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A Labeled Deduction System for the Logic UB

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***Abstract*—We propose an approach for defining labeled natural deduction systems for the class of Peircean branching temporal logics, seen as logics in their own right rather than as sublogics of Ockhamist systems. In particular, we give a system for the logic UB, i.e., the until-free fragment of CTL, and show that it is sound and complete. We also study normalization and discuss how derivations may reduce to a normal form using an appropriate management of proof contexts. Finally, we briefly discuss how to extend our system in order to capture full CTL.**

***Keywords*-Labeled deduction; branching temporal logics; natural deduction.**

I. Introduction

In labeled deduction systems for modal and other non-classical logics (e.g., [3], [13], [15]), labels are typically used to denote worlds in the underlying Kripke-style semantics. For example, the following introduction and elimination rules for the modal operators are defined in [13], [15] for the modal logic K:

$$\dfrac{\begin{matrix}[xRy]\\ \vdots \\ y : A\end{matrix}}{x : \Box A}\ \Box I \qquad \dfrac{x : \Box A \quad xRy}{y : A}\ \Box E \qquad \dfrac{y : A \quad xRy}{x : \Diamond A}\ \Diamond I \qquad \dfrac{x : \Diamond A \quad \begin{matrix}[xRy]\ [y : A]\\ \vdots \\ z : B\end{matrix}}{z : B}\ \Diamond E$$

where x, y are labels, R denotes the accessibility relation, and, in $\Box I$, y is required to be *fresh*, i.e., it must be different from x and not occur in any assumption on which $y : A$ depends other than the discharged assumption xRy. Analogously, y must be fresh in $\Diamond E$.

If we consider a logic whose models are defined by using more than one accessibility relation and whose language allows for quantifying over relations, then it seems convenient to introduce a second sort of labels for denoting such relations. This is the case of branching temporal logics. Depending on the underlying language, we distinguish between Ockhamist branching temporal logics, where any combination of linear-time operators and path quantifiers is allowed, and Peircean branching temporal logics, where the only temporal operators admitted are those obtained as a combination of one single linear-time operator immediately preceded by one single path quantifier. In this paper, we will show that having two sorts of labels allows for defining well-behaved rules for the composed temporal operators of Peircean logics.

Consider, for instance, the path quantifiers $\forall$ and $\exists$, and the linear-time operator $\Box$. By introducing not only the labels $x, y, \ldots$ for time instants but also the labels R_1, R_2, $\ldots$ for representing accessibility relations along the different paths, and by properly tuning the freshness conditions on such labels, we can, e.g., define the following rules for the Peircean operators $\forall\Box$ and $\exists\Box$, which follow the same patterns as the ones seen above for the logic K:

$$\dfrac{\begin{matrix}[xR_1y]\\ \vdots \\ y : A\end{matrix}}{x : \forall\Box A}\ \forall\Box I \qquad \dfrac{x : \forall\Box A \quad xR_1y}{y : A}\ \forall\Box E \qquad \dfrac{x : \exists\Box A \quad x\, r(x, \Box A)\, y}{y : A}\ \exists\Box E \qquad \dfrac{\begin{matrix}[xR_1y]\\ \vdots \\ y : A\end{matrix} \quad xR_1z}{x : \exists\Box A}\ \exists\Box I$$

with y and R_1 fresh in $\forall\Box I$, and y fresh in $\exists\Box I$ (but not R_1).

In the rule $\exists\Box E$, the nesting of a "universal" modality inside an "existential" quantifier does not allow for designing a rule that follows the standard pattern of a $\Diamond E$. In our approach, we tackle this problem by introducing Skolem functions for the accessibility relations: in particular, in this case, we use a Skolem function $r(x, \Box A)$ to name a path starting from x and such that A holds in all the points along that path.

Based on this novel idea of "Skolemized accessibility relations", in this paper, we build a sound and complete labeled deduction system for the Peircean logic *UB*, i.e., the until-free fragment of *CTL*. As a further contribution, we also study normalization and discuss how derivations in our system may reduce to a normal form using an appropriate management of so-called proof contexts. Finally, we briefly discuss how to extend our system to the class of Peircean branching temporal logics, and focus in particular on the extension required in order to capture full *CTL*.

We proceed as follows. In Section II, we recall the details of *UB*. In Section III, we formalize our deduction system, which we show to be sound and complete in Section IV. Section V is dedicated to the study of normalization. In Section VI, we discuss how to extend the approach to the full *CTL* language and then, in Section VII, we draw conclusions and point towards future work.

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II. The Logic *UB*

We introduce the logic *UB*, i.e., the until-free fragment of *CTL*, by briefly presenting its syntax and semantics, along with a Hilbert-style axiomatization; see [1] for further details.

Definition 1 *Given a set* $\mathcal{P}$ *of propositional symbols, the set of* (well-formed) *UB* formulas *is defined by the grammar*

$$A ::= p \mid \perp \mid A \supset A \mid \forall\Box A \mid \exists\Box A \mid \exists\bigcirc A\,,$$

where $p \in \mathcal{P}$. *Other propositional connectives and temporal operators can be defined as abbreviations in the usual way.*

Definition 2 *A UB* model *is a triple* $\mathcal{M} = (\mathcal{S}, \mathcal{R}, \mathcal{V})$ *such that* $\mathcal{S}$ *is a set of states;* $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S}$ *is a serial binary relation, i.e., for all* $s \in \mathcal{S}$, *there exists* $t \in \mathcal{S}$ *such that* $\mathcal{R}(s,t)$, *also denoted as* $s\mathcal{R}t$; *and* $\mathcal{V} : \mathcal{S} \to 2^{\mathcal{P}}$ *is a valuation function assigning a subset of* $\mathcal{P}$ *to each state in* $\mathcal{S}$.

When $s\mathcal{R}t$ *holds, we say that* t *is an* immediate successor *of* s. *An* $\mathcal{R}$-path *is a sequence of states* $(s_0, s_1, \ldots, s_n)$ *such that* $s_i\mathcal{R}s_{i+1}$ *for* $0 \le i \le n-1$. *An* s-branch b *is an infinite* $\mathcal{R}$*-path* $b = (s_0, s_1, \ldots)$ *such that* $s = s_0$. *We write* $\mathcal{R}_b$ *to denote the restriction of* $\mathcal{R}$ *to a given* s*-branch* b*, i.e.,* $t\mathcal{R}_b t'$ *iff* $t = s_i$ *and* $t' = s_{i+1}$ *for some* i*, and denote with* $\mathcal{R}_{\mathcal{B}}$ *the set of all such* $\mathcal{R}_b$. *We denote with* $\mathcal{R}^*$ *(respectively,* $\mathcal{R}_b^*$*) the reflexive and transitive closure of* $\mathcal{R}$ *(respectively,* $\mathcal{R}_b$*).*

Given a UB model $\mathcal{M}$ *and a state* $s \in \mathcal{S}$, truth *for a formula* A *is the relation* $\models^{\mathcal{M},s}$ *defined inductively as follows:*

$\not\models^{\mathcal{M},s} \perp$		
$\models^{\mathcal{M},s} p$	*iff*	$p \in \mathcal{V}(s)$
$\models^{\mathcal{M},s} A \supset B$	*iff*	$\models^{\mathcal{M},s} A$ *implies* $\models^{\mathcal{M},s} B$
$\models^{\mathcal{M},s} \forall\Box A$	*iff*	*for all* t, $s\mathcal{R}^*t$ *implies* $\models^{\mathcal{M},t} A$
$\models^{\mathcal{M},s} \exists\Box A$	*iff*	*there exists an* s*-branch* b *such that for all* t, $s\mathcal{R}_b^*t$ *implies* $\models^{\mathcal{M},t} A$
$\models^{\mathcal{M},s} \exists\bigcirc A$	*iff*	*there exists* t *such that* $s\mathcal{R}t$ *and* $\models^{\mathcal{M},t} A$

Such a definition can be extended in a standard way to the notions of satisfiability *by a model* $\mathcal{M}$*, denoted by* $\models^{\mathcal{M}}$*, and* validity *for all models, denoted by* $\models$.

