A System for Reasoning with Nonconvex Intervals

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Abstract

In this paper, we describe a new system of relations for reasoning about general, i.e., both convex and nonconvex, intervals of time. This system is essentially an integration of Allen's theory [1,2] and a one-dimensional version of the Region Connection Calculus [6,9] of Randall, Cui and Cohn. When dealing solely with convex intervals, this system reduces to Allen's set of relations. Fundamental to the definition of our system is the concept of convexity with respect to an interval, an extension of the concept of convexity.

1. Introduction

On of the most widely used systems for temporal reasoning is the interval-based theory of Allen [1,2]. Allen's theory is clear and easy to use, but it is restricted to convex intervals, i.e., intervals composed of a single stretch of time with no gaps. In this paper, we describe an extension of Allen's theory which allows us to reason about general intervals, i.e., both convex and nonconvex temporal intervals. This extension is accomplished primarily through the use of the concept of ***convexity with respect to an interval.***

The general outline of this paper is as follows: in the next section, we introduce Allen's system and in the following section we discuss the Region Connection Calculus (RCC) [6,9]. We then integrate a one-dimensional version of RCC with Allen's theory to produce a new theory of temporal reasoning with general intervals. We then describe how this new theory can cover intersections, sums and differences of general intervals. and we briefly compare this system to those of Ladkin [7] and Ligozat [8]. Finally, we illustrate the utility of our system with some examples.

2. Allen's System

There are thirteen basic relations between convex intervals of time in Allen's theory: before (b), after (a), meets (m), met-by (mi), during (d), contains (c), starts (s), started-by (si), finishes (f), finished-by (fi), equals (eq), overlaps (o) and overlapped-by (oi). These relations are mutually exhaustive and pairwise disjoint. Ambiguous temporal relations are expressed by using sets of basic relations in which relations are separated by a vertical bar, I, which can be read as ***or.*** Ladkin [7] refers to these relations as the convex ***relations.*** In Allen and Hayes [3], all of these relations are shown to be definable in terms of the ***meets*** relation. In addition, the temporal logic is shown to be reducible to five axioms defining the logical properties of meets.

3. The Region Connection Calculus

The Region Connection Calculus (RCC) [6,9] is a widely used theory of spatial reasoning based on the merestopological calculus of Clarke [4,5]. Analogously to Allen's temporal ontology, RCC takes regions (as opposed to points) of space as primitive. The eight basic spatial relations of RCC are:

(1) region x is ***externally connected (ec)*** to region y.
(2) region x is ***disconnected*** (dc) from region y.
(3) region x ***partially overlaps*** (po) region y.
(4) region x is a ***tangential proper part*** (tpp) of region y.
(5) the inverse of tpp.
(6) region x is a ***nontangential proper part*** (ntpp) of region y.
(7) the inverse of ntpp.
(8) region x is equal (eq) to region y.

These relations are mutually exhaustive and pairwise disjoint. There are seven additional relations which, together with the basic relations, form an is-a lattice. These can be defined in terms of the basic relations as follows:

connected (c,,) $=_{def}$ tpp I ntpp I eq I tppi I ntppi I po I ec
(not to be confused with Allen's ***contains)***
part (p) $=_{def}$ tpp I ntpp I eq
proper part (pp) $=_{def}$ tpp I ntpp
overlaps (o_{gen}) $=_{def}$ tpp I ntpp I eq I tppi I ntppi I po

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(generalized overlaps, not to be confused with Allen's strict overlaps)

discrete (dr) $=_{def}$ dc I ec

inverse of ***part*** (pi) $=_{def}$ tppi | ntppi | eq

inverse of ***proper part*** (ppi) $=_{def}$ tppi | ntppi

4. An Integration of Allen's System and RCC

If we restrict our attention to one-dimensional regions, RCC can be understood as a system for temporal representation and reasoning. As such, it has some severe shortcomings of course, such as the inability to distinguish between ***before*** and ***after,*** but it has the advantage of not being restricted to convex intervals. In the general case, time intervals can be understood as unions of maximal convex subintervals [7] (called ***components), so*** that

∀x, t1,t2. comp(t1, x) A comp(t2, x) → tl b I a t2

where *comp(t, x)* means that t is a component of x.

