An optimal tableau system for the logic of temporal neighborhood over the reals

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***Abstract*—The propositional logic of temporal neighborhood (PNL) features two modalities that make it possible to access intervals adjacent to the right and to the left of the current one. PNL has been extensively studied in the last years. In particular, decidability and complexity of its satisfiability problem have been systematically investigated, and optimal decision procedures have been developed, for various (classes of) linear orders, including $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$. The only missing piece is that for $\mathbb{R}$. It is possible to show that PNL is expressive enough to separate $\mathbb{Q}$ and $\mathbb{R}$. Unfortunately, there is no way to reduce the satisfiability problem for PNL over $\mathbb{R}$ to that over $\mathbb{Q}$. In this paper, we first prove the NEXPTIME-completeness of the satisfiability problem for PNL over $\mathbb{R}$, and then we devise an optimal tableau system for it.**

***Keywords*-Interval Temporal Logics; Real Numbers; Decidability; Complexity; Tableaux Methods.**

Introduction

Interval temporal logics (ITLs) are a family of modal logics for reasoning about relational interval structures over linear orders, which are well suited for a number of applications [1] The 13 *binary ordering relations* between two intervals on a linear order form the set of the so-called *Allen's interval relations* [2]. Halpern and Shoham's modal logic of time intervals (HS) features a distinct unary modality for each Allen's relation and it is interpreted over frames where intervals are primitive entities [3]. While formulas of point-based temporal logics are evaluated at time points, formulas of ITLs are evaluated at time intervals. This results in a substantially higher expressive power and computational complexity of ITLs as compared to point-based ones. Hence, it does not come as a surprise that, while decidability is a common feature of point-based temporal logics, undecidability dominates among HS and its fragments [3], [4], [5]. For a long time, such a situation has discouraged further theoretical investigations on ITLs and prevented people from systematically using them in practical applications. This bleak picture started lightening up in the last few years when various non-trivial decidable ITLs have been identified [6], [7], [8], [9], [10], [11], [12]. (Un)decidability of an ITL depends on two main factors: (i) the set of its modalities, and (ii) the class of interval models (the linear order) over which it is interpreted. Gradually, it became evident that the trade-off between expressiveness and computational feasibility in ITLs is rather subtle and sometimes unpredictable, with the border between decidability and undecidability cutting right across the core of that family.

A prominent role in the family of ITLs is played by Propositional Neighborhood Logic (PNL). It features two modalities $\langle A \rangle$ and $\langle \overline{A} \rangle$ that make it possible to access intervals adjacent to the right (future) and to the left (past) of the reference one, respectively. Representation theorems and complete axiomatic systems for PNL with respect to various classes of interval neighborhood frames have been given by Goranko et al. in [13]. Expressiveness issues have been addressed in [6], [14]. Decidability and complexity of the satisfiability problem for PNL have been studied by Bresolin et al. in [6], where PNL decidability with respect to the classes of all linearly-ordered domains, well-ordered domains, and finite linearly-ordered domains, and the linear order of $\mathbb{N}$ has been proved via a reduction to the satisfiability problem for the two-variable fragment of first-order logic for binary relational structures over ordered domains [15]. Additional decidability results for PNL over the classes of dense, discrete, and finite linear orders, as well as over the linear orders of $\mathbb{Z}$ and $\mathbb{Q}$ have been obtained in [7], [8]. In all the considered (classes of) linear orders, the satisfiability problem for PNL turns out to be NEXPTIME-complete. Optimal tableau systems for PNL over various classes of linear orders, including $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$, have been developed in [7], [8], [10]. The only missing piece of information is that about PNL over $\mathbb{R}$. PNL is expressive enough to distinguish between satisfiability over different classes of linear orders. In [8], Bresolin et al. have shown that PNL can separate the class of discrete (resp., dense) linear orders from that of all linear orders. Additional separation results for specific linear orders have been given in [14], where Della Monica et al. have proved that PNL is able to separate $\mathbb{Z}$ from $\mathbb{N}$ (easy) and $\mathbb{R}$ from $\mathbb{Q}$ (difficult). As for the latter, they have shown that if a PNL-formula is satisfiable over $\mathbb{R}$, then it is satisfiable over $\mathbb{Q}$ as well, but the converse does not hold. In most separation results, the past modality $\langle \overline{A} \rangle$ plays an essential role. As an example, the future fragment of PNL, featuring the modality $\langle A \rangle$ only, is not able to distinguishing between $\mathbb{R}$ and $\mathbb{Q}$. The relationships between $\mathbb{Z}$ and $\mathbb{N}$ and between $\mathbb{R}$ and $\mathbb{Q}$ are, however, quite different. In [14], the authors give a log-space reduction from the satisfiability problem for PNL over $\mathbb{Z}$ to that over $\mathbb{N}$, that is, the satisfiability problem

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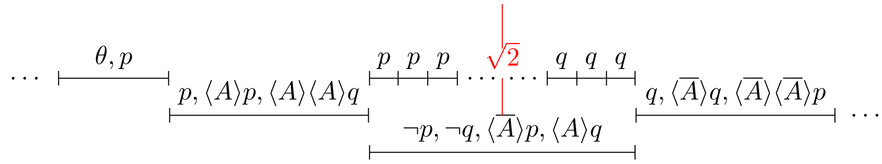

Figure 1. A model for the PNL-formula θ over $\mathbb{Q}$.

for PNL over $\mathbb{Z}$ has been reduced to that for a suitable class of PNL formulas over $\mathbb{N}$. Unfortunately, there is no way to reduce the satisfiability problem for PNL over $\mathbb{R}$ to that over $\mathbb{Q}$. To define such a reduction, we should be able to provide a characterization of the class of $\mathbb{Q}$-models corresponding to $\mathbb{R}$-models in PNL, and we are not (a simple game-theoretic argument can be used to prove it).

In this paper, we address the satisfiability problem for PNL over $\mathbb{R}$. First, we define syntax and semantics of PNL, and we briefly discuss separability of $\mathbb{R}$ and $\mathbb{Q}$ in PNL. Then, we introduce the key notion of fulfilling labeled interval structure. Next, we give a model-theoretic proof of the decidability of PNL over $\mathbb{R}$. Finally, an optimal tableau system for PNL over $\mathbb{R}$ is developed.

THE LOGIC OF TEMPORAL NEIGHBORHOOD

In this section, we introduce syntax and semantics of PNL, and we provide a formula that separates $\mathbb{R}$ and $\mathbb{Q}$ in PNL. Let D be a set of points and $\mathbb{D} = \langle D, <\rangle$ be a linear order on it. An *interval* on $\mathbb{D}$ is a pair $[d_i, d_j]$, with $d_i, d_j \in D$ and $d_i < d_j$ (strict semantics)[1]. The set of all intervals over $\mathbb{D}$ (*interval structure*) is denoted by $\mathbb{I}(\mathbb{D})$. For every $[d_i, d_j], [d'_i, d'_j] \in \mathbb{I}(\mathbb{D})$, we say that $[d'_i, d'_j]$ is a *right* (resp., *left*) *neighbor* of $[d_i, d_j]$ iff $d_j = d'_i$ (resp., $d'_j = d_i$). The vocabulary of PNL consists of a set $\mathcal{AP}$ of proposition letters, the logical connectives $\neg$ and $\vee$, and the modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$. The other connectives and the logical constants $\top$ (true) and $\bot$ (false) are defined as usual. Moreover, $[A]$ and $[\overline{A}]$ stand for the duals $\neg\langle A\rangle\neg$ and $\neg\langle\overline{A}\rangle\neg$ of $\langle A\rangle$ and $\langle\overline{A}\rangle$, respectively. *Formulae* of PNL, denoted by $\varphi, \psi, \ldots$, are recursively defined by the grammar: $\varphi ::= p \mid \neg\varphi \mid \varphi\vee\varphi \mid \langle A\rangle\varphi \mid \langle\overline{A}\rangle\varphi$. We denote by $|\varphi|$ the length of φ (number of symbols in φ). A formula of the form $\langle A\rangle\psi$, $\neg\langle A\rangle\psi$, $\langle\overline{A}\rangle\psi$, or $\neg\langle\overline{A}\rangle\psi$ is called a *temporal formula*.

An *interval model* for a PNL formula φ is a pair $\mathbf{M} = \langle\mathbb{I}(\mathbb{D}), V\rangle$, where $V : \mathbb{I}(\mathbb{D}) \longrightarrow 2^{\mathcal{AP}}$ is a *valuation function* assigning to every interval the set of proposition letters true over it. Given an interval model $\mathbf{M} = \langle\mathbb{I}(\mathbb{D}), V\rangle$ and an interval $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$, the semantics of PNL is defined recursively by the *satisfiability relation* $\Vdash$ as follows:

- for $p \in \mathcal{AP}$, $\mathbf{M}, [d_i, d_j] \Vdash p$ iff $p \in V([d_i, d_j])$;
- $\mathbf{M}, [d_i, d_j] \Vdash \psi_1 \vee \psi_2$ (resp., $\neg\psi$) iff $\mathbf{M}, [d_i, d_j] \Vdash \psi_1$ or $\mathbf{M}, [d_i, d_j] \Vdash \psi_2$ (resp., $\mathbf{M}, [d_i, d_j] \nVdash \psi$);
- $\mathbf{M}, [d_i, d_j] \Vdash \langle A\rangle\psi$ iff $\exists d_k \in D$ such that $d_k > d_j$ and $\mathbf{M}, [d_j, d_k] \Vdash \psi$;
- $\mathbf{M}, [d_i, d_j] \Vdash \langle\overline{A}\rangle\psi$ iff $\exists d_k \in D$ such that $d_k < d_i$ and $\mathbf{M}, [d_k, d_i] \Vdash \psi$.