The Hilbert-style axiomatization $\mathcal{H}(UB)$ for the logic *UB*, taken from [1], consists of the following set of axioms:

(*CL*)	Any tautology instance of classical propositional logic
(*A1*)	$\forall\Box(A \supset B) \supset (\forall\Box A \supset \forall\Box B)$
(*A2*)	$\forall\bigcirc(A \supset B) \supset (\forall\bigcirc A \supset \forall\bigcirc B)$
(*A3*)	$\forall\Box A \supset (\forall\bigcirc A \wedge \forall\bigcirc\forall\Box A)$
(*A4*)	$\forall\Box(A \supset \forall\bigcirc A) \supset (A \supset \forall\Box A)$
(*E1*)	$\forall\Box(A \supset B) \supset (\exists\Box A \supset \exists\Box B)$
(*E2*)	$\exists\Box A \supset (A \wedge \exists\bigcirc\exists\Box A)$
(*E3*)	$\forall\Box A \supset \exists\Box A$
(*E4*)	$\forall\Box(A \supset \exists\bigcirc A) \supset (A \supset \exists\Box A)$

and of the following two rules of inference:

(*MP*) If A and $A \supset B$ then B (*Nec*) If A then $\forall\Box A$

The set of theorems of $\mathcal{H}(UB)$ is the smallest set containing this set of axioms and closed with respect to these rules of inference. Soundness and weak completeness of this axiomatization have been proven in [1].

III. The System $\mathcal{N}(UB)$

A. Syntax and Semantics of $\mathcal{N}(UB)$

In order to formalize our labeled deduction system $\mathcal{N}(UB)$, we extend the syntax and semantics of *UB* as follows.

First of all, we extend the syntax by introducing labels that represent states and paths in the underlying semantics. More specifically, let us assume given two fixed denumerable *sets of labels* $\mathcal{L}_S$ and $\mathcal{L}_B$. Intuitively, the labels $x, y, z, \ldots$ in $\mathcal{L}_S$ refer to *states* and the labels $\lhd$, $\lhd_1$, $\lhd_2$, … in $\mathcal{L}_B$ refer to *path relations*.

We also introduce a third kind of labels. We define $\mathcal{L}_B^+ = \mathcal{L}_B \cup \{r(x, \star A) \mid x \in \mathcal{L}_S, \star \in \{\Box, \bigcirc\}, A \text{ is a } UB \text{ formula}\}$, where we will write $r(x, \star A)$ to denote labels in $\mathcal{L}_B^+ \setminus \mathcal{L}_B$, which refer to *accessibility relations corresponding to distinct paths of computation*. Further, we will write R, R_1, R_2, … to denote generic elements of $\mathcal{L}_B^+$.

We can then define the formulas of $\mathcal{N}(UB)$: if $x, y \in \mathcal{L}_S$, $R \in \mathcal{L}_B^+$ and A is a *UB* formula, then

- xRy and xR^*y are *relational well-formed formulas* (*relational formulas* or *rwffs* for short), where the superscript * denotes the reflexive and transitive closure, and
- $x : A$ is a *labeled well-formed formula* (*labeled formula* or *lwff* for short).

In order to give a semantics for our labeled system, we thus need to define explicitly an interpretation of the labels.

Definition 3 *Let* $\mathcal{L} = \mathcal{L}_S \cup \mathcal{L}_B^+$ *and let* $\mathcal{M}$ *be a UB model* $(\mathcal{S}, \mathcal{R}, \mathcal{V})$. *An* interpretation *is a function* $\mathcal{I} : \mathcal{L} \to \mathcal{S} \cup \mathcal{R}_{\mathcal{B}}$ *such that:*

- *for all* $x \in \mathcal{L}_S$, $\mathcal{I}(x) \in \mathcal{S}$;
- *for all* $R \in \mathcal{L}_B^+$, $\mathcal{I}(R) \in \mathcal{R}_{\mathcal{B}}$;
- *if* $r(x, \star A) \in \mathcal{L}_B^+ \setminus \mathcal{L}_B$, *then*
  - $\mathcal{I}(r(x, \star A)) = \mathcal{R}_b$ *for some* $\mathcal{I}(x)$*-branch* b*;*
  - *if* $\models^{\mathcal{M},\mathcal{I}(x)} \exists\star A$, *then for all* $t \in \mathcal{S}$:
    - *if* $\star = \bigcirc$, *then* $\mathcal{I}(x)\mathcal{I}(r(x, \star A))\, t$ *implies* $\models^{\mathcal{M},t} A$*;*
    - *if* $\star = \Box$, *then* $\mathcal{I}(x)\mathcal{I}(r(x, \star A))^*\, t$ *implies* $\models^{\mathcal{M},t} A$*;*

The notion of interpretation allows us to extend the truth relation of Definition 2 to labeled formulas, as well as define truth of relational formulas.

Definition 4 *Given a UB model* $\mathcal{M} = (\mathcal{S}, \mathcal{R}, \mathcal{V})$ *and an interpretation* $\mathcal{I}$ *on it,* truth *for a formula* φ *(labeled or relational) is the relation* $\models^{\mathcal{M},\mathcal{I}}$ *defined as follows:*

$\models^{\mathcal{M},\mathcal{I}} x : A$	*iff*	$\models^{\mathcal{M},\mathcal{I}(x)} A$
$\models^{\mathcal{M},\mathcal{I}} xRy$	*iff*	$\mathcal{I}(x)\mathcal{I}(R)\mathcal{I}(y)$
$\models^{\mathcal{M},\mathcal{I}} xR^*y$	*iff*	$\mathcal{I}(x)\mathcal{I}(R)^*\mathcal{I}(y)$

When $\models^{\mathcal{M},\mathcal{I}} \varphi$, *we say that* φ *is* true *in* $\mathcal{M}$ *according to* $\mathcal{I}$. *By extension:*

$$\begin{array}{lll}
\models^{\mathcal{M},\mathcal{I}} \Gamma & \textit{iff} & \models^{\mathcal{M},\mathcal{I}} \varphi \textit{ for all } \varphi \in \Gamma \\
\Gamma \models^{\mathcal{M},\mathcal{I}} \varphi & \textit{iff} & \models^{\mathcal{M},\mathcal{I}} \Gamma \textit{ implies } \models^{\mathcal{M},\mathcal{I}} \varphi \\
\models^{\mathcal{M}} \varphi & \textit{iff} & \textit{for every interpretation } \mathcal{I}, \models^{\mathcal{M},\mathcal{I}} \varphi \\
\models^{\mathcal{M}} \Gamma & \textit{iff} & \textit{for every interpretation } \mathcal{I}, \models^{\mathcal{M},\mathcal{I}} \Gamma \\
\Gamma \models \varphi & \textit{iff} & \textit{for every UB model } \mathcal{M} \textit{ and} \\
 & & \textit{interpretation } \mathcal{I}, \Gamma \models^{\mathcal{M},\mathcal{I}} \varphi
\end{array}$$

B. The Rules of $\mathcal{N}(UB)$

The rules of $\mathcal{N}(UB)$ are given in Fig. 1. We can classify them into four categories: (*i*) rules for classical connectives, (*ii*) rules for temporal operators, (*iii*) relational rules, and (*iv*) induction rules.