Understood as a system for temporal representation and reasoning, the relations of RCC can be defined in terms of Allen's relations as follows:

(1) x dry ≡ ∀t1,t2. comp(t1, x) A comp(t2, y)
  → tl blalmlmi t2
(2) x ec y ≡ (x dry) A ∃t1,t2. comp(t1, x) A comp(t2, y)
  A (tl mlmi t2)
(3) x dc y ≡ (x dry) A ¬∃t1,t2. comp(t1, x) A comp(t2, y)
  A (tl mlmi t2)
(4) x c y ≡ ∃t1,t2. comp(t1, x) A comp(t2, y)
  A (t1 mlmi|dlclslsilflfileqlolοi t2)
(5) x 0 y ≡ ∃t1,t2. comp(t1, x) A comp(t2, y)
  A (tl dlclslsilflfileqlolοi t2)
(6) x p y ≡ Vtl. comp(t1, x)
  → ∃t2. comp(t2, y) A (t1 dlslfleq t2)
(7) x pi Y ≡ (Y P x)
(8) x PO Y ≡ (x o Y) A ¬(x P Y) A ¬(x Pi Y)
(9) x pp y ≡ (x p y) A ¬(x eq y)
(10) x ppi y ≡ y pp x
(11) x eq y ≡ ∀t. comp(t, x) ↔ comp(t, y)
(12) x ntpp y ≡ (x pp y) A ∀t1. comp(t1, x)
  → ∃t2. comp(t2, y) A t1 d t2
(13) x ntppi y ≡ y ntpp x
(14) x tpp Y ≡ (x pp y) A ¬(x ntpp y)
(15) x tppi y ≡ y tpp x

Since all of Allen's relations can be defined in terms of meets, this means that all of the above relations are also definable in terms of meets.

In order to expand the definitions of Allen's relations to cover general intervals, we need to look more closely at convexity. First, however, we need to define the extended version of ***before:***

(1) x b y $\equiv_{def}$ ∀t1,t2. comp(t1, x) A comp(t2, y) → tl b t2

Convexity of temporal intervals can now be defined as follows: interval x is convex iff

∀t1,t2. (t1 ⊆ x) A (t2 ⊆ X) A (tl b t2)
  → [∀t3. (tl b t3) A (t3 b t2) → (t3 ⊆ x)]

where ***t ⊆ x*** means that t is a general subinterval of x, a notion which is primitive to our system. We can generalize this definition to the notion of convexity with respect to an interval as follows: interval x is convex with ***respect to*** interval y iff

∀t1,t2. (t1 ⊆ x) A (t2 ⊆ X) A (t1 b t2)
  → [∀t3. (tl b t3) A (t3 b t2) A (t3 ⊆ y) → (t3 ⊆ x)]

If ***x is convex*** means that x has no gaps with respect to the time line, then ***x is convex with respect to y*** means that x has no gaps with respect to y. For example, let x be the interval below with two components using parentheses and let y be the interval with two components using square brackets. Then, x is convex-wrt y but y is not convex-wit x.

(ex) ------[----(----]--)-----(--[-----)----]------

We have the following axioms:

(1) convex(x) → convex-wrt(x, y)
(2) convex-wrt(x, x)
(3) convex-wrt(x, y) A convex-wrt(y, z)
  → convex-wrt(x, z)

Intervals x and y are defined to be ***mutually convex*** iff x is ***convex-wrr y*** and y is ***convex-wrt x.***

We can now redefine the rest of Allen's relations to apply to general intervals as follows:

(2) x a y $\equiv_{def}$ y b x
(3) x m y $\equiv_{def}$ ∃t1,t2. comp(t1, x) A comp(t2, y) A tl m t2
  A ∀t.[comp(t, x) A ¬(t eq t1) → t b tl]
  A vt. [comp(t, y) A ¬(t eq t2) → t a t2]
(4) xmi y $\equiv_{def}$ y m x
(5) x d y $\equiv_{def}$ (x pp y) A mutually-convex(x, y)
  ∧ ∃t. (t b x) ∧ (t sld y) ∧ ∃t. (t a x) ∧ (t fld y)
(6) x c y $\equiv_{def}$ y d x
(7) x s y $\equiv_{def}$ (x pp y) A mutually-convex(x, y)
  A ¬∃t. (t b x) A (t sld y)
(8) x si y $\equiv_{def}$ y s x
(9) x f y $\equiv_{def}$ (x pp y) A mutually-convex(x, y)
  ∧ ¬∃t. (t a x) ∧ (t fld y)

(10) $x \text{ fi } y \equiv_{def} y \text{ f } x$
(11) $x \text{ o } y \equiv_{def} (x \text{ po } y) \wedge \text{mutually-convex}(x, y) \wedge \exists t.\ (t \text{ b } y) \wedge (t \text{ s|d } x)$
(12) $x \text{ oi } y \equiv_{def} y \text{ o } x$
(13) $x \text{ eq } y \equiv_{def} \forall t.\ \text{comp}(t, x) \leftrightarrow \text{comp}(t, y)$

The primed relations are the new, extended versions while the unprimed relations are the original versions. These extended convex relations are pairwise disjoint but not mutually exhaustive. This is because intervals which are not mutually convex cannot be related using these relations. In order to have a set of basic relations which are mutually exhaustive and pairwise disjoint, we need to add the following five new relations:

(14) $x \text{ idc } y \equiv_{def} (x \text{ dc } y) \wedge \neg(x \text{ b } y) \wedge \neg(x \text{ a } y)$
(15) $x \text{ iec } y \equiv_{def} (x \text{ ec } y) \wedge \neg(x \text{ m } y) \wedge \neg(x \text{ mi } y)$
(16) $x \text{ ipo } y \equiv_{def} (x \text{ po } y) \wedge \neg(x \text{ o } y) \wedge \neg(x \text{ oi } y)$
(17) $x \text{ ipp } y \equiv_{def} (x \text{ pp } y) \wedge \neg(x \text{ s } y) \wedge \neg(x \text{ f } y) \wedge \neg(x \text{ d } y)$
(18) $x \text{ ippi } y \equiv_{def} y \text{ ipp } x$

It will be noticed that these five relations are based on, but not identical to, relations from RCC. They are forms of ***dc***, *ec*, *po*, *pp* and *ppi*, respectively, which apply solely to intervals which are not mutually convex and which are therefore "intermingled' to some extent. This is the reason for the "i" prefix. Combined with others of RCC's relations, these eighteen basic relations form the is-a lattice shown in Figure 1. Conspicuously absent from this set of relations is any version of *tpp* or *ntpp*. Distinguishing between these two relations would split ***during*** into two relations. Thus while our system is an extension of Allen's system, it is not an extension of RCC.

The complete transitive closure table for the set of eighteen basic relations is given in Tables 1-3 at the end of this paper. In these tables and in the rest of the paper, the primes have been dropped from the relation names; unless stated otherwise, it is always the extended versions of the relations which are intended.

