Grupa 344, Seminar 6, EDDP, 09.11.2020

① Fie problema Cauchy $\begin{cases} x' = 2tx \\ x(0) = 1 \end{cases}$ (1) , $(t,x) \in \mathbb{R}^2$

a) Arătați că sunt verificate ipotezele TEU.
b) Determinați soluția problemei (1).
c) Determinați șirul de aproximații succesive $(\varphi_n)_{n \geq 0}$.

② Fie problema Cauchy $\begin{cases} x' = 2t \sin x, \\ x(0) = \frac{\pi}{6} \end{cases}$ $(t,x) \in [-1,1] \times [0, \frac{\pi}{2}]$ (2)

a) Ipotezele TEU pt (2)
b) Soluția prob (2)
c) Determinați 3 aproximații succesive $(\varphi_0, \varphi_1, \varphi_2)$.

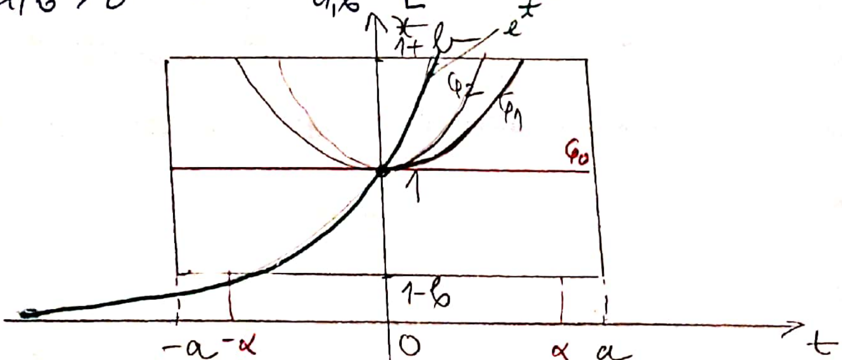
③ Fie problema Cauchy $\begin{cases} x' = \sqrt[3]{x^2} \\ x(t_0) = x_0. \end{cases}$ , $(t,x) \in \mathbb{R}^2$ (3)

a) Determinați mulț. sol. ec $x' = \sqrt[3]{x^2}$.
b) În ce condiții sunt verificate ipotezele TEU pt. problema (3)?
c) Câte soluții are problema (3) pt $x_0 = 0$?

① $\begin{cases} x' = 2tx \\ x(0) = 1 \end{cases}$ $\qquad \begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$

$f : D = \mathbb{R}^2 \to \mathbb{R}$
$f(t,x) = 2tx$
$t_0 = 0$
$x_0 = 1$

a) 1) $\exists a, b > 0$ a.î. $D_{a,b} = [0-a, 0+a] \times [1-b, 1+b] \subset D = \mathbb{R}^2$




2) f continuă în variabile $(t,x)$ ca produs de funcții continue.

$$M = \sup_{(t,x) \in D_{a,b}} |f(t,x)| = 2 \sup_{\substack{t \in [-a,a] \\ x \in [1-b, 1+b]}} |t| \cdot |x| = 2a(1+b)$$

3) $\frac{\partial f}{\partial x}(t,x) = \frac{\partial}{\partial x}(2tx) = 2t$ este continuă ca funcție elementară

$L = \sup_{\substack{t\in[-a,a]\\ x\in[1-b,1+b]}} |2t| = 2a.$ Deci, sunt verificate ipotezele TEU $\Rightarrow$ $\Rightarrow$ prob. Cauchy (1) are soluție unică: $\forall \alpha \in (0, \min(a, \frac{b}{2a(1+b)}))$ $\exists\, \varphi : [-\alpha, \alpha] \to [1-b, 1+b]$ sol a prob. Cauchy (1)

b) Soluția problemei $\begin{cases} \frac{dx}{dt} = 2t\,x \\ x(0) = 1 \end{cases}$

Se determină mulțimea soluțiilor ec. $\frac{dx}{dt} = 2tx$ și apoi soluția care verifică $x(0) = 1$.

