README: `superkerr`
Continuum fitting in the full Kerr metric

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1 Introduction

`superkerr` is an `XSPEC` model which performs standard continuum fitting procedures in the fully general Kerr metric, with spin parameter a unconstrained. A detailed description of the physics of this model can be found in the paper [?]. In this README we simply briefly describe the input parameters of the model, how to load the model into `XSPEC`, and provide some general statements about the properties of this model.

If anything remains unclear, or you can think of any way in which `superkerr` can be improved, please do get in touch with the authors.

2 Basic physical properties of the model

Any relativistic continuum fitting model consists of a description of three distinct physical process. These are

1. A description of the temperature profile of the disc in terms of the free parameters of the theory $T(r, \Theta)$. Here Θ describes a list of the free parameters.
2. A description of the fluid rest-frame specific intensity $I_E(E, T)$
3. A relativistic ray-tracing calculation which computes disc-observer energy shifts for photons travelling from the disc plane to a distant observer

We now briefly describe the physics of each three processes, as modelled in `superkerr` .

2.1 Disc temperature profile

The temperature profile of a thin accretion flow fed by a constant mass flux $\dot{M}$ in the general Kerr metric $g^{\mu\nu}$ is given by [1]

$$\sigma T^4 = -\frac{1}{2\sqrt{|g|}} \left(U^t\right)^2 \frac{\mathrm{d}\Omega}{\mathrm{d}r} \left[\frac{\dot{M}}{2\pi} \int^r \frac{U_\phi'}{U^t} \,\mathrm{d}r + \frac{\mathcal{F}_\mathcal{J}}{2\pi} \right], \tag{2.1}$$

where

$$\Omega \equiv \frac{U^\phi}{U^t}, \tag{2.2}$$

g is the metric determinant, and $\dot{M}$ is the constant mass flux through the disc, while $\mathcal{F}_\mathcal{J}$ is an integration constant relating to the angular momentum flux through the disc. Clearly then, determining the temperature profile boils down to solving the integral

$$\mathcal{I}(r, a) = \int^r \frac{U_\phi'}{U^t} \,\mathrm{d}r, \tag{2.3}$$

where

$$U^t(r, a) = \frac{1 + a\sqrt{r_g/r^3}}{\left(1 - 3r_g/r + 2a\sqrt{r_g/r^3}\right)^{1/2}}, \tag{2.4}$$

$$U_\phi'(r, a) = \frac{(GM)^{1/2}}{2r^{1/2}\mathcal{D}} \left(1 + \frac{a r_g^{1/2}}{r^{3/2}}\right) \left(1 - \frac{6r_g}{r} - \frac{3a^2}{r^2} + \frac{8 a r_g^{1/2}}{r^{3/2}}\right), \tag{2.5}$$

and

$$\mathcal{D} \equiv \left(1 - 3r_g/r + 2a r_g^{1/2}/r^{3/2}\right)^{3/2}. \tag{2.6}$$

It turns out this integral can be solved in closed form for all values of the Kerr spin parameter a. This was described in detail in the paper [?]. The solution of this problem has the following general form

$$\sigma T^4 = \frac{3GM\dot{M}}{8\pi r^3} \left(\frac{1}{1 - 3r_g/r + 2a\sqrt{r_g/r^3}}\right) \left[f(r, a) - (1 - \delta_\mathcal{J}) f(r_I, a)\right]. \tag{2.7}$$

The function $f(r, a)$ is in general rather complicated, and depends on whether the spacetime describes a black hole $|a| < r_g$, an extremal black hole $|a| = r_g$, or a naked singularity $|a| > r_g$. See the paper for further details.

The above functional form introduces many of the free parameters of the theory. Namely the Kerr metric mass M and spin a parameters, the accretion rate $\dot{M}$ and an ISCO stress parameter $\delta_\mathcal{J}$. Physically $\delta_\mathcal{J}$ corresponds to the fraction of angular momentum the fluid passes back to the disc upon crossing the ISCO, where a value of $\delta_\mathcal{J} = 0$ corresponds physically to no communication between the disc and plunging region and a vanishing ISCO stress. Numerical simulations [2] place $\delta_\mathcal{J}$ somewhere in the range $\delta_\mathcal{J} \sim 0.02 - 0.2$. `superkerr` only computes the disc temperature at radii exterior to the ISCO radius r_I.

