Quantified Propositional Temporal Logic with Repeating States

Tim French
School of Computer Science and Software Engineering
University of Western Australia
tim@cs.uwa.edu.au

Abstract

Quantified Propositional Temporal Logic (QPTL) is a linear temporal logic that allows quantification over propositional variables. In the usual semantics for QPTL, a model is an infinite discrete linear sequence of states, with each state having some propositional interpretation. The effect of this is that the interpretation of a proposition at one point in time is independent from its interpretation at another point in time. In this paper we examine the expressivity and decidability of an of QPTL, given generalizations of the usual semantics that do not have this restriction. We introduce the repeating semantics *(QPTL^R), which allows states to be repeated throughout a model. While this semantic interpretation does not affect the unquantified fragment of QPTL it significantly increases the expressive power in the presence of propositional quantification. In the main result of this paper we show that QPTL^R makes the satisfiability problem highly undecidable through a complicated encoding of a tiling problem. We also investigate two less expressive semantics which still allow states to be repeated. We prove the satisfiability problem for one is undecidable, and decidable for the other.*

1. Introduction

Quantified Propositional Temporal Logic (QPTL) [14], extends traditional propositional linear temporal logic, PLTL [12], and has the ability to quantify over propositional atoms. The set of formulas of QPTL is built up recursively from the atoms via classical negation and disjunction, temporal next-time and future-always connectives (from the standard PLTL) along with quantification over atoms. The formulas are evaluated over a linear sequence of states, with each atom being true or false at each state. The logic is expressively equivalent to Buchi's monadic second-order logic of one successor ($S1S$) and consequently it has been shown to be decidable (albeit non-elementary).

In this paper we define new semantic interpretations (QPTL^R, QPTL^F, QPTL^P) of QPTL by removing the restriction that the model be a linear sequence of states. We do however retain the basic concept of a linear model by distinguishing the states of the model from "temporal moments", (the time at which a state is visited). That is, the temporal operators of the language are interpreted with respect to countable, discrete, linear moments in time; while the atoms are interpreted with respect to the (possibly recurring) states of the model. We cannot assume that the interpretation of an atom at one temporal moment is independent of its interpretation at another moment. This way we can allow the temporal proposition *"now will never happen again"* to be false, (however to avoid confusion is is best not to associate the states with such temporal notions as *now*).

Our motivation for doing this is largely to investigate the expressive power of propositional quantification in modal structures. This has been studied extensively in the context of branching modal structures [4], [3], [5], and also in the context of strictly linear structures [14], [10]. In this paper we investigate languages which neither have branching operators, nor a strictly linear structure. Effectively this allows us to apply quantification over the states of the model, which results in a dramatic increase in the expressive power of QPTL.

The language QPTL has been used to reason about finite state machines and automata. By associating a state of the model with a state of the machine the repeating semantics allows us to succinctly reason about properties of finite state machines, without having to assign an atom to each state of the machine. For example, the repeating semantics allows us to detect if a machine will loop infinitely in a single state, independant of what the actual state is.

Another possible application of such a language would be reasoning about a trajectory of an agent through some environment. The states of a model would represent spatial entities, and the temporal moments represent time. The language can express such concepts as whether the trajectory of the agent is eventually periodic, or whether there are some states that the agent will only visit once.

Proceedings of the 10th International Symposium on Temporal Representation and Reasoning and Fourth International Conference on Temporal Logic (TIME-ICTL'03) 1530-1311/03 $17.00 © 2003 IEEE

2. Semantics for QPTL

The language QPTL consists of an infinite set of atomic variables $\mathcal{V} = \{x_0, x_1, ...\}$, the boolean operations $\neg, \vee$, (**not** and **or** respectively) and the future temporal operators $\bigcirc$, $\Box$ (**next** and **generally** respectively), along with the quantifier $\forall$ (**for all**). We will use the convention that all unary operators (including $\forall$) bind weaker than binary operators, unless otherwise indicated by brackets. A formula of QPTL is defined inductively as follows.

- For all $x \in \mathcal{V}$, x is a formula.
- If α and β are formulas then so are $\neg\alpha,\ \alpha \vee \beta,\ \bigcirc\alpha$, and $\Box\alpha$.
- If α is a formula and $x \in \mathcal{V}$ then $\forall x\alpha$ is also a formula.

We will define additional operators in terms of those above. The propositional operators $\wedge, \rightarrow, \leftrightarrow$ are well known to be expressible in terms of $\neg$ and $\vee$. We will also consider the formulas $\top, \bot$ (respectively "true" and "false") to be abbreviations for, respectively $(x_0 \vee \neg x_0)$ and $\neg(x_0 \vee \neg x_0)$, and we will use $\exists X$ as an abbreviation for $\exists x_0 ... \exists x_m$ where $X = \{x_0, ..., x_m\}$. The other abbreviations in QPTL are:

$$\begin{aligned}
&\exists x\alpha = \neg\forall x\neg\alpha \\
&\Diamond\,\alpha = \neg\,\Box\neg\alpha \\
&\alpha\, W\beta = \exists u(u \wedge \Box(u \rightarrow ((\alpha \wedge \bigcirc u) \vee \beta))) \\
&\alpha \mathrm{U}\beta = \Diamond\,\beta \wedge (\alpha\, W\beta)
\end{aligned}$$

where $u = x_i$ for the least i such that x_i does not appear in α or β. These operators will be referred to as, respectively: existential quantification, future, waiting, and until.

We will next present the definition of a structure σ that evaluates a formula in the language QPTL. This semantic interpretation will be referred to as QPTL^R and is more general than the one given in [6].

A structure, σ, is given by the tuple $(S, \mu : \omega \rightarrow S, \pi : s \rightarrow \wp(\mathcal{V}))$. Given σ we say the structure $\sigma' = (S, \mu, \pi')$ is an X-*variant* of σ for some finite $X \subset \mathcal{V}$, if $\forall s \in S\, \pi(s)\backslash X = \pi'(s)\backslash X$.

We say a structure $\sigma = (S, \mu, \pi)$ is a QPTL$^\omega$ model if μ is a bijection. In this case we usually let σ be an infinite sequence $(\sigma_0, \sigma_1, ...)$ where $\sigma_i = \pi(\mu(i))$. In this paper we investigate the general case, where there is no restriction on the function μ. This is the only difference between QPTL$^\omega$ and QPTL^R, and the semantic interpretation of the formulas is this same for both languages.

Given some structure σ we define the j^{th} moment of σ be the tuple (σ, j) for $j \in \omega$ and inductively define a formula α to "hold" at the j^{th} moment of σ (denoted $\sigma, j \models \alpha$) as follows.

$$\begin{aligned}
&\sigma, j \models x \Leftrightarrow x \in \pi(\mu(i)),\ \text{for all } x \in \mathcal{V}. \\
&\sigma, j \models \neg\alpha \Leftrightarrow \sigma, j \not\models \alpha. \\
&\sigma, j \models \alpha_1 \vee \alpha_2 \Leftrightarrow \sigma, j \models \alpha_1 \text{ or } \sigma, j \models \alpha_2. \\
&\sigma, j \models \bigcirc\alpha \Leftrightarrow \sigma, j+1 \models \alpha. \\
&\sigma, j \models \Box\alpha \Leftrightarrow \forall k \geq j \quad \sigma, k \models \alpha. \\
&\sigma, j \models \forall x\alpha \Leftrightarrow \sigma', j \models \alpha,\ \text{for all } x\text{-variants } \sigma' \text{ of } \sigma.
\end{aligned}$$

We say σ is a model of α(written $\sigma \models \alpha$) if σ,0 $\models \alpha$. If for every structure σ and for every $i \in \omega$ we have $\sigma, i \models \alpha$, we say α is valid (written $\models \alpha$), and if $\neg\alpha$ is not valid, we say α is satisfiable.