Este ec. liniară omogenă: $\frac{dx}{dt} = a(t)\cdot x \Rightarrow x(t) = Ce^{A(t)}$

unde $A(t)$ este primitiva a lui $a(t) = 2t \Rightarrow$

$\Rightarrow \int 2t\,dt = t^2 + C \Rightarrow A(t) = t^2 \Rightarrow \boxed{x(t) = Ce^{t^2}, C\in\mathbb{R}}$

determinăm $C$ a.î. $x(0) = 1 \Rightarrow 1 = Ce^0 \Rightarrow C = 1 \Rightarrow$

$\Rightarrow \boxed{x(t) = \varphi(t) = e^{t^2}}$

c) $\varphi_0(t) = 1$

$\varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\,ds = 1 + \int_0^t 2s\,\varphi_n(s)\,ds$, $\forall n$

$\varphi_1(t) = 1 + \int_0^t 2s\cdot 1\cdot ds = 1 + s^2\Big|_0^t = 1 + t^2.$

$\varphi_2(t) = 1 + \int_0^t 2s(1+s^2)\,ds = 1 + \int_0^t 2s\,ds + \int_0^t 2s^3\,ds =$

$= 1 + 2\frac{s^2}{2}\Big|_0^t + 2\frac{s^4}{4}\Big|_0^t = 1 + 2\left(\frac{s^2}{2} + \frac{s^4}{2\cdot 2}\right)$

$\varphi_3(t) = 1 + 2\int_0^t s\cdot\left(1 + 2\frac{s^2}{2} + 2\frac{s^4}{2\cdot 2}\right)ds =$

$= 1 + 2\frac{s^2}{2}\Big|_0^t + 2^2\cdot\frac{s^4}{2\cdot 4}\Big|_0^t + 2^2\cdot\frac{s^6}{2\cdot 2\cdot 2\cdot 3}\Big|_0^t =$

$= 1 + \cancel{2}\frac{t^2}{\cancel{2}} + \cancel{2^2}\frac{t^4}{\cancel{2^2}\cdot 2!} + \cancel{2^2}\frac{t^6}{\cancel{2^2}\cdot 3!} = 1 + \frac{t^2}{1!} + \frac{t^4}{2!} + \frac{t^6}{3!}$

Dem prin inducție: $\varphi_n(t) = 1 + \frac{t^2}{1!} + \frac{t^4}{2!} + \dots + \frac{t^{2n}}{n!}$.

Presup. adev pt $n$ și dem pt $n+1$; adică, dem că:

$$\varphi_{n+1}(t) = 1+\frac{t^2}{1!}+\frac{t^4}{2!}+\dots+\frac{t^{2(n+1)}}{(n+1)!}$$

Din rel. de recurență:

$$\varphi_{n+1}(t) = 1+\int_0^t 2s\,\varphi_n(s)\,ds =$$

$$= 1+\int_0^t 2s\left(1+\frac{s^2}{1!}+\frac{s^4}{2!}+\dots+\frac{s^{2n}}{n!}\right)ds =$$

$$= 1+\int_0^t \left(2s+\frac{2s^3}{1!}+\frac{2s^5}{2!}+\dots+\frac{2s^{2n+1}}{n!}\right)ds =$$

$$= 1+\cancel{2}\frac{s^2}{\cancel{2}}\Big|_0^t+\frac{\cancel{2}s^4}{1!\cdot\cancel{4}_2}\Big|_0^t+\frac{\cancel{2}s^6}{2!\cancel{6}_3}\Big|_0^t+\dots+\frac{\cancel{2}s^{2n+2}}{n!(\cancel{2}(n+1))}\Big|_0^t =$$

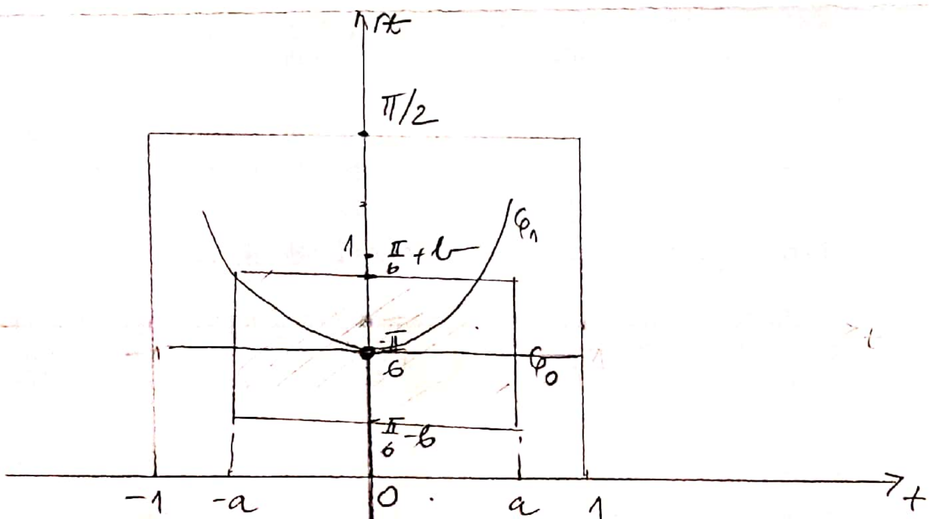
$$= 1+\frac{t^2}{1!}+\frac{t^4}{2!}+\frac{t^6}{3!}+\dots+\frac{t^{2(n+1)}}{(n+1)!}$$

Deci: $\varphi_n(t) = 1+\frac{t^2}{1!}+\frac{t^4}{2!}+\dots+\frac{t^{n}}{n!}$, $\forall n\in\mathbb{N}^*$.

