathbb{Z}}$ making the following diagram commute

$$\begin{tikzcd} \dots \arrow[r] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"'] & A^{n+2} \arrow[r, "d_A^{n+2}"] \arrow[d, "f^{n+2}"'] & \dots \\ \dots \arrow[r] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r, "d_B^{n+1}"'] & B^{n+2} \arrow[r, "d_B^{n+2}"'] & \dots \end{tikzcd} . \tag{8.62}$$

Proposition 8.39. *Given an abelian category* A*, then we can define a functor*

$$L\colon \mathsf{S} \to \mathsf{L}, \tag{8.63}$$

that, on objects, maps each short exact sequence of complexes to its corresponding exact sequence in (co)homology. In particular a given morphism in S

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r, "\alpha"] \arrow[d, "f"'] & B^\bullet \arrow[r, "\beta"] \arrow[d, "g"'] & C^\bullet \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & X^\bullet \arrow[r, "\alpha'"'] & Y^\bullet \arrow[r, "\beta'"'] & W^\bullet \arrow[r] & 0 \end{tikzcd} \tag{8.64}$$

gets mapped to the following morphism of long exact sequences, in L

$$\begin{tikzcd} \dots \arrow[r] & H^n(B^\bullet) \arrow[r, "H^n(\beta)"] \arrow[d, "H^n(g)"'] & H^n(C^\bullet) \arrow[r, "\partial_1^n"] \arrow[d, "H^n(h)"'] \arrow[dr, phantom, "\circlearrowright"] & H^{n+1}(A^\bullet) \arrow[r, "H^{n+1}(\alpha)"] \arrow[d, "H^{n+1}(f)"] & H^{n+1}(B^\bullet) \arrow[r, "H^{n+1}(\beta)"] \arrow[d, "H^{n+1}(g)"] & \dots \\ \dots \arrow[r] & H^n(Y^\bullet) \arrow[r, "H^n(\beta')"'] & H^n(W^\bullet) \arrow[r, "\partial_2^n"'] & H^{n+1}(X^\bullet) \arrow[r, "H^{n+1}(\alpha')"'] & H^{n+1}(Y^\bullet) \arrow[r, "H^{n+1}(\beta')"'] & \dots \end{tikzcd} .$$

In particular also the squares involving the connecting morphisms ∂^n commute, in other words we have $H^{n+1}(f) \circ \partial_1^n = \partial_2^n \circ H^n(h)$.

Remark 8.40: Notation.
The long exact (co)homology sequence associated to

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0 \tag{8.65}$$

can be visualized by the following diagram, called the exact triangle

$$\begin{tikzcd} H^\bullet(A^\bullet) \arrow[rr] & & H^\bullet(B^\bullet) \arrow[dl] \\ & H^\bullet(C^\bullet) \arrow[ul, "\partial"] & \end{tikzcd} . \tag{8.66}$$

Lemma 8.41 (3×3 lemma)**.** *Let* A *be an abelian category. Consider the following commutative diagram with exact columns*

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, tail] & B_1 \arrow[r] \arrow[d, tail] & C_1 \arrow[r] \arrow[d, tail] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] \arrow[d, two heads] & B_2 \arrow[r] \arrow[d, two heads] & C_2 \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & A_3 \arrow[r] & B_3 \arrow[r] & C_3 \arrow[r] & 0 \end{tikzcd} \tag{8.67}$$

.

1. *If the* 2*nd and* 3*rd rows are exact, then so is the* 1*st.*

2. *If the* 1*st and* 2*nd rows are exact, then so is the* 3*rd.*

3. *If the* 1*st and* 3*rd rows are exact, and the* 2*nd is a complex, then the* 2*nd is also exact.*

Lemma 8.42 (5 lemma)**.** *Let* A *be an abelian category. Consider the following commutative diagram with exact rows*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "a"] & B_1 \arrow[r] \arrow[d, "b"] & C_1 \arrow[r] \arrow[d, "c"] & D_1 \arrow[r] \arrow[d, "d"] & E_1 \arrow[d, "e"] \\ A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & D_2 \arrow[r] & E_2. \end{tikzcd} \tag{8.68}$$

1. *If* b *and* d *are mono and* a *is epi, then* c *is mono.*

2. *If* b *and* d *are epi and* e *is mono, then* c *is epi.*

Definition 8.43: quasi-isomorphism.
Let A be an abelian category. Let $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ be a cochain map in $\mathrm{Ch}(\mathsf{A})$. We say that f is a *quasi-isomorphism* iff the induced cohomology morphism

$$H^n(f)\colon H^n(X^\bullet) \to H^n(Y^\bullet) \tag{8.69}$$

is an isomorphism for every $n \in \mathbb{Z}$.

Lemma 8.44. *An homotopy equivalence* $f\colon X^\bullet \to Y^\bullet$*, i.e. an iso in* $K(\mathsf{A})$*, is a* quasi-isomorphism.

Lemma 8.45. *One can find examples of quasi-isomorphism, which is not an homotopy equivalence. (Look at morphisms of exact sequences).*

Lemma 8.46. *Let* A *be an abelian category and consider* $(X^\bullet, d_X) \in \mathrm{Ch}(\mathsf{A})$*. Define* $(Z^\bullet, d_Z)$ *by:*

$$Z^n \coloneqq Z^n(X^\bullet) \coloneqq \ker d_X^n \qquad \textit{and} \qquad d_Z^n = 0 \qquad \forall\, n \in \mathbb{Z}. \tag{8.70}$$

Analogously define the complex $(B^\bullet, d_B)$ *by*

$$B^n \coloneqq B^n(X^\bullet) \coloneqq \operatorname{im} d_X^{n-1} \qquad \textit{and} \qquad d_B^n = 0 \quad \forall\, n \in \mathbb{Z}. \tag{8.71}$$

Then there is a short exact sequence of complexes

$$0 \to Z^\bullet \to X^\bullet \to B^\bullet[1] \to 0, \tag{8.72}$$

whose associated long exact sequence breaks into short exact sequences in A*.*

Proof. Apart from the exactness of the short exact sequence of complexes, notice that: $H^n(Z^\bullet) = Z^n$ and $H^n(B^\bullet[1]) = H^{n+1}(B^\bullet) = B^{n+1}$ for all $n \in \mathbb{Z}$. Then the associated long exact sequence is

$$\dots B^n \xrightarrow{\partial} Z^n \to H^n(X^\bullet) \to B^{n+1} \xrightarrow{\partial} Z^{n+1} \to H^{n+1}(X^\bullet) \to \dots. \tag{8.73}$$

But, for each n, the above breaks into the short exact sequences

$$0 \to B^n \to Z^n \to H^n(X^\bullet) \to 0. \qquad \blacksquare$$

Lemma 8.47. *Let* $f\colon (X^\bullet, d_X) \to (Y^\bullet, d_Y)$ *be a cochain map in* $\mathrm{Ch}(\mathsf{A})$*, for an abelian category* A*. Assume that* $(\ker f^\bullet, d^\bullet)$ *and* $(\operatorname{coker} f^\bullet, d^\bullet)$ *are acyclic. Then* f *is a quasi-isomorphism.*

Remark 8.48 Notice that the converse of the above lemma is not true: for example the complexes

$$X^\bullet = Y^\bullet = 0 \to \mathbb{Z} \xrightarrow{\dot{2}} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0 \tag{8.74}$$

are both acyclic. Then the map $f = (\dot{4}, \dot{4}, 0)$, represented by

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\dot{2}"] \arrow[d, "\dot{4}"'] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\dot{4}"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "0"'] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\dot{2}"'] & \mathbb{Z} \arrow[r, "\pi"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{8.75}$$

is a quasi-isomorphism. Moreover the cochains $(\ker f)^\bullet$ and $(\operatorname{coker} f)^\bullet$ are

$$(\ker f)^\bullet = 0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0 \tag{8.76}$$

$$(\operatorname{coker} f)^\bullet = 0 \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\dot{2}} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0. \tag{8.77}$$

Then we can compute that $H^2((\ker f)^\bullet) = \mathbb{Z}/2\mathbb{Z}$ and $H^0((\operatorname{coker} f)^\bullet) = \mathbb{Z}/2\mathbb{Z}$.

8.3 Operations on complexes

Definition 8.49: Canonical truncation.
Let $(X^\bullet, d_X)$ be a cochain complex and $n \in \mathbb{Z}$. We define the *canonical truncation* of $(X^\bullet, d_X)$ to be the complex $\big([\tau_{\leq n}(X^\bullet)]^\bullet, d_{[\tau_{\leq n}(X^\bullet)]}\big)$, whose objects are

$$[\tau_{\leq n}(X^\bullet)]^i := \begin{cases} X^i & \text{if } i < n \\ \ker d_X^n & \text{if } i = n \\ 0 & \text{if } i > n \end{cases}, \tag{8.78}$$

and differentials given by the induced ones. Denoted by $\epsilon^n\colon \ \ker d_X^n \to X^n$ the Kernel, then we have a natural cochain map $\epsilon\colon \tau_{\leq n}(X^\bullet) \to X^\bullet$, given by

$$\begin{tikzcd} \dots \arrow[r] & X^{n-2} \arrow[r, "d^{n-2}"] \arrow[d, "1_{X^{n-2}}"'] & X^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "1_{X^{n-1}}"'] & Z^n(X^\bullet) \arrow[r, "0"] \arrow[d, "\epsilon^n"'] & 0 \arrow[r, "0"] \arrow[d, "0"'] & \dots \\ \dots \arrow[r] & X^{n-2} \arrow[r, "d^{n-2}"'] & X^{n-1} \arrow[r, "d^{n-1}"'] & X^n \arrow[r, "d^n"'] & X^{n+1} \arrow[r, "d^{n+1}"'] & \dots \end{tikzcd}, \tag{8.79}$$

which is clearly a mono. Moreover we can compute the associated cohomology groups (assuming A is abelian, or that we can compute them) and they are

$$H^i\left(\tau_{\leq n}(X^\bullet)\right) = \begin{cases} 0 & \text{if } i > n \\ H^i(X^\bullet) & \text{if } i \leq n \end{cases}. \tag{8.80}$$

Moreover, since ϵ is an embedding, we can define the quotient complex, which we denote by $\big([X^\bullet/\tau_{\leq n}(X^\bullet)]^\bullet, d_{[X^\bullet/\tau_{\leq n}(X^\bullet)]}\big)$, whose objects are

$$[X^\bullet/\tau_{\leq n}(X^\bullet)]^i := \begin{cases} X^i & \text{if } i > n \\ X^n/\ker d_X^n & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}, \tag{8.81}$$

and differentials given by the induced one. Then, as expected

$$H^i\left(X^\bullet/\tau_{\leq n}(X^\bullet)\right) = \begin{cases} 0 & \text{if } i \leq n \\ H^i(X^\bullet) & \text{if } i > n \end{cases}. \tag{8.82}$$

And we obtain a short exact sequence of complexes

$$0 \longrightarrow \tau_{\leq n}(X^\bullet) \xrightarrow{\epsilon} X^\bullet \longrightarrow X^\bullet/\tau_{\leq n}(X^\bullet) \longrightarrow 0.$$

Definition 8.50: Stupid truncation.
Given, as before, a cochain complex $(X^\bullet, d_X)$ and $n \in \mathbb{Z}$, one defines its *stupid truncation* as the cochain complex with objects

$$[\sigma_{\leq n}(X^\bullet)]^i = \begin{cases} X^i & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases} \tag{8.83}$$

and induced differentials. Then one can construct a canonical map $X^\bullet \to \sigma_{\leq n}(X^\bullet)$ as

$$\begin{tikzcd} \dots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "1_{X^{n-1}}"] & X^n \arrow[r] \arrow[d, "1_{X^n}"] & X^{n+1} \arrow[r] \arrow[d, "0"] & \dots \\ \dots \arrow[r] & X^{n-1} \arrow[r] & X^n \arrow[r, "0"'] & 0 \arrow[r] & \dots \end{tikzcd}. \tag{8.84}$$

Moreover we can compute its cohomology groups, and obtain that they are

$$H^i\left(\sigma_{\leq n}(X^\bullet)\right) = \begin{cases} 0 & \text{if } i > n \\ X^n/\operatorname{im} d_X^{n-1} & \text{if } i = n \\ H^i(X^\bullet) & \text{if } i < n \end{cases}. \tag{8.85}$$

Definition 8.51: Mapping cone.
Let $f \in \operatorname{Hom}_{\mathsf{Ch(A)}}(X^\bullet, Y^\bullet)$ an arbitrary cochain map. We define the *mapping cone* of f as the cochain complex, denoted by $(\operatorname{Cone} f)^\bullet$, whose objects are

$$[\operatorname{Cone} f]^n \coloneqq Y^n \oplus X^{n+1} \tag{8.86}$$

and differentials $d^n_{\operatorname{Cone} f} \colon Y^n \oplus X^{n+1} \to Y^{n+1} \oplus X^{n+2}$ given by the following matrix

$$d^n_{\operatorname{Cone} f} \coloneqq \begin{bmatrix} d_Y^n & f^{n+1} \\ 0 & -d_X^{n+1} \end{bmatrix}. \tag{8.87}$$

This really is a complex, since we have the identity

$$d^2_{\operatorname{Cone} f} = \begin{bmatrix} d_Y^n & f^{n+1} \\ 0 & -d_X^{n+1} \end{bmatrix} \begin{bmatrix} d_Y^{n-1} & f^n \\ 0 & -d_X^n \end{bmatrix} = \begin{bmatrix} 0 & d_Y^n f^n - f^{n+1} d_X^n \\ 0 & 0 \end{bmatrix} \tag{8.88}$$

and f is a cochain map (hence the last matrix is zero).

