An Application of Monodic First-Order Temporal Logic to Reasoning about Knowledge

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Abstract

Using a result on the decidability of a certain monodic fragment of first-order linear time temporal logic due to Hodkinson we show that the hybrid subset space logic of knowledge over dense flows of time is decidable, too.

1 Introduction

The idea of *knowledge* has proved to be of fundamental importance to several branches of computer science and AI, eg, analyzing distributed systems and reasoning about agents, respectively. Concerning this the reader may consult the textbooks [4] and [11], which give comprehensive introductions to the field; not least many questions of topical interest are brought up there.

Roughly a decade ago, a remarkable connection between knowledge and *topology* was discovered and exploited for both the logic of knowledge and topological reasoning. The first paper regarding this is [12], and [3] is the standard one. In these papers several systems were studied that are sensitive to the semantics underlying any *knowledge acquisition* procedure.

MOSS and PARIKH's systems appear as *bi-modal logics of subset spaces,* including unary connectives K representing knowledge and $\Box$ representing (computational *effort* in the course of) *time,* respectively. To be a bit more precise here, subset spaces are triples $(X, \mathcal{O}, V)$ consisting of a non-empty set X (of states), a set $\mathcal{O}$ of distinguished subsets of X, and a valuation V. The operator K quantifies then across any *knowledge state* (i.e., any element of $\mathcal{O}$), and $\Box$ quantifies 'downward' across $\mathcal{O}$. With that, time is rather implicit in subset spaces, viz given by *evolving* knowledge states, actually (that is, by *shrinking* sets).

While this framework is very suitable for dealing with *branching* time structure, cf [5], serious difficulties arise in case *linear* flows of time underly the scenario to be modelled. (Such a scenario could be, for instance, a *partial query system,* providing answers only in certain cases.) In fact, not even a complete axiomatization of the set of validities is known in this case up to now. Apparently, one cannot have purely modal control of linearity occurring in connection with subset spaces.

Making the specification language more expressive can help one out of this dilemma. For instance, if we add the usual temporal operators *nexttime* and *until* to the modal language, then a reasonable system having nice meta-properties results for *discrete linear time;* cf [6]. But how to proceed in case of *dense* flows of time? Well, another extension of modal logic turns out to be useful here: hybrid logic. (As to the corresponding basics cf [2, Section 7.3].) Letting nominals designate the real semantic objects of the modal logic of subsets spaces (viz *neighbourhood situations* consisting of a state and a neighbourhood thereof) enables one to provide an appropriate hybrid axiomatization and prove its soundness and 'canonical' completeness; cf [7].

At the time the paper [7] was written we could not say whether or not the system considered there is decidable. The reason for this was hybrid expressiveness: unlike the modal correspondent to density, the hybrid one forces the underlying time structure to be in fact dense. Thus the usual modal techniques based on finite models are not applicable to answer that question affirmatively.

At this point monodic first-order temporal logic comes into play. It is, by definition, a fragment of first-order temporal logic we are dealing with here where application of temporal quantifiers is rigorously limited: the latter is allowed only to formulas having at most one free variable. In the landmark paper [9] it was proved that such a monodic fragment turns out to be decidable if this is true for a suitably associated fragment of first-order logic. Several other decidability results followed presently; see [10] for a survey. We exploit below decidability of the *guarded* fragment of FOTL *with equality,* which is a consequence of a theorem of HODKINSON; cf [8, Theorem 1].

The paper is organized as follows. First we hybridize the

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logic of subset spaces. We define the underlying language and have a brief look at the list of axioms and rules for the validities, regarding linearity and dense flows of time. Afterwards we state the above mentioned soundness and completeness theorem and touch on a couple of points crucial to its proof. (The latter is done for the reader's convenience, to get to know concisely the logic we consider.) All this is contained in Section 2. The core of the proof of decidability for the subset space logic of knowledge over dense flows of time follows then, in Section 3: we reduce the corresponding satisfiability problem to the satisfiability problem of the monodic guarded fragment of first-order temporal logic with equality over dense linear orders. Finally, Section 4 concludes the paper with some additional remarks.

It should be mentioned that the problem solved in this paper was raised during the FST TCS 1999 conference.[1]

2 The Hybrid Logic of Subset Spaces

First in this section we extend the modal language of subset spaces in a hybrid way. We add nominals as names of neighbourhood situations and assign a jump operator to every nominal of this kind. Then we describe the hybrid logic arising from the new language.