We can see the function μ maintains the relationship between the temporal moments and the states of the structure. Given a structure $\sigma = (S, \mu, \pi)$ we can see some state $s \in S$ could be repeated if there is some $i \neq j$ such that $\mu(i) = \mu(j) = s$. However the actual moment σ, i will never be repeated. The interpretation of formulas is derived from the moment i, and only the interpretation of atoms are derived from the state. For example, if $\mu(i) = \mu(j) = s$ then $\sigma, i \models x$ if and only if $\sigma, j \models x$, however it may be that $\sigma, i \models \bigcirc x$ while $\sigma, j \models \bigcirc\neg x$.

The language QPTL is decidable, though non-elementarily complex. The process to determine the satisfiability of some formula α involves constructing a non-deterministic ω-automaton that accepts exactly the models that satisfy α. The satisfiability of α is therefore equivalent to the non-emptiness of the automaton. For details on an optimal decision procedure for QPTL, see [13] or [6].

We will refer to these generalized semantics as the *repeating semantics* (QPTL^R). The states are the elements of S, and the temporal moments are the pairs σ, j. The most basic example of the expressive difference between QPTL and QPTL^R if the formula:

$$\exists x(x \wedge \bigcirc\,\Box\neg x) \tag{1}$$

which appeared as an axiom in [6] and is equivalent to the statement *"now will never happen again"*. This is a validity of QPTL but its negation, $\forall x(x \rightarrow \bigcirc\Diamond\, x)$, is satisfiable in any model σ where there is some $j > 0$ such that $\mu(j) = \mu(0)$.

Another example of the expressive difference is being able to succinctly express such concepts as whether the model is deterministic (i.e. every state in the model will always be followed by the same state):

$$\Box\forall x\forall y((x \wedge \bigcirc y) \rightarrow \Box(x \rightarrow \bigcirc y)) \tag{2}$$

3. Undecidability

We prove that QPTL^R is highly undecidable by showing that QPTL^R can encode a highly undecidable tiling problem. This technique was used in [8] to prove undecidability of modal logics and has since been used in many similar scenarios [15], [5], [11], [7]. The important step in encoding a tiling is to specify the model to be grid-like, and all these approaches use branching or bi-modal languages. This proof is unique in that it only uses non-branching temporal operators.

The $\omega \times \omega$ *Tiling Problem* is as follows: We are given a finite set $\Gamma = \{\gamma_i | i = 1, ..., m\}$ of tiles. Each tile γ_i has four coloured sides: left, right, top and bottom, written γ_i^l, γ_i^r, γ_i^t, and γ_i^b. Each side can be one of n colours c_j for $j = 1, ..., n$. Given any set of these tiles, we would like to know if we can cover the plane $\omega \times \omega$ with these tiles such that adjacent sides share the same colour.

DEFINITION 1 *Given some set of tiles, Γ, we say $\lambda : \omega \times \omega \to \Gamma$ is a* tiling function for Γ *if for all $(x, y) \in \omega \times \omega$*

1. $\lambda(x, y)^r = \lambda(x + 1, y)^l$
2. $\lambda(x, y)^t = \lambda(x, y + 1)^b$.

If such a function exists we say Γ tiles the plane.

Given some finite set of tiles Γ, the tiling problem is to decide whether or not Γ tiles the plane. This problem was shown to be Π_1^0-complete (the complement of the recursively enumerable set, or co-RE) [2].

We will also use the $\omega \times \omega$ *Recurrent Tiling Problem*, which is the same as the $\omega \times \omega$ tiling problem with the following additional requirement: The tile γ_0 must occur infinitely often in the bottom row (i.e. $\lambda(x, 0) = \gamma_0$ for infinitely many x). In such a case we say Γ *recurrently tiles the plane*. In [9] this problem was shown to be highly undecidable, or Σ_1^1.

THEOREM 1 *The satisfiability problem for QPTL^R is highly undecidable.*

Given the set of tiles Γ we give a formula, $Tile^\Gamma$, of QPTL that is satisfiable given the QPTL$^\omega$ semantics if and only if Γ recurrently tiles the plane. As with many approaches, to show the satisfiability problem for a language is highly undecidable it is enough to show that there is a formula that specifies the under-lying Kripke structure to be "grid-like". This is exactly what we will do here. What makes this approach different is that the language is syntactically a one dimensional language: we do not so much as specify the properties of the under-lying Kripke structure, as we specify the properties of a path through the structure. Examples of the properties of the path we will be specifying are:

- whether we have visited a state before;
- how many times have we visited a state;

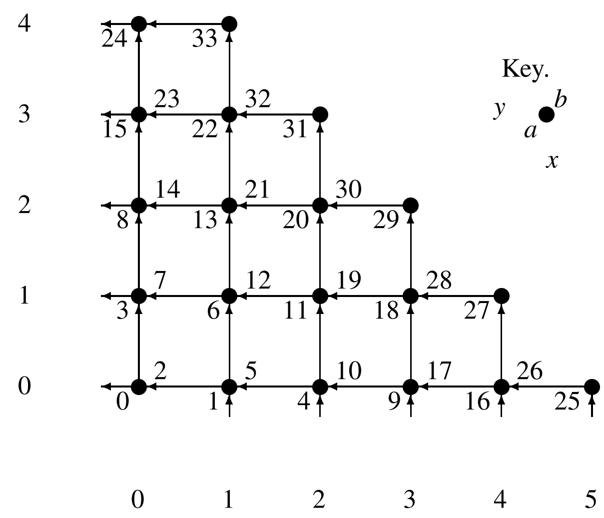

Figure 1. An illustration of the function ρ, where $\rho(a) = \rho(b) = (x, y)$

- how many time have we visited some state since we last visited some other state.

These properties could in fact be represented in a weaker semantic than QPTL^R, and we discuss this later. The properties mentioned above will be used to "wrap" a path through a grid. To describe how that is done we must first describe a new pairing function.

DEFINITION 2 *We define the function $\rho : \omega -\!\!\to \omega \times \omega$ recursively as for all $n \in \omega$, $\rho(n^2) = (n, 0)$ and for all m, $n^2 < m < (n + 1)^2$*

1. $\rho(m-1) = (x, y) \Longrightarrow \rho(m) = (x-1, y)$ *if* $x+y = n$,
2. $\rho(m - 1) = (x, y) \Longrightarrow \rho(m) = (x, y + 1)$ *otherwise.*

We also define $\rho^+ : \omega -\!\!\to \omega$ by $\rho(m) = (x, y) \Longrightarrow \rho^+(m) = x + y$

The function ρ is illustrated in Figure 3. This is a relatively complicated pairing function, however it has a number of useful properties which will allow us to encode this function in QPTL^R. These properties are given in the following definition.

DEFINITION 3 *Let Δ be some set and let $\lambda : \omega -\!\!\to \Delta$ be some function. Given any $d \in \Delta$ let $d_1 < d_2$ be two integers such that $\lambda(d_1) = \lambda(d_2) = d$, or ∞ if no such integer exists. We say λ is a* grid-path *if $\forall d \in \Delta$:*

1. *there are exactly two integers, d_1 and d_2, such that $\lambda(d_1) = \lambda(d_2) = d$,*
2. *d_1 is even if and only if d_2 is even,*
3. *for all $e,\ f \in \Delta$ such that $d_1,\ e_1,\ f_1$ are congruent modulo 2:*
   (a) *$d_1 < e_2 < f_1 \Longrightarrow d_2 < f_1$;*
   (b) *$d_1 < e_1 < d_2 \Longrightarrow d_2 < e_2$;*
4. *if there exists $e \in \Delta$ such that $e_1 = d_1 + 1$ then there is some $f \in \Delta$ such that $e_1 + 1 = f_2$,*
5. *there exists $e \in \Delta$ such that $e_1 = d_2 + 1$.*

We will refer to the five conditions in Definition 3 as $C1 - C5$.

