Grupa 344, EDDP, Seminar 8, 23.11.2020

Determinarea unui sistem fundamental de soluții pentru un sistem liniar cu coeficienți constanți.

$$x' = Ax \quad , \quad A \in \mathcal{M}_n(\mathbb{R})$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

S_A = mulțimea soluțiilor sistemului $x' = Ax$

$\underbrace{\{\varphi_1, \dots, \varphi_n\}}$, bază, adică, sistem fundamental de soluții în S_A.

Exerciții: Să se determine mulțimea soluțiilor sistemelor diferențiale următoare:

✓ 1) $\begin{cases} x_1' = 3x_1 - x_2 \\ x_2' = 2x_1 \end{cases}$ 2) $\begin{cases} x_1' = 3x_1 + 2x_2 \\ x_2' = -2x_1 - x_2 \end{cases}$

✓ 3) $\begin{cases} x_1' = x_1 - 3x_2 \\ x_2' = 3x_1 + x_2 \end{cases}$ 4) $\begin{cases} x_1' = 2x_1 + x_2 \\ x_2' = -x_1 + 4x_2 \end{cases}$

5) $\begin{cases} x_1' = 4x_1 - x_2 \\ x_2' = 3x_1 + x_2 - x_3 \\ x_3' = x_1 + x_3 \end{cases}$ 6) $\begin{cases} x_1' = 2x_1 - x_2 + 2x_3 \\ x_2' = x_1 + 2x_3 \\ x_3' = -2x_1 + x_2 - x_3 \end{cases}$

7) $\begin{cases} x_1' = 2x_1 - x_2 - x_3 \\ x_2' = 3x_1 - 2x_2 - 3x_3 \\ x_3' = -x_1 + x_2 + 2x_3 \end{cases}$ ✓ 8) $\begin{cases} x_1' = x_2 - x_3 \\ x_2' = 2x_1 + x_2 + x_3 \\ x_3' = -x_2 + x_3 \end{cases}$

1) $\begin{cases} x_1' = 3x_1 - x_2 \\ x_2' = 2x_1 \end{cases}$

Se scrie sistemul în formă matricială:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad , n = 2$$

Se determină valorile proprii ale matricei A, adică, se rezolvă ec:

$$\det(A-\lambda I_2)=0 \Rightarrow \begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix}=0 \Rightarrow$$

$$\Rightarrow \lambda^2-3\lambda+2=0 \,.\Rightarrow (\lambda-1)^1(\lambda-2)^1=0$$

$$\lambda_1=1\,,\ m_1=1$$
$$\lambda_2=2,\ m_2=1\quad ;\ m_1+m_2=2=n$$

Pt. fiecare valoare proprie se determină sistemul fundamental de soluții, atâtea soluții cât este multiplicitatea valorii proprii.

Pt $\boxed{\lambda_1=1,\ m_1=1}$ $\to$ determinăm $u\in\mathbb{R}^2$, $u\neq 0_{\mathbb{R}^2}$ a.î.

$$Au=\lambda_1 u\,.\Rightarrow$$

$$\Rightarrow \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=1\cdot\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \begin{cases} 3u_1-u_2=u_1 \\ 2u_1=u_2 \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{u_2=2u_1} \Rightarrow u=\begin{pmatrix} u_1 \\ 2u_1 \end{pmatrix}=u_1\begin{pmatrix} 1 \\ 2 \end{pmatrix},\ u_1\in\mathbb{R}\Rightarrow$$

$$\Rightarrow \varphi_1(t)=e^{\lambda_1 t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \boxed{\varphi_1(t)=\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{t}}$$

Pt. $\boxed{\lambda_2=2,\ m_1=1}$ $\Rightarrow$ determinăm $u\in\mathbb{R}^2$, $u\neq 0_{\mathbb{R}^2}$ a.î.

$$Au=\lambda_2 u$$

$$\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \begin{cases} 3u_1-u_2=2u_1 \\ 2u_1=2u_2 \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{u_2=u_1} \Rightarrow u=u_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \boxed{\varphi_2(t)=\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}}$$

