Tehnici de Optimizare

Facultatea de Matematica si Informatica

Universitatea Bucuresti

- Department Informatica-

2021

Evaluare

- 4 teme laborator = 40%;
- 2 saptamani pentru fiecare

- Cel putin 1 pct bonus pt activitate seminar!

- Proiect (2-3 in echipa) = 60%
- Predare dupa vacanta

Notatii

In cadrul cursului vom utiliza urmatoarele notatii:

- Vectori (considerati intotdeauna vector coloana) cu litere mici, i.e. $x \in \mathbb{R}^n$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
- Produs scalar in spatiul Euclidian: $\langle x, y \rangle = x^T y = \sum_{i=1}^n x_i y_i$
- Norma Euclidiana standard $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + \cdots + x_n^2}$
- Multimi cu litere mari: $S, Q, U \subseteq \mathbb{R}^n$ ($\mathbb{R}^n_+$ - orthantul negativ, S^n_+ - multimea matricelor pozitiv semidefinite)
- Matrice cu litere mari: $A, B, C, H \in \mathbb{R}^{n \times m}$
- Norma spectrala a unei matrici $\|A\| = \sqrt{\lambda_{max}(A^T A)}$
- Matrice pozitiv definita: $A \succ 0$, si pozitiv semidefinita $A \succeq 0$

Multimi afine

$S \subseteq \mathbb{R}^n$ este afina daca $\forall x_1, x_2 \in S$ si $\forall \alpha \in \mathbb{R}$ avem

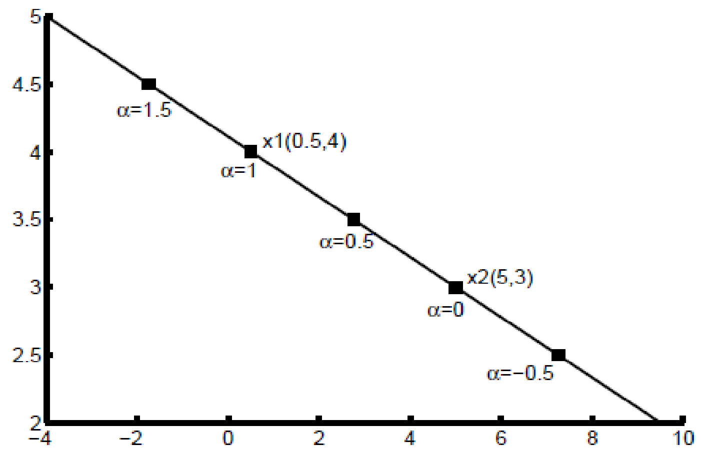
$$\alpha x_1 + (1 - \alpha)x_2 \in S$$

Exemplu multime afina:

- Multimea solutiilor unui sistem de ecuatii liniare $\{x : \ Ax = b\}$

Interpretare:

- Orice punct pe dreapta definita de x_1 si x_2 se afla in multime.



Exemple

- **Multimea hiperplan**:

$$H = \{x \in R^n\colon\ a^T x = b\}$$

$$a^T(\alpha x + (1-\alpha)y) = \alpha a^T x + (1-\alpha)a^T y = \alpha b + (1-\alpha)b = b$$

- Multimea solutiilor unui sistem liniar:

$$S = \{x \in R^n\colon\ Ax = b\}$$

Multimi convexe

$S \subseteq \mathbb{R}^n$ este convexa daca $\forall x_1, x_2 \in S$ si $\forall \alpha \in [0, 1]$ avem

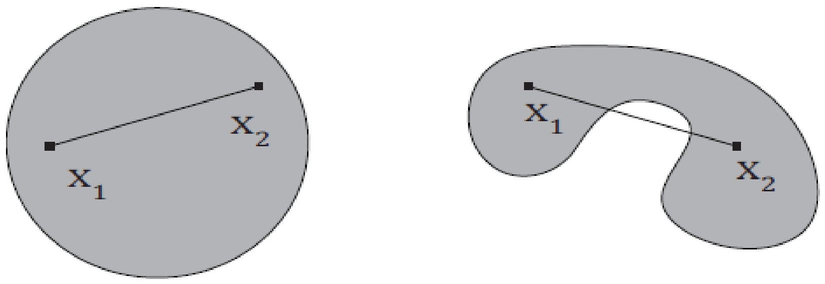
$$\alpha x_1 + (1 - \alpha) x_2 \in S$$

Interpretare:

