Tehnici de Optimizare

Facultatea de Matematica si Informatica

Universitatea Bucuresti

- Department Informatica-

2021

Proiect

- Alegere 1-2 lucrari; maxim 2 alegeri per lucrare
- Documentatie (rezultate ale lucrarii); limita pagini: 5-20
- Simulari (algoritmi din lucrare); grafice de analiza
- Echipa: maxim 2 - 3 studenti
- Pondere nota: 60%
- Termen: in sesiune
- Evaluare: online fata-in-fata

Cursul de azi

- **Problema de fezabilitate convexa**

- Probleme de optimizare cu constrangeri de egalitate
  - Functia Lagrange
  - Multiplicatori Lagrange
  - Conditii de optimalitate

Problema de fezabilitate convexa

$$Fie\ C_1\ si\ C_2\ multimi\ simple\ convexe, calculati$$
$$x \in Q\ = C_1 \cap C_2$$

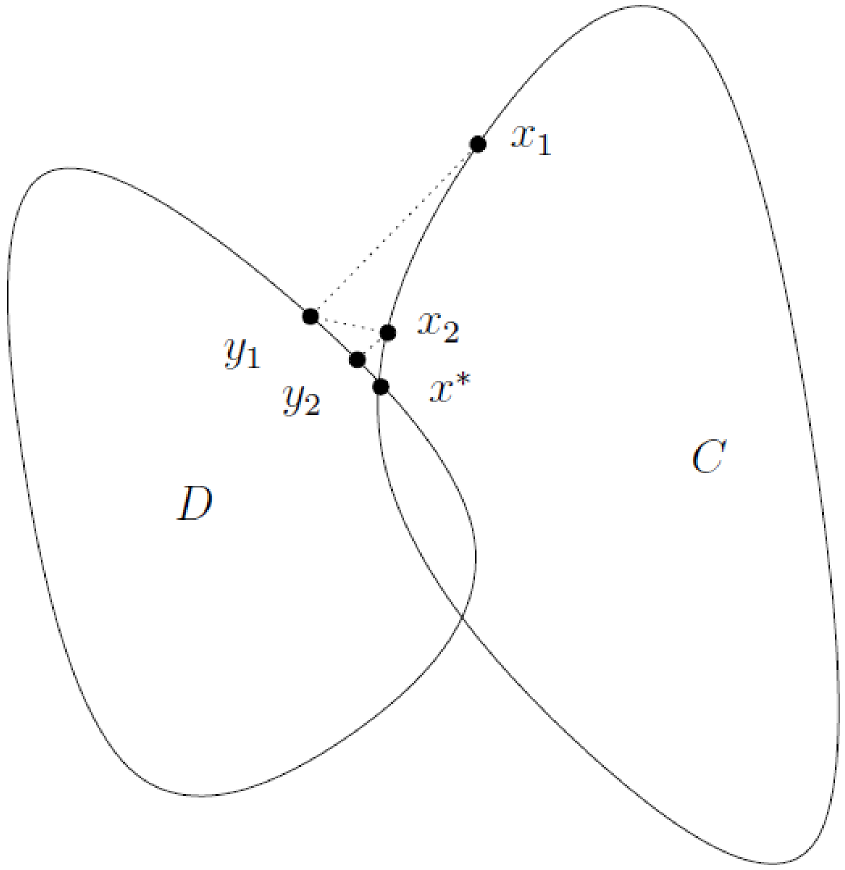
Algoritmul proiectiilor alternative:

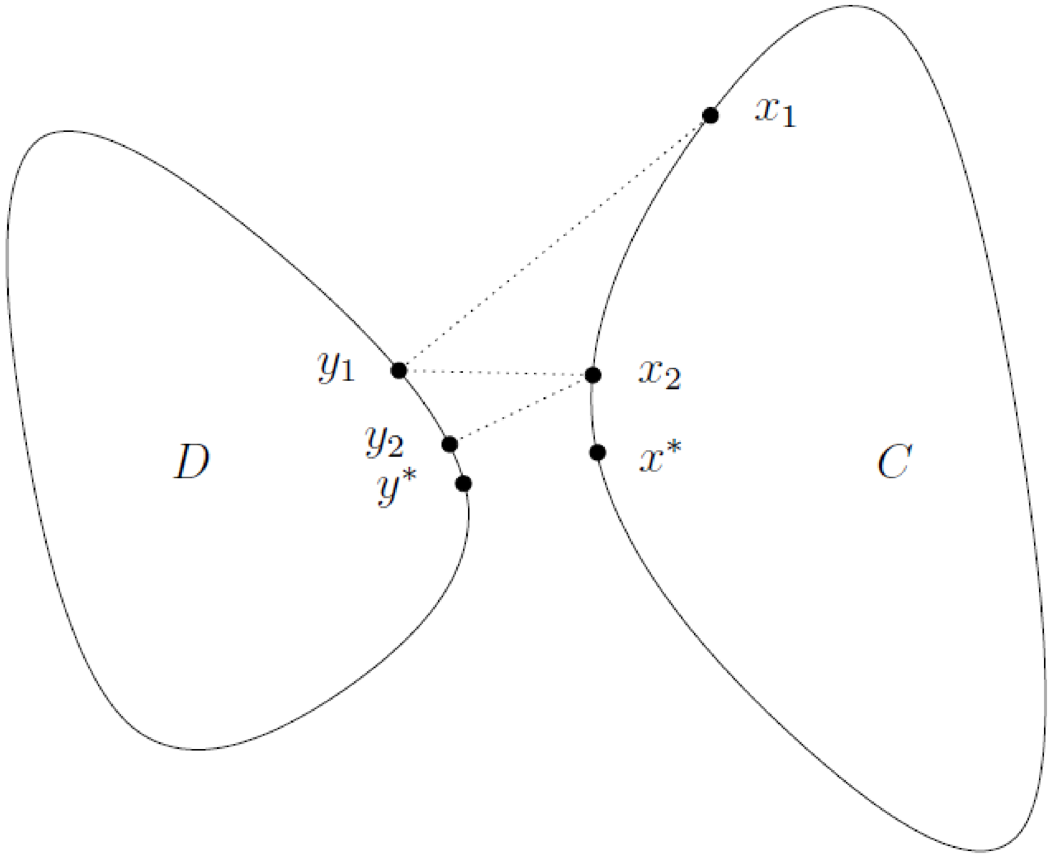
1. $x^k = \pi_{C_1}(y^k)$

2. $y^{k+1} = \pi_{C_2}(x^k)$

Daca $Q \neq \emptyset$, se arata ca $x^k \to x^* \in Q$

Problema de fezabilitate convexa

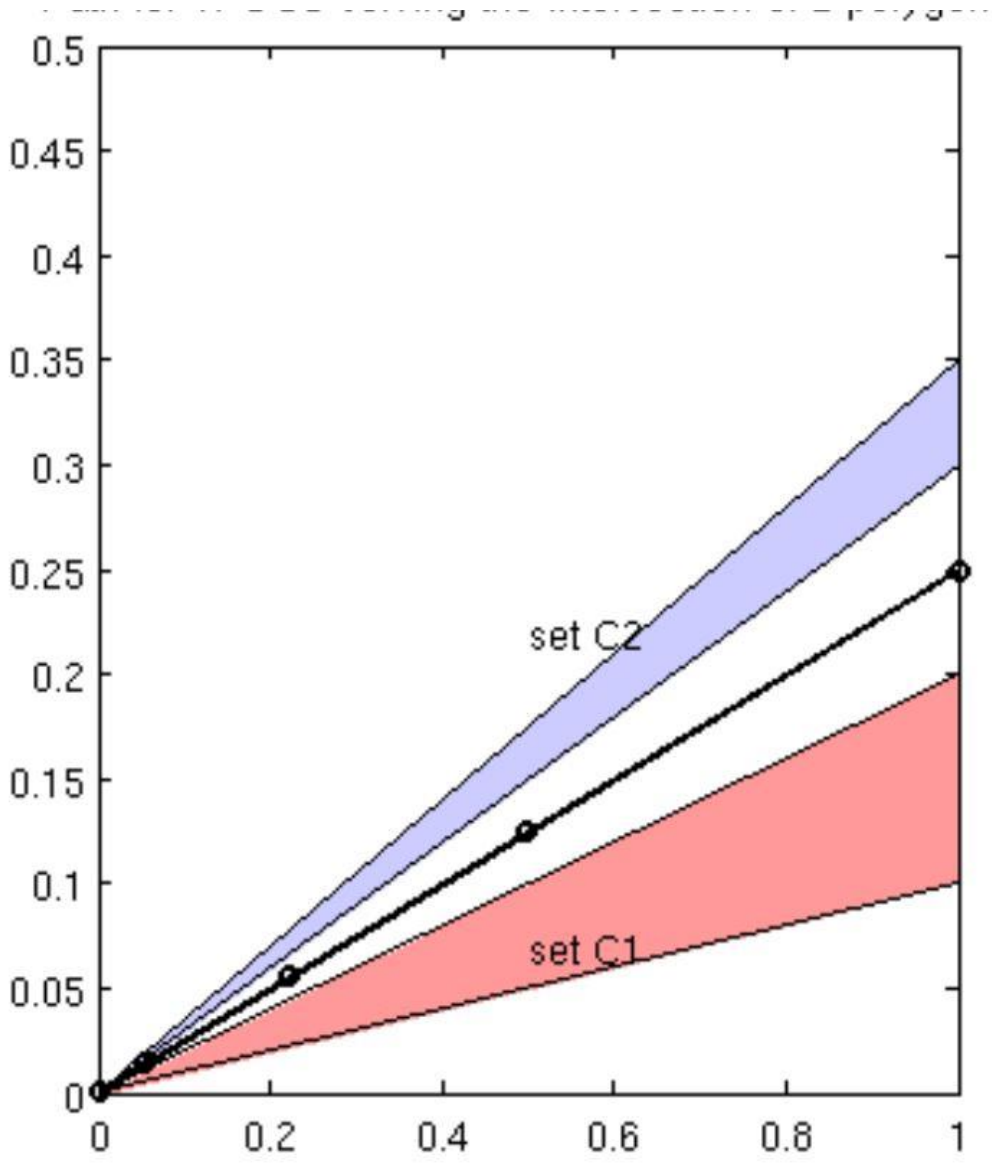




Problema de fezabilitate convexa

- Daca C_1 si C_2 sunt subspatii/poligoane liniare, atunci APA are convergenta liniara

- Convergenta este strict legata de unghiul de la intersectia celor doua: **$\theta \approx 0$ implica un nr. mare de iteratii**.



Cursul de azi

$$\min f(x) \quad s.l. \quad x \in Q$$

- Problema de fezabilitate convexa

- **Probleme de optimizare cu constrangeri de egalitate**
    - Functia Lagrange
    - Multiplicatori Lagrange
    - Conditii de optimalitate

Optimizare cu constrangeri de egalitate

$$\min_{\mathrm{x}} f(x) \quad s.l.\ \ g_i(x) = 0, \qquad i = 1, \cdots, m$$

- f, g_i functii diferentiabile, notam $g(x) = \begin{bmatrix} g_1(x) \\ \dots \\ g_m(x) \end{bmatrix}$
- Pentru $g(x) = Ax - b$ avem $Q = \{x : g(x) = 0\}$ convexa (altfel neconvexa!)
- Exemplu: $Q = \{x \in R^2 \colon x_1^2 + x_2^2 = 1\}$

Optimizare cu constrangeri de egalitate

$$\min_{\mathrm{x}} f(x) \quad s.l. \;\; g_i(x) = 0, \qquad i = 1, \cdots, m$$

- In forma de mai sus, nu se intrevad conditii de optimalitate

Functia Lagrangian: $\mathrm{L}: \mathrm{R}^{\mathrm{n}} \times R^m \to R, L(x, \lambda) = f(x) + \sum_i \lambda_i g_i(x)$

- Vom folosi functia Lagrange pentru a exprima conditii de optimalitate!