Definition 8.52: Cone of a complex.
Given a complex $(X^\bullet, d_X)$, we define the cocahin $(\operatorname{Cone} X)^\bullet$ as the mapping cone of the cochain map $1_{X^\bullet} \colon X^\bullet \to X^\bullet$.

Remark 8.53 From the definition of mapping cone we obtain the short exact sequence of complexes

$$0 \longrightarrow Y^\bullet \xrightarrow{\ \alpha\ } (\operatorname{Cone} f)^\bullet \xrightarrow{\ \beta\ } X^\bullet[1] \longrightarrow 0, \tag{8.89}$$

where the maps α and β (check they are indeed cochain maps) are defined by the matrices

$$\alpha := \begin{bmatrix} 1_Y \\ 0 \end{bmatrix} \qquad \text{and} \qquad \beta := \begin{bmatrix} 0 & 1_{X^\bullet[1]} \end{bmatrix}. \tag{8.90}$$

In particular, for each degree, the short exact sequence splits, in fact it is

$$0 \to Y^n \to Y^n \oplus X^{n+1} \to X^{n+1} \to 0, \tag{8.91}$$

and the maps are induced by α and β (hence the splitting).

Lemma 8.54. *Let the following be a degree-wise splitting short exact sequence*

$$0 \longrightarrow Y^\bullet \longrightarrow C^\bullet \longrightarrow W^\bullet \longrightarrow 0.$$

Then there is a cochain map $f\colon W^\bullet[-1] \to Y^\bullet$ s.t. $C^\bullet \simeq \operatorname{Cone} f$.

Lemma 8.55. *Let $f\colon X^\bullet \to Y^\bullet$ be a cochain map in $\operatorname{Ch}(\mathsf{A})$. Then f is a quasi-isomorphism iff the complex $(\operatorname{Cone} f)^\bullet$ is acyclic.*

Proof. From the short exact sequence for the Cone of f, see (8.89), and the fundamental theorem in cohomology, one obtains the long exact cohomology sequence

$$\ldots \to H^{n-1}(X^\bullet[1]) \xrightarrow{\partial^n} H^n(Y^\bullet) \to H^n(\operatorname{Cone} f) \to H^n(X^\bullet[1]) \to \ldots. \tag{8.92}$$

One can show that $H^n(f) = \partial^n$, then $H^n(f)$ is an isomorphism iff $H^n(\operatorname{Cone} f) = 0$. ■

Definition 8.56: Split complex.
A complex $(X^\bullet, d_X)$ is *split* iff there exist maps $s^n\colon X^{n+1} \to X^n$, for all $n \in \mathbb{Z}$, s.t. $d_X^n \circ s^n \circ d_X^n = d_X^n$ for all $n \in \mathbb{Z}$ (shortly $d = d \circ s \circ d$). The maps s^n are called *splitting maps*.

Lemma 8.57. *Let $(X^\bullet, d_X)$ be a complex, with cycles Z^n and boundaries B^n. $X^\bullet$ is split iff, for every $n \in \mathbb{Z}$, there exist decompositions*

$$X^n = Z^n \oplus C^n \qquad \textit{and} \qquad Z^n = B^n \oplus K^n, \tag{8.93}$$

with $K^n \simeq H^n(X^\bullet)$.

Proof. In this proof we use the general fact, for R-modules, that given an idempotent endomorphism $e\colon M \to M$ (i.e. s.t. $e^2 = e$), then

$$M = \ker e \oplus \operatorname{im} e. \tag{8.94}$$

In fact for any $x \in M$, then $x = e(x) + (x - e(x))$ and $e(x - e(x)) = 0$. Moreover, given $x \in \ker e \cap \operatorname{im} e$, there exists y s.t. $x = e(y)$, then

$$0 = e(x) = e(e(y)) = e(y) = x.$$ ■

Definition 8.58: Split exact/contractible complex.
A complex $(X^\bullet, d_X)$ is called *split exact* or *contractible* iff it is both *split* and *acyclic* (i.e. exact).

Remark 8.59 By the above lemma, the complex $(X^\bullet, d_X)$ is contractible iff there exist decompositions $X^n = Z^n \oplus C^n$ and $B^n = Z^n$, for every $n \in \mathbb{Z}$.

Lemma 8.60. $(\operatorname{Cone} X)^\bullet$ *is contractible.*

Proof. $(\operatorname{Cone} C)^\bullet$ is exact, since $1_{X^\bullet}$ is a quasi-isomorphism. Then we define the splitting maps by

$$s^n := \begin{bmatrix} 0 & 0 \\ 1_{X^{n+1}} & 0. \end{bmatrix}$$ ■

Lemma 8.61. *A complex $(X^\bullet, d_X)$ is contractible iff $1_{X^\bullet}$ is nullhomotopic.*

Remark 8.62 This lemma can be stated as: any contractible complex is isomorphic to the 0 complex in the homotopy category.

Lemma 8.63. *Let $f \colon X^\bullet \to Y^\bullet$ be a cochain map. Then $f \sim 0$ iff f extends to*

$$\begin{bmatrix} f & s \end{bmatrix} : \operatorname{Cone} X \to Y, \tag{8.95}$$

where $\{s^n\}_{n \in \mathbb{Z}}$ are the contractions.

Lemma 8.64. *Let $(X^\bullet, d_X)$ be a split complex with splitting maps $s := \{s^n\}_{n\in\mathbb{Z}}$. Then $f = s \circ d + d \circ s$ is a cochain map (clearly, then $f \sim 0$).*

Remark 8.65 for all $A \in \operatorname{Ob}(\mathsf{A})$, we define the following complex

$$D^n(A) := \ldots \to 0 \to A \xrightarrow{1_A} A \to 0 \to \ldots, \tag{8.96}$$

where the non-zero elements are in degree n and $n+1$. Clearly $D^n(A)$ contractible. In fact:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "1_A"] \arrow[d, "1_A"'] \arrow[dl, "0"'] & A \arrow[r] \arrow[d, "1_A"] \arrow[dl, "1_A"'] & 0 \arrow[dl, "0"] \\ 0 \arrow[r] & A \arrow[r, "1_A"'] & A \arrow[r] & 0 \end{tikzcd} . \tag{8.97}$$

Lemma 8.66. *for all $A \in \operatorname{Ob}(\mathsf{A})$ and $X^\bullet \in \operatorname{Ch}(\mathsf{A})$ we have, naturally in both components,*

$$\operatorname{Hom}_{\operatorname{Ch}(\mathsf{A})}(D^n(A), X^\bullet) \simeq \operatorname{Hom}_{\mathsf{A}}(A, X^n). \tag{8.98}$$

In other words the pair $(D^n, (-)^n)$ is an adjoint pair for every $n \in \mathbb{Z}$, for the functors

$$\begin{aligned} D^n \colon \mathsf{A} &\longrightarrow \operatorname{Ch}(\mathsf{A}) \\ A &\longmapsto D^n(A) \end{aligned} \qquad \textit{and} \qquad \begin{aligned} (-)^n \colon \operatorname{Ch}(\mathsf{A}) &\longrightarrow \mathsf{A} \\ X^\bullet &\longmapsto x^n. \end{aligned}$$

Proposition 8.67. *Let A be an abelian category. A complex $(P^\bullet, d_P)$ is a projective object of $\operatorname{Ch}(\mathsf{A})$ iff P^i is projective in A for all $i \in \mathbb{Z}$ and $(P^\bullet, d_P)$ is contractible.*

A complex $(I^\bullet, d_I)$ is an injective object of $\operatorname{Ch}(\mathsf{A})$ iff I^i is injective in A for all $i \in \mathbb{Z}$ and $(I^\bullet, d_I)$ is contractible.

Lemma 8.68. *Assume that A is an abelian category, with enough projectives (i.e. $\forall\, A \in \operatorname{Ob}(\mathsf{A})$ there is a projective object $P \in \operatorname{Ob}(\mathsf{A})$, with an epi $P \xrightarrow{\varphi} A \to 0$). Then $\operatorname{Ch}(\mathsf{A})$ has enough projectives.*

Remark 8.69 Given a cochain complex $(X^\bullet, d_X)$, we can define an associated chain complex $\left(X_\bullet, d^X\right)$ by setting $X_n := X^{-n}$.

9 Derived functors

9.1 Resolutions

Definition 9.1: (Co)homological ∂-functor.
Let A, B be abelian categories. A *(co)homological ∂-functor* between A and B is the data of a sequence of functors $\{T^n\}_{n\in\mathbb{Z}}$, with $T^n \colon \mathsf{A} \to \mathsf{B}$ for every n, ($\{T_n\}_{n\in\mathbb{Z}}$ for the homological functors) s.t. $T^i = 0$ for all $i < 0$ ($T_i = 0$ for all $i > 0$) and for any short exact sequence $S \in \mathsf{S}(\mathsf{A})$

$$0 \to A \to B \to C \to 0 \tag{9.1}$$

for all $n \in \mathbb{Z}$ there is a connecting morphism $\partial^n \colon T^n(C) \to T^{n+1}(A)$ (resp. $\partial_n \colon T_n(C) \to T_{n-1}(A)$) satisfying

1. there is a long exact sequence

$$\ldots \to T^{n-1}(C) \xrightarrow{\partial^{n-1}} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\partial^n} T^{n+1}(A) \to \ldots =: T(S), \tag{9.2}$$

respectively the long exact sequence

$$\ldots \to T_{n+1}(C) \xrightarrow{\partial_{n+1}} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\partial_n} T_{n-1}(A) \to \ldots =: T(S), \tag{9.3}$$

2. For any $S' \in \mathsf{S}(\mathsf{A})$ and any morphism $S \to S'$ in $\mathsf{S}(\mathsf{A})$, i.e.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0, \end{tikzcd} \tag{9.4}$$

there is an associated morphism between the long exact sequences, i.e. a commutative diagram (with a clear dual for the homological case)

$$\begin{tikzcd} T^{n-1}(C) \arrow[r, "\partial^{n-1}"] \arrow[d, "T^{n-1}(h)"] & T^n(A) \arrow[r] \arrow[d, "T^n(f)"] & T^n(B) \arrow[r] \arrow[d, "T^n(g)"] & T^n(C) \arrow[r, "\partial^n"] \arrow[d, "T^n(h)"] & T^{n+1}(A) \arrow[d, "T^{n+1}(f)"] \\ T^{n-1}(C') \arrow[r, "\partial^{n-1}"'] & T^n(A') \arrow[r] & T^n(B') \arrow[r] & T^n(C') \arrow[r, "\partial^n"'] & T^{n+1}(A') \end{tikzcd} . \tag{9.5}$$

Then the family $T := \{T^n\}_{n\in\mathbb{Z}} : \mathsf{S}(\mathsf{A}) \to \mathsf{L}(\mathsf{B})$ actually is a functor.

Remark 9.2 T^0 is always left exact, for a cohomological ∂-functor. In fact given a short exact sequence in A

$$0 \to A \to B \to C \to 0, \tag{9.6}$$

the associated long exact sequence, since $T^{-1} = 0$, is

$$T^{-1}(C) = 0 \xrightarrow{\partial^{-1}} T^0(A) \to T^0(B) \to T^0(C) \to \ldots . \tag{9.7}$$

Analogously, one checks that T_0 is right exact, for any homological ∂-functor.

Example Consider $\mathsf{A} := \mathsf{Mod}\text{-}R$, for a ring R. Consider U^{-1} and U^0 free (in particular projective) R-modules and the morphism of modules $u\colon U^{-1} \to U^0$. Consider functor T, given by the family $\{T^0, T^1\}$ (i.e. $T^i = 0$ for all $i \neq 1, 0$), where

$$T^0 := \ker \operatorname{Hom}_R(u, -) \qquad \text{and} \qquad T^1 := \operatorname{coker} \operatorname{Hom}_R(u, -)\,. \tag{9.8}$$

Let's show that $T\colon \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}R$ is a cohomological ∂-function. Consider a short exact sequence of modules $0 \to A \to B \to C \to 0$. We need to show that the following is exact:

$$0 \to T^0(A) \to T^0(B) \to T^0(C) \to T^1(A) \to T^1(B) \to T^1(C) \to 0. \tag{9.9}$$

In fact we can apply the covariant hom functor $\operatorname{Hom}_R(u, -)$ and obtain

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(U^0, A) \arrow[r] \arrow[d, "\operatorname{Hom}_R(u,A)"'] & \operatorname{Hom}_R(U^0, B) \arrow[r] \arrow[d, "\operatorname{Hom}_R(u,B)"'] & \operatorname{Hom}_R(U^0, C) \arrow[r] \arrow[d, "\operatorname{Hom}_R(u,C)"'] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(U^{-1}, A) \arrow[r] & \operatorname{Hom}_R(U^{-1}, B) \arrow[r] & \operatorname{Hom}_R(U^{-1}, C) \arrow[r] & 0
\end{tikzcd} \tag{9.10}$$

which clearly is commutative and with exact rows (both U^0 and U^{-1} are free, hence projective, i.e. both $\operatorname{Hom}_R(U^0, -)$ and $\operatorname{Hom}_R(U^{-1}, -)$ are exact functors), then by the snake lemma we obtain exactness of the long sequence.

