Math 337 Homework 16

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1. I stick with MATLAB for this one, and the following code produces a plot with 6 stages of the evolution of the solution. I'll note that MATLAB's `quad` fails to properly integrate the piecewise function, so I use `integral` with the option `ArrayValued` set to `true`.

   Since the solution is supported at any x that overlaps $[0,2]$ at time t, we have the solution is always supported by $[0,2]$. The solution is support ct to the right of 2, since the lower integration bound $(x-ct)$ will overlap, and similarly supported ct to the left of 0. Therefore, the width of the support is $2ct+2$.

```
% HW 16 Number 1
%
% Plot the d'Alembert Solution with c=1
%
% Andy Reagan

c = 1;
t0 = 0.1;
tf = 10;
k = 0.2;
tvec = t0:k:tf;
h = 0.1;
x = -10:h:10;
u = zeros(length(x),1);
width = zeros(length(t),1);

for i=1:length(tvec)
    t = tvec(i);
    for j=1:length(x)
        u(j) = 1/(2*c)*integral(@phi,x(j)-c*t,x(j)+c*t,'ArrayValued',true);
    end
    figure(16010101);
    plot(x,u);
    title(sprintf('t␣=␣%g',t));
    % pause;
    disp(t);
    disp(i);
    width(i) = (length(find(u>0))-1)*h;
    if t == 0.1
        figure(16010201);
        subplot(321);
        plot(x,u);
        ylim([0,1]);
        ylabel('y')
        title(sprintf('t␣=␣%g',t));
    end
    if t == 0.9
        figure(16010201);
        subplot(322);
```

```
        plot(x,u);
        ylim([0,1]);
        title(sprintf('t␣=␣%g',t));
    end
    if i == 12
        figure(16010201);
        subplot(323);
        plot(x,u);
        ylim([0,1]);
        ylabel('y')
        title(sprintf('t␣=␣%g',t));
    end
    if i == 43
        figure(16010201);
        subplot(324);
        plot(x,u);
        ylim([0,1]);
        title(sprintf('t␣=␣%g',t));
    end
    if i == 46
        figure(16010201);
        subplot(325);
        plot(x,u);
        ylim([0,1]);
        ylabel('y')
        xlabel('x')
        title(sprintf('t␣=␣%g',t));
    end
    if t == 9.9
        figure(16010201);
        subplot(326);
        plot(x,u);
        ylim([0,1]);
        xlabel('x')
        title(sprintf('t␣=␣%g',t));
    end
end
```

