Ultimately Periodic Simple Temporal Problems (UPSTPs)

Jean-François Condotta [1] Gérard Ligozat [2] Mahmoud Saade [1] Stavros Tripakis [3]

[1] CRIL-CNRS, Université d'Artois, Rue de l'Université, 62307 Lens, France
[2] LIMSI-CNRS, Université de Paris-Sud, 91403 Orsay, France
[3] VERIMAG, Centre Équation, 2, avenue de Vignate, 38610 Gieres, France
{condotta,saade}@cril.univ-artois.fr, ligozat@limsi.fr, tripakis@imag.fr

Abstract

In this paper, we consider quantitative temporal or spatial constraint networks whose constraints evolve over time in an ultimately periodic fashion. These constraint networks are an extension of STP*s (Simple Temporal Problems). We study some properties of these new types of constraint networks. We also propose a constraint propagation algorithm. We show that this algorithm decides the consistency problem in some particular cases.*

1 Introduction

In many areas of Computer Science, in particular in Artificial Intelligence, it is necessary to reason about temporal information. Numerous formalisms for representing and reasoning about time with constraints have been proposed. The constraint networks considered by these formalisms differ, on the one hand, in terms of the temporal entities represented by the variables: these entities can be temporal points, temporal intervals, durations or distances, for example; on the other hand, they differ in terms of the nature of the constraints they use: these constraints can be qualitative [1, 2, 3], metric/quantitative [4, 5] or both [6, 7, 8].

Simple temporal problems (STPs) [5] belong to the class of quantitative constraint networks. They represent temporal entities by points on the line and allow to constrain the distances/durations between these points using numeric values specified by intervals. It is well known that STPs can be solved in polynomial time, which is one of the reasons why these constraint networks are very used. On many occasions the STPs have been extended to define more expressive constraint networks [9, 10, 11].

In this paper, we consider temporal quantitative constraint networks whose constraints evolve over time in a ultimately periodic fashion. These constraint networks are extensions of the STPs and can be used to represent "the cyclic constraints" defined by Tripakis [12] for example. We call them ultimately periodic STPs (UPSTPs in short). These networks, interpreted in a spatial context, can be seen as temporalized STPs. More precisely, consider a set of punctual objects on the line whose spatial locations change over time. At each instant, an object has a given location. Then a UPSTP makes it possible to express constraints on the relative locations of the objects over time, such as constraints which have to be satisfied at each new occurrence of a particular time (periodic constraints), as well as constraints involving different instants.

In a temporal context, a recurrent activity or event can have a finite or infinite number of occurrences over time. In some applications these occurrences may have to satisfy a set of quantitative constraints on their relative durations. A UPSTP allows to specify such constraints.

The goal of this paper is to study various properties of these networks. We also propose a specific constraint propagation algorithm for UPSTPs. We show that this algorithm decides the consistency problem in polynomial time for interesting particular cases.

The remainder of this paper is organized as follows. Section 2 recalls some basic facts about

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STPs. In Section 3 we introduce the ultimately periodic simple temporal problems (UPSTPs). In Section 4 we relate the consistency problem for UPSTPs to the consistency problem for classical STPs. Section 5 is devoted to the study of particular UPSTPs, namely closed UPSTPs. A constraint propagation algorithm is proposed in Section 6. Section 7 ends the paper with concluding remarks.

2 Preliminaries on STPs

We denote the set of intervals on the line of rational numbers by $\mathsf{INT}_{\mathbb{Q}}$. These intervals can be finite or infinite, and they can include or not their lower and upper bounds, if any. In particular, this set contains *the empty interval*, denoted by $\emptyset$ in the sequel. Given two intervals I and J, $-I$ denotes the interval opposite to I, *i.e.* the interval deduced from I by the symmetry $x \mapsto -x$ *w.r.t.* the origin. $I \cap J$ denotes the interval corresponding to the intersection of I and J, $I + J$ is the sum of the intervals I and J, *i.e.* the interval which is the union of all translated intervals $i + J = \{i + x \mid x \in J\}$, for $i \in I$. Given an integer number c, $c.I$ is the image of I under the transform $x \rightarrow c.x$. As an illustration, consider the intervals $]-\infty, 3]$ and $]1, 5]$ (this last interval corresponds to the rational numbers strictly greater than 1 and less than 5). We have $-]-\infty, 3] = [-3, +\infty[$, $-]1, 5] = [-5, -1[$, $]-\infty, 3] \cap]1, 5] =]1, 3]$, $]-\infty, 3] +]1, 5] =]-\infty, 8]$ and $2.]1, 5] =]2, 10]$. STPs [5] are binary quantitative constraint networks where the constraints involve distances between points on the line: Each constraint is defined by an interval which represents the admissible values for the distance between the two points involved. All constraints of the STPs considered in this paper will be defined by intervals belonging to $\mathsf{INT}_{\mathbb{Q}}$:

Definition 1 *A* STP $\mathcal{S}$ *is a pair* (V, C) *where:*

- V *is a finite set of variables* $\{v_0, \ldots, v_{n-1}\}$*, where* n *is a positive integer;*
- C *is a map from* $V \times V$ *to* $\mathsf{INT}_{\mathbb{Q}}$*, associating to each* $(v_i, v_j) \in V \times V$ *an interval* $C(v_i, v_j)$ *belonging to* $\mathsf{INT}_{\mathbb{Q}}$ *(also denoted by* C_{ij} *in the sequel) such that* $C(v_i, v_i) \subseteq [0, 0]$ *(actually* $C(v_i, v_i)$ *can be* $[0, 0]$ *or the empty interval) and* $C(v_j, v_i) = -C(v_i, v_j)$ *for all* $v_i, v_j \in V$.

