Extending Topological Nexttime Logic

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Abstract

Subsequently, we provide an extension of topological nexttime logic, see [10], by an operator expressing increasing *of sets. The resulting formalism enables one to reason about the change of sets in the course of (discrete) linear time. We establish completeness and decidability of the new system, and we determine its complexity. As to the latter, we obtain a 'low' upper complexity bound of the corresponding satisfiability problem: NP; this is due to the fact that the time operators involved in our logic are comparatively weak. It is intended that the system is applicable to diverse fields of temporal reasoning.*

1. Introduction

In this paper we are concerned with the change of sets in the course of time. This seems to be a rather general approach since changing sets actually occur in many fields of computer science and AI. Let us mention two different topics.

First, consider a multi–agent system and an agent involved in it, for instance a processor in a distributed system; see [5], [8]. The set of states representing the *knowledge* of the agent *changes* during a run of the system. Thus, in order to specify the behaviour of such a system one should have at one's disposal a tool dealing with changing sets (of this particular kind) formally. The language we introduce below in fact originates from this context, and a *knowledge operator* is retained in it. However, its semantics differs from that of the common logic of knowledge considerably.

The second topic is spatio–temporal modelling and reasoning which currently represents a very active branch of research. It is one of the major goals of the efforts undertaken in this field to express adequately and handle formally the growing and shrinking, respectively, of various objects, and the temporal change of geometric shapes in particular.

In fact, the purpose of the present paper is to provide a simple general framework subsuming dynamics of sets of the kind indicated by these examples.

To this end we consider a certain system admitting *topological reasoning*, which has been proposed in [2]; let us just call it *TR*. According to the definition of its semantics the system *TR* captures *shrinking* of sets at least, offering access to our topic thus; moreover, it is related to the logic of knowledge. This system serves as our starting point presently. One should notice, however, that there are different formalisms of computational logic dealing with dynamic aspects of sets, but having other objectives; see [3] and [9], for instance.

We mention the very basic features of *TR*, for convenience of the reader. The system is derived from *modal logic*. (We presuppose the fundamentals of modal and temporal logic in this paper; they can be found in [1] or in [7], e.g.) Confining oneself to the single–agent case there are two modalities included in *TR:* the first, designated K, quantifies 'horizontally' over the elements of a set, and the second, $\Box$, quantifies 'vertically' over certain of its subsets expressing 'shrinking' in this way. The two modalities also interact (and their duals respectively). This interaction depends on the class of the semantical structures to be examined. It is a demanding question in general to describe the interplay of K and $\Box$ axiomatically.

Several classes of *set spaces*, i.e., pairs $(X, \mathcal{O})$ such that X is a non–empty set and $\mathcal{O}$ is a distinguished set of subsets of X, have been treated on the basis of the just indicated logical language up to now, topological spaces in particular. (Notice that certain connections have already been discovered between topology and modal logic many years ago; see [14]. Quite recently, they have been utilized for qualitative spatial reasoning in AI; see [15].) This and further examples have been worked out in [2] and [6]. In view of what follows let us mention a particular variation of *TR* where shrinking of sets proceeds in *discrete steps*. The resulting system can therefore be viewed as a generalization of propositional linear time temporal logic; see [10] and [11] for a closer examination.

In this paper we deal with the *growth* of sets additionally. It is our idea to modify the $\Box$–operator of the logic

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Figure 1. A discretely changing set

of set spaces appropriately so that the just mentioned temporal meaning is achieved. In order to guarantee that this works the expressiveness of $\Box$ has to be reduced in a sense. Actually, $\Box$ will be substituted by two *nextstep* operators corresponding to growing and shrinking respectively.

By that it is in fact possible to express modally a dynamic change of sets of the kind depicted in Figure 1. One can, for instance, describe the development of knowledge of an agent in this way, where phases of 'learning' are followed by those of 'forgetting' and vice versa. Systems regarding each of these properties separately have been examined from an implementation–oriented point of view in [4]. Note, however, that possible worlds outside the knowledge state of the agent are not considered; so, we can only speak of a description 'from the agent's point of view'.

The outline of the paper is as follows. We define the basic logical language in Section 2. Afterwards, in Section 3, we introduce the system and prove its semantical completeness. In Section 4 we prove that the set of all theorems of the system is decidable. Moreover, we show that the corresponding satisfiability problem is NP–complete.

We omit many details in this extended abstract, especially routine arguments in proofs like structural inductions; those concerning the fragment of shrinking sets can be found in [10].

2. Prerequisites

In this section we introduce a language designated *CS*. With the aid of *CS* one can speak about the **c**hange of **s**ets in the course of time.

First we determine the *syntax* of *CS*. Starting out with a suitable alphabet and a recursive set of *propositional variables*, PV, we let the set $\mathcal{F}$ of *CS–formulas* be defined as the smallest set of strings satisfying

- $\mathrm{PV} \subseteq \mathcal{F}$
- $\alpha, \beta \in \mathcal{F} \Longrightarrow \neg\,\alpha, K\alpha, \bigcirc_{\mathrm{g}}\alpha, \bigcirc_{\mathrm{s}}\alpha, (\alpha \wedge \beta) \in \mathcal{F}.$

The single step operator $\bigcirc_{\mathrm{g}}$ corresponds to growing of sets and the operator $\bigcirc_{\mathrm{s}}$ to their shrinking; furthermore, K is intended to move inside sets.

