Using Temporal Logic for Spatial Reasoning: Spatial Propositional Neighborhood Logic *

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Abstract

It is widely accepted that spatial reasoning plays a central role in artificial intelligence, for it has a wide variety of potential applications, e.g., in robotics, geographical information systems, medical analysis and diagnosis. As noticed by many authors, spatial and temporal reasoning have a close connection. In this paper we propose a new, semidecidable, modal logic for spatial reasoning through directional relations, which is able to express meaningful spatial statements. Spatial Propositional Neighborhood Logic can be polynomially reduced to a decidable temporal logic based on time intervals preserving, at least, valid formulas. Thanks to such a reduction, we are able to re-use a sound and complete tableaux method in order to reason with Spatial Propositional Neighborhood Logic; to the best of our knowledge, there are practically no previous attempts of devising automatic reasoning methods for spatial reasoning.

1 Introduction

It is widely accepted that spatial reasoning plays a central role in artificial intelligence, for it has a wide variety of potential applications e.g. in robotics, geographical information systems, medical analysis and diagnosis. As for other qualitative reasoning formalisms (e.g., temporal reasoning), spatial reasoning can be viewed under three different, somehow complementary, points of view. We may distinguish between the *algebraic* level, that is, purely existential theories formulated as constraint satisfaction systems over jointly exclusive and mutually disjoint set of topological, directional, or combined relations; the *first-order* level, that is, first-order theories of topological, directional, or combined relations; and the *modal logic* level, where a (usually propositional) modal language is interpreted over opportune Kripke structures representing space. For a comprehensive survey on the various formalisms (topological, directional, and combined constraint systems and relations) see [8].

As for modal logics for spatial reasoning, it is worth mentioning Bennet's work [3, 4], later extended by Bennet, Cohn, Wolter and Zakharyaschev in [5]. In [3], Bennet proposes to interpret regions as subsets of the topological space, and shows how it is possible to exploit both the classical propositional calculus and the intuitionistic propositional calculus, together with certain meta-level constraint concerning entailments between formulas, for reasoning about space with topological relations. In such a way, a spatial topological constraints problem can be solved by checking the satisfiability of a logical formula. In [4] Bennet extends his approach by the use of modal languages. Bennet takes into consideration the modal logic S4, and interprets the modal operator in a topological sense, as the interior operator of a given topology. Moreover, in the same work, a modal convex-hull operator is defined and studied, by translating a first-order axiomatization into a modal schemata. In [5], the authors consider a multi-modal system for spatio-temporal reasoning, based on Bennet's previous work. Fur-

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ther research on this issue can be found in [14], where Nutt gives a rigorous foundation of the translation of topological relations into modal logic, introducing generalized topological relations. It is worth to point out that Bennet, Cohn, Wolter, Zakharyaschev, and Nutt's results basically exploit the finite model property and decidability of the classical propositional logic, the modal logic S4, and of some of their extensions. For a recent investigation concerning the major mathematical theories of space from a modal standpoint, see [1].

Unlike the above work, an important attempt to exploit the whole expressive power of modal logic for reasoning about space (instead of using it for constraint solving) is that of Lutz and Wolter's modal logic for topological relations [11]. Lutz and Wolter present a new propositional modal logic, where propositional variables are interpreted in the regions of topological space, and references to other regions are enabled by modal operators interpreted as topological relations. There are many possible choices for the set of relations. The set RCC8, for example, contains the relations *equal* (eq), *disconnected* (di), *externally connected* (ec), *tangential proper part* (tpp), *inverse of tangential proper part* ($tppi$), *non-tangential proper part* ($ntpp$), *inverse of non-tangential proper part* ($ntppi$), and *partially overlap* (po). Among other possibilities, we mention a refinement of RCC8 into 23 relations, and the set RCC5, obtained from RCC8 by keeping the relations eq and po, but coarsening the relations tpp and $ntpp$ into a new relation (*proper part*), the relations $tppi$ and $ntppi$ into a new relation (*inverse of proper part*), and the relations ec and dc into a new relation *disconnected* (see e.g. [8] for a detailed discussion). Lutz and Wolter's modal logic of topological relations has been studied either for the set RCC8 and the set RCC5, and for different choices of the topological space and the set of regions. For the matter of a comparison with the present work, Lutz and Wolter have shown that the satisfiability problem for the modal logic of RCC8 relations interpreted in the Euclidean space $\mathbb{R}^n$ ($n > 1$), where $\mathbb{R}$ is the set of real numbers, when the set of basic regions is exactly the set of all (hyper)-rectangles on it is not even recursively enumerable. Moreover, it is quite simple to see that such an undecidability result holds also in the Euclidean space $\mathbb{D}^n$ ($n > 1$), where $\mathbb{D}$ is any linearly ordered set. This means that it is not possible even to devise a semi-decidability method for it. As for directional relations, on the other hand, we mention Venema's Compass Logic introduced in [16] and further studied in [12]. Compass logic features four modal operators, namely $\diamondright$, $\diamondminus$, $\diamondup$, and $\diamonddown$, and propositional variables are interpreted as points in the Euclidean two-dimensional space. The modalities are interpreted as the natural *north*, *south*, *east*, and *west* relations between two given points. As for example, given a point with coordinates (d_x, d_y) such that p holds on it, one is able to reach a point with coordinates (d'_x, d_y), where $d_x < d'_x$, such that q holds on it, by the formula $p \wedge \diamondright q$. In [12], Marx and Reynolds show that Compass Logic is undecidable even in the class of all two-dimensional frames. Moreover, Balbiani, Condotta, and Del Cerro [2] study an algebra called Rectangle Algebra (RA) which allows one to express any relation between two rectangles in an Euclidean space $\mathbb{D}^2$. To our knowledge, the set of all 169 relations between any two rectangles has not been studied at the logical level. Nevertheless, it easy to see that the natural propositional modal logic based on RA is not recursively enumerable at least when interpreted in the same classes of frames as Lutz and Wolter's modal logic of topological relations. Indeed, by a straightforward translation it is possible to express the RCC8 relations in full RA, which means that the modal logic of topological relations of the topological space of all rectangles in the Euclidean space $\mathbb{D}^2$ is a fragment of the modal logic based on RA.