Each variable v_i represents a point on the rational line. An interval $C(v_i, v_j)$ gives the admissible values for the distance $(v_j - v_i)$ between the two points represented by v_i and v_j. A solution of a STP is formally defined in the following way:

Definition 2 *Let* $\mathcal{S} = (V, C)$ *be a* STP.

- *An instantiation* σ *of* $\mathcal{S}$ *is a map from* V *to* $\mathbb{Q}$ *associating to each variable* v_i *of* V *a rational number* $\sigma(v_i)$ *(also denoted by* σ_i*).*
- *An instantiation* σ *of* $\mathcal{S}$ *is a solution iff for all* $v_i, v_j \in V$, $\sigma_j - \sigma_i \in C_{ij}$.

A STP is consistent iff it has a solution. The consistency problem for STPs consists in determining, given a STP, whether it is consistent or not. This problem is a polynomial problem. A STP $\mathcal{S} = (V, C)$ is a subSTP of $\mathcal{S}' = (V', C')$, which is denoted by $\mathcal{S} \subseteq \mathcal{S}'$, if $V = V'$ and $C(v_i, v_j) \subseteq C'(v_i, v_j)$ for all $v_i, v_j \in V$ ($\mathcal{S} \subset \mathcal{S}'$ denotes the case where $\mathcal{S} \subseteq \mathcal{S}'$ and for at least a pair of variables $v_i, v_j \in V$ we have $C(v_i, v_j) \subset C'(v_i, v_j)$). We will say that a STP $\mathcal{S} = (V, C)$ is PC-closed iff for all $v_i, v_j, v_k \in V$, $C(v_i, v_j) \subseteq C(v_i, v_k) + C(v_k, v_j)$. It is well known that a PC-closed STP which does not contain the empty interval as a constraint is a consistent STP. It is also globally consistent (each partial solution on a subset of variables can be extended to a solution). Given a STP $\mathcal{S}$, there exists a unique equivalent STP which is PC-closed; we denote it by $\mathsf{PC}(\mathcal{S})$. Polynomial methods consisting in iterating the operation of triangulation: $C_{ij} \leftarrow C_{ij} \cap (C_{ik} + C_{kj})$ for each triple of variables v_i, v_j and v_k until a fix-point is reached are used to obtain this equivalent STP. For example, we can use the algorithm $PC1$ [13] which only uses one main loop for STPs and whose complexity is hence $0(|V|^3)$ for STPs. Such a method will be generically called a path-consistency method in the sequel.

3 Ultimately Periodic Simple Temporal Problems

In a temporal context, a STP expresses quantitative constraints between a set of activities or events represented by points. A STP can also be used in a spatial context to represent constraints on the relative positions of a set of punctual objects on the line.

Now we define the main notion of this paper which

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we call *ultimately periodic simple temporal problems* or *ultimately periodic* STPs (UPSTPs in brief). This new notion, interpreted in a spatial context, can be seen as a temporalized STP. More precisely, consider a set of punctual objects on the line whose spatial locations may change over the time. At each instant, an object has a given location. With a UPSTP we can express three kinds of constraints: constraints between the locations of the objects at one given instant, constraints between the locations of the objects at different instants, constraints between the locations of the objects which have to be satisfied at each instant following an initial instant. We assume that time is modeled by the natural integers. Hence, each integer $t \geq 0$ corresponds to an instant in time.
In a temporal context, a recurrent activity or event can have a finite or infinite number of occurrences over time. In some applications these occurrences may have to satisfy a set of quantitative constraints on the durations between two of them. A UPSTP allows to specify such constraints.
Formally, we define an *ultimately periodic* STP in the following way:

Definition 3 *A* UPSTP *is a structure* $\mathcal{U} = (V, C, t_{min}, t_{max})$ *where:*

- $V = \{v_0, \ldots, v_{n-1}\}$ *is a set of* n *variables ;*
- t_{min} *and* t_{max} *are two positive integers such that* $t_{min} \leq t_{max}$ *;*
- C *is a map from* $V \times \{0, \ldots, t_{max}\} \times V \times \{0, \ldots, t_{max}\}$ *to* $\mathsf{INT}_{\mathbb{Q}}$ *such that* $C(v_i, t_i, v_j, t_j) = -C(v_j, t_j, v_i, t_i)$ *and* $C(v_i, t_i, v_i, t_i) \subseteq [0, 0]$ *for all* $v_i, v_j \in V$ *and* $t_i, t_j \in \{0, \ldots, t_{max}\}$.

