Risk and Fixed Income Project

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The aim of this project is not to analyze in depth the concepts of Modern Portfolio Theory nor the Black-Litterman Methodology. Here, their computational implementation will be presented. The interested reader is referred to the seminal works [Markowitz, 1952] and [Black and Litterman, 1952], as well the detailed article [Meucci, 2008].

1 Modern Portfolio Theory

1.1 Formulation

Following the assumptions of the Modern Portfolio Theory, the portfolio Expected Return and Expected Risk can be calculated as:

$$\mu_p = \mathbf{w}^T \boldsymbol{\mu} \tag{1.1}$$

$$\sigma_p^2 = \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \tag{1.2}$$

Where $\mathbf{w}$, $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ are the vector of weights, vector of returns and covariance matrix of the constituent risk assets, respectively.

There is a infinite number of possible portfolios (in practice, finitely many, as assets are not arbitrarily divisible), we can call this the attainable portfolios set. In the Risk vs Return plane, the set is bounded by the Efficient Frontier. This mean, is not possible to have a portfolio with moments outside this frontier.

This frontier is the solution of the problem:

$$\min_{\mathbf{w}} \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \tag{1.3}$$

Subject to:

$$\begin{aligned} \mathbf{w}^T \mathbf{1} &= 1 \\ \mathbf{w}^T \boldsymbol{\mu} &= \mu \end{aligned} \tag{1.4}$$

Where μ (not in bold) is the desired return.

The solution is:

$$\sigma_p^2 = \frac{c}{d}\mu^2 - \frac{2a}{d}\mu + \frac{b}{d} \tag{1.5}$$

The auxiliary functions:

$$\begin{aligned} a &= \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-} \mathbf{1} \mathbf{1} \\ b &= \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \\ c &= \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1} \\ d &= bc - a^2 \end{aligned} \tag{1.6}$$

Laying in the Frontier, a portfolios is of special interest, the Minimum Variance Portfolio. Which is The portfolio with least risk.

$$\begin{aligned} \mu^* &= \frac{a}{c} \\ \sigma^* &= \frac{1}{\sqrt{c}} \\ \mathbf{w}^* &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}} \end{aligned} \tag{1.7}$$

By adding a riskless asset, like a short term government bond, yielding r_f the investor can expand the set of attainable portfolios. But before explore it, let's look another portfolio of interest, the Tangency Portfolio.

Define the Sharpe Ratio as:

$$S_p = \frac{\mu_p - r_f}{\sigma_p} \tag{1.8}$$

The Tangency Portfolio is the one with the highest Sharpe Ratio.

$$\begin{aligned} \mu^* &= \frac{a r_f - b}{c r_f - a} \\ \sigma^* &= \sqrt{\frac{c}{d}(\mu^*)^2 - \frac{2a}{d}\mu^* + \frac{b}{d}} \\ \mathbf{w}^* &= \frac{\mathbf{\Sigma}^{-1}[\boldsymbol{\mu} - \mathbf{1} r_f]}{\mathbf{1}^T \mathbf{\Sigma}^{-1}[\boldsymbol{\mu} - \mathbf{1} r_f]} \end{aligned} \tag{1.9}$$

By investing part of the wealth in the Tangency Portfolio and the rest in the Risk Free Asset, or by borrowing at risk free and leveraging the portfolio, a new frontier is achieved. The Capital Asset Line is the line of all portfolios combinations of Risk Free and Tangency. This line is tangent to the Efficient Frontier, hence the portfolio name.

In economy, the investors are said to have Utility Functions. In our context, the level of "satisfaction" with a certain investment. One possible formulation is the Quadratic Utility function:

$$U = \mu - \frac{1}{2}\gamma\sigma^2 \tag{1.10}$$

The γ represents the level of risk aversion, [Bodie et al., 2014] suggest a typical value of 2.85.

