Gr. 344, Seminar 2, EDDP, 12.10.2020

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1 din temă

$$\int \frac{1}{\cos x}\,dx = \int \frac{(1+tg^2\frac{x}{2})}{1-tg^2\frac{x}{2}}\,dx = I$$

$$\cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

$$tg\frac{x}{2} = y \Rightarrow \left(tg\frac{x}{2}\right)' dx = dy \Rightarrow$$

$$\Rightarrow \left(1+tg^2\frac{x}{2}\right)\cdot\left(\frac{x}{2}\right)' dx = dy \Rightarrow$$

$$\Rightarrow \frac{1}{2}\left(1+tg^2\frac{x}{2}\right)dx = dy \Rightarrow$$

$$\Rightarrow \left(1+tg^2\frac{x}{2}\right)dx = 2\,dy$$

$$\int \frac{2dy}{1-y^2} = 2(-1)\int\frac{dy}{y^2-1^2} = -\cancel{2}\cdot\frac{1}{\cancel{2}\cdot 1}\ln\left|\frac{y-1}{y+1}\right| + C$$

$$\Rightarrow I = -\ln\left|\frac{tg\frac{x}{2}-1}{tg\frac{x}{2}+1}\right| + C$$

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$$(tg\,x)' = \frac{1}{\cos^2 x} = 1+tg^2 x$$

$$(ctg\,x)' = \frac{1}{\sin^2 x} = 1+ctg^2 x$$

1. Să se determine mulțimea soluțiilor următoarelor ecuații diferențiale:

✓ 1) $\frac{dx}{dt} = \frac{x-x^3}{t(1+x^2)}$, $t\in(0,\infty)$, $x\in\mathbb{R}$.

2) $\frac{dx}{dt} = \frac{t\sqrt{x^2+1}}{x\sqrt{1-t^2}}$, $t\in(-1,1)$, $x\in\mathbb{R}$.

3) $\frac{dx}{dt} = \frac{1}{(tg^2x+1)(t^2-1)}$, $t\in(1,+\infty)$, $x\in\mathbb{R}$.

4) $\frac{dx}{dt} = \frac{\cos x\cdot\sin t}{(\sin^2x-4)(\cos^2t+9)}$; $x\in\mathbb{R}$, $t\in\mathbb{R}$.

5) $\frac{dx}{dt} = \frac{x(\ln x)\cdot e^t}{\ln(\ln x)\cdot(e^{2t}+1)}$, $t \in \mathbb{R}$, $x \in (3, +\infty)$

✓ 6) $\frac{dx}{dt} = \frac{(x^2-2x-3)\,t}{\sqrt{t^2+1}}$, $t \in \mathbb{R}$, $x \in \mathbb{R}$

7) $\frac{dx}{dt} = \frac{(x^2-8)(t+1)}{\sqrt{t^2-1}}$, $t \in (1, +\infty)$, $x \in \mathbb{R}$.

8) $\frac{dx}{dt} = \frac{x-4}{t^2+3t-4}$, $t \in (1, +\infty)$, $x \in \mathbb{R}$.

9) $\frac{dx}{dt} = \frac{e^{e^t+t+2}\cdot(x+2)}{x^2+4x+5}$, $x \in \mathbb{R}$, $t \in \mathbb{R}$

✓ 10) $\frac{dx}{dt} = \frac{(t+1)\sin x}{(t^2+4t+3)}$, $x \in \left(0, \frac{\pi}{2}\right)$, $t \in (-1, +\infty)$

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(6) $\frac{dx}{dt} = \frac{(x^2-2x-3)t}{\sqrt{t^2+1}}$ ; $t \in \mathbb{R}$, $x \in \mathbb{R}$.

ec. cu variabile separabile $\frac{dx}{dt} = a(t)\,b(x)$

$a : I \to \mathbb{R}$
$b : J \to \mathbb{R}$.

