Adjoint Equation

Looking at an optimal control problem

$$\min_{y,u} J(y,u) \text{ subject to } E(y,u) = 0$$

Where

$$J(y,u) = \frac{1}{2}\int_0^T u^2 dt + \frac{1}{2}(y(T) - y^T)^2 \tag{1}$$

and

$$E(y,u) = y' - \alpha y - u \tag{2}$$
$$y(0) = y_0 \tag{3}$$

We solve this problem by reducing (1) to $\hat{J}(u) = J(y(u),u)$ and computing the gradient of $\hat{J}$ in direction s $\langle \hat{J}'(u), s\rangle$.

$$\langle \hat{J}'(u), s\rangle = \langle \frac{\partial J(y(u),u)}{\partial u}, s\rangle \tag{4}$$
$$= \langle \frac{\partial y(u)}{\partial u}^* J_y(y(u),u), s\rangle + \langle J_u(y(u),u), s\rangle \tag{5}$$
$$= \langle y'(u)^* J_y(u), s\rangle + \langle J_u(u), s\rangle \tag{6}$$

Here $\langle \cdot, \cdot \rangle$ is the L^2 inner product. The difficult term in (6) is $y'(u)^*$,so lets first differentiate $E(y(u),u) = 0$ with respect to u, and try to find expression for $y'(u)^*$:

$$\frac{\partial}{\partial u} E(y(u),u) = 0 \Rightarrow E_y(y(u),u)y'(u) = -E_u(y(u),u) \tag{7}$$
$$\Rightarrow y'(u) = -E_y(y(u),u)^{-1} E_u(y(u),u) \tag{8}$$
$$\Rightarrow y'(u)^* = -E_u(y(u),u)^* E_y(y(u),u)^{-*} \tag{9}$$

This means that

$$y'(u)^* J_y(u) = -E_u(y(u),u)^* E_y(y(u),u)^{-*} J_y(u) = -E_u(y(u),u)\lambda \tag{10}$$

were λ is the solution of the adjoint equation

$$E_y(y(u),u)^* \lambda = J_y(u) \tag{11}$$

This again means that

$$\langle \hat{J}'(u), s\rangle = \langle -E_u(y(u),u)\lambda, s\rangle + \langle J_u(u), s\rangle \tag{12}$$

Now lets try to derive the adjoint equation using (1),(2) and (3). See that

$$E_u(y(u),u) = -1 \tag{13}$$