Deci:
$$S_A=\langle \varphi_1,\varphi_2\rangle=\{C_1\varphi_1+C_2\varphi_2 \mid C_1, C_2\in\mathbb{R}\} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=C_1\cdot\begin{pmatrix} e^t \\ 2e^t \end{pmatrix}+C_2\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} \Rightarrow \boxed{\begin{cases} x_1(t)=C_1e^t+C_2e^{2t} \\ x_2(t)=2C_1e^t+C_2e^{2t} \end{cases}}$$

(8) $\begin{cases} x_1'=x_2-x_3 \\ x_2'=2x_1+x_2+x_3 \\ x_3'=-x_2+x_3 \end{cases} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}=\underbrace{\begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ n=3.$

$$\det(A-\lambda I_3)=0 \Rightarrow \begin{vmatrix} 0-\lambda & 1 & -1 \\ 2 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0 .$$

$L2 \leftarrow L2+L1$
$L3 \leftarrow L3+L1$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & -1 \\ 2-\lambda & 2-\lambda & 0 \\ -\lambda & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(2-\lambda)\begin{vmatrix} -\lambda & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \Rightarrow$$

$$\Rightarrow -\lambda(2-\lambda)(-\lambda+0+0+\cancel{1}-0-\cancel{1})=0$$

$$\Rightarrow \boxed{\lambda^2(2-\lambda)=0} \Rightarrow \begin{cases} \lambda_1=0, & m_1=2 \\ \lambda_2=2, & m_2=1 \end{cases}$$

Pt $\boxed{\lambda_2=2, \quad m_2=1}$ $\Rightarrow$ se det. $u\in\mathbb{R}^3$, nenul a.î. $Au=\lambda_2 u \Rightarrow$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 2\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \begin{cases} u_2-u_3=2u_1 \\ 2u_1+u_2+u_3=2u_2 \Rightarrow \\ -u_2+u_3=2u_3 \end{cases}$$

$$\Rightarrow \begin{cases} -2u_1+u_2-u_3=0 \\ 2u_1-u_2+u_3=0 . \\ -u_2-u_3=0 \end{cases}$$

det matricii sist este nul $\Rightarrow$ căutăm minor principal

$$\Delta_p = \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = 2 \neq 0 \Rightarrow$$ ec. 2 și 3 sunt principale; x_1 nec sec $\Rightarrow$

$$\Rightarrow \begin{cases} -u_2+u_3=-2u_1 \\ -u_2-u_3=0 \end{cases} (+)$$
$$-2u_2 \quad / \quad = -2u_1 \Rightarrow \boxed{u_2=u_1} \quad \boxed{u_3=-u_2=-u_1}$$

$$\Rightarrow u=\begin{pmatrix} u_1 \\ u_1 \\ -u_1 \end{pmatrix} = u_1\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \boxed{\varphi_1(t)=e^{2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}$$

Pt. $\boxed{\lambda_1=0, \quad m_1=2}$ $\Rightarrow$ determinăm vectorii $p_0, p_1\in\mathbb{R}^3$ nu amândoi nuli, a.î. $\varphi(t)=(p_0+p_1t)e^{0t}$

să fie soluție a sistemului $x' = Ax$. $\Rightarrow$

$$\Rightarrow \left((p_0 + p_1 t)\right)' = A(p_0 + p_1 t) \Rightarrow$$

$$\left.\begin{array}{l}\Rightarrow p_1 = Ap_0 + Ap_1 t \\ \text{Identificăm coef. puterilor lui } t\end{array}\right\} \Rightarrow \begin{cases} 0_{\mathbb{R}^3} = Ap_1 \\ p_1 = Ap_0 \end{cases} \Big|\cdot A \text{ în stg}$$

$$\Rightarrow \underbrace{Ap_1}_{0_{\mathbb{R}^3}} = A^2 p_0 \Rightarrow A^2 p_0 = 0_{\mathbb{R}^3} \Rightarrow p_0 \in \ker(A^2) \overset{def}{=} \{v \in \mathbb{R}^3 \mid A^2 v = 0_{\mathbb{R}^3}\}$$