We omit brackets and use abbreviations as usual; furthermore, we let

$$\begin{array}{lcl} L\alpha & :\equiv & \neg K \neg\, \alpha, \\ \boxtimes_{\mathrm{g}}\alpha & :\equiv & \neg \bigcirc_{\mathrm{g}} \neg\, \alpha, \text{ and} \\ \boxtimes_{\mathrm{s}}\alpha & :\equiv & \neg \bigcirc_{\mathrm{s}} \neg\, \alpha, \end{array}$$

for all $\alpha \in \mathcal{F}$.

The idea to define the *semantics* of *CS* is as follows. We would like to express the change of a given set, Y. Since the set may grow or shrink in the course of time, certain subsets and supersets of Y have to be considered in the formal model. Thus we take a universe, X, in which all these sets are contained, and the system of these sets, $\mathcal{O}$, as the basic components of the domains by means of which we give meaning to formulas. But $\mathcal{O}$ is *structured* by phases of growing and shrinking respectively. This is captured by a suitable *segmentation* of $\mathcal{O}$; i.e., a partition such that on every segment the sets *either* descend *or* ascend. Finally, we want to assign a truth value with respect to 'locus' and 'time' to all propositions, by means of a *valuation* σ. Consequently, we consider certain triples $(X, (\mathcal{O}_\iota)_{\iota \in J}, \sigma)$ as the relevant semantical structures in essence, where J is a set indexing a partition of $\mathcal{O}$. Actually we choose J to be an initial segment of $\mathbb{N}$, and we let $\mathcal{O}$ be indexed in the same way.

For any set X we let $\mathcal{P}(X)$ denote the powerset of X.

Definition 2.1 *1. Let I be an initial segment of $\mathbb{N}$. A subset $I' \subseteq I$ is called* connected, *iff there is no $j \in I \setminus I'$ strictly between any two elements of I'. A partition of I into connected subsets is called a* segmentation *of I.*

2. Let X be a non–empty set, I an initial segment of $\mathbb{N}$ and

$$f : I \longrightarrow \mathcal{P}(X) \setminus \emptyset$$

a mapping. Furthermore, let I' be a connected subset of I. Then f is called increasing *on I', iff*

$$i \leq j \iff f(i) \subseteq f(j) \text{ holds for all } i, j \in I',$$

and decreasing *on I', iff*

$$i \leq j \iff f(i) \supseteq f(j) \text{ holds for all } i, j \in I'.$$

3. Let X, I and f be as above. Moreover, let

$$\mathcal{I} := (I_k^{\mathrm{c}_k})_{k \in J}$$

be a segmentation of I, where J is a suitable initial segment of $\mathbb{N}$ and $\mathrm{c}_k \in \{\mathrm{g}, \mathrm{s}\}$ for all $k \in J$. (The letter g indicates 'growing', and the letter s 'shrinking'.) Then f is called faithful *on $\mathcal{I}$, iff for all $k \in J$*

$$\begin{array}{l} f \text{ is increasing on } I_k^{\mathrm{c}_k}, \text{ if } \mathrm{c}_k = \mathrm{g}, \text{ and} \\ f \text{ is decreasing on } I_k^{\mathrm{c}_k}, \text{ if } \mathrm{c}_k = \mathrm{s}. \end{array}$$

4. *Let X, I, $\mathcal{I} = (I_k^{c_k})_{k \in J}$ and f be as in 3.*

   (a) *The pair $\mathcal{S} := (X, f)$ is called a* set frame (corresponding with $\mathcal{I}$), *iff f is faithful on $\mathcal{I}$.*

   (b) *Let*
   $$I_{\mathrm{g}} := \bigcup \{ I_k^{c_k} \mid c_k = \mathrm{g} \},$$
   and let
   $$I_{\mathrm{s}} := \bigcup \{ I_k^{c_k} \mid c_k = \mathrm{s} \}.$$
   (Note that these sets form a disjoint union of I.)

   (c) *We denote the range of f by $\mathcal{O}$ or by*
   $$\{ U_i \mid i \in I \}$$
   occasionally.

5. *Let $\mathcal{S} := (X, f)$ be a set frame, I the domain of f, and*
   $$\sigma : \mathrm{PV} \times X \times I \longrightarrow \{0, 1\}$$
   a mapping which is defined exactly for the triples (A, x, i) such that $x \in f(i)$. Then σ is called a valuation, *and the triple $\mathcal{M} := (X, f, \sigma)$ is called a* model *(based on $\mathcal{S}$).*

This definition formalizes the change of sets along the lines of *TR*.

Now we define the validity relation between formulas and so–called *situations* of set frames, which are simply pairs x, U_i (designated without brackets mostly) such that $x \in U_i \in \mathcal{O}$ $(i \in I)$. The set component U_i of a situation qualitatively measures the actual 'degree of closeness' to x.