Each choice of investment in and bellow the Capital Allocation Line will have an associated utility. But one of them will have the highest level of satisfaction. This is the Optimal Portfolio.

$$\begin{aligned} \mu^* &= r_f + \frac{1}{\gamma} S_p^2 \\ \sigma^* &= \frac{1}{\gamma} S_p \\ y &= \frac{1}{\gamma}\frac{S_p}{\sigma_p} \\ \mathbf{w}^* &= [y\mathbf{w_p}, (1-y) r_f] \end{aligned} \tag{1.11}$$

Using S_p, σ_p and $\mathbf{w_p}$ the values of the Tangency Portfolio. The allocation is part in the $\mathbf{w_p}$ and the remainder in (or borrowed from) the Risk Free asset

Aside the Modern Portfolio Theory, but a corollary, is the Kelly Portfolio. It has many interpretations, but a convenient one is that if you continuous rebalance the weights to the targets, the Kelly is the allocation that maximize the final wealth, regardless the investor risk aversion.

$$\mathbf{w}^* = (1 + r_f)\mathbf{\Sigma^{-1}}(\boldsymbol{\mu} - r_f) \tag{1.12}$$

With return and risk obtained from 1.1 and 1.2.

1.2 Risk Metrics

Two risk metris were considered in this report: VaR and CVar.

$$VaR_\$ = F_p^{-1}(\alpha) \cdot Notional \tag{1.13}$$

$$CVaR_\$ = \mu_p - \frac{\sigma_p}{\alpha}\phi[\Phi^{-1}(\alpha)] \cdot Notional \tag{1.14}$$

Where F_p^{-1} is the quantile function for the returns distribution of the portfolio. ϕ the standard Normal PDF and Φ the standard Normal CDF.

1.3 Implementation

This work was implemented in Python, as speed was not a concern. The MPT formulas were implemented in the *Portfolio* class. Bellow the class highlights, see companion code for details.

```
class Portfolio:

    def __init__(self, riskFreeRate, expecReturn,
    covMatrix, VarStdConv = True, names = [])

    def abcd(self):

    def efficFront(self,mu):

    def efficientFrontier(self, variance = True, vMu = np.arange(0,0.5,0.01)):

     def minVar(self, variance = True):

     def tangent(self):

     def optimal(self, riskAver):

     def kelly(self):

     def CAL(self, mult = 2):

     def util(self, riskAv, mult = 2):
```

The risk measures are in the *RiskMetrics* class.

1.4 Results - Simulated Data

Using below inputs the portfolios were calculatad and ploted in the (μ x σ) space.

$$r_f = 5\% \tag{1.15}$$

$$\gamma = 2.85 \tag{1.16}$$

$$\boldsymbol{\mu} = \begin{bmatrix} 8\% \\ 13\% \\ 18\% \\ 25\% \\ 30\% \end{bmatrix} \tag{1.17}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 12\% & 6\% & -1\% & 3\% & 5\% \\ 6\% & 20\% & -1\% & 2\% & 4\% \\ -1\% & -1\% & 30\% & 5\% & -1\% \\ 3\% & 2\% & 5\% & 40\% & 6\% \\ 5\% & 4\% & -1\% & 6\% & 50\% \end{bmatrix} \tag{1.18}$$