Moreover consider $\mathsf{C} := \{X \in \operatorname{Ob}(\mathsf{Mod}\text{-}R) \mid T^0(X) = T^1(X) = 0\} \subset \mathsf{Mod}\text{-}R$. Then this subcategory of $\mathsf{Mod}\text{-}R$ is closed under kernel, cokernel, extension and products. In particular C is an abelian full subcategory of $\mathsf{Mod}\text{-}R$. In fact, given $X, Y \in \operatorname{Ob}(\mathsf{C})$, and a morphism $f\colon X \to Y$. Let $K := \ker f$, $I := \operatorname{im} f$, and $C := \operatorname{coker} f$. Then we have the short exact sequences $0 \to K \to X \to I \to 0$ and $0 \to I \to Y \to C \to 0$. The functor T associates to them the long exact sequences

$$0 \to T^0(K) \to 0 \to T^0(I) \to T^1(K) \to 0 \to T^1(I) \to 0 \tag{9.11}$$

$$0 \to T^0(I) \to 0 \to T^0(C) \to T^1(I) \to 0 \to T^1(C) \to 0. \tag{9.12}$$

With simple computations one shows that $I, C, K \in \operatorname{Ob}(\mathsf{C})$.

Definition 9.3: Left resolution.
Let A be an abelian category, and $M \in \operatorname{Ob}(\mathsf{A})$. A *left resolution* of M is a chain-complex:

$$X_\bullet := \ldots \to X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \to 0 \tag{9.13}$$

s.t. there exists $\pi\colon X_0 \to M$ with which the augmented complex

$$X_\bullet \xrightarrow{\pi} M \to 0 := \ldots \to X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\pi} M \to 0 \tag{9.14}$$

is exact. We can view these conditions as stating that π is a quasi-isomorphism between $X_\bullet$ and M, viewed as a complex concentrated in degree 0. In fact $H_i(X_\bullet) = 0$ for all $i \neq 0$ and $H_0(X_\bullet) \simeq M$.

If, moreover, each X_i is a projective object in A, then the left resolution $X_\bullet \xrightarrow{\pi} M \to 0$ is called a *projective resolution* of M.

Lemma 9.4. *Let A be an abelian category, with enough projectives. Then every $M \in \operatorname{Ob}(\mathsf{A})$ admits a projective resolution.*

Proof. One obtains an exact resolution by taking, each time, a projection onto the kernel of the previous map (it can be done, since A has enough projectives)

$$\ldots \to P_2 \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} M \to 0. \tag{9.15}$$

We can define, for each $n \in \mathbb{N}$, $K_n := \ker \pi_{n-1}$. Then K_n is called n-th syzygy of M, sometimes denoted by $\Omega_n(M)$.

Moreover the projective resolution is usually denoted by $P_\bullet \xrightarrow{\pi_0} M \to 0$. ■

Theorem 9.5 (Comparison). *Let A be an abelian category. Let $f_{-1}\colon M \to N$ be a morphism in A. Consider the chain complex (not necessairily exact)*

$$\ldots \to P_3 \to P_2 \to P_1 \to P_0 \xrightarrow{\pi} M \to 0, \tag{9.16}$$

with P_i projective for all $i \geq 0$. Let $Y_\bullet \xrightarrow{\sigma} N \to 0$ be a left resolution of N. Then there is a chain map $f\colon P_\bullet \to Y_\bullet$ lifting f_{-1}, i.e.

$$\begin{tikzcd}
\ldots \arrow[r] & P_3 \arrow[r, "d_3^P"] \arrow[d, "f_3"'] & P_2 \arrow[r, "d_2^P"] \arrow[d, "f_2"'] & P_1 \arrow[r, "d_1^P"] \arrow[d, "f_1"'] & P_0 \arrow[r, "\pi"] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f_{-1}"'] & 0 \\
\ldots \arrow[r] & Y_3 \arrow[r, "d_3^Y"'] & Y_2 \arrow[r, "d_2^Y"'] & Y_1 \arrow[r, "d_1^Y"'] & Y_0 \arrow[r, "\sigma"'] & N \arrow[r] & 0
\end{tikzcd} . \tag{9.17}$$

Moreover given any other chain map $g := \{g_n\}_{n\geq 0}$ lifting f_{-1}, then $f \sim g$ the two chain maps are homotopic. In other words the lift of f_{-1} is unique up to homotopy.

Lemma 9.6. *Let A be an abelian category, and $P_\bullet$ an acyclic chain complex bounded below (i.e. s.t. $\exists\, m \in \mathbb{Z}$ for which $P_i = 0$ for all $i < m$) with projective components. Then $P_\bullet$ is contractible (hence it is projective in the category of complexes).*

Lemma 9.7 (Horseshoe). *Let A be an abelian category. Consider the short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0, \tag{9.18}$$

and the projective resolutions $P_\bullet \to A \to 0$ and $Q_\bullet \to C \to 0$ for A and C. Then we can complete the diagram with the red arrows.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\ldots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & 0 \\
\ldots \arrow[r] & P_1 \oplus Q_1 \arrow[r] & P_0 \oplus Q_0 \arrow[r] & B \arrow[r] \arrow[d, "g"] & 0 \\
\ldots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd} . \tag{9.19}$$

In particular $\left(P_\bullet \oplus Q_\bullet, d_\bullet^P \oplus d_\bullet^Q\right)$ gives a projective resolution of B, completing the diagram in the second row.

Let's now dualize everything we obtained up to now:

Definition 9.8: Right coresolution.
Let A be an abelian category and $M \in \mathrm{Ob}(\mathsf{A})$. A *right coresolution* of M is a cochain complex

$$Y^\bullet := 0 \to Y_0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} Y^2 \to \dots \tag{9.20}$$

s.t. there exists a morphism $\delta^0 \colon M \to Y^0$ with which the augmented complex

$$0 \to M \xrightarrow{\delta^0} Y^\bullet := 0 \to M \xrightarrow{\delta^0} Y_0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} Y^2 \to \dots \tag{9.21}$$

is exact. In other words we ask that δ^0 is a quasi-isomorphism between $Y^\bullet$ and M concentrated in degree 0. Then $H^i(Y^\bullet) = 0$ for all $i \neq 0$ and $H^0(Y^\bullet) \simeq M$.

If, moreover, each Y^i is an injective object in A, then the right coresolution $0 \to M \xrightarrow{\delta^0} Y^\bullet$ is called an *injective coresolution* of M.

Lemma 9.9. *Let A be an abelian category, with enough injectives. Then every object $M \in \mathrm{Ob}(\mathsf{A})$ admits an injective coresolution.*

Proof. One obtains an exact resolution by taking, each time, the cokernel of the previous map

$$0 \to M \xrightarrow{\delta^0} I^0 \xrightarrow{\delta^1} I^1 \xrightarrow{\delta^1} I^2 \to \dots . \tag{9.22}$$

We can define, for each $n \in \mathbb{N}$, $C^n := \operatorname{coker} \delta^{n-1}$. Then C^n is called n-th cosyzygy of M, sometimes denoted by $\Omega^n(M)$.

Moreover the projective resolution is usually denoted by $0 \to M \xrightarrow{\delta^0} I^\bullet$. ∎

Theorem 9.10 (Comparison)**.** *Le A be an abelian category. Let $f^{-1} \colon M \to N$ a morphism in A. Consider the cochain complex (not necessairily exact)*

$$0 \to N \xrightarrow{\eta} I^0 \to I^1 \to I^2 \to I^3 \to \dots, \tag{9.23}$$

with I^i injective for all $i \geq 0$. Let $0 \to M \xrightarrow{\delta^0} Y^\bullet$ a right coresolution of M. Then there exists a cochain map $f \colon Y^\bullet \to I^\bullet$ extending f^{-1}, i.e.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\delta^0"] \arrow[d, "f^{-1}"'] & Y^0 \arrow[r, "d_Y^0"] \arrow[d, "f^0"'] & Y^1 \arrow[r, "d_Y^1"] \arrow[d, "f^1"'] & Y^2 \arrow[r, "d_Y^2"] \arrow[d, "f^2"'] & Y^3 \arrow[r, "d_Y^3"] \arrow[d, "f^3"'] & \dots \\
0 \arrow[r] & N \arrow[r, "\eta"'] & I^0 \arrow[r, "d_I^0"'] & I^1 \arrow[r, "d_I^1"'] & I^2 \arrow[r, "d_I^2"'] & I^3 \arrow[r, "d_I^3"'] & \dots
\end{tikzcd} . \tag{9.24}$$

Moreover, given any other cochain map $g := \{g^n\}_{n \geq 0}$ extending f^{-1}, then $f \sim g$, the two cochain maps are homotopic. In other words the extension of f^{-1} is unique up to homotopy.

9.2 Left derived functors

Remark 9.11 A functor $F \colon \mathsf{A} \to \mathsf{B}$ between additive categories induces a functor, again denoted by F,

$$F \colon \mathrm{Ch}(\mathsf{A}) \to \mathrm{Ch}(\mathsf{A}) \tag{9.25}$$

$$(X^\bullet, d_X) \mapsto \big(F(X^\bullet), d_{F(X^\bullet)}\big) , \tag{9.26}$$

where $[F(X^\bullet)]^n := F(X^n)$ and $d^n_{F(X^\bullet)} := F(d^n_{X^\bullet})$. Given a morphism $f\colon X^\bullet \to Y^\bullet$ in $\mathrm{Ch}(\mathsf{A})$, one has $d_Y \circ f = f \circ d_X$. Then $F(f) \circ F(d_X) = f(d_Y) \circ F(f)$, i.e. $F(f)$ is a morphism in $\mathrm{Ch}(\mathsf{B})$.

Moreover, if F is an additive functor, then $f = s \circ d_X + d_Y \circ s$ implies $F(f) = F(s) \circ F(d_X) + F(d_Y) \circ F(s)$, hence F induces a functor

$$F\colon K(\mathsf{A}) \to K(\mathsf{B}). \tag{9.27}$$

Definition 9.12: Left derived functors.
Let A and B be abelian categories. Assume that A has enough projectives and $F\colon \mathsf{A} \to \mathsf{B}$ is a right exact functor. We define the left derived functor $L_iF\colon \mathsf{A} \to \mathsf{B}$ s.t. $L_iF(A) := H_i(F(P_\bullet))$, for $P_\bullet \to A \to 0$ a projective resolution of A and $i \geq 0$.

Remark 9.13 One actually needs to prove that the above is a good definition, i.e. that L_iF does not depend on the projective resolution $P_\bullet \to A \to 0$.

Moreover one can prove that $L_0F \simeq F$ as functors. In fact consider any projective resolution $P_\bullet \to A \to 0$ of A. Then

$$\ldots \to P_2 \to P_1 \xrightarrow{d_1} P_0 \to A \to 0 \tag{9.28}$$

is exact, with F right exact. Then

$$F(P_1) \xrightarrow{F(d_1)} F(P_0) \to F(A) \to 0 \tag{9.29}$$

is also exact. In particular $\operatorname{coker} F(d_1) \simeq F(A)$. But then $L_0F(A) = H_0(F(P_\bullet))$. We know that

$$F(P_\bullet) = \ldots \to F(P_1) \xrightarrow{F(d_1)} F(P_0) \to 0 \tag{9.30}$$

hence that $H_0(F(P_\bullet)) = \operatorname{coker} F(d_1) = F(A)$.

Lemma 9.14.

(a) For each $i \in \mathbb{N}$, L_iF is well defined, up to natural isomorphism.

(b) Let $\alpha\colon A \to C$ be a morphism in A. Then there are natural maps

$$L_iF(\alpha)\colon L_iF(A) \to L_iF(C). \tag{9.31}$$

(c) For any $i \geq 0$, the functor L_iF is additive.

Lemma 9.15. *Let $f\colon A \to C$ be a morphism in A. Then $L_0F(f) = F(f)$.*

Proposition 9.16. *Let F and L_iF be as in the above definition. If $A \in \mathsf{A}$ is projective, then $L_iF(A) = 0$ for all $i > 0$ (recall that $L_0F(A) \simeq F(A)$).*

Definition 9.17: F-acyclic object.
Let A be an abelian category with enough projectives and $F\colon \mathsf{A} \to \mathsf{B}$ be a right exact functor. An object $A \in \mathrm{Ob}(\mathsf{A})$ is called *F-acyclic* iff $L_iF(A) = 0$ for all $i > 0$.

Definition 9.18: F-acyclic resolution.
Let A be an abelian category with enough projectives, $F\colon \mathsf{A} \to \mathsf{B}$ be a right exact functor and $A \in \mathrm{Ob}(\mathsf{A})$. A left resolution $Q_\bullet \to A \to 0$ of A is called an *F-acyclic resolution* iff Q_i are F-acyclic for all $i \geq 0$.

Remark 9.19 Any projective object $A \in \mathrm{Ob}\,(\mathsf{A})$ is F-acyclic for any right exact functor F.