$$\underset{(A - \lambda_1 I_3)^2}{\overset{\parallel}{A^2}} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ -2 & -2 & 0 \end{pmatrix}$$

$$A^2 v = 0_{\mathbb{R}^3} \Rightarrow \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ -2 & -2 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2v_1 + 2v_2 = 0 \\ 2v_1 + 2v_2 = 0 \\ -2v_1 - 2v_2 = 0. \end{cases}$$

$$\Rightarrow \begin{cases} v_1 + v_2 = 0 \Rightarrow v_1 = -v_2 \\ v_3 \in \mathbb{R} \end{cases} \Rightarrow \ker(A^2) = \left\{ v = \begin{pmatrix} -v_2 \\ v_2 \\ v_3 \end{pmatrix}, \mid v_2, v_3 \in \mathbb{R} \right\}$$

$$\Rightarrow p_0 \in \ker(A^2) = \left\{ v = v_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \operatorname{Sp\,gen}\left\{\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)\right\}.$$

<u>OBS</u>: $\ker\left((A - \lambda_j I_n)^{m_j}\right)$ este subspațiu în $\mathbb{R}^n$ și dimensiunea m_j.

Pentru p_0 se iau vectorii unei baze din $\ker(A^2) \Rightarrow$

$$\Rightarrow p_0 \in \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

<u>Pt $p_0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$</u> $\Rightarrow p_1 = Ap_0 = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow$

$$\Rightarrow \varphi_2(t) = (p_0 + p_1 t)\, e^{0t} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} t \Rightarrow \boxed{\varphi_2(t) = \begin{pmatrix} -1+t \\ 1-t \\ -t \end{pmatrix}}$$

<u>Pt $p_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$</u> $\Rightarrow$ $p_1 = Ap_0 = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$

$$\Rightarrow \varphi_3(t) = (p_0 + p_1 t)\, e^{0t} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} t \Rightarrow \boxed{\varphi_3(t) = \begin{pmatrix} -t \\ t \\ 1+t \end{pmatrix}}$$

Avem $\{\varphi_1, \varphi_2, \varphi_3\}$ sistem fundamental de soluții $\Rightarrow$

$$\Rightarrow S_A = \{ c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 \mid c_1, c_2, c_3 \in \mathbb{R} \} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ e^{2t} \\ -e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} -1+t \\ 1-t \\ -t \end{pmatrix} + c_3 \begin{pmatrix} -t \\ t \\ 1+t \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1(t) = c_1 e^{2t} + c_2(-1+t) - c_3 t \\ x_2(t) = c_1 e^{2t} + c_2(1-t) + c_3 t \\ x_3(t) = -c_1 e^{2t} - c_2 t + c_3(1+t) \end{cases}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

<u>OBS</u>: Verificarea că sistemul fundamental de soluții obținut este bun, este:

$$\det(\varphi_1(t), \varphi_2(t), \varphi_3(t)) \neq 0. \quad \forall t \in \mathbb{R}$$

În acest caz:

$$\begin{vmatrix} e^{2t} & -1+t & -t \\ e^{2t} & 1-t & t \\ -e^{2t} & -t & 1+t \end{vmatrix} \underset{C2 \leftarrow C2+C3}{=} e^{2t} \begin{vmatrix} 1 & -1 & -t \\ 1 & 1 & t \\ -1 & 1 & 1+t \end{vmatrix} \underset{C1 \leftarrow C1+C2}{=}$$

$$= e^{2t} \begin{vmatrix} 0 & -1 & -t \\ 2 & 1 & t \\ 0 & 1 & 1+t \end{vmatrix} = e^{2t} \cdot 2(-1)^{2+1} \begin{vmatrix} -1 & -t \\ 1 & 1+t \end{vmatrix} =$$

$$= -2e^{2t}(-1 \cancel{-t} \cancel{+t}) = 2e^{2t} \neq 0$$

$\forall t \in \mathbb{R}.$

③ $\begin{cases} x_1' = x_1 - 3x_2 \\ x_2' = 3x_1 + x_2 \end{cases} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $n=2$