In this work we present a new modal logic for reasoning about two-dimensional space by means of directional relations. Regions are approximated by their minimum bounding box, and four modal operators allow one to move along the x- and the y-axis. Spatial Propositional Neighborhood Logic is presented here, and the semi-decidability of its validity problem is shown by means of a polynomial translation into a decidable modal logic for temporal intervals over linearly ordered sets. Such a translation preserves valid formulas, and, thus, it results a semi-decidability method. The logic used for the the translation is a well-studied propositional logic for temporal interval, and, among other results, a sound, complete (non-terminating) tableaux method for it has been presented. Our translation allows one to use such a tableaux method for spatial reasoning. It is worth noticing that comparing our approach with the previous (modal) ones presents some difficulties, basically because most of the previous work is based on topological relations instead of directional relations. Nevertheless, the expressive power of SpPNL can be compared with Lutz and Wolter's modal logic of topological relation in the Euclidean space $\mathbb{D}^2$; to some extent, SpPNL can be considered as a fragment of it.

The present paper is organized as follows. In the next section we briefly discuss some initial choices, and in Section 3 we present the syntax and semantics of SpPNL. Some simple examples of possible applications are presented in Section 4. Section 5 is centered on the main result, while in the last section we conclude and discuss possible future developments of this work.

2 Some Initial Choices

As for what concerns basic objects, in this paper we consider *extended* objects, instead of points. According to

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the majority of the authors, we define a *region* in the two-dimensional space as a *closed, connected and with connected boundaries* non-empty set of points in $\mathbb{D}^2$, or a *finite union* of such sets. Notice that such a general definition allow us to model, for example, the variety and complexity of geographical entities, allowing holes and and disconnected regions. A common way to deal with regions is by approximating them by theirs minimum bounding boxes. The *minimum bounding box* (MBB) of a region is the rectangle identified by the pair $\langle(min(x), min(y)), (max(x), max(y))\rangle$, where $min(x)$ (resp., $max(x)$) is the minimum (resp., maximum) point of the projection on the x-axis (and, similarly, for the y-axis) of the considered region. For us, a *basic object* will be the MBB of a region. There are 169 possible different relations (see Figure 1) between two given MBBs [10]. As noticed in [15], MBBs suffer of an important limitation for real application, basically because by approximating two objects with their MBBs it is not always guaranteed that the topological relation between them is respected. Nevertheless, most work about directional relation has been focused on MBBs, and in many practical applications MBBs can be considered as the real objects of interest instead of a simple approximation of them. Moreover, as we have seen above, spatial reasoning can be tackled at least by means of two distinct categories of relations, namely *topological* relations and *directional* relations. In the topological approach, a topological space is considered, that is, a pair $\mathcal{T} = \langle U, I\rangle$, where U is a set and I is the *interior operator*, which meets opportune closure properties. Given any two subsets $O_1, O_2 \subseteq I$, one can ask himself which is the binary relation holding between them. Typical sets of relations are RCC8 and RCC5. On the contrary, in the present work we focus our attention on directional relations. Thus, we fix a (bidimensional) *reference frame* $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$, and we consider the possible cardinal relations between any two rectangles, in order to describe where such rectangles are placed relative to one another; among all possible cardinal relations, we consider the four relations *east*, *west*, *north*, and *south*.

3 Syntax and Semantics of SpPNL

The language for SpPNL consists of a set of propositional variables $\mathcal{AP}$, the logical connectives $\neg$ and $\vee$, and the modalities $\langle \mathrm{E}\rangle, \langle \mathrm{W}\rangle, \langle \mathrm{N}\rangle, \langle \mathrm{S}\rangle$ (*east*, *west*, *north*, and *south*). The other logical connectives, as well as the logical constants $\top$ and $\bot$, can be defined in the usual way. As for example, $\phi \wedge \psi \equiv \neg(\neg\phi \vee \neg\psi)$, and $\top \equiv p \vee \neg p$ for any $p \in \mathcal{AP}$. SpPNL well formed *formulas*, denoted by $\phi, \psi, \ldots$, are recursively defined as follows (where $p \in \mathcal{AP}$):

$$\phi = p \mid \neg\phi \mid \phi \vee \psi \mid \langle \mathrm{E}\rangle\phi \mid \langle \mathrm{W}\rangle\phi \mid \langle \mathrm{N}\rangle\phi \mid \langle \mathrm{S}\rangle\phi.$$

Given any linearly ordered set $\mathbb{D} = \langle D, <\rangle$, we call *spatial frame* the pair $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$, and we denote by $\mathbb{O}(\mathbb{D})$ the set of all *objects*, that is, $\mathbb{O}(\mathbb{D}) = \{\langle(d_x, d_y), (d'_x, d'_y)\rangle \mid d_x < d'_x,\ d_y < d'_y,\ d_x, d'_x,\ d_y, d'_y \in D\}$. The semantics of SpPNL is given in terms of *models* of the type $M = \langle \mathbb{D}^2, \mathbb{O}(\mathbb{D}), V\rangle$, where $\mathbb{D}^2$ is a spatial frame, and $V : \mathbb{O}(\mathbb{D}) \mapsto 2^{\mathcal{AP}}$ is a *spatial valuation function*, assigning to any object the set of all and only propositional letters which are true on that object. The *truth* relation for a well formed SpPNL-formula ϕ in a model M and an object $\langle(d_x, d_y), (d'_x, d'_y)\rangle$ is given by the following clauses:

- $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash p$ if and only if $p \in V(\langle(d_x, d_y), (d'_x, d'_y)\rangle)$, for any $p \in \mathcal{AP}$;
- $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash \neg\phi$ if and only if it is not the case that $M, \langle(d_x, d'_x), (d_y, d'_y)\rangle \Vdash \phi$;
- $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash \phi \vee \psi$ if and only if $M, \langle(d_x, d_y),\ (d'_x, d'_y)\rangle \Vdash \phi$ or $M, \langle(d_x, d'_x), (d_y, d'_y)\rangle \Vdash \psi$;
- $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash \langle \mathrm{E}\rangle\phi$ if only if there exists $d''_x \in \mathbb{D}$ such that $d'_x < d''_x$, and $M, \langle(d'_x, d_y), (d''_x, d'_y)\rangle \Vdash \phi$;
- $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash \langle \mathrm{W}\rangle\phi$ if only if there exists $d''_x \in \mathbb{D}$ such that $d''_x < d_x$, and $M, \langle(d''_x, d_y), (d_x, d'_y)\rangle \Vdash \phi$;
- $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash \langle \mathrm{N}\rangle\phi$ if only if there exists $d''_y \in \mathbb{D}$ such that $d'_y < d''_y$, and $M, \langle(d_x, d'_y), (d'_x, d''_y)\rangle \Vdash \phi$;
- $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash \langle \mathrm{S}\rangle\phi$ if only if there exists $d''_y \in \mathbb{D}$ such that $d''_y < d'_y$, and $M, \langle(d_x, d''_y), (d'_x, d_y)\rangle \Vdash \phi$.

As usual, we denote by $[\mathrm{X}]$ the dual operator of the modality $\langle X\rangle$, where $\langle X\rangle \in \{\langle \mathrm{E}\rangle, \langle \mathrm{W}\rangle, \langle \mathrm{N}\rangle, \langle \mathrm{S}\rangle\}$, and by $M \Vdash \phi$ the fact that ϕ is *valid* on M.

As an aside, it is interesting to notice that Spatial Propositional Neighborhood Logic can be studied in a slightly different version, that we call Non-Strict Spatial Propositional Neighborhood Logic (SpPNL$^+$). The mail difference with the above system is that in the non-strict version the objects can be also lines and points. In the non-strict version, it is useful to have three modal constant π_p, π_h, π_v which hold, respectively, only over points, over horizontal lines, and over vertical lines, that is:

- $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash \pi_p$ if and only if $d_x = d'_x$ and $d_y = d'_y$;
- $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash \pi_h$ if and only if $d_x < d'_x$ and $d_y = d'_y$;

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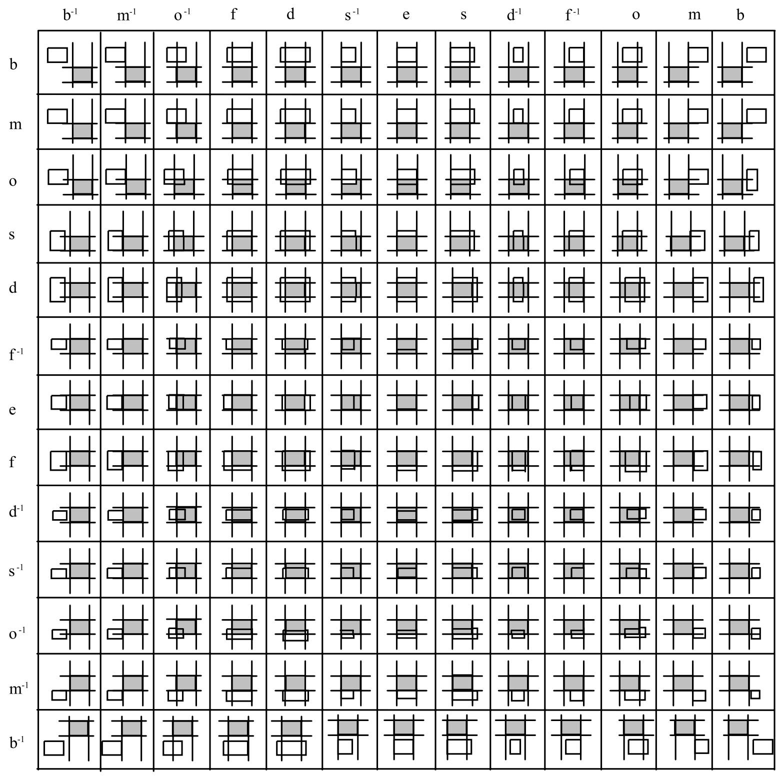


Figure 1. The basic relations between two rectangles.

- $M, \langle (d_x, d_y), (d'_x, d'_y) \rangle \Vdash \pi_v$ if and only if $d_x = d'_x$ and $d_y < d'_y$.

In what follows, we will concentrate on SpPNL only, except for Section 5.3 below.

