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Consistency of qualitative constraint networks from tree decompositions

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***Abstract*—A common way to decide the consistency problem of a qualitative constraint network (QCN) is to encode it as a boolean formula in order to benefit from the efficiency of SAT solvers. In recent works, a decomposition method of QCNs have been proposed to reduce the amount of boolean formulae. In this paper, we first show that the decompositions used can be expressed by particular tree decompositions. Furthermore, for some classes of relations, we prove that the consistency problem of a QCN can be decided by applying the method of the closure by weak composition on the clusters of a tree decomposition. This result allows us to extend the approach recently proposed to tree decompositions of QCNs.**

***Keywords*-Temporal qualitative constraints ; Consistency problem ; Tree decomposition**

I. Introduction

Reasoning about temporal or spatial knowledge is a major task in many domains of Artificial Intelligence, such as Geographic Information System (GIS), natural language processing, temporal/spatial scheduling, and so on. Qualitative reasoning is a way to express and process the qualitative aspect of knowledge about temporal or spatial entities. A qualitative calculus considers a domain from temporal or spatial entities and a finite set of base relations over these entities. Each base relation symbolizes a relative position between the entities, and is a factoring for configurations given as numeric information. These previous decades, many qualitative calculi have been studied. The Interval Algebra [1] represents temporal entities by intervals and considers thirteen base relations describing each possible relative position between two temporal entities (see Figure 1). Many qualitative calculi for temporal knowledge are derived from the Interval Algebra [2], [3], [4]. In spatial reasoning, the well-known qualitative calculus RCC [5], [6] is quite possibly the most studied. RCC is based on eight base relations between entities defined over all regions of a topological space. In the qualitative calculi frameworks, the set of temporal or spatial information may be represented by some specific constraint networks called qualitative constraint networks (QCNs). In a QCN, each variable stands for a temporal or spatial entity and each constraint restricts the possible configurations between entities by using a set of base relations. Given a QCN, the main decision problem is the consistency problem. In the general case, this problem is NP-complete. To solve it, we can cite two kinds of approaches can be used.

The first, introduced by Nebel [7], consists in a synchronous backtracking algorithm by maintaining a local consistency. This algorithm exploits certain tractable classes of relations with multi-valued assignments and the weak composition closure allowing to obtain a local consistency close to path-consistency. At each step during search, a constraint relation is split into sub-relations belonging to the tractable class. The constraint is iteratively replaced by each sub-relation. This cutting allows us to decrease the branching factor during the search. Most of the QCN solvers exploit this solving method, in particular GQR* [8] which is currently the most efficient solver.

The second approach [9], [10] consists in encoding the consistency problem of QCNs as boolean formulae in order to benefit from the efficiency of SAT solvers. However, the counter-part of this approach is the large amount of the boolean formulae obtained. Recently, in order to overcome this drawback for QCNs of the Interval Algebra, Li *et al.* [11] proposed a method of decomposition allowing to discard some constraints of the QCN. This method splits recursively the QCN considered in two equivalent sub-QCNs. Each constraint of the initial QCN which is absent from one of the two QCNs is considered as useless to decide the consistency, and is not considered in the encoding. The amount of the boolean formulae is greatly reduced compared to the full encoding, and the experimental results show an improvement in term of solving time.

In this paper, we define particular decompositions called RecPart decompositions. These specific decompositions formalize the decompositions of QCNs used by Li *et al.* in their process of SAT encoding. Then, we show that the RecPart decompositions can be equivalently defined as tree decompositions [12], widely studied in the framework of finite CSPs. Moreover, we study the consistency problem for the QCN through the tree decompositions. In particular, we show that, given a tree decomposition for a QCN defined over a tractable class of relations, the weak composition closure applied on each cluster is sufficient to decide the consistency of the QCN. Finally, by exploiting this result we propose to decide the consistency of QCNs by using a tractable class of relations and a tree decomposition in the framework of boolean formula encodings.

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II. PRELIMINARY NOTIONS ON QUALITATIVE CALCULI

A. Qualitative calculi

A qualitative calculus considers a finite set of relations B, called *base relations*, over an infinite domain D representing the temporal or spatial entities. In our study, we focus on binary relations (a large part of qualitative calculi considers this kind of base relations). Each base relation of B represents a certain relative position between spatial or temporal entities. They are *Jointly Exhaustive and Pairwise Disjoint* (JEPD), *i.e.* each element $(x, y) \in \mathsf{D} \times \mathsf{D}$ belongs to one and only one $a \in \mathsf{B}$. The set B has some properties [13] : (1) B is a partition of $\mathsf{D} \times \mathsf{D}$, (2) B contains the identity relation id and, (3) B is closed by converse, *i.e.* the converse of an base relation in B is also in B. As an illustration, the *Interval Algebra* (IA), also known as *Allen's calculus* [1], considers intervals of the line to represent the temporal entities. The domain D is defined by $\mathsf{D} = \{(x^-, x^+) \in \mathbb{Q} \times \mathbb{Q} \mid x^- < x^+\}$. The base relations correspond to the set $\mathsf{B} = \{eq, p, pi, m, mi, o, oi, s, si, d, di, f, fi\}$. Each of these base relations symbolizes a relative position between two temporal intervals, which is illustrated in Figure 1.

A *complex relation*, also called *relation*, for a qualitative calculus is an union of base relations. It is customary to represent a relation by the set of the base relations which compose it. Hence, in the sequel we make no distinction between the set of relations and the set 2^{B} which will represent the set of relations of a qualitative calculus based on the set of base relations B. In the Interval Algebra, the relation $r = p \cup mi \cup eq$ will be represented by the set $\{p, mi, eq\}$. The usual set-theoretic operations *union* ($\cup$), *intersection* ($\cap$) and *converse* ($\cdot^{-1}$) are defined over 2^{B}. For a relation $r \in 2^{\mathsf{B}}$, the converse is defined as $r^{-1} = \bigcup\{a^{-1} | a \in r\}$. Among the relations of 2^{B}, Ψ denotes the relation that contains all the base relations of B. The set 2^{B} is also equipped with the *weak composition* operation, denoted by $\diamond$, defined as: $a \diamond b = \{c \in \mathsf{B} : \exists x, y, z \in \mathsf{D}$ with $x\ a\ z, z\ b\ y$ and $x\ c\ y\}$, with $a, b \in \mathsf{B}$, $r \diamond s = \bigcup_{a \in r, b \in s}\{a \diamond b\}$, with $r, s \in 2^{\mathsf{B}}$. The relation $r \diamond s$ is also defined as the strongest relation in 2^{B} which contains the usual composition $r \circ s = \{(x, y) \in \mathsf{D} \times \mathsf{D} : \exists x, y, z \in \mathsf{D}$ with $x\ a\ z, z\ b\ y$ and $x\ c\ y\}$. For some qualitative calculi, $r \circ s$ and $r \diamond s$ are equivalent.

