Grupa 344, Seminar 4, EDDP, 26.10.2020

① Să se determine mulțimea sol. ec:

$$x' = \frac{t-x+1}{-t+x}, \quad (t,x) \in D \subseteq \{(t,x) \in \mathbb{R} \mid x-t>0\}$$

Ec de tipul $x' = g\left(\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$.

$a_1 = 1;\ b_1 = -1;\ c_1 = 1$
$a_2 = -1;\ b_2 = 1;\ c_2 = 0$.

$d = a_1 b_2 - a_2 b_1 = 1 - 1 = 0$. $\Rightarrow$ se face schimbarea de variabile: $t - x = y$

$$(t,x) \xrightarrow[x = t-y]{} (t,y)$$

Ec. devine:

$$(t-y)' = \frac{\cancel{t} - \cancel{t} + y + 1}{-\cancel{t} + \cancel{t} - y} \Rightarrow 1 - y' = -\frac{y+1}{y} \Rightarrow$$

$$\Rightarrow y' = \frac{y}{1} + \frac{y+1}{y} \Rightarrow \frac{dy}{dt} = \frac{2y+1}{y}$$

ec. cu var. separabile

$a(t) = 1$
$b(y) = \frac{2y+1}{y}$
(temă!).

② Să se integreze ec:

a) $x' = \frac{4x}{t} + t\sqrt{x}, \quad t \in (0,\infty), \ x \in [0,\infty)$

b) $x' = xt + (\sin t)x^2, \ t \in (0, \frac{\pi}{2}), \ x \in \mathbb{R}$

c) $x' = \frac{x}{t} + \frac{1}{t^2 x^2}; \ t, x \in (0,+\infty)$

③ a) Fie ecuatia: $x' - \frac{3t^2}{t^5-1} - \frac{t^4}{t^5-1}x + \frac{2t}{t^5-1}x^2 = 0$. $t \in (1,+\infty)$, $x \in \mathbb{R}$.

i) Să se determine perechile (m,n) de numere reale a.î. $\varphi_0(t) = mt^n$ a.î. fie soluție a ecuației

ii) Să se determine mulțimea soluțiilor ec.

b) Fie ecuația: $x' = -\frac{1}{t}x^2 + \frac{4}{t}x - \frac{3}{t}$, $t > 0$, $x \in \mathbb{R}$.

i) Să se determine o soluție particulară de forma $\varphi_0(t) = \alpha$, cu α constantă.

ii) Determinați mulț. soluțiilor ec.

(4) Se cer soluțiile parametrice ale ecuațiilor:

a) $x = (1+x')t + (x')^2$

b) $x = 2tx' - x'^2$.

c) $t + x = \left(\frac{x'+1}{x'-1}\right)^2$

d) $x = tx' + \frac{1}{(x')^2}$.

(2) (a) $x' = \frac{4}{t}x + t\,x^{1/2}$

ec. Bernoulli: $a, b : (0, \infty) \to \mathbb{R}$

$a(t) = \frac{4}{t}$

$b(t) = t$

$\alpha = \frac{1}{2}$

• ec. liniară omogenă atașată: $\frac{d\bar{x}}{dt} = \frac{4}{t}\bar{x} \Rightarrow$

$\Rightarrow \bar{x}(t) = C \cdot e^{A(t)}$, A = prim. lui a $\Rightarrow$

$\Rightarrow \int \frac{4}{t}dt = 4\int \frac{1}{t}dt = 4\ln|t| + C \Rightarrow$

$\Rightarrow A = 4\ln t = \ln(t^4) \Rightarrow$

$\Rightarrow \underline{\bar{x}(t)} = C \cdot e^{\ln|t^4|} = \underline{Ct^4}$

• aplicăm metoda variației constantelor: determinăm $C : (0, \infty) \to \mathbb{R}$ a.î. $\boxed{x(t) = C(t)t^4}$ să fie soluția ec. Bernoulli.

$$\left(c(t)t^4\right)' = \frac{4}{t}\cdot c(t)t^4 + t\cdot\left(c(t)\,t^4\right)^{1/2}$$

$$c'(t)\cdot t^4 + \cancel{c(t)\cdot 4t^3} = \cancel{\frac{4}{t}c(t)t^4} + c^{1/2}\cdot t^3 \Rightarrow$$

$$\Rightarrow c' = c^{1/2}\cdot\frac{1}{t} \rightarrow \frac{dc}{dt} = \underbrace{c^{1/2}}_{b_1(c)}\cdot\underbrace{\frac{1}{t}}_{a_1(t)}.$$

