Regarding Overlapping as a Basic Concept of Subset Spaces

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Abstract

An operator describing overlapping is substituted for the effort modality of Moss and Parikh's logic of knowledge and topology. This means that the spatial idea underlying that system, viz closeness, is replaced with covering. We ask what properties of frames can be expressed by means of the new language. It turns out that the language is, for example, capable of characterizing directed spaces. After adding some expressive power originating from hybrid logic we are able to deal with linearity as well. This makes a temporal interpretaion of the new system possible. – The main issues of this paper are first corresponding completeness theorems, and second the decidability of the arising logics.

1 Introduction

This paper revolves around two prominent interpretations of the modal $\Box$-operator which seem to be completely different at first glance, *knowledge* and *space*. Seen in the light of history, both approaches have their origins in different sciences. While mathematicians brought out the relationship between the topological interior operator and the modal system S4 (cf, eg, [13]), philosophers first came out with a corresponding treatment of knowledge ([12]). From the beginning of the eighties, the concept of knowledge was then utilized for computer science. The extensive work related to this culminated in the textbooks [5] and [14], respectively. And more recently, the spatial calculi invented in AI (cf, eg, [16]) initiated a revitalization of the modal view of space; see the recent handbook [1] for an overview of the state of the art.

Essentially, knowledge and space *are* related. In the paper [15], and more detailedly in [4], Moss and Parikh developed a suitable connecting framework. For convenience of the reader, we briefly recall the ideas being basic to the accompanying logic. The knowledge of an agent in question is described by the space of all *knowledge states,* which are the sets of states the agent considers possible at a time. Any effort that is made to acquire knowledge results in a *shrinking procedure* regarding that space of sets. Exactly this is captured by the formal system. In particular, the formulas of the underlying language speak about knowledge and effort, which are both represented by a corresponding modality. And while these aspects of (evolving) knowledge principally appear in the syntax, the spatial aspect is prevailing at the semantics. In fact, the spaces of knowledge states represent the adequate semantic domains, which sometimes turn out to be real topological spaces. The most significant example is Cantor's space of all infinite binary sequences here, which models 'generically' the acquisition of knowledge about computable functions.

As shrinking subsets and gaining knowledge correspond to each other, topological ideas like closeness, neighbourhood, or approximation, are not only linked to knowledge-theoretic ones like acquisition or (in the case of discrete linear time) perfect recall,[1] but are in fact fundamental to the arising logic. This point of view is changed in the present paper.

An *overlap operator* was integrated into the language of general subset spaces in the paper [10], and the interplay between this operator and the effort modality was studied there. It turned out that one cannot get beyond the original system very far in this way. Now, the question naturally comes up whether the overlap operator and the effort operator are comparably expressive. Or, in other words, what spatial properties can be captured by the overlap operator alone? Giving some answers to this question, we regard no longer closeness, but overlapping as a basic concept of subset spaces in this paper. From the viewpoint of knowledge this means that no longer acquisition, but *changing the knowledge state* is taken as a basic action.

The subsequent technical part of the paper is organized as follows. In the next section, we introduce the logical language our theory is based on. We give some examples concerning the use of this language there, too. In Section 3, we briefly examine the most general development of the new logic. It turns out that this system is very easy to handle

[1] Concerning this notion, see [6], or [5], Sec. 4.4.4.

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since no interaction between knowledge and overlapping is imposed. Afterwards, we deal with the class of all *directed spaces;* cf [17]. Somewhat surprisingly, the treatment here proceeds much more smoothly than there. Directed spaces already comprise a certain temporal component, which is made more explicit in the final technical section. There we expand the allowed methods by integrating some basic features from *hybrid logic;* cf [3], Ch. 14. This enables us to handle *linear subset spaces,* too. Concluding the paper, we sum up and point to future research.

2 The basic language

In this section, we first define the bi-modal language for subset spaces which was indicated above. Second, we give examples of generally valid formulas. And finally, we use the new language for specifying the defining property of directed spaces.

