Subiecte de teorie pentru Analiza

1. Definiti notiunile de corp ordonat, corp complet ordonat, corp arhimedian.

$(S, +, \cdot, \leq)$ se numeste corp ordonat daca:

1) este corp comutativ;
2) $(S, \leq)$ este spatiu total ordonat ($\forall x, y \in S \rightarrow x \leq y\ sau\ y \leq x$);
3) $x \leq y\ si\ z \in S \rightarrow x + y \leq y + z$ si $x \leq y\ si\ z \geq 0 \rightarrow xz \leq yz$

Un corp ordonat $(S, +, \cdot, \leq)$ se numeste arhimedian daca $\forall x \in S \rightarrow \exists n \in \mathbb{N}\ a.i. x \leq n$

Un corp ordonat S se numeste complet ordonata daca pentru $\forall A \subset S$ care este marginita superior, atunci $\exists \sup A$.

2. Propozitie privind caracterizarea corpurilor arhimediene.

Fie $(S, +, \cdot, \leq)$. Atunci urmatoarele afirmatii sunt echivalente:

1) S arhimedian;
2) pentru $\forall x \in S\ si\ u \geq 0 \rightarrow \exists n \in \mathbb{N}\ ai\ n \cdot u \geq x$;
3) pentru $\forall x > 0 \rightarrow \exists n \in \mathbb{N}\ ai\ 0 \leq \frac{1}{n} \leq x$;
4) pentru $\forall x, y \in S, x \leq y \rightarrow \exists r \in \mathbb{Q}\ ai\ x < r < y$

3. Legatura dintre corpurile arhimediene si corpurile complet ordonate (2 teoreme).

Teorema: Orice corp complet ordonat este arhimedian.

Teorema: Fie $(S, +, \cdot, \leq)$ un corp arhimedian si $(R, +, \cdot, \leq)$ un corp complet ordonat. Atunci exista un morfism de corpuri ordonate $\phi: S \rightarrow R$. Daca S este complet ordonat, atunci ϕ este izomorfism.

4. Definitia unui sir convergent (a unui sir Cauchy) in $\mathbb{R}$.

Spunem ca sirul x_n de numere reale converge la un numar real a si notam $\lim_{n \to \infty} x_n = a$ sau $x_n \xrightarrow[n \to \infty]{} a$ daca $\forall \epsilon > 0\ \exists n_\epsilon \in \mathbb{N}$ a.i. $\forall n \geq n_\epsilon \rightarrow |x_n - a| < \epsilon$.

5. Proprietatea sirurilor convergente. <u>Demonstratie:</u> produsul a doua siruri convergente este sir convergent.

Fie $(x_n)_n$ si $(y_n)_n$ doua siruri de numere reale si $a, b, c \in \mathbb{R}$.

1) Daca $x_n \xrightarrow[n \to \infty]{} a$, atunci sirul $(x_n)_n$ este marginit;
2) Daca $x_n \xrightarrow[n \to \infty]{} a$ si $y_n \xrightarrow[n \to \infty]{} b$, atunci $\begin{cases} x_n + y_n \rightarrow a + b \\ x_n y_n \rightarrow ab \end{cases}$;
3) Daca $x_n \xrightarrow[n \to \infty]{} a$, atunci $|x_n| \xrightarrow[n \to \infty]{} |a|$;

4) Daca $x_n \xrightarrow[n\to\infty]{} a$ si $a \neq 0$, atunci $x_n \neq 0\ \forall n \to \frac{1}{x_n} \to \frac{1}{a}$.

Demonstratie (produsul a doua siruri convergente este sir convergent):

$$x_n \to a: \forall \epsilon > 0 \to \exists n'_\epsilon\ ai\ \forall n \geq n'_\epsilon \to |x_n - a| < \frac{\epsilon}{2}$$

$$y_n \to b: \forall \epsilon > 0 \to \exists n''_\epsilon\ ai\ \forall n \geq n''_\epsilon \to |y_n - b| < \frac{\epsilon}{2}$$

$$|x_n y_n - ab| = |x_n y_n - x_n b + x_n b - ab| \leq |x_n y_n - x_n b| + |x_n b - ab|$$
$$= |x_n(y_n - b)| + |b(x_n - a)|$$

Cum $(x_n)_n$ este marginit, atunci

$$\exists M > 0\ a.i. |x_n| \leq M, \forall n \in \mathbb{N} \to |x_n y_n - ab| \leq M|y_n - b| + |b||x_n - a|\ \forall n \geq n_\epsilon$$

unde $n_\epsilon = \max\{n'_\epsilon, n''_\epsilon\} \to |x_n y_n - ab| \leq \frac{\epsilon}{2}(M + |b|)$ ■

6. Teorema privind convergenta sirurilor monotone + Demonstratie.

Enunt: Orice sir monoton si marginit este convergent.