We say that (i) φ is *satisfiable* if there exist $\mathbf{M}$ and $[d_i, d_j]$ such that $\mathbf{M}, [d_i, d_j] \Vdash \varphi$ and (ii) φ is *valid*, denoted $\models \varphi$, if it is true on every interval in every interval model. We do not impose any constraint on the valuation function, thus placing ourselves in the most general (and difficult) interval setting. As an example, given an interval $[d_i, d_j]$, it may happen that $p \in V([d_i, d_j])$ and $p \notin V([d'_i, d'_j])$ for all $[d'_i, d'_j] \subset [d_i, d_j]$.

As already pointed pointed out, PNL is expressive enough to separate different classes of linear orders as well as specific ones. Here, we focus our attention on $\mathbb{Q}$ and $\mathbb{R}$. On the one hand, it holds that, for any PNL-formula φ, if φ is satisfiable over $\mathbb{R}$, then it is also satisfiable over $\mathbb{Q}$ [14]. Roughly speaking, given an $\mathbb{R}$-model $\mathbf{M} = \langle\mathbb{I}(\mathbb{R}), V\rangle$ for φ, a $\mathbb{Q}$-model $\mathbf{M}' = \langle\mathbb{I}(\mathbb{Q}), V'\rangle$ for it can be obtained by defining a suitable (strictly monotonic) mapping from $\mathbb{Q}$ to $\mathbb{R}$ that mimicks the original valuation V over $\mathbb{R}$ by a valuation V' over $\mathbb{Q}$. The key observation is that it is always possible to replace every $d \in \mathbb{R} \setminus \mathbb{Q}$ by a suitable $d' \in \mathbb{Q}$ without affecting the truth value of a PNL (sub)formula. On the other hand, the opposite implication does not hold: there exist PNL-formulas which are satisfiable over $\mathbb{Q}$, but not over $\mathbb{R}$ [14]. Let θ be the PNL-formula: $p \wedge \langle A\rangle\langle A\rangle q \wedge [G]((p \rightarrow \langle A\rangle p) \wedge (q \rightarrow \langle\overline{A}\rangle q) \wedge (p \rightarrow [A]([\overline{A}]p \wedge [\overline{A}][\overline{A}]p)) \wedge (q \rightarrow [\overline{A}]([A]q \wedge [A][A]q)) \wedge \neg(p \wedge q) \wedge (\neg p \wedge \neg q \rightarrow \langle\overline{A}\rangle p \wedge \langle A\rangle q))$, where $[G]$ is the (PNL-definable) universal modality. θ is satisfiable over $\mathbb{Q}$, but not over $\mathbb{R}$. A $\mathbb{Q}$-model $\mathbf{M} = \langle\mathbb{I}(\mathbb{Q}), V\rangle$ for θ can be built by taking a *fictitious* point $d \in \mathbb{R} \setminus \mathbb{Q}$, say, $d = \sqrt{2}$, and by forcing p to be true over all (and only) the intervals of rational numbers $[d_i, d_j]$ to the left of d and q to be true over all (and only) the intervals of rational numbers $[d_i, d_j]$ to the right of d. It can be easily checked that $\mathbf{M}$ satisfies θ. A graphical account of $\mathbf{M}$ is given in Figure 1. The unsatisfiability of θ over $\mathbb{R}$ can be proved by a reductio ad absurdum. Intuitively, θ forces the existence of an infinite descending sequence of intervals where q holds and of an infinite ascending ones where p holds such that both the infimum of (the endpoints of) the first sequence and the supremum of (the endpoints of) the second one exist and must coincide (Dedekind-completeness of $\mathbb{R}$ is exploited

[1] As an alternative, one may assume a non-strict semantics which admits point intervals, that is, intervals of the form $[d_i, d_i]$. All results in the paper can be adapted to the case in which non-strict semantics is assumed.

here). Let d be such a point. It is possible to show that there is no way to consistently define the truth value of p and q over intervals with d as their left endpoint (θ excludes the existence of an interval where both p and q hold).

The unsatisfiability of θ over $\mathbb{R}$ can be interpreted as a plus of $\mathbb{R}$-models: structural properties of $\mathbb{R}$ exclude pathological models like the above-described $\mathbb{Q}$-model satisfying θ. Hence, $\mathbb{R}$-models can be viewed as the most appropriate models for practical applications where density is an essential ingredient of the temporal domain.

FULFILLING LABELED INTERVAL STRUCTURES

We introduce some basic notions and results. In particular, we show that the satisfiability of a PNL formula over a linearly-ordered domain $\mathbb{D}$ can be reduced to the existence of a suitable labeling of the interval structure $\mathbb{I}(\mathbb{D})$.

Let φ be a PNL-formula to be checked for satisfiability. The *closure* $\mathrm{CL}(\varphi)$ of φ is the set of all sub-formulas of φ and of their negations (we identify $\neg\neg\psi$ with ψ, $\neg\langle A\rangle\psi$ with $[A]\neg\psi$, and $\neg\langle\overline{A}\rangle\psi$ with $[\overline{A}]\neg\psi$). Among the formulas in $\mathrm{CL}(\varphi)$, a special role is played by the set of *temporal formulas* $\mathrm{TF}(\varphi) = \{\langle A\rangle\psi,\ [A]\psi, \langle\overline{A}\rangle\psi,\ [\overline{A}]\psi \in \mathrm{CL}(\varphi)\}$. By induction on the structure of φ, one can prove that, for every formula φ, $|\mathrm{CL}(\varphi)| \leq 2\cdot|\varphi|$, and $|\mathrm{TF}(\varphi)| \leq 2\cdot(|\varphi|-1)$. A *maximal set of requests* for φ is a set $S \subseteq \mathrm{TF}(\varphi)$ such that (i) for each $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$, $\langle A\rangle\psi \in S$ iff $[A]\neg\psi \notin S$, and (ii) for each $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$, $\langle\overline{A}\rangle\psi \in S$ iff $[\overline{A}]\neg\psi \notin S$. We denote by REQ_φ the set of all maximal sets of requests. $|\mathrm{REQ}_\varphi|$ is equal to $2^{\frac{|\mathrm{TF}(\varphi)|}{2}}$. We define a *$\varphi$-atom* (atom for short) as a set $A \subseteq \mathrm{CL}(\varphi)$ such that (i) for each $\psi \in \mathrm{CL}(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$, and (ii) for each $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$. Let $\mathcal{A}_\varphi$ be the set of all atoms. It holds that $|\mathcal{A}_\varphi| \leq 2^{|\varphi|}$. We connect atoms by a *binary relation* LR_φ such that for each pair of atoms $A_1, A_2 \in \mathcal{A}_\varphi$, $A_1\ LR_\varphi\ A_2$ iff (i) for each $[A]\psi \in \mathrm{CL}(\varphi)$, if $[A]\psi \in A_1$, then $\psi \in A_2$, and (ii) for each $[\overline{A}]\psi \in \mathrm{CL}(\varphi)$, if $[\overline{A}]\psi \in A_2$, then $\psi \in A_1$.

We now introduce a suitable labeling of interval structures based on atoms that will play an important role in the following proofs. We define a *φ-labeled interval structure* (LIS) as a pair $\mathbf{L} = \langle\mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$, where $\mathbb{I}(\mathbb{D})$ is an interval structure and $\mathcal{L} : \mathbb{I}(\mathbb{D}) \rightarrow \mathcal{A}_\varphi$ is a labeling function such that, for every pair of neighboring intervals $[d_i, d_j], [d_j, d_k] \in \mathbb{I}(\mathbb{D})$, $\mathcal{L}([d_i, d_j])\ LR_\varphi\ \mathcal{L}([d_j, d_k])$. If we interpret $\mathcal{L}$ as a valuation function, LISs can be viewed as *candidate models* for φ: truth of formulas devoid of temporal operators follows from the definition of atom, and universal temporal conditions, imposed by $[A]$ and $[\overline{A}]$ operators, are forced by the relation LR_φ. To turn a LIS into a *model* for φ, we must also guarantee the satisfaction of existential temporal conditions, imposed by $\langle A\rangle$ and $\langle\overline{A}\rangle$ operators. To this end, we introduce the notion of fulfilling LIS. A LIS $\mathbf{L} = \langle\mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ is *fulfilling* iff (i) for each $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$ and every $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$, if $\langle A\rangle\psi \in \mathcal{L}([d_i, d_j])$, then there exists $d_k > d_j$ such that $\psi \in \mathcal{L}([d_j, d_k])$ and (ii) for each $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$ and every $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$, if $\langle\overline{A}\rangle\psi \in \mathcal{L}([d_i, d_j])$, then there exists $d_k < d_i$ such that $\psi \in \mathcal{L}([d_k, d_i])$. The next theorem proves that for any PNL-formula φ and any linearly-ordered domain $\mathbb{D}$, the satisfiability of φ is equivalent to the existence of a fulfilling LIS with an interval labeled by φ.

