

Math 337 Homework 07

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1. Solve the BVP

$$y'' + xy' - 3y = 3x, y(0) = 1, y(2) = 5$$

by the shooting method. Use the modified Euler method with $h = 0.1$ as the IVP solver. Plot your solution.

Solution: My code and a solution plot follow.

```
% HW07 Problem 01
%
% solve the BVP using the shooting method
% - I define the homogeneous and non-homogeneous in separate files
% - the IC are both Nuemann

% IC, BC
y0 = 1; yf = 5;
h = 0.1;
tvec = 0:h:2;

i=1;
shot1 = andy_ME(@andy_hw07_prb01_ODE,tvec,[y0;0],h,[]);
shot2 = andy_ME(@andy_hw07_prb01_ODEh,tvec,[0;1],h,[]);

theta = (yf-shot1(1,end))/shot2(1,end);

soln = shot1+theta.*shot2;
plot(tvec,soln(1,:));
xlabel('x','FontSize',20);
ylabel('y','FontSize',20);
set(gcf,'units','inches','position',[1 1 10 10])
set(gcf,'PaperPositionMode','auto')
print('-depsc2','-zbuffer','-r200',sprintf('andy_hw07_prb01_%02g.eps',i))
system(sprintf('epstopdf\andy_hw07_prb01_%02g.eps',i));
```

2. Solve the BVP

$$x^3 y''' + xy' - y = -3 + \ln x, y(1) = 1, y'(2) = 1/2, y''(2) = 1/4$$

by “shooting” from the left end point and using the example given in the notes. Use the ME method with $h = 0.02$. Plot your solution

Solution: My code and a solution plot follow.

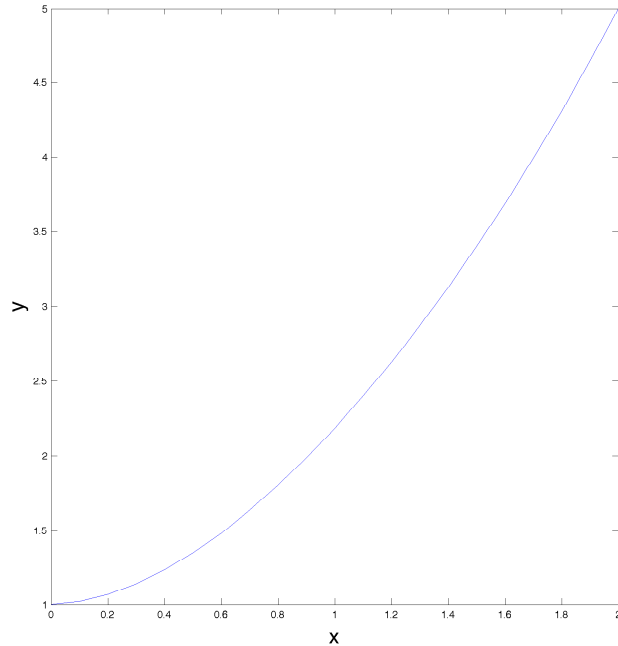


Figure 1: Solution of the BVP with the shooting method.

```
% HW07 Problem 01
%
% solve the BVP using the shooting method
% - I define the homogeneous and non-homogeneous in separate files
% - the IC are both Nuemann

% IC, BC
y0 = 1; yfp = 1/2; yfpp = 1/4;
h = 0.02;
tvec = 1:h:2;

i=1;
% solve the three IVP's
shot1 = andy_ME(@andy_hw07_prb02_ODE,tvec,[y0;0;0],h,[]);
shot2 = andy_ME(@andy_hw07_prb02_ODEh,tvec,[0;1;0],h,[]);
shot3 = andy_ME(@andy_hw07_prb02_ODEh,tvec,[0;0;1],h,[]);

% construct z
z = [shot1(:,end) shot2(:,end) shot3(:,end)];
% take just the bottom two
z = z(2:3,:);

% solve for theta,psi. call them both theta
theta = z(:,2:3)\[yfp-z(1,1);yfpp-z(2,1)];

soln = shot1+theta(1).*shot2+theta(2).*shot3;
plot(tvec,soln);
legend('y','y','ypp')
xlabel('x','FontSize',20);
ylabel('y','FontSize',20);
set(gcf, 'units', 'inches', 'position', [1 1 10 10])
set(gcf, 'PaperPositionMode','auto')
print('-depsc2','-zbuffer','-r200',sprintf('andy_hw07_prb02_%02g.eps',i))
system(sprintf('epstopdf_andy_hw07_prb02_%02g.eps',i));
```