The application C expresses explicitly the constraints between the locations of the different occurrences of the variables of V for the instants belonging to $\{0, \ldots, t_{max}\}$. The map C also expresses constraints which have to be satisfied at each future instant. Indeed, the constraints given for the instants $\{t_{min}, \ldots, t_{max}\}$ have to be also satisfied on all future periods, *i.e.* on each interval $\{t_{min} + i, \ldots, t_{max} + i\}$ with $i \geq 0$.
Intuitively, in a spatial context, each variable $v_i \in V$ represents a point on the rational line whose location evolves over time. The pair (v_i, t_i), with $t_i \in \mathbb{N}$, represents this location at time t_i. The constraint $C(v_i, t_i, v_j, t_j)$ constrains the distance between the point v_i at time t_i and the point v_j at time t_j. In a temporal context, the variable v_i no longer represents the punctual spatial component of the object, but instead a recurrent activity or event. The pair $(v_i, t_i) \in V \times \mathbb{N}$ represents then the $(t_i + 1)^{\text{th}}$ occurrence of the event represented by v_i.
In accordance with the preceding interpretations, we define a solution of a UPSTP in the following way:

Definition 4 *A solution* σ *of a* UPSTP $\mathcal{U} = (V, C, t_{min}, t_{max})$ *is a map from* $V \times \mathbb{N}$ *to* $\mathbb{Q}$ *such that, for all* $v_i, v_j \in V$ *and* $t_i, t_j \in \mathbb{N}$ *:*

1. *if* $t_i, t_j \leq t_{max}$ *then* $\sigma(v_j, t_j) - \sigma(v_i, t_i) \in C(v_i, t_i, v_j, t_j)$ *;*
2. *if* $t_{min} \leq t_i \leq t_j$ *and* $t_j - t_i \leq t_{max} - t_{min}$ *then for all* t'_i, t'_j *such that* $t_{min} \leq t'_i \leq min\{t_{max}, t_i\}$ *and* $t_{min} \leq t'_j \leq min\{t_{max}, t_j\}$ *and* $t_j - t_i = t'_j - t'_i$ *we have* $\sigma(v_j, t_j) - \sigma(v_i, t_i) \in C(v_i, t'_i, v_j, t'_j)$.

We extend in an obvious way the notions of consistency and equivalence for ordinary constraint networks to the case of UPSTPs. The following examples illustrate the preceding definitions.

Example 1 *In a spatial context, consider three objects* O_0*,* O_1 *and* O_2*, whose spatial locations are represented by three variables* v_0*,* v_1 *and* v_2 *which stand for rational numbers. Assume that the objects change positions over time with the following constraints:*

- *At time* 0*,* O_0 *is left of* O_1 *and is left of* O_2 *at a distance comprised between* 3 *and* 5*;*
- *the location of* O_2 *at time* 0 *is left of its location at time* 1 *;*
- *at time* 1*, and for all future instants,* O_0 *is right of* O_1 *at a maximal distance of* 10*;*
- *after time* 1*,* O_0 *moves left, and* O_1 *moves right ;*
- *after time* 2*,* O_2 *moves left and stays away* O_1 *at a maximal distance of* 4.

These constraints can be expressed by the UPSTP $\mathcal{U} = (V, C, t_{min}, t_{max})$*, where* $V = \{v_0, v_1, v_2\}$*,* $t_{min} = 1$*,* $t_{max} = 3$*. The constraints defined by* C *are represented in Figure 1. As usual, the following constraints are not given: the constraint* *between a variable and itself, the universal constraint ($]-\infty,+\infty[$), the opposite constraint of a constraint which is already given. This* UPSTP $\mathcal{U}$ *is a consistent* UPSTP*, a solution is depicted in Figure 1.*

Example 2 *Consider now two recurrent punctual events E_0 and E_1 represented by two variables v_0 and v_1 which stand for rational numbers. Assume that the different occurrences of these events are constrained in the following way:*

- *for all $i \geq 0$, the i^{th} occurrence of E_1 is always after the i^{th} occurrence of E_0. At least one second elapses between these occurrences.*
- *For all $i \geq 0$, at least two seconds elapse between the $(i+1)^{th}$ occurrence and the i^{the} occurrence of E_0, exactly one second elapses between those of E_1.*

These constraints can be expressed by the UPSTP $\mathcal{U} = (V, C, t_{min}, t_{max})$*, where $V = \{v_0, v_1\}$, $t_{min} = 0$, $t_{max} = 1$. The constraints defined by C are represented in Figure 2. We leave it to the reader to check that this* UPSTP *is not consistent.*

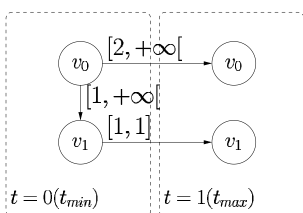

Figure 2. The constraints C of the UPSTP $\mathcal{U}$ **of Example 2.**

To close this section we show that the consistency problem of any UPSTP can be can be reduced (in polynomial time) to the consistency problem of a UPSTP whose constraints have either infinite bounds, or bounds defined by integers:

Proposition 1 *Let $\mathcal{U} = (V, C, t_{min}, t_{max})$ be a* UPSTP*, and d be product of the denominators of the finite bounds* [1] *of the intervals defining C ($d = 1$ in the case where all bounds are infinite). Let $\mathcal{U}' = (V, C', t_{min}, t_{max})$ be the* UPSTP *defined by $C'(v_i, t_i, v_j, t_j) = d.C(v_i, t_i, v_j, t_j)$ for all $v_i, v_j \in V$ and $t_i, t_j \in \{0, \dots, t_{max}\}$. Then $\mathcal{U}$ is consistent iff $\mathcal{U}'$ is consistent.*

Proof Let σ be a solution of $\mathcal{U}$. Let σ' be a map from $V \times \mathbb{N}$ to $\mathbb{Q}$ defined from $\sigma'(v_i, t_i) = d(\sigma(v_i, t_i) - \sigma(v_0, 0))$ for all $v_i \in V$ and $t_i \in \mathbb{N}$. We can show that σ' is a solution of $\mathcal{U}'$. Suppose now that a solution σ' of $\mathcal{U}'$ is given. By defining a map σ from $V \times \mathbb{N}$ to $\mathbb{Q}$ with $\sigma(v_i, t_i) = (\sigma'(v_i, t_i) - \sigma'(v_0, 0))/d$ for all $v_i \in V$ and $t_i \in \mathbb{N}$ we obtain a solution of $\mathcal{U}$. $\dashv$

Because of this fact, we can assume without loss of generality that all UPSTPs have constraints whose finite bounds are integers.

4 Implicit constraints *versus* explicit constraints

In this section we relate the consistency problems for UPSTPs – which potentially express an infinite number of constraints – to the consistency problems for the STPs. In order to do this we take the following steps :

1. We associate to each UPSTP a STP corresponding to its periodic constraints. This STP is called the *motif* of the UPSTP.
2. Based on the use of the motif, we define a finite sequence of STPs "with increasing temporal support". Each one of these STPs makes explicit the constraints of the UPSTP on a number of initial points in time. We call them the strengthenings of the UPSTP.
3. Finally, we relate the consistency problem of the UPSTPs to properties of its strengthenings.

We now proceed to implement these steps in detail. The first finite networks we consider are the motifs of the UPSTPs:

Definition 5 *Let $\mathcal{U} = (V, C, t_{min}, t_{max})$ be a* UPSTP*. The motif of $\mathcal{U}$, denoted by* $\mathsf{motif}(\mathcal{U})$*, is the* STP $\mathcal{S}_m = (V_m, C_m)$ *where $V_m = V \times \{0, \dots, \mathsf{lg}\}$ (with $\mathsf{lg} = t_{max} - t_{min}$) and $C_m((v_i, t_i), (v_j, t_j)) = C(v_i, t_i + t_{min}, v_j, t_j + t_{min})$ for all $v_i, v_j \in V$ and for all $t_i, t_j \in \{0, \dots, \mathsf{lg}\}$.*

[1] We can assume without loss of generality that each value of a finite bound of a constraint is defined by a fraction p/q, where p is an integer and q is a strictly positive integer

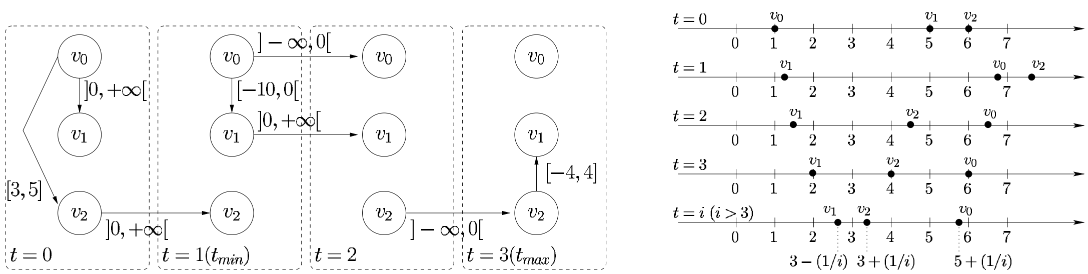

Figure 1. The UPSTP $\mathcal{U}$ corresponding to Example 1 and a solution of $\mathcal{U}$.

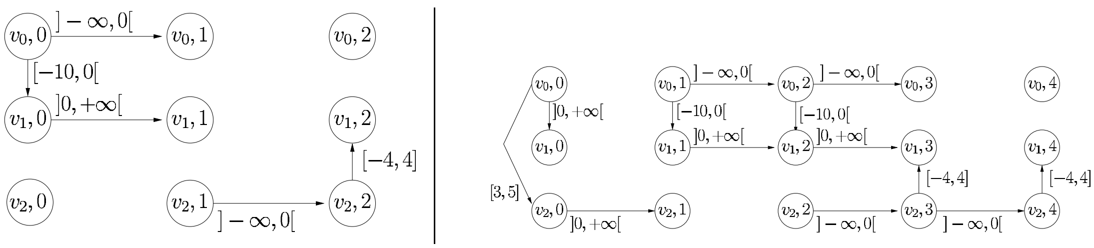

Figure 3. The motif of the UPSTP $\mathcal{U}$ and its 4-strengthening.