4 Expressive Power of Spatial Propositional Neighborhood Logic

It is easy to see that in SpPNL, only 25 out of 169 possible basic RA-relations are directly expressible (see Fig. 2). Nevertheless, despite its simplicity, SpPNL is surprisingly powerful. Consider the following shorthand:

$$\begin{aligned} hor(\phi) \quad = \quad & [\mathrm{W}][\mathrm{W}][\mathrm{E}]\phi \wedge [\mathrm{W}][\mathrm{E}][\mathrm{E}]\phi \wedge \\ & [\mathrm{E}][\mathrm{E}][\mathrm{W}]\phi \wedge [\mathrm{E}][\mathrm{W}][\mathrm{W}]\phi \end{aligned}$$

The operator $hor(\phi)$ states that the formula ϕ holds everywhere (i.e., in any rectangle) such that its y-coordinate is the same as that of the current rectangle. Similarly, an operator $ver(\phi)$ can be defined. This means that in SpPNL it is possible to express the *difference* operator:

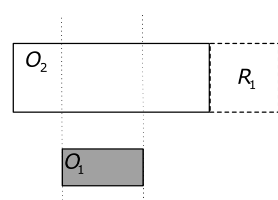
$$[\neq](\phi) = hor(ver(\phi) \wedge \phi),$$

and, thus, to simulate the *universal modality* and *nominals*:

$$u(\phi) = \phi \wedge [\neq]\phi \;\text{ and }\; n(p) = p \wedge [\neq](\neg p),$$

where $n(p)$ states that p holds in the current rectangle and nowhere else.

By exploiting the above definitions, we are able to express non-trivial spatial statements. Suppose for example that the rectangular objects O_1 and O_2 are placed in an Euclidean space as shown below:



The propositional variable denoted by R_1 represents the (simulation of a) nominal that can be used in order to express the desired relation. Consider the formula $\phi(O_1, O_2) = p(O_2) \rightarrow \langle \mathrm{E} \rangle\, n(p_{R_1}) \wedge \langle \mathrm{W} \rangle\, \langle \mathrm{E} \rangle \langle \mathrm{E} \rangle\, (\langle \mathrm{E} \rangle \langle \mathrm{E} \rangle\, n(p_{R_1}) \wedge\, \langle \mathrm{S} \rangle \langle \mathrm{S} \rangle\, (p(O_1)))$, where $p(O_1)$ and $p(O_2)$ are propositional variables representing objects, and p_{R_1} is a propositional variable used here to simulate a nominal.

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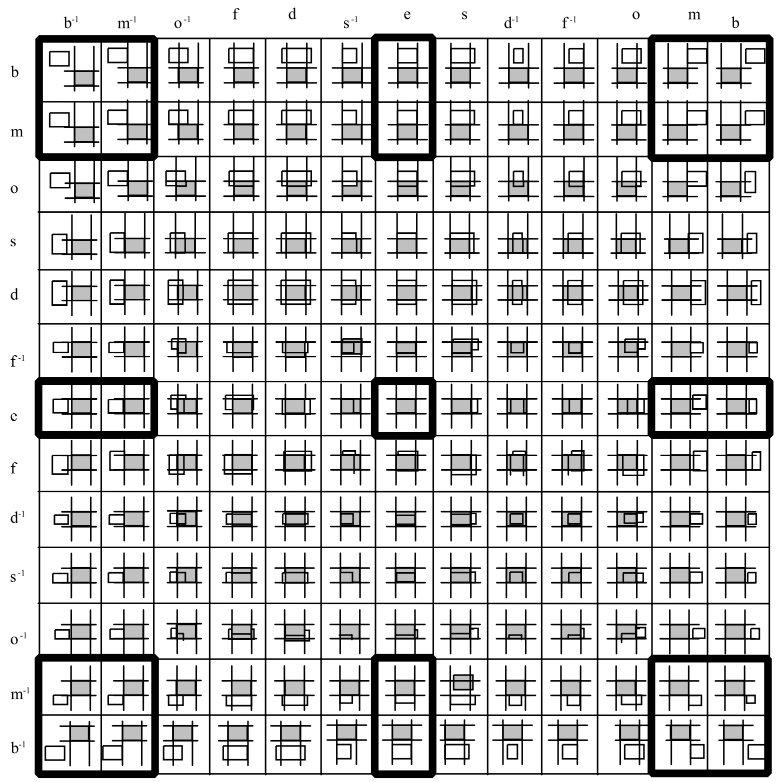

Figure 2. The basic relations between two rectangles directly expressible in SpPNL**.**

Clearly, situations like the above one cannot be generalized (it is not possible to express any binary relation between two rectangles as a modality). Anyway, very natural relations such as *southeast* or *northwest* can be easily expressed; as for example, we can define the modal operator for *southeast* as follows:

$$\begin{aligned}\langle \mathrm{SE}\rangle\phi \quad = \quad & \langle \mathrm{E}\rangle\langle \mathrm{S}\rangle\phi \vee \langle \mathrm{E}\rangle\langle \mathrm{S}\rangle\langle \mathrm{S}\rangle\phi \vee \langle \mathrm{E}\rangle\langle \mathrm{E}\rangle\langle \mathrm{S}\rangle\phi \vee \\ & \langle \mathrm{E}\rangle\langle \mathrm{E}\rangle\langle \mathrm{S}\rangle\langle \mathrm{S}\rangle\phi.\end{aligned}$$