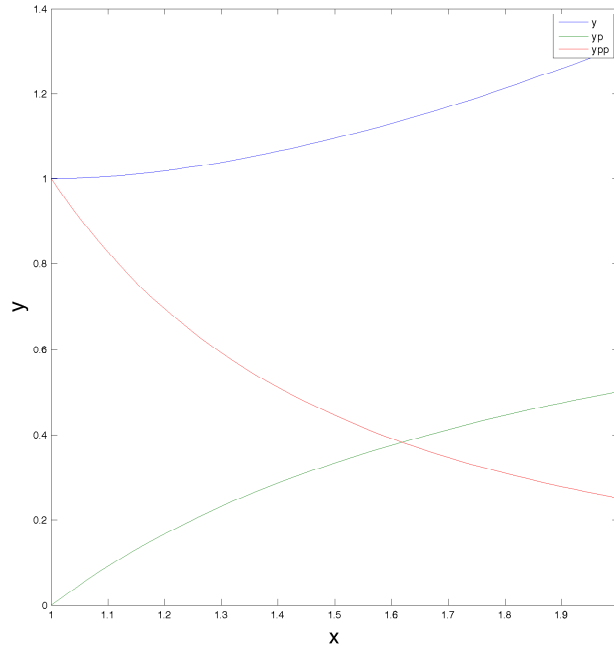


Figure 2: Solution of the BVP with the shooting method. Here y_p denotes y' and y_{pp} denotes y'' .

3. Solve problem 2, in reverse.

Solution: My code and a solution plot follow. I found it easiest to write a ME function that goes backwards.

```
% HW07 Problem 01
%
% solve the BVP using the shooting method
% - I define the homogeneous and non-homogeneous in separate files
% - the IC are both Nuemann

% IC, BC
y0 = 1; yfp = 1/2; yfpp = 1/4;
h = 0.02;
tvec = 1:h:2;

i=1;
% solve the three IVP's
shot1 = andy_MEr(@andy_hw07_prb02_ODE,tvec,[0;yfp;yfpp],h,[]);
shot2 = andy_MEr(@andy_hw07_prb02_ODEh,tvec,[1;0;0],h,[]);

% solve for theta,psi. call them both theta
theta = (y0-shot1(1,1))/shot2(1,1);

soln = shot1+theta.*shot2;
plot(tvec,soln);
legend('y','yp','ypp')
xlabel('x','FontSize',20);
ylabel('y','FontSize',20);
set(gcf,'units','inches','position',[1 1 10 10])
set(gcf,'PaperPositionMode','auto')
print('-depsc2','-zbuffer','-r200',sprintf('andy_hw07_prb03_%02g.eps',i))
system(sprintf('epstopdf\andy_hw07_prb03_%02g.eps',i));
```

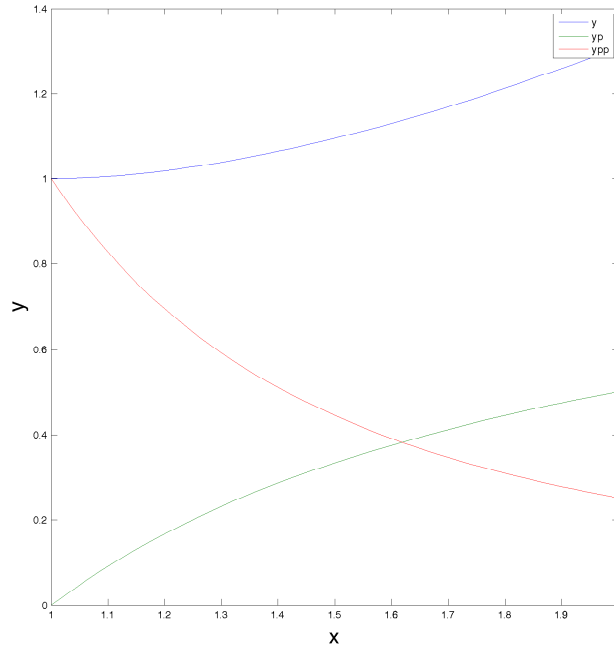


Figure 3: Solution of the BVP with the shooting method. Here yp denotes y' and ypp denotes y'' .