• rez. întâi $b_1(C)=0 \Rightarrow c^{1/2}=0 \Rightarrow c=0 \Rightarrow c(t)=0,\ t\in(0,\infty) \Rightarrow$

$$\Rightarrow x(t) = 0\cdot t^4 = 0,\quad t\in(0,\infty).$$

$$\boxed{x(t)=0,\ \forall t\in(0,\infty)}$$

• $b_1(c)\neq 0 \Rightarrow$ separăm variabilele: $\dfrac{dc}{c^{1/2}} = \dfrac{1}{t}dt$

$$\int\frac{dc}{c^{1/2}} = \int c^{-\frac{1}{2}}dC = \frac{c^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C_1 = 2\sqrt{c} + C_1 \Rightarrow$$

$$\left.\begin{array}{l}\Rightarrow B(c) = 2\sqrt{c}\\ \int\frac{1}{t}dt = \ln|t| + C_1\end{array}\right\} \Rightarrow 2\sqrt{c} = \ln t + C_1 \Rightarrow$$

$A(t)=\ln t.$
$t\in(0,+\infty)$

$$\Rightarrow \sqrt{c} = \frac{\ln t + C_1}{2} \Rightarrow$$

$$\rightarrow c(t) = \left(\frac{\ln t + C_1}{2}\right)^2 \Rightarrow$$

$$\Rightarrow \boxed{x(t) = \left(\frac{\ln t + C_1}{2}\right)^2\cdot t^4,\ C_1\in\mathbb{R}}$$

③ ⓐ $$x' = \frac{-2t}{t^5-1}x^2 + \frac{t^4}{t^5-1}x + \frac{3t^2}{t^5-1}$$

i) $\varphi_0(t) = m t^n$ să fie soluție a ec $\Rightarrow$

$$\Rightarrow (mt^n)' = \frac{-2t}{t^5-1}(mt^n)^2 + \frac{t^4}{t^5-1}mt^n + \frac{3t^2}{t^5-1} \quad \Big|\cdot(t^5-1)$$

$$mnt^{n-1}(t^5-1) = -2tm^2t^{2n} + mt^{4+n} + 3t^2$$

$$mn\,t^{n+4} - mn\,t^{n-1} = -2m^2t^{2n+1} + mt^{4+n} + 3t^2.$$

c1) $\boxed{2=n-1} \Rightarrow \boxed{n=3}$

$$3m\,t^7 - 3m\,t^2 = -2m^2 t^7 + m\,t^7 + 3t^2$$

$$\begin{cases} 3m = -2m^2 + m \\ -3m = 3 \end{cases} \Rightarrow \boxed{m = -1} \quad \text{verifică} \quad 3(-1) = -2(-1)^2 + (-1)$$

$-3 = -3$ Adev.

$$\Rightarrow \boxed{\varphi_0(t) = -t^3}$$

c2) $\boxed{2 = n+4} \Rightarrow \boxed{n = -2}$

$$-2m\,t^2 + 2m\,t^{-3} = -2m^2 t^{-3} + m\,t^2 + 3t^2 \Rightarrow$$

$$\Rightarrow \begin{cases} -2m = m+3 \Rightarrow -3m = 3 \Rightarrow \boxed{m = -1} \\ 2m = -2m^2 \quad \text{verifică} \quad m = -1 \end{cases} \Rightarrow$$

$\Rightarrow$ Soluția particulară: $\boxed{\varphi_0(t) = -t^{-2} = -\frac{1}{t^2}}$

c3) $\boxed{2n+1 = 2} \Rightarrow \boxed{n = \frac{1}{2}} \Rightarrow$

$$\Rightarrow \frac{1}{2}m\,t^{\frac{?}{2}} - \frac{1}{2}m\,t^{-\frac{1}{2}} = -2m^2 t^2 + m\,t^2 + 3t^2.$$

$$\begin{cases} \frac{1}{2}m = m \Rightarrow \frac{1}{2}m = 0 \Rightarrow m = 0 \\ -\frac{1}{2}m = 0 \Rightarrow m = 0 \\ 0 = -2m^2 + 3. \text{ nu e verificată pt } m = 0. \end{cases} \Rightarrow$$

$\Rightarrow$ nu obținem soluție particulară.

ii) ec Riccati

alegem soluția particulară $\boxed{\varphi_0(t) = -t^3}$

$$(t, x) \xrightarrow{x = y - t^3;\ (x(t) = y(t) - t^3)} (t, y)$$

$$(y - t^3)' = \frac{-2t}{t^5-1}(y - t^3)^2 + \frac{t^4}{t^5-1}(y - t^3) + \frac{3t^2}{t^5-1} \Big/ \cdot (t^5-1)$$

$$(t^5-1)(y' - 3t^2) = -2t(y^2 - 2yt^3 + t^6) + t^4(y - t^3) + 3t^2$$

$$(t^5-1)y' - \cancel{3t^7} + \cancel{3t^2} = -2ty^2 + 4yt^4 - 2\cancel{t^7} + yt^4 - t^7 + \cancel{3t^2} \quad \Big| : t^5-1$$

$$\frac{dy}{dt} = \underbrace{\frac{5t^4}{t^5-1}}_{=a_1(t)}\,y - \underbrace{\frac{2t}{t^5-1}}_{b_1(t)}\,y^2 \qquad \text{ec. Bernoulli cu } \alpha = 2.$$