Let PROP and NNS be two disjoint sets of symbols called *propositional letters* and *names of neighbourhood situations,* respectively. We use p, q, ... to denote typical elements of PROP, and $\mathfrak{n}, \mathfrak{m}, \ldots$ for the same purpose in case of neighbourhood situations. The set WFF of *well-formed formulas* with typical elements α, β, ... is defined by the generative rule

$$\alpha \ ::= \ p \mid \mathfrak{n} \mid \neg\alpha \mid \alpha \wedge \beta \mid K\alpha \mid \Box\alpha \mid @_{\mathfrak{n}}\alpha.$$

The missing boolean connectives $\top, \bot, \vee, \rightarrow, \leftrightarrow$ will be treated as abbreviations, as needed. The duals of the modalities K and $\Box$ are written L and $\Diamond$, respectively. (Our use of the letter L differs from that in traditional modal logic, but is common for the usual modal logic of subset spaces; cf [3].) Note that there is no need to distinguish between $@_{\mathfrak{n}}$ and its dual because $@_{\mathfrak{n}}$ turns out to be self-dual (as it is for the basic hybrid logic).

We give meaning to formulas next. To start with we define the relevant structures. These are essentially the same domains as those of the modal logic of subset spaces. However, the valuations, which determine the states where the atomic propositions are true, have to be extended to NNS suitably.

Definition 2.1 (Frames and hybrid subset spaces) *Let X be a non-empty set, and let $\mathcal{P}(X)$ designate the powerset of X.*

1. *Let $\mathcal{O} \subseteq \mathcal{P}(X)$ be a system of non-empty subsets of X. Then the pair $\mathcal{F} := (X, \mathcal{O})$ is called a* subset frame.

2. *Let $\mathcal{F} := (X, \mathcal{O})$ be a subset frame. The set of* neighbourhood situations *of $\mathcal{F}$ is*

   $$\mathcal{N} := \{x, U \mid (x, U) \in X \times \mathcal{O} \text{ and } x \in U\}.$$

   (Neighbourhood situations are written without brackets here and subsequently.)

3. *A* hybrid valuation *is a mapping*

   $$V : \mathrm{PROP} \cup \mathrm{NNS} \longrightarrow \mathcal{P}(X) \cup \mathcal{N}$$

   such that

   - $V(p) \in \mathcal{P}(X)$ *for all* $p \in \mathrm{PROP}$, *and*
   - $V(\mathfrak{n}) \in \mathcal{N}$ *for all* $\mathfrak{n} \in \mathrm{NNS}$.

4. *A* hybrid subset space *is a triple $\mathcal{M} := (X, \mathcal{O}, V)$, where $(X, \mathcal{O})$ is a subset frame and V a hybrid valuation. $\mathcal{M}$ is said to* be based on $(X, \mathcal{O})$.

It should be stressed here that a valuation maps a propositional letter to a *subset of* X. As we will define the semantics of formulas with respect to *neighbourhood situations* immediately, this forces a special axiom involving only propositions. The set of validities can, therefore, not be closed under substitution.

For a given hybrid subset space we define now the satisfaction relation $\models$ between neighbourhood situations of the underlying frame and formulas in WFF.

Definition 2.2 (Satisfaction and validity) *Let $\mathcal{F} := (X, \mathcal{O})$ be a subset frame, $\mathcal{M} := (X, \mathcal{O}, V)$ a hybrid subset space based on $\mathcal{F}$, and x, U a neighbourhood situation of $\mathcal{F}$. Then we let*

$$\begin{array}{lll}
x, U \models_{\mathcal{M}} p & :\Longleftrightarrow & x \in V(p) \\
x, U \models_{\mathcal{M}} \mathfrak{n} & :\Longleftrightarrow & V(\mathfrak{n}) = x, U \\
x, U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & x, U \not\models_{\mathcal{M}} \alpha \\
x, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & \left\{ \begin{array}{l} x, U \models_{\mathcal{M}} \alpha \text{ and} \\ \quad x, U \models_{\mathcal{M}} \beta \end{array} \right. \\
x, U \models_{\mathcal{M}} K\alpha & :\Longleftrightarrow & y, U \models_{\mathcal{M}} \alpha \text{ for all } y \in U \\
x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & \left\{ \begin{array}{l} x, \hat{U} \models_{\mathcal{M}} \alpha \text{ for all} \\ \quad \hat{U} \in \mathcal{O} \text{ such that} \\ \qquad x \in \hat{U} \subset U \end{array} \right. \\
x, U \models_{\mathcal{M}} @_{\mathfrak{n}}\alpha & :\Longleftrightarrow & V(\mathfrak{n}) \models_{\mathcal{M}} \alpha,
\end{array}$$

for all $p \in \mathrm{PROP}$, $\mathfrak{n} \in \mathrm{NNS}$, and $\alpha, \beta \in \mathrm{WFF}$.

In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that α holds in $\mathcal{M}$ *at the neighbourhood situation x, U, and α is called* valid in $\mathcal{M}$ *iff it holds in $\mathcal{M}$ at every neighbourhood situation. (Manner of writing: $\mathcal{M} \models \alpha$.)*

[1] I am grateful to R. Ramanujam for corresponding questions there.

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Note that the meaning of propositional letters is regardless of neighbourhoods, thus stable with respect to $\Box$. Moreover, the relation $\subset$ used in the clause for $\Box$ means *proper* set inclusion. This is due to the special classes of frames we will be interested in later on in the paper, viz *dense* and *linear* ones.

Definition 2.3 (Dense and linear subset frames) *Let* $\mathcal{F} := (X, \mathcal{O})$ *be a subset frame. Then* $\mathcal{F}$ *is called*

1. dense, *iff for all* $U, \hat{U} \in \mathcal{O}$ *such that* $\hat{U} \subset U$ *there exists* $U' \in \mathcal{O}$ *satisfying* $\hat{U} \subset U' \subset U$;
2. linear, *iff for all* $U, \hat{U} \in \mathcal{O}$ *it holds that* $U \subset \hat{U}$ *or* $\hat{U} \subset U$ *or* $U = \hat{U}$.

We turn now to a corresponding hybrid logic of subset spaces. We briefly introduce the system to the reader, for convenience. More details can be found in the paper [7].[2] For a start, we list the bulk of the axiom system for hybrid logic that is given in [2, p 438 ff] and slightly adapted to the present bi-modal case. Note that all of the following axiom and rule schemata are sound on hybrid subset spaces.

1. $@_{\mathrm{n}}(\alpha \to \beta) \to (@_{\mathrm{n}}\alpha \to @_{\mathrm{n}}\beta)$
2. $@_{\mathrm{n}}\alpha \leftrightarrow \neg @_{\mathrm{n}}\neg\alpha$
3. $\mathrm{n} \wedge \alpha \to @_{\mathrm{n}}\alpha$
4. $@_{\mathrm{n}}\mathrm{n}$
5. $@_{\mathrm{n}}\mathrm{m} \wedge @_{\mathrm{m}}\alpha \to @_{\mathrm{n}}\alpha$
6. $@_{\mathrm{n}}\mathrm{m} \leftrightarrow @_{\mathrm{m}}\mathrm{n}$
7. $@_{\mathrm{m}}@_{\mathrm{n}}\alpha \leftrightarrow @_{\mathrm{n}}\alpha$,

where $\mathrm{n}, \mathrm{m} \in \mathrm{NNS}$ and $\alpha, \beta \in \mathrm{WFF}$. The axiom schema called (*back*) in [2] occurs twice here because we have to take into account both modalities, K and $\Box$ (viewed as K–modalities at the moment):

8. $L@_{\mathrm{n}}\alpha \to @_{\mathrm{n}}\alpha$
9. $\Diamond @_{\mathrm{n}}\alpha \to @_{\mathrm{n}}\alpha$.

The same is true for the rule (PASTE) so that we have three additional rules (besides *modus ponens* and *necessitation*):

$$\text{(NAME)}\ \frac{\mathrm{m} \to \beta}{\beta}$$

$$\text{(PASTE)}_K\ \frac{@_{\mathrm{n}}L\mathrm{m} \wedge @_{\mathrm{m}}\alpha \to \beta}{@_{\mathrm{n}}L\alpha \to \beta}$$

$$\text{(PASTE)}_\Box\ \frac{@_{\mathrm{n}}\Diamond\mathrm{m} \wedge @_{\mathrm{m}}\alpha \to \beta}{@_{\mathrm{n}}\Diamond\alpha \to \beta},$$

where $\alpha, \beta \in \mathrm{WFF}$, $\mathrm{n}, \mathrm{m} \in \mathrm{NNS}$, and m is 'new' each time.

The axioms and rules stated so far enable one to pursue both the naming and pasting technique of hybrid logic in the bi-modal case equally well; cf [2, p 441 ff]. This easily gives us completeness. But we can get even more: axiomatizing special properties of the accessibility relations belonging to K and $\Box$, respectively, by means of *pure* formulas, i.e., without using propositional letters, yields that the frame underlying the hybrid canonical model shares these properties; cf [2, Theorem 7.29]. This fact leads to *extended* hybrid completeness, which will be used heavily below.

The first application of that concerns the hybrid version of most of the usual modal axioms for subset spaces, cf [3]:

10. $\mathrm{n} \to L\mathrm{n}$
11. $LL\mathrm{n} \to L\mathrm{n}$
12. $L\mathrm{n} \to KL\mathrm{n}$
13. $(p \to \Box p) \wedge (\neg p \to \Box\neg p)$
14. $\Diamond\Diamond\mathrm{n} \to \Diamond\mathrm{n}$
15. $\Diamond L\mathrm{n} \to L\Diamond\mathrm{n}$,

where p and n vary across PROP and NNS, respectively. (We have left out, in particular, the distribution schemata for K and $\Box$, which were implicitly used above already.)

Some remarks are to be inserted here. First, the schemata 10 – 12 represent the hybrid axioms of knowledge, corresponding to the well-known modal S5. Second, Axiom 13 is *not* pure (and it is the only one with this property, apart from the distribution schemata). This axiom corresponds to evaluating propositional letters with respect to points only; cf Definition 2.2 and the remark subsequent to Definition 2.1. It turns out that this kind of non-purity does not cause any problems concerning hybrid completeness. Third, the axiom for reflexivity is missing; this must clearly be the case because we deal with *strict* inclusion of sets. But also the hybrid axiom of irreflexivity does not occur, as it proves to be a theorem of the system.

Because of purity Axioms 10 – 12 and 14 – 15, respectively, give us right off the following basic properties required of the accessibility relations $\xrightarrow{L}$ (induced by K) and $\xrightarrow{\Diamond}$ (induced by $\Box$) of the canonical model:

- the relation $\xrightarrow{L}$ is an equivalence,
- the relation $\xrightarrow{\Diamond}$ is transitive, and
- for all points s, t, u of the canonical model such that $s \xrightarrow{\Diamond} t \xrightarrow{L} u$ there exists some point v satisfying $s \xrightarrow{L} v \xrightarrow{\Diamond} u$. (Or, in other words,

$$\xrightarrow{\Diamond} \circ \xrightarrow{L} \subseteq \xrightarrow{L} \circ \xrightarrow{\Diamond}.)$$

[2] The observant reader will notice that there are slight differences between the presentations here and there. This is due to the fact that we confine ourselves to the linear case right from the beginning presently.

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The property stated in the previous item was introduced in [3] and called the *cross property* there. It is easy to see that this property corresponds to Axiom 15, actually. The cross property is, therefore satisfied on the hybrid canonical model.

The relations $\xrightarrow{L}$ and $\xrightarrow{\Diamond}$ should satisfy also the following more complex properties:

- *exclusivity:* $\forall\, w, v, u : \left(w \xrightarrow{\Diamond} v \Rightarrow \neg (w \xrightarrow{L} v) \right)$,
- *functionality:*
$$\forall\, w, v, u : \left(w \xrightarrow{\Diamond} v \wedge w \xrightarrow{\Diamond} u \xrightarrow{L} v \Rightarrow v = u \right),$$
- *injectivity:*
$$\forall\, w, v, u : \left(w \xrightarrow{\Diamond} u \wedge v \xrightarrow{\Diamond} u \wedge w \xrightarrow{L} v \Rightarrow w = v \right),$$
- *faithfulness:* $\forall\, w, v, u, t :$
$$w \xrightarrow{\Diamond} v \wedge w \xrightarrow{L} u \xrightarrow{\Diamond} t \wedge \forall\, x : \neg \left(v \xrightarrow{L} x \xrightarrow{\Diamond} t \right) \Rightarrow \exists\, s, r : w \xrightarrow{L} s \xrightarrow{\Diamond} r \xrightarrow{L} t \wedge \forall\, y : \neg \left(s \xrightarrow{\Diamond} y \xrightarrow{L} v \right),$$
- *factor property:* $\forall\, w, v, u, t :$
$$w \xrightarrow{\Diamond} v \wedge w \xrightarrow{L} u \xrightarrow{\Diamond} t \xrightarrow{\Diamond} v \Rightarrow \exists\, s : \left(w \xrightarrow{\Diamond} s \xrightarrow{\Diamond} v \wedge s \xrightarrow{L} t \right),$$
- *authenticity:* $\forall\, w, v :$
$w \xrightarrow{\Diamond} v \Rightarrow \exists\, u : \left(w \xrightarrow{L} u \wedge \forall\, t : \neg (u \xrightarrow{\Diamond} t \xrightarrow{L} v) \right)$;

for, by means of these properties we are able to ensure a subset space structure on the canonical model and position the denotation of every nominal unequivocally there. And in fact, each of them can be captured by a pure correspondent:

16. $\Diamond \mathrm{n} \rightarrow \neg L \mathrm{n}$
17. $\Diamond \mathrm{n} \wedge \Diamond (\mathrm{m} \wedge L\mathrm{n}) \rightarrow \Diamond (\mathrm{n} \wedge \mathrm{m})$
18. $\neg \mathrm{n} \wedge \Diamond \mathrm{m} \rightarrow K(\mathrm{n} \rightarrow \neg \Diamond \mathrm{m})$
19. $\Diamond (\mathrm{n} \wedge \neg \Diamond L\mathrm{m}) \wedge L \Diamond \mathrm{m} \rightarrow L(\Diamond L\mathrm{m} \wedge \neg \Diamond L\mathrm{n})$
20. $\Diamond \mathrm{m} \wedge L \Diamond (\mathrm{n} \wedge \Diamond \mathrm{m}) \rightarrow \Diamond (L\mathrm{n} \wedge \Diamond \mathrm{m})$
21. $\Diamond \mathrm{n} \rightarrow L \neg \Diamond L \mathrm{n}$.

It remains to provide for density and linearity of the desired hybrid subset space. For this purpose we can take the formulas expressing density of the relation $\xrightarrow{\Diamond}$ and trichotomy of the composite relation $\xrightarrow{L} \circ \xrightarrow{\Diamond}$, respectively:

22. $\Diamond \mathrm{n} \rightarrow \Diamond \Diamond \mathrm{n}$
23. $@_{\mathrm{m}} L \Diamond \mathrm{n} \vee @_{\mathrm{m}} L\mathrm{n} \vee @_{\mathrm{n}} L \Diamond \mathrm{m}$.

All in all, the above announced soundness and completeness theorem results via a suitable *truth lemma* (not stated here):

Theorem 2.4 (Soundness and Completeness) *Let the above schemata of axioms and rules constitute the logical system* DL. *Then, a formula* $\alpha \in$ WFF *is* DL*–derivable, iff it is valid in every hybrid subset space based on a dense and linear subset frame.*

3 Decidability for Dense Linear Time

The aim of this main section of the paper is to prove that the DL–satisfiability problem is decidable. To this end we revisit first the comparison language significant to our problem, viz the monodic guarded fragment of first-order temporal logic with equality. Then we embed DL suitably and show that in fact a reduction of the respective satisfiability problems is induced. This point is decisive for the desired decidability.

The first-order syntax needed here is built in the usual way on

- the equality symbol $\doteq$,
- three pairwise disjoint sets of unary predicate symbols: $\{P\}$, $\{P_p \mid p \in \mathrm{PROP}\}$ and $\{P_{\mathrm{n}} \mid \mathrm{n} \in \mathrm{NNS}\}$,
- a set of individual variables: $\{x_i \mid i \in \mathbb{N}\}$,
- a set of individual constants $\{c_{\mathrm{n}} \mid \mathrm{n} \in \mathrm{NNS}\}$,
- the boolean connectives $\neg$ and $\wedge$,
- the universal quantifier $\forall x$ for each individual variable x,
- the temporal operators $\mathcal{U}$ (*until*) and $\mathcal{S}$ (*since*).

Let TLF designate the corresponding set of formulas. We will use, in particular, the following abbreviations:

$$\begin{array}{rcl} \Diamond_{\mathrm{f}} \alpha :\equiv \top \mathcal{U} \alpha & \text{and} & \Diamond_{\mathrm{p}} \alpha :\equiv \top \mathcal{S} \alpha, \\ \Box_{\mathrm{f}} \alpha :\equiv \neg \Diamond_{\mathrm{f}} \neg \alpha & \text{and} & \Box_{\mathrm{p}} \alpha :\equiv \neg \Diamond_{\mathrm{p}} \neg \alpha, \\ \Diamond \alpha :\equiv \Diamond_{\mathrm{f}} \alpha \vee \Diamond_{\mathrm{p}} \alpha & \text{and} & \Box \alpha :\equiv \Box_{\mathrm{f}} \alpha \wedge \Box_{\mathrm{p}} \alpha, \\ \Diamond^{+} \alpha :\equiv \alpha \vee \Diamond \alpha & \text{and} & \Box^{+} \alpha :\equiv \alpha \wedge \Box \alpha. \end{array}$$

Note that it is sufficient for our purposes to have merely *unary* predicate symbols to hand.

A *first-order temporal model* is a triple $\mathfrak{M} := (\mathfrak{L}, D, \mathfrak{I})$ where $\mathfrak{L} := (T, <)$ is a strict linear order, D a non-empty set and $\mathfrak{I}$ a mapping assigning to every time $t \in T$ a realization $\mathfrak{I}_t$ of the predicate symbols on D. D is called the *domain* of $\mathfrak{M}$. Note that the domain remains constant in the course of time; moreover, the individual constants are interpreted rigidly whereas the interpretation of the predicate symbols may vary.

The evaluation of a TLF–formula in a model $\mathfrak{M} = (\mathfrak{L}, D, \mathfrak{I})$ depends on *assignments* giving variables a value from D. The corresponding satisfaction relation $\models^{\mathfrak{a}}$, $\mathfrak{a}$ an assignment, is defined recursively with respect to points of time.[3] We only mention the clause for *until* here:

$$t \models^{\mathfrak{a}} \alpha \mathcal{U} \beta : \Longleftrightarrow \exists u : \begin{cases} t < u \wedge u \models^{\mathfrak{a}} \beta \wedge \forall v : \\ t < v < u \Rightarrow v \models^{\mathfrak{a}} \alpha \end{cases}$$

According to [9] a formula $\alpha \in$ TLF is called *monodic,* iff every subformula of α of the form $\beta \mathcal{U} \gamma$ or $\beta \mathcal{S} \gamma$ contains at most one free variable. And α is called *guarded,* iff it is built from the atomic formulas using $\neg, \wedge, \mathcal{U}, \mathcal{S}$ and $\forall$–quantification, but the latter restricted in the following way actually:

- if $\overline{x}, \overline{y}$ are tuples of variables, $\gamma(\overline{x}, \overline{y})$ is atomic, $\beta(\overline{x}, \overline{y})$ is guarded, and every variable occurring free in $\beta(\overline{x}, \overline{y})$ also occurs in $\gamma(\overline{x}, \overline{y})$, then

  $$\forall \overline{y} \left(\gamma(\overline{x}, \overline{y}) \rightarrow \beta(\overline{x}, \overline{y})\right)$$

  is guarded as well; cf [9, Definition 72].

Now let

$$\mathrm{TLF}^1 := \{\alpha \in \mathrm{TLF} \mid \alpha \text{ monodic}\}$$

and

$$\mathrm{TLF}^{1\mathrm{g}} := \{\alpha \in \mathrm{TLF}^1 \mid \alpha \text{ guarded}\}.$$

Then the *monodic guarded fragment of first-order temporal logic with equality over dense flows of time,* abbreviated

$$\mathit{TLF}_d^{1g},$$

is the set of formulas from $\mathrm{TLF}^{1\mathrm{g}}$ that are true at every time of all first-order temporal models $\mathfrak{M} = (\mathfrak{L}, D, \mathfrak{I})$ such that $\mathfrak{L} = (T, <)$ is dense, for all assignments. We will use below the following result stated in [10, Theorem 13], which originates from [8]:

Theorem 3.1 *TLF_d^{1g} is decidable.*

We define next a 'non-standard translation' from the hybrid language for subset spaces to the set of monodic guarded TLF–formulas.

Definition 3.2 (Translation function) *Fix an individual variable x. Let a mapping*

$$NT_x : \mathrm{WFF} \longrightarrow \mathrm{TLF}^{1\mathrm{g}}$$

be given recursively by

$$\begin{array}{lcl}
NT_x(p) & :\Longleftrightarrow & P_p(x) \\
NT_x(\neg\alpha) & :\Longleftrightarrow & \neg NT_x(\alpha) \\
NT_x(\alpha \wedge \beta) & :\Longleftrightarrow & NT_x(\alpha) \wedge NT_x(\beta) \\
NT_x(\mathfrak{n}) & :\Longleftrightarrow & P_{\mathfrak{n}}(x) \\
NT_x(K\alpha) & :\Longleftrightarrow & \begin{cases} P(x) \wedge \\ \forall y : (P(y) \rightarrow NT_y(\alpha)) \end{cases} \\
NT_x(\Box\alpha) & :\Longleftrightarrow & \Box_{\mathrm{f}}\,(P(x) \rightarrow NT_x(\alpha)) \\
NT_x(@_{\mathfrak{n}}\alpha) & :\Longleftrightarrow & \begin{cases} P(x) \wedge \Diamond^{+} \\ (P_{\mathfrak{n}}(c_{\mathfrak{n}}) \wedge NT_x(\alpha)\,[c_{\mathfrak{n}}/x]) \end{cases}
\end{array}$$

where $p \in$ PROP, $\mathfrak{n} \in$ NNS, $\alpha, \beta \in$ WFF, y is 'fresh' for α, and $[c_{\mathfrak{n}}/x]$ means substitution of x with $c_{\mathfrak{n}}$.

Compared to the standard translation of hybrid (modal) logic, NT_x is non-standard in three respects actually. First, the translation of nominals and formulas involving a jump operator is quite different from usual hybrid logic; cf [1, 3.4]. Second, the accessibility relation belonging to K is encoded by a *unary* predicate symbol here; this is due to the fact that subsets are nothing but unary relations. Finally, the accessibility relation belonging to $\Box$ is to be essentially the flow of time under consideration; thus $\Box$ 'remains untranslated'.

Note that the computation of the translation of a WFF–formula can take exponential time in the worst case. This is due to the operator $\Diamond^{+}$ occurring in the clause for $@_{\mathfrak{n}}\alpha$ in Definition 3.2.

It is not hard to formulate and prove a *correspondence lemma* à la [2, 2.47] for hybrid subset spaces based on linear frames. We do not carry out that in this paper, but see the proof of Lemma 3.3 below.

Now we stipulate the required conditions on first-order temporal models. As it turns out, this can really be done by means of $\mathrm{TLF}^{1\mathrm{g}}$–formulas:

(p–PERSISTENCY)

$$\Box^{+} \forall x_0 : (P_p(x_0) \rightarrow \Box\, P_p(x_0))\,.$$

This formula expresses the fact that the truth value of $p \in$ PROP does not change in the course of time.

The next formula captures single-valuedness of the denotation of a nominal $\mathfrak{n}$:[4]

($\mathfrak{n}$–UNIQUENESS)

$$\Box^{+} \forall x_0 : (P_{\mathfrak{n}}(x_0) \rightarrow \Box \neg P_{\mathfrak{n}}(x_0) \wedge x_0 \doteq c_{\mathfrak{n}}) \\ \wedge \Diamond^{+} P_{\mathfrak{n}}(c_{\mathfrak{n}}).$$

[3] Denoting this relation we suppress its dependence on $\mathfrak{M}$.

[4] One of the referees pointed a possible simplification concerning this part of the translation into monodic logic out to us.

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Note that we use equality only in this place.

Finally, the formula forcing (strict) shrinking of sets reads

(SHRINKING)

$$\Box^+ \forall x_0 : (P(x_0) \to \Box_{\mathrm{p}} P(x_0)) \wedge$$
$$\Box^+ \exists x_1 : (P(x_1) \wedge \Box_{\mathrm{f}} \neg P(x_1)) .$$

Let

$$\delta_p :\equiv (p\text{–PERSISTENCY}),$$

$$\delta_{\mathfrak{n}} :\equiv (\mathfrak{n}\text{–UNIQUENESS}),$$

and

$$\delta :\equiv (\text{SHRINKING}).$$

We have then the following *reduction lemma.*

Lemma 3.3 *A formula $\alpha \in$ WFF is satisfiable in a hybrid subset space based on a dense and linear subset frame, iff there exists a first-order temporal model over a dense flow of time validating the formula*

$$\exists x NT_x(\alpha) \wedge \chi \in \mathrm{TLF}^{1\mathrm{g}}$$

at some point of time, where

$$\chi :\equiv \delta \wedge \bigwedge_{p \text{ occurring in } \alpha} \delta_p \wedge \bigwedge_{\mathfrak{n} \text{ occurring in } \alpha} \delta_{\mathfrak{n}} .$$

Proof. First assume that there is a dense and linear subset frame $\mathcal{F} = (X, \mathcal{O})$, a hybrid subset space $\mathcal{M} = (X, \mathcal{O}, V)$ based on $\mathcal{F}$, and a neighbourhood situation y, U of $\mathcal{F}$ such that $y, U \models_{\mathcal{M}} \alpha$. Then we define a first-order temporal model $\mathfrak{M} = (\mathfrak{L}, D, \mathfrak{I})$ over a dense flow of time as follows. We let

- $\mathfrak{L} := (\mathcal{O}, \supset)$,
- $D := X$,
- $c_{\mathfrak{n}}^{\mathfrak{M}} := z$ where z is determined by

$$V(\mathfrak{n}) = z, \hat{U} \text{ for some } \hat{U} \in \mathcal{O},$$

  and
- $\mathfrak{I}$ given by
  - $\mathfrak{I}_U(P) := U$,
  - $\mathfrak{I}_U(P_p) := V(p)$,
  - $\mathfrak{I}_U(P_{\mathfrak{n}}) := \begin{cases} \{c_{\mathfrak{n}}^{\mathfrak{M}}\} & \text{if } V(\mathfrak{n}) = c_{\mathfrak{n}}^{\mathfrak{M}}, U \\ \emptyset & \text{otherwise,} \end{cases}$

for all $U \in \mathcal{O}$, $p \in$ PROP and $\mathfrak{n} \in$ NNS. We get from this that

$$U \models^{\mathfrak{a}} NT_x(\alpha) \wedge \chi ,$$

for all assignments $\mathfrak{a}$ mapping x to y. This proves the left-to-right direction.

As to the opposite direction, suppose that

$$\exists x NT_x(\alpha) \wedge \chi$$

is satisfied at some time t_0 of a first-order temporal model $\mathfrak{M} = (\mathfrak{L}, D, \mathfrak{I})$ over a dense flow $\mathfrak{L} = (T, <)$. We define the components of a corresponding hybrid subset space, letting

- $X := D$,
- $\mathcal{O} := \{\mathfrak{I}_t(P) \mid t \in T\}$,
- $V(p) := \begin{cases} \mathfrak{I}_{t_0}(P_p) & \text{if } p \text{ occurs in } \alpha \\ \emptyset & \text{otherwise} \end{cases}$

  for all $p \in$ PROP, and
- $V(\mathfrak{n}) := \begin{cases} c_{\mathfrak{n}}^{\mathfrak{M}}, \mathfrak{I}_t(P) & \text{if } \begin{cases} \mathfrak{I}_t(P_{\mathfrak{n}}) \neq \emptyset \\ \text{and} \\ \mathfrak{n} \text{ occurs in } \alpha \end{cases} \\ \text{arbitrary} & \text{if } \mathfrak{n} \text{ does not occur in } \alpha \end{cases}$

  for all $\mathfrak{n} \in$ NNS.

According to the validity of χ the mapping V is well-defined, and the structure

$$\mathcal{M} := (X, \mathcal{O}, V)$$

is really based on a dense and linear subset frame. Furthermore, we have

$$y, \mathfrak{I}_{t_0}(P) \models_{\mathcal{M}} \alpha$$

where y witnesses the existence asserted by $\exists x NT_x(\alpha)$. Thus we obtain that α is DL–satisfiable, as desired. □

Since

$$\alpha \mapsto \exists x NT_x(\alpha) \wedge \chi$$

can be realized by a *computable* function we get in fact a reduction of the set of satisfiable WFF–formulas to the set of $\mathrm{TLF}^{1\mathrm{g}}$–formulas satisfiable in a first-order temporal model over a dense flow of time. Thus, combining Theorem 3.1 and Lemma 3.3 yields the following theorem, which is the main issue of this paper.

Theorem 3.4 *The* DL*–satisfiability problem is decidable.*

A corresponding outcome for the original modal case can be derived from that without any bother.

Corollary 3.5 *The modal theory of dense linear subset spaces is decidable.*

4 Concluding Remarks

Disregarding nominals and jump operators we could also have proceeded directly, i.e., as for the richer language in the preceeding section (and even without using *since*) in order to obtain the result stated in Corollary 3.5. In this case equality is actually not necessary, the comparison language is the monodic *monadic* fragment of first-order temporal logic, and decidability follows from [9, Theorem 71].

It can be expected that effectivity theorems concerning monodic fragments of first-order temporal logics have further applications to open decidability questions, not least of the modal logic of subset spaces.

The *complexity* of DL has to be determined by future research. In connection with this it should be mentioned that the satisfiability problem of *linear tense logic* over dense flows of time is NP–complete; cf [13, Theorem 8]. Maybe this is true for the modal logic of dense linear subset spaces as well.

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References

[1] P. Blackburn. Representation, Reasoning, and Relational Structures: a Hybrid Logic Manifesto. *Logic Journal of the IGPL*, 8:339–365, 2000.

[2] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 2001.

[3] A. Dabrowski, L. S. Moss, and R. Parikh. Topological Reasoning and The Logic of Knowledge. *Annals of Pure and Applied Logic*, 78:73–110, 1996.

[4] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning about Knowledge*. MIT Press, Cambridge, MA, 1995.

[5] K. Georgatos. Knowledge on Treelike Spaces. *Studia Logica*, 59:271–301, 1997.

[6] B. Heinemann. About the Temporal Decrease of Sets. In C. Bettini and A. Montanari, editors, *Temporal Representation and Reasoning, 8th International Workshop, TIME-01*, pages 234–239, Los Alamitos, CA, 2001. IEEE Computer Society Press.

[7] B. Heinemann. Knowledge over Dense Flows of Time (from a Hybrid Point of View). In M. Agrawal and A. Seth, editors, *FST TCS 2002: Foundations of Software Technology and Theoretical Computer Science*, volume 2556 of *Lecture Notes in Computer Science*, pages 194–205, Berlin, 2002. Springer.

[8] I. Hodkinson. Monodic Packed Fragment with Equality is Decidable. *Studia Logica*, 72(2):185–197, 2002.

[9] I. Hodkinson, F. Wolter, and M. Zakharyaschev. Decidable Fragments of First–Order Temporal Logics. *Annals of Pure and Applied Logic*, 106(1–3):85–134, 2000.

[10] I. Hodkinson, F. Wolter, and M. Zakharyaschev. Monodic Fragments of First–Order Temporal Logics: 2000–2001 A.D. In R. Nieuwenhuis and A. Voronkov, editors, *Logic for Programming, Artificial Intelligence and Reasoning*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 1–23, Berlin, 2001. Springer.

[11] J.-J. C. Meyer and W. van der Hoek. *Epistemic Logic for AI and Computer Science*, volume 41 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 1995.

[12] L. S. Moss and R. Parikh. Topological Reasoning and The Logic of Knowledge. In Y. Moses, editor, *Proceedings of the 4th Conference on Theoretical Aspects of Reasoning about Knowledge (TARK 1992)*, pages 95–105, San Francisco, CA, 1992. Morgan Kaufmann.

[13] H. Ono and A. Nakamura. On the Size of Refutation Kripke Models for Some Linear Modal and Tense Logics. *Studia Logica*, 39(4):325–333, 1980.

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