$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{1}{SS_x} \sum \left[(x_i - \bar{x}) y_i - (x_i - \bar{x}) \bar{y} \right]$$

$$= \frac{1}{SS_x} \left[\sum (x_i - \bar{x}) y_i - \cancel{\bar{y} \sum (x_i - \bar{x})} \right]$$

$$= \frac{1}{SS_x} \sum (x_i - \bar{x}) y_i$$

$$x_1 - \bar{x} + x_2 - \bar{x} + x_3 - \bar{x}$$

$$x_1 + x_2 + x_3 - n \frac{1}{n}(x_1 + x_2 + x_3)$$

$$Var(\hat{\beta}_1) = \frac{1}{(SS_x)^2} \sum Var\left[(x_i - \bar{x}) y_i \right]$$

$$= \frac{1}{(SS_x)^2} \sum (x_i - \bar{x})^2 Var(y_i)$$

$$= \frac{\sigma^2}{SS_x}$$

$$\cancel{\widehat{Var}(\beta_1)}$$

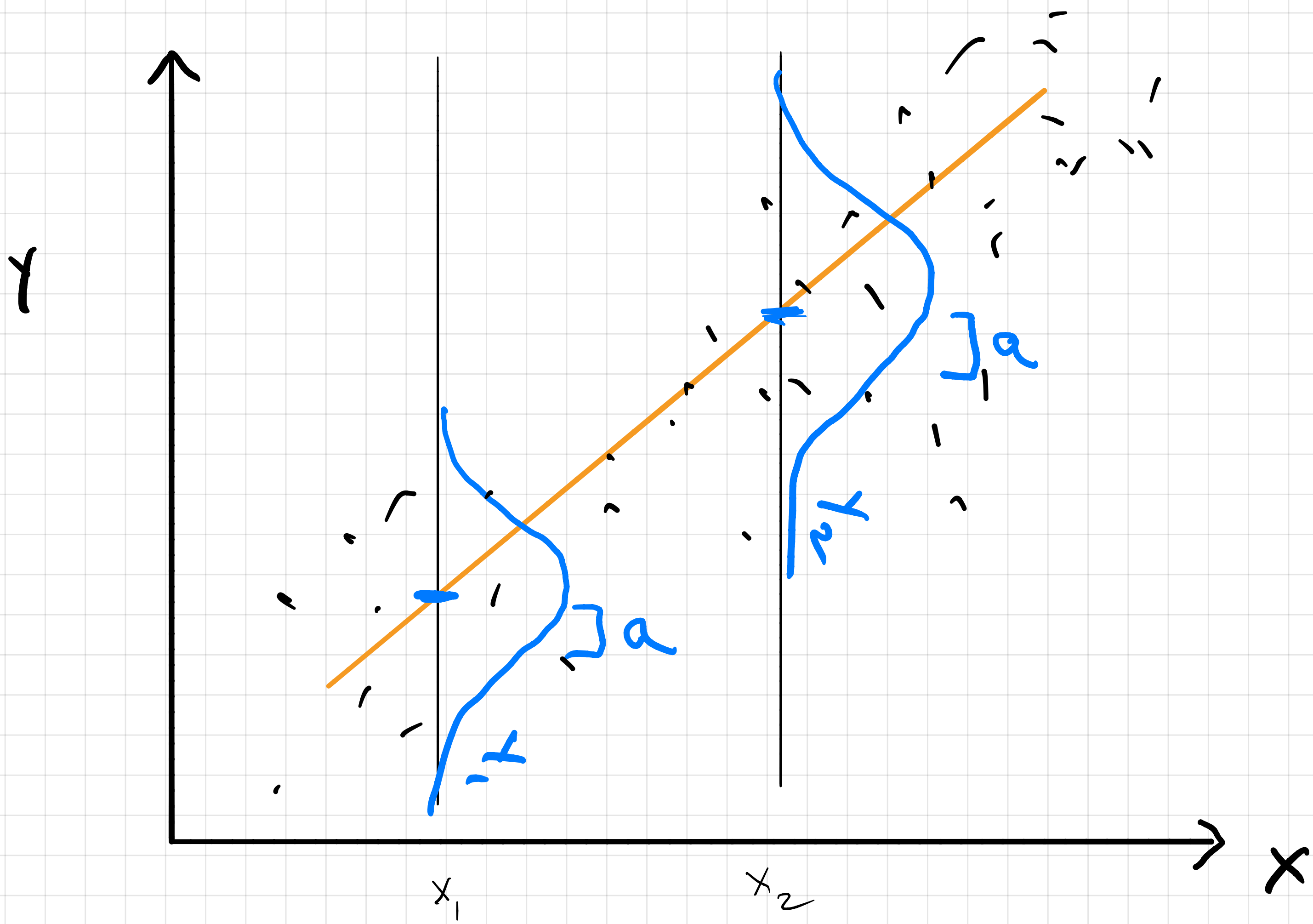
$$\widehat{Var}(\hat{\beta}_1)$$

Inference for the Simple Linear Model

The Simple Linear Model: $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$; $\varepsilon \sim N(0, \sigma^2)$