Demonstratie: Fie $(x_n)_n$ un sir crescator (analog si pentru descrescator) si marginit, atunci:

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq M$$

Fie $a = \sup_{n\geq 1} x_n \in \mathbb{R} \to a \geq x_n$ si $\epsilon > 0 \to \exists n_\epsilon\ a.i. a \leq x_{n\epsilon} + \epsilon$, deci

$$\forall n \geq n_\epsilon \to a - \epsilon \leq x_{n\epsilon} \leq x_n \leq a \leq a + \epsilon \to |x_n - a| \leq \epsilon, \forall n \geq n_\epsilon$$ ■

7. Definiti distanta, spatiul metric, bila intr-un spatiu metric, notiunea de sir Cauchy si de sir convergent intr-un spatiu metric.

Distanta:

O functie $d: X \times X \to [0, \infty)$ se numeste distanta daca:

1) $d(a,b) = 0 \leftrightarrow a = b, \forall a, b \in X$;
2) $d(a,b) = d(b,a), \forall a, b \in X$;
3) $d(a,b) + d(b,c) \geq d(a,c), \forall a, b, c \in X$.

Spatiul metric:

In acest caz, perechea (X, d) se numeste spatiu metric.

Sirul convergent:

Fie (X, d) un spatiu metric. Spunem ca $x_n \xrightarrow[n\to\infty]{} a$ si notam $\lim_{n\to\infty} x_n = a$ daca

$$\forall\epsilon > 0\ \exists n_\epsilon\ ai\ \forall n \geq n_\epsilon \to d(x_n, a) \leq \epsilon \leftrightarrow \lim_{n\to\infty} d(x_n, a) = 0$$

Bila:

Notam $B(a,r)(a \in X, r > 0)$ multimea punctelor x cu proprietatea ca $d(x,a) \leq r$:

$$\{x \mid d(x,a) < r| \leftrightarrow x_n \in B(a,\epsilon)$$

O multime $A \subset X$ se numeste marginita daca $\exists B(a,r)\ a.i. A \subset B(a,r)$.

Sirul Cauchy:

Fie (X,d) un spatiu metric. Un sir $(x_n)_n \subset X$ se numeste sir Cauchy daca

$$\forall\epsilon > 0 \to \exists n_\epsilon\ a.i. \forall n,m \geq n_\epsilon \to d(x_n, x_m) < \epsilon$$

8. Propozitia privind sirurile convergente si sirurile Cauchy intr-un spatiu metrtic + Demonstratie.

Intr-un spatiu metric, orice sir convergent este sir Cauchy.

Demonstratie: In $x_n \xrightarrow[n\to\infty]{} a$ se verifica conditiile din definitie:

$$n,m \geq n_\epsilon\ a.i. d(x_n, x_m) \leq d(x_n, a) + d(a, x_m) \leq \epsilon + \epsilon \leq 2\epsilon$$

9. Limita superioara, limita inferioara: definitii si proprietati.

Fie $(x_n)_n \subset \mathbb{R}, u_n = \sup_{k\geq n} x_k \geq v_n = \inf_{k\geq n} x_k, u_n \geq u_{n+1} \geq v_{n+1} \geq v_n$. Atunci

Se numeste limita superioara: $\lim\sup_{n\to\infty} x_n =^{notatie} \overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} u_n = \inf_{n\geq 1} u_n$

Se numeste limita inferioara: $\lim\inf_{n\to\infty} x_n =^{notatie} \underline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} v_n = \sup_{n\geq 1} u_n$

Proprietati:

(folosesc L(x) pentru a spune limita superioara din sirul xn, l(x) pentru limita inferioara si lim x pentru a spune doar limita)

