Qualitative Temporal Representation and Reasoning about Points, Intervals and Durations

Silvana Badaloni
University of Padova
Dept. of Electronics and Computer Science
Via Gradenigo 6A - 35100 Padova - Italy
badaloni@ladseb.pd.cnr.it

Massimiliano Giacomin
University of Brescia
Dept. of Electronics and Automation
Via Branze 38 - 25123 Brescia - Italy
giacomin@ing.unibs.it

Claudio Masolo
University of Padova
Dept. of Electronics and Computer Science
Via Gradenigo 6A - 35100 Padova - Italy
masolo@ladseb.pd.cnr.it

Abstract

Recently, an elegant framework called $\mathcal{INDU}$ [8] has been proposed for representing qualitative information about time intervals and durations. $\mathcal{INDU}$ is a single network, therefore it avoids typical problems of bi-networks, and in addition it has interesting computational properties. In this paper, we extend $\mathcal{INDU}$ in two directions: we enrich its expressive power introducing points and maintaining the same computational properties, and we provide it with an axiomatic theory able to handle qualitative temporal information about points, intervals and durations in a unified framework. This theory is based on general interval *entities and two relations:* general meets *and* not longer than.

1. Introduction

In many domains, it is more important to consider the relative information concerning the durations of two temporal intervals instead of considering the absolute knowledge about a single interval. If durations are taken as distances between endpoints of intervals then qualitative constraints as "*the duration of the interval I is shorter than the duration of the interval J*" cannot be represented in a binary constraint network since they concern four points. This problem is usually overcome encoding duration informations in a network orthogonal to the interval relationship network, but this strategy presents some limits. In the IA framework [1] it does not guarantee the total consistency of the network [7]. In the same line, there are other bi-network based works proposed for representing information about durations, PDN [7] and APDN [12]. PDN is a point duration network defined as a structure formed by two point-algebra networks separately but not independently, as ternary constraints are introduced for relating point and duration information. APDN extends PDN where only qualitative information is considered, by representing both qualitative and quantitative information about point events. Like other point-based approaches, they do not handle the disjointedness of interval algebra.

The PIDN framework [9] considers a single network able to handle both qualitative and quantitative temporal informations about points, intervals and durations, but its high expressive power is achieved at the price of its computational properties.

Recently, an elegant subframework of PIDN called $\mathcal{INDU}$ [8] has been proposed. $\mathcal{INDU}$ represents only qualitative information about time intervals and durations and it has interesting computational properties. The set of interval relations considered is an extension of 13 Allen's relations [1] integrated with qualitative information about durations $\{<, =, >\}$. For example, the relation b is extended to the three relations $\{b^<, b^=, b^>\}$, while other relations such as $\{eq, s, si, d, di, f, fi\}$ which include implicit information about interval durations, are left unchanged. Then, 25 basic relations between pairs of intervals are considered. The $\mathcal{INDU}$ -network exhibits similar characteristics as the IA-network concerning tractable subclasses (Convex as well as Pre-convex). But the level of local consistency to achieve global consistency is 4-consistency in-

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stead of 3-consistency.

In this paper we extend $\mathcal{INDU}$ in two directions: we enrich its expressive power introducing points and maintaining the same computational properties, and we provide it with an axiomatic theory capable of handling qualitative temporal information about points, intervals and durations in a unified framework. In this theory we introduce both points and intervals as first-class domain entities (called *general intervals*)[1]. As a consequence, we generalize the *meets* relation introduced in [1] defining a new primitive *general meets* that is not irreflexive. Furthermore, to handle qualitative informations about general interval durations, we add the primitive *not longer than.*

This theory allows one to make explicit the properties of entities and of relations and provides a direct formal explanation of composition table. This type of explanation is often given in terms of point structures: an interval is reduced to a pair of points and all the relations between intervals are defined in terms of relations among their endpoints. To better understand the "nature" of general intervals and points we analyze the links between general interval structures and point structures. This type of analysis is important also from the ontological point of view: if one can formally reduce general interval structures to point structures and vice versa, the choice of basing time either on points or on general intervals is only a matter of ontological adequacy as well as of elegance/economy of expression.

