Math 337 Homework 09

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1. Solve the BVP
$$y'' - \frac{2y}{(1+x)^2} = -\frac{4}{(1+x)^2}, \qquad y(0) = 0, \quad y(1) = 1$$
using the collocation method with $\phi_j = \sin(j\pi x)$ for $j = 1, \ldots, M$ and $M = 10$. Use the equidistant collocation points $x_k = x_0 + k \cdot h$. Compare you finite-element solution with the exact solution $y_{\text{exact}} = 2x/(1+x)$ by plotting them together, and also by plotting, in a separate figure, the error for $x \in [0,1]$.

Solution: Using the hint, we make the change of variables $z(x) = y(x) - x$, and therefore the new problem has boundary conditions $z(0) = y(0) - 0 = 0$ and $z(1) = y(1) - 1 = 0$. Noting that $y'' = z''$ and the definition $z = y - x$ we rewrite the BVP as
$$z'' - \frac{2z}{(1+x)^2} = -\frac{4}{(1+x)^2} + \frac{2x}{(1+x)^2}, \qquad z(0) = 0, \quad z(1) = 0.$$

The solution is obtained by the following code, and I include plots of the numerical solution and exact solution (Figure 1) and the scaling of error with $(1/M)$ (Figure 2).

```
% HW09 Problem 1
%
% solve BVP by collocation method

Mvec = [10,15,20,30,40,60,80,100,120]';
errorvec = zeros(size(Mvec));
for j=1:length(Mvec)
    %% setup
    M = Mvec(j);

    % collocation points
    x = linspace(0,1,M+2);

    % set r
    r = ((2.*x(2:end-1)-4)./((1+x(2:end-1)).^2))';

    % build A
    % A = sparse(M,M);
    A = zeros(M,M);
    for i=1:M
        A(:,i) = -pi^2*i^2.*sin(i*pi.*x(2:end-1)) + (-2./((1+x(2:end-1)).^2)).*sin(i*
            pi.*x(2:end-1));
    end

    %% solve
```

```
    c = A\r;

    %% add all of the functions
    y = x';
    for i=1:M
        y = y+c(i).*sin(i*pi.*x');
    end

    yexact = 2.*x./(1+x);

    %% quick plot of all of the theta's
    figure;
    for i=1:M
        plot(x,c(i).*sin(i*pi.*x'));
        hold on;
    end

    %% compute max error (and save)
    maxerror = max(abs(yexact'-y));
    fprintf('maximum␣error␣for␣M␣=␣%g␣is␣%g\n',M,maxerror);
    errorvec(j) = maxerror;

    %%  plot the solution, and exact
    figure;
    tmpfigh = gcf;
    clf;
    figshape(600,600);
    set(gcf,'Color','none');
    set(gcf,'InvertHardCopy', 'off');
    set(gcf,'DefaultAxesFontname','helvetica');
    set(gcf,'DefaultLineColor','r');
    set(gcf,'DefaultAxesColor','none');
    set(gcf,'DefaultLineMarkerSize',5);
    set(gcf,'DefaultLineMarkerEdgeColor','k');
    set(gcf,'DefaultLineMarkerFaceColor','g');
    set(gcf,'DefaultAxesLineWidth',0.5);
    set(gcf,'PaperPositionMode','auto');

    plot(x',y,'LineWidth',2,'Color','b');
    hold on;
    plot(x,yexact,'LineWidth',2,'Color','r')
    legend({'numerical','exact'},'Location','NorthWest');
    legend boxoff;
    set(gca, 'fontsize',18)
    xlabel('x','FontSize',20)
    ylabel('y','FontSize',20)

    psprintcpdf_keeppostscript(sprintf('andy_hw09_prb01_%02g_m%02g',1,M));
end

%% plot error versus 1/M

figure;
tmpfigh = gcf;
clf;
figshape(600,600);
set(gcf,'Color','none');
set(gcf,'InvertHardCopy', 'off');
set(gcf,'DefaultAxesFontname','helvetica');
set(gcf,'DefaultLineColor','r');
set(gcf,'DefaultAxesColor','none');
set(gcf,'DefaultLineMarkerSize',5);
set(gcf,'DefaultLineMarkerEdgeColor','k');
set(gcf,'DefaultLineMarkerFaceColor','g');
set(gcf,'DefaultAxesLineWidth',0.5);
```

```
set(gcf,'PaperPositionMode','auto');

plot(1./Mvec,errorvec,'LineWidth',2,'Color','b')

set(gca, 'fontsize',18)
xlabel('1/M','FontSize',20)
ylabel('Max␣error','FontSize',20)

psprintcpdf_keeppostscript(sprintf('andy_hw09_prb01_%02g',2));

% close all;
```

