Mondache Andrei 244 8.12.2019

Tema 3

1) A_K - evenimentul că în primele k extrageri avem $n-1$ elemente de tip T extrase

B_K - evenimentul că la extragerea k am extras un element de tip T

$$\mathbb{P}(Z=K)=\mathbb{P}(A_{K-1}\cap B_K)=\mathbb{P}(A_{K-1})\cdot\mathbb{P}(B_K\mid A_{K-1})$$

$$\mathbb{P}(A_{K-1})=\mathcal{H}(N,N_1,K-1)=\frac{C_{N_1}^{n-1}C_{N-N_1}^{(K-1)-(n-1)}}{C_N^{K-1}}$$

$$\mathbb{P}(B_K\mid A_{K-1})=\frac{N_1-(n-1)}{N-(K-1)}$$

$$\Rightarrow \mathbb{P}(Z=K)=\frac{C_{N_1}^{n-1}C_{N-N_1}^{K-n}}{C_N^{K-1}}\cdot\frac{N_1-(n-1)}{N-(K-1)}=\frac{N_1!\,(N-N_1)!\,(K-1)!\,(N-K+1)!\,(N_1-n+1)}{(n-1)!\,(N_1-n+1)!\,(K-n)!\,(N-N_1-K+n)!\,N!\,(N-K+1)}$$

$$\mathbb{P}(Z=K)=\frac{N_1!\,(N-N_1)!\,(K-1)!\,(N-K)!}{N!\,(n-1)!\,(K-n)!\,(N_1-n)!\,(N-N_1-K+n)!}=\boxed{\frac{C_{K-1}^{n-1}\cdot C_{N-K}^{N_1-n}}{C_N^{N_1}}=\frac{n}{K}\,\frac{C_K^{n}\cdot C_{N-K}^{N_1-n}}{C_N^{N_1}}}$$

$$E[Z]=\sum_{K=n}^{N-N_1+n}K\cdot\frac{n}{K}\cdot\frac{C_K^{n}C_{N-K}^{N_1-n}}{C_N^{N_1}}=\frac{n}{C_N^{N_1}}\sum_{K=n}^{N-N_1+n}C_K^{n}C_{N-K}^{N_1-n}$$

(Variabila aleatoare Z ia valori între n și (sunt necesare cel puțin n extrageri pentru a extrage n obiecte de tip T) și cel mult $N-N_1+n$ (în cel mai rău caz le extragem pe toate care nu sunt T și după cele n obiecte T rămase din cele N_1))

$$Z:\begin{pmatrix} n & n+1 & \cdots & K & \cdots & N-N_1+n \\ & & & \frac{C_{K-1}^{n-1}C_{N-K}^{N_1-n}}{C_N^{N_1}} & & \end{pmatrix}$$

$$\mathbb{P}(Z=K)=\frac{C_{K-1}^{n-1}C_{N-K}^{N_1-n}}{C_N^{N_1}}\Rightarrow\boxed{\mathbb{P}(Z=K)\,C_N^{N_1}=C_{K-1}^{n-1}\cdot C_{N-K}^{N_1-n}}$$

$$E[Z]=\frac{n}{C_N^{N_1}}\sum_{K=n}^{N-N_1+n}C_K^{n}C_{N-K}^{N_1-n}$$

Aplicăm schimbarea $j\Leftarrow K+1$

$$E[Z]=\frac{m}{C_N^{N_1}}\sum_{j=m+1}^{N-N_1+n+1} C_{j-1}^{(m+1)-1}\, C_{(N+1)-j}^{(N_1+1)-(m+1)}$$

Aceasta e ~~sum~~ media unei variabile aleatoare similare cu cea din ipoteză – avem o situație cu populația de N+1 indivizi, din care N_1+1 de tip T și avem $n+1$ de astfel de indivizi care trebuie extrași (fie X variabila aleatoare în cazul acesta)