Let $\mathrm{Prop} = \{p, q, \ldots\}$ be a denumerable set of symbols. The elements of Prop are called *proposition letters*. We define the set Form of *formulas over* Prop by the rule

$$\alpha \ ::= \ p \mid \neg\alpha \mid \alpha \wedge \beta \mid \mathsf{K}\alpha \mid \mathsf{O}\alpha.$$

The operators K and O represent *knowledge* and *overlapping,* respectively. The duals of K and O are denoted L and P, respectively. The missing boolean connectives $\top$, $\bot$, $\vee$, $\rightarrow$ and $\leftrightarrow$ are treated as abbreviations, as needed.

We now turn to the semantics of the new language. For a start, we fix the relevant domains. We let $\mathcal{P}(X)$ designate the powerset of a given set X.

Definition 2.1 (Subset frames and spaces) *1. Let X be a non-empty set and $\mathcal{O} \subseteq \mathcal{P}(X)$ be a set of subsets of X. Then, $\mathcal{F} := (X, \mathcal{O})$ is called a* subset frame.

2. *Let $\mathcal{F} = (X, \mathcal{O})$ be a subset frame. The* set $\mathcal{N}_{\mathcal{F}}$ of neighbourhood situations *of $\mathcal{F}$ is defined by $\mathcal{N}_{\mathcal{F}} := \{(x, U) \mid x \in U$ and $U \in \mathcal{O}\}$.*

3. *Let $\mathcal{F}$ be a subset frame. A mapping $V : \mathrm{Prop} \rightarrow \mathcal{P}(X)$ is called an $\mathcal{F}$*-valuation.

4. *A* subset space *is a triple $\mathcal{M} := (X, \mathcal{O}, V)$, where $\mathcal{F} = (X, \mathcal{O})$ is a subset frame and V an $\mathcal{F}$-valuation; $\mathcal{M}$ is then called* based on $\mathcal{F}$.

It is sometimes assumed that the whole space or the empty set are members of $\mathcal{O}$; cf [4], Sec. 1.1. We will mention this explicitly when required.

The neighbourhood situations introduced in Definition 2.1.2 make up the atomic semantic entities of our language. They are used for evaluating formulas; see the next definition. In a sense, the set component of a neighbourhood situation measures the uncertainty about the associated state component at any one time.

We are mainly interested in *interpreted systems,* which are formalized by means of subset spaces here (Definition 2.1.4). The assignment of sets of *states* to proposition letters by means of valuations (see item 3 above) is in accordance with the usual logic of knowledge; cf [5] or [14].

The next definition concerns the relation of satisfaction, which is defined with regard to subset spaces now. In the following, neighbourhood situations are written without brackets.

Definition 2.2 (Satisfaction; validity) *Consider a subset space $\mathcal{M} = (X, \mathcal{O}, V)$. Let x, U be a neighbourhood situation of the underlying frame $\mathcal{F} = (X, \mathcal{O})$. Then*

$$\begin{array}{lcl}
x, U \models_{\mathcal{M}} p & :\Longleftrightarrow & x \in V(p) \\
x, U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & x, U \not\models_{\mathcal{M}} \alpha \\
x, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & x, U \models_{\mathcal{M}} \alpha \text{ and } x, U \models_{\mathcal{M}} \beta \\
x, U \models_{\mathcal{M}} \mathsf{K}\alpha & :\Longleftrightarrow & \forall\, y \in U : y, U \models_{\mathcal{M}} \alpha \\
x, U \models_{\mathcal{M}} \mathsf{O}\alpha & :\Longleftrightarrow & \left\{ \begin{array}{l} \forall\, \tilde{U} \in \mathcal{O} : (x \in \tilde{U} \\ \ \Rightarrow x, \tilde{U} \models_{\mathcal{M}} \alpha), \end{array} \right.
\end{array}$$

where $p \in \mathrm{Prop}$ and $\alpha, \beta \in \mathrm{Form}$. In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that α holds in $\mathcal{M}$ at *the neighbourhood situation x, U.*

A formula α is called valid in $\mathcal{M}$ *(written '$\mathcal{M} \models \alpha$'), iff it holds in $\mathcal{M}$ at every neighbourhood situation of the frame $\mathcal{M}$ is based on.*

Note that the meaning of proposition letters is independent of neighbourhoods by definition, thus 'stable' with respect to O. This fact will also find expression in the logical system considered later on; see Section 3.

