Manolache Andrei 244 28.12.2019

Tema 4

1) a) $E[X] = \sum_{n\geq 1} \mathbb{P}(X \geq n)$

Rutem scrie $\mathbb{P}(X \geq n) = \mathbb{P}(X=n) + \mathbb{P}(X=n+1) + \mathbb{P}(X=n+2) + \dots$
$= \sum_{k=n}^{\infty} \mathbb{P}(X=k)$

$$\Rightarrow \sum_{n\geq 1} \mathbb{P}(X\geq n) = \sum_{n\geq 1}^{\infty} \sum_{k=n}^{\infty} \mathbb{P}(X=k) = (\mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3) + \dots +) + (\mathbb{P}(X=2) + \mathbb{P}(X=3) + \dots) + \dots$$

$$= 1\cdot \mathbb{P}(X=1) + 2\cdot \mathbb{P}(X=2) + 3\mathbb{P}(X=3) + \dots$$

$$= \sum_{k=1}^{\infty} k\cdot \mathbb{P}(X=k) = \underline{\underline{E[X]}}$$

b) $E[X] = \int_0^{\infty} \mathbb{P}(X\geq x) = \int_0^{\infty} E[\mathbb{1}_{\{X\geq x\}}]\,dx \overset{\text{Fubini}}{=}$
$E[\int_0^{\infty} \mathbb{1}_{\{X\geq x\}} dx]$

Dar avem că: $X = \int_0^{X} dx = \int_0^{\infty} \mathbb{1}_{\{X\geq x\}}\, dx \Big\} \Rightarrow$

$E[X] = E[\int_0^{X} dx] = E[X]$ A

1/

2) $f(x) = \begin{cases} \alpha x^2 e^{-kx} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$ $k > 0$

a) $\alpha = ?$

$f(x) > 0 \ \forall x \in \mathbb{R} \iff \alpha x^2 e^{-kx} > 0 \Rightarrow \boxed{\alpha > 0}$

$\int_{-\infty}^{\infty} f(x) = 1 \iff \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx = 1$

$\iff \int_0^{\infty} \alpha x^2 e^{-kx}\,dx = 1$

Facem schimbarea $z = kx$
$dz = k\,dx$

$\int_0^{\infty} \alpha \frac{z^2}{k^2} e^{-z} \frac{dz}{k} = 1 \iff \frac{\alpha}{k^3} \underbrace{\int_0^{\infty} z^2 e^{-z}\,dz}_{\Gamma(3) = 2! = 2} = 1$

$\iff \frac{\alpha}{k^3} \cdot 2 = 1 \Rightarrow \boxed{\alpha = \frac{k^3}{2}} > 0$ (A)

b) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$f(x) = \begin{cases} \frac{k^3}{2} x^2 e^{-kx} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$ $k > 0$

Dacă $x < 0 \Rightarrow \int_{-\infty}^{x} f(t)\,dt = 0$

Dacă $x \in (0, \infty)$

$$I = \int_0^x f(t)\,dt = \int_0^x \frac{k^3}{2} t^2 e^{-kt}\,dt = \frac{k^3}{2}\int_0^x t^2 e^{-kt}\,dt$$

$f' = e^{-kt}$ $\quad f = \frac{e^{-kt}}{-k}$

$g = t^2$ $\quad g' = 2t$

$$I = \frac{k^3}{2}\left(\frac{e^{-kt}}{-k}\cdot t^2\Big|_0^x + \frac{2}{k}\int_0^x e^{-kt}\cdot t\,dt\right)$$

$$= \frac{k^3}{2}\left(\frac{e^{-kt}\cdot x^2}{-k} + \frac{2}{k}\underbrace{\int_0^x e^{-kt}\cdot t\,dt}_{J}\right)$$