$N \leftarrow N+1$
$N_1 \leftarrow N_1+1$
$m \leftarrow n+1$

$$E[Z]=\frac{m}{C_N^{N_1}}\sum_{j=n+1}^{N-N_1+n+1} P(X=j)\, C_{N+1}^{N_1+1}$$

$$E[Z]=\frac{m}{C_N^{N_1}}\cdot C_{N+1}^{N_1+1}\underbrace{\sum_{j=n+1}^{N-N_1+n+1} P(X=j)}_{1}=\frac{m\cdot C_{N+1}^{N_1+1}}{C_N^{N_1+1}}=\boxed{\frac{m(N+1)}{N_1+1}}$$

$$Var(Z)=E[Z^2]-E[Z]^2=E[Z(Z+1)]-E[Z]-E[Z^2]$$

$$E[Z(Z+1)]=\sum_{k=n}^{N-N_1+n} k(k+1)\frac{C_{k-1}^{m-1}C_{N-k}^{N_1-n}}{C_N^{N_1}}=\sum_{k=n}^{N-N_1+n} k(k+1)\cdot\frac{m(n+1)}{k(k+1)}\,\frac{C_{k+1}^{m+1}C_{N-k}^{N_1-n}}{C_N^{N_1}}$$

$$=\sum_{k=n}^{N-N_1+n} m(n+1)\frac{C_{k+1}^{n+1}C_{N-k}^{N_1-n}}{C_{N_1}^{N_1}}$$

$j \leftarrow k+2$

$$E[Z(Z+1)]=\sum_{j=m+2}^{N-N_1+n+2} m(m+1)\frac{C_{j-1}^{m+1}\,C_{(N+2)-j}^{N_1-n}}{C_N^{N_1}}=\frac{m(m+1)}{C_N^{N_1}}\underbrace{\sum_{j=m+2}^{N-N_1+2} C_{j-1}^{m+1}\,C_{(N+2)-j}^{N_1-n}}_{S}$$

Notăm $\begin{cases} N_1' = N+2 \\ N_1' = N_1+2 \\ m' = m+2 \end{cases}$

$$S=\sum_{j=m'}^{N'-N_1'+m'} C_{j-1}^{m'-1}\, C_{N'-j}^{N_1'-j}$$

Similar, obținem $S=C_{N'}^{N_1'}=C_{N+2}^{N_1+2}$

Deci $E[Z(Z+1)]=\dfrac{m(m+1)}{C_N^{N_1}}\cdot C_{N+2}^{N_1+2}=\dfrac{m(m+1)(N+1)(N+2)}{(N_1+1)(N_1+2)}$

$$Var(Z) = E[Z(Z+1)] - E[Z] - E[Z]^2$$

$$Var(Z) = \frac{\overset{N_1+1}{} m(m+1)(N+1)(N+2)}{(N_1+1)(N_1+2)} - \frac{\overset{(N_1+1)(N_1+2)}{} m(N+1)}{N_1+1} - \overset{N_1+2}{}\left(\frac{m(N+1)}{(N_1+1)}\right)^2$$

$$= \frac{(N_1+1)m(m+1)(N+1)(N+2) - m(N+1)(N_1+1)(N_1+2) - m^2(N+1)^2(N_1+2)}{(N_1+1)^2(N_1+2)}$$

$$= \frac{m(N+1)\left[(m+1)(N+2)(N_1+1) - (N_1+1)(N_1+2) - m(N+1)(N_1+2)\right]}{(N_1+1)^2(N_1+2)}$$

$$= \frac{m(N+1)\left(\cancel{mNN_1} + mN + 2N_1m + \cancel{2m} + NN_1 + N + 2N_1 \cancel{+2} - N_1^2 - 3N_1 \cancel{-2} - \cancel{mNN_1} - 2mN - mN_1 - \cancel{2m}\right)}{(N_1+1)^2(N_1+2)}$$

$$= \frac{m(N+1)(-mN + mN_1 + NN_1 + N - N_1 - N_1^2)}{(N_1+1)^2(N_1+2)}$$

$$= \frac{m(N+1)\left[N_1(m-1-N_1) + N(-m+N_1+1)\right]}{(N_1+1)^2(N_1+2)} = \frac{m(N+1)(N_1-N)(m-1-N_1)}{(N_1+1)^2(N_1+2)}$$

$$Var(Z) = \frac{m(N+1)(N_1-m)(m-1-N_1)}{(N_1+1)^2(N_1+2)}$$

2) $X \sim Pois(\lambda)$
$Y \sim Pois(\mu)$
$X \perp\!\!\!\perp Y$

$\mathbb{P}(X \mid X+Y=n) = ?$

$$P_{X \mid X+Y=n}(k) = \frac{\mathbb{P}(X=k,\ X+Y=n)}{\mathbb{P}(X+Y=n)} = \frac{\mathbb{P}(X=k,\ Y=n-X)}{\mathbb{P}(X+Y=n)}$$

$$= \frac{\mathbb{P}(X=k,\ Y=n-k)}{\mathbb{P}(X+Y=n)} \overset{X \perp\!\!\!\perp Y}{=} \frac{\mathbb{P}(X=k)\cdot\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)}$$