$a(t) = \frac{t}{\sqrt{t^2+1}}$ ; $a : \mathbb{R} \to \mathbb{R}$

$b(x) = x^2-2x-3$ ; $b : \mathbb{R} \to \mathbb{R}$.

• rez ec. $b(x) = 0 \Rightarrow x^2-2x-3=0$

$\Delta = 4 - 4\cdot 1(-3) = 4+12 = 16 = 4^2$

$x_{1,2} = \frac{2 \pm 4}{2}$ ⟨ $x_1 = 3 \in \mathbb{R}$, $x_2 = -1 \in \mathbb{R}$ $\Rightarrow$

$\Rightarrow$ $\left.\begin{array}{l} \varphi_1 : \mathbb{R} \to \mathbb{R} \\ \varphi_1(t) = 3, \ t \in \mathbb{R} \\ \varphi_2 : \mathbb{R} \to \mathbb{R} \\ \varphi_2(t) = -1 \end{array}\right\}$ soluții staționare (1)

• pt $b(x) \neq 0$, $x \in \mathbb{R} \setminus \{-1, 3\}$ se separă variabilele.

$$\frac{dx}{x^2-2x-3} = \frac{t\,dt}{\sqrt{t^2+1}}$$

- $$\int \frac{dx}{x^2-2x-3} = \int \frac{dx}{(x-1)^2-4} = I$$

$$x-1=t$$
$$dx=dt$$

$$\int \frac{dt}{t^2-4} = \frac{1}{2\cdot 2}\ln\left|\frac{t-2}{t+2}\right| + C \Rightarrow$$

$$\Rightarrow I = \frac{1}{4}\ln\left|\frac{x-3}{x+1}\right| + C \Rightarrow \boxed{B(x)=\frac{1}{4}\ln\left|\frac{x-3}{x+1}\right|}$$

- $$\int \frac{t\,dt}{\sqrt{t^2+1}} = \sqrt{t^2+1} + C \Rightarrow \boxed{A(t)=\sqrt{t^2+1}}$$

- O mulțime de soluții implicite: $B(x)=A(t)+C \Rightarrow$

$$\Rightarrow \frac{1}{4}\ln\left|\frac{x-3}{x+1}\right| = \sqrt{t^2+1} + C, \quad C\in\mathbb{R}. \quad (2)$$

- Mulțimea soluțiilor ec. date este $(1)\cup(2)$.
- În acest caz se poate explicita:

$$\ln\left|\frac{x-3}{x+1}\right| = 4\sqrt{t^2+1} + 4C$$

$$\left|\frac{x-3}{x+1}\right| = e^{4\sqrt{t^2+1}+4C}$$

$$\left|\frac{x-3}{x+1}\right| = e^{4\sqrt{t^2+1}}\cdot \underbrace{e^{4C}}_{>0} \Rightarrow \frac{x-3}{x+1} = \underbrace{\pm e^{4C}}_{C_1\in\mathbb{R}^*}\cdot e^{4\sqrt{t^2+1}} \Rightarrow$$

$$\Rightarrow x-3 = xC_1 e^{4\sqrt{t^2+1}} + 1\cdot C_1 e^{4\sqrt{t^2+1}} \Rightarrow$$

$$\Rightarrow x\left(1-C_1 e^{4\sqrt{t^2+1}}\right) = 3 + C_1 e^{4\sqrt{t^2+1}} \Rightarrow$$

$$\Rightarrow x(t) = \frac{3+C_1 e^{4\sqrt{t^2+1}}}{1-C_1 e^{4\sqrt{t^2+1}}}, \quad C_1\in\mathbb{R}^+ \quad (3).$$

- Avem altă exprimare a mulțimii soluțiilor prin $(1)\cup(3)$.

✓ (10) $\dfrac{dx}{dt} = \dfrac{(t+1)\sin x}{t^2+4t+3}$ , $x \in \left(0, \frac{\pi}{2}\right)$, $t \in (-1, +\infty)$

$a : (-1, +\infty) \to \mathbb{R}$

$a(t) = \dfrac{t+1}{t^2+4t+3}$

$b : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$

$b(x) = \sin x$.