$$J = \int_0^x e^{-kt}\cdot t\,dt$$

$f' = e^{-kt}$ $\quad f = -\frac{e^{-kt}}{k}$

$g = t$ $\quad g' = 1$

$$J = -\frac{e^{-kt}}{k}t\Big|_0^x + \int_0^x e^{-kt}\,dt = \frac{e^{-kx}\cdot x}{-k} + \frac{1}{k}\int_0^x e^{-kt}\,dt = \frac{e^{-kx}\cdot x}{-k} + \frac{1}{k}\left(\frac{e^{-kt}}{-k}\right)\Big|_0^x$$

$$= \frac{e^{-kx}\cdot x}{-k} + \frac{1}{k}\left(\frac{e^{-kx}}{-k} - \frac{1}{-k}\right) = \frac{e^{-kx}\cdot kx + e^{-kx} - 1}{-k^2}$$

$$I = \frac{k^3}{2}\left(\frac{e^{-kt}\cdot x^2}{-k} + \frac{2}{k}\cdot\frac{e^{-kx}\cdot kx + e^{-kx} - 1}{-k^2}\right) = -\frac{1}{2}\left(e^{-kx}\left(x^2k^2 + 2xk + 2\right) - 2\right)$$

$$= 1 - \frac{e^{-xk}\left(x^2k^2 + 2xk + 2\right)}{2}$$

Deci $F(x) = \begin{cases} 0, & x < 0 \\ 1 - \frac{e^{-xk}\left(x^2k^2 + 2xk + 2\right)}{2} & x > 0 \end{cases}$

Pentru $x \to \infty$, $\lim_{x \to \infty} F(x) = 1$

c) $\mathbb{P}(0<X<k^{-1}) = ?$

$$\mathbb{P}(0<X<k^{-1}) = F(k^{-1}) - \overset{0}{\overset{\shortparallel}{F(0)}}$$

$$= F\left(\frac{1}{k}\right) = 1 - \frac{e^{-1}}{2}\cdot(1+2+2) = 1 - \frac{5}{2e}$$

3) a) $X \sim Exp(\alpha) \Rightarrow f(x) = \begin{cases} \alpha e^{-\alpha x}, & x>0 \\ 0, & \text{altfel} \end{cases}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \alpha \cdot e^{-\alpha t}\,\mathbb{1}_{[0,\infty)}(t)\,dt = \begin{cases} 0 & x \le 0 \\ \int_0^x \alpha e^{-\alpha t}\,dt = -e^{-\alpha t}\Big|_0^x = 1-e^{-\lambda x} & x>0 \end{cases}$$

$$\mathbb{P}(X>x) = 1 - \underbrace{F(x)}_{\mathbb{P}(X<x)} = e^{-\alpha x}$$

$$\Rightarrow \mathbb{P}(X>t+s \mid X>s) = \frac{\mathbb{P}(X>t+s,\ X>s)}{\mathbb{P}(X>s)} = \frac{\mathbb{P}(X>t+s)}{\mathbb{P}(X>s)}$$

$$= \frac{e^{-\alpha(t+s)}}{e^{-\alpha \cdot s}} = e^{-\alpha \cdot t} = \mathbb{P}(X>t), \quad t>0$$

b) $\mathbb{P}(X>s+t \mid X>s) = \dfrac{\mathbb{P}(X>s+t,\ X>s)}{\mathbb{P}(X>s)} = \dfrac{\mathbb{P}(X>s+t)}{\mathbb{P}(X>s)} = \mathbb{P}(X>t)$

$\Rightarrow \mathbb{P}(X>s+t) = \mathbb{P}(X>t)\,\mathbb{P}(X>s)$

Notăm $f(t) = \mathbb{P}(X>t)$, avem $f(s+t) = f(s)\cdot f(t)$ $(s>0, t>0)$

Avem că $f(s+s) = f(2s) = f(s)\cdot f(s) = f^2(s)$.

