WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. Let $\vec{a} = 5\vec{i} \quad \vec{j} + 8\vec{k}$ and $\vec{b} = \vec{i} + \vec{j} + \lambda\vec{k}$. If the vectors $\vec{a} + \vec{b}$ and $\vec{a} \quad \vec{b}$ are orthogonal to each other then $|\lambda|$ is equal to

(A) $\sqrt{80}$ (B) $\sqrt{90}$ (C) $\sqrt{88}$ (D) $\sqrt{99}$

Solution: By hypothesis

$$(\vec{a} + \vec{b}) \times (\vec{a} \quad \vec{b}) = 0$$

Therefore

$$|\vec{a}|^2 \quad |\vec{b}|^2 = 0 \qquad 90 = 2 + \lambda^2$$

This implies

$$|\lambda| = \sqrt{88}$$

Answer: (C)

2. If $\vec{a}$ is a unit vector and $\vec{x}$ is any vector such that

$$(\vec{x} + \vec{a}) \times (\vec{x} \quad \vec{a}) = 48$$

then $|\vec{x}|$ is equal to

(A) 4 (B) 5 (C) 6 (D) 7

Solution: We have

$$\begin{aligned} 48 &= (\vec{x} + \vec{a}) \times (\vec{x} \quad \vec{a}) \\ &= |\vec{x}|^2 \quad |\vec{a}|^2 \\ &= |\vec{x}|^2 \quad 1 \end{aligned}$$

Therefore

$$|\vec{x}| = 7$$

Answer: (D)

3. If θ is the angle between the vectors $\vec{i} + 3\vec{j} + 7\vec{k}$ and $\vec{i} \quad 3\vec{j} + 7\vec{k}$, then $\cos\theta$ is equal to

(A) $\dfrac{41}{59}$ (B) $\dfrac{40}{59}$ (C) $\dfrac{42}{59}$ (D) $\dfrac{39}{59}$

Solution: Let

$$\vec{a} = \vec{i} + 3\vec{j} + 7\vec{k}$$

and

$$\vec{b} = \vec{i} \quad 3\vec{j} + 7\vec{k}$$

By Theorem 6.7,

$$\cos\theta = \frac{1 \quad 9 + 49}{\sqrt{59}\sqrt{59}} = \frac{41}{59}$$

Answer: (A)

4. The vectors $\lambda x\vec{i} \quad 6\vec{j} + 3\vec{k}$ and $x\vec{i} + 2\vec{j} + 2\lambda x\vec{k}$ make an obtuse angle with each other for all real x. Then λ belongs to the interval

(A) $\frac{4}{3}, 0 \div$ (B) $\frac{4}{3}, 0$

(C) $\frac{4}{3}, 0$ (D) $0, \frac{4}{3}$

Solution: By hypothesis,

$$(\lambda x\vec{i} \quad 6\vec{j} + 3\vec{k}) \times (x\vec{i} + 2\vec{j} + 2\lambda x\vec{k}) < 0$$

for all real x. Therefore

$$\lambda x^2 + 6\lambda x \quad 12 < 0$$

for all real x. This implies

$$\begin{aligned} 36\lambda^2 + 48\lambda &< 0 \\ 3\lambda^2 + 4\lambda &< 0 \\ \frac{4}{3} < \lambda &< 0 \end{aligned}$$

When $\lambda = 0$, then clearly

$$(\quad 6\vec{j} + 3\vec{k}) \times (x\vec{i} + 2\vec{j}) = \quad 12 < 0$$

Therefore

$$\frac{4}{3} < \lambda \quad 0$$

Answer: (B)

5. Let $\vec{a}$ and $\vec{b}$ two vectors such that $|\vec{a}| = 2\sqrt{2}$ and $|\vec{b}| = 3$ and contain angle 45°. A parallelogram is constructed with $\vec{a} \quad 3\vec{b}$ and $5\vec{a} + 2\vec{b}$ as adjacent sides. Then the length of the larger diagonal is

(A) $\sqrt{393}$ (B) $\sqrt{493}$ (C) $\sqrt{593}$ (D) $\sqrt{693}$

Solution: The diagonal vectors are

$$(5\vec{a} + 2\vec{b}) + (\vec{a} \quad 3\vec{b}) = 6\vec{a} \quad \vec{b}$$

and

$$(5\vec{a} + 2\vec{b}) \quad (\vec{a} \quad 3\vec{b}) = 4\vec{a} + 5\vec{b}$$

Now

$$\begin{aligned} |6\vec{a} \quad \vec{b}|^2 &= 36|\vec{a}|^2 \quad 12\vec{a} \times \vec{b} + |\vec{b}|^2 \quad \text{(see Corollary 6.4)} \\ &= 36(8) \quad 12(2\sqrt{2})(3) \; \frac{1}{\sqrt{2}} \div + 9 \\ &= 288 \quad 72 + 9 = 225 \end{aligned}$$

Therefore

$$|6\vec{a} \quad \vec{b}| = 15$$

Also

$$|4\vec{a} + 5\vec{b}|^2 = 16|\vec{a}|^2 + 40(\vec{a}\cdot\vec{b}) + 25|\vec{b}|^2$$

$$= 16(8) + 40(2\sqrt{2})(3)\frac{1}{\sqrt{2}} + 25(9)$$

$$= 128 + 240 + 225$$

$$= 593$$

Therefore

$$|4\vec{a} + 5\vec{b}| = \sqrt{593}$$

Hence this is the larger diagonal.

Answer: (C)

6. In the two-dimensional analytical plane having rectangular coordinate system, A and B are two points on the curve $y = 2^{x+2}$ such that $\overline{OA}\cdot\vec{i} = \quad 1$ and $\overline{OB}\cdot\vec{i} = 2$, where $\vec{i}$ is the unit vector in the positive direction of the X-axis. Then $|4\overline{OA} \quad \overline{OB}|$ is equal to

(A) 10 (B) 12 (C) 8 (D) 6

Solution: Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Therefore

$$y_1 = 2^{x_1+2}$$

and

$$y_2 = 2^{x_2+2}$$

Now

$$\overline{OA}\cdot\vec{i} = \quad 1 \qquad x_1 = \quad 1$$

and

$$OB\cdot\vec{i} = 2 \qquad x_2 = 2$$

Therefore $y_1 = 2$ and $y_2 = 16$. Hence

$$\overline{OA} = (\quad 1, 2) = \quad \vec{i} + 2\vec{j}$$

and

$$\overline{OB} = (2, 16) = 2\vec{i} + 16\vec{j}$$

Therefore

$$4\overline{OA} \quad \overline{OB} = (\quad 4\vec{i} + 8\vec{j}) \quad (2\vec{i} + 16\vec{j})$$

$$= \quad 6\vec{i} \quad 8\vec{j}$$

Hence

$$|4\overline{OA} \quad \overline{OB}| = \sqrt{(\quad 6)^2 + (\quad 8)^2} = 10$$

Answer: (A)

7. Let $\vec{a} = \vec{i} + \vec{j}$ and $\vec{b} = \vec{j} + \vec{k}$ and the angle between $\vec{a}$ and $\vec{b}$ be θ. If $\vec{c} = x\vec{i} + y\vec{j} + z\vec{k}$ such that $y \quad 0, |\vec{c}| = \sqrt{2}$ and makes angle θ with $\vec{a}$ and $\vec{b}$, then $\vec{c}$ is equal to

(A) $\frac{1}{3}(\vec{i} + \vec{j} + 4\vec{k})$ (B) $\frac{1}{3}(4\vec{i} + \vec{j} + \vec{k})$

(C) $\frac{1}{3}(\vec{i} \quad 4\vec{j} + \vec{k})$ (D) $\frac{1}{3}(\vec{i} + 2\vec{j} + \vec{k})$

Solution: We have $|\vec{c}| = \sqrt{2}$ which implies

$$x^2 + y^2 + z^2 = 1 \tag{6.9}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{2} \tag{6.10}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{b}|} = \frac{x+y}{2} \tag{6.11}$$

Also

$$\cos\theta = \frac{y+z}{2} \tag{6.12}$$

From Eqs. (6.10), (6.11) and (6.12),

$$x + y = 1$$

and

$$y + z = 1$$

Substituting these values in Eq. (6.9) we get

$$x^2 + (1 \quad x)^2 + x^2 = 2$$

Therefore

$$x = 1, \frac{1}{3}$$

Now $x = 1 \qquad y = 0$. But by hypothesis $y \quad 0$. Therefore

$$x = \frac{1}{3}, y = \frac{4}{3}, z = \frac{1}{3}$$

and so

$$\vec{c} = \frac{1}{3}(\vec{i} \quad 4\vec{j} + \vec{k})$$

Answer: (C)

8. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4$ and $|\vec{c}| = \sqrt{24}$ and sum of any two vectors is orthogonal to the third vector, then $|\vec{a} + \vec{b} + \vec{c}|$ is equal to

(A) 7 (B) $5\sqrt{2}$ (C) $7\sqrt{2}$ (D) $6\sqrt{2}$

Solution: By hypothesis

$$(\vec{a} + \vec{b})\cdot\vec{c} = 0$$

$$(\vec{b} + \vec{c})\cdot\vec{a} = 0$$

$$(\vec{c} + \vec{a})\cdot\vec{b} = 0$$

Therefore

$$\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = 0$$

Now

$$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$$
$$= 9+16+24+2(0)$$
$$= 49$$

Hence $|\vec{a}+\vec{b}+\vec{c}| = 7$.

Answer: (A)

9. If $|\vec{a}| = |\vec{b}| = |\vec{a}+\vec{b}| = 1$, then $|\vec{a}-\vec{b}|$ equals

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $\sqrt{3}$

Solution: By hypothesis

$$1 = |\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b}) = 1+1+2\cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. Therefore

$$\cos\theta = -\frac{1}{2}$$

Hence

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\cdot\vec{b})$$
$$= 1+1-2(1)(1)\left(-\frac{1}{2}\right)$$
$$= 3$$

Answer: (D)

10. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors having same magnitude, then the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to each of $\vec{a}, \vec{b}$ and $\vec{c}$ at an angle

(A) $\mathrm{Cos}^{-1}\left(\frac{1}{3}\right)$ (B) $\mathrm{Cos}^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(C) $\mathrm{Cos}^{-1}\left(\frac{2}{3}\right)$ (D) $\mathrm{Cos}^{-1}\left(\frac{1}{2}\right)$

Solution: We have

$$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2 \quad (\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0)$$
$$= 3K^2$$

where $K = |\vec{a}| = |\vec{b}| = |\vec{c}|$. Therefore

$$|\vec{a}+\vec{b}+\vec{c}| = K\sqrt{3}$$

Now, let α, β and γ be the angles of inclination of $\vec{a}+\vec{b}+\vec{c}$ with $\vec{a}$, $\vec{b}$, $\vec{c}$, respectively. Therefore

$$\cos\alpha = \frac{\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \frac{K^2}{K(K\sqrt{3})} = \frac{1}{\sqrt{3}}$$

Answer: (B)

11. Let $\vec{a} = \vec{i}+\vec{j}+2\vec{k}$ and $\vec{b} = \vec{i}+2\vec{j}+\vec{k}$ and $\vec{c}$ be a unit vector in the plane determined by $\vec{a}$ and $\vec{b}$. If $\vec{c}$ is perpendicular to the vector $\vec{i}+\vec{j}+\vec{k}$ and makes an obtuse angle with $\vec{a}$, then $\vec{c}$ is

(A) $\frac{\vec{j}-\vec{k}}{\sqrt{2}}$ (B) $\frac{\vec{j}+\vec{k}}{\sqrt{2}}$

(C) $\frac{\vec{i}-\vec{k}}{\sqrt{2}}$ (D) $\frac{\vec{i}+\vec{k}}{\sqrt{2}}$

Solution: Since $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$ (notice that $\vec{a}$ and $\vec{b}$ are not collinear vectors), let $\vec{c} = x\vec{a}+y\vec{b}$ where x and y are scalars. Now

$$|\vec{c}| = 1 \Rightarrow \sqrt{(x+y)^2+(x+2y)^2+(2x+y)^2} = 1 \quad (6.13)$$

Also $\vec{c}$ is perpendicular to $\vec{i}+\vec{j}+\vec{k}$, we have

$$\vec{c}\cdot(\vec{i}+\vec{j}+\vec{k}) = 0$$
$$(x+y)+(x+2y)+(2x+y) = 0$$
$$4x+4y = 0$$
$$x = -y$$

Substituting the value of $y = -x$ in Eq. (6.13), we have

$$\sqrt{2}x = \pm 1$$

so that

$$x = \pm\frac{1}{\sqrt{2}}$$

Therefore

$$x = \frac{1}{\sqrt{2}} \Rightarrow \vec{c} = -\frac{\vec{j}}{\sqrt{2}}+\frac{\vec{k}}{\sqrt{2}}$$

and

$$x = -\frac{1}{\sqrt{2}} \Rightarrow \vec{c} = \frac{\vec{j}}{\sqrt{2}}-\frac{\vec{k}}{\sqrt{2}}$$

But $\vec{c}$ makes obtuse angle with $\vec{a}$. This means

$$\vec{c} = \frac{\vec{j}-\vec{k}}{\sqrt{2}}$$

because

$$\vec{a}\cdot\vec{c} = 0+\frac{1}{\sqrt{2}}-\frac{2}{\sqrt{2}} = -\frac{1}{\sqrt{2}} < 0$$

Answer: (A)

12. Consider the following two statements:

S_1: Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ and further the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$. Then $\vec{a} = \pm 2(\vec{b}\times\vec{c})$

S_2: The points with position vectors $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$ and $\vec{a}+\lambda\vec{b}$ are collinear for all real values of λ where $\vec{a}$ and $\vec{b}$ are non-collinear vectors.
Then,

(A) Both S_1 and S_2 are true
(B) S_1 is true, but S_2 is false
(C) S_1 is false and S_2 is true
(D) Both S_1 and S_2 are false

Solution: $S_1: \vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$. This implies

$$\vec{a} = \lambda(\vec{b}\times\vec{c})$$

for some scalar λ. Therefore

$$\begin{aligned}1 = |\vec{a}|^2 &= \lambda^2|\vec{b}\times\vec{c}|^2 \\ &= \lambda^2[|\vec{b}|^2|\vec{c}|^2 - (\vec{b}\cdot\vec{c})^2] \quad \text{(see Theorem 6.26)} \\ &= \lambda^2\left(1-\cos^2\frac{\pi}{6}\right) \\ &= \lambda^2\left(1-\frac{3}{4}\right) \\ &= \frac{\lambda^2}{4}\end{aligned}$$

Therefore $\lambda = \pm 2$ and so

$$\vec{a} = \pm 2(\vec{b}\times\vec{c})$$

Hence S_1 is true. Let A, B and C be the points whose position vectors are $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$ and $\vec{a}+\lambda\vec{b}$, respectively. Now, A, B and C are collinear if and only if $\overline{AB} = x\overline{BC}$ for some scalar x. That is,

$$(\vec{a}-\vec{b})-(\vec{a}+\vec{b}) = x[\vec{a}+\lambda\vec{b}-(\vec{a}-\vec{b})]$$

That is,

$$-2\vec{b} = x(\lambda+1)\vec{b}$$

Therefore $x(\lambda+1) = -2$ for all real λ and for some real x. This is not possible when $\lambda = -1$, in which case, $C = B$. Hence S_2 is also true.

Answer: (A)

13. The number of distinct real values of λ for which the vectors $-\lambda^2\vec{i}+\vec{j}+\vec{k}$, $\vec{i}-\lambda^2\vec{j}+\vec{k}$ and $\vec{i}+\vec{j}-\lambda^2\vec{k}$ are coplanar is

(A) 0 (B) 1 (C) 2 (D) 3

Solution: The given vectors are coplanar if and only if

$$\begin{vmatrix}-\lambda^2 & 1 & 1\\ 1 & -\lambda^2 & 1\\ 1 & 1 & -\lambda^2\end{vmatrix} = 0 \quad \text{(see Corollary 6.16)}$$

That is

$$\lambda^6 - 3\lambda^2 - 2 = 0$$

$$(\lambda^2+1)^2(\lambda^2-2) = 0 \quad (\because \pm 1 \text{ and } \pm i \text{ are repeated roots})$$

This gives

$$\lambda = \pm\sqrt{2}$$

Answer: (C)

14. Let $\vec{a} = x\vec{i}-\vec{j}+\vec{k}$ (x is a scalar) and $\vec{b} = 2\vec{i}-\vec{j}+5\vec{k}$. If the scalar projection of $\vec{a}$ on $\vec{b}$ is $1/\sqrt{30}$, then x is equal to

(A) $\frac{2}{5}$ (B) $\frac{3}{5}$ (C) $\frac{-5}{3}$ (D) $\frac{-5}{2}$

Solution: By hypothesis

$$\frac{1}{\sqrt{30}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{2x+6}{\sqrt{30}}$$

Therefore

$$x = \frac{-5}{2}$$

Answer: (D)

15. If $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 1$ and $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, then the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ is

(A) 12 (B) −12 (C) −13 (D) 13

Solution: We have

$$\begin{aligned}\vec{a}+\vec{b}+\vec{c} = \vec{0} &\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = 0\\ &\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|+2\sum\vec{a}\cdot\vec{b} = 0\\ &\Rightarrow 9+16+1+2\sum\vec{a}\cdot\vec{b} = 0\\ &\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -13\end{aligned}$$

Answer: (C)

16. Let $\vec{a} = 2\vec{i}+3\vec{j}+\vec{k}$, $\vec{b} = 4\vec{i}+\vec{j}$ and $\vec{c} = \vec{i}+3\vec{j}+2\vec{k}$. If $\vec{d}$ is such that $\vec{d}\cdot\vec{a} = 9$, $\vec{d}\cdot\vec{b} = 7$ and $\vec{d}\cdot\vec{c} = 6$, then $\vec{d}$ is equal to

(A) $\vec{i}+3\vec{j}-2\vec{k}$ (B) $\vec{i}-3\vec{j}+2\vec{k}$
(C) $-\vec{i}+3\vec{j}-2\vec{k}$ (D) $\vec{i}+3\vec{j}+2\vec{k}$

Solution: Let $\vec{d} = x\vec{i}+y\vec{j}+z\vec{k}$.

$$\vec{d}\cdot\vec{a} = 9 \Rightarrow 2x+3y+z = 9 \quad (6.14)$$

$$\vec{d}\cdot\vec{b} = 7 \Rightarrow 4x+y = 7 \quad (6.15)$$

$$\vec{d}\cdot\vec{c} = 6 \Rightarrow x+3y+2z = 6 \quad (6.16)$$

Solving Eqs. (6.14), (6.15) and (6.16) we have $x = 1$, $y = 3$ and $z = \ \ 2$. Therefore

$$\vec{d} = \vec{i} + 3\vec{j} \quad 2\vec{k}$$

Answer: (A)

17. The points $A(2,\ \ 1, 1)$, $B(1,\ \ 3,\ \ 5)$ and $C(3,\ \ 4,\ \ 4)$ form the vertices of

(A) an equilateral triangle
(B) isosceles but not equilateral
(C) right angled
(D) right angled isosceles

Solution: We have

$$\overline{AB} = \quad \vec{i} \quad 2\vec{j} \quad 6\vec{j}$$
$$\overline{CA} = \quad \vec{i} + 3\vec{j} + 5\vec{j}$$
$$\overline{BC} = 2\vec{i} \quad \vec{j} + \vec{k}$$
$$\overline{AB} + \overline{BC} + \overline{CA} = \vec{0}$$

Therefore A, B, C form a triangle and

$$\overline{BC} \times \overline{CA} = \quad 2 \quad 3 + 5 = 0$$

Therefore $\underline{|C} = 90°$.

Answer: (C)

18. If the vectors $2\vec{i} + a\vec{j} + 4\vec{k}$ and $\ \ 3\vec{i} + 5\vec{j} \quad 3\vec{k}$ are at right angles, then the value of a is

(A) $\frac{18}{5}$ (B) 1 (C) $\frac{1}{5}$ (D) 1

Solution: We have

$$(2\vec{i} + a\vec{j} + 4\vec{k}) \times (\ \ 3\vec{i} + 5\vec{j} \quad 3\vec{k}) = 0$$
$$6 + 5a \quad 12 = 0$$
$$a = \frac{18}{5}$$

Answer: (A)

19. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors, then the value of

$$|\vec{a} \quad \vec{b}|^2 + |\vec{b} \quad \vec{c}|^2 + |\vec{c} \quad \vec{a}|^2$$

is less than or equal to

(A) 8 (B) 12 (C) 9 (D) 6

Solution: We have

$$|\vec{a} + \vec{b} + \vec{c}| \quad 0$$
$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 \quad 2 \quad \vec{a} \times \vec{b} \quad 0$$
$$\frac{3}{2} \quad \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} \qquad (6.17)$$

Now

$$|\vec{a} \quad \vec{b}|^2 + |\vec{b} \quad \vec{c}|^2 + |\vec{c} \quad \vec{a}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2) \quad 2 \quad (\vec{a} \times \vec{b})$$
$$= 6 \quad 2 \quad (\vec{a} \times \vec{b}) \quad 6 + 2 \ \frac{3}{2} \div \quad \text{[by Eq. (6.17)]}$$

Answer: (C)

20. The angle between the two vectors $\vec{a} = 2\vec{i} + 2\vec{j} \quad \vec{k}$ and $\vec{b} = 6\vec{i} \quad 3\vec{j} + 2\vec{k}$ is

(A) $\text{Cos}^{\ 1}\frac{1}{3}$ (B) $\text{Cos}^{\ 1}\frac{1}{2}$
(C) $\text{Cos}^{\ 1}\ \frac{5\sqrt{7}}{21} \div$ (D) $\text{Cos}^{\ 1}\ \frac{4}{21} \div$

Solution: We have

$$\vec{a} \times \vec{b} = 12 \quad 6 \quad 2 = 4$$
$$|\vec{a}||\vec{b}| = \sqrt{36 + 9 + 4}\sqrt{4 + 4 + 1} = 21$$

Therefore the angle between the vectors, by Theorem 6.7, is

$$\text{Cos}^{\ 1}\ \frac{\vec{a} \times \vec{b}}{|\vec{a}||\vec{b}|} \div = \text{Cos}^{\ 1}\ \frac{4}{21} \div$$

Answer: (D)

21. If the scalar product of the vector $\vec{i} + \vec{j} + \vec{k}$ with the unit vector in the direction of the resultant of the vectors $2\vec{i} + 4\vec{j} \quad 5\vec{k}$ and $\lambda\vec{i} + 2\vec{j} + 3\vec{k}$ is unity, then λ equals

(A) 5 (B) 2 (C) 1 (D) 1

Solution: Let

$$\vec{r} = (2\vec{i} + 4\vec{j} \quad 5\vec{k}) + (\lambda\vec{i} + 2\vec{j} + 3\vec{k})$$
$$= (2 + \lambda)\vec{i} + 6\vec{j} \quad 2\vec{k}$$
$$\vec{e} = \text{unit vector in the direction of } \vec{r}$$
$$= \frac{(2 + \lambda)\vec{i} + 6\vec{j} \quad 2\vec{k}}{\sqrt{(2 + \lambda)^2 + 36 + 4}}$$

Therefore

$$\vec{e} \times (\vec{i} + \vec{j} + \vec{k}) = 1$$
$$(2 + \lambda) + 6 \quad 2 = \sqrt{(2 + \lambda)^2 + 40}$$
$$(\lambda + 6)^2 = (2 + \lambda)^2 + 40$$
$$12\lambda + 36 = 4\lambda + 44$$
$$8\lambda = 8$$
$$\lambda = 1$$

Answer: (C)

22. Let $\vec{a} = 2\vec{i} + 3\vec{j} + 2\vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$. The vector component of $\vec{a}$ perpendicular to the direction of $\vec{b}$ is

(A) $\frac{1}{3}(\vec{i} + \vec{j} + \vec{k})$ (B) $\frac{1}{3}(\vec{i} \quad \vec{j} + \vec{k})$

(C) $\frac{1}{3}(\vec{i} + \vec{j} \quad \vec{k})$ (D) $\frac{1}{3}(\quad \vec{i} + \vec{j} + \vec{k})$

Solution: Vector component of $\vec{a}$ perpendicular to the direction of $\vec{b}$ is given by (see Quick Look 3)

$$\vec{a} \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = (2\vec{i} + 3\vec{j} + 2\vec{k}) \quad \frac{5}{3}(\vec{i} + 2\vec{j} + \vec{k})$$

$$= \frac{1}{3}(\vec{i} \quad \vec{j} + \vec{k})$$

Answer: (B)

23. If $\vec{a}$ is collinear with $\vec{b} = 3\vec{i} + 6\vec{j} + 6\vec{k}$ and $\vec{a}\cdot\vec{b} = 27$, then $\vec{a}$ is

(A) $3(\vec{i} + \vec{j} + \vec{k})$ (B) $\vec{i} + 2\vec{j} + 2\vec{k}$

(C) $2\vec{i} + 2\vec{j} + 2\vec{k}$ (D) $\vec{i} + 3\vec{j} + 3\vec{k}$

Solution: $\vec{a}$ is collinear with $\vec{b}$. This implies

$$\vec{a} = \lambda\vec{b}$$

for some scalar λ. Now

$$\vec{a}\cdot\vec{b} = 27$$

$$\lambda(\vec{b}\cdot\vec{b}) = 27$$

$$\lambda(9 + 36 + 36) = 27$$

$$\lambda = \frac{1}{3}$$

Therefore

$$\vec{a} = \frac{1}{3}\vec{b} = \vec{i} + 2\vec{j} + 2\vec{k}$$

Answer: (B)

24. Let $\vec{a}$ and $\vec{b}$ be unit vectors. If the vectors $\vec{x} = \vec{a} + 2\vec{b}$ and $\vec{y} = 5\vec{a} \quad 4\vec{b}$ are orthogonal vectors, then the angle between $\vec{a}$ and $\vec{b}$ is

(A) 60° (B) 30° (C) 45° (D) 75°

Solution: By hypothesis

$$\vec{x}\cdot\vec{y} = 0$$

$$5|\vec{a}|^2 \quad 8|\vec{b}|^2 + 6\vec{a}\cdot\vec{b} = 0$$

Assuming θ is the angle between $\vec{a}$ and $\vec{b}$ we get

$$5 \quad 8 + 6\cos\theta = 0$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = 60^\circ$$

Answer: (A)

25. Angle between $\vec{a}$ and $\vec{b}$ is $2\pi/3$. If $|\vec{b}| = 2|\vec{a}|$ and the vectors $\vec{a} + x\vec{b}$ and $\vec{a} \quad \vec{b}$ are at right angles, then x is equal to

(A) $\frac{2}{5}$ (B) $\frac{2}{3}$ (C) $\frac{1}{5}$ (D) $\frac{1}{3}$

Solution: We have

$$(\vec{a} + x\vec{b})\cdot(\vec{a} \quad \vec{b}) = 0$$

$$|\vec{a}|^2 \quad x|\vec{b}|^2 + (x \quad 1)(\vec{a}\cdot\vec{b}) = 0$$

$$|\vec{a}|^2 \quad 4x|\vec{a}|^2 + (x \quad 1)2|\vec{a}|^2\cos\frac{2\pi}{3} = 0$$

$$1 \quad 4x \quad (x \quad 1) = 0$$

$$x = \frac{2}{5}$$

Answer: (A)

26. A plane is at a distance of 5 units from the origin and perpendicular to the vector $2\vec{i} + \vec{j} + 2\vec{k}$. Then its equation is

(A) $\vec{r}\cdot(2\vec{i} + \vec{j} + 2\vec{k}) = 15$

(B) $\vec{r}\cdot(2\vec{i} + \vec{j} + \vec{k}) = 5$

(C) $\vec{r}\cdot(\vec{i} + 2\vec{j} + 2\vec{k}) = 15$

(D) $\vec{r}\cdot(2\vec{i} + \vec{j} + 2\vec{k}) = 5$

Solution: Unit normal to the given plane is

$$\vec{n} = \frac{2\vec{i} + \vec{j} + 2\vec{k}}{3}$$

By Theorem 6.13, the plane equation

$$\vec{r}\cdot\frac{(2\vec{i} + \vec{j} + 2\vec{k})}{3} = 5$$

That is

$$\vec{r}\cdot(2\vec{i} + \vec{j} + 2\vec{k}) = 15$$

Answer: (A)

27. M is the foot of the perpendicular drawn from the point $A(2, 4, 3)$ onto the line joining the points $P(1, 2, 4)$ and $Q(3, 4, 5)$. Then the vector representing the point M is

(A) $\frac{1}{9}(19\vec{i}+28\vec{j}+40\vec{k})$

(B) $\frac{1}{9}(28\vec{i}+9\vec{j}+41\vec{k})$

(C) $\frac{1}{9}(21\vec{i}+28\vec{j}+19\vec{k})$

(D) $\frac{1}{9}(19\vec{i}+28\vec{j}+41\vec{k})$

Solution: By Theorem 5.29 (Chapter 5), the equation of the line PQ is

$$\vec{r}=(1-t)(\vec{i}+2\vec{j}+4\vec{k})+t(3\vec{i}+4\vec{j}+5\vec{k})$$

That is

$$\vec{r}=(1+2t)\vec{i}+(2+2t)\vec{j}+(4+t)\vec{k} \qquad (6.18)$$

Therefore, the position vector of M is given by Eq. (6.18). So

$$\overline{AM}=(-1+2t)\vec{i}+(-2+2t)\vec{j}+(1+t)\vec{k}$$

Since $\overline{AM}$ is perpendicular to $\overline{PQ}=2\vec{i}+2\vec{j}+\vec{k}$, we have

$$\overline{AM}\cdot\overline{PQ}=0$$

Therefore

$$2(-1+2t)+2(-2+2t)+(1+t)=0$$
$$-5+9t=0$$
$$t=\frac{5}{9}$$

Substituting the value $t = 5/9$ in Eq. (6.18), we have

$$M=\frac{1}{9}(19\vec{i}+28\vec{j}+41\vec{k})$$

Answer: (D)

28. Let p be real and $|p| \geq 2$. If A, B and C are variable angles such that

$$\sqrt{p^2-4}\tan A+p\tan B+\sqrt{p^2+4}\tan C=6p$$

then the minimum value of $\tan^2 A+\tan^2 B+\tan^2 C$ is

(A) 8 (B) 12 (C) 18 (D) 6

Solution: Consider the vectors

$$\vec{a}=\sqrt{p^2-4}\vec{i}+p\vec{j}+\sqrt{p^2+4}\vec{k}$$

and $$\vec{b}=(\tan A)\vec{i}+(\tan B)\vec{j}+(\tan C)\vec{k}$$

so that

$$\vec{a}\cdot\vec{b}=\sqrt{p^2-4}\tan A+p\tan B+\sqrt{p^2+4}\tan C$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then

$$\begin{aligned}36p^2&=(\vec{a}\cdot\vec{b})^2\\&=|\vec{a}|^2|\vec{b}|^2\cos^2\theta\leq|\vec{a}|^2|\vec{b}|^2\\&=(p^2-4+p^2+p^2+4)(\tan^2 A+\tan^2 B+\tan^2 C)\\&=3p^2(\tan^2 A+\tan^2 B+\tan^2 C)\end{aligned}$$

Hence

$$\tan^2 A+\tan^2 B+\tan^2 C\geq 12$$

and is equal to 12 when θ is either 0 or π.