A *class* of relations $\mathcal{C}$ is a subset of 2^{B} which contains the relation Ψ, all of the singleton relations of 2^{B}, and which is closed under converse, intersection and weak composition. Given $r \in 2^{\mathsf{B}}$ and a class $\mathcal{C}$, the smallest relation of $\mathcal{C}$ which contains r is denoted by $r^{\mathcal{C}}$ and is called the closure of r in $\mathcal{C}$. In IA, let us consider the relation $\{p, m\} \in 2^{\mathsf{B}}$ between two entities X and Y. $X\ \{p, m\}\ Y$ means that X *precedes* or *meets* Y. The converse relation $\{p, m\}^{-1} = \{p^{-1}, m^{-1}\} = \{pi, mi\}$ expresses the relation between Y and X, so Y *is preceded* or *is met by* X. Finally, let us introduce $Y\ \{m, s, eq\}\ Z$. We have some information about the relations between X, Y and Y, Z, so we can deduce information about the relation between X, Z by using the weak composition. Since $\{p, m\} \diamond \{m, s, eq\} = \{p, m\}$, we have $X\ \{p, m\}\ Z$.

Relation	Symbol	Converse	Illustration
precedes	p	pi	
meets	m	mi	
overlaps	o	oi	
starts	s	si	
during	d	di	
finishes	f	fi	
equals	eq	eq	

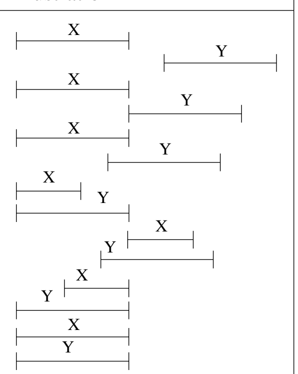

Figure 1. The base relations of the Interval Algebra.

B. Qualitative Constraint Networks

A Qualitative Constraint Network (QCN) consists of a finite set of m variables $V = \{v_1, \ldots, v_m\}$ which represent the spatial or temporal entities, and a map C from $V \times V$ to 2^{B} such that $C(v_i, v_i) \subseteq \{\mathsf{Id}\}$ for each $v_i \in V$, with Id the base relation corresponding to the identity relation over D, and $C(v_i, v_j) = C(v_j, v_i)^{-1}$ for all $v_i, v_j \in V$. In the rest of this paper, we will also denote $C(v_i, v_j)$ by $\mathcal{N}[v_i, v_j]$. The figure 2 illustrates a QCN $\mathcal{N}$ of the Interval Algebra. In this figure, a variable is represented by a node, and a constraint by an arc labelled with the associated relation. Note that, for simplicity, there is no arc going from v_i to v_j when either there is already an arc going from v_j to v_i or $i = j$.

Given a QCN $\mathcal{N} = (V, C)$, a *partial instantiation* of $\mathcal{N}$ on $V' \subseteq V$ is a map s from V' to D. A *partial solution* on $\mathcal{N}$ on $V' \subseteq V$ is a partial instantiation on V' such that $(s(v_i), s(v_j))$ satisfies $C(v_i, v_j)$ for all $v_i, v_j \in V'$, *i.e.* there exists a base relation $b \in C(v_i, v_j)$ such that $(s(v_i), s(v_j)) \in b$ for all $v_i, v_j \in V'$. A *solution* of $\mathcal{N}$ is a partial solution of $\mathcal{N}$ on V. $\mathcal{N}$ is *consistent* if, and only if,

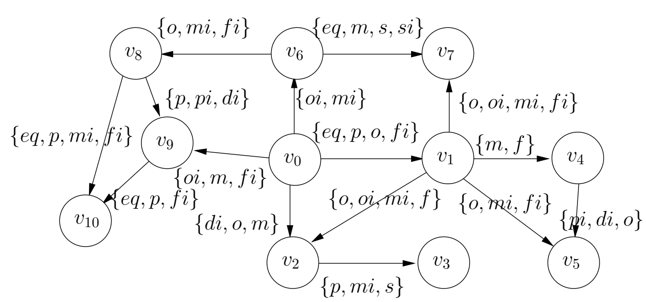

Figure 2. A QCN $\mathcal{N} = (V, C)$ of the Interval Algebra.

there exists a solution of $\mathcal{N}$. $\mathcal{N}$ is *trivially inconsistent* when there exist two variables $v, v' \in V$ such that $\mathcal{N}[v, v'] = \emptyset$. $\mathcal{N}$ is *globally consistent* if, and only if, each partial solution of $\mathcal{N}$ can be extended to a solution of $\mathcal{N}$. The *projection* of the QCN $\mathcal{N}$ to V' with $V' \subseteq V$, denoted by $\mathcal{N}_{V'}$, is the QCN (V', C_{proj}) with C_{proj} the restriction of C to the set V'. A sub-QCN $\mathcal{N}'$ of $\mathcal{N}$ is a QCN (V, C') such that $C'(v_i, v_j) \subseteq C(v_i, v_j)$, for all $v_i, v_j \in V$. Let $\mathcal{N}^1$ and $\mathcal{N}^2$ be two QCNs defined respectively on the sets of variables V^1 and V^2, with for each pair of variables $v, v' \in V^1 \cap V^2$, $\mathcal{N}^1[v, v'] = \mathcal{N}^2[v, v']$. We denote by $\mathcal{N}^1 \cup \mathcal{N}^2$ the unique QCN $\mathcal{N}$ defined on $V^1 \cup V^2$ such that $\mathcal{N}[v, v'] = \mathcal{N}^1[v, v']$ for all $v, v' \in V^1$, $\mathcal{N}[v, v'] = \mathcal{N}^2[v, v']$ for all $v, v' \in V^2$, $\mathcal{N}[v, v'] = \Psi$ for all $v \in V^2 \setminus V^1$ and $v' \in V^1 \setminus V^2$.