Demonstrăm prin inducție $f(ns) = f^n(s)$ $(s>0)$

P_0: $f(1s) = f^1(s)$ (A)

P_n: $n \to n+1$

$f(ns) = f^n(s)$

P_{n+1}: $f((n+1)s) = f(ns)\,f(s) = f^n(s)\cdot f(s) = f^{n+1}(s)$ (A)

Dacă luăm $s = \frac{1}{2} \Rightarrow$ ~~P(X ≥ 1)~~ =

$$f(1) = f(2 \cdot \tfrac{1}{2}) = f^2(\tfrac{1}{2})$$

$$\Rightarrow f(\tfrac{1}{2}) = \sqrt{f(1)}$$

Dacă luăm $s = \frac{1}{k} \Rightarrow f(\frac{1}{k}) = f^{\frac{1}{k}} f(1)$ analog rezultatului anterior

Utilizând cele 2 relații anterioare, obținem:

$$f(\tfrac{m}{n}) = f(m \cdot \tfrac{1}{n})\ f^m(\tfrac{1}{n}) = f^{\frac{m}{n}}(1)$$

Avem astfel că dacă $q := \frac{m}{n} \in \mathbb{Q} \Rightarrow f(q) = f^q(1)$

Deci $\forall q \in \mathbb{Q}_+$, $f(q) = f^q(1)$. $\left(P(X>q) = P^{\frac{m}{n}}(X>1) \right)$

Dacă $r \in \mathbb{R}_+ \setminus \mathbb{Q}_+$, știm $\mathbb{Q}$ densă în $\mathbb{R}$, deci $\exists (q_n)_n \subseteq \mathbb{Q}_+$ a.î. $q_n \searrow r$.

Folosind continuitatea la dreapta obținem $f(q_n) \searrow f(r)$

$\Rightarrow f(r) = f^r(1)$.

Deci $\forall t \in \mathbb{R}$, $f(t) = f^t(1) \Leftrightarrow f(t) = e^{\ln f^t(1)}$

$f(t) = e^{t \ln f(1)} \Leftrightarrow f(t) = e^{-t \ln \frac{1}{f(1)}} \Leftrightarrow P(X>t) = e^{-t \ln \frac{1}{P(X>1)}}$

$\Leftrightarrow P(X \leq t) = 1 - e^{-t \ln \frac{1}{P(X>1)}}$

Deci $X \sim Exp\left(\ln \frac{1}{P(X>1)}\right)$

④. a) X – nivelul de zgomot produs de o mașină de spălat
$M(X(10))$ – media unui eșantion de 10 mașini

$$M(X(10)) \sim N\left(44, \frac{5^2}{10}\right) \quad \text{(Aplicând Teorema de limită centrală)}$$

$$\mathbb{P}(M(X(10)) > 48) \simeq \mathbb{P}\Big(\underbrace{Z}_{\sim N(0,1)} > \frac{48-44}{\sqrt{\frac{5^2}{10}}}\Big) = \mathbb{P}(Z > 2{,}53) = 1 - \mathbb{P}(Z < 2{,}53)$$
$$= 1 - 0{,}9943$$
$$= 0{,}0057$$

b) X – greutatea unei persoane
$M(X(100))$ – media unui eșantion de 100 persoane

$$M(X(100)) \sim N\left(66{,}3, \frac{15{,}6^2}{100}\right) \quad \text{(Aplicând Teorema de limită centrală)}$$

$$\mathbb{P}(100 \cdot M(X(100)) > 7000) = \mathbb{P}(M(X(100)) > 70) \simeq \mathbb{P}\Big(\underbrace{Z}_{\sim N(0,1)} > \frac{70-66{,}3}{\sqrt{\frac{15{,}6^2}{100}}}\Big)$$
$$\simeq \mathbb{P}\left(Z > \frac{3{,}7 \cdot 10}{15{,}6}\right) = \mathbb{P}(Z > 2{,}37) = 1 - \mathbb{P}(Z \leq 2{,}37)$$
$$= 1 - 0{,}9911 = 0{,}0089$$