4. Solve the BVP

$$y'' = 30^2(y - 1 + 2x), y(0) = 1, y(1.62) = -2.24$$

by the usual (i.e., not multiple) shooting method. Use any appropriate IVP-solving method and any reasonable value for h . Plot your numerical solution along with the exact solution, found in the notes.

Solution: My code and a solution plot follow.

```
% HW07 Problem 04
%
% solve the BVP using the shooting method
% - I define the homogeneous and non-homogeneous in separate files
% - the IC are both Nuemann

% IC, BC
y0 = 1; yf = -2.24;
h = 0.0001;
tvec = 0:h:1.62;

i=1;
shot1 = andy_ME(@andy_hw07_prb04_ODE,tvec,[y0;0],h,[]);
shot2 = andy_ME(@andy_hw07_prb04_ODEh,tvec,[0;1],h,[]);

theta = (yf-shot1(1,end))/shot2(1,end);

soln = shot1+theta.*shot2;
plot(tvec,soln(1,:));
xlabel('x','FontSize',20);
ylabel('y','FontSize',20);
```

```

set(gcf, 'units', 'inches', 'position', [1 1 10 10])
set(gcf, 'PaperPositionMode', 'auto')
print('-depsc2', '-zbuffer', '-r200', sprintf('andy_hw07_prb04_%02g.eps', i))
system(sprintf('epstopdf_andy_hw07_prb04_%02g.eps', i));

```

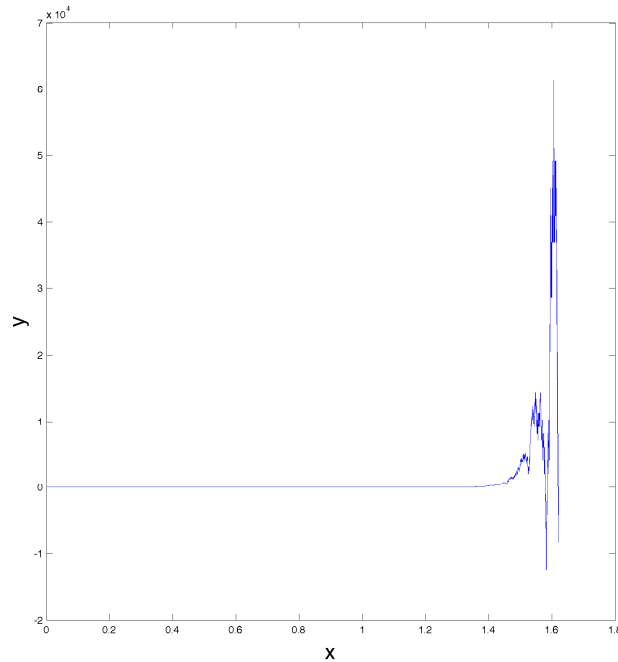


Figure 4: Solution of the BVP with the (regular) shooting method.

5. Solve the nonlinear BVP

$$y'' = \frac{y^2}{2+x}, y(0) = 1, y(2) = 1$$

following the outline given in Sec. 7.5. Find (and, of course, plot) the solutions of this BVP corresponding to both $\bar{\theta}$ and $\bar{\bar{\theta}}$.

Solution: My code and a solution plot follow.

```

% HW07 Problem 05
%
% solve the nonlinear BVP using the shooting method
% shoot iteratively
% use the secant method for solving

% IC, BC
y0 = 1; yf = 1;
h = 0.02;
tvec = 0:h:2;
tol = 10^(-3);

% initialize theta and f
thetacell = {[ -2, -1], [2, 1]};
for j=1:2

```

```

err = 1;
i=2;
theta = thetacell{j};
% disp(theta);
shot1 = andy_ME(@andy_hw07_prb05_ODE,tvec,[y0;theta(1)],h,[]);
shot2 = andy_ME(@andy_hw07_prb05_ODE,tvec,[y0;theta(2)],h,[]);
f = [shot1(1,end),shot2(1,end)];
while err > tol
    % generate new theta
    theta = [theta theta(end)-f(end)/((f(end)-f(end-1))/(theta(end)-theta(end-1)))
    ];
    % solve the ODE
    shot = andy_ME(@andy_hw07_prb05_ODE,tvec,[y0;theta(end)],h,[]);
    % f is y(2) at that theta
    f = [f shot(1,end)-yf];
    % count the iterations
    i=i+1;
    % error is just f
    err = abs(f(end));
end
fprintf('took %g iterations for theta %g\n',i,j);
% disp(theta);

soln = shot;
figure;
plot(tvec,soln(1,:));
xlabel('x','FontSize',20);
ylabel('y','FontSize',20);
set(gcf,'units','inches','position',[1 1 10 10])
set(gcf,'PaperPositionMode','auto')
print('-depsc2','-zbuffer','-r200',sprintf('andy_hw07_prb05_%02g.eps',j))
system(sprintf('epstopdf_andy_hw07_prb05_%02g.eps;\rm_andy_hw07_prb05_%02g.eps',
j,j));
end