A QCN $\mathcal{N} = (V, C)$ is $\diamond$-consistent or closed by weak composition if, and only if, $C(v_i, v_j) \subseteq C(v_i, v_k) \diamond C(v_k, v_j)$ for all $v_i, v_j, v_k \in V$. The weak composition closure of the QCN $\mathcal{N}$, denoted by $\diamond(\mathcal{N})$ is the largest (for $\subseteq$) $\diamond$-consistent sub-QCN of $\mathcal{N}$. This closure may be obtained by iterating the operation $C(v_i, v_j) \leftarrow C(v_i, v_j) \cap (C(v_i, v_k) \diamond C(v_k, v_j))$ for all $v_i, v_j, v_k \in V$ until a fixpoint is reached. The worst-case time-complexity of this method is $O(m^3)$, with m the number of variables. For some classes of relations, such as the set of the ORD-Horn relations or the set of the convex relations [14], [15] of IA, the consistency problem of a QCN can be decided by enforcing $\diamond$-consistency. Hence, a $\diamond$-consistent ORD-Horn QCN with no empty constraint is a consistent QCN. The class of convex relations admits a stronger property : each $\diamond$-consistent convex QCN non trivially inconsistent is globally consistent [16]. We conclude this section with some definitions about trees. Given a rooted tree (a connected acyclic graph with a root) $T = (X, F)$ and a node $X_i \in X$, we denote by $\mathsf{desc}(X_i)$ (resp. $\mathsf{asc}(X_i)$) the set of the descendant nodes (resp. the ancestor nodes) of X_i (note that X_i belongs to $\mathsf{desc}(X_i)$ and $\mathsf{asc}(X_i)$). Given $X' \subseteq X$ a non-empty subset of nodes, $\mathsf{lca}(X')$ denote the node of T which is the lowest common ancestor of the nodes belonging to X'. The set $\mathsf{leaves}(T)$ corresponds the set of the leaf nodes of T. Finally, given a $X_i \in X$, T_{X_i} denotes the sub-tree of T rooted in X_i.

III. Decompositions of QCNs

A. Tree decompositions

The relation Ψ consists of all the possible base relations of B and is satisfied by any pair of elements of the domain D. A constraint between two variables of a QCN defined by the relation Ψ specifies that locally there is no constraint concerning the relative position of both entities represented. So, in a natural way, we define the graph of constraints of a QCN $\mathcal{N} = (V, C)$, by the undirected graph $\mathsf{G}(\mathcal{N}) = (V, E)$ with $(v, v') \in E$ if, and only if, $\mathcal{N}[v, v'] \neq \Psi$ and $v \neq v'$. In the sequel we suppose that given a QCN $\mathcal{N}$, $\mathsf{G}(\mathcal{N})$ is connected. In the contrary case, $\mathcal{N}$ may be trivially split in two independent QCNs without common variables. As

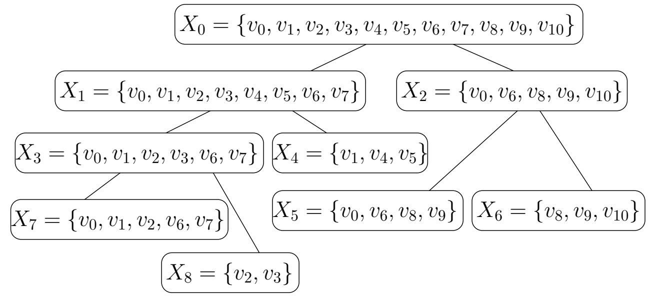

Figure 3(a). A RecPart $T = (X, F)$ decomposition of $\mathcal{N}$.

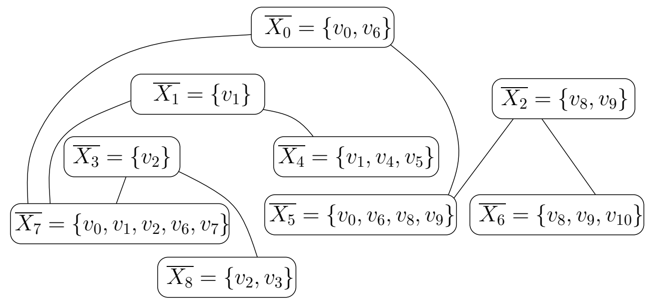

Figure 3(b). A tree decomposition of $\mathcal{N}$ corresponding to $\overline{T}$.

within the framework of the discrete CSPs we define a tree decomposition of a QCN as a decomposition of its constraint graph:

Definition 1: Let $\mathcal{N} = (V, C)$ be a QCN and $\mathsf{G}(\mathcal{N}) = (V, E)$ be its constraint graph. A tree decomposition of $\mathcal{N}$ is a tree $T = (X = \{X_0, \ldots, X_n\}, F)$ with n a positive integer, where X is a family of subsets of variables of V ($X_i \subseteq V$), such that :

(1) $\bigcup\{X_i \in X\} = V$;
(2) $\forall (v, v') \in E$, there exists $X_i \in X$ with $v, v' \in X_i$;
(3) for all $X_i, X_j, X_k \in X$, if X_j is on the unique path between X_i and X_k then $X_i \cap X_k \subseteq X_j$.

Given a tree decomposition $T = (X = \{X_0, \ldots, X_n\}, F)$ of a QCN, the treewidth of T is equal to $max\{|X_i| - 1 : X_i \in X\}$. Furthermore, every set of variables X_i is called a cluster. In Figure 3(b), a tree decomposition of the QCN $\mathcal{N}$ of Figure 2 is represented.

B. The RecPart *decompositions*

Recently, Li *et al.* [11] proposed a method allowing to translate QCNs of the Interval Algebra into boolean satisfiability problem (SAT instances). This method recursively decomposes at each step a QCN $\mathcal{N} = (V, C)$ into two QCNs $\mathcal{N}^1$ and $\mathcal{N}^2$ such that $\mathcal{N} = \mathcal{N}^1 \cup \mathcal{N}^2$. The constraints defined by the relation Ψ in the QCN $\mathcal{N}$ not belonging to $\mathcal{N}^1$ and $\mathcal{N}^2$ are characterized as not necessary in the search of a solution of the initial QCN and thus not translated. The advantage of such a translation is that the obtained SAT instance is of smaller size than an instance stemming from a complete translation. Taking inspiration from this method, we define particular decompositions of QCNs, called RecPart decompositions (for recursive partitioning), in the following way:

Definition 2: Let $\mathcal{N}$ be a QCN $= (V, C)$. A RecPart decomposition of $\mathcal{N}$ is a rooted tree $T = (X = \{X_0, \ldots, X_n\}, F)$, with n a positive integer, where X is a family of subsets of V ($X_i \subseteq V$) and $X_r = V$ with X_r the root of T. Moreover, each node of T has two or no child nodes and, given $X_i, X_j, X_k \in X$ such that X_j and X_k are child nodes of X_i, T must satisfy the following properties :

(1) $\mathcal{N}_{X_i} = \mathcal{N}_{X_j} \cup \mathcal{N}_{X_k}$, $X_j \setminus X_k \neq \emptyset$ and $X_k \setminus X_j \neq \emptyset$.
(2) For every $X_m \in \mathsf{desc}(X_i)$, if $X_m \cap (X_j \cap X_k) \neq \emptyset$ then $(X_j \cap X_k) \subseteq X_m$.