2. General interval structures

In this section, first we introduce a structure based on one set of entities, called *general intervals* and on one primitive relation, called *general meets*. In this structure, we can define "points" as *general intervals* that *meet* themselves and "intervals" as *general intervals* that do not *meet* themselves. We show that all the standard relations between intervals and points introduced in [6] can be defined, and we propose an axiomatization that permits to prove (*i*) the axioms of Allen and Hayes for *meet* as given in [3], (*ii*) the axioms for relations between points and intervals introduced in [11].

Furthermore, in order to deal with qualitative relations on durations, we extend this structure by including the primitive *not longer than* that permits to define all the relations between pairs of intervals considered in $\mathcal{INDU}$.

We propose an axiomatization that permits to prove the composition table of Interval-Interval binary relations assumed in [8] and the full standard composition table of binary relations (Point-Point, Interval-Interval, Interval-Point, and Point-Interval relations), as Table 2 in [6], properly modified to include qualitative relations about interval durations.

[1] General intervals must be not confused with "generalized intervals". In the first case we refer to convex intervals that can have null length, in the second case, in the literature, one refers to intervals that can be non-convex.

Definition 2.1 *A* general interval structure *is an ordered pair* $\langle I, \wr\wr \rangle$ *where* I *is a non empty set of* "general intervals" *and* $\wr\wr$ *is a binary relation on* $I \times I$ *called* "general meets".

A general interval structure with qualitative duration *is an ordered triple* $\langle I, \wr\wr, \preceq \rangle$ *where* $\langle I, \wr\wr \rangle$ *is a general interval structure and* $\preceq$ *is a binary relation on* $I \times I$ *called* "is not longer than".

Definition 2.2 *A binary relation is a* linear order (LO) *if it is reflexive, transitive, antisymmetric, and linear. A binary relation is a* strict linear order (SLO) *if it is irreflexive, transitive, and linear.*

Definition 2.3 *A* GI *structure is a general interval structure satisfying the following axioms:*

Uniqueness of "meeting place" (UNI$-\wr\wr$)
$(\forall x, y, z, v \in I)((x \wr\wr z \wedge x \wr\wr v \wedge y \wr\wr z) \rightarrow y \wr\wr v)$

Uniqueness of "end places" (UEND$-\wr\wr$)
$(\forall x, y, z, v \in I)((x \wr\wr z \wr\wr y \wedge x \wr\wr v \wr\wr y) \rightarrow z = v)$

Existence of sum (SUM$-\wr\wr$)
$(\forall x, y, z, v \in I)(x \wr\wr y \wr\wr z \wr\wr v \rightarrow (\exists s \in I)(x \wr\wr s \wr\wr v))$

Weak linearity (WLIN$-\wr\wr$)
$(\forall x, y, z, v \in I)((x \wr\wr y \wedge z \wr\wr v) \rightarrow (\exists s \in I)(x \wr\wr s \wr\wr v \vee z \wr\wr s \wr\wr y))$

Unboundedness (UNB$-\wr\wr$)
$(\forall x \in I)(\exists y, z \in I)(y \wr\wr x \wr\wr z \wedge \neg(y \wr\wr y) \wedge \neg(z \wr\wr z))$

Existence of degenerate intervals (EXM$-\wr\wr$)
$(\forall x \in I)(\exists y, z \in I)(y \wr\wr x \wr\wr z \wedge y \wr\wr y \wedge z \wr\wr z)$

Identity of degenerate intervals (IDM$-\wr\wr$)
$(\forall x, y \in I)((x \wr\wr y \wedge y \wr\wr x) \rightarrow x = y)$

In comparison with Allen and Hayes's theory as given in [3], (*i*) (UNI$-\wr\wr$), (UEND$-\wr\wr$), and (SUM$-\wr\wr$) are already present; (*ii*) (WLIN$-\wr\wr$) and (UNB$-\wr\wr$) are direct modifications of Allen and Hayes's linearity and unboundedness axioms in order to manage the introduction of degenerate intervals; (*iii*) (EXM$-\wr\wr$) and (IDM$-\wr\wr$) are new axioms for degenerate intervals, the first one ensures the existence of two degenerate intervals ending a given interval (this existence condition is necessary to prove the second part of Theorem 2.1) and the second one ensures the identity of all the intervals that meet themselves.