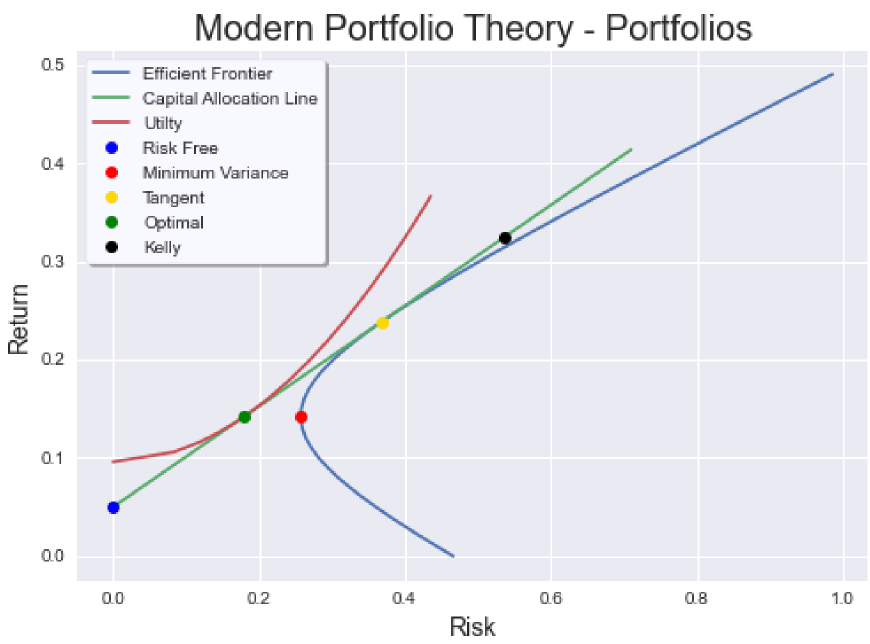

Figure 1: (μ x σ) space

The weights are:

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Risk Free
Minimum Variance	42.41%	19.28%	22.92%	8.50%	6.89%	0.00%
Tangent	42.41%	19.28%	22.92%	8.50%	6.89%	0.00%
Optimal	20.73%	9.42%	11.20%	4.15%	3.37%	51.13%
Kelly	-18.06%	36.00%	41.05%	39.80%	47.47%	-46.26%

Table 1: Weights

The VaR and CVar for the Optimal portfolio are:

VaR	\$4,334.54
CVaR	\$5,450.00

Table 2: Risk Metrics

1.5 Results - Real Data

In this section, the same results will be presented, but using real data. Ten stock were selected, their returns and covariances were estimated from 20/07/2017 to 20/07/2022. In respect of full disclosure, some stocks were changed from the initial selection to give a better final result. The final selection is in table 3.

The data were obtained using the library *investpy* which in turn, uses Yahoo Finance Data. A class *market* was defined. This object retrieves the historical price, calculate the historical return and the mean and covariance matrix.

ITUB4	PETR4	B3SA3	RENT3	WEGE3
TAEE11	VALE3	BPAC11	LCAM3	EQTL3

Table 3: Selected Stocks

The annualized returns and covariances in the period were:

	Return
ITUB4	12.20%
PETR4	29.03%
B3SA3	20.71%
RENT3	35.45%
WEGE3	34.64%
TAEE11	17.39%
VALE3	33.93%
BPAC11	51.41%
LCAM3	40.96%
EQTL3	19.87%

Table 4: Returns

	ITUB4	PETR4	B3SA3	RENT3	WEGE3	TAEE11	VALE3	BPAC11	LCAM3	EQTL3
ITUB4	10.67%	9.06%	7.35%	7.54%	4.33%	2.94%	4.76%	8.36%	6.35%	4.70%
PETR4	9.06%	23.45%	9.39%	10.36%	6.55%	3.59%	8.73%	11.28%	10.26%	5.73%
B3SA3	7.35%	9.39%	15.89%	10.41%	7.24%	3.76%	5.28%	11.95%	9.17%	6.31%
RENT3	7.54%	10.36%	10.41%	21.68%	7.53%	3.95%	5.65%	12.41%	16.78%	6.69%
WEGE3	4.33%	6.55%	7.24%	7.53%	14.53%	2.65%	4.02%	7.16%	7.54%	4.37%
TAEE11	2.94%	3.59%	3.76%	3.95%	2.65%	5.04%	1.90%	4.00%	4.01%	2.91%
VALE3	4.76%	8.73%	5.28%	5.65%	4.02%	1.90%	16.44%	6.77%	5.98%	3.23%
BPAC11	8.36%	11.28%	11.95%	12.41%	7.16%	4.00%	6.77%	24.64%	12.50%	6.78%
LCAM3	6.35%	10.26%	9.17%	16.78%	7.54%	4.01%	5.98%	12.50%	34.55%	6.89%
EQTL3	4.70%	5.73%	6.31%	6.69%	4.37%	2.91%	3.23%	6.78%	6.89%	8.61%

Table 5: Covariance Matrix

With 10 assets, the Efficient Frontier is expanded (in sample) due to the broader diversification. Is interesting to note it is broadened to the point the Efficient Frontier and the Capital Asset Line are indistinguishable after a certain point.