```

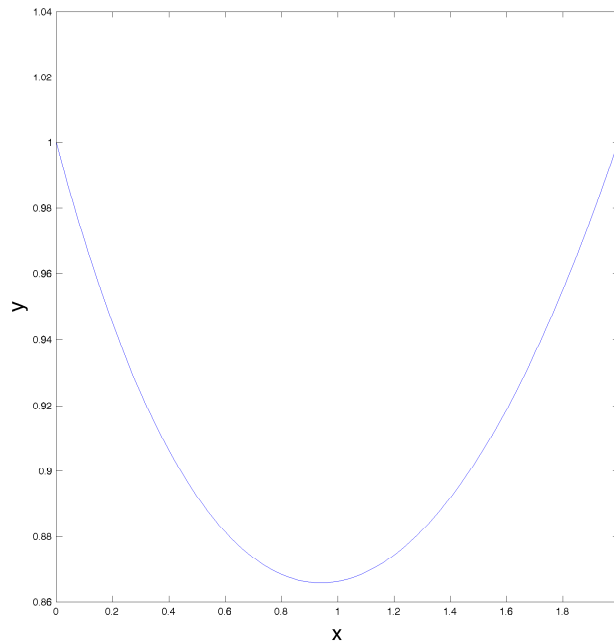


Figure 5: Solution of the BVP with the (regular) shooting method corresponding to $\bar{\theta}$.

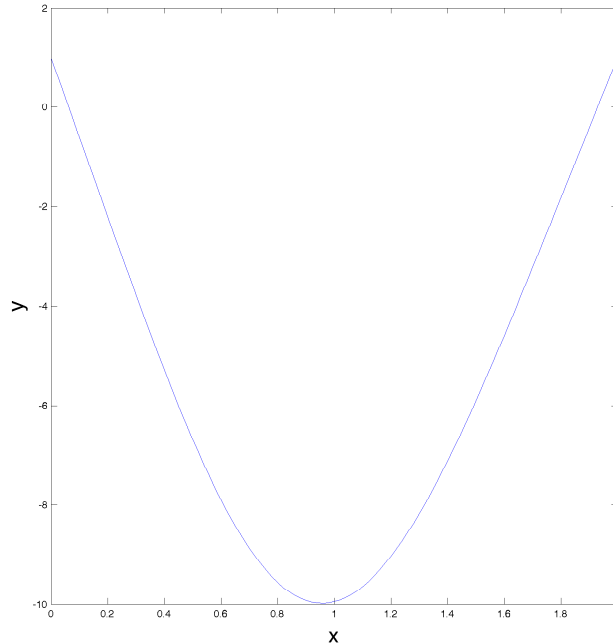


Figure 6: Solution of the BVP with the (regular) shooting method corresponding to $\bar{\bar{\theta}}$.

6. Solve the eigenvalue problem

$$y'' + (2\text{sech}^2 x - \lambda^2)y = 0, x \in (-\infty, \infty), y(|x| \rightarrow \infty) \rightarrow 0.$$

As is explained in the notes, solve the IVP on the interval $[-R, R]$ for some reasonable large R (say $R = 10$). Choose $y(-R) = \exp[-cR]$ and $y'(-R) = \lambda v(-R)$, where the constant c is of order one (so you may just take $c = 1$). The follow the algorithm outlined in the notes.

