Manolache Andrei
344

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Tema 2

1. $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1,x_2) = x_1^3 x_2^2 (a - x_1 - x_2) = a x_1^3 x_2^2 - x_1^4 x_2^2 - x_1^3 x_2^3$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 3a x_1^2 x_2^2 - 4x_1^3 x_2^2 - 3x_1^2 x_2^3 \\ 2a x_1^3 x_2 - 2x_1^4 x_2 - 3x_1^3 x_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1^2 x_2^2 (3a - 4x_1 - 3x_2) \\ x_1^3 x_2 (2a - 2x_1 - 3x_2) \end{bmatrix}$$

Calculăm punctele staționare:

$$\begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1^2 x_2^2 (3a - 4x_1 - 3x_2) = 0 \\ x_1^3 x_2 (2a - 2x_1 - 3x_2) = 0 \end{cases}$$

I. $x_1 = 0$, $x_2 \in \mathbb{R}$

II. $x_2 = 0$, $x_1 \in \mathbb{R}$

III. $\begin{cases} 3a - 4x_1 - 3x_2 = 0 \\ 2a - 2x_1 - 3x_2 = 0 \end{cases}$

Alte posibilități precum $\begin{cases} x_1 = 0 \\ 2a - 2x_1 - 3x_2 = 0 \end{cases}$ sunt, de fapt, doar cazuri particulare pt. I sau II

$$\begin{cases} 3a-4x_1-3x_2=0 \\ 2a-2x_1-3x_2=0 \end{cases} \iff \begin{cases} x_2 = \dfrac{3a-4x_1}{3} \\ x_2 = \dfrac{2a-2x_1}{3} \end{cases}$$

$$3a-4x_1 = 2a-2x_1 \iff a = 2x_1 \iff \boxed{x_1 = \frac{a}{2}}$$

și $\boxed{x_2 = \frac{a}{3}}$

Deci, punctele critice : $\{ (0,z), (z,0), (\frac{a}{2}, \frac{a}{3}) \mid z \in \mathbb{R} \}$
sunt puncte critice.

$$\nabla^2 f = \begin{bmatrix} \dfrac{\partial f}{\partial x_1 \partial x_1} & \dfrac{\partial f}{\partial x_1 \partial x_2} \\[2ex] \dfrac{\partial f}{\partial x_1 x_2} & \dfrac{\partial f}{\partial x_2 \partial x_2} \end{bmatrix}$$

$$\nabla^2 f = \begin{bmatrix} 6x_1x_2^2(a-2x_1-x_2) & x_1^2x_2(6a-8x_1-9x_2) \\[2ex] x_1^2x_2(6a-8x_1-9x_2) & 2x_1^3(a-x_1-3x_2) \end{bmatrix}$$

I. Fie $(x_1,x_2) \in \{ (0,z) \mid z \in \mathbb{R} \}$

$\Rightarrow \nabla^2 f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Criteriul lui Sylvester nu decide

II. Fie $(x_1,x_2) \in \{ (z,0) \mid z \in \mathbb{R} \}$

$$\Rightarrow \nabla^2 f = \begin{bmatrix} 0 & 0 \\ 0 & 2z^3(a-z) \end{bmatrix}$$

Notăm $h(z) = 2z^3(a-z)$. Căutăm pt ce valori, f ~~este~~ pozitiv semidefinită sau ~~semipoz~~ ~~semnegativ~~ negativ semidefinită.

$h(x) = 2x^3(a-x)$.

• Pt $a>0$

x	$-\infty$	0	a	∞
$2x^3$	$- - - -$	0	$+ + + +$	
$a-x$	$+ + + + + +$		0	$- -$
$h(x)$	$- - - - -$	0	$+$ 0	$- - - - -$

concavă

⇑

Deci, pt $x \in (-\infty, 0) \cup (a, \infty)$, f negativ semidefinită, deci $(x_1, x_2) \in \{ (z, 0) \mid a>0, z \in (-\infty, 0) \cup (a, \infty) \}$ punct de maxim.

$\Rightarrow$ convexă

Pt $x \in (0, a)$, f. pozitiv semidefinită, deci $(x_1, x_2) \in \{ (z, 0) \mid a>0, z \in (0, a) \}$ punct de minim.

• Pt $a=0 \Rightarrow h(x) = -2x^4 \leq 0 \ \forall x \in \mathbb{R} \Rightarrow$ toate punctele sunt de maxim.

