Math 337 Assignment 1

Andy Reagan

January 17, 2014

1. Find the explicit form of the cubic term (i.e. the term with $(\Delta x)^n(\Delta y)^m), n+m=3)$ in expansion (0.6).

   Solution: The explicit form is given by (assuming the equality of mixed partials):

$$\frac{1}{3!}\left(\Delta x\frac{\partial}{\partial\overline{x}}+\Delta y\frac{\partial}{\partial\overline{y}}\right)^3 f(\overline{x},\overline{y})|_{\overline{x}=x_0,\overline{y}=y_0}$$
$$=\frac{1}{3!}\left((\Delta x)^3\frac{\partial^3}{\partial\overline{x}^3}+3\Delta x(\Delta y)^2\frac{\partial^2}{\partial\overline{y}^2}\frac{\partial}{\partial\overline{x}}+3\Delta y(\Delta x)^2\frac{\partial^2}{\partial\overline{x}^2}\frac{\partial}{\partial\overline{y}}+(\Delta y)^3\frac{\partial^3}{\partial\overline{y}^3}\right) f(\overline{x},\overline{y})|_{\overline{x}=x_0,\overline{y}=y_0}$$
$$=\frac{1}{3!}\left((\Delta x)^3 f_{xxx}(x_0,y_0)+3\Delta x(\Delta y)^2 f_{yyx}(x_0,y_0)+3\Delta y(\Delta x)^2 f_{xxy}(x_0,y_0)+(\Delta y)^3 f_{yyy}(x_0,y_0)\right).$$

2. Find the Lipschitz constant L for:
   (a) $f(x,y)=xy^2$ on $R: 0\le x\le 3, 1\le y\le 5$;
   (b) $f(x,y)=x+|\sin 2y|$ on $R: 0\le x\le 3, -\pi\le y\le\pi$;

   Solution: (a)We have that in general $L=\max_R|f_y(x,y)|$ where $f_y=2xy$ such that $L=2\cdot 3\cdot 5=30$ here.
   (b)Again let $L=\max_{R'}|f_y(x,y)|$ where R' is the interection of R and the domain of f_y. Then we have that $f_y=1+\max|(\pm\cos y)|$ so $L=2$.

3. Solve the IVP

$$y'=2y+e^{3x},\quad y(-1)=4.$$

   Solution: We can solve this using "variation of parameters". We first solve the homogeneous ODE

$$y'_{\text{hom}}=2y_{\text{hom}}$$

   to obtain $y=e^{2(x+1)}$. Putting this homogeneous solution (times c) into the nonhomogenous problem for y we have

$$c'y_{\text{hom}}=e^{3x}$$

   such that

$$c=\int_{-1}^{x}e^{3z}e^{-2(z+1)}dz=\int_{-1}^{x}e^{z-2}dz=e^{x-2}-e^{-3}$$

   and therefore the solution is given by

$$y=e^{2x+2}\left(4+e^{x-2}-e^{-3}\right)=4e^{2x+2}+e^{3x}-e^{2x-1}.$$

4. Find

$$\lim_{h\to 0}\left(\frac{1}{1-2h}\right)^{\pi/h}.$$

Solution: First take exponential of the natural logarithm of the limit. This gives us the form

$$e^{\lim_{h\to 0}(\pi/h)\log\left(\frac{1}{1-2h}\right)}.$$

Pulling out the constant π and using L'Hopital's rule on the remaining limit, we have

$$e^{\pi \lim_{h\to 0} 2/(1-2h)}.$$

The constant 2 comes out of the limit and we see that as $h \to 0$ the remaining limit goes to 1. Alternatively, let $h = 1/n$ and as $n \to \infty$ we see the same. Therefore, we have that

$$\lim_{h\to 0}\left(\frac{1}{1-2h}\right)^{\pi/h} = e^{2\pi}.$$

Alternativy, we have that the limit is of the form

$$\lim_{h\to 0}(1+ah)^{b/h}$$

for $a = -2$ and $b = -\pi$. We then know from the notes that this limit is equal to

$$e^{ab} \quad \to \quad e^{2\pi}.$$