Theorem 9.20. *Let* A *and* B *be abelian categories. Assume that* A *has enough projectives and* $F\colon \mathsf{A} \to \mathsf{B}$ *is a right exact functor. Then the left derived functors* $\{L_iF\}_{i\geq 0}$ *form a homological* ∂*-functor.*

Definition 9.21: Morphism of (co)homological ∂-functor.
Let $S, T\colon \mathsf{A} \to \mathsf{B}$ be cohomological ∂-functors. A morphism $S \to T$ is a sequence of natural transformations $\eta^n\colon S^n \to T^n$ commuting with ∂. More explicitly, given any short exact sequence $0 \to A \to B \to C \to 0$ in A, the following diagram commutes

$$\begin{array}{ccc} S^n(C) & \xrightarrow{\partial^n_S} & S^{n+1}(A) \\ {\scriptstyle \eta^n_C}\downarrow & & \downarrow{\scriptstyle \eta^{n+1}_A} \\ T^n(C) & \xrightarrow[\partial^n_T]{} & T^{n+1}(A) \end{array} . \tag{9.32}$$

(Clearly for homological ∂-functors one only has to dualize).

Definition 9.22: Universal cohomological ∂-functor.
A cohomological ∂-functor T is called *universal* iff given any cohomological ∂-functor S, and any natural transformation $\eta^0\colon T^0 \to S^0$, then $\exists\,!\ \{\eta^n\colon T^n \to S^n\}_{n\geq 0}$ a natural transformation of ∂-functors extending η^0. (Analogously of homological ∂-functors).

Lemma 9.23. *Consider an exact functor* $F\colon \mathsf{A} \to \mathsf{B}$*. Setting* $T^0 := F$ *and* $T^n := 0$ *for all* $n > 0$ *defines a universal cohomological* ∂*-functor* $\{T^n\}_{n\in\mathbb{N}}$*. (Analogously setting* $T_0 := F$ *and* $T_n := 0$*, for a universal homological* ∂*-functor).*

Theorem 9.24. *Let* A *and* B *be abelian categories. Assume that* A *has enough projectives and* $F\colon \mathsf{A} \to \mathsf{B}$ *is a right exact functor. Then the left derived functors* $\{L_iF\}_{i\geq 0}$ *form a universal homological* ∂*-functor.*

Lemma 9.25. *Let* A *and* B *be abelian categories. Assume that* A *has enough projectives and* $F\colon \mathsf{A} \to \mathsf{B}$ *is a right exact functor. Consider* $G\colon \mathsf{B} \to \mathsf{C}$ *an exact functor, then:*

$$L_i\,(G \circ F) \simeq_{\textit{nat.}} G \circ L_iF \qquad \forall\, i \geq 0. \tag{9.33}$$

Lemma 9.26. *Consider* $G\colon \mathsf{A} \to \mathsf{B}$ *an exact functor between abelian categories. Consider* $X_\bullet \in \mathrm{Ch}(\mathsf{A})$*, then for every* $i \in \mathbb{Z}$

$$G\,(H_i\,(X_\bullet)) = H_i\,(G(X_\bullet))\,. \tag{9.34}$$

Lemma 9.27 (Dimension shifting)**.** *Let* A *and* B *be abelian categories. Assume that* A *has enough projectives and* $F\colon \mathsf{A} \to \mathsf{B}$ *is a right exact functor. Consider a short exact sequence* $0 \to K \to Q \to A \to 0$ *in* A*, with* Q *an* F*-acyclic object (e.g. if* Q *is projective). Then*

1. $L_1F(A) = \ker\,(F(K) \to F(Q))$,
2. $L_iF(A) \simeq L_{i-1}F(K)$ *for all* $i \geq 2$.

Remark 9.28 Let A be an abelian category. We define

$$\mathrm{Ch}_{\geq 0}(\mathsf{A}) := \left\{ \left(X_\bullet, d^X\right) \in \mathrm{Ch}(\mathsf{A}) \,\middle|\, X_n = 0\,\forall\, n < 0 \right\}. \tag{9.35}$$

By the fundamental theorem on homology, we know that $\{H_n\}_{n\in\mathbb{Z}}$, for $H_n\colon \mathrm{Ch}_{\geq 0}(\mathsf{A}) \to \mathsf{A}$, is a homological ∂-functor

Lemma 9.29. *Moreover one can prove that* $\{H_n\}_{n\in\mathbb{Z}}$ *is a* universal *homological* ∂*-functor.*

Lemma 9.30. *Let* A *and* B *be abelian categories, s.t.* A *has enough projectives. Consider* $F\colon \mathsf{A} \to \mathsf{B}$ *an exact functor, then*

$$L_iF(A) = 0 \qquad \forall\, A \in \mathrm{Ob}\,(\mathsf{A})\,,\, \forall\, i > 0. \tag{9.36}$$

Moreover we also know that $L_0F \simeq F$.

9.3 Right derived functors

Remark 9.31: Standard assumption.
In the following section we will assume the following: A and B are abelian categories. Moreover we assume that A has enough injectives, and $F\colon \mathsf{A} \to \mathsf{B}$ is a left exact functor.

Definition 9.32: Right derived funtors.
Let A, B and F be as in remark 9.31. We define the right derived functors $R^iF\colon \mathsf{A} \to \mathsf{B}$ s.t. $R^iF(A) := H^i\left(F(I^\bullet)\right)$, for $0 \to A \to I^\bullet$ an injective coresolution of A, and $i \geq 0$.

Remark 9.33: Important!.
Recall that $A \in \mathrm{Ob}\,(\mathsf{A})$ is injective iff A is projective in A^{op}. Then, given an injective coresolution $0 \to A \to I^\bullet$ for A, then $I_\bullet \to A \to 0$ becomes a projective resolution in A^{op}.

Then, given $F\colon \mathsf{A} \to \mathsf{B}$ a left exact functor, we define $F^{op}\colon \mathsf{A}^{op} \to \mathsf{B}^{op}$ a covariant functor. Clearly A^{op} has enough projectives, moreover F^{op} is right exact (in fact F is left exact iff F^{op} is right exact). Then we can define the left derived functor $L_iF^{op}(A)$. Finally we have the equality

$$\left(L_iF^{op}\right)^{op}(A) = R^iF(A). \tag{9.37}$$

In particular $\left\{R^iF\right\}_{i\geq 0}$ form a universal cohomological ∂-functor. Moreover, dualizing the previous results, we obtain that:

- $R^0F \simeq F$,
- Given a short exact sequence in A

$$0 \to A \to B \to C \to 0 \tag{9.38}$$

there is an associated long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \xrightarrow{\partial^0} R^1F(A) \to R^1F(B) \to R^iF(C) \xrightarrow{\partial^1} \dots. \tag{9.39}$$

Definition 9.34: F-acyclic objects.
Let A, B and F be as in remark 9.31. An object $A \in \mathrm{Ob}\,(\mathsf{A})$ is F-*acyclic* iff

$$R^iF(A) = 0 \qquad \forall\, i > 0. \tag{9.40}$$

Remark 9.35 Any injective object $Q \in \mathrm{Ob}\,(\mathsf{A})$ is F-acyclic for any left-exact functor F.

Lemma 9.36. *Consider* A *an abelian category with enough injectives, and* $F\colon \mathsf{A} \to \mathsf{B}$ *an exact functor, then* $R^iF = 0$ *for all* $i > 0$.

Example Let A an abelian category with enough injectives. Fix $M \in \mathrm{Ob}\,(\mathsf{A})$, then consider

$$H_M := \mathrm{Hom}_{\mathsf{A}}\,(M, -)\,:\, \mathsf{A} \to \mathsf{Ab} \tag{9.41}$$

the covariant Hom functor. We know that H_M is left exact. Then we can define the right derived functors of H_M. In particular they are defined as follows. For an object $A \in \mathrm{Ob}(\mathsf{A})$, take an injective coresolution of A, $0 \to A \to I^\bullet$, then

$$R^i H_M(A) = H^i\left(\mathrm{Hom}_{\mathsf{A}}(M, I^\bullet)\right). \tag{9.42}$$

Moreover one introduces the notation (which is especially useful in the category of modules)

$$\mathrm{Ext}^i_{\mathsf{A}}(M, A) := R^i H_M(A). \tag{9.43}$$

Proposition 9.37. *Let A be abelian with enough injectives (e.g. $\mathsf{A} = \mathsf{Mod}\text{-}R$). Fix $A \in \mathrm{Ob}(\mathsf{A})$, then the following are equivalent:*

1. *A is injective, i.e. $\mathrm{Hom}_{\mathsf{A}}(-, A)$ is exact;*
2. *$\mathrm{Ext}^i_{\mathsf{A}}(M, A) = 0$ for all $M \in \mathrm{Ob}(\mathsf{A})$ and for all $i \geq 0$;*
3. *$\mathrm{Ext}^1_{\mathsf{A}}(M, A) = 0$ for all $M \in \mathrm{Ob}(\mathsf{A})$.*

We can dualize the above proposition and obtain

Proposition 9.38. *Let A be abelian with enough injectives (e.g. $\mathsf{A} = \mathsf{Mod}\text{-}R$). Fix $M \in \mathrm{Ob}(\mathsf{A})$, then the following are equivalent:*

1. *M is projective, i.e. $\mathrm{Hom}_{\mathsf{A}}(M, -)$ is exact;*
2. *$\mathrm{Ext}^i_{\mathsf{A}}(M, A) = 0$ for all $A \in \mathrm{Ob}(\mathsf{A})$ and for all $i \geq 0$;*
3. *$\mathrm{Ext}^1_{\mathsf{A}}(M, A) = 0$ for all $A \in \mathrm{Ob}(\mathsf{A})$.*

9.4 Derived functors of contravariant functors

Remark 9.39: Right derived functors of a contravariant functor.
Let A and B be abelian categories and $F\colon \mathsf{A} \to \mathsf{B}$ a contravariant left-exact functor (e.g. $F = H^M := \mathrm{Hom}_{\mathsf{A}}(-, M)$ for $M \in \mathrm{Ob}(\mathsf{A})$). Then $F\colon \mathsf{A}^{op} \to \mathsf{B}$ is covariant and, still, left-exact. If A^{op} has enough injectives (iff A has enough projectives) we can define the right derived functors $R^i F\colon \mathsf{A}^{op} \to \mathsf{B}$, for $i \geq 0$. In particular this is computed by taking a projective resolution of $A \in \mathrm{Ob}(\mathsf{A})$: $P_\bullet \to A \to 0$, which gives an injective coresolution $0 \to A \to P^\bullet$ of A in A^{op}. Then we define

$$R^i F(A) := H^i(F(P_\bullet)). \tag{9.44}$$

Notice that given a chain complex $P_\bullet$, then $F(P_\bullet)$ is a cochain complex.

Remark 9.40 Let A be an abelian category with enough injectives and projectives (e.g. for $\mathsf{A} = \mathsf{Mod}\text{-}R$). Then, fixed $M \in \mathrm{Ob}(\mathsf{A})$, $H_M := \mathrm{Hom}_{\mathsf{A}}(M, -)$ is a covariant, left-exact, functor. In particular it admits right-derived funtors

$$R^i H_M(A) = \mathrm{Ext}^i_{\mathsf{A}}(M, A) = H^i\left(H_M(I^\bullet)\right), \tag{9.45}$$

for an injective coresolution $0 \to A \to I^\bullet$ of A. Moreover we can consider $H^A := \mathrm{Hom}_{\mathsf{A}}(-, A)$, which is a contravariant, left-exact, functor. Also this admits right-derived functors

$$R^i H^A(M) = H^i\left(H^A(P_\bullet)\right), \tag{9.46}$$

for $P_\bullet \to M \to 0$ a projective resolution of M.

