Adjoint Equation

Looking at an optimal control problem

$$\min_{y,u} J(y,u) \text{ subject to } E(y,u) = 0$$

Where

$$J(y,u) = \frac{1}{2}\int_0^T u^2 dt + \frac{1}{2}(y(T) - y^T)^2 \tag{1}$$

and

$$E(y,u) = y' - \alpha y - u \tag{2}$$

$$y(0) = y_0 \tag{3}$$

We solve this problem by reducing (1) to $\hat{J}(u) = J(y(u), u)$ and computing the gradient of $\hat{J}$, $\hat{J}'$.

$$\hat{J}'(u) = \frac{\partial J(y(u),u)}{\partial u} = J_y(y(u),u)\frac{\partial y(u)}{\partial u} + J_u(y(u),u) = J_y\frac{\partial y}{\partial u} + J_u$$

The difficult term here is $\frac{\partial y}{\partial u}$,so lets first look at