LEMMA 1 *The function ρ is a grid path.*

PROOF:

We will show that all the conditions of Definition 3 are satisfied

For $C1$, it can be seen from Figure 3 that if $n^2 \leq m < (n+1)^2$, then $\rho^+(m) \in \{n, n-1\}$. Suppose $\rho(m) = (x, y)$. For all i, if $\rho(i) = (u, v)$ and $n^2 \leq i < m$ then $u > x$ or $v < y$, and if $m < i < (n+1)^2$ then $u < x$ or $v > y$. Therefore $\rho^+(m) = n$ implies $\forall k < m$, $\rho(k) \neq \rho(m)$ and $\rho^+(m) = (n-1)$ implies $\forall k > m$, $\rho(k) \neq \rho(m)$. It follows that there is at most one $j \neq m$ such that $\rho(j) = \rho(m)$.

To prove that there is at least one $j \neq m$ such that $\rho(j) = \rho(m)$ suppose, without loss of generalization, that $n^2 \leq m < (n+1)^2$ and $\rho^+(m) = n$. Then it can be seen that $\rho(m+2n+2) = \rho(m)$ (the calculation is left to the reader).

The condition $C2$ follows trivially from the proof for $C1$, and the fact that $\rho(m + 2n + 2) = \rho(m)$.

Having proven the first two conditions, for all $d \in \omega \times \omega$ we let d_1 be the least integer such that $\rho(d_1) = d$, and let d_2 be the greatest such integer.

For the first part of $C3$, it can be shown that if d_1 and e_2 are congruent modulo 2 then there is some n such that $d_1 < n^2 < e_2$. Similarly since e_2 and f_1 are congruent modulo 2 there is some j such that $e_2 < j^2 \geq f_1$. Therefore $\rho^+(d_1) < n$ and $\rho^+(i) \geq n$ for all $i \geq f_1$. Therefore $d_2 < f_1$.

For the second part of $C3$, suppose $n^2 < m < k < (n+1)^2$ and (without loss of generalization) $\rho^+(m) = \rho^+(k) = n$. Then the result follows from the fact that $m + 2n + 2 < k + 2n + 2$ (from the proof for $C1$).

The proof for $C4$ follows from the fact that given $n^2 \leq d_1 < (n+1)^2 - 1$, $\rho^+(d_1) = n$. This can be seen from the fact that $d_2 > (n+1)^2$ and therefore $n \geq \rho^+(d_1) = \rho^+(d_2) \geq n$. The same reasoning applies to e_1 so we can deduce that $e_1 = (n+1)^2$ (from the fact that $\rho^+(e_1) \neq \rho^+(d_1)$). Therefore $\rho^+(e_1 + 1) = n$ so the same reasoning will show $e_1 + 1 = f_2$ for some f.

The proof for $C3$ follows trivially from the proof given for $C4$. $\Box$

The following proof is required to show that any grid-path represents a grid structure in exactly the same way ρ does.

LEMMA 2 *Given some set Δ and a grid-path $\lambda : \omega \dashrightarrow \Delta$, there is a bijection $\delta : \Delta \dashrightarrow \omega \times \omega$ such that $\forall n \in \omega$, $\delta(\lambda(n)) = \rho(n)$.*

PROOF:

Given any integer n, let the subset $S_n \subset \Delta$ be the set $\{d \in \Delta | \exists m < (n+1)^2 \text{ s.t. } \lambda(\mathrm{m}) = \mathrm{d}\}$. We prove the lemma by induction over n where the induction hypothesis is:

Given any integer n, there is some bijection $\delta_{S_n} : S_n \to \omega \times \omega$ such that for all $m < (n+1)^2$, $\delta_{S_n}(\lambda(m)) = \rho(m)$.

We begin with $n = 0$. This is trivial, as $S_0 = \{\lambda(0)\}$ and all that is required is to define $\delta_{S_0}(\lambda(0)) = (0, 0)$.

For the inductive step, suppose that for some integer n there is a bijection $\delta_{S_n} : S_n \to \omega \times \omega$ such that for all $(m < n+1)^2$, $\delta_{S_n}(\lambda(m)) = \rho(m)$.

We construct δ_{S_n} as follows. Let $\delta_{S_{n+1}}(d) = \delta_{S_n}(d)$ for all $d \in S_n$. We would like to define $\delta_{S_{n+1}}$ such that for all $m < (n+2)^2$, $\delta_{S_{n+1}}(\lambda(m)) = \rho(m)$, however we must ensure that if $\rho(m) = \rho(\ell)$ for $\ell < (n+1)^2$, then $\lambda(m) = \lambda(\ell)$.

For all m, $(n+1)^2 \leq m < (n+2)^2$ we note that $\rho^+(m) \in \{n, n+1\}$ and furthermore, if $m = d_1$ for some $d \in \omega \times \omega$ if and only if $\rho^+(m) = n+1$. Therefore we are required to show that for all such m if $\rho^+(m) = n+1$ then for all $k < m$ we have $\lambda(k) \neq \lambda(m)$, and if $\rho^+(m) = n$ then there is some $\ell < m$ such that $\rho(m) = \rho(\ell)$ and $\lambda(m) = \lambda(\ell)$. This is proven in Lemma 3 below.

Therefore we can define the bijection $\delta_{S_{n+1}}$ such that for all $m < (n+2)^2$, $\delta_{S_{n+1}}(\lambda(m)) = \rho(m)$. Furthermore, by construction, the bijection $\delta_{S_{n+1}}$ agrees with δ_{S_n} over the set S_n so by induction there is some bijection δ such that for all m, $\delta(\lambda(m)) = \rho(m)$. $\Box$

Below is a technical sub-lemma required to complete the proof of Lemma 2.

LEMMA 3 *Suppose that there is some bijection δ_{S_n} : $S_n \to \omega\times\omega$ such that for all $i < (n+1)^2$ $\delta_{S_n}(\lambda(i)) = \rho(i)$. Then for all m where $(n+1)^2 \leq m < (n+2)^2$:*

1. $\rho^+(m) = n+1 \Longrightarrow \forall k < m,\ \lambda(k) \neq \lambda(m)$
2. $\rho^+(m) = n \Longrightarrow \exists k < m$ s.t. $\lambda(\mathrm{k}) = \lambda(\mathrm{m})$ and $\rho(\mathrm{k}) = \rho(\mathrm{m})$

PROOF:

We will prove the lemma by strong induction. As the base case, let $m = (n+1)^2$. Therefore $\rho^+(m) = n+1$ and we must show for all $k < m$, $\lambda(k) \neq \lambda(m)$. Suppose for contradiction that there is some $k < m$ such that $\lambda(k) \neq \lambda(m)$. If $\rho^+(k) < n$ then $\rho(j) = \rho(k)$ implies $j < (n+1)^2$, and since δ_{S_n} is a bijection we would have $\lambda(j) = \lambda(k) = \lambda(m)$ contradicting Definition 3.1. Since $k < (n+1)^2$ we have $\rho^+(k) = n$. By the construction of ρ we can see that $\rho^+(i)$ is even if and only if i is even. Therefore m is even if and only if k is odd, contradicting Definition 3.2 and thus for all $k < m$, $\lambda(k) \neq \lambda(m)$. This completes the base case of the induction.

