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Tema 2

1) Fie evenimentele: A_i - la aruncarea "i" suma fețelor este 5
B_i - la aruncarea "i" suma fețelor este 7

E_n se poate scrie:

$$E_n = (A_1^c \cap B_1^c) \cap (A_2^c \cap B_2^c) \cap \dots \cap (A_{n-1}^c \cap B_{n-1}^c) \cap A_n$$
$$= \left(\bigcap_{i=1}^{n-1} (A_i^c \cap B_i^c) \right) \cap A_n$$

$$\Rightarrow \mathbb{P}(E_n) = \mathbb{P}((A_1^c \cap B_1^c) \cap (A_2^c \cap B_2^c) \cap \dots (A_{n-1}^c \cap B_{n-1}^c) \cap A_n)$$

Dar evenimentele sunt independente
(aruncările)

$$\Rightarrow \mathbb{P}(E_n) = \mathbb{P}(A_1^c \cap B_1^c) \cdot \mathbb{P}(A_2^c \cap B_2^c) \cdot \mathbb{P}(A_3^c \cap B_3^c) \cdot \dots \cdot \mathbb{P}(A_{n-1}^c \cap B_{n-1}^c) \cdot \mathbb{P}(A_n)$$

Suma fețelor 5 se poate obține în 4 moduri ((1,4), (2,3), (3,2), (4,1))

$$\Rightarrow \mathbb{P}(A_n) = \frac{4}{6 \cdot 6} = \frac{4}{36} = \frac{1}{9}$$

Suma fețelor 7 se poate obține în 6 moduri ((1,6), (2,5), (3,4), (4,3), (5,2), (6,1))

$$\Rightarrow \mathbb{P}(B_i) = \frac{6}{36}$$

$$\Rightarrow \mathbb{P}(A_i^c \cap B_i^c) = 1 - \mathbb{P}(A_i \cup B_i) = 1 - (\mathbb{P}(A_i) + \mathbb{P}(B_i) - \underbrace{\mathbb{P}(A_i \cap B_i)}_{0})$$
$$= 1 - \frac{1}{9} - \frac{1}{6} = \frac{13}{18}$$

Știm că $\mathbb{P}(A_1^c \cap B_1^c) = \mathbb{P}(A_2^c \cap B_2^c) = \dots = \mathbb{P}(A_{n-1}^c \cap B_{n-1}^c)$.

$$\Rightarrow \mathbb{P}(E_n) = \mathbb{P}(A_{n-1}^c \cap B_{n-1}^c)^{n-1} \cdot \mathbb{P}(A_n) = \left(\frac{13}{18}\right)^{n-1} \cdot \frac{1}{9}$$

Fie X= evenimentul că suma fețelor celor 2 zaruri este 5 (să apare înaintea sumei 7)

$$\mathbb{P}(X)=\mathbb{P}(E_1\cup E_2\cup E_3\cup\ldots\cup E_n)=\cancel{\sum_{i=1}^{n-1}\left(\frac{13}{18}\right)^{n-1}\cdot\frac{1}{9}=\frac{1}{9}\sum_{i=1}^{n-1}\left(\frac{13}{18}\right)^{n-1}}$$

$$=\mathbb{P}(E_1)+\mathbb{P}(E_2)+\mathbb{P}(E_3)+\ldots+\mathbb{P}(E_n)=\sum_{i=1}^{n}\left(\frac{13}{18}\right)^{i-1}\cdot\frac{1}{9}$$

$$=\lim_{n\to\infty}\frac{1}{9}\sum_{i=1}^{n}\left(\frac{13}{18}\right)^{i-1}=\lim_{n\to\infty}\frac{1}{9}\cdot\frac{1-\left(\frac{13}{18}\right)^n}{1-\frac{13}{18}}=\lim_{n\to\infty}\frac{1}{9}\cdot\frac{18}{5}\cdot\left(1-\underbrace{\left(\frac{13}{18}\right)^n}_{0}\right)$$

$$=\frac{2}{5}=0{,}4$$

2) Notăm F_n- evenimentul că în primele $n-1$ aruncări nu a apărut nici suma 2, nici 7, iar la a n-a aruncare am obținut suma 2.