Definition 2.2 *Let a model $\mathcal{M} = (X, f, \sigma)$ and a situation x, U_i of the underlying set frame (X, f) be given. Then we let*

$$\begin{array}{lcl}
x, U_i \models_{\mathcal{M}} A & :\Longleftrightarrow & \sigma(A, x, i) = 1 \\
x, U_i \models_{\mathcal{M}} \neg\, \alpha & :\Longleftrightarrow & x, U_i \not\models_{\mathcal{M}} \alpha \\
x, U_i \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & \left\{ \begin{array}{l} x, U_i \models_{\mathcal{M}} \alpha \textit{ and} \\ \quad x, U_i \models_{\mathcal{M}} \beta \end{array} \right. \\
x, U_i \models_{\mathcal{M}} K\alpha & :\Longleftrightarrow & \left\{ \begin{array}{l} y, U_i \models_{\mathcal{M}} \alpha \\ \textit{for all } y \in U_i \end{array} \right. \\
x, U_i \models_{\mathcal{M}} \bigcirc_{\mathrm{g}}\alpha & :\Longleftrightarrow & \left\{ \begin{array}{l} i+1 \in I_{\mathrm{g}} \textit{ and} \\ x, U_{i+1} \models_{\mathcal{M}} \alpha \end{array} \right. \\
x, U_i \models_{\mathcal{M}} \bigcirc_{\mathrm{s}}\alpha & :\Longleftrightarrow & \left\{ \begin{array}{c} i+1 \in I_{\mathrm{s}}, \\ x \in U_{i+1} \textit{ and} \\ x, U_{i+1} \models_{\mathcal{M}} \alpha, \end{array} \right.
\end{array}$$

for all $A \in \mathrm{PV}$ and $\alpha, \beta \in \mathcal{F}$.

In case $x, U_i \models_{\mathcal{M}} \alpha$ is valid we say that α *holds in $\mathcal{M}$ at the situation* x, U_i; moreover, we say that the formula $\alpha \in \mathcal{F}$ *holds in $\mathcal{M}$* (denoted by $\models_{\mathcal{M}} \alpha$), iff it holds in $\mathcal{M}$ at every situation.

Note that a formula $\bigcirc_{\mathrm{g}}\alpha$ can only hold a given situation of some model, if the actual time point lies in a 'growing phase' and is not its endpoint, or is immediately followed by such a phase; an analogous statement holds for $\bigcirc_{\mathrm{s}}\alpha$.

For convenience of the reader we also present the semantics of the duals of the single step operators:

$$\begin{array}{lcl}
x, U_i \models_{\mathcal{M}} \boxtimes_{\mathrm{g}}\alpha & :\Longleftrightarrow & \left\{ \begin{array}{c} i+1 \in I_{\mathrm{g}} \\ \Longrightarrow \\ x, U_{i+1} \models_{\mathcal{M}} \alpha \end{array} \right. \\
x, U_i \models_{\mathcal{M}} \boxtimes_{\mathrm{s}}\alpha & :\Longleftrightarrow & \left\{ \begin{array}{c} i+1 \in I_{\mathrm{s}} \wedge x \in U_{i+1} \\ \Longrightarrow \\ x, U_{i+1} \models_{\mathcal{M}} \alpha. \end{array} \right.
\end{array}$$

3. The Logic

First in this section we present a list of axioms holding in every model. Adding approriate rules gives a logical system then. Later on we prove that the given axiomatization is *complete* with respect to models based on set frames. To this end we will show that every formula that is not derivable in the system can be falsified at some situation of some model.

We divide the axioms into four groups. The first one only consists of a single scheme embedding propositional logic in the present framework.

(P) All $\mathcal{F}$–instances of propositional tautologies.

The next group concerns those axioms involving the modality K alone. They are well–known from the logic of knowledge of a single agent; see [5]. Presently they stipulate quantification inside sets. Let $\alpha, \beta \in \mathcal{F}$ be formulas.

(K1) $K(\alpha \to \beta) \to (K\alpha \to K\beta)$

(K2) $K\alpha \to \alpha$

(K3) $K\alpha \to KK\alpha$

(K4) $L\alpha \to KL\alpha$

Now a group of schemes follows, speaking about exactly one of the modalities $\bigcirc_{\mathrm{g}}$ and $\bigcirc_{\mathrm{s}}$, respectively. Let $\mathrm{c} \in \{\mathrm{g}, \mathrm{s}\}$.

($\mathrm{C1_c}$) $\boxtimes_{\mathrm{c}}(\alpha \to \beta) \to (\boxtimes_{\mathrm{c}}\alpha \to \boxtimes_{\mathrm{c}}\beta)$

($\mathrm{C2_c}$) $\bigcirc_{\mathrm{c}}\alpha \to \boxtimes_{\mathrm{c}}\alpha$

Subsequently those axioms are listed which reflect the interaction of the different modal operators.

(I 1) $L \bigcirc_{\mathrm{g}} \alpha \to \bigcirc_{\mathrm{g}} L\alpha$

(I 2) $\bigcirc_{\mathrm{s}} L\alpha \to L \bigcirc_{\mathrm{s}} \alpha$

(I 3) $K \boxtimes_{\mathrm{s}} (\alpha \to L\beta) \vee K \boxtimes_{\mathrm{s}} (\beta \to L\alpha)$

(I 4) $K \boxtimes_{\mathrm{g}} \alpha \vee K \boxtimes_{\mathrm{s}} \beta$

In modal logic some of these schemes determine certain properties of the accessibility relations of the frames in which they are valid; e.g., (K2) corresponds with reflexivity in this sense, (K3) with transitivity, (K4) with weak symmetry[1], and (C2$_{\mathrm{c}}$) with partial functionality. While the scheme (I 4) of the last group in particular implies exclusion of simultaneous growing and shrinking, the first three axioms of this group are in accordance with certain requirements on the *composition* of the corresponding accessibility relations on the canonical model of the logical system which we are going to introduce now.

This system is obtained by adding four derivation rules to the axioms, which read

$$\frac{\alpha \to \beta, \alpha}{\beta} \qquad \frac{\alpha}{K\alpha} \qquad \frac{\alpha}{\boxtimes_{\mathrm{c}}\alpha},$$

for all $\mathrm{c} \in \{\mathrm{g}, \mathrm{s}\}$ and $\alpha, \beta \in \mathcal{F}$. As usual they are called *modus ponens, K–necessitation* and *$\boxtimes_{\mathrm{c}}$–necessitation* respectively.

