Grupa 344, Seminar 3, EDDP, 19.10.2020

4 (din tema seminar 2) $\dfrac{dx}{dt} = \underbrace{\dfrac{\sin t}{\cos^2 t + 9}}_{a(t)} \cdot \underbrace{\dfrac{\cos x}{\sin^2 x - 4}}_{b(x)}$; $t, x \in \mathbb{R}$

ec. cu var. separabile

- $b(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pm \arccos 0 + 2k\pi = \pm \frac{\pi}{2} + 2k\pi \Rightarrow$

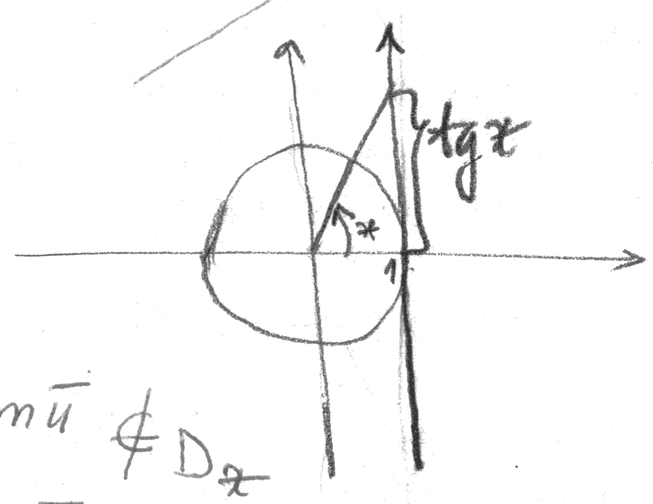
$\Rightarrow$ o infinitate de soluții staționare:

$$\begin{cases} \varphi_{k1}, \varphi_{k2} : \mathbb{R} \to \mathbb{R} \\ \varphi_{k1}(t) = \frac{\pi}{2} + 2k\pi \\ \varphi_{k2}(t) = -\frac{\pi}{2} + 2k\pi \end{cases}, t \in \mathbb{R}, k \in \mathbb{Z}.$$

- $b(x) \neq 0$; $x \in \mathbb{R} \setminus \{\pm \frac{\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\}$ se separă variabilele și se integrează.

3. $\dfrac{dx}{dt} = \dfrac{1}{(\operatorname{tg}^2 x + 1)(t^2 - 1)}$, $t \in (-1, 1)$; $x \in \overbrace{\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}}^{D_x}$

$\dfrac{dx}{dt} = \dfrac{\cos^2 x}{t^2 - 1}$, $t \in (-1, 1)$



- $b(x) = \cos^2 x = 0$

$x = \pm \frac{\pi}{2} + 2k\pi = \frac{\pi}{2} + m\pi \notin D_x$, $m \in \mathbb{Z}$

$\Rightarrow$ ec. nu are soluții staționare

I Să se determine soluțiile următoarelor ec. diferențiale:

ec. dif omogene

1) $\dfrac{dx}{dt} = \dfrac{2xt}{x^2 - t^2}$; $t \in (0, 1)$, $x \in (1, \infty)$

2) $\dfrac{dx}{dt} = \dfrac{x + t}{x - t}$; $t \in (0, 1)$, $x \in (1, +\infty)$

3) $\dfrac{dx}{dt} = \dfrac{3tx^2 + x^3}{2t^3}$, $t \in (0, \infty)$, $x \in \mathbb{R}$.

4) $\dfrac{dx}{dt} = \dfrac{x}{t} + e^{\frac{x}{t}}$, $t \in (-\infty, 0)$, $x \in \mathbb{R}$.

ec. afine (liniare omogene/ neomogene)

