Math 337 TEST 1

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1. (16 points) Show that

$$y_n''' = \frac{1}{2h^3}\left(y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}\right) + O(h^2).$$

Solution: To show this, I will expand each term on the RHS and then combine them after. I expand each term as

$$\begin{aligned}
\frac{1}{2h^3}y_{n+2} &= \frac{1}{2h^3}\left[y_n + 2hy_n' + \frac{(2h)^2}{2}y_n'' + \frac{(2h)^3}{6}y_n''' + \frac{(2h)^4}{24}y_n^{(4)} + \frac{(2h)^5}{120}y_n^{(5)} + O(h^6)\right] \\
&= \frac{1}{2h^3}y_n + \frac{1}{h^2}y_n' + \frac{1}{h}y_n'' + \frac{2}{3}y_n''' + \frac{1}{3}hy_n^{(4)} + \frac{2}{15}h^2y_n^{(5)} + O(h^3) \\
\frac{-1}{h^3}y_{n+1} &= \frac{-1}{h^3}\left[y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + \frac{h^4}{24}y_n^{(4)} + \frac{h^5}{120}y_n^{(5)} + O(h^6)\right] \\
&= -\frac{y_n}{h^3} - \frac{y_n'}{h^2} - \frac{y_n''}{2h} - \frac{1}{6}y_n''' - \frac{h}{24}y_n^{(4)} - \frac{h^2}{120}y_n^{(5)} + O(h^3) \\
\frac{1}{h^3}y_{n-1} &= \frac{1}{h^3}\left[y_n - hy_n' + \frac{h^2}{2}y_n'' - \frac{h^3}{6}y_n''' + \frac{h^4}{24}y_n^{(4)} - \frac{h^5}{120}y_n^{(5)} + O(h^6)\right] \\
&= \frac{y_n}{h^3} - \frac{y_n'}{h^2} + \frac{y_n''}{2h} - \frac{1}{6}y_n''' + \frac{h}{24}y_n^{(4)} - \frac{h^2}{120}y_n^{(5)} + O(h^3) \\
\frac{-1}{2h^3}y_{n-2} &= \frac{-1}{2h^3}\left[y_n - (2h)y_n' + \frac{(2h)^2}{2}y_n'' - \frac{(2h)^3}{6}y_n''' + \frac{(2h)^4}{24}y_n^{(4)} - \frac{(2h)^5}{120}y_n^{(5)} + O(h^6)\right] \\
&= -\frac{y_n}{2h^3} + \frac{y_n'}{h^2} - \frac{y_n''}{h} + \frac{2}{3}y_n''' - \frac{h}{3}y_n^{(4)} + \frac{2}{15}h^2y_n^{(5)} + O(h^3)
\end{aligned}$$

Now we group terms by order of the derivative y_n, y_n', etc. So we have

$$\begin{aligned}
y_n &: \frac{1}{2h^3} - \frac{1}{h^3} + \frac{1}{h^3} - \frac{1}{2h^3} = 0 \\
y_n' &: \frac{1}{h^2} - \frac{1}{h^2} - \frac{1}{h^2} + \frac{1}{h^2} = 0 \\
y_n'' &: \frac{1}{h} - \frac{1}{2h} + \frac{1}{2h} - \frac{1}{h} = 0 \\
y_n''' &: \frac{2}{3} - \frac{1}{6} + \frac{1}{6} + \frac{2}{3} = 1 \\
y_n^{(4)} &: \frac{1}{3}h - \frac{1}{24}h + \frac{1}{24}h - \frac{1}{3}h = 0 \\
y_n^{(5)} &: \frac{2}{15}h^2 - \frac{h^2}{120} - \frac{h^2}{120} + \frac{2}{15}h^2 = \frac{1}{4}h^2 = O(h^2)
\end{aligned}$$

Thus, we have reduced the RHS of the given equation to $y''' + O(h^2)$, as desired.

2. (12 points) State the main reason on emay want to use a Runge-Kutta-Fehlberg method instead of the classical Runge-Kutta method. Provide as much *relevant* detail about the Runge-Kutta-Fehlberg method as possible.