Let us call the resulting system **C**. Soundness of **C** with respect to the intended structures can easily be established.

Proposition 3.1 *All of the above axioms hold in every model, and the rules preserve validity.*

Proof. Straightforward. □

We are going to show next how completeness of the system **C** with respect to the class of models based on set frames can be proved.

We start out with the *canonical model* $\widetilde{\mathcal{M}}$ of **C**. This is formed in the usual way (see [7], §5); i.e., the domain C of $\widetilde{\mathcal{M}}$ consists of all maximal **C**–consistent sets of formulas, and the accessibility relations induced by the modal operators K and $\boxtimes_{\mathrm{c}}$ ($\mathrm{c} \in \{\mathrm{g}, \mathrm{s}\}$) are defined as follows:

$$\begin{aligned} s \xrightarrow{L} t \quad &:\Longleftrightarrow \quad \{\alpha \in \mathcal{F} \mid K\alpha \in s\} \subseteq t \\ s \xrightarrow{\bigcirc_{\mathrm{c}}} t \quad &:\Longleftrightarrow \quad \{\alpha \in \mathcal{F} \mid \boxtimes_{\mathrm{c}}\alpha \in s\} \subseteq t, \end{aligned}$$

for all $s, t \in C$. Finally, the distinguished valuation of the canonical model is defined by

$$\sigma(A, s) = 1 : \Longleftrightarrow A \in s,$$

[1] Also called the *euclidean property*, in [7] (p. 12) for instance. A binary relation R on a set X fulfills this property by definition, iff $s\, R\, t$ and $s\, R\, u$ implies $t\, R\, u$, for all $s, t, u \in X$.

for all $A \in \mathrm{PV}$ and $s \in C$.

The subsequent *truth lemma* is well–known.

Lemma 3.2 *Let us denote the usual satisfaction relation of multimodal logic by $\models$, and let $\Vdash$ designate* **C***–derivability. Then, for all $\alpha \in \mathcal{F}$ and $s \in C$, it holds that*

(a) $\widetilde{\mathcal{M}} \models \alpha[s]$ *iff* $\alpha \in s$, *and*

(b) $\widetilde{\mathcal{M}} \models \alpha$ *iff* $\Vdash \alpha$.

The next lemma is commonly known as well. The axiom schemes (K2) – (K4) are responsible for (a), whereas (C2$_{\mathrm{c}}$) is used to establish (b) ($\mathrm{c} \in \{\mathrm{g}, \mathrm{s}\}$).

Lemma 3.3 *(a) The relation $\xrightarrow{L}$ is an equivalence relation on the set C.*

(b) For each $\mathrm{c} \in \{\mathrm{g}, \mathrm{s}\}$, the relation $\xrightarrow{\bigcirc_{\mathrm{c}}}$ is a partial function on C.

We now utilize the schemes (I 1) and (I 2).

Proposition 3.4 *Let $s, t, u \in C$ be given.*

(a) Assume that $s \xrightarrow{L} t \xrightarrow{\bigcirc_{\mathrm{g}}} u$ holds. Then there exists a point $v \in C$ satisfying

$$s \xrightarrow{\bigcirc_{\mathrm{g}}} v \xrightarrow{L} u.$$

(b) Now assume that $s \xrightarrow{\bigcirc_{\mathrm{s}}} t \xrightarrow{L} u$ is valid. Then there exists a point $v \in C$ satisfying

$$s \xrightarrow{L} v \xrightarrow{\bigcirc_{\mathrm{s}}} u.$$

Proof. We only prove (a) because the proof of (b) proceeds analogously.

We consider the following set of formulas

$$S := \{\alpha \in \mathcal{F} \mid \boxtimes_{\mathrm{g}}\alpha \in s\} \cup \{L\beta \in \mathcal{F} \mid \beta \in u\},$$

and we assume towards a contradiction that this set is inconsistent. Then some finite subset

$$\{\alpha_1, \ldots, \alpha_n, L\beta_1, \ldots, L\beta_m\}$$

of S is inconsistent as well. Using standard techniques from modal proof theory we first get

$$\Vdash L\beta \to \neg\, \alpha,$$

where $\beta := \beta_1 \wedge \ldots \wedge \beta_m$ and $\alpha := \alpha_1 \wedge \ldots \wedge \alpha_n$. This implies

$$\Vdash \bigcirc_{\mathrm{g}} L\beta \to \bigcirc_{\mathrm{g}} \neg\, \alpha.$$

With the aid of axiom (I 1) we obtain

$$\Vdash L \bigcirc_{\mathrm{g}} \beta \rightarrow \bigcirc_{\mathrm{g}} \neg \alpha.$$

Now we apply well–known properties of maximal consistent sets: Because of the maximality of u, β is contained in u. Thus, according to

$$s \xrightarrow{L} t \xrightarrow{\bigcirc_{\mathrm{g}}} u,$$

the formula $L \bigcirc_{\mathrm{g}} \beta$ is a member of s. We conclude that $\bigcirc_{\mathrm{g}} \neg \alpha \in s$ holds, too. But this contradicts $\boxtimes_{\mathrm{g}} \alpha \in s$. Consequently, our assumption is false; i.e., the above considered set S is consistent. Therefore S is contained in some maximal **C**–consistent set v. The point $v \in C$ fulfills the desired property. □

Following a common manner of speaking (in [2], e.g.) the assertion of (b) is called the *cross property*. Accordingly, let us call the assertion of (a) the *modified cross property*.

