# A Topological Transition Based Logic for the Qualitative Motion of Objects

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**Abstract**

*We present a spatio-temporal ontology suitable for representing and reasoning about the qualitative motion of rigid bodies. This simple ontology provides a uniform treatment of motion in one, two, and three dimensional space. A succinct axiomatization is provided capturing the ontology. This first order logic is based on the transition of topological relations between objects.*

## 1 Introduction

Most interesting real world problems have a temporal and spatial component. Traditionally, these components have been studied in isolation. There are few AI approaches that combine both space and time. One useful application of such approaches is the study of motion in naive physics.

We are interested in the qualitative study of the motion and collision of objects in one, two, and three dimensional space. We restrict our attention to n-dimensional objects moving in n-dimensional space. Namely, intervals in one dimension, planes in two dimension, and non-zero volume objects in three dimension.

We assume that the objects are convex and rigid. We refer to the common sense notion of rigidity where objects tend to maintain their shape (e.g. lines maintain their length, planes maintain their area, and non zero-volume objects maintain their volume). This notion excludes liquids and gases. Examples of rigid objects are bricks, balls, pencils, and cars.

We assume fluid motion. An object's trajectory follows connected lines, and cannot disappear and reappear. But, the trajectory may be erratic. For example, the trajectory of a squash ball during a game. An object is allowed to be at rest either permanently, or intermitently.

We are not interested in the relative directions of the objects. For example, we do not distinguish between an object approaching another from the right and from the left. What is important is that they are approaching each other and may collide. When two objects collide, they can either repel (e.g., two billiard balls) or go through one another (e.g., two cones of light).

The first half of the paper describes the ontology. The second half presents a first order axiomatization of it.

## 2 Motion Tree

At the basis of our ontology are topological relations between objects in one, two and three dimensions. We restrict our attention to n-dimensional objects moving in n-dimensional space. The transition between topological relations leads to a uniform graphical representation for motion in all three dimensions. We represent motion with a common tree struture called a *motion tree*. Nodes of a motion tree are spatial relationships, and the edges represent motion as the transition between relations. The motion of objects must follow a path in the motion tree.

Below we present the motion tree for each dimension, then conclude the section with a description of motion tree properties.

### 2.1 One Dimensional Motion

An interval is one type of object that populates one dimensional space. In [1], Allen identifies 13 possible relations between temporal intervals:

| Relation | Symbol | Inverse |
|---|---|---|
| Precedes | p | pi |
| Meets | m | mi |
| Overlaps | o | oi |
| Starts | s | si |
| During | d | di |
| Finishes | f | fi |
| Equals | eq | |

These relations can also be used to describe the one dimensional spatial relationships between two intervals.

Assume we have two intervals $i$ and $j$ in one dimensional space where at least one of the intervals is moving towards the other. At impact, they either repel or go through each other. If they repel one another, we get the sequence of relations shown in figure 1 A and B. The arrowhead represents the direction of travel. In sequence A, $i$ approaches $j$ from the left, and in B $i$ approaches from the right. In both A and B, they approach, collide, then move away. Note that the relations are independent of the relative sizes of $i$ and $j$.

