Mek4250 - Mandatory assignment 2
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Exercise 1
7.1) Given the following weak formulation of Stokes problem:

$$a(u, v) + b(p, v) = (f, v) \; \forall \; v \in H_0^1(\Omega)$$
$$b(q, u) = 0 \; \forall q \in L^2(\Omega)$$

Where $a(u, v) = \int_\Omega \nabla u : \nabla v dx$ and $b(p, v) = \int_\Omega p \nabla \cdot v dx$.
We then want to show the following conditions required for well-posedness:

$$a(u, v) \leq C_1 ||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)} \; \forall \; v, u \in H_0^1(\Omega) \tag{1}$$
$$a(u, u) \geq D ||u||^2_{H^1(\Omega)} \; \forall \; u \in H_0^1(\Omega) \tag{2}$$
$$b(q, u) \leq C_2 ||u||_{H^1(\Omega)} ||q||_{L^2(\Omega)} \; \forall \; u \in H_0^1(\Omega), \; \forall q \in L^2(\Omega) \tag{3}$$

Note that u and v are vector functions. To show the three conditions I will use the notation $u(x) = [u^1(x), ..., u^d(x)]$, for $x \in \mathbb{R}^d$. We must also remember that the inner product " : " in the a form is defined as

$$\nabla u : \nabla v = \Sigma_{i=1}^d \Sigma_{j=1}^d u^i_{x_j}(x) v^i_{x_j}(x)$$

Inequality (1)

$$\begin{aligned}
a(u, v) = \int_\Omega \nabla u : \nabla v dx &= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d u^i_{x_j} v^i_{x_j} dx \\
&= \sum_{i=1}^d \sum_{j=1}^d (u^i_{x_j}, v^i_{x_j})_{L^2(\Omega)} \\
&\leq \sum_{i=1}^d \sum_{j=1}^d ||u^i_{x_j}||_{L^2(\Omega)} ||v^i_{x_j}||_{L^2(\Omega)} \quad \text{using Cauchy-Schwartz inequality} \\
&\leq d \sum_{i=1}^d |u^i|_{H^1(\Omega)} |v^i|_{H^1(\Omega)} \quad \text{since } ||u^i_{x_j}||_{L^2} \leq |u^i|_{H^1} \\
&\leq d^2 |u|_{H^1(\Omega)} |v|_{H^1(\Omega)} \quad \text{using } |u^i|_{H^1} \leq |u|_{H^1} \\
&\leq d^2 ||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)} \quad \text{since } |u|_{H^1} \leq ||u||_{H^1}
\end{aligned}$$

Inequality (2)

Using the Poincares inequality for $H_0^1(\Omega)$, given by:

$$|u|^2_{H^1(\Omega)} \geq C||u||^2_{L^2(\Omega)}$$

we get

$$\begin{aligned}
a(u,u) &= \int_\Omega \nabla u : \nabla u dx = \int_\Omega \sum_{i=1}^d \sum_{j=1}^d u^i_{x_j} u^i_{x_j} dx \\
&= \sum_{i=1}^d \sum_{j=1}^d ||u^i_{x_j}||^2_{L^2(\Omega)} = \sum_{i=1}^d |u^i|^2_{H^1(\Omega)} = |u|^2_{H^1(\Omega)} \\
&= \frac{1}{2}(|u|^2_{H^1(\Omega)} + |u|^2_{H^1(\Omega)}) \\
&\geq \frac{1}{2}(|u|^2_{H^1(\Omega)} + C||u||^2_{L^2(\Omega)}) \\
&\geq \frac{min\{1,C\}}{2}(|u|^2_{H^1(\Omega)} + ||u||^2_{L^2(\Omega)}) \\
&= D||u||^2_{H^1(\Omega)}
\end{aligned}$$

Inequality (3)

$$\begin{aligned}
b(q,u) &= \int_\Omega q \nabla \cdot u dx = \int_\Omega \sum_{i=1}^d q u^i_{x_i} dx \\
&\leq \sum_{i=1}^d ||q||_{L^2(\Omega)} ||u^i_{x_i}||_{L^2(\Omega)} \quad \text{using Cauchy-Schwartz inequality} \\
&\leq ||q||_{L^2(\Omega)} \sum_{i=1}^d |u^i|_{H^1(\Omega)} \\
&\leq d||q||_{L^2(\Omega)} |u|_{H^1(\Omega)} \\
&\leq d||q||_{L^2(\Omega)} ||u||_{H^1(\Omega)}
\end{aligned}$$

7.6) Want to solve the the stokes problem with a manufactured solution $ue(x,y) = (\sin(\pi y), \cos(\pi x))$ and $pe(x,y) = \sin 2\pi x)$ on $\Omega = (0,1)^2$. This means solving the problem:

$$\begin{aligned}
-\Delta u - \nabla p &= f \\
\nabla \cdot u &= 0 \\
u &= ue \text{ on } \partial\Omega_D \\
\frac{\partial u}{\partial n} + pn &= h \text{ on } \partial\Omega_N
\end{aligned}$$

To be able to solve the equation we need to define $\partial\Omega_D$ and $\partial\Omega_N$, as well as deriving f and h. Setting $\partial\Omega_N = \{x = 1\}$ and $\partial\Omega_D = \partial\Omega - \partial\Omega_N$, gives us $h = (0, 0)$, which makes life easier. We see this by noting that $n = (0, 1)$ on $\partial\Omega_N$, and calculating:

$$
\begin{aligned}
h &= \frac{\partial u(1,y)}{\partial n} + p(1,y)n = (\nabla u_1 \cdot n, \nabla u_2 \cdot n) + \sin(2\pi)n \\
&= ([0, \pi\cos(\pi y)] \cdot [1,0], ([\pi\sin(\pi), 0] \cdot [1,0]) + (0,0) \\
&= (0,0)
\end{aligned}
$$

Now lets find f:

$$
\begin{aligned}
f &= -\Delta u - \nabla p \\
&= (\pi^2 \sin(\pi y), \pi^2 \cos(\pi x)) - (2\pi\cos(2\pi x), 0)
\end{aligned}
$$

The last step before implementing the numerics is to derive the variational form:

$$
\begin{aligned}
-\int_\Omega \Delta u \cdot v dx - \int_\Omega \nabla p \cdot v dx &= \int_\Omega \nabla u : \nabla v dx + \int_\Omega p \nabla \cdot v dx - \int_{\partial\Omega_n} (\frac{\partial u}{\partial N} + pn) \cdot v dS \\
&= \int_\Omega \nabla u : \nabla v dx + \int_\Omega p \nabla \cdot v dx - \int_{\partial\Omega_N} h \cdot v dS \\
&= \int_\Omega \nabla u : \nabla v dx + \int_\Omega p \nabla \cdot v dx
\end{aligned}
$$

Together with the deivergence term this gives us the following variational form:

$$
\int_\Omega \nabla u : \nabla v dx + \int_\Omega p \nabla \cdot v dx + \int_\Omega q \nabla \cdot u dx = \int_\Omega f \cdot v dx
$$