Solution: My code and a solution plot follow. I find that the method in the notes of “sweeping” values of λ does not produce good results within the realm where this search is possible. This could be due to error in my solving method. A more efficient implementation, using the secant method on $y(R) = f = 0$ produces satisfactory results.

```
% HW07 Problem 06
%
% solve the eigenvalue problem using the shooting method

% IC, BC
R = 10;
y0 = exp(-R);
h = 0.02;
tvec = -R:h:R;

% initialize lambda and f
% can set the search to .001 for f=-.34
% searching with .00001 we find f=-0.012, better!
lambdas = 0.98:.001:1.02;
f = zeros(1,length(lambdas));
% now shoot for all lambda
```

```

for i=1:length(lambdas)
    % set lambda
    lambda = lambdas(i);
    yf = lambda*y0;
    % shoot
    shot = andy_ME(@andy_hw07_prb06_ODE,tvec,[y0;yf],h,[lambda]);
    % save shot
    f(i) = shot(1,end);
end
disp(f);

% find where f crosses 0
for i=1:length(f)-1
    % where it crosses
    if f(i+1)*f(i) < 0
        disp(i);
        % take the middle value
        lambdastar = (lambdas(i)+lambdas(i+1))/2;
        disp(lambdastar);
    end
end

% now take a shot with lambda*
yf = lambdastar*y0;
shot = andy_ME(@andy_hw07_prb06_ODE,tvec,[y0;yf],h,[lambdastar]);

j=1;
soln = shot;
plot(tvec,soln(1,:));
xlabel('x','FontSize',20);
ylabel('y','FontSize',20);
set(gcf,'units','inches','position',[1 1 10 10])
set(gcf,'PaperPositionMode','auto')
print('-depsc2','-zbuffer','-r200',sprintf('andy_hw07_prb06_%02g.eps',j))
system(sprintf('epstopdf\andy_hw07_prb06_%02g.eps;\rm\andy_hw07_prb06_%02g.eps',j,j)
);

```

```

% HW07 Problem 06...using smarter search
%
% solve the eigenvalue problem using the shooting method

% IC, BC
R = 10;
y0 = exp(-R);
h = 0.02;
tvec = -R:h:R;

% initialize lambda and f
lambdas = [0.98,1.02];
TOL = 10^-7;
yf = lambdas(1)*y0;
shot1 = andy_ME(@andy_hw07_prb06_ODE,tvec,[y0;yf],h,[lambdas(1)]);
yf = lambdas(2)*y0;
shot2 = andy_ME(@andy_hw07_prb06_ODE,tvec,[y0;yf],h,[lambdas(2)]);
f = [shot1(1,end),shot2(1,end)];
% now shoot for all lambda
while abs(lambdas(end)-lambdas(1)) > TOL
    % set lambda
    lambdas = [lambdas lambdas(end)-f(end)/((f(end)-f(end-1))/(lambdas(end)-lambdas(
        end-1)))];
    lambdanew = lambdas(end);
    yf = lambdanew*y0;
    % shoot
    shot = andy_ME(@andy_hw07_prb06_ODE,tvec,[y0;yf],h,[lambdanew]);

