Mek4250 - Mandatory assignment 1

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Exercise 1

a) The H^p norm of a function $u(x, y)$ of two variables on $\Omega = (0,1)^2$ is defined as follows:

$$||u||^2_{H^p(\Omega)} = \sum_{i=0}^{p}\sum_{j=0}^{i}\binom{i}{j}||\frac{\partial^i u}{\partial^j x \partial^{i-j} y}||^2_{L^2(\Omega)} \tag{1}$$

In our case $u(x, y) = sin(k\pi x)cos(l\pi y)$. We easily see that the derivative of this function can be expressed with the following formula:

$$\frac{\partial^i u(x,y)}{\partial^j x \partial^i y} = (k\pi)^j(l\pi)^i f_j(k\pi x)g_j(l\pi y) \tag{2}$$

were f_j is the j-th derivative of $sin(x)$ and g_i is the i-th derivative of $cos(y)$. Now lets look at the L^2 norm of (2).

$$||\frac{\partial^i u(x,y)}{\partial^j x \partial^i y}||^2_{L^2(\Omega)} = (k\pi)^{2j}(l\pi)^{2i}\int\int_{\Omega} f_j(k\pi x)^2 g_i(l\pi y)^2 dxdy \tag{3}$$

$$= (k\pi)^{2j}(l\pi)^{2i}\int_0^1 f_j(k\pi x)^2 dx \int_0^1 g_i(l\pi y)^2 dy \tag{4}$$

Since f_j^2 and g_i^2 are:

$$f_j(x)^2 = \begin{cases} sin^2(x) & \text{j even} \\ cos^2(x) & \text{j odd} \end{cases}$$

and

$$g_i(y)^2 = \begin{cases} cos^2(y) & \text{i even} \\ sin^2(y) & \text{i odd} \end{cases}$$

and since

$$\int_0^1 sin^2(l\pi y)dy = \int_0^1 cos^2(l\pi y)dy = \frac{1}{2}$$

we get the following expression for the square of the L^2 norm of a general derivative of u:

$$||\frac{\partial^i u(x,y)}{\partial^j x \partial^i y}||^2_{L^2(\Omega)} = \frac{1}{4}(k\pi)^{2j}(l\pi)^{2i} \tag{5}$$

If we plug this into (1), we get

$$||u||^2_{H^p(\Omega)} = \frac{1}{4}\sum_{i=0}^{p}\sum_{j=0}^{i}\binom{i}{j}(k\pi)^{2j}(l\pi)^{2(i-j)} \tag{6}$$

Exercise 2

a) Assume our solution is on the form $u(x,y) = X(x)Y(y)$. If we plug this into our equation and assume that both X and Y are nonzero, we get:

$$-\mu(X''Y + XY'') + X'Y = 0 \iff \frac{-\mu X'' + X'}{X} - \mu\frac{Y''}{Y} = 0 \tag{7}$$

$$\iff -\mu X'' + X' = \lambda X \text{ and } \mu Y'' = \lambda Y \tag{8}$$

For some λ. Together with the Neumann boundary conditions look at the equation for Y:

$$\begin{cases} \mu Y'' = \lambda Y \\ Y'(0) = Y'(1) = 0 \end{cases}$$

There are three possibilities: $\lambda < 0$, $\lambda > 0$ and $\lambda = 0$. Start with first:

Case (i) $\lambda < 0$:

$\lambda < 0$ implies that $Y(y) = A\sin(ny) + B\cos(ny)$ where $n = \sqrt{|\frac{\lambda}{\mu}|}$, and A and B are constants. $Y'(y) = n(A\cos(ny) - B\sin(ny))$ and $Y'(0) = 0 \Rightarrow A = 0$. $Y'(1) = 0$ means that either $B = 0$, or $n = k\pi$ for an integer k. Since we assumed $Y \neq 0$, $B = 0$ is not allowed, and $Y(y) = B\cos(ny)$.

Case (ii) $\lambda > 0$:

Now $Y(y) = Ae^{ny} + Be^{-ny}$ where $n = \sqrt{\frac{\lambda}{\mu}}$, and A and B are constants.