About the Temporal Decrease of Sets

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Abstract

This paper is about a simple general framework for modelling decrease. *We provide an extension of propositional linear time temporal logic with operators expressing 'nexttime' and 'henceforth', by a modality that quantifies over the points of some set X. This set changes in the course of time; actually, X is assumed to* shrink *little by little. We develop an appropriate logical language, give a sound and complete axiomatization of the set of validities, and prove decidability of this set. Moreover, the computational complexity of the logic is determined.*

1 Introduction

At TIME-00 we presented a formalism modelling qualitatively the notion of *change* of some set in the course of time; cf [12]. Although having nice computational properties that system lacks sufficient *expressiveness*. This deficiency is overcome by the present paper, at least concerning *decrease*. We actually provide a generalization of propositional linear time temporal logic where the transition between states is replaced by *gradual shrinking* of a given set.

To put such shrinking procedures in concrete terms one may think of the *development of knowledge of an agent*. For convenience, we briefly remind the reader of the relevance of this to computer science: The notion of knowledge has proved to be quite useful in modelling *distributed systems;* cf [9] and [4], e.g. The knowledge of an agent involved in such a system is represented by the set of alternatives the agent considers possible at any time. Actually, this set called the agent's *knowledge state shrinks* in the course of time in case the system is *synchronous* and the agents have *perfect recall;* as to the latter notions cf also the recent paper [3].

In usual logic of knowledge and time it is presupposed that *runs* of systems are functions defined on $\mathbb{N}$, i.e., infinite in particular. As a consequence one gets that, from a 'global' point of view, the actual knowledge state of an agent splits up in a tree–like manner step by step. In this paper, however, we take a 'local' point of view describing only the shrinking of the agent's knowledge state. This leads to a *chain structure* of the sets involved. We clearly have to require then runs to be partial.

We believe that our approach is not only interesting in connection with knowledge, but also in other contexts. For instance, the proposed system is basically a contribution to pure temporal logic. Moreover, it belongs to 'topological modal logic' in a sense; cf [6], [2]. Finally, it could provide the core of a temporal reasoning formalism; see corresponding remarks in the introduction to [12].

We now get to the details of our logic. First, we present the underlying language. Afterwards we axiomatize the set of theorems of the logic, in Section 3, where also semantical completeness of the given axiomatization is proved; this takes up the main part of the paper. The final technical section is devoted to discussing effectiveness properties of the proposed system.

We assume acquaintance of the reader with basic modal and temporal logic. All that we need is contained in the standard textbooks; cf [1], [5] and [8], for instance.

2 The Language

From a technical point of view we essentially link the well–known modal system S5 with propositional linear time temporal logic involving the operators *nexttime* and *henceforth*. The type of this combination is determined by our goals viz. describing steadily decreasing sets formally, and is reflected in a couple of peculiar axiom schemata.

Thus, three non–boolean connectives will be present in our language: a *location operator* L addressing points inside the set being under discourse (or rather the 'universal' dual K of L), a *shrinking operator* $\boxtimes$ modelling gradual decrease, and an operator $\Box$ intended to capture the reflexive and transitive closure of $\boxtimes$.

In order to define the set WFF of well–formed formulas of the logical language we let $\mathrm{PV} = \{p, q, r, \ldots\}$ be an enumerable set of *propositional variables*. Designating

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formulas by lower case Greek letters the set WFF is determined by the following recursive conditions then:

$$\alpha :\equiv p \mid \neg\alpha \mid \alpha \wedge \beta \mid \boxtimes\alpha \mid \Box\alpha \mid K\alpha.$$

Concerning the *duals* of the modalities, we let

$$\begin{array}{lll} \bigcirc\alpha & :\equiv & \neg \boxtimes \neg\alpha, \\ \Diamond\alpha & :\equiv & \neg\Box\neg\alpha, \\ L\alpha & :\equiv & \neg K \neg\alpha. \end{array}$$

Moreover, the boolean operators $\vee$ and $\rightarrow$ are treated as abbreviations.