Theorem 9.41 (Balancing of Ext).

$$R^i H_M(A) = \mathrm{Ext}^i_{\mathsf{A}}(M, A) \simeq R^i H^A(M). \tag{9.47}$$

Remark 9.42: Consequence.
This theorem means that $\mathrm{Ext}^i_{\mathsf{A}}(M, A)$ can be computed in two equivalent ways: We can consider $0 \to A \to I^\bullet$ an injective coresolution of A, or $P_\bullet \to M \to 0$ a projective resolution of M and

$$H^i\left(\mathrm{Hom}_{\mathsf{A}}\left(M, I^\bullet\right)\right) \simeq \mathrm{Ext}^i_{\mathsf{A}}(M, A) \simeq H^i\left(\mathrm{Hom}_{\mathsf{A}}\left(P_\bullet, A\right)\right). \tag{9.48}$$

Remark 9.43 Let A be an abelian category with arbitrary coproducts. Consider $(X_i^\bullet, d_{X_i})_{i \in I}$ a family of cochain complexes in $\mathrm{Ch}(\mathsf{A})$. Then the cochain complex $(\widetilde{X}^\bullet, d_{\widetilde{X}})$, with objects $(\widetilde{X}^\bullet)^n := \coprod_{i \in I} X_i^n$ and differentials $d^n_{\widetilde{X}} := \coprod_{i \in I} d^n_{X_i}$, is a coproduct of $X_i^\bullet$ in $\mathrm{Ch}(\mathsf{A})$. Then one checks that

$$H^n(\widetilde{X}) = \coprod_{i \in I} H^n\left(X_i\right) \qquad \forall\, n \in \mathbb{Z}. \tag{9.49}$$

Analogously, if A admits arbitrary products, consider $(X_i^\bullet, d_{X_i})_{i \in I}$ a family of cochain complexes in $\mathrm{Ch}(\mathsf{A})$. Then the cochain complex $(\widetilde{X}^\bullet, d_{\widetilde{X}})$, with objects $(\widetilde{X}^\bullet)^n := \prod_{i \in I} X_i^n$ and differentials $d^n_{\widetilde{X}} := \prod_{i \in I} d^n_{X_i}$, is a product of $X_i^\bullet$ in $\mathrm{Ch}(\mathsf{A})$. Then one checks that

$$H^n(\widetilde{X}) = \prod_{i \in I} H^n\left(X_i\right) \qquad \forall\, n \in \mathbb{Z}. \tag{9.50}$$

Lemma 9.44. *Let (L, R) be an adjoint pair of functors $L\colon \mathsf{A} \to \mathsf{B}$ and $R\colon \mathsf{B} \to \mathsf{A}$, between additive categories. Then (L, R), thanks to naturality of the adjunction, induces an adjoint pair of morphisms*

$$L\colon \mathrm{Ch}(\mathsf{A}) \to \mathrm{Ch}(\mathsf{B}) \qquad \textit{and} \qquad R\colon \mathrm{Ch}(\mathsf{B}) \to \mathrm{Ch}(\mathsf{A}). \tag{9.51}$$

Proposition 9.45. *Let A and B be abelian categories. Consider an adjoint pair of functors (F, G), for $F\colon \mathsf{A} \to \mathsf{B}$ and $G\colon \mathsf{B} \to \mathsf{A}$. Assume that A has enough projectives and arbitrary coproducts, whereas B has enough injectives and arbitrary products. Let $\{A_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathrm{Ob}\left(\mathsf{A}\right)$ be a family of objects of A and $\{B_\beta\}_{\beta \in \mathcal{B}} \subset \mathrm{Ob}\left(\mathsf{B}\right)$ be a family of objects of B. Then*

$$L_i F\left(\coprod_{\alpha \in \mathcal{A}} A_\alpha\right) \simeq \coprod_{\alpha \in \mathcal{A}} L_i F(A_\alpha) \tag{9.52}$$

and

$$R^i F\left(\prod_{\beta \in \mathcal{B}} B_\beta\right) \simeq \prod_{\beta \in \mathcal{B}} R^i G\left(B_\beta\right). \tag{9.53}$$

9.5 Derived functors of tensor product functors

Recall that, for a ring R, and $M_R \in \mathsf{Mod}\text{-}R$, then, as seen in proposition 7.11,

$$M_R \otimes_R -\colon R\text{-}\mathsf{Mod} \to \mathsf{Ab} \qquad \text{and} \qquad \mathrm{Hom}_{\mathbb{Z}}\left(M, -\right) : R\text{-}\mathsf{Mod} \to \mathsf{Ab} \tag{9.54}$$

constitute an adjoint pair $(M_R \otimes_R -, \mathrm{Hom}_{\mathbb{Z}}\left(M, -\right))$.

As a consequence $T_M := M_R \otimes_R -$ is a left adjoint, hence it is right exact and preserves arbitrary coproducts and $\varinjlim$.

Definition 9.46: Flat module.
Consider $M_R \in \mathsf{Mod}\text{-}R$. We say that M_R is *flat* iff $T_M := M_R \otimes_R -$ is exact (i.e. iff T_M is also left exact). Simmetrically ${}_R N \in R\text{-}\mathsf{Mod}$ is flat iff $- \otimes_R N$ is exact.

Proposition 9.47. *Let $M_R \in \mathsf{Mod}\text{-}R$. The following are equivalent:*

1. *M_R is flat;*
2. *for every mono $0 \to {}_RA \xrightarrow{\mu} {}_RB$ of left R-modules, then $\mathrm{id}_M \otimes \mu \colon M \otimes A \to M \otimes B$ is mono (in Ab);*
3. *$L_i(M \otimes_R -)(N) = 0$ for all $i \geq 1$ and for all $N \in R\text{-}\mathsf{Mod}$;*
4. *$L_1(M \otimes_R -)(N) = 0$ for all $N \in R\text{-}\mathsf{Mod}$.*

Dually:

Proposition 9.48. *Let ${}_RN \in R\text{-}\mathsf{Mod}$. The following are equivalent:*

1. *${}_RN$ is flat;*
2. *for every mono $0 \to A_R \xrightarrow{\mu} B_R$ of right R-modules, then $\mu \otimes \mathrm{id}_N \colon A \otimes N \to B \otimes N$ is mono (in Ab);*
3. *$L_i(- \otimes_R N)(M) = 0$ for all $i \geq 1$ and for all $M \in \mathsf{Mod}\text{-}R$;*
4. *$L_1(- \otimes_R N)(M) = 0$ for all $M \in \mathsf{Mod}\text{-}R$.*

Remark 9.49 Combining the above propositions we obtain that M_R is flat iff M_R is $(- \otimes_R N)$-acyclic for all ${}_RN$ left R-modules. Analogously ${}_RN$ is flat iff ${}_RN$ is $(M \otimes_R -)$-acyclic for all M_R right R-modules.

Definition 9.50: Notation.
Called $T_M \coloneqq M \otimes_R -$, then we define

$$\mathrm{Tor}_i^R(M, N) \coloneqq L_i(M \otimes_R -)(N). \tag{9.55}$$

Theorem 9.51 (Balancing of Tor)**.**

$$\mathrm{Tor}_i^R(M, N) = L_i(M \otimes_R -)(N) = L_i(- \otimes_R N)(M) \tag{9.56}$$

for all $i \geq 0$, all $M \in \mathsf{Mod}\text{-}R$ and all $N \in R\text{-}\mathsf{Mod}$.

Remark 9.52: Consequence.
The above theorem means that $\mathrm{Tor}_i^R(M, N)$ can be computed in two equivalent ways: Consider $P_\bullet \to {}_RN \to 0$ a projective resolution of ${}_RN$ or $Q_\bullet \to M_R \to 0$ a projective resolution of M_R, then

$$H_i(M \otimes_R P_\bullet) \simeq \mathrm{Tor}_i^R(M, N) \simeq H_i(Q_\bullet \otimes_R N). \tag{9.57}$$

Proposition 9.53. *Let $\{M_i\}_{i \in I}$ be a family of right R-modules. Then*

1. *$\bigoplus_{i \in I} M_i$ is flat iff M_i is flat for all $i \in I$,*
2. *If $\{M_i\}_{i \in I}$ is a direct system of flat R-modules, then the filtered direct limit $\varinjlim_{i \in I} M_i$ is flat.*

Remark 9.54 For every ${}_RN$ $T_N \coloneqq - \otimes_R N$ is a left adjoint. This means that T_N preserves colimts, in particular, for every direct system $\{M_i, F_{ij}\}_{i \leq j}$, then

$$\Big(\varinjlim_{i \in I} M_i\Big) \otimes_R N \simeq \varinjlim_{i \in I} (M_i \otimes_R N). \tag{9.58}$$

Remark 9.55

$$\varinjlim_{i\in I} M_i \text{ flat} \quad \not\Rightarrow \quad M_i \text{ flat}. \tag{9.59}$$

In fact every module is the filtered direct limit of its finitely generated submodules. Though it is not true that, given M flat, then its finitely generated submodules are flat.

As an example any ring R is a free, hence flat, R-module. Though this doesn't imply that its (finitely generated) ideals are flat. For instance, take $R := \mathbb{K}[x, y]$, for a field $\mathbb{K}$. Consider $\mathfrak{m} := (x, y)$ the maximal ideal generated by x and y. Consider the mono $0 \to \mathfrak{m} \xrightarrow{\epsilon} R$, and

$$\mathrm{id}_{\mathfrak{m}} \otimes \epsilon \colon \mathfrak{m} \otimes_R \mathfrak{m} \to \mathfrak{m} \otimes_R R \simeq \mathfrak{m} \tag{9.60}$$

$$a \otimes b \mapsto a \cdot b. \tag{9.61}$$

In fact $0 \neq x \otimes y - y \otimes x \mapsto xy - yx = 0$, then $\mathfrak{m}$ is finitely generated, but not flat.

Proposition 9.56. *Let* A *and* B *be abelian categories with enough projectives. Consider* $F\colon \mathsf{A} \to \mathsf{B}$ *a right exact functor. Then* $L_i F$ *can be computed using* F*-acyclic resolutions, instead of projective resolutions. More explicitly, given* $Q_\bullet \to A \to 0$ *a resolution of* A *s.t.* Q_i *is* F*-acyclic for each* i*, then*

$$L_i F(A) \simeq H_i\left(F(Q_\bullet)\right). \tag{9.62}$$

Remark 9.57 In particular $\mathrm{Tor}_i^R(-,-)$ can be computed using flat resolutions.

Remark 9.58: Flat modules.
Clearly any projective P_R right R-module is flat, since it is $- \otimes_R N$-acyclic for all ${}_RN$ modules. Analogously a projective left R-module ${}_RP$ is flat. In particular any free module is flat.

Example: Flat modules. Recall the definition of localization: given a commutative ring R and a multiplicatively closed subset $S \subset R$, i.e. s.t. $0 \notin S, 1 \in S$ and $st \in S$ for all $s, t \in S$, we can consider the localization

$$R_S = R\left[S^{-1}\right] := \left\{ \frac{r}{s} \,\middle|\, r \in R,\ s \in S \text{ and } \frac{r}{s} = \frac{r'}{s'} \iff \exists\, t \in S \text{ s.t. } t\left(rs' - r's\right) = 0 \right\}. \tag{9.63}$$

Notice, moreover, that given any module M, then

$$M \otimes_R R_S =: M_S = \left\{ \frac{x}{s} \,\middle|\, x \in M,\ s \in S \right\} \tag{9.64}$$

and $x/s = x'/s'$ iff there exists $t \in S$ s.t. $t\left(xs' - x's\right) = 0$. In particular $x/1 = 0$ iff $\exists\, t \in S$ s.t. $tx = 0$. Moreover any element $\zeta \in M_R \otimes_R R_S$ can be represented as $y \otimes 1/s$, for $s \in S$ and $y \in M$.

Let's prove that R_S is a flat R-module. Consider a mono $\mu\colon A_R \rightarrowtail B_R$, we have to prove that

$$\mu \otimes 1_{R_S} \colon A_R \otimes_R R_S \longrightarrow B_R \otimes_R R_S$$

is still mono. Let's consider $x/t := x \otimes 1/t \in A_R \otimes_R R_S$, then $\mu(x/t) := \mu(x)/t$. Assume $\mu(x)/t = 0$, i.e. there exists $s \in S$ s.t. $s\mu(x) = 0$, which means $\mu(sx) = 0$, hence $sx = 0$, since μ is mono. But this means that $x/t = 0$.

As a consequence, for any M_R flat R-module, we obtain that its localization at S, i.e. $M_S := M_R \otimes_R R_S$, is still flat. This is because, for all $N_R \in \mathsf{Mod}\text{-}R$, associativity of tensor product implies

$$N \otimes_R M_S := N \otimes_R \left(M \otimes_R R_S\right) \simeq \left(N \otimes_R M\right) \otimes_R R_S.$$

Theorem 9.59 (Lazard)**.** *A module is flat iff it is a filtered direct limit of projective modules, or a direct limit of finitely generated free modules. (It can be specialized to left or right modules, then every module in the statement has to be either left or right, accordingly).*

Lemma 9.60. *Let C and D be abelian categories, and $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$ be an adjoint pair (L, R). Assume that L is an exact functor. Then, if I is an injective object of D, then $R(I)$ is injective in C. Dually, if R is exact, and P is a projective object of C, then $L(P)$ is a projective object of D.*

Proposition 9.61. *Let ${}_SF_R$ be an S-R-bimodule and ${}_SE$ be an injective left S-module, then*

- *If F_R is flat, then $\operatorname{Hom}_S({}_SF_R, {}_SE)$ is an injective left R-module.*
- *Conversely, if ${}_SE$ is an injective cogenerator of S-Mod and $\operatorname{Hom}_S({}_SF_R, {}_SE)$ is an injective left R-module, then F_R is flat.*

Corollary 9.62. *Since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in the category $\mathsf{Ab} = \mathsf{Mod}$-$\mathbb{Z}$: it is the direct sum of the injective envelopes of the simple modules $\mathbb{Z}/p\mathbb{Z}$:*

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in P} E(\mathbb{Z}/p\mathbb{Z}) . \tag{9.65}$$

The module ${}_\mathbb{Z}F_R$ is flat iff $\operatorname{Hom}_\mathbb{Z}(F_R, \mathbb{Q}/\mathbb{Z})$ is an injective left R-module. We introduce the following notation

$$F_R^* \coloneqq \operatorname{Hom}_\mathbb{Z}(F_R, \mathbb{Q}/\mathbb{Z}) \tag{9.66}$$

and call this important module, the character module *of F_R.*

Theorem 9.63 (Dimension shifting for right derived functors)**.**

1. *Let $F\colon \mathsf{A} \to \mathsf{B}$ be a covariant left-exact functor between abelian categories, with A having enough injectives. Let Q be an F-acyclic object (e.g. Q injective) and*

   $$0 \to K \to Q \to A \to 0 \tag{9.67}$$

   be a short exact sequence. Then, for all $i \geq 1$,

   $$R^iF(A) \simeq R^{i+1}F(K). \tag{9.68}$$

2. *Let $F\colon \mathsf{A} \to \mathsf{B}$ be a contravariant left-exact functor between abelian categories, with A having enough projectives. Let Q be an F-acyclic object (e.g. Q projective) and*

   $$0 \to K \to Q \to A \to 0 \tag{9.69}$$

   be a short exact sequence. Then, for all $i \geq 1$,

   $$R^iF(K) \simeq R^{i+1}F(A). \tag{9.70}$$

Proof.