Notice that the above definition captures any region to the south-east of the current one, no matters if their MBB meet (on either of the two axes) or not. Also, in SpPNL it is possible to express 2 out of the 8 RCC8 topological relations (namely *disconnected* and *equal*) in the topological space of all rectangles in $\mathbb{D}^2$. As a final example, we can translate in SpPNL a natural language statement borrowed from the geographical context such as: *suppose that at the southeast of the current region there exists a region containing water (w) at the northeast of which there are no trees (t) at all; so we can deduce that there exists at least one region at the east of the current one (with no side in common with it) with no trees.* Such a statement can be expressed by means of the following (valid) formula:

$$\langle \mathrm{SE}\rangle(w \wedge \neg\langle \mathrm{NE}\rangle t) \rightarrow \langle \mathrm{E}\rangle\langle \mathrm{E}\rangle\neg \mathrm{t}.$$

5 Recursive Enumerability

5.1 Propositional Neighborhood Logic

The Propositional Neighborhood Logic (PNL for short) has been presented in [9]. PNL is a propositional logic for temporal intervals interpreted over linearly ordered sets. It features two modal operators, namely $\langle A\rangle$ and $\langle \overline{A}\rangle$, which are interpreted as Allen's relations *met by* and *meets*, respectively. Given a linearly ordered set $\mathbb{D} = \langle D, <\rangle$, an (strict[1]) *interval neighborhood model* is a tuple of the type $\mathrm{M} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{V}\rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all strict intervals $[d, d']$ where $d < d'$.The valuation function is a mapping $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(D)}$ in such a way that, for any $p \in \mathcal{AP}$, $[d_0, d_1] \in V(p)$ if (and only if) p is *true* over $[d_0, d_1]$. The *truth* relation at a given interval in a model M is defined by induction on the structural complexity of formulas:

- $M, [d_0, d_1] \Vdash p$ iff $[d_0, d_1] \in V(p)$, for all $p \in \mathcal{AP}$;
- $M, [d_0, d_1] \Vdash \neg\psi$ iff it is not the case that $M, [d_0, d_1] \Vdash \psi$;

[1]Two different semantics have been given to interval logics, namely, a *non-strict* one, which includes intervals with coincident endpoints (*point-intervals*), and a *strict* one, which excludes them. In this paper, we focus on the strict one.

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- $M, [d_0, d_1] \Vdash \phi \vee \psi$ iff $M, [d_0, d_1] \Vdash \phi$ or $M, [d_0, d_1] \Vdash \psi$;
- $M, [d_0, d_1] \Vdash \langle A \rangle \psi$ iff there exists d_2 s.t. $d_1 < d_2$ and $M, [d_1, d_2] \Vdash \psi$;
- $M, [d_0, d_1] \Vdash \langle \overline{A} \rangle \psi$ iff there exists d_2 s.t. $d_2 < d_0$ and $M, [d_2, d_0] \Vdash \psi$.

PNL has been shown to be complete for nearly all interesting classes of linearly ordered sets in [9], and to be decidable in NEXPTIME when interpreted at least in the class of all linearly ordered sets in [13]. An effective procedure to decide satisfiability of PNL-formulas is still missing, even though such an algorithm has been devised for a fragment of PNL called RPNL, featuring only one modality, and interpreted over natural numbers, in [6, 7]. Nevertheless, an original tableau method for PNL interpreted in the class of all linearly ordered sets, which combines features of classical first-order tableau and point-based temporal tableaux, has been presented in [9].

5.2 The Translation

Since we are going to exploit the properties of PNL by means of a polynomial translation between SpPNL and PNL formulas, we first have to assure that we are able to translate PNL-models into SpPNL-models, and, conversely, to identify those PNL-models which are compatible with a spatial representation. Consider a SpPNL-model $M = \langle \mathbb{D}^2, \mathbb{O}(\mathbb{D}), V \rangle$, and an object $\langle (d_{1,x}, d_{1,y}), (d_{2,x}, d_{2,y}) \rangle$ such that p holds on it. In order to represent such a situation, we consider a PNL-model $\mathrm{M} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{V} \rangle$ built up on $\mathbb{D} = \langle D, < \rangle$, that is, the same linearly ordered set used for M; the propositional letter p can be represented by using two propositional letters p_x and p_y, that is, by using their projection on the x and y-axis. Thus, we place p_x onto the interval $[d_{1,x}, d_{2,x}]$, and p_y onto the interval $[d_{1,y}, d_{2,y}]$. Notice that there is no relation between the elements representing the x-axis projection of an object and those representing the y-axis projection of the same object. Nevertheless, a problem may occur. In fact, suppose now that, in the same spatial model, we place $\neg p$ onto the object $\langle (d_{3,x}, d_{3,y}), (d_{4,x}, d_{4,y}) \rangle$, and that $d_{3,x} = d_{1,x}$, $d_{4,x} = d_{2,x}$, and $d_{2,y} < d_{3,y}$. This is clearly a non-contradictive situation. But, at the same time, we cannot put $\neg p_x$ over the interval $[d_{3,x}, d_{4,x}]$. Thus, in order to translate a SpPNL-model based on a set of propositional letters $\mathcal{AP}$ into a PNL-model, we will use a set $\mathcal{AP}'$ containing four groups of propositional letters, namely, for each $p \in \mathcal{AP}$, we put $p_x, p_y, p'_x, p'_y \in \mathcal{AP}'$, together with a set of conditions that we well see below. The intended meaning is that p_x (resp., p_y) represents the projection on the x-axis (resp., y-axis) of an object labeled with p, and p'_x (resp., p'_y) represents the same projection of an object labeled with $\neg p$.