Theorem 1. *A PNL-formula φ is satisfiable over a linearly-ordered domain $\mathbb{D}$ iff there exists a fulfilling* LIS $\mathbf{L} = \langle\mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ *such that $\varphi \in \mathcal{L}([d_i, d_j])$ for some $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$.*

The left-to-right implication is straightforward; the opposite one is proved by induction on the structure of φ [8]. The statement of the theorem is parametric in $\mathbb{D}$: it holds whatever linearly-ordered domain we take as $\mathbb{D}$. From now on, we say that a fulfilling LIS $\mathbf{L} = \langle\mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ *satisfies* φ iff there exists $[d_i, d_j] \in \mathbb{I}(\mathbb{D})$ such that $\varphi \in \mathcal{L}([d_i, d_j])$.

To reason about LISs, we associate with each point the set of its temporal requests. Given $\mathbf{L} = \langle\mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ and $d_i \in D$, the set of *future temporal requests* of d_i is $\mathrm{REQ}^{\mathbf{L}}_f(d_i) = \{\langle A\rangle\xi \in \mathrm{TF}(\varphi) : \exists d_j \in D(\langle A\rangle\xi \in \mathcal{L}([d_j, d_i]))\} \cup \{[A]\xi \in \mathrm{TF}(\varphi) : \exists d_j \in D([A]\xi \in \mathcal{L}([d_j, d_i]))\}$. *Past temporal requests* are defined in a similar way. The set of *temporal requests* of d_i is $\mathrm{REQ}^{\mathbf{L}}(d_i) = \mathrm{REQ}^{\mathbf{L}}_f(d_i) \cup \mathrm{REQ}^{\mathbf{L}}_p(d_i)$. Given a PNL-formula φ and a LIS $\mathbf{L}$, we define $\mathrm{REQ}^{\mathbf{L}} = \{R \in \mathrm{REQ}_\varphi \mid R = \mathrm{REQ}^{\mathbf{L}}(d) \text{ for some } d \in D\}$ and, for each $d \in D$, $Future^{\mathbf{L}}(d) = \{R \in \mathrm{REQ}^{\mathbf{L}} \mid R = \mathrm{REQ}^{\mathbf{L}}(d') \text{ for some } d' > d\}$ and $Past^{\mathbf{L}}(d) = \{R \in \mathrm{REQ}^{\mathbf{L}} \mid R = \mathrm{REQ}^{\mathbf{L}}(d') \text{ for some } d' < d\}$.

The general notion of fulfilling LIS can be applied to a single point as follows. Given $\mathbf{L} = \langle\mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$, $d_i \in D$, and $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_f(d_i)$, we say that $\langle A\rangle\psi$ is *fulfilled for* d_i in $\mathbf{L}$ if $\psi \in \mathcal{L}([d_i, d_j])$ for some $d_j > d_i$ in D. The same for $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_p(d_i)$. We say that d_i is *fulfilled* in $\mathbf{L}$ if all $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_f(d_i)$ and $\langle\overline{A}\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}_p(d_i)$ are fulfilled for d_i in $\mathbf{L}$. We also need to count the number of occurrences of $R \in \mathrm{REQ}_\varphi$ in a (portion of a) LIS. Given $\mathbf{L} = \langle\mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$, $D' \subseteq D$, and $R \in \mathrm{REQ}_\varphi$, we say that R *occurs* n *times* in D' if there exist exactly n distinct points $d_{i_1}, \ldots, d_{i_n} \in D'$ such that $\mathrm{REQ}^{\mathbf{L}}(d_{i_j}) = R$, for each $1 \leq j \leq n$. In particular, given $\mathbf{L} = \langle\mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$, $D' \subseteq D$, and $d_i \in D'$, we say that $\mathrm{REQ}^{\mathbf{L}}(d_i)$ (resp., d_i) is *unique* in D' if for every $d_j \in D'$, with $d_j \neq d_i$, $\mathrm{REQ}^{\mathbf{L}}(d_j) \neq \mathrm{REQ}^{\mathbf{L}}(d_i)$.

DECIDABILITY OF PNL OVER $\mathbb{R}$

We now show that the satisfiability problem for PNL over $\mathbb{R}$ is decidable (NEXPTIME-complete).

Definition 1. *Let φ be a PNL formula, A be an atom, and $S_1, S_2 \in \mathrm{REQ}_\varphi$. The triple $\langle S_1, A, S_2\rangle$ is an* interval-tuple *iff (i) for each $[A]\psi \in S_1$, $\psi \in A$; (ii) for each $[\overline{A}]\psi \in S_2$, $\psi \in A$; (iii) for each $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$, $\langle A\rangle\psi \in A$ iff $\langle A\rangle\psi \in S_2$; (iv) for each $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$, $\langle\overline{A}\rangle\psi \in A$ iff $\langle\overline{A}\rangle\psi \in S_1$; (v) for each $\psi \in A$, if $\langle A\rangle\psi \in \mathrm{TF}(\varphi)$, then $\langle A\rangle\psi \in S_1$; and (vi) for each $\psi \in A$, if $\langle\overline{A}\rangle\psi \in \mathrm{TF}(\varphi)$, then $\langle\overline{A}\rangle\psi \in S_2$.*

Interval-tuples act as the basic building blocks in the construction of a (pseudo-)model for a PNL formula.

Proposition 1. *Let $\mathbf{L} = \langle \mathbb{I}(\mathbb{R}), \mathcal{L}\rangle$ be a* LIS *for a PNL formula φ. For every $d_i, d_j \in \mathbb{R}$, the triple $\langle \mathrm{REQ}^{\mathbf{L}}(d_i), \mathcal{L}([d_i, d_j]), \mathrm{REQ}^{\mathbf{L}}(d_j)\rangle$ is an interval-tuple.*

Given a LIS $\mathbf{L}$, we identify the interval-tuples that occur in $\mathbf{L}$, and, among them, those which are fulfilled in $\mathbf{L}$.

Definition 2. *Let $\mathbf{L}$ be a* LIS *and $\langle R, A, R'\rangle$ be an interval-tuple. We say that $\langle R, A, R'\rangle$* occurs *in $\mathbf{L}$ if there exists $[d_i, d_j] \in \mathbb{I}(\mathbb{R})$ such that $\mathcal{L}([d_i, d_j]) = A$, $\mathrm{REQ}^{\mathbf{L}}(d_i) = R$, and $\mathrm{REQ}^{\mathbf{L}}(d_j) = R'$. If $\langle R, A, R'\rangle$ occurs in $\mathbf{L}$ and there exists $[d_i, d_j]$ such that $\mathcal{L}([d_i, d_j]) = A, \mathrm{REQ}^{\mathbf{L}}(d_i) = R$, $\mathrm{REQ}^{\mathbf{L}}(d_j) = R'$, and both d_i and d_j are fulfilled in $\mathbf{L}$, then we say that $\langle R, A, R'\rangle$ is* fulfilled *in $\mathbf{L}$ (via $[d_i, d_j]$).*

We show now how to reduce the problem of checking the existence of a fulfilling LIS for φ to that of checking the existence of a finite pseudo-model of bounded size for it.

Definition 3. *Let $\mathbf{L} = \langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ be a* LIS*, with $\mathbb{D}$ being (a restriction of) $\mathbb{R}$. A function $\mathcal{F}_{inf} : \mathrm{REQ}^{\mathbf{L}} \mapsto D \cup \{-\infty\}$ (resp., $\mathcal{F}_{sup} : \mathrm{REQ}^{\mathbf{L}} \mapsto D \cup \{+\infty\}$) is an* infimum *(resp.,* supremum*)* region function *if, for each $R \in \mathrm{REQ}^{\mathbf{L}}$, $\mathcal{F}_{inf}(R) = d$, with $d \neq -\infty$ (resp., $\mathcal{F}_{sup}(R) = d$, with $d \neq +\infty$), implies $\mathrm{REQ}^{\mathbf{L}}(d') \neq R$ for every $d' \in D$, with $d' < d$ (resp., $d' > d$).*

Definition 4. *Let $\mathbf{L} = \langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ be a* LIS*, with $\mathbb{D}$ being (a restriction of) $\mathbb{R}$, $\mathcal{F}_{inf}$ (resp., $\mathcal{F}_{sup}$) be an infimum (resp., supremum) region function, and $d \in D$. The set of sets of requests that $\mathcal{F}_{inf}$ (resp., $\mathcal{F}_{sup}$)* accumulates *on d is $\mathcal{R}_{inf}(d) = \{R \in \mathrm{REQ}^{\mathbf{L}} \mid \mathcal{F}_{inf}(R) = d \wedge R \neq \mathrm{REQ}^{\mathbf{L}}(d)\}$ (resp., $\mathcal{R}_{sup}(d) = \{R \in \mathrm{REQ}^{\mathbf{L}} \mid \mathcal{F}_{sup}(R) = d \wedge R \neq \mathrm{REQ}^{\mathbf{L}}(d)\}$). We put $\mathcal{R}_{-\infty} = \{R \in \mathrm{REQ}^{\mathbf{L}} \mid \mathcal{F}_{inf}(R) = -\infty\}$ (resp., $\mathcal{R}_{+\infty} = \{R \in \mathrm{REQ}^{\mathbf{L}} \mid \mathcal{F}_{sup}(R) = +\infty\}$).*

Let $\mathbb{D}_{\mathcal{F}} = \{\overline{d}_1 < \ldots < \overline{d}_n\}$ be the restriction of $\mathbb{D}$ to the elements which are 'suprema' or 'infima' of some $R \in \mathrm{REQ}^{\mathbf{L}}$, i.e., $\mathbb{D}_{\mathcal{F}} = \langle (range(\mathcal{F}_{inf}) \cup range(\mathcal{F}_{sup})) \cap D, <\rangle$. For each $1 \leq i \leq n-1$ and $R \in \mathrm{REQ}^{\mathbf{L}}$, R *occurs in region* i if there is $\overline{d}_i < d < \overline{d}_{i+1}$ such that $\mathrm{REQ}^{\mathbf{L}}(d) = R$, and R *occurs in region* 0 (resp., n) if there is $d < \overline{d}_1$ (resp., $d > \overline{d}_n$) such that $\mathrm{REQ}^{\mathbf{L}}(d) = R$. Given a PNL formula φ, an $\mathbb{R}$*-pseudo-model* for φ is defined as follows.