We define the semantics of our language next. As we would like to treat the decrease of a given set X, certain subsets of X have to be considered in the formal model. Consequently, we take X and the system of those subsets, $\mathcal{O}$, as basic ingredients of the domains with respect to which formulas are shortly to be interpreted. However, the set $\mathcal{O}$ carries a time structure which is made explicit by the following definition. — Subsequently, let $\mathcal{P}(A)$ denote the powerset of a given set A.

Definition 2.1 *1. Let X be a non-empty set, $(I, \leq)$ an initial segment of $(\mathbb{N}, \leq)$, and d an order-reversing mapping from $(I, \leq)$ into $\mathcal{P}(X)$. Then $\mathcal{S} := (X, d)$ is called a* flow of decreasing sets.

2. Let $\mathcal{S} = (X, d)$ be a flow of decreasing sets and $V : \mathrm{PV} \longrightarrow \mathcal{P}(X)$ a mapping. Then V is called a valuation, *and the triple $\mathcal{M} := (X, d, V)$ is called a* model (based on $\mathcal{S}$).

Notice that 'the future includes the present', i.e., the ordering of time is reflexive, as it is common in computer science.

We are going to evaluate formulas in models at *situations* of a flow of decreasing sets, (X, d). Such situations are simply pairs x, U_i satisfying $x \in U_i = d(i)$, where $i \in I$; these pairs are designated without brackets.

Definition 2.2 *Let be given a model $\mathcal{M} = (X, d, V)$ and a situation x, U_i of the flow of sets which $\mathcal{M}$ is based on. Then we define*

$$\begin{array}{lll}
x, U_i \models_{\mathcal{M}} p & \mathit{iff} & x \in V(p) \\
x, U_i \models_{\mathcal{M}} \neg\alpha & \mathit{iff} & x, U_i \not\models_{\mathcal{M}} \alpha \\
x, U_i \models_{\mathcal{M}} \alpha \wedge \beta & \mathit{iff} & \left\{ \begin{array}{c} x, U_i \models_{\mathcal{M}} \alpha \\ \mathit{and} \\ x, U_i \models_{\mathcal{M}} \beta \end{array} \right. \\
x, U_i \models_{\mathcal{M}} K\alpha & \mathit{iff} & \left\{ \begin{array}{c} \forall\, y \in U_i : \\ y, U_i \models_{\mathcal{M}} \alpha \end{array} \right. \\
x, U_i \models_{\mathcal{M}} \boxtimes\alpha & \mathit{iff} & \left\{ \begin{array}{c} i+1 \in I \ \mathit{and} \\ x \in U_{i+1} \\ \mathit{imply} \\ x, U_{i+1} \models_{\mathcal{M}} \alpha \end{array} \right. \\
x, U_i \models_{\mathcal{M}} \Box\alpha & \mathit{iff} & \left\{ \begin{array}{c} \forall\, j \geq i : \\ j \in I \ \mathit{and} \\ x \in U_j \ \mathit{imply} \\ x, U_j \models_{\mathcal{M}} \alpha, \end{array} \right.
\end{array}$$

for all $p \in \mathrm{PV}$ and $\alpha, \beta \in \mathrm{WFF}$.

In case $x, U_i \models_{\mathcal{M}} \alpha$ is satisfied we say that α *holds in $\mathcal{M}$ at the situation x, U_i;* moreover, the formula $\alpha \in \mathrm{WFF}$ is said to be *valid in $\mathcal{M}$*, iff it holds in $\mathcal{M}$ at every situation.