- Orice punct pe segmentul de dreapta definit de x_1 si x_2 se afla in multime

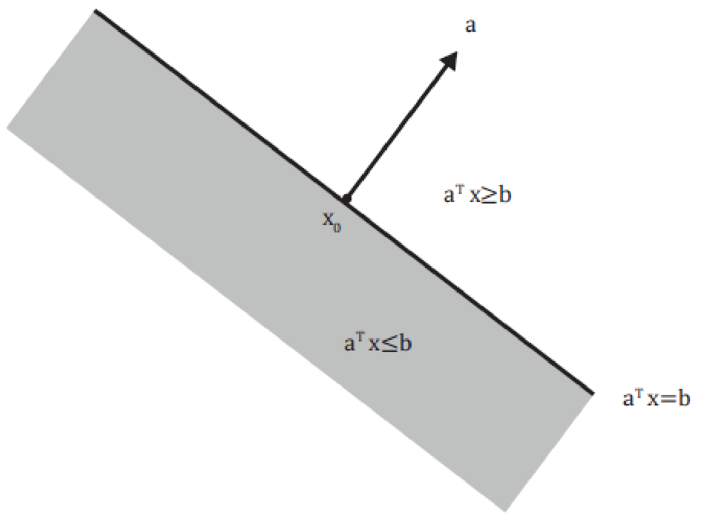
Exemplu de multime convexa si neconvexa:

- Orice multime afina este multime convexa (orice combinatie convexa intre x_1 si x_2 se afla in multime)



Exemple

- **Hiperplan si semiplan**: multimi convexe definite astfel
  - hiperplan: $\{x \in \mathbb{R}^n : a^T x = b\}$, $a \neq 0, a \in \mathbb{R}^n, b \in \mathbb{R}$
  - semiplan: $\{x \in \mathbb{R}^n : a^T x \geq b\}$ sau $\{x \in \mathbb{R}^n : a^T x \leq b\}$, $a \neq 0, a \in \mathbb{R}^n, b \in \mathbb{R}$
- Exemplu de hiperplan si semiplanele aferente:



$$\{x \in \mathbb{R}^2 : \; x_1 + 2x_2 = 1\} \Rightarrow a = [1 \;\; 2]^T \;\&\; b = 1$$

Exemple

- **Poliedru** = multime determinata de mai multe hiperplane/semiplane

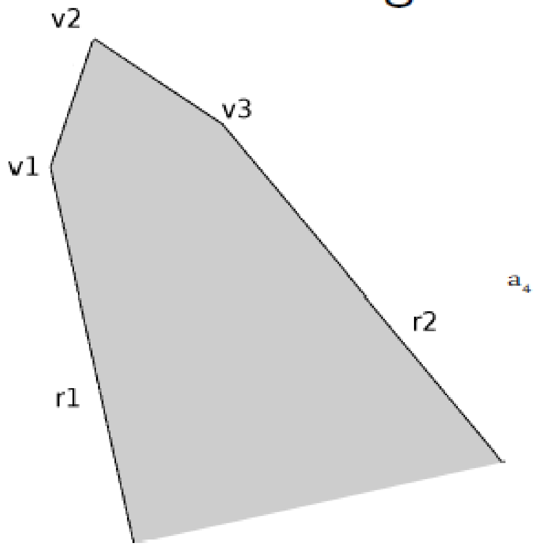
$$\left\{x \in \mathbb{R}^n : a_i^T x = b_i \ \forall i = 1, \ldots, p, \quad c_j^T x \leq d_j \ \forall j = 1, \ldots, m\right\}$$ sau

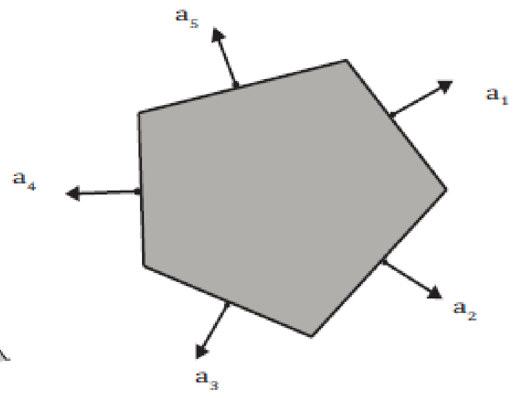
mai compact $\{x : Ax = b, \ Cx \leq d\}$