1) *L(x) = -l(x);*
2) *L(x+y) ≤ L(x) + L(y);*
3) *L(x+y) ≥ L(x) + l(y);*
4) *Daca exista lim yn, atunci L(x+y) = L(x) + lim y;*
5) *l(x+y) ≥ l(x) + l(y);*
6) *l(x+y) ≤ L(x) + l(y);*
7) *analoagele la produs;*

10. Limita superioara ca un punct limita + <u>Demonstratie.</u>

<u>Propozitie:</u> Fie $(x_n)_n \subset \mathbb{R}$ marginit si $a = \lim\sup x_n$, atunci $a \in \mathbb{R}$ si $\exists x_{nk} \to a$.

<u>Demonstratie:</u> Fie k=1. $a = \inf\limits_{n\geq 1} u_n \to \exists m_1\ a.i.\ a \leq u_{m1} < a + \frac{1}{k+1}$

$$u_{m1} = \sup_{k\geq m_1} x_k \to \exists n_1 \geq m_1\ a.i.\ a - 1 < u_{m1} - 1 < x_{n1} \leq u_{m1} < a + \frac{1}{k+1} \to |x_{n1} - a| < 1$$

Presupunem ca $x_{nl}, l = \overline{1,k}\ a.i.\ |x_{nl} - a| < \frac{1}{l}\ si\ n_{l+1} > n_l$

$$a - \frac{1}{k+1} < u_{m_{k+1}} - \frac{1}{k+1} < x_{nk} \leq u_{m_{k+1}} < a + \frac{1}{k+1} \to \left|x_{n_{k+1}} - a\right| < \frac{1}{k+1}, \forall n_{k+1} \geq m_{k+1} > u_k$$

11. Caracterizarea limitei superioare.

Fie $(x_n)_n \subset \mathbb{R}$ marginit si $a \in \mathbb{R}$. Atunci numarul $a = \overline{\lim\limits_{n\geq 1}}\, x_n \leftrightarrow$

1) $\forall \epsilon > 0 \to \exists n_\epsilon\ a.i.\ \forall n \geq n_\epsilon \to x_n < a + \epsilon;$
2) $\exists (x_{nk})_k\ a.i.\ (x_{nk})_k \to a.$

12. Doua consecinte de la punctul 10 (2 teoreme).

<u>Teorema:</u> Orice sir marginit are un subsir convergent.

<u>Teorema:</u> Spatiul metric ($\mathbb{R}$, d), d(x, y) = |x-y| este complet.

13. Criteriile de convergenta pentru serii (enunturi).

<u>1. Criteriul Cauchy</u>

O serie $\sum_{n\geq 1} x_n$ este convergenta daca pentru $\forall \epsilon > 0\ \exists n_\epsilon\ a.i.\ \forall n \geq n_\epsilon\ si\ \forall p \in \mathbb{N} \to \left|\sum_{k=n}^{n+p} x_k\right| < \epsilon.$

<u>2.</u> Daca seria $\sum_{n\geq 1} x_n$ este convergenta, atunci termenul general $x_n \xrightarrow[n\to\infty]{} 0.$

<u>3.</u> O serie absolut convergenta este convergenta.

<u>4. Criteriul lui Abel</u>

Fie $(a_n)_{n\geq 1}$ si $(x_n)_{n\geq 1}$ doua siruri de numere reale, astfel incat:

1) $a_n \searrow 0;$
2) $\exists M > 0\ a.i.\ |\sum_{k=1}^{n} x_k| \leq M.$

Atunci $\sum_{n\geq 1} a_n x_n$ este convergenta.

5. Daca $a_n \searrow 0$, atunci seria $\sum_{n\geq 1}(-1)^n a_n$ este convergenta.

<u>6. Criteriul condensarii</u>

Fie $(a_n)_{n\geq 1}$ a.i. $a_{n+1} \leq a_n$ si $a_n \geq 0, \forall n \in \mathbb{N}$. Atunci $\sum_{n\geq 1} a_n \sim \sum_{n\geq 1} 2^n a_{2^n}$.

7. Criteriul comparatiei

Fie $\sum_{n\geq 1} a_n$ si $\sum_{n\geq 1} b_n$ cu termeni pozitivi. Presupunem ca $\exists n_0$ si $M \in \mathbb{N}^*$ $a.i.$ $\forall n \geq n_0 \to a_n \leq M b_n$.