In Figure 3(a) is represented a RecPart decomposition of the QCN $\mathcal{N}$ illustrated in Figure 2. We define the treewidth of a RecPart decomposition $T = (X, F)$ by $max\{|X_i| - 1 : X_i \in \mathsf{leaves}(T)\}$. Note that from the property (1) of the previous definition, $X_i \neq \emptyset$ for every $X_i \in X$. A RecPart decomposition satisfies the following properties :

Proposition 1: Let $T = (X = \{X_0, \ldots, X_n\}, F)$ be a RecPart decomposition of a QCN $\mathcal{N} = (V, C)$. For each $X_i \in X$, we have :

(1) For each $X_j \in X$ such that $i \neq j$, $X_i \neq X_j$ and moreover, if $X_j \in \mathsf{desc}(X_i)$ then $X_j \subset X_i$;
(2) T_{X_i} is a RecPart decomposition of $\mathcal{N}_{X_i}$;
(3) $\mathcal{N}_{X_i} = \bigcup\{\mathcal{N}_{X_j} : X_j \in \mathsf{leaves}(T_{X_i})\}$;
(4) $X_i = \bigcup\{X_j \in \mathsf{leaves}(T_{X_i})\}$.

The proofs of these properties are omitted, they can be directly established from Definition 2. Note that from the previous property (1), we have for all $X_i, X_j \in X$, the sets X_i and X_j which are distinct sets if, and only if, $i \neq j$. Hence, for each $X_i \in X$, the set of variables belonging to X_i characterises one and only one node of T. Let $T = (X, F)$ be a RecPart decomposition of a QCN $\mathcal{N} = (V, C)$. Given $X_i \in X$, $\overline{X_i}$ will denote the set X_i in the case where $X_i \in \mathsf{leaves}(T)$. In the case where $X_i \notin \mathsf{leaves}(T)$, $\overline{X_i}$ is defined by the set $X_j \cap X_k$ with X_j and X_k the two child nodes of X_i. Note that $\overline{X_i} \subseteq X_i$ since $X_j \subseteq X_i$ and $X_k \subseteq X_i$. Moreover, we have the following properties :

Proposition 2: Let $T = (X = \{X_0, \ldots, X_n\}, F)$ be a RecPart decomposition of a QCN $\mathcal{N} = (V, C)$.

(1) For each $X_i \in X$, $\overline{X_i} \neq \emptyset$.
(2) Given $X_i, X_j, X_k \in X$ with $X_k = \mathsf{lca}(\{X_i, X_j\})$, if $X_k \neq X_i$ and $X_k \neq X_j$ then $X_i \cap X_j \subseteq \overline{X_k}$.
(3) For each $X_i \in X$, there exists $X_j \in \mathsf{leaves}(X)$ such that $\overline{X_i} \subseteq X_j$.

Proof.

(1) Consider the case where $X_i \notin \mathsf{leaves}(T)$ (for the case where $X_i \in \mathsf{leaves}(T)$ the property is trivially satisfied since $\overline{X_i} = X_i$ and $X_i \neq \emptyset$). Let $X_i, X_j, X_k \in X$ such that X_i is the parent node of X_j and X_k. Suppose by contradiction that $X_j \cap X_k = \emptyset$. From Definition 2, we have $\mathcal{N}_{X_i} = \mathcal{N}_{X_j} \cup \mathcal{N}_{X_k}$. Hence, for all $v \in X_j$ and $v' \in X_k$, we can assert that $\mathcal{N}[v, v'] = \Psi$ and that (v, v') is not an edge of $\mathsf{G}(\mathcal{N})$. Moreover, we have previously assumed that for every QCN $\mathcal{N}$ considered, $\mathsf{G}(\mathcal{N})$ is connected. Consequently, for all $v \in X_j$ and $v' \in X_k$, there exists a path of $\mathsf{G}(\mathcal{N})$ between v and v'. Moreover this path passes necessarily through two edges (v, v'') and (v, v''') with $v'' \in V \setminus \{X_j \cup X_k\}$ and $v''' \in V \setminus \{X_j \cup X_k\}$. As $X_j \cup X_k = X_i$, we have $v'' \notin X_i$. Let X_l be the nearest common ancestor of X_i such that $v'' \in X_l$. Let X_m and X_o be the two child nodes of X_l with $X_m \in \mathsf{asc}(X_i)$. We have $v'' \notin X_m$ and $v \in X_m$. Moreover, $v'' \in X_o$ since $X_o \cup X_m = X_l$. We also know $\mathcal{N}_{X_l} = \mathcal{N}_{X_m} \cup \mathcal{N}_{X_o}$. Consequently, we have $\mathcal{N}[v, v''] = \Psi$. We know that (v, v'') is an edge of $\mathsf{G}(\mathcal{N})$. There is a contradiction. We conclude that $X_j \cap X_k \neq \emptyset$.

(2) Let $X_i, X_j, X_k \in X$ with $X_k = \mathsf{lca}(\{X_i, X_j\})$. Assume that $X_k \neq X_i$ and $X_k \neq X_j$. We have necessarily $X_k \notin \mathsf{leaves}(T)$. Let us denote by X_l and X_m the two child nodes of X_k. We have $X_i \in \mathsf{desc}(X_l)$ and $X_j \in \mathsf{desc}(X_m)$ or, $X_i \in \mathsf{desc}(X_m)$ and $X_j \in \mathsf{desc}(X_l)$. Hence, from Proposition 1 (1), we have $X_i \subseteq X_l$ and $X_j \subseteq X_m$ or, $X_i \subseteq X_m$ and $X_j \subseteq X_l$. Consequently, we have $X_i \cap X_j \subseteq X_m \cap X_l$. We conclude that $X_i \cap X_j \subseteq \overline{X_k}$.