We will now prove the inductive step. Suppose that for some $m < (n+2)^2$, we have defined $\delta_{S_{n+1}}(k)$ for all $k < m$. If $\rho^+(m) = n+1$, then by the construction of ρ, $\rho^+(m-1) = n$. By the induction hypothesis there exists $k < m-1$ such that $\lambda(k) = \lambda(m-1)$, and consequently by Definition 3.5 for all $k < m$, $\lambda(k) \neq \lambda(m)$. This completes the proof of the first case.

For the second case of the inductive step, suppose for some $m < (n+2)^2$ we have defined $\delta_{S_{n+1}}(k)$ for all $k < m$ and $\rho^+(m) = n$. To continue with the strong induction, we must include another base case, corresponding to $m = (n+1)^2 + 1$. For such a case we must show that there is some $k < m$ such that $\lambda(k) = \lambda(m)$. In this case $\rho^+(m-1) = n+1$ so $\forall k < m-1,\ \lambda(k) \neq \lambda(m-1)$. Also, $\rho^+(m-2) = n$ and $m-2 < (n+1)^2$ so $\forall k < m-2,\ \lambda(k) \neq \lambda(m-2)$. Therefore by Definition 3.4 there must be some $k < m$ such that $\lambda(k) = \lambda(m)$.

We will complete the induction in two steps. First we will show that for any $m > (n+1)^2$ with $\rho^+(m) = n$ there is some $k < m$ such that $\lambda(k) = \lambda(m)$. The next step will be to show that in such a case, $\rho(k) = \rho(m)$.

There are exactly $n+1$ distinct $k < (n+1)^2$ such that $\rho^+(k) = n$, and since δ_{S_n} is a bijection we can see that for each such k there is no $j < (n+1)^2$ such that $\lambda(k) = \lambda(j)$. By Definition 3.1, for each such k there must exist some unique $j > (n+1)^2$ such that $\lambda(k) = \lambda(j)$, and k, j, m, and $(n+1)^2 + 1$ are congruent modulo 2 (since $\rho^+(k) = \rho^+(m) = \rho^+(j) = \rho(n^2+2n+2)$). We now apply Definition 3.3.a, where $k = d_1$, $m = e_2$, $n^2+2n+2 = f_1$ and $j = d_2$. Therefore we must have $j < n^2+2n+2$. Since there is such a j for each of the $n+1$ distinct k, there must exist $j_0, ..., j_n$ such that $(n+1)^2 < j_0 < j_1 < ... < j_n < (n+2)^2$ where $j_{i+1} - j_i \geq 2$. Since $(n+2)^2 - (n+1)^2 = 2n+1$, it must be that $m = j_i$ for some i, and thus there is some $k < m$ such that $\lambda(k) = \lambda(m)$.

It remains to show that $\delta_{S_{n+1}}(\lambda(m)) = \rho(m)$. Suppose that for some $j < m$, $\lambda(j) = \lambda(m)$ but $\rho(j) \neq \rho(m)$. By the induction hypothesis, $\delta_{S_{n+1}}(\lambda(j)) = \rho(j)$ and there is some $k < m$ such that $\rho(k) = \rho(m)$.

If $k < j$ then by Definition 3.3.b for any $\ell > k$ such that $\lambda(\ell) = \lambda(k)$ we must have $\ell < m$. However if $\ell < m$ then by the induction hypothesis $\lambda(\ell) = \lambda(k)$ implies $\rho(\ell) = \rho(k) = \rho(m)$ contradicting the fact the ρ is a grid-path.

If $k > j$ then by Definition 3.3.b for any $\ell > j$ such that $\rho(\ell) = \rho(j)$ we must have $\ell < m$. However if ℓ, m then by the induction hypothesis $\rho(\ell) = \rho(k)$ implies $\lambda(\ell) = \lambda(k) = \lambda(m)$ contradicting the fact the λ is a grid path (Definition 3.1.. □

To complete the proof of undecidability we must encode the tiling. This relies on some convenient properties of the function ρ. With respect to QPTL^R, the main property that we are interested in is that if a state occurs twice in a model, its interpretation with respect to propositional variables remains unchanged. We wish to create a formula which forces the model to have a grid path, and then use propositions to represent the different tiles in Γ. We can then make sure that the sides of adjacent tiles match up.

We begin by defining the formula $Tile^\Gamma$. This consists of two parts: the first $gridpath$ is true at the initial state in a model $\sigma = (S, \mu, \pi)$ if and only if ρ is a grid-path; and the second t^Γ ensures that the sides of adjacent tiles agree.

DEFINITION 4 *Let the formula $2nd = \exists x(x \wedge \Box\neg x)$ and $1st = \neg 2nd$. The formula $gridpath$ is given by the following definitions:*

$$C1 \quad = \quad \Box\exists x(x \to (\bigcirc\Box(x \to \bigcirc\Box\neg x)) \wedge \forall x(\Diamond x \to \Diamond(x \wedge \bigcirc\Diamond x))$$

$$
\begin{aligned}
C2 &= e \wedge \Box(e \leftrightarrow \bigcirc\neg e) \\
C3a(e) &= \exists x(e \wedge x \wedge \Box((2nd \wedge e) \rightarrow \\
&\qquad \Box((1st \wedge e) \rightarrow \Box\neg x))) \\
C3b(e) &= \exists x(e \wedge x \wedge \forall y(\Diamond(1st \wedge e \wedge y) \rightarrow \\
&\qquad \Box(x \rightarrow \Diamond y))) \\
C3 &= \Box(1st \rightarrow ((C3a(\neg e) \wedge C3b(\neg e)) \vee \\
&\qquad (C3a(e) \wedge C3b(e))) \\
C4 &= \Box(2nd \rightarrow \bigcirc 1st) \\
C5 &= \Box(1st \wedge \bigcirc 1st \rightarrow \bigcirc\bigcirc 2nd)) \\
gridpath &= \exists e(C1 \wedge C2 \wedge C3 \wedge C4 \wedge C5).
\end{aligned}
$$

In this definition we use the variable e to mark even states. That is there is some e-variant of π, π' such that $e \in \pi'(s)$ if and only if there is some even k such that $\mu(k) = s$. The formula $C2$ also enforces that $e \notin \pi'(s)$ if and only if there is some odd k such that $\mu(k) = s$. This is enough to enforce Definition 3.2. It can also be seen that $(\sigma, j) \models 2nd$ if and only if for all $k > j$, $\mu(k) \neq \mu(j)$. Otherwise for any x-variant π' of π, $x \in \pi'(\mu(k))$ if and only if $x \in \pi'(\mu(j))$.

LEMMA 4 *Let $\sigma = (S, \mu, \pi)$. Then $(\sigma, 0) \models gridpath$ if and only if μ is a grid-path.*

PROOF:

This follows directly from the semantic definitions given in Section 2, and comparison with Definition 3. $\Box$

We will now define the formula t^Γ such that if $\sigma \models t^\Gamma \wedge gridpath$, then the function $\lambda : \omega \times \omega \rightarrow \Gamma$ is a recurrent tiling function, where $\lambda(x, y) = \gamma_i$ if and only if $\gamma_i \in \pi(\delta^{-1}(x, y))$. To do this we require the following lemma which associates horizontal and vertical directions in the grid with properties definable in QPTL.