C_i - evenimentul că la aruncarea i am obținut aruncarea suma fețelor 2.

$$F_n=(B_1^c\cap C_1^c)\cap(B_2^c\cap C_2^c)\cap\ldots\cap(B_{n-1}^c\cap C_{n-1}^c)\cap C_n$$

$$\Rightarrow\mathbb{P}(F_n)=\mathbb{P}((B_1^c\cap C_1^c)\cap(B_2^c\cap C_2^c)\cap\ldots\cap(B_{n-1}^c\cap C_{n-1}^c)\cap C_n)$$

Evenimentele sunt idependente între ele:

$$\Rightarrow\mathbb{P}(F_n)=\mathbb{P}(B_1^c\cap C_1^c)\,\mathbb{P}(B_2^c\cap C_2^c)\ldots\mathbb{P}(B_{n-1}^c\cap C_{n-1}^c)\cdot\mathbb{P}(C_n)$$

$\mathbb{P}(C_n)=\frac{1}{36}$ (Singurul caz favorabil este perechea $(1,1)$)

$$\mathbb{P}(B_1^c\cap C_1^c)=\mathbb{P}(B_2^c\cap C_2^c)=\ldots=\mathbb{P}(B_{n-1}^c\cap C_{n-1}^c)$$

$$\mathbb{P}(B_{n-1}^c\cap C_{n-1}^c)=1-\mathbb{P}(B_{n-1}\cup C_{n-1})=1-\frac{1}{36}-\frac{6}{36}=\frac{29}{36}$$

$$\Rightarrow\mathbb{P}(F_n)=\left(\frac{29}{36}\right)^{n-1}\cdot\frac{1}{36}$$

Fie Y - evenimentul că suma 2 să apară înaintea sumei 7 ...

$\mathbb{P}(Y) = \mathbb{P}(F_1 \cup F_2 \cup F_3 \cup \_ \cup F_n) = \mathbb{P}(E_1) + \mathbb{P}(E_2) + \_ + \mathbb{P}(E_n)$

$$= \sum_{i=1}^{n} \left(\frac{29}{36}\right)^{i-1} \cdot \frac{1}{9} = \frac{1}{9} \sum_{i=1}^{n} \left(\frac{29}{36}\right)^{i-1}$$

$$= \lim_{n \to \infty} \frac{1}{\cancel{36}} \cdot \frac{1-\left(\frac{29}{36}\right)^n}{1-\frac{29}{36}} = \frac{36}{7} \cdot \frac{1}{\cancel{36}} \left(1 - \left(\frac{29}{36}\right)^n\right) \to 0$$

$$= \frac{1}{7} = 0{,}1428$$

2) Fie A_i - evenimentul că Maria este la zi cu materia în săpt. i
A_i^c — " — NU — " —

$\mathbb{P}(A_{i+1} | A_i) = 0{,}8$
$\mathbb{P}(A_{i+1} | A_i^c) = 0{,}4$
$\mathbb{P}(A_{i+1}^c | A_i) = 0{,}2$
$\mathbb{P}(A_{i+1}^c | A_i^c) = 0{,}6$

Considerăm $\mathbb{P}(A_0) = 1$ (era cu materia la zi la începutul semestrului)

$$\mathbb{P}(A_3) = \mathbb{P}(A_3 | A_2) \cdot \mathbb{P}(A_2) + \mathbb{P}(A_3 | A_2^c) \cdot \mathbb{P}(A_2^c)$$
$$= 0{,}8 \cdot \mathbb{P}(A_2) + 0{,}4\, \mathbb{P}(A_2^c)$$

$$\mathbb{P}(A_2) = \mathbb{P}(A_2 | A_1)\, \mathbb{P}(A_1) + \mathbb{P}(A_2 | A_1^c) \cdot \mathbb{P}(A_1^c)$$
$$= 0{,}8\, \mathbb{P}(A_1) + 0{,}4\, \mathbb{P}(A_1^c)$$

$$\mathbb{P}(A_2^c) = 1 - \mathbb{P}(A_2) = 0{,}2\, \mathbb{P}(A_1) + 0{,}6\, \mathbb{P}(A_1^c)$$

Cum a început semestrul cu materia la zi $\Rightarrow \mathbb{P}(A_1) = 0{,}8 \cdot 1 = 0{,}8$
$\mathbb{P}(A_1^c) = 0{,}2 \; (= 1 - 0{,}8)$

$\Rightarrow \mathbb{P}(A_2) = 0{,}8 \cdot 0{,}8 + 0{,}4 \cdot 0{,}2 = 0{,}72$
$\mathbb{P}(A_2^c) = 1 - \mathbb{P}(A_2) = 0{,}28$

$\Rightarrow \mathbb{P}(A_3) = 0{,}8 \cdot 0{,}72 + 0{,}4 \cdot 0{,}28 = \underline{0{,}688}$