Answer: (B)

29. In ΔABC (Figure 6.28), $AB = BC = 8$ and $AC = 12$. P is a point on the side AB such that $AP:PB = 1:3$. Then the angle between the vectors $\overline{CP}$ and $\overline{CA}$ is

(A) $\text{Cos}^{-1}\left(\frac{3\sqrt{5}}{8}\right)$ (B) $\text{Cos}^{-1}\left(\frac{3\sqrt{3}}{8}\right)$

(C) $\text{Cos}^{-1}\left(\frac{3\sqrt{7}}{8}\right)$ (D) $\text{Cos}^{-1}\left(\frac{3\sqrt{2}}{8}\right)$

Solution: Take A as origin and let $\overline{AB}=\vec{\alpha}$ and $\overline{AC}=\vec{\beta}$ so that

$$|\vec{\alpha}|=8=|\vec{\beta}-\vec{\alpha}|$$

and $$|\vec{\beta}|=12$$

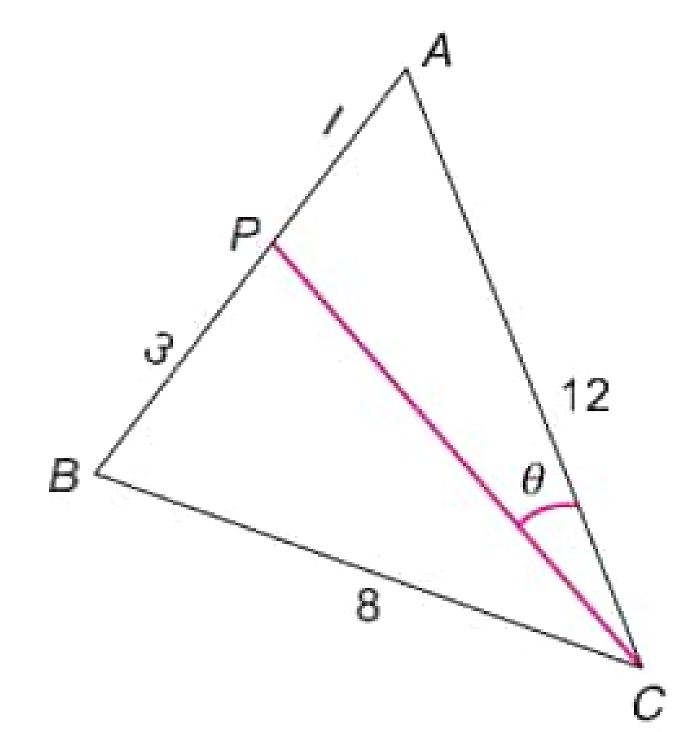

FIGURE 6.28 Single correct choice type question 29.

Now $AB = BC = 8$ implies

$$\begin{aligned}&|\vec{\alpha}|=|\vec{\beta}-\vec{\alpha}|\\&\Rightarrow|\vec{\alpha}|^2=|\vec{\beta}|^2+|\vec{\alpha}|^2-2\vec{\alpha}\cdot\vec{\beta}\\&\Rightarrow 2\vec{\alpha}\cdot\vec{\beta}=|\vec{\beta}|^2\\&\Rightarrow 2|\vec{\alpha}||\vec{\beta}|\cos A=|\vec{\beta}|^2\\&\Rightarrow 2(8)(12)\cos A=144\\&\Rightarrow\cos A=\frac{3}{4}\end{aligned} \qquad (6.19)$$

Now

$$\overline{AP} = \frac{\vec{\alpha}}{4}$$

and $$\overline{CP} = \frac{\vec{\alpha}}{4} \quad \vec{\beta} = \frac{1}{4}(\vec{\alpha} \quad 4\vec{\beta})$$

Again

$$|\overline{CP}|^2 = \frac{1}{16}|\vec{\alpha} \quad 4\vec{\beta}|^2$$

$$= \frac{1}{16}[8^2 + 16 \quad 12^2 \quad 8(\vec{\alpha}\cdot\vec{\beta})]$$

$$= \frac{1}{16} \quad 64 + 16 \quad 144 \quad 8(8)(12)\frac{3}{4} \qquad \because \cos A = \frac{3}{4}\div$$

$$= 4 + 144 \quad 36$$

$$= 112$$

Therefore

$$|\overline{CP}| = \sqrt{112} = 4\sqrt{7}$$

Let θ be the angle between $\overline{CP}$ and $\overline{CA}$. Then

$$\cos\theta = \frac{\overline{CP}\cdot\overline{CA}}{|\overline{CP}||\overline{CA}|}$$

$$= \frac{(1/4)(\vec{\alpha} \quad 4\vec{\beta})\cdot(\quad \vec{\beta})}{4\sqrt{7} \quad 12}$$

$$= \frac{4|\vec{\beta}|^2 \quad \vec{\alpha}\cdot\vec{\beta}}{16\sqrt{7} \quad 12}$$

$$= \frac{4 \quad 144 \quad 8(12)(3/4)}{16\sqrt{7} \quad 12}$$

$$= \frac{144 \quad 18}{4\sqrt{7} \quad 12}$$

$$= \frac{126}{4\sqrt{7} \quad 12}$$

$$= \frac{21}{8\sqrt{7}}$$

$$= \frac{3\sqrt{7}}{8}$$

Therefore

$$\theta = \text{Cos}^{\ 1} \quad \frac{3\sqrt{7}}{8}\div$$

Answer: (C)

30. Let $\overline{OA} = \vec{i} + 2\vec{j} + 2\vec{k}$. In the plane of $\overline{OA}$ and $\vec{i}$, rotate $\overline{OA}$ through 90° about the origin O such that the new position of $\overline{OA}$ makes an acute angle with the positive X-axis. Then the new position of $\overline{OA}$ is

(A) $\frac{1}{\sqrt{2}}(4\vec{i} \quad 2\vec{j} \quad 2\vec{k})$

(B) $\frac{1}{\sqrt{2}}(\quad 4\vec{i} + 2\vec{j} + 2\vec{k})$

(C) $2\vec{j} \quad 2\vec{k}$

(D) $6\vec{i} \quad 3\vec{k}$

Solution: Let the new position of $\overline{OA}$ be $\vec{r}$. Since $\vec{r}$ is coplanar with $\overline{OA}$ and $\vec{i}$, let

$$\vec{r} = x(\overline{OA}) + y\vec{i}$$

$$= (x + y)\vec{i} + 2x\vec{j} + 2x\vec{k}$$

Since $\vec{r}$ is perpendicular to $\overline{OA}$, we have

$$\vec{r}\cdot\overline{OA} = 0$$

$$(x + y) + 4x + 4x = 0$$

$$y = \quad 9x \tag{6.20}$$

Also

$$|\vec{r}| = 3 \quad (x + y)^2 + 4x^2 + 4x^2 = 9$$

$$64x^2 + 8x^2 = 9 \quad \text{[by Eq. (6.20)]}$$

$$x = \pm\frac{1}{2\sqrt{2}}$$

so that

$$y = \mp\frac{9}{2\sqrt{2}}$$

Now

$$\vec{r} = \frac{1}{\sqrt{2}}(4\vec{i} \quad \vec{j} \quad \vec{k})$$

or $$\vec{r} = \frac{1}{\sqrt{2}}(\quad 4\vec{i} + \vec{j} + \vec{k})$$

But $\vec{r}$ makes acute angle with the positive X-axis. Hence

$$\vec{r} = \frac{1}{\sqrt{2}}(4\vec{i} \quad \vec{j} \quad \vec{k})$$

Answer: (A)

31. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are vectors such that $\vec{a} + \vec{b} + 2\vec{c} = \vec{0}$ and $|\vec{a}| = 1$, $|\vec{b}| = 4$ and $|\vec{c}| = 2$, then the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$ is

(A) $\frac{13}{2}$ (B) $\frac{13}{2}$ (C) $\frac{17}{2}$ (D) $\frac{17}{2}$

Solution: By hypothesis

$$\vec{a}+\vec{b}+\vec{c}=\;\;\vec{c}$$

Therefore

$$4=|\vec{c}|^2=|\vec{a}+\vec{b}+\vec{c}|^2$$
$$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\quad(\vec{a}\times\vec{b})$$
$$=1+16+4+2\quad(\vec{a}\times\vec{b})$$

Therefore

$$\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\;\;\frac{17}{2}$$

Answer: (D)

32. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $|\vec{a}+\vec{b}|=\sqrt{3}$, then $(3\vec{a}\quad 4\vec{b})\times(2\vec{a}+5\vec{b})$ is equal to

(A) $\frac{15}{2}$ (B) $\frac{15}{2}$ (C) $\frac{21}{2}$ (D) $\frac{21}{2}$

Solution: We have

$$|\vec{a}+\vec{b}|^2=3$$
$$1+1+2(\vec{a}\times\vec{b})=3$$
$$\vec{a}\times\vec{b}=\frac{1}{2}$$

Now

$$(3\vec{a}\quad 4\vec{b})\times(2\vec{a}+5\vec{b})=6|\vec{a}|^2\quad 20|\vec{b}|^2+7(\vec{a}\times\vec{b})$$
$$=6\quad 20+\frac{7}{2}$$
$$=\;\;\frac{21}{2}$$

Answer: (C)

33. In quadrilateral $ABCD$, if

$$|\overline{AB}|^2+|\overline{CD}|^2=|\overline{BC}|^2+|\overline{AD}|^2$$

then the angle between the diagonals AC and BD is

(A) 60° (B) 75° (C) 90° (D) 120°

Solution: Take A as origin and let $\overline{AB}=\vec{b}$, $\overline{AC}=\vec{c}$ and $\overline{AD}=\vec{d}$. Then

$$|\overline{AB}|^2+|\overline{CD}|^2=|\overline{BC}|^2+|\overline{AD}|^2$$
$$|\vec{b}|^2+|\vec{d}\quad\vec{c}|^2=(\vec{c}\quad\vec{b})^2+|\vec{d}|^2$$
$$|\vec{b}|^2+|\vec{d}|^2+|\vec{c}|^2\quad 2(\vec{d}\times\vec{c})=|\vec{c}|^2+|\vec{b}|\quad 2(\vec{b}\times\vec{c})+|\vec{d}|^2$$
$$\vec{d}\times\vec{c}=\vec{b}\times\vec{c}$$
$$(\vec{d}\quad\vec{b})\times\vec{c}=0$$
$$\overline{BD}\times\overline{AC}=0$$

Therefore angle between AC and BD is 90°.

Answer: (C)

34. In ABC (Figure 6.29), if $BC=a$, $CA=b$ and $AB=c$ and the internal bisector of the angle A meets the side BC in L, then $|\overline{AL}|^2$ is equal to

(A) $bc\quad\frac{bca^2}{(b+c)^2}$ (B) $bc\quad\frac{abc}{(b+c)^2}$

(C) $a^2bc+\frac{ab}{(b+c)^2}$ (D) $a^2bc\quad\frac{bc}{(b+c)^2}$

Solution: It is known that $BL:LC=c:b$. Take A as origin, let $\overline{AB}=\vec{\alpha}$ and $\overline{AC}=\vec{\beta}$ so that $|\vec{\alpha}|=c, |\vec{\beta}|=b$ and the angle between $\vec{\alpha}$ and $\vec{\beta}$ is A.

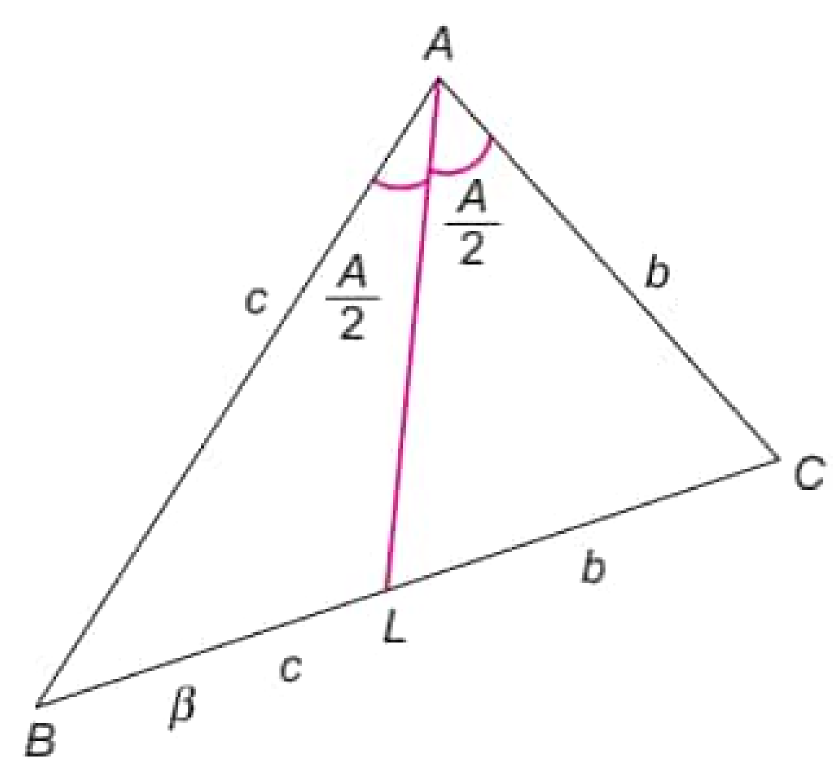

FIGURE 6.29 Single correct choice type question 34.

Now $BL:LC=c:b$. This means that the position vector $\overline{AL}$ of L is given by

$$\overline{AL}=\frac{b\vec{\alpha}+c\vec{\beta}}{b+c}$$

Therefore

$$|\overline{AL}|^2=\frac{1}{(b+c)^2}[b^2|\vec{\alpha}|^2+c^2|\vec{\beta}|^2+2bc(\vec{\alpha}\times\vec{\beta})]$$
$$=\frac{1}{(b+c)^2}[b^2c^2+b^2c^2+2b^2c^2\cos A]$$
$$=\frac{1}{(b+c)^2}\quad 2b^2c^2+2b^2c^2\frac{(b^2+c^2\quad a^2)}{2bc}$$
$$=\frac{2b^2c^2}{(b+c)^2}\quad\frac{2bc+b^2+c^2\quad a^2}{2bc}$$
$$=\frac{bc}{(b+c)^2}[(b+c)^2\quad a^2]$$
$$=bc\quad\frac{bc\boldsymbol{a}^2}{(b+c)^2}$$

Answer: (A)

35. $\vec{a}, \vec{b}, \vec{c}$ are unit vectors. If $\vec{r}$ is a vector such that $|\vec{r}| = 2$, $\vec{a} + \vec{b} + \vec{c} = \vec{r}$, $\vec{a}\cdot\vec{r} = 1$, $\vec{b}\cdot\vec{r} = 3/2$, then the angle between $\vec{c}$ and $\vec{r}$ is

(A) $\text{Cos}^{-1}\left(\frac{1}{4}\right)$ (B) $\text{Cos}^{-1}\left(\frac{3}{4}\right)$

(C) $\text{Cos}^{-1}\left(\frac{1}{2\sqrt{2}}\right)$ (D) $\text{Cos}^{-1}\left(\frac{1}{3}\right)$

Solution: We have

$$(\vec{a} + \vec{b} + \vec{c})\cdot\vec{r} = \vec{r}\cdot\vec{r} = 4$$

$$\vec{a}\cdot\vec{r} + \vec{b}\cdot\vec{r} + \vec{c}\cdot\vec{r} = 4$$

$$1 + \frac{3}{2} + 2\cos\theta = 4$$

where θ is the angle between $\vec{c}$ and $\vec{r}$. Therefore

$$2\cos\theta = \frac{3}{2}$$

$$\theta = \text{Cos}^{-1}\left(\frac{3}{4}\right)$$

Answer: (B)

36. In $\triangle ABC$ (Figure 6.30), if "O" is the circumcentre and R is the circumradius and G is the centroid, then $|\overline{OG}|^2$ equals

(A) $R - \frac{1}{9}(a^2 + b^2 + c^2)$

(B) $R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$

(C) $R^2 - \frac{1}{6}(a^2 + b^2 + c^2)$

(D) $R^2 - \frac{1}{3}(a^2 + b^2 + c^2)$

Solution: Take "O" as the origin. Let $\overline{OA} = \vec{\alpha}$, $\overline{OB} = \vec{\beta}$ and $\overline{OC} = \vec{\gamma}$. Therefore $(\vec{\beta}, \vec{\gamma}) = 2A$, $(\vec{\gamma}, \vec{\alpha}) = 2B$ and $(\vec{\alpha}, \vec{\beta}) = 2C$.

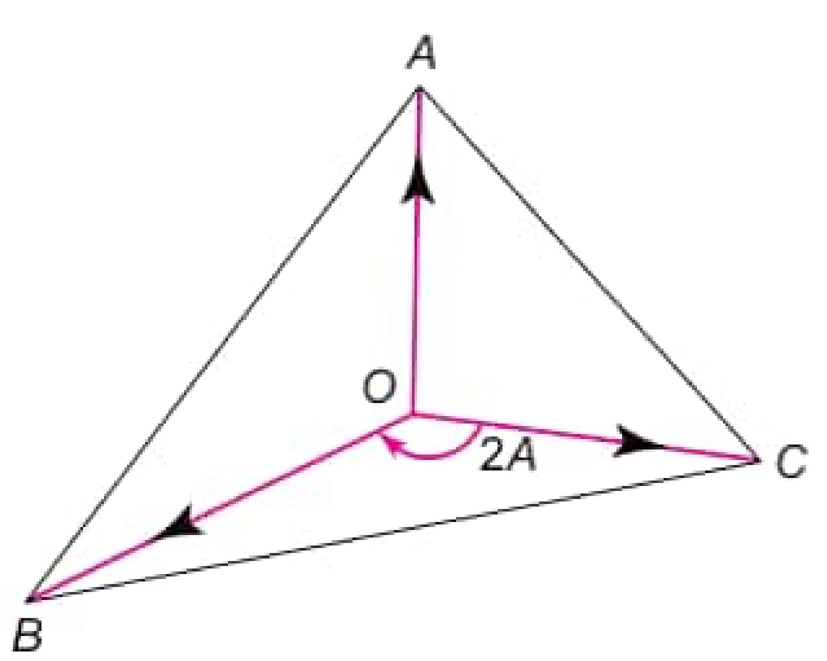

FIGURE 6.30 Single correct choice type question 36.

It is known that

$$\overline{OG} = \frac{\vec{\alpha} + \vec{\beta} + \vec{\gamma}}{3} \quad \text{(see Corollary 5.4)}$$

Therefore

$$\begin{aligned} 9|\overline{OG}|^2 &= |\vec{\alpha} + \vec{\beta} + \vec{\gamma}|^2 \\ &= |\vec{\alpha}|^2 + |\vec{\beta}|^2 + |\vec{\gamma}|^2 + 2\sum(\vec{\beta}\cdot\vec{\gamma}) \\ &= 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C) \\ &= 3R^2 + 2R^2[1 - 2\sin^2 A + 1 - 2\sin^2 B + 1 - 2\sin^2 C] \\ &= 9R^2 - 4R^2\sin^2 A - 4R^2\sin^2 B - 4R^2\sin^2 C \\ &= 9R^2 - a^2 - b^2 - c^2 \quad \left(\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right) \end{aligned}$$

Hence

$$|\overline{OG}|^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

Answer: (B)

37. Quadrilateral $ABCD$ is inscribed in a circle of radius R (Figure 6.31). If $(AB)^2 + (CD)^2 = 4R^2$ then the angle between the diagonals AC and BD is

(A) 120° (B) 90° (C) 75° (D) 60°

Solution: Let "O" be the centre of the circle. Let $\overline{OA} = \vec{a}$, $\overline{OB} = \vec{b}$, $\overline{OC} = \vec{c}$ and $\overline{OD} = \vec{d}$ so that

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{d}| = R$$

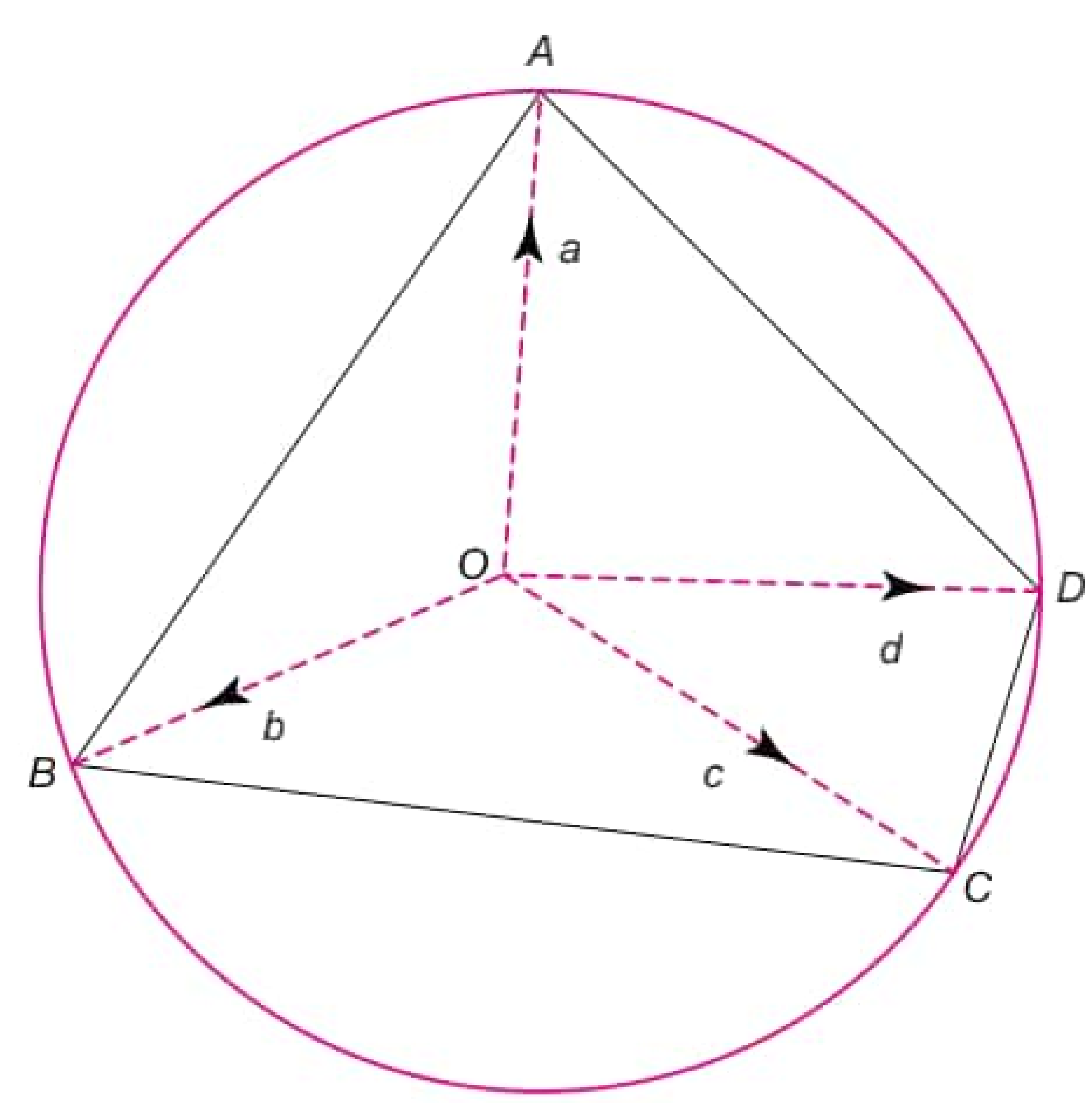

FIGURE 6.31 Single correct choice type question 37.

By hypothesis,

$$|\overline{AB}|^2 + |\overline{CD}|^2 = 4R^2$$

$$\Rightarrow |\vec{b} - \vec{a}|^2 + |\vec{d} - \vec{c}|^2 = 4R^2$$

$$\Rightarrow -2(\vec{a}\cdot\vec{b}) - 2(\vec{c}\cdot\vec{d}) = 0 \quad (6.21)$$

$$\Rightarrow -2R^2[\cos\underline{|AOB} + \cos\underline{|COD}] = 0$$

$$\Rightarrow \cos\underline{|AOB} + \cos\underline{|COD} = 0$$

$$\Rightarrow \underline{|AOB} + \underline{|COD} = \pi \quad (6.22)$$

Therefore

$$\underline{|BOC} + \underline{|AOD} = \pi$$

Now

$$\overline{AC}\cdot\overline{BD} = (\vec{c} - \vec{a})\cdot(\vec{d} - \vec{b})$$

$$= (\vec{c}\cdot\vec{d}) - (\vec{b}\cdot\vec{c}) - (\vec{a}\cdot\vec{d}) + (\vec{a}\cdot\vec{b})$$

$$= -(\vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{d})$$

(because from Eq. (6.21) $\vec{a}\cdot\vec{b} + \vec{c}\cdot\vec{d} = 0$)

$$= -R^2[\cos\underline{|BOC} + \cos\underline{|AOD}] \quad \text{[by Eq. (6.22)]}$$

$$= -R^2(0)$$

$$= 0$$

Therefore AC and BD are at right angles.

38. "O" is the circumcentre and H is the orthocentre of a ΔABC. If P is any arbitrary point, then the vector $\overline{PA} + \overline{PB} + \overline{PC} - \overline{PH}$ is equal to

(A) $\overline{PO}$ (B) $2\overline{PO}$ (C) $\overline{OP}$ (D) $2\overline{OP}$

Solution: With respect to the circumcentre 'O' as origin, let $\overline{OA} = \vec{a}, \overline{OB} = \vec{b}$ and $\overline{OC} = \vec{c}$.
Therefore by Example 5.3 (Chapter 5)

$$\overline{OH} = \vec{a} + \vec{b} + \vec{c}$$

Let $\overline{OP} = \vec{p}$. Therefore

$$\overline{PA} + \overline{PB} + \overline{PC} - \overline{PH} = (\vec{a} - \vec{p}) + (\vec{b} - \vec{p}) + (\vec{c} - \vec{p}) - (\vec{a} + \vec{b} + \vec{c} - \vec{p})$$

$$= -2\vec{p}$$

$$= 2\overline{PO}$$

Answer: (B)

39. In ΔABC (Figure 6.32), angle B is a right angle. If the medians AD and BE are perpendicular to each other, then angle C is

(A) $\text{Tan}^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (B) $\text{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (C) $\text{Tan}^{-1}(\sqrt{2})$ (D) $\text{Tan}^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Solution: With respect to origin B, let $\overline{BA} = \vec{a}$ and $\overline{BC} = \vec{c}$ so that $\vec{c}\cdot\vec{a} = 0$.