Definition 5. *Let φ be a PNL formula, $\mathbf{L} = \langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ be a finite* LIS *for φ, $\mathcal{F}_{inf}$ and $\mathcal{F}_{sup}$ be respectively an infimum and a supremum region function, and $\mathbb{D}_{\mathcal{F}} = \{\overline{d}_1 < \ldots < \overline{d}_n\}$ be the above-defined suborder of $\mathbb{D}$. The triple $\langle \mathbf{L}, \mathcal{F}_{inf}, \mathcal{F}_{sup}\rangle$ is an* $\mathbb{R}$-pseudo-model *for φ if it satisfies the following conditions:*

1) *there exists $[d, d'] \in \mathbb{I}(\mathbb{D})$ such that $\varphi \in \mathcal{L}([d, d'])$;*
2) *each interval-tuple that occurs in $\mathbf{L}$ is fulfilled;*
3) *for each $0 \leq i \leq n$ and each $R \in \mathrm{REQ}^{\mathbf{L}}$ that occurs in region i, there exists a fulfilled point d in region i such that $\mathrm{REQ}^{\mathbf{L}}(d) = R$;*
4) *each point $d \in D_{\mathcal{F}}$ is fulfilled in $\mathbf{L}$;*
5) *for all $d, d' \in D_{\mathcal{F}}$, with $d < d'$, there exists $d'' \in D$ such that $d < d'' < d'$;*
6) *for each $d \in D$ such that $\mathcal{R}_{inf}(d) = \{R_1, \ldots, R_n\}$ ($\neq \emptyset$), $\mathcal{R}_{inf}(d) \subseteq Future^{\mathbf{L}}(d)$ and, for each $1 \leq i \leq n$, there is d' such that $\mathrm{REQ}^{\mathbf{L}}(d') = R_i$ and $Future^{\mathbf{L}}(d') = Future^{\mathbf{L}}(d)$; if $\mathcal{R}_{-\infty} = \{R_1, \ldots, R_n\}$, then for each $1 \leq i \leq n$, there is d' such that $\mathrm{REQ}^{\mathbf{L}}(d') = R_i$ and $Future^{\mathbf{L}}(d') = range(\mathrm{REQ}^{\mathbf{L}})$;*
7) *the symmetric condition for $\mathcal{R}_{sup}(d)$ and $\mathcal{R}_{+\infty}$.*

Even though all interval-tuples are fulfilled in an $\mathbb{R}$-pseudo-model $\mathbf{L}$ for φ, $\mathbf{L}$ is not necessarily fulfilling, since there can be multiple occurrences of the same interval-tuple associated with different intervals. Thus, to turn $\mathbf{L}$ into a fulfilling LIS (for φ) some additional effort is needed. The next definition introduces an important ingredient of such a process.

Definition 6. *Let φ be a PNL formula, $\mathbf{L} = \langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle$ be a* LIS*, and $d \in D$ be a fulfilled point in $\mathbf{L}$. We say that:*

- *a set $ES^d_f \subseteq D$ is a* future essential set *for d if (i) for each $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$, there is $d' \in ES^d_f$ such that $\psi \in \mathcal{L}([d, d'])$ (fulfilling condition) and (ii) for each $d' \in ES^d_f$ there is $\langle A\rangle\psi \in \mathrm{REQ}^{\mathbf{L}}(d)$ such that, for each $d'' \in ES^d_f \setminus \{d'\}$, $\neg\psi \in \mathcal{L}([d, d''])$ (minimality);*
- *a set $ES^d_p \subseteq D$ is a* past essential set *for d if it satisfies the symmetric fulfilling and minimality conditions.*

From Definition 6, it follows that for each $d' \in ES^d_f$ (resp., $d' \in ES^d_p$), there exists at least one formula ψ that belongs to $\mathcal{L}([d, d'])$ (resp., $\mathcal{L}([d', d])$ only, while we cannot exclude the existence of formulas ψ that belong to the labeling of more than one interval $[d, d']$ (resp., $[d', d]$), with $d' \in ES^d_f$ (resp., $d' \in ES^d_p$).

Lemma 1. *Let φ be a PNL formula and $\mathbf{L} = \langle \mathbb{I}(\mathbb{R}), \mathcal{L}\rangle$ be a fulfilling* LIS *that satisfies it. Then, there exists an $\mathbb{R}$-pseudo-model $\mathbf{L}_{\mathbb{R}} = \langle\langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle, \mathcal{F}_{inf}, \mathcal{F}_{sup}\rangle$ for φ with $|D| \leq \left(\frac{2^{2\cdot|\varphi|} + 3\cdot 2^{|\varphi|} - 2}{2}\right) \cdot (2 \cdot |\varphi| + 1) + 2 \cdot |\varphi| \cdot 2^{3\cdot|\varphi|+1}$.*

Lemma 2. *Let φ be a PNL formula and $\mathbf{L}_{\mathbb{R}} = \langle\langle \mathbb{I}(\mathbb{D}), \mathcal{L}\rangle, \mathcal{F}_{inf}, \mathcal{F}_{sup}\rangle$ be an $\mathbb{R}$-pseudo-model for it. Then, there exists a fulfilling* LIS *$\mathbf{L}$ over $\mathbb{R}$ that satisfies φ.*

Proof: First, we build a fulfilling LIS over $\mathbb{Q}$ satisfying φ; then, we turn it into a fulfilling LIS over $\mathbb{R}$ satisfying φ.

Building a fulfilling LIS $\mathbf{L}'$ over $\mathbb{Q}$. The fulfilling LIS $\mathbf{L}' = \langle \mathbb{I}(\mathbb{Q}), \mathcal{L}_{\mathbb{Q}}\rangle$ is obtained as the 'limit' of an infinite sequence of $\mathbb{R}$-pseudo-models $\mathbf{L}_0 (= \mathbf{L}_{\mathbb{R}}), \mathbf{L}_1, \mathbf{L}_2, \ldots$. We show how to transform $\mathbf{L}_i$ into $\mathbf{L}_{i+1}$, for every $i \geq 0$, by a 3-step process. We make use of an auxiliary queue to collect points to be checked for fulfillment. Formally, for any $i \geq 0$, let Q_i be the queue of all points $d \in D_i$ that must be checked for fulfillment. Q_0 consists of all and only those $d \in D$ such that d is not fulfilled in $\mathbf{L}_{\mathbb{R}}$.

Step 1. First, we observe that $Q_i \cap D_{\mathcal{F}} = \emptyset$, as points in $D_{\mathcal{F}}$ are already fulfilled in $\mathbf{L}_{\mathbb{R}}$ (condition 4 of Definition 5) and, as we will see, points which are fulfilled in $\mathbf{L}_i$ remain fulfilled in $\mathbf{L}_{i+1}$. If Q_i is empty, then we directly move to step 2. Otherwise, we build $\mathbf{L}_{i+1}$ as follows. Let d be the element at the head of Q_i. If d is fulfilled, we remove it from Q_i and put $\mathbf{L}_{i+1} = \mathbf{L}_i$ (every point is not fulfilled when added to the queue, but subsequent expansions of the domain can make it fulfilled before the time at which it will be taken into consideration). Otherwise, there exist a $\langle A \rangle$- or a $\langle \overline{A} \rangle$-formula in $REQ^{\mathbf{L}_i}(d)$ which are not fulfilled. Let us consider the former case. By condition 3 of Definition 5, there exists d' (in the same region as d) such that $\mathrm{REQ}^{\mathbf{L}_i}(d') = \mathrm{REQ}^{\mathbf{L}_i}(d)$ and d' is fulfilled. We distinguish two cases, namely, $d' > d$ and $d' < d$, and we show how to make d fulfilling in both of them (details in the Appendix). The case of a $\langle \overline{A} \rangle$-formula in $\mathrm{REQ}^{\mathbf{L}_i}(d)$ is symmetric. Since d is fulfilled in the resulting structure $\mathbf{L}_{i+1}$, it can be safely removed from the queue. As the insertion of new points does not affect the properties of $\mathcal{F}_{inf}$ and $\mathcal{F}_{sup}$, we keep them unchanged during the whole construction. It can be easily shown that $\mathbf{L}_{i+1}$ is an $\mathbb{R}$-pseudo-model for φ.