Rules for Classical Connectives: The rules for the logical connectives mirror those of other labeled natural deduction systems for modal logics, e.g., [13], [15]. The rules $\supset I$ and $\supset E$ are the labeled version of the standard [9] natural deduction rules for implication introduction and elimination. The rule $\perp E$ is a labeled version of *reductio ad absurdum*, where we do not enforce Prawitz's side condition that $A \neq \perp$ and we do not constrain the world in which we derive a contradiction to be the same as in the assumption.

Rules for Temporal Operators: The rules for the introduction and the elimination of $\forall\Box$, $\exists\Box$ and $\exists\bigcirc$ follow the same structure as the rules for introduction and elimination of $\Box$ in labeled systems for modal logics. Let us consider $\forall\Box I$; the idea is that the meaning of $x : \forall\Box A$ is given by the metalevel implication $x \lhd^* y \Longrightarrow y : A$ for an arbitrary path denoted by the relation $\lhd$ and an arbitrary y $\lhd^*$-accessible from x. The arbitrariness of both $\lhd$ and y is ensured by the side-conditions of the rule.

Introductions of $\exists\Box$ and $\exists\bigcirc$ follow the same principle, but relax the freshness condition on the label denoting the relation, thus allowing us to reason on a single specific path. Note that in this case a further premise (xRz) is required: such a premise works as a "witness", in the sense that it ensures that the relation R considered is indeed a relation passing through the state x.

For what concerns the elimination rules, the intuition behind $\forall\Box E$ is that if $\forall\Box A$ holds in a state x and y is accessible from x (along some path), then it is possible to conclude that A holds in y. The case of $\exists\Box E$ and $\exists\bigcirc E$ is similar but complicated by the fact that the universal linear-time operator ($\Box$ or $\bigcirc$) is preceded by an existential path quantifier ($\exists$), which prevents us from inferring the conclusion for a successor along an arbitrary relation. Our solution is based on the novel idea of using Skolem functions as names for particular relations, e.g., $r(x, \Box A)$ denotes a relation passing at x and such that if $\exists\Box A$ holds in x, then A holds at each successor of x along $r(x, \Box A)$.

Relational Rules: Relational rules allow for modeling properties of the accessibility relations.[1] In particular, the rule *base* expresses the fact that for each relation R, R^* contains R. *refl* and *trans* model reflexivity and transitivity of each relation, respectively, whereas *comp* states that it is possible to compose two relations, i.e., if xR_1^*y and yR_2^*z, then there exists a third relation $\lhd^*$ such that $x \lhd^* z$. Finally, we have two rules capturing two different aspects of the seriality of the relations. *ser* captures the fact that, given a state x, there is at least a relation passing through x and a successor along that relation. ser_{sk} says that, given a state x and a Skolem function $r(x, \star B)$, there exists a successor of x along that relation.

Induction Rules: Finally, we have two rules modeling the induction principle underlying the relation between R and R^*. Intuitively, $ind\forall$ corresponds to the axiom ($A4$) of $\mathcal{H}(UB)$, whereas $ind\exists$ corresponds to ($E4$) (cf. the proof of ($E4$) in Fig. 2, which makes crucial use of the $ind\exists$ rule).

[1]Note that in these rules we use rwffs as auxiliary formulas in order to derive lwffs. Rules treating rwffs as full-fledged first class formulas, which can be assumed and derived, could also be defined in the style of [15].

Given the rules in Fig. 1, the notions of *derivation*, *conclusion*, *open* and *discharged assumption* are the standard ones (see, e.g., [4], pp. 127-129) for natural deduction systems. We write

$$\Gamma \vdash_{\mathcal{N}(UB)} x : A$$

to say that there exists a derivation of $x : A$ in the system $\mathcal{N}(UB)$ whose open assumptions are all contained in the set of formulas Γ. A derivation of $x : A$ in $\mathcal{N}(UB)$ where all the assumptions are discharged is a *proof* of $x : A$ in $\mathcal{N}(UB)$ and we then say that $x : A$ is a *theorem* of $\mathcal{N}(UB)$ and write $\vdash_{\mathcal{N}(UB)} x : A$. To denote that Π is a derivation of $x : A$ whose set of assumptions may contain the formulas $\varphi_1, \ldots, \varphi_n$, we write

$$\begin{array}{c} \varphi_1 \ldots \varphi_n \\ \Pi \\ x : A \end{array}$$

As an example, which will also be useful for the proof of completeness later on, in Fig. 2 we give derivations for the labeled versions of the axioms ($E2$) and ($E4$), which are obtained simply by prefixing the axioms with the state label x. For simplicity, in these derivations, we use a rule $\wedge I$, a labeled version of the standard rule for introduction of $\wedge$ [10], which is not included in the presentation of the system but is easily derivable from the rules of Fig. 1.

IV. Soundness and completeness

The system $\mathcal{N}(UB)$ is sound and (weakly) complete with respect to the semantics of Section II, as can be shown by adapting standard proof techniques for labeled natural deduction systems.

Theorem 5 (Soundness) For every set Γ of labeled and relational formulas and every labeled formula $x : A$, it holds that $\Gamma \vdash_{\mathcal{N}(UB)} x : A \ \Rightarrow\ \Gamma \models x : A$.

Proof (sketch): The proof proceeds by induction on the structure of the derivation of $x : A$. The base case is when $x : A \in \Gamma$ and is trivial. There is one step case for every

$$\dfrac{\begin{array}{c}[x : A \supset \bot]\\ \vdots\\ y : \bot\end{array}}{x : A}\ \bot E \qquad \dfrac{\begin{array}{c}[x : A]\\ \vdots\\ x : B\end{array}}{x : A \supset B}\ \supset I \qquad \dfrac{x : A \supset B \quad x : A}{x : B}\ \supset E \qquad \dfrac{\begin{array}{c}[x \lhd^* y]\\ \vdots\\ y : A\end{array}}{x : \forall\Box A}\ \forall\Box I \qquad \dfrac{x : \forall\Box A \quad xR^*y}{y : A}\ \forall\Box E \qquad \dfrac{\begin{array}{c}[x \lhd y]\\ \vdots\\ z : A\end{array}}{z : A}\ ser$$

$$\dfrac{\begin{array}{c}[xR^*y]\\ \vdots\\ y : A\end{array} \quad xRz}{x : \exists\Box A}\ \exists\Box I \qquad \dfrac{x : \exists\Box A \quad x\, r(x, \Box A)^*\, y}{y : A}\ \exists\Box E \qquad \dfrac{\begin{array}{c}[xRy]\\ \vdots\\ y : A\end{array} \quad xRz}{x : \exists\bigcirc A}\ \exists\bigcirc I \qquad \dfrac{x : \exists\bigcirc A \quad x\, r(x, \bigcirc A)\, y}{y : A}\ \exists\bigcirc E \qquad \dfrac{\begin{array}{c}[x\, r(x, \star B)\, y]\\ \vdots\\ z : A\end{array}}{z : A}\ ser_{sk}$$

$$\dfrac{xRy \quad \begin{array}{c}[xR^*y]\\ \vdots\\ z : A\end{array}}{z : A}\ base \qquad \dfrac{xRz \quad \begin{array}{c}[xR^*x]\\ \vdots\\ y : A\end{array}}{y : A}\ refl \qquad \dfrac{xR^*y \quad yR^*z \quad \begin{array}{c}[xR^*z]\\ \vdots\\ w : A\end{array}}{w : A}\ trans \qquad \dfrac{xR_1^*y \quad yR_2^*z \quad \begin{array}{c}[x \lhd^* z]\\ \vdots\\ w : A\end{array}}{w : A}\ comp$$

$$\dfrac{x : A \quad xR^*y \quad \begin{array}{c}[x \lhd_1^* z] \quad [z \lhd_2 w] \quad [z : A]\\ \vdots\\ w : A\end{array}}{y : A}\ ind\forall \qquad \dfrac{x : A \quad x\, r(x, \Box A)^*\, w \quad \begin{array}{c}[x \lhd^* y] \quad [y\, r(y, \bigcirc A)\, z] \quad [y : A]\\ \vdots\\ z : A\end{array}}{w : A}\ ind\exists$$