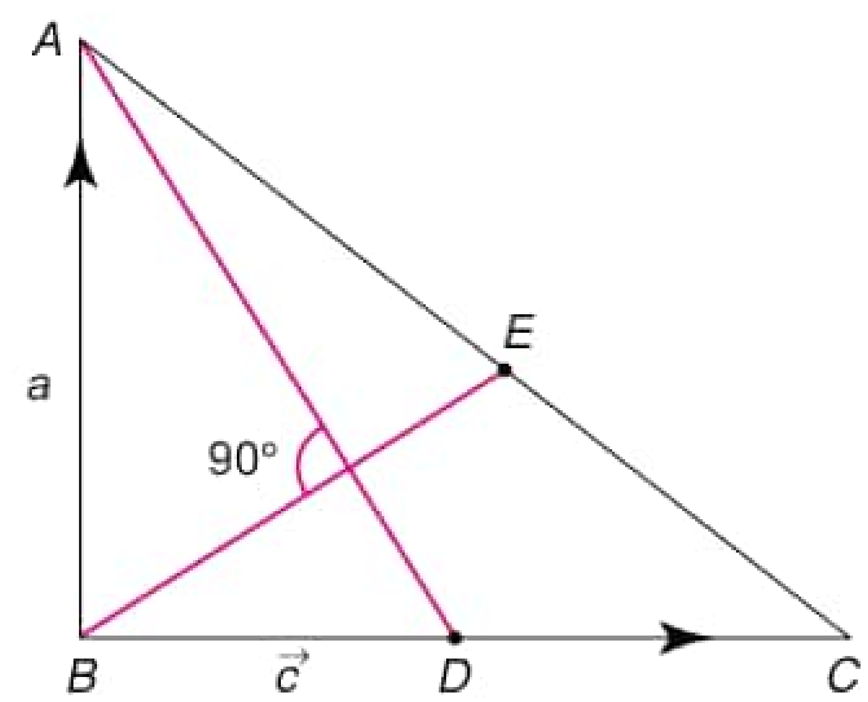

FIGURE 6.32 Single correct choice type question 39.

Now,

$$\overline{BD} = \frac{\vec{c}}{2}$$

and

$$\overline{BE} = \frac{\vec{c} + \vec{a}}{2}$$

BE and AD are perpendicular to each other. This implies

$$\left(\frac{\vec{c} + \vec{a}}{2}\right)\cdot\left(\frac{\vec{c}}{2} - \vec{a}\right) = 0$$

$$\Rightarrow (\vec{c} + \vec{a})\cdot(\vec{c} - 2\vec{a}) = 0$$

$$\Rightarrow |\vec{c}|^2 - 2|\vec{a}|^2 - \vec{c}\cdot\vec{a} = 0$$

$$\Rightarrow |\vec{c}|^2 - 2|\vec{a}|^2 = 0 \quad (\because \vec{c}\cdot\vec{a} = 0)$$

$$\Rightarrow \frac{|\vec{a}|^2}{|\vec{c}|^2} = \frac{1}{2}$$

$$\Rightarrow \frac{(AB)^2}{(BC)^2} = \frac{1}{2}$$

$$\Rightarrow \frac{AB}{BC} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \tan C = \frac{1}{\sqrt{2}}$$

$$\Rightarrow C = \text{Tan}^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Answer: (A)

40. In a ΔABC, if $a^2 + b^2 = 2c^2$ and m_a, m_b and m_c are the lengths of the medians through the vertices A, B and C, respectively then $am_b + bm_a$ is equal to

(A) $c\,m_c$ (B) $2c\,m_c$ (C) $3c\,m_c$ (D) $\sqrt{3}c\,m_c$

Solution: In Chapter 4, we have proved (see Theorem 4.25) that

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$m_b = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$

$$m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

Therefore

$$a\,m_b + b\,m_a = \frac{a}{2}\sqrt{2c^2 + 2a^2 - b^2} + \frac{b}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$= \frac{a}{2}\sqrt{3a^2} + \frac{b}{2}\sqrt{3b^2} \quad (\because a^2 + b^2 = 2c^2)$$

$$= \frac{\sqrt{3}}{2}(a^2 + b^2)$$

$$= \frac{\sqrt{3}}{2}(2c^2)$$

$$= \sqrt{3}c^2$$

$$= 2c\left(\frac{\sqrt{2a^2 + 2b^2 - c^2}}{2}\right)$$

$$= 2c\,m_c$$

Answer: (B)

41. In ΔABC (Figure 6.33), the median CM is perpendicular to the angle bisector AL of angle A. If $|\overline{CM}|:|\overline{AL}| = m:1$, then

(A) $m^2 = \frac{9}{4}\left(\frac{1+\cos A}{1-\cos A}\right)$ (B) $m^2 = \frac{9}{4}\left(\frac{1-\cos A}{1+\cos A}\right)$

(C) $m^2 = \frac{4}{9}\left(\frac{1-\cos A}{1+\cos A}\right)$ (D) $m^2 = \frac{4}{9}\left(\frac{1+\cos A}{1-\cos A}\right)$

Solution: Taking "A" as origin, let $\overline{AB} = \vec{\alpha}, \overline{AC} = \vec{\beta}$ and so that $|\vec{\alpha}| = c, |\vec{\beta}| = b$ and $(\vec{\alpha}, \vec{\beta}) = A$. Therefore

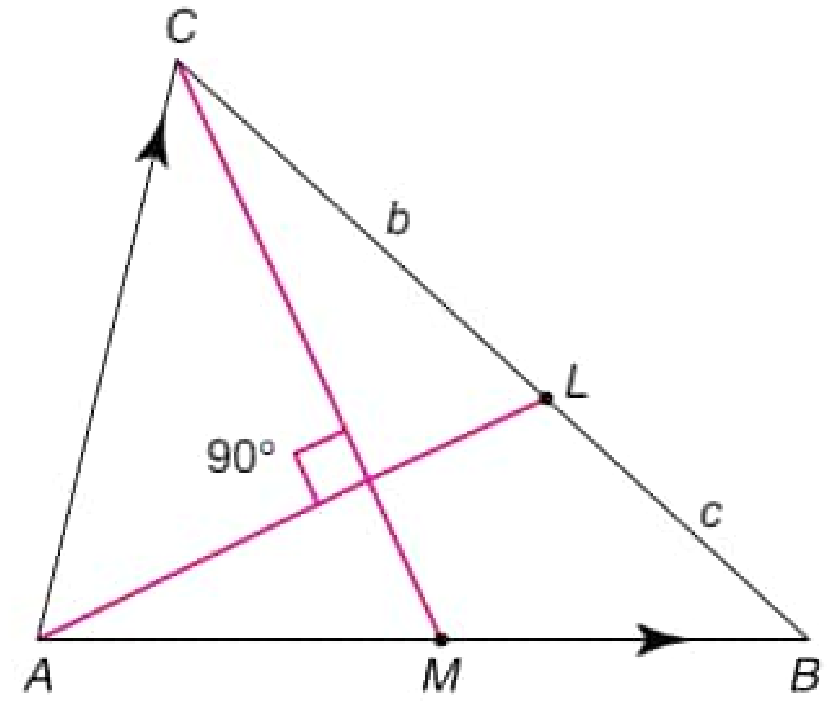

FIGURE 6.33 Single correct choice type question 41.

$$\overline{AM} = \frac{1}{2}\vec{\alpha}$$

$$\overline{AL} = \frac{b\vec{\alpha} + c\vec{\beta}}{b+c} \quad (\because BL:LC = c:b)$$

$$\overline{CM} = \frac{\vec{\alpha}}{2} - \vec{\beta}$$

By hypothesis,

$$\overline{AL}\cdot\overline{CM} = 0$$

$$\Rightarrow \left(\frac{b\vec{\alpha} + c\vec{\beta}}{2}\right)\cdot\left(\frac{\vec{\alpha} - 2\vec{\beta}}{2}\right) = 0$$

$$\Rightarrow b|\vec{\alpha}|^2 - 2c|\vec{\beta}|^2 + (c-2b)(\vec{\alpha}\cdot\vec{\beta}) = 0$$

$$\Rightarrow bc^2 - 2cb^2 + bc(c-2b)\cos A = 0$$

$$\Rightarrow c - 2b + (c-2b)\cos A = 0$$

$$\Rightarrow (c-2b)(1+\cos A) = 0$$

$$\Rightarrow c = 2b \qquad (6.23)$$

Again

$$|\overline{CM}|:|\overline{AL}| = m:1$$

$$\Rightarrow |\overline{CM}|^2 = m^2|\overline{AL}|^2$$

$$\Rightarrow \frac{|\vec{\alpha} - 2\vec{\beta}|^2}{4} = m^2\frac{|b\vec{\alpha} + c\vec{\beta}|^2}{(b+c)^2}$$

Therefore

$$\frac{|\vec{\alpha} - 2\vec{\beta}|^2}{4} = \frac{m^2}{9}|\vec{\alpha} + 2\vec{\beta}|^2 \quad \text{[By Eq. (6.23)]}$$

$$9[|\vec{\alpha}|^2 + 4|\vec{\beta}|^2 - 4(\vec{\alpha}\cdot\vec{\beta})] = 4m^2[|\vec{\alpha}|^2 + 4\vec{\beta}^2 + 4(\vec{\alpha}\cdot\vec{\beta})]$$

$$9[c^2 + 4b^2 - 4bc\cos A] = 4m^2[c^2 + 4b^2 + 4bc\cos A]$$

$$9[4b^2 + 4b^2 - 8b^2\cos A] = 4m^2[4b^2 + 4b^2 + 8b^2\cos A]$$

[by Eq. (6.23)]

$$9(1-\cos A) = 4m^2(1+\cos A)$$

$$m^2 = \frac{9}{4}\left(\frac{1-\cos A}{1+\cos A}\right)$$

Answer: (B)

42. Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors of same magnitude. If the vectors $\vec{a} + 3\vec{b}$ and $5\vec{a} + 3\vec{b}$ are at right angles to each other, then the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\text{Cos}^{-1}\left(\frac{7}{9}\right)$ (B) $\pi - \text{Cos}^{-1}\left(\frac{7}{9}\right)$

(C) $\text{Cos}^{-1}\left(\frac{1}{3}\right)$ (D) $\pi - \text{Cos}^{-1}\left(\frac{1}{3}\right)$

Solution: It is given that $|\vec{a}| = |\vec{b}|$. Let $\theta = (\vec{a}, \vec{b})$. Now

$$(\vec{a} + 3\vec{b})\cdot(5\vec{a} + 3\vec{b}) = 0$$

$$5|\vec{a}|^2 + 9|\vec{b}|^2 + 18(\vec{a}\cdot\vec{b}) = 0$$

$$5|\vec{a}|^2 + 9|\vec{b}|^2 + 18|\vec{a}||\vec{b}|\cos\theta = 0$$

$$5 + 9 + 18\cos\theta = 0 \quad (\because |\vec{a}| = |\vec{b}|)$$

$$\cos\theta = \quad \frac{7}{9}$$

$$\theta = \pi \quad \text{Cos}^{\ 1} \ \frac{7}{9}÷$$

Answer: (B)

43. a, b, c are positive reals and are the lth, mth and nth terms of a geometric progression, respectively. If

$$\vec{A} = (\log a)\vec{i} + (\log b)\vec{j} + (\log c)\vec{k}$$

and $$\vec{B} = (m \quad n)\vec{i} + (n \quad l)\vec{j} + (l \quad m)\vec{k}$$

then the angle between $\vec{A}$ and $\vec{B}$ is

(A) 60° (B) 45° (C) 30° (D) 90°

Solution: Let α be the first term and x be the common ratio of the GP. Then

$$a = \alpha x^{l \ 1}$$

$$b = \alpha x^{m \ 1}$$

$$c = \alpha x^{n \ 1}$$

Therefore

$$\log a = (l \quad 1)\log x + \log\alpha$$

$$\log b = (m \quad 1)\log x + \log\alpha$$

$$\log c = (n \quad 1)\log x + \log\alpha$$

Now

$$\vec{A}\cdot\vec{B} = (m \quad n)\log a + (n \quad l)\log b + (l \quad m)\log c$$

$$= (\log x)[(m \quad n)(l \quad 1) + (n \quad l)(m \quad 1) + (l \quad m)(n \quad 1)]$$

$$+ (\log\alpha)[(m \quad n) + (n \quad l) + (l \quad m)]$$

$$= (\log x)(0) + (\log\alpha)(0) = 0$$

Answer: (D)

44. $ABCD$ is a trapezium (Figure 6.34) in which the sides BC and AD are parallel and the adjacent sides AB and AD are at right angles and AC, BD are at right angles. If $BC:AD = n:1$, then the ratio of the diagonals $AC:BD$ is equal to

(A) $(n+1):1$ (B) $n:1$ (C) $\sqrt{n}:1$ (D) $n^2:1$

Solution: Take A as origin and let $\overline{AB} = \vec{b}$ and $\overline{AD} = \vec{d}$ so that $\overline{BC} = n\vec{d}$, $\vec{b}\cdot\vec{d} = 0$. Now AC and BD are at right angles. This implies

$$\overline{AC}\cdot\overline{BD} = 0$$

$$(\vec{b} + n\vec{d})\cdot(\vec{d} \quad \vec{b}) = 0$$

$$\vec{b}\cdot\vec{d} \quad |\vec{b}|^2 + n|\vec{d}|^2 \quad n(\vec{b}\cdot\vec{d}) = 0$$

$$n|\vec{d}|^2 \quad |\vec{b}|^2 = 0 \quad (\because \vec{b}\cdot\vec{d} = 0)$$

$$n|\vec{d}|^2 = |\vec{b}|^2 \qquad (6.24)$$

Now,

$$\frac{|\overline{AC}|^2}{|\overline{BD}|^2} = \frac{|\vec{b} + n\vec{d}|^2}{|\vec{d} \quad \vec{b}|^2}$$

$$= \frac{|\vec{b}|^2 + n^2|\vec{d}|^2 + 2n(\vec{b}\cdot\vec{d})}{|\vec{d}|^2 + |\vec{b}|^2 \quad \vec{b}\cdot\vec{d}}$$

$$= \frac{|\vec{b}|^2 + n^2|\vec{d}|^2}{|\vec{d}|^2 + |\vec{b}|^2} \quad (\because \vec{b}\cdot\vec{d} = 0)$$

$$= \frac{n|\vec{d}|^2 + n^2|\vec{d}|^2}{|\vec{d}|^2 + n|\vec{d}|^2} \quad \text{[by Eq. (6.24)]}$$

$$= n\frac{(|\vec{d}|^2 + n|\vec{d}|^2)}{|\vec{d}|^2 + n|\vec{d}|^2}$$

$$= n$$

Therefore

$$AC:BD = \sqrt{n}:1$$

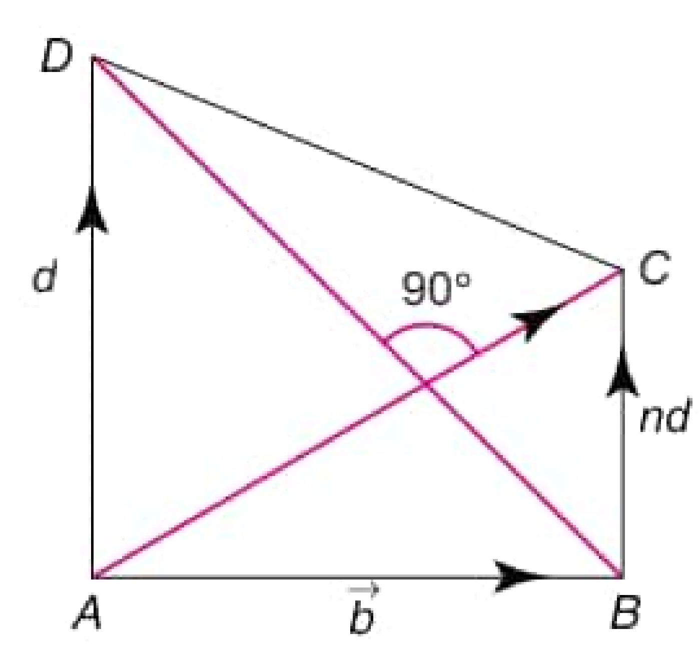

FIGURE 6.34 Single correct choice type question 44.

Answer: (C)

45. If $|\vec{a}| = 13, |\vec{b}| = 5$ and $\vec{a}\cdot\vec{b} = 30$, then $|\vec{a} \quad \vec{b}|$ is equal to

(A) 30 (B) $\frac{65}{23}\sqrt{493}$

(C) $\frac{30}{23}\sqrt{493}$ (D) $\frac{65}{13}\sqrt{133}$

Solution: We have

$$30 = \vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$= (65)\cos\theta$$

where $\theta = (\vec{a}, \vec{b})$. Therefore

$$\cos\theta = \frac{30}{65} = \frac{6}{13}$$

$$\sin\theta = \sqrt{1 \quad \cos^2\theta} = \sqrt{1 \quad \frac{36}{169}} = \frac{\sqrt{133}}{13}$$

So

$$|\vec{a} \quad \vec{b}| = |\vec{a}||\vec{b}|\sin\theta = \frac{65 \times \sqrt{133}}{13}$$

Answer: (D)

46. For any vector $\vec{a}$,

$$|\vec{a} \quad \vec{i}|^2 + |\vec{a} \quad \vec{j}|^2 + |\vec{a} \quad \vec{k}|^2 =$$

(A) $|\vec{a}|^2$ (B) $2|\vec{a}|^2$ (C) $\frac{1}{2}|\vec{a}|^2$ (D) $3|\vec{a}|^2$

Solution: Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$. Then

$$\vec{a} \quad \vec{i} = a_2\vec{k} + a_3\vec{j}$$

$$\vec{a} \quad \vec{j} = a_1\vec{k} \quad a_3\vec{i}$$

$$\vec{a} \quad \vec{k} = \quad a_1\vec{j} + a_2\vec{i}$$

Therefore

$$|\vec{a} \quad \vec{i}|^2 + |\vec{a} \quad \vec{j}|^2 + |\vec{a} \quad \vec{k}|^2$$

$$= (a_2^2 + a_3^2) + (a_1^2 + a_3^2) + (a_1^2 + a_2^2)$$

$$= 2(a_1^2 + a_2^2 + a_3^2)$$

$$= 2|\vec{a}|^2$$

Answer: (B)

47. $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B and C of $\quad ABC$, respectively. Then the vector perpendicular to the plane of $\quad ABC$ is

(A) $\vec{b} \quad \vec{c}$

(B) $\vec{a} + \vec{b} + \vec{c}$

(C) $\vec{b} \quad \vec{c} + \vec{c} \quad \vec{a} + \vec{a} \quad \vec{b}$

(D) $(\vec{a} \quad \vec{b}) \quad (\vec{a} + \vec{b} + \vec{c})$

Solution: We have $\overline{AB} = \vec{b} \quad \vec{a}$, $\overline{BC} = \vec{c} \quad \vec{b}$, $\overline{AC} = \vec{c} \quad \vec{a}$. The vector perpendicular to the plane of the triangle ABC is

$$\overline{AB} \quad \overline{AC} = (\vec{b} \quad \vec{a}) \quad (\vec{c} \quad \vec{a})$$

$$= \vec{b} \quad \vec{c} \quad \vec{b} \quad \vec{a} + \vec{c} \quad \vec{a}$$

$$= \vec{b} \quad \vec{c} + \vec{c} \quad \vec{a} + \vec{a} \quad \vec{b}$$

Answer: (C)

48. $\vec{a}$ and $\vec{b}$ are non-collinear vectors such that $|\vec{a}| = 2\sqrt{2}$ and $|\vec{b}| = 3$ and the angle between $(\vec{a}, \vec{b})$ is $\pi/4$. If $5\vec{a} + 2\vec{b}$ and $\vec{a} \quad 3\vec{b}$ are adjacent sides of a parallelogram, then its area in square units is

(A) 72 (B) 92 (C) 102 (D) 112

Solution: The area of a the parallelogram with $\vec{\alpha}$ and $\vec{\beta}$ as adjacent sides is (see Corollary 6.13)

$$|\vec{\alpha} \quad \vec{\beta}|$$

Therefore area of the parallelogram whose adjacent sides are $5\vec{a} + 2\vec{b}$ and $\vec{a} \quad 3\vec{b}$ is

$$|(5\vec{a} + 2\vec{b}) \quad (\vec{a} \quad 3\vec{b})| = | \quad 17(\vec{a} \quad \vec{b})|$$

$$= 17|\vec{a} \quad \vec{b}|$$

$$= 17|\vec{a}||\vec{b}|\sin \quad \frac{\pi}{4} \div$$

$$= 17(2\sqrt{2})(3) \quad \frac{1}{\sqrt{2}} \div$$

$$= 102 \text{ sq. units}$$

Answer: (C)

49. The vector $\vec{r}$ is perpendicular to the vectors $\vec{a} = 3\vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{b} = 18\vec{i} \quad 22\vec{j} \quad 5\vec{k}$ and makes an obtuse angle with $\vec{i}$. If $|\vec{r}| = 14$, then $\vec{r}$ equals

(A) $\quad 4\vec{i} \quad 6\vec{j} + 12\vec{k}$ (B) $\quad 4\vec{i} + 6\vec{j} + 6\vec{k}$

(C) $4\vec{i} + 6\vec{j} \quad 6\vec{k}$ (D) $6\vec{i} \quad 4\vec{j} \quad 12\vec{k}$

Solution: $\vec{r}$ is parallel to $\vec{a} \quad \vec{b}$. This implies

$$\vec{r} = \lambda(\vec{a} \quad \vec{b})$$

$$= \lambda \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 2 \\ 18 & 22 & 5 \end{vmatrix}$$

$$= (\lambda)17(2\vec{i} + 3\vec{j} \quad 6\vec{k})$$

Now $\vec{r}$ makes obtuse angle with $\vec{i}$. This means $\lambda < 0$. Hence

$$|\vec{r}| = 14 \qquad 14^2 = 17^2(\lambda^2)(49)$$

$$\lambda^2 = \frac{4}{17^2}$$

$$\lambda = \quad \frac{2}{17} \quad (\because \lambda < 0)$$

$$\vec{r} = \quad 2(2\vec{i} + 3\vec{j} \quad 6\vec{k})$$

Answer: (A)

50. $A(1, y_1)$ and $B(x_2, 11)$ are two points on a plane curve $y = x^2 \quad 2x + 3$ with $x_2 > 0$. Then the area of the parallelogram with $\overline{OA}$ and $\overline{OB}$ ("O" is the origin) as adjacent sides is

(A) 3 (B) 9 (C) 6 (D) 4

Solution: We have

$$y_1 = 1 \quad 2 + 3 = 2$$

and

$$11 = x_2^2 \quad 2x_2 + 3$$

$$x_2^2 \quad 2x_2 \quad 8 = 0$$

$$(x_2 \quad 4)(x_2 + 2) = 0$$

$$x_2 = 4, \quad 2$$

Therefore $B = (4, 11)$ and $A = (1, 2)$. Hence

$$\overline{OA} \quad \overline{OB} = (\vec{i} + 2\vec{j}) \quad (4\vec{i} + 11\vec{j})$$

$$= 11\vec{k} \quad 8\vec{k} = 3\vec{k}$$

So

$$\text{Area} = |\overline{OA} \quad \overline{OB}| = 3$$

Answer: (A)

51. The area of the parallelogram whose diagonals are represented by the vectors $\vec{a} = 3\vec{i} + \vec{j} \quad 2\vec{k}$ and $\vec{b} = \vec{i} \quad 3\vec{j} + 4\vec{k}$ is

(A) $4\sqrt{3}$ (B) $5\sqrt{3}$ (C) $6\sqrt{3}$ (D) $3\sqrt{3}$

Solution: The area of the parallelogram $ABCD$ in terms of its diagonal vectors is (see Theorem 6.30)

$$\frac{1}{2}|\overline{AC} \quad \overline{BD}|$$

Now

$$\vec{a} \quad \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= \quad 2\vec{i} \quad 14\vec{k} \quad 10\vec{k}$$

$$= \quad 2(\vec{i} + 7\vec{k} + 5\vec{k})$$

Therefore

$$\text{Area} = \frac{1}{2}|\vec{a} \quad \vec{b}|$$

$$= \sqrt{1^2 + 7^2 + 5^2}$$

$$= \sqrt{75} = 5\sqrt{3}$$

Answer: (B)

52. Let $\vec{a} = 3\vec{i} + 2\vec{j} + 2\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} \quad 2\vec{k}$. Then a unit vector perpendicular to both $\vec{a} \quad \vec{b}$ and $\vec{a} + \vec{b}$ is

(A) $\frac{1}{3}(\quad 2\vec{i} + 2\vec{j} + \vec{k})$ (B) $\frac{1}{3}(\quad 2\vec{i} + 2\vec{j} \quad \vec{k})$

(C) $\frac{1}{3}(2\vec{i} \quad 2\vec{j} + \vec{k})$ (D) $\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$

Solution: We have

$$\vec{a} + \vec{b} = 4\vec{i} + 4\vec{j}$$

$$\vec{a} \quad \vec{b} = 2\vec{i} + 4\vec{k}$$

Then

$$(\vec{a} + \vec{b}) \quad (\vec{a} \quad \vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= 16\vec{i} \quad 16\vec{j} \quad 8\vec{k}$$

$$= 8(2\vec{i} \quad 2\vec{j} \quad \vec{k})$$

Unit vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} \quad \vec{b}$ is

$$\pm\frac{1}{3}(2\vec{i} \quad 2\vec{j} \quad \vec{k})$$

One of them is

$$\frac{1}{3}(\quad 2\vec{i} + 2\vec{j} + \vec{k})$$

Answer: (A)

53. The perpendicular distance of a vertex of a unit cube from a diagonal not passing through it is

(A) $\frac{2}{3}$ (B) $\sqrt{\frac{2}{3}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{2}{\sqrt{3}}$

Solution: Let $\overline{OA} = (1, 0, 0)$, $\overline{OB} = (0, 1, 0)$ and $\overline{OC} = (0, 0, 1)$ be three coterminus edges of the unit cube (see Figure 6.35). Clearly the vector $\overline{OR} = \vec{i} + \vec{j} + \vec{k}$. OR is a diagonal not passing through A.

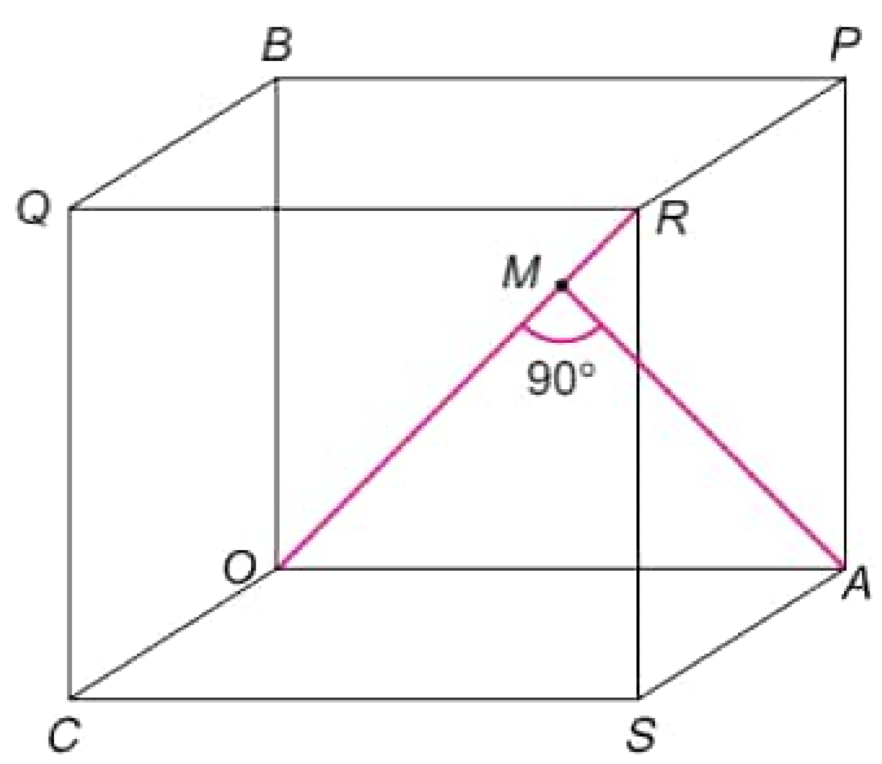

FIGURE 6.35 Single correct choice type question 53.