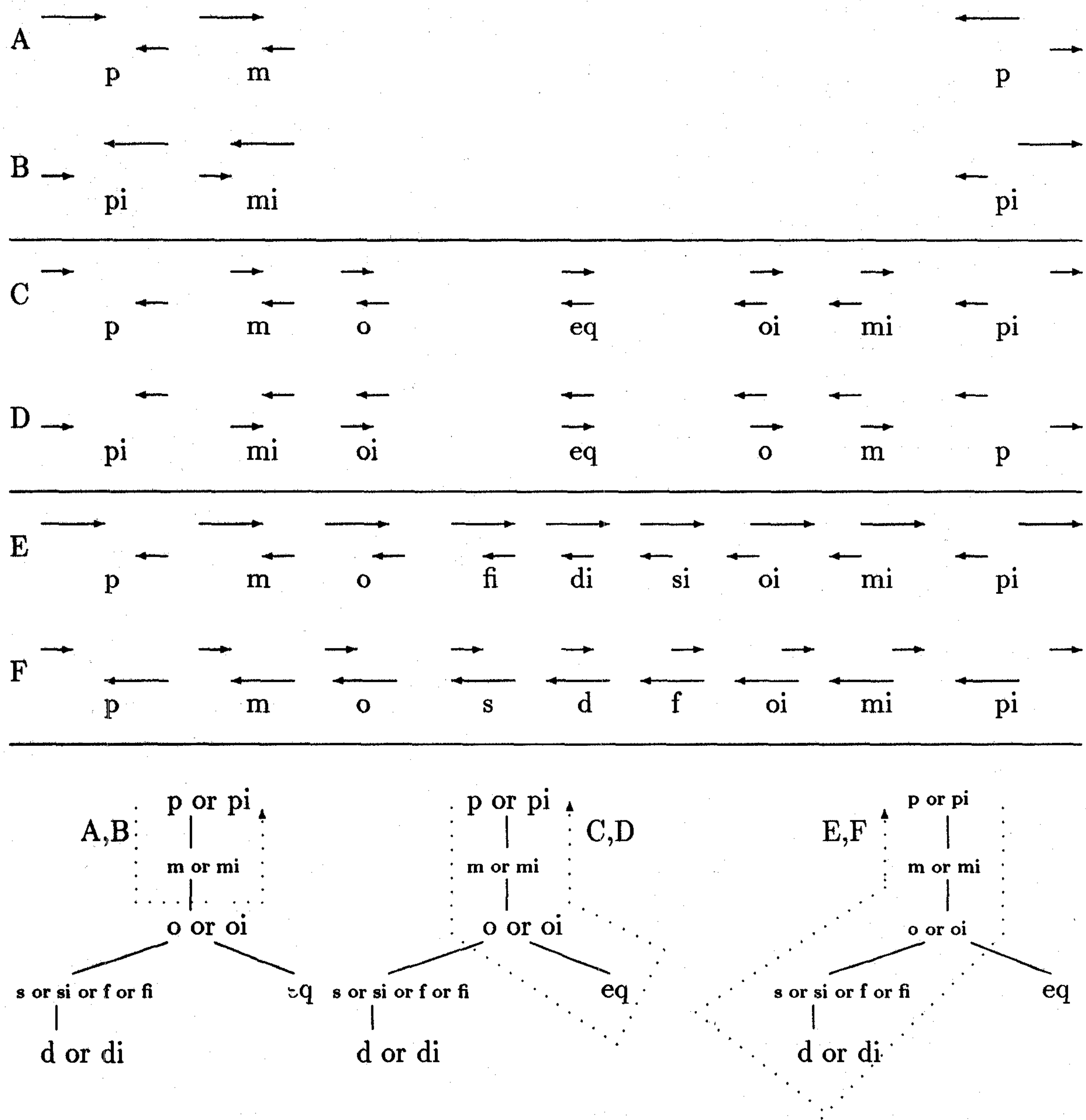


Figure 1: 1-D Example

If at impact $i$ and $j$ go through one another, we have two sub-cases to consider. Either the intervals are equal or different in size. If $i$ and $j$ are equal, we get the sequence of relations shown in figure 1 C and D. In C $i$ approaches from the left, and from the right in D. Note that the middle relation is eq.

The sequences in figure 1 E and F show the sub-case where $i$ and $j$ are of different lengths. In both E and F, $i$ approaches from the left. For $i$ approaching from the right, we simply read the sequence E and F backwards (and reverse the arrow directions).

There are similarities between the sequences A–F. They all begin and end with p or pi. The second and penultimate relation is m or mi. In C–F, the third relation from the front and end is o or oi. Only the center relations differ in each sequence.