1) Daca $\sum_{n\geq 1} a_n = \infty \to \sum_{n\geq 1} b_n = \infty$;
2) Daca $\sum_{n\geq 1} b_n$ $converge \to \sum_{n\geq 1} a_n$ $converge$.

8. Criteriul raportului

Fie $\sum_{n\geq 1} a_n \leq C + M \sum_{n\geq 1} b_n$.

1) Daca $\exists l < 1$ si n_0 $a.i.$ $\forall n \geq n_0 \to \frac{a_{n+1}}{a_n} < l$ $\left(\leftrightarrow \overline{\lim_n} \frac{a_{n+1}}{a_n} < 1\right) \to \sum_{n\geq 1} a_n$ $converge$;
2) Daca $\exists l$ si n_0 $a.i.$ $\forall n \geq n_0 \to \frac{a_{n+1}}{a_n} > l$ $\left(\leftrightarrow \underline{\lim_n} \frac{a_{n+1}}{a_n} > 1\right) \to \sum_{n\geq 1} a_n$ $diverge$;

9. Criteriul radicalului

Fie $\sum_{n\geq 1} a_n$, $a_n > 0$. Atunci

1) $\overline{\lim} \sqrt[n]{a_n} < 1$ ($\leftrightarrow \exists l < 1$ si n_0 $a.i.$ $\forall n \geq n_0 \to \sqrt[n]{a_n} < l$ => seria este convergenta;
2) $\overline{\lim} \sqrt[n]{a_n} > 1$ => seria este divergenta.

10. Criteriul Raabe-Duhamel

Fie $\sum_{n\geq 1} a_n$ si $l = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right)$. Daca l > 1, atunci seria este convergenta, iar daca l < 1, atunci seria este divergenta.

14. Topologie (ce este o topologie, spatiu topologic, topologia asociata unui spatiu metric, multime deschisa, multime inchisa, vecinatate).

Fie X o multime si $\tau \subset \mathcal{P}(X)$ $a.i.$

1) $\Phi, x \in \tau$;
2) $D_1, D_2 \in \tau \to D_1 \cap D_2 \in \tau$;
3) $(D_i)_{i\in I} \subset \tau \to \bigcup_{i\in I} D_i \in \tau$.

Atunci τ se numeste topologie.

Perechea (X, τ) se numeste spatiu topologic.

O multime $D \in \tau$ se numeste multime deschisa.

O multime $F \subset X$ $a.i.$ $X \setminus F \in \tau$ se numeste multime inchisa.

Fie $a \in X$. Atunci $V \subset X$ se numeste vecinatate a lui X daca $\exists D \in \tau$ $a.i.$ $a \in D \subset V$.

Notam $\mathcal{V}_a = \{V \mid V$ $este$ o $vecinatate$ a lui $a\}$:

1) $V_1, V_2 \in \mathcal{V}_a \to V_1 \cap V_2 \in \mathcal{V}_a$;

2) $V_1 \in \mathcal{V}_a, V_1 \subset V \to V \in \mathcal{V}_a;$
3) $\forall V \in \mathcal{V}_a \to \exists V_1 \in \mathcal{V}_a, V_1 \subset V\ si\ V_1 \in \mathcal{V}_x, \forall x \in V_1.$

Spatiul topologic asociat unui spatiu metric:

Fie (X, d) un spatiu metric, $a \in X\ si\ r > 0$. Fie $\mathcal{B}(a, r) \coloneqq \{x | d(a, x) < r\}$.

Atunci $V \in \mathcal{V}_a \leftrightarrow \exists r > 0\ a.i. \mathcal{B}(a, r) \subset V$.

15. Definitii: multimea punctelor de acumulare, inchiderea multimii A, frontiera lui A.

Multimea punctelor de acumulare $A' = \{a \in X | \forall V \in \mathcal{V}_a => V \cap A - \{a\} \neq \Phi\}$

Inchiderea multimii $\bar{A} = \{a \in X | \forall V \in \mathcal{V}_a \to V \cap A \neq \Phi\} = A' \cup A = \bigcap_{F \in \mathcal{F}, A \subset F} F$.