Definition 2.4 *Let* $\langle I, \wr\wr \rangle$ *be a general interval structure. We define the following predicates and relations:*

$$
\begin{array}{rcll}
\mathcal{P}(x) & \triangleq & x \wr x & x \textit{ is a point} \\
\mathcal{I}(x) & \triangleq & \neg(x \wr x) & x \textit{ is an interval} \\
i \,\|\, i' & \triangleq & \mathcal{I}(i) \wedge \mathcal{I}(i') \wedge i \wr i' & i \textit{ meets } i' \\
\mathsf{begin}(p, i) & \triangleq & \mathcal{P}(p) \wedge \mathcal{I}(i) \wedge p \wr i & p \textit{ begins } i \\
\mathsf{end}(p, i) & \triangleq & \mathcal{P}(p) \wedge \mathcal{I}(i) \wedge i \wr p & p \textit{ ends } i \\
p < p' & \triangleq & (\exists i \in I)(\mathsf{begin}(p, i) & \\
 & & \wedge \mathsf{end}(p', i)) & p \textit{ precedes } p'
\end{array}
$$

Theorem 2.1 *Let $\langle I, \wr \rangle$ be a* GI *structure. Then:*

(i) the "meets" *relation $\|$ satisfies all the axioms in the Allen and Hayes's theory as given in [3] (theory $\mathcal{T}_A$);*

(ii) the $<$, begin, *and* end *relations satisfy all the axioms given in [11] (in particular $<$ is a* SLO).

Definition 2.5 *A* GID *structure is a general interval structure with qualitative duration $\langle I, \wr, \preceq \rangle$ such that $\langle I, \wr \rangle$ is a* GI *structure; $\preceq$ is reflexive, transitive and linear; the following links between $\preceq$ and $\wr$ hold:*

$$
\begin{array}{l}
(\mathsf{ADD}-\wr) \\
(\forall a, b, x, y, z \in I)((a \wr x \wr y \wr b \wedge a \wr z \wr b) \\
\qquad\qquad\qquad\qquad \rightarrow x \preceq z \wedge y \preceq z) \\
(\mathsf{ADD}^*-\wr) \\
(\forall a, b, x, y, z \in I)((a \wr x \wr y \wr b \wedge a \wr z \wr b \wedge x \approx z) \\
\qquad\qquad\qquad\qquad \rightarrow y \wr y) \\
(\forall a, b, x, y, z \in I)((a \wr x \wr y \wr b \wedge a \wr z \wr b \wedge y \approx z) \\
\qquad\qquad\qquad\qquad \rightarrow x \wr x) \\
\textit{where } x \approx y \triangleq x \preceq y \wedge y \preceq x
\end{array}
$$

2.1. Interval-Interval (I-I) relations

Definition 2.6 *Let $\langle I, \wr, \preceq \rangle$ be a general interval structure with qualitative duration. By means of $\|$ relation, following [1], we define the 13 Interval-Interval Allen's relations ($\{b, bi, m, mi, s, si, d, di, f, fi, a, ai, eq\}$). By means of the $\preceq$ primitive we define the following 3 relations between intervals:*

$$
\begin{array}{rcl}
i <_I i' & \triangleq & \mathcal{I}(i) \wedge \mathcal{I}(i') \wedge i \preceq i' \wedge \neg(i' \preceq i) \\
 & & \qquad\qquad i \textit{ is shorten than } i' \\[1em]
i \equiv_I i' & \triangleq & \mathcal{I}(i) \wedge \mathcal{I}(i') \wedge i \preceq i' \wedge i' \preceq i \\
 & & \qquad\qquad i \textit{ is congruent to } i' \\[1em]
i >_I i' & \triangleq & \mathcal{I}(i) \wedge \mathcal{I}(i') \wedge \neg(i \preceq i') \wedge i' \preceq i \\
 & & \qquad\qquad i \textit{ is longer than } i'
\end{array}
$$

