Gr. 344: Seminar 1, EDDP, 05.10.2020.

Determinarea primitivelor unei funcții $f : D \to \mathbb{R}$, $D \subset \mathbb{R}$ înseamnă rezolvarea unei ecuații diferențiale:

$$F'(x) = f(x) \qquad (1)$$

Mulțimea soluțiilor ec. (1) este mulțimea primitivelor funcției f: $\int f(x)\,dx$

• Operații cu mulțimi de primitive:

$$\int (f_1(x) \pm f_2(x))\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx$$

• Tabel de primitive ale funcțiilor elementare:

1) $\int 1\,dx = x + \mathcal{C}$, $\mathcal{C}$ este mulțimea funcțiilor constante. Avem: $\mathcal{C} + \mathcal{C} = \mathcal{C}$, $\alpha\mathcal{C} = \mathcal{C}$, $\alpha \in \mathbb{R}$.

2) $\int x^r\,dx = \frac{x^{r+1}}{r+1} + \mathcal{C}$, $r \in \mathbb{R} \setminus \{-1\}$

3) $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + \mathcal{C}$

4) $\begin{cases} \int a^x\,dx = \frac{a^x}{\ln a} + \mathcal{C}, & a \in (0,\infty) \setminus \{1\} \\ \int e^x\,dx = e^x + \mathcal{C}, & e \simeq 2{,}71\ldots \end{cases}$

5) $\begin{cases} \int \sin x\,dx = -\cos x + \mathcal{C} \\ \int \cos x\,dx = \sin x + \mathcal{C} \\ \int \frac{1}{\cos^2 x}\,dx = \int (1 + tg^2 x)\,dx = tg\,x + \mathcal{C} \\ \int \frac{1}{\sin^2 x}\,dx = \int (1 + ctg^2 x)\,dx = -ctg + \mathcal{C} \\ \int tg\,x\,dx = -\ln|\cos x| + \mathcal{C} \\ \int ctg\,x\,dx = \ln|\sin x| + \mathcal{C} \end{cases}$

6) $\begin{cases} \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\, arctg\,\frac{x}{a} + \mathcal{C} \\ \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + \mathcal{C} \end{cases}$

7) $$\begin{cases} \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)+C \\ \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x-\sqrt{x^2-a^2}\right|+C \\ \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}+C \end{cases}$$

8) $$\begin{cases} \int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2}+C \\ \int \frac{x}{\sqrt{x^2-a^2}}\,dx = \sqrt{x^2-a^2}+C \\ \int \frac{x}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2}+C \end{cases}$$

9) $$\begin{cases} \int \frac{2x}{x^2+a^2}\,dx = \ln(x^2+a^2)+C \\ \int \frac{2x}{x^2-a^2}\,dx = \ln|x^2-a^2|+C \end{cases}$$

Metode de integrare

1) prin reducere la formule din tabelul de primitive.

2) Metoda de integrare prin părți

$$\boxed{\int \underbrace{u(x)\,v'(x)}_{f(x)}\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx}$$

3) Prima schimbare de variabilă

$$\boxed{\int \overbrace{g(u(x))\,u'(x)}^{f(x)}\,dx = G(u(x))+C}$$, unde G este o primitivă pt g.

$u(x)=t$

$u'(x)\,dx=dt$

$$\int g(t)\,dt = G(t)+C$$

4) A doua schimbare de variabilă

$$\boxed{\int \underbrace{g(u(x))}_{f(x)=g\circ u}\,dx = H(u(x))+C}$$, unde H este o primitivă pt $g\cdot\varphi'$

notam $u(x) = t \Longleftrightarrow x = u^{-1}(t) \overset{not}{=} \varphi(t)$

(pt-că e schimbare de variabilă)

$x = \varphi(t)$

$dx = \varphi'(t)\,dt$

Se obține: $\int \underbrace{g(t)\cdot\varphi'(t)}_{\overset{not}{=}}dt = \int h(t)dt = H(t) + C$

$h(t) = g(t)\varphi'(t)$

<u>Aplicații</u>: Să se determine mulțimea primitivelor funcțiilor următoare:

1) $f(x) = 3x^4 + x^2 - x + 1$

2) $f(x) = \sqrt[3]{x} - 2\sqrt{x}$

3) $f(x) = x^2\sqrt{x} - 2^x$

4) $f(x) = 2^x \cdot 3^x$

5) $f(x) = \dfrac{(x-1)^3}{\sqrt{x}}$

✓ 6) $f(x) = \dfrac{1}{\sin^2 x \cdot \cos^2 x}$

✓ 7) $f(x) = \dfrac{1}{9x^2 - 1}$

8) $f(x) = \dfrac{1}{3x^2 + 12}$

✓ 9) $f(x) = \dfrac{1}{(x^2-1)(x^2+3)}$

10) $f(x) = \dfrac{\sqrt{x^2-3} + 2\sqrt{x^2+3}}{\sqrt{x^4-9}}$

11) $f(x) = \dfrac{1}{\sqrt{8-2x^2}}$

12) $f(x) = \mathrm{ctg}^2 x$

13) $f(x) = \dfrac{x+2}{\sqrt{4-x^2}}$

14) $f(x) = 2x \cdot e^x$

✓ 15) $f(x) = x \ln x$

16) $f(x) = x \cos x$

17) $f(x) = e^x \sin x$

18) $f(x) = e^{2x} \sin 3x$

✓ 19) $f(x) = \sqrt{x^2 - 1}$

20) $f(x) = x\sqrt{x^2+1}$

21) $f(x) = x\, e^{x^2+1}$

22) $f(x) = \sin^2 x$

23) $f(x) = \dfrac{\ln(\ln x)}{x \cdot \ln x}$

✓ 24) $f(x) = \dfrac{x+1}{x^2+2x+3}$

✓ 25) $f(x) = \dfrac{1}{\cos x}$

26) $f(x) = \dfrac{1}{\sin x}$

(6) $\int \frac{1}{\sin^2 x \cdot \cos^2 x}\,dx = \int \frac{1}{(\sin x \cos x)^2}\,dx = \int \frac{1}{\left(\frac{\sin 2x}{2}\right)^2}\,dx =$

$\sin x \cdot \cos x = \frac{\sin 2x}{2}$

$= 4 \int \frac{1}{(\sin 2x)^2}\,dx = 2\int \frac{2\,dx}{\sin^2 2x} \overset{not}{=} I$

$2x = t$
$2\,dx = dt$

$\Rightarrow$

Obținem: $2\int \frac{dt}{\sin^2 t} = -2\,ctg\,t + C$

$\Rightarrow I = -2\,ctg(2x) + C$

Sau, altă soluție

$I = \int \frac{1}{\sin^2 x \cos^2 x}\,dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x}\,dx =$

$= \int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x} \cos^2 x}\,dx + \int \frac{\cancel{\cos^2 x}}{\sin^2 x \cancel{\cos^2 x}}\,dx =$

$= \int \frac{1}{\cos^2 x}\,dx + \int \frac{1}{\sin^2 x}\,dx = tg\,x - ctg\,x + C$

OBS: $\boxed{-2\,ctg(2x)} = -\frac{2}{tg(2x)} = -\frac{2}{\frac{2\,tg\,x}{1 - tg^2 x}} =$

$tg\,y = \frac{2\,tg\frac{y}{2}}{1 - tg^2\frac{y}{2}}$

$= -\frac{\cancel{2}}{1} \cdot \frac{1 - tg^2 x}{\cancel{2}\,tg\,x} = -\left(\frac{1}{tg\,x} - \frac{\cancel{tg^2 x}}{\cancel{tg\,x}}\right) =$

$= -(ctg\,x - tg\,x) = \boxed{tg\,x - ctg\,x}$

(7). $\int \frac{1}{\sqrt{9x^2 - 1}}\,dx = \int \frac{1}{\sqrt{9\left(x^2 - \frac{1}{9}\right)}}\,dx = \int \frac{1}{3\sqrt{x^2 - \left(\frac{1}{3}\right)^2}}\,dx =$

$= \frac{1}{3} \ln\left| x - \sqrt{x^2 - \frac{1}{9}} \right| + C$

(9) $\int \frac{1}{(x^2-1)(x^2+3)}dx = \frac{1}{4}\int \frac{(x^2+3)-(x^2-1)}{(x^2-1)(x^2+3)}dx =$

$= \frac{1}{4}\left(\int \frac{\cancel{x^2+3}}{(x^2-1)\cancel{(x^2+3)}}dx - \int \frac{\cancel{x^2-1}}{\cancel{(x^2-1)}(x^2+3)}dx\right) =$

$= \frac{1}{4}\left(\int \frac{1}{x^2-1}dx - \int \frac{dx}{x^2+(\sqrt{3})^2}\right) =$

$= \frac{1}{4}\left(\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}\right) + C$

(19) $I = \int\sqrt{x^2-1}\,dx = \int\frac{(\sqrt{x^2-1})^2}{\sqrt{x^2-1}}dx = \int\frac{x^2-1}{\sqrt{x^2-1}}dx =$

$= \underbrace{\int \frac{x^2}{\sqrt{x^2-1}}dx} - \int\frac{1}{\sqrt{x^2-1}}dx \Rightarrow$