5) $x' + \frac{2x}{t^2-1} = 2t+2$; $t \in (-1,1)$, $x \in \mathbb{R}$

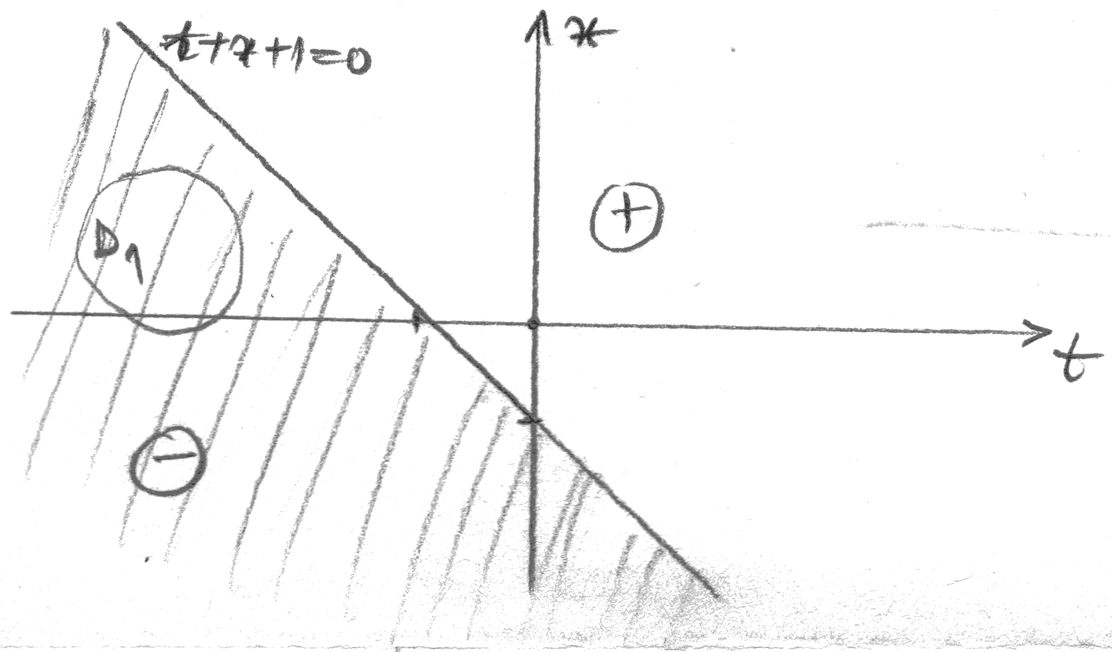
6) $x' + \frac{x}{t-2} = 0$; $t \in (2,+\infty)$, $x \in \mathbb{R}$.

7) $x' = \frac{x \cdot e^{2t} + 4t}{e^{2t}}$, $t, x \in \mathbb{R}$

8) $x' = x(\operatorname{tg} t) + \sin^2 t$, $t \in (0, \frac{\pi}{2})$, $x \in \mathbb{R}$.

ec. dif de tipul: $\frac{dx}{dt} = g\left(\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$

9) $x' = \frac{3t+3x-1}{t+x+1}$, $(t,x) \in D_1 \subset \{(t,x) \mid t+x+1<0\}$.



10) $\frac{dx}{dt} = \frac{t+3x-5}{-t+2x}$, $(t,x) \in D_1 \subset \{(t,x) \mid -t+2x>0\}$.

(3) $\frac{dx}{dt} = \underbrace{\frac{3tx^2+x^3}{2t^3}}_{f(t,x)}$, $t>0$, $x \in \mathbb{R}$.

$f(\alpha t, \alpha x) = \frac{3\alpha t(\alpha x)^2 + (\alpha x)^3}{2(\alpha t)^3} = \frac{\cancel{\alpha^3}(3tx^2+x^3)}{2\cancel{\alpha^3}t^3} = f(t,x)$. $\Rightarrow$

$\Rightarrow$ ec. omogenă $\Rightarrow$ aplicăm schimbare de variabile:

$x = ty$

$(t,x) \xrightarrow[\boxed{x=ty}]{} (t,y)$

Ec. devine: $(ty)' = \frac{3t \cdot t^2y^2 + t^3y^3}{2t^3} \Rightarrow$

$\Rightarrow t'y + ty' = \frac{\cancel{t^3}(3y^2+y^3)}{2\cancel{t^3}} \Rightarrow y + ty' = \frac{3y^2+y^3}{2}$