(3) Let $X_i \in X$. First, consider the case where $X_i \in \mathsf{leaves}(X)$. By definition, $\overline{X_i} = X_i$, hence the property is satisfied. Now, assume that $X_i \notin \mathsf{leaves}(X)$ and let X_k and X_l be the two child nodes of X_i. We have $X_k \cap X_l = \overline{X_i} \neq \emptyset$. Let $v \in \overline{X_i}$, we have $v \in X_i$. From Proposition 1 (4), there exists $X_j \in \mathsf{leaves}(T) \cap \mathsf{desc}(X_i)$ such that $v \in X_j$. From the property (2) of Definition 2, we have $X_k \cap X_l \subseteq X_j$ since $(X_k \cap X_l) \cap X_j \neq \emptyset$. $\dashv$

IV. From RecPart decompositions to tree decompositions

In this section, we are going to show that from a RecPart decomposition T of a QCN we can define a tree decomposition T' with same treewidth and considering the same clusters of variables (more exactly the same binary constraints). Before this, we introduce a new notation : given

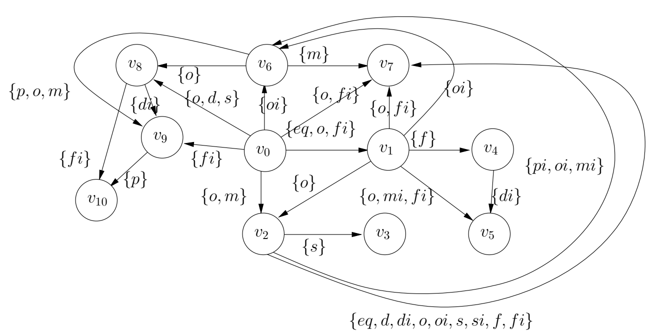

Figure 4. $\mathcal{N}'$ a ORD-Horn $\overset{\diamond}{X}$–consistent sub-QCN of $\mathcal{N}$.

$X_i, X_j \in X$ with X_j child node of X_i, $\widetilde{X_iX_j}$ will denote the set $\{X_k : X_k \in \mathsf{desc}(X_j)$ and $\overline{X_i} \subseteq \overline{X_k}\}$. Concerning $\widetilde{X_iX_j}$, we have the following properties :

Proposition 3: Let $T = (X, F)$ be a RecPart decomposition of a QCN $\mathcal{N} = (V, C)$ and let $X_i, X_j \in X$ with X_j a child node of X_i. We have : (1) $\widetilde{X_iX_j} \neq \emptyset$, (2) $\mathsf{lca}(\widetilde{X_iX_j}) \in \widetilde{X_iX_j}$.

Proof.
(1) We know that $\overline{X_i} \neq \emptyset$ (Proposition 2 (1)), hence there exists $v \in \overline{X_i}$. By definition of $\overline{X_i}$, we know that $v \in X_j$. From Proposition 1 (4), there exists $X_k \in \mathsf{leaves}(T) \cap \mathsf{desc}(X_j)$ such that $v \in X_k$. From the property (3) of Definition 2, we can assert that $\overline{X_i} \subseteq X_k$. Hence, $X_k \in \widetilde{X_iX_j}$ and we can conclude that $\widetilde{X_iX_j} \neq \emptyset$.
(2) Let $X_l = \mathsf{lca}(\widetilde{X_iX_j})$. There exists $X_k, X_m \in \widetilde{X_iX_j}$ such that $X_l = \mathsf{lca}(\{X_k, X_m\})$. In the case where $X_l = X_k$ or $X_l = X_m$ the property is trivially satisfied. In the case where $X_l \neq X_k$ and $X_l \neq X_m$, from Proposition 2 (2) we have $X_k \cap X_m \subseteq \overline{X_l}$. As $\overline{X_i} \subseteq \overline{X_k}$ and $\overline{X_i} \subseteq \overline{X_m}$ we can assert that $\overline{X_i} \subseteq \overline{X_l}$. Moreover, since $X_l = \mathsf{lca}(\{X_k, X_m\})$, $X_k \in \mathsf{desc}(X_j)$ and $X_m \in \mathsf{desc}(X_j)$, we have $X_l \in \mathsf{desc}(X_j)$. We conclude that $X_l \in \widetilde{X_iX_j}$. $\dashv$

Now, we define from a RecPart decomposition T of a QCN $\mathcal{N}$ a tree denoted by $\overline{T}$. We will prove in the sequel that this tree is a tree decomposition of the QCN $\mathcal{N}$ satisfying some particular properties.

Definition 3: Let $\mathcal{N}$ be a QCN and $T = (X = \{X_0, \ldots, X_n\}, F)$ a tree decomposition of $\mathcal{N}$. From T and an element $X_i \in X$, we inductively define a rooted tree $\overline{T}_{X_i} = (X_{X_i}, F_{X_i})$ with root $\overline{X_i}$ in the following way :

- Base case : $X_i \in \mathsf{leaves}(T)$, $\overline{T}_{X_i} = (\{\overline{X_i}\}, \emptyset)$.
- Inductive case : $X_i \notin \mathsf{leaves}(T)$. By considering T, let X_j and X_k the child nodes of X_i, $X_l = \mathsf{lca}(\widetilde{X_iX_j})$ and $X_m = \mathsf{lca}(\widetilde{X_iX_k})$. X_{X_i} and F_{X_i} are defined by $X_{X_i} = X_{X_j} \cup X_{X_k} \cup \{\overline{X_i}\}$, $F_{X_i} = F_{X_j} \cup F_{X_k} \cup \{(\overline{X_i}, \overline{X_l}), (\overline{X_i}, \overline{X_m})\}$.

$\overline{T}$ is defined by the rooted tree $\overline{T}_{X_r}$ with X_r the root of T. Let us show that the tree $\overline{T}$ is a tree decomposition of the QCN $\mathcal{N}$ for which T is a RecPart decomposition.

Proposition 4: Let $\mathcal{N} = (V, C)$ be a QCN and a RecPart decomposition $T = (X = \{X_0, \ldots, X_n\}, F)$. We have $\overline{T} = (\overline{X} = \{\overline{X_0}, \ldots, \overline{X_n}\}, \overline{F})$ which is a tree decomposition of $\mathcal{N}$ such that for each $\overline{X_i} \in \overline{X}$ there exists $X_j \in \mathsf{leaves}(T)$ with $\overline{X_i} \subseteq X_j$.