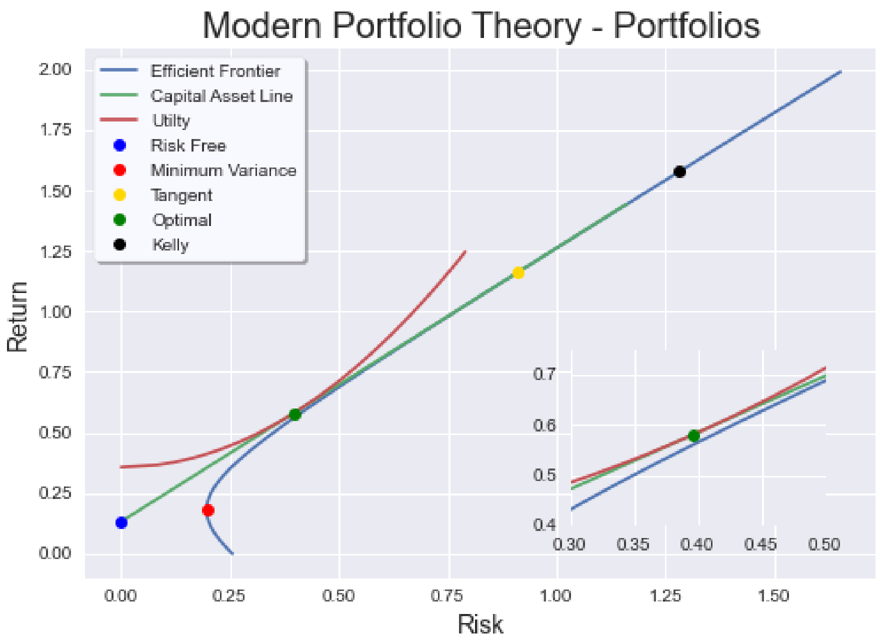

Figure 2: (μ x σ) space

The weights in each portfolio are:

	Minimum Variance	Tangent	Optimal	Kelly
ITUB4	14.77%	14.77%	6.44%	-238.16%
PETR4	-5.31%	-5.31%	-2.31%	18.31%
B3SA3	-4.67%	-4.67%	-2.03%	-144.77%
RENT3	-3.59%	-3.59%	-1.56%	51.26%
WEGE3	10.76%	10.76%	4.69%	134.31%
TAEE11	60.05%	60.05%	26.17%	0.36%
VALE3	13.44%	13.44%	5.86%	104.39%
BPAC11	-4.37%	-4.37%	-1.91%	221.41%
LCAM3	-1.02%	-1.02%	-0.45%	20.55%
EQTL3	19.93%	19.93%	8.69%	-26.97%
Risk Free	0.00%	0.00%	56.41%	-40.68%

Table 6: Weights

The VaR and CVar for the Optimal portfolio are:

VaR	$6,294.65
CVaR	$7,952.29

Table 7: Risk Metrics

2 Black-Litterman

The estimations of Covariance Matrix from the sample is somewhat informative and can be easily improved. The estimation of the Returns from the sample is more complicated, almost useless. This is a pitfall in the use of Modern Portfolio Theory.

Also, asset managers are usually proud persons full of opinions, some of then even have good information. But in the MPT, this opinion can't be passed to his portfolio.