⑤ (X,Y) $\quad f_{(X,Y)}: \mathbb{R}^2 \to \mathbb{R}$

$$f_{(X,Y)}(x,y) = \begin{cases} k(x+y+1) & , x \in [0,1], y \in [0,2] \\ 0 & , \text{altfel} \end{cases}$$

a) $k = ?$

$$f_{(X,Y)}(x,y) \geq 0 \ \forall x,y \in \mathbb{R} \iff k(\underbrace{x+y+1}_{\geq 0}) \geq 0 \quad x \in [0,1], y \in [0,2]$$
$$\Rightarrow \boxed{k \geq 0}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\,dx\,dy = 1 \iff$$

$$\int_0^1\int_0^2 k(x+y+1)\,dx\,dy = 1$$

$$\iff k\int_0^1\left(\int_0^2 x+y+1\,dy\right)dx = 1$$

$\Leftrightarrow k \int_0^1 (x+1)y\Big|_0^2 + \frac{y^2}{2}\Big|_0^2 \, dx = 1$

$\Leftrightarrow k\int_0^1 2x+2+2 \, dx = 1 \quad \Leftrightarrow k\left(\frac{2x^2}{2}\Big|_0^1 + 4\right) = 1$

$k \cdot 5 = 1 \Rightarrow \boxed{k = \frac{1}{5}} > 0$ (A)

b) $f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\, dy$

$= \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\, dy = \int_{-\infty}^{0} f_{(X,Y)}(x,y)\, dy + \int_0^2 f_{(X,Y)}(x,y)\, dy + \int_2^{\infty} f_{(X,Y)}(x,y)\, dy$

$= \int_0^2 \frac{1}{5}(x+y+1)\, dy$

$= \frac{1}{5}\left(\frac{y^2}{2}\Big|_0^2 + (x+1)y\Big|_0^2\right)$

$= \frac{2x+2+2}{5} = \frac{2x+4}{5}$ – densitatea marginală pt X pt $x \geq 0$, $x \leq 1$

$f_X(x) = \begin{cases} \frac{2x+4}{5}, & x \in [0,1] \\ 0, & \text{altfel} \end{cases}$

$f_Y(y) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\, dx$

$= \int_{-\infty}^{0} f_{(X,Y)}(x,y)\, dx + \int_0^1 f_{(X,Y)}(x,y)\, dx + \int_1^{\infty} f_{(X,Y)}(x,y)\, dx$

$= \int_0^1 \frac{x+y+1}{5}\, dx = \frac{1}{5}\left((y+1)x\Big|_0^1 + \frac{x^2}{2}\Big|_0^1\right) = \frac{1}{5}\left(y+1+\frac{1}{2}\right)$

$= \frac{2y+3}{10}$ – densitatea marginală pt Y pt $y \geq 0$, $y \leq 2$

$f_Y(y) = \begin{cases} \frac{2y+3}{10}, & y \in [0,2] \\ 0, & \text{altfel} \end{cases}$

c) $f_{(X,Y)}(x,y) = f_X(x) \cdot f_Y(y) \quad \forall x,y \in \mathbb{R}$

$\frac{1}{5}(x+y+1) = \frac{2x+4}{5} \cdot \frac{2y+3}{10}$

$10(x+y+1) = (2x+4)(2y+3) \quad \Rightarrow 10x+10y+10 = 4xy+6x+8y+12 \;\; \forall x,y \in \mathbb{R}$ (F)

$\Rightarrow$ X și Y nu sunt independente

7/

d) $F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f_{(x,y)}(t,u)\,dt\,du$