It is known that the area of the triangle whose vertices have the position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is (see Corollary 6.14)

$$\frac{1}{2}|\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}|$$

Therefore the area of ΔOAR is

$$\frac{1}{2}|\vec{i}\times(\vec{i}+\vec{j}+\vec{k}) + (\vec{i}+\vec{j}+\vec{k})\times\vec{0} + \vec{0}\times\vec{i}| = \frac{1}{2}|\vec{k}-\vec{j}| = \frac{1}{\sqrt{2}}$$

But,

$$\text{Area of } \Delta OAR = \frac{1}{2}|\overline{OR}|\cdot AM$$

where AM is the perpendicular distance of the vertex A from the side OR given by

$$\frac{1}{2}\sqrt{3}\cdot(AM)$$

Therefore

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}AM$$

$$AM = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$$

Answer: (B)

54. The area of the triangle whose vertices are $\vec{i} + \vec{j} + \vec{k}$, $\vec{i} - \vec{j} + \vec{k}$ and $\vec{k}$ is

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $\frac{1}{2}$

Solution: Let A, B and C be represented by $\vec{i} + \vec{j} + \vec{k}$, $\vec{i} - \vec{j} + \vec{k}$ and $\vec{k}$, respectively. Then

$$\overline{AB} = -2\vec{j}$$

$$\overline{AC} = -\vec{i} - \vec{k}$$

Now, the area of the ΔABC (see Theorem 6.29) is

$$\frac{1}{2}|\overline{AB}\times\overline{AC}| = \frac{1}{2}|2\vec{j}\times(\vec{i}+\vec{k})|$$

$$= |-\vec{k} + \vec{i}| = \sqrt{2}$$

Answer: (B)

55. Let $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}, \vec{b} = -\vec{i} + 2\vec{j} - 4\vec{k}$ and $\vec{c} = \vec{i} + \vec{j} + \vec{k}$. Then $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})$ is equal to

(A) 26 (B) 24 (C) −24 (D) −26

Solution: We have (see Theorem 6.48)

$$(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c}) = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{c} \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 2 \\ 4 & -3 \end{vmatrix}$$

$$= -26$$

Answer: (D)

56. Let $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ be non-coplanar vectors. If $x\vec{\alpha} + y\vec{\beta} + z\vec{\gamma}$ is equal to $a(\vec{\alpha}+\vec{\gamma}) + b(-\vec{\alpha}+\vec{\gamma}) + c(\vec{\alpha}+\vec{\beta}+\vec{\gamma})$, then a is equal to

(A) $-\frac{x}{2} + y - \frac{z}{2}$ (B) $-\frac{x}{2} + \frac{z}{2}$

(C) $\frac{x}{2} - y + \frac{z}{2}$ (D) $x - \frac{y}{2} + \frac{z}{2}$

Solution: We have

$$x\vec{\alpha} + y\vec{\beta} + z\vec{\gamma} = (a-b+c)\vec{\alpha} + c\vec{\beta} + (a+b+c)\vec{\gamma}$$

Therefore

$$a - b + c = x$$

$$y = c$$

$$a + b + c = z$$

Now

$$a - b = -y + x$$

and

$$a + b = z - y$$

Hence

$$a = \frac{x - 2y + z}{2} = \frac{x}{2} - y + \frac{z}{2}$$

Answer: (C)

57. Let $\overline{OA}=\vec{a}$, $\overline{OB}=10\vec{a}+2\vec{b}$ and $\overline{OC}=\vec{b}$ where O, A and C are non-collinear points. Let p denote the area of the quadrilateral $OABC$ and q denote the area of the parallelogram with $\overline{OA}$ and $\overline{OC}$ as adjacent sides. If $p=\lambda q$, then λ equals

(A) 6 (B) 8 (C) 10 (D) 9

Solution: $q=$ area of the parallelogram $=|\vec{a}\quad\vec{b}|$.

$$p=\text{Area of the Quadrilateral } OABC=\frac{1}{2}|\overline{OB}\quad\overline{AC}|$$

$$=\frac{1}{2}|(10\vec{a}+2\vec{b})\quad(\vec{b}\quad\vec{a})|$$

$$=\frac{1}{2}|10(\vec{a}+\vec{b})\quad 2(\vec{b}\quad\vec{a})|$$

$$=\frac{12}{2}|\vec{a}\quad\vec{b}|=6q$$

Answer: (A)

58. Let $\vec{a}=\vec{i}\quad\vec{k}, \vec{b}=x\vec{i}+\vec{j}+(1\quad x)\vec{k}$ and $\vec{c}=y\vec{i}+x\vec{j}+(1+x\quad y)\vec{k}$. Then $[\vec{a}\,\vec{b}\,\vec{c}]$ depends on

(A) only x (B) only y

(C) neither x nor y (D) both x and y

Solution: Using Theorem 6.35 we know that

$$[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}1 & 0 & 1\\ x & 1 & 1\quad x\\ y & x & 1+x\quad y\end{vmatrix}$$

$$=(1+x\quad y)\quad x(1\quad x)\quad (x^2\quad y)$$

$$=1+x\quad y\quad x+x^2\quad x^2+y$$

$$=1$$

Answer: (C)

59. Let $\vec{a},\vec{b},\vec{c}$ be unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Which one of the following is correct?

(A) $\vec{a}\quad\vec{b}=\vec{b}\quad\vec{c}=\vec{c}\quad\vec{a}=\vec{0}$

(B) $\vec{a}\quad\vec{b}=\vec{b}\quad\vec{c}=\vec{c}\quad\vec{a}\quad\vec{0}$

(C) $\vec{a}\quad\vec{b}=\vec{b}\quad\vec{c}=\vec{a}\quad\vec{c}\quad\vec{0}$

(D) $\vec{a}\quad\vec{b},\vec{b}\quad\vec{c},\vec{c}\quad\vec{a}$ are mutually perpendicular

Solution: We have $\vec{a}+\vec{b}=\quad\vec{c}$. Therefore

$$(\vec{a}+\vec{b})\quad\vec{c}=\vec{0}$$

$$\vec{b}\quad\vec{c}=\quad\vec{a}\quad\vec{c}=\vec{c}\quad\vec{a}$$

Again

$$(\vec{b}+\vec{c})\quad\vec{a}=\vec{0}$$

$$\vec{c}\quad\vec{a}=\quad\vec{b}\quad\vec{a}=\vec{a}\quad\vec{b}$$

Therefore

$$\vec{a}\quad\vec{b}=\vec{b}\quad\vec{c}=\vec{c}\quad\vec{a}\quad\vec{0}$$

Answer: (B)

60. A non-zero vector $\vec{a}$ is parallel to the line of intersection of the planes determined by the vectors $\vec{i},\vec{i}+\vec{j}$ and the plane determined by the vectors $\vec{i}\quad\vec{j},\vec{i}+\vec{k}$. The angle between $\vec{a}$ and the vector $\vec{i}\quad 2\vec{j}+2\vec{k}$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

Solution: $\vec{n}_1=\vec{i}\quad(\vec{i}+\vec{j})=\vec{k}$ is normal to the first plane $\vec{n}_2=(\vec{i}\quad\vec{j})\quad(\vec{i}+\vec{k})=\quad\vec{j}+\vec{k}\quad\vec{i}$ is normal to second plane. Since $\vec{a}$ is parallel to the line of intersection of the planes, it is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$ and hence $\vec{a}$ is parallel to the vector

$$\vec{n}_1\quad\vec{n}_2=\vec{k}\quad(\quad\vec{j}+\vec{k}\quad\vec{i})=\vec{i}\quad\vec{j}$$

The angle between $\vec{a}$ and $\vec{i}\quad 2\vec{j}+2\vec{k}$ is same as the angle between $\vec{n}_1\quad\vec{n}_2=\vec{i}\quad\vec{j}$ and $\vec{i}\quad 2\vec{j}+2\vec{k}$. If θ is the angle between these two vectors, then

$$\cos\theta=\frac{(\vec{i}\quad\vec{j})\cdot(\vec{i}\quad 2\vec{j}+2\vec{k})}{|\vec{i}\quad\vec{j}||\vec{i}\quad 2\vec{j}+2\vec{k}|}=\frac{1+2}{\sqrt{2}(3)}=\frac{1}{\sqrt{2}}$$

Therefore

$$\theta=\frac{\pi}{4}$$

Answer: (A)

61. The perpendicular distance of the point $\vec{i}+\vec{j}+\vec{k}$ from the line joining the points $3\vec{i}+4\vec{j}\quad\vec{k}$ and $2\vec{i}+\vec{j}\quad\vec{k}$ is

(A) $\frac{3}{\sqrt{10}}$ (B) $\frac{\sqrt{10}}{2}$ (C) $\frac{7}{\sqrt{10}}$ (D) $2\sqrt{10}$

Solution: Let A and B be the points $3\vec{i}+4\vec{j}\quad\vec{k}$ and $2\vec{i}+\vec{j}\quad\vec{k}$, respectively, and P the point $\vec{i}+\vec{j}+\vec{k}$. Draw AM perpendicular to the line AB (Figure 6.36). Then

$$\overline{PA}=2\vec{i}+3\vec{j}\quad 2\vec{k}$$

$$\overline{PB}=\vec{i}\quad 2\vec{k}$$

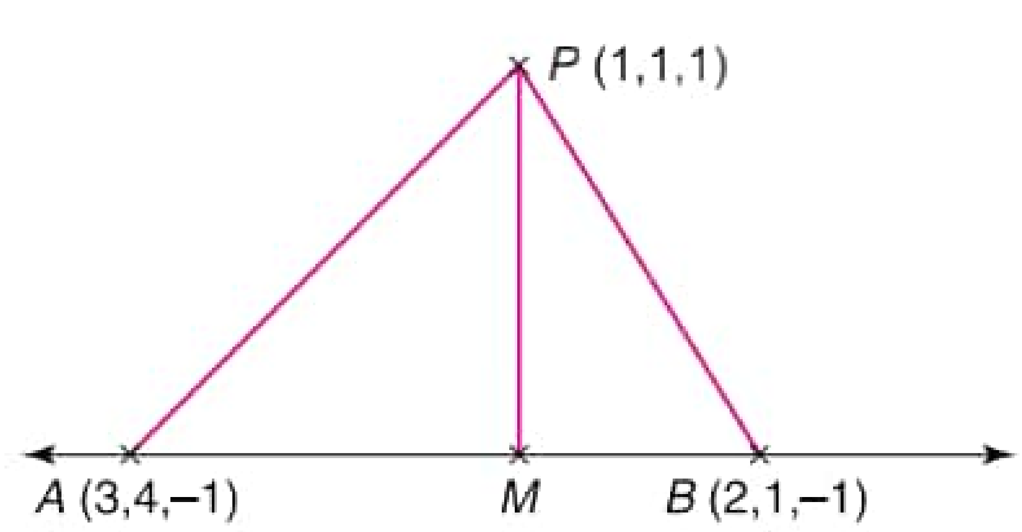

FIGURE 6.36 Single correct choice type question 61.

Now

$$\overline{PA}\quad \overline{PB}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & \ \ 2\\ 1 & 0 & \ \ 2\end{vmatrix}=\ \ 6\vec{i}+2\vec{j}\quad 3\vec{k}$$

and $\qquad \overline{AB}=\ \ \vec{i}\quad 3\vec{j}$

Now (see Theorem 6.29)

$$\frac{1}{2}(AB)(PM)=\text{Area of}\quad PAB=\frac{1}{2}|\overline{PA}\quad \overline{PB}|$$

$$=\frac{1}{2}\sqrt{(\ \ 6)^2+2^2+(\ \ 3)^2}$$

$$=\frac{7}{2}$$

Hence

$$PM=\frac{7}{AB}=\frac{7}{\sqrt{(\ \ 1)^2+(\ \ 3)^2}}=\frac{7}{\sqrt{10}}$$

Answer: (C)

62. If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the vertices A, B and C of a triangle, then, the angle between the vectors $\vec{b}\quad \vec{c}+\vec{c}\quad \vec{a}+\vec{a}\quad \vec{b}$ and $\overline{BC}$ is

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{3}$

(C) $\text{Cos}^{\ 1}\ \dfrac{1}{3}\div$ (D) $\text{Cos}^{\ 1}\ \dfrac{1}{\sqrt{3}}\div$

Solution: It is known that $\overline{AB}\quad \overline{AC}$ is perpendicular to the plane of $\quad ABC$. But

$$\overline{AB}\quad \overline{AC}=(\vec{b}\quad \vec{a})\quad (\vec{c}\quad \vec{a})$$

$$=\vec{b}\quad \vec{c}\quad \vec{b}\quad \vec{a}\quad \vec{a}\quad \vec{c}$$

$$=\vec{b}\quad \vec{c}+\vec{c}\quad \vec{a}+\vec{a}\quad \vec{b}$$

Answer: (A)

63. Let $\vec{a}$ be the position vector of a point and $\vec{b}$ be a non-zero vector. Then, the locus of the point P whose position vector $\vec{r}$ satisfies the relation $\vec{r}\quad \vec{b}=\vec{a}\quad \vec{b}$ is

(A) a straight line

(B) a line passing through $\vec{a}$ and perpendicular to the line $\vec{r}=\vec{a}+t\vec{b}, t\quad \mathbb{R}$

(C) a line making an angle of 60° with the line $\vec{r}=\vec{a}+t\vec{b}$

(D) is a circle with centre at the point $\vec{a}$.

Solution: We have

$$\vec{r}\quad \vec{b}=\vec{a}\quad \vec{b}$$

$$(\vec{r}\quad \vec{a})\quad \vec{b}=\vec{0}$$

$$\vec{r}\quad \vec{a}=t\vec{b}, t\quad \mathbb{R}$$

$$\vec{r}=\vec{a}+t\vec{b}, t\quad \mathbb{R}$$

Answer: (A)

64. Let p be the area of a quadrilateral $ABCD$ and q the area of a parallelogram inscribed in the quadrilateral with its sides drawn parallel to the diagonals of the quadrilateral. Then

(A) $p\quad 2q$ (B) $p>3q$

(C) $p=3q$ (D) $p<2q$

Solution: Take A as origin and let $\overline{AB}=\vec{b}$, $\overline{AC}=\vec{c}$ and $\overline{AD}=\vec{d}$. Let $PQRS$ be the parallelogram inscribed in the quadrilateral with P, Q, R and S on the segments AB, BC, CD and DA, respectively (Figure 6.37). Let

$$AP:PB=CQ:QB=\lambda:1$$

Therefore

$$AS:SD=CR:RD=\lambda:1$$

Hence

$$\overline{AP}=\frac{\lambda\vec{b}}{\lambda+1}$$

$$\overline{AQ}=\frac{\lambda\vec{b}+\vec{c}}{\lambda+1}$$

$$\overline{AS}=\frac{\lambda\vec{d}}{\lambda+1}$$

$$\overline{PQ}=\frac{\vec{c}}{\lambda+1}$$

$$\overline{PS}=\frac{\lambda}{\lambda+1}(\vec{d}\quad \vec{b})=\frac{\lambda}{\lambda+1}(\overline{BD})$$

Now q = Area of the parallelogram $PQRS$ is given by

$$|\overline{PQ}\quad \overline{PS}|=\frac{\lambda}{(\lambda+1)^2}|\overline{AC}\quad \overline{BD}|$$

$$=\frac{2\lambda}{(\lambda+1)^2}(\text{Area of ABCD})$$

$$=\frac{2\lambda}{(\lambda+1)^2}(p)$$

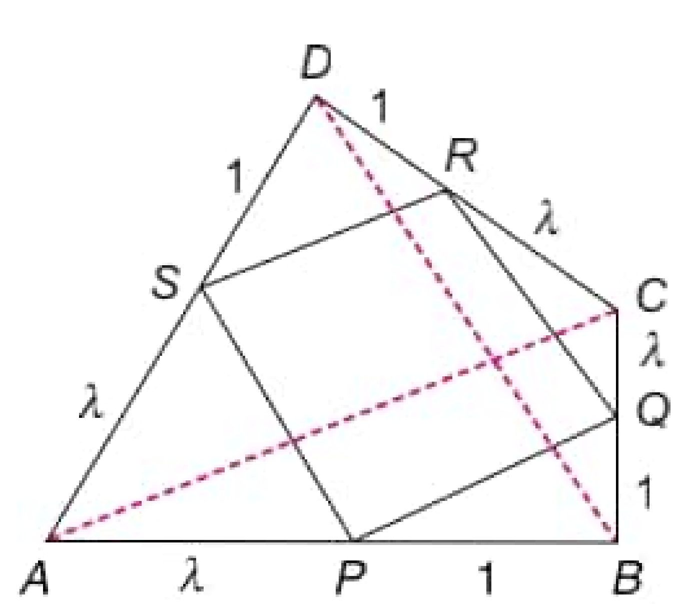

FIGURE 6.37 Single correct choice type question 64.

Therefore

$$p = \frac{(\lambda+1)^2}{2\lambda}(q) = \left[\left(\frac{\lambda}{2} + \frac{1}{2\lambda}\right) + 1\right](q) \geq 2q$$

Equality holds when $\lambda = 1$, that is, P, Q, R and S are the mid-points of the sides.

Answer: (A)

65. In the rhombus $ABCD$, $\underline{|A} = 60°$. K is any point on the segment AC. L and M are points on the segments AB and BC, respectively, such that $KLBM$ is a parallelogram. Then the angles of ΔLMD are

(A) $90°, 45°, 45°$ (B) $60°, 60°, 60°$
(C) $90°, 60°, 30°$ (D) $75°, 30°, 75°$

Solution: Take A as origin. Let $\overline{AB} = \vec{b}$ and $\overline{AD} = \vec{d}$ so that $\overline{AC} = \vec{b} + \vec{d}, |\vec{b}| = |\vec{d}| = a$ (suppose) and $(\vec{b}, \vec{d}) = 60°$. Let

$$\overline{AK} = \lambda\overline{AC} = \lambda(\vec{b} + \vec{d})$$

and $0 < \lambda < 1$. $\overline{LK}$ is parallel to BC implies that

$$\frac{AL}{AB} = \frac{AK}{AC} = \lambda \quad \text{(suppose)}$$

Therefore

$$\overline{AL} = \lambda\vec{b}$$

Equation of the line KM is

$$\vec{r} = \lambda(\vec{b} + \vec{d}) + t\vec{b} \tag{6.25}$$

where $t \in \mathbb{R}$ (see Theorem 5.27, Chapter 5). Equation of the line BC is

$$\vec{r} = \vec{b} + s\vec{d} \tag{6.26}$$

Since M is the point of intersection of the lines KM and BC, on equating the coefficients of Eqs. (6.25) and (6.26) we have

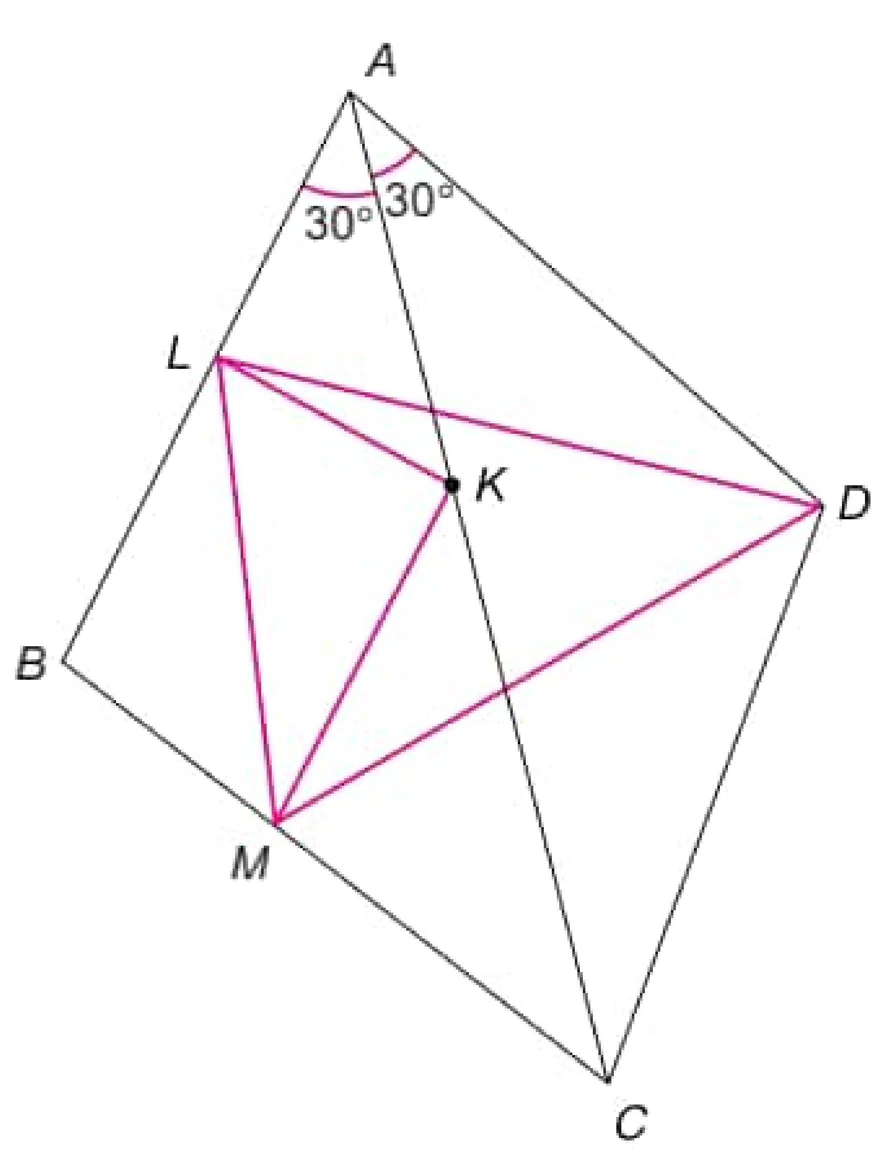

FIGURE 6.38 Single correct choice type question 65.

$$\lambda + t = 1 \quad \text{and} \quad \lambda = s$$

Therefore the position vectors of M, L and K are, respectively, $\vec{b} + \lambda\vec{d}$, $\lambda\vec{b}$ and $\lambda(\vec{b} + \vec{d})$. Hence

$$|\overline{LM}|^2 = |\vec{b} + \lambda\vec{d} - \lambda\vec{b}|^2$$
$$= |(1-\lambda)\vec{b} + \lambda\vec{d}|^2$$
$$= (1-\lambda)^2|\vec{b}|^2 + \lambda^2|\vec{d}|^2 + 2\lambda(1-\lambda)(\vec{b}\cdot\vec{d})$$
$$= a^2[(1-\lambda)^2 + \lambda^2 + \lambda(1-\lambda)] \quad \left(\because \vec{b}\cdot\vec{d} = a^2\cos 60° = \frac{a^2}{2}\right)$$
$$= a^2(\lambda^2 - \lambda + 1)$$

Note that $\lambda^2 - \lambda + 1 > 0$. Now

$$|\overline{MD}|^2 = |\vec{b} + \lambda\vec{d} - \vec{d}|^2$$
$$= |\vec{b} + (\lambda-1)\vec{d}|^2 = |\vec{b}|^2 + (\lambda-1)^2|\vec{d}|^2 + 2(\lambda-1)(\vec{b}\cdot\vec{d})$$
$$= a^2[1 + (\lambda-1)^2 + \lambda - 1]$$
$$= a^2(\lambda^2 - \lambda + 1)$$
$$|\overline{LD}|^2 = |\lambda\vec{b} - \vec{d}|^2$$
$$= \lambda^2|\vec{b}|^2 - 2\lambda(\vec{b}\cdot\vec{d}) + |\vec{d}|^2$$
$$= a^2(\lambda^2 - \lambda + 1)$$

Therefore

$$LM = DL = DM$$

and so ΔLMD is equilateral.

Answer: (B)

66. P is a point on the circumcircle of ΔABC other than the vertices A, B and C. H is the orthocentre, M is the mid-point of PH and D is the mid-point of BC. Then the angle between the vectors $\overline{DM}$ and $\overline{AP}$ is

(A) $75°$ (B) $60°$ (C) $120°$ (D) $90°$

Solution: Take the circumcentre "O" of the triangle as origin and let $\overline{OA} = \vec{a}, \overline{OB} = \vec{b}$ and $\overline{OC} = \vec{c}$. See Figure 6.39.

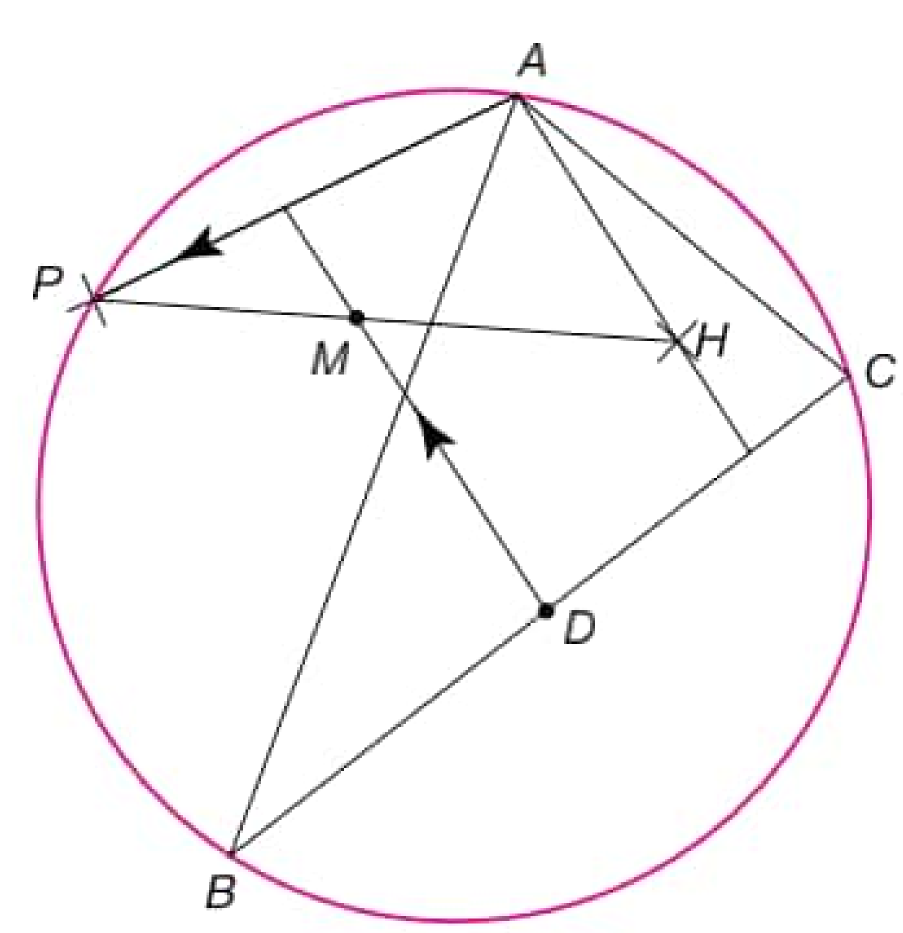

FIGURE 6.39 Single correct choice type question 66.

Therefore by Example 5.3 (Chapter 5)

$$\overline{OH} = \vec{a} + \vec{b} + \vec{c}$$

Let $\overline{OP} = \vec{p}$ so that

$$\overline{OM} = \frac{\vec{p} + \vec{a} + \vec{b} + \vec{c}}{2}$$

and

$$\overline{DM} = \frac{\vec{p} + \vec{a} + \vec{b} + \vec{c}}{2} - \frac{\vec{b} + \vec{c}}{2}$$

$$= \frac{\vec{p} + \vec{a}}{2}$$

Now

$$\overline{DM} \cdot \overline{AP} = \left(\frac{\vec{p} + \vec{a}}{2}\right) \cdot (\vec{p} - \vec{a})$$

$$= \frac{1}{2}(|\vec{p}|^2 - |\vec{a}|^2)$$

$$= \frac{1}{2}(0) \quad (\because |\vec{p}| = |\vec{a}| = R)$$

$$= 0$$

Therefore $\overline{DM}$ and $\overline{AP}$ are at right angles.

