Math 337 Homework 06

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1. Verify that the *homogeneous* versions of Problems I and III in the notes have nontrivial solutions, while the homogeneous version of Problem II has only the trivial solution.

   Solution: The homogeneous version of Problem I is the same the homogeneous version of Problem III. This is:

   $$y'' + \pi^2 y = 0, \qquad y(0) = 0, \qquad y(1) = 0.$$

   From the notes, equation 6.6, we have the general solution given by

   $$y = A \sin \pi x + B \cos \pi x.$$

   For the first BC, we see that $y(0) = B = 0$. Similarly, $y(1) = -B = 0$. Therefore, $y = A \sin \pi x$ is a solution to the ODE and both BC, for arbitrary A, so there are nontrivial (infinitely many) solutions.

   The homogeneous version of Problem II is:

   $$y'' + \pi^2 y = 0, \qquad y(0) = 0, \qquad y'(1) = 0.$$

   Again, for the first BC, we see that $y(0) = B = 0$. For the second BC, $y'(1) = A = 0$. Therefore, the only solution is $y(x) = 0$, the trivial solution.

2. Find the solution of Problem I with the π^2 replaced by $(\pi + \epsilon)^2$, where $\epsilon << 1$. Based on your answer, explain why such a problem is ill-posed.

   Solution: We have the solution from equation 6.6 as

   $$y = A \sin(\pi + \epsilon)x + B \cos(\pi + \epsilon)x + \frac{1}{(\pi + \epsilon)^2}.$$

   The first BC simply gives us that $B = -\frac{1}{(\pi+\epsilon)^2}$. The second BC is more complicated, so I write it out (using the appropriate trig identities followed by the Maclaurin series):

   $$\begin{aligned} 0 = y(1) &= A \sin(\pi + \epsilon) - \frac{1}{(\pi + \epsilon)^2} \cos(\pi + \epsilon) + \frac{1}{(\pi + \epsilon)^2} \\ &= -A \sin \epsilon + \frac{\cos \epsilon}{(\pi + \epsilon)^2} + \frac{1}{(\pi + \epsilon)^2} \\ &= -A\epsilon - \frac{\epsilon^2}{2(\pi + \epsilon)^2} + \frac{1}{(\pi + \epsilon)^2} + O(\epsilon^3) \end{aligned}$$

Dropping the $O(\epsilon^3)$ terms and solving this for A, we have

$$A \approx \frac{2+\epsilon^2}{2\epsilon(\pi+\epsilon)^2}$$

We observe here that for ϵ small, A is very large, and very sensitive to the value of epsilon. As $\epsilon \to 0$, we have $A \to 2/0 \to \infty$.

Alternatively, we can write this as a linear system

$$\begin{pmatrix} 0 & 1 \\ -\epsilon & -\epsilon/2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{(\pi+\epsilon)^2} \\ -\frac{1}{(\pi+\epsilon)^2} \end{pmatrix}$$

The condition number of solving this system (the ratio of the singular values) is given by the eigenvalues of the LHS matrix multiplied by its transpose. That is, we compute the eigenvalues of

$$\begin{pmatrix} 0 & -\epsilon \\ 1 & -\epsilon/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\epsilon & -\epsilon/2 \end{pmatrix} = \begin{pmatrix} \epsilon^2 & \epsilon^3/2 \\ \epsilon^3/2 & 1-\epsilon^4/4 \end{pmatrix}$$

We find

$$\lambda_{1,2} = \frac{4+4\epsilon^2+\epsilon^4+\sqrt{16-32\epsilon^2+24\epsilon^4+8\epsilon^6+\epsilon^8}}{4+4\epsilon^2+\epsilon^4-\sqrt{16-32\epsilon^2+24\epsilon^4+8\epsilon^6+\epsilon^8}}$$

This is not very helpful, so I make a plot of this for $\epsilon \in [0, 0.1]$. We see the same as above, for small ϵ the condition number of the matrix explodes, making this problem ill-posed.

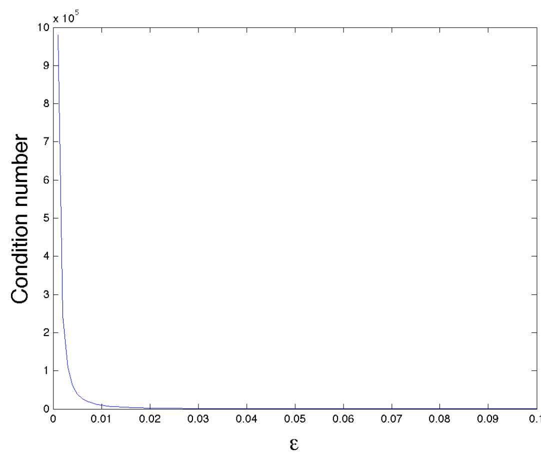

Figure 1: Condition number of the linear BVP for small ϵ.

```
a=linspace(0,.1,100);
c=(1/8*(4+4*a.^2+a.^4+sqrt(16-32*a.^2+24*a.^4+8*a.^6+a.^8)))./(1/8*(4+4*a.^2+a.^4-sqrt
    (16-32*a.^2+24*a.^4+8*a.^6+a.^8)));
plot(a,c);
xlabel('\epsilon','FontSize',20)
ylabel('Condition number','FontSize',20)
saveas(gcf,'andy_hw06_prb2_01.png')
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