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Tema 1

1. $f(x) = \sum_{i=1}^{m} \log(a_{(i)}^T x + b_i)$

$$\nabla f(x) = \cancel{\begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}} \quad \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$\nabla f(x) = \begin{bmatrix} \sum_{i=1}^{m} \frac{a_{(i)1}}{a_{(i)}^T x + b_i} \\ \sum_{i=1}^{m} \frac{a_{(i)2}}{a_{(i)}^T x + b_i} \\ \cdots \\ \sum_{i=1}^{n} \frac{a_{(i)m}}{a_{(i)}^T x + b_i} \end{bmatrix}$$

$a_{(i)j}$ – val j-lea element din al i-lea vector

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1 \partial x_1} & \frac{\partial f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial f}{\partial x_1 \partial x_n} \\ \vdots & & & \\ \frac{\partial f}{\partial x_n \partial x_1} & & \cdots & \frac{\partial f}{\partial x_n \partial x_n} \end{bmatrix}$$

$$\nabla^2 f = \begin{bmatrix} \sum_{i=1}^{m} \frac{-a_{(i)1}^2}{(a_{(i)}^T x + b_i)^2} & \sum_{i=1}^{m} \frac{-a_{(i)1} \cdot a_{(i)2}}{(a_{(i)}^T x + b_i)^2} & \cdots & \sum_{i=1}^{m} \frac{-a_{(i)1} \cdot a_{(i)n}}{(a_{(i)}^T x + b_i)^2} \\ \vdots & & & \\ \sum_{i=1}^{m} \frac{-a_{(i)n} \cdot a_{(i)1}}{(a_{(i)}^T x + b_i)^2} & & \cdots & \sum_{i=1}^{m} \frac{-a_{(i)n}^2}{(a_{(i)}^T x + b_i)^2} \end{bmatrix}$$

Desfacem fiecare sumă, scriem $\nabla^2 f$ ca sumă de mai multe matrici:

$$\nabla^2 f = \begin{bmatrix} \frac{-a_{(1)1}^2}{(a_{(1)}^T x + b_1)^2} & \frac{-a_{(1)1} \cdot a_{(1)2}}{(a_{(1)}^T x + b_1)^2} & \cdots & \frac{-a_{(1)n} \cdot a_{(1)1}}{(a_{(1)}^T x + b_1)^2} \\ \vdots & & & \\ \frac{-a_{(1)n} \cdot a_{(1)1}}{(a_{(1)}^T x + b_1)^2} & & \cdots & \frac{-a_{(1)n}^2}{(a_{(1)}^T x + b_1)^2} \end{bmatrix} + \begin{bmatrix} \frac{-a_{(2)1}^2}{(a_{(2)}^T x + b_2)^2} & & \cdots & \frac{-a_{(2)n} \cdot a_{(2)1}}{(a_{(2)}^T x + b_2)^2} \\ \vdots & & & \vdots \\ \frac{-a_{(2)n} \cdot a_{(2)1}}{(a_{(2)}^T x + b_2)^2} & & \cdots & \frac{-a_{(2)n}^2}{(a_{(2)}^T x + b_2)^2} \end{bmatrix}$$

$$+ \quad \cdots$$

$$+ \begin{bmatrix} \dfrac{-a_{(m)1}^2}{(a_{(m)}^T x + b_m)^2} & \cdots & \dfrac{-a_{(m)n} \cdot a_{(m)1}}{(a_{(m)}^T x + b_m)^2} \\ \vdots & & \\ \dfrac{-a_{(m)n} \cdot a_{(m)1}}{(a_{(m)}^T x + b_m)^2} & \cdots & \dfrac{-a_{(m)n}^2}{(a_{(m)}^T x + b_m)^2} \end{bmatrix}$$

$$= \sum_{i=1}^{m} \frac{-a_{(i)} \cdot a_{(i)}^T}{\left(a_{(i)}^T x + b_i\right)^2}$$

$$\|\nabla^2 f(x)\|_2 = \left\| \sum_{i=1}^{m} \frac{-a_{(i)} a_{(i)}^T}{a_{(i)}^T x + b_i} \right\|_2 \leq \sum_{i=1}^{m} \left\| \frac{-a_{(i)} a_{(i)}^T}{(a_{(i)}^T x + b_i)^2} \right\|$$

$$= \sum_{i=1}^{m} \frac{\|-a_{(i)} a_{(i)}^T\|}{\left(a_{(i)}^T x + b_i\right)^2}$$

Fie $M = \max_{i=1}^{n} \|a_{(i)}^T a_{(i)}\|$

$$\Rightarrow \|\nabla^2 f(x)\|_2 \leq M \sum_{i=1}^{n} \frac{1}{\left(a_{(i)}^T x + b_i\right)^2}$$

- Pentru $Q = \{x \in \mathbb{R}^n \mid a_{(i)}^T x + b_i > 0, \ i = \overline{1,m}\}$

Pentru $\sum_{i=1}^{n} \frac{1}{(a_{(i)}^T x + b_i)^2} \to \infty$ ($x \in \mathbb{R}^n$ are $x_i \to 0$, $i = \overline{1,n}$ și $b = 0^n$)

nu avem mărginire, deci nu există constantă Lipschitz

• Pentru $Q = \{x \in \mathbb{R}^n \mid a_{(i)}^T x + b_i \geq 1\}$, toate fracțiile sunt subunitare, deci va exista un $M \in \mathbb{R}$ care mărginește superior $\Rightarrow L = M = \max_{i=1}^{n} \| a_i^T a_i \|$

b) Alegem $m = 2$, $n = 1$

$$f(x) = \ln(a_{(1)} x + b_1) + \ln(a_{(2)} x + b_2)$$

$$f'(x) = \frac{a_{(1)}}{a_{(1)} x + b_1} + \frac{a_{(2)}}{a_{(2)} x + b_2}$$

$$f'(x) = 0 \iff a_{(1)}(a_{(2)} x + b_2) + a_{(2)}(a_{(1)} x + b_1) = 0$$

$$a_{(1)} a_{(2)} x + a_{(1)} b_2 + a_{(2)} a_{(1)} x + a_{(2)} b_1 = 0$$

$$x (a_{(1)} a_{(2)} + a_{(2)} a_{(1)}) = -a_{(1)} b_2 - a_{(2)} b_1$$

$$x = \frac{-a_{(1)} b_2 - a_{(2)} b_1}{a_{(1)} a_{(2)} + a_{(2)} a_{(1)}} = \frac{-a_{(1)} b_2 - a_{(2)} b_1}{2 a_{(1)} a_{(2)}}$$

punct de extrem

c) $$\nabla^2 f(x) = \sum_{i=1}^{m} \frac{-a_{(i)} a_{(i)}^T}{(a_{(i)}^T x + b_i)^2}$$

Știm că $\displaystyle\sum_{i=1}^{m} \frac{a_{(i)} a_{(i)}^T}{(a_{(i)}^T x + b_i)^2}$ semipozitiv definită

(produsul ~~[illegible]~~ $a_{(i)} a_{(i)}^T$ ~~[illegible]~~ este semipozitiv definit), deci toate punctele de extrem sunt puncte de minim

Dar cum $\nabla^2 f(x) = -\displaystyle\sum_{i=1}^{m} \frac{-a_{(i)} a_{(i)}^T}{(a_{(i)}^T x + b_i)^2}$, orice punct care era de minim , în cazul acesta va fi de maxim(din cauza schimbării semnului)