LEMMA 5 *Given a model $\sigma = (S, \mu, \pi)$ where μ is a grid-path, let $\delta : S \rightarrow \omega \times \omega$ be some bijection such that for all n, $\delta(\mu(n)) = \rho(n)$. For all n, let $\delta(\mu(n)) = (x_n, y_n)$. Then*

- *if for all $j > n$, $\mu(j) \neq \mu(n)$ then $x_{n+1} = x_n$ and $y_{n+1} = y_n + 1$,*
- *if there is some $j > n$ such that $\mu(j) = \mu(n)$ then $x_{n+1} = x_n - 1$ and $y_{n+1} = y_n$.*

PROOF:

Since δ is a bijection and for all m, $\delta(\mu(m)) = \rho(m)$, all we have to show is that given $\rho(m) = (x, y)$

- if for all $j > m$, $\rho(j) \neq \rho(m)$ then $\rho(m + 1) = (x, y + 1)$,
- if there is some $j > m$ such that $\rho(j) = \rho(m)$ then $\rho(m + 1) = (x - 1, y)$.

Suppose $n^2 \leq m < (n + 1)^2$. Since we know from the proof of Lemma 2 that either $\rho^+(m) = n$ or $\rho^+(m) = n-1$. We also know $\rho^+(m) = n$ if and only if there is some $j > m$ such that $\rho(j) = \rho(m)$. Therefore if $\rho^+(m) = n$, then $\rho^+(m + 1) = n - 1$ and hence $\rho(m + 1) = (x - 1, y)$, (from Definition 2). The case for $\rho^+(m) = n - 1$ is similar. $\Box$

Let $\Gamma = \{\gamma_0, ..., \gamma_n\}$ be the set of tiles, and for each $i \leq n$ let $L_i = \{j \leq n | \gamma_j^r = \gamma_i^l\}$ and $T_i = \{j \leq n | \gamma_j^b = \gamma_i^t\}$. Finally let $\tau_0, ..., \tau_n$ be propositions corresponding to the tiles in Γ.

DEFINITION 5 *The formula $Tile^\Gamma$ is given as*

$$
\begin{aligned}
\gamma^i &= \tau_i \wedge \bigwedge_{j \neq i} \neg\tau_j \\
left_i &= 1st \rightarrow \bigcirc(2nd \rightarrow \bigvee_{j \in L_i} \tau_j) \\
up_i &= 2nd \rightarrow \bigcirc \bigvee_{j \in T_i} \tau_j \\
inf &= \Box\Diamond(1st \wedge \bigcirc(1st \wedge \gamma_0)) \\
t^\Gamma &= \Box \bigvee_{i=0}^{n} (\gamma^i \wedge left_i \wedge up_i) \wedge inf \\
Tile^\Gamma &= t^\Gamma \wedge gridpath.
\end{aligned}
$$

LEMMA 6 *The formula $Tile^\Gamma$ is satisfiable if and only if Γ recurrently tiles the plane.*

PROOF:

Suppose there is some function $\lambda : \omega \times \omega \rightarrow \Gamma$ satisfying the properties of a tiling. We define a model , σ, of $Tile^\Gamma$ as follows. Let $\sigma = (\omega \times \omega, \rho, \pi)$, where $\tau_i \in \pi(n)$ if and only if $\lambda(n) = \gamma_i$.

Since ρ is a grid-path (Lemma 1), we have $\sigma \models gridpath$ (Lemma 4).

For all x, y there is some unique i such that $\lambda(x, y) = \gamma_i$, and hence for all x, y, $\tau_i \in \pi(x, y)$ for some unique i, so γ^i will always be satisfied.

Suppose that $\tau_i \in \pi(\rho(n))$ for some n, and $\tau_j \in \pi(\rho(n + 1))$. By Lemma 5, if $\sigma, n \models 1st \wedge \bigcirc 2nd$ then $j \in L_i$ and if $\sigma, n \models 2nd$ then $j \in T_i$, so $left_i$ and up_i will be satisfied.

Finally, for infinitely many x, $\lambda(x, 0) = \gamma_0$. From the proof of Lemma 3 we know that $\rho(n) = (x, 0)$ only if $\sigma, n - 1 \models 1st$ and $\sigma, n \models 1st$, so it follows that $\sigma, 0 \models inf$ and thus $\sigma, 0 \models Tile^\Gamma$.

Now suppose there is some model $\sigma = (S, \mu, \pi)$ such that $\sigma, 0 \models Tile^{\Gamma}$. We construct a recurrent tiling function λ as follows.

As $\sigma, 0 \models gridpath$, it follows from Lemma 4 that μ is a grid-path. Therefore, by Lemma 2, there is some bijection $\delta : S \twoheadrightarrow \omega \times \omega$ such that $\delta(\mu(n)) = \rho(n)$. We define the tiling function $\lambda : \omega \times \omega \to \Gamma$ by $\lambda(x, y) = \gamma_i$ if and only if $\tau_i \in \pi(\delta^{-1}(x, y))$. Since $\sigma \models \Box \bigvee_{i=0}^{n} \gamma^i$, λ is well defined.

Suppose $\lambda(x, y) = \gamma_i$, and $x + y = n$. By examination of the function ρ we can see $\rho(n^2 + 2y) = (x, y) = \rho((n + 1)^2 + 2y + 1)$. Let $k = (n+1)^2+2y+1$. Since μ is a grid-path, it follows from the definition of λ and the proof of Lemma 3 that $\sigma, k \models 2nd \wedge \gamma^i$. It therefore follows that $\sigma, k \models \bigvee_{j \in L_i} \tau^j$. Hence $\lambda(\rho(k)) = \gamma_j$ for some $j \in T_i$, so $\gamma_j^b = \gamma_i^t$. Since $x+y = n$, and $n+1^2 \leq k < (n + 2)^2$ we find that $\rho(k + 1) = (x - 1, y)$, and thus $\lambda(x, y)^l = \lambda(x - 1, y)^r$.

Now suppose that $k = n^2 + 2y$. We apply a similar process to show $\lambda(\rho(k))^l = \lambda(\rho(k + 1))^r$ provided $\sigma, k \models \bigcirc 2nd$. If $\sigma, k \models \bigcirc 1st$ we must have $\forall j < k$, $\rho(j) \neq \rho(k)$ and $\rho(j) \neq \rho(k + 1)$. By the proof of Lemma 1 we must have $k + 1 = (n + 1)^2$, and hence $\rho(k) = (0, n)$ so there is no tile to the left. This can be seen by inspection of Figure 3.

Finally, since $\sigma, 0 \models inf$, we must have $\sigma, k \models 1st \wedge \bigcirc(1st \wedge \tau^0)$ for infinitely many k. If $\sigma, k \models 1st \wedge \bigcirc 1st$ we must have $k+1 = (n+1)^2$ for some n. Therefore $\rho(k + 1) = (0, n)$, and $\tau_i \in \pi(\mu(k + 1))$, thus $\lambda(x, 0) = \gamma_0$ for infinitely many x. $\Box$

We have shown that QPTL with repeating states is undecidable when we allow the function $\mu : \omega \to S$ to be arbitrary and thus we have proven Theorem 1. The repeating semantics are too expressive to to allow any form of automated deduction. However, we can see that the result could be stronger, since we only required μ to belong to a class of functions that could represent a grid-path (for example, the class of functions where each state is repeated only a finite number of times). The satisfiability problem for QPTL^R could be even harder than Σ_1^1.