OR

$$[Y_i \mid X = x_i] \sim N(\mu = \beta_0 + \beta_1 x_i, \sigma^2 = \sigma^2)$$



Some Inferential results

(A) $\hat{\beta}_1 \sim N\left(\beta, \frac{\sigma^2}{SS_x}\right)$

(B) If σ^2 is unknown, estimate w/

$$s^2 = \frac{1}{n-2} \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2$$

- $E(s^2) = \sigma^2$
- $s^2 \sim \chi^2$-ish

(C) $$\frac{\hat{\beta}_1 - \beta}{\sqrt{s^2 / SS_x}} \sim t_{df=n-2}$$

$$\frac{\bar{X} - \mu}{\sqrt{s^2/n}} \begin{array}{l} \leftarrow Z \\ \leftarrow \chi^2 \end{array}$$

Classic Inference on $\hat{\beta}_1$

H-test on $\hat{\beta}_1$

$H_0: \beta_1 = 0$

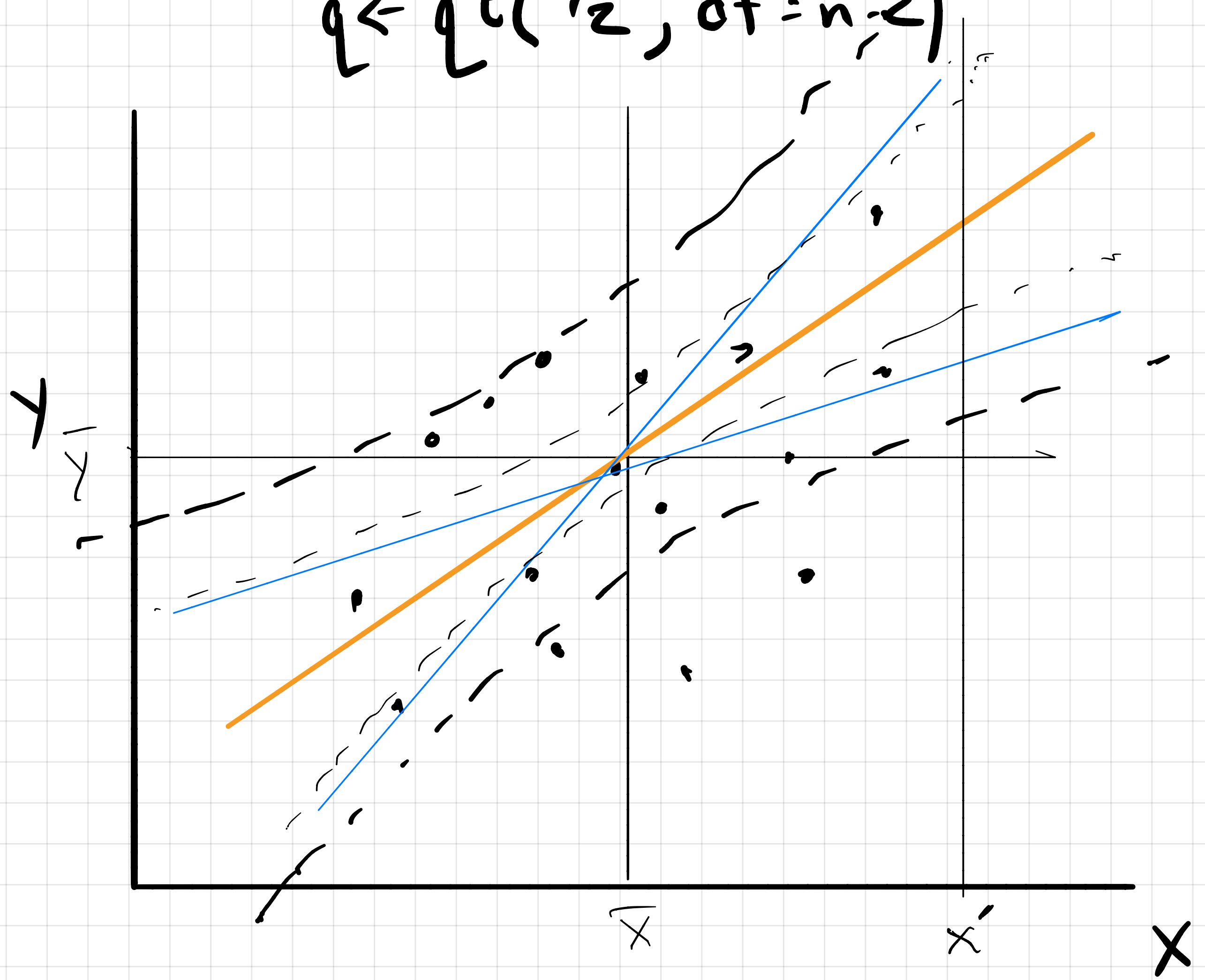
$H_1: \beta_1 \neq 0$

$$\underbrace{\frac{\hat{\beta}_1}{s/\sqrt{SS_x}}}_{\text{test statistic}} \sim t_{n-2}$$

