MATH 392 Problem Set 5

Exercise from the book

8.5: 4

Case Study: German Tank Problem

Let's pick up the example that we began in class but make the simplifying assumption that we're studying a process where our sample is drawn from the continuous distribution, $X_1, X_2, \ldots, X_n \sim \text{Unif}(0, \theta)$, but we're still interested in estimating θ. The MLE and Method of Moments estimators are the same:

$$\hat{\theta}_{MLE} = \max(X_1, X_2, \ldots, X_n) = X_{max}$$
$$\hat{\theta}_{MOM} = 2\bar{X}$$

1. Calculate the bias of each estimator. If either one is biased, propose an additional estimator that corrects that bias (in the spirit of how s^2 is the bias-corrected version of $\hat{\sigma}^2$) . What happens to the bias of these estimators as sample size grows? Plot the relationship between sample size and bias for each estimator (two lines on one plot).
2. Calculate the variance of each estimator (including any new bias-corrected ones). What happens as sample size grows? Create an analogous plot to the one above.
3. Combine the notions of bias and variance into a third plot that shows how the Mean Squared Error changes as a function of sample size. Based on this plot, which estimator would you use and why?
4. Using the method that we saw in class based on Markov's Inequality, assess whether each of these estimators is consistent.
5. What is the sampling distribution of each statistic? For the MLE, consider both the Irwin-Hall distribution and a sensible approximation based on the Central Limit Theorem.
6. Create a plot of the sampling distribution of each estimator using $n = 10$. Construct the empirical distribution via simulation and overlay the appropriate exact or approximate analytical form (each plot should be a curve overlayed on a histogram. See slides.)
7. Form two different 95% confidence intervals for θ by using pivotal statistics inspired by each estimator.