4. QPTL with finite repeating states

We will now examine a class of functions which does not allow μ to be a grid-path. We define the semantic interpretation QPTL^F to be the same as the semantic interpretation QPTL^R, except now we require that the range of μ be finite (i.e. there are only a finite number of states). These semantics retain much of the expressive power of QPTL^R, however the set of states is no longer an infinite domain. Formulas such as (2) are still expressible and meaningful. However it is worth noting that these semantics are not a generalization of the usual semantic interpretation QPTL$^\omega$. In fact, every model of QPTL$^\omega$ is disallowed in QPTL^F since the model will necessarily have an infinite number of states. Consequently, contradictions in QPTL might be validities in QPTL^F, such as

$$\Diamond \Box \forall x(x \to \bigcirc \Diamond x). \tag{3}$$

Since the models for QPTL^F can be enumerated, we might expect the satisfiability problem to be recursively enumerable. However, we will now show that the validity problem (determining whether a formula is valid) is undecidable, and consequently the satisfiability problem remains undecidable.

To show that QPTL is undecidable given the interpretation, QPTL^F, we will use the $\omega \times \omega$ tiling problem. Since the range of μ is finite, we cannot represent the entire plane. However, we can use the following definition and lemma to get around this problem.

DEFINITION 6 *The* n^{th} corner of the plane*,* $\omega \times \omega$*, is the set* $C^n = \{(x, y) | x + y \leq n\}$*. We say a set of tiles,* Γ*,* tiles C^n if there is some function $\lambda : C^n \to \Gamma$ such that for all x, y where $x + y < n$,

1. $\lambda(x, y)^r = \lambda(x + 1, y)^l$
2. $\lambda(x, y)^t = \lambda(x, y + 1)^b$

LEMMA 7 *A set of tiles,* Γ*, satisfies the* $\omega \times \omega$ *tiling problem, if and only if for all* $n \in \omega$*,* Γ *tiles the* n^{th} *corner of the plane.*

PROOF:

If there is some tiling function, $\lambda : \Gamma \to \omega \times \omega$, then restricting the domain of Λ to C^n will show that Γ tiles C^n.

Now suppose that for all $n \in \omega$, λ^n is a tiling function for C^n. We will define a tiling function, $\lambda : \omega \times \omega \to \Gamma$ by induction over n. We first define $\kappa^0 : C^0 \to \Gamma$ to be some function such that $\kappa^0(0, 0) = \lambda^i(0, 0)$ for infinitely many i. Clearly such a function exists. For the inductive step suppose that we have defined $\kappa^n : C^n \to \Gamma$ such that there are infinitely many i where for all $c \in C^n$, $\kappa^n(c) = \lambda^i(c)$. We define κ^{n+1} such that:

1. $\forall c \in C^n$, $\kappa^{n+1}(c) = \kappa^n(c)$;
2. for infinitely many i, for all $c \in C^{n+1}$, $\kappa^{n+1}(c) = \lambda^i(c)$.

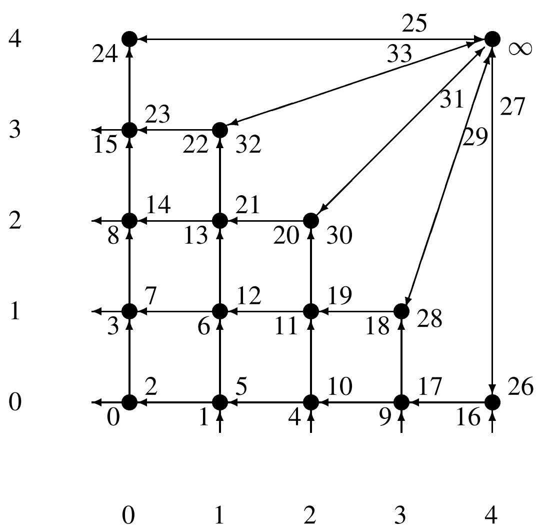

Figure 2. An illustration of the function ρ^n, where $n = 5$

It is a consequence of Ramsey's theorem that some such κ^{n+1} must exist. We define the function $\kappa : \omega \times \omega \to \Gamma$ by $\kappa(x, y) = \kappa^{x+y}(x, y)$, and it is not hard to see that κ is a tiling function. $\square$

We can now generalize the results of the previous section to apply only to the n^{th} corner of the plane. We define the function $\theta : \omega \times \omega \to C^n \cup \{\infty\}$ by $\theta(c) = c$ if $c \in C^n$ and $\theta(c) = \infty$ otherwise. We define $\rho^n : \omega \times \omega \to C^n \cup \{\infty\}$ by $\rho^n(c) = \theta(\rho(c))$ where ρ is defined in Definition 2. The function ρ^n restricts the gridpath the the n^{th} corner of the plane and is illustrated in Figure 4.

We can now proceed as we did in the previous section. For reasons of brevity we will not reproduce the full proofs, but rather skip straight to encoding the restricted grid-path in QPTL. We will define the formula $GPC(z)$ (*grid-path corner*) which forces the structure to represent a grid-path up to some arbitrary point, and all other states in the grid-path are collapsed to a single point, the final state. We will use the free variable z to mark the final state. We retain the definitions $1st$, $2nd$, $C3a(e)$ and $C3b(e)$ given in Definition 4. We also add two new conditions. The first requires that z be true at only one state, and the second requires that after the first occurrence of z, only states that have previously appeared in the model can be used. Although the final state appears infinitely often in the model, the formula $1st$ will always be true at the final state, simply because it is not the last occurrence of that state.

DEFINITION 7 *The formula $GPC(z)$ is given by the following definitions:*

$$
\begin{aligned}
C1 &= \Box\exists x(x \to (z \vee (\bigcirc\Box(x \to \bigcirc\Box\neg x))) \wedge \\
&\qquad \forall x(\Diamond x \to \Diamond(x \wedge \bigcirc\Diamond x)) \\
C2 &= e \wedge \Box(z \vee (e \leftrightarrow \bigcirc\neg e)) \\
C3 &= \Box(1st \to ((C3a(e) \wedge C3b(e)) \vee \\
&\qquad (C3a(\neg e) \wedge C3b(\neg e))) \\
C4 &= \Box(2nd \to \bigcirc 1st) \\
C5 &= \Box(1st \wedge \bigcirc 1st \to (z \vee \bigcirc\bigcirc 2nd)) \\
C6 &= \forall x(\Box(z \wedge x) \to \Box(z \to x)) \\
C7 &= \Box(z \to \bigcirc(2nd \vee z) \\
GPC(z) &= \exists e(C1 \wedge C2 \wedge C3 \wedge C4 \wedge C5 \wedge C6 \wedge C7)
\end{aligned}
$$

Proofs similar to those given in the previous section will show that if $GPC(z)$ is satisfied by some model, that model will act like a grid-path until z is true. Particularly, if $z \notin \pi(\mu(i))$ for all $i < (n+2)^2$, then it follows that μ defines a grid-path over the n^{th} corner of the plane. The only difference between $Tile_2^\gamma$ and $Tile^\gamma$ is that we do not require the tiling to continue after the first moment that z is true.

We can now define the formula $Tile_2^\Gamma$, which is similar to the formula $Tile^\Gamma$, however it is not required to encode a recurrent tiling. Again, a proof of soundness can be inferred from the previous section.

DEFINITION 8 *The formula $Tile_2^\Gamma$ is given as*

$$
\begin{aligned}
\gamma^i &= \tau_i \wedge \bigwedge_{j \neq i} \neg\tau_j \\
left_i &= 1st \to \bigcirc(2nd \to \bigvee_{j \in L_i} \tau_j) \\
up_i &= 2nd \to \bigcirc(z \vee \bigvee_{j \in T_i} \tau_j) \\
t_2^\Gamma(z) &= \Box\bigvee_{i=0}^{n}(z \vee (\gamma^i \wedge left_i \wedge up_i)) \\
Tile_2^\Gamma &= \forall z((GPC(z)) \to \exists\Gamma t_2^\Gamma(z)).
\end{aligned}
$$

We can now sketch a proof of the undecidability of the satisfiability problem. The form of the proof is slightly different since given any satisfiaible formula, we can always show it is satisfiable by enumerating all QPTL^F models. However this process will not halt if the formula is unsatisfiable. Several technical sub-lemma's are omitted, however these results are similar to results given in the previous section and are not hard to reproduce in the context of this section.