Fischer Black and Bob Litterman proposed a novel approach [Black and Litterman, 1952] to circumvent this problems. their approach has three steps:

- Market Equilibrium Returns
- Bayesian Update of Estimations
- Portfolio Optimization

2.1 Market Equilibrium Returns

A Efficient Market in Equilibrium should allocate in the resources in the Optimal Portfolio, with the Risk Aversion being the investors aggregate. In the market, we have the assets weights, i.e, their Market Capitalization, using the estimated covariance matrix, we can invert 1.11 and extract the Market Expected Returns.

$$\boldsymbol{\mu}_{prior} = \gamma_{mkt} \boldsymbol{\Sigma} \boldsymbol{w}_{mkt} \tag{2.1}$$

2.2 Bayesian Update

The asset manager should provide three new matrix:

$\boldsymbol{P}$: A matrix with one row for each opinion and as many columns as assets. The opinions can be absolute, put a 1 in the asset in attention and 0 in the rest of columns. Or relative, with 1 in the outperforming asset, -1 in the underperforming and 0 elsewhere.

$\boldsymbol{Q}$: A vector with the return of each opinion. "Asset A return 10%, Asset B will outperform Asset C in 3%", Q = [0.1, 0.03].

$\boldsymbol{\Omega}$: The Covariance Matrix of each opinion. Usually, is a diagonal matrix, but off-diagonal elements are allowed.

A shrinkage factor τ should be provided, too, usually around 0.025 to 0.05, but its precise determination is a open problem. It indicates the manager confidence in the opinion.

The posterior expected returns are itself a multivariate Normal Distribution. Defined as:

$$
\begin{aligned}
\boldsymbol{\mu} &\sim N(\boldsymbol{m}, \boldsymbol{S}) \\
\boldsymbol{m} = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1}&[(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{Q}] \\
\boldsymbol{S} &= [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}]
\end{aligned}
\tag{2.2}
$$

The posterior returns distribution is:

$$
\boldsymbol{\mu}_{post} \sim (\boldsymbol{m}, \boldsymbol{S} + \boldsymbol{\Sigma}) \tag{2.3}
$$

2.3 Portfolio Optimization

The portfolio optimization is the standard MPT procedure.

2.4 Implementation and Results

The model was implemented in the *BlackLitterman* class.

The Opinios were arbitrarially set to:

$$
\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{2.4}
$$

$$
\boldsymbol{Q} = \begin{bmatrix} 0.2 & 0.05 & 0.08 \end{bmatrix}^T \tag{2.5}
$$

$$
\boldsymbol{\Omega} = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & -0.004 & 0 \\ 0 & 0 & 0.00225 \end{bmatrix} \tag{2.6}
$$

The τ was set at 0.025 and the Market Risk Aversion at 2.5.

The prior and posterior expected estimations are:

	Exp. Returns
ITUB4	32.17%
PETR4	47.43%
B3SA3	34.29%
RENT3	36.18%
WEGE3	29.13%
TAEE11	20.98%
VALE3	35.34%
BPAC11	39.54%
LCAM3	35.63%
EQTL3	25.96%

Table 8: Prior Returns

	Exp. Returns
ITUB4	28.82%
PETR4	39.55%
B3SA3	30.86%
RENT3	32.40%
WEGE3	26.88%
TAEE11	19.44%
VALE3	37.75%
BPAC11	36.38%
LCAM3	32.03%
EQTL3	22.41%

Table 9: Posterior Returns

It's interesting to notice the Market implied returns are much more close to each other than to the historical estimations, which leads to less extreme allocated portfolios.