Due to the final clause of Definition 2.2, overlapping really means 'overlapping with respect to a fixed point x', which is modelled by quantifying over all subsets from $\mathcal{O}$ containing x.

We now look for formulas which are valid in all subset spaces. Unlike the usual logic of subset spaces, we here do not have any interaction of knowledge and the spatial operator from the outset.

Proposition 2.3 *Let $\mathcal{M}$ be any subset space. Then, for all $\alpha \in \mathrm{Form}$, we have $\mathcal{M} \models (\mathsf{O}\alpha \rightarrow \alpha) \wedge (\mathsf{O}\alpha \rightarrow \mathsf{O}\mathsf{O}\alpha) \wedge (\alpha \rightarrow \mathsf{O}\mathsf{P}\alpha)$.*

As a consequence of this proposition we get that the accessibility relation associated with O in an arbitrary Kripke model is an equivalence. – Concluding this section, we consider *directed spaces*. These spaces reflect a kind of convergence of acquisition processes.

Definition 2.4 (Directed frames; directed spaces) *A subset frame $\mathcal{F} = (X, \mathcal{O})$ is called* directed, *iff $\forall\, U_1, U_2 \in \mathcal{O}$, $\forall x \in X$: if $x \in U_1 \cap U_2$, then $\exists\, U \in \mathcal{O} : x \in U \subseteq U_1 \cap U_2$. A subset space $\mathcal{M}$ is called* directed, *iff it is based on a directed frame.*

Directed spaces were studied in [17]. Their theory based on the original language is recursively, but not finitely axiomatizable. This is the main result of that paper. – The next proposition contains the characteristic validity of directed spaces; see Section 4.

Proposition 2.5 *Let $\mathcal{M}$ be any directed space. Then, for all $\alpha, \beta \in \mathrm{Form}$, $\mathcal{M} \models \alpha \wedge \mathsf{P}\beta \rightarrow \mathsf{PK}(\mathsf{PL}\alpha \wedge \mathsf{PL}\beta)$.*

3 The logic of overlapping

It turns out that the basic logic LO resulting from the new language is extremely simple, viz $\mathrm{S5}_2$ with rigidity of proposition letters regarding one of the modalities. This implies that, for every Kripke model M validating these axioms,

(A) the accessibility relations $\overset{\mathsf{K}}{\rightarrow}$ and $\overset{\mathsf{O}}{\rightarrow}$ of M belonging to knowledge and overlapping, respectively, are equivalences, and

(B) the valuation of M is constant along every $\overset{\mathsf{O}}{\rightarrow}$-path, for all proposition letters.

Since the soundness of LO (with respect to the class of all subset spaces) is immediately clear, we now strive for completeness. For these purposes, we consider the accompanying canonical model, i.e., the set S of all maximal LO-consistent sets of formulas. Let the canonical accessibility relations be denoted $\overset{\mathsf{K}}{\rightarrow}$ and $\overset{\mathsf{O}}{\rightarrow}$ as well (and the canonical valuation V). Then we obtain the following lemma, which can be proved by induction on the structure of formulas.

Lemma 3.1 *Let $s, t \in S$ satisfy $s \overset{\mathsf{K}}{\rightarrow} t$ and $s \overset{\mathsf{O}}{\rightarrow} t$. Then $s = t$.*

Consequently, restricting the relation $\overset{\mathsf{O}}{\rightarrow}$ to the $\overset{\mathsf{K}}{\rightarrow}$-equivalence classes of s in the domain and t in the range, respectively, induces a left-and-right deterministic relation, for all $s, t \in S$. We are, therefore, able to define a subset space on the canonical model in the following way.