Ştim că dacă $X \perp\!\!\!\perp Y$ $X \sim Pois(\lambda)$ $Y \sim Pois(\mu) \Rightarrow X+Y \sim Pois(\lambda+\mu)$

$$\Rightarrow \mathbb{P}_{X \mid X+Y=n}(k) = \frac{e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{e^{-\lambda_1} \cdot e^{-\lambda_2}}{e^{-(\lambda_1+\lambda_2)}} \cdot \frac{\lambda_1^k}{(\lambda_1+\lambda_2)^k} \cdot \frac{\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^{n-k}}$$

$$= C_n^k \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$\Rightarrow X \mid X+Y=n \sim B\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$X \mid X+Y=n \sim \begin{pmatrix} 0 & 1 & & n \\ C_n^0\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^n & C_n^1\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-1} & \cdots & C_n^n\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^n \end{pmatrix}$$

3) X_i - variabilă aleatoare ce reprezintă suma cheltuită de clientul i

$E[X_i]=30$

N - numărul de clienți

$E[N]=50$

$X_i \perp\!\!\!\perp X_j$

$X_i \perp\!\!\!\perp N$

$C=\sum_{i=1}^{N} X_i$ - cifra de afaceri

$$E[C]=E\left[\sum_{i=1}^{N}X_i\right]=E\left[E\left[\sum_{i=1}^{N}X_i\mid N\right]\right]=\cancel{\sum_{n\geq 1}E\left[\sum_{i=1}^{N}X_i\mid N=n\right]}\cdot\mathbb{P}(N=n)$$

$$=\sum_{n\geq 1}E\left[\sum_{i=1}^{n}X_i\mid N=n\right]\mathbb{P}(N=n)=\sum_{n\geq 1}\left(\sum_{i=1}^{n}E[X_i\mid N=n]\right)\mathbb{P}(N=n)$$

$$\overset{\text{IND}}{=}\sum_{n\geq 1}n\,E[X_n]\,\mathbb{P}(N=n)=E[X_n]\underbrace{\sum_{n\geq 1}n\,\mathbb{P}(N=n)}_{E[N]}$$

$$=E[X_n]\,E[N]=30\cdot 50=1500.$$

Cifra de afaceri este 1500.

4) N_1 - numărul de teste necesare pentru identificarea primului rezistor

N_2 - numărul de teste suplimentare necesare pentru identificarea celui de-al doilea rezistor

Știm că $2 \le N_1 + N_2 \le 4$

Fie $\mathbb{P}((T_i, T_j))$ probabilitatea că cei 2 rezistori defecți se găsesc la a i-a încercare, respectiv a j-a.

~~$\mathbb{P}(N_1 = 1, N_2 = 1) = \mathbb{P}((T_1, T_2)) =$~~

$\mathbb{P}((T_i, T_j)) = \frac{1}{C_5^2} = \frac{1}{10}$ (presupunem că tranzistorii au aceeași șansă să fie defecți).

$\mathbb{P}(N_1 = 1, N_2 = 1) = \mathbb{P}((T_1, T_2)) = \frac{1}{10}$

$\mathbb{P}(N_1 = 1, N_2 = 2) = \mathbb{P}((T_1, T_3)) = \frac{1}{10}$

$\mathbb{P}(N_1 = 1, N_2 = 3) = \mathbb{P}((T_1, T_4) \cup (T_1, T_5)) = \frac{1}{10} + \frac{1}{10} = \frac{2}{10} = \frac{1}{5}$

(dacă la a 3-a încercare suplimentară nu am găsit ~~un~~ tranzistorul defect, el sigur este singurul neîncercat, adică al 5-lea)

$\mathbb{P}(N_1 = 2, N_1 = 1) = \mathbb{P}((T_2, T_3)) = \frac{1}{10}$

$\mathbb{P}(N_1 = 2, N_2 = 2) = \mathbb{P}((T_2, T_4) \cup (T_2, T_5)) = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$