Interiorul multimii $\dot{A} = \{a \in X | A \in \mathcal{V}_a\} = \bigcup_{D \in \tau, D \subset A} D \in \tau$.

Frontiera lui A, $\mathcal{F}r(A) = \bar{A} \backslash \dot{A}$. Multimea punctelor izolate $iz(A) = A \backslash A'$.

16. Definitia functiei continue si definitia limitei intr-un punct.

Def. functiei continue: Fie (X, τ_X) si (Y, τ_Y) doua spatii topologice, $a \in X$ si $f: X \to Y$. Spunem ca functia f este continua in punctul a daca $\forall V \in \mathcal{V}_{f(a)} \to f^{-1}(V) = \mathrm{W} \in \mathcal{V}_a$ $(\leftrightarrow \forall V \in \mathcal{V}_a \to \exists W \in \mathcal{V}_a\ a.i. f(W) \subset V)$.

Def. limitei intr-un punct: Fie (X, τ_X) si (Y, τ_Y) doua spatii topologice. Fie o multime $A \subset X, f: A \to Y, a \in X, \alpha \in Y$. Daca a este punct de acumulare pentru A, spunem ca functia f are limita α in a si notam $\lim_{x \to a} f(x) = \alpha$ daca $\forall V \subset \mathcal{V}_\alpha \to \exists W = \mathcal{V}_a\ a.i. \forall x \in V \backslash \{\alpha\} \to f(x) \in V$.

17. Pastrarea continuitatii prin compunerea functiilor + <u>Demonstratie</u>.

Fie $(X, \tau_1), (X, \tau_2), (X, \tau_3)$ spatii topologice, $f_1: X_1 \to X_2, f_2: X_2 \to X_3, a \in X_1$. Daca f_1 este continua in a si f_2 este continua in $f_1(a) => f_2 \circ f_1$ este continua in a.

Demonstratie: Fie $V \in \mathcal{V}_{f_2 \circ f_1(a)}$. Deoarece f_2 este continua in a $\to f_2^{-1}(V) \in \mathcal{V}_{f_{1(a)}}$. Deoarece f_1 este continua in a $\to f_1^{-1}\left(f_2^{-1}(V)\right) = (f_2 \circ f_1)^{-1}(V) \in \mathcal{V}_a \blacksquare$.

18. Caracterizarea continuitatii in spatii metrice si a derivabilitatii in spatii topologice (nu a predat) + <u>Demonstratii</u>.

Fie $(X_1, d_1), (X_2, d_2)$ doua spatii metrice si $f: X_1 \to X_2$ si $a \in X_1$. Atunci urmatoarele afirmatii sunt echivalente:

1) f este continua in a $(\leftrightarrow \forall V \in \mathcal{V}_{f(a)} \to f^{-1}(V) \in \mathcal{V}_a;$

2) $\forall \epsilon > 0 \rightarrow \exists \delta_\epsilon > 0\ a.i.\ d_1(x,a) < \delta_\epsilon\ si\ d_2\big(f(a), f(x)\big) < \epsilon$;
3) $\forall (x_n)_n \subset X_1\ a.i.\ x_1 \rightarrow a \rightarrow f(x_n) \rightarrow f(a)$.

Demonstratie:

(1) => (2)

$$d_1(x,a) < \epsilon \rightarrow x \in \mathcal{B}_{x1}(a,\epsilon) \rightarrow \mathcal{B}(f(a),\epsilon) \in \mathcal{V}_{f(a)} \rightarrow f^{-1}\big(\mathcal{B}(f(a),\epsilon)\big) \in \mathcal{V}_a$$

$$\rightarrow \exists \delta_\epsilon > 0\ a.i.\ \mathcal{B}(a,\delta_\epsilon) \subset f^{-1}\big(\mathcal{B}(f(a),\epsilon)\big) \rightarrow f\left(\mathcal{B}_{d_1}(a,\delta_\epsilon)\right) \subset \mathcal{B}_{d_2}(f(a),\epsilon)$$
$$\forall x \in \mathcal{B}(a,\delta_\epsilon) \rightarrow f(x) \in \mathcal{B}(a,\epsilon)\ \big(\leftrightarrow d_2\big(f(x), f(a)\big) < \epsilon\big)$$