In $\forall\Box I$ (respectively $\exists\Box I$, $\exists\bigcirc I$), y is *fresh*, i.e., it is different from x and does not occur in any assumption on which $y : A$ depends other than the discharged assumption $x \lhd^* y$ (respectively xR^*y, xRy). Moreover, in $\forall\Box I$, $\lhd$ is fresh, i.e., it does not occur in any assumption on which $y : A$ depends other than the discharged assumption $x \lhd^* y$.
In *ser*, y and $\lhd$ are fresh, i.e., they do not occur in any assumption on which $z : A$ depends other than the discharged assumption $x \lhd y$; y is different from z.
In ser_{sk}, y is fresh, i.e., it is different from x and z and does not occur in any assumption on which $z : A$ depends other than the discharged assumption.
In *comp*, $\lhd$ is fresh, i.e., it is different from R_1 and R_2 and does not occur in any assumption on which $w : A$ depends other than the discharged assumption $x \lhd^* z$.
In $ind\forall$, z, w, $\lhd_1$ and $\lhd_2$ are fresh, i.e., they are different from each other and from x, y and R, and do not occur in any assumption on which $w : A$ depends other than the discharged assumptions of the rule.
In $ind\exists$, y, z and $\lhd$ are fresh, i.e., they are different from each other and from x and w, and do not occur in any assumption on which $z : A$ depends other than the discharged assumptions of the rule.

Figure 1. The rules of $\mathcal{N}(UB)$

$$\dfrac{\dfrac{\dfrac{[x\, r(x,\Box A)\, w]^2 \quad \dfrac{[x : \exists\Box A]^1 \quad [x\, r(x,\Box A)^*\, x]^3}{x : A}\ \exists\Box E}{x : A}\ refl^3}{x : A}\ {ser_{sk}}^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[xr(x,\Box A)y]^5 \quad \dfrac{[x\, r(x,\Box A)^*\, y]^7 \quad [y\, r(x,\Box A)^*\, z]^6 \quad \dfrac{[x : \exists\Box A]^1 \quad [x\, r(x,\Box A)^*\, z]^8}{z : A}\ \exists\Box E}{z : A}\ trans^8}{z : A}\ base^7 \quad [x\, r(x,\Box A)\, w]^4}{y : \exists\Box A}\ \exists\Box I^6 \quad [x\, r(x,\Box A)\, w]^4}{x : \exists\bigcirc\exists\Box A}\ \exists\bigcirc I^5}{x : \exists\bigcirc\exists\Box A}\ {ser_{sk}}^4}{\dfrac{x : A \wedge \exists\bigcirc\exists\Box A}{x : \exists\Box A \supset (A \wedge \exists\bigcirc\exists\Box A)}\ \supset I^1}\ \wedge I$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : A]^2 \quad [x\, r(x,\Box A)^*\, w]^4 \quad \dfrac{\dfrac{\dfrac{[x : \forall\Box(A \supset \exists\bigcirc A)]^1 \quad [x \lhd^* y]^5}{y : A \supset \exists\bigcirc A}\ \forall\Box E \quad [y : A]^5}{y : \exists\bigcirc A}\ \supset E \quad [y\, r(y,\bigcirc A)\, z]^5}{z : A}\ \exists\bigcirc E}{w : A}\ ind\exists^5 \quad [x\, r(x,\Box A)\, v]^3}{x : \exists\Box A}\ \exists\Box I^4}{\dfrac{\dfrac{x : \exists\Box A}{x : A \supset \exists\Box A}\ \supset I^2}{x : \forall\Box(A \supset \exists\bigcirc A) \supset (A \supset \exists\Box A)}\ \supset I^1}\ {ser_{sk}}^3$$

Figure 2. Derivations of the labeled versions of $(E2)$ and $(E4)$

rule; we show here two representative cases as the other ones follow in a similar way.

Consider the case when the last rule applied is $\exists\Box I$:

$$\dfrac{\begin{array}{c}[xR^*y]\\ \Pi\\ y:A\end{array}\quad xRz}{x:\exists\Box A}\ \exists\Box I$$

where Π is a proof of $y : A$ from hypotheses in $\Gamma_2 = \Gamma_1 \cup \{xR^*y\}$ for some set Γ_1 of formulas, with y fresh. By the induction hypothesis, for each model $\mathcal{M}$ and interpretation $\mathcal{I}$, if $\models^{\mathcal{M},\mathcal{I}} \Gamma_2$ then $\models^{\mathcal{M},\mathcal{I}} y : A$. We consider an $\mathcal{I}$ and a $\mathcal{M} = (\mathcal{S}, \mathcal{R}, \mathcal{V})$ such that $\models^{\mathcal{M},\mathcal{I}} \Gamma = \Gamma_1 \cup \{xRz\}$, and show that $\models^{\mathcal{M},\mathcal{I}} x : \exists\Box A$. Let $\mathcal{I}(x) = s$ and $\mathcal{I}(R) = \mathcal{R}_b$. Since $\models^{\mathcal{M},\mathcal{I}} \Gamma$, we have $\models^{\mathcal{M},\mathcal{I}} xRz$, i.e., $s\mathcal{R}_b^*\mathcal{I}(z)$, b is a branch passing through s. Now let s' be an arbitrary successor of s along the branch b, i.e., $s\mathcal{R}_b^*s'$. We can define an interpretation $\mathcal{I}'$ such that $\mathcal{I}'(y) = s'$ and $\mathcal{I}'(\delta) = \mathcal{I}(\delta)$ if $\delta \neq y$. Since y is fresh and $\mathcal{I}'$ differs from $\mathcal{I}$ only in the value assigned to y, we have $\models^{\mathcal{M},\mathcal{I}'} \Gamma_1$. Moreover, $\models^{\mathcal{M},\mathcal{I}'} xRz$ and this implies $\models^{\mathcal{M},\mathcal{I}'} \Gamma_2$. By the induction hypothesis, $\models^{\mathcal{M},\mathcal{I}'} y : A$, which yields $\models^{\mathcal{M},s'} A$. But since s' was arbitrary, we can conclude $\models^{\mathcal{M},s} \exists\Box A$, i.e., $\models^{\mathcal{M},\mathcal{I}(x)} \exists\Box A$, and thus $\models^{\mathcal{M},\mathcal{I}} x : \exists\Box A$.