Answer: (D)

67. $ABCD$ is a parallelogram. Through the vertex A, an arbitrary line is drawn to meet the diagonal BD in P, the side BC in Q and the side DC produced in R. Then AP is

(A) AM between PR and PQ

(B) GM between PR and PQ

(C) HM between PR and PQ

(D) $AR = PQ \cdot PR$

Solution: Take A as origin and let $\overline{AB} = \vec{b}$ and $\overline{AD} = \vec{d}$. Suppose $BP : PD = \lambda : 1$ so that

$$\overline{AP} = \frac{\vec{b} + \lambda\vec{d}}{\lambda + 1}$$

Therefore the equation of the line AP is (see Theorem 5.29)

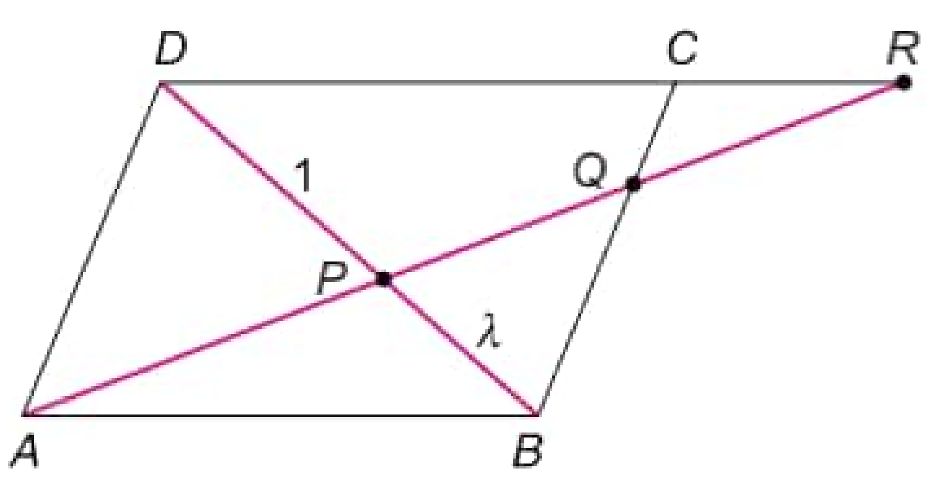

FIGURE 6.40 Single correct choice type question 67.

$$\vec{r} = t\frac{(\lambda\vec{d} + \vec{b})}{\lambda + 1} \tag{6.27}$$

Equation of the line BC is (see Theorem 5.27)

$$\vec{r} = \vec{b} + s\vec{d}, \quad s \in \mathbb{R} \tag{6.28}$$

Equation of the line DC is (see Theorem 5.27)

$$\vec{r} = \vec{d} + u\vec{b}, \quad u \in \mathbb{R} \tag{6.29}$$

From Eqs. (6.27) and (6.28), we have

$$\frac{t}{\lambda + 1} = 1 \quad \text{and} \quad \frac{t\lambda}{\lambda + 1} = s$$

and hence $s = \lambda$, $t = \lambda + 1$. Therefore

$$\overline{AQ} = \vec{b} + \lambda\vec{d} \tag{6.30}$$

Also from Eqs. (6.27) and (6.29), we have

$$\frac{t\lambda}{\lambda + 1} = 1 \quad \text{and} \quad \frac{t}{\lambda + 1} = u$$

Therefore

$$\lambda u = 1 \quad \text{or} \quad u = \frac{1}{\lambda}$$

Hence

$$\overline{AR} = \vec{d} + \frac{1}{\lambda}\vec{b} \tag{6.31}$$

Therefore

$$\overline{PQ} = \overline{AQ} - \overline{AP}$$

$$= (\vec{b} + \lambda\vec{d}) - \left(\frac{\vec{b} + \lambda\vec{d}}{\lambda + 1}\right)$$

$$= \frac{\lambda}{\lambda + 1}(\vec{b} + \lambda\vec{d})$$

$$= \lambda(\overline{AP})$$

This gives

$$\frac{PQ}{AP} = \lambda \tag{6.32}$$

Also

$$\overline{PR} = \overline{AR} - \overline{AP}$$

$$= \left(\vec{d} + \frac{1}{\lambda}\vec{b}\right) - \frac{\vec{b} + \lambda\vec{d}}{\lambda + 1} \quad \text{[see Eq. (6.31)]}$$

$$= (\vec{b} + \lambda\vec{d})\left(\frac{1}{\lambda} - \frac{1}{\lambda + 1}\right)$$

$$= \frac{\vec{b} + \lambda\vec{d}}{\lambda(\lambda+1)}$$

$$= \frac{1}{\lambda}(\overline{AP})$$

Hence

$$\frac{AP}{PR} = \lambda \qquad (6.33)$$

Equations (6.32) and (6.33) give

$$\frac{PQ}{AP} = \frac{AP}{PR}$$

$$(AP)^2 = PQ \times PR$$

Answer: (B)

68. On the sides BC, CA and AB of ABC, points D, E and F are taken, respectively, such that $BD:DC = CE:EA = AF:FB = \lambda:1$. Then the ratio of the area of DEF to the area of ABC is

(A) $(\lambda^2 \quad \lambda + 1):(\lambda+1)^2$

(B) $(\lambda^2 \quad \lambda + 1):(\lambda+1)$

(C) $(\lambda^2 + \lambda + 1):(\lambda+1)^2$

(D) $(\lambda^2 + \lambda + 1):(\lambda+1)$

Solution: Take A as origin and let $\overline{AB} = \vec{b}$ and $\overline{AC} = \vec{c}$. Therefore

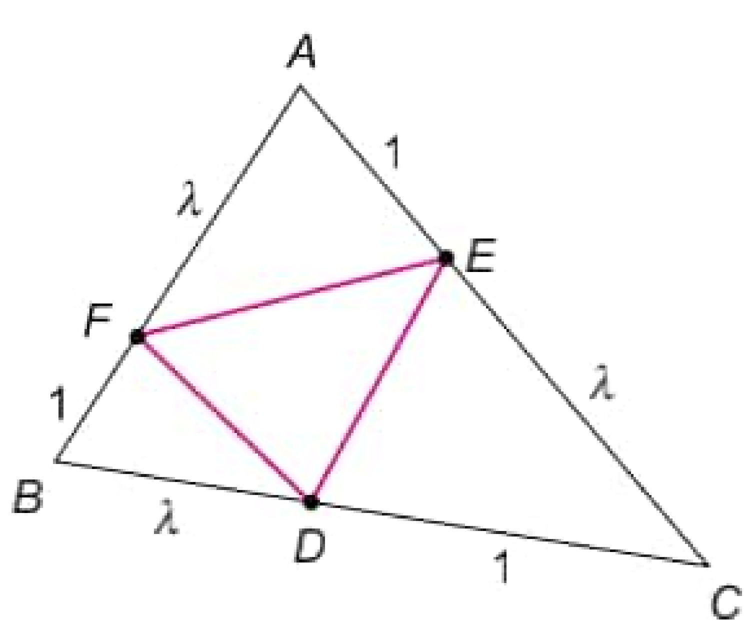

FIGURE 6.41 Single correct choice type question 68.

$$\overline{AD} = \frac{\vec{b} + \lambda\vec{c}}{\lambda+1}$$

$$\overline{AE} = \frac{\vec{c}}{\lambda+1}$$

$$\overline{AF} = \frac{\lambda\vec{b}}{\lambda+1}$$

$$\overline{DE} = \overline{AE} \quad \overline{AD} = \frac{\vec{b} + (1 \quad \lambda)\vec{c}}{\lambda+1}$$

$$\overline{DF} = \frac{(\lambda \quad 1)\vec{b} \quad \lambda\vec{c}}{\lambda+1}$$

Therefore (using Theorem 6.29)

$$\text{Area of } \quad DEF = \frac{1}{2}|\overline{DE} \quad \overline{DF}|$$

$$= \frac{1}{2}\left| \frac{\vec{b} + (1 \quad \lambda)\vec{c}}{\lambda+1} \quad \frac{(\lambda \quad 1)\vec{b} \quad \lambda\vec{c}}{\lambda+1} \right|$$

$$= \frac{1}{2(\lambda+1)^2}|\lambda(\vec{b} \quad \vec{c}) + (\lambda \quad 1)^2(\vec{b} \quad \vec{c})|$$

$$= \frac{\lambda^2 \quad \lambda + 1}{2(\lambda+1)^2}|\vec{b} \quad \vec{c}|$$

$$= \frac{\lambda^2 \quad \lambda + 1}{(\lambda+1)^2}(\text{Area of } \quad ABC)$$

$$\because \frac{1}{2}|\vec{b} \quad \vec{c}| = \text{Area } \quad ABC \div$$

Answer: (A)

69. A vector of magnitude $\sqrt{2}$ and coplanar with the vectors $3\vec{i} \quad \vec{j} \quad \vec{k}$ and $\vec{i} + \vec{j} \quad 2\vec{k}$ which is perpendicular to the vector $2\vec{i} + 2\vec{j} + \vec{k}$ is one of the following vectors. Which one is it?

(A) $\frac{\sqrt{2}}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$ (B) $\frac{1}{5}(3\vec{i} \quad 5\vec{j} + 4\vec{k})$

(C) $\frac{1}{5}(3\vec{i} + 5y + 4\vec{k})$ (D) $\frac{2}{\sqrt{6}}(\vec{i} \quad \vec{j} + \vec{k})$

Solution: Let $\vec{r}$ be the required vector. Since $\vec{r}$ is coplanar with the vectors $\vec{b} = 3\vec{i} \quad \vec{j} \quad \vec{k}$ and $\vec{c} = \vec{i} + \vec{j} \quad 2\vec{k}$ (observe that $\vec{b}$ and $\vec{c}$ are non-collinear), let

$$\vec{r} = x\vec{b} + y\vec{c}$$

$$= (3x+y)\vec{i} + (y \quad x)\vec{j} \quad (x+2y)\vec{k} \qquad (6.34)$$

$$|\vec{r}| = \sqrt{2} \quad (3x+y)^2 + (y \quad x)^2 + (x+2y)^2 = 2 \quad (6.35)$$

where x and y are scalars. Now $\vec{r}$ is perpendicular to $2\vec{i} + 2\vec{j} + \vec{k}$ implies

$$2(3x+y) + 2(y \quad x) \quad x \quad 2y = 0$$

$$3x + 2y = 0$$

$$y = \quad \frac{3x}{2}$$

Putting this values of $y = \quad 3x/2$ in Eq. (6.35) we get

$$3x \quad \frac{3x}{2}^{2}\div + \quad \frac{3x}{2} \quad x\div^{2} + (x \quad 3x)^2 = 2$$

$$\frac{9x^2}{4} + \frac{25x^2}{4} + 4x^2 = 2$$

$$50x^2 = 8$$

$$x = \pm\frac{2\sqrt{2}}{5\sqrt{2}} = \pm\frac{2}{5}$$

Now

$$x = \frac{2}{5} \Rightarrow y = -\frac{3x}{2} = -\frac{3}{5}$$

$$x = \frac{-2}{5} \Rightarrow y = \frac{3}{5}$$

In tabular form we have

x	$y = -\frac{3x}{2}$
$\frac{2}{5}$	$-\frac{3}{5}$
$-\frac{2}{5}$	$\frac{3}{5}$

Answer: (B)

70. The position vectors of the points A, B, C and D are, respectively, $3\vec{i} - 2\vec{j} - \vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$, $-\vec{i} + \vec{j} + 2\vec{k}$ and $4\vec{i} + 5\vec{j} + \lambda\vec{k}$. If all the four points lie in a plane, then the value of λ is

(A) $-\frac{146}{17}$ (B) $\frac{146}{7}$ (C) $\frac{98}{17}$ (D) $-\frac{98}{17}$

Solution: We have

$$\overline{AB} = -\vec{i} + 5\vec{j} - 3\vec{k}$$

$$\overline{AC} = -4\vec{i} + 3\vec{j} + 3\vec{k}$$

$$\overline{AD} = \vec{i} + 7\vec{j} + (\lambda + 1)\vec{k}$$

Now

A, B, C and D are coplanar

$\Leftrightarrow \overline{AB}, \overline{AC}$ and $\overline{AD}$ are coplanar

$$\Leftrightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & (\lambda+1) \end{vmatrix} = 0 \quad \text{(see Corollary 6.16)}$$

$$\Leftrightarrow 17\lambda + 146 = 0$$

Answer: (A)

71. The value of a so that the volume of the parallelopiped formed by the vectors $\vec{i} + a\vec{j} - \vec{k}$, $\vec{j} + a\vec{k}$, $\vec{i} + \vec{j} + \vec{k}$ becomes minimum is

(A) $\sqrt{3}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{7}{4}$

Solution: We have

$$V = \text{Volume of the parallelopiped} = |\det A|$$

(see Theorem 6.33 and Theorem 6.35) where A is the matrix

$$\begin{bmatrix} 1 & a & -1 \\ 0 & 1 & a \\ 1 & 1 & 1 \end{bmatrix}$$

Now

$$\det A = 1 - a + a^2 + 1$$
$$= a^2 - a + 2$$
$$= \left(a - \frac{1}{2}\right)^2 + \frac{7}{4} \geq \frac{7}{4}$$

Therefore

$$V = \det A = \left(a - \frac{1}{2}\right)^2 + \frac{7}{4} \geq \frac{7}{4}$$

V is equal to 7/4 when $a = 1/2$. Hence V is minimum when $a = 1/2$ and the minimum value of V is 7/4.

Answer: (B)

72. If the volume of the parallelopiped whose set of coterminus edges are represented by the vectors $-12\vec{i} + \lambda\vec{k}$, $3\vec{j} - \vec{k}$ and $2\vec{i} + \vec{j} - 15\vec{k}$ is 546 cubic units, then a value of λ is

(A) -3 (B) 3 (C) -179 (D) 178

Solution: We have

$$V = \text{Volume}$$

$$= \text{Absolute value of} \begin{vmatrix} -12 & 0 & \lambda \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix}$$

$$= |12 \times 44 - 6\lambda|$$

Therefore

$$88 - \lambda = \pm 91$$

Now

$$88 - \lambda = 91 \Rightarrow \lambda = -3$$

and $$88 - \lambda = -91 \Rightarrow 179$$

Answer: (A)

73. Unit vector orthogonal to $\vec{a} = 3\vec{i} + 2\vec{j} + 6\vec{k}$ and coplanar with $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} - \vec{j} + \vec{k}$ is

(A) $\frac{6\vec{i} - 5\vec{k}}{\sqrt{61}}$ (B) $\frac{3\vec{j} - \vec{k}}{\sqrt{10}}$

(C) $\frac{2\vec{i} - 5\vec{j}}{\sqrt{29}}$ (D) $\frac{2\vec{i} + \vec{j} - 2\vec{k}}{3}$

Solution: By the definition of cross product, the vector $\vec{a}\times(\vec{b}\times\vec{c})$ is coplanar with $\vec{b}$ and $\vec{c}$ and is orthogonal to $\vec{a}$. Now (using Theorem 6.46) we have

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$
$$= 7\vec{b} - 14\vec{c} = 7(3\vec{j} - \vec{k})$$

Therefore the required unit vector is

$$\frac{7(3\vec{j}-\vec{k})}{7\sqrt{9+1}} = \frac{3\vec{j}-\vec{k}}{\sqrt{10}}$$

Answer: (B)

74. If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{b}$ is a vector such that $\vec{a}\cdot\vec{b} = 1$ and $\vec{a}\times\vec{b} = \vec{j} - \vec{k}$, then $\vec{b}$ is

(A) $\vec{i} - \vec{j} + \vec{k}$ (B) $2\vec{j} - \vec{k}$ (C) $\vec{i}$ (D) $2\vec{i}$

Solution: We have

$$(\vec{a}\times\vec{b})\times\vec{a} = (\vec{j}-\vec{k})\times(\vec{i}+\vec{j}+\vec{k})$$

Therefore (using Theorem 6.46)

$$(\vec{a}\cdot\vec{a})\vec{b} - (\vec{b}\cdot\vec{a})\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$
$$3\vec{b} - \vec{a} = 2\vec{i} - \vec{j} - \vec{k} \quad (\because \vec{a}\cdot\vec{b} = 1)$$
$$3\vec{b} = (\vec{i}+\vec{j}+\vec{k}) + (2\vec{i}-\vec{j}-\vec{k})$$
$$\vec{b} = \frac{1}{3}(3\vec{i}) = \vec{i}$$

Answer: (C)

75. Let $\vec{a} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = \vec{i} + \vec{j}$. If $\vec{c}$ is a vector such that $\vec{a}\cdot\vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$ is 30°, then $|(\vec{a}\times\vec{b})\times\vec{c}|$ is equal to

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) 2 (D) 3

Solution: We have

$$8 = |\vec{c} - \vec{a}|^2$$
$$= |\vec{c}|^2 - 2(\vec{c}\cdot\vec{a}) + |\vec{a}|^2$$
$$= |\vec{c}|^2 - 2|\vec{c}| + 9$$

Therefore

$$(|c| - 1)^2 = 0 \quad \text{or} \quad |\vec{c}| = 1$$

Now

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\vec{i} - 2\vec{j} + \vec{k}$$

Hence

$$|\vec{a}\times\vec{b}| = 3$$

So

$$|(\vec{a}\times\vec{b})\times\vec{c}| = |\vec{a}\times\vec{b}||\vec{c}|\sin 30°$$
$$= 3(1)\left(\frac{1}{2}\right) = \frac{3}{2}$$

Answer: (B)

76. The vectors $\vec{a} = (\sec^2\alpha)\vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + (\sec^2\beta)\vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + \vec{j} + (\sec^2\gamma)\vec{k}$ are coplanar only if

(A) α, β, γ are distinct such that $\alpha + \beta + \gamma = \pi$

(B) $\alpha = \beta = \gamma = n\pi$, $n \in \mathbb{R}$

(C) $\alpha = \beta = \gamma = \frac{\pi}{4}$

(D) no real values exist

Solution: The vector are coplanar only if

$$\begin{vmatrix} \sec^2\alpha & 1 & 1 \\ 1 & \sec^2\beta & 1 \\ 1 & 1 & \sec^2\gamma \end{vmatrix} = 0$$

Replacing $\sec^2\alpha$, $\sec^2\beta$ and $\sec^2\gamma$ with $1+\tan^2\alpha$, $1+\tan^2\beta$ and $1+\tan^2\gamma$ and simplifying we have

$$\sum(\tan^2\beta\tan^2\gamma) + \tan^2\alpha\tan^2\beta\tan^2\gamma = 0$$

which is possible only when

$$\tan^2\alpha = \tan^2\beta = \tan^2\gamma = 0$$

Answer: (B)

77. Let $\vec{a}$, $\vec{b}$ be unit vectors. If $\vec{c}$ is a vector such that $\vec{b} = \vec{c} + \vec{c}\times\vec{a}$ and if

$$|(\vec{a}\times\vec{b})\cdot\vec{c}| = \frac{1}{2}$$

then the angle between $\vec{a}$ and $\vec{b}$ is

(A) 60° (B) 30° (C) 45° (D) 90°

Solution: Let θ be the angle between $\vec{a}$ and $\vec{c}$. Now

$$|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\vec{b}\cdot(\vec{c}\times\vec{a})| = |(\vec{c}+\vec{c}\times\vec{a})\cdot(\vec{c}\times\vec{a})|$$
$$= |\vec{c}\times\vec{a}|^2 = |\vec{c}|^2\sin^2\theta \qquad (6.36)$$

Also

$$|\vec{b}|^2 = |\vec{c}+\vec{c}\times\vec{a}|^2 = |\vec{c}|^2 + |\vec{c}\times\vec{a}|^2 = |\vec{c}|^2 + |\vec{c}|^2\sin^2\theta$$

Therefore

$$1 = |\vec{b}|^2 = (1+\sin^2\theta)|\vec{c}|^2 \quad (\because |\vec{b}| = 1)$$

So

$$|\vec{c}|^2 = \frac{1}{1+\sin^2\theta} \qquad (6.37)$$

Therefore from Eqs. (6.36) and (6.37) we get

$$|(\vec{a} \quad \vec{b})\times\vec{c}| = \frac{\sin^2\theta}{1+\sin^2\theta} = \frac{1}{2}$$

if and only if $\theta = 90°$. Now

$$\vec{b}\times\vec{a} = \vec{c}\times\vec{a} + (\vec{c} \quad \vec{a})\times\vec{a}$$

$$= 0 + 0 \quad (\because \vec{c}\times\vec{a} = 0)$$

Therefore $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

Answer: (D)

78. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors of magnitudes 1, 1 and 2, respectively. If $\vec{a} \quad (\vec{a} \quad \vec{c}) + \vec{b} = \vec{0}$, then acute angle between $\vec{a}$ and $\vec{c}$ is

(A) 90° (B) 60° (C) 45° (D) 30°

Solution: We have (using Theorem 6.46)

$$\vec{b} = \vec{a} \quad (\vec{a} \quad \vec{c}) = (\vec{a}\cdot\vec{c})\vec{a} \quad (\vec{a}\cdot\vec{a})\vec{c}$$

Therefore

$$1 = |\vec{b}|^2 = (\vec{a}\cdot\vec{c})^2|\vec{a}|^2 + |\vec{c}|^2 \quad 2(\vec{a}\cdot\vec{c})^2 \quad (\because |\vec{a}| = 1 = |\vec{b}|)$$

$$= (\vec{a}\cdot\vec{c})^2 + 4 \quad 2(\vec{a}\cdot\vec{c})^2$$

$$= 4 \quad (\vec{a}\cdot\vec{c})^2$$

$$= 4 \quad 4\cos^2\theta$$

where $\theta = (\vec{a}, \vec{c})$. Therefore

$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$

Hence θ is acute $\quad \theta = 30°$.

Answer: (D)

79. Let $A = (1, 2, 1)$, $B = (2, 1, 2)$, $C = (0, \quad 4, 4)$, $D = (2, \quad 2, 2)$ and $E = (4, 1, 2)$ be five points. The line AB meets the plane CDE at the points

(A) $(3, \quad 1, 3)$ (B) $(3, 0, 3)$

(C) $(1, 1, 1)$ (D) $(2, 0, 2)$

Solution: Equation of the line AB is (using Theorem 5.29)

$$\vec{r} = (1 \quad t)(2\vec{i} + \vec{j} + 2\vec{k}) + t(\vec{i} + 2\vec{j} + \vec{k}),\ t \quad \mathbb{R}$$

$$\vec{r} = (2 \quad t)\vec{i} + (1+t)\vec{j} + (2 \quad t)\vec{k} \qquad (6.38)$$

For the plane CDE, $\overline{CD} \quad \overline{CE}$ is a normal. So

$$\overline{CD} \quad \overline{CE} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 2 \\ 4 & 5 & 2 \end{vmatrix}$$

$$= 6\vec{i} \quad 4\vec{j} + 2\vec{k}$$

Therefore $3\vec{i} \quad 2\vec{j} + \vec{k}$ is also normal to the plane CDE. The plane passes through the point C. By Theorem 6.17, the equation of the plane is

$$[\vec{r} \quad (\quad 4\vec{j} + 4\vec{k})]\cdot(3\vec{i} \quad 2\vec{j} + \vec{k}) = 0$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is any point in the plane CDE. That is

$$[x\vec{i} + (y+4)\vec{j} + (z \quad 4)\vec{k}]\cdot(3\vec{i} \quad 2\vec{j} + \vec{k}) = 0$$

$$3x \quad 2(y+4) + z \quad 4 = 0$$

$$3x \quad 2y + z \quad 12 = 0 \qquad (6.39)$$

From Eqs. (6.38) and (6.39), we have

$$3(2 \quad t) \quad 2(1+t) + 2 \quad t \quad 12 = 0$$

$$6 \quad 3t \quad 2 \quad 2t + 2 \quad t \quad 12 = 0$$

$$6t \quad 6 = 0$$

$$t = \quad 1$$

Substituting the value of $t = \quad 1$ in Eq. (6.38), we have

$$\vec{r} = 3\vec{i} + 3\vec{k}$$

Therefore (3, 0, 3) is the point where the line AB meets the plane CDE.

Answer: (B)

80. Let $\vec{a} = \vec{i} \quad \vec{j}$, $\vec{b} = \vec{j} \quad \vec{k}$, $\vec{c} = \vec{k} \quad \vec{i}$. $\vec{d}$ is a unit vector such that $\vec{a}\cdot\vec{d} = 0 = [\vec{b}\ \vec{c}\ \vec{d}]$, then $\vec{d}$ equals

(A) $\pm\dfrac{(\vec{i} + \vec{j} \quad 2\vec{k})}{\sqrt{6}}$ (B) $\pm\dfrac{(\vec{i} \quad \vec{j} \quad 2\vec{k})}{\sqrt{6}}$

(C) $\pm\dfrac{(2\vec{i} + \vec{j} \quad \vec{k})}{\sqrt{6}}$ (D) $\pm\dfrac{(\vec{i} + 2\vec{j} \quad \vec{k})}{\sqrt{6}}$

Solution: Let $\vec{d} = x\vec{i} + y\vec{j} + z\vec{k}$ and $|\vec{d}| = 1$.

$$\vec{a}\cdot\vec{d} = 0 \quad x \quad y = 0 \qquad (6.40)$$

$$[\vec{b}\ \vec{c}\ \vec{d}] = 0 \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ x & y & z \end{vmatrix} = 0$$

$$x + y + z = 0 \qquad (6.41)$$

$$|\vec{d}| = 1 \quad x^2 + y^2 + z^2 = 1 \qquad (6.42)$$

From Eqs. (6.40) and (6.41) we get

$$y = x \quad \text{and} \quad z = \quad 2x$$

Substituting these values in Eq. (6.42), we have

$$6x^2 = 1 \quad \text{or} \quad x = \pm\frac{1}{\sqrt{6}}$$

Therefore

$$\vec{d} = \pm\frac{1}{\sqrt{6}}(\vec{i} + \vec{j} \quad 2\vec{k})$$

Answer: (A)

81. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-coplanar vectors and $2\vec{a} + 3\vec{b} \quad \vec{c}$, $\vec{a} \quad 2\vec{b} + 3\vec{c}$, $3\vec{a} + 4\vec{b} \quad 2\vec{c}$ and $\vec{a} \quad 6\vec{b} + 6\vec{c}$ be position vectors of the points A, B, C and D, respectively. Then the scalar triple product $[\overline{AB}\ \overline{AC}\ \overline{AD}]$ is equal to

(A) $9[\vec{a}\ \vec{b}\ \vec{c}]$ (B) $6[\vec{a}\ \vec{b}\ \vec{c}]$

(C) $4[\vec{a}\ \vec{b}\ \vec{c}]$ (D) 0

Solution: We have

$$\overline{AB} = \quad \vec{a} \quad 5\vec{b} + 4\vec{c}$$

$$\overline{AC} = \vec{a} + \vec{b} \quad \vec{c}$$

$$\overline{AD} = \quad \vec{a} \quad 9\vec{b} + 7\vec{c}$$

Therefore (using Quick Look 11)

$$[\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix} 1 & 5 & 4 \\ 1 & 1 & 1 \\ 1 & 9 & 7 \end{vmatrix}[\vec{a}\ \vec{b}\ \vec{c}]$$

$$= (2 + 30 \quad 32)[\vec{a}\ \vec{b}\ \vec{c}]$$

$$= 0$$

Answer: (D)

QUICK LOOK

The points A, B, C and D are coplanar.

82. In Figure 6.42, $\overline{AB} = 3\vec{i} \quad \vec{j}$, $\overline{AC} = 2\vec{i} + 3\vec{j}$ and $\overline{DE} = 4\vec{i} \quad 2\vec{j}$. Then the area of the shaded region in square units is

(A) 5 (B) 6 (C) 7 (D) 8

Solution: We have

$$\overline{BC} = \overline{BA} + \overline{AC}$$

$$= (\quad 3\vec{i} + \vec{j}) + (2\vec{i} + 3\vec{j})$$

$$= \quad \vec{i} + 4\vec{j}$$

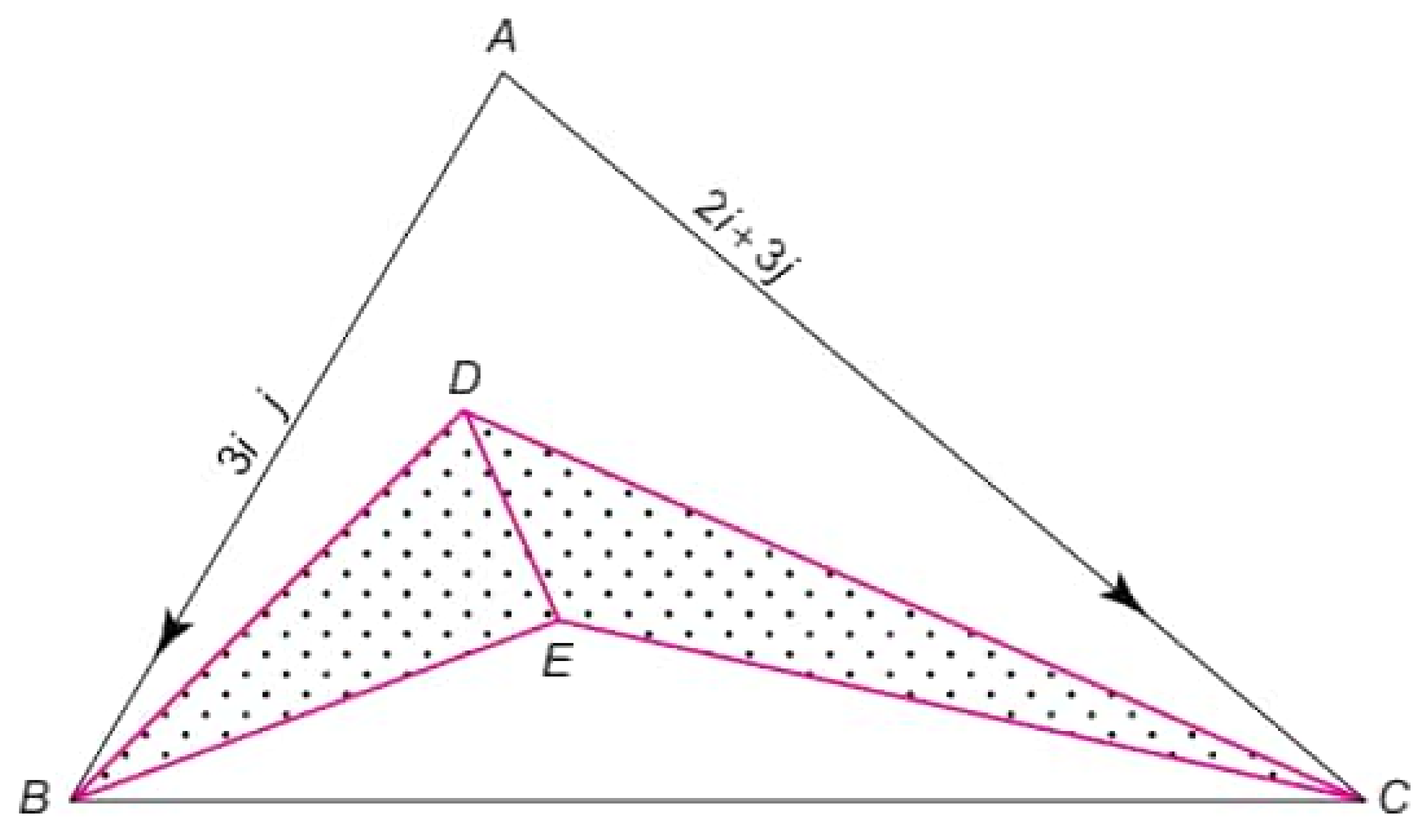

FIGURE 6.42 Single correct choice type question 82.