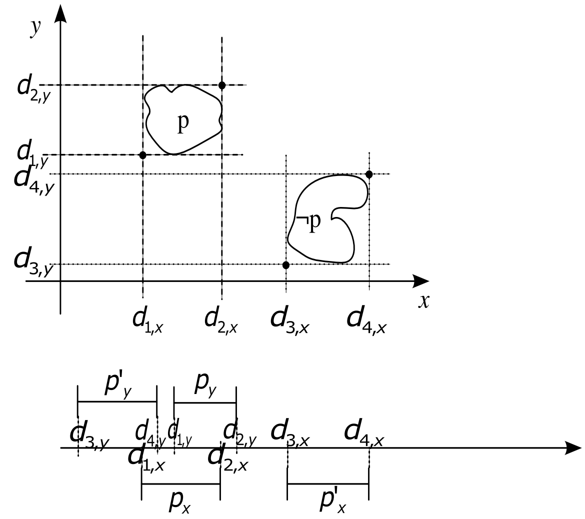

Figure 3. An example of model translation.

(see Fig. 3). From now on, we will assume as fixed the set $\mathcal{AP}$ for spatial models and spatial formulas, and the corresponding set $\mathcal{AP}'$ for propositional neighborhood models and formulas.

Definition 1 *Given a set of propositional letters $\mathcal{AP}$, we say that a* PNL*-model* $\mathrm{M} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{V} \rangle$ *is* plane-compatible *(with respect to $\mathcal{AP}'$) if and only if it is not the case that there exist two intervals $[d, d'], [d'', d''']$ such that $p_x, p'_x \in \mathcal{V}([d, d'])$ and $p_y, p'_y \in \mathcal{V}([d'', d'''])$, for any $p \in \mathcal{AP}$.*

As shown in [9], in PNL it is possible to define the *difference* modality $[\neq]$. Thus, it is also possible to define an operator $E(\phi) = \phi \vee \neg[\neq]\neg\phi$, such that $E(\phi)$ holds on a given interval $[d, d']$ if and only if ϕ holds on $[d, d']$ or over some other interval of the model. It is not difficult to see that the following PNL-formula only admits models which are plane-compatible with respect to $\mathcal{AP}'$:

$$\Gamma = \neg \bigvee_{p \in \mathcal{AP}} (E(p_x \wedge p'_x) \wedge E(p_y \wedge p'_y)).$$

Now, we formalize the translation between SpPNL-models and PNL-models.

Definition 2 *Given any* SpPNL*-model* $M = \langle \mathbb{D}^2, \mathbb{O}(\mathbb{D}), V \rangle$, *we define its* interval-correspondent *model $\mu(M)$ as the PNL-model $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{V} \rangle$, where, for every $p \in \mathcal{AP}$, we set $p_x \in \mathcal{V}([d, d'])$ and $p_y \in \mathcal{V}([d'', d'''])$ if and only if $p \in V(\langle (d, d''), (d', d''') \rangle)$, and $p'_x \in \mathcal{V}([d, d'])$ and $p'_y \in \mathcal{V}([d'', d'''])$ if and only if $p \notin V(\langle (d, d''), (d', d''') \rangle)$. Similarly, given any* PNL*-model* $\mathrm{M} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{V} \rangle$, *we define its* spatial-correspondent *model $\mu^{-1}(M)$ as the* SpPNL*-model $\langle \mathbb{D}^2, \mathbb{O}(\mathbb{D}), V \rangle$, where, for every $p \in \mathcal{AP}$, we put*

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τ_x translation clauses	τ_y translation clauses
$\tau_x(p) = p_x$	$\tau_y(p) = p_y$
$\tau_x(\neg p) = p'_x$	$\tau_y(\neg p) = p'_y$;
$\tau_x(\phi\wedge\psi) = \tau_x(\phi)\wedge\tau_x(\psi)$	$\tau_y(\phi\wedge\psi) = \tau_y(\phi)\wedge\tau_y(\psi)$
$\tau_x(\phi\vee\psi) = \tau_x(\phi)\vee\tau_x(\psi)$	$\tau_y(\phi\vee\psi) = \tau_y(\phi)\vee\tau_y(\psi)$
$\tau_x(\langle \mathrm{E}\rangle\phi) = \langle \mathrm{A}\rangle\tau_{\mathrm{x}}(\phi)$	$\tau_y(\langle \mathrm{E}\rangle\phi) = \tau_{\mathrm{y}}(\phi)$
$\tau_x(\langle \mathrm{N}\rangle\phi) = \tau_{\mathrm{x}}(\phi)$	$\tau_y(\langle \mathrm{N}\rangle\phi) = \langle \mathrm{A}\rangle\tau_{\mathrm{y}}(\phi)$
$\tau_x(\langle \mathrm{W}\rangle\phi) = \langle \overline{\mathrm{A}}\rangle\tau_{\mathrm{x}}(\phi)$	$\tau_y(\langle \mathrm{W}\rangle\phi) = \tau_{\mathrm{y}}(\phi)$
$\tau_x(\langle \mathrm{S}\rangle\phi) = \tau_{\mathrm{x}}(\phi)$	$\tau_y(\langle \mathrm{S}\rangle\phi) = \langle \overline{\mathrm{A}}\rangle\tau_{\mathrm{y}}(\phi)$

Figure 4. Translation clauses.

$p \in V(\langle (d, d''), (d', d''')\rangle)$ *if and only if* $p_x \in \mathcal{V}([d, d'])$ *and* $p_y \in \mathcal{V}([d'', d'''])$.

The following observations will be useful for the rest of this section:

(a) If M is a SpPNL-model and $\mu(M)$ its interval correspondent PNL-model, then $\mu(M) \Vdash \Gamma$;

(b) If M is a PNL-model and $\mathrm{M} \Vdash \Gamma$, then its spatial correspondent model $\mu^{-1}(\mathrm{M})$ is well-defined.