1. Consider the long exact sequence

   $$R^1F(K) \to R^1F(Q) = 0 \to R^1F(A) \to R^2F(K) \to R^2F(Q) = 0 \to \dots . \tag{9.71}$$

   Since $R^iF(Q) = 0$ for all i, we have our thesis.

2. Consider the long exact sequence

$$R^1F(A) \to R^1F(Q) = 0 \to R^1F(K) \to R^2F(A) \to R^2F(Q) = 0 \to \dots \quad (9.72)$$

Since $R^iF(Q) = 0$ for all i, we have our thesis. ■

Remark 9.64 Assume that A is an abelian category with enough projectives. Consider $M \in \mathrm{Ob}(\mathsf{A})$ such that, for all $N \in \mathrm{Ob}(\mathsf{A})$,

$$\mathrm{Ext}^{n+i}_{\mathsf{A}}(M, N) = 0 \qquad \forall\, i \geq 0. \quad (9.73)$$

By dimension shifting $\mathrm{Ext}^1_{\mathsf{A}}(K_n, N) = \mathrm{Ext}^2_{\mathsf{A}}(K_{n-1}, N) = \dots = \mathrm{Ext}^{n+1}_{\mathsf{A}}(M, N)$ for all $N \in \mathrm{Ob}(\mathsf{A})$. And, moreover, for $K_n = \Omega_n(M)$, the n-th syzygy of M, we have

$$\mathrm{Ext}^{n+1}_{\mathsf{A}}(M, N) \simeq \mathrm{Ext}^1_{\mathsf{A}}(K_n, N). \quad (9.74)$$

In particular the above condition holds iff K_n is projective.

Analogously, if A has enough injectives we obtain: By dimension shifting $\mathrm{Ext}^1_{\mathsf{A}}(M, C_n) = \mathrm{Ext}^2_{\mathsf{A}}(M, C_{n-1}) = \dots = \mathrm{Ext}^{n+1}_{\mathsf{A}}(M, N)$ for all $N \in \mathrm{Ob}(\mathsf{A})$. And, moreover, for $C_n = \Omega^n(N)$, the n-th cosyzygy of N, we have

$$\mathrm{Ext}^{n+1}_{\mathsf{A}}(M, N) \simeq \mathrm{Ext}^1_{\mathsf{A}}(M, C_n). \quad (9.75)$$

In particular $\mathrm{Ext}^{n+i}_{\mathsf{A}}(M, N) = 0\, \forall\, i \geq 0$ iff C_n is injective.

Lemma 9.65 (Schanuel)**.** *Let A be an abelian category. Let $P, Q \in \mathrm{Ob}(\mathsf{A})$ be projective objects. Assume that the following are short exact sequences*

$$0 \to K \to P \to M \to 0 \qquad \textit{and} \qquad 0 \to H \to Q \to M \to 0. \quad (9.76)$$

Then $K \oplus Q \simeq H \oplus P$. In particular K is projective iff H is projective.

Corollary 9.66. *Consider the two long exact sequences with P_i, Q_i projective*

$$0 \to K_n \to P_{n-1} \to P_{n-2} \to \dots \to P_1 \to P_0 \to M \to 0 \quad (9.77)$$

and

$$0 \to H_n \to Q_{n-1} \to Q_{n-2} \to \dots \to Q_1 \to Q_0 \to M \to 0. \quad (9.78)$$

Then

$$K_n \oplus Q_{n-1} \oplus P_{n-2} \oplus \dots \simeq K_n \oplus P_{n-1} \oplus Q_{n-2} \oplus \dots. \quad (9.79)$$

In particular K_n is projective iff H_n is projective.

Definition 9.67: Projective dimension.
Let A be an abelian category, with enough proectives. Consider $M \in \mathrm{Ob}(\mathsf{A})$. We define the *projective dimension* of M, denoted by $\mathrm{p.d.}(M)$, as the smallest integer $n \in \mathbb{N}$ s.t. there exist $P_i \in \mathrm{Ob}(\mathsf{A})$ projective and an exact sequence

$$0 \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \to M \to 0, \quad (9.80)$$

i.e. it is the minimal length of a projective resolution of M. Equivalently n is the minimal index s.t. the n-th syzygy of M is already a projective object. If no finite resolution exists, we define $\mathrm{p.d.}\, M = \infty$.

Remark 9.68 The projective dimension is well defined thanks to our corollary to Schanuel lemma.

Example: Infinite projective dimension. Let $R \coloneqq \mathbb{Z}/2\mathbb{Z}$ and $M \coloneqq \mathbb{Z}/2\mathbb{Z}$ as an R-module. Then

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0 \tag{9.81}$$

is exact. This means that $\Omega_1(M) = M$, hence $\mathrm{p.d.}\, M = \infty$.

Analogously for $R \coloneqq \mathbb{K}[x]/\left(x^2\right)$, for a field $\mathbb{K}$, R is called the ring of dual numbers. Then $M_R \coloneqq (x)/(x^2)$ has infinite projective dimension.

Proposition 9.69. *Let A be an abelian category with enough projectives. Let $M \in \mathrm{Ob}(\mathsf{A})$, then the following are equivalent:*

1. $\mathrm{p.d.}\, M \leq n$*;*
2. $\mathrm{Ext}^{n+i}_{\mathsf{A}}(M, N) = 0$ *for all $N \in \mathsf{A}$, and all $i \geq 1$;*
3. $\mathrm{Ext}^{n+1}_{\mathsf{A}}(M, N) = 0$ *for all $N \in \mathsf{A}$.*

Corollary 9.70. *If $M \in \mathrm{Ob}(\mathsf{A})$ (for A as above) has $\mathrm{p.d.}\, M = n$, then $\mathrm{Ext}^{n+1}_{\mathsf{A}}(M, N) = 0$ for all $N \in \mathrm{Ob}(\mathsf{A})$ and $\exists\, N_0 \in \mathrm{Ob}(\mathsf{A})$ s.t. $\mathrm{Ext}^{n}_{\mathsf{A}}(M, N_0) \neq 0$.*

Let's now dualize everything for injectives

Lemma 9.71 (Schanuel for injectives)**.** *Let A an abelian category and $M \in \mathrm{Ob}(\mathsf{A})$. Let $I, E \in \mathrm{Ob}(\mathsf{A})$ be injective objects. Assume the following are short exact sequences*

$$0 \to M \to I \to C \to 0 \qquad \textit{and} \qquad 0 \to M \to E \to D \to 0. \tag{9.82}$$

Then $C \oplus E \simeq I \oplus D$. In particular D is injective iff C is injective.

Corollary 9.72. *Consider the two long exact sequences with I^n, E^n injective*

$$0 \to M \to I^0 \to I^1 \to \ldots \to I^{n-1} \to C \to 0 \tag{9.83}$$

and

$$0 \to M \to E^0 \to E^1 \to \ldots \to E^{n-1} \to D \to 0. \tag{9.84}$$

Then

$$C \oplus E^{n-1} \oplus I^{n-2} \oplus \ldots \simeq D \oplus I^{n-1} \oplus E^{n-2} \oplus \ldots. \tag{9.85}$$

In particular C is injective iff D is injective.

Definition 9.73: Injective dimension.
Let A be an abelian category with enough injectives. Consider $M \in \mathrm{Ob}(\mathsf{A})$. We define the *injective dimension* of M, denoted by $\mathrm{i.d.}\, M$, as the smallest integer $n \in \mathbb{N}$ s.t. there exist $I^j \in \mathrm{Ob}(\mathsf{A})$ injective and an exact sequence

$$0 \to M \to I^0 \to I^1 \to \ldots \to I^{n-1} \to I^n \to 0, \tag{9.86}$$

i.e. n is the minimal length of an injective coresolution of M. Equivalently n is the minimal index s.t. the n-th cosyzygy of M is already an injective object. If no finite resolution exists, we define $\mathrm{i.d.}\, M = \infty$.

Example: Infinite injective dimension. Consider $R \coloneqq \mathbb{Z}/4\mathbb{Z}$. Prove that R is self injective, i.e. R is injective as an R-module (you can prove using Baer's criterion). Let $M_R \coloneqq \mathbb{Z}/2\mathbb{Z}$. Prove that $\mathrm{i.d.}\, M_R = \infty$.

Proposition 9.74. *Let* A *be an abelian category with enough injectives. Let* $M \in \mathrm{Ob}(\mathsf{A})$*, then the following are equivalent:*

1. $\text{i.d.}\, M \leq n$*;*
2. $\mathrm{Ext}^{n+i}_{\mathsf{A}}(N, M) = 0$ *for all* $N \in \mathsf{A}$*, and all* $i \geq 1$*;*
3. $\mathrm{Ext}^{n+1}_{\mathsf{A}}(N, M) = 0$ *for all* $N \in \mathsf{A}$*.*

Definition 9.75: Right global dimension of R.
Let R be a ring. We define the *right global dimension* of R, denoted by $\text{r.gld}\, R$, as

$$\text{r.gld}\, R := \sup \{\text{p.d.}\, M_R \mid M_R \in \mathsf{Mod}\text{-}R\}. \tag{9.87}$$

Theorem 9.76 (Global dimension)**.** *Consider the following numbers:*

$$(2) := \sup \{\text{i.d.}\, M_R \mid M_R \in \mathsf{Mod}\text{-}R\} \tag{9.88}$$

$$(3) := \sup \{\text{p.d.}\, R/I_R \mid I_R \lhd R \textit{ is a right ideal}\} \tag{9.89}$$

$$(4) := \sup \{n \in \mathbb{N} \mid \mathrm{Ext}^n_{\mathsf{A}}(M, N) \neq 0 \textit{ for some } M_R, N_R \in \mathsf{Mod}\text{-}R\}. \tag{9.90}$$

Let's call $(1) := \text{r.gld}\, R$*. Then, if finite,* $(1) = (2) = (3) = (4)$*. Moreover, if any is infinite, also all the others are.*

Lemma 9.77. *Let* R *be a ring. Consider the short exact sequences*

$$0 \to K \to F \to M \to 0 \qquad \textit{and} \qquad 0 \to H \to G \to M \to 0, \tag{9.91}$$

with F, G *flat. Then* K *is flat iff* H *is flat.*

Definition 9.78: Flat (weak) dimension.
Let R be a ring and $M_R \in \mathsf{Mod}\text{-}R$. We define the *flat (or weak) dimension* of M_R, denoted by $\text{f.d.}\, M_R$ or $\text{w.d.}\, M_R$, as the minimum length of a flat resolution of M.

Remark 9.79 As before, by the above lemma, this is a good definition.

Proposition 9.80. *For* $M_R \in \mathsf{Mod}\text{-}R$*, the following are equivalent:*

1. $\text{w.d.}\, M \leq n$*;*
2. $\mathrm{Tor}^R_{n+i}(N, M) = 0$ *for all* $N \in \mathsf{A}$*, and all* $i \geq 1$*;*
3. $\mathrm{Tor}^R_{n+1}(N, M) = 0$ *for all* $N \in \mathsf{A}$*.*

Definition 9.81: Right weak-global dimension.
Let R be a ring. We define the *right weak global dimension* of R, denoted by $\text{r.w.gld}\, R$, as

$$\text{r.w.gl.dim}\, R := \sup \{\text{w.d.}\, M_R \mid M_R \in \mathsf{Mod}\text{-}R\}. \tag{9.92}$$

Remark 9.82 Notice that $\text{r.w.gl.dim}\, R = \sup \{\text{w.d.}\, {}_R N \mid {}_R N \in R\text{-}\mathsf{Mod}\}$. Then we can analogously define the *left weak global dimension* of R and $\text{r.w.gl.dim}\, R = \text{l.w.gl.dim}\, R$.

Remark 9.83 Consider $M_R \in \mathsf{Mod}\text{-}R$ and its character module $M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. We proved that M_R is flat iff M^* is injective. Moreover we can define a canonical map $\mu\colon M \to M^{**}$ that acts sending an element of M to the corresponding valuation map on M^*. More explicitly, given $x \in M_R$, $\mu(x) \in \mathrm{Hom}_{\mathbb{Z}}(M^*, \mathbb{Q}/\mathbb{Z})$ is the map which, on $f \in M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, acts by $\mu(x)(f) := f(x)$. Notice that μ is mono, since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of Ab. In fact, for any $0 \neq x \in M$, we can define a nonzero map $g\colon \langle x \rangle_{\mathbb{Z}} \to \mathbb{Q}/\mathbb{Z}$ which can be extended to the whole M.