2.2 Fluid rest-frame specific intensity

`superkerr` is designed to model sources in the so-called "soft state". In the soft state the specific intensity of the locally emitted radiation is typically given by a modified Planck function B_E, of the form

$$\begin{aligned} I_E(E_{\rm emit}) &= f_{\rm col}^{-4}\, B_E(E_{\rm emit}, f_{\rm col}T) &(2.8)\\ &= \frac{2E_{\rm emit}^3}{h^2c^2 f_{\rm col}^4}\left[\exp\left(\frac{E_{\rm emit}}{kf_{\rm col}T}\right) - 1\right]^{-1}. &(2.9)\end{aligned}$$

The colour correction factor, denoted $f_{\rm col}$, is included here so as to model the effects of radiative transfer in the disc atmosphere.

In brief, while the surface temperature of the disc $T(r,t)$ is given by the constraints of energy conservation (e.g., equation 2.7), this temperature in fact corresponds physically to the temperature of the disc surface at a height above the midplane where the optical depth of the disc equals 1. It is important to note, however, that the disc's central temperature is given by [3]

$$T_c^4 = \frac{3}{8}\kappa\Sigma T^4, \tag{2.10}$$

where for standard astrophysical parameters $\kappa\Sigma \gg 1$ [4]. This result highlights that the energy of the disc photons produced in the disc midplane is higher than the surface temperature, taking a value roughly $E_\gamma \sim kT_c$. Only if the liberated disc energy can be fully thermalised in the disc atmosphere do the photons emerge with temperature T (eq. 2.7). On their path through the disc atmosphere, photons can either be absorbed and re-emitted (thus totally thermalising their energy), or they can undergo elastic scattering. Elastic scattering however, by definition, does not change the energy of the photon, and so if this process dominates in the disc atmosphere, photons will be observed to have the "hotter" temperatures associated with the altitudes closer to the disc midplane, not the disc's $\tau = 1$ surface. This modifies the emergent disc spectrum, a result which is typically modelled with a so-called colour-correction factor $f_{\rm col}$, which can be thought of as quantifying the relative dominance of these two different opacities in the disc atmosphere.

Note that the normalising factor $1/f_{\rm col}^4$ here ensures that, despite the temperature of the emission being modified by the colour-correction factor, the total (integrated over all energies) emitted luminosity remains σT^4 (and therefore energy is conserved). The value of the colour-correction factor $f_{\rm col}$ in general depends on the local properties of the emitting region [5]. In this work we use the Done et al. [6] model for $f_{\rm col}(T)$.

For the lowest disc temperatures, below a critical temperature $T = 3\times10^4$K, Hydrogen is neutral and the Hydrogen absorption opacity is extremely large. This results in the full thermalisation of the liberated disc energy, meaning that the emitted disc spectrum is well described by a pure blackbody function with temperature T, i.e.,

$$f_{\rm col}(T) = 1, \quad T(r,t) < 3\times10^4\,{\rm K}. \tag{2.11}$$

As the temperature increases above 3×10^4 K, the colour correction factor begins to increase. This results from the growing ionisation fraction's of both Hydrogen and Helium, which acts to reduce the total disc absorption opacity. As a result the electron scattering opacity begins to dominate, more photons are scattered out of the disc atmosphere, and the typical temperature of observed photons increases. [6] model the colour correction factor in this regime as

$$f_{\rm col}(T) = \left(\frac{T}{3\times 10^4 {\rm K}}\right)^{0.82}, \quad 3\times 10^4\,{\rm K} < T(r,t) < 1\times 10^5\,{\rm K}. \tag{2.12}$$

It should be noted that the choice of temperature index and normalisation in this expression were set so that the colour-correction factor was equal to 1 at $T = 3\times 10^4$ K, and was continuous in joining onto the Compton-scattering regime discussed below, and was not determined by fundamental atomic physics. This parameterisation did however accurately reproduce the results of full radiative transfer simulations [6]. For the highest disc temperatures $T > 1\times 10^5$K, electron scattering completely dominates the absorption opacity and the colour correction factor begins to saturate, a result of Compton down-scattering in the disc atmosphere, to

$$f_{\rm col}(T) = \left(\frac{72\,{\rm keV}}{k_B T}\right)^{1/9}, \quad T(r,t) > 1\times 10^5\,{\rm K}. \tag{2.13}$$

This saturation leads to a maximum value of $f_{\rm col} \approx 2.7$. As X-ray binary discs are hot, we are typically always in this final regime and the colour correction value is only weakly dependent on disc temperature.

