Grupa 344, EDDP, Seminar 10, 14.12.2020

Ecuații liniare neomogene (afine) de ordin n

$$x^{(n)} = \sum_{k=0}^{n-1} a_k(t)\, x^{(k)} + g(t)$$

$$a_0, \dots, a_{n-1}, g : I \subset \mathbb{R} \to \mathbb{R}.$$

I) Să se determine soluțiile generale ale următoarelor ecuații:

✓ 1) $x''' - x'' + x' - x = e^{2t}$, $t \in \mathbb{R}$

✓ 2) $x'' - 2x' + x = 2t e^{t}$, $t \in \mathbb{R}$

3) $t^3 x''' + t x' - x = t^2$, $t > 0$

4) $(2t+3)^3 x''' + 4(2t+3)^2 x'' + 4(2t+3) x' - 8x = 8(2t+3)^2$, $t > -\frac{3}{2}$.

✓ 5) $t^3 x'' - 2t x = 3 \ln t$, $t > 0$

1) $x''' - x'' + x' - x = e^{2t}$

$a_2 = 1$, $a_1 = -1$, $a_0 = 1$ $\Rightarrow$ ec. are coef. constanți

$g : \mathbb{R} \to \mathbb{R}$

$g(t) = e^{2t}$.

• se scrie ec. liniară omogenă (cu coef. constanți)

$$\bar{x}''' - \bar{x}'' + \bar{x}' - \bar{x} = 0.$$

fiind cu coef constanți se rezolvă prin determinarea unui sistem fundamental de soluții folosind rădăcinile ec. caracteristice:

$$\underbrace{r^3 - r^2}_{} + r - 1 = 0$$

$$r^0.$$

$$r^2(r-1) + (r-1) = 0 \Rightarrow (r-1)(r^2+1) = 0 \Rightarrow$$

$$\Rightarrow \begin{cases} r_1 = 1, & m_1 = 1 \Rightarrow \boxed{\varphi_1(t) = e^{t}} \\ r_2 = i, & m_2 = 1 \\ r_3 = -i = \bar{r}_2, & m_3 = 1 \end{cases}$$

$$\Rightarrow \begin{array}{l} \varphi_2(t) = \mathrm{Re}(e^{r_2 t}) \\ \varphi_3(t) = \mathrm{Im}(e^{r_2 t}) \end{array}$$

$$e^{r_2 t} = e^{it} = \cos t + i \sin t \Rightarrow \boxed{\begin{array}{l} \varphi_2(t) = \cos t \\ \varphi_3(t) = \sin t \end{array}}$$

Deci: $\overline{x}(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + C_3\varphi_3(t)$.

$C_1, C_2, C_3 \in \mathbb{R}$.

• se aplică metoda variației constantelor:

determinăm $C_1, C_2, C_3 : \mathbb{R} \to \mathbb{R}$ a.î.

$$x(t) = C_1(t)\varphi_1(t) + C_2(t)\varphi_2(t) + C_3(t)\varphi_3(t)$$

sol. a ec. afine.

Știu din teorie că C_1', C_2', C_3' verifică sistemul algebric liniar:

$$\left\{\begin{array}{l} C_1'\varphi_1(t) + C_2'\varphi_2(t) + C_3'\varphi_3(t) = 0 \\ C_1'\varphi_1'(t) + C_2'\varphi_2'(t) + C_3'\varphi_3'(t) = 0 \\ C_1'\varphi_1''(t) + C_2'\varphi_2''(t) + C_3'\varphi_3''(t) = e^{2t} \end{array}\right. \Rightarrow$$

$\varphi_1(t) = e^t \Rightarrow \varphi_1'(t) = e^t$; $\varphi_1''(t) = e^t$

$\varphi_2(t) = \cos t \Rightarrow \varphi_2'(t) = -\sin t$; $\varphi_2''(t) = -\cos t$

$\varphi_3(t) = \sin t \Rightarrow \varphi_3'(t) = \cos t$; $\varphi_3''(t) = -\sin t$

$$\Rightarrow \left\{\begin{array}{l} C_1'e^t + C_2'\cos t + C_3'\sin t = 0 \\ C_1'e^t - C_2'\sin t + C_3'\cos t = 0 \\ C_1'e^t - C_2'\cos t - C_3'\sin t = e^{2t} \end{array}\right. \xrightarrow[(+)]{} 2C_1'e^t = e^{2t} \Rightarrow$$