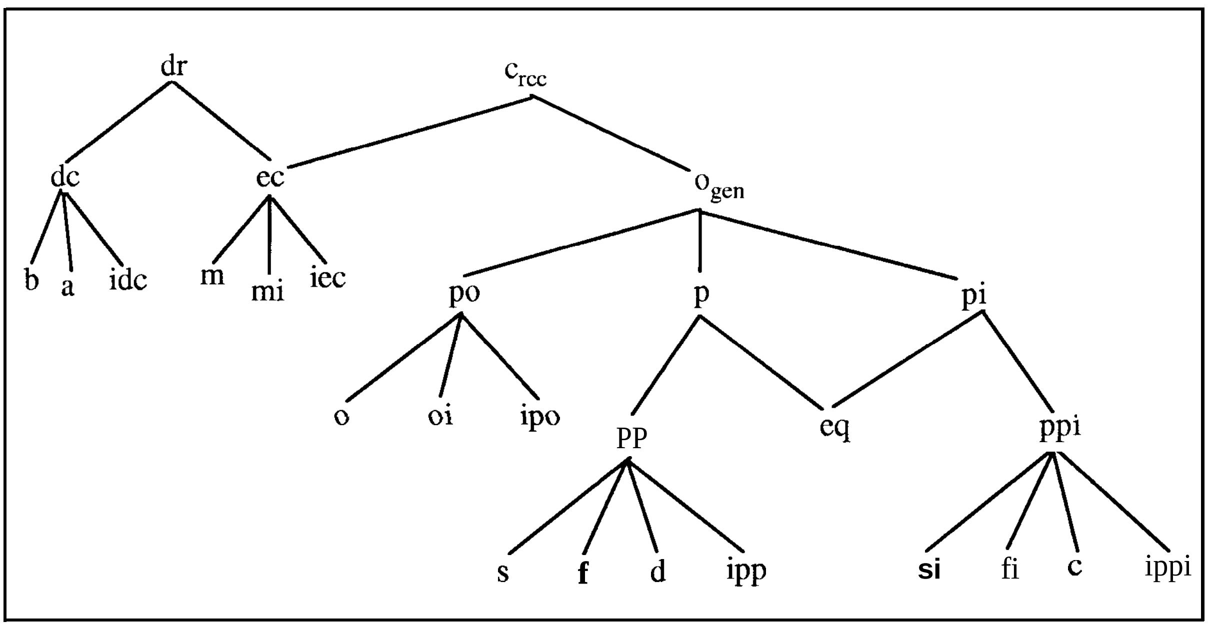

Figure 1. Lattice of relations

5. Some Extensions to the Theory

In this section, we describe some extensions to the theory as presented so far, starting with extensions to finite intersections, sums and differences of intervals, and then giving some rules for inferring convexity.

The intersection of two general intervals (tl ∩ t2) is defined iff tl dlclslsilflfiloloileqlipolipplippi t2. The relations between intersections and their argument intervals are given by the following rules:

(1) tl dlslfleqlipp t2 → (tl ∩ t2) eq tl
(2) t1 filo t2 → (t1 ∩ t2) f t1
(3) tl c t2 → (tl ∩ t2) d tl
(4) tl siloi t2 → (tl ∩ t2) s tl
(5) tl ipo t2 → (tl ∩ t2) dlslflipp tl
(6) tl ippi t2 → (tl ∩ t2) ippi tl
(7) (tl R1 t2 → (tl ∩ t2) *R2* tl)
→ (tl $R1^{-1}$ t2 → (tl ∩ t2) R2 t2)

The union or sum or join of two general intervals (tl ∪ t2) is defined for all pairs of intervals. The relations between sums and their argument intervals are given by the following rules:

(1) tl clsilfileqlippi t2 → (tl *u t2)* eq tl
(2) tl almilfloi t2 → (tl *u* t2) fi tl
(3) tl d t2 → (tl u t2) c tl
(4) tl blmlslo t2 → (tl u t2) si tl
(5) tl idclieclipo t2 → (tl u t2) clippi tl
(6) tl ipp t2 → (tl u t2) ippi tl
(7) (tl R1 t2 → (tl u t2) R2 tl)
→ (tl $R1^{-1}$ t2 → (tl u t2) R2 t2)

The difference of two general intervals (tl – t2) is defined iff tl blalmlmilclsilfiloloilidclieclipolippi t2. The relations between differences and their argument intervals are given by the following rules:

(1) tl blalmlmilidcliec t2 → (tl – t2) eq tl
(2) tl c t2 → (tl – t2) ipp tl
(3) tl siloi t2 → (tl – t2) f tl
(4) t1 filo t2 → (t1 – t2) s tl
(5) tl ipo t2 → (tl – t2) ipplslfld tl
(6) tl ippi t2 → (tl – t2) ippld tl
(7) if R equals b or a or m or mi or idc or iec then
tl R t2 → (tl – t2) R t2
(8) tl clipolippi t2 → (tl – t2) iec t2
(9) tl siloi t2 → (tl – t2) mi t2
(10) tl filo t2 → (tl – t2) m t2

Rules involving assertions of convexity/nonconvexity:

(1) tl dlslfleq t2 ∧ ¬convex(t1) → ¬convex(t2)
(2) tl idclieclipo t2 ∧ convex(t1) → ¬convex(t2)
(3) tl ipp t2 → ¬convex(t1)
(4) convex(t1) ∧ convex(t2) → convex(t1 ∩ t2)
(5) tl idc t2 → ¬convex(t1 u t2)
(6) convex(t1) ∧ convex(t2) ∧ ¬(t1 idc t2)
→ convex(t 1 u t2)
(7) convex(t1) ∧ convex(t2) ∧ (tl c t2)
→ ¬convex(t1 ~ t2)
(8) convex(t1) ∧ convex(t2) ∧ ¬(t1 c t2)
→ convex(t1 – t2)

6. Ladkin's Theory

Ladkin [7] describes a set of approximately thirty relations for reasoning about general intervals. In his examples, he is primarily concerned with reasoning about the relationships between periodic events. Ladkin's relations are a heterogeneous group produced by introducing functors, e.g., *mostly, always* and *sometimes,* that generate nonconvex relations from convex relations, by producing new subclassifications of relations that aren't there in the convex case, and by enumerating relations that are based on the first and last components of the intervals.