I. $x\in(0,1],\ y\in(0,2]$

$$F(x,y)=\int_0^x\int_0^y \frac{u+t+1}{5}\,du\,dt=\frac{1}{5}\int_0^x\int_0^y u+t+1\,dt\,du$$

$$=\frac{1}{5}\int_0^x \frac{t^2}{2}\Big|_0^y+(u+1)t\Big|_0^y\,du$$

$$=\frac{1}{5}\int_0^x \frac{y^2}{2}+(u+1)y\,du$$

$$=\frac{1}{5}\left(\frac{y^2}{2}\cdot x+\frac{u^2}{2}y\Big|_0^x+y\cdot x\right)$$

$$=\frac{1}{5}\left(\frac{xy^2}{2}+\frac{x^2y}{2}+\frac{2xy}{2}\right)$$

$$=\frac{1}{10}\left(xy^2+xy^2+2xy\right)$$

II. $x\in(0,1]\ y\in(2,\infty)$

$$F(x,y)=\int_0^x\int_0^2 \frac{t+u+1}{5}\,du\,dt=\frac{1}{5}\int_0^x (t+1)u\Big|_0^2+\frac{u^2}{2}\Big|_0^2\,dt$$

$$=\frac{1}{5}\int_0^x 2(t+1)+2\,dt$$

$$=\frac{1}{5}\int_0^x 2t+4\,dt=\frac{1}{5}\left(2\frac{t^2}{2}\Big|_0^x+4x\right)$$

$$=\frac{1}{5}\left(x^2+4x\right)$$

III. $x\in(1,\infty)\ y\in(0,2]$

$$F(x,y)=\int_0^1\int_0^y \frac{t+u+1}{5}\,du\,dt=\frac{1}{5}\int_0^1 (t+1)u\Big|_0^y+\frac{u^2}{2}\Big|_0^y\,dt$$

$$=\frac{1}{5}\int_0^1 (t+1)y+\frac{y^2}{2}\,dt$$

$$=\frac{1}{5}\left(\frac{y^2}{2}\cdot t\Big|_0^1+y+\frac{t^2}{2}y\Big|_0^1\right)$$

$$=\frac{1}{5}\left(\frac{y^2}{2}+y+\frac{y}{2}\right)=\frac{y^2+3y}{10}$$

IV. $x\in(1,\infty)\ y\in(2,\infty)$

$$F(x,y)=\int_0^1\int_0^2 \frac{t+u+1}{5}\,du\,dt=\frac{1}{5}\int_0^1 (t+1)u\Big|_0^2+\frac{u^2}{2}\Big|_0^2\,dt$$

$$=\frac{1}{5}\int_0^1 2(t+1)+2\,dt$$

$$= \frac{1}{5}\int_0^1 2t+4\,dt = \frac{1}{5}\left(2\frac{t^2}{2}\Big|_0^1 + 4t\Big|_0^1\right) = \frac{1}{5}(1+4) = 1$$

Pentru funcțiile de repartiție marginale folosim relațiile:

$$F_X(x) = \lim_{y\to\infty} F(x,y)$$

$$F_Y(y) = \lim_{x\to\infty} F(x,y).$$

deci
$$F_X(x) = \begin{cases} 0 & x\leq 0 \\ \frac{1}{5}(x^2+4x) & x\in(0,1] \\ 1 & x>1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y\leq 0 \\ \frac{1}{10}(y^2+3y) & y\in[0,2] \\ 1 & y>2 \end{cases}$$

e) $$f_{X|Y}(x,y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(x,y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)}$$

Pentru $x\in[0,1]$, $y\in[0,2]$

$$f_{X|Y}(x,y) = \frac{\frac{x+y+1}{5}}{\frac{2y+3}{10}} = \frac{2(x+y+1)}{2y+3}$$

$$f_{Y|X}(x,y) = \frac{\frac{x+y+1}{5}}{\frac{2x+4}{5}} = \frac{x+y+1}{2x+4}$$

Altfel, $f_{X|Y}(x,y) = 0$ și $f_{Y|X}(x,y) = 0$.