$$\Rightarrow \boxed{C_1' = \frac{1}{2}e^t}$$

ec.3 − ec1 $\Rightarrow \left\{\begin{array}{ll} -2C_2'\cos t - 2C_3'\sin t = e^{2t} & \mid \cdot \cos t \\ -C_2'\sin t + C_3'\cos t = -\frac{1}{2}e^{2t} & \mid \cdot 2\sin t \end{array}\right.$ $\left(\begin{array}{l} \mid \cdot \sin t \\ \mid \cdot 2\cos t \end{array}\right)$

$$\Rightarrow \left\{\begin{array}{l} -2C_2'\cos^2 t - 2C_3'\sin t\cos t = e^{2t}\cos t \\ -2C_2'\sin^2 t + 2C_3'\cos t\sin t = -e^{2t}\sin t \end{array}\right. \quad (+)$$

$$-2C_2'\underbrace{(\sin^2 t + \cos^2 t)}_{1} = e^{2t}(\cos t - \sin t) \Rightarrow$$

$$\Rightarrow \boxed{C_2' = \frac{1}{2}e^{2t}\sin t - \frac{1}{2}e^{2t}\cos t}$$

$$\left\{\begin{array}{l} +2C_2'\cos t\sin t + 2C_3'\sin^2 t = -e^{2t}\sin t \\ -2C_2'\sin t\cos t + 2C_3'\cos^2 t = -e^{2t}\cos t \end{array}\right. \quad (+)$$

$$2C_3'(\cos^2 t + \sin^2 t) = -e^{2t}\sin t - e^{2t}\cos t \Rightarrow \boxed{C_3' = -\frac{1}{2}e^{2t}\sin t - \frac{1}{2}e^{2t}\cos t}$$

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$$C_1' = \frac{1}{2}e^t \Rightarrow \boxed{C_1(t) = \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t + K_1}$$

$$C_2(t) = \frac{1}{2}\underbrace{\int e^{2t}\sin t\,dt}_{I_1} - \frac{1}{2}\underbrace{\int e^{2t}\cos t\,dt}_{I_2}$$

$$I_1 = \int e^{2t}\sin t\,dt = \int e^{2t}(-\cos t)'\,dt = -e^{2t}\cos t + \int (e^{2t})'\cos t\,dt =$$

$$= -e^{2t}\cos t + 2\int e^{2t}(\sin t)'\,dt =$$

$$= -e^{2t}\cos t + 2\Big(e^{2t}\sin t - 2\underbrace{\int e^{2t}\sin t\,dt}_{I_1}\Big) \Rightarrow$$

$$\Rightarrow I_1 = -e^{2t}\cos t + 2e^{2t}\sin t - 4I_1 \Rightarrow$$

$$\Rightarrow \boxed{I_1 = -\frac{1}{5}e^{2t}\cos t + \frac{2}{5}e^{2t}\sin t + \mathcal{C}}$$

$$I_2 = \int e^{2t}\cos t\,dt = \int e^{2t}(\sin t)'\,dt = e^{2t}\sin t - \int (e^{2t})'\sin t\,dt =$$

$$= e^{2t}\sin t - 2\underbrace{\int e^{2t}\sin t\,dt} = e^{2t}\sin t - 2\Big(e^{2t}(-\cos t) -$$

$$- 2\int e^{2t}(-\cos t)\,dt\Big] \Rightarrow$$

$$\Rightarrow I_2 = e^{2t}\sin t + 2e^{2t}\cos t - 4I_2 \Rightarrow$$

$$\Rightarrow \boxed{I_2 = \frac{1}{5}e^{2t}\sin t + \frac{2}{5}e^{2t}\cos t + \mathcal{C}.}$$

$$C_2(t) = \frac{1}{2}\Big(-\frac{1}{5}e^{2t}\cos t + \frac{2}{5}e^{2t}\sin t\Big) -$$

$$- \frac{1}{2}\Big(\frac{1}{5}e^{2t}\sin t + \frac{2}{5}e^{2t}\cos t\Big) + K_2 \Rightarrow$$

$$\Rightarrow \boxed{C_2(t) = -\frac{3}{10}e^{2t}\cos t + \frac{1}{10}e^{2t}\sin t + K_2}$$

$$C_3(t) = -\frac{1}{2}I_1 - \frac{1}{2}I_2 = -\frac{1}{2}\Big(-\frac{1}{5}e^{2t}\cos t + \frac{2}{5}e^{2t}\sin t\Big) -$$

$$- \frac{1}{2}\Big(\frac{1}{5}e^{2t}\sin t + \frac{2}{5}e^{2t}\cos t\Big) + K_3$$

$$\boxed{C_3(t) = -\frac{1}{10}e^{2t}\cos t - \frac{3}{10}e^{2t}\sin t + K_3}$$

Deci, sol. ec. afine este:

$$x(t) = \Big(\frac{1}{2}e^t + K_1\Big)e^t + \Big(-\frac{3}{10}e^{2t}\cos t + \frac{1}{10}e^{2t}\sin t + K_2\Big)\cos t +$$

$$+ \Big(-\frac{1}{10}e^{2t}\cos t - \frac{3}{10}e^{2t}\sin t + K_3\Big)\sin t \Rightarrow$$

$$\Rightarrow x(t) = \underbrace{K_1e^t + K_2\cos t + K_3\sin t} + \Big(\frac{1}{2}e^{2t} - \frac{3}{10}e^{2t}\cos^2 t +$$