(Același raționament ca la $\mathbb{P}(N_1 = 1, N_2 = 3)$)

$\mathbb{P}(N_1 = 3, N_2 = 1) = \mathbb{P}((T_3, T_4) \cup (T_3, T_5)) = \frac{1}{5}$

$\mathbb{P}(N_1 = 3, N_2 = 0) = \mathbb{P}((T_4, T_5)) = \frac{1}{10}$

(Dacă în primele 3 încercări nu am găsit niciun rezistor, cu siguranță rezistorii defecți sunt cei 2 rămași, deci nu mai este nevoie de niciun test suplimentar)

Restul : $\mathbb{P}((N_1=1, N_2=0))=0$, $\mathbb{P}((N_1=2, N_2=0))=0$, $\mathbb{P}((N_1=3, N_2=2))=0$,
$\mathbb{P}((N_1=3, N_2=3))=0$, $\mathbb{P}((N_1=2; N_2=3))=0$.

$N_1 \backslash N_2$	0	1	2	3	
1	0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{2}{5}$
2	0	$\frac{1}{10}$	$\frac{1}{5}$	0	$\frac{3}{10}$
3	$\frac{1}{10}$	$\frac{1}{5}$	0	0	$\frac{3}{10}$
	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{5}$	

Ştim că $N_1 \sim \begin{pmatrix} k \\ \sum_{i=0}^{3} \pi_{ki} \end{pmatrix}$ $N_2 \sim \begin{pmatrix} k \\ \sum_{i=1}^{3} \pi_{ik} \end{pmatrix}$

$\Rightarrow N_1 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{2}{5} & \frac{3}{10} & \frac{3}{10} \end{pmatrix}$

$\Rightarrow N_2 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{10} & \frac{2}{5} & \frac{3}{10} & \frac{1}{5} \end{pmatrix}$

$E[N_1] = 1\cdot\frac{2}{5} + 2\cdot\frac{3}{10} + 3\cdot\frac{3}{10} = \frac{19}{10}$

$E[N_2] = 0\cdot\frac{1}{10} + 1\cdot\frac{2}{5} + 2\cdot\frac{3}{10} + 3\cdot\frac{1}{5} = \frac{16}{10}$

5)

X|Y	1	2	3
1	0,22	0,11	0,02
2	0,2	0,15	0,1
3	0,06	0,07	0,07

a) $X: \begin{pmatrix} 1 & 2 & 3 \\ 0{,}22+0{,}11+0{,}02 & 0{,}2+0{,}15+0{,}1 & 0{,}06+0{,}07+0{,}07 \end{pmatrix} \Rightarrow X: \begin{pmatrix} 1 & 2 & 3 \\ 0{,}35 & 0{,}45 & 0{,}2 \end{pmatrix}$

$Y: \begin{pmatrix} 1 & 2 & 3 \\ 0{,}22+0{,}2+0{,}06 & 0{,}11+0{,}15+0{,}07 & 0{,}02+0{,}1+0{,}07 \end{pmatrix} \Rightarrow Y: \begin{pmatrix} 1 & 2 & 3 \\ 0{,}48 & 0{,}33 & 0{,}19 \end{pmatrix}$

b) $E[X] = 1 \cdot 0{,}35 + 2 \cdot 0{,}45 + 3 \cdot 0{,}2 = 1{,}85$

$E[Y] = 1 \cdot 0{,}48 + 2 \cdot 0{,}33 + 3 \cdot 0{,}19 = 1{,}71$

$X^2: \begin{pmatrix} 1^2 & 2^2 & 3^2 \\ 0{,}35 & 0{,}45 & 0{,}2 \end{pmatrix} \quad X^2: \begin{pmatrix} 1 & 4 & 9 \\ 0{,}35 & 0{,}45 & 0{,}2 \end{pmatrix}$