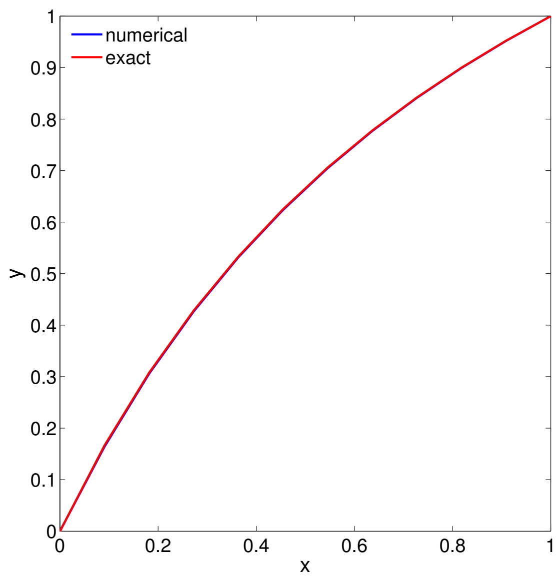

Figure 1: Exact and numerical solutions of the BVP using the collocation method with $M = 10$.

2. Obtain Eqs. (9.20) and (9.22) of the notes.

   Solution: First we have that

$$\phi_j' = \begin{cases} \frac{1}{h} & x_{j-1} < x < x_j \\ \frac{-1}{h} & x_j < x < x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

   For $|k - j| > 1$ it is clear from the above function that the $\phi_k'(x)\phi_j'(x) = 0$. Therefore I consider the remaining three cases: $|k - j| = 1$ and $k = j$.

   First, for $k = j$ we have that

$$\phi_j'(x)\phi_j'(x) = \begin{cases} \frac{1}{h^2} & x_{j-1} < x < x_j \\ \frac{1}{h^2} & x_j < x < x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

   We write this more concisely as

$$\phi_j'(x)\phi_j'(x) = \begin{cases} \frac{1}{h^2} & x_{j-1} < x < x_{j+1}, x \neq x_j \\ 0 & \text{otherwise} \end{cases}$$

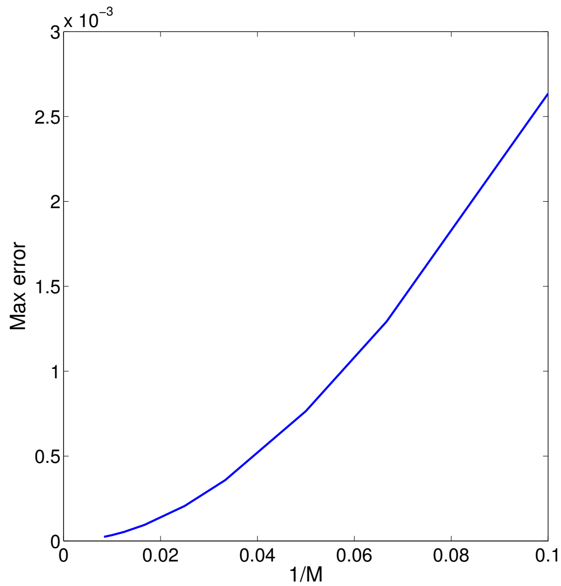

Figure 2: Scaling of the maximum error the collocation solution with $(1/M)$. We observe that error scales super-linearly (perhaps quadratically) with $1/M$, indicating that the error decrease scales sub-linearly with the decrease in h (where by h I mean the point spacing, which makes sense here, where the points are evenly spaced).

The integral of this function is x/h^2, we don't worry about the single point x_j, and so we take the integral over the nonzero part of this function (the rest is 0) to obtain

$$\int_{x_{j-1}}^{x_{j+1}} h^{-2}dx = h^{-2}x\Big|_{x_j-h}^{x_j+h} = \frac{2}{h}. \tag{1}$$

Now consider $k = j - 1$, and we have

$$\phi'_{j-1}(x)\phi'_j(x) = \begin{cases} \frac{1}{h} & x_{j-2} < x < x_{j-1} \\ \frac{-1}{h^2} & x_{j-1} < x < x_j \\ \frac{1}{h} & x_j < x < x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

