Math 337 Homework 06

Andy Reagan

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1. Verify that the *homogeneous* versions of Problems I and III in the notes have nontrivial solutions, while the homogeneous version of Problem II has only the trivial solution.

   Solution: The homogeneous version of Problem I is the same the homogeneous version of Problem III. This is:
   $$y'' + \pi^2 y = 0, \qquad y(0) = 0, \qquad y(1) = 0.$$
   From the notes, equation 6.6, we have the general solution given by
   $$y = A \sin \pi x + B \cos \pi x.$$
   For the first BC, we see that $y(0) = B = 0$. Similarly, $y(1) = -B = 0$. Therefore, $y = A \sin \pi x$ is a solution to the ODE and both BC, for arbitrary A, so there are nontrivial (infinitely many) solutions.

   The homogeneous version of Problem II is:
   $$y'' + \pi^2 y = 0, \qquad y(0) = 0, \qquad y'(1) = 0.$$
   Again, for the first BC, we see that $y(0) = B = 0$. For the second BC, $y'(1) = A = 0$. Therefore, the only solution is $y(x) = 0$, the trivial solution.

2. Find the solution of Problem I with the π^2 replaced by $(\pi + \epsilon)^2$, where $\epsilon << 1$. Based on your answer, explain why such a problem is ill-posed.

   Solution: We have the solution from equation 6.6 as
   $$y = A \sin(\pi + \epsilon)x + B \cos(\pi + \epsilon)x + \frac{1}{(\pi + \epsilon)^2}.$$
   The first BC simply gives us that $B = -\frac{1}{(\pi+\epsilon)^2}$. The second BC is more complicated, so I write it out (using the appropriate trig identities followed by the Maclaurin series):
   $$\begin{aligned}
   0 = y(1) &= A \sin(\pi + \epsilon) - \frac{1}{(\pi + \epsilon)^2} \cos(\pi + \epsilon) + \frac{1}{(\pi + \epsilon)^2} \\
   &= -A \sin \epsilon + \frac{\cos \epsilon}{(\pi + \epsilon)^2} + \frac{1}{(\pi + \epsilon)^2} \\
   &= -A\epsilon + \frac{1}{(\pi + \epsilon)^2} + \frac{1}{(\pi + \epsilon)^2} + O(\epsilon^3) \\
   &\simeq -A\epsilon + \frac{2}{(\pi + \epsilon)^2}
   \end{aligned}$$

Solving this for A, we have

$$A \simeq \frac{2}{\epsilon(\pi + \epsilon)^2}$$

We observe here that for ϵ small, A is very large, and very sensitive to the value of epsilon and behaves as $1/\epsilon$. As $\epsilon \to 0$, we have $A \to \infty$.