$+ \cancel{\frac{1}{10}e^{2t}\sin t\cos t} - \cancel{\frac{1}{10}e^{2t}\cos t\sin t} - \frac{3}{10}e^{2t}\sin^2 t) \Rightarrow$

$\Rightarrow x(t) = \bar{x}(t) + \varphi_0(t)$

$\boxed{\varphi_0(t) = \frac{1}{2}e^{2t} - \frac{3}{10}e^{2t}\underbrace{(\sin^2 t + \cos^2 t)}_{1} = \frac{1}{5}e^{2t}}$

$\hookrightarrow$ sol. particulară.

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(2) $x'' - 2x' + x = 2te^t$, $t \in \mathbb{R}$.

$\boxed{n=2}$; $\boxed{x'' = a_1 x' + a_0 x + g(t)}$

$a_1 = 2$; $a_0 = -1$; $g(t) = 2te^t$

$g: \mathbb{R} \to \mathbb{R}$

• ec. liniară omogenă atașată:

$\bar{x}'' - 2\bar{x}' + \bar{x} = 0$ $\boxed{\bar{x} = x^{(0)}}$

• ec. caracteristică $r^2 - 2r + 1 = 0 \Rightarrow$

$\Rightarrow (r-1)^2 = 0 \Rightarrow \boxed{r_1 = 1,\ m_1 = 2} \Rightarrow \begin{cases} \varphi_1(t) = e^t \\ \varphi_2(t) = te^t. \end{cases}$

$\Rightarrow \bar{x}(t) = C_1\varphi_1(t) + C_2\varphi_2(t).$

• aplicăm metoda variației constantelor:

determinăm $C_1, C_2 : \mathbb{R} \to \mathbb{R}$ a.î.

$x(t) = C_1(t)\varphi_1(t) + C_2(t)\varphi_2(t).$

să fie sol. ec. afine.

Știm că C_1', C_2' sunt sol. sistem algebric liniar:

$\begin{cases} C_1'\varphi_1(t) + C_2'\varphi_2(t) = 0 \\ C_1'\varphi_1'(t) + C_2'\varphi_2'(t) = 2te^t \end{cases} \Rightarrow \begin{cases} C_1'e^t + C_2'te^t = 0 \\ C_1'e^t + C_2'(1+t)e^t = 2te^t \end{cases}$

$\varphi_1'(t) = e^t$

$\varphi_2'(t) = (1+t)e^t$

$\Downarrow (-)$ ec 2 − ec 1

$C_2'e^t = 2te^t \,/ : e^t$

$\Rightarrow \boxed{C_2' = 2t}$

$C_1'e^t + 2t \cdot te^t = 0 \,/ : e^t \Rightarrow \boxed{C_1' = -2t^2}$

$\Rightarrow C_1(t) = \int(-2t^2)\,dt = -\frac{2}{3}t^3 + K_1$

$C_2(t) = \int 2t\,dt = t^2 + K_2$

$$\Rightarrow x(t) = \left(-\frac{2}{3}t^3 + k_1\right)e^t + (t^2 + k_2)te^t =$$

$$= \underbrace{k_1 e^t + k_2 t e^t}_{\bar{x}(t)} + \left(-\frac{2}{3}t^3 e^t + t^3 e^t\right) = \bar{x}(t) + \varphi_0(t)$$

$$\boxed{\varphi_0(t) = \frac{1}{3}t^3 e^t}$$

soluția particulară.

Temă:
(6) $x^{(4)} + x^{(2)} = 2\cos t$
(7) $x^{(5)} + 8x^{(3)} + 16x^{(1)} = 32$. (verificați că $\varphi_0(t) = 2t$ este soluție)

(5) $t^3 x'' - 2tx = 3\ln t$, $t > 0$.