Consider the case in which the last rule applied is *ind*$\exists$:

$$\dfrac{\begin{array}{c}\Pi'\\ x:A\end{array}\quad x\, r(x,\Box A)^*\, w \qquad \begin{array}{c}[x \lhd^* y]\quad [y\, r(y,\bigcirc A)\, z]\quad [y:A]\\ \Pi\\ z:A\end{array}}{w:A}\ ind\exists$$

where Π is a proof of $z : A$ from hypotheses in Γ_2 and Π' is a proof of $x : A$ from hypotheses in Γ_1, with $\Gamma = \Gamma_1 \cup \{x\ r(x, \Box A)^*\ w\}$ and $\Gamma_2 = \Gamma_1 \cup \{x \lhd^* y,\ y\ r(y, \bigcirc A)\ z,\ y : A\}$ for some set Γ_1 of formulas. By applying the induction hypothesis on Π and Π', we have: $\Gamma_2 \models z : A$ and $\Gamma_1 \models x : A$. We proceed by considering a generic model $\mathcal{M} = (\mathcal{S}, \mathcal{R}, \mathcal{V})$ and a generic interpretation $\mathcal{I}$ on it such that $\models^{\mathcal{M},\mathcal{I}} \Gamma$ and showing that this entails $\models^{\mathcal{M},\mathcal{I}} w : A$. Let $\mathcal{I}(x) = s$. For each $\mathcal{R}$-successor s' of s such that $\models^{\mathcal{M},s'} A$, we can define an interpretation $\mathcal{I}'$ such that $\mathcal{I}'(y) = s'$ and $\mathcal{I}'(z)$ is an $\mathcal{R}$-successor s'' of s' along a "privileged" branch $\mathcal{I}'(r(y, \bigcirc A))$ (that makes A hold in s'' if $\bigcirc A$ holds in s'). For the other labels, $\mathcal{I}'$ behaves as $\mathcal{I}$. By induction hypothesis on Π, we get $\models^{\mathcal{M},s''} A$ and thus $\models^{\mathcal{M},s'} \bigcirc A$. But s' is just an arbitrary $\mathcal{R}$-successor of s such that $\models^{\mathcal{M},s'} A$. By induction, and by using the fact that in *UB* models it holds in general that if $s_1\mathcal{R}_b s_2$ and $s_2\mathcal{R}_{b'} s_3$ then there exists a $\mathcal{R}_{b''}$ such that $s_1\mathcal{R}_{b''} s_3$, we can prove that $\models^{\mathcal{M},s} \exists\Box A$, i.e., $\models^{\mathcal{M},\mathcal{I}} x : \exists\Box A$. Since $\models^{\mathcal{M},\mathcal{I}} \Gamma$, we also have $\models^{\mathcal{M},\mathcal{I}} xr(x, \Box A)w$. By Definition 3, we conclude $\models^{\mathcal{M},\mathcal{I}} w : A$. ∎

The proposed system $\mathcal{N}(UB)$ consists of only finitary rules; consequently, it cannot be strongly complete for the logic *UB*, which does not enjoy compactness. We can prove weak completeness, i.e., that for each valid formula of *UB*, we can derive in $\mathcal{N}(UB)$ the corresponding labeled formula.

Theorem 6 (Weak completeness) For every *UB* formula A and $x \in \mathcal{L}_S$, it holds that $\models x : A \;\Rightarrow\; \vdash_{\mathcal{N}(UB)} x : A$.

Proof (sketch): We prove the thesis by showing that $\mathcal{N}(UB)$ is complete with respect to the axiomatization $\mathcal{H}(UB)$, which is known to be sound and complete for the logic *UB* [1]. In other words, we need to prove that: (i) (the labeled versions) of all of the axioms of $\mathcal{H}(UB)$ are provable in $\mathcal{N}(UB)$ and (ii) $\mathcal{N}(UB)$ is closed under the (labeled equivalent of the) rules of inference of $\mathcal{H}(UB)$. Showing (ii) is straightforward and we omit it here. As for (i), see the derivations that we have given for the labeled versions of two of the axioms in Fig. 2; the other axioms follow similarly. ∎

V. Normalization

The presence of rules *ind*$\forall$ and *ind*$\exists$ modeling an induction principle suggests an analogy with deduction systems for Peano arithmetic (e.g., [10], [14], [4]). Although it can be proved that the standard subformula property does not hold for such systems, it is possible to consider forms of (even strong) normalization that allow us to obtain a purely syntactical proof of consistency.

In [6], where a labeled natural deduction system for the logic $BCTL^*_-$, i.e. the until-free version of CTL^* with a bundled semantics, is presented, the authors define a normalization procedure based on the use of candidates of reducibility, from which, in analogy with systems for Arithmetic, they infer a proof of the consistency of the system. We have been working on a full presentation of a normalization procedure for $\mathcal{N}(UB)$, which we expect to be able to obtain by imposing some restrictions on the system and by properly adapting the results in [6]. In this paper, we focus on the treatment of the most interesting cases, i.e., those involving the introduction of Skolem functions, which constitute the backbone of the full normalization proof. Namely, we show how to define a proper reduction rule for the case of a detour, i.e., an introduction immediately followed by an elimination of the same operator, involving the operator $\exists\Box$ (the case of $\exists\bigcirc$ is analogous).

As we discussed in the introduction, "existential" operators are typically eliminated in an indirect way (see, e.g., the rule $\Diamond E$ shown in the introduction or the rule for elimination of $\exists$ in natural deduction systems for first-order logic [9]): a "parallel" derivation is started where further assumptions providing a witness for the existential (e.g., the formulas xRy and $y : A$ in $\Diamond E$) are available; the conclusion of such a parallel derivation is then used as a conclusion of the elimination rule and we go back to the "main" derivation.

If we wanted to define an elimination rule with a similar structure for $\exists\Box$, the universal quantification expressed by the linear-time operator $\Box$ would require the possibility of assuming a rule (and not just a formula) in the parallel derivation, as allowed in the framework of higher-level natural

$$\dfrac{\begin{array}{c}[x : A \supset \bot]\\ \vdots\\ y : \bot, \Sigma\end{array}}{x : A, \Sigma}\ \bot E_N \qquad \dfrac{\begin{array}{c}[x : A]\\ \vdots\\ x : B, \Sigma\end{array}}{x : A \supset B, \Sigma}\ \supset I_N \qquad \dfrac{x : A \supset B, \Sigma_1 \quad x : A, \Sigma_2}{x : B, \Sigma_1 \cup \Sigma_2}\ \supset E_N \qquad \dfrac{\begin{array}{c}[x \lhd^* y]\\ \vdots\\ y : A, \Sigma\end{array}}{x : \forall\Box A, \Sigma}\ \forall\Box I_N \qquad \dfrac{x : \forall\Box A, \Sigma \quad xR^*y}{y : A, \Sigma}\ \forall\Box E_N \qquad \dfrac{\begin{array}{c}[x \lhd y]\\ \vdots\\ z : A, \Sigma\end{array}}{z : A, \Sigma}\ ser_N$$

$$\dfrac{\begin{array}{cc}[xR^*y] & \\ \vdots & \\ y : A, \Sigma & xRz\end{array}}{x : \exists\Box A, \Sigma}\ \exists\Box I_N \qquad \dfrac{x : \exists\Box A, \Sigma \quad x\, r(x, \Box A)^*\, y}{y : A, \Sigma \cup \{\langle r(x, \Box A), \{\varphi_1, \ldots, \varphi_n\}\rangle\}}\ \exists\Box E_N \qquad \dfrac{\begin{array}{cc}[xRy] & \\ \vdots & \\ y : A, \Sigma & xRz\end{array}}{x : \exists\bigcirc A, \Sigma}\ \exists\bigcirc I_N \qquad \dfrac{x : \exists\bigcirc A, \Sigma \quad x\, r(x, \bigcirc A)\, y}{y : A, \Sigma \cup \{\langle r(x, \bigcirc A), \{\varphi_1, \ldots, \varphi_n\}\rangle\}}\ \exists\bigcirc E_N$$