(2) => (1)

$$V \in \mathcal{V}_{f(a)} \rightarrow \exists \epsilon > 0\ a.i.\ \mathcal{B}(f(a),\epsilon) \subset V \xrightarrow{din\ (2)} \exists \delta_\epsilon > 0\ a.i.\ d_1(x,a) < \delta_\epsilon$$
$$\rightarrow d_2\big(f(a), f(x)\big) < \epsilon \rightarrow\ \mathcal{B}(a,\delta_\epsilon) \subset f^{-1}\big(\mathcal{B}(f(a),\epsilon)\big) \rightarrow f^{-1}\big(\mathcal{B}(a,\epsilon)\big) \in \mathcal{V}_a$$

(2) => (3)

$$\forall \epsilon > 0\ \exists \delta_\epsilon > 0\ a.i.\ d_1(x,a) < \delta_\epsilon \rightarrow d_2\big(f(x), f(a)\big) < \epsilon$$
$$x_n \rightarrow a \rightarrow \forall \eta > 0 \rightarrow \exists n_\eta\ a.i.\ \forall n > n_\eta \rightarrow d_1(x_n, a) < \eta$$
$$alegem\ n_\epsilon = n_{\delta_\epsilon} \rightarrow \forall n \geq n_{\delta_\epsilon} \rightarrow d_1(x_n,a) < \delta_\epsilon \rightarrow d_2\big(f(x_n), f(a)\big) < \epsilon$$

(3) => (2)

$$pp\ RA\ ca\ (3)\ este\ falsa: \exists \epsilon > 0\ a.i.\ \forall \delta > 0 \rightarrow x_\delta\ cu\ propr.\ d_1(x,a) < \delta\ si\ d_2\big(f(x_\delta), f(a)\big) \geq \epsilon$$

$$Alegem: \delta = \frac{1}{n}, y_n = x\frac{1}{n}, d_1(y_n, a) < \frac{1}{n} \rightarrow y_n \rightarrow a$$

$$d_2\big(f(y_n), f(a)\big) \leq \epsilon \rightarrow f(y_n) \nrightarrow f(a)\ contradictie\ \blacksquare$$

19. Definitia derivabilitatii + caracterizare (cu omega(x)).

Fie $f: (a,b) \rightarrow \mathbb{R}, c \in (a,b)$. Spunem ca functia f este derivabila in punctul c

$$\leftrightarrow \exists \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \in \mathbb{R}\ (este\ finita)$$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \leftrightarrow 0 = \lim_{x \to c} \frac{f(x) - f(c) - f'(c)(x-c)}{x - c},$$

$$\omega(x) = \frac{f(x) - f(c) - f'(c)(x-c)}{x - c}$$

Altfel, spunem ca functia f este derivabila in punctul c $\leftrightarrow \exists \alpha \in \mathbb{R}\ \big(\alpha = f'(c)\big)$ astfel incat:

1) $f(x) = f(c) + f'(c)(x-c) + \omega(x)(x-c)$;
2) $\lim_{x \to c} \omega(x) = 0$.

***de facut* 20. Teoremele Fermat, Rolle, Lagrange, Cauchy, L'Hospital si proprietate Darboux cu <u>Demonstratii</u>.**

Teorema lui Fermat: Fie $f:(a,b) \to \mathbb{R}, c \in (a,b)$. Daca $\exists f'(c)$ si c este punct de extrem local pentru f, atunci $f'(c) = 0$.

Demonstratie: Presupunem ca c este punct de minim local

$$\to \exists \epsilon > 0 \text{ a. i. } (c-\epsilon, c+\epsilon) \subset (a,b) \text{ si } \forall x \in (c-\epsilon, c+\epsilon) \to f(x) > f(c)$$

Daca $x < c \to x - c < 0$ si $f(x) - f(c) \geq 0 \to \frac{f(x)-f(c)}{x-c} \leq 0 \to \lim\limits_{\substack{x \to c \\ x<c}} \frac{f(x)-f(c)}{x-c} = f'(c) \leq 0$

Daca $x > c \to x - c > 0$ si $f(x) - f(c) \geq 0 \to \frac{f(x)-f(c)}{x-c} \geq 0 \to \lim\limits_{\substack{x \to c \\ x>c}} \frac{f(x)-f(c)}{x-c} = f'(c) \geq 0$

Deci $f'(c) = 0$ ■

Teorema lui Rolle: Fie $f:[a,b] \to \mathbb{R}$ a. i.:

1) f derivabila pe (a,b);
2) f continua pe $[a,b]$;
3) $f(a) = f(b)$.