  - poate fi reprezentat si prin varfuri v_i si raze afine r_j:

    $$\left\{\sum_{i=1}^{n_1} \alpha_i v_i + \sum_{j=1}^{n_2} \beta_j r_j : \ \sum_{i=1}^{n_1} \alpha_i = 1, \ \alpha_i \geq 0, \beta_j \geq 0 \ \ \forall i, j\right\}$$

  - exemplu de poliedru nemarginit: $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : \ x_1, x_2 \geq 0\}$





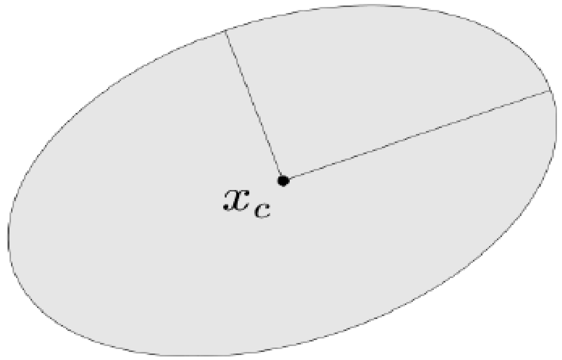
Exemple

Bila:
multime convexa definita de o norma $\|\cdot\|$, un centru x_c si o raza r:

$$B(x_c, r) = \{x \in \mathbb{R}^n : \ \|x - x_c\| \leq r\}$$

Exemplu, bila Euclidiana de centru zero si raza 1 in $\mathbb{R}^2$:
$B_2(0,1) = \left\{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} \leq 1\right\}$

Elipsoid:
multimea convexa definita de un centru x_c si o matrice $Q \succ 0$
$\{x \in \mathbb{R}^n : \ (x - x_c)^T Q^{-1}(x - x_c) \leq 1\}$



Multimi

- **Multime compacta** = o multime se numeste compacta daca este inchisa si marginita

- **Exemple**:

  [1,2], [a,b]

  $\Delta_n = \{x \in R^n : \sum_i x_i = 1, x_i \geq 0\}$

  $C = \{x \in R^n : a_i \leq x_i \leq b_i\}$

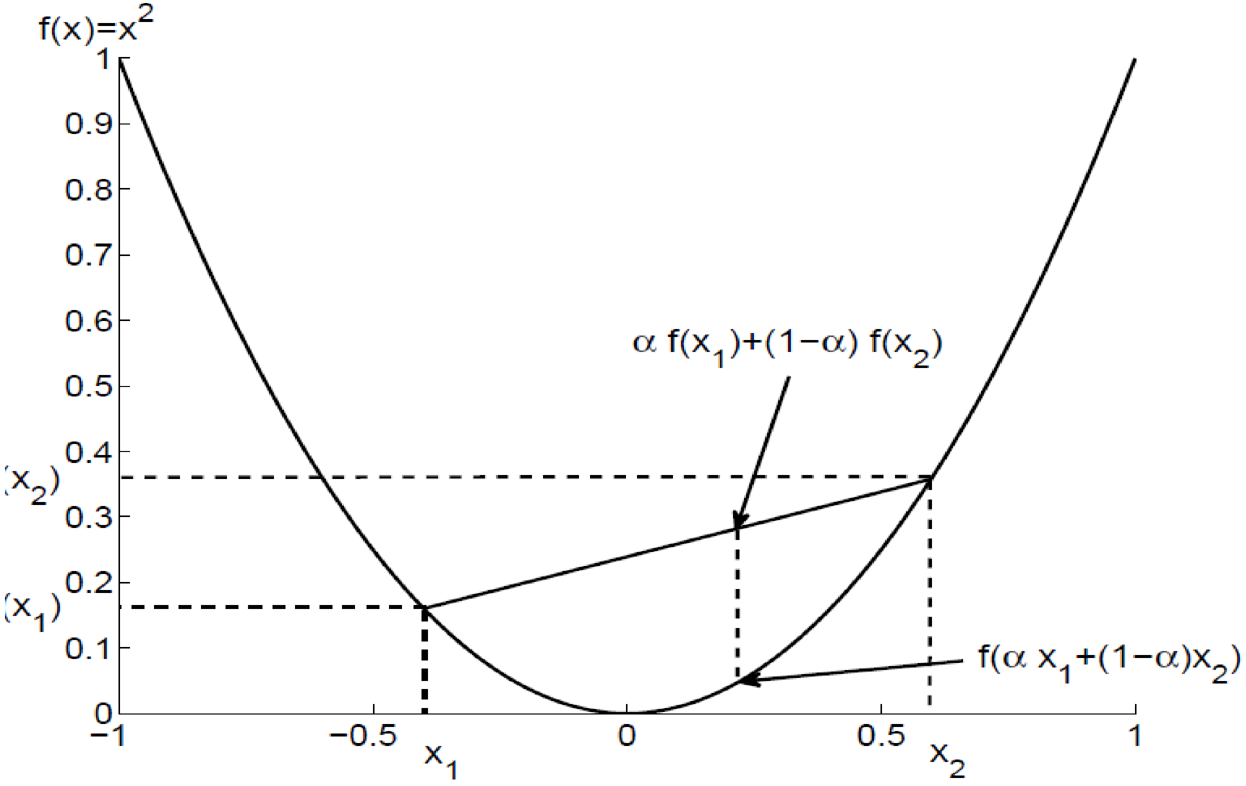
  $B = \{x \in R^n : \|x - c\| \leq r\}$