3 Axioms

In this section we first present a list of schemata aimed at providing a *sound* and *complete* axiomatization of all the formulas which are valid in every model based on a flow of decreasing sets. The schemata are divided into three groups, called *axioms of location, time,* and *decrease,* respectively. Secondly, we sketch the crucial steps of the completeness proof of the logical system **S** determined by these schemata and the usual rules *modus ponens* and *necessitation.*

- *Axioms of location:*

$$\begin{array}{ll}
(L1) & \left\{ \begin{array}{c} \text{All instances of} \\ \text{propositional tautologies} \end{array} \right. \\
(L2) & K(\alpha \rightarrow \beta) \rightarrow (K\alpha \rightarrow K\beta) \\
(L3) & K\alpha \rightarrow \alpha \\
(L4) & K\alpha \rightarrow KK\alpha \\
(L5) & L\alpha \rightarrow KL\alpha
\end{array}$$

- *Axioms of time:*

$$\begin{array}{ll}
(T0) & (p \rightarrow \boxtimes p) \wedge (\neg p \rightarrow \boxtimes\neg p) \\
(T1) & \boxtimes(\alpha \rightarrow \beta) \rightarrow (\boxtimes\alpha \rightarrow \boxtimes\beta) \\
(T2) & \bigcirc\alpha \rightarrow \boxtimes\alpha \\
(T3) & \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta) \\
(T4) & \Box\alpha \rightarrow \alpha \wedge \boxtimes\Box\alpha \\
(T5) & \Box(\alpha \rightarrow \boxtimes\alpha) \rightarrow (\alpha \rightarrow \Box\alpha)
\end{array}$$

- *Axioms of decrease:*

$$\begin{array}{ll}
(D1) & K \boxtimes \alpha \rightarrow \boxtimes K\alpha \\
(D2) & \boxtimes K\alpha \rightarrow K \boxtimes \alpha \vee \boxtimes\beta
\end{array}$$

While the schemata of the first two groups are well-known, the third group of axioms is the decisive one. Here the interaction of the modal connectives involved in the system is described. $(D1)$ is the real 'axiom of shrinking' and is typical of the various systems of topological reasoning;

see [2]. $(D2)$ is responsible for the linear structure of flows of decreasing sets. The effect of this axiom is discussed more detailedly below.

In the paper [10], the chain property of the distinguished set of subsets of a given set has been expressed by the schema

$$(*) \quad K \boxtimes (\alpha \rightarrow L\beta) \vee K \boxtimes (\beta \rightarrow L\alpha).$$

As follows from our completeness theorem below, this schema is **S**–derivable.

We state the soundness of the system **S** with respect to the intended semantic domains as our first result.

Proposition 3.1 *Let $\alpha \in$ WFF be an **S**–derivable formula. Then α is valid in every flow of decreasing sets.*

We now point out the crucial steps proving semantic completeness of **S**. Clearly, we use the canonical model $\mathcal{M}_{\text{can}}$ of **S**. Let $\xrightarrow{\bigcirc}$, $\xrightarrow{\Diamond}$ and $\xrightarrow{L}$ denote the accessibility relations of $\mathcal{M}_{\text{can}}$ belonging to the connectives $\boxtimes$, $\Box$ and K, respectively. The above axioms then provide for the required properties of these relations. It is well–known from basic modal and temporal logic (see [8], §9, e.g.) that $\xrightarrow{L}$ is an equivalence relation, $\xrightarrow{\bigcirc}$ is partially functional, and $\xrightarrow{\Diamond}$ is reflexive and transitive. Maybe it is less known that for all points s, t, u of the canonical model such that $s \xrightarrow{\bigcirc} t \xrightarrow{L} u$ there exists a point v such that $s \xrightarrow{L} v \xrightarrow{\bigcirc} u$; this is valid because of $(D1)$ and is called the *cross property.*

Let $[s]$ denote the $\xrightarrow{L}$–equivalence class of a point s of $\mathcal{M}_{\text{can}}$. Define a binary relation $\succ$ on the set of all such equivalence classes by

$$[s] \succ [t] :\Longleftrightarrow \begin{cases} \text{there are } s' \in [s],\ t' \in [t] \\ \text{such that } s' \xrightarrow{\bigcirc} t'. \end{cases}$$

The following fact about $\succ$ was proved in [10], Proposition 3.5, using the schema $(*)$. But, the schema $(D2)$ in connection with $(T2)$ is also sufficient for this.