Optimizare cu constrangeri de egalitate

$$\min_{x} f(x) \quad s.l. \;\; g_i(x) = 0, \qquad i = 1, \cdots, m$$

Vecinatate $\mathrm{V}(x) = \{s: \left|\left|x - s\right|\right| \leq r\}$

Pct. de minim: $x^* \in Q,\; a.i.\; f(x^*) \leq f(x) \quad \forall x \in Q \cap V(x^*)$

Pct. de minim regulat : $x^* \in Q,\; a.i.\; f(x^*) \leq f(x) \quad \forall x \in Q \cap V(x^*)$

+

$$\nabla g_i(x^*)\ liniar\ independenti$$

$$v_1, \cdots, v_m\ l.d. daca\ exista\ \alpha \neq 0\ a.i. \sum \alpha_i v_i = 0$$

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\ l.i., \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\ l.d.$$

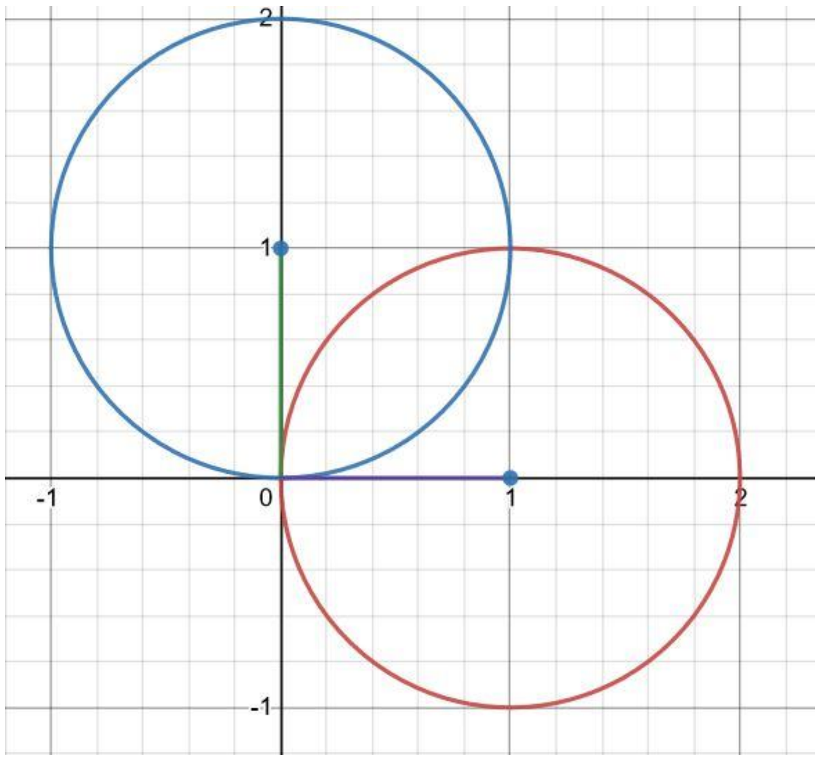
Exemplu minim regulat

$$\min x_2$$
$$\text{s.l.}(x_1 - 1)^2 + x_2^2 = 1$$
$$x_1^2 + (x_2 - 1)^2 = 1$$

$$\mathbf{\nabla g_1(0,0)} = \begin{bmatrix} \mathbf{-2} \\ \mathbf{0} \end{bmatrix}$$

$$\mathbf{\nabla g_2(0,0)} = \begin{bmatrix} \mathbf{0} \\ \mathbf{-2} \end{bmatrix}$$