   Solution: The principal advantage of the Runge-Kutta-Fehlberg (RKF) method of the Runge-Kutta (RK) method is the control of local error. Since in practice it is impractical (too time expensive) to control global error by running multiple simulations, local error control is often sufficient. In addition, by using an adaptive step size, the RKF method uses a larger step size when this is appropriate (i.e. the solution changes very smoothly on some subinterval), avoiding unncessary computation in this region.

   In particular, the RKF method requires 6 function evaluations per timestep, whereas a 4-th order RK (the cRK) method would require 4. This is indeed an advantage, since to control error by running another simulation with cRK would require twice the function evaluation (and perhaps 3 times, if the timestep is halved), to control for error. Also, as mentioned above, to increase the accuracy of the cRK to within a certain tolerance globally may require a very small timestep in some places (i.e. skydiver with parachute, piecewise ODE), but this small timestep is not necessary for the rest of the solution. This would result in the nieve cRK incurring unnecessary time expense.

3. (48 points) A popular predictor-corrector method, called the Hamming method after R.W. Hamming, is:

$$\text{Predictor:} \quad Y_{n+1}^{(p)} = Y_{n-3} + \frac{4h}{3}\left(2f_{n-2} - f_{n-1} + 2f_n\right), \tag{1}$$

$$\text{Corrector:} \quad Y_{n+1}^{(c)} = \frac{1}{8}\left(9Y_n + Y_{n-2}\right) + \frac{3h}{8}\left(-f_{n-1}2f_n + f_{n+1}^{(p)}\right),$$

   where $f_n = f(x_n, Y_n)$ etc., and $f_{n+1}^{(p)} = f\left(x_{n+1}, Y_{n+1}^{(p)}\right)$.

   (a) Show that the predictor equation is a fourth-order method. More precisely, show that in the leading order, its local truncation error is:

$$y_{n+1} - Y_{n+1}^{(p)} = \frac{112}{3}\frac{h^5}{5!}y_n^{(v)} \tag{2}$$

Similarly, one can show (you do not to do that) that the corrector equation is also fourth-order, and its local truncation error satisfies

$$y_{n+1} - Y_{n+1}^{(c)} = -3\frac{h^5}{5!}y_n^{(v)} \tag{3}$$

(b) Use Eqs (2) and (3) and, following the lines of Sec 3.6, show that

$$Y_{n+1} = \frac{1}{121}\left(9Y_{n+1}^{(p)} + 112Y_{n+1}^{(c)}\right) \tag{4}$$

along with Eqs. (1), provides a 5-th order method. Also, obtain an estimate for the correctors local truncation error given the computed values $Y_{n+1}^{(p)}$ and $Y_{n+1}^{(c)}$.

(c) In order to start the calculations, you will need points $Y_{1,2,3}$ (Y_0 is given by the initial condition). What (minimal) order of the single-step method do you need to use to obtain those points so as to be consistent with the order of method (1) alone? Explain your answer. Now answer (with explanation) the same question about the combined method (1) and (4). (See a remark in Sec. 3.6).

(d) Show *analytically* that the Hamming method (1) is partially stable (i.e. that it is stable for sufficiently small step sizes).

Solution:

(a) For local truncation error we assume that $Y_{n-k} = y_{n-k}$ for $k = 0, 1, 2, \ldots$. Expanding the RHS of Eq (1), we have

$$\begin{aligned}
Y_{n+1}^{(p)} &= y_n - 3hy_n' + \frac{(3h)^2}{2}y_n'' - \frac{(3h)^3}{6}y^{(3)} + \frac{(3h)^4}{24}y^{(4)} + \frac{(3h)^5}{120}y^{(5)} + O(h^6) \\
&+ \frac{8h}{3}\left(y_n' - 2hy_n'' + \frac{(2h)^2}{2}y_n''' - \frac{(2h)^3}{6}y^{(4)} + \frac{(2h)^4}{24}y^{(5)} + O(h^5)\right) \\
&- \frac{4h}{3}\left(y_n' - hy_n'' + \frac{h^2}{2}y_n''' - \frac{h^3}{6}y^{(4)} + \frac{h^4}{24}y^{(5)} + O(h^5)\right) + \frac{8}{3}hy_n'
\end{aligned}$$