$$\dfrac{\begin{array}{cc} & [xR^*y]\\ & \vdots\\ xRy & z : A, \Sigma\end{array}}{z : A, \Sigma}\ base_N \qquad \dfrac{\begin{array}{cc} & [xR^*x]\\ & \vdots\\ xRz & y : A, \Sigma\end{array}}{y : A, \Sigma}\ refl_N \qquad \dfrac{\begin{array}{ccc} & & [xR^*z]\\ & & \vdots\\ xR^*y & yR^*z & w : A, \Sigma\end{array}}{w : A, \Sigma}\ trans_N \qquad \dfrac{\begin{array}{c}[x\, r(x, \star B)\, y]\\ \vdots\\ z : A, \Sigma\end{array}}{z : A, \Sigma \setminus \Sigma_D}\ ser_{skN}$$

$$\dfrac{\begin{array}{ccc} & & [x \lhd_1^* z] \quad [z \lhd_2 w] \quad [z : A]\\ & & \vdots\\ x : A, \Sigma_1 & xR^*y & w : A, \Sigma_2\end{array}}{y : A, \Sigma_1 \cup \Sigma_2}\ ind\forall_N \qquad \dfrac{\begin{array}{ccc} & & [x \lhd^* z]\\ & & \vdots\\ xR_1^*y & yR_2^*z & w : A, \Sigma\end{array}}{w : A, \Sigma}\ comp_N \qquad \dfrac{\begin{array}{ccc} & & [x \lhd^* y] \quad [y\, r(y, \bigcirc A)\, z] \quad [y : A]\\ & & \vdots\\ x : A, \Sigma_1 & x\, r(x, \Box A)^*\, w & z : A, \Sigma_2\end{array}}{w : A, \Sigma_1 \cup \Sigma_2 \cup \{\langle r(x, \Box A), \{\varphi_1, \ldots, \varphi_n\}\rangle\} \setminus \Sigma_D}\ ind\exists_N$$

In all the rules above, an assumption can be discharged only if the formula in the assumption does not occur in the context of the conclusion.
In $\forall\Box I_N$ (respectively $\exists\Box I_N$, $\exists\bigcirc I_N$), y is *fresh*, i.e., it is different from x and does not occur in any assumption on which $y : A$ depends other than the discharged assumption $x \lhd^* y$ (respectively xR^*y, xRy). Moreover, in $\forall\Box I_N$, $\lhd$ is fresh, i.e., it does not occur in any assumption on which $y : A$ depends other than the discharged assumption $x \lhd^* y$.
In ser_N, y and $\lhd$ are fresh, i.e., they do not occur in any assumption on which $z : A$ depends other than the discharged assumption $x \lhd y$; moreover y is different from z.
In $\exists\Box E_N$ ($\exists\bigcirc E_N$), $\varphi_1, \ldots, \varphi_n$ are the open assumptions on which $x : \exists\Box A, \Sigma$ ($x : \exists\bigcirc A, \Sigma$) depends.
In ser_{skN}, y is fresh, i.e., it is different from x and z and does not occur in any assumption on which $z : A$ depends other than the discharged assumption. If the conclusion of the rule does not depend on any open assumption where $r(x, \star B)$ occurs, then Σ_D represents any set of dependences of the form $\langle r(x, \star B), \Gamma\rangle$ for Γ a set of formulas; Σ_D is empty otherwise.
In $ind\forall_N$, z, w, $\lhd_1$ and $\lhd_2$ are fresh, i.e., they are different from each other and from x, y and R, and do not occur in any assumption on which $w : A$ depends other than the discharged assumptions of the rule.
In $comp_N$, $\lhd$ is fresh, i.e., it is different from R_1 and R_2 and does not occur in any assumption on which $w : A$ depends other than the discharged assumption $x \lhd^* z$.
In $ind\exists_N$, y, z and $\lhd$ are fresh, i.e., they are different from each other and from x and w, and do not occur in any assumption on which $z : A$ depends other than the discharged assumptions of the rule. $\varphi_1, \ldots, \varphi_n$ are the open assumptions on which $x : A, \Sigma_1$ and $z : A$ depend other than the ones discharged by the rule itself. If the conclusion of the rule does not depend on any open assumption where $r(y, \bigcirc A)$ occurs, then Σ_D represents any set of dependences of the form $\langle r(y, \bigcirc A), \Gamma\rangle$ for Γ a set of formulas; Σ_D is empty otherwise.

Figure 3. The rules of $\mathcal{N}(UB_N)$

deduction systems introduced in [12][2], i.e.

$$\dfrac{x : \exists\Box A \quad \begin{array}{c}[\frac{x \lhd^* y}{y : A}]\\ \vdots\\ z : B\end{array}}{z : B}\ \exists\Box E_{SH}$$

Inside the derivation of $z : B$, and only there, we are allowed to use a further rule that, given a fresh $\lhd$ and for any y such that $x \lhd^* y$, concludes $y : A$.

A reduction for an $\exists\Box I/\exists\Box E_{SH}$ detour is the following:

$$\dfrac{\dfrac{\begin{array}{cc}[xR_1^*w] & \\ \Pi_1 & \\ w : A & xR_1z\end{array}}{x : \exists\Box A}\ \exists\Box I \quad \begin{array}{c}[\frac{x \lhd_2^* y}{y : A}]\\ \Pi_2\\ z : B\end{array}}{z : B}\ \exists\Box E_{SH} \quad \rightsquigarrow \quad \begin{array}{c}\Pi_2^*\\ z : B\end{array}$$

where Π_2^* is obtained from Π_2 by replacing each application of $\frac{x \lhd_2^* y}{y : A}$ by a proper instance of Π_1.

Instead of adding such higher-level rules, in $\mathcal{N}(UB)$ we rather prefer to instantiate—by means of Skolem functions—the relations over which we quantify universally and define a "direct" elimination rule for $\exists\Box$ (and $\exists\bigcirc$), because this instantiation allows us to model the induction step in the induction rule $ind\exists$ (which we would not be able to express if $\exists\bigcirc$ was eliminated indirectly).

This makes our normalization procedure a bit more involved. In reducing $\exists\Box$ detours, we want to use the same idea seen with the rule $\exists\Box E_{SH}$ but, in order to do that, we need to identify, in our derivation, the subderivation that corresponds to the parallel derivation of the previous example. In such a subderivation, we will replace the Skolem function involved in the elimination rule by the relation used to introduce $\exists\Box$. Further, as in the previous example, in such a subderivation, we will possibly reuse the derivation

[2]Similarly, we might define an indirect elimination rule for $\exists\Box$ and $\exists\bigcirc$ by using Skolemization and the pattern of general elimination rules [8].

that introduced $\exists\Box$ (Π_1 in the example above) in all the cases where the Skolem function was used in a significant way, i.e., in applications of rules that specifically require the presence of a Skolem function. However, such a replacement is justified only if all the assumptions of Π_1 are still open in this subderivation. While this requirement is satisfied by construction in the case of indirect elimination rules, in our case we need to enforce it by introducing some restrictions on the applicability of rules.[3]

This leads us to define a variant $\mathcal{N}(UB_N)$ of the system $\mathcal{N}(UB)$. Some further terminology is required:

- $\langle r(x, \star A), \{\varphi_1, \ldots, \varphi_n\}\rangle$, where $r(x, \star A) \in \mathcal{L}_B^+ \setminus \mathcal{L}_B$ and φ_i is an lwff or a rwff for $1 \le i \le n$, is a *dependence*;
- $\{\sigma_1, \ldots, \sigma_n\}$, where σ_i is a dependence for $1 \le i \le n$, is a *context*;
- (φ, Σ), where φ is a lwff and Σ is a context, is a *formula with context* (for simplicity, we will often omit the parentheses).