$t'=1$

$$y' = \frac{1}{t}\left(\frac{3y^2+y^3}{2} - \frac{2}{y}\right) \Rightarrow y' = \frac{1}{t}\cdot\frac{y^3+3y^2-2y}{2}$$

ec. cu variabile separabile (t,y)

$a(t) = \frac{1}{t}$; $b(y) = \frac{y^3+3y^2-2y}{2}$

$t>0$; $y = \frac{x}{t} \in \mathbb{R}$ $(x \in \mathbb{R})$

- $b(y) = 0 \Rightarrow y^3+3y^2-2y = 0$

$$y(y^2+3y-2) = 0$$

$y_1 = 0$

$y^2+3y-2 = 0$

$y_{2,3} = \frac{-3\pm\sqrt{17}}{2}$

$\Rightarrow$ 3 sol. staționare pt ec. în (t,y) $\Rightarrow$

$\Rightarrow$ 3 soluții particulare pt ec. în (t,x):

(1) $\varphi_1, \varphi_2, \varphi_3 : (0,+\infty) \to \mathbb{R}$

$\varphi_1(t) = 0\cdot t = 0$

$\varphi_2(t) = \frac{-3+\sqrt{17}}{2}\cdot t$; $\varphi_3(t) = \frac{-3-\sqrt{17}}{2}t$

- $b(y) \neq 0 \Rightarrow y \in \mathbb{R}\setminus\left\{\frac{-3\pm\sqrt{17}}{2}, 0\right\}$ separăm variabilele:

$$\frac{2\,dy}{y(y^2+3y-2)} = \frac{1}{t}\,dt$$

$$\int \frac{2\,dy}{y(y^2+3y-2)} = J$$

$$\frac{2}{y(y^2+3y-2)} = \frac{A}{y} + \frac{By+C}{y^2+3y-2}; \quad A,B,C \in \mathbb{R}.$$

$$2 = Ay^2 + 3Ay - 2A + By^2 + Cy$$

$$2 = y^2(A+B) + y(3A+C) - 2A \Rightarrow \begin{cases} A+B=0 \\ 3A+C=0 \\ -2A=2 \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{\begin{array}{l} A=-1 \\ B=1 \\ C=3 \end{array}}$$

$$J = -\int \frac{1}{y}\,dy + \int \frac{y+3}{y^2+3y-2}\,dy = -\ln|y| + J$$

$$J = \int \frac{y+3}{\left(y+\frac{3}{2}\right)^2 - \frac{9}{4} - 2}\, dy$$

$$y + \frac{3}{2} = v \Rightarrow y = v - \frac{3}{2}.$$
$$dy = dv$$

$$\int \frac{v - \frac{3}{2} + \frac{3}{2}}{v^2 - \frac{17}{4}}\, dv = \int \frac{v}{v^2 - \frac{17}{4}}\, dv + \frac{3}{2} \int \frac{1}{v^2 - \left(\frac{\sqrt{17}}{2}\right)^2}\, dv =$$

$$= \frac{1}{2} \ln\left|v^2 - \frac{17}{4}\right| + \frac{3}{2} \cdot \frac{1}{\cancel{2}\frac{\sqrt{17}}{\cancel{2}}} \cdot \ln\left|\frac{v - \frac{\sqrt{17}}{2}}{v + \frac{\sqrt{17}}{2}}\right| + C$$

$$J = \frac{1}{2} \ln\left|y^2 + 3y - 2\right| + \frac{3}{2\sqrt{17}} \ln\left|\frac{y + \frac{3-\sqrt{17}}{2}}{y + \frac{3+\sqrt{17}}{2}}\right| \Rightarrow$$

$$\Rightarrow \boxed{B(y) = -\ln|y| + \frac{1}{2} \ln\left|y^2 + 3y - 2\right| + \frac{3}{2\sqrt{17}} \ln\left|\frac{y + \frac{3-\sqrt{17}}{2}}{y + \frac{3+\sqrt{17}}{2}}\right|},$$

$$\int \frac{1}{t}\, dt = \ln|t| + C \xrightarrow[t>0]{} \boxed{A(t) = \ln t}$$

Multimea sol. implicite pt ec. în y este:

$$B(y) = A(t) + C, \quad C \in \mathbb{R}.$$

iar pt. ec în x:

$$B\left(\frac{x}{t}\right) = A(t) + C, \quad C \in \mathbb{R}.$$

(4) $\frac{dx}{dt} = \underbrace{\frac{x}{t} + e^{\frac{x}{t}}}_{f(t,x) = g\left(\frac{x}{t}\right)} \Rightarrow$ ec. este omogenă.