Combining the 13 Allen's relations and the 3 relations above, we define 39 relations between intervals introducing for each Allen's relation R the following 3 relations:

$$
\begin{array}{rcl}
iR^{<}i' & \triangleq & iRi' \wedge i <_I i' \\
iR^{\equiv}i' & \triangleq & iRi' \wedge i \equiv_I i' \\
iR^{>}i' & \triangleq & iRi' \wedge i >_I i'
\end{array}
$$

Theorem 2.2 *Let $\langle I, \wr, \preceq \rangle$ be a* GID *structure. Then only 25 relations (among the 39 defined in Definition 2.6) are admissible and they form a JEPD (Joint Exhaustive Pairwise Disjoint) set. The 14 relations excluded are:$\{eq^{<}, eq^{>}, s^{\equiv}, s^{>}, si^{\equiv}, si^{<}, d^{\equiv}, d^{>}, di^{\equiv}, di^{<}, f^{\equiv}, f^{>}, fi^{\equiv}, fi^{<}\}$. These 25 relations are the same introduced in $\mathcal{INDU}$ and for this reason they will be called $R_{\mathcal{INDU}}$.*

Observation 2.1 *Given two relations $R_1^{r_1}$ and $R_2^{r_2}$ among the above 25 relations, one can directly see that: $R_1^{r_1} \circ R_2^{r_2} = \{R_x^{r_x} | R_x \in \{R_1 \circ R_2\} \text{and} r_x \in \{r_1 \circ r_2\}\} \cap R_{\mathcal{INDU}}$.*

Theorem 2.3 *Let $\langle I, \wr, \preceq \rangle$ be a* GID *structure. Then $(<_I \circ <_I) = \{<_I\}$, $(<_I \circ \equiv_I) = (\equiv_I \circ <_I) = \{<_I\}$, $(>_I \circ >_I) = \{>_I\}$ and $(<_I \circ >_I) = (>_I \circ <_I) = \{<_I, \equiv_I, >_I\}$. From this, the Observation 2.1 and Allen's composition table proof given in [2], it follows that the composition table of these 25 relations corresponds to the composition table assumed in $\mathcal{INDU}$.*

2.2. Point-Point (P-P) relations

The binary relations between points already introduced are the identity and the precedence defined in Definition 2.4. We have already shown that in a GI structure the precedence is a SLO, and then one can easily prove the standard composition table for precedence, identity and the inverse of precedence (these 3 relation clearly form a JEPD set).

2.3. Point-Interval (P-I) relations

Definition 2.7 *Let $\langle I, \wr, \preceq \rangle$ be a general interval structure with qualitative duration. We define the standard 5 binary relations between points and intervals as follows:*

$$
\begin{array}{rcl}
\mathsf{before}(p, i) & \triangleq & (\exists p' \in I)(\mathsf{begin}(p', i) \wedge p < p') \\
\mathsf{starts}(p, i) & \triangleq & \mathsf{begin}(p, i) \\
\mathsf{during}(p, i) & \triangleq & (\exists p', p'' \in I)(\mathsf{begin}(p', i) \\
 & & \wedge\ \mathsf{end}(p'', i) \wedge p' < p < p'') \\
\mathsf{finishes}(p, i) & \triangleq & \mathsf{end}(p, i) \\
\mathsf{after}(p, i) & \triangleq & (\exists p' \in I)(\mathsf{end}(p', i) \wedge p' < p)
\end{array}
$$

Theorem 2.4 *Let $\langle I, \wr, \preceq \rangle$ be a* GID *structure. Then the relations introduced in Definition 2.7 form a JEPD set and the composition tables between P-P and P-I relations and between P-I and I-I relations are the standard ones.*

The composition table between P-P and P-I relations is the standard one because P-P and P-I relations do not carry any information on durations. In the case of the composition table between P-I and I-I relations, given the P-I relation R_{P-I} and the three I-I relations $R^{<}_{I-I}, R^{\overline{=}}_{I-I}, R^{>}_{I-I}$ one can see that, because P-I relations do not carry any information about durations, $R_{P-I} \circ R^{<}_{I-I} = R_{P-I} \circ R^{\overline{=}}_{I-I} = R_{P-I} \circ R^{>}_{I-I}$; then they coincide with $R_{P-I} \circ R_{I-I}$ in the standard composition table.

2.4. Interval-Point (I-P) relations

Definition 2.8 *Let $\langle I, \wr\wr, \preceq\rangle$ be a general interval structure with qualitative duration. We define the standard 5 binary relations between intervals and points as the inverses of the 5 binary relations between points and intervals introduced in Definition 2.7, i.e.:*

$$i\, Ri\, p \;\triangleq\; p\, R\, i$$

Theorem 2.5 *Let $\langle I, \wr\wr, \preceq\rangle$ be a* GID *structure. Then the relations introduced in Definition 2.8 form a JEPD set and the composition table between I-I and I-P relations is the standard one (in the sense described above for the composition of P-I and I-I relations).*

The composition tables between I-P and P-P relations and between P-I and I-P relations are the standard ones because P-I and I-P relations do not carry any information on durations. For the same reason, the composition table between I-P and P-I is obtained from the standard one by adding information about duration when possible: if in the standard table we have $R_a \circ R_b = \{R_1, \ldots, R_n\}$ then, in the new composition table, we have $R_a \circ R_b = \{R^{<}_1, R^{\overline{=}}_1, R^{>}_1, \ldots, R^{<}_n, R^{\overline{=}}_n, R^{>}_n\} \cap R_{\mathcal{INDU}}$.

Therefore, in a GID structure we are able to prove the standard full composition table in [6] in which the composition tables P-I $\circ$ I-I, I-I $\circ$ I-P, I-I $\circ$ I-I, and I-P $\circ$ P-I, are extended to handle the 25 I-I relations which involve qualitative information about interval durations[2].

3. Point structures

In this section we introduce two kinds of point structures and we show their equivalence with respect to the general interval structures GI and GID presented in Section 2. This equivalence is established defining relations between intervals in terms of relations between their endpoints and/or their distances and, vice versa, relations between points in terms of relations between classes of general intervals which meet at these points (cf [10]).

Besides pointing out the equivalence of the ontological choices between points and general intervals, our analysis reveals that, as far as qualitative reasoning is concerned, only some weak properties must necessarily hold in point structures. In particular, we show that, defining general intervals as pairs of points (x, y) such that $x \leq y$, it is not necessary to define durations as $y - x$: to prove the full composition table, it is sufficient to consider points and distances between points as disjoint sets and to take the weak structure PD (see Definition 3.3). It is not already clear whether this structure is minimal or not.

[2] Note that in a GI structure we can prove the standard full composition table in [6].

Definition 3.1 *A* point structure *is an ordered pair $\langle P, \leq_P\rangle$ where P is a non empty set of* "points" *and $\leq_P$ is a binary relation on $P \times P$ called* "precedence"*. A* point structure with distances *is an ordered 5uple $\langle P, D, \leq_P, \leq_D, dist\rangle$ where $\langle P, \leq_P\rangle$ is a point structure; D is a non empty set of* "distances"*; $\leq_D$ is a binary relation on $D \times D$; $dist$ is a ternary relation on $D \times P \times P$ called* "distance".

Definition 3.2 *A* P *structure is a point structure where the precedence relation is a* LO *and the following axiom holds:*

Unboundedness (ILL)
$(\forall x \in P)(\exists y, z \in P)(y \leq_P x \leq_P z \land x \neq y \land x \neq z)$

Definition 3.3 *A* PD *structure is a point structure with distances $\langle P, D, \leq_P, \leq_D, dist\rangle$ such that $\langle P, \leq_P\rangle$ is a* P *structure; $\leq_D$ is a* LO*; $dist$ is a commutative total function and the following links between $\leq_P$, $\leq_D$ and $dist$ hold:*

(DISTR$-dist$)
$$(\forall a, b \in D)(\forall x, y, z \in P)((x \leq_P y \leq_P z \land a\, dist\langle x, y\rangle \land b\, dist\langle y, z\rangle \land c\, dist\langle x, z\rangle) \rightarrow a \leq_D c \land b \leq_D c)$$

(DISTR*$-dist$)
$$(\forall a \in D)(\forall x, y, z \in P)((x \leq_P y \leq_P z \land a\, dist\langle x, y\rangle \land a\, dist\langle x, z\rangle) \rightarrow y = z)$$
$$(\forall a \in D)(\forall x, y, z \in P)((x \leq_P y \leq_P z \land a\, dist\langle y, z\rangle \land a\, dist\langle x, z\rangle) \rightarrow x = y)$$

3.1. From points to general intervals

Definition 3.4 *Let $\mathcal{PS} = \langle P, \leq_P\rangle$ be a point structure. We define the general interval structure associated with $\mathcal{PS}$ ($\Phi(\mathcal{PS}) = \langle I, \wr\wr\rangle$) in the following way:*

- $I = \{(x, y) \mid x, y \in P$ *and* $x \leq_P y\}$,
- $(x, y) \wr\wr (z, v)$ *iff* $y = z$.

Theorem 3.1 *The general interval structure $\Phi(\mathcal{PS})$ associated with a* P *structure $\mathcal{PS}$ is a* GI *structure.*

Definition 3.5 *Let $\mathcal{PSD} = \langle P, D, \leq_P, \leq_D, dist\rangle$ be a point structure with distances. We define the general interval structure with qualitative durations associated with $\mathcal{PSD}$ ($\Phi(\mathcal{PSD}) = \langle I, \wr\wr, \preceq\rangle$) in the following way:*

- $\langle I, \between \rangle$ *is the structure associated with* $\langle P, \leq_P \rangle$ *as in definition 3.4,*
- $(x,y) \preceq (z,v)$ *iff* $(\exists a,b \in D)(a\ dist\langle x,y\rangle \wedge b\ dist\langle z,v\rangle \wedge a \leq_D b)$.

Theorem 3.2 *The general interval structure with qualitative durations* $\Phi(\mathcal{PSD})$ *associated with a* PD *structure* $\mathcal{PSD}$ *is a* GID *structure.*

3.2. From general intervals to points

Definition 3.6 *Let* $\mathcal{IS} = \langle I, \between \rangle$ *be a general interval structure. We define the point structure associated with* $\mathcal{IS}$ $(\Sigma(\mathcal{IS}) = \langle P, \leq_P \rangle)$ *in the following way:*

- $P = \{[x,y] \mid x,y \in I \text{ and } x \between y\}$ *where* $[x,y] = \{(z,v) \mid z,v \in I \text{ and } x \between v \text{ and } z \between y\}$
- $[x,y] \leq_P [z,v]$ *iff* $(\exists w \in I)(x \between w \between v)$.

Theorem 3.3 *The point structure* $\Sigma(\mathcal{IS})$ *associated with a* GI *structure* $\mathcal{IS}$ *is a* P *structure.*

Definition 3.7 *Let* $\mathcal{ISD} = \langle I, \between, \preceq \rangle$ *be a general interval structure with qualitative durations. We define the point structure with distance associated with* $\mathcal{ISD}(\Sigma(\mathcal{ISD}) = \langle P, D, \leq_P, \leq_D, dist \rangle)$ *in the following way:*

- $\langle P, \leq_P \rangle$ *is the structure associated with* $\langle I, \between \rangle$ *as in definition 3.6,*
- $D = \{|x|_\approx \mid x \in I\}$ *where* $|x|_\approx = \{y \mid y \in I \text{ and } y \approx x\}$,
- $|x|_\approx \leq_D |y|_\approx$ *iff* $x \preceq y$,
- $|a|_\approx\ dist\langle [x,y],[z,v]\rangle$ *iff* $(\exists s \in I)((x \between s \between v \vee z \between s \between y) \wedge s \approx a)$.

Theorem 3.4 *The point structure with distances* $\Sigma(\mathcal{ISD})$ *associated with a* GID *structure* $\mathcal{ISD}$ *is a* PD *structure.*

4. Computational properties

In order to perform reasoning, it is useful to represent qualitative temporal information by means of a binary constraint network, consisting of a set of variables representing points, a set of variables representing intervals, and a set of binary constraints between them. These are the four kinds of binary constraints introduced in section 2.

In [8] it is shown that for two subclasses of $\mathcal{INDU}$ relations, namely convex and pre-convex relations, the tasks of checking consistency and computing the minimal network benefit from particular computational properties. These properties can be seen as a generalization of the results presented in [5], though the introduction of the relations concerning duration does not allow a direct extension of the earlier proof.

In order to define convex and pre-convex relations in our framework, we order the atomic IP and PI relations as in Figure 1, where those represented by black circles are said to be black atomic relations.

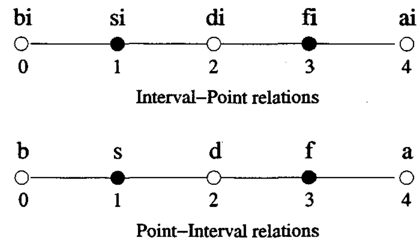

Figure 1. Linear order of IP and PI relations.

Definition 4.1 *A convex relation is a point-point relation except* $\neq$*, or an interval-interval convex relation as defined in [8], or an interval-point (or point-interval) relation corresponding to an interval in Figure 1.*

Definition 4.2 *A pre-convex relation is a point-point relation, or an interval-interval pre-convex relation as defined in [8], or an interval-point (or point-interval) relation obtained from a convex relation by removing at most black atomic relations.*

As in [9, 4] we interpret intervals and points as elements of the half plane **H** defined by $X \leq Y$ in the (X,Y)-plane. In this representation, all the points belong to the line $Y = X$. As shown in Figure 2, given a point (a,a), we can associate to each interval-point relation R that region in **H** which is admissible for an interval I such that I is in relation R with (a,a). In the same way, we can associate to each point-interval relation and point-point relation a region of the line $Y = X$. The dimension of a relation is defined as the dimension of the associated region: it turns out that interval-point relations are of dimension 2 or 1, while point-interval relations are of dimension 1 or 0, as well as point-point relations. In this geometric representation, convex relations correspond to convex regions, while pre-convex relations correspond to convex regions with possible discontinuities of lower dimension.

We can extend the results concerning the computational properties of $\mathcal{INDU}$ to our framework, in which we can handle both points and intervals.

Theorem 4.1 *A strongly 4-consistent network such that all the relations are convex is also strongly* n*-consistent, hence minimal.*

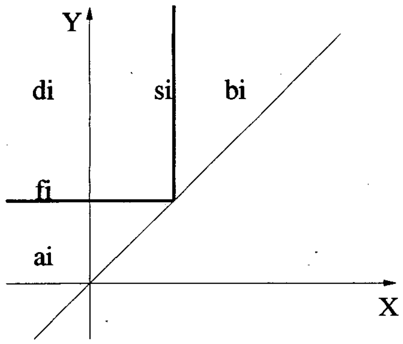

Figure 2. Geometric representation of IP relations.

Theorem 4.2 *A strongly 4-consistent network such that all the relations are pre-convex is consistent.*

The extension of these results to our framework follows from the fact that, given a network involving points and intervals, it is possible to consider a corresponding $\mathcal{INDU}$ network where each point P is represented by an interval XP, assuming X as a reference point. The two networks are equivalent, since they have the same set of solutions apart from the relevant transformation, therefore consistency holds for one network if and only if it holds for the other one.

5. Conclusions

We have proposed an axiomatic theory capable of handling qualitative information about points, intervals and durations in a unified framework. The relevant temporal network corresponds to $\mathcal{INDU}$ enriched with points as temporal primitives. The theory is based on the notion of *general interval* and two relations: *general meets* and *not longer than*. In this structure, points can be defined as general intervals that meet themselves, and intervals as general intervals that do not meet themselves.

We have proved that the relevant computational results of the $\mathcal{INDU}$ framework can be extended to our framework, where both points and intervals are handled in a unified network.

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