• $b(x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = k\pi,\ k \in \mathbb{Z}$. dar $k\pi \notin \left(0, \frac{\pi}{2}\right), \forall k \in \mathbb{Z}$ $\Big\} \Rightarrow$

$\Rightarrow$ ec. nu are soluții staționare $\Rightarrow$ separăm variabilele: $\dfrac{dx}{\sin x} = \dfrac{(t+1)\,dt}{t^2+4t+3}$.

• $\displaystyle\int \frac{dx}{\sin x} = \int \frac{(1+tg^2\frac{x}{2})}{2\,tg\frac{x}{2}}\,dx = I$

$\sin x = \dfrac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}}$

$tg\frac{x}{2} = y \Rightarrow \left(1+tg^2\frac{x}{2}\right)\left(\frac{x}{2}\right)' dx = dy \Rightarrow$

$\Rightarrow \frac{1}{2}\left(1+tg^2\frac{x}{2}\right)dx = dy$

$\displaystyle\int \frac{dy}{y} = \ln|y| + C \Rightarrow I = \ln\left|tg\frac{x}{2}\right| + C$

dar $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \frac{x}{2} \in \left(0, \frac{\pi}{4}\right) \Rightarrow$
$\Rightarrow tg\frac{x}{2} > 0$ $\Big\} \Rightarrow$

$\Rightarrow B(x) = \ln\left(tg\frac{x}{2}\right)$

• $\displaystyle\int \frac{(t+1)\,dt}{t^2+4t+3} = \int \frac{t+1}{(t^2+4t+4)-4+3}\,dt = \int \frac{t+1}{(t+2)^2-1}\,dt = I$

$t+2 = u \Rightarrow dt = du$; $t = u-2$

$$\int \frac{y-2+1}{y^2-1}\,dy = \int \frac{y}{y^2-1}\,dy - \int \frac{1}{y^2-1}\,dy =$$

$$= \frac{1}{2}\ln|y^2-1| - \frac{1}{2\cdot 1}\ln\left|\frac{y-1}{y+1}\right| + C \Rightarrow$$

$$\Rightarrow J = \frac{1}{2}\ln|(t+2)^2-1| - \frac{1}{2}\ln\left|\frac{t+1}{t+3}\right| + C \Rightarrow$$

$$A(t) = \frac{1}{2}\ln|t^2+4t+3| - \frac{1}{2}\ln\left|\frac{t+1}{t+3}\right| =$$

$$= \frac{1}{2}\ln\left|\frac{t^2+4t+3}{\frac{t+1}{t+3}}\right| = \frac{1}{2}\ln\left(\cancel{(t+1)}(t+3)\left(\frac{t+3}{\cancel{t+1}}\right)\right)$$

$$= \frac{1}{2}\ln(t+3)^2 = \frac{\cancel{1}}{\cancel{2}}\cdot\cancel{2}\ln|t+3| =$$

$$= \ln|t+3|$$

dar $t \in (-1,+\infty)$ $\Big\} \Rightarrow t+3>0$ $\quad A(t) = \ln(t+3)$

<u>OBS</u>: Funcția putea fi simplificată de la început:

$$\frac{t+1}{t^2+4t+3} = \frac{\cancel{t+1}}{\cancel{(t+1)}(t+3)} =$$

$$= \frac{1}{t+3} \Rightarrow$$

$$\Rightarrow \int \frac{1}{t+3}\,dt = \underbrace{\ln|t+3|}_{A(t)} + C$$

Mulțimea soluțiilor implicite este

$$B(x) = A(t) + C$$

$$\boxed{\ln\left(tg\frac{x}{2}\right) = \ln(t+3) + \ln C, \quad C>0}$$

$$\ln\left(tg\frac{x}{2}\right) = \ln(C(t+3)) \Rightarrow tg\left(\frac{x}{2}\right) = C(t+3) \Rightarrow$$

$$\Rightarrow \frac{x}{2} = arctg(C(t+3)) \Rightarrow x = 2\,arctg(C(t+3))$$
$$C \in (0,\infty)$$