2.3 Photon ray tracing

The specific flux density F_E of the disc radiation, as observed by a distant observer at rest (subscript obs), is given by

$$F_E(E_{\rm obs}) = \int I_E(E_{\rm obs})\,{\rm d}\Theta_{\rm obs}. \tag{2.14}$$

Here, $E_{\rm obs}$ is the observed photon energy and $I_E(E_{\rm obs})$ the specific intensity, both measured at the location of the distant observer. The differential element of solid angle subtended on the observer's sky by the disk element is ${\rm d}\Theta_{\rm obs}$. Since I_E/E^3 is a relativistic invariant [7], we may write

$$F_E(E_{\rm obs}) = \int f_\gamma^3 I_E(E_{\rm emit})\,{\rm d}\Theta_{\rm obs}, \tag{2.15}$$

where the photon energy ratio factor f_γ is the ratio of $E_{\rm obs}$ to the emitted local rest frame photon energy $E_{\rm emit}$:

$$f_\gamma(r,\phi) \equiv \frac{E_{\rm obs}}{E_{\rm emit}} = \frac{1}{U^t}\left(1+\frac{p_\phi}{p_t}\Omega\right)^{-1}. \tag{2.16}$$

In this expression p_ϕ and $-p_t$ are the angular momentum and energy of the emitted photon. For an observer at a large distance D from the source, the differential solid angle into which the radiation is emitted is

$${\rm d}\Theta_{\rm obs} = \frac{{\rm d}\alpha\,{\rm d}\beta}{D^2}, \tag{2.17}$$

where α and β are the impact parameters at infinity [8]. For a given set of theory parameters we know $I_E(E_{\rm emit})$ and all that remains is to compute f_γ and ${\rm d}\Theta_{\rm obs}$.

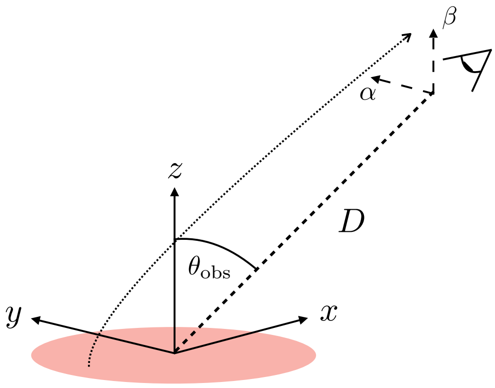

Figure 1. Ray tracing geometry.

We assume a distant observer orientated at an inclination angle i from the disc plane at a distance D. We set up an image plane perpendicular to the line of sight centred at $\phi = 0$, with image plane cartesian coordinates (α, β). We trace the rays back from the observer to the disc by solving the null-geodesics of the Kerr metric (using the code `YNOGK` [9], which is based on `GEOKERR` [10]).

Starting from a finely spaced grid of points (α, β) in the image plane, we trace the geodesics of each photon back to the disc plane, recording the location at which the photon intercepts the disc plane (r_f), and the ratio p_ϕ/p_t for each photon. The parameter r_f allows the disc temperature T to be calculated. The parameters r_f and p_ϕ/p_t together uniquely define the energy-shift factor f_γ. The integrand (of eq. 2.15) can therefore be calculated at every grid point in the image plane, and the integral (2.15) is then calculated numerically. This model can then be fit to data.

3 Model input parameters

The total set of model parameter for `superkerr` is listed in Table 1. The different parameters pertain to both the disc physics and observer. The first two parameter describe the central compact object, namely the:

- Kerr mass M parameter.
- *Dimensionless* Kerr spin parameter $a_\star$. Black hole metrics correspond to dimensionless spins $-1 \leq a_\star \leq 1$.