Vector area of the shaded region is given by

$$\frac{1}{2}\overline{ED} \quad \overline{EB} + \frac{1}{2}\overline{EC} \quad \overline{ED} = \frac{1}{2}[\overline{ED} \quad \overline{EB} + \overline{ED} \quad \overline{CE}]$$

$$= \frac{1}{2}[\overline{ED} \quad (\overline{CE} + \overline{EB})]$$

$$= \frac{1}{2}(\overline{ED} \quad \overline{CB})$$

$$= \frac{1}{2}[(\quad 4\vec{i} + 2\vec{j}) \quad (\vec{i} \quad 4\vec{j})]$$

$$= \frac{1}{2}(16\vec{k} \quad 2\vec{k})$$

$$= 7\vec{k}$$

Therefore

$$\text{Area} = |\text{Vector area}| = 7$$

Answer: (C)

83. The angle between two unit vectors $\vec{a}$ and $\vec{b}$ is $\text{Cos}^{\ 1}(1/4)$. If $\vec{c}$ is a vector such that $|\vec{c}| = 4$ and $\vec{c} \quad 2\vec{b} = \lambda\vec{a}$, then value of λ is

(A) 3, 4 (B) 3, 4 (C) 3, 4 (D) 3, 4

Solution: We have

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\cos \quad \cos^{\ 1}\frac{1}{4}\div = \frac{1}{4} \qquad (6.43)$$

$$\vec{c} = \lambda\vec{a} + 2\vec{b}$$

$$16 = |\vec{c}|^2$$

$$= |\lambda\vec{a} + 2\vec{b}|^2$$

$$= \lambda^2|\vec{a}|^2 + 4\lambda(\vec{a}\times\vec{b}) + 4|\vec{b}|^2$$

$$= \lambda^2 + 4\lambda \quad \frac{1}{4}\div + 4 \quad [\text{by Eq. (6.43)}]$$

Therefore

$$\lambda^2+\lambda \quad 12=0$$

$$(\lambda+4)(\lambda \quad 3)=0$$

$$\lambda=3 \quad \text{or} \quad 4$$

Answer: (A)

84. In ABC (Figure 6.43), if$(BC)^2+(AC)^2=5(AB)^2$, then the angle between the medians AD and BE is

(A) 60° (B) 90° (C) 120° (D) 45°

Solution: Take A as origin. Let $\overline{AB}=\vec{\alpha}$ and $\overline{AC}=\vec{\beta}$ so that $|\vec{\alpha}|=c$, $|\vec{\beta}|=b$ and $(\vec{\alpha},\vec{\beta})=A$. Therefore

$$\overline{AD}=\frac{\vec{\alpha}+\vec{\beta}}{2}$$

and

$$\overline{AE}=\frac{1}{2}\vec{\beta}$$

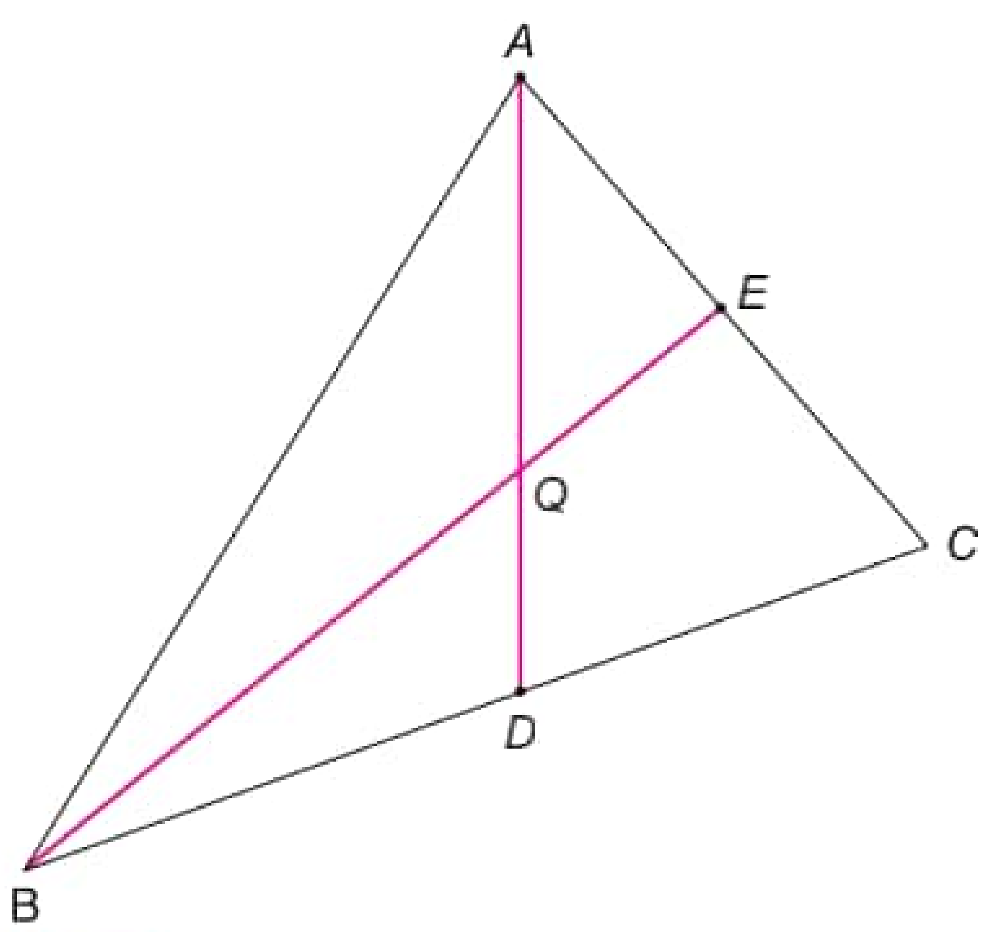

FIGURE 6.43 Single correct choice type question 84.

Now

$$\overline{AD}\times\overline{BE}=\frac{\vec{\alpha}+\vec{\beta}}{2}\div\times\frac{[(\vec{\beta}/2) \quad \vec{\alpha}]}{2}$$

$$=\frac{1}{8}(|\vec{\beta}|^2 \quad 2|\vec{\alpha}|^2 \quad \vec{\alpha}\times\vec{\beta}) \qquad (6.44)$$

But by hypothesis

$$|\vec{\beta} \quad \vec{\alpha}|^2+|\vec{\beta}|^2=5|\vec{\alpha}|^2$$

$$2|\vec{\beta}|^2 \quad 2(\vec{\alpha}\times\vec{\beta})=4|\vec{\alpha}|^2$$

$$|\vec{\beta}|^2 \quad (\vec{\alpha}\times\vec{\beta}) \quad 2|\vec{\alpha}|^2=0 \qquad (6.45)$$

From Eqs. (6.44) and (6.45), it follows that $\overline{AD}\times\overline{BE}=0$. Therefore the medians AD and BE are at right angles.

Answer: (B)

85. In ABC, $AB=BC$ and the median AD is perpendicular to the bisector of the angle C (Figure 6.44). Then $\underline{|ACB}$ is

(A) 60° (B) 45°

(C) $\text{Cos}^{\,1}\,\frac{1}{3}\div$ (D) $\text{Cos}^{\,1}\,\frac{1}{4}\div$

Solution: Take A as origin. Let $\overline{AB}=\vec{\alpha}$ and $\overline{AC}=\vec{\beta}$ so that $|\vec{\alpha}|=c$, $|\vec{\beta}|=b$ and $(\vec{\alpha},\vec{\beta})=A$. Let CE be the bisector of $\underline{|ACB}$ so that $AE:EB=b:a$. Therefore

$$\overline{AD}=\frac{\vec{\alpha}+\vec{\beta}}{2}$$

and

$$\overline{AE}=\frac{b\vec{\alpha}}{a+b}$$

By hypothesis,

$$AB=BC$$

$$|\overline{AB}|=|\overline{BC}|$$

$$|\vec{\alpha}|^2=|\vec{\alpha} \quad \vec{\beta}|^2$$

$$|\vec{\beta}|^2 \quad 2(\vec{\alpha}\times\vec{\beta})=0$$

$$|\vec{\beta}|^2 \quad 2|\vec{\alpha}||\vec{\beta}|\cos A=0$$

$$b=2c\cos A \qquad (6.46)$$

Now

$$\overline{AD}\times\overline{CE}=0 \qquad \frac{\vec{\alpha}+\vec{\beta}}{2}\div\times\frac{b\vec{\alpha}}{a+b} \quad \vec{\beta}\div=0$$

$$(\vec{\alpha}+\vec{\beta})\times(b\vec{\alpha} \quad (a+b)\vec{\beta})=0$$

$$b|\vec{\alpha}|^2 \quad (a+b)(\vec{\alpha}\times\vec{\beta})+b(\vec{\alpha}\times\vec{\beta}) \quad (a+b)|\vec{\beta}|^2=0$$

$$b|\vec{\alpha}|^2 \quad a(\vec{\alpha}\times\vec{\beta}) \quad (a+b)|\vec{\beta}|^2=0$$

$$bc^2 \quad abc\cos A \quad (a+b)b^2=0$$

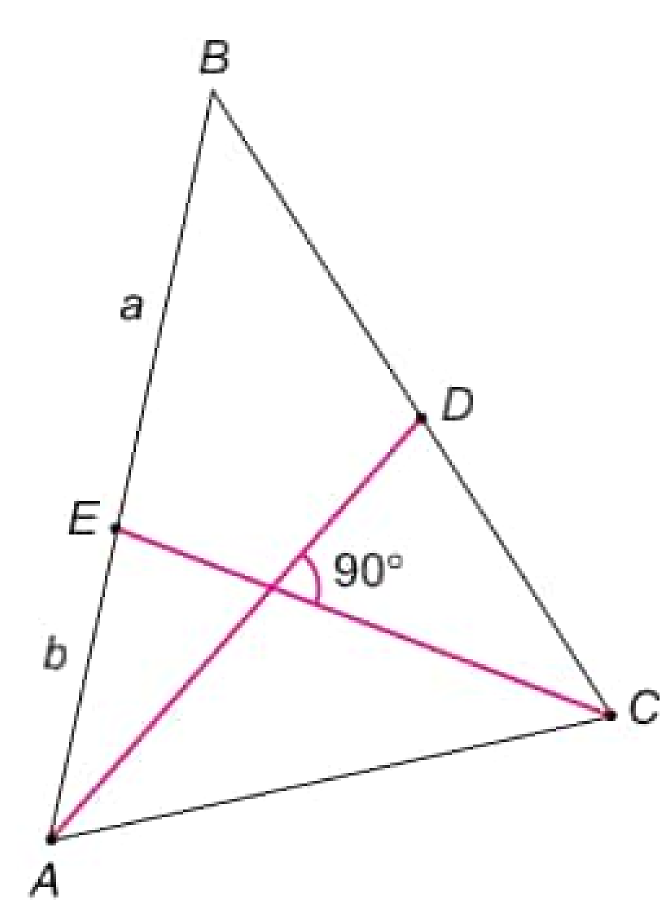

FIGURE 6.44 Single correct choice type question 85.

$$ba^2 \quad ba^2\cos A \quad (a+b)b^2 = 0 \quad (\because\ AB = BC \quad c = a)$$

$$a^2 \quad a^2\cos A \quad ab \quad b^2 = 0$$

$$a^2 \quad a^2\cos A \quad a(2a\cos A) \quad 4a^2\cos^2 A = 0$$

[from Eq. (6.46) and c = a]

$$1 \quad 3\cos A \quad 4\cos^2 A = 0$$

$$4\cos^2 A + 3\cos A \quad 1 = 0$$

$$(4\cos A \quad 1)(\cos A + 1) = 0$$

$$\cos A = \frac{1}{4}$$

But

$$\underline{|A} = \underline{|ACB} \quad (\because\ AB = BC)$$

Therefore

$$\cos C = \frac{1}{4} \quad \text{or} \quad \underline{|ACB} = \mathrm{Cos}^{\ 1}\ \frac{1}{4}\div$$

Answer: (D)

86. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors and $\vec{r}$ be a non-zero vector such that

$$\vec{r} = x(\vec{b} \quad \vec{c}) + y(\vec{c} \quad \vec{a}) + z(\vec{a} \quad \vec{b})$$

where x, y, z are scalars and $[\vec{a}\,\vec{b}\,\vec{c}](x+y+z) = 1$. Then $\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ is equal to

(A) 3 (B) 2 (C) 1 (D) $\frac{1}{3}$

Solution: We have

$$\vec{r} = x(\vec{b} \quad \vec{c}) + y(\vec{c} \quad \vec{a}) + z(\vec{a} \quad \vec{b})$$

Therefore

$$\vec{r}\cdot\vec{a} = x[\vec{a}\,\vec{b}\,\vec{c}]$$

$$\vec{r}\cdot\vec{b} = y[\vec{a}\,\vec{b}\,\vec{c}]$$

$$\vec{r}\cdot\vec{c} = z[\vec{a}\,\vec{b}\,\vec{c}]$$

So

$$\vec{r}\cdot\vec{a} + \vec{r}\cdot\vec{b} + \vec{r}\cdot\vec{c} = (x+y+z)[\vec{a}\,\vec{b}\,\vec{c}] = 1$$

This implies

$$\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c}) = 1$$

Answer: (C)

87. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of magnitudes 1, 2 and 2, respectively, then the value of $|[\vec{a}\,\vec{b}\,\vec{c}]|$ is

(A) 4 (B) 5 (C) 5 (D) 2

Solution: Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors, the parallelopiped formed by them is a rectangular box. Therefore

$$|[\vec{a}\,\vec{b}\,\vec{c}]| = |\vec{a}||\vec{b}||\vec{c}| = 4$$

Answer: (A)

88. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar unit vectors such that

$$\vec{a} \quad (\vec{b} \quad \vec{c}) = \frac{1}{\sqrt{2}}(\vec{b}+\vec{c})$$

and $\vec{b}$, $\vec{c}$ are non-collinear vectors, then the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

Solution: By Theorem 6.46

$$\vec{a} \quad (\vec{b} \quad \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} \quad (\vec{a}\cdot\vec{b})\vec{c}$$

Therefore

$$(\vec{a}\cdot\vec{c})\vec{b} \quad (\vec{a}\cdot\vec{b})\vec{c} = \frac{1}{\sqrt{2}}(\vec{b}+\vec{c})$$

This implies (using Theorem 5.15)

$$\vec{a}\cdot\vec{c} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \vec{a}\cdot\vec{b} = \quad \frac{1}{\sqrt{2}}$$

Therefore

$$\frac{1}{\sqrt{2}} = \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

where $\theta = (\vec{a}, \vec{b})$. Hence

$$\frac{1}{\sqrt{2}} = \cos\theta \quad \theta = \frac{3\pi}{4}$$

Answer: (B)

89. If $\vec{a} \quad (\vec{b} \quad \vec{c}) = (\vec{a} \quad \vec{b}) \quad \vec{c}$ and $\vec{a}, \vec{c}$ are not collinear, then

(A) $(\vec{a} \quad \vec{c}) \quad \vec{b} = \vec{0}$ (B) $(\vec{a} \quad \vec{c})\cdot\vec{b} = 0$

(C) $\vec{a} \quad \vec{b} = \vec{a} \quad \vec{c}$ (D) $\vec{b} \quad \vec{c} = \vec{c} \quad \vec{a} = \vec{a} \quad \vec{b}$

Solution: We have

$$\vec{a} \quad (\vec{b} \quad \vec{c}) = (\vec{a} \quad \vec{b}) \quad \vec{c}$$

This implies

$$(\vec{a}\cdot\vec{c})\vec{b} \quad (\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} \quad (\vec{b}\cdot\vec{c})\vec{a} \quad \text{(see Theorem 6.46)}$$

which further gives

$$(\vec{a}\cdot\vec{b})\vec{c} = (\vec{b}\cdot\vec{c})\vec{a}$$

But $\vec{a}$ and $\vec{c}$ are non-collinear vectors. Therefore

$$\vec{a} \cdot \vec{b} = 0 = \vec{b} \cdot \vec{c}$$

Hence $\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{c}$ and so $\vec{b}$ is parallel to $\vec{a} \quad \vec{c}$. Therefore

$$(\vec{a} \quad \vec{c}) \quad \vec{b} = \vec{0}$$

Answer: (A)

90. In $\quad ABC$ (Figure 6.45), if $|\overline{AC}| = 3$ and $|\overline{BC}| = 4$ and the medians AD and BE are at right angles, then the area of $\quad ABC$ is

(A) 12 (B) $\sqrt{7}$ (C) 7 (D) $\sqrt{11}$

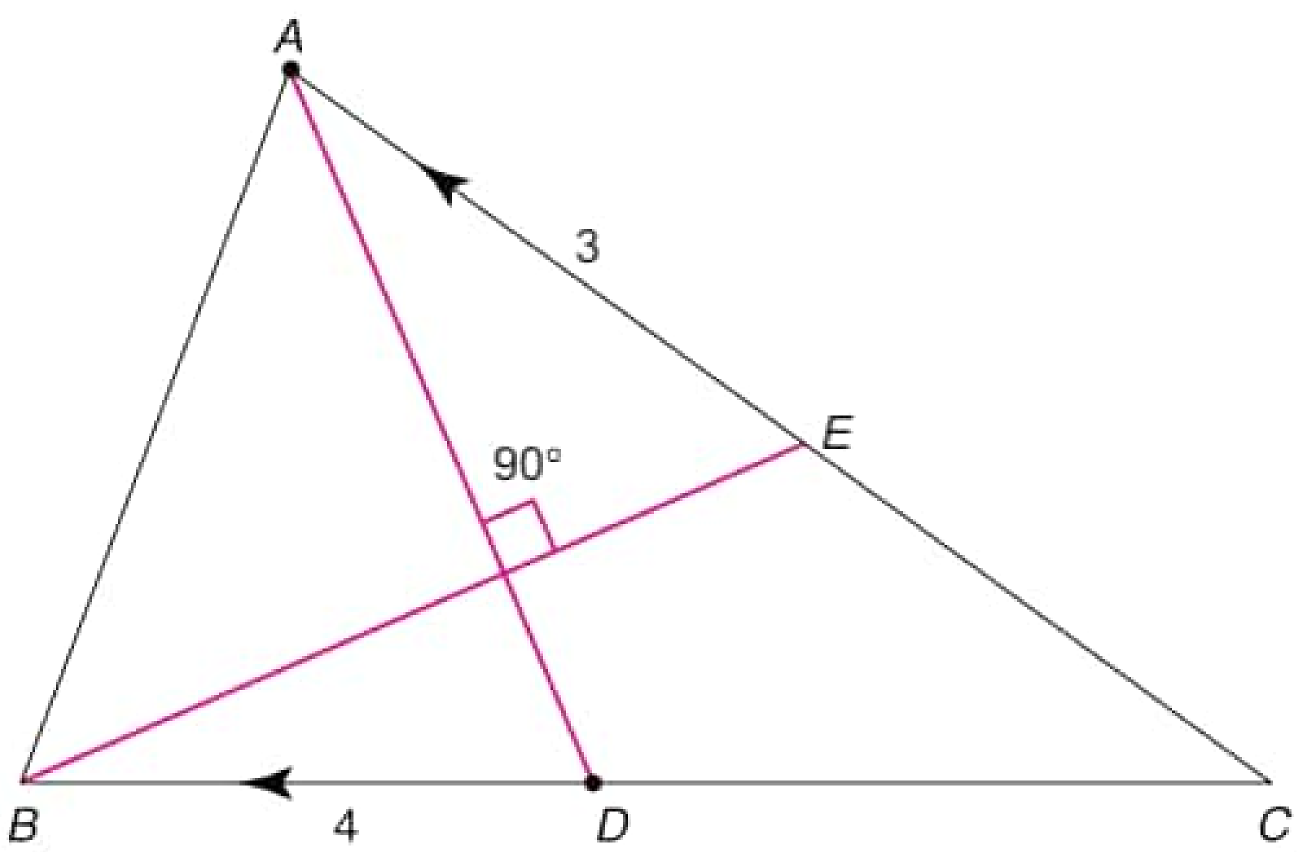

FIGURE 6.45 Single correct choice type question 90.

Solution: Take C as origin and let $\overline{CA} = \vec{a}$ and $\overline{CB} = \vec{b}$ so that $|\vec{a}| = 3, |\vec{b}| = 4$ and $(\vec{a}, \vec{b}) = \underline{|C}$. Now

$$\overline{AD} \cdot \overline{BE} = 0$$

$$\frac{\vec{b}}{2} \quad \vec{a}_{\div} \times \frac{\vec{a}}{2} \quad \vec{b}_{\div} = 0$$

$$(\vec{b} \quad 2\vec{a}) \cdot (\vec{a} \quad 2\vec{b}) = 0$$

$$2|\vec{a}|^2 \quad 2|\vec{b}|^2 + 5(\vec{a} \cdot \vec{b}) = 0$$

$$2(9) \quad 2(16) + 5(3)(4)\cos C = 0$$

$$60 \cos C = 50$$

$$\cos C = \frac{5}{6}$$

Therefore

$$\sin C = \sqrt{1 \quad \frac{25}{36}} = \frac{\sqrt{11}}{6}$$

Now area of $\quad ABC$

$$= \frac{1}{2}|\overline{CA} \quad \overline{CB}| = \frac{1}{2}|\vec{a}||\vec{b}|\sin C$$

$$= \frac{1}{2}(3)(4)\frac{\sqrt{11}}{6}$$

$$= \sqrt{11}$$

Answer: (D)

91. If the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors, then the value of the determinant

$$\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} =$$

(A) 0 (B) 2 (C) 2 (D) 1

Solution: In Quick Look 11, part (2), take $\vec{l} = \vec{a}, \vec{m} = \vec{b}$ and $\vec{n} = \vec{c}$. Then

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{a} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

But $\vec{a}, \vec{b}, \vec{c}$ are coplanar. This implies

$$[\vec{a}\,\vec{b}\,\vec{c}] = 0$$

Direct Solution:
Let

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

$$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

Therefore

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad [\because \det A = \det(A^{\mathrm{T}})]$$

$$= \begin{vmatrix} a_1^2 & a_1b_1 & a_1c_1 \\ b_1\boldsymbol{a}_1 & b_1^2 & b_1c_1 \\ c_1a_1 & c_1b_1 & c_1^2 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{c}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

Answer: (A)

92. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then the value of $[\vec{b}\times\vec{c} \quad \vec{c}\times\vec{a} \quad \vec{a}\times\vec{b}]$ is

(A) 2 (B) 1 (C) 0 (D) $|\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}|$

Solution: We have

$$\begin{aligned}[\vec{b}\times\vec{c} \quad \vec{c}\times\vec{a} \quad \vec{a}\times\vec{b}] &= ((\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}))\cdot(\vec{a}\times\vec{b})\\ &= ([\vec{b}\;\vec{c}\;\vec{a}]\vec{c}-[\vec{c}\;\vec{c}\;\vec{a}]\vec{b})\cdot(\vec{a}\times\vec{b})\\ &= [\vec{b}\;\vec{c}\;\vec{a}][\vec{c}\;\vec{a}\;\vec{b}]-0\\ &= [\vec{a}\;\vec{b}\;\vec{c}]^2 \quad (\because [\vec{a}\;\vec{b}\;\vec{c}]=[\vec{b}\;\vec{c}\;\vec{a}]=[\vec{c}\;\vec{a}\;\vec{b}])\\ &= 0 \quad (\because \vec{a}, \vec{b}, \vec{c} \text{ are coplanar})\end{aligned}$$

For deriving this we have used Theorem 6.48 part (2) and Theorem 6.36.

Answer: (C)

93. In the ΔABC (Figure 6.46), points K and L are taken respectively on the segments AB and BC such that $AK:KB = 1:2$ and $BL:LC = 1:2$. Let P be the point of intersection of the lines AL and CK. If the area of the ΔBCP is 2 sq. units, then the area of the ΔABC in sq. units is

(A) $\frac{7}{4}$ (B) $\frac{7}{2}$ (C) 7 (D) $\sqrt{7}$

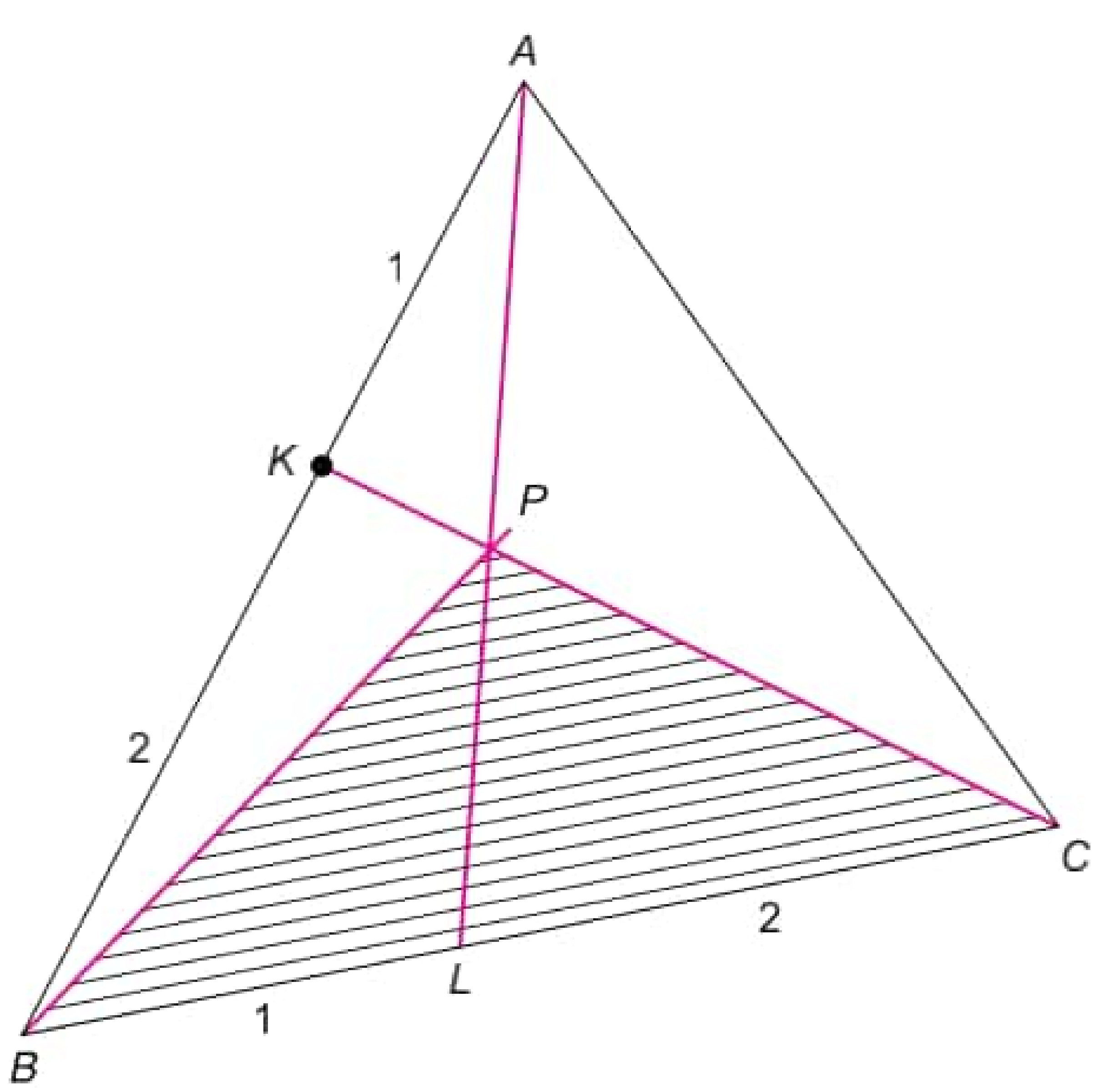

FIGURE 6.46 Single correct choice type question 93.

