Mek4250 - Mandatory assignment 1
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Exercise 1

a) The H^p norm of a function $u(x,y)$ of two variables on $\Omega = (0,1)^2$ is defined as follows:

$$||u||^2_{H^p(\Omega)} = \sum_{i=0}^{p}\sum_{j=0}^{i}\binom{i}{j}||\frac{\partial^i u}{\partial^j x \partial^{i-j} y}||^2_{L^2(\Omega)} \tag{1}$$

In our case $u(x,y) = sin(k\pi x)cos(l\pi y)$. We easily see that the derivative of this function can be expressed with the following formula:

$$\frac{\partial^i u(x,y)}{\partial^j x \partial^i y} = (k\pi)^j (l\pi)^i f_j(k\pi x) g_j(l\pi y) \tag{2}$$

were f_j is the j-th derivative of $sin(x)$ and g_i is the i-th derivative of $cos(y)$. Now lets look at the L^2 norm of (2).

$$||\frac{\partial^i u(x,y)}{\partial^j x \partial^i y}||^2_{L^2(\Omega)} = (k\pi)^{2j}(l\pi)^{2i}\int\int_\Omega f_j(k\pi x)^2 g_i(l\pi y)^2 dxdy$$
$$= (k\pi)^{2j}(l\pi)^{2i}\int_0^1 f_j(k\pi x)^2 dx \int_0^1 g_i(l\pi y)^2 dy$$

Since f_j^2 and g_i^2 are:

$$f_j(x)^2 = \begin{cases} sin^2(x) & \text{j even} \\ cos^2(x) & \text{j odd} \end{cases}$$

and

$$g_i(y)^2 = \begin{cases} cos^2(y) & \text{i even} \\ sin^2(y) & \text{i odd} \end{cases}$$

and since

$$\int_0^1 sin^2(l\pi y)dy = \int_0^1 cos^2(l\pi y)dy = \frac{1}{2}$$

we get the following expression for the square of the L^2 norm of a general derivative of u:

$$||\frac{\partial^i u(x,y)}{\partial^j x \partial^i y}||^2_{L^2(\Omega)} = \frac{1}{4}(k\pi)^{2j}(l\pi)^{2i}$$

If we plug this into (1), we get

$$||u||^2_{H^p(\Omega)} = \frac{1}{4}\sum_{i=0}^{p}\sum_{j=0}^{i}\binom{i}{j}(k\pi)^{2j}(l\pi)^{2(i-j)}$$

Exercise 2

a) Assume our solution is on the form $u(x,y) = X(x)Y(y)$. If we plug this into our equation and assume that both X and Y are nonzero, we get:

$$
\begin{aligned}
&-\mu(X''Y + XY'') + X'Y = 0 \iff \frac{-\mu X'' + X'}{X} - \mu\frac{Y''}{Y} = 0 \\
&\iff -\mu X'' + X' = \lambda X \text{ and } \mu Y'' = \lambda Y
\end{aligned}
$$

Now lets look at the boundary conditions, starting withe the Dirichlet conditions:

$$
u(0,y) = 0 \iff X(0)Y(y) = 0 \Rightarrow X(0) = 0
$$

Since $Y(y) = 0$ would be a contradiction contradiction

$$
u(1,y) = 1 \iff X(1)Y(y) = 1 \Rightarrow Y(y) = 1/X(1)
$$

This means that $Y(y)$ is a constant. This does not contradict our Neumann boundary conditions, since they say that the y-derivative is zero at $y = 0$ and $y = 1$. This means that our PDE is really an ODE on the form:

$$
\begin{cases} -\mu X''(x) + X'(x) = 0 \\ X(0) = 0,\ X(1) = 1 \end{cases}
$$

This gives us:

$$
\begin{aligned}
\mu X'(x) = X(x) + C &\iff (X(x)e^{\frac{-x}{\mu}})' = Ce^{\frac{-x}{\mu}} && (3) \\
&\iff X(x) = C' + De^{\frac{x}{\mu}} && (4)
\end{aligned}
$$

Our boundary terms yields

$$
\begin{aligned}
C' &= -D \\
C' + De^{\frac{1}{\mu}} &= 1
\end{aligned}
$$

The solution to this system is:

$$
\begin{aligned}
C' &= \frac{1}{1 - e^{\frac{1}{\mu}}} \\
D &= -\frac{1}{1 - e^{\frac{1}{\mu}}}
\end{aligned}
$$

Putting this into (8) gives us:

$$
X(x) = \frac{1 - e^{\frac{x}{\mu}}}{1 - e^{\frac{1}{\mu}}}
$$

and

$$
u(x,y) = \frac{1 - e^{\frac{x}{\mu}}}{1 - e^{\frac{1}{\mu}}}
$$