Proof. Properties (1) and (2) of the definition 1 arise from the fact that for each $X_i \in \mathsf{leaves}(X)$, $\overline{X_i} = X_i$ and from Proposition 1. Now, let us prove that the property (3) of Definition 1 is satisfied by $\overline{T}$. Let $\overline{X_i}, \overline{X_j}, \overline{X_k} \in \overline{X}$ with $\overline{X_j}$ on the unique path between $\overline{X_i}$ and $\overline{X_k}$ *w.r.t.* $\overline{T}$. We are going to show that if $v \in \overline{X_i}$ and $v \in \overline{X_k}$ then $v \in \overline{X_j}$. If the length of the path between $\overline{X_i}$ and $\overline{X_k}$ is 0 the property is trivially satisfied. Now, assume that the property is satisfied for each path with a length $l \geq 0$ and let us show in an inductive way that the property holds for each path between $\overline{X_i}$ and $\overline{X_k}$ of length $l+1$. First, assume that $X_i \notin \mathsf{desc}(X_k)$ and $X_k \notin \mathsf{desc}(X_i)$ *w.r.t.* T. By examining the definition 3 we notice that for $\overline{T}$ a path between $\overline{X_i}$ and $\overline{X_k}$ necessarily passes through $\overline{X_o}$ with $X_o = \mathsf{lca}(\{X_i, X_k\})$. From Proposition 2 (2) we have $X_i \cap X_k \subseteq \overline{X_o}$. Consequently, $v \in \overline{X_o}$. By considering the two paths $\overline{X_i}, \ldots, \overline{X_o}$ and $\overline{X_o}, \ldots, \overline{X_k}$ which have a length lower or equal to l, and by using the induction hypothesis, we can assert that the nodes on the path $\overline{X_i}, \ldots, \overline{X_o}$ and the nodes on the path $\overline{X_o}, \ldots, \overline{X_k}$ contain v since $\overline{X_i}, \overline{X_o}$ and $\overline{X_k}$ contain v. Now, assume that $X_i \in \mathsf{desc}(X_k)$ in T (the case $X_k \in \mathsf{desc}(X_i)$ can be handled in a similar way). The case $X_i = X_k$ is a trivial case, assume that $X_i \neq X_k$. We have $X_i \in \mathsf{desc}(X_o)$ with X_o one of the child nodes of X_k in T. By examining Definition 3 we remark that for $\overline{T}$, a path between $\overline{X_k}$ and $\overline{X_i}$ passes necessarily through $\overline{X_m}$ with $X_m = \mathsf{lca}(\widetilde{X_kX_o})$. From Proposition 3 (2), we know that $\overline{X_k} \subseteq \overline{X_m}$. It results that $v \in \overline{X_m}$. By considering the paths $\overline{X_i}, \ldots, \overline{X_m}$ and $\overline{X_m}, \ldots, \overline{X_k}$ which have a length lower or equal to l, and by using the induction hypothesis, we can assert that the nodes of the path $\overline{X_i}, \ldots, \overline{X_m}$ and the nodes of the path $\overline{X_m}, \ldots, \overline{X_k}$ contain v since $\overline{X_i}, \overline{X_m}$ and $\overline{X_k}$ contain v.

The fact that for each $\overline{X_i} \in \overline{X}$, there exists $X_j \in \mathsf{leaves}(T)$ such that $\overline{X_i} \subseteq X_j$ results from Proposition 2 (3). $\dashv$

V. Tree decompositions and consistency of QCNs

In this section we are going to show that to decide the consistency of a QCN from one of its tree decomposition we can leave aside some of its constraints. In particular, we show that for some classes of relations, the closure by weak composition restricted to constraints of the clusters stemming from a tree decomposition is complete for the consistency problem. First of all, we introduce a new local consistency corresponding to the property of $\diamond$-consistency restricted to some subsets of variables of a QCN:

Definition 4: Let $\mathcal{N} = (V, C)$ be a QCN and $X = \{X_0, \ldots, X_n\}$ a family of subsets of V. $\mathcal{N}$ is $\overset{\diamond}{X}$-consistent if, and only if, for each $X_i \in X$, the QCN $\mathcal{N}_{X_i}$ is a $\diamond$-consistent QCN.

Given a QCN $\mathcal{N} = (V, C)$ and $X = \{X_0, \ldots, X_n\}$ a family of subsets of V, we will denote by $\overset{\diamond}{X}(\mathcal{N})$ the larger (for $\subseteq$) $\overset{\diamond}{X}$-consistent sub-QCN of $\mathcal{N}$.

The following result extends the one of Li *et al.* on atomic networks. It concerns QCNs whose constraints are defined by relations stemming from a class $\mathcal{C}$ for which any QCN closed by weak composition is globally consistent. As illustration, we can consider the QCNs defined by relations belonging to the class of the convex relations of the Interval Algebra which admits this property.

Theorem 1: Let $\mathcal{N} = (V, C)$ be a QCN defined on a class of relations $\mathcal{C}$ for which each QCN $\diamond$-consistent is globally consistent, and let $T = (X = \{X_0, \ldots, X_n\}, F)$ be a tree decomposition of $\mathcal{N}$. If $\mathcal{N}$ is a non trivially inconsistent and $\overset{\diamond}{_X}$-consistent QCN then $\mathcal{N}$ is a consistent QCN.

Proof. We suppose without loss of generality that T has a root. Let $X_i \in X$ and $T_{X_i} = (X_{X_i}, F_{X_i})$ the sub-tree of T. Given a partial instantiation s on V' with $V' \cap X'_i \subseteq X_i$ for each $X'_i \in X_{X_i}$ and such that for each $X_j \in X$ with $X_j \subseteq V'$, s_{X_j} is a solution of $\mathcal{N}_{X_j}$. We are going to prove the following property : s can be extended to a partial instantiation s' on $V'' = V' \cup \bigcup\{X'_i \in X_{X_i}\}$ such that for each $X_j \in X$ with $X_j \subseteq V''$, s'_{X_j} is a solution of $\mathcal{N}_{X_j}$. We are going to prove this property in an inductive way on the size of X_{X_i}.

• Base case: $|X_{X_i}| = 1$. We have $X_{X_i} = \{X_i\}$. Since $\mathcal{N}$ is a $\overset{\diamond}{_X}$-consistent QCN, we have the QCN $\mathcal{N}_{X_i}$ which is a QCN $\diamond$-consistent and hence globally consistent. $s_{V' \cap X_i}$ is a partial solution of $\mathcal{N}_{X_i}$ which can be extended to a solution s'' of $\mathcal{N}_{X_i}$. We define by s' the partial instantiation on $V' \cup X_i$ in the following way : if $v \in V'$ then $s'(v) = s(v)$ else $s'(v) = s''(v)$. We have s'_{X_i} which is a solution of $\mathcal{N}_{X_i}$ and more generally s'_{X_k} is a solution of $\mathcal{N}_{X_k}$ for each $X_k \in X$ and $X_k \subseteq V' \cup X_i$.