Deci prin inducție: $\mathbb{P}(A_n) = (0{,}4)^{n-1} \cdot 1{,}8 + (0{,}4)^{n-2} + \ldots + (0{,}4)^1$, $n \geq 2$
$\mathbb{P}(A_n) = 0{,}8 \quad \underline{n=1}$

$\underline{n=2}$ $\mathbb{P}(A_2) = 0{,}8\,\mathbb{P}(A_1) + 0{,}4\,\mathbb{P}(A_1^c) = 0{,}8\,\mathbb{P}(A_1) + 0{,}4(1 - \mathbb{P}(A_1))$

$$\mathbb{P}(A_2) = 0{,}4\,\mathbb{P}(A_1) + 0{,}4 = 0{,}4(\mathbb{P}(A_1)+1) = 0{,}4(0{,}8+1) = 0{,}4 \cdot 1{,}8 \quad (A)$$

Presupunem adevărat $P(k)$: $\mathbb{P}(A_k) = 0{,}4^{k-1} \cdot 1{,}8 + 0{,}4^{k-2} + \dots + 0{,}4$

Demonstrăm $\mathbb{P}(k+1)$: $\mathbb{P}(A_{k+1}) = 0{,}4^{k} \cdot 1{,}8 + 0{,}4^{k-1} + \dots + 0{,}4$

$$\mathbb{P}(A_{k+1}) = \mathbb{P}(A_{k+1} | A_k) \cdot \mathbb{P}(A_k) + \mathbb{P}(A_{k+1} | A_k^c) \cdot \mathbb{P}(A_k^c)$$

$$= 0{,}8 \cdot \mathbb{P}(A_k) + 0{,}4\,\mathbb{P}(A_k^c)$$

$$= 0{,}8 \cdot \mathbb{P}(A_k) + 0{,}4(1 - \mathbb{P}(A_k))$$

$$= 0{,}8\,\mathbb{P}(A_k) + 0{,}4 - 0{,}4\,\mathbb{P}(A_k) = 0{,}4\,\mathbb{P}(A_k) + 0{,}4$$

$$= 0{,}4(\mathbb{P}(A_k)+1)$$

$$\Rightarrow \mathbb{P}(A_{k+1}) = 0{,}4(0{,}4^{k-1} \cdot 1{,}8 + 0{,}4^{k-2} + \dots + 0{,}4 + 1)$$

$$= 0{,}4^{k} \cdot 1{,}8 + 0{,}4^{k-1} + \dots + 0{,}4^2 + 0{,}4 \quad \checkmark$$

$\Rightarrow$ La sfârșitul cursului:

$$\mathbb{P}(A_{14}) = 0{,}4^{13} \cdot 1{,}8 + \underbrace{0{,}4^{12} + 0{,}4^{11} + \dots + 0{,}4}_{= \, 0{,}4 \cdot \frac{1-(0{,}4)^{12}}{1-0{,}4} = \frac{2}{3}(1-0{,}4^{12})}$$

$$\mathbb{P}(A_{14}) = 0{,}4^{13} \cdot 1{,}8 + \frac{2}{3}(1-0{,}4^{12}) = (0{,}4)^{13} \cdot 1{,}8 + \frac{2}{3} - \frac{2}{3}\,0{,}4^{12}$$

$$= (0{,}4)^{12}\left(0{,}4 \cdot 1{,}8 - \frac{2}{3}\right) + \frac{2}{3}$$

$$= \frac{2}{3} + (0{,}4)^{12} \cdot \frac{4}{75} \sim 0{,}66666667$$

$$\lim_{n \to \infty} \mathbb{P}(A_n) = \frac{2}{3}$$

3) $X \sim \begin{pmatrix} -1 & 0 & 1 \\ 0,3 & 0,2 & 0,5 \end{pmatrix}$

Folosim faptul că dacă $X \sim \begin{pmatrix} x_1 & \dots & x_n \\ p_1 & & p_n \end{pmatrix}$ variabilă aleatoare independentă și $g(x)$ continuă, atunci $g(X) \sim \begin{pmatrix} g(x_1) & \dots & g(x_n) \\ p_1 & - - & p_n \end{pmatrix}$

a) $3X+7$

$3x+7 \in \{4, 7, 10) \Rightarrow 3X+7 \sim \begin{pmatrix} 4 & 7 & 10 \\ 0,3 & 0,2 & 0,5 \end{pmatrix}$