Concluzie: minim regulat $\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$**!**



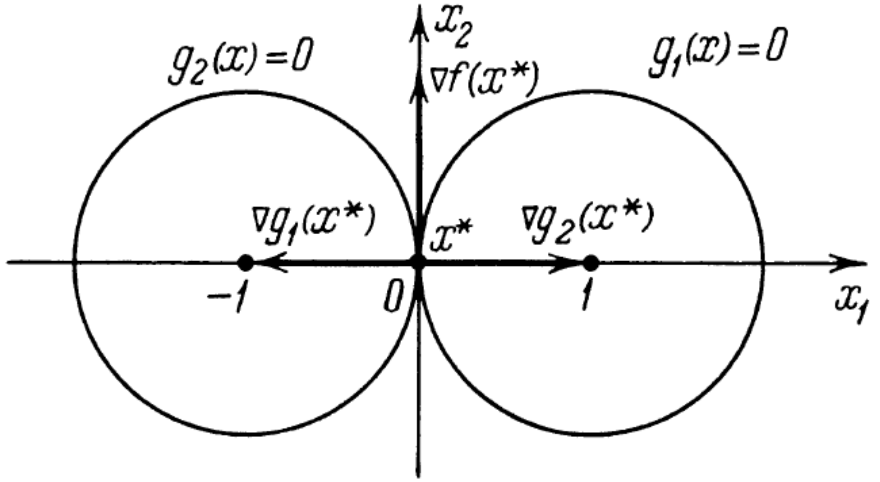
Exemplu minim neregulat

$$\min x_2$$
$$\text{s.l.}(x_1 - 1)^2 + x_2^2 = 1$$
$$(x_1 + 1)^2 + x_2^2 = 1$$

$$\nabla \mathbf{g_1}(\mathbf{0}, \mathbf{0}) = \begin{bmatrix} \mathbf{-2} \\ \mathbf{0} \end{bmatrix}$$

$$\nabla \mathbf{g_2}(\mathbf{0}, \mathbf{0}) = \begin{bmatrix} \mathbf{2} \\ \mathbf{0} \end{bmatrix}$$

Concluzie: minim neregulat $\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$**!**



Conditii necesare de ordin I

THEOREM 2 (The rule of Lagrange multipliers). If x^* is a regular minimum point, then we can find $y_1^*, \ldots, y_m^*$ such that

$$\nabla f(x^*) + \sum_{i=1}^{m} y_i^* \nabla g_i(x^*) = 0 . \qquad (2)$$

Echivalent cu $\nabla_x L(x^*, \lambda^*) = 0$ (conditii de optimalitate, caz neconstrans!)
λ^* se numesc Multiplicatori Lagrange!

Conditii necesare de ordin I - demonstratie

Pentru simplitate pp. ca x^* minim unic si global in vecinatatea V(x^*).

Formam functia penalitate: $f_\rho(x) = f(x) + \frac{\rho}{2}\left||g(x)|\right|^2$ (param. ρ de penalitate); O noua problema:

$$\min_{\mathrm{x} \in V(x^*)} f_\rho(x) = f(x) + \frac{\rho}{2}\left||g(x)|\right|^2$$

cu solutia x^ρ. Observam ca:

$$f(x^\rho) + \frac{\rho}{2}\left||g(x^{\rho})|\right|^2 = f_\rho(x^\rho) \leq f_\rho(x^*) = f(x^*) + \frac{\rho}{2}\left||g(x^{*})|\right|^2 = f(x^*)$$

$$\left||g(x^{\rho})|\right|^2 \leq \frac{2}{\rho}(f(x^*) - f(x^\rho))$$

Termenul $\frac{2}{\rho}\left(f(x^*) - f(x^\rho)\right) \rightarrow 0$ cand $\rho \rightarrow \infty$ ($x^\rho \in V(x^*)$)