```

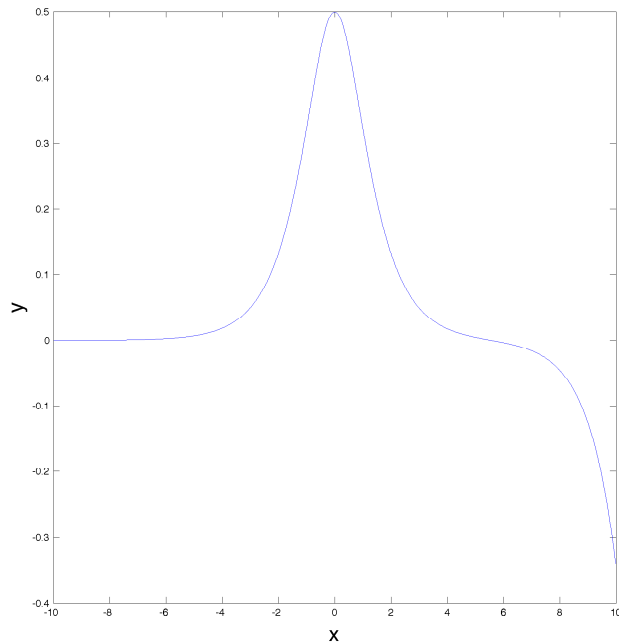



Figure 7: Solution of the eigenvalue problem with the shooting method, by scanning.

```

        % save shot
        f = [f,shot(1,end)];
    end
    disp(f(end));
    lambdafinal = lambdas(end)-f(end)/((f(end)-f(end-1))/(lambdas(end)-lambdas(end-1)));
    disp(lambdafinal);

    % now take a shot with lambda*
    yf = lambdafinal*y0;
    shot = andy_ME(@andy_hw07_prb06_ODE,tvec,[y0;yf],h,[lambdafinal]);
    disp(shot(1,end));

    j=2;
    soln = shot;
    plot(tvec,soln(1,:));
    xlabel('x','FontSize',20);
    ylabel('y','FontSize',20);
    set(gcf, 'units', 'inches', 'position', [1 1 10 10])
    set(gcf, 'PaperPositionMode', 'auto')
    print('-depsc2','-zbuffer','-r200',sprintf('andy_hw07_prb06_%02g.eps',j))
    system(sprintf('epstopdf\andy_hw07_prb06_%02g.eps;\rm\andy_hw07_prb06_%02g.eps',j,j)
    );

```

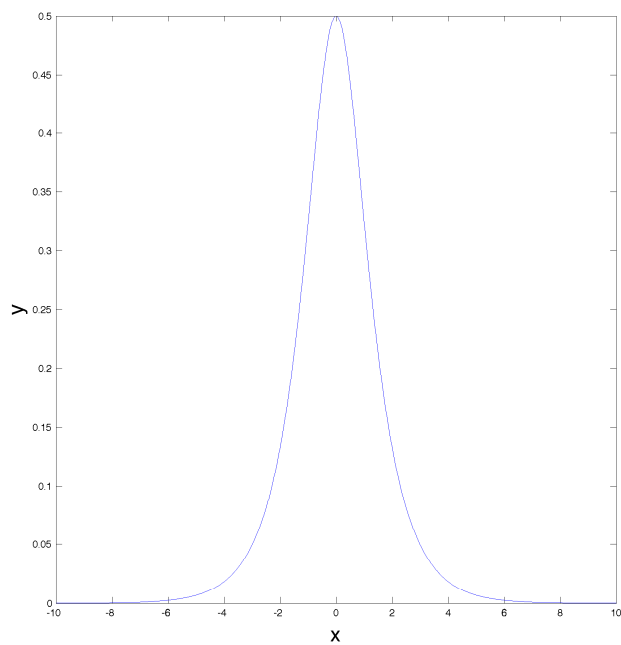


Figure 8: Solution of the eigenvalue problem with the shooting method, by applying the secant method on finding the zero of f .

7. Bonus: I code up the multiple shooting method and plot it below. The method is successful.

```
% HW07 Problem 07 (Bonus 1)
%
% solve the BVP using the (multiple) shooting method
% - I define the homogeneous and non-homogeneous in separate files
% - the IC are both Nuemann

% IC, BC
y0 = 1; yf = -2.24;
h = 0.01;
tvec = 0:h:1.62;
% divide the interval in half
tvec1 = 0:h:1.62/2;
tvec2 = 1.62/2:h:1.62;

% shoot on the first half
i=1;
shot1 = andy_ME(@andy_hw07_prb04_ODE,tvec1,[y0;0],h,[]);
shot11 = andy_ME(@andy_hw07_prb04_ODEh,tvec1,[0;1],h,[]);

shot2 = andy_ME(@andy_hw07_prb04_ODE,tvec2,[0;0],h,[]);
shot21 = andy_ME(@andy_hw07_prb04_ODEh,tvec2,[1;0],h,[]);
shot22 = andy_ME(@andy_hw07_prb04_ODEh,tvec2,[0;1],h,[]);

% now solve for all three thetas
r = [-shot1(1,end);-shot1(2,end);yf-shot2(1,end)];
A = [shot11(1,end),-1,0; shot11(2,end),0,-1; 0,shot21(1,end),shot22(1,end)];
% solve by elimination
thetavec = A\r;

soln = [shot1(:,1:end-1)+thetavec(1).*shot11(:,1:end-1),shot2+thetavec(2).*shot21+
        thetavec(3).*shot22];
plot(tvec,soln(1,:));
hold on;
truth = 1-2.*tvec;
plot(tvec,truth,'r--')
legend('numerical','exact')
xlabel('x','FontSize',20);
ylabel('y','FontSize',20);
set(gcf, 'units', 'inches', 'position', [1 1 10 10])
set(gcf, 'PaperPositionMode', 'auto')
print('-depsc2','-zbuffer','-r200',sprintf('andy_hw07_prb07_%02g.eps',i))
system(sprintf('epstopdf\andy_hw07_prb07_%02g.eps',i));
```