While some of Ladkin's relations cannot be reproduced in our system, many of them can be expressed in terms of our relations in combination with the convex closures of the intervals. The *convex* closure of a general interval is the minimal convex interval which has that interval as a subinterval. An example of one of his relations is:

i *disjointly-contains j* iff i and j have no common subobjects, and some subobject of i precedes all subobjects of j and some subobject of i follows all subobjects of j.

This relation can be expressed in terms of our relations as follows:

x disjointly-contains y ≡ (x idcliec y)
∧ (convex-closure(x) c convex-closure(y))

7. Ligozat's Theory

In the approach of Ligozat [8], general intervals are represented by sequences of the endpoints of their convex components. (This system can also include isolated time points as components, but this possibility will be ignored in this discussion.) Therefore the q-interval y = (y_1, ... y_q) defines a partition of time into 2q+ 1 zones, numbered 0 to 2q, as follows:

zone 0 is {t<y_1};
zone 1 is y_1;

zone 2 is $\{y_1<t<y_2\}$;

...

zone 2q is $\{t>y_q\}$.

A relation between a p-interval x and a q-interval y is represented by a sequence of integers giving the zone of y occupied by each of the p endpoints of x. This is equivalent to having each component of x related to each component of y by one of Allen's relations. This system is capable of representing any possible relation between two general intervals. Consequently this system is far more fine-grained than Ladkin's theory or the theory described in this paper. However, in this straightforward form, Ligozat's system has two (related) problems: (1) it is too fine-grained for many purposes and (2) it requires complete information about the intervals being related.

8. Examples

Example 1: "There was an eight-hour meeting which started at 9:00 a.m. and ended at 7:00 p.m. There were two one-hour breaks, one at 12:00 and the other at 4:O0. For the first six hours of the meeting we discussed temporal reasoning and for the last two hours we discussed spatial reasoning."

There are three relevant events in this example: a meeting occupying an interval with three convex components, a discussion of temporal reasoning occupying an interval with two components, and a discussion of spatial reasoning occupying a convex interval. The special feature of this example is that we know exactly the makeup of all of its intervals and all of the relationships between the components of these intervals. We can represent this example as follows:

time(meeting, tl) A **duration**(t1) = S-hours
A ¬**convex(t1)**
A start-time(convex-closure(tl), 9:OOam)
A end-time(convex-closure(tl), **7:00pm)**
A comp(t2, **t** 1) A comp(t3, **t1)** A comp(t4, **t** 1)
A **t2** b t3 A **t3** b t4
A time(temporal-discussion, **t2∪t3)**
A **duration(t2∪t3)** = 6-hours
A ¬**convex(t2∪t3)**
A **start-time(convex-closure(t2∪t3), 9:00am)**
A **end-time(convex-closure(t2∪t3),** 4:OOpm)
A comp(t2, **t2∪t3)** A comp(t3, **t2∪t3)**
A time(spatial-discussion, t4) A **duration(t1)** = 2-hours
A **convex(t4)**
A **(t2∪t3)** b t4
A **(t2∪t3) s t1** A **t4** f **t1**

In general, however, we will not know whether intervals are convex and/or we will not know the precise relationships holding between the components of one interval and the components of another.

Example 2: "There was an eight-hour meeting which started at 9:00 a.m. and ended at 7:00 p.m. The first part of the meeting was occupied with a discussion of temporal reasoning and the remainder of the meeting was occupied with a discussion of spatial reasoning."