$Y^2: \begin{pmatrix} 1 & 4 & 9 \\ 0{,}48 & 0{,}33 & 0{,}19 \end{pmatrix}$

$E[X^2] = 1 \cdot 0{,}35 + 4 \cdot 0{,}45 + 9 \cdot 0{,}2 = 3{,}95$

$E[Y^2] = 1 \cdot 0{,}48 + 4 \cdot 0{,}33 + 9 \cdot 0{,}19 = 3{,}51$

$Var(X) = E[X^2] - E[X]^2 = 3{,}95 - 1{,}85^2 = 0{,}5275$

$Var(Y) = E[Y^2] - E[Y]^2 = 3{,}51 - 1{,}71^2 = 0{,}5859$

c) $\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$

$cov(X,Y) = E[XY] - E[X]E[Y]$

$E[XY] = 1 \cdot 1 \cdot 0{,}22 + 1 \cdot 2 \cdot 0{,}11 + 1 \cdot 3 \cdot 0{,}02 + 2 \cdot 1 \cdot 0{,}2 + 2 \cdot 2 \cdot 0{,}15 + 2 \cdot 3 \cdot 0{,}1 +$
$3 \cdot 1 \cdot 0{,}06 + 3 \cdot 2 \cdot 0{,}07 + 3 \cdot 3 \cdot 0{,}07 = 3{,}33$

$cov(x,y) = 3{,}33 - 1{,}85 \cdot 1{,}71 = 0{,}1665$

$$\rho(x,y) = \frac{0{,}1665}{\sqrt{0{,}5275 \cdot 0{,}5859}} = 0{,}2994$$

d) ~~Şirul cu $X|Y=y_j : \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ f_{X|Y}(x_1|y_j) & & \dots & f_{X|Y}(x_n|y_j) \end{pmatrix}$~~

Şirul cu $X|Y=y_j : \begin{pmatrix} x_i \\ p(x_i|y_j) \end{pmatrix}$ $\quad P(x_i|y_j) = \dfrac{\mathbb{P}(X=x_i, Y=y_j)}{\mathbb{P}(Y=y_j)}$

deci $X|Y=2 : \begin{pmatrix} 1 & 2 & 3 \\ \frac{0{,}11}{0{,}33} & \frac{0{,}15}{0{,}33} & \frac{0{,}07}{0{,}33} \end{pmatrix}$

$Y|X=2 : \begin{pmatrix} 1 & 2 & 3 \\ \frac{0{,}2}{0{,}45} & \frac{0{,}15}{0{,}45} & \frac{0{,}1}{0{,}45} \end{pmatrix}$

e) $E[X|Y=2] = 1 \cdot \dfrac{0{,}11}{0{,}33} + \dfrac{2 \cdot 0{,}15}{0{,}33} + \dfrac{3 \cdot 0{,}07}{0{,}33} = \dfrac{0{,}62}{0{,}33}$

$E[Y|X=2] = \dfrac{1 \cdot 0{,}2}{0{,}45} + \dfrac{2 \cdot 0{,}15}{0{,}45} + \dfrac{3 \cdot 0{,}1}{0{,}45} = \dfrac{0{,}8}{0{,}45}$

$X^2|Y=2 : \begin{pmatrix} 1 & 4 & 9 \\ \frac{0{,}11}{0{,}33} & \frac{0{,}15}{0{,}33} & \frac{0{,}07}{0{,}33} \end{pmatrix}$ $\quad Y^2|X=2 : \begin{pmatrix} 1 & 4 & 9 \\ \frac{0{,}2}{0{,}45} & \frac{0{,}15}{0{,}45} & \frac{0{,}1}{0{,}45} \end{pmatrix}$

$E[X^2|Y=2] = \dfrac{1}{0{,}33}(1 \cdot 0{,}11 + 4 \cdot 0{,}15 + 9 \cdot 0{,}07) = \dfrac{1{,}34}{0{,}33}$

$E[Y^2|X=2] = \dfrac{1}{0{,}45}(0{,}2 \cdot 1 + 0{,}15 \cdot 4 + 9 \cdot 0{,}1) = \dfrac{1{,}7}{0{,}45}$

$Var(X|Y=2) = E[X^2|Y=2] - E[X|Y=2]^2 = \dfrac{1{,}34}{0{,}33} - \left(\dfrac{0{,}62}{0{,}33}\right)^2 = \dfrac{0{,}0578}{0{,}33^2} = 0{,}5307$

$Var(Y|X=2) = E[Y^2|X=2] - E[Y|X=2]^2 = \dfrac{1{,}7}{0{,}45} - \left(\dfrac{0{,}8}{0{,}45}\right)^2 = 0{,}6172$

6)