Considering the integral of all three pieces, we have

$$\begin{aligned} \int_{x_{j-2}}^{x_{j+1}} \phi'_{j-1}(x)\phi'_j(x)dx &= \int_{x_{j-2}}^{x_{j-1}} \frac{1}{h}dx + \int_{x_{j-1}}^{x_j} \frac{-1}{h^2}dx + \int_{x_j}^{x_{j+1}} \frac{-1}{h}dx \\ &= \frac{x}{h}\Big|_{x_j-2h}^{x_j-h} - \frac{x}{h^2}\Big|_{x_j-h}^{x_j} - \frac{x}{h}\Big|_{x_j}^{x_j+h} \\ &= 1 + \frac{1}{h} - 1 = \frac{1}{h} \end{aligned}$$

The above applies to case $k = j + 1$ by replacing all of the subscripts $j - 2$ to $j + 2$, $j + 1$ to $j - 1$ and $j - 1$ to $j + 1$ (or simply setting $h \to -h$ in the replacement).

Therefore we have Eq 9.20, as desired.

To compute the integral in Eq 9.22, first we (again) note that for $|k-j|>1$ that $\phi_j(x)\phi_k(x) = 0$ for all x. Again we consider the three remaining cases (and will argue that WLOG we only need show $k = j-1$ without $k = j+1$).

For $k = j$ we have

$$\phi_j(x)\phi_j(x) = \begin{cases} \left(1 - \frac{|\Delta x_j|}{h}\right)^2 & x_{j-1} < x < x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

Therefore we take the integral only on the nonzero part (the rest is 0) to obtain (changing variables $z = x - x_j$)

$$\begin{aligned}
\int_{x_{j-1}}^{x_{j+1}} \phi_j(x)\phi_j(x)dx &= \int_{x_{j-1}}^{x_{j+1}} \left(1 - \frac{|\Delta x_j|}{h}\right)^2 dx \\
&= \int_{x_j-h}^{x_j+h} \left(1 - \frac{|x - x_j|}{h}\right)^2 dx \\
&= \int_{-h}^{h} \left(1 - \frac{2\,|z|}{h} + \frac{z^2}{h^2}\right) dz \\
&= \int_{0}^{h} \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right) dz + \int_{-h}^{0} \left(1 - \frac{2|z|}{h} + \frac{z^2}{h^2}\right) dz \\
&= 2\int_{0}^{h} \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right) dz \\
&= 2\left(z - \frac{z^2}{h} + \frac{z^3}{3h^2}\right)\Bigg|_0^h \\
&= 2\left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = 2\left(h - h + \frac{h}{3}\right) = \frac{2}{3}h
\end{aligned}$$

For $k = j-1$ we have the following integral:

$$\begin{aligned}
\int_{x_{j-2}}^{x_{j+1}} \phi_{j-1}(x)\phi_j(x)dx = &\int_{x_{j-2}}^{x_{j-1}} \left(1 - \frac{|x - x_{j-1}|}{h}\right)(0)dx + \int_{x_{j-1}}^{x_j} \left(1 - \frac{|x - x_{j-1}|}{h}\right)\left(1 - \frac{|x - x_j|}{h}\right) dx \\
&+ \int_{x_j}^{x_{j+1}} \left(1 - \frac{|x - x_j|}{h}\right)(0)dx
\end{aligned}$$

The first and last integral are zero, so we are only concerned with the middle one. The middle integral, transforming $z = x - x_{j-1}$ becomes

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \left(1 - \frac{|x - x_{j-1}|}{h}\right)\left(1 - \frac{|x - x_j|}{h}\right) dx &= \int_0^h \left(1 - \frac{|z|}{h}\right)\left(1 - \frac{|z - h|}{h}\right) dz \\
&= \int_0^h \left(1 - \frac{z}{h}\right)\left(1 + \frac{z - h}{h}\right) dz \\
&= \int_0^h \left(1 + \frac{z - h}{h} - \frac{z}{h} - \frac{z}{h}\frac{z - h}{h}\right) dz \\
&= -\int_0^h \left(\frac{z^2}{h^2} - \frac{z}{h}\right) dz \\
&= \left.\frac{z^2}{2h} - \frac{z^3}{3h^2}\right|_0^h = \frac{h}{6}
\end{aligned}$$

3. Solve the BVP in Problem 1 by the Galerkin method with the hat functions and $M = 10$. Compare your result with the exact solution. As in Problem 1, investigate how the error of the Galerkin method scales with $(1/M)$.

   Solution:

Appendix 1: ODE Functions

Appendix 2: Numerical Methods