$\frac{x}{2} \in \left(0,\frac{\pi}{4}\right) \subset \left(0,\frac{\pi}{2}\right)$

(1) $\dfrac{dx}{dt} = \dfrac{x-x^3}{t(1+x^2)}$ ; $t \in (0,\infty)$, $x \in \mathbb{R}$.

$$f(x)=\frac{x-x^3}{1+x^2}, \quad f: \mathbb{R}\to\mathbb{R}$$

$$a(t)=\frac{1}{t}, \quad a:(0,\infty)\to\mathbb{R}.$$

• $f(x)=0 \Rightarrow \frac{x-x^3}{1+x^2}=0 \Rightarrow x-x^3=0 \Rightarrow$

$\Rightarrow x(1-x^2)=0 \begin{cases} x=0\in\mathbb{R} \\ x=\pm 1 \in\mathbb{R} \end{cases} \Rightarrow$

$(1-x^2=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1)$

$\Rightarrow$ 3 soluții staționare: $\varphi_1,\varphi_2,\varphi_3:(0,\infty)\to\mathbb{R}$

$$\left.\begin{aligned} &\varphi_1(t)=0 \\ &\varphi_2(t)=1 \quad \forall t\in(0,\infty) \\ &\varphi_3(t)=-1 \end{aligned}\right] (4)$$

• pt $f(x)\neq 0 \Rightarrow x\in\mathbb{R}\setminus\{0,1,-1\}$ se separă variabilele:

$$\frac{(1+x^2)\,dx}{x-x^3}=\frac{1}{t}dt$$

$\int\frac{1}{t}dt=\ln|t|+C \Rightarrow A(t)=\ln t,\ t\in(0,\infty)$

$$\int\frac{1+x^2}{x(1-x)(1+x)}dx=I.$$

Determinăm $A,B,C\in\mathbb{R}$ a.î.

$$\frac{1+x^2}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{1-x}+\frac{C}{1+x}$$

$$1+x^2=A(1-x)(1+x)+Bx(1+x)+Cx(1-x)$$

$$1+x^2=A(1-x^2)+Bx+Bx^2+Cx-Cx^2$$

$$1+x^2=x^2(-A+B-C)+x(B+C)+A \Rightarrow$$

$\Rightarrow \begin{cases} -A+B-C=1 \\ B+C=0 \\ \boxed{A=1} \end{cases} \Rightarrow \begin{cases} B-C=2 \\ B+C=0 \end{cases}$ $\quad 2B=2 \quad (+) \Rightarrow$

$\Rightarrow \boxed{B=1} \Rightarrow \boxed{C=-1}$

$$I=\int\frac{1}{x}dx+\int\frac{1}{1-x}dx-\int\frac{1}{1+x}dx=$$

$$=\ln|x|-\ln|1-x|-\ln|1+x|=\ln\left|\frac{x}{1-x^2}\right|+C \Rightarrow$$

$$\Rightarrow B=\ln\left|\frac{x}{1-x^2}\right|$$

Mulțimea soluțiilor implicite ale ec. este:

$$\ln\left|\frac{x}{1-x^2}\right| = \ln t + \ln C, \quad C>0 \qquad (5)$$

$$\ln\left|\frac{x}{1-x^2}\right| = \ln(Ct) \Rightarrow \left|\frac{x}{1-x^2}\right| = Ct \Rightarrow$$

$$\Rightarrow \frac{x}{1-x^2} = \pm C t \Rightarrow \frac{x}{1-x^2} = C_1 t \qquad (6)$$

$C_1 \in \mathbb{R}^*$ $\qquad$ $C_1 \in \mathbb{R}^*$

Mulț. sol. ec este (4) $\cup$ (5) sau (4) $\cup$ (6).

<u>Temă</u>: restul de exerciții (în afară de 1, 6, 10).