b) X^2

$x^2 \in \{0,1\} \Rightarrow \mathbb{P}(\cancel{X}^2=0) = \mathbb{P}(X=0) = 0,2$

$\mathbb{P}(X^2=1) = \mathbb{P}(X=1) + \mathbb{P}(X=-1) = 0,8$

$X^2 \sim \begin{pmatrix} 0 & 1 \\ 0,2 & 0,8 \end{pmatrix}$

c) X^3

$x^3 \in \{-1,0,1\} \Rightarrow X^3 \sim \begin{pmatrix} -1 & 0 & 1 \\ 0,3 & 0,2 & 0,5 \end{pmatrix}$

d) $X+X^2$

$x+x^2 \in \{0,2\} \Rightarrow \mathbb{P}(X^2+X=0) = \mathbb{P}(X=-1) + \mathbb{P}(X=0) = 0,5$

$\mathbb{P}(X^2+X=2) = \mathbb{P}(X=1)$

$X^2+X \sim \begin{pmatrix} 0 & 2 \\ 0,5 & 0,5 \end{pmatrix}$

e) $\mathbb{P}(X > -\frac{1}{3})$

$M = \{x \in X \mid x > -\frac{1}{3}\} = \{0,1\}$

$\mathbb{P}(M) \overset{ind}{=} \mathbb{P}(\cancel{X}=0) + \mathbb{P}(X=1) = 0,7$

f) $\mathbb{P}(X < \frac{1}{4} \mid X \geq -\frac{1}{2}) = \dfrac{\mathbb{P}(-\frac{1}{2} \leq X < \frac{1}{4})}{\mathbb{P}(X \geq -\frac{1}{2})}$

$\mathbb{P}(-\frac{1}{2} \leq X < \frac{1}{4}) =$

$x \in \{-1, 0, 1\}$

$-\frac{1}{2} \leq 0 < \frac{1}{4}$

$\Rightarrow \mathbb{P}(-\frac{1}{2} \leq X < \frac{1}{4}) = \mathbb{P}(X=0) = 0,2$

$\mathbb{P}(X \geq -\frac{1}{2})$

$x \in \{-1, 0, 1\} \quad \Rightarrow \mathbb{P}(X \geq -\frac{1}{2}) = \mathbb{P}(X=0) + \mathbb{P}(X=1) = 0,7$

$\Rightarrow \mathbb{P}(X < \frac{1}{4} \mid X \geq -\frac{1}{2}) = \frac{0,2}{0,7} = \frac{2}{7}$

4) $X \sim \begin{pmatrix} 1 & \dots & n \\ p_1 & \dots & p_n \end{pmatrix} \quad p_i > 0 \quad i = \overline{1,n} \quad p_i = \mathbb{P}(X = i)$

a) $\lambda > 0$

i) $X \sim Pois(\lambda)$

ii) $\frac{p_n}{p_{n-1}} = \frac{\lambda}{n} \quad n \geq 1$ $\quad (\Leftrightarrow)$

i) $\Rightarrow$ ii)

$$\left.\begin{aligned} p_n &= e^{-\lambda} \cdot \frac{\lambda^n}{n!} \\ p_{n-1} &= e^{-\lambda} \frac{\lambda^{n-1}}{(n-1)!} \end{aligned}\right\} \Rightarrow \frac{p_n}{p_{n-1}} = \frac{\cancel{e^{-\lambda}} \cdot \frac{\lambda^n}{n!}}{\cancel{e^{-\lambda}} \frac{\lambda^{n-1}}{(n-1)!}} = \frac{\lambda}{n} \quad \text{A}$$

ii) $\Rightarrow$ i) $\quad \frac{p_n}{p_{n-1}} = \frac{\lambda}{n} \quad n \geq 1$

$$\frac{p_1}{p_0} \cdot \frac{p_2}{p_1} \cdot \frac{p_3}{p_2} \cdot \ldots \cdot \frac{p_n}{p_{n-1}} = \frac{p_n}{p_0} = \frac{\lambda}{1} \cdot \frac{\lambda}{2} \cdot \frac{\lambda}{3} \cdot \ldots \cdot \frac{\lambda}{n} = \frac{\lambda^n}{n!}$$