Atunci $\exists c \in (a,b)$ a. i. $f'(c) = 0$.

Demonstratie: Functia $f:[a,b] \to \mathbb{R}$ este continua, deci este marginita.

Notam $M = \sup\limits_{a \in [a,b]} f(x), m = \inf\limits_{x \in [a,b]} f(x), m \leq f(a) = f(b) \leq M$

I. $M > f(a) = f(b) \to \exists c \in (a,b)$ a. i. $M = f(c) \to$ c este punct de maxim, deci $f'(c) = 0$;
II. $m < f(a) = f(b) \to \exists d \in (a,b)$ a. i. $m = f(d) \to$ d este punct de minim, deci $f'(c) = 0$;
III. $m = f(a) = f(b) = M \to f$ const $\to \forall c \in (a,b), f'(c) = 0$. ■

Teorema lui Lagrange: Fie $f:[a,b] \to \mathbb{R}$ derivabila pe (a,b) si continua pe [a,b]. Atunci

$$\exists c \in (a,b) \text{ a. i. } \frac{f(b)-f(a)}{b-a} = f'(c)$$

Demonstratie: Fie $h:[a,b] \to \mathbb{R}, h(x) = f(x) - \alpha x$ care sa verifice Teorema lui Rolle (T.R.) pe [a,b], h continua pe [a,b], derivabila pe (a,b) si h(a)=h(b).

$$f(a) - \alpha a = f(b) - \alpha b \leftrightarrow f(b) - f(a) = \alpha(b-a) \leftrightarrow \alpha = \frac{f(b)-f(a)}{b-a}$$

Din T.R. $\exists c \in (a,b)$ a. i. $h'(c) = 0 \leftrightarrow h'(x) = f'(x) - \alpha \leftrightarrow h'(c) = \alpha = \frac{f(b)-f(a)}{b-a}$ ■

Teorema lui Cauchy: Fie $f, g:[a,b] \to \mathbb{R}$ continue pe [a,b] si derivabile pe (a,b), astfel incat $g'(x) \neq 0, \forall x \in (a,b)$. Atunci $g(b) \neq g(a)$ si $\exists c \in (a,b)$ a. i. $\frac{(f(b)-f(a))}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Demonstratie: Aplicam Teorema lui Lagrange pe [a,b]

$$\rightarrow \exists d \in (a,b) \text{ a.i.} \frac{\big(g(b)-g(a)\big)}{b-a} = g'(c) \neq 0 \rightarrow g(b) \neq g(c).$$

Fie $h: [a,b] \rightarrow \mathbb{R}m\ h(x) = f(x) - \alpha g(x)$. Functia h este continua pe [a,b], derivabila pe (a,b) si h(a)=h(b).

$$f(a) - \alpha g(a) = f(b) - \alpha g(b) \rightarrow \alpha = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Aplicam T.R. pentru h pe intervalul [a, b]

$$\rightarrow \exists c \in (a,b) \text{ a.i.} h'(c) = 0 \rightarrow f'(c) - \alpha f'(c) = 0 \rightarrow \frac{f'(c)}{g'(c)} = \alpha = \frac{f(b)-f(a)}{g(b)-g(a)} \ \blacksquare$$

Teorema privind Proprietatea lui Darboux: Fie $f: (a,b) \rightarrow \mathbb{R}$ derivabila pe (a,b). Atunci f' are Proprietatea lui Darboux.

Demonstratie: Fie a < c < d < b. Notam $\alpha = f'(c), \beta = f'(d)\ (\alpha < \beta)$. Aratam ca

$$\forall \gamma \in (\alpha, \beta) \rightarrow \exists y \in [c,d] \text{ a.i. } f'(y) = \gamma.$$

Daca $\alpha = \beta \rightarrow y = c$ sau d.