LEMMA 8 Γ *can tile the plane if and only if $Tile_2^\Gamma$ is a validity.*

PROOF:

If Γ can tile the plane, then Γ will clearly be able to tile any portion of the plane. The formula GPC requires that the model be a grid-path up to some point, and all moments after that point share a single state. Since Γ tiles the plane we can easily find an assignment of the atoms in Γ that will satisfy the formula t_2^Γ. Since this process can be repeated for any assignment of z satisfying $GPC(z)$, it must be that $Tile_2^\Gamma$ is a validity.

If $\forall z((\Diamond \Box z \wedge GPC(z)) \rightarrow tile2^\Gamma(z))$ is a validity, then it must be true for every model. It thus follows that Γ can tile the n^{th} corner of the plane for every n, and thus Γ can tile the plane. $\Box$

We have shown that even with finite models these semantic interpretations make QPTL too expressive to be axiomatized. We will now examine one more semantic definition which is more general than the linear models allowed by QPTL$^\omega$, but has a decidable satisfiability problem.

5. QPTL with deterministic successors

We now examine the semantic QPTL^P, a restriction of QPTL^R which allows states to be repeated throughout a model, however it restricts the successor function to be deterministic. The semantic interpretation QPTL^P is the same as the semantic interpretation QPTL^R except that we now require μ to be defined such that if $\mu(i) = \mu(j)$, then $\mu(i+1) = \mu(j+1)$. These semantics allow all models of QPTL, however they also allow models that loop infinitely. The resulting theory is decidable as we show below.

The cost of this decidability is expressivity. We find that QPTL^P is only slightly less restrictive than QPTL$^\omega$ (in fact we show a QPTL^P is satisfiable by translating formulas and models to agree with QPTL$^\omega$). This semantic still allows us to examine periodic models, and formulate such concepts as the period of cycle for an automata. Importantly, the complexity of satisfiability problem for QPTL^P is no harder than the satisfiability problem for QPTL$^\omega$, so any increase in expressivity is an increase we get for free.

We will show that the satisfiability problem for QPTL^P is reducible to the satisfiability problem for QPTL$^\omega$, and hence decidable. To do this we simply unwind the QPTL^P model and allow the variable x_0 to mark the beginning and the end of the loop.

DEFINITION 9 *Given any QPTL^P model $\sigma = (S, \mu, \pi)$, we say $s \in S$ is the* first looping state *if there exists integers $j > i$ such that $\mu(i) = s = \mu(j)$ and for all $k < i$, $\mu(k) = \mu(\ell)$ implies $k = \ell$.*

The ω-marked version *of σ is the QPTL$^\omega$ model ($\sigma' = (S', \mu', \pi')$ where*

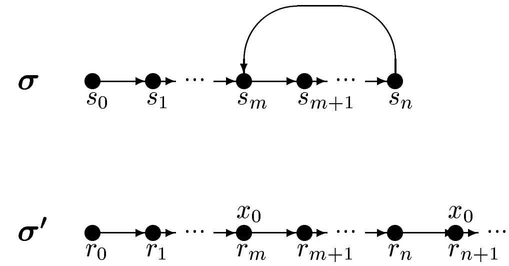

Figure 3. An illustration of σ', the ω-version of σ.

1. *$\pi'(\mu'(i)) = \pi(\mu(i)) \cup \{x_0\}$ if $\mu(i)$ is a first looping state,*
2. *$\pi'(\mu'(i)) = \pi(\mu(i)) \backslash \{x_0\}$ otherwise.*

This is illustrated in Figure 5, where $\mu(i) = s_i$, $\mu'(i) = r_i$, and r_m and r_{n+1} are marked with x_0.

For formulas without quantification or the variable x_0, it is clear that satisfiability is preserved in the ω-marked version of the model. To extend this to include formulas containing quantifiers we must transform the formula so that it only refers to moments up to (and including) the last moment in the loop, and infers all infinite behavior from that finite set of moments. This is done in the following definition.

DEFINITION 10 *The* ω-version of α *is defined in the following stages. First we define a translation for the sub-formulas of α. For every sub-formula, γ, of α let y_γ describe a unique variable that does not appear in α. Furthermore, suppose that the variable x_0 does not appear in α. We define by induction:*

$$\begin{aligned}
\gamma = \bigcirc\beta &\implies \overline{\gamma} = \bigcirc((x_0 \wedge y_\gamma) \vee (\neg x_0 \wedge \overline{\beta})) \\
\gamma = \Box\beta &\implies \overline{\gamma} = \overline{\beta}\, W (y_\gamma \wedge x_0) \\
\gamma = \neg\beta &\implies \overline{\gamma} = \neg\overline{\beta} \\
\gamma = \beta \vee \phi &\implies \overline{\gamma} = \overline{\beta} \vee \overline{\phi} \\
\gamma = \exists x \beta &\implies \overline{\gamma} = \exists x \overline{\beta} \\
\gamma \in \mathcal{V} &\implies \overline{\gamma} = \gamma.
\end{aligned}$$

1. *Given $\gamma = \bigcirc\beta$, some sub-formula of α, define the formula:*

$$f_\gamma = \neg y_\gamma \wedge \Box(((x_0 \wedge \overline{\beta}) \vee y_\gamma) \leftrightarrow \bigcirc y_\gamma) \quad (4)$$

2. *Given $\gamma = \Box\beta$, some sub-formula of α, define the formula:*

$$f_\gamma = \neg y_\gamma \wedge \Box(((x_0 \vee y_\gamma) \wedge \overline{\beta}) \leftrightarrow \bigcirc y_\gamma) \quad (5)$$

Finally we enumerate all the sub-formulas of α which have the form $\bigcirc\beta$ or $\Box\beta$, as $\gamma_1, \gamma_2, ..., \gamma_m$. Then

$$\alpha^* = \exists y_{\gamma_1} ... \exists y_{\gamma_m} \left(\overline{\alpha} \wedge \bigwedge_{i=1}^{m} f_{\gamma_i} \right) \quad (6)$$

The definition of α^* requires the temporal operators to be modified so they never refer to the same state at different moments. This way α^* can be interpreted in a linear model (i.e. a QPTL$^\omega$ model), however it will simulate the looping behavior of a QPTL^P model. The variable x_0 is used to mark the start and end of the loop (if there is a loop). The atoms, y_γ, are used to reference what formulas have been true at, or since the start of the loop. All sub-formulas of α^* only refer to moments up to the end of the loop and infer the infinite behavior of the the model from those moments.

For every sub-formula, $\bigcirc\beta$, of α the interpretation of the operator $\bigcirc$ will remain unchanged at all moments except for the moment before the second occurrence of x_0. At this moment we must ensure that the operator $\bigcirc$ does not refer to the next moment, but rather the first moment that x_0 was true. This way we can ensure that the properties of the loop are maintained. For example if $\alpha = \forall x(x \rightarrow \Diamond \bigcirc x$ and $\beta = x$, to make α^* satisfiable in QPTL we must ensure that $\bigcirc x$ can refer to previous moments. The formula $f_{\bigcirc\beta}$ allows us to refer to previous moments by ensuring that $y_{\bigcirc\beta}$ is true in all moments after the first moment x_0 was true if and only if β was true at that moment. Just before the second moment x_0 is true, $\overline{\bigcirc\beta}$ does not refer to the next state in the model (since x_0 is true there). Instead α^* "interprets" $\bigcirc\beta$ as true if and only if $y_{\bigcirc\beta}$ is true, which is true if and only if β was true at the start of the loop. In this way α^* behaves the same way on the ω-marked version of a model, as α behaves on the model itself.