Step 2. This step acts on the $\mathbb{R}$-pseudo-model generated by step 1 to guarantee that, in the 'limit' LIS $\mathbf{L}'$, for each $d \in range(\mathcal{F}_{inf})$ (resp., $range(\mathcal{F}_{sup})$) and each $R \in \mathcal{R}_{inf}(d)$ (resp., $\mathcal{R}_{sup}(d)$), $\mathrm{REQ}^{\mathbf{L}'}(d') \neq R$, for every $d' < d$ (resp., $d' > d$), and for every $\epsilon > 0$, there is $d < d' < d + \epsilon$ (resp., $d - \epsilon < d' < d$) such that $\mathrm{REQ}^{\mathbf{L}'}(d') = R$. For each $d \in range(\mathcal{F}_{inf})$, let $\mathcal{R}_{inf}(d) = \{R_1, ..., R_n\}$ (if $\mathcal{R}_{inf}(d) = \emptyset$, the property trivially holds) and let $\overline{d}$ be the immediate successor of d in (the current value of) $\mathbb{D}_{i+1}$ (we refer to the output of Step 1). We add n new points $d < \widehat{d}_1 < .. < \widehat{d}_n < \overline{d}$ to D_{i+1}, at positions $d + (d - \overline{d})/(n+1), .., d + n \cdot (d - \overline{d})/(n+1)$, and we force $\mathrm{REQ}^{\mathbf{L}_{i+1}}(\widehat{d}_i)$ to be equal to R_i, for $1 \leq i \leq n$. By condition 6 of Definition 5, for each $1 \leq i \leq n$, there is $d'_i \in D_{i+1}$ such that $\mathrm{REQ}^{\mathbf{L}_{i+1}}(d'_i) = R_i$ and $\mathrm{Future}^{\mathbf{L}_{i+1}}(d'_i) = \mathrm{Future}^{\mathbf{L}_{i+1}}(d)$, and thus $\mathcal{R}_{inf}(d) \subseteq \mathrm{Future}^{\mathbf{L}_{i+1}}(d'_i)$. Then, for each $i < j < n$, there is $d''_j > d'_i$ such that $\mathrm{REQ}^{\mathbf{L}_{i+1}}(d''_j) = R_j$. We put $\mathcal{L}_{i+1}([\widehat{d}_i, \widehat{d}_j]) = \mathcal{L}_{i+1}([d'_i, d''_j])$. Moreover, for every $\overline{d}' \geq \overline{d}$, there is $d'' > d'_i$ such that $\mathrm{REQ}^{\mathbf{L}_{i+1}}(d'') = \mathrm{REQ}^{\mathbf{L}_{i+1}}(\overline{d}')$. We put $\mathcal{L}_{i+1}([\widehat{d}_i, \overline{d}']) = \mathcal{L}_{i+1}([d'_i, d''])$. Finally, by definition of $\mathcal{F}_{inf}$, $d'_i > d$, and thus, for every $\overline{d}' \leq d$, we put $\mathcal{L}_{i+1}([\overline{d}', \widehat{d}_i]) = \mathcal{L}_{i+1}([\overline{d}', d'_i])$. For each $R \in \mathcal{R}_{-\infty}$, we proceed in the same way, the only difference being that new points are inserted to the left of the least element of D_{i+1}. A completely symmetric set of operations must be peformed for each $d \in range(\mathcal{F}_{sup})$ and $R \in \mathcal{R}_{sup}(d)$, and for $\mathcal{R}_{+\infty}$.

Step 3. This step forces the 'limit' LIS $\mathbf{L}'$ to be dense by simply adding a point in between any pair of consecutive points. Let d, d' be a pair of consecutive points in D_{i+1} (we refer to the output of Step 2). Since $\mathbf{L}_{i+1}$ is an $\mathbb{R}$-pseudo-model for φ, by condition 5 of Definition 5, at most one between d, d' belongs to $D_{\mathcal{F}}$. Let $d \notin D_{\mathcal{F}}$. It can be easily shown that $R = \mathrm{REQ}^{\mathbf{L}_{i+1}}(d)$ is not unique (the proof is by contradiction), and thus there is $\overline{d} \in D_{i+1}$, with $\overline{d} \neq d$ (assume $\overline{d} < d$), such that $\mathrm{REQ}^{\mathbf{L}_{i+1}}(\overline{d}) = R$. We add a new point $\widehat{d}$ to D_{i+1}, with $d < \widehat{d} < d'$, and we force $\mathrm{REQ}^{\mathbf{L}_{i+1}}(\widehat{d})$ to be equal to R. To this end, for each $d'' \in D_{i+1}$, if $d'' < d$, we put $\mathcal{L}_{i+1}([d'', \widehat{d}]) = \mathcal{L}_{i+1}([d'', d])$, otherwise ($d'' > \widehat{d}$), we put $\mathcal{L}_{i+1}([\widehat{d}, d'']) = \mathcal{L}_{i+1}([d, d''])$. Moreover, we put $\mathcal{L}_{i+1}([d, \widehat{d}]) = \mathcal{L}_{i+1}([\overline{d}, d])$. The case in which $d \in D_{\mathcal{F}}$ and $d' \notin D_{\mathcal{F}}$ is symmetric.

First, we observe that the above construction introduces infinitely many new points, but it does not remove any point. The choice of a queue to manage points which are not fulfilled guarantees that their defects are sooner or later fixed. We show now that the fulfilling LIS $\mathbf{L}'$ for φ we were looking for is the limit of this infinite construction. Let $\mathbf{L}_i^-$ be equal to $\mathbf{L}_i$ devoid of the labeling of all intervals consisting of a (non-unique) point in Q_i and a unique point in $D_i \setminus Q_i$. We define $\mathbf{L}'$ as the infinite union $\cup_{i \geq 0} \mathbf{L}_i^-$. It trivially holds that $D_i \subseteq D_{i+1}$, for every $i \geq 0$. To prove that $\mathcal{L}_i^- \subseteq \mathcal{L}_{i+1}^-$, for every $i \geq 0$, we observe that: (i) the set of unique points never changes, that is, the set of unique points in D_0 is equal to that in D_i, for every $i \geq 0$, (ii) the labeling of intervals whose endpoints are both non-unique (resp., unique) never changes, that is, it is fixed once and for all, and (iii) for each pair of points $d, d' \in D_i \setminus Q_i$ such that d is non-unique and d' is unique, if $d < d'$ (resp., $d' < d$), then $\mathcal{L}_j([d, d']) = \mathcal{L}_i([d, d'])$ (resp., $\mathcal{L}_j([d', d]) = \mathcal{L}_i([d', d])$) for all $j \geq i$, that is, the labeling of $[d, d']$ (resp., $[d', d]$) may possibly change when the non-unique point is removed from the queue, but then it remains unchanged forever (non-unique points which are fulfilled from the beginning never change "their labeling"). Finally, to prove that all points are fulfilled in $\cup_{i \geq 0} \mathbf{L}_i^-$, it suffices to observe that: (i) all unique points are fulfilled in the restriction of $\mathbf{L}_0$ to $D_0 \setminus Q_0$ (and thus in $\mathbf{L}_0^-$), and (ii) for every $i \geq 0$, all points in $D_i \setminus Q_i$ are fulfilled in $\mathbf{L}_i^-$ and the first element of Q_i may be not fulfilled in $\mathbf{L}_i$ (and thus in $\mathbf{L}_i^-$), but it is fulfilled in $\mathbf{L}_{i+1}^-$. Every point is indeed either directly inserted into $D_i \setminus Q_i$ or added to Q_i (and thus it becomes the first element of Q_j for some $j > i$) for some $i \geq 0$. To conclude, it suffices to observe that, by condition 1 of Definition 5, $\varphi \in \mathcal{L}([d, d'])$ for some $[d, d'] \in \mathbb{I}(\mathbb{D})$, and such a condition is preserved by the construction (from a given iteration on, the interval over which φ holds does not change anymore). Hence, $\mathbf{L}'$ is a fulfilling LIS for φ over $\mathbb{Q}$ (it can be easily shown that D' is isomorphic to $\mathbb{Q}$).

Turning $\mathbf{L}'$ into a fulfilling LIS $\mathbf{L}$ over $\mathbb{R}$. The fulfilling LIS $\mathbf{L} = \langle \mathbb{I}(\mathbb{R}), \mathcal{L}_{\mathbb{R}} \rangle$ for φ can be obtained from $\mathbf{L}'$ as follows. First, we put $\mathcal{L}_{\mathbb{R}}([d, d']) = \mathcal{L}_{\mathbb{Q}}([d, d'])$ for every $d, d' \in \mathbb{Q}$. Next, we define a function $\mathcal{F}_{\mathbb{R}} : \mathbb{R} \setminus \mathbb{Q} \to \mathrm{REQ}_{\varphi}$ such that for every $d_r \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{F}_{\mathbb{R}}(d_r) = R$, for some R such that $\mathcal{F}_{inf}(R) < d_r < \mathcal{F}_{sup}(R)$ (the existence of

R is guaranteed by the construction of $\mathbf{L}'$, which forces all sets of requests to accumulate on rational points - set $D_{\mathcal{F}}$). For every $d_r \in \mathbb{R} \setminus \mathbb{Q}$, we force $\mathrm{REQ}^{\mathbf{L}}(d_r)$ to be equal to $\mathcal{F}_{\mathbb{R}}(d_r)$. By construction, there exists a pair of rational points $d < d_r < d'$ such that $\mathrm{REQ}^{\mathbf{L}}(d) = \mathrm{REQ}^{\mathbf{L}}(d_r) = \mathrm{REQ}^{\mathbf{L}}(d')$ (= R). Then, for every $\overline{d} \in \mathbb{Q}$, we put (i) $\mathcal{L}_{\mathbb{R}}([\overline{d}, d_r]) = \mathcal{L}_{\mathbb{R}}([\overline{d}, d])$ if $\overline{d} < d$, (ii) $\mathcal{L}_{\mathbb{R}}([\overline{d}, d_r]) = \mathcal{L}_{\mathbb{R}}([\overline{d}, d'])$ if $d < \overline{d} < d_r$, (iii) $\mathcal{L}_{\mathbb{R}}([d_r, \overline{d}]) = \mathcal{L}_{\mathbb{R}}([d, \overline{d}])$ if $d_r < \overline{d} < d'$, and (iv) $\mathcal{L}_{\mathbb{R}}([d_r, \overline{d}]) = \mathcal{L}_{\mathbb{R}}([d', \overline{d}])$ if $\overline{d} > d'$. Moreover, we put $\mathcal{L}_{\mathbb{R}}([d, d_r]) = \mathcal{L}_{\mathbb{R}}([d_r, d']) = \mathcal{L}_{\mathbb{R}}([d, d'])$. From (i) and (iv), it immediately follows that d_r is fulfilled. To complete the construction, we must define the labeling of intervals $[d_r, d'_r]$ for $d'_r \in \mathbb{R} \setminus \mathbb{Q}$ (intervals of the form $[d'_r, d_r]$, with $d'_r \in \mathbb{R} \setminus \mathbb{Q}$, are taken into consideration in the labeling of d'_r). By definition of $\mathcal{F}_{\mathbb{R}}$, there exist $d, d' \in \mathbb{Q}$, with $d < d_r$ and $d'_r < d'$, such that $\mathrm{REQ}_{\mathbb{R}}(d) = \mathrm{REQ}_{\mathbb{R}}(d_r)$ and $\mathrm{REQ}_{\mathbb{R}}(d') = \mathrm{REQ}_{\mathbb{R}}(d'_r)$. We put $\mathcal{L}_{\mathbb{R}}([d_r, d'_r]) = \mathcal{L}_{\mathbb{R}}([d, d'])$. It is easy to check that $\mathbf{L}$ is a fulfilling LIS for φ over $\mathbb{R}$. ■

Theorem 2. *The satisfiability problem for PNL over* $\mathbb{R}$ *is decidable.*

The thesis follows from Lemma 1, Lemma 2, and Theorem 1. Theorem 2 allows us to reduce the problem of checking whether a PNL formula φ is satisfiable to that of checking the existence of an $\mathbb{R}$-*pseudo-model for* it of bounded size (the bound is given by Lemma 1). A NEXPTIME decision procedure for such a problem can be easily developed. NEXPTIME-hardness can be proved as in [10].

A TABLEAU SYSTEM FOR PNL OVER $\mathbb{R}$

In this section, we develop a tableau-based decision procedure for PNL over $\mathbb{R}$. First, we give the rules of the tableau system; then, we describe expansion strategies and blocking conditions; finally, we state termination, soundness, and completeness. Optimality easily follows.

Let us introduce basic notation and definitions. A tableau for a PNL formula φ is a special *decorated tree* $\mathcal{T}$. We associate a finite linear order $\mathbb{D}_B = \langle D_B, < \rangle$ and a *request function* $\mathrm{REQ}_B : D_B \mapsto \mathrm{REQ}_{\varphi}$ with each branch B of $\mathcal{T}$. We define an *accumulation constraint* as a triple $\langle R, direction, d \rangle$, where $R \in \mathrm{REQ}_{\varphi}$, $direction \in \{sup, inf\}$, and $d \in D_B \cup \{-\infty, +\infty\}$. Each node n in B is labeled either by a pair $\langle [d_i, d_j], A_n \rangle$, when $\langle \mathrm{REQ}_B(d_i), A_n, \mathrm{REQ}_B(d_j) \rangle$ is an interval-tuple (*expansion nodes*), or by an accumulation constraint (*accumulation nodes*). Accumulation nodes in B define two functions $\mathcal{F}^B_{inf}$ and $\mathcal{F}^B_{sup}$ as follows: for each $R \in range(\mathrm{REQ}_B)$, $\mathcal{F}^B_{inf}(R) = d$ (resp., $\mathcal{F}^B_{sup}(R) = d$) iff the accumulation node $\langle R, inf, d \rangle$ (resp., $\langle R, sup, d \rangle$) belongs to B (during the construction of B, $\mathcal{F}^B_{inf}$ and $\mathcal{F}^B_{sup}$ are partial functions on $range(\mathrm{REQ}_B)$). Likewise, we define $\mathcal{R}^B_{inf}$, $D_{\mathcal{F}^B}$, Future^B, and Past^B.

The *initial tableau* for φ consists of a single expansion node (and thus of a single branch B) labeled by the pair $\langle [d_0, d_1], A_{\varphi} \rangle$, where $\mathbb{D}_B = \{d_0 < d_1\}$ and $\varphi \in A_{\varphi}$. Given $d \in D_B$ and $\langle A \rangle \psi \in \mathrm{REQ}_B(d)$ (resp., $\langle \overline{A} \rangle \psi \in \mathrm{REQ}_B(d)$), $\langle A \rangle \psi$ (resp., $\langle \overline{A} \rangle \psi$) is *fulfilled in* B *for* d if there is a node $n' \in B$ labeled by $\langle [d, d'], A_{n'} \rangle$ (resp., $\langle [d', d], A_{n'} \rangle$) such that $\psi \in A_{n'}$. Given $d \in D_B$, d is *fulfilled in* B if each $\langle A \rangle \psi$ (resp., $\langle \overline{A} \rangle \psi$) in $\mathrm{REQ}_B(d)$ is fulfilled in B for d.

Let $\mathcal{T}$ be a tableau and B be a branch of $\mathcal{T}$, with $\mathbb{D}_B = \{d_0 < .. < d_k\}$. We denote by $B \cdot n$ (resp., $B \cdot n_1|..|n_h$) the expansion of B with an immediate successor node n (resp., h immediate successor nodes $n_1, .., n_h$). To possibly expand B, we apply one of the following *expansion rules*:

$\langle A \rangle$-rule. If there are $d_j \in D_B$ and $\langle A \rangle \psi \in \mathrm{REQ}_B(d_j)$ such that $\langle A \rangle \psi$ is not fulfilled in B for d_j, we proceed as follows. If there is not an interval-tuple $\langle \mathrm{REQ}_B(d_j), A_{\psi}, S \rangle$, with $\psi \in A_{\psi}$, we *close* B. If for every interval-tuple $\langle \mathrm{REQ}_B(d_j),\ A_{\psi}, S \rangle$, with $\psi \in A_{\psi}$, $\mathcal{F}^B_{sup}(S)$ is defined and $\mathcal{F}^B_{sup}(S) = d'$ for some $d' \leq d_j$, we *close* B as well. Otherwise, let $\langle \mathrm{REQ}_B(d_j), A_{\psi}, S \rangle$ be the selected interval-tuple. Two cases must be considered:

1) $\mathcal{F}^B_{sup}(S) = +\infty$ or $\mathcal{F}^B_{sup}(S)$ is not defined. We choose a new point d and we expand B with $h = k - j + 1$ immediate successor nodes $n_1, \ldots, n_h$ such that, for each $1 \leq l < h$, $\mathbb{D}_{B \cdot n_l} = \mathbb{D}_B \cup \{d_{j+l-1} < d < d_{j+l}\}$ (for $l = h$, we add $d > d_k$ to the linear order), $n_l = \langle [d_j, d], A_{\psi} \rangle$, with $\psi \in A_{\psi}$, $\mathrm{REQ}_{B \cdot n_l}(d) = S$, and $\mathrm{REQ}_{B \cdot n_l}(d') = \mathrm{REQ}_B(d')$ for each $d' \in D_B$;
2) there exists $h > 0$ such that $\mathcal{F}^B_{sup}(S) = d_{j+h}$. We choose a new point d and we expand B with h immediate successor nodes $n_1, \ldots, n_h$ such that, for each $1 \leq l \leq h$, $\mathbb{D}_{B \cdot n_l} = \mathbb{D}_B \cup \{d_{j+l-1} < d < d_{j+l}\}$ $n_l = \langle [d_j, d], A_{\psi} \rangle$, with $\psi \in A_{\psi}$, $\mathrm{REQ}_{B \cdot n_l}(d) = S$, and $\mathrm{REQ}_{B \cdot n_l}(d') = \mathrm{REQ}_B(d')$ for each $d' \in D_B$.

$\langle \overline{A} \rangle$-rule. Symmetric to the *$\langle A \rangle$-rule*.

Fill-in rule. If there are $d_i, d_j \in D_B$, with $d_i < d_j$, such that no node in B is decorated with $[d_i, d_j]$ and there is an interval-tuple $\langle \mathrm{REQ}_B(d_i), A, \mathrm{REQ}_B(d_j) \rangle$, then we expand B with a node $n = \langle [d_i, d_j], A \rangle$. If such an interval-tuple does not exist, we *close* B.

Dense rule. If there are two consecutive points $d_i, d_{i+1} \in D_B$, with $d_i, d_{i+1} \in D_{\mathcal{F}^B}$, we proceed as follows. If there is not an interval-tuple $\langle \mathrm{REQ}_B(d_i), A, S \rangle$ for some $S \in \mathrm{REQ}_{\varphi}$ and $A \in \mathcal{A}_{\varphi}$, we *close* B. If for each interval-tuple $\langle \mathrm{REQ}_B(d_i), A, S \rangle$, $\mathcal{F}^B_{inf}(S) = d'$ for some $d' \geq d_{i+1}$ or $\mathcal{F}^B_{sup}(S) = d'$ for some $d' \leq d_i$, we *close* B. Otherwise, let $\langle \mathrm{REQ}_B(d_i), A, S \rangle$ be the selected interval-tuple. We choose a new point d and we expand B with a node n, labeled by $\langle [d_i, d], A \rangle$, such that $\mathbb{D}_{B \cdot n} = \mathbb{D}_B \cup \{d_i < d < d_{i+1}\}$, $\mathrm{REQ}_{B \cdot n}(d) = S$, and $\mathrm{REQ}_{B \cdot n}(d') = \mathrm{REQ}_B(d')$ for every $d' \in D_B$.

Inf-rule. Let $R \in range(\mathrm{REQ}_B)$ be such that $\mathcal{F}^B_{inf}$ is undefined, d_i be the least point in D_B with $\mathrm{REQ}_B(d_i) = R$, and S be a set of requests. We expand B with $h = 2i + 3$

accumulation nodes $n_0, .., n_{h-1}$, where $n_0 = \langle R, \mathit{inf}, -\infty\rangle$ and, for each $0 \leq j \leq i$, $n_{2j+2} = \langle R, \mathit{inf}, d_j\rangle$ and $n_{2j+1} = \langle R, \mathit{inf}, d\rangle$, where $d_{j-1} < d < d_j$ ($d < d_0$ for $j = 0$) is a new point. Moreover, for $0 \leq j \leq i$, $\mathrm{REQ}_{B\cdot n_{2j+2}}(d') = \mathrm{REQ}_B(d')$ for each $d' \in D_B$, and $\mathrm{REQ}_{B\cdot n_{2j+1}}(d) = S$, $\mathbb{D}_{B\cdot n_{2j+1}} = \mathbb{D}_B \cup \{d_{j-1} < d < d_j\}$ ($\{d < d_0\}$ for $j = 0$), and $\mathrm{REQ}_{B\cdot n_{2j+1}}(d') = \mathrm{REQ}_B(d')$ for each $d' \in D_B$.

Sup-rule. Let $R \in range(\mathrm{REQ}_B)$ be such that $\mathcal{F}^B_{\mathit{sup}}$ is undefined, d_i be the greatest point in D_B with $\mathrm{REQ}_B(d_i) = R$, and S be a set of requests. We expand B with $h = 2k-2i+3$ accumulation nodes $n_0, .., n_{h-1}$, where $n_{2k-2i+2} = \langle R, \mathit{sup}, +\infty\rangle$ and, for each $0 \leq j \leq k-i$, $n_{2j} = \langle R, \mathit{sup}, d_{i+j}\rangle$ and $n_{2j+1} = \langle R, \mathit{sup}, d\rangle$, where $d_{i+j} < d < d_{i+j+1}$ ($d_k < d$ for $j = k-i$) is a new point. Moreover, for $0 \leq j \leq i$, $\mathrm{REQ}_{B\cdot n_{2j}}(d') = \mathrm{REQ}_B(d')$ for each $d' \in D_B$, and $\mathrm{REQ}_{B\cdot n_{2j+1}}(d) = S$, $\mathbb{D}_{B\cdot n_{2j+1}} = \mathbb{D}_B \cup \{d_{i+j} < d < d_{i+j+1}\}$ ($\{d_k < d\}$ for $j = k-i$), and $\mathrm{REQ}_{B\cdot n_{2j+1}}(d') = \mathrm{REQ}_B(d')$ for each $d' \in D_B$.

Inf-chain rule. Let $d \in D_B$ be such that $\mathcal{R}^B_{\mathit{inf}}(d) = \{R_1, \ldots, R_n\}$ ($\neq \emptyset$) and $1 \leq i \leq n$ be such that for each $d' \in D_B$, with $\mathrm{REQ}_B(d') = R_i$, $\mathrm{Future}^B(d') \neq \mathrm{Future}^B(d)$. If there is not an interval-tuple of the form $\langle \mathrm{REQ}_B(d), A, R_i\rangle$, for some A, we *close* B. Otherwise, let $\langle \mathrm{REQ}_B(d), A, R_i\rangle$ be such an interval-tuple and $\overline{d}$ be the immediate successor of d in $\mathbb{D}_B$. We choose a new point d' and we expand B with a node $n = \langle [d, d'], A\rangle$ such that $\mathrm{REQ}_{B\cdot n}(d') = R_i$, $\mathbb{D}_{B\cdot n} = \mathbb{D}_B \cup \{d < d' < \overline{d}\}$, and $\mathrm{REQ}_{B\cdot n}(d'') = \mathrm{REQ}_B(d'')$ for each $d'' \in D_B$.

Sup-chain rule. Symmetric to the *inf-chain rule*.

The application of any of the above rules results in the replacement of the branch B with one or more new branches, each one featuring a new node n. However, while the *fill-in rule* decorates such a node with a new interval whose endpoints already belong to D_B, the other rules add a new point which becomes the left or right endpoint of the interval associated with the new node.

We say that $d \in D_B$ is *active* iff one of the following three conditions occurs: (i) there is a node n in B, with $n = \langle R, *, d\rangle$ for some $R \in \mathrm{REQ}_\varphi$ and $* \in \{\mathit{inf}, \mathit{sup}\}$; (ii) there is a node n in B, with $n = \langle [d', d''], A\rangle$, such that either $d' = d$ or $d'' = d$, and for each expansion node $n' = \langle [\overline{d}', \overline{d}''], A'\rangle$, which is an ancestor of n in B, $\langle \mathrm{REQ}_B(\overline{d}'), A', \mathrm{REQ}_B(\overline{d}'')\rangle \neq \langle \mathrm{REQ}_B(d'), A, \mathrm{REQ}_B(d'')\rangle$; (iii) for each point d', with $\mathcal{F}^B_{\mathit{inf}}(\mathrm{REQ}^B(d)) < d' < \mathcal{F}^B_{\mathit{sup}}(\mathrm{REQ}^B(d))$, inserted in D_B before d, $\mathrm{REQ}^B(d') \neq \mathrm{REQ}^B(d)$.

Let B be a non-closed branch. We say that B is *complete* if for each $d_i, d_j \in D_B$, with $d_i < d_j$, there is a node n in B labeled by $n = \langle [d_i, d_j], A\rangle$, for some A. It can be easily shown that if B is complete, then the pair $\langle \mathbb{I}(\mathbb{D}_B), \mathcal{L}_B\rangle$ such that, for each $[d_i, d_j] \in \mathbb{I}(\mathbb{D}_B)$, $\mathcal{L}_B([d_i, d_j]) = A$ iff there is a node n in B labeled by $\langle [d_i, d_j], A\rangle$ is a LIS. We say that B is *blocked* if B is complete and for each active point $d \in B$, d is fulfilled in B.

We start from an initial tableau for φ and we apply the expansion rules to all the non-blocked and non-closed branches B. The expansion strategy is defined as follows:

1) apply the *inf-rule* until it generates no new node in B;
2) apply the *sup-rule* until it generates no new node in B;
3) apply the *Fill-in rule* until it generates no new node in B;
4) if there exist an active point $d \in D_B$ and $\langle A\rangle\psi \in \mathrm{REQ}_B(d)$ such that $\langle A\rangle\psi$ is not fulfilled in B for d, then apply the $\langle A\rangle$*-rule* on d, and then go back to step 1;
5) if there exist an active point $d \in D_B$ and $\langle \overline{A}\rangle\psi \in \mathrm{REQ}_B(d)$ such that $\langle \overline{A}\rangle\psi$ is not fulfilled in B for d, then apply the $\langle \overline{A}\rangle$*-rule* on d, and then go back to step 1;
6) apply the *inf-chain rule* until it generates no new node in B;
7) apply the *sup-chain rule* until it generates no new node in B;
8) apply the *Dense rule* until it generates no new node in B.

A tableau $\mathcal{T}$ for φ is *final* iff each branch B of $\mathcal{T}$ is closed or blocked.

Theorem 3 (Termination)**.** *Let $\mathcal{T}$ be a final tableau for a PNL formula φ and B be a branch of $\mathcal{T}$. It holds that* $|B| \leq (\binom{2^{2|\varphi|}+3\cdot 2^{|\varphi|}-2}{2} \cdot (2|\varphi|+1) + 2\cdot|\varphi|\cdot 2^{3\cdot|\varphi|+1}) \cdot (\binom{2^{2|\varphi|}+3\cdot 2^{|\varphi|}-2}{2} \cdot (2|\varphi|+1) + 2\cdot|\varphi|\cdot 2^{3\cdot|\varphi|+1} - 1)/2.$

Theorem 4 (Soundness and completeness)**.** *Let φ be a PNL formula. If $\mathcal{T}$ is a final tableau for φ that features one blocked branch, then φ is satisfiable over $\mathbb{R}$ and, conversely, if φ is satisfiable over $\mathbb{R}$, then there exists a final tableau for φ with at least one blocked branch.*

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APPENDIX

Proof of **Lemma 2** (**step 1**).