A compact representation for the different sequences of spatial relations is a motion tree (see figure 2(a)). The root node has the label "p or pi". We are at this node whenever the spatial relationship between two intervals is either p or pi (we do not care which). Similarly for the other nodes. When we are at a node, the only allowable spatial transitions are to adjacent nodes. For example, when the spatial relation is "m or mi", the only valid transitions are to "p or pi" and "o or oi".

The motion of two intervals must follow a path in the tree. For example, the motion tree paths for sequences A–F are shown at the bottom of figure 1.

## 2.2 Two Dimensional Motion

Randall, Cui, and Cohn [3, 4] identify 8 possible relationships between two planes in two dimensional space:

| Relation | Symbol | Inverse |
|---|---|---|
| Disconnected | DC | |
| Externallay Connected | EC | |
| Partially Overlaps | PO | |
| Equals | = | |
| Tangential Proper Part | TPP | $TPP^{-1}$ |
| Non Tangential Proper Part | NTPP | $NTPP^{-1}$ |

Six of the relations are shown in figure 3. There are also the inverses of TPP and NTPP denoted by $TPP^{-1}$ and $NTPP^{-1}$ respectively.

As in the one dimensional case, we construct a motion tree. Assume we have two disconnected (DC) planes and at least one is moving towards the other. If they collide, we have EC holding between them. If they bounce or brush off each other, we then return to DC. Otherwise, they go through each other and we have PO. At this point, we have four possibilities for the next spatial relation: EC, =, TPP or $TPP^{-1}$. Which transition happens next depends on the relative motion and sizes of the objects. The different choices are shown by the edges coming out of PO in the motion tree in figure 2(b). If the next transition is TPP or $TPP^{-1}$ then we can either go up or down in the tree. Once at a leaf (i.e., NTPP or $NTPP^{-1}$, = ) the only choice is to go up.

A specific example of two dimensional motion is shown at the top of figure 4. Here we have two identical size circles of light crossing paths. As in one dimensional space, the motion of two objects must follow a path in the motion tree in figure 2(b). For example, the specific path for the collision at the top of figure 4 is depicted by the dotted line in the motion tree at the bottom of the same figure.

## 2.3 Three Dimensional Motion

For the three dimensional case, we have two rigid, convex, non-zero volume objects. The spatial relations are the same as in the two dimensional case. The motion tree is also identical (i.e., the tree in figure 2(b)).

A 3-dimensional example is shown at the top of figure 5, where two billiard balls come into collision. The path traversed by this example in the motion tree is represented by the dotted line in the tree at the bottom of the same figure.

## 2.4 Observations

Except for the node labels, the motion trees are identical in all three dimensions. For the sake of uniformity, we define a generic motion tree and use it in the remainder of the paper. The generic tree is shown in figure 2 (c). The nodes in the generic tree represent the spatial relations shown in table 1. The node INST is shorthand for "INSide and Touching a boundary", and INS for "completely INSide".

Motion must follow a path in the motion tree. Note that the path only captures motion. Given a path, we cannot infer the relative sizes or directions of the objects.

Oscillations can occur within the motion tree. For example, consider shaking a ball which contains a smaller ball. We get an oscillation between INST and INS.

## 2.5 Time

So far, we have ignored the temporal aspect. A spatial relation can remain true indefinitely if the objects involved are stationary. But, we get interesting restrictions when the objects are in motion.

The relation DC must hold over a temporal interval. The relation cannot be true at an isolated point in time. But, the next relation in the motion tree (i.e., EC) can be true at an isolated point. For example, the collision of two billiard balls (as in figure 5). EC can also be true over an interval (e.g., rolling a billiard ball on top of a box). This pattern percolates throughout the motion tree. The root and alternating nodes (i.e., DC, PO, and INS) must be true over a temporal interval. The other nodes can be true either at an isolated point or over an interval (see figure 6).

# 3 Axiomatization

We provide a first order axiomatization of the generic motion tree and its properties.