In $\mathcal{N}(UB_N)$, we reason on formulas with contexts, where the dependences in such contexts are used to keep track of which assumptions cannot be discharged along a derivation. Namely, when eliminating $\exists\Box$ and $\exists\bigcirc$, we add in the context a dependence between the Skolem function introduced and the set of assumptions on which the eliminated formula depends. Such a dependence will be removed from the context when all the assumptions where the Skolem function occurs have been discharged.

Inference rules are, of course, modified to account for contexts. The full set of rules of the labeled natural deduction system $\mathcal{N}(UB_N)$ is given in Fig. 3.

We use $\cup$ to denote an operation on the syntax which, given two contexts Σ_1 and Σ_2, corresponding to the sets of dependences Δ_1 and Δ_2 respectively, returns any context $\Sigma_1 \cup \Sigma_2$ which corresponds to the set union of Δ_1 and Δ_2. Similarly, $\setminus$ operates on contexts as the set-minus operator does on the corresponding sets of dependences. The side-conditions of Fig. 1 still apply. Moreover, in all the rules of $\mathcal{N}(UB_N)$, an assumption can be discharged only if the formula in the assumption does not occur in the context of the conclusion (and this highlights the main use of contexts). The rules $\exists\Box E_N$, $\exists\bigcirc E_N$ and $ind\exists_N$ introduce dependences, while ser_{skN} and $ind\exists_N$ allow one to eliminate dependences from a context.

It is not difficult to prove that

Theorem 7 $\mathcal{N}(UB_N)$ and $\mathcal{N}(UB)$ are equivalent.

The truth relation for formulas with contexts can be simply defined as:

$$\models^{\mathcal{M},\mathcal{I}} (x : A, \Sigma) \quad \text{iff} \quad \models^{\mathcal{M},\mathcal{I}(x)} A$$

[3]This is similar to what is done in those systems for first-order logic in which a rule of existential instantiation is used instead of existential elimination (see, e.g., [7]).

Since $\mathcal{N}(UB_N)$ only introduces restrictions on the applicability of the rules of $\mathcal{N}(UB)$, the soundness of $\mathcal{N}(UB_N)$ is a direct consequence of the soundness of $\mathcal{N}(UB)$. Furthermore, one can show that all the $\mathcal{N}(UB)$ derivations of the axioms of $\mathcal{H}(UB)$ shown in Section IV are still legal in $\mathcal{N}(UB_N)$, thus proving that $\mathcal{N}(UB_N)$ is also complete (with respect to formulas with empty contexts).

We say that a derivation Π is *pure* if for all $R \in \mathcal{L}_B^+ \setminus \mathcal{L}_B$, R does not occur in the conclusion nor in any open assumption of Π. When normalizing, we restrict to consider pure derivations. Without loss of generality, we can assume that each pure derivation containing a detour $\exists\Box I_N/\exists\Box E_N$ has the following form:

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1 \ \ldots \ \varphi_n \quad [xR^*y] \\ \Pi_1 \\ y : A, \Sigma\end{array} \qquad xRx'}{x : \exists\Box A, \Sigma}\ \exists\Box I_N \qquad [x\, r(x, \Box A)^* z]}{\begin{array}{c} z : A, \Sigma \cup \{\langle r(x, \Box A), \{\varphi_1, \ldots, \varphi_n\}\rangle\} \\ \Pi_2 \\ w : B, \Sigma' \\ \Pi_3 \end{array}}\ \exists\Box E_N$$

where Π_2 is the smallest subderivation having $z : A, \Sigma \cup \{\langle r(x, \Box A), \{\varphi_1, \ldots, \varphi_n\}\rangle\}$ as one of its assumptions and concluding with a formula $(w : B, \Sigma')$ such that Σ' does not contain the dependence $\langle r(x, \Box A), \{\varphi_1, \ldots, \varphi_n\}\rangle$ (such a Π_2 exists since we deal with pure derivations). The derivation above can be reduced to the following one:

$$\begin{array}{c}\varphi_1 \ \ldots \ \varphi_n \quad [xR^*z] \\ \Pi_1[z/y] \\ z : A, \Sigma[z/y] \\ \Pi_2^*[R/r(x, \Box A)] \\ w : B, \Sigma' \\ \Pi_3\end{array}$$

where the assumption xR^*z can be discharged by the application that discharges $x\, r(x, \Box A) z$ in Π_2 (or by its transformation in Π_2^*, see below). The idea is that, since R is the relation that made it possible to derive $x : \exists\Box A$, we can safely replace any occurrence of $r(x, \Box A)$ by R itself.

However, a blind replacement of each occurrence of $r(x, \Box A)$ by R in Π_2 does not necessarily give rise to a well-defined $\mathcal{N}(UB_N)$ derivation. In particular, this is not the case if $r(x, \Box A)$ is used in Π_2 in a significant way, i.e., in an application of ser_{skN}, $\exists\Box I_N$ or $ind\exists_N$. In such cases, it is necessary to further modify the structure of Π_2 by means of an iterative process whose result we denote by Π_2^*. Such a process consists in: (i) replacing each application of ser_{skN} that uses $r(x, \Box A)$ as follows:

$$\dfrac{\begin{array}{c}[x\, r(x, \Box A) x''] \\ \Pi'' \\ v : C, \Sigma''\end{array}}{v : C, \Sigma'' \setminus \Sigma_D}\ ser_{skN} \qquad \rightsquigarrow \qquad \begin{array}{c} xRx' \\ \Pi''^*[R/r(x, \Box A), x'/x''] \\ v : C, \Sigma''[R/r(x, \Box A)]\end{array}$$

(*ii*) replacing each application of $\exists\Box E_N$ that uses $r(x, \Box A)$ by an instance of Π_1 as follows:

$$\dfrac{\begin{matrix}\Pi'' \\ x : \exists\Box A, \Sigma''\end{matrix} \quad xr(x, \Box A)^* v}{v : A, \Sigma'' \cup \Sigma'''} \exists\Box E_N \qquad \rightsquigarrow \qquad \begin{matrix} xR^* v \\ \Pi_1[v/x] \\ v : A, \Sigma \end{matrix}$$

(*iii*) replacing each application of $ind\exists_N$ that uses $r(x, \Box A)$ by an instance of Π_1 as in (*ii*). Note that the contexts in the conclusions are in some cases different in the derivations on the left and on the right. However, we can show that such differences are not relevant, in the sense that dependences that are present on the left but not on the right cannot contain significant restrictions for the derivation on the right; on the contrary, dependences on the right but not on the left can always be eliminated afterwards in Π_2^*.

With regard to (*ii*) and (*iii*), the fact that Π_2 is the smallest subderivation concluding with a formula $(w : B, \Sigma')$ such that Σ' does not contain $\langle r(x, \Box A), \{\varphi'_1, \ldots, \varphi'_n\}\rangle$ ensures that all the assumptions $\varphi_1, \ldots, \varphi_n$ on which the conclusion of Π_1 possibly depends are not discharged in Π_2. Thus Π_1 can be safely reused inside Π_2^*, i.e., all the assumptions $\varphi_1, \ldots, \varphi_n$ that were discharged by a rule application in Π_3 in the original derivation can be also discharged in any copy of Π_1, in the reduced derivation.