Case 1. There exists $d' > d$ such that $\mathrm{REQ}^{\mathbf{L}_i}(d') = \mathrm{REQ}^{\mathbf{L}_i}(d)$ and d' is fulfilled. Let $ES_f^{d'} = \{d_1, ..., d_k\}$. For $j = 1, ..., k$, we proceed as follows:

Case 1.a. If d_j is unique, then we put $\mathcal{L}_{i+1}([d, d_j]) = \mathcal{L}_i([d', d_j])$. Such a replacement does not introduce new defects for d_j. By contradiction, suppose that there is a formula $\langle \overline{A} \rangle \theta \in REQ^{\mathbf{L}_i}(d_j)$ that is fulfilled in $\mathbf{L}_i$ by $[d, d_j]$ only. Since by condition 2 of Definition 5 $\langle REQ^{\mathbf{L}_i}(d), \mathcal{L}_i([d, d_j]), \mathrm{REQ}^{\mathbf{L}_i}(d_j)\rangle$ is fulfilled in $\mathbf{L}_i$, there is $[d'', d''']$ such that $\langle REQ^{\mathbf{L}_i}(d), \mathcal{L}_i([d, d_j]), \mathrm{REQ}^{\mathbf{L}_i}(d_j)\rangle$ is fulfilled in $\mathbf{L}_i$ via $[d'', d''']$. Since d_j is unique, $d''' = d_j$. However, since d is not fulfilled in $\mathbf{L}_i$, $d'' \neq d$. Hence, $[d'', d_j]$, with $d'' \neq d$, fulfills $\langle \overline{A} \rangle \theta$, thus contradicting the hypothesis that changing the labeling of $[d, d_j]$ causes a defect for d_j.

Case 1.b. If d_j is not unique, then there is $\overline{d} \neq d_j$ such that $\mathrm{REQ}^{\mathbf{L}_i}(\overline{d}) = \mathrm{REQ}^{\mathbf{L}_i}(d_j)$. Three cases are possible: (i) $d_j \neq \mathcal{F}_{inf}(\mathrm{REQ}^{\mathbf{L}_i}(d_j))$ and $d_j \neq \mathcal{F}_{sup}(\mathrm{REQ}^{\mathbf{L}_i}(d_j))$, (ii) $d_j = \mathcal{F}_{inf}(\mathrm{REQ}^{\mathbf{L}_i}(d_j))$, and (iii) $d_j = \mathcal{F}_{sup}(\mathrm{REQ}^{\mathbf{L}_i}(d_j))$ (since d_j is not unique, $\mathcal{F}_{inf}(\mathrm{REQ}^{\mathbf{L}_i}(d_j)) \neq \mathcal{F}_{sup}(\mathrm{REQ}^{\mathbf{L}_i}(d_j))$). In cases (i) and (ii), we insert a new point $\widehat{d}$ immediately after d_j and we force $\mathrm{REQ}^{\mathbf{L}_{i+1}}(\widehat{d})$ to be equal to $\mathrm{REQ}^{\mathbf{L}_i}(d_j)$ as follows: we put $\mathcal{L}_{i+1}([d'', \widehat{d}]) = \mathcal{L}_i([d'', d_j])$ (if $d'' < \widehat{d}$) and $\mathcal{L}_{i+1}([\widehat{d}, d'']) = \mathcal{L}_i([d_j, d''])$ (if $d'' > \widehat{d}$) for each d'', with $d'' \neq d$, $d'' \neq d'$, and $d'' \neq d_j$; moreover, we put $\mathcal{L}_{i+1}([d, \widehat{d}]) = \mathcal{L}_i([d', d_j])$ and $\mathcal{L}_{i+1}([d', \widehat{d}]) = \mathcal{L}_i([d, d_j])$. In such a way, d satisfies over $[d, \widehat{d}]$ the request that d' satisfies over $[d', d_j]$. Moreover, $\widehat{d}$ satisfies the same past requests that d_j satisfies: $\widehat{d}$ satisfies over $[d, \widehat{d}]$ (resp., $[d', \widehat{d}]$) the request that d_j satisfies over $[d', d_j]$ (resp., $[d, d_j]$) and it satisfies the remaining past requests over intervals that start at the same points where the intervals over which d_j satisfies them start. Finally, if $\overline{d} > d_j$, we put $\mathcal{L}_{i+1}([d_j, \widehat{d}]) = \mathcal{L}_i([d_j, \overline{d}])$; $\mathcal{L}_{i+1}([d_j, \widehat{d}]) = \mathcal{L}_i([\overline{d}, d_j])$ otherwise. The labeling remains unchanged for all the remaining pairs d_l, d_r t, that is, $\mathcal{L}_{i+1}([d_l, d_r]) = \mathcal{L}_i([d_l, d_r])$. Now, by definition of $\mathcal{L}_{i+1}$, if d_j is fulfilled (in $\mathbf{L}_i$), then $\widehat{d}$ is fulfilled (in $\mathbf{L}_{i+1}$), while if d_j is not fulfilled (in $\mathbf{L}_i$), being $\widehat{d}$ fulfilled or not (in $\mathbf{L}_{i+1}$) depends on the labeling of the interval $[d_j, \widehat{d}]$. If $\widehat{d}$ is not fulfilled (in $\mathbf{L}_{i+1}$), we insert it into Q_{i+1}. The case (iii) is completely symmetric, and thus its description is omitted.

Case 2. Every $d' > d$ such that $\mathrm{REQ}^{\mathbf{L}_i}(d') = \mathrm{REQ}^{\mathbf{L}_i}(d)$ (if any) is not fulfilled. By condition 3 of Definition 5, there is $d' < d$ such that (i) $\mathrm{REQ}^{\mathbf{L}_i}(d') = \mathrm{REQ}^{\mathbf{L}_i}(d)$, (ii) d' is fulfilled, (iii) for each $d' \leq d'' \leq d$, $d'' \notin D_{\mathcal{F}}$ (d' belongs to the same region as d), and (iv) for each $d' < d'' < d$, if $\mathrm{REQ}^{\mathbf{L}_i}(d'') = \mathrm{REQ}^{\mathbf{L}_i}(d)$, then d'' is not fulfilled (d' is the greatest fulfilling point with the same requests as d). We first prove that $\mathrm{Past}^{\mathbf{L}_i}(d') = \mathrm{Past}^{\mathbf{L}_i}(d)$. The proof is by reductio ad absurdum. Suppose that there is $d' < d'' < d$ such that $\mathrm{REQ}^{\mathbf{L}_i}(d'') \notin \mathrm{Past}^{\mathbf{L}_i}(d')$. Since $\mathbf{L}^i$ is an $\mathbb{R}$-pseudo-model, by condition 2 of Definition 5 there are $\overline{d}, \overline{d}' \in D_i$ such that $\langle \mathrm{REQ}^{\mathbf{L}_i}(d''), \mathcal{L}_i([d'', d]), \mathrm{REQ}^{\mathbf{L}_i}(d)\rangle$ is fulfilled in $\mathbf{L}^i$ via $[\overline{d}, \overline{d}']$. By definition, both $\overline{d}$ and $\overline{d}'$ are fulfilled; moreover, $\mathrm{REQ}^{\mathbf{L}_i}(\overline{d}) = \mathrm{REQ}^{\mathbf{L}_i}(d'')$, $\mathrm{REQ}^{\mathbf{L}_i}(\overline{d}') = \mathrm{REQ}^{\mathbf{L}_i}(d)$, and $\mathcal{L}_i([\overline{d}, \overline{d}']) = \mathcal{L}_i([d'', d])$; finally, since $\mathrm{REQ}^{\mathbf{L}_i}(d'') \notin \mathrm{Past}^{\mathbf{L}_i}(d')$, $d' < \overline{d} < \overline{d}'$. However, since by condition (iv) d' is the greatest fulfilled element in D_i with $\mathrm{REQ}^{\mathbf{L}_i}(d') = \mathrm{REQ}^{\mathbf{L}_i}(d)$, $\overline{d}'$ cannot be greater than d' (contradiction). Hence, $\mathrm{Past}^{\mathbf{L}_i}(d') = \mathrm{Past}^{\mathbf{L}_i}(d)$. Now, let $ES_f^{d'} = \{d_1, .., d_k\}$. For $j = 1, .., k$, we proceed as follows. If d_j is unique, then $d_j > d$, since $\mathrm{Past}^{\mathbf{L}_i}(d') = \mathrm{Past}^{\mathbf{L}_i}(d)$, and thus Case 1a applies. If d_j is not unique and $d_j > d$, Case 1b applies. If d_j is not unique and $d' < d_j < d$, we insert a new point $\widehat{d}$ immediately after d and we force $\mathrm{REQ}^{\mathbf{L}_{i+1}}(\widehat{d})$ to be equal to $\mathrm{REQ}^{\mathbf{L}_i}(d_j)$. For each d'', with $d'' < d_j$ (resp., $d'' > \widehat{d}$), we put $\mathcal{L}_{i+1}([d'', \widehat{d}]) = \mathcal{L}_i([d'', d_j])$ (resp., $\mathcal{L}_{i+1}([\widehat{d}, d'']) = \mathcal{L}_i([d_j, d''])$). Since $\mathrm{Past}^{\mathbf{L}_i}(d') = \mathrm{Past}^{\mathbf{L}_i}(d)$, for each $d_j \leq d'' < d$, there is $d''' < d'$ such that $\mathrm{REQ}^{\mathbf{L}_i}(d''') = \mathrm{REQ}^{\mathbf{L}_i}(d'')$, and thus we put $\mathcal{L}_{i+1}([d'', \widehat{d}]) = \mathcal{L}_i([d''', d_j])$. Finally, we put $\mathcal{L}_{i+1}([d, \widehat{d}]) = \mathcal{L}_i([d', d_j])$. As in case 1, if $\widehat{d}$ is not fulfilled, we insert it into Q_{i+1}.