- For every $s \in S$, let $\overline{s}$ denote the $\overset{\mathsf{K}}{\rightarrow}$-equivalence class and $\underline{s}$ denote the $\overset{\mathsf{O}}{\rightarrow}$-equivalence class of s. Then, define $X := \{\underline{s} \mid s \in S\}$.
- For every $t \in S$, let $U_t := \{\underline{s} \in X \mid \underline{s} \cap \overline{t} \neq \emptyset\}$. Let $\mathcal{O} := \{U_t \mid t \in S\}$.
- Finally, let $\underline{s} \in V(p)$ iff $p \in s$, for all $s \in S$ and $p \in \mathrm{Prop}$.

Then, the desired completeness result follows immediately from the subsequent *Truth Lemma*.

Lemma 3.2 *The structure $\mathcal{M} := (X, \mathcal{O}, V)$ is a subset space. Moreover, for all $\alpha \in \mathrm{Form}$ and $s, t \in S$ so that $\underline{s} \in U_t$, we have $(\underline{s}, U_t \models_{\mathcal{M}} \alpha \iff \alpha \in s_t)$, where s_t is the unique element of $\underline{s} \cap \overline{t}$.*

Theorem 3.3 (Completeness) *The logic* LO *is sound and complete for subset spaces.*

Using well-known methods from modal logic one can easily prove that LO is decidable (in PSPACE).

Theorem 3.4 (Decidability) LO *is a decidable set of formulas.*

Note that LO satisfies in addition the finite model property with respect to subset spaces.

4 The logic of directed spaces

In this section, we first prove that the axioms of LO together with the schema from Proposition 2.5 yield a complete axiomatization of the logic of directed spaces, LOD. We secondly show that LOD is decidable, too. To begin with, we bring out a property of accessibility relations corresponding to that 'Axiom of Directedness'.

Lemma 4.1 *Let $M = \left(W, \overset{\mathsf{K}}{\rightarrow}, \overset{\mathsf{O}}{\rightarrow}, V\right)$ be a Kripke model. Suppose that M satisfies the following property* (D)*:*

> *For all $u, v \in W$ so that $u \overset{\mathsf{O}}{\rightarrow} v$, there exists some $w \in W$ so that $u \overset{\mathsf{O}}{\rightarrow} w$ and, for all $w' \in W$ satisfying $w \overset{\mathsf{K}}{\rightarrow} w'$, there are $u', v' \in W$ so that $w' \overset{\mathsf{O}}{\rightarrow} u' \overset{\mathsf{K}}{\rightarrow} u$ and $w' \overset{\mathsf{O}}{\rightarrow} v' \overset{\mathsf{K}}{\rightarrow} v$.*

Then $M \models \alpha \wedge \mathsf{P}\beta \rightarrow \mathsf{PK}(\mathsf{PL}\alpha \wedge \mathsf{PL}\beta)$, for all $\alpha, \beta \in$ Form.

It is our goal to establish the property (D) on the canonical model of LOD now. To this end, we need an auxiliary result.

Proposition 4.2 *Let s, t be maximal* LOD*-consistent sets so that $s \overset{\mathsf{O}}{\rightarrow} t$. Then the set $\{\mathsf{P}\gamma \mid \gamma \in t\} \cup \{\mathsf{K}(\mathsf{PL}\alpha \wedge \mathsf{PL}\beta) \mid \alpha \in s, \beta \in t\}$ is* LOD*-consistent.*

Some modal proof theory is used for the proof of this proposition.

Corollary 4.3 *The canonical model satisfies the property* (D).

With the aid of the property (D) one can now show that the construction from Section 3 in fact yields a directed space. With that and Proposition 2.5 we obtain the next theorem.