$(x_1, x_2) \in \{ (z, 0) \mid a=0 \}$ punct de maxim

• Pt $a<0$

x	$-\infty$	a	0	∞
$2x^3$	$- - -$		0	$+ + + +$
$a-x$	$+ +$	0	$-$	$- -$
$h(x)$	$- -$	0	$+$ 0	$- - -$

minim $(x_1, x_2) \in \{ (z, 0) \mid a<0, z \in (a, 0) \}$ punct de
~~maxim~~ și $(x_1, x_2) \in \{ (z, 0) \mid a<0, z \in (-\infty, a) \cup (0, \infty) \}$ punct de
~~minim~~; maxim

III. $(x_1, x_2) = \left(\frac{a}{2}, \frac{a}{3}\right)$ ($a \neq 0$, pt $a = 0$ obținem cazul I)

$$\nabla^2 f = \begin{bmatrix} 6\cdot\frac{a}{2}\cdot\left(\frac{a}{3}\right)^2\left(a - 2\cdot\frac{a}{2} - \frac{a}{3}\right) & \left(\frac{a}{2}\right)^2\frac{a}{3}\left(6\cdot a - 8\frac{a}{2} - \dots\right. \\ \left(\frac{a}{2}\right)^2\frac{a}{3}\left(6\cdot a - 8\frac{a}{2} - 9\frac{a}{3}\right) & 2\left(\frac{a}{2}\right)^3\left(a - \frac{a}{2} - 3\cdot\frac{a}{3}\right) \end{bmatrix}$$

$$H = \nabla^2 f = \begin{bmatrix} -\frac{a^4}{9} & -\frac{a^4}{12} \\ -\frac{a^4}{12} & -\frac{a^4}{8} \end{bmatrix}$$

$$\left|-\frac{a^4}{9}\right| < 0$$

$$\begin{vmatrix} -\frac{a^4}{9} & -\frac{a^4}{12} \\ -\frac{a^4}{12} & -\frac{a^4}{8} \end{vmatrix} = \frac{a^8}{72} - \frac{a^8}{144} > 0$$

H negativ semidefinită $\Rightarrow$ f concavă $\Rightarrow$ f are punct de maxim pt $(x_1, x_2) = \left(\frac{a}{2}, \frac{a}{3}\right)$.

c) Alegem $x_0 = (1, 1)$

$$x_1 = x_0 - \alpha \nabla f(x_0)$$

$$\nabla f(x_0) = \begin{bmatrix} -3-4-3 \\ -2-2-3 \end{bmatrix} = \begin{bmatrix} -10 \\ -7 \end{bmatrix}$$

$$\Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\cdot\begin{bmatrix} -10 \\ -7 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$\nabla f(x_1) = \begin{bmatrix} 11^2 8^2 (3\cdot(-1) - 4\cdot 11 - 3\cdot 8) \\ 11^3\cdot 8 (2\cdot(-1) - 2\cdot 11 - 3\cdot 8) \end{bmatrix} = \begin{bmatrix} -88^2\cdot 71 \\ -384\cdot 11^3 \end{bmatrix}$$

$$\cancel{x_2 = x_1 - \alpha \cdot \nabla f(x_1)}$$

$$\cancel{= \begin{bmatrix} 11 \\ 8 \end{bmatrix} - \begin{bmatrix} -88^2 \cdot 71 \\ -384 \cdot 11^3 \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} \\ 384 \cdot 11^3 + 8 \end{bmatrix}}$$

c) Alegem $x_0 = (1, -1)$

$$x_1 = x_0 - \alpha \cdot \nabla f(x_0)$$

$$\nabla f(x_0) = \begin{bmatrix} 3 \cdot (-1) - 4 + 3 \\ (-1)(-2 - 2 + 3) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\Rightarrow x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$x_2 = x_1 - \alpha \cdot \nabla f(x_1)$$

$$\nabla f(x_1) = \begin{bmatrix} 25 \cdot 4(-3 - 20 + 6) \\ -250(-2 - 10 + 6) \end{bmatrix} = \begin{bmatrix} -17 \cdot 100 \\ 1500 \end{bmatrix}$$

$$\Rightarrow x_2 = \begin{bmatrix} 5 \\ -2 \end{bmatrix} - \begin{bmatrix} -1700 \\ 1500 \end{bmatrix} = \begin{bmatrix} 1705 \\ -1502 \end{bmatrix}$$