Now, we devise a polynomial translation between SpPNL-formulas, based on a set of propositional variables $\mathcal{AP}$ and PNL-formulas based on the corresponding set $\mathcal{AP}'$. Consider any SpPNL-formula ϕ in *negated normal form*, and the translations functions τ_x and τ_y whose clauses are given in Fig. 4. Clearly, we identify $\neg\neg p$ with p.

Definition 3 *Given any* SpPNL-*formula* ϕ *in negated normal form, we define its* interval-correspondent *formula* $\tau(\phi)$ *as the* PNL-*formula* $\tau_x(\phi) \wedge E(\tau_y(\phi)) \wedge \Gamma$.

Lemma 4 *Given any* SpPNL-*formula* ϕ *in negated normal form, if the* PNL-*formula* $\tau(\phi)$ *is satisfiable, then* ϕ *is satisfiable.*

Proof.

We proceed on the structural complexity of ϕ proving a stronger result: *given any* SpPNL-*formula* ϕ *in negated normal form, if* $\mathrm{M}, [d, d'] \Vdash \tau_x(\phi)$*,* $\mathrm{M}, [d'', d'''] \Vdash \tau_y(\phi)$*, and* $\mathrm{M} \Vdash \Gamma$*, then* $\mu^{-1}(\mathrm{M}), \langle (d, d'')(d', d''')\rangle \Vdash \phi$. Notice that $\mathrm{M} \Vdash \Gamma$ implies that $\mu^{-1}(\mathrm{M})$ is well-defined. We show only some significant cases. Suppose $\phi = p$. If $\mathrm{M}, [d, d'] \Vdash \tau_x(\phi)$, and $\mathrm{M}, [d'', d'''] \Vdash \tau_y(\phi)$, then $p_x \in \mathcal{V}([d, d'])$ and $p_y \in \mathcal{V}([d'', d'''])$. By construction, $p \in V(\langle (d, d'')(d', d''')\rangle)$, and, thus, $\mu^{-1}(\mathrm{M}), \langle (d, d'')(d', d''')\rangle \Vdash \phi$. Similarly, suppose $\phi = \neg p$. If $\mathrm{M}, [d, d'] \Vdash \tau_x(\phi)$, and $\mathrm{M}, [d'', d'''] \Vdash \tau_y(\phi)$, then $p'_x \in \mathcal{V}([d, d'])$ and $p'_y \in \mathcal{V}([d'', d'''])$. By construction, $\neg p \in V(\langle (d, d'')(d', d''')\rangle)$, and, thus, $\mu^{-1}(\mathrm{M}), \langle (d, d'')(d', d''')\rangle \Vdash \phi$. Now, suppose $\phi = \psi \wedge \varphi$. If $\mathrm{M}, [d, d'] \Vdash \tau_x(\psi \wedge \varphi)$ and $\mathrm{M}, [d'', d'''] \Vdash \tau_y(\psi \wedge \varphi)$, then we have that $\mathrm{M}, [d, d'] \Vdash \tau_x(\psi) \wedge \tau_x(\varphi)$, and $\mathrm{M}, [d'', d'''] \Vdash \tau_y(\psi) \wedge \tau_y(\varphi)$. Thus, since $\mathrm{M}, [d, d'] \Vdash \tau_x(\psi)$ and $\mathrm{M}, [d'', d'''] \Vdash \tau_y(\psi)$, by inductive hypothesis, $\mu^{-1}(\mathrm{M}), \langle (d, d'')(d', d''')\rangle \Vdash \psi$, and similarly, $\mu^{-1}(\mathrm{M}), \langle (d, d'')(d', d''')\rangle \Vdash \varphi$. This means that $\mu^{-1}(\mathrm{M}), \langle (d, d'')\ (d', d''')\rangle \Vdash \phi$. Otherwise, assume $\phi = \langle \mathrm{E}\rangle\psi$. If $\mathrm{M}, [d, d'] \Vdash \tau_x(\langle \mathrm{E}\rangle\psi)$ and $\mathrm{M}, [d'', d'''] \Vdash \tau_y(\langle \mathrm{E}\rangle\psi)$, then we have that $\mathrm{M}, [d, d'] \Vdash \langle A\rangle\psi$ and $\mathrm{M}, [d'', d'''] \Vdash \psi$. By definition, we have that that there exists a point $\overline{d}$ such that $d' < \overline{d}$ and $\mathrm{M}, [d', \overline{d}] \Vdash \psi$). By inductive hypothesis, $\mu^{-1}(\mathrm{M}), \langle (d', d'')(\overline{d}, d''')\rangle \Vdash \psi$. By definition, we conclude that $\mu^{-1}(\mathrm{M}), \langle (d, d')(d'', d''')\rangle \Vdash \phi$. Analogously, assume $\phi = [\mathrm{E}]\psi$. If $\mathrm{M}, [d, d'] \Vdash \tau_x([\mathrm{E}]\psi)$ and $\mathrm{M}, [d'', d'''] \Vdash \tau_y([\mathrm{E}]\psi)$, then we have that $\mathrm{M}, [d, d'] \Vdash [A]\psi$ and $\mathrm{M}, [d'', d'''] \Vdash \psi$. So, we have that that for all points $\overline{d}$ such that $d' < \overline{d}$ it is the case that $\mathrm{M}, [d', \overline{d}] \Vdash \psi$). By inductive hypothesis, $\mu^{-1}(\mathrm{M}), \langle (d', d'')(\overline{d}, d''')\rangle \Vdash \psi$. By definition, we conclude that $\mu^{-1}(\mathrm{M}), \langle (d, d')(d'', d''')\rangle \Vdash \phi$. Finally, suppose $\phi = \langle \mathrm{N}\rangle\psi$. If $\mathrm{M}, [d, d'] \Vdash \tau_x(\langle \mathrm{N}\rangle\psi)$ and $\mathrm{M}, [d'', d'''] \Vdash \tau_y(\langle \mathrm{N}\rangle\psi)$, then we have that $\mathrm{M}, [d, d'] \Vdash \psi$ and $\mathrm{M}, [d'', d'''] \Vdash \langle A\rangle\psi$. By definition, we have that that there exists a point $\overline{d}$ such that $d''' < \overline{d}$ and $\mathrm{M}, [d''', \overline{d}] \Vdash \psi$. By inductive hypothesis, $\mu^{-1}(\mathrm{M}), \langle (d, d''')(d', \overline{d})\rangle \Vdash \psi$. By definition, we conclude that $\mu^{-1}(\mathrm{M}), \langle (d, d'')(d', d''')\rangle \Vdash \phi$. In the same way, suppose $\phi = [\mathrm{N}]\psi$. If $\mathrm{M}, [d, d'] \Vdash \tau_x([\mathrm{N}]\psi)$ and $\mathrm{M}, [d'', d'''] \Vdash \tau_y([\mathrm{N}]\psi)$, then we have that $\mathrm{M}, [d, d'] \Vdash \psi$ and $\mathrm{M}, [d'', d'''] \Vdash [A]\psi$. We have that for all points $\overline{d}$ such that $d''' < \overline{d}$ and $\mathrm{M}, [d''', \overline{d}] \Vdash \psi$. By inductive hypothesis, $\mu^{-1}(\mathrm{M}), \langle (d, d''')(d', \overline{d})\rangle \Vdash \psi$. By definition, we conclude that $\mu^{-1}(\mathrm{M}), \langle (d, d'')(d', d''')\rangle \Vdash \phi$. ∎