Theorem 4.4 (Completeness II) LOD *is sound and complete for directed spaces.*

The proof of the decidability of LOD is much more involved than that for the basic logic. Following well-established approaches, the strategy for it could be as follows. In order to realize a consistent formula α in a finite model of the axioms, we first define a suitable filter set $\Sigma \subseteq \mathrm{Form}$ arising from the set $\mathrm{sf}(\alpha)$ of all subformulas of α. Then we show that the minimal filtration, corresponding to Σ, of the canonical model satisfies (A) and (B) from the preceding section as well as the property (D). According to Lemma 4.1 and well-known facts from modal logic, this would suffice for our goal. – Unfortunately, the transitivity of $\xrightarrow{\mathsf{O}}$ cannot be proved in this way. But LOD is decidable nevertheless.

Theorem 4.5 (Decidability II) LOD *is a decidable set of formulas.*

For a proof of this theorem, we apply the methods developed in the subsequent section. (In fact, Theorem 4.5 is obtained as a corollary to Theorem 5.3.)

The results of this section should be compared to the outcome of the paper [17]. There the authors proved that the usual logic of directed spaces is not finitely axiomatizable. By contrast, the logic of overlapping *is*. Moreover, the decidability of that logic was left as an open problem.[2] Here we can show that our logic of directed spaces is decidable.

5 Hybridization

Directed frames represent an interesting class of spatial models which can be dealt with by means of the concept of overlapping. However, it seems to be difficult to characterize other important classes of frames. So we expand the expressive means in this section. Encouraged by the successful approach regarding the usual logic of subset spaces, cf [11], we develop a *hybrid logic of overlapping* in this section. This system will be applied to *linear* subset spaces, which may be conceived as linear time temporal structures, afterwards.

Technically speaking, we have two options. First, we could base our development on the $\Box$-free fragment of the system from [11] by simply forgetting the effort modality there. This means that we would have to take the global modality (cf [2], Sec. 7.1) into account as well. In this case, we would obtain finitely axiomatizable and decidable logics. However, due to the previous work just quoted all this is more or less obvious and not very interesting thus.

[2] In the paper [9], it was proved that the subset space logic of directed spaces is in fact decidable.

The second possibility is setting the global modality aside. This idea was too implemented once; see [10], where it was applied to the original logic of subset spaces including overlapping. But we had to pay a price for that. We had to accept that an infinite (but at least recursive) set of unorthodox proof rules is required then. In the present paper, we shall additionally need an infinite (but still recursive) set of axiom schemata. The resulting theory is rather simple and has nice properties nevertheless, particularly with regard to the applications we have in mind.

We now turn to our hybrid logic of overlapping of the second type. For a start, we define the extended language. We merely add two sets of so-called *nominals* to the language underlying LO, but compared to usual hybrid logic (and to former hybridizations of the language of subset spaces as well) the concept of nominal has to be weakened. Here the denotation of each nominal should be either a unique state contained in some set or a distinguished set of states. Let $\mathrm{N}_{stat} = \{i, j, \ldots\}$ and $\mathrm{N}_{sets} = \{I, J, \ldots\}$ be the corresponding sets of symbols, which we call *names of states* and *names of sets,* respectively.

Definition 5.1 (Subset spaces with names) *1. Let $\mathcal{F} = (X, \mathcal{O})$ be a subset frame so that $\emptyset \in \mathcal{O}$. A* hybrid *$\mathcal{F}$-*valuation *is a mapping $V : \mathrm{PROP} \cup \mathrm{N}_{stat} \cup \mathrm{N}_{sets} \to \mathcal{P}(X)$ so that*

- *(a) $V(i) \cap U$ is either $\emptyset$ or a singleton, for every $i \in \mathrm{N}_{stat}$ and $U \in \mathcal{O}$, and*
- *(b) $V(I) \in \mathcal{O}$ for every $I \in \mathrm{N}_{sets}$.*

2. A subset space with names *(or, in short, an* SSN*) is a triple $(X, \mathcal{O}, V)$, where $\mathcal{F} = (X, \mathcal{O})$ is a subset frame as in item 1 and V a hybrid $\mathcal{F}$-valuation.*

The definition of the satisfaction relation for nominals is as expected, i.e., $x, U \models_{\mathcal{M}} i :\iff x \in V(i)$ and $x, U \models_{\mathcal{M}} I :\iff V(I) = U$, respectively, where $\mathcal{M} = (X, \mathcal{O}, V)$ is an SSN, x, U a neighbourhood situation of the underlying frame, $i \in \mathrm{N}_{stat}$ and $I \in \mathrm{N}_{sets}$.