Proposition 3.2 *The relation $\succ$ is partially functional.*

We mention the following *modified cross property* corresponding with Axiom $(D2)$, for later purposes: for all s, t, u, v such that

$$s \xrightarrow{L} t \xrightarrow{\bigcirc} u \text{ and } s \xrightarrow{\bigcirc} v$$

there exists a point x such that

$$s \xrightarrow{\bigcirc} x \xrightarrow{L} u;$$

see Figure 1. It turns out that this property, which is valid on the canonical model in particular, can be transmitted to certain *filtrations.*

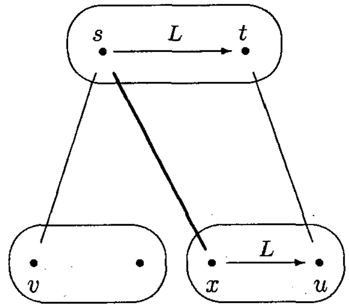

Figure 1. *Modified cross property*

From the transitivity of the relation $\xrightarrow{\Diamond}$ we infer

$$(\xrightarrow{\bigcirc})^* \subseteq \xrightarrow{\Diamond},$$

where $(\ldots)^*$ designates the reflexive and transitive closure. In order to establish equality here, the method of filtration has to be used. By means of the following derivabilities it can be guaranteed that passing to a filtration does not violate the relevant semantic properties. It is well–known (see [8], [4]) that all these formulas are theorems of ordinary linear time temporal logic where the schema

$$\bigcirc\alpha \leftrightarrow \boxtimes\alpha$$

represents an axiom, but it turns out that they are even provable with the aid of the weaker schema $(T2)$. — We designate **S**–derivability '$\Vdash$'.

Proposition 3.3 *1. The schema* Dum *is **S**–derivable:*

$$\Vdash \Box(\Box(\alpha \rightarrow \Box\alpha) \rightarrow \alpha) \rightarrow (\Diamond\Box\alpha \rightarrow \Box\alpha)$$

2. *Also, the schema* L *guaranteeing weak connectedness of the relation $\xrightarrow{\Diamond}$ is **S**–derivable:*

$$\Vdash \Box(\Box\alpha \rightarrow \beta) \vee \Box(\Box\beta \rightarrow \alpha)$$

We now introduce a suitable filtration of the submodel $\mathcal{M}'$ of $\mathcal{M}_{\text{can}}$ which is generated by a point s_0 containing the negation α of a given non–derivable formula γ.

Let $\text{sf}(\alpha)$ denote the set of subformulas of α. Then certain finite sets of formulas are defined as follows:

$$\begin{array}{rcl}
\Gamma_{\Box} & := & \{\boxtimes\Box\beta \mid \Box\beta \in \text{sf}(\alpha)\} \\
\Gamma_{\boxtimes} & := & \{\boxtimes\neg\beta \mid \boxtimes\beta \in \Gamma_{\Box} \cup \text{sf}(\alpha)\} \\
\Gamma & := & \text{sf}(\alpha) \cup \Gamma_{\Box} \cup \Gamma_{\boxtimes} \\
\Gamma^{\neg} & := & \Gamma \cup \{\neg\beta \mid \beta \in \Gamma\} \\
\Gamma^{\wedge} & := & \begin{cases} \Gamma^{\neg} \text{ joined with the set of} \\ \text{all finite conjunctions of} \\ \text{distinct elements of } \Gamma^{\neg} \end{cases} \\
\Gamma^{L} & := & \{L\beta \mid \beta \in \Gamma^{\wedge}\} \\
\widetilde{\Gamma} & := & \Gamma^{\wedge} \cup \Gamma^{L}.
\end{array}$$

The special form of these sets facilitates the subsequent results. Note that $\widetilde{\Gamma}$ is in fact finite and closed under subformulas.

For all points s, t of $\mathcal{M}'$, we let

$$s \sim t :\iff s \cap \widetilde{\Gamma} = t \cap \widetilde{\Gamma}.$$

Moreover, we let $\bar{s}$ designate the $\sim$–equivalence class of s, and $\bar{C}$ the set of all such classes. Since $\widetilde{\Gamma}$ is finite, $\bar{C}$ is a finite set as well.