Daca $\alpha < \gamma < \beta$: $g: [a,b] \rightarrow \mathbb{R},\ g(x) = f(x) - \gamma x$.

$$g'(c) = f'(c) - \gamma = \alpha - \gamma < 0; g'(d) = f'(d) - \gamma = \beta - \gamma > 0$$

$y \in [c,d]$ → y este punct de minim local → $g'(y) = 0$ (Fermat) → $f'(y) - \gamma = 0 \rightarrow f'(y) = \gamma$ ■

21. Convergenta simpla si uniforma: definitii.

Fie $f_n: (a,b) \rightarrow \mathbb{R}$.

Convergenta simpla: $f_n \xrightarrow{s} l \leftrightarrow \forall x \in (a,b)\ si\ \forall \epsilon > 0 \rightarrow \exists n_{\epsilon x}\ a.i. \forall n \geq n_{\epsilon x} \rightarrow |f_n(x) - f(x)| < \epsilon$

Convergenta uniforma: $f_n \xrightarrow{u} f \leftrightarrow \forall \epsilon > 0\ \exists n_\epsilon\ a.i. \forall n \geq n_\epsilon \rightarrow |f_n(x) - f(x)| < \epsilon, \forall x \in (a,b)$

22. Pastrarea continuitatii prin convergenta uniforma. (dem. In cartea lui R. Miculescu, p. 219)

Teorema: Fie $f_n, f: (a,b) \rightarrow \mathbb{R}$ si $c \in (a,b)$ a.i. $f_n \xrightarrow{u} f$ si f_n sa fie continua in c, $\forall n \geq 1$. Atunci f este continua.

23. Teorema privind derivabilitatea limitei unui sir de functii.

Fie $f_n, g: (a,b) \rightarrow \mathbb{R}, a, b \in \mathbb{R}$ astfel incat

(1) $f_n' \xrightarrow{u} g$;
(2) $\exists c \in (a,b)\ a.i. \big(f_n(c)\big)$ sa fie convergent,

atunci $\exists f:(a,b)\to\mathbb{R}$ derivabila astfel incat $f_n \overset{u}{\to} f$ si $f' = g$.

24. Teorema marginirii unei functii continue + Demonstratie.

Teorema: Fie $f:[a,b]\to\mathbb{R}$ continua, atunci $\exists c\in[a,b]\ a.i.\ f(c)=\sup\limits_{x\in[a,b]} f(x)$.

Demonstratie:

Pasul 1: demonstram ca $\sup\limits_{x\in[a,b]} f(x) < +\infty$.

Presupunem prin reducere la absurd

$$\sup_{x\in[a,b]} f(x)=\infty \to \forall n, \exists x_n \in [a,b] \text{ a.i. } f(x_n)\geq n$$

$$(x_n)_n \subset [a,b] \to \exists (x_{nk})_k, x_{nk} \xrightarrow[n\to\infty]{} d\in[a,b]$$

$$\lim_{n\to\infty} f(x_n)=\infty\ si\ \lim_{n\to\infty} f(x_{nk})=f(d) \text{ (contradictie)}$$

Pasul 2: Aratam $M=\sup\limits_{x\in[a,b]} f(x)$

$$\forall n\in\mathbb{N}^* \to \exists x_n\in[a,b] \text{ a.i. } M-\frac{1}{n}\leq f(x_n)\leq M \to \lim_{n\to\infty} f(x_n)=M$$

$$(x_n)_n \subset [a,b] \to \exists (x_{nk})_k, x_{nk} \xrightarrow[n\to\infty]{} c\in[a,b]$$

$$\to \begin{cases} f(x_{nk})=f(c) \\ f(x_{nk})\to M \end{cases} \to f(c)=M \blacksquare$$

25. Functii uniform continue: definitie.

Fie $(X_1,d_1),(X_2,d_2)$ spatii metrice si o functie $f:X_1\to X_2$. Atunci f este uniform continua daca

$$\forall \epsilon>0 \to \exists \delta_\epsilon>0\ a.i. \forall x,y\in X_1\ cu\ d_1(x,y)<\delta_\epsilon \to d_2\big(f(x),f(y)\big)<\epsilon$$

26. Teorema privind uniform continuitatea functiilor continue.

Fie $f:[a,b]\to\mathbb{R}$ continua. Atunci functia f este uniform continua.

***Dupa vacanta* 27. Polinomul Taylor asociat unei functii derivabile de ordin n, teorema locala a lui Taylor si teorema globala a lui Taylor.**

***Dupa vacanta* 28. Serii de puteri (enunt + teorema).**