• determinăm val. proprii pt A: $\det(A - \lambda I_2) = 0 \Rightarrow$

$\Rightarrow \det\left(\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 + 9 = 0.$

$\lambda^2 - 2\lambda + 10 = 0$

$\Delta = (-2)^2 - 4 \cdot 1 \cdot 10 = 4 - 40 = -36$

$\lambda_{1,2} = \dfrac{2 \pm 6i}{2}$; $\boxed{i^2 = -1}$

$\lambda_1 = \dfrac{2+6i}{2} = \dfrac{\cancel{2}(1+3i)}{\cancel{2}} = 1 + 3i$, $m_1 = 1$

$\lambda_2 = 1 - 3i$, $m_2 = 1$

$\boxed{\lambda_1 \in \mathbb{C} \setminus \mathbb{R},\ m_1 = 1}$

$\lambda_1 = 1 + 3i$
$\lambda_2 = \bar{\lambda}_1$ $\Rightarrow$ 2 soluții în sistemul fundamental coresp. pt λ_1 și $\lambda_2 = \bar{\lambda}_1$.

• determinăm $u \in \mathbb{C}^2$, nenul ai $Au = \lambda_1 u \Rightarrow$

$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (1+3i) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases} u_1 - 3u_2 = (1+3i)u_1 \\ 3u_1 + u_2 = (1+3i)u_2 \end{cases} \Rightarrow \begin{cases} u_1(\cancel{1} - \cancel{1} - 3i) - 3u_2 = 0 \\ 3u_1 + u_2(\cancel{1} - \cancel{1} - 3i) = 0 \end{cases}$

$\Rightarrow \begin{cases} -3i u_1 - 3u_2 = 0 & | \cdot i \Rightarrow 3u_1 - 3i u_2 = 0 \\ 3u_1 - 3i u_2 = 0 \end{cases}$ $\Big/ \Rightarrow$

$\Rightarrow \boxed{u_1 = i u_2} \Rightarrow u = \begin{pmatrix} i u_2 \\ u_2 \end{pmatrix} = u_2 \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\varphi_1(t) = \mathrm{Re}\left(e^{\lambda_1 t} \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$; $\varphi_2(t) = \mathrm{Im}\left(e^{\lambda_1 t} \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$

$e^{\lambda_1 t} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^{t + i \cdot 3t} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) =$

$$= e^{t}\cdot\underbrace{(\cos 3t + i \sin 3t)}\left(\begin{pmatrix}0\\1\end{pmatrix} + i\begin{pmatrix}1\\0\end{pmatrix}\right) \Rightarrow$$

$$\boxed{e^{i\alpha} = \cos\alpha + i\sin\alpha}$$

$$\Rightarrow \varphi_1(t) = e^{t}\left(\cos 3t\begin{pmatrix}0\\1\end{pmatrix} - \sin 3t\begin{pmatrix}1\\0\end{pmatrix}\right) \Rightarrow$$

$$\Rightarrow \boxed{\varphi_1(t) = \begin{pmatrix}-e^{t}\sin 3t\\ e^{t}\cos 3t\end{pmatrix}}$$

$$\varphi_2(t) = e^{t}\left(\cos 3t\cdot\begin{pmatrix}1\\0\end{pmatrix} + \sin 3t\begin{pmatrix}0\\1\end{pmatrix}\right) \Rightarrow \boxed{\varphi_2(t) = \begin{pmatrix}e^{t}\cos 3t\\ e^{t}\sin 3t\end{pmatrix}}$$

Deci:

$$\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = c_1\begin{pmatrix}-e^{t}\sin 3t\\ e^{t}\cos 3t\end{pmatrix} + c_2\begin{pmatrix}e^{t}\cos 3t\\ e^{t}\sin 3t\end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1(t) = -c_1e^{t}\sin 3t + c_2e^{t}\cos 3t\\ x_2(t) = c_1e^{t}\cos 3t + c_2e^{t}\sin 3t \end{cases},\ c_1, c_2 \in \mathbb{R}.$$

<u>Tema</u>: 2, 4, 5, 6, 7.