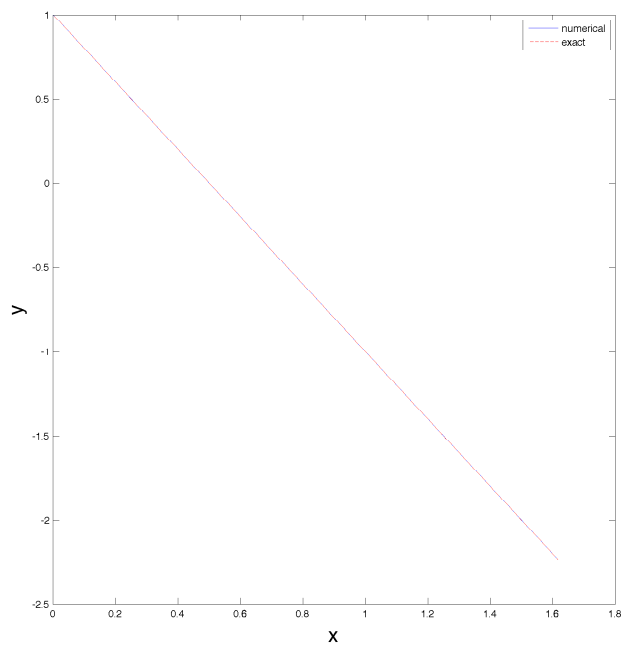


Figure 9: The multiple shooting method on the BVP from Problem 4.

Appendix 1: ODE Functions

```
function dy = andy_hw07_prb01_ODE(t,y,params)
% [y';v']
% where y' = v, v' = -xv+3y-3x
dy = [y(2);-t*y(2)+3*y(1)-3*t];
```

```
function dy = andy_hw07_prb01_ODEh(t,y,params)
% [y';v']
% where y' = v, v' = -xv+3y
dy = [y(2);-t*y(2)+3*y(1)];
```

```
function dy = andy_hw07_prb02_ODE(t,y,params)
% [y1';y2';y3']
% where y1 = y, y2 = y', y3 = y''
dy = [y(2);y(3);(y(1)-t*y(2)-3+log(t))/t^3];
```

```
function dy = andy_hw07_prb02_ODEh(t,y,params)
% [y1';y2';y3']
% where y1 = y, y2 = y', y3 = y''
dy = [y(2);y(3);(y(1)-t*y(2))/t^3];
```

```
function dy = andy_hw07_prb04_ODE(t,y,params)
% [y';v']
% where y' = v, v' = -xv+3y-3x
dy = [y(2);30^2*(y(1)-1+2*t)];
```

```
function dy = andy_hw07_prb04_ODEh(t,y,params)
% [y';v']
% where y' = v, v' = -xv+3y
dy = [y(2);30^2*(y(1))];
```

```
function dy = andy_hw07_prb05_ODE(t,y,params)
% [y';v']
% where y' = v, v' = -xv+3y-3x
dy = [y(2);y(1)^2/(2+t)];
```

```
function dy = andy_hw07_prb06_ODE(t,y,params)
% [y';v']
% where y' = v, v' = (lambda^2 - 2sech^2(x))y
dy = [y(2);y(1)*(params(1)^2-2*sech(t)^2)];
```

Appendix 2: Numerical Methods

```
function yvec = andy_ME(func,tspan,y0,h,params)
% modified Euler

yvec = [];
yvec = [yvec y0];
t = tspan(1);
for i=2:length(tspan)
    k1 = func(t,yvec(:,i-1),params);
    k2 = func(t+h,yvec(:,i-1)+h*k1,params);
    yvec = [yvec yvec(:,i-1)+h/2*(k1+k2)];
    t=t+h;
end
```

```
function yvec = andy_MEr(func,tspan,y0,h,params)
% modified Euler

yvec = [];
yvec = [yvec y0];
t = tspan(end);
for i=2:length(tspan)
    k1 = func(t,yvec(:,i-1),params);
    k2 = func(t-h,yvec(:,i-1)-h*k1,params);
    yvec = [yvec yvec(:,i-1)-h/2*(k1+k2)];
    t=t-h;
end
% IMPORTANT
yvec=fliplr(yvec);
```