We only mention the most interesting axiom schema for nominals here. It reads $(\mathsf{KO})^n(I \wedge \alpha \to \mathsf{L}\beta) \vee (\mathsf{KO})^m(I \wedge \beta \to \mathsf{L}\alpha)$, where $n, m \in \mathbb{N}$, $I \in \mathrm{N}_{sets}$, and $\alpha, \beta \in \mathrm{Form}$,[3] and it captures the property that every element of N_{sets} denotes at most one $\xrightarrow{\mathsf{K}}$-class. Note that this is the point where the non-finiteness of the axiomatization appears.

By fixing the hybrid proof rules we get to the extended logical system, HLO. To get an idea of these rules, the reader is referred to Sec. 5 of the paper [10]. Technically speaking, the hybrid rules are used for establishing an appropriate *Lindenbaum Lemma,* which makes a corresponding *Existence Lemma* possible. These lemmata represent the first steps of the proof of the following theorem.

[3] Form denotes the set of formulas of the enriched language here.

Theorem 5.2 (Hybrid completeness) HLO *is sound and complete for subset spaces with names.*

In addition, we obtain the decidability of HLO.

Theorem 5.3 (Hybrid decidability) HLO *is a decidable set of formulas.*

Finally in this section, we apply the apparatus developed so far to the class of all linear subset spaces.

Definition 5.4 (Linear frames; linear spaces) *Let $\mathcal{F} = (X, \mathcal{O})$ be a subset frame. $\mathcal{F}$ is called* linear, *iff $\mathcal{O}$ is linearly ordered with respect to inclusion. A subset space (an SSN) $\mathcal{M}$ is called* linear, *iff it is based on a linear frame.*

The just defined structures can be taken as a *temporal* form of subset spaces. Thus a model of knowledge and linear time appears, which is complementary to existing ones. On this account, the logic of linear subset spaces was already studied previously (though in other contexts); see [8] and [7].[4] Nevertheless, the fact that linearity can be captured by overlapping in the presence of names is worth mentioning.

Proposition 5.5 *Let $\mathcal{M}$ be any linear SSN. Then, for all $I, J \in \mathrm{N}_{sets}$, $\mathcal{M} \models I \wedge \mathsf{P}J \rightarrow \mathsf{KP}J \vee \mathsf{O}(J \rightarrow \mathsf{KP}I)$.*

The last but one result of this paper is the characterization theorem just indicated.

Theorem 5.6 (Completeness III) *Let* HLOL *be* HLO *plus the schema from the preceding proposition. Then,* HLOL *is sound and complete for linear subset spaces with names.*

Using the same techniques as in the proof of Theorem 5.3 we obtain that HLOL is as well decidable.

Theorem 5.7 (Decidability III) HLOL *is a decidable set of formulas.*

Note that we did not use names of states for this particular application. With other examples coming it might be useful to have too such nominals at our disposal.

6 Concluding remarks

We regarded overlapping as a basic concept of subset spaces in this paper. Concerning knowledge, this means that we have got a tool for changing the knowledge state of an agent under consideration to hand. We proved several completeness and decidability results for the most general and for more special classes of frames. In some cases, we could do this only after hybridizing the ground language. This applies, in particular, to the linear time version of our approach.

We did not study complexity issues here. Apart from the treatment of further frame classes this should be a topic of future research.

[4]Note that it is unknown whether a purely modal treatment of linear subset spaces is possible.

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