X/Y	2	4	6
0	0,1	0,2	0,1
1	0,1	0,1	0,1
2	0,1	0,1	0
3	0,05	0	0,05

a) $Y: \begin{pmatrix} 2 & 4 & 6 \\ 0{,}35 & 0{,}4 & 0{,}25 \end{pmatrix}$

$0{,}35 = 0{,}1+0{,}1+0{,}1+0{,}05$; $0{,}4 = 0{,}2+0{,}1+0{,}1$; $0{,}25 = 0{,}1+0{,}1+0{,}05$

$E[Y] = 2\cdot 0{,}35 + 4\cdot 0{,}4 + 6\cdot 0{,}25 = 3{,}8$

$Y^2: \begin{pmatrix} 2^2 & 4^2 & 6^2 \\ 0{,}35 & 0{,}4 & 0{,}25 \end{pmatrix} \Rightarrow E[Y^2] = 4\cdot 0{,}35 + 16\cdot 0{,}4 + 36\cdot 0{,}25 = 16{,}8$

$Var(y) = E[Y^2] - E[Y]^2 = 16{,}8 - 3{,}8^2 = 2{,}36$

b) $E[Y|X] = ?$

~~$Y|X=0: \begin{pmatrix} 2 & 4 & 6 \\ 0{,}1 \end{pmatrix}$~~

$X: \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0{,}4 & 0{,}3 & 0{,}2 & 0{,}1 \end{pmatrix}$

Ştim că $X|Y=y_j : \begin{pmatrix} x_i \\ P(x_i/y_j) \end{pmatrix}$ $\quad P(x_i/y_j) = \dfrac{P(X=x_i, Y=y_j)}{P(Y=y_j)}$

$\Rightarrow Y|X=0: \begin{pmatrix} 2 & 4 & 6 \\ \frac{0{,}1}{0{,}4} & \frac{0{,}2}{0{,}4} & \frac{0{,}1}{0{,}4} \end{pmatrix}$ $\quad Y|X=1: \begin{pmatrix} 2 & 4 & 6 \\ \frac{0{,}1}{0{,}3} & \frac{0{,}1}{0{,}3} & \frac{0{,}1}{0{,}3} \end{pmatrix}$

$Y|X=2: \begin{pmatrix} 2 & 4 & 6 \\ \frac{0{,}1}{0{,}2} & \frac{0{,}1}{0{,}2} & 0 \end{pmatrix}$ $\quad Y|X=3: \begin{pmatrix} 2 & 4 & 6 \\ \frac{0{,}05}{0{,}1} & 0 & \frac{0{,}05}{0{,}1} \end{pmatrix}$

$$E[Y/X=0] = \frac{2\cdot 0{,}1 + 4\cdot 0{,}2 + 6\cdot 0{,}1}{0{,}4} = 4$$

$$E[Y|X=1] = \frac{2\cdot 0{,}1 + 4\cdot 0{,}1 + 6\cdot 0{,}1}{0{,}3} = 4$$

$$E[Y|X=2] = \frac{2\cdot 0{,}1 + 4\cdot 0{,}1 + 6\cdot 0}{0{,}2} = 3$$

$$E[Y|X=3] = \frac{2\cdot 0{,}05 + 4\cdot 0 + 6\cdot 0{,}05}{0{,}1} = 4$$

$$\Rightarrow E[Y|X]: \begin{pmatrix} 3 & 4 \\ 0{,}2 & 0{,}8 \end{pmatrix}$$

$E[Y|X] = 3$ cu probabilitatea $\mathbb{P}(X=2)$

$E[Y|X] = 4$ cu probabilitatea $\mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=3)$

$$Var(Y|X) = E[Y^2|X] - E[Y|X]^2$$

$$Y^2|X=0: \begin{pmatrix} 4 & 16 & 36 \\ \frac{0{,}1}{0{,}4} & \frac{0{,}2}{0{,}4} & \frac{0{,}1}{0{,}4} \end{pmatrix} \qquad Y|X=1: \begin{pmatrix} 4 & 16 & 36 \\ \frac{0{,}1}{0{,}3} & \frac{0{,}1}{0{,}3} & \frac{0{,}1}{0{,}3} \end{pmatrix}$$

$$Y^2|X=2: \begin{pmatrix} 4 & 16 & 36 \\ \frac{0{,}1}{0{,}2} & \frac{0{,}1}{0{,}2} & 0 \end{pmatrix} \qquad Y^2|X=3: \begin{pmatrix} 4 & 16 & 36 \\ \frac{0{,}05}{0{,}1} & 0 & \frac{0{,}05}{0{,}1} \end{pmatrix}$$