• Inductive step: $|X_{X_i}| > 1$. We assume that the property holds for each sub-tree $T_{X_j} = (X_{X_j}, F_{X_j})$ with $|X_{X_j}| < |X_{X_i}|$. As in the previous case, we extend s to a partial instantiation s' on $V' \cup X_i$ such that s'_{X_k} is a solution of $\mathcal{N}_{X_k}$ for each $X_k \in X$ and $X_k \subseteq V' \cup X_i$. By the induction hypothesis, this partial instantiation s' can be extended to the set of variables belonging to the descendant nodes of X_i. Indeed, consider X_l a child node of X_i. First, we remark that by denoting by $T_{X_l} = (X_{X_l}, F_{X_l})$ the sub-tree of T, we have $|X_{X_l}| < |X_{X_i}|$. Moreover, as T is a tree decomposition, we have for each $X_m \in X_{X_l}$, $X_m \cap (V' \cup X_i) \subseteq X_l$ (from the property (3) of the definition 1). Hence, the induction hypothesis can be applied on $T_{X_l} = (X_{X_l}, F_{X_l})$.

By applying the previous property on X_r the root of T, we know that there exists an instantiation s on the set of variables $\bigcup\{X_i : X_i \in X\}$ such that s_{X_i} is a solution of the QCN $\mathcal{N}_{X_i}$ for each $X_i \in X$. First, from the property (1) of Definition 1 we can assert that $V = \bigcup\{X_i : X_i \in X\}$. Hence, s is an instantiation on V. Moreover we can show that s is a solution of $\mathcal{N}$. Indeed, let $v, v' \in V$, if $\mathcal{N}[v, v'] = \Psi$ we have $s(v)$ and $s(v')$ which satisfy the constraint $\mathcal{N}[v, v']$. Now, assume that $\mathcal{N}[v, v'] \neq \Psi$. From the property (2) of Definition 1, there exists $X_i \in X$ such that $v \in X_i$ and $v' \in X_i$. We know that $s(v)$ and $s(v')$ satisfy $\mathcal{N}_{X_i}[v, v']$. As $\mathcal{N}_{X_i}[v, v'] = \mathcal{N}[v, v']$ we can assert that $s(v)$ and $s(v')$ satisfy $\mathcal{N}[v, v']$. We can conclude that s is a solution of $\mathcal{N}$ ⊣

We are going to characterize a similar result for the particular class of the ORD-Horn relations of the Interval Algebra. In [15], Ligozat attributes a dimension (an integer included between 0 and 2) to each base relations of the Interval Algebra : the dimension of the base relations p, pi, o, oi, d, di is 2, this one of the base relations m, mi, s, si, f, fi is 1, and the dimension of eq is 0. A partial solution of maximal dimension is a solution satisfying for every pair of variables a base relation of maximal dimension with regard to the dimensions of the base relations belonging to the constraint. For illustration, consider the QCN $\mathcal{N}'$ in Figure 4, a maximal instantiation s of $\mathcal{N}'_{\{v_0,v_1,v_2,v_6,v_7\}}$ is represented in Figure 5(b), the atomic QCN corresponding to this solution is given in Figure 5(a). For example, the base relation satisfied between $s(v_0)$ and $s(v_1)$ is the base relation o of dimension 2, it is a maximal dimension *w.r.t.* to the dimensions of the base relations of the relation $\mathcal{N}'[v_0, v_1] = \{eq, o, fi\}$. Given a QCN $\mathcal{N}$ closed by weak composition defined by ORD-Horn relations, in the general case $\mathcal{N}$ is not a globally consistent QCN. Nevertheless we have a nearest property which is satisfied : each partial solution of maximal dimension of $\mathcal{N}$ can be extended to a maximal solution of $\mathcal{N}$ (see Proposition 6 in [15]). From this property we can establish the following result :

Theorem 2: Let $\mathcal{N} = (V, C)$ be a QCN defined by relations of the ORD-Horn class of the Interval Algebra and let $T = (X = \{X_0, \ldots, X_n\}, F)$ be a tree decomposition of $\mathcal{N}$. If $\mathcal{N}$ is a $\overset{\diamond}{_X}$-consistent QCN non trivially inconsistent then $\mathcal{N}$ is consistent.

Proof. The proof is similar to the proof of Theorem 1 except that the manipulated partial instantiations are maximal partial instantiations. ⊣

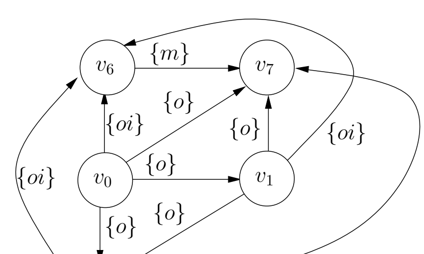

Figure 5(a). An atomic sub-QCN of $\mathcal{N}'_{\{v_0,v_1,v_2,v_6,v_7\}}$

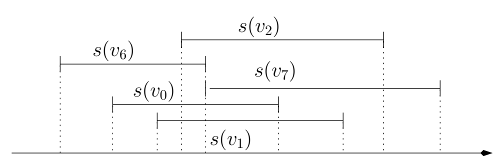

Figure 5(b). A maximal solution of $\mathcal{N}'_{\{v_0,v_1,v_2,v_6,v_7\}}$

We proved in the previous section that given a QCN $\mathcal{N}$ and a RecPart decomposition $T = (X = \{X_0, \ldots, X_n\}, F)$ of this QCN we can define a tree decomposition $T' = (X' = \{X'_0, \ldots, X'_n\}, F) = \overline{T}$ such that for each $X'_i \in X'$ there exists $X_j \in \mathsf{leaves}(T)$ such that $X'_i \subseteq X_j$. From this property and the previous theorems we can establish the following properties :

Corollary 1: Let $\mathcal{N} = (V, C)$ be a QCN defined on a class of relations $\mathcal{C}$ and let $T = (X = \{X_0, \ldots, X_n\}, F)$ a RecPart decomposition of $\mathcal{N}$.

- If $\mathcal{C}$ is such that each $\diamond$-consistent QCN defined on $\mathcal{C}$ is globally consistent and if $\mathcal{N}$ is a $\overset{\diamond}{}_{\mathsf{leaves}(T)}$-consistent QCN then $\mathcal{N}$ is consistent.
- If $\mathcal{C}$ is the ORD-Horn class of the Interval Algebra and if $\mathcal{N}$ is a $\overset{\diamond}{}_{\mathsf{leaves}(T)}$-consistent QCN then $\mathcal{N}$ is consistent.