Functii convexe

O functie $f: R^n \rightarrow (-\infty, \infty]$ se numeste **convexa** daca:

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

pentru orice x,y si $\alpha \in [0,1]$.



Exemple

- Functia definita de orice norma este convexa, i.e. $f(x) = \|x\|$ este convexa pe $\mathbb{R}^n$ (folositi definitia).
- Functia $f(x) = -\log(x)$ este convexa pe $\mathbb{R}_{++}$.
- Functia $f(x) = \max\{x_1, \ldots, x_n\}$ este convexa pe $\mathbb{R}^n$.

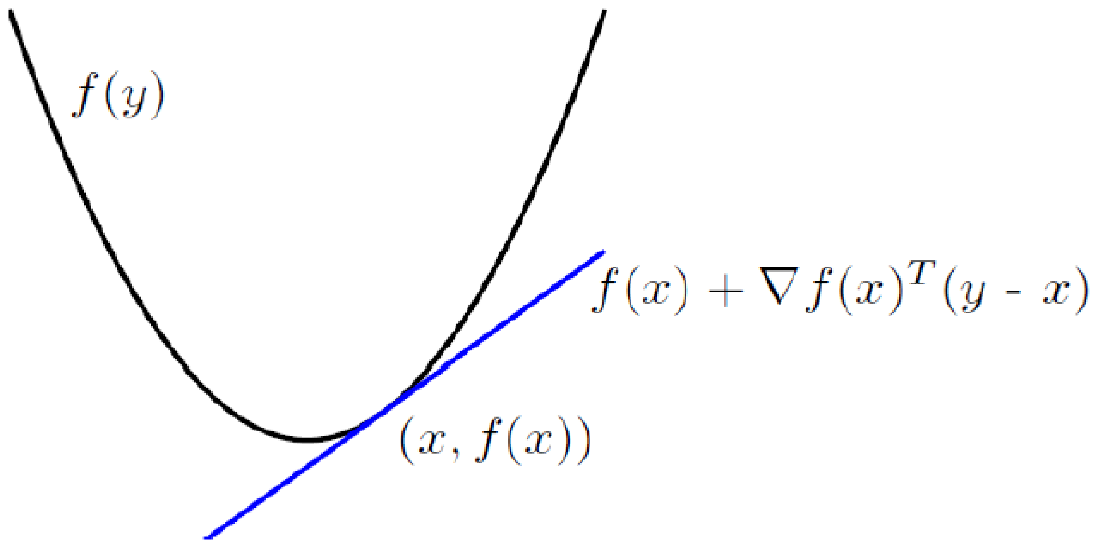
Functii convexe

Conditii de convexitate pentru functii continuu diferentiabile.

Conditii de ordinul I: daca $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ este continuu diferentiabila si dom$f$ este o multime convexa. Atunci, f este convexa daca si numai daca:

$$f(x_2) \geq f(x_1) + \nabla f(x_1)^T (x_2 - x_1) \quad \forall x_1, x_2 \in \mathrm{dom} f.$$

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$



Functii convexe

Conditii de convexitate pentru functii continuu diferentiabile.

Conditii de ordinul II: daca $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ este de doua ori continuu diferentiabila si dom$f$ este multime convexa. Atunci f este convexa daca si numai daca pentru orice $x \in$ dom$f$ matricea Hessiana este pozitiv semidefinita, adica:

$$\nabla^2 f(x) \succeq 0 \quad \forall x \in \mathrm{dom} f.$$

$$\nabla^2 f(x) \succeq 0.$$