In this example, we know that the meeting occupies a nonconvex interval, but we do not know how many components it has. We also know that the discussion of temporal reasoning starts the meeting and the discussion of spatial reasoning finishes the meeting, but we don't know of either of these intervals whether it is convex or nonconvex. However, we do know that the sum of these intervals equals the time of the meeting. This case could be represented as:

time(temporal-discussion, **t5)** A duration(d) < S-hours
A time(spatial-discussion, t6) A **duration(t6)** c 8-hours
A **time(meeting, t5∪t6)** A **duration(t5∪t6)** = S-hours
A ¬**convex(t5∪t6)**
A **start-time(convex-closure(t5∪t6),** 9:OOam)
A **end-time(convex-closure(t5∪t6),** 7:OOpm)
A **t5 s (t5∪t6)** A t6 f **(t5∪t6)** A **t5** mlb t6

The reason that t5 is *before* or meets t6 is that we don't know whether there was a break in the meeting between these two discussions.

Example 3: "There was a meeting, part of which was a discussion of temporal reasoning and part of which was a discussion of spatial reasoning."

In this case, we have no idea how many components any of the relevant intervals has, nor do we know much about the relationships between the two discussions or between the discussions and the overall meeting. This case can be represented as follows:

time(meeting, t7)
A time(temporal-discussion, **t8)**
A time(spatial-discussion, **t9)**
A t8 slfldlipp t7 A t9 slfldlipp t7 A **t8** mlmilieclblalidc t9

9. Conclusions

In this paper, we have introduced a new system of relations for reasoning about general temporal intervals. This set of relations is a direct extension of Allen's original relations and when dealing solely with convex intervals it reduces to Allen's system. We showed how this system can reason about intersections, sums and differences of intervals, and we briefly compared it to the theories of **Ladkin [7]** and Ligozat **[8]**. We believe that a major advantage of our theory relative to these others is its simplicity and conciseness. Currently, we are implementing this system in CLIPS, a forward-chaining, rule-based programming language.

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Table 1. Transitive closure table (part 1)

	b	a	m	mi	d	c
b	b	all	b	b m o s d idc iec ipo ipp	b m o s d idc iec ipo ipp	b
a	all	a	a mi oi f d idc iec ipo ipp	a	a mi oi f d idc iec ipo ipp	a
m	b	a mi oi si c idc iec ipo ippi	b	f fi eq ipo ipp ippi	d s o ipo ipp	b
mi	b m o fi c idc iec ipo ippi	a	s si eq ipo ipp ippi	a	d f oi iec ipo ippi	a
d	b	a	b	a	d	all
c	b m o fi c idc iec ipo ippi	a mi c oi si idc iec ipo ippi	c fi o iec ipo ippi	c si oi ipo ippi	d c s si f fi o oi eq ipo ipp ippi	c
s	b	a	b	mi	d	b m c o fi
si	b m c fi o idc iec ipo ippi	a	c fi o ipo ipp	mi	d f oi ipo ipp	c
f	b	a	m	a	d	a mi c si oi
fi	b	a mi c si oi idc iec ipo ippi	m	c si oi ipo ippi	d s o	c
o	b	a mi c si oi idc iec ipo ipp	b	c si oi ipo ipp	d s o ipo ipp	b m c fi o
oi	b m c f i o idc iec ipo ippi	a	c f i o ipo ippi	a	d foi ipo ipp	a c si si oi
idc	b m c o f i idc iec ipo ipp ippi	a mi c si oi idc iec ipo ippi	b idc iec ipo	idc iec ipo	idc iec ipo ipp	idc
iec	b m c o f i idc iec ipo ippi	a mi c si oi idc iec ipo ippi	b c f i idc iec ipo ippi	a c idc iec ipo ippi	iec ipo i pp	idc iec
ipo	b m c o idc iec ipo ippi	a mi c si oi idc iec ipo ippi	b m c o f i idc iec ipo ippi	a m i c s i idc ipo ippi	ipo i pp	a b m mi c si fi 0 oi idc iec ipo ippi
ipp	b	a	b m	a mi	ipp	b a c m mi si fi 0 oi idc iec ipo i p p i p p i
ippi	b m c fi o idc iec ipo ippi	a mi c si oi idc iec ipo ippi	m c f i o ipo ippi	mi c si oi iec ipo ippi	ipo ippi	a ippi