Ec. diferențiala Euler:

$$t^n x^{(n)} = \sum_{k=0}^{n-1} t^k \alpha_k x^{(k)} + g(t)$$

$$a_k(t) = \frac{t^k \alpha_k}{t^n} = \frac{\alpha_k}{t^{n-k}}, \quad k = \overline{0, n-1}, \quad \alpha_k \in \mathbb{R}$$

Se face schimbarea de variabilă: $\boxed{|t| = e^s}$

$$t^3 x'' - 2tx = 3\ln t \quad |:t$$

$$t^2 x'' - 2x = \frac{3\ln t}{t} \quad \text{ec. Euler}$$

$$(t, x) \xrightarrow[t = e^s \Leftrightarrow s = \ln t]{} (s, y)$$

$$\boxed{x(t) = y(s(t))}$$

$$x'(t) = y'(s(t)) \cdot s'(t) = y'(s) \cdot \frac{1}{t} \Rightarrow \boxed{tx' = y'}$$

$$x''(t) = \left(\frac{1}{t}y'(s)\right)' = \frac{-1}{t^2}y'(s) + \frac{1}{t}y''(s)\cdot\frac{1}{t} \Rightarrow \quad | \cdot t^2$$

$$\Rightarrow \boxed{t^2 x''(t) = y'' - y'}$$

Ec. în y este: $y'' - y' - 2y = \frac{3s}{e^s}$

$\boxed{y'' - y' - 2y = 3se^{-s}}$, ec. afină cu coef. constanți.

Varianta 1: Rezolvăm ec. afină în y, după care sol ec. afine în x este: $x(t) = y(s(t)) = y(\ln t)$.

Varianta 2: Sistem ec. liniară omogenă atașată ec. în y: $\bar{y}'' - \bar{y}' - 2\bar{y} = 0$

$$r^2 - r - 2 = 0$$
$$r^2 + r - 2r - 2 = 0$$
$$r(r+1) - 2(r+1) = 0 \Rightarrow (r+1)(r-2) = 0 \Rightarrow$$
$$\Rightarrow \begin{cases} r_1 = -1, & m_1 = 1 \Rightarrow \varphi_1(s) = e^{-s} \\ r_2 = 2, & m_2 = 1 \Rightarrow \varphi_2(s) = e^{2s}. \end{cases}$$

$\Rightarrow \{\varphi_1, \varphi_2\}$ sistem fundamental de soluții pt ec liniară atașată ec. în y $\Rightarrow$

$$\Rightarrow \begin{cases} \psi_1(t) = \varphi_1(\ln t) = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1} = \frac{1}{t} \\ \psi_2(t) = \varphi_2(\ln t) = e^{2\ln t} = e^{\ln t^2} = t^2 \end{cases}$$

$\Rightarrow \left\{ \psi_1(t) = \frac{1}{t}, \psi_2(t) = t^2 \right\}$ sist. fundam. de soluții pt ec. liniară omogenă atașată ec. în x:

$t^2 \bar{x}'' - 2\bar{x} = 0$; $\bar{x}(t) = c_1 \frac{1}{t} + c_2 t^2$

Se aplică pt ec. afină în x variația constantelor: determinăm $c_1, c_2 : (0, +\infty) \to \mathbb{R}$

a.î. $x(t) = c_1(t)\psi_1(t) + c_2(t)\psi_2(t)$ să fie sol sist. afin.

Știm că c_1', c_2' verifică: $\begin{cases} c_1'\psi_1(t) + c_2'\psi_2(t) = 0 \\ c_1'\psi_1'(t) + c_2'\psi_2'(t) = \frac{3\ln t}{t} \end{cases} \Rightarrow$

$\psi_1'(t) = \frac{-1}{t^2}$; $\psi_2'(t) = 2t$

$$\Rightarrow \begin{cases} c_1' \frac{1}{t} + c_2' \cdot t^2 = 0 & \Big| : t \\ -c_1' \frac{1}{t^2} + c_2' \cdot 2t = \frac{3\ln t}{t} \end{cases} \Rightarrow \begin{cases} c_1' \cdot \frac{1}{t^2} + t c_2' = 0 \\ -c_1' \frac{1}{t^2} + c_2' \cdot 2t = \frac{3\ln t}{t} \end{cases}$$

$$\cancel{3} c_2' t = \frac{\cancel{3}\ln t}{t} \Rightarrow$$

$$\Rightarrow \boxed{c_2' = \frac{\ln t}{t^2} = t^{-2}\ln t}$$

$$c_1' \frac{1}{t} + \frac{\ln t}{\cancel{t^2}} \cancel{t^2} = 0. \Rightarrow \boxed{c_1' = -\frac{\ln t}{t}}$$

$\Rightarrow$ ec. de tip primitivă $\Rightarrow c_1(t), c_2(t)$ (de terminat)

Temă: 3) $t = e^s$
4) $2t + 3 = e^s$ (schimbări de variabilă pt ex. 3 & 4).