$\Rightarrow p_n = \frac{p_0 \cdot \lambda^n}{n!}$

$$p_0+p_1+p_2+\dots+p_n=p_0+\frac{p_0\cdot\lambda}{1!}+\frac{p_0\cdot\lambda^2}{2!}+\dots+\frac{p_0\,\lambda^n}{n!}=1$$

$$p_0\left(\underbrace{1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\dots+\frac{\lambda^n}{n!}}_{e^{\lambda}}\right)=1 \Rightarrow p_0\cdot e^{\lambda}=1 \Rightarrow \boxed{p_0=e^{-\lambda}}$$

$$\Rightarrow p_n=e^{-\lambda}\frac{\lambda^n}{n!} \quad n\geq 1 \rightarrow X\sim Pois(\lambda)$$

e1) $X\sim P(\lambda)$

i) $k=?$ a.î. $\mathbb{P}(X=k)$ maxim

$$\mathbb{P}(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}$$

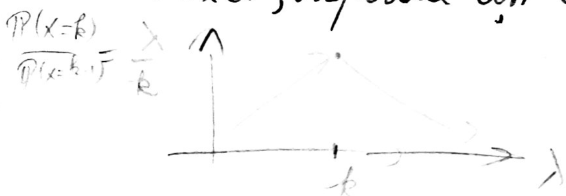
$$\frac{\mathbb{P}(X=k)}{\mathbb{P}(X=k-1)}=\frac{\lambda}{k} \ (a) \Rightarrow \begin{array}{l}\mathbb{P}(X=k)\geq\mathbb{P}(X=k-1) \text{ pt } \lambda\geq k\\ \mathbb{P}(X=k)\leq\mathbb{P}(X=k-1) \text{ pt } \lambda<k\end{array}$$

$$\Rightarrow k=[\lambda] \ (k\in\mathbb{N}) \Rightarrow \mathbb{P}(X=[\lambda])=e^{-\lambda}\frac{\lambda^{[\lambda]}}{[\lambda]!} \text{ val. maxim}$$

ii) $$\frac{\mathbb{P}(X=k)}{\mathbb{P}(X=k-1)}=\frac{\lambda}{k}$$

Pt a maximiza, știm că $\mathbb{P}(X=k)\geq\mathbb{P}(X=k-1)$ pt $\lambda\geq k$
$\mathbb{P}(X=k)\leq\mathbb{P}(X=k-1)$ pt $\lambda<k$

Deci, raportul își întâlnește maximul pt $\lambda=k$



sau

$\mathbb{P}(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}$. Pt a maxima:

$$\frac{\partial\,\mathbb{P}(X=k)}{\partial\lambda}=0 \quad \text{și} \quad \frac{\partial^2\mathbb{P}(X=k)}{\partial^2\lambda}<0$$

$$\frac{\partial\,\mathbb{P}(X=k)}{\partial\lambda}=\frac{1}{k!}\left(e^{-\lambda}k\cdot\lambda^{k-1}-\lambda^k e^{-\lambda}\right)=\frac{e^{-\lambda}}{k!}\left(k\cdot\lambda^{k-1}-\lambda^k\right)$$

$$=\frac{e^{-\lambda}}{k!}\lambda^{k-1}(k-\lambda)$$

$$\frac{\partial\,\mathbb{P}(X=k)}{\partial\lambda}=0 \Leftrightarrow e^{-\lambda}\lambda^{k-1}(k-\lambda)=0 \Rightarrow \lambda=0 \text{ sau } \lambda=k$$

$$\frac{\partial^2 \mathbb{P}(X=k)}{\partial^2 \lambda} = \frac{\partial}{\partial \lambda}\left(\frac{e^{-\lambda}}{k!}\lambda^{k-1}(k-\lambda)\right) = \frac{e^{-\lambda}}{k!}\lambda^{k-2}\left(\lambda^2-2\lambda k+k^2-k\right)$$

$$\frac{\partial^2 \mathbb{P}(X=k)}{\partial^2 \lambda} = 0 \iff \lambda^2-2\lambda k+k^2-k=0$$

$$\Delta = 4k^2-4(k^2-k) = 4k \Rightarrow \sqrt{\Delta} = 2\sqrt{k} \quad \lambda_1 = k+\sqrt{k}, \quad \lambda_2 = k-\sqrt{k}$$

λ	0	$k-\sqrt{k}$	$k+\sqrt{k}$	∞
$\lambda^2-2\lambda k+k^2-k$	$+$	$0 \; -$	$0 \; +$	

Cum $\lambda=0$ sau $\lambda=k$ și $\frac{\partial^2 \mathbb{P}(X=\lambda)}{\partial^2 \lambda} < 0 \Rightarrow \boxed{\lambda=k}$