**Time:** We use a totally ordered set of dense points for time. Fluent $x$ is true at time $t$ is written as $true(t, x)$. If $x$ is true over the interval $(t_1, t_2)$ (assume $t_1 < t_2$) we write: $true(t_1, t_2, x)$. For example, if the relation between planes $p_1$ and $p_2$

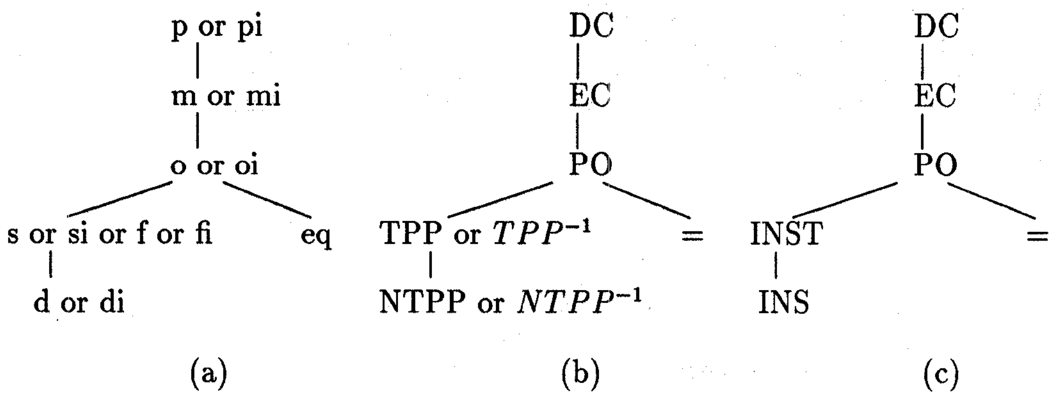


Figure 2: Motion trees

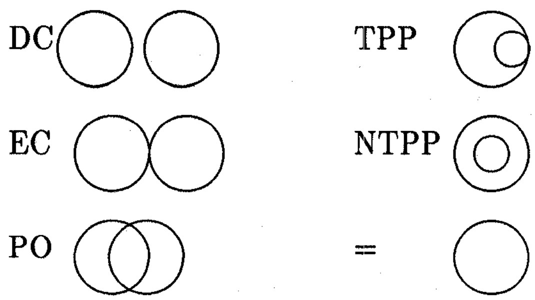


Figure 3: 2-D spatial relations

over the temporal interval (0,10) is EC we have: $true(0, 10, rel(p_1, p_2, EC))$. The order of *rel*'s first two arguments is not important:

$$\begin{aligned}\forall t_1, t_2, p_1, p_2, r \quad . \quad & [true(t_1, t_2, rel(p_1, p_2, r)) \leftrightarrow \\ & true(t_1, t_2, rel(p_2, p_1, r))] \wedge \\ & [true(t_1, rel(p_1, p_2, r)) \leftrightarrow \\ & true(t_1, rel(p_2, p_1, r))].\end{aligned}$$

Only a node from the motion tree can appear as the third argument to *rel*. Something is true over an interval if and only if it is true at each point in the interval:

$$\begin{aligned}\forall x, t_1, t_2 \quad . \quad & true(t_1, t_2, x) \leftrightarrow \\ & [\forall t \;.\; t_1 < t < t_2 \rightarrow true(t, x)].\end{aligned}$$

Note that no commitments are made at the endpoints of the interval.

**Motion tree:** The edges are:

$$\begin{aligned}& edge(DC, EC) \wedge edge(EC, PO) \wedge edge(PO, INST) \wedge \\ & \quad edge(INST, INS) \wedge edge(PO, =).\end{aligned}$$

Edges are bi-directional:

$$\forall x, y \;.\; edge(y, x) \leftrightarrow edge(x, y).$$

**Existence and Uniqueness:** At any point in time, there exists a spatial relation between two objects:

$$\forall t, x, y \; \exists r \;.\; true(t, rel(x, y, r))$$

and the relation is unique:

$$\begin{aligned}\forall t, x, y, r_1, r_2 \quad . \quad & [true(t, rel(x, y, r_1)) \wedge \\ & true(t, rel(x, y, r_2))] \rightarrow r_1 = r_2.\end{aligned}$$