• ec. lin. omog. atasată:

$$\frac{d\bar{y}}{dt} = \underbrace{\frac{5t^4}{t^5-1}}_{a_1(t)}\bar{y} \Rightarrow \bar{y} = C e^{A_1(t)}$$

$$\int \frac{5t^4}{t^5-1}dt = \int \frac{(t^5-1)'}{t^5-1}dt = \ln|t^5-1| + C \underset{t>1}{\Rightarrow}$$

$$\Rightarrow A_1(t) = \ln(t^5-1) \Rightarrow \boxed{\bar{y}(t) = C(t^5-1)}$$

• var. const $\Rightarrow$ determinăm $C:(1,\infty)\to\mathbb{R}$ a.î.

$y(t) = C(t)(t^5-1)$ sol. ec. Bernoulli $\Rightarrow$

$$\Rightarrow (C(t)(t^5-1))' = \frac{5t^4}{\cancel{t^5-1}}\cdot C(t)\cancel{(t^5-1)} - \frac{2t}{\cancel{(t^5-1)}}\cdot C^2(t)(t^5-1)^{\cancel{2}}$$

$$\Rightarrow C'(t)\cancel{(t^5-1)} + \cancel{C(t)\cdot 5t^4} = \cancel{5t^4C(t)} - 2t\,C^2(t)\cancel{(t^5-1)}$$

$$\Rightarrow \frac{dC}{dt} = -2t\,C^2(t)$$

sol. staționară $C=0 \Rightarrow y=0 \Rightarrow$

$\Rightarrow x(t) = -t^3$ (3)

$$C\neq 0 \Rightarrow \frac{dC}{C^2} = -2t\,dt \Rightarrow$$

$$\Rightarrow \int C^{-2}dC = \int -2t\,dt \Rightarrow \frac{C^{-1}}{-1} = -t^2 + K \Rightarrow$$

$$\Rightarrow -\frac{1}{C} = K - t^2 \Rightarrow \boxed{C(t) = \frac{1}{t^2-K}} \Rightarrow$$

$K\in\mathbb{R}$

$$\Rightarrow y(t) = \frac{t^5-1}{t^2-K} \Rightarrow \boxed{x(t) = \frac{t^5-1}{t^2-K} - t^3} \quad (4)$$

$K\in\mathbb{R}$.

Mult. sol ec. Riccati: (3) $\cup$ (4).

Obs: Pt $\underline{K=0}$: $x(t) = \dfrac{t^5-1}{t^2} - \dfrac{t^5}{t^2} = \dfrac{-1}{t^2}$.

② ⓐ $x' = \frac{4}{t}x - t\,x^{\frac{1}{2}}$

ec. Bernoulli: $x = y^{\frac{1}{1-\frac{1}{2}}} \Rightarrow x = y^2$

$\alpha = \frac{1}{2}$

$$(t,x) \xrightarrow[x=y^2]{} (t,y)$$

ec. devine: $(y^2)' = \frac{4}{t}y^2 - t(y^2)^{\frac{1}{2}} \Rightarrow$

$$2yy' = \frac{4}{t}y^2 - ty \quad /:2y$$

$$y' = \underbrace{\frac{2}{t}}_{a_1(t)} y \underbrace{- \frac{t}{2}}_{b_1(t)}.$$ ec. afină → se rezolvă cu var. constantelor.

OBS: $\int e^{4e^t}\,dt = \int (e^{e^t})^4\,dt = \frac{1}{4}\int \frac{4(e^{e^t})^4 \cdot e^t\,dt}{e^t} = J.$

$(e^{e^t})^4 = v$

$4(e^{e^t})^3 \cdot e^{e^t} \cdot e^t\,dt = dv.$

$4(e^{e^t})^4 \cdot e^t\,dt = dv.$

$4 \ln(e^{e^t}) = \ln v$

$4e^t \underbrace{\ln e}_{=1} = \ln v \Rightarrow e^t = \frac{\ln v}{4}$

$\cancel{\frac{1}{4}}\int \frac{dv}{\frac{\ln v}{\cancel{4}}} = \int \frac{1}{\ln v}\,dv.$ $\quad ; v \neq 1, v > 0.$

$\int \frac{1}{\ln v}\,dv = \int \frac{1}{v \ln v} \cdot v\,dv = \int (\ln(\ln v))' \cdot v\,dv =$

$= \ln(\ln v) \cdot v - \int \ln(\ln v)\,dv.$

Temă: 2(b, c)
3(b)
4(a, b, c, d).