So far we have formed a filtration of the carrier set of $\mathcal{M}'$. Next we introduce filtrations of the accessibility relations of this model (which actually are the restrictions of the accessibility relations of $\mathcal{M}_{\text{can}}$). For convenience of the reader, we repeat the definition of this notion for the present case. Let $\triangle$ be a modal operator (i.e., $\triangle = K, \boxtimes$, or $\Box$) and ∇ its dual. Then, a binary relation $\overset{\nabla}{\longmapsto}$ on $\bar{C}$ is called a $\widetilde{\Gamma}$–*filtration* of $\overset{\nabla}{\longrightarrow}$, iff the following two conditions are satisfied for all points s, t of $\mathcal{M}'$:

- $s \overset{\nabla}{\longrightarrow} t$ implies $\bar{s} \overset{\nabla}{\longmapsto} \bar{t}$;
- $\bar{s} \overset{\nabla}{\longmapsto} \bar{t}$ implies $\{\beta \mid \triangle\beta \in s \cap \widetilde{\Gamma}\} \subseteq t$.

Let the relation $\overset{\bigcirc}{\longmapsto}$ on $\bar{C}$ be a filtration of $\overset{\bigcirc}{\longrightarrow}$. Let $\overset{\bigcirc^*}{\longmapsto}$ designate the transitive closure of $\overset{\bigcirc}{\longmapsto}$. Then the following lemma gives the precise circumstances surrounding our previous remark that the accessibility relation corresponding with $\Box$ represents the reflexive and transitive closure of the accessibility relation corresponding with $\boxtimes$.

Lemma 3.4 *The relation* $\overset{\bigcirc^*}{\longmapsto}$ *is a filtration of the relation* $\overset{\diamond}{\longrightarrow}$.

This is Goldblatt's 'Ancestral Lemma' 9.8 in [8]; regarding the modifications caused by the partiality of $\overset{\bigcirc}{\longrightarrow}$, the proof of 3.4 can be done analogously.

The appropriate version of the '*Fun*–Lemma' 9.9 in [8], reads as follows.

Lemma 3.5 *Let* $\overset{\bigcirc}{\longmapsto}$ *be the minimal filtration of the relation* $\overset{\bigcirc}{\longrightarrow}$. *Suppose that* $\boxtimes\beta \in \Gamma$, *and let* s, t *be points of the model* $\mathcal{M}'$ *such that* $\bar{s} \overset{\bigcirc}{\longmapsto} \bar{t}$. *Then* $\boxtimes\beta \in s$, *iff* $\beta \in t$.

Clearly, the proof of Lemma 3.5 requires Axiom $(T2)$.

Note that functionality of the relation $\overset{\bigcirc}{\longrightarrow}$ is lost by passing to a filtration in general. Lemma 3.5 represents a corresponding substitute.

Some rather technical part in the completeness proof of propositional linear time temporal logic can be transferred literally to the present case. It deals with a meticulous analysis of the *clusters* of the relation $\overset{\bigcirc^*}{\longmapsto}$ and a related one, respectively, in which (1) and (2) of Proposition 3.3 are used decisively; see [8], 9.11 and 9.12, for the details.

So, we may now proceed to the real peculiarities of the system **S**. Subsequently, we exclusively use the minimal filtration of the relation $\overset{\bigcirc}{\longrightarrow}$, which we designate $\overset{\bigcirc}{\longmapsto}$. We also consider only the minimal filtration of $\overset{L}{\longrightarrow}$, from now on. This relation is denoted $\overset{L}{\longmapsto}$. Due to the special choice of the set Γ, $\overset{L}{\longmapsto}$ is an equivalence relation and the *cross property* is valid with respect to $\overset{\bigcirc}{\longmapsto}$ and $\overset{L}{\longmapsto}$.

Lemma 3.6 *1. The relation* $\overset{L}{\longmapsto}$ *is an equivalence relation.*

2. Suppose that we have

$$\bar{s} \overset{\bigcirc}{\longmapsto} \bar{t} \overset{L}{\longmapsto} \bar{u}.$$

Then there exists a point v *of* $\mathcal{M}'$ *such that*

$$\bar{s} \overset{L}{\longmapsto} \bar{v} \overset{\bigcirc}{\longmapsto} \bar{u}.$$

A proof of the lemma can be found in the paper [11].

While Lemma 3.5 enables us to select an arbitrary $\overset{\bigcirc}{\longmapsto}$–successor of the $\sim$–class $\bar{s}$ under consideration without loss of any semantic property concerning the formula α, we actually need a further, formally similar assertion: we have to be able to select a suitable successor of the whole $\overset{L}{\longmapsto}$–equivalence class of $\bar{s}$, at least as far as the realization of $\boxtimes$–formulas contained in Γ is concerned. That this faithfully is possible can be guaranteed by the following variant of the *modified cross property* involving the schema $(D2)$ (and going beyond the issues of [11] thus).

Proposition 3.7 *Suppose that*

$$\bar{s} \overset{\bigcirc}{\longmapsto} \bar{v} \text{ and } \bar{s} \overset{L}{\longmapsto} \bar{t} \overset{\bigcirc}{\longmapsto} \bar{u}$$

is valid for the $\sim$*–classes* $\bar{s}, \bar{t}, \bar{u}, \bar{v}$. *Furthermore, assume that for some formula* $\boxtimes\beta \in \Gamma$ *it holds that* $\boxtimes\beta \in s$. *Then there exists a* $\sim$*–class* $\bar{x}$ *such that*

$$\bar{s} \overset{\bigcirc}{\longmapsto} \bar{x} \overset{L}{\longmapsto} \bar{u}.$$

Proof. Let $\boxtimes\beta$ be an element of $s \cap \Gamma$. Due to the second filtration condition we have that $\boxtimes\neg\beta \notin s'$ whenever $s' \sim s$, since otherwise we would get $\beta \in v$ and $\neg\beta \in v$. Consequently, $\bigcirc\beta \in s'$ for all such points s'.

From $\bar{t} \overset{\bigcirc}{\longmapsto} \bar{u}$ we conclude that there are $t' \in \bar{t}$ and $u' \in \bar{u}$ such that $t' \overset{\bigcirc}{\longrightarrow} u'$. Because of the appropriate definition of the set $\widetilde{\Gamma}$ we can find a point $s' \sim s$ of the model $\mathcal{M}'$ such that $t' \xrightarrow{L} s'$. We infer $s' \xrightarrow{L} t' \xrightarrow{\bigcirc} u'$, since the relation $\xrightarrow{L}$ is symmetrical. As $\bigcirc\beta \in s'$, there exists an $\xrightarrow{\bigcirc}$–successor of s'. But then, the *modified cross property* applies yielding a point x such that $s' \xrightarrow{\bigcirc} x \xrightarrow{L} u'$. It follows that $\bar{s} \overset{\bigcirc}{\longmapsto} \bar{x} \overset{L}{\longmapsto} \bar{u}$. This proves the proposition. □

Consider the structure

$$\overline{\mathcal{M}} := \left(\bar{C}, \overset{\bigcirc}{\longmapsto}, \overset{\bigcirc^*}{\longmapsto}, \overset{L}{\longmapsto}, \bar{V}\right),$$

where $\bar{V}$ is induced by the distinguished valuation of the canonical model. As we have shown above, $\overline{\mathcal{M}}$ is a filtration of the model $\mathcal{M}'$. Thus

$$(\overline{\mathcal{M}}, \bar{s}) \models \beta \iff (\mathcal{M}', s) \models \beta$$

holds for all points s of $\mathcal{M}'$ and formulas β contained in $\widetilde{\Gamma}$.

We are going to select a 'chain' of $\overset{L}{\longmapsto}$ –equivalence classes, as indicated above. Let $[\,\bar{s}\,]$ denote the $\overset{L}{\longmapsto}$ –equivalence class of $\bar{s}$. Define a relation $\succ$ on the set of all such classes by

$$[\,\bar{s}\,] \succ [\,\bar{t}\,] :\iff \begin{cases} \text{there are } \bar{s}' \in [\,\bar{s}\,],\ \bar{t}' \in [\,\bar{t}\,] \\ \text{such that } \bar{s}' \overset{\bigcirc}{\longmapsto} \bar{t}'. \end{cases}$$

The first class we choose is $[\,\bar{s}_0\,]$. The selection of the successor class of the current one is in accordance with Proposition 3.7: Given $[\,\bar{s}\,]$, we choose, if possible, any $\succ$–successor $[\,\bar{t}\,] \neq [\,\bar{s}\,]$ of $[\,\bar{s}\,]$ such that whenever some formula $\boxtimes\beta \in \Gamma$ is valid at some point of $\bar{s}' \in [\,\bar{s}\,]$ and $\bar{s}'$ is extendable at all, then this formula is 'realized' at some $\overset{\bigcirc}{\longmapsto}$–successor $\bar{t}'$ of $\bar{s}'$ contained in $[\,\bar{t}\,]$. Because of 3.7 we have in fact taken into account *all* such formulas from Γ then.

After a finite number of steps the procedure does not make any headway, or reaches a class processed already. In the latter case we are finished. Otherwise we unwind the last (and possibly only non–simple) $\overset{\bigcirc^*}{\longmapsto}$–clusters contained in the last class simultaneously, just as in propositional linear time temporal logic.

During the construction we 'forget' those $\overset{\bigcirc}{\longmapsto}$–arrows of $\overline{\mathcal{M}}$ which do not point to the class chosen at a time; additionally, points not belonging to any of the selected classes may be forgotten afterwards as well.

Designating the resulting (Kripke) model

$$\overline{\mathcal{M}}' := \left(\bar{C}', \overset{\bigcirc}{\longmapsto}{}', \overset{\bigcirc^*}{\longmapsto}{}', \overset{L}{\longmapsto}{}', \bar{V}'\right)$$

we obtain:

Proposition 3.8 *For all $\beta \in \widetilde{\Gamma}$ and $\bar{s} \in \bar{C}'$,*

$$(\overline{\mathcal{M}}', \bar{s}) \models \beta \iff (\overline{\mathcal{M}}, \bar{s}) \models \beta$$

is valid.

The proposition can be proved by structural induction. The cases $\beta = \boxtimes\gamma$ and $\beta = \Box\gamma$ are the critical ones. In the first case 3.7 and 3.5 have to be applied. For the second case certain matters related to the remark following Lemma 3.5 are relevant.

To eventually get the desired model based on a flow of decreasing sets, we now use the same techniques as in the completeness proof for topological nexttime logic; see [10], Section 3. This model gets carrier set a suitable *space of functions* over $\bar{C}'$. (To be more precise, we take all $\overset{\bigcirc}{\longmapsto}{}'$–paths *through* $\overline{\mathcal{M}}'$.) Furthermore, a function f belongs to the i–th subset iff $f(i)$ exists, for all $i \in \mathbb{N}$.

By means of an appropriate *truth lemma*, see 3.8 of [10], the first of our main results follows:

Theorem 3.9 *Let $\gamma \in$ WFF be a formula. Then γ is* **S**–*derivable, iff γ is valid in every flow of decreasing sets.*

4 Effectiveness Properties

First in this section we sketch how *decidability* of the set of **S**–theorems can be obtained. Afterwards we determine the complexity of the corresponding *satisfiability problem.*

The above completeness proof does not obviously yield the finite model property of our logic; so, we have to proceed differently in order to achieve this result. Some prerequisite notions are introduced below.