With the aid of the preceding proposition we immediately get the following corollary.

Corollary 3.5 *Let a natural number $n > 1$ and points $s_1, \ldots, s_n, t \in C$ be given.*

(a) Assume that

$$t \xrightarrow{L} s_1 \xrightarrow{\bigcirc_{\mathrm{g}}} s_2 \xrightarrow{\bigcirc_{\mathrm{g}}} \ldots \xrightarrow{\bigcirc_{\mathrm{g}}} s_n$$

holds. Then there are $s'_2, \ldots, s'_n \in C$ satisfying

$$t \xrightarrow{\bigcirc_{\mathrm{g}}} s'_2 \xrightarrow{\bigcirc_{\mathrm{g}}} s'_3 \xrightarrow{\bigcirc_{\mathrm{g}}} \ldots \xrightarrow{\bigcirc_{\mathrm{g}}} s'_n,$$

and $s_i \xrightarrow{L} s'_i$ is valid for $i = 2, \ldots, n$.

(b) Now assume that we have

$$s_1 \xrightarrow{\bigcirc_{\mathrm{s}}} s_2 \xrightarrow{\bigcirc_{\mathrm{s}}} \ldots \xrightarrow{\bigcirc_{\mathrm{s}}} s_n \xrightarrow{L} t.$$

Then there are $s'_1, \ldots, s'_{n-1} \in C$ satisfying

$$s'_1 \xrightarrow{\bigcirc_{\mathrm{s}}} s'_2 \xrightarrow{\bigcirc_{\mathrm{s}}} \ldots \xrightarrow{\bigcirc_{\mathrm{s}}} s'_{n-1} \xrightarrow{\bigcirc_{\mathrm{s}}} t$$

and $s_i \xrightarrow{L} s'_i$ for $i = 1, \ldots, n-1$.

Next we examine the part that axiom (I 3) plays. Let $[s]$ denote the $\xrightarrow{L}$–equivalence class of $s \in C$. Define the following binary relation $\succ_{\mathrm{s}}$ on the set

$$\bar{C} := \{[s] \mid s \in C\}$$

of all $\xrightarrow{L}$–equivalence classes:

$$[s] \succ_{\mathrm{s}} [t] :\Longleftrightarrow \begin{cases} \text{there are } s' \in [s], t' \in [t] \\ \quad \text{such that } s' \xrightarrow{\bigcirc_{\mathrm{s}}} t', \end{cases}$$

for all $[s], [t] \in \bar{C}$. Actually this relation turns out to be *functional*.

Proposition 3.6 *The just defined relation $\succ_{\mathrm{s}}$ is a partial function on the set $\bar{C}$.*

Proof. See [10], Proposition 3.5. □

Let us also define a corresponding relation $\succ_{\mathrm{g}}$ on $\bar{C}$.

$$[s] \succ_{\mathrm{g}} [t] :\Longleftrightarrow \begin{cases} \text{there are } s' \in [s], t' \in [t] \\ \quad \text{such that } s' \xrightarrow{\bigcirc_{\mathrm{g}}} t', \end{cases}$$

for all $[s], [t] \in \bar{C}$. — Although the *cross property* and the *modified cross property* look completely symmetrical, their effect on the canonical model concerning the development of $\xrightarrow{L}$–equivalence classes with respect to $\succ_{\mathrm{s}}$ and $\succ_{\mathrm{g}}$, respectively, is different. In fact, in order to obtain the analogue of Proposition 3.6 with respect to $\succ_{\mathrm{g}}$ axiom (I 1) is already sufficient.

Proposition 3.7 *The binary relation $\succ_{\mathrm{g}}$ is a partial function on the set $\bar{C}$.*

Proof. Follows immediately from Lemma 3.3 (b) and Proposition 3.4 (a). □

Finally, the relations $\succ_{\mathrm{s}}$ and $\succ_{\mathrm{g}}$ mutually exclude each other in the following sense.

Proposition 3.8 *Let $s, t \in C$ be given. Then it cannot happen that both*

$$[s] \succ_{\mathrm{s}} [t] \textit{ and } [s] \succ_{\mathrm{g}} [t]$$

holds.

Proof. Suppose on the contrary that there exist $s', s'' \in [s]$ and $t', t'' \in [t]$ such that

$$s' \xrightarrow{\bigcirc_{\mathrm{g}}} t' \text{ and } s'' \xrightarrow{\bigcirc_{\mathrm{s}}} t''.$$

Choose formulas $\gamma \in t'$ and $\delta \in t''$. Then

$$L \bigcirc_{\mathrm{g}} \gamma \in s \text{ and } L \bigcirc_{\mathrm{s}} \delta \in s$$

holds, hence also

$$L \bigcirc_{\mathrm{g}} \gamma \wedge L \bigcirc_{\mathrm{s}} \delta \in s.$$

Letting $\alpha := \neg \gamma$ and $\beta := \neg \delta$ we obtain

$$\neg (K \boxtimes_{\mathrm{g}} \alpha \vee K \boxtimes_{\mathrm{s}} \beta) \in s,$$

violating axiom (I 4). Consequently, the assumption is false; i.e., the assertion of the proposition is valid. □

As a consequence of this proposition we get that even $s \xrightarrow{\bigcirc_{\mathrm{s}}} t$ and $s \xrightarrow{\bigcirc_{\mathrm{g}}} t$ cannot hold simultaneously.