In the sequel lg will denote the difference $t_{max} - t_{min}$. The motif of the UPSTP $\mathcal{U}$ in Figure 1 is represented in Figure 3.

Using the notion of motif, we are going to define a STP which, in some way, makes explicit constraints of a UPSTP which must be satisfied during the $(k+1)$ first instants (where $k \geq t_{max}$). For such a k, the STP we define is called the k-strengthening of the UPSTP. Before giving a formal definition, we describe them in an intuitive way. Consider a picture representing the constraints of the UPSTP over the instants $\{0, \ldots, t_{max}\}$. Imagine that a picture of the motif is drawn independently on a transparent sheet. Given an integer number $k \geq t_{max}$, we superpose the transparent sheet on the original picture, then we move this motif with a shift of one, then of two, and so on, until the instant k is reached. Each time, we add the constraints of the transparent sheet to the existing ones (taking intersections). Then the k-strengthening is the STP we get when reaching instant k. We now give a formal definition:

Definition 6 *Let $\mathcal{U} = (V, C, t_{min}, t_{max})$ be a* UPSTP *and $\mathcal{S}_m = (V \times \{0, \ldots, \mathsf{lg}\}, C_m)$ its motif. Given an integer number $k \geq t_{max}$, the k-strengthening of $\mathcal{U}$, denoted by* k–strenthening($\mathcal{U}$), *is the* STP *$\mathcal{S}^k = (V^k, C^k)$ defined recursively* [2] *by:*

- $V^k = V \times \{0, \ldots, k\}$;
- $C^{t_{max}}((v_i, t_i), (v_j, t_j)) = C(v_i, t_i, v_j, t_j)$ *for all* $v_i, v_j \in V$ *and* $t_i, t_j \in \{0, \ldots, t_{max}\}$;
- *for* $k \geq t_{max}$ *and for all* $v_i, v_j \in V$ *and* $t_i, t_j \in \{0, \ldots, k+1\}$ *with* $t_i \leq t_j$,
  - $C^{k+1}((v_i, t_i), (v_j, t_j)) = C^k((v_i, t_i), (v_j, t_j))$, *if* $t_i < (k+1) - \mathsf{lg}$ *and* $t_j < k+1$,
  - $C^{k+1}((v_i, t_i), (v_j, t_j)) = C^k((v_i, t_i), (v_j, t_j)) \cap C_m((v_i, t_i - ((k+1) - \mathsf{lg})), (v_j, t_j - ((k+1) - \mathsf{lg})))$, *if* $t_i \geq (k+1) - \mathsf{lg}$ *and* $t_j < k+1$,
  - $C^{k+1}((v_i, t_i), (v_j, t_j)) = C_m((v_i, t_i - ((k+1) - \mathsf{lg})), (v_j, t_j - ((k+1) - \mathsf{lg})))$, *if* $t_j = k+1$ *and* $t_j - t_i \leq \mathsf{lg}$,
  - $C^{k+1}((v_i, t_i), (v_j, t_j)) =]-\infty, +\infty[$, *if* $t_j = k+1$ *and* $t_j - t_i > \mathsf{lg}$,
  - $C^{k+1}((v_j, t_j), (v_i, t_i)) = -C^{k+1}((v_i, t_i), (v_j, t_j))$.

[2] Firstly, we define C^k for $k = t_{max}$ then, we define C^{k+1} from C^k for an integer $k \geq t_{max}$.

Figure 3 shows the 4-strengthening of the UPSTP $\mathcal{U}$ depicted in Figure 1. In the sequel we also use the notion of window of a k-strengthening which is a STP capturing its constraints over $(\mathsf{lg}+1)$ consecutive time points:

Definition 7 *Let $\mathcal{S}^k = (V \times \{0,\dots,k\}, C^k)$ the k-strengthening of a* UPSTP *$\mathcal{U} = (V, C, t_{min}, t_{max})$, with $k \geq t_{max}$. The t-*window *of $\mathcal{S}^k$, with $t_{min} \leq t \leq k - \mathsf{lg}$, denoted by* $\mathsf{t} - \mathsf{window}(\mathcal{S})$, *is the* STP *$\mathcal{S}_t = (V_t, C_t)$ where: $V_t = V \times \{0,\dots,\mathsf{lg}\}$ and $C_t((v_i,t_i),(v_j,t_j)) = C^k((v_i,t_i+t),(v_j,t_j+t))$, for all $v_i, v_j \in V$ and $t_i, t_j \in \{0,\dots,\mathsf{lg}\}$.*

The *end* of $\mathcal{S}^k$ corresponds to its last window, *i.e.* its $(k - \mathsf{lg})$-window. Figure 4 depicts the end of the 4-strengthening of Figure 3.
We give a last definition before beginning the study of the interactions between the various consistency properties of the constraint networks previously introduced.