și $\lambda \in (k-\sqrt{k};\ k+\sqrt{k})$

5) $\mathbb{P}(X=k) = \frac{(1-p)^k}{-k\log p}$ $\quad k \geq 1$

a) $E[X] = \sum_{i=0}^{n} i\cdot \mathbb{P}(X=i) = \sum_{i=1}^{n} i\mathbb{P}(X=i) = \sum_{i=1}^{n} i\,\frac{(1-p)^i}{-i\cdot\log(p)} = \frac{-1}{\log p}\left(\left(\sum_{i=0}^{n}(1-p)^i\right)-1\right)$

$= \frac{-1}{\log p}\left(\frac{1-(1-p)^{n+1}}{1-(1-p)}-1\right) = \frac{-1}{\log p}\,\frac{1-(1-p)^{n+1}-1+(1-p)}{p} = \frac{1-p+(1-p)^{n+1}}{-p\cdot\log p}$

Cum $0<p<1 \Rightarrow 1-p \in (0,1)$

$\Rightarrow \lim_{n\to\infty}\frac{1-p+(1-p)^{n+1}}{-p\log p} = \frac{1-p}{-p\log p} = \frac{p-1}{p\log p}$

b) $E[X^2] = \sum_{i=0}^{n} i\,\mathbb{P}(X=i) = \sum_{i=1}^{n} i\mathbb{P}(X=i) = \sum_{i=1}^{n} i^2\,\frac{(1-p)^i}{-i\log p}$

$= \sum_{i=1}^{n}\frac{(1-p)^i\, i}{-\log p} = \frac{1}{-\log p}\sum_{i=1}^{n}(1-p)^i\cdot i = \frac{-1}{\log p}(1-p)\sum_{i=1}^{n}(1-p)^{i-1}\cdot i$

$= \frac{-1}{\log p}(1-p)\underbrace{\sum_{i=1}^{n}\left((1-p)^i\right)'}_{S}$

$$\sum_{i=1}^{n}((1-p)^i) = (1-p)\cdot\frac{(1-p)^n-1}{1-(1-p)} = \frac{1-p}{p}[(1-p)^n-1]$$

$$\lim_{n\to\infty}\frac{1-p}{p}[(1-p)^n-1] = \frac{p-1}{p}$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}((1-p)^i)' = \frac{p-(p-1)}{p^2} = \frac{1}{p^2}$$

$$\Rightarrow E[X^2] = \lim_{n\to\infty}\frac{1}{\log p}(1-p)\cdot\frac{1}{p^2} = \frac{p-1}{p^2}\cdot\frac{1}{\log p}$$

c) $Var[X] = E[X^2] - E[X]^2$ $\log p$

$$\lim_{n\to\infty} Var[X] = \frac{p-1}{p^2}\cdot\frac{1}{\log p} - \left(\frac{p-1}{p\log p}\right)^2 = \frac{p-1}{p^2\log p} - \frac{(p-1)^2}{p^2\log p}$$

$$= \frac{p\log p - \log p - p^2+2p-1}{(p\log p)^2} = \frac{\log p(p-1)-(p-1)^2}{(p\log p)^2}$$

$$= \frac{(p-1)\log p - (p-1)^2}{(p\log p)^2} = \frac{(p-1)[\log p - (p-1)]}{(p\cdot\log p)^2}$$

6) $\mathbb{P}$(Fischer câștigă meciul) = $\cancel{\sum}$ $\mathbb{P}(\bigcup_{i=1}^{10}$ Fischer câștigă a i-a partidă și toate celelalte i-1 partide au fost remize).

evenimentele sunt independente $\sum_{i=1}^{10}\mathbb{P}(\underbrace{\text{Fischer câștigă a i-a partidă și toate celelalte sunt remize}}_{A_i})$

$$\mathbb{P}(A_i) = (0{,}3)^{i-1}\cdot 0{,}4$$

$$\Rightarrow \mathbb{P}(\text{Fischer câștigă meciul}) = \sum_{i=1}^{10}(0{,}3)^{i-1}\cdot 0{,}4 = 0{,}4\cdot\sum_{i=1}^{10}(0{,}3)^{i-1} = 0{,}4\sum_{0}^{9}(0{,}3)^i$$

$$= 0{,}4\cdot\frac{1-(0{,}3)^{10}}{1-0{,}3}$$

$$= \frac{4}{7}\cdot(1-(0{,}3)^{10}) = 0{,}5714251$$

4) Fie n ~~numărul~~ durata meciului. Discutăm 2 cazuri:
- Meciul s-a încheiat cu ~~o remiză~~ egal $\Rightarrow$ Au fost 10 remize
- Meciul ~~s-a câștigat prin~~ a fost câștigat de cineva