Corollary 3.9 *Let $s, t \in C$ be given. Then*

$$s \xrightarrow{\bigcirc_s} t \implies \textit{not } s \xrightarrow{\bigcirc_g} t$$

is valid.

Proof. Otherwise we would also have $[s] \succ_s [t]$ and $[s] \succ_g [t]$, by the definition of $\succ_s$ and $\succ_g$ respectively. □

Now we are in a position to define a model $\mathcal{M}$ falsifying a given non–derivable formula $\alpha \in \mathcal{F}$. For this purpose we first choose a maximal **C**–consistent set $s_\alpha \in C$ containing $\neg\,\alpha$.

Let $\succ^l$ designate the l–fold iteration of the relation $\succ_g \cup \succ_s$, for every $l \in \mathbb{N}$. For all $k \in \mathbb{N}$ and points $t \in C$ such that $[s_\alpha] \succ^k [t]$ consider the function

$$f_t^k : I_t \longrightarrow C$$

which is inductively defined on a suitable initial segment I_t of $\mathbb{N}$ by

$$f_t^k(0) := t$$

$$f_t^k(n+1) := \begin{cases} t' \in C & \text{satisfying} \\ & f_t^k(n) \xrightarrow{\bigcirc_c} t', \text{ if} \\ & f_t^k(n),\ c \in \{g, s\}, \\ & \text{and } t' \text{ exist}, \\ \text{undefined} & \text{otherwise}, \end{cases}$$

for all $n \in \mathbb{N}$; f_t^k is well–defined because of Lemma 3.3 (b) and Corollary 3.9.

Taking advantage of Proposition 3.6 and Proposition 3.7 the following statement can be proved by an induction argument.

Proposition 3.10 *Let $t, t' \in C$ and $k, k' \in \mathbb{N}$. Assume that for some $j \in \mathbb{N}$ both $f_t^k(j)$ and $f_{t'}^{k'}(j)$ are defined. Then $f_t^k(j) \xrightarrow{L} f_{t'}^{k'}(j)$ is valid.*

Call a function f_t^k of the type introduced above *maximal*, iff either $k = 0$ holds, or there do not exist $u \in C$ and $c \in \{g, s\}$ such that

$$[s_\alpha] \succ^{k-1} [u] \text{ and } u \xrightarrow{\bigcirc_c} t.$$

Define

$$X := \left\{ f_t^k \mid \left\{ \begin{array}{c} k \in \mathbb{N},\ t \in S,\ [s_\alpha] \succ^k [t], \\ \text{and } f_t^k \text{ is maximal} \end{array} \right\} \right\}.$$

The set X will be the carrier of $\mathcal{M}$.

In order to define $\mathcal{O}$ let I be the one initial segment of the natural numbers having

$$m := \max\{1, \max\{k + \mathrm{card}(I_t) - 1 \mid f_t^k \in X\}$$

as largest element, if this maximum exists, and being equal to $\mathbb{N}$ otherwise. Let

$$U_i := \{f_t^k \in X \mid k \leq i \text{ and } f_t^k(i) \text{ exists}\},$$

for all $i \in I$.

According to Proposition 3.8 a segmentation

$$\mathcal{I} := (I_k^{c_k})_{k \in J}$$

of I is induced by $\succ_g$ and $\succ_s$ respectively, where J is a suitable set of indices and $c_k \in \{g, s\}$; furthermore, the mapping f given by $i \mapsto U_i$ for all $i \in I$ is faithful on $\mathcal{I}$ because of Proposition 3.4 and the definition of U_i. Thus we have:

Proposition 3.11 *The just defined structure $\mathcal{S} := (X, f)$ is a set frame.*

It remains to define a valuation on $\mathcal{S}$. This is done by

$$\sigma(A, f_t^k, i) = 1 :\Longleftrightarrow A \in f_t^k(i),$$

for all $A \in \mathrm{PV}$, $f_t^k \in X$ and $i \in I$ such that $f_t^k(i)$ exists.

The following *truth lemma* is crucial.

Lemma 3.12 *Let X, f and σ be as above. Define $\mathcal{M} := (X, f, \sigma)$. Then, for all $\beta \in \mathcal{F}$, $f_t^k \in X$ and $k \leq i \in I$ such that $f_t^k(i)$ exists, we have*

$$f_t^k, U_i \models_{\mathcal{M}} \beta \iff \beta \in f_t^k(i).$$

Proof. By induction on the structure of α ; the interesting case $\beta = K\gamma$ needs Corollary 3.5 and Proposition 3.10, among other things. □

Letting $\beta = \neg\,\alpha$, $t = s_\alpha$, and $i = 0$, we obtain the desired completeness result.

Theorem 3.13 *Every formula that is not derivable in the system* **C** *can be falsified at some situation of some model based on a certain set frame.*

Combining Proposition 3.1 and Theorem 3.13 we get:

Corollary 3.14 *A formula $\alpha \in \mathcal{F}$ is* **C***–derivable, iff it holds in every model.*

4. Decidability and Complexity

Starting out from the above completeness proof *decidability* of our logic can be obtained rather easily now.

In fact, the logical system **C** satisfies the *finite model property* in the sense that every non–derivable formula $\alpha \in \mathcal{F}$ can be refuted in some finite model of the axioms. The reason for this fact is the bounded 'scope' of a formula holding at some situation of a given model.