Definition 8 *Let $\mathcal{U} = (V, C, t_{min}, t_{max})$ be a* UPSTP *and $\mathcal{S}^k = (V \times \{0,\dots,k\}, C^k)$ its k-strengthening for an integer number $k \geq t_{max}$. A map σ from $V \times \mathbb{N}$ to $\mathbb{Q}$ is* a solution *of $\mathcal{S}^k$ iff the restriction of σ to $V \times \{0,\dots,k\}$ is a solution of $\mathcal{S}^k$,* i.e. *iff $\sigma(v_j,t_j) - \sigma(v_i,t_i) \in C^k((v_i,t_i),(v_j,t_j))$ for all $v_i, v_j \in V$ and $t_i, t_j \in \{0,\dots,k\}$.*

A solution of a UPSTP provides solutions for its k-strengthenings moreover, a solution of all k-strengthenings corresponds to a solution of the UPSTP:

Proposition 2 *Let $\mathcal{U}$ be an* UPSTP *and a map σ from $V \times \mathbb{N}$ to $\mathbb{Q}$. The map σ is a solution of the* $\mathsf{k-strenthening}(\mathcal{U})$ *for all $k \geq t_{max}$ iff σ is a solution of $\mathcal{U}$.*

Notice that the consistency of each k-strengthening of a UPSTP does not imply, in the general case, the consistency of the UPSTP. An counter-example is provided by the UPSTP represented in Figure 2: despite its non-consistency, we can define a solution for each one of its k-strengthenings. The final proposition of this section can be straightforwardly proved using the definitions of a k-strengthening and of its end:

Proposition 3 *Let $\mathcal{U}$ be a* UPSTP. *For all $k \geq t_{max}$, the* STP $\mathsf{end}(\mathsf{k-strenthening}(\mathcal{U}))$ *is a subnetwork of* $\mathsf{motif}(\mathcal{U})$.

5 The consistency problem for closed UPSTPs

In this section we consider particular UPSTPs, namely, closed UPSTPs. We have shown that the consistency problem of these constraint networks can be reduced to the consistency problem of its t_{max}-strengthenings and hence is "easy" to solve. In the following section we will introduce a constraint propagation algorithm aiming to transform any UPSTP into a closed UPSTP.
Before defining the property of closure for a UPSTP we introduce an operation called the translation operation. From the motif of a UPSTP, the translation operation gives a STP whose constraints are the constraints of the motif on which are superposed the constraints which must be satisfied on "the next period" (those that must be satisfied at the next instant by shifting by one the constraints of the motif). This operation is used by Tripakis [12] in the framework of periodic constraints which can be expressed by a UPSTP $\mathcal{U} = (V, C, t_{min}, t_{max})$ where $t_{min} = 0$ and $t_{max} = 1$. Formally, we define the translation operation in the following way:

Definition 9 *Let $\mathcal{S} = (V', C)$ be a* STP *with $V' = V \times \{0,\dots,max\}$ (where V is a finite set $\{v_0,\dots,v_m\}$ and $m, max \geq 0$). The* translation *of $\mathcal{S}$, denoted by* $\mathsf{translation}(\mathcal{S})$, *is the* STP *$\mathcal{S}_{tr} = (V_{tr}, C_{tr})$ where $V_{tr} = V'$, and for all $v_i, v_j \in V$ and for all $t_i, t_j \in \{0,\dots,max\}$, $C_{tr}((v_i,t_i),(v_j,t_j)) = C((v_i,t_i),(v_j,t_j)) \cap C((v_i,t_i-1),(v_j,t_j-1))$ if $t_i > 0$ and $t_j > 0$, and $C_{tr}((v_i,t_i),(v_j,t_j)) = C((v_i,t_i),(v_j,t_j))$ else.*

The translation of the motif depicted in Figure 3 is represented in Figure 4. We extend the notion of PC-closure and translation to UPSTPs in the following way:

Definition 10 *Let $\mathcal{U} = (V, C, t_{min}, t_{max})$ a* UPSTP. *The PC-closure of $\mathcal{U}$ (resp. the translation of $\mathcal{U}$), denoted by* $\mathsf{PC}(\mathcal{U})$ *(resp.* $\mathsf{translation}(\mathcal{U})$*), is the* UPSTP *$(V, C', t_{min}, t_{max})$ where C' in the map from $V \times \{0,\dots,t_{max}\} \times V \times \{0,\dots,t_{max}\}$ to $\mathsf{INT}_{\mathbb{Q}}$ defined by:*

- *$C'(v_i,t_i,v_j,t_j) = C(v_i,t_i,v_j,t_j)$ for all $v_i, v_j \in V$ and $t_i, t_j \in \{0,\dots,t_{max}\}$ such that $t_i < t_{min}$ or $t_j < t_{min}$,*
- *$C'(v_i,t_i,v_j,t_j) = C^*_m((v_i,t_i -$*