(6) Notăm X și Y cele 2 măsurători

$X \sim \mathcal{N}(0,1)$
$Y \sim \mathcal{N}(0,1)$ $\quad X \perp\!\!\!\perp Y$

Notăm $M = \max(X,Y)$ măsurătoarea mai mare
$m = \min(X,Y)$ măsurătoarea mai mică

$M + m = x + y \qquad M - m = |x - y|$

$E[M+m] = E[m] + E[M] = E[X+Y] = E[X] + E[Y] = 0$

$E[M] - E[m] = E[M-m] = E[|X-Y|]$

Cum $\begin{cases} Var(X) + Var(Y) = Var(X-Y) = Var(X+Y), \text{ pt } X \perp\!\!\!\perp Y \\ E[X-Y] = E[X] - E[Y] \quad X \perp\!\!\!\perp Y \end{cases}$

$\Rightarrow X - Y \sim \mathcal{N}(0,2)$

Cum $Z = \dfrac{X-\mu}{\sigma}$, $Z \sim \mathcal{N}(0,1) \Rightarrow X = \mu + \sigma Z$

$\Rightarrow X - Y = Z\sqrt{2} \qquad Z \sim \mathcal{N}(0,1)$

$\Rightarrow E[|X-Y|] = E[|Z\sqrt{2}|] = \sqrt{2}\, E[|Z|] = \sqrt{2} \int_{-\infty}^{\infty} |z| \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

$\overset{\text{simetrie}}{=} 2\sqrt{2} \int_0^{\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{2}{\sqrt{\pi}} \int_0^{\infty} z \cdot e^{-\frac{z^2}{2}} dz = \frac{2}{\sqrt{\pi}} \cdot \left(-e^{-\frac{z^2}{2}}\right)\Big|_0^{\infty}$

$= \frac{2}{\sqrt{\pi}} (0+1) = \frac{2}{\sqrt{\pi}}$

$\begin{cases} E[M] + E[m] = 0 \\ E[M] - E[M] = \frac{2}{\sqrt{\pi}} \end{cases} \Rightarrow \quad E[M] = \frac{1}{\sqrt{\pi}}$

$E[m] = -\frac{1}{\sqrt{\pi}}$

$E[XY] = E[X] \cdot E[Y] \underset{(X \perp\!\!\!\perp Y)}{=} 0 \cdot 0 = 0$

$E[XY] = \cancel{E[M]\,E[m]}\ E[Mm]$

$\Rightarrow Cov(M, m) = E[Mm] - E[M]\,E[m] = E[\overset{0}{XY}] + \frac{1}{\pi} = \frac{1}{\pi}$ 10/

$M+m=X+Y$

$Var(M+m)=Var(X+Y)=Var(X)+Var(Y)=1+1=2$

$Var(M+m)=Var(M)+Var(m)+2\,Cov(M,m)=2$

$\Rightarrow Var(M)+Var(m)=2-2\cdot\frac{1}{\pi}=2\left(1-\frac{1}{\pi}\right)$

Dar: $\max(X,Y)=-\min(-X,-Y)$ și (X,Y) este repartizat ca $(-X,-Y)$

$\Rightarrow \underline{Var(M)}=Var(\max(X,Y))=Var(-\min(-X,-Y))=Var(\min(-X,-Y))$

$=Var(\min(X,Y))=\underline{Var(m)}.$

$\Rightarrow Var(M)=Var(m)=\frac{2\left(1-\frac{1}{\pi}\right)}{2}=1-\frac{1}{\pi}$

$\Rightarrow \rho(M,m)=\frac{Cov(M,m)}{\sqrt{Var(M)}\sqrt{Var(m)}}=\frac{\frac{1}{\pi}}{\sqrt{\left(1-\frac{1}{\pi}\right)^2}}=\frac{\frac{1}{\pi}}{1-\frac{1}{\pi}}=\frac{1}{\pi-1}$

7) a) $T\sim\mathcal{N}(0,4^2)$

$\mathbb{P}(T\geq 0, T<1)=\mathbb{P}\left(\frac{0-0}{4}<\frac{T-0}{4}<\frac{1-0}{4}\right)=\mathbb{P}\left(0<\underbrace{\frac{T}{4}}_{\sim\mathcal{N}(0,1)}<\frac{1}{4}\right)$