Lemma 5 *The translation* τ *can be performed in polynomial time.*

Theorem 6 *The validity problem for* SpPNL *interpreted in the class of all Euclidean spaces* $\mathbb{D}^2$*, where* $\mathbb{D}$ *is any linearly ordered set, is recursively enumerable.*

Finally, we recall that in [9] an implementable (non-terminating) tableaux method for PNL has been implemented. Thus, we are able to use it in order to reason with SpPNL by simply applying the translation τ.

5.3 Undecidability

In this section, as a final step, we show which are the main limits of Spatial Propositional Neighborhood Logic. We will see that, at least in the non-strict version, our logic is undecidable when interpreted in the class of all Euclidean spaces $\mathbb{D}^2$. For, we will use Marx and Reynolds's result about the undecidability of Compass Logic [12].

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Compass Logic	SpPNL$^+$
$\Diamond_{\rightarrow} f$	$\langle \mathrm{E} \rangle(\pi_\mathrm{h} \wedge \langle \mathrm{E} \rangle(\pi_\mathrm{p} \wedge \xi(\mathrm{f})))$
$\Diamond_{\leftarrow} f$	$\langle \mathrm{W} \rangle(\pi_\mathrm{h} \wedge \langle \mathrm{W} \rangle(\pi_\mathrm{p} \wedge \xi(\mathrm{f})))$
$\Diamond_{\uparrow} f$	$\langle \mathrm{N} \rangle(\pi_\mathrm{v} \wedge \langle \mathrm{N} \rangle(\pi_\mathrm{p} \wedge \xi(\mathrm{f})))$
$\Diamond_{\downarrow} f$	$\langle \mathrm{S} \rangle(\pi_\mathrm{h} \wedge \langle \mathrm{S} \rangle(\pi_\mathrm{p} \wedge \xi(\mathrm{f})))$

Figure 5. Compass Logic Translation.

Consider the following translation ξ from SpPNL$^+$-formulas to Compass Logic formulas (denoted here by $f, g, \ldots$) in Figure 5, of which we only give the modal clauses.

Theorem 7 *The satisfiability/validity problem for* SpPNL$^+$ *interpreted in the class of all Euclidean spaces $\mathbb{D}^2$, where $\mathbb{D}$ is any linearly ordered set, is undecidable.*

Proof.

The proof is almost straightforward. Suppose the contrary, and consider any Compass Logic formula f. Clearly, f is satisfiable if and only if the SpPNL$^+$-formula $\xi(f)$ is satisfiable. Since the translation ξ is polynomial, this is in contradiction with Marx and Reynolds's result [12]. Thus, the satisfiability/validity problem for SpPNL$^+$ is undecidable. $\blacksquare$

6 Conclusions and Future Work

In this paper we presented a new modal logic for spatial reasoning in directional relations. We approximate objects in a two-dimensional space by means of their minimum bounding boxes, each one of which is labeled by the set of propositional variables holding over it. Spatial Propositional Neighborhood Logic is endowed with four modal operators, which allow one to express 25 out of 169 possible basic relations between two rectangles. Then, we showed that the validity problem for SpPNL is semi-decidable, by means of a polynomial translation to a propositional logic for temporal intervals, namely PNL. As future work, it is worth to mention that a sound and complete axiomatic system for SpPNL is still missing, as well as a representation theorem for abstract spatial frames (a similar result can be found in [11]). Moreover, we noticed that SpPNL can be viewed as a (very small) fragment of Luzt and Wolter's modal logic of topological relations, which is highly undecidable (not recursively axiomatizable) in most cases, thus we believe that a systematic analysis for maximal recursively enumerable fragments is worth. Finally, we recall that in [9] an implementable (non-terminating) tableaux method for PNL has been implemented. Thus, we are able to use it in order to reason with SpPNL by simply applying the translation τ.

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