Conditii necesare de ordin I - demonstratie

$$\min_{x \in V(x^*)} f_\rho \ (x) = f(x) + \frac{\rho}{2} \left|\left|g(x)\right|\right|^2$$

cu solutia x^ρ. Observam ca:

$$f(x^\rho) + \frac{\rho}{2} \left|\left|g(x^\rho)\right|\right|^2 = f_\rho(x^\rho) \le f_\rho(x^*) = f(x^*) + \frac{\rho}{2} \left|\left|g(x^*)\right|\right|^2 = f(x^*)$$

$$\left|\left|g(x^\rho)\right|\right|^2 \le \frac{2}{\rho} (f(x^*) - f(x^\rho))$$

Termenul $\frac{2}{\rho}\left(f(x^*) - f(x^\rho)\right) \to 0$ cand $\rho \to \infty$ ($x^\rho \in V(x^*)$)

Deci $g(x^\rho) \to 0$ cand $\rho \to \infty$. Orice pct. limita $x^{\rho_k} \to \bar{x}$ are $g(\bar{x}) = 0$.

Din inegalitatea de mai sus: $f(\bar{x}) \le f(x^*)$, dar si $f(\bar{x}) \ge f(x^*)$, deci $\bar{x} = x^*$

Conditii necesare de ordin I - demonstratie

$$\min_{\mathrm{x}\in V(x^*)} f_\rho(x) = f(x) + \frac{\rho}{2}\left|\left|g(x)\right|\right|^2$$

cu solutia x^ρ.

Cand $\rho \to \infty$, $x^\rho = x^*$, deci pt ρ suf. de mare avem x^ρ in interiorul lui $V(x^*)$

Pe de alta parte x^ρ satisfice $\nabla \mathrm{f}_\rho(x^\rho) = 0$, echivalent

$$\nabla f(x^\rho) + \rho \sum_i g_i(x^\rho)\nabla \mathrm{g_i}(\mathrm{x}^\rho) = 0$$

Impartim prin $\sqrt{1+\rho^2\sum_i g_i^2(x^\rho)}$(suma tuturor ponderilor) si obtinem:

$$\lambda_0^\rho \nabla f(x^\rho) + \sum_i \lambda_i^\rho \nabla \mathrm{g_i}(\mathrm{x}^\rho) = 0\,;\ \ \lambda_0^\rho = \frac{1}{\sqrt{1+\rho^2\sum_i g_i^2(x^\rho)}}, \lambda_i^\rho = \frac{\rho g_i(x^\rho)}{\sqrt{1+\rho^2\sum_i g_i^2(x^\rho)}}$$

Conditii necesare de ordin I - demonstratie

Impartim prin $1 + \rho \sum_i g_i(x^\rho)$ (suma tuturor ponderilor) si obtinem:

$$\lambda_0^\rho \nabla f(x^\rho) + \sum_i \lambda_i^\rho \nabla \mathrm{g_i}(\mathrm{x}^\rho) = 0$$

Pentru ca $\sum_i (\lambda_i^\rho)^2 \quad = 1$, deci λ^ρ marginit, are un pct limita λ^*, care confirma:

$$\lambda_0^* \nabla f(x^*) + \sum_i \lambda_i^* \nabla \mathrm{g_i}(\mathrm{x}^*) = 0$$

$$\nabla f(x^*) + \sum_i \frac{\lambda_i^*}{\lambda_0^*} \nabla \mathrm{g_i}(\mathrm{x}^*) = 0$$

Metoda penalitate

$$x^k = \arg\min f_{\rho_k}(x) = f(x) + \frac{\rho_k}{2} ||g(x)||^2$$

$$\rho_{k+1} = 2\rho_k$$

C.N. ordin I sugereaza: $x^k \rightarrow x^* \ (punct\ stationar)$