Definition 4.1 *Let $\mathcal{I} := (I, \leq)$ be an initial segment of $(\mathbb{N}, \leq)$.*

1. *A subset $\emptyset \neq I' \subseteq I$ is called a* segment *of $\mathcal{I}$, iff there is no $i \in I \setminus I'$ strictly between any two elements of I'.*
2. *A partition of I into segments is called a* segmentation *of $\mathcal{I}$.*

Let be given a formula α and a model $\mathcal{M} = (X, d, V)$ based on a flow of decreasing sets. We will have to consider segmentations of $\mathcal{I} = (I, \leq)$, where I is the domain of d, such that the truth value of α remains unaltered on every segment, in the following sense.

Definition 4.2 *Let $\alpha \in$ WFF be a formula and $\mathcal{M} = (X, d, V)$ a model based on a flow of decreasing sets. Furthermore, let Λ be an indexing set and $\mathcal{P} := \{\mathcal{I}_\lambda \mid \lambda \in \Lambda\}$ a segmentation of $\mathcal{I} = (I, \leq)$. Then α is called* stable *on $\mathcal{P}$, iff for all $\lambda \in \Lambda$ and $x \in X$ we have*

$$x, U_i \models \alpha \text{ for all } i \in \mathcal{I}_\lambda \text{ such that } x \in U_i,$$

or

$$x, U_i \models \neg\alpha \text{ for all } i \in \mathcal{I}_\lambda \text{ such that } x \in U_i.$$

It turns out that we can always get a *finite* segmentation of $\mathcal{I}$ on which α is stable.

Proposition 4.3 *Let $\mathcal{M} = (X, d, V)$, $\mathcal{I}$ and α be as above. Then there exists a finite segmentation $\mathcal{P}_\alpha := \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ of $\mathcal{I}$ such that α is stable on $\mathcal{P}_\alpha$. Moreover, $\mathcal{P}_\alpha$ can be chosen such that it refines $\mathcal{P}_\beta$ for every subformula β of α.*

In fact, the segmentation $\mathcal{P}_\alpha$ can be constructed by induction on the structure of formulas starting with the trivial segmentation $\{\mathcal{I}\}$ in case α is a propositional variable (mind Axiom $(T0)$).

According to Proposition 4.3 the question whether a given formula α is satisfiable can be reduced to models of 'finite depth'. By a standard procedure of the logic of set spaces this question can then be whittled down to models which are of 'finite width' additionally; cf [7], 3.35, e.g. This eventually yields the finite model property, which implies the claimed decidability result.

Theorem 4.4 *The set of formulas $\alpha \in$ WFF being valid in every model, is decidable.*

A careful analysis of the proof of Proposition 4.3, i.e., counting segments carefully along the tree structure of α and thereby utilizing the S5–properties of K, yields the following strengthening of that assertion.

Proposition 4.5 *The segmentation $\mathcal{P}_\alpha$ can be chosen such that the number of segments is polynomial in the length of α.*

Finally, the linearity of flows of decreasing sets plays its part once again, among other things. We obtain that the satisfiability problem of our logic is NP–complete. In view of [13], Theorem 4.1, this result is somewhat surprising at first glance, according to the presence of the temporal operators. However, the axioms $(T0)$ and $(D2)$ are responsible for breaking down the complexity.

Theorem 4.6 *The set of formulas $\alpha \in$ WFF satisfiable at some situation of some model, is* NP*–complete.*

5 Concluding Remarks

We have provided a logical framework modelling the temporal decrease of sets qualitatively. Our results include soundness, completeness, decidability and complexity of the proposed logical system.

The reader might wonder why the *until*–operator has not been considered presently, which really makes up the full expressive power of propositional linear time temporal logic. The only reason for this is to be brief as far as features of the standard systems involved are concerned. In fact, enriching **S** by that operator is relatively easy, i.e., goes on as in ordinary temporal logic (see [8], § 9), and does not change the issues of this paper.

Future work will be concerned with elaborating a *temporal logic of change,* in order to obtain the desired strengthening of the system studied in [12]. One has to treat the case of *growth* separately and add suitable axioms of *change.* Maybe the corresponding axiom of the paper [12] works for this more general setting as well.

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