**Temporal restrictions:** The root and alternating nodes:

$$int(DC) \wedge int(PO) \wedge int(INS)$$

must be true over an interval:

$$\begin{aligned}\forall x, y, r, t \quad . \quad & true(t, rel(x, y, r)) \wedge int(r) \rightarrow \\ [\exists t_1, t_2 \quad . \quad & t_1 \leq t \leq t_2 \wedge t_1 < t_2 \wedge \\ & true(t_1, t_2, rel(x, y, r))].\end{aligned}$$

**Smooth motion:** The transition of spatial relations must follow a path in the motion tree:

$$\begin{aligned}\forall x, y, r_1, r_2, t_1, t_2, t_3 \quad . \quad & true(t_1, t_2, rel(x, y, r_1)) \wedge \\ & true(t_2, t_3, rel(x, y, r_2)) \wedge\end{aligned}$$

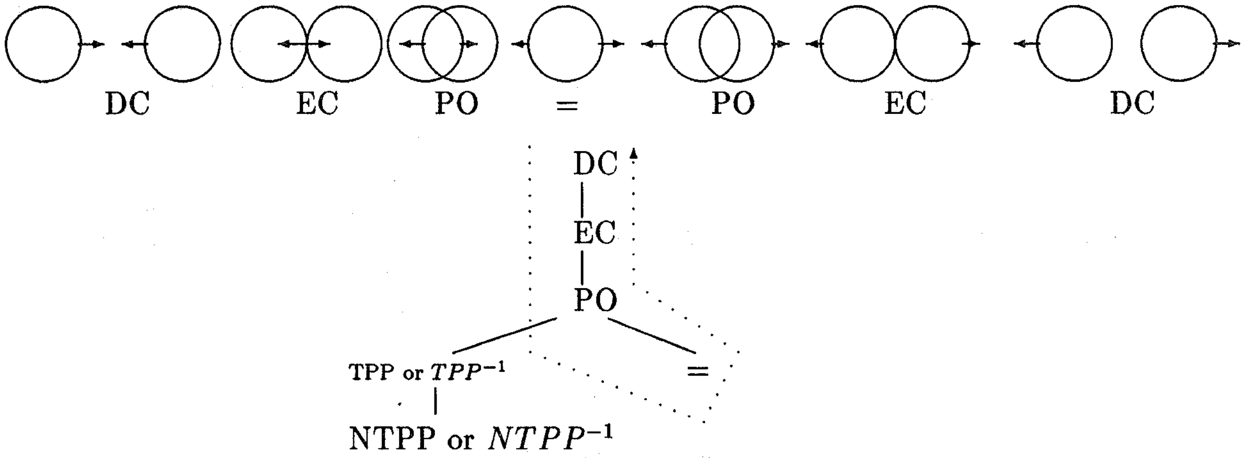


Figure 4: 2-D Example

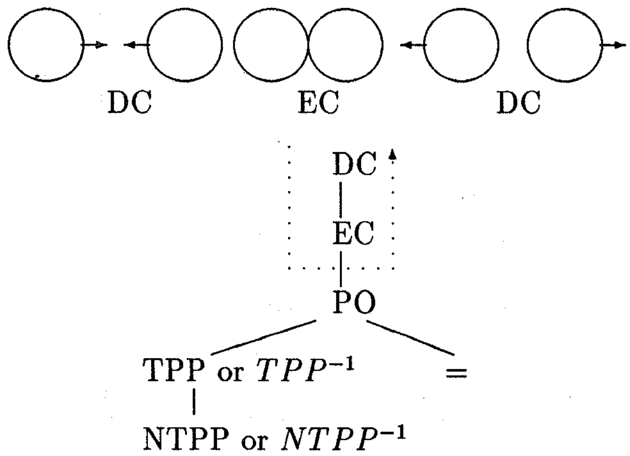


Figure 5: 3-D Example

$$\begin{aligned}
&r_1 \neq r_2 \rightarrow \\
&edge(r_1, r_2) \vee \\
&[\exists r_3 \,.\, true(t_2, rel(x, y, r_3)) \wedge \\
&edge(r_1, r_3) \wedge edge(r_3, r_2)].
\end{aligned}$$