(6) $x' = \underbrace{-\frac{1}{t-2}}_{a(t)} \cdot x \Rightarrow$ ec. liniară omogenă $\Rightarrow$

$$\Rightarrow x(t) = C \cdot e^{A(t)}, \quad C \in \mathbb{R}$$

A este primitiva pt a

$$\int a(t)\, dt = -\int \frac{1}{t-2}\, dt = -\ln|\underbrace{t-2}_{>0}| + C = -\ln(t-2) + C \Rightarrow$$

$$\Rightarrow A(t) = -\ln(t-2) = \ln\left((t-2)^{-1}\right) = \ln\left(\frac{1}{t-2}\right).$$

Deci, mulț. soluțiilor ec. date:

$$x(t) = C \cdot e^{\ln \frac{1}{t-2}} = C \cdot \frac{1}{t-2}, \ C \in \mathbb{R}.$$

$$\boxed{e^{\ln y} = y, \ \forall y > 0}$$

$$\boxed{\alpha^{\log_\alpha y} = y, \ \forall y > 0, \ \alpha > 0, \alpha \neq 1}$$

(7) $x' = \dfrac{x e^{2t} + 4t}{e^{2t}}$, $t, x \in \mathbb{R}$.

ec dif afină: $\dfrac{dx}{dt} = a(t)x + b(t)$

$$\frac{dx}{dt} = \frac{x \cancel{e^{2t}}}{\cancel{e^{2t}}} + \frac{4t}{e^{2t}} \Rightarrow \frac{dx}{dt} = 1 \cdot x + 4te^{-2t}$$

$$\begin{cases} a, b : \mathbb{R} \to \mathbb{R} \\ a(t) = 1 \\ b(t) = 4te^{-2t} \end{cases}$$

• se rezolvă ec. liniară <u>omogenă atașată</u>:

$$\frac{d\bar{x}}{dt} = \bar{x}$$

$$a(t) = 1 \Rightarrow \int 1 \, dt = t + C \Rightarrow A(t) = t \Rightarrow$$

$$\Rightarrow \bar{x}(t) = C \cdot e^t, \ C \in \mathbb{R}$$

$\nearrow$ φ_1 • dacă știm o soluție φ_0 pt ec. afină, atunci mulț sol. ei este:

$$x(t) = \varphi_0(t) + \bar{x}(t).$$

$\searrow$ φ_2 • dacă nu cunoaștem soluție particulară, atunci aplicăm <u>metoda variației constantelor</u>:

determinăm o funcție $C : \mathbb{R} \to \mathbb{R}$ a.î.

$x(t) = C(t) e^t$ să verifice ec. afină $\Rightarrow$

$$\Rightarrow (C(t) e^t)' = C(t) e^t + 4te^{-2t}$$

$$C'(t) e^t + \cancel{C(t) e^t} = \cancel{C(t) e^t} + 4te^{-2t} \Rightarrow$$

$$\Rightarrow C'(t) = \frac{4te^{-2t}}{e^t} \Rightarrow \boxed{C'(t) = 4te^{-3t}} \Rightarrow$$

ec. de tip primitivă

$$\Rightarrow C(t) = \int 4te^{-3t} dt$$

$f(t)=4t \rightarrow f'(t)=4$

$g'(t)=e^{-3t} \Rightarrow g(t)=-\frac{1}{3}e^{-3t}$

$\boxed{\int e^{mt}dt=\frac{e^{mt}}{m}+c}$

$$C(t)=4t\left(-\frac{1}{3}e^{-3t}\right)-\int 4\cdot\left(-\frac{1}{3}e^{-3t}\right)dt=$$

$$=-\frac{4}{3}te^{-3t}+\frac{4}{3}\left(-\frac{1}{3}\right)e^{-3t}+c_1 \Rightarrow$$

$$\Rightarrow \boxed{C(t)=-\frac{4}{3}e^{-3t}\left(t+\frac{1}{3}\right)+c_1,\ c_1\in\mathbb{R}} \Rightarrow$$