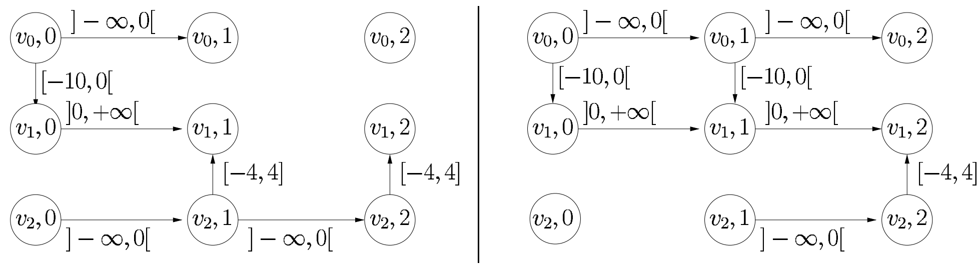

Figure 4. The end of the 4-strengthening of the UPSTP $\mathcal{U}$ **and the translation of** $\mathsf{motif}(\mathcal{U})$**.**

$t_{min}), (v_j, t_j - t_{min}))$ *for all* $v_i, v_j \in V$ *and* $t_i, t_j \in \{t_{min}, \ldots, t_{max}\}$*, where* C^*_m *denotes the constraints of the PC-closure (resp. the translation) of the motif of* $\mathcal{U}$*.*

Using these operations we can now define the closure property:

Definition 11 *Let* $\mathcal{U} = (V, C, t_{min}, t_{max})$ *be a* UPSTP. *The* UPSTP $\mathcal{U}$ *(resp. the motif* $\mathsf{motif}(\mathcal{U})$*) is closed iff* $\mathcal{U} = \mathsf{PC}(\mathsf{translation}(\mathcal{U}))$ *(resp. iff* $\mathsf{motif}(\mathcal{U}) = \mathsf{PC}(\mathsf{translation}(\mathsf{motif}(\mathcal{U})))$.

Concerning the closure property we have the following properties:

Proposition 4 *Let* $\mathcal{U}$ *be a* UPSTP. *We have :* $\mathcal{U}$ *is closed iff* $\mathsf{motif}(\mathcal{U})$ *is closed ;* $\mathcal{U}$ *is closed iff* $\mathcal{U} = \mathsf{translation}(\mathcal{U})$ *and* $\mathcal{U} = \mathsf{PC}(\mathcal{U})$ *;* $\mathsf{motif}(\mathcal{U})$ *is closed iff* $\mathsf{motif}(\mathcal{U}) = \mathsf{translation}(\mathsf{motif}(\mathcal{U}))$ *and* $\mathsf{motif}(\mathcal{U}) = \mathsf{PC}(\mathsf{motif}(\mathcal{U}))$.

We also have the following result:

Proposition 5 *Let* $\mathcal{U} = (V, C, t_{min}, t_{max})$ *be a closed* UPSTP. *Let* $\mathcal{S}^k = (V \times \{0, \ldots, k\}, C_k)$ *and* $\mathcal{N}^{k+1} = (V \times \{0, \ldots, k+1\}, C_{k+1})$ *be the* k*-strengthening and the* $(k+1)$*-strengthening of* $\mathcal{U}$*, respectively, with* $k \geq t_{max}$*. The restriction of the map* C_{k+1} *to* $V \times \{0, \ldots, k\} \times V \times \{0, \ldots, k\}$ *is the map* C_k*.*

Now, we give a fundamental result about the consistency problem for closed UPSTPs.

Theorem 1 *Let* $\mathcal{U} = (V, C, t_{min}, t_{max})$ *be a closed* UPSTP. *Each solution of* $\mathsf{k}-\mathsf{strenthening}(\mathcal{U})$*, with* $k \geq t_{max}$*, can be extended to a solution of* $(\mathsf{k}+1)-\mathsf{strenthening}(\mathcal{U})$.

Proof(sketch) Starting from a solution σ of $\mathsf{k}-\mathsf{strenthening}(\mathcal{U})$ we can extract a partial solution of $\mathsf{end}((\mathsf{k}+1)-\mathsf{strenthening}(\mathcal{U}))$. $\mathsf{end}((\mathsf{k}+1)-\mathsf{strenthening}(\mathcal{U}))$ is a subnetwork of the STP $\mathsf{motif}(\mathcal{U})$ which is closed and hence PC-closed. $\mathsf{motif}(\mathcal{U})$ is hence also globally consistent. Hence we can extend the partial solution to a solution of $\mathsf{motif}(\mathcal{U})$. Since the constraints concerning the instant lg are the same constraints for $\mathsf{motif}(\mathcal{U})$ and for $\mathsf{end}((\mathsf{k}+1)-\mathsf{strenthening}(\mathcal{R}))$, this solution is also a solution of $\mathsf{end}((\mathsf{k}+1)-\mathsf{strenthening}(\mathcal{U}))$. This solution can be used to complete the solution σ to obtain a solution of $(\mathsf{k}+1)-\mathsf{strenthening}(\mathcal{U})$. $\dashv$

A corollary of this theorem is the following result:

Corollary 1 *The consistency problem for closed* UPSTP*s can be solved in polynomial time; more precisely, it can be solved by applying the path-consistency method on the* t_{max}*-strengthening of the* UPSTP*, which can be achieved in* $O((t_{max} * |V|)^3)$.