Grouping by derivatives of y_n, we have

$$\begin{aligned}
y_n &: 1 \\
y_n' &: \frac{8}{3}h - \frac{4}{3}h - \frac{8}{3}h - 3h = h \\
y_n'' &: -\frac{16}{3}h^2 + \frac{4}{3}h^2 + \frac{9}{2}h^2 = \frac{h^2}{2} \\
y_n''' &: \frac{8}{3}\frac{4h^3}{2} - \frac{4}{3}\frac{h^3}{2} - \frac{27}{6}h^3 = \frac{h^3}{6} \\
y_n^{(4)} &: -\frac{8}{3}\frac{8h^3}{6} - \frac{4}{3}\left(\frac{-h^3}{6}\right) + \frac{81}{24}h^2 = \frac{1}{24}h^4 \\
y_n^{(5)} &: \frac{8}{3}\frac{16}{24}h^4 - \frac{4}{3}\frac{h^4}{24} - \frac{243}{120}h^5 = \frac{h^5}{5!}\cdot\left(\frac{-109}{3}\right)
\end{aligned}$$

Observe that in the above, the coefficient on y_n through $y_n^{(4)}$ are the same as those in the Taylor expansion of y_{n+1} about y_n. Therefore, these terms drop in $y_{n+1} - Y_{n+1}^{(p)}$. The order $O(h^5)$ term is therefore $(1-(-109/3))h^5/5! = 112/3 \cdot h^5/5!$, as desired.

(b) To see that Eq (4) produces a 5-th order method, we simply require that the the coefficent of h^5 on the RHS of Eq 4 agrees with the Taylor expansion about y_n, that is, it is $h^5/5!$. From Equations 2 and 3 we have that the h^5 term in the Taylor expansion of the RHS of equation 4 is:

$$\frac{9}{121} \cdot \frac{-109}{3} \cdot \frac{h^5}{5!} + \frac{112}{121} \cdot 4 \cdot \frac{h^5}{5!} = \frac{h^5}{5!}\left(\frac{-327}{121} + \frac{448}{122}\right) = \frac{h^5}{5!}.$$

I should also explicitly note that the agreement of the Taylor expansion agrees up to 4-th order by the form of (3), since both $Y_{n+1}^{(p,c)}$ agree (they are the same) and the sum of the coefficients RHS of (3) will add to 1, so it will still agree.

Now to obtain an estimate of the corrector's local truncation error, using computed values, we start by noting that

$$\left|\epsilon_{i+1}^c\right| \approx 3\frac{1}{5!}h^5 \left|y_n^{(5)}\right|.$$

We also have that from both methods

$$\left|Y_{i+1}^p - Y_{i+1}^c\right| \approx \left(\frac{14}{45} + \frac{1}{40}\right) h^5 \left|y_n^{(5)}\right|.$$

Therefore

$$\left|\epsilon_{i+1}^c\right| \approx \frac{9}{121}\left|Y_{i+1}^p - Y_{i+1}^c\right|.$$

(c) To start the predictor equation alone, we require a starting method that is third order. This is the result of Sec. 3.4 in the notes.

To start the combined method, we require a fourth-order starting method. This is because starting with a third order method would invalidate the derivation of the local truncation error, thus causing (3) to no longer approximate the local error.

(d) Substituting the model problem into Eqs (1), we have:

$$Y_{n+1} = Y_n\left(\frac{9}{8} + \frac{3}{4}h\lambda + \lambda^2 h^2\right) + Y_{n-1}\left(\frac{-3}{8}h\lambda - \frac{1}{2}h^2\lambda^2\right) + Y_{n-2}\left(\frac{-1}{8} + h^2\lambda^2\right) + Y_{n-3}\left(\frac{3}{8}h\lambda\right)$$

I cannot determine a "special" value of λh that makes any more than one of the coefficients here equal to 0. Choosing $\lambda h = -3/4$ makes the Y_{n-1} term drop, but no others. Regardless, I carry on. We solve this as a linear ODE with the ansatz $y = e^{\rho x}$, set $\lambda h = z$ and have:

$$\rho^2 = \rho\left(\frac{9}{8} + \frac{3}{4}z + z^2\right) + \left(\frac{-3}{8}z - \frac{1}{2}z^2\right) + \rho^{-1}\left(\frac{-1}{8} + z^2\right) + \frac{3}{8}z\rho^{-2}$$