Two further parameters describe the accretion disc itself, these are:

- The mass accretion rate through the disc is denoted $\dot{M}$. `superkerr` expects mass accretion rates input in units of $L_{\rm edd}/c^2$, where $L_{\rm edd}$ is the Eddington luminosity, which in cgs units is equal to $L_{\rm edd} = 1.26 \times 10^{38} (M/M_\odot)$ erg/s. Note that many authors instead normalise their accretion rate with an additional factor of $1/\eta(a_\star)$, where

Parameter	Units	Allowed range
M	$M_\odot$	$0 < M$
$a_\star$		$-\infty < a_\star < \infty$
i	Degrees	$0 < i < 90$
$\dot{M}$	$L_{\rm edd}/c^2$	$0 < \dot{M}$
$\delta_{\mathcal{J}}$		$0 < \delta_{\mathcal{J}} < 1$
`norm`	$1/{\rm kpc}^2$	$0 <$ `norm`

Table 1. The parameters of the `superkerr` model, their units and allowed ranges. The accretion rate parameter is defined in terms of the Eddington luminosity, which in cgs units is equal to $L_{\rm edd} = 1.26 \times 10^{38}(M/M_\odot)$ erg/s.

$\eta(a_\star) = 1 - U_t(r_I, a_\star)$ is the accretion efficiency. *We do not follow this convention.* This is because the efficiency is discontinuous across $a_\star = 1$, and this discontinuous jump would affect standard fitting procedures.

- The parameter $\delta_{\mathcal{J}}$ is an ISCO stress parameter. Physically $\delta_{\mathcal{J}}$ corresponds to the fraction of angular momentum the fluid passes back to the disc upon crossing the ISCO, where a value of $\delta_{\mathcal{J}} = 0$ corresponds physically to no communication between the disc and plunging region and a vanishing ISCO stress. Numerical simulations [2] place $\delta_{\mathcal{J}}$ somewhere in the range $\delta_{\mathcal{J}} \sim 0.02 - 0.2$. `superkerr` only computes the disc temperature at radii exterior to the ISCO radius r_I.

Finally there are two parameters which describe the observer. These are

- i, the disc-observer inclination angle (input in degrees). We use the convention that $i = 90°$ is an edge-on disc, while $i = 0$ is a face-on disc (see also Fig 1).
- The default `XSPEC` parameter `norm` sets the source-observer distance. We use a convention where `norm` $= 1/D^2_{\rm kpc}$, where $D_{\rm kpc}$ is the source-observer distance in kiloparsec.

4 Loading into `XSPEC`

The `superkerr` model can be loaded into `XSPEC` in the usual manner. This requires both `amodules.f90` and `superkerr.f90` to be pre-compiled. Instructions on how to load a local model into `XSPEC` can be found at the following link: https://heasarc.gsfc.nasa.gov/xanadu/xspec/manual/XSappendixLocal.html. The scripts `load.xcm` and `lmodel.dat` are included to do this for you. The user should simply be able to type

```
@load.xcm
```

and `XSPEC` should take care of the rest. The model will then be loadable within `XSPEC` in the usual way.

5 Some thoughts on parameter fitting

There are intrinsic degeneracies between some black hole and naked singularity metrics. This is discussed in detail in the paper, but in brief: black hole spacetimes all have a one-to-one correspondence with a naked singularity spacetime which has an identical ISCO location. These same-ISCO spacetimes produce very similar thermal continuum emission.

The range of naked singularity spin values which share black hole ISCO-locations are $5/3 \leq a_\star \leq 9$. As this degeneracy is strong, and fits to fake data are often equally good for both spin values (see paper), it may be of interest therefore to restrict the Kerr spin parameter to lie in the range $-\infty \leq a_\star \leq 5/3$. This can be done in `XSPEC` with the `newpar` command[1]:

$$\texttt{newpar superkerr} \;\; 2 \;\; 0 \;\; 0.01 \;\; -10 \;\; -1 \;\; 1.67 \;\; 1.67$$

which will reset the spin parameter to zero, with a fitting δ of 0.01. The hard lower limit is then -10, the soft lower limit is -1, and the soft and hard upper limits are 5/3.

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