Proposition 9.84. *Let M_R be a right R-module, then the following are equivalent:*

1. *M_R is flat;*
2. *M^* is an injective left R-module;*
3. *for all ${}_RI \triangleleft {}_RR$ (a left ideal)*

$$M_R \otimes_R I \simeq MI = \left\{ \sum_{i=1}^{n} x_i a_i \;\middle|\; x_i \in M_R,\, a_i \in {}_RI,\, n \in \mathbb{N} \right\}; \tag{9.93}$$

4. *$\mathrm{Tor}_1^R(M, R/{}_RI) = 0$ for all ${}_RI \triangleleft {}_RR$.*

(Clearly all of the above holds true even for left R-modules).

Remark 9.85 Consider the embedding $\epsilon\colon I \hookrightarrow R$ of I into R and $M_R \in \mathsf{Mod}\text{-}R$, then we can take the tensor $\mathrm{id}_M \otimes \epsilon\colon M \otimes_R I \to M \otimes_R R \simeq M$ acting as $x \otimes a \mapsto xa$. Then

$$\mathrm{im}(\mathrm{id}_M \otimes \epsilon) = \left\{ \sum_{i=1}^{n} x_i a_i \;\middle|\; x_i \in M, a_i \in I \right\} = MI. \tag{9.94}$$

Thus $M \otimes I \simeq MI$ iff $id_M \otimes \epsilon$ is mono.

Lemma 9.86. *Consider $f\colon M_R \to N_R$ a morphism in the catogory of right R-modules. Let $f^* := \mathrm{Hom}_{\mathbb{Z}}(f, \mathbb{Q}/\mathbb{Z})$.*

- *f is mono iff f^* is epi,*
- *f is epi iff f^* is mono.*

Lemma 9.87. *Consider M_R and ${}_RN$. Then there are canonical isomorphisms:*

- *$M_R \otimes_R R \simeq R$ as right R-modules (resp. $R \otimes_R N \simeq {}_RN$ as left R-modules),*
- *$\mathrm{Hom}_R(R, M) \simeq M$ as right R-modules (resp. $\mathrm{Hom}_R(R, N) \simeq N$ as left R-modules),*
- *$M \otimes_R R/{}_RI \simeq M/MI$ as abelian groups (resp. $R/I_R \otimes N \simeq N/IN$ as abelian groups).*

Exercise 1 Let $\mathbb{K}$ be a field. Consider $R := \mathbb{K}[x, y]$ and $\mathfrak{m} := (x, y)$. Then $R/\mathfrak{m} \simeq \mathbb{K}$.

- Show that $\mathbb{K}$ has a projective resolution

$$0 \to R \xrightarrow{\beta} R \oplus R \xrightarrow{\alpha} R \to R/\mathfrak{m} \simeq \mathbb{K} \to 0, \tag{9.95}$$

  where $\beta = \begin{bmatrix} -y \\ x \end{bmatrix}$ and $\alpha(e_1) = x, \alpha(e_2) = y$.

- Show that $\mathrm{Tor}_2^R(\mathbb{K}, \mathbb{K}) \simeq \mathrm{Tor}_1^R(\mathfrak{m}, \mathbb{K}) \simeq \mathbb{K}$, sot that $\mathfrak{m}$ is torsion-free and not flat.
- $\mathrm{p.d.}\,\mathfrak{m} = 1$, $\mathrm{p.d.}\,\mathbb{K} = 2$ and $\mathrm{w.d.}\,\mathbb{K} = 2$.

Proposition 9.88.

1. $\mathrm{Tor}_n^R\left(\bigoplus_{i\in I} M_i, N\right) \simeq \bigoplus_{i\in I} \mathrm{Tor}_n^R(M_i, R)$

2. *For a (from the proof I guess it is filtered) direct system of modules $\{M_i, f_{ji}\}_{i\leq j}$*

$$\mathrm{Tor}_n^R\big(\varinjlim_{i\in I} M_I, N\big) \simeq \varinjlim_{i\in I} \mathrm{Tor}_n^R(M_i, N). \tag{9.96}$$

This theorem holds also for the second component of Tor.

Lemma 9.89. *Let $M_R \in \mathsf{Mod}\text{-}R$. M is flat iff*

$$\mathrm{Tor}_1^R(M, R/I) = 0 \tag{9.97}$$

for all ${}_RI \lhd R$ finitely generated left ideal.

Proposition 9.90. *Let A be an abelian category with products, coproducts, enough injectives and projectives. For any family $\{M_i\}_{i\in I}$, $\{N_i\}_{i\in I}$ and any object M, N of A and any $n \in \mathbb{N}$:*

1. $\mathrm{Ext}^n_{\mathsf{A}}\left(M, \prod_{i\in I} N_i\right) \simeq \prod_{i\in I} \mathrm{Ext}^n_{\mathsf{A}}(M, N_i)$
2. $\mathrm{Ext}^n_{\mathsf{A}}\left(\bigoplus_{i\in I} M_i, N\right) \simeq \prod_{i\in I} \mathrm{Ext}^n_{\mathsf{A}}(M_i, N)$

Remark 9.91 Let $\{M_i, f_{ji}\}_{i\leq j}$ be a directed system of modules. In general

$$\mathrm{Ext}_n^R\big(\varinjlim_{i\in I} M_i, N\big) \not\simeq \varinjlim_{i\in I} \mathrm{Ext}_n^R(M_i, N). \tag{9.98}$$

Example Let F_R be a flat, but not projective right R-module. Then there exists a module N s.t. $\mathrm{Ext}^1_R(F, N) \neq 0$. Moreover $F = \varinjlim_{i\in I} G_i$, for G_i finitely generated free modules (by Lazard theorem). Then, for all $i \in I$, $\mathrm{Ext}^1_R(G_i, N) = 0$. In other words we have a counterexample to the above "equality".

(Notice that there exist module such as F_R, in fact $\mathbb{Q}$ is a flat, but not projective, module; in particular it is flat, since it is a localization of $\mathbb{Z}$).

Definition 9.92: Right (left) hereditary ring.
A ring R is right (resp left) *hereditary* iff every submodule of a projective right (resp left) R-module is projective.

Proposition 9.93 (Characterization of hereditary rings)**.** *Let R be a ring, then the following are equivalent*

1. *R is right hereditary;*
2. $\mathrm{r.gl.dim}\, R \leq 1$*;*
3. $\mathrm{Ext}^2_R(M, N) = 0$ *for all $M, N \in \mathsf{Mod}\text{-}R$;*
4. $\mathrm{Ext}^2_R(R/I, N) = 0$ *for all $N_R \in \mathsf{Mod}\text{-}R$ and $I_R \lhd R$ right ideal;*
5. *I_R is projective for every right ideal $I_R \lhd R$.*

Clearly there exists also a left version of this proposition.

Example Being right or left hereditary is not symmetrical. In particular Kaplansky constructed an example of a ring R which is right hereditary, but has $\mathrm{l.gl.dim}\, R = 2$, i.e. it is not left hereditary. Small gave another example of a right hereditary ring R, with $\mathrm{l.gl.dim}\, R = 3$.

Recall the definition of weak global dimension of a ring R:

$$\text{w.gl.dim}\, R = \sup \{\text{w.d. } M_R \mid M_R \in \mathsf{Mod}\text{-}R\}\,. \tag{9.99}$$

(By symmetry of Tor functor this coincides with the left weak global dimension).

Proposition 9.94. *Let R be a ring. The following are equivalent:*

1. $\text{w.gl.dim}\, R \leq 1$*;*
2. $\mathrm{Tor}_2^R(M, N) = 0$ *for all* $M \in \mathsf{Mod}\text{-}R$ *and* $N \in R\text{-}\mathsf{Mod}$*;*
3. *Every submodule of a flat module is flat;*
4. $\mathrm{Tor}_R^2(R/I_R, N) = 0$ *for all* $I_R \lhd R$ *right ideals and* $N \in R\text{-}\mathsf{Mod}$*;*
5. *Every right ideal $I_R \lhd R$ is a flat R-module.*

Remark 9.95 Recall that we have the following implications: choose a ring R, and an R-module M, then

- M free $\implies$ M projective;
- M projective $\implies$ M flat;
- filtered direct limits of projective modules are flat;
- direct limits of finitely generated free modules are flat;
- in general, it is not true that any flat module is projective.

We now want to show that, in the particular case where M is finitely presented, then it is projective as soon as it is flat.

Definition 9.96: Finitely presented module.
Recall that $M \in R\text{-}\mathsf{Mod}$ is finitely presented iff there is a short exact sequence

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0,$$

with $n \in \mathbb{N}$ and K a finitely generated R-module.

Lemma 9.97. *Let M_R be a finitely presented module and consider the short exact sequence*

$$0 \longrightarrow H \longrightarrow P \longrightarrow M \longrightarrow 0,$$

with P projective and finitely generated. Then H is finitely generated.

Remark 9.98 The above implies that M_R is finitely presented iff there is an exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0,$$

for some $m, n \in \mathbb{N}$.

Remark 9.99: Pano's guess.
I guess that any finitely generated projective R-module P is also finitely presented. Let's, in fact, give a presentation of M:

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

By projectivity of M it splits. Then K is a direct summand of a free module, hence it is finitely generated.

Remark 9.100 Fix a pair of right R-modules $M_R, N_R \in \mathsf{Mod}\text{-}R$, then the character module $\operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) = (N_R)^* \in R\text{-}\mathsf{Mod}$ is a left R-module. Moreover there exists a morphism in Ab

$$\begin{aligned} \sigma_{M,N} \colon M \otimes N^* &\longrightarrow [\operatorname{Hom}_R(M,N)]^* = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_R(M,N), \mathbb{Q}/\mathbb{Z}) \\ x \otimes f &\longmapsto g, \end{aligned}$$

where g acts as follows on $\alpha \in \operatorname{Hom}_R(M,N)$

$$g(\alpha) \coloneqq f(\alpha(x)). \tag{9.100}$$

Lemma 9.101. *Let M_R be a finitely presented R-module, then $\sigma_{M,N}$, as defined above, is an isomorphism for all $N \in \mathsf{Mod}\text{-}R$.*

Theorem 9.102. *A finitely presented flat module M_R is projective.*

9.6 Change of rings

Tor under change of rings

Let R, S be rings, and $f \colon R \to S$ a ring homomorphism. Then S is an R-R bimodule via f and every S-module is an R-module via restriction of scalars. Moreover, given any $M_R \in \mathsf{Mod}\text{-}R$, then $M_R \otimes_R S$ is a right S-module via extension of scalars.

Proposition 9.103. *Let $f \colon R \to S$ be a ring homomorphism. Assume that ${}_R S$ is a flat left R-module. Then, for all $M_R \in \mathsf{Mod}\text{-}R$, all $n \in \mathbb{N}$ and ${}_S C \in S\text{-}\mathsf{Mod}$ (hence ${}_S C \in R\text{-}\mathsf{Mod}$), we have*

$$\operatorname{Tor}_n^R(M_R, {}_S C) \simeq \operatorname{Tor}_n^S(M \otimes_R S, {}_S C). \tag{9.101}$$

Proposition 9.104. *Let $f \colon R \to S$ be a ring homomorphism. Assume that ${}_R S$ is a flat left R-module. Then, for all $M_R \in \mathsf{Mod}\text{-}R$, $C_S \in \mathsf{Mod}\text{-}S$ and $n \in \mathbb{N}$, we have*

$$\operatorname{Ext}_R^n(M, C) \simeq \operatorname{Ext}_S^n(M \otimes_R S, C). \tag{9.102}$$

Proposition 9.105. *Let R, S be commutative rings, and $f \colon R \to S$ a ring homomorphism. Assume that S is a flat R-module. Then for all modules M and N, and $n \in \mathbb{N}$, we have*

$$\operatorname{Tor}_n^R(M, N) \otimes_R S \simeq \operatorname{Tor}_n^S(M \otimes_R S, N \otimes_R S). \tag{9.103}$$

Corollary 9.106. *Let R be a commutative ring, M and N be R-modules and $n \in \mathbb{N}$. The following are equivalent:*

1. $\operatorname{Tor}_n^R(M, N) = 0$;
2. $\operatorname{Tor}_n^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$;
3. $\operatorname{Tor}_n^{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m})$ *for all* $\mathfrak{m} \in \operatorname{MaxSpec} R$.

Hom and Ext with finitely presented modules

Let R, S be commutative rings, $\varphi\colon R \to S$ be a ring homomorphism and S be a flat R-module (e.g. T a multiplicatively closed subset of R and $S := R_T = R[T^{-1}]$).

Proposition 9.107. *Let R, S be commutative rings, $\varphi\colon R \to S$ be a ring homomorphism. Assume that S is a flat R-module and consider M_R a finitely presented R-module. Then, for any $N \in \mathsf{Mod}\text{-}R$,*

$$\operatorname{Hom}_S(M \otimes_R S, N \otimes_R S) \simeq \operatorname{Hom}_R(M, N) \otimes_R S. \tag{9.104}$$

Let's now extend this result for the Ext functor:

Definition 9.108 We denote by $\mathsf{mod}\text{-}R$ (using the lowercase m to differentiate from the bigger category) the category of right R-modules M with a projective resolution of finitely generated projective modules (i.e. all the syzygys $\Omega_n(M)$ are finitely generated: at each point the kernel [i.e. the syzygy] is an epimorphic image of a finitely generated module).