6 The closure method for the UPSTPs

In the previous section we have shown that the consistency problem of closed UPSTPs is a polynomial problem. We will make use of this result, and introduce a constraint propagation algorithm which tries to transform an arbitrary UPSTP into an equivalent UPSTP which is closed. This algorithm corresponds to the algorithm Closure (see Algorithm 1). We will see that this algorithm is sound, in the sense that if the algorithm Closure terminates, then the resulting UPSTP is closed and equivalent to the initial UPSTP. However, this algorithm is not complete; indeed, we will see that there are cases where the algorithm Closure cannot terminate. In spite of this, we will characterize two particular interesting cases for which the algorithm

Algorithm 1 Closure

Compute the closure of UPSTP $\mathcal{U} = (V, C, t_{min}, t_{max})$

Do
  $\mathcal{U}' := \mathcal{U}$
  $\mathcal{U} := \mathsf{translation}(\mathcal{U})$
  $\mathcal{U} := \mathsf{PC}(\mathcal{U})$
While $(\mathcal{U} \neq \mathcal{U}')$
return $\mathcal{U}$

Closure is complete. This algorithm uses the translation operation and the path-consistency method to compute the closure of a UPSTP. As a first step, we will show that this algorithm is sound. The following proposition asserts the equivalence between a UPSTP and its transform under PC-closure and translation and hence allows us to assert that the algorithm Closure computes a UPSTP which is equivalent to the initial UPSTP:

Proposition 6 *The PC-closure of a* UPSTP $\mathcal{U}$ *and its translation are equivalent to* $\mathcal{U}$*.*

The algorithm Closure does not always terminate. Indeed, consider the UPSTP represented in Figure 2 and apply the algorithm Closure to it. We notice that this algorithm does not terminate and loops indefinitely. The resulting UPSTP after the i^{th} loop of the algorithm Closure is shown in Figure 5. Actually, the non termination of the algorithm Closure allows us to decide the consistency problem of the UPSTP, indeed we have the following property:

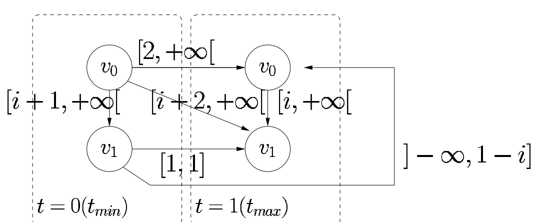

Figure 5. The resulting UPSTP **after the** i^{th} **loop of the algorithm** Closure**.**

Proposition 7 *The non termination of the algorithm* Closure *on a* UPSTP $\mathcal{U}$ *implies the non consistency of the* UPSTP $\mathcal{U}$*.*

Proof In the case where the algorithm Closure loops indefinitely, we can assert that there exists one of the intervals defining a constraint $C(v_i, t_i, v_j, t_j)$ of $\mathcal{U}$ which decreases indefinitely by intersection. This interval decreases at least by one (the finite bounds of the intervals are integer number) for each intersection. This interval has necessarily a finite bound and an infinite bound. Indeed, in the contrary case $C(v_i, t_i, v_j, t_j)$ will become the empty interval. Using these observations and from the fact that the algorithm Closure is sound, we can assert that it is not possible to define a map σ solution of $\mathcal{U}$. Indeed, for any distance $\sigma(v_j, t_j) - \sigma(v_i, t_i)$ there exists a loop of the algorithm from which $\sigma(v_j, t_j) - \sigma(v_i, t_i) \notin C(v_i, t_i, v_j, t_j)$. $\dashv$

An open question is: in the general case, does exist a way to detect that the algorithm Closure will indefinitely loop after a particular number of loops? For particular kinds of constraints we are sure that this algorithm will terminate after a finite number of iterations. For example, we can cite the two following particular cases:

1. the intervals used as constraints have uniquely finite bounds ;
2. the intervals used as constraints have infinite bounds or finite bounds (open or closed) associated with the value 0.

For the first kind of constraints, the number of iterations done by the algorithm Closure is bounded by $m(|V| * (\lg + 1))^2$ where m is the size of the largest interval. This is a consequence of the fact that at each loop, at least one constraint decreases at least one unity. Concerning the second type of constraints we can notice that the number of iterations is bounded by $3(|V| * (\lg + 1))^2$ as each constraint can decrease at most three times. The second kind of constraints is used to represent qualitative constraints stemming from formalisms such that the Allen's Calculus [1] or the point calculus [2]. Hence, the consistency problem for the UPSTPs with these two kinds of constraints is polynomial in time.

7 Conclusion

In this paper, we have introduced the notion of ultimately periodic simple temporal problems (UPSTPs). This notion allows to express (spatial or temporal) quantitative constraints which, after an initial period, evolve in a periodic way by repeating the same pattern that we called motif. For this kind of constraint network, we propose a constraint propagation algorithm for deciding the problem of consistency. In the general case, this algorithm is not complete. Despite it, we showed that for particular interesting cases. We are currently developing an implementation of this algorithm[3]. This work also opens new perspectives for future work. One of them consists in the characterization of new cases where the consistency problem of the UPSTP is polynomial in time. Another one consists in determining the class of complexity to which the consistency problem for UPSTPs belongs, and to define algorithms allowing to solve it in the general case.

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[3]The current implementation can be found at http://www.cril.univ-artois.fr/∼condotta/upstp/.