The above states that if relation $r_1$ holds over $(t_1, t_2)$ and $r_2$ over $(t_2, t_3)$ then either $r_1$ and $r_2$ are neighbors in the motion tree, or there was another relation $r_3$ true at $t_2$ which is a neighbor of both $r_1$ and $r_2$.

## 4 Comparison with Previous Work

Galton [2] considers the motion of bodies passing over a fixed region in two dimensional space. Note that Galton restricts his attention to two dimensional space. But, he does consider the motion of points.

Galton's ontology is also based on the transition of spatial relations. He constructs what are called *pertubation diagrams* to restrict the transition between relations. A perturbation diagram is equivalent to a path in our motion tree. Because he is dealing with a set of paths, his axiomatization is not as succinct as ours. He requires 49 axioms versus our 9.

Galton places temporal restrictions on the spatial relations by dividing them into what he defines as states of motion and position. This division is artificial. For example, let two objects be disjoint over the time interval (a,b), and touching over the interval (b,c). Since touching (EC) is defined to be a state of position, the objects are forced to be touching at time point b according to Galton. There is no reason to prefer one relation over another. We should make no commitment at time b.

Galton also defines a "perturbation principle" which states that spatial relations alternate between states of motion and position. A re-statement of this principle is to say that a transition must be to a neigh-

| Generic | 1-D | 2-D | 3-D |
|---|---|---|---|
| DC | p or pi | DC | DC |
| EC | m or mi | EC | EC |
| PO | o or oi | PO | PO |
| = | eq | = | = |
| INST | s or si or f or fi | TPP or $TPP^{-1}$ | TPP or $TPP^{-1}$ |
| INS | d or di | NTPP or $NTPP^{-1}$ | NTPP or $NTPP^{-1}$ |

Table 1: Associated spatial relations

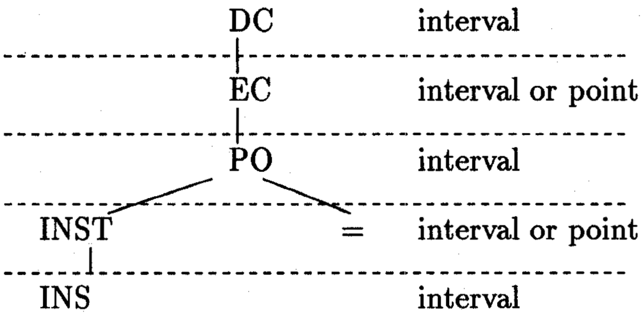


Figure 6: Temporal restrictions

boring node in the motion tree.

We do not represent spatial points or the relative sizes of objects (although it would be simple to implement). Galton does.

## 5 Conclusion

We presented an ontology suitable for representing and reasoning about the qualitative motion of n-dimensional objects in n-dimensional space. It is important for the dimensions of the objects and space to be equal. Otherwise, the motion tree does not apply. For example, assume we have two parallel equal lines in two dimensional space that are moving towards each other. The transition goes from DC directly to =. This is not a path in the motion tree.

The representation of motion is based on the transition of spatial relations. The transitions form a tree which is isomorphic across dimensions. The smooth motion of two objects must follow a path in the tree.

We further presented a very succinct axiomatization of the ontology leading to a first order logic for representing and reasoning about the qualitative motion of n-dimensional objects in n-dimensional space. Our choice of using a first order logic is arbitrary, one could easily use any procedural or declarative language to capture the ontology.

Note that all our examples involved two objects. We represent worlds with more than two objects by considering pair wise relations.

## Acknowledgments

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