Remark 9.109 If R is a right Noetherian ring, then the objects of $\mathrm{mod}\text{-}R$ are exactly the finitely generated R-modules.

Definition 9.110: Right coherent ring.
A ring R is *right coherent* iff every finitely generated right ideal is also finitely presented. Equivalently iff every finitely generated submodule of a finitely presented right R-module is finitely presented.

Remark 9.111 Let R be right coherent, then $\mathsf{mod}\text{-}R$ is the category of finitely presented right R-modules.

Proposition 9.112. *Let R, S be commutative rings and $\varphi\colon R \to S$ be a ring homomorphism. Assume that S is a flat R-module and consider $M \in \mathsf{mod}\text{-}R$ and $N \in \mathsf{Mod}\text{-}R$. Then, for all $n \in \mathbb{N}$,*

$$\operatorname{Ext}^n_S(M \otimes_R S, N \otimes_R S) \simeq \operatorname{Ext}^n_R(M, N) \otimes_R S. \tag{9.105}$$

Corollary 9.113. *Let R be a commutative ring, $M_R \in \mathsf{mod}\text{-}R$, $N \in \mathsf{Mod}\text{-}R$ and $n \in \mathbb{N}$. The following are equivalent:*

1. $\operatorname{Ext}^n_R = 0$*;*
2. $\operatorname{Ext}^n_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$*;*
3. $\operatorname{Ext}^n_{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) = 0$ *for all* $\mathfrak{m} \in \operatorname{MaxSpec} R$*.*

9.7 Homological formulas relating Ext and Hom

Proposition 9.114. *Consider R and S rings. Let ${}_R N_S$ be an S-R bimodule and $M_R \in \mathsf{Mod}\text{-}R$. Consider C_S an injective right S-module. Then, for all $n \geq 0$,*

$$\operatorname{Ext}^n_R(M_R, \operatorname{Hom}_S(N_S, C_S)_R) \simeq \operatorname{Hom}_S\left(\operatorname{Tor}^R_n(M, N)_S, C_S\right). \tag{9.106}$$

In particular, if we pick $S := \mathbb{Z}$ and $C := \mathbb{Q}/\mathbb{Z}$, we obtain, for all $n \in\geq 0$

$$\operatorname{Ext}^n_R(M_R, N^*) \simeq \left[\operatorname{Tor}^R_n(M, N)\right]^*. \tag{9.107}$$

Proposition 9.115. *Consider R, S rings. Let ${}_R N_S$ be an S-R bimodule and $M_R \in \mathsf{mod}\text{-}R$. Consider ${}_S C$ an injective left S-module. Then, for all $n \geq 0$,*

$$\operatorname{Tor}_n^R\left(M_R, \operatorname{Hom}_S\left({}_S N_R, {}_S C\right)\right) \simeq \operatorname{Hom}_S\left(\operatorname{Ext}_R^n\left(M_R, {}_S N_R\right), {}_S C\right). \tag{9.108}$$

In particular, if $S := \mathbb{Z}$ and $C := \mathbb{Q}/\mathbb{Z}$, then

$$\operatorname{Tor}_n^R\left(M, N^*\right) \simeq \left[\operatorname{Ext}_R^n\left(M, N\right)\right]^*. \tag{9.109}$$

Example Let M_R be a right R-module, ${}_R G_S$ an R-S bimodule and C_S a right S-module. Assume that $\operatorname{Tor}_1^R\left(M, G\right) = 0$, then there is a monomorphism (of abelian groups)

$$\operatorname{Ext}_R^1\left(M_R, \operatorname{Hom}_S\left({}_R G_S, C_S\right)\right) \rightarrowtail \operatorname{Ext}_S^1\left(M \otimes_R G, C_S\right).$$

10 Yoneda extension

Our next aim is, given an abelian category A and objects $A, B \in \operatorname{Ob}(\mathsf{A})$, to define $\operatorname{Ext}_{\mathsf{A}}(A, B)$ even though A might not have enough injectives nor projectives.

Definition 10.1: Extension.
Let A be an abelian category, $A, B \in \operatorname{Ob}(\mathsf{A})$. An extension of A by B is a short exact sequence

$$\zeta := 0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0.$$

We say that two extensions ζ and ζ' are equivalent, denoted by $\zeta \sim \zeta'$, iff there is a commutative diagram such that the nontrivial vertical arrow is an isomorphism

$$\begin{tikzcd}
\zeta\colon & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\wr"] & A \arrow[r] \arrow[d, equal] & 0 \\
\zeta'\colon & 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}. \tag{10.1}$$

Remark 10.2: Split extensions.
Recall the characterization of splitting short exact sequences: an extension

$$\zeta := 0 \longrightarrow B \xrightarrow{\mu} X \xrightarrow{p} A \longrightarrow 0$$

splits iff it is equivalent to the following extension of A by B:

$$0 \longrightarrow B \xrightarrow{\epsilon_B} A \oplus B \xrightarrow{\pi_A} A \longrightarrow 0.$$

Equivalently iff there is $f\colon X \to B$ s.t. $f \circ \mu = 1_B$, iff there is $g\colon A \to X$ s.t. $p \circ g = 1_A$.

Remark 10.3: Class of extensions and Ext.
Denote by $\mathrm{E}(A, B)$ the class of all extensions of A by B. If we denote by $\sim$ the above equivalence relation and we can define $\operatorname{Ext}_{\mathsf{A}}^1(A, B)$ (i.e. if A has enough injectives of projectives), then we want to construct an isomorphism θ of abelian groups

$$\operatorname{Ext}_{\mathsf{A}}^1(A, B) \simeq_\theta \frac{\mathrm{E}(A, B)}{\sim}. \tag{10.2}$$

Let's define θ. Fix $A, B \in \operatorname{Ob}(\mathsf{A})$ and consider an extension of A by B

$$\zeta := 0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0.$$

Assume that $\operatorname{Ext}^n_{\mathsf{A}}(-,B)$ exist, forming a cohomological ∂-functor (e.g. if A has enough projectives). Apply $\operatorname{Hom}_{\mathsf{A}}(-,B)$ to ζ and obtain

$$0 \longrightarrow \operatorname{Hom}_{\mathsf{A}}(A,B) \longrightarrow \operatorname{Hom}_{\mathsf{A}}(X,B) \longrightarrow \operatorname{Hom}_{\mathsf{A}}(B,B) \xrightarrow{\partial} \operatorname{Ext}^1_{\mathsf{A}}(A,B)\,.$$

Finally we define $\theta(\zeta) := \partial(1_B) \in \operatorname{Ext}^1_{\mathsf{A}}(A,B)$.

Lemma 10.4. *Fixed a category A and objects A, B of A as before, if $\zeta \sim \zeta' \in \mathrm{E}(A,B)$, then $\theta(\zeta) = \theta(\zeta')$. In other words $\theta\colon \mathrm{E}(A,B) \to \operatorname{Ext}^1_{\mathsf{A}}(A,B)$ induces a map on the quotient $\frac{\mathrm{E}(A,B)}{\sim}$.*

Theorem 10.5. *Given A, A, B as before, θ gives a bijective correspondence*

$$\frac{\mathrm{E}(A,B)}{\sim} \xleftrightarrow{\theta} \operatorname{Ext}^1_{\mathsf{A}}(A,B)\ . \tag{10.3}$$

Lemma 10.6. *Let A be an abelian category. Consider a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\nu"] \arrow[d, "\beta"'] & M \arrow[r] \arrow[d, "h"] \arrow[dl, dashed, "g"'] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & Y \arrow[r] & A \arrow[r] & 0 \end{tikzcd}\ , \tag{10.4}$$

with exact rows and where the leftmost square is a pushout. Then there is $g\colon M \to B$ s.t. $g \circ \nu = \beta$ iff the second row splits.

Lemma 10.7. *Let A, A, B be as before, then $\operatorname{Ext}^1_{\mathsf{A}}(A,B) = 0$ (as abelian groups) iff every extension of A by B splits.*

Remark 10.8 Consider $\zeta \in \mathrm{E}(A,B)$, then $\gamma \in \operatorname{Hom}_{\mathsf{A}}(A',A)$ gives $\zeta\gamma \in \mathrm{E}(A',B)$

$$\begin{tikzcd} \zeta\gamma\colon & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "\gamma"] & 0 \\ \zeta\colon & 0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0 \end{tikzcd}\ , \tag{10.5}$$

where X' is a pullback of the diagram

$$\begin{tikzcd} & A' \arrow[d, "\pi"] \\ X \arrow[r] & A \end{tikzcd}\ . \tag{10.6}$$

Analogously $\beta \in \operatorname{Hom}_{\mathsf{A}}(B,B')$ gives $\beta\zeta \in \mathrm{E}(A,B')$:

$$\begin{tikzcd} \zeta\colon & 0 \arrow[r] & B \arrow[r] \arrow[d, "\beta"] & X \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ \beta\zeta\colon & 0 \arrow[r] & B' \arrow[r] & X' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}\ , \tag{10.7}$$

where X' is a pushout of the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "p"'] & X \\ B' & \end{tikzcd}\ . \tag{10.8}$$

Definition 10.9: Baer sum in $\mathrm{E}(A,B)/\sim$**.**
Consider A, A, B as before. Let $[\zeta], [\zeta']$ be the equivalence classes of $\zeta, \zeta' \in \mathrm{E}(A,B)$ with respect to $\sim$:

$$\zeta := 0 \longrightarrow B \xrightarrow{i} X \xrightarrow{\pi} A \longrightarrow 0 \quad \text{and} \quad \zeta' := 0 \longrightarrow B \xrightarrow{i'} X' \xrightarrow{\pi'} A \longrightarrow 0.$$

Now consider the extension of $A \oplus A$ by $B \oplus B$, given by the direct sum

$$\zeta \oplus \zeta' := 0 \longrightarrow B \oplus B \xrightarrow{i \oplus i'} X \oplus X' \xrightarrow{\pi \oplus \pi'} A \oplus A \longrightarrow 0.$$

Let moreover $\Delta_A \colon A \to A \oplus A$ be the diagonal map, i.e. $\Delta_A = \begin{bmatrix} 1_A \\ 1_A \end{bmatrix}$, and $\nabla_B \colon B \to B \oplus B$ the codiagonal map, i.e. $\nabla_B = \begin{bmatrix} 1_B & 1_B \end{bmatrix}$. By the above remark we can construct

$$\begin{tikzcd}
\nabla_B (\zeta \oplus \zeta') \Delta_A \colon & 0 \arrow[r] & B \arrow[r] & Z \arrow[r] & A \arrow[r] & 0 \\
(\zeta \oplus \zeta') \Delta_A \colon & 0 \arrow[r] & B \oplus B \arrow[r] \arrow[u, "\nabla_B"] \arrow[d, equal] & Y \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, "\Delta_A"] & 0 \\
\zeta \oplus \zeta' \colon & 0 \arrow[r] & B \oplus B \arrow[r] & X \oplus X' \arrow[r] & A \oplus A \arrow[r] & 0
\end{tikzcd} \cdot$$

Finally we define $[\zeta] + [\zeta'] := [\nabla_B (\zeta \oplus \zeta') \Delta_A]$. We can also see that this definition is independent of the choice of representatives.

Proposition 10.10. $\mathrm{E}(A,B)/\sim$ *endowed with the Baer sum (i.e. the sum we just defined) is an abelian group. Moreover the map*

$$\theta \colon \tfrac{\mathrm{E}(A,B)}{\sim} \longrightarrow \mathrm{Ext}^1_{\mathsf{A}}(A,B)$$

is a group homomorphism.

By this proposition, one can also define Ext^1_R to be $\mathrm{E}(A,B)/\sim$, and this definition does not require the abelian category A to have enough injectives or projectives.

Analogously this construction can be carried out for every $n \in \mathbb{N}$:

Definition 10.11: n **extension.**
Let A be an abelian category, $A, B \in \mathrm{Ob}(\mathsf{A})$. An n extension of A by B, denoted by $\zeta \in \mathrm{E}_n(A,B)$, is an exact sequence

$$\zeta := 0 \longrightarrow B \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \dots \longrightarrow X_1 \longrightarrow A \longrightarrow 0.$$

We say that two extensions ζ and ζ' are equivalent, denoted by $\zeta \sim \zeta'$, iff there is a commutative diagram s.t. the nontrivial vertical arrows are all isomorphisms

$$\begin{tikzcd}
\zeta \colon 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X_n \arrow[r] \arrow[d, "f_n"] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \arrow[r] & X_1 \arrow[r] \arrow[d, "f_1"] & A \arrow[r] \arrow[d, equal] & 0 \\
\zeta' \colon 0 \arrow[r] & B \arrow[r] & X'_n \arrow[r] & X'_{n-1} \arrow[r] & \dots \arrow[r] & X'_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd} .$$

And, as expected, it gives the desired result:

Proposition 10.12. *For every* $n \in \mathbb{N}$*, if* $\mathrm{Ext}^n_{\mathsf{A}}(A,B)$ *is defined in* A *abelian, we have the following isomorphism of abelian groups*

$$\frac{\mathrm{E}_n(A,B)}{\sim} \simeq \mathrm{Ext}^n_{\mathsf{A}}(A,B). \tag{10.9}$$