Table 2. Transitive closure table (part 2)

	s	si	f	fi	o	oi
b	b	b	b m o s d idc iec ipo ipp	b	b	b m o s d idc iec ipo ipp
a	a mi oi f d idc iec ipo ipp	a	a	a	a mi oi f d idc iec ipo ipp	a
m	m	m	d s o ipo ipp	b	b	d s o ipo iec ipp
mi	d f oi ipo ipp	a	mi	mi	d f oi ipo ipp	a
d	d	a mi d f oi	d	b m d s o	b m d s o	a mi d f oi
c	c fi o ipo ippi	c	c si oi	c	c fi o ipo ippi	c si oi ipo ippi
s	s	s si eq	d	b m o	b m o	d f oi
si	s si eq ipo ipp ippi	si	oi	c	c fi o ipo ipp	oi
f	d	a mi oi	f	f fi eq	d s o	a mi oi
fi	o	c	f fi eq	fi	o	c si oi ipo ippi
o	o	c fi o	d s o ipo ipp	b m o	b m o	d c s si f fi o oi eq ipo ipp ippi
oi	d f oi ipo ipp	a mi oi	oi	c si oi	d c s si f fi o oi eq ipo ipp ippi	a mi oi
idc	idc iec ipo	a idc	idc iec ipo	b idc	b idc iec ipo	a idc iec ipo
iec	iec ipo	a idc iec	iec	b m idc iec	b idc iec ipo	a idc iec ipo
ipo	ipo	a mi c si oi idc iec ipo ippi	ipo	idc iec ipo	b m o idc iec ipo	a mi oi idc iec ipo
ipp	ipp	a mi d si f oi idc ipo ipp	ipp	b m fi o ipo ipp	d s o ipo ipp	d f oi ipo ipp
ippi	ipo ippi	c si ippi	ipo ippi	c fi ippi	c fi o ipo ippi	c si oi ipo ippi

Table 3. Transitive closure table (part 3)

	idc	iec	ipo	ipp	ippi
b	b m d s o idc iec ipo ipp	b m d s o idc iec ipo ipp	b m d s o idc iec ipo ipp	b m d s o idc iec ipo ipp	b
a	a mi d f oi idc iec ipo ipp ippi	a mi d f oi idc iec ipo ipp	a mi d oi idc iec ipo ipp	a mi d f oi idc iec ipo ipp	a
m	b idc iec ipo	b d idc iec ipo ipp	b m d s idc ipo ipp	m d s o iec ipo ipp	b m
mi	a idc iec ipo	a d f idc iec ipo ipp	a mi d f oi idc iec ipo ipp	mi d f oi idc iec ipo ipp	a mi
d	idc	idc iec	b a m mi d s f o oi idc iec ipo ipp	d ipp	b a m mi d s f o oi idc iec ipo ipp ippi
c	idc iec ipo ippi	iec ipo ippi	ipo ippi	ipo ipp	ippi
s	b idc	b idc iec	b m d s o idc iec ipo ipp	d s ipp	b m s o ipo ippi
si	idc iec ipo	iec ipo	ipo	d f oi ipo ipp	ippi
f	a idc	a mi idc iec	idc iec ipo	d f ipp	a mi oi ippi
fi	idc iec ipo	iec	ipo	ipo ipp	ippi
o	b idc iec ipo	b idc iec ipo	b m o idc iec ipo	d s o ipo ipp	b m o ipo ippi
oi	a idc iec ipo	a idc iec ipo	a mi oi idc iec ipo	d f oi ipo ipp	c si oi ipo ippi
idc	b idc iec ipo	b a m mi d s fo oi idc iec ipo ipp	b a m mi d s fo oi idc iec ipo ipp	d idc iec ipo ipp	idc
iec	b a m mi c si fi o oi idc iec ipo ippi	b a m mi s si f fi o oi eq idc iec ipo ipp ippi	b a m mi d s f o oi iec ipo ipp	iec ipo ipp	idc iec
ipo	b a m mi c si fi o oi idc iec ipo ippi	b a m mi c si fi o oi idc iec ipo ippi	all	d ipo ipp	all
ipp	idc	idc iec	all	ipp	all
ippi	c idc iec ipo ippi	iec ipo ippi	a ipo ippi	d c s si f fi o oi eq ipo ipp ippi	ippi