For every sub-formula, $\Box\beta$, of α the interpretation of the operator $\Box$ is altered so it only refers to moments up to the end of the loop (the moment before x_0 is true for the second time). The formula $f_{\Box\beta}$ will ensure that the atom $y_{\Box\beta}$ is true at some moment if and only if β has been true at every moment since the first moment x_0 was true. The formula $\overline{\Box\beta}$ is then interpreted to be true if and only if $\overline{\beta}$ is true up to the end of the loop, and $y_{\Box\beta}$ is true at the end of the loop also. This will only happen if $\overline{\beta}$ was true at every moment in the loop, and would therefore for true at every subsequent moment in the QPTL^P model. Note that since the interpretation of $\overline{\Box\beta}$ is defined in terms of the operator W, the interpretation remains valid even for models where x_0 is false for every moment.

It is now a simple matter to show the satisfiability problem is decidable. We will give a sketch of the proof below.

LEMMA 9 *Suppose $x_0 \notin \mathrm{var}(\alpha)$. Then α is satisfied by some QPTL^P model σ if and only if α^* is satisfied by the ω-marked version of σ.*

PROOF:

The reasoning above can be applied inductively to show that if α is satisfied by some QPTL^P model σ, then α^* will be satisfied by the ω-marked version of σ.

If α^* is satisfied by some QPTL model σ, where x_0 is true for at most one moment, then clearly σ will also be a model for α. If α^* is satisfied by some model, $\sigma = (S, \mu, \pi)$, where x_0 is true for at least two moments, we can find a QPTL^P model of α by defining a function μ' that agrees with μ up to (but not including) the second moment that x_0 is true and loops back to the first state where x_0 was true. Since the successor function is deterministic, this is enough to define the model. A simple inductive argument will show that α will be satisfied by the model (S, μ, π). $\Box$

Consequently α is satisfiable in QPTL^P if and only if α^* is satisfiable in QPTL. Since the size of α^* is polynomial in the size of α the complexity of the decision procedure is roughly equivalent (bearing in mind that the decision procedure for QPTL is non-elementary [13]). From this we can infer an optimal decision procedure for QPTL^P.

It is also worth noting that a recent, complete axiomatization of QPTL, [6] specified the axiom QX0, (given as the equation 1), which is not necessarily valid in QPTL^P. However all the other axioms and rules given in that axiomatization are sound for QPTL^P, and the models of QPTL which are not models of QPTL^P are exactly the models, σ, such that $\sigma \models \mathrm{QX0}$. It is therefore worth considering whether that axiomatization, without the axiom QX0, is complete for QPTL^P. However the proof of completeness for QPTL was very complicated given the QPTL$^\omega$ semantics, and a proof of completeness given the QPTL^P semantics has not yet been investigated.

6. Conclusion

In this paper we have investigated several alternative semantics for QPTL which allow states to be repeated throughout a model. These semantics were generated by varying the properties of the function μ, which maps moments of time to the states of the model. While we have not systematically investigated the decidability of all possible variations of μ, we have identified three semantics which give a good indication of the relationship between the restrictions on μ and the complexity of the satisfiability problem.

For the least restrictive semantic, QPTL^R, we have shown that the satisfiability problem is highly undecidable, via an encoding of the recurrent tiling problem. While this is enough to show that the language is too expressive to be of practical use, it only scratches the surface of the true complexity of the language. When encoding the tiling problem we only required the function μ to allow a state to be repeated twice, rather than arbitrarily. Also, the formulation of the tiling problem only required the nesting of three propositional quantifiers. The effect of quantifier depth on the satisfiability problem and the true complexity of QPTL^R are also worth investigating.

We have also shown that the satisfiability problem is co-RE complete when the models are restricted to have a finite number of states (QPTL^F). It should be noted the same proof will suffice to show that the satisfiability problem is co-RE complete when μ is restricted to have a finite number of states, and each state can have at most two successors, (i.e. $\mu(i) = \mu(j) = \mu(k)$ implies either $\mu(i+1) = \mu(j+1)$, $\mu(i+1) = \mu(k+1)$ or $\mu(j+1) = \mu(k+1)$). This shows that even quite strong restrictions on μ are not enough to make the satisfiability problem for QPTL decidable.

The last semantic we investigated was QPTL^P which required the successor function to be deterministic. The satisfiability problem was shown to be decidable, and a decision procedure was given. There are still some semantics that are less restrictive than QPTL^P and are not known to be decidable, however these are quite artificial and we will not go into them here.

Finally it is worth noting some comparisons with work done in hybrid logic. We have seen that allowing repeating states allows us to quantify not only propositional atoms, but also the states of the model. Quantifying the states of a model has been studied extensively in hybrid logic [1], [7]. The proof of undecidability given in this paper is comparable to the one given in [7], since it essentially uses propositional quantification to simulate Goranko's reference pointers. The significance of the proof given here, is that it uses a syntactically linear language, and does not rely on branching modal operators.

References

[1] C. Areces, P. Blackburn, and M. Marx. The computational complexity of hybrid temporal logics. *L.J. of the IGPL*, 8(5):653–679, 2000.

[2] R. Berger. The undecidability of the dominoe problem. *Mem. Amer. Math. Soc.*, 66, 1966.

[3] K. Englehardt, R. van der Meyden, and Y. Moses. Knowledge and the logic of local propositions. In *Theoretical Aspects of Rationality and Knowledge, Proceedings of the Seventh Conference*, pages 29–41, 1998.

[4] K. Fine. Propositional quantifiers in modal logic. *Theoria*, 36:336–346, 1970.

[5] T. French. Decidability of quantified propositional branching time logics. In *Proceedings of the 14th Australian Joint Conference on Artificial Intelligence*, pages 165–176, 2001.

[6] T. French and M. Reynolds. A sound and complete proof system for qptl. In *Proceedings of AiML 2002*, to appear.

[7] V. Goranko. Hierarchies of modal and temporal logics with reference pointers. *Journal of Logic Language and Information*, 5(1):1–24, 1996.

[8] D. Harel. Recurring dominoes: making the highly undecidable highly understandable. In *Proc. of the Conference on Foundations of Computing Theory*, pages 177–194, 1983.

[9] D. Harel. Effective transformations on infinite trees, with applications to high undecidability, dominoes, and fairness. *Journal A.C.M.*, 33(1):224–248, 1986.

[10] Y. Kesten and A. Pnueli. A complete proof systems for QPTL. In *Proceedings, Tenth Annual IEEE Symposium on Logic in Computer Science*, pages 2–12, 1995.

[11] M. Marx and M. Reynolds. Undecidability of compass logic. *Journal of Logic and Computation*, 9(6):897–914, 1999.

[12] A. Pnueli. The temporal logic of programs. In *Proceedings of the Eighteenth Symposium on Foundations of Computer Science*, pages 46–57, 1977.

[13] A. Sistla, M. Vardi, and P. Wolper. The complementation problem for Buchi automata with applications to temporal logic. *Theoretical Computer Science*, 49:217–237, 1987.

[14] A. P. Sistla. *Theoretical Issues in the Design and Verification of Distributed Systems*. PhD thesis, Harvard University, 1983.

[15] E. Spaan. *Complexity of Modal Logics*. PhD thesis, Universiteit van Amsterdam, 1993.