Multiplying through by ρ^2 and moving everything to the RHS we have

$$0 = \rho^4 + \rho^3\left(-\frac{9}{8} - \frac{3}{4}z - z^2\right) + \rho^2\left(\frac{3}{8}z + \frac{1}{2}z^2\right) + \rho^1\left(\frac{1}{8} - z^2\right) + \frac{3}{8}z$$

As in the notes, equation 4.37, we consider $z \to 0$ as $h \to 0$ for the lowest order approximation. I tried many other ways to reduce the equation (settin $z = -1/2, 2$,etc to cancel terms) with no success, so I looked to the notes for this lowest order approximation used in Sec 4.4. Therefore we have the quartic

$$0 = \rho^4 + -\frac{9}{8}\rho^3 + \frac{1}{8}\rho$$

Cancelling ρ, we have

$$0 = \rho^3 + -\frac{9}{8}\rho^2 + \frac{1}{8}$$

An obvious root of this cubic is $\rho = 1$, so we factor this out to obtain

$$0 = (\rho - 1)(\rho^2 - \frac{1}{8}\rho - \frac{1}{8})$$

and use the quadratic formula to obtain

$$\rho_{1,2} = \frac{1}{16} \pm \frac{1}{16}\sqrt{33}.$$

Since $\sqrt{33} < 8$, we can conclude that the roots are both less than 1 in magnitude and that the Hamming method is stable as $h \to 0$.

4. (a) Integrate the harmonic oscillator model (5.25) using the simple implicit Euler method. (You may use any meaningful values for the initial conditions and also $h = 0.1$ and $x_{\max} = 20$; the exact numbers are not very important in this problem exvept for the Bonus part below.) Attach the printouts of your code and the phase plot of your solution. Qualitatively explain your results.

   (b) Will the numerical solution change *qualitatively* if you use a modified implicit Euler method instead of the simple implicit one? Please explain.

Solution:

(a) Code and phase plot below. Qualitatively, we see that the solution decays. This is expected, since the eigenvalues of the harmonic oscillator system are $\pm i\omega$ and the simple Euler is "stable" on the imaginary axis, meaning that the magnitude of the model problem's solution decreases.

```
% Exam 1 problem 4
%
% integrate the simple harmonic oscillator

h = 0.1;
tmax = 20;
tvec = 0:h:tmax;
omega = 1;
y0 = [0;2];

y = andy_IE_ho(@dummy,tvec,y0,h,[omega]);

plot(y(1,:),y(2,:))
xlabel('y','FontSize',20)
ylabel('v','FontSize',20)
```

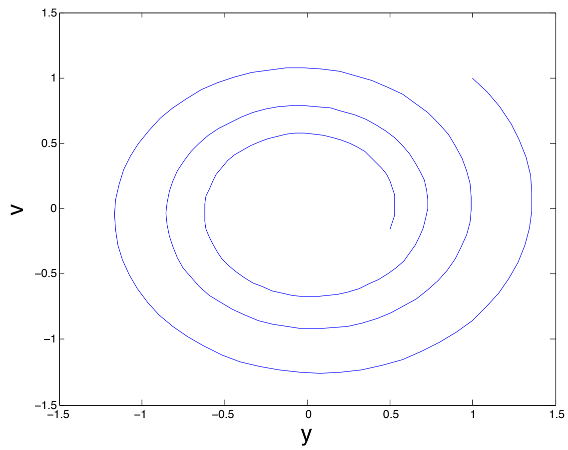

Figure 1: Phase plot of the harmonic oscillator solved with simple implicit Euler method.

```
function yvec = andy_IE_ho(func,tspan,y0,h,params)
% implicit Euler
%
% solves the 2D harmonic oscillator

omega = params(1);
t = tspan(1);

yvec =  [];
yvec = [yvec y0];
for i=2:length(tspan)
    y1up = (yvec(1,i-1)+h*yvec(2,i-1))/(1+h^2*omega^2);
    y2up = yvec(2,i-1)-h*omega^2*y1up;
    yvec = [yvec [y1up;y2up]];
    t = t+h;
end
```

(b) No. In a previous HW assignment, I showed that the stability region of the modified implicit Euler scheme is the left half of the complex plane. This region is smaller than that for the simple implicit Euler scheme, but still contains the imaginary axis and so I would also expect the solution to decay. Qualitatively, therefore, the results will be the same.