$$E[Y^2|X=0] = \frac{4\cdot 0{,}1 + 16\cdot 0{,}2 + 36\cdot 0{,}1}{0{,}4} = \frac{7{,}2}{0{,}4} = 18$$

$$E[Y^2/X=1] = \frac{4\cdot 0{,}1 + 16\cdot 0{,}1 + 36\cdot 0{,}1}{0{,}3} = 18{,}66$$

$$E[Y^2/X=2] = \frac{4\cdot 0{,}1 + 16\cdot 0{,}1 + 36\cdot 0}{0{,}2} = 10$$

$$E[Y^2|X=3] = \frac{4\cdot 0{,}05 + 16\cdot 0 + 36\cdot 0{,}05}{0{,}1} = 20$$

$$\Rightarrow Var(Y/X=0) = E[Y^2|X=0] - E[Y|X=0]^2 = 18 - 4^2 = 2$$

$$Var(Y|X=1) = E[Y^2|X=1] - E[Y/X=1]^2 = 18{,}66 - 4^2 = 2{,}66$$

$Var(Y|X=2) = E\{Y^2|X=2\} - E\{Y|X=2\}^2 = 10 - 3^2 = 1$

$Var(Y|X=3) = E\{Y^2|X=3\} - E\{Y|X=3\}^2 = 20 - 4^2 = 4$

Obum $Var(Y|X) = 2$ cu probabilitatea $\mathbb{P}(X=0)$

$Var(Y|X) = 2{,}66$ cu probabilitatea $\mathbb{P}(X=1)$

$Var(Y|X) = 1$ cu probabilitatea $\mathbb{P}(X=2)$

$Var(Y|X) = 4$ cu probabilitatea $\mathbb{P}(X=3)$

$\Rightarrow Var(Y|X) : \begin{pmatrix} 1 & 2 & 2{,}66 & 4 \\ 0{,}2 & 0{,}4 & 0{,}3 & 0{,}1 \end{pmatrix}$

c) $Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$

$E\{Var(Y|X)\} = 1 \cdot 0{,}2 + 2 \cdot 0{,}4 + 2{,}66 \cdot 0{,}3 + 4 \cdot 0{,}1 = 2{,}2$

$E[Y|X] : \begin{pmatrix} 3 & 4 \\ 0{,}2 & 0{,}8 \end{pmatrix} \Rightarrow E[E[Y|X]] = 0{,}6 + 3{,}2 = 3{,}8$

$E[Y^2|X] : \begin{pmatrix} 9 & 16 \\ 0{,}2 & 0{,}8 \end{pmatrix}$

$\Rightarrow Var(E[Y|X]) = E[E[Y|X]^2] - E[E[Y|X]]^2$

$E[E[Y|X]^2] = 9 \cdot 0{,}2 + 16 \cdot 0{,}8 = 1{,}8 + 12{,}8 = 14{,}6$

$E[E[Y|X]] = 3 \cdot 0{,}2 + 4 \cdot 0{,}8 = 3{,}8$

$\Rightarrow Var(E[Y|X]) = 14{,}6 - 3{,}8^2 = 0{,}16$

$Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$

$2{,}36 = 2{,}2 + 0{,}16$ A

$\Rightarrow Var(Y) = E\{Var(Y|X)\} + Var(E[Y|X])$

În plus:

$$Var(Y|X) = E[Y^2|X] - E[Y|X]^2$$

$$E[Var(Y|X)] = E[E[Y^2|X] - E[Y|X]^2]$$

$$E[Var(Y|X)] = E[E[Y^2|X]] - E[E[Y|X]^2]$$

Știm că $E[E[Y|X]] = E[Y]$

$$\Rightarrow E[Var(Y|X)] = E[Y^2] - E[E[Y|X]^2] \quad (1)$$

$$Var(E[Y|X]) = E[E[Y|X]^2] - E[\underbrace{E[Y|X]}]^2$$

$$Var(E[Y|X]) = E[E[Y|X]^2] - E[Y]^2 \quad (2) \qquad E[Y]^2$$

$$Var(Y) = E[Var(Y|X)] + Var(E[Y|X]) \overset{(1),(2)}{\Longrightarrow}$$

$$Var(Y) = E[Y^2] - E[Y]^2 \quad \text{A}$$

Deci este adevărată afirmația $Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$