(2) $\begin{cases} x' = 2t\sin x \\ x(0) = \frac{\pi}{6} \end{cases}$, $(t,x)\in[-1,1]\times[0,\frac{\pi}{2}]$.

a) temă!

b) soluția temă!
ec. cu variabile separabile.




c) $D = [-1,1]\times[0,\frac{\pi}{2}]$

$f: D\to\mathbb{R}$

$f(t,x) = 2t\sin x$

$t_0 = 0$, $x_0 = \frac{\pi}{6}$.

$\boxed{\varphi_0(t) = \frac{\pi}{6}}$

$$\varphi_{n+1}(t) = \frac{\pi}{6}+\int_0^t f(s,\varphi_n(s))\,ds = \frac{\pi}{6}+\int_0^t 2s\sin(\varphi_n(s))\,ds,\ \forall n\in\mathbb{N}.$$

$$\varphi_1(t) = \frac{\pi}{6}+\int_0^t 2s\cdot\sin\frac{\pi}{6}\,ds = \frac{\pi}{6}+\int_0^t \cancel{2}s\cdot\frac{1}{\cancel{2}}\,ds = \frac{\pi}{6}+\frac{s^2}{2}\Big|_0^t = \frac{\pi}{6}+\frac{t^2}{2}$$

$$\boxed{\varphi_1(t) = \frac{\pi}{6} + \frac{t^2}{2}}.$$

$$\varphi_2(t) = \frac{\pi}{6} + \int_0^t 2s \cdot \sin\left(\frac{\pi}{6} + \frac{s^2}{2}\right) ds \Rightarrow$$

$$\frac{\pi}{6} + \frac{s^2}{2} = u$$

$$\frac{2s}{2} ds = du \Rightarrow \boxed{s\,ds = du}$$

| $s$ | $0$ | $t$ |
|---|---|---|
| $u$ | $\frac{\pi}{6}$ | $\frac{\pi}{6} + \frac{t^2}{2}$ |

$$\Rightarrow \varphi_2(t) = \frac{\pi}{6} + 2\int_{\frac{\pi}{6}}^{\frac{\pi}{6}+\frac{t^2}{2}} \sin u\, du = \frac{\pi}{6} + 2(-\cos u)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{6}+\frac{t^2}{2}} =$$

$$= \frac{\pi}{6} - 2\cos\left(\frac{\pi}{6} + \frac{t^2}{2}\right) + 2\cos\frac{\pi}{6} \Rightarrow$$

$$\Rightarrow \boxed{\varphi_2(t) = \frac{\pi}{6} + \sqrt{3} - 2\cos\left(\frac{\pi}{6} + \frac{t^2}{2}\right)}$$

$$\varphi_3(t) = \frac{\pi}{6} + \int_0^t 2s \sin\left(\frac{\pi}{6} + \sqrt{3} - 2\cos\left(\frac{\pi}{6} + \frac{s^2}{2}\right)\right) ds.$$

3) a) $x' = \sqrt[3]{x^2}$, $(t, x) \in \mathbb{R}^2$

$$\frac{dx}{dt} = \sqrt[3]{x^2}.$$

$$\frac{dx}{dt} = a_1(t) \cdot b_1(x) \quad ; \begin{cases} a_1(t) = 1 & ; a_1 : \mathbb{R} \to \mathbb{R} \\ b_1(x) = \sqrt[3]{x^2} & ; b_1 : \mathbb{R} \to \mathbb{R} \end{cases}$$

- $b_1(x) = 0 \Rightarrow \sqrt[3]{x^2} = 0 \Rightarrow x = 0 \Rightarrow$ $\boxed{x(t) = 0 \text{ sol. staționară}}$
- $b_1(x) \neq 0 \Rightarrow \underline{\underline{x \neq 0}} \Rightarrow$ separăm variabilele:

$$\frac{dx}{\sqrt[3]{x^2}} = dt$$

$$\int x^{-\frac{2}{3}} dx = \int dt \Rightarrow \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} = t + C.$$

$$\Rightarrow 3x^{\frac{1}{3}} = t + C \Rightarrow x^{\frac{1}{3}} = \frac{t+C}{3} \Rightarrow \boxed{x(t) = \left(\frac{t+C}{3}\right)^3, \quad C \in \mathbb{R}}$$

b, c - temă!