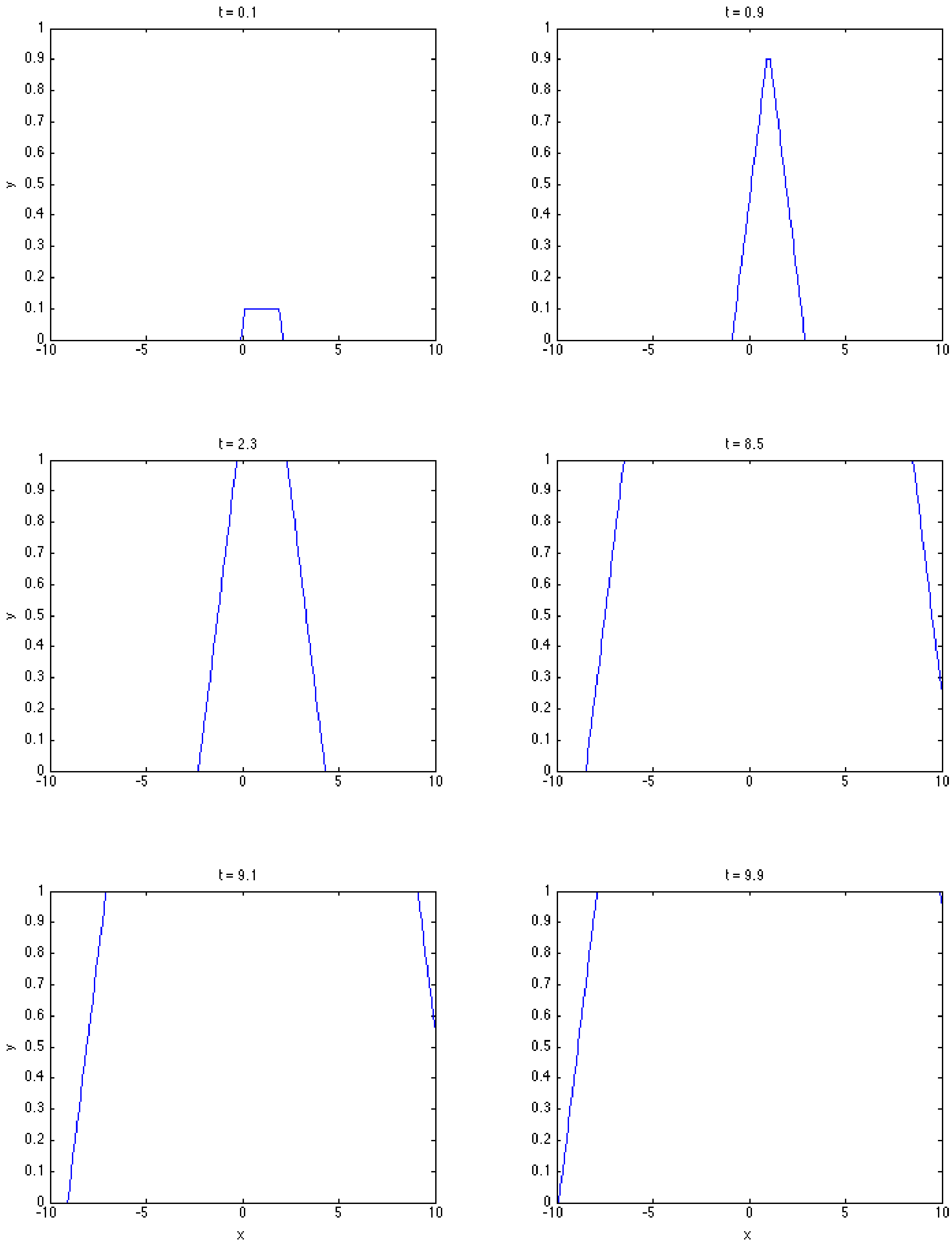

Figure 1: Six panels of evolution of the D'Alembert solution. I observe three qualitatively different profiles. Namely, in the first phase, the height and support of the solution are both growing. When the height has reached 1, the solution attains a peak. And when $t > 1$, the solution remains with a plateau at 1, and grows in width. My code (above) plots the time evolution.

2. Plugging 16.7 into 16.6 we have the two equations:

$$F(x) + G(x) = \phi(x) \qquad - cF_x(x) + cG_x(x) = \psi(x).$$

Differentiating the first equation wrt x, these equations become

$$F_x(x) + G_x(x) = \phi_x(x) \qquad - cF_x(x) + cG_x(x) = \psi(x).$$

Solving the first for F, and plugging into the second, we have:

$$\begin{aligned} -c\left(\phi_x(x) - G_x(x)\right) + cG_x(x) &= \psi(x) \\ -\phi_x(x) + G_x(x) + G_x(x) &= \frac{1}{c}\psi(x) \\ G_x(x) &= \frac{1}{2c}\psi(x) + \frac{1}{2}\phi_x(x) \\ G(x) &= \frac{1}{2c}\int_{-\infty}^{x} \psi(s)ds + \frac{1}{2}\phi(x) \end{aligned}$$

where in the last step we have integrated from $-\infty$ to x, applied the Fundamental Theorem of Calculus, and rely upon the assumption that there no disturbance coming in from $x = -\infty$, such that $G(-\infty) = \phi(-\infty) = 0$.

Similarly, we can solve for F:

$$\begin{aligned} c\left(\phi_x(x) - F_x(x)\right) - cF_x(x) &= \psi(x) \\ \phi_x(x) - F_x(x) - F_x(x) &= \frac{1}{c}\psi(x) \\ F_x(x) &= -\frac{1}{2c}\psi(x) + \frac{1}{2}\phi_x(x) \\ F(x) &= -\frac{1}{2c}\int_{-\infty}^{x} \psi(s)ds + \frac{1}{2}\phi(x) \end{aligned}$$

This verifies Eq (16.28a). Simply evaluating G and F at $(x \pm ct)$, respectively, we have:

$$F(x-ct) = -\frac{1}{2c}\int_{-\infty}^{x-ct} \psi(s)ds + \frac{1}{2}\phi(x-ct) \qquad G(x+ct) = \frac{1}{2c}\int_{-\infty}^{x+ct} \psi(s)ds + \frac{1}{2}\phi(x+ct)$$

which verifies Eq (16.28b).

Finally, we plug the above (16.28b) into Eq (16.6) and we have

$$\begin{aligned} u(x,t) &= -\frac{1}{2c}\int_{-\infty}^{x-ct} \psi(s)ds + \frac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_{-\infty}^{x+ct} \psi(s)ds + \frac{1}{2}\phi(x+ct) \\ &= \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \end{aligned}$$

where we have cancelled the overlapping portions of the integral (formally, splitting the integral to $x + ct$ into the overlapping and non-overlapping parts, then cancelling). This verifies Eq (16.8).