	ITUB4	PETR4	B3SA3	RENT3	WEGE3	TAEE11	VALE3	BPAC11	LCAM3	EQTL3
ITUB4	0.108907	0.092188	0.074895	0.076790	0.044076	0.029956	0.048781	0.085578	0.064706	0.047555
PETR4	0.092188	0.238603	0.095399	0.105208	0.066516	0.036423	0.089920	0.114485	0.104231	0.057870
B3SA3	0.074895	0.095399	0.162043	0.105856	0.073663	0.038233	0.053952	0.121251	0.093114	0.063796
RENT3	0.076790	0.105208	0.105856	0.221310	0.076603	0.040169	0.057708	0.125874	0.171009	0.067594
WEGE3	0.044076	0.066516	0.073663	0.076603	0.148563	0.026971	0.041025	0.072621	0.076699	0.044206
TAEE11	0.029956	0.036423	0.038233	0.040169	0.026971	0.051548	0.019407	0.040574	0.040735	0.029484
VALE3	0.048781	0.089920	0.053952	0.057708	0.041025	0.019407	0.167922	0.068981	0.061049	0.032764
BPAC11	0.085578	0.114485	0.121251	0.125874	0.072621	0.040574	0.068981	0.249655	0.126644	0.068399
LCAM3	0.064706	0.104231	0.093114	0.171009	0.076699	0.040735	0.061049	0.126644	0.353096	0.069698
EQTL3	0.047555	0.057870	0.063796	0.067594	0.044206	0.029484	0.032764	0.068399	0.069698	0.087165

Table 10: Posterior Covariance

At last, but not least, the portfolios:

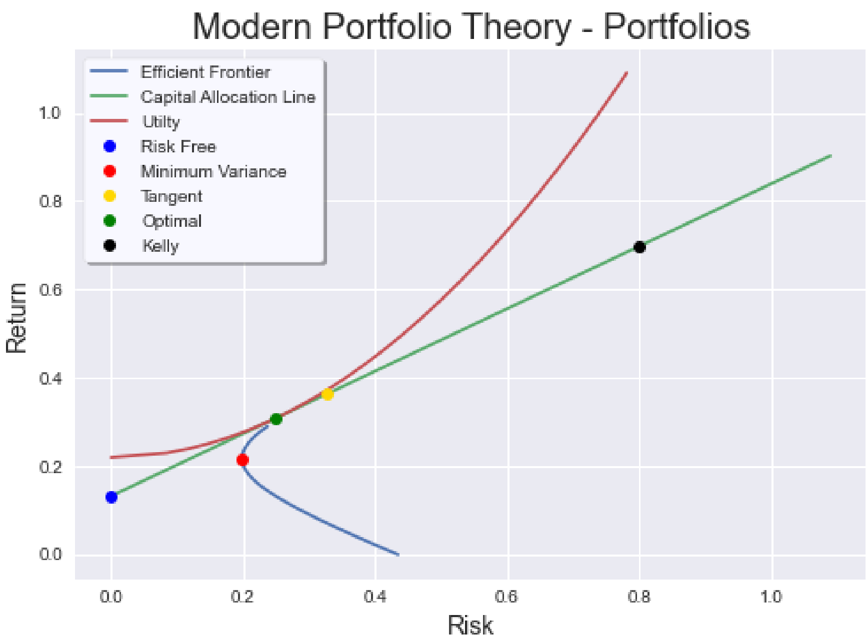

Figure 3: (μ x σ) space

Risk:

VaR	\$5,036.32
CVaR	\$6,346.75

Table 11: Risk Metrics

And compositions:

	Minimum Variance	Tangent	Optimal	Kelly
ITUB4	14.73%	14.73%	11.15%	35.88%
PETR4	-5.31%	-5.31%	-4.02%	44.21%
B3SA3	-4.64%	-4.64%	-3.51%	13.02%
RENT3	-3.57%	-3.57%	-2.70%	10.98%
WEGE3	10.70%	10.70%	8.10%	23.00%
TAEE11	59.70%	59.70%	45.20%	2.72%
VALE3	13.39%	13.39%	10.14%	109.55%
BPAC11	-4.39%	-4.39%	-3.32%	29.12%
LCAM3	-1.02%	-1.02%	-0.77%	2.39%
EQTL3	20.40%	20.40%	15.44%	-26.51%
Risk Free	0.00%	0.00%	24.29%	-144.37%

Table 12: Weights

References

[Black and Litterman, 1952] Black, F. and Litterman, R. B. (1952). Asset Allocation: Combining Investor Views with Market Equilibrium. *The Journal of Fixed Income*.

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