$=\Phi\left(\frac{1}{4}\right)-\Phi(0)=0{,}5987-0{,}5=0{,}0987$

b) Notăm X ~~fe~~ prima femeie
Notăm Y a doua femeie $\quad X \perp\!\!\!\perp Y$

$X\sim\mathcal{N}(0,4^2)$

$Y\sim\mathcal{N}(0,4^2)$

Notăm $T_1=\max(X,Y)$ data de naștere mai mare (ultima)
$T_0=\min(X,Y)$ —"— mică (prima)

$T_0 + T_1 = X+Y$

$T_1 - T_0 = |X-Y|$

$E[T_1] + E[T_0] = E[T_1+T_0] = E[X+Y] \overset{\perp\!\!\!\perp}{=} E[X] + E[Y] = 0$

$E[T_1] - E[T_0] = E[T_1 - T_0] = E[|X-Y|]$

Oles cō $X-Y \sim N(0,32)$

$(X \perp\!\!\!\perp Y,\ Var(X-Y) = Var(X+Y) = Var(X) + Var(Y))$

$E[X-Y] = E[X] - E[Y]$

Wotōm $Z\sqrt{32} = X-Y \quad Z \sim N(0,1) \quad \left(\begin{matrix} X-Y = \mu + \sigma Z \\ Z \sim N(0,1) \end{matrix} \right)$

$E[|X-Y|] = \sqrt{32}\, E[|Z|]$

$= \sqrt{32} \int_{-\infty}^{\infty} |Z| \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} dZ = 2\sqrt{32} \int_0^{\infty} Z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} dZ$

$= \frac{8}{\sqrt{\pi}} \int_0^{\infty} Z \cdot e^{-\frac{Z^2}{2}} dZ = \frac{8}{\sqrt{\pi}} \left(e^{-\frac{Z^2}{2}} \right) \Big|_0^{\infty} = \frac{8}{\sqrt{\pi}}$

$$\begin{cases} E[T_1] + E[T_0] = 0 \\ E[T_1] - E[T_0] = \frac{8}{\sqrt{\pi}} \end{cases}$$

$$\Rightarrow \begin{cases} E[T_1] = \frac{4}{\sqrt{\pi}} \\ E[T_0] = -\frac{4}{\sqrt{\pi}} \end{cases}$$

$Cov(T_1, T_0) = E[T_0 T_1] - E[T_0]\, E[T_1]$

$E[XY] = E[T_0 T_1] = E[X] \cdot E[Y] = 0 \cdot 0 = 0 \quad (X \perp\!\!\!\perp Y)$

$\Rightarrow Cov(T_1, T_0) = -E[T_0]\, E[T_1] = \frac{16}{\pi}$

~~Var(Min)~~

$Var(X+Y) = Var(T_1+T_0) \underset{(X \perp Y)}{=} Var(X) + Var(Y) = 16 + 16 = 32$

$Var(T_1+T_0) = Var(T_1) + Var(T_0) + 2Cov(T_1, T_0) = 32$

$\Rightarrow Var(T_1) + Var(T_0) = 32 - \frac{32}{\pi}$

Știm că $\overset{T_1}{\overset{\shortparallel}{max(X,Y)}} = -min(-X,-Y)$ și (X,Y) repartizate ca $(-X,-Y)$

$\Rightarrow \underline{Var(T_1)} = Var(max(X,Y)) = Var(-min(-X,-Y)) = Var(min(-X,-Y))$

$= Var(min(X,Y)) = \underline{Var(T_0)}$

$\Rightarrow Var(T_0) = Var(T_1) = \frac{32 - \frac{32}{\pi}}{2} = 16 - \frac{16}{\pi}$

$\boxed{Var(T_0) = 16 - \frac{16}{\pi}}$