$\Rightarrow$ primele $n-1$ partide au fost remize, a n-a partidă fiind adjudecată de unul dintre jucători.

$$\text{Deci } \mathbb{P}(n=x) = \begin{cases} 0{,}3^{9}\cdot 1 & x=10 \text{ (Jocul are lungimea 10 dacă au fost 9 remize anterior)} \\ 0{,}3^{x-1}\cdot 0{,}7 & x=1,2,\dots 9. \end{cases}$$

(Am considerat că se joacă cel puțin o partidă și vor putea fi maxim 10)

am ales $0{,}7=0{,}3+0{,}4$ (suma probabilităților să câștige unul dintre jucători ~~la~~ partida)

7) N - nr. de mașini comandate

~~Cum mașinile au aceeași probabilitate, înseamnă că X~~

$$\text{Câștig} = a\cdot \underbrace{X}_{\text{mașini vândute}} - b\,(\underbrace{N-X}_{\text{mașini rămase nevândute}}).$$

Dar, dacă $X \geq N$, atunci câștigul este $a\cdot X$.

$$\text{Deci } G = \begin{cases} a\cdot N & X\geq N \\ a\cdot X - b\,(N-X) & X<N \end{cases}$$

$$E[G] = \sum_{i=0}^{N-1} [a i - b(N-i)]\cdot \mathbb{P}(X=i) + \sum_{i=N}^{n} a\cdot N\cdot \mathbb{P}(X=i)$$
$$= \sum_{i=0}^{N-1} [a i - b(N-i)]\cdot \mathbb{P}(X=i) + aN\cdot \mathbb{P}(X\geq N)$$

Cum mașinile au aceeași probabilitate să fie vândute $\Rightarrow$

$$\mathbb{P}(X=x) = \frac{1}{n+1} \quad (x\in\{0,1,2,\dots n\})$$

$$\Rightarrow E[G] = \sum_{i=0}^{N-1} \frac{ai-b(N-i)}{n+1} + aN\sum_{i=N}^{n}\frac{1}{n+1}$$

$$E[G] = \sum_{i=0}^{N-1} \frac{ai + bi - bN}{n+1} + aN \sum_{i=N}^{n} \frac{1}{n+1}$$

$$= \sum_{i=0}^{N-1} \frac{i(a+b) - bN}{n+1} + aN \sum_{i=N}^{n} \frac{1}{n+1}$$

$$= \frac{(a+b)(N-1)N}{2(n+1)} - \frac{bN^2}{n+1} + \frac{aN}{n+1}(n-N+1)$$

$$= \frac{(a+b)N(N-1) - 2bN^2 + 2aN(n-N+1)}{2(n+1)}$$

$$= \frac{N[(a+b)(N-1) - 2bN + 2a(n-N+1)]}{2(n+1)}$$

$$= \frac{N(aN - a + bN - b - 2bN + 2an - 2aN + 2a}{2(n+1)}$$

$$= \frac{N(-aN + a - bN - b + 2an)}{2(n+1)} = \frac{N[-N(a+b) - b + a(2n+1)]}{2(n+1)}$$

$$= \frac{N[a(2n+1) - b - N(a+b)]}{2(n+1)}$$

Comanda optimă:

În mod clar pt a obține comanda optimă, trebuie maximizat ~~numitorul~~ numărătorul

Notăm $f(N) = N[a(2n+1) - b - N(a+b)] = -N^2(a+b) + N[a(2n+1) - b]$

$f'(N) = -2N(a+b) + a(2n+1) - b$

$f'(N) = 0 \iff a(2n+1) - b = 2N(a+b) \Rightarrow N = \frac{a(2n+1) - b}{2(a+b)}$

$\cancel{\lim_{N\to\infty} f(N) = -\infty < 0}$

$\cancel{\lim_{N\to 0} f(N) = a(2n+1) - b}$

$\cancel{f''(N) = -2(a+b) < 0 \Rightarrow}$

$f'(N) = -2N(a+b) + a(2u+1) - b$

$f''(N) = -2(a+b) < 0$

N	0	$\frac{(2u+1)a-b}{2(a+b)}$	∞
f''	$-\ -\ -$	$-$	$-\ -$
f'	$++++$	0	$-\ -$
f	$\nearrow$		$\searrow$

Cum $f''(N) < 0 \Rightarrow f'$ str. descresc.