Let a formula β, a model $\mathcal{M} = (X, f, \sigma)$, a point $x \in X$, and a natural number $i \in \mathbb{N}$ be given such that $x, U_i \models_{\mathcal{M}} \beta$. Without loss of generality we may assume that $i = 0$ holds additionally. Let $\mathrm{r}(\beta)$ be the $\bigcirc$*–rank* of β, i.e., the degree of nesting the $\bigcirc_{\mathrm{c}}$–operators in β ($\mathrm{c} \in \{\mathrm{g}, \mathrm{s}\}$). Finally, let $\mathcal{M}'$ be the structure obtained from $\mathcal{M}$ by restricting f to the initial segment

$$\{0, \ldots, \mathrm{r}(\beta)\}$$

of its domain (hence 'cutting' the image of f after the initial segment of length $\mathrm{r}(\beta)$), letting the carrier X be unaltered, and restricting the valuation σ appropriately. Then the following *coincidence lemma* is valid, which can be proved by a structural induction.

Lemma 4.1 *For all subformulas γ of β, natural numbers i such that $0 \leq i \leq \mathrm{r}(\beta)$, and points $x \in X$, we have that if $\mathrm{r}(\gamma) \leq \mathrm{r}(\beta) - i$, then*

$$x, U_i \models_{\mathcal{M}'} \gamma \textit{ iff } x, U_i \models_{\mathcal{M}} \gamma.$$

Now let $\alpha \in \mathcal{F}$ be not derivable in **C**. According to Theorem 3.13 and Lemma 4.1 which is applied to $\beta = \neg\, \alpha$, the formula α fails at some situation of a model

$$\mathcal{M}' = (X, f', \sigma')$$

such that f' has finite domain $I_{f'}$, thus finite image, too, which we designate $\mathcal{O}$. Taking advantage of the following equivalence relation on X a *finite* model can be obtained, falsifying α as well:

$$x \sim y :\Longleftrightarrow \left\{ \begin{array}{c} \text{for all } i \in I_{f'},\ U_i \in \mathcal{O} \text{ and} \\ A \in \mathrm{PV} \text{ occurring in } \alpha : \\ x \in U_i \text{ iff } y \in U_i, \text{ and} \\ \sigma(A, x, i) = \sigma(A, y, i). \end{array} \right.$$

In fact, the new model $\mathcal{M}_\sim$ consists of the set of all $\sim$–equivalence classes $x_\sim$ of $x \in X$, the induced set

$$\mathcal{O}_\sim = \{(U_i)_\sim \mid i \in I_{f'}\}$$

inherits the linear structure from $\mathcal{O}$, and the valuation is given by

$$\sigma_\sim(A, x_\sim, i) = 1 :\Longleftrightarrow \exists y \in x_\sim : \sigma(A, y, i) = 1,$$

for all $A \in \mathrm{PV}$, $x \in X$ and $i \in I_{f'}$ such that $x \in f'(i)$. According to the following lemma the validity of subformulas of α is preserved by passing from $\mathcal{M}'$ to $\mathcal{M}_\sim$.

Lemma 4.2 *$\mathcal{M}_\sim$ is a model. Moreover, for all subformulas β of α, natural numbers $i \in I_{f'}$, and points $x \in X$, we have*

$$x, U_i \models_{\mathcal{M}'} \beta \textit{ iff } x_\sim, (U_i)_\sim \models_{\mathcal{M}_\sim} \beta.$$

Proof. Due to the definition of $\sim$, the valuation $\sigma_\sim$ is well–defined. Everything else is more or less obvious. □

Consequently, it suffices to consider finite models based on set spaces in order to refute a given non–derivable formula. This gives decidability in a well–known manner.

Theorem 4.3 *The set of formulas derivable in* **C** *is decidable.*

Let us call a formula $\alpha \in \mathcal{F}$ *satisfiable*, iff it holds at some situation of some model. Lemma 4.1 and Lemma 4.2 show that a satisfiable formula can always be realized in a 'flat' finite model, in which the chain of distinguished subsets is short. Unfortunately the model is possibly too 'broad' at the same time. We are going to show that we can guarantee satisfaction even in a 'slender' model if the given formula is satisfiable at all.

Proposition 4.4 *Let $\alpha \in \mathcal{F}$ be satisfiable. Then there exists a model $\widetilde{\mathcal{M}} = (\widetilde{X}, \tilde{f}, \tilde{\sigma})$ such that*

- $\mathrm{r}(\alpha) \cdot \mathrm{length}(\alpha)$ *is an upper bound of the size of $\widetilde{X}$,*
- $\mathrm{r}(\alpha)$ *is an upper bound of the size of the domain of $\tilde{f}$, and*
- *α holds in $\widetilde{\mathcal{M}}$ at some situation.*

Proof. We start with the finite model of α that has been obtained in the decidability proof and call it $\mathcal{M} = (X, f, \sigma)$ presently. We thus have $x, U_0 \models_{\mathcal{M}} \alpha$ for some $x \in X$ and $U_0 = f(0)$.