$\Rightarrow$ mulț. soluțiilor ec. afine este:

$$x(t)=C(t)e^{t}=\left(-\frac{4}{3}e^{-3t}\left(t+\frac{1}{3}\right)+c_1\right)e^{t}\Rightarrow$$

$$\Rightarrow x(t)=\underbrace{c_1e^{t}}_{\text{sol. ec. liniare omogene}}\underbrace{-\frac{4}{3}e^{-2t}\left(t+\frac{1}{3}\right)}_{\text{soluția particulară}};\ c_1\in\mathbb{R}$$

(10) $\frac{dx}{dt}=\frac{t+3x-5}{-t+2x}$, $(t,x)\in D_1\subset\{(t,x)\mid -t+2x>0\}$

ec. de forma $\frac{dx}{dt}=g\left(\frac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2}\right)$

a_1, b_1, c_1
$a_2, b_2, c_2 \in\mathbb{R}$

$g(u)=u$

$a_1=1,\ b_1=3,\ c_1=-5$

$a_2=-1,\ b_2=2,\ c_2=0,$

$d=a_1b_2-a_2b_1=1\cdot 2-(-1)\cdot 3=2+3=5\neq 0.\Rightarrow$

$\Rightarrow$ se trece de la variabile (t,x) la variabile (s,y) prin schimbarea de variabile $\begin{cases}t=s+t_0\\ x=y+x_0\end{cases}$, unde (t_0,x_0) este soluția sistemului: $\begin{cases}t+3x-5=0\\ -t+2x=0\end{cases}$

$5x-5=0 \Rightarrow \boxed{x_0=1}$

$t=2x \Rightarrow \boxed{t_0=2}$

$$(t,x) \xrightarrow[\begin{cases} t = s+2 \\ x = y+1 \end{cases}]{} (s,y)$$

sau $\begin{cases} s = t-2 = s(t) \\ y = x-1 \end{cases} \Rightarrow \dfrac{ds}{dt} = 1$

$$x(t) = y(s(t)) \Rightarrow$$

$$\Rightarrow \frac{dx}{dt} = \frac{dy}{ds} \cdot \underbrace{\frac{ds}{dt}}_{1} \Rightarrow \boxed{\frac{dx}{dt}(t) = \frac{dy}{ds}(s(t))}$$

Ec devine: $\dfrac{dy}{ds} = \dfrac{s+\cancel{2}+3y+\cancel{3}-\cancel{5}}{-s-\cancel{2}+2y+\cancel{2}}$

$$\boxed{\frac{dy}{ds} = \frac{s+3y}{-s+2y}}$$ ec. omogenă în (s,y)

$$\frac{dy}{ds} = \frac{\cancel{s}(1+3\frac{y}{s})}{\cancel{s}(-1+2\frac{y}{s})} \Rightarrow \frac{dy}{ds} = \frac{1+3\frac{y}{s}}{-1+2\frac{y}{s}}$$

Se face o nouă schimbare de variabilă în ec în (s,y):

$$(s,y) \xrightarrow[y = sz]{} (s,z)$$

care conduce la o ec. cu var. separabile:

$$(sz)' = \frac{1+3z}{-1+2z} \Rightarrow z + s\cdot z' = \frac{1+3z}{-1+2z} \Rightarrow$$

$$\Rightarrow z' = \underbrace{\frac{1}{s}}_{a(s)} \underbrace{\left(\frac{1+3z}{-1+2z} - z\right)}_{b(z)}$$ ec. var. separabile (Acasă de terminat)

<u>Temă</u>: De integrat ec. rămase.