Rejection Sampling

Lessons from Buffon's Needle

Math 392

Buffon's Needle (1733)

Dorp a needle of unit length on to a wood floor of infinite expanse with plane of unit length. What is the probability the needle crosses a line between planks?

Parameterization

D: random distance from the needle midpoint to the line, $0 < D < \frac{1}{2}$

θ: random angle between the needle and the line, $0 < \theta < \pi$.

Observe: the needle will cross when $D' > D$

$$\sin(\theta) = \frac{D'}{1/2}; \quad D' = \frac{\sin(\theta)}{2}$$

that is, when $\frac{\sin(\theta)}{2} > D$.

Geometric probability

$$P(\text{needle crosses line}) = P\left(\frac{\sin(\theta)}{2} > D\right)$$

Monte Carlo approximation

```
theta <- runif(1e6, 0, pi)
D <- runif(1e6, 0, 1/2)
mean(sin(theta)/2 > D)
```

```
## [1] 0.635898
```

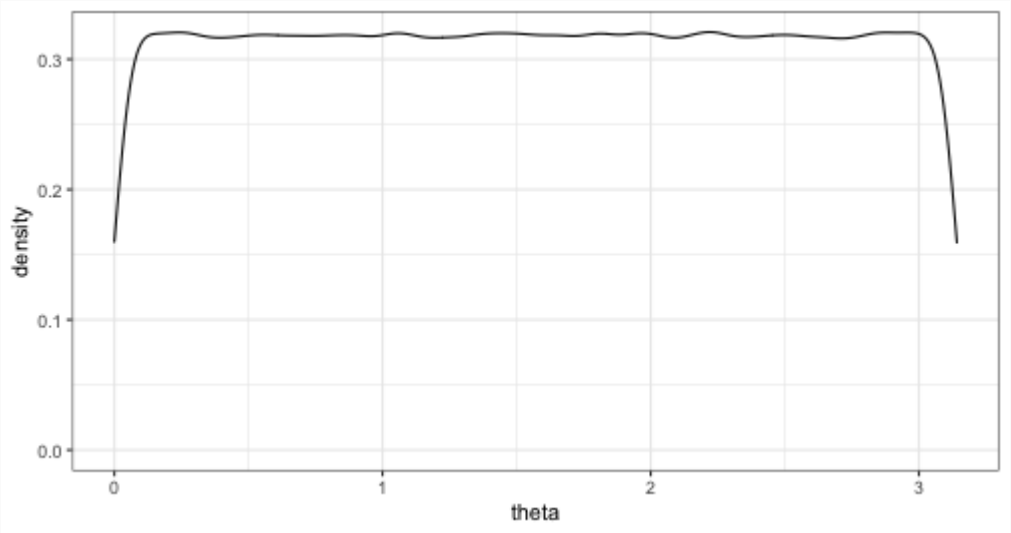
```
2/pi
```

```
## [1] 0.6366198
```

Generating from $f(\theta)$

Our original sample of θ.

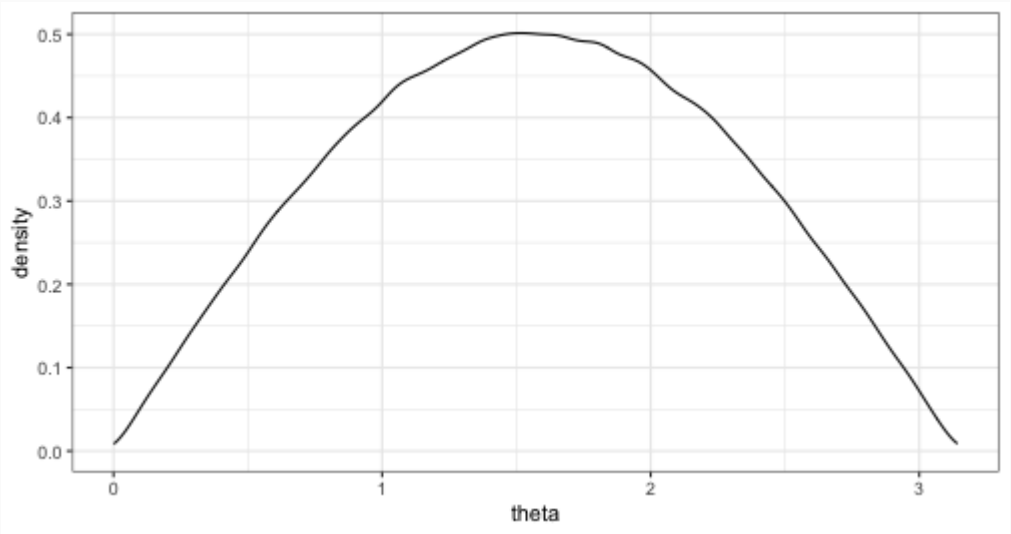
```
data.frame(theta) %>%
  ggplot(aes(x = theta)) +
  geom_density() +
  theme_bw()
```



Generating from $f(\theta)$, cont.

The *theta* that meet our condition.

```
data.frame(theta) %>%
  filter(sin(theta)/2 > D) %>%
  ggplot(aes(x = theta)) +
  geom_density() +
  theme_bw()
```



Monte Carlo approximation rephrased

1. Generate $\theta_i \sim Unif(0, pi)$
2. Generate $D \sim Unif(0, 1/2)$
3. Retain θ_i if D < f(\theta_i)$

Or:

1. Generate $\theta_i \sim Unif(0, pi)$
2. Retain θ_i with probability $\frac{f(\theta_i)}{1/2}$

Rejection sampling

Let $f(\theta)$ be the density we wish to sample from and $q(\theta)$ another density that we are able to sample from. Select an $M > 0$ such that $Mq(\theta) \geq f(\theta)$ for all values of θ.

1. Generate $\theta_i \sim q(\theta)$
2. Generate $U_i \sim Unif(0, 1)$
3. Retain $\theta_i \sim \frac{f(\theta)}{Mq(\theta)}$

Practice: Sampling from the Gamma

In groups, use rejection sampling to approximate Gamma(2, 3). Plot your approximation against the true density.