$1-\alpha$ CI for β_1

$$\hat{\beta}_1 \pm q \, s/\sqrt{SS_x}$$

$$q \leftarrow qt(\alpha/2, df = n-2)$$



I. Assessing Uncertainty in a Prediction

We want to form a confidence interval for $\hat{Y}_{\cdot} = E(Y | X = x')$

$$E(\hat{E}(Y|X=x')) = E(\hat{\beta}_0 + \hat{\beta}_1 x')$$
$$= E(\hat{\beta}_0) + x' E(\hat{\beta}_1)$$
$$= \beta_0 + \beta_1 x'$$

if we try the same approach w/ $\widehat{Var}$ we'd need $\hat{\beta}_0 \perp\!\!\!\perp \hat{\beta}_1$ (it's not).

We do know $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$, and $\hat{\beta}_1 \perp\!\!\!\perp \bar{Y}$

$$V(\hat{E}(Y|X=x')) = Var(\hat{\beta}_0 + \hat{\beta}_1 x')$$
$$= Var(\bar{Y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x')$$
$$= Var(\bar{Y} + \hat{\beta}_1 (x' - \bar{x}))$$
$$= \frac{\sigma^2}{n} + (x' - \bar{x})^2 Var(\hat{\beta}_1)$$
$$= \frac{\sigma^2}{n} + (x' - \bar{x})^2 \frac{\sigma^2}{SSx}$$

II. Assessing Uncertainty in a new value

We want to form a CI for $[Y|X=x']$ ⇐ "Prediction Interval".

$$E(Y|X=x') = E(\hat{\beta}_0 + \hat{\beta}_1 x' + \varepsilon)$$

$$= E(\hat{\beta}_0) + x' E(\hat{\beta}_1) + E(\varepsilon)$$

$$= \beta_0 + \beta_1 x'$$

$$V(Y|X=x') = V(\hat{\beta}_0 + \hat{\beta}_1 x' + \varepsilon)$$

$$= \frac{\sigma^2}{n} + (x'-\bar{x})^2 \frac{\sigma^2}{SS_x} + \sigma^2$$

$$= \sigma^2\left(\frac{1}{n} + \frac{(x'-\bar{x})^2}{SS_x} + 1\right)$$

Find $\mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1)$

$$\mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \mathrm{Cov}(\bar{Y} - \hat{\beta}_1 \bar{x}, \hat{\beta}_1)$$

$$= 0 - \mathrm{Cov}(\hat{\beta}_1 \bar{x}, \hat{\beta}_1)$$

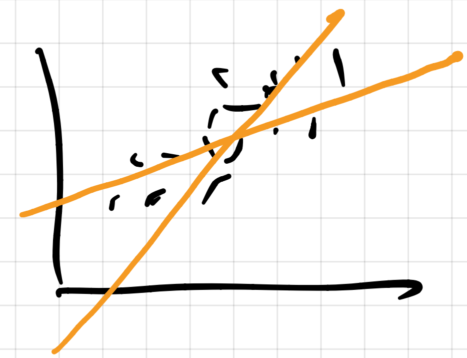
$$= 0 - \bar{x}\,\mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_1)$$

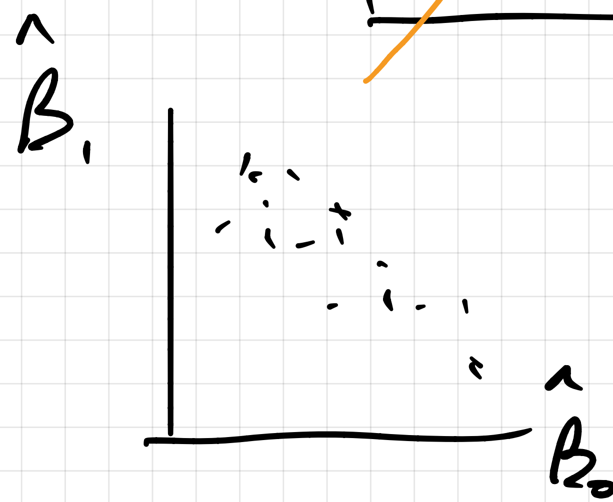
$$= 0 - \bar{x}\,\mathrm{Var}(\hat{\beta}_1)$$

$$= -\bar{x}\frac{\sigma^2}{SS_x}$$

- $\mathrm{Cov}(A - B, C) = \mathrm{Cov}(A, C) - \mathrm{Cov}(B, C)$
- $\hat{\beta}_1 \perp\!\!\!\perp \bar{Y}$

$\mathrm{Cov}(X, Y) = E\big((X - E(X))(Y - E(Y))\big)$





$$-\bar{x}\frac{\sigma^2}{SS_x}$$