VI. From QCNs to boolean formulae

To decide the consistency problem of QCNs, recent studies [9], [10] propose to exploit the theoretical and practical framework of the propositional logic, by using SAT encodings. Given a QCN $\mathcal{N} = (V, C)$, a first part of these encodings allows to represent the possible base relations of $C(v_i, v_j)$ for each pair of variables $v_i, v_j \in V$. A second part is defined by a set of clauses allowing to the SAT solver to enforce the property of $\diamond$-consistency during search. Intuitively, these clauses represent the possible configurations for each triple of variables $v_i, v_j, v_k \in V$ with regard to the weak composition operation. Hence, a SAT instance resulting of these encodings will be consistent if, and only if, there exists a $\diamond$-consistent sub-QCN of $\mathcal{N}$. The encoding proposed in [9] leads to a sub-QCN defined by singleton relations whereas the approach proposed in [10] leads to a convex sub-QCN.

From Theorem 1, we can restrict these encodings to the constraints belonging to clusters of a tree decomposition of $\mathcal{N}$. For illustration, we consider two kinds of decompositions : RecPart decompositions obtained by a method similar as in [11], and tree decompositions obtained from triangulation of the constraint graphs of the QCNs by using the lexBFS algorithm [17]. We have focused on QCNs of the Interval Algebra, randomly generated by following the model $A(n, d, s)$ [7]. This model involves the generation of QCNs according to three parameters: n the number of variables, d the density of constraints not defined by the relation Ψ (the average degree of the nodes in the constraint graph) and s the average number of base relations in each constraint. The results presented concern QCN instances from series $A(100, d, 6.5)$ for d varying from 4 to 24 with a step of 0.25, for each point We generated 100 networks for each serie.

In Figure 6(c) are given the percentages of pairs and triples of variables belonging to clusters for each kind of tree decompositions. Clearly, we can observe that the lexBFS-based tree decompositions discard much more constraints than the RecPart decompositions. Remark that the less is the density of non-trivial constraints, the more is the number of discarded constraints. In Figure 6(d) are given the ratios of the size of the SAT instances using tree decompositions to the size of the SAT instances using the complete encoding. The SAT encodings used are based on the SAT encoding defined in [10]. Unsurprisingly, the using of the LexBFS-based tree decompositions always performs the using of RecPart decompositions.

Figure 6 shows the number of solved instances against CPU time. The results are given for QCN instances with the parameter d varying from 8 to 12, and *Minisat 2.2* [18] was used to solve generated SAT instances. CPU time is restricted to solving time, and QCN instances are not preprocessed before encoding into SAT instances. As we can see, the lexBFS-based tree decompositions allows to improve the performance for solving SAT instances.

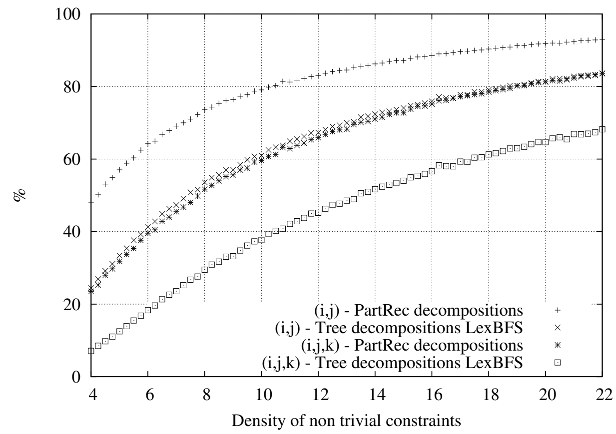

Figure 6(c). Percentages of pairs and triples of variables belonging to the clusters

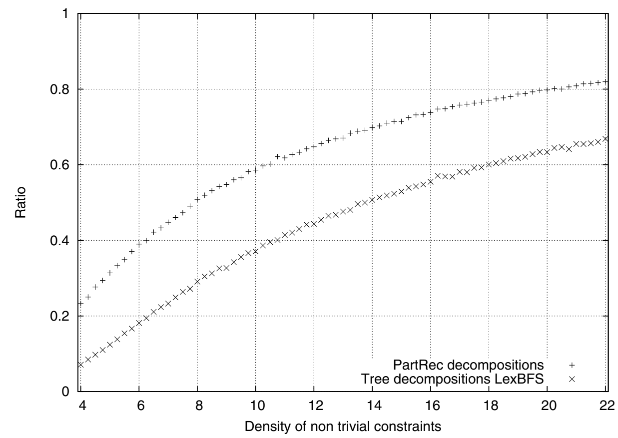

Figure 6(d). Ratios of the size of the SAT instances by using tree decompositions to the size of the SAT instances by using the complete encoding.

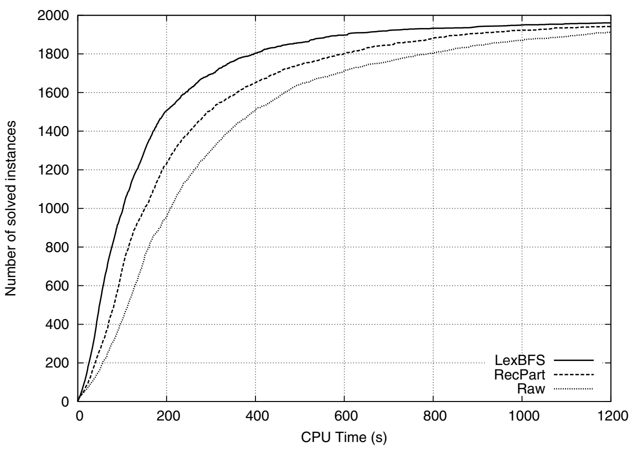

Figure 6. Number of solved instances against CPU time.

VII. Conclusion and future works

In this paper, we have introduced and studied the RecPart decompositions. We proved that these decompositions are equivalent to particular tree decompositions. Moreover, we have studied the consistency problem of QCNs with regard to tree decompositions. We proved that, for some tractable classes of relations such as ORD-Horn class, we can decide the consistency problem of a QCN by enforcing the $\diamond$-consistency restricted to the constraints belonging to clusters of a tree decomposition. In order to illustrate these results, we have compared two kinds of decompositions : RecPart decompositions and tree decompositions obtained from triangulation of the constraint graphs of QCNs by using the lexBFS algorithm. A future work is to conduct extensive experiments in order to compare more tree decompositions into the framework of SAT encodings of QCNs.

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