$u(x) = x \qquad u'(x) = 1$

$v'(x) = \frac{x}{\sqrt{x^2-1}} \qquad v(x) = \sqrt{x^2-1}$

$I = x\sqrt{x^2-1} - \underbrace{\int\sqrt{x^2-1}\,dx}_{I} - \ln|x-\sqrt{x^2-1}| \Rightarrow$

$\Rightarrow 2I = x\sqrt{x^2-1} - \ln|x-\sqrt{x^2-1}| \Rightarrow$

$\Rightarrow I = \frac{1}{2}\left(x\sqrt{x^2-1} - \ln|x-\sqrt{x^2-1}|\right) + C$

sau

$I = \int\sqrt{x^2-1}\,dx = \int x'\cdot\sqrt{x^2-1}\,dx = \ldots$ temă!

prin părți

$u(x) = \sqrt{x^2-1} \qquad u'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}$

$v'(x) = 1 \qquad v(x) = x$

(15) $\int x\ln x\,dx = I$

$u(x) = \ln x \qquad u'(x) = \frac{1}{x}$

$v'(x) = x \qquad v(x) = \frac{x^2}{2}$

$\Rightarrow I = \frac{1}{2}x^2\ln x - \int\frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}}{2}dx =$

$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx =$

$= \frac{1}{2}x^2\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + C = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C$

(24) $\int \frac{x+1}{x^2+2x+3}dx = \int \frac{x+1}{(x+1)^2+2}dx = I$

$x+1=t$
$dt = dx$

$\int \frac{t}{t^2+2}dt = \frac{1}{2}\int \frac{2t}{t^2+2}dt$

$k=t^2+2$
$dk = 2t\,dt$

$\frac{1}{2}\int \frac{dk}{k} = \frac{1}{2}\ln|k|+c$

$\Rightarrow I = \frac{1}{2}\ln|t^2+2| + C = \frac{1}{2}\ln|(x+1)^2+2|+C$

(Sau)

$x^2+2x+3 = t$

$dt = (2x+2)dx = 2(x+1)dx$

$I = \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+3}dx \Rightarrow \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t|+C \Rightarrow$

$\Rightarrow I = \frac{1}{2}\ln|x^2+2x+3|+C$

(24') $\int \frac{x+4}{x^2+2x+2}dx = \int \frac{x+4}{(x+1)^2+1}dx = I$

$x+1=t \Rightarrow x = t-1$
$dx = dt$

Se obține: $\int \frac{t-1+4}{t^2+1}dt = \int \frac{t+3}{t^2+1}dt =$

$= \int \frac{t}{t^2+1}dt + 3\int \frac{dt}{t^2+1} = \frac{1}{2}\ln(t^2+1) + 3\,\mathrm{arctg}\,t + C \Rightarrow$

$\Rightarrow I = \frac{1}{2}\ln(x^2+2x+2) + 3\,\mathrm{arctg}\,(x+1) + C$

(25) $\int \frac{1}{\cos x}dx = \int \frac{\cos x}{\cos^2 x}dx = \int \frac{\cos x}{1-\sin^2 x}dx = I$

$\sin x = t$
$\cos x\,dx = dt$ $\Rightarrow \int \frac{dt}{1-t^2} = -\int \frac{1}{t^2-1^2}dt = -\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C$

$\Rightarrow I = -\frac{1}{2}\ln\left|\frac{\sin x-1}{\sin x+1}\right| + C \underset{\sin x\in[-1,1]}{=} -\frac{1}{2}\ln\left(\frac{1-\sin x}{1+\sin x}\right)+C.$

Temă: 1) Rezolvaţi 25, folosind formulele:

$$\boxed{\cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}}$$

resp. (la 26): $\boxed{\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}}$

2) Fie $I = \int \frac{\sin x}{\sin x - \cos x}\,dx$; $J = \int \frac{\cos x}{\sin x - \cos x}\,dx$.

Calculaţi $I+J$, $I-J$ şi apoi I şi J.

3) Calculaţi: $\int \frac{\sin x}{e^x + \sin x + \cos x}\,dx$.

Indicaţie: Pt $\int \frac{g(x)}{h(x)}\,dx$ căutaţi $\alpha, \beta \in \mathbb{R}$ a.î. $g(x) = \alpha h(x) + \beta h'(x)$. Dacă se poate, atunci:

$$\int \frac{g(x)}{h(x)}\,dx = \alpha \int 1\,dx + \beta \int \frac{h'(x)}{h(x)}\,dx = \alpha x + \beta \ln|h(x)| + C.$$