Let I be the domain of f and $\mathcal{I} = (I_k^{c_k})_{k \in J}$ be the segmentation of I which (X, f) corresponds with. For every $k \in J$ let i_k be the minimal element of $I_k^{c_k}$ if $\mathrm{c}_k = \mathrm{g}$, and the maximal one if $\mathrm{c}_k = \mathrm{s}$. Choose some $y_k \in f(i_k)$ for each $k \in J$, and let

$$L_k := \left\{ (i, K\beta) \mid \left\{ \begin{array}{c} i \in I_k,\ K\beta \text{ a subformula} \\ \text{of } \alpha, \text{ and } y_k, U_i \not\models K\beta \end{array} \right\} \right\}.$$

For all $k \in J$ and $j = (i, K\beta) \in L_k$ choose a point $x_j \in X$ such that $x_j, U_i \not\models \beta$. Define

$$\widetilde{X} := \{x_j \mid k \in J \text{ and } j \in L_k\} \cup \{x\},$$

and let $\tilde{f}$ be given by

$$\tilde{f}(i) := U_i \cap \widetilde{X} =: \widetilde{U}_i,$$

for all $i \in I$; moreover, let $\tilde{\sigma}$ be the restriction of σ in the first component of its domain to $\widetilde{X}$. Then

$$\widetilde{\mathcal{M}} := (\widetilde{X}, \tilde{d}, \tilde{\sigma})$$

is clearly a model based on some set space.

By induction on the structure of formulas the following assertion can be proved:

For all subformulas γ of α, indices $k \in J$, $i \in I_k$, and points $\tilde{x} \in \widetilde{X}$ such that $\tilde{x} \in \widetilde{U}_i$, it holds that

$$\tilde{x}, \widetilde{U}_i \models_{\widetilde{\mathcal{M}}} \gamma, \text{ iff } \tilde{x}, U_i \models_{\mathcal{M}} \gamma.$$

The third assertion of the proposition can be inferred from this assertion now, whereas the bounds specified above are clear from the construction. □

It is not hard to obtain the following main result of this section with the aid of Proposition 4.4.

Theorem 4.5 *The problem to decide whether a formula $\alpha \in \mathcal{F}$ is satisfiable is* NP*–complete.*

5. Concluding Remarks

In the present paper we have examined a modal logic describing the change of sets in the course of time. Here ‘change’ means growing and shrinking respectively, neither alteration in shape nor in quantity, and ‘time’ proceeds in single steps. We have proposed a sound and complete axiomatization of the set S of formulas which are valid in all models based on set spaces. Such set spaces precisely represent our intended semantical structures. Furthermore, we have shown decidability of S and determined the complexity of the associated satisfiability problem. The latter is complete in NP, thus no worse than the corresponding problem of the standard modal system S5 [13], linear tense logics [16], and topological nexttime logic [10] (and, clearly, propositional logic). This is essentially due to the weakness of the temporal connectives.

It is very desirable to get a corresponding system with more expressive operators speaking about time. However, this seems to be difficult. Only restricting to a single property, shrinking *or* growing, yields satisfactory results; as to increasing sets, see [12] for instance.

On the other hand, a generalization of our approach to the case of more than one K–operator is obvious; however, the complexity assertion 4.5 is clearly no longer valid then.

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References

[1] B. F. Chellas. *Modal Logic: An Introduction*. Cambridge University Press, Cambridge, 1980.

[2] A. Dabrowski, L. S. Moss, and R. Parikh. Topological Reasoning and The Logic of Knowledge. *Annals of Pure and Applied Logic*, 78: 73–110, 1996.

[3] J. Davoren. *Modal Logics for Continuous Dynamics*. PhD dissertation, Department of Mathematics, Cornell University, 1998.

[4] C. Dixon and M. Fisher. Tableaux for Synchronous Systems of Knowledge and Time with Interactions. In G. Grahne (ed.) *SCAI'97 – Sixth Scandinavian Conference on Artificial Intelligence*. Frontiers in Artificial Intelligence and Applications, Vol. 40:28–39, IOS Press, Amsterdam, August 1997.

[5] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning about Knowledge*. MIT Press, Cambridge (Mass.), 1995.

[6] K. Georgatos. Knowledge on Treelike Spaces. *Studia Logica*, 59(2): 271–301, 1997.

[7] R. Goldblatt. *Logics of Time and Computation*. Center for the Study of Language and Information, Stanford, CA, 1987.

[8] J. Y. Halpern and Y. Moses. Knowledge and Common Knowledge in a Distributed Environment. *Journal of the ACM*, 37(3):549–587, 1990.

[9] M. R. Hansen and Z. Chaochen. Duration Calculus: Logical Foundations. *Formal Aspects of Computing*, 9:283–330, 1997.

[10] B. Heinemann. Topological nexttime logic. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev (eds.) *Advances in Modal Logic*, Vol. 1:99–113, CSLI Publications 87, Stanford, CA, 1998.

[11] B. Heinemann. Temporal Aspects of the Modal Logic of Subset Spaces. *Theoretical Computer Science*, 224(1-2):135–155, 1999.

[12] B. Heinemann. On Sets Growing Continuously. In C. Pandu Rangan, V. Raman, and R. Ramanujam (eds.) *Fundamentals of Software Technology & Theoretical Computer Science (19th Conference, FST&TCS'99)*. Lecture Notes in Computer Science, Vol. 1738:420–431, Springer, Berlin, December 1999.

[13] R. E. Ladner. The Computational Complexity of Provability in Systems of Modal Propositional Logic. *SIAM Journal of Computing*, 6(3):467–480, 1977.

[14] J. C. C. McKinsey. A Solution to the Decision Problem for the Lewis Systems S2 and S4, with an Application to Topology. *Journal of Symbolic Logic*, 6(3):117–141, 1941.

[15] W. Nutt. On the Translation of Qualitative Spatial Reasoning Problems into Modal Logics. Lecture Notes in Artificial Intelligence, Vol. 1701:113–125, Springer, Berlin, 1999.

[16] H. Ono and A. Nakamura. On the Size of Refutation Kripke Models for Some Linear Modal and Tense Logics. *Studia Logica* 39(4):325–333, 1980.