Solution: Take A as origin and let $\overline{AB}=\vec{\alpha}$, $\overline{AC}=\vec{\beta}$, so that the angle between $\vec{\alpha}$ and $\vec{\beta}$ is A. Note

$$AK:KB = 1:2 \Rightarrow \overline{AK} = \frac{1}{3}\vec{\alpha}$$

$$BL:LC = 1:2 \Rightarrow \overline{AL} = \frac{2\vec{\alpha}+\vec{\beta}}{3}$$

Equation of the line AL is

$$\vec{r} = t\frac{(2\vec{\alpha}+\vec{\beta})}{3}, t\in\mathbb{R} \qquad (6.47)$$

Equation of the line CK is

$$\vec{r} = \frac{s}{3}\vec{\alpha}+(1-s)\vec{\beta} \qquad (6.48)$$

[For Eqs. (6.47) and (6.48) see Theorem 5.30.] Equating the corresponding coefficients of $\vec{\alpha}$ and $\vec{\beta}$ in Eqs. (6.47) and (6.48) we have

$$\frac{2t}{3}=\frac{s}{3} \quad \text{and} \quad \frac{t}{3}=1-s$$

Therefore

$$\frac{s}{6}=\frac{t}{3}=1-s$$

$$\Rightarrow s=\frac{6}{7} \quad \text{and} \quad t=\frac{3}{7}$$

Therefore position vector $\overline{AP}$ of P is

$$\frac{1}{7}(2\vec{\alpha}+\vec{\beta})$$

Hence

$$\overline{PB} = \vec{\alpha}-\frac{1}{7}(2\vec{\alpha}+\vec{\beta}) = \frac{1}{7}(5\vec{\alpha}-\vec{\beta})$$

and

$$\overline{PC} = \vec{\beta}-\frac{1}{7}(2\vec{\alpha}+\vec{\beta}) = \frac{1}{7}(6\vec{\beta}-2\vec{\alpha}) = \frac{2}{7}(3\vec{\beta}-\vec{\alpha})$$

Now

$$\begin{aligned}2 &= \text{Area of } \Delta ABC\\ &= \frac{1}{2}|\overline{PB}\times\overline{PC}|\\ &= \frac{1}{2}\times\frac{2}{49}|(5\vec{\alpha}-\vec{\beta})\times(3\vec{\beta}-\vec{\alpha})|\\ &= \frac{1}{49}|15\vec{\alpha}\times\vec{\beta}-\vec{\alpha}\times\vec{\beta}| = \frac{2}{7}|\vec{\alpha}\times\vec{\beta}|\end{aligned}$$

Therefore $|\vec{\alpha}\times\vec{\beta}|=7$. Now

$$\text{Area of } \Delta ABC = \frac{1}{2}|\overline{AB}\times\overline{AC}| = \frac{1}{2}|\vec{\alpha}\times\vec{\beta}| = \frac{7}{2}$$

Answer: (B)

94. Let $\vec{a}=3\vec{i}-6\vec{j}-\vec{k}$, $\vec{b}=\vec{i}+4\vec{j}-3\vec{k}$ and $\vec{c}=3\vec{i}-4\vec{j}-12\vec{k}$ then the projection of $\vec{a}\times\vec{b}$ on $\vec{c}$ in vector form is

(A) $\dfrac{14\vec{c}}{13}$ (B) $\dfrac{14\vec{c}}{13}$ (C) $\dfrac{12\vec{c}}{13}$ (D) $\dfrac{15\vec{c}}{13}$

Solution: Projection vector of $\vec{\beta}$ on $\vec{\alpha}$ is [see Theorem 6.1 part (2)]

$$\frac{\vec{\beta}\times\vec{\alpha}}{|\vec{\alpha}|^2}\div\vec{\alpha}$$

Therefore vector projection of $\vec{a}\quad\vec{b}$ on $\vec{c}$ is

$$\frac{(\vec{a}\quad\vec{b})\times\vec{c}}{|\vec{c}|^2}(\vec{c})=\frac{1}{169}\begin{vmatrix}3 & 6 & 1\\ 1 & 4 & 3\\ 3 & 4 & 12\end{vmatrix}\vec{c}$$

$$=\frac{1}{169}[3(\ 48\quad 12)+6(\ 12+9)\quad 1(\ 4\quad 12)]\vec{c}$$

$$=\frac{1}{169}(\ 180\quad 18+16)\vec{c}$$

$$=\frac{14\vec{c}}{13}$$

Answer: (B)

95. If $|\vec{a}|=\sqrt{6}$ and $|\vec{b}|=\sqrt{5}$, then $[(\vec{a}\quad\vec{b})\quad\vec{b}]\quad\vec{b}$ is equal to

(A) $5(\vec{b}\quad\vec{a})$ (B) $5(\vec{a}\quad\vec{b})$

(C) $6(\vec{a}\quad\vec{b})$ (D) $6(\vec{b}\quad\vec{a})$

Solution: We have

$$[(\vec{a}\quad\vec{b})\quad\vec{b}]\quad\vec{b}=[(\vec{a}\quad\vec{b})\times\vec{b}]\vec{b}\quad(\vec{b}\times\vec{b})(\vec{a}\quad\vec{b})$$

$$=0(\vec{b})\quad 5(\vec{a}\quad\boldsymbol{b})$$

$$=5(\vec{b}\quad\vec{a})$$

Answer: (A)

96. Let the angle between $\vec{a}$ and $\vec{b}$ be 120° and $|\vec{a}|=1$, $|\vec{b}|=2$. Then

$$|(\vec{a}+3\vec{b})\quad(3\vec{a}\quad\vec{b})|^2=$$

(A) 300 (B) 225 (C) 275 (D) 325

Solution: We have

$$|(\vec{a}+3\vec{b})\quad(3\vec{a}\quad\vec{b})|^2=|\ (\vec{a}\quad\vec{b})+9(\vec{b}\quad\vec{a})|^2$$

$$=10^2|\vec{a}\quad\vec{b}|^2$$

$$=100|\vec{a}|^2|\vec{b}|^2\sin^2 120^\circ$$

$$=100\quad(1)(4)\ \frac{3}{4}\div$$

$$=300$$

Answer: (A)

97. If $(\vec{a}\quad\vec{b})\quad\vec{c}=\ 5\vec{a}+4\vec{b}$ and $\vec{a}\times\vec{b}=3$, where $\vec{a},\vec{b},\vec{c}$ are non-coplanar, then $\vec{a}\quad(\vec{b}\quad\vec{c})$ is equal to

(A) $4\vec{b}\quad 3\vec{c}$ (B) $3\vec{b}\quad 5\vec{c}$

(C) $3\vec{c}\quad 4\vec{b}$ (D) $5\vec{b}\quad 3\vec{c}$

Solution: We have

$$5\vec{a}+4\vec{b}=(\vec{a}\quad\vec{b})\quad\vec{c}=(\vec{a}\times\vec{c})\vec{b}\quad(\vec{b}\times\vec{c})\vec{a}$$

Therefore

$$\vec{a}\times\vec{c}=4\quad\text{and}\quad\vec{b}\times\vec{c}=5$$

Now

$$\vec{a}\quad(\vec{b}\quad\vec{c})=(\vec{a}\times\vec{c})\vec{b}\quad(\vec{a}\times\vec{b})\vec{c}$$

$$=4\vec{b}\quad 3\vec{c}$$

Answer: (A)

98. Let $\vec{a}=\vec{i}+\vec{j}+2\vec{k}$ and $\vec{b}=2\vec{i}\quad\vec{k}$. The point of intersection of the lines represented by the equations $\vec{r}\quad\vec{a}=\vec{b}\quad\vec{a}$ and $\vec{r}\quad\vec{b}=\vec{a}\quad\vec{b}$ is

(A) $3\vec{i}\quad\vec{j}+\vec{k}$ (B) $3\vec{i}+\vec{j}+\vec{k}$

(C) $\vec{i}\quad\vec{j}\quad\vec{k}$ (D) $\quad\vec{i}+\vec{j}+\vec{k}$

Solution: We have

$$\vec{r}\quad\vec{a}=\vec{b}\quad\vec{a}$$

$$(\vec{r}\quad\vec{b})\quad\vec{a}=\vec{0}$$

$$\vec{r}\quad\vec{b}\text{ is collinear with }\vec{a}$$

$$\vec{r}\quad\vec{b}=t\vec{a},t\quad\mathbb{R}$$

$$\vec{r}=\vec{b}+t\vec{a}\tag{6.49}$$

Similarly

$$\vec{r}\quad\vec{b}=\vec{a}\quad\vec{b}\qquad\vec{r}=\vec{a}+s\vec{b}\tag{6.50}$$

where $s\quad\mathbb{R}$. The two lines given by Eqs. (6.49) and (6.50) intersect if $t=s=1$. Therefore the point of intersection is

$$\vec{a}+\vec{b}=3\vec{i}+\vec{j}+\vec{k}$$

Answer: (B)

99. $\vec{a}\times[(\vec{b}+\vec{c})\quad(\vec{a}+\vec{b}+\vec{c})]$ is equal to

(A) 0 (B) $2[\vec{a}\,\vec{b}\,\vec{c}]$

(C) $[\vec{a}\,\vec{b}\,\vec{c}]$ (D) $\quad[\vec{a}\,\vec{b}\,\vec{c}]$

Solution: We have

$$\vec{a}\times[(\vec{b}+\vec{c})\quad(\vec{a}+\vec{b}+\vec{c})]=\vec{a}\times[(\vec{b}+\vec{c})\quad\vec{a}]$$

$$=\vec{a}\times[\vec{b}\quad\vec{a}+\vec{c}\quad\vec{a}]$$

$$=[\vec{a}\,\vec{b}\,\vec{a}]+[\vec{a}\,\vec{c}\,\vec{a}]$$

$$=0+0=0$$

Answer: (A)

100. $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors. Then

$$(\vec{a}+\vec{b}+\vec{c})\cdot[(\vec{a}+\vec{b})\quad(\vec{a}+\vec{c})]=$$

(A) 0 (B) $[\vec{a}\,\vec{b}\,\vec{c}]$
(C) $2[\vec{a}\,\vec{b}\,\vec{c}]$ (D) $\quad[\vec{a}\,\vec{b}\,\vec{c}]$

Solution: Given scalar is

$$(\vec{a}+\vec{b}+\vec{c})\cdot[\;\vec{c}\quad\vec{a}\quad\vec{a}\quad\vec{b}+\vec{b}\quad\vec{c}]=[\vec{a}\,\vec{b}\,\vec{c}]\quad[\vec{b}\,\vec{c}\,\vec{a}]\quad[\vec{c}\,\vec{a}\,\vec{b}]$$

$$=\quad[\vec{a}\,\vec{b}\,\vec{c}]$$

Answer: (D)

101. Let $\vec{a}=x\vec{i}+y\vec{j}+z\vec{k}$ be a vector of magnitude $2\sqrt{3}$ and makes an obtuse angle with y-axis. If $\vec{a}$ makes equal angles with the vectors $\vec{b}=y\vec{i}\quad 2z\vec{j}+3x\vec{k}$ and $\vec{c}=2z\vec{i}+3x\vec{j}\quad y\vec{k}$ and is perpendicular to $\vec{d}=\vec{i}\quad\vec{j}+2\vec{k}$, then $\vec{a}$ is equal to

(A) $2\vec{i}\quad 2\vec{j}\quad 2\vec{k}$ (B) $2\vec{i}+2\vec{j}\quad 2\vec{k}$
(C) $2\vec{i}\quad 2\vec{j}+2\vec{k}$ (D) $2\vec{i}+2\vec{j}+2\vec{k}$

Solution: We have $|\vec{a}|=2\sqrt{3}$. Therefore

$$x^2+y^2+z^2=12 \tag{6.51}$$

Since $|\vec{b}|=|\vec{c}|$ and $\vec{a}$ makes equal angles with $\vec{b}$ and $\vec{c}$, we have

$$\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$$

Therefore

$$xy\quad 2yz+3zx=2zx+3xy\quad yz$$
$$2xy+yz\quad zx=0 \tag{6.52}$$

Also

$$\vec{a}\cdot\vec{d}=0\qquad x\quad y+2z=0 \tag{6.53}$$

Also $\vec{a}$ makes an obtuse angle with y-axis. This implies

$$y<0 \tag{6.54}$$

From Eqs. (6.52) and (6.53) we have

$$2x(x+2z)+(x+2z)z\quad zx=0$$
$$2x^2+4zx+2z^2=0$$
$$(z+x)^2=0$$
$$z=\quad x$$

From Eq. (6.53) we have $y=\quad x$. Therefore, from Eq. (6.51)

$$x^2+x^2+x^2=12\qquad x^2=4$$
$$x=\pm 2$$

Since from Eq. (6.54) $y<0$, we have $x=2$, $y=\quad 2$ and $z=\quad 2$. Therefore

$$\vec{a}=2\vec{i}\quad 2\vec{j}\quad 2\vec{k}$$

Answer: (A)

102. If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes and $\vec{x}$ is a vector satisfying the equation

$$\vec{a}\quad[(\vec{x}\quad\vec{b})\vec{a}]+\vec{b}\quad[(\vec{x}\quad\vec{c})\quad\vec{b}]+\vec{c}\quad[(\vec{x}\quad\vec{a})\quad\vec{c}]=\vec{0}$$

then $\vec{x}$ equals

(A) $\frac{1}{2}(\vec{a}+\vec{b}\quad 2\vec{c})$ (B) $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c})$
(C) $\frac{1}{3}(\vec{a}+\vec{b}+\vec{c})$ (D) $\frac{1}{3}(2\vec{a}+\vec{b}\quad\vec{c})$

Solution: We have

$$\vec{a}\quad[(\vec{x}\quad\vec{b})\quad\vec{a}]=(\vec{a}\cdot\vec{a})(\vec{x}\quad\vec{b})\quad\vec{a}\cdot(\vec{x}\quad\vec{b})\vec{a}$$
$$=|\vec{a}|^2(\vec{x}\quad\vec{b})\quad(\vec{a}\cdot\vec{x}\quad\vec{a}\cdot\vec{b})\vec{a} \tag{6.55a}$$

Similarly

$$\vec{b}\quad[(\vec{x}\quad\vec{c})\quad\vec{b}]=|\vec{b}|^2(\vec{x}\quad\vec{c})\quad(\vec{b}\cdot\vec{x}\quad\vec{b}\cdot\vec{c})\vec{b} \tag{6.54b}$$
$$\vec{c}\quad[(\vec{x}\quad\vec{a})\quad\vec{c}]=|\vec{c}|^2(\vec{x}\quad\vec{a})\quad(\vec{c}\cdot\vec{x}\quad\vec{c}\cdot\vec{a})\vec{c} \tag{6.55c}$$

Let $|\vec{a}|=|\vec{b}|=|\vec{c}|=\lambda$. Since $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$, by adding Eqs. (6.55a), (6.55b) and (6.55c), we have

$$3\lambda^2\vec{x}\quad\lambda^2(\vec{a}+\vec{b}+\vec{c})\quad[(\vec{a}\cdot\vec{x})\vec{a}+(\vec{b}\cdot\vec{x})\vec{b}+(\vec{c}\cdot\vec{x})\vec{c}]=0 \tag{6.56}$$

Suppose

$$\vec{x}=a_1\vec{a}+b_1\vec{b}+c_1\vec{c}$$

(This is possible, since $\vec{a},\vec{b},\vec{c}$ are non-coplanar.) Therefore

$$\vec{a}\cdot\vec{x}=a_1|\vec{a}|^2=\lambda^2a_1$$
$$\vec{b}\cdot\vec{x}=\lambda^2b_1$$
$$c\cdot\vec{x}=\lambda^2c_1$$

Substituting these values in Eq. (6.56) we have

$$3\lambda^2\vec{x}\quad\lambda^2(\vec{a}+\vec{b}+\vec{c})\quad\lambda^2\vec{x}=0$$
$$2\vec{x}=\vec{a}+\vec{b}+\vec{c}$$
$$\vec{x}=\frac{1}{2}(\vec{a}+\vec{b}+\vec{c})$$

Note: Since $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of same magnitude, one can take $\vec{a}=p\vec{i}$, $\vec{b}=p\vec{j}$ and $\vec{c}=p\vec{k}$ and proceed.

Answer: (B)

103. Let the vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ be such that $(\vec{a}\quad\vec{b})\quad(\vec{c}\quad\vec{d})=\vec{0}$. Let P_1 and P_2 be the planes determined by the pairs of vectors $\vec{a},\vec{b}$ and $\vec{c},\vec{d}$ respectively. Then the angle between P_1 and P_2 is

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution: $(\vec{a} \quad \vec{b}) \quad (\vec{c} \quad \vec{d}) = \vec{0} \quad \vec{a} \quad \vec{b}$ and $\vec{c} \quad \vec{d}$ are parallel. But $\vec{a} \quad \vec{b}$ and $\vec{c} \quad \vec{d}$ are normals to the planes P_1 and P_2, respectively. Also, since $(\vec{a}, \vec{b}, \vec{a} \quad \vec{b})$ and $(\vec{c}, \vec{d}, \vec{c} \quad \vec{d})$ are right-handed systems, it follows that $\vec{a} \quad \vec{b}$ and $\vec{c} \quad \vec{d}$ are like vectors. Since the angle between P_1 and P_2 is the angle between their normals (see Definition 6.3) it follows that the angle between P_1 and P_2 is 0.

Answer: (A)

104. A plane P_1 is parallel to the vectors $\vec{a} = 2\vec{j} + 3\vec{k}$ and $\vec{b} = 4\vec{j} \quad 3\vec{k}$ and a plane P_2 is parallel to the vectors $\vec{c} = \vec{j} \quad \vec{k}$ and $\vec{d} = 3\vec{i} + 3\vec{k}$. Vector $\vec{A}$ is parallel to the line of intersection of P_1 and P_2. Then the angle θ between $\vec{A}$ and the vector $2\vec{i} + \vec{j} \quad 2\vec{k}$, $0 \quad \theta \quad \pi/2$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{3}$

Solution: Normal to $P_1 = \vec{a} \quad \vec{b} = \quad 18\vec{i}$ and normal to $P_2 = \vec{c} \quad \vec{d} = 3\vec{i} \quad 3\vec{j} \quad 3\vec{k}$. By hypothesis, $\vec{A}$ is perpendicular to both $\vec{a} \quad \vec{b}$ and $\vec{c} \quad \vec{d}$ and hence we can take

$$\vec{A} = (\vec{a} \quad \vec{b}) \quad (\vec{c} \quad \vec{d}) = 54(\quad \vec{j} + \vec{k})$$

Therefore

$$\cos\theta = \frac{54(\vec{j} \quad \vec{k}) \cdot (2\vec{i} + \vec{j} \quad 2\vec{k})}{54\sqrt{2} \quad 3} = \frac{54 \quad 3}{54\sqrt{2} \quad 3} = \frac{1}{\sqrt{2}}$$

and so

$$\theta = \frac{\pi}{4}$$

Note: If $\vec{A} = (\vec{c} \quad \vec{d}) \quad (\vec{a} \quad \vec{b})$, then $\theta = \frac{3\pi}{4}$.

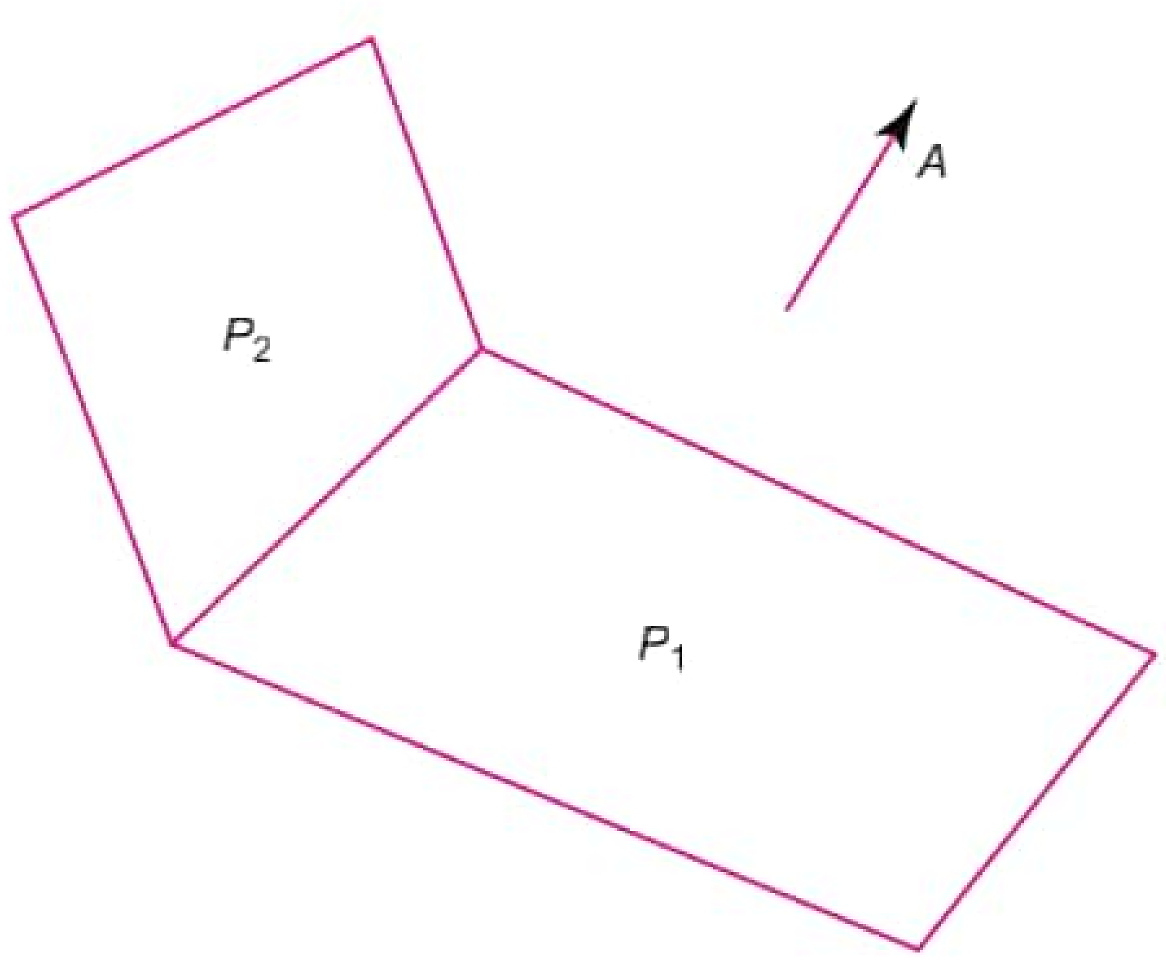

FIGURE 6.47 Single correct choice type question 104.

Answer: (B)

105. In ABC (Figure 6.48), D is the mid-point of AC and E is a point on the side BC such that $BE:EC = 2:1$. If AE and BD intersect in P, then the ratio $AP:PE$ is

(A) 2:1 (B) 2:3 (C) 3:2 (D) 1:2

Solution: Take A as origin, let $\overline{AB} = \vec{b}$ and $\overline{AC} = \vec{c}$. Therefore

$$\overline{AD} = \frac{1}{2}\vec{c}$$

and
$$\overline{AE} = \frac{\vec{b} + 2\vec{c}}{3}$$

Suppose the ratio $AP:PE = \lambda:1$ and the ratio $BP:PD = \mu:1$. Then

$$\frac{\vec{b} + \mu(\vec{c}/2)}{\mu + 1} = \frac{\lambda(\vec{b} + 2\vec{c})}{3(\lambda + 1)}$$

$\vec{b}$ and $\vec{c}$ are non-collinear vectors. This implies

$$\frac{1}{\mu + 1} = \frac{\lambda}{3(\lambda + 1)}$$

and
$$\frac{\mu}{2(\mu + 1)} = \frac{2\lambda}{3(\lambda + 1)}$$

Solving these we get that

$$\mu = 4 \quad \text{and} \quad \lambda = \frac{3}{2}$$

Therefore

$$AP:PE = 3:2$$

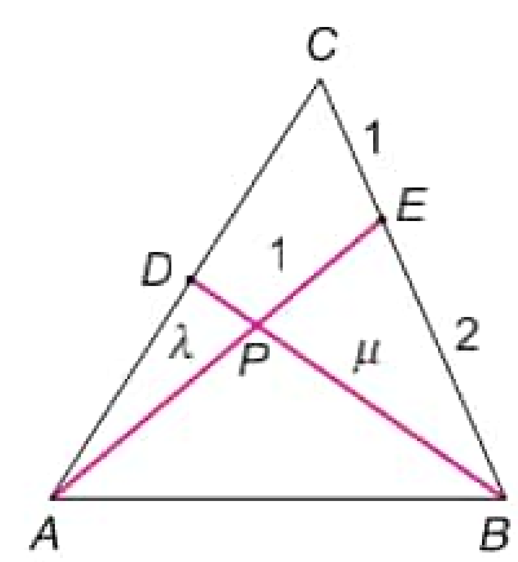

FIGURE 6.48 Single correct choice type question 105.

Answer: (C)

106. If $\vec{a}, \vec{b}, \vec{c}$ are unit coplanar vectors, then the scalar triple product $[2\vec{a} \quad \vec{b} \; 2\vec{b} \quad \vec{c} \; 2\vec{c} \quad \vec{a}]$ equals

(A) $\sqrt{2}$ (B) 1 (C) $\sqrt{3}$ (D) 0

Solution: The given scalar triple product is

$$= (2\vec{a} \quad \vec{b}) \cdot [(2\vec{b} \quad \vec{c}) \quad (2\vec{c} \quad \vec{a})]$$
$$= (2\vec{a} \quad \vec{b}) \cdot [4(\vec{b} \quad \vec{c}) + 2(\vec{a} \quad \vec{b}) + (\vec{c} \quad \vec{a})]$$
$$= 8[\vec{a}\,\vec{b}\,\vec{c}] \quad [\vec{b}\,\vec{c}\,\vec{a}]$$
$$= 7[\vec{a}\,\vec{b}\,\vec{c}]$$
$$= 7(0) = 0$$

Answer: (D)

107. Let $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, $\vec{b} = \vec{i} + 5\vec{j} + 2\vec{k}$ and $\vec{c} = 3\vec{i} + 15\vec{j} + 6\vec{k}$. Then

$$\begin{vmatrix} \vec{a}\times\vec{a} & \vec{a}\times\vec{b} & \vec{a}\times\vec{c} \\ \vec{b}\times\vec{a} & \vec{b}\times\vec{b} & \vec{b}\times\vec{c} \\ \vec{c}\times\vec{a} & \vec{c}\times\vec{b} & \vec{c}\times\vec{c} \end{vmatrix} =$$

(A) 0 (B) 8 (C) 4 (D) 6

Solution: Under Quick Look 11, part (2), put $\vec{l} = \vec{a}, \vec{m} = \vec{b}$ and $\vec{n} = \vec{c}$. Then the given determinant value is $[\vec{a}\,\vec{b}\,\vec{c}]^2$. But by hypothesis $\vec{c} = 3\vec{b}$, so that $[\vec{a}\,\vec{b}\,\vec{c}]$ is zero.

Answer: (A)

108. Let A, B, C and D be any four points in the space. Then the minimum value of $(AC)^2 + (BD)^2 + (AD)^2 + (BC)^2$ is

(A) $(AD)^2 + (BC)^2$
(B) $(AB)^2 + (CD)^2$
(C) $(AC)^2 + (BD)^2$
(D) no minimum value

Solution: Take A as origin and let $\overline{AB} = \vec{b}, \overline{AC} = \vec{c}$ and $\overline{AD} = \vec{d}$. Then

$$\begin{aligned}(AC)^2 + (BD)^2 + (AD)^2 + (BC)^2 &= |\vec{c}|^2 + |\vec{d} - \vec{b}|^2 + |\vec{d}|^2 + |\vec{c} - \vec{b}|^2 \\ &= |\vec{c}|^2 + |\vec{b}|^2 + |\vec{d}|^2 - 2(\vec{b}\times\vec{d}) + |\vec{d}|^2 + |\vec{b}|^2 + |\vec{c}|^2 - 2(\vec{b}\times\vec{c}) \\ &= |\vec{b} - \vec{c} - \vec{d}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{d}|^2 - 2(\vec{c}\times\vec{d}) \\ &= |\vec{b} - \vec{c} - \vec{d}|^2 + |\vec{b}|^2 + |\vec{c} - \vec{d}|^2 \geq |\vec{b}|^2 + |\vec{c} - \vec{d}|^2\end{aligned}$$

where the equality occurs if and only if $\vec{b} = \vec{c} + \vec{d}$ which is equivalent to $\vec{b} - \vec{c} = \vec{d}$ or $\overline{CB} = \overline{AD}$. Hence the minimum value is

$$|\vec{b}|^2 + |\vec{c} - \vec{d}|^2 = (AB)^2 + (CD)^2$$

Answer: (B)

109. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be non-coplanar unit vectors and the angles $(\vec{u}, \vec{v}) = \alpha, (\vec{v}, \vec{w}) = \beta$ and $(\vec{w}, \vec{u}) = \gamma$. Suppose $\vec{a}, \vec{b}$ and $\vec{c}$ be the unit vectors along the bisectors of α, β and γ respectively. Then the value of the scalar triple product

$$[\vec{a} - \vec{b} \quad \vec{b} - \vec{c} \quad \vec{c} - \vec{a}] = \frac{1}{K}[\vec{u}\,\vec{v}\,\vec{w}]^2 \sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$$

where the value of K is

(A) 2 (B) 4 (C) 8 (D) 16

Solution: As per Theorem 5.33

$$\vec{a} = \frac{\vec{u} + \vec{v}}{|\vec{u} + \vec{v}|}$$

$$\vec{b} = \frac{\vec{v} + \vec{w}}{|\vec{v} + \vec{w}|}$$

$$\vec{c} = \frac{\vec{w} + \vec{u}}{|\vec{w} + \vec{u}|}$$

Now

$$\begin{aligned}[\vec{a} - \vec{b}\ \ \vec{b} - \vec{c}\ \ \vec{c} - \vec{a}] &= (\vec{a} - \vec{b})\times[(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})] \\ &= (\vec{a} - \vec{b})\times([\vec{b}\,\vec{c}\,\vec{a}]\vec{c} - [\vec{b}\,\vec{c}\,\vec{c}]\vec{c}) \\ &\quad \text{(by Theorem 6.48)} \\ &= [(\vec{a} - \vec{b})\times\vec{c}][\vec{b}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2\end{aligned}$$

Substituting the values of $\vec{a}, \vec{b}$ and $\vec{c}$ we have

$$\begin{aligned}&[\vec{a} - \vec{b} \quad \vec{b} - \vec{c} \quad \vec{c} - \vec{a}] \\ &= \frac{1}{|\vec{u} + \vec{v}|^2|\vec{v} + \vec{w}|^2|\vec{w} + \vec{u}|^2}[\vec{u} + \vec{v} \quad \vec{v} + \vec{w} \quad \vec{w} + \vec{u}] \\ &= \frac{1}{4\cos^2\frac{\alpha}{2} \div 4\cos^2\frac{\beta}{2} \div 4\cos^2\frac{\gamma}{2} \div}(2[\vec{u}\,\vec{v}\,\vec{w}])^2 \\ &= \frac{1}{16\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}}[\vec{u}\,\vec{v}\,\vec{w}]^2\end{aligned}$$

Answer: (D)

110. In a Quadrilateral $ABCD$, if $(AB)^2 + (CD)^2 = (BC)^2 + (AD)^2$ then the angle between the diagonals AC and BD is

(A) 120° (B) 90° (C) 60° (D) 45°

Solution: Take A as origin and let $\overline{AB} = \vec{b}, \overline{AC} = \vec{c}$ and $\overline{AD} = \vec{d}$. Now

$$(AB)^2 + (CD)^2 = (BC)^2 + (AD)^2$$

$$|\vec{b}|^2 + |\vec{d} - \vec{c}|^2 = |\vec{c} - \vec{b}|^2 + |\vec{d}|^2$$

$$\vec{c}\times\vec{d} = \vec{c}\times\vec{b} \tag{6.57}$$

Again

$$\begin{aligned}\overline{AC}\times\overline{BD} &= \vec{c}\times(\vec{d} - \vec{b}) \\ &= \vec{c}\times\vec{d} - \vec{c}\times\vec{b} \\ &= 0 \quad \text{[by Eq. (6.57)]}\end{aligned}$$

Therefore AC and BD are at right angles.