$f'\left(\frac{(2u+1)a-b}{2(a+b)}\right) = 0 \Big\} \Rightarrow f' > 0$ pe $\left(0, \frac{(2u+1)a-b}{2(a+b)}\right)$

și $f' < 0$ pe $\left(\frac{(2u+1)a-b}{2(a+b)}, \infty\right)$

$\Rightarrow$ Maximul funcției f este atins în $N = \frac{(2u+1)a-b}{2(a+b)}$

$\Rightarrow$ Comanda optimă este $N = \frac{(2u+1)a-b}{2(a+b)}$

⑧ $\mathbb{P}(X < E[X]) = ?$ $\quad X \sim B(n,p)$

$E[X] \notin \mathbb{N}$

$E[X] = 2\,Var(X)$

$X \in \{0, 1, 2, \dots n\}$

$\mathbb{P}(X=k) = C_n^k p^k (1-p)^{n-k}$

$$E[X] = \sum_{k=0}^{n} k\,\mathbb{P}(X=k) = \sum_{k=0}^{n} k \cdot C_n^k p^k (1-p)^{n-k} = \sum_{k=1}^{n} k\, C_n^k p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= n \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} = n \sum_{k=1}^{n} C_{n-1}^{k-1} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} C_{n-1}^{k-1} p^{k-1} (1-p)^{n-k} = \underline{np}$$

$$\text{Var}(x) = E[x^2] - E\{x\}^2$$

$$E\{x^2\} = \sum_{k=0}^{n} k^2 \, \mathbb{P}(x=k) = \sum_{k=0}^{n} k^2 C_n^k p^k (1-p)^{n-k} = \sum_{k=1}^{n} k^2 C_n^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k \frac{n!}{(k-1)!(n-k)!} p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} (k-1+1) \frac{n!}{(k-1)!(n-k)!} p^k \cdot (1-p)^{n-k} = \sum_{k=1}^{n} (k-1) \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k}$$

$$+ \underbrace{\sum_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!} p^k \cdot (1-p)^{n-k}}_{np}$$

$$= \sum_{k=2}^{n} (k-1) \frac{n!}{(n-k)!\underbrace{(k-1)!}_{(k-2)!}} \cdot p^k \cdot (1-p)^{n-k} + np$$

$$= \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} + np$$

$$= n(n-1) \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} + np$$

$$= n(n-1)p^2 \underbrace{\sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2} (1-p)^{n-k}}_{1} + np$$

$$= n(n-1)p^2 + np = \cancel{np(1-p)+np)}$$

$$\Rightarrow \text{Var}(x) = n(n-1)p^2 + np - (np)^2 = np[(n-1)p + 1 - np]$$

$$= np(1-p)$$

$E[X] = 2 Var(X)$

$np = 2np(1-p)$

$np[2(1-p)-1] = 0$

$np(1-2p) = 0$

Cum $E[X] = np \notin N$ $\Big\} \Rightarrow 1-2p = 0 \Rightarrow p = \frac{1}{2}$

$E[X] = \frac{n}{2} \notin N \Rightarrow n$ impar

$$\mathbb{P}(X < E[X]) = \sum_{x=0}^{[\frac{n}{2}]} \mathbb{P}(x) = \sum_{x=0}^{\frac{n-1}{2}} C_n^x p^x \cdot (1-p)^{n-x} = \sum_{x=0}^{\frac{n-1}{2}} C_n^x \left(\frac{1}{2}\right)^x \cdot \left(\frac{1}{2}\right)^{n-x}$$

$$= \left(\frac{1}{2}\right)^n \sum_{x=0}^{\frac{n-1}{2}} C_n^x = \frac{1}{2^n}\Big(\underbrace{C_n^0 + C_n^1 + C_n^2 + \dots + C_n^{\frac{n-1}{2}}}_{S}\Big)$$

$S = C_n^0 + C_n^1 + \dots + C_n^{\frac{n-1}{2}}$

Cum $C_n^m = C_n^{n-m} \Rightarrow S = C_n^n + C_n^{n-1} + \dots + C_n^{\frac{n+1}{2}}$

Știm că $C_n^{\frac{n-1}{2}}$ și $C_n^{\frac{n+1}{2}}$ (n impar) sunt termeni consecutivi în suma $C_n^0 + C_n^1 + \dots + C_n^n = 2^n$

$\Rightarrow 2S = \left(C_n^0 + C_n^1 + \dots + C_n^{\frac{n-1}{2}}\right) \cdot 2 = 2^n$

$\Rightarrow S = 2^{n-1}$

Deci $\mathbb{P}(X < E[X]) = \frac{1}{2^n} \cdot 2^{n-1} = \frac{1}{2}$