An example of a reduction of a derivation containing an $\exists\Box I_N/\exists\Box E_N$ detour is shown in Fig. 4. In the derivation above, the $\exists\Box E_N$ application concluding with $z : A, \Sigma_1$ opens a subderivation with a dependence for the Skolem function $r(x, \Box A)$. Such a subderivation terminates with the application of ser_{skN}, which discharges all the assumptions containing $r(x, \Box A)$ and eliminates the corresponding dependences from the context. The reduction replaces R for $r(x, \Box A)$ all along the subderivation. The example also shows how the structure of the subderivation needs to be modified when it contains rules that use the Skolem function in a significant way. In particular, as an instance of case (*ii*) of the description above, the $\exists\Box E_N$ application concluding with $z : A, \Sigma_2$ in the original derivation is replaced by a copy of the derivation concluding $y : A$ ($z : A$ after the substitution); while, as an instance of case (*i*) above, the application of ser_{skN} is removed in the reduced derivation and the assumption that was discharged by ser_{skN} is replaced by the assumption xRx'.

The contraction rule defined here for the case $\exists\Box I_N/\exists\Box E_N$ can be seen as a key reduction step in the definition of a complete normalization procedure, to be developed along the lines of the one presented in [6]. A full treatment is left for future work.

VI. Towards *CTL*

The framework based on the use of path labels and Skolemization, described in the previous sections for the logic *UB*, can be adapted to cover the whole class of Peircean branching-time logics. In particular, the use of path labels allows us to reason on path quantifiers in a modular way and the use of Skolemization allows us to design proper rules even in the case when the nesting of operators with an existential and a universal nature (like $\exists\Box$) makes it impossible to define a standard natural deduction rule.

The treatment of the operator *until* is notoriously complex from a proof-theoretical point of view (see [5] for a detailed discussion). In this paper, in order to simplify the presentation of our approach, we decided to focus on the until-free fragment of *CTL*. In this section, we just give an insight of how the idea of Skolemization can be used also to express introduction and elimination rules for the temporal operators containing until.

Let us consider the operator $\exists\mathsf{U}$ (the operator $\forall\mathsf{U}$ can be treated similarly). First, we recall the semantics of the operator:

$$\models^{\mathcal{M},s} \exists(A\mathsf{U}B) \;\text{ iff }\; \text{there exist an } s\text{-branch } b \text{ and a } t \in b \text{ such that } \models^{\mathcal{M},t} B \text{ and for all } v,\; s\mathcal{R}_b^* v \mathcal{R}_b t \text{ implies } \models^{\mathcal{M},v} A.$$

As we did in the cases of $\exists\Box E$ and $\exists\bigcirc E$, we can introduce Skolem functions of the form $r(x, A\mathsf{U}B)$ to denote a relation passing through x and such that $A\mathsf{U}B$ holds over the path represented by that relation. Moreover, given a state x and a path b starting from x and such that $A\mathsf{U}B$ holds on b, we can use a similar mechanism to give a name to a state along b where B actually holds and such that A was always true before. Such a name will be a function of three arguments: the state x, the relation R corresponding to b and the formula $A\mathsf{U}B$; we will denote it $w(x, R, A\mathsf{U}B)$.

Rules for the elimination of $\exists\mathsf{U}$ will have the form:

$$\frac{x : \exists(A\mathsf{U}B)}{w(x, r(x, A\mathsf{U}B), A\mathsf{U}B) : B} \exists\mathsf{U}E1$$

$$\frac{x : \exists(A\mathsf{U}B) \quad x\, r(x, A\mathsf{U}B)^* y \quad y\, r(x, A\mathsf{U}B)^* w(x, r(x, A\mathsf{U}B), A\mathsf{U}B)}{y : A} \exists\mathsf{U}E2$$

$\exists\mathsf{U}E1$ creates a state, denoted $w(x, r(x, A\mathsf{U}B), A\mathsf{U}B)$, where B holds. $\exists\mathsf{U}E2$ expresses the fact that in each state y, placed between x and $w(x, r(x, A\mathsf{U}B), A\mathsf{U}B)$ along $r(x, A\mathsf{U}B)$, A must hold.

The corresponding introduction rule will have the form:

$$\dfrac{xR^* y \quad y : B \quad \begin{matrix}[xR^* z] \quad [zR^* y] \\ \vdots \\ z : A\end{matrix}}{x : \exists(A\mathsf{U}B)} \exists\mathsf{U}I$$

where z is required to be fresh.

The rules above show that the use of Skolem functions gives us also enough expressivity to capture the semantics of temporal operators based on U. This paves the way to an extension of the system $\mathcal{N}(UB)$ able to cover full *CTL*. A complete treatment is left for future work.

VII. Conclusions

Different labeled natural deduction approaches for Peircean branching-time logics have been proposed, e.g., in [2], [11].

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x:\forall\Box A\wedge\exists\Box A]^1}{x:\forall\Box A}\,\wedge E_N \quad [xR^*y]^5}{y:A}\,\forall\Box E_N \quad [xRx']^2}{x:\exists\Box A}\,\exists\Box I^5_N \quad [x\ r(x,\Box A)^*\ z]^4}{z:A,\Sigma_1}\,\exists\Box E_N \qquad \dfrac{\dfrac{[x:\forall\Box A\wedge\exists\Box A]^1}{x:\exists\Box A}\,\wedge E_N \quad [x\ r(x,\Box A)^*\ z]^4}{z:A,\Sigma_2}\,\exists\Box E_N}{z:A\wedge A,\Sigma}\,\wedge I_N \quad [x\ r(x,\Box A)\ x'']^3}{x:\exists\Box(A\wedge A),\Sigma}\,\exists\Box I^4_N}{x:\exists\Box(A\wedge A)}\,ser_{skN}{}^3}{x:\exists\Box(A\wedge A)}\,ser_N{}^2}{x:(\forall\Box A\wedge\exists\Box A)\supset\exists\Box(A\wedge A)}\,\supset I^1_N
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x:\forall\Box A\wedge\exists\Box A]^1}{x:\forall\Box A}\,\wedge E_N \quad [xR^*z]^4}{z:A}\,\forall\Box E_N \qquad \dfrac{\dfrac{[x:\forall\Box A\wedge\exists\Box A]^1}{x:\forall\Box A}\,\wedge E_N \quad [xR^*z]^4}{z:A}\,\forall\Box E_N}{z:A\wedge A}\,\wedge I_N \quad [xRx']^2}{x:\exists\Box(A\wedge A)}\,\exists\Box I^4_N}{x:\exists\Box(A\wedge A)}\,ser_N{}^2}{x:(\forall\Box A\wedge\exists\Box A)\supset\exists\Box(A\wedge A)}\,\supset I^1_N
$$

where $\Sigma_1 = \{\langle r(x,\Box A),\{x:\forall\Box A\wedge\exists\Box A, xRx'\}\rangle\}$, $\Sigma_2 = \{\langle r(x,\Box A),\{x:\forall\Box A\wedge\exists\Box A\}\rangle\}$ and $\Sigma = \Sigma_1\cup\Sigma_2$.

Figure 4. An example of a contraction of a $\exists\Box I/\exists\Box E$ detour: the derivation above reduces to the one below

Both focus on the logic *CTL*, although [11] presents a set of operators different from the standard one. In [11], normalization is proved for the subsystem obtained by removing rules for induction, while in [2] there is no treatment of normalization.

In our approach, the use of Skolemized accessibility relations has allowed us to give a sound and complete labeled deduction system for the Peircean logic *UB*, and set up a proof of how derivations in our system may reduce to a normal form using an appropriate management of so-called proof contexts. Moreover, this approach is open to extensions: we have discussed the extension required to capture full *CTL* and, consequently, the whole class of Peircean branching temporal logics. All these issues will be discussed in detail in future work.

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