Answer: (B)

Multiple Correct Choice Type Questions

1. Let $\vec{a} = 2\vec{i} + \vec{j} \quad \vec{k}, \vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{c} = \quad 3\vec{i} \quad \vec{j} + \vec{k}$. Then

(A) the volume of the parallelopiped whose coterminus edges are $\vec{a}, \vec{b}, \vec{c}$ is 5 cubic units.

(B) $|\vec{a} \quad (\vec{b} \quad \vec{c})| = 5\sqrt{5}$

(C) area of the face with $\vec{a}, \vec{b}$ as adjacent sides is $\sqrt{83}$

(D) $[\vec{a} \quad \vec{b}\,\vec{b} \quad \vec{c}\,\vec{c} \quad \vec{a}] = 25$

Solution:

(A) Volume of the parallelopiped $= |[\vec{a}\,\vec{b}\,\vec{c}]|$ where

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= 2(2+3) \quad 1(1+9) \quad 1(\quad 1+6)$$

$$= \quad 5$$

Therefore

$$|[\vec{a}\,\vec{b}\,\vec{c}]| = 5$$

Hence (A) is correct.

(B) We have

$$\vec{a} \quad (\vec{b} \quad \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} \quad (\vec{a}\cdot\vec{b})\vec{c}$$

$$= (\quad 6 \quad 1 \quad 1)(\vec{i} + 2\vec{j} + 3\vec{k}) \quad (2+2 \quad 3)(\quad 3\vec{i} \quad \vec{j} + \vec{k})$$

$$= \quad 8(\vec{i} + 2\vec{j} + 3\vec{k}) \quad (\quad 3\vec{i} \quad \vec{j} + \vec{k})$$

$$= \quad 5\vec{i} \quad 15\vec{j} \quad 25\vec{k}$$

$$= \quad 5(\vec{i} + 3\vec{j} + 5\vec{k})$$

Therefore

$$|\vec{a} \quad (\vec{b} \quad \vec{c})| = 5\sqrt{1+9+25} = 5\sqrt{35}$$

Hence (B) is not correct.

(C) Area of the parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides $= |\vec{a} \quad \vec{b}|$. Now

$$\vec{a} \quad \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 5\vec{i} \quad 7\vec{j} + 3\vec{k}$$

Therefore

$$|\vec{a} \quad \vec{b}| = \sqrt{25+49+9} = \sqrt{83}$$

Hence (C) is correct.

(D) We have

$$[\vec{a} \quad \vec{b} \quad \vec{b} \quad \vec{c} \quad \vec{c} \quad \vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 = (\quad 5)^2 = 25$$

So (D) is correct.

Answers: (A), (C), (D)

2. Let λ be a real scalar and

$$\vec{a} = \lambda\vec{i} + 2\lambda\vec{j} \quad 3\lambda\vec{k}$$

$$\vec{b} = (2\lambda+1)\vec{i} + (2\lambda+3)\vec{j} + (\lambda+1)\vec{k}$$

$$\vec{c} = (3\lambda+5)\vec{i} + (\lambda+5)\vec{j} + (\lambda+2)\vec{k}$$

Then $\vec{a}, \vec{b}, \vec{c}$ are coplanar

(A) for exactly one real values of λ

(B) for exactly two real values of λ

(C) for exactly three real values of λ

(D) for no non-zero real value of λ

Solution: The vectors are coplanar if and only if

$$= \begin{vmatrix} \lambda & 2\lambda & 3\lambda \\ 2\lambda+1 & 2\lambda+3 & \lambda+1 \\ 3\lambda+5 & \lambda+5 & \lambda+2 \end{vmatrix} = 0$$

The operations $C_2 \quad 2C_1$ and $C_3 + 3C_1$ imply

$$= \begin{vmatrix} \lambda & 0 & 0 \\ 2\lambda+1 & 2\lambda+1 & 7\lambda+4 \\ 3\lambda+5 & 5\lambda \quad 5 & 10\lambda+17 \end{vmatrix}$$

$$= \lambda[(\quad 2\lambda+1)(10\lambda+17) + 5(\lambda+1)(7\lambda+4)]$$

$$= \lambda[\quad 20\lambda^2 \quad 24\lambda + 17 + 35\lambda^2 + 55\lambda + 20]$$

$$= \lambda(15\lambda^2 + 31\lambda + 37)$$

Clearly $\lambda = 0 \qquad = 0$ and the vectors are coplanar. In this case $\vec{a} = \vec{0}$ which is coplanar with any two vectors. Now

$$15\lambda^2 + 31\lambda + 37 \quad 0$$

for all $\lambda \quad 0$ because its discriminant is negative. Therefore (A) and (D) are correct.

Answers: (A), (D)

3. Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{b}$ is a vector such that $\vec{a}\cdot\vec{b} = 1$ and $\vec{a} \quad \vec{b} = \vec{j} \quad \vec{k}$. Then

(A) $\vec{b} = 2\vec{i}$

(B) $\vec{b} = \vec{i}$

(C) $(\vec{a} \quad \vec{b}) \quad \vec{b} = \quad (\vec{j} + \vec{k})$

(D) $(\vec{a} \quad \vec{b}) \quad \vec{b} = \vec{k} \quad \vec{j}$

Solution: We have

$$\vec{j}-\vec{k}=\vec{a}\times\vec{b}\Rightarrow\vec{a}\times(\vec{j}-\vec{k})=\vec{a}\times(\vec{a}\times\vec{b})$$
$$\Rightarrow(\vec{i}+\vec{j}+\vec{k})\times(\vec{j}-\vec{k})=(\vec{a}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{a})\vec{b}$$
$$\Rightarrow -2\vec{i}+\vec{j}+\vec{k}=\vec{a}-3\vec{b}$$
$$\Rightarrow\vec{b}=\vec{i}$$

Hence (B) is correct.
Also

$$(\vec{a}\times\vec{b})\times\vec{b}=(\vec{a}\times\vec{i})\times\vec{i}$$
$$=(\vec{a}\cdot\vec{i})\vec{i}-(\vec{i}\cdot\vec{i})\vec{a}$$
$$=\vec{i}-\vec{a}=-\vec{j}-\vec{k}$$

So (C) is correct.

Answers: (B), (C)

4. Let $\vec{a},\vec{b}$ and $\vec{c}$ be the position vectors of three non-collinear points A, B and C, respectively, and $\vec{p}=\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$. Then

(A) $\vec{p}$ is perpendicular to the plane of ΔABC

(B) $|\vec{p}|$ is twice the area of ΔABC

(C) $[\vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\ \vec{a}\times\vec{b}]=2[\vec{a}\ \vec{b}\ \vec{c}]$

(D) $\vec{a}\cdot\vec{p}+\vec{b}\cdot\vec{p}+\vec{c}\cdot\vec{p}=3[\vec{a}\ \vec{b}\ \vec{c}]$

Solution: We have $\overline{AB}=\vec{b}-\vec{a}$, $\overline{AC}=\vec{c}-\vec{a}$. Then

$$\overline{AB}\times\overline{AC}=(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})$$
$$=\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}$$
$$=\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$$

But $\overline{AB}\times\overline{AC}$ is perpendicular to the plane of ΔABC. Therefore $\vec{p}$ is perpendicular the plane of ΔABC. Hence (A) is correct.

Also $(1/2)|\overline{AB}\times\overline{AC}|$ is the area of ΔABC. So (B) is correct. Again

$$[\vec{b}\times\vec{c}\quad\vec{c}\times\vec{a}\quad\vec{a}\times\vec{b}]=[\vec{a}\ \vec{b}\ \vec{c}]^2$$

This implies (C) is not correct. Finally

$$\vec{a}\cdot\vec{p}+\vec{b}\cdot\vec{p}+\vec{c}\cdot\vec{p}=[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]+[\vec{c}\ \vec{a}\ \vec{b}]=3[\vec{a}\ \vec{b}\ \vec{c}]$$

This means (D) is correct.

Answers: (A), (B), (D)

5. Let $\vec{a}$ and $\vec{b}$ be unit vectors and θ be the angle between $\vec{a}$ and $\vec{b}$. Let $\vec{x}=\vec{a}+\vec{b}$ and $\vec{y}=\vec{a}-\vec{b}$. Then

(A) $\vec{x}\cdot\vec{y}=1$

(B) $\vec{x}\cdot\vec{y}=0$

(C) $|\vec{x}\times\vec{y}|=2\sin\theta$

(D) $|\vec{x}\times\vec{y}|$, two times the area of the parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides

Solution: We have

$$\vec{x}\cdot\vec{y}=(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$$
$$=|\vec{a}|^2-|\vec{b}|^2$$
$$=1-1=0$$

So (B) is correct. Also

$$\vec{x}\times\vec{y}=(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$$
$$=-2(\vec{a}\times\vec{b})$$

Therefore

$$|\vec{x}\times\vec{y}|=2|\vec{a}\times\vec{b}|$$
$$=2|\vec{a}||\vec{b}|\sin\theta$$
$$=2\sin\theta$$

So (C) is correct. Also

$$|\vec{x}\times\vec{y}|=2|\vec{a}\times\vec{b}|$$

This implies (D) is correct.

Answers: (B), (C), (D)

6. $ABCD$ is a regular tetrahedron. P and Q are the mid-points of the edges AC and AB, respectively, G is the centroid of the face BCD and θ is the angle between the vectors $\overline{PG}$ and $\overline{DG}$. Then

(A) the angle between $\overline{AB}$ and $\overline{CD}$ is 90°

(B) the angle θ is $\pi-\mathrm{Cos}^{-1}\left(\dfrac{5}{6\sqrt{3}}\right)$

(C) θ is $\mathrm{Cos}^{-1}\left(\dfrac{5}{6\sqrt{3}}\right)$

(D) angle between $\overline{AB}$ and $\overline{CD}$ is 120°

Solution: Take D as the origin and let $\overline{DA}=\vec{a}$, $\overline{DB}=\vec{b}$ and $\overline{DC}=\vec{c}$. See Figure 6.49. Then

$$\overline{DP}=\frac{(\vec{a}+\vec{c})}{2}$$
$$\overline{DQ}=\frac{\vec{a}+\vec{b}}{2}$$
$$\overline{DG}=\frac{\vec{b}+\vec{c}}{3}$$

Since $ABCD$ is regular all the faces are equilateral triangles with equal edges and hence the angles of each triangular face are 60°. In a tetrahedron $ABCD$, the pairs of edges (AD, BC), (CD, AB) and (BD, AC) are called opposite pairs of edges. Now

$$\begin{aligned}\overline{AD}\cdot\overline{BC} &= (-\vec{a})\cdot(\vec{c}-\vec{b}) = \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c}\\ &= |\vec{a}||\vec{b}|\cos 60^\circ\\ &= -|\vec{a}||\vec{c}|\cos 60^\circ = 0\end{aligned}$$

because $|\vec{a}| = |\vec{b}| = |\vec{c}|$.

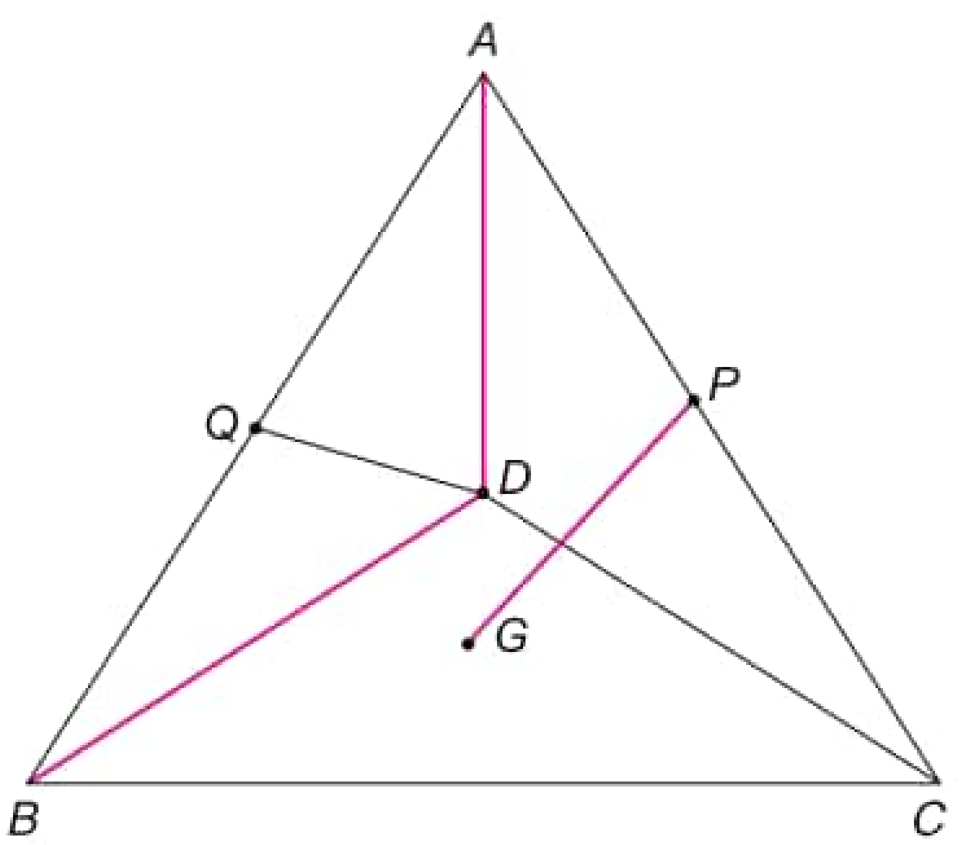

FIGURE 6.49 Multiple correct choice type question 6.

Therefore AD and BC are at right angles. Hence opposite pairs of edges are at right angles. (A) is correct. Now

$$\begin{aligned}\overline{PG} &= \frac{1}{3}(\vec{b}+\vec{c}) - \frac{1}{2}(\vec{a}+\vec{c})\\ &= \frac{1}{6}(2\vec{b}-\vec{c}-3\vec{a}) = -\frac{1}{6}(3\vec{a}-2\vec{b}+\vec{c})\end{aligned}$$

$$\overline{DQ} = \frac{1}{2}(\vec{a}+\vec{b})$$

$$\begin{aligned}36|\overline{PG}|^2 &= 9|\vec{a}|^2 + 4|\vec{b}|^2 + |\vec{c}|^2 - 12(\vec{a}\cdot\vec{b}) - 4(\vec{b}\cdot\vec{c}) + 6(\vec{c}\cdot\vec{a})\\ &= 9|\vec{a}|^2 + 4|\vec{b}|^2 + |\vec{c}|^2 - 12|\vec{a}||\vec{b}|\cos 60^\circ\\ &\quad - 4|\vec{b}||\vec{c}|\cos 60^\circ + 6|\vec{c}||\vec{a}|\cos 60^\circ\end{aligned}$$

It is known that $|\vec{a}| = |\vec{b}| = |\vec{c}| = K$ (suppose). Therefore

$$36|\overline{PG}|^2 = K^2(14-6-2+3) = 9K^2 \tag{6.58}$$

Again

$$\begin{aligned}4|\overline{DQ}|^2 &= |\vec{a}+\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos 60^\circ\\ &= 3K^2\end{aligned} \tag{6.59}$$

From Eqs. (6.58) and (6.59) we have

$$|\overline{PG}| = \frac{K}{2} \quad\text{and}\quad |\overline{DQ}| = \frac{\sqrt{3}K}{2} \tag{6.60}$$

Now

$$\begin{aligned}\overline{PG}\cdot\overline{DQ} &= -\frac{1}{6}(3\vec{a}-2\vec{b}+\vec{c})\cdot\frac{1}{2}(\vec{a}+\vec{b})\\ &= -\frac{1}{12}\left[3K^2 + \frac{3}{2}K^2 - K^2 - 2K^2 + \frac{K^2}{2} + \frac{K^2}{2}\right]\\ &= -\frac{5K^2}{24}\end{aligned} \tag{6.61}$$

Therefore, using Eqs. (6.60) and (6.61) we have

$$\begin{aligned}\cos\theta &= \frac{\overline{PG}\cdot\overline{DQ}}{|\overline{PG}||\overline{DQ}|} = -\frac{5K^2}{24}\times\frac{4}{\sqrt{3}K^2}\\ &= -\frac{5}{6\sqrt{3}}\end{aligned}$$

Hence

$$\begin{aligned}\theta &= \cos\left(-\frac{5}{6\sqrt{3}}\right)\\ &= \pi - \mathrm{Cos}^{-1}\left(\frac{5}{6\sqrt{3}}\right)\end{aligned}$$

(B) is correct.

Answers: (A), (B)

7. A, B, C and D are fixed points in space or in plane and M is any arbitrary point in the space. Which of the following true?

(A) $\overline{MA}\cdot\overline{MC} = \overline{MB}\cdot\overline{MD}$ if $ABCD$ is a rectangle.

(B) $|\overline{MA}|^2 + |\overline{MC}|^2 = |\overline{MB}|^2 + |\overline{MD}|^2$ if $ABCD$ is rectangle.

(C) If $\overline{AM}\cdot\overline{CM} \neq \overline{BM}\cdot\overline{DM}$ for all positions of M, then $ABCD$ is a parallelogram without being a rectangle.

(D) If $\overline{AM}\cdot\overline{CM} \neq \overline{BM}\cdot\overline{DM}$ for all positions of M, then $ABCD$ is a trapezium which is not a parallelogram.

Solution: Take A as origin and let $\overline{AB} = \vec{b}$, $\overline{AD} = \vec{d}$ and $\overline{AM} = \vec{m}$. See Figure 6.50. Suppose $ABCD$ is a rectangle so that $\vec{b}\cdot\vec{d} = 0$ and $\overline{AC} = \vec{b} + \vec{d}$. Now

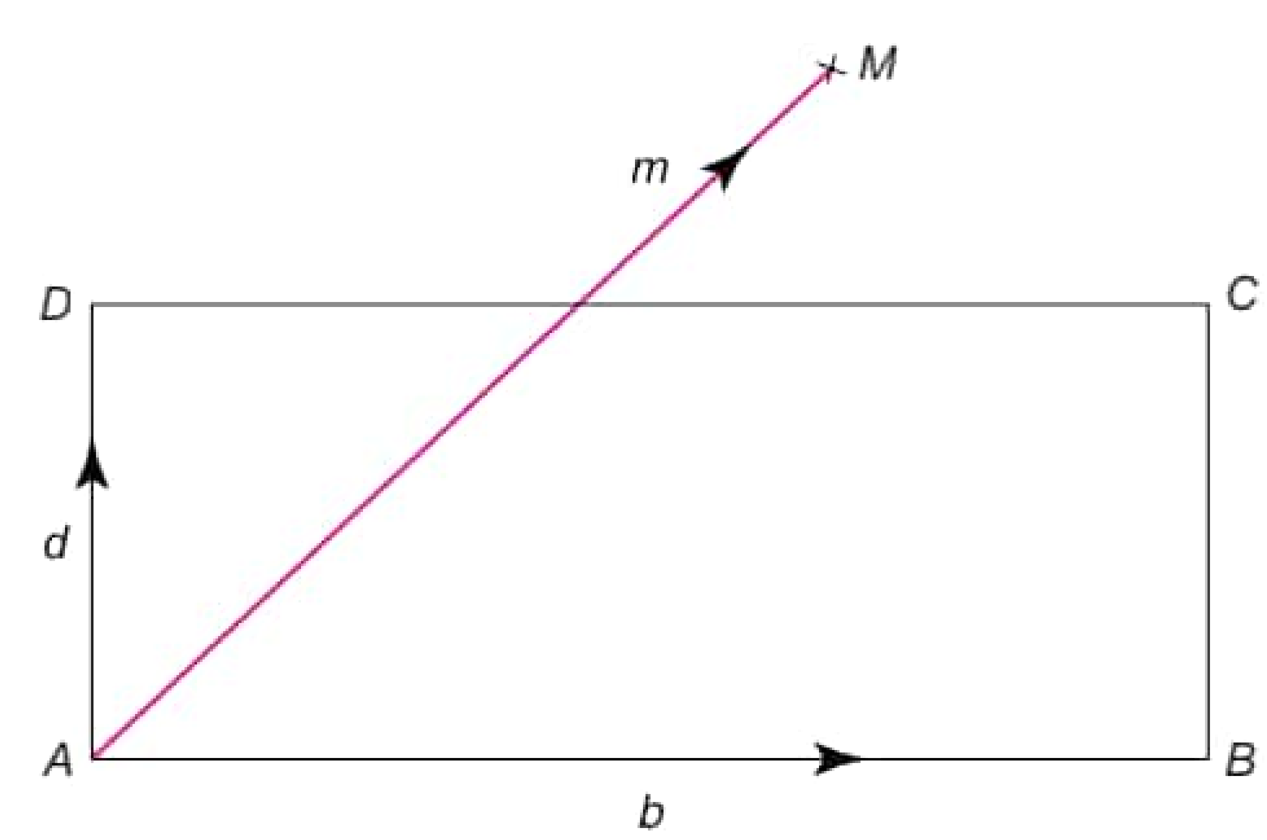

FIGURE 6.50 Multiple correct choice type question 7.

$$\overline{MA}\times\overline{MC} = (\quad \vec{m})\times(\vec{b}+\vec{d} \quad \vec{m})$$

$$= \quad \vec{m}\times\vec{b} \quad \vec{m}\times\vec{d} + |\vec{m}|^2 \qquad (6.62)$$

$$\overline{MB}\times\overline{MD} = (\vec{b} \quad \vec{m})\times(\vec{d} \quad \vec{m})$$

$$= \vec{b}\times\vec{d} \quad \vec{b}\times\vec{m} \quad \vec{m}\times\vec{d} + |\vec{m}|^2$$

$$= \quad \vec{m}\times\vec{b} \quad \vec{m}\times\vec{d} + |\vec{m}|^2 \quad (\because\ \vec{b}\times\vec{d} = 0) \quad (6.63)$$

Therefore Eqs. (6.62) and (6.63) imply

$$\overline{MA}\times\overline{MC} = \overline{MB}\times\overline{MD}$$

Hence (A) is correct.
Again

$$|\overline{MA}|^2 + |\overline{MC}|^2 = |\vec{m}|^2 + |\vec{b}+\vec{d} \quad \vec{m}|^2$$

$$= 2|\vec{m}|^2 + |\vec{b}|^2 + |\vec{d}|^2 \quad 2\vec{m}\times(\vec{b}+\vec{d}) \quad (\because\ \vec{b}\times\vec{d} = 0) \quad (6.64)$$

Also

$$|\overline{MB}|^2 + |\overline{MD}|^2 = |\vec{b} \quad \vec{m}|^2 + |\vec{d} \quad \vec{m}|^2$$

$$= 2|\vec{m}|^2 + |\vec{b}|^2 + |\vec{d}|^2 \quad 2\vec{m}\times(\vec{b}+\vec{d}) \quad (6.65)$$

Equations (6.64) and (6.65) imply

$$|\overline{MA}|^2 + |\overline{MC}|^2 = |\overline{MB}|^2 + |\overline{MD}|^2$$

Hence (B) is correct.
Suppose $\overline{AM}\times\overline{CM} \quad \overline{BM}\times\overline{DM}$ for all points M. Then

$$\vec{m}\times(\vec{m} \quad \vec{c}) \quad (\vec{m} \quad \vec{b})\times(\vec{m} \quad \vec{d})$$

$$\vec{m}\times(\vec{d}+\vec{b} \quad \vec{c}) \quad (\vec{b}\times\vec{d}) \qquad (6.66)$$

Let $\vec{\alpha} = \vec{b}+\vec{d} \quad \vec{c}$. We prove that $\vec{\alpha} = \vec{0}$ so that $\vec{b} \quad \vec{c} = \vec{d}$ in which case $\overline{BC} = \overline{AD}$ and hence $ABCD$ is a parallelogram. Suppose $\vec{\alpha} \quad \vec{0}$. Then the vector

$$\vec{m} = \frac{\vec{b}\times\vec{d}}{|\vec{\alpha}|^2} \div \vec{\alpha}$$

satisfies the equation $\vec{m}\times\vec{\alpha} = \vec{b}\times\vec{d}$ which is a contradiction to Eq. (6.66). Therefore $\vec{\alpha} = \vec{0}$. Now $\overline{AM}\times\overline{CM} \quad \overline{BM}\times\overline{DM}$ assures that $ABCD$ is not a rectangle because of (A).

Answers: (A), (B), (C)

8. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-coplanar vectors such that $(\vec{a}, \vec{b}, \vec{c})$ is a right-handed system. Let

$$\vec{\alpha} = \frac{\vec{b} \quad \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\vec{\beta} = \frac{\vec{c} \quad \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\vec{\gamma} = \frac{\vec{a} \quad \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

and $x > 0$. Then

(A) $x[\vec{a}\ \vec{b}\ \vec{c}] + \dfrac{[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]}{x}$ has least value 2

(B) $[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]$ is a positive scalar

(C) $\vec{a}\times\vec{\alpha} + \vec{b}\times\vec{\beta} + \vec{c}\times\vec{\gamma} = 3$

(D) $\vec{b}\times\vec{\alpha} + \vec{c}\times\vec{\beta} + \vec{a}\times\vec{\gamma} = 3$

Solution: We have

$$[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}[\vec{b} \quad \vec{c}\ \vec{c} \quad \vec{a}\ \vec{a} \quad \vec{b}]$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]^2}{[\vec{a}\ \vec{b}\ \vec{c}]^3} = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]} \quad 0$$

Also $(\vec{a}, \vec{b}, \vec{c})$ is a right-handed system $\quad (\vec{\alpha}, \vec{\beta}, \vec{\gamma})$ is also right-handed system.

$$[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}][\vec{a}\ \vec{b}\ \vec{c}] = 1$$

and $$[\vec{a}\ \vec{b}\ \vec{c}] > 0 \quad [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] > 0$$

Hence (B) is correct.
Now AM GM implies

$$x[\vec{a}\ \vec{b}\ \vec{c}] + \frac{[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]}{x} \quad 2\sqrt{x[\vec{a}\ \vec{b}\ \vec{c}]\frac{[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]}{x}} = 2$$

and hence least value is 2. So (A) is correct.
Now

$$\vec{a}\times\vec{\alpha} + \vec{b}\times\vec{\beta} + \vec{c}\times\vec{\gamma} = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{[\vec{b}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{[\vec{c}\ \vec{a}\ \vec{b}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 1+1+1 = 3$$

So (C) is correct.
Also, $\vec{b}\times\vec{\alpha} = 0$, $\vec{c}\times\vec{\beta} = 0$ and $\vec{a}\times\vec{\gamma} = 0$, therefore (D) is not correct.

Answers: (A), (B), (C)

9. $\vec{b}$ and $\vec{c}$ are non-collinear vectors. If $\vec{a}$ is a vector such that $|\vec{c}|^2\vec{a} = \vec{c}$ and

$$\vec{a} \quad (\vec{b} \quad \vec{c}) + (\vec{a}\times\vec{b})\vec{b} = (4 \quad 2x \quad \sin y)\vec{b} + (x^2 \quad 1)\vec{c}$$

then

(A) $x = \quad 1$

(B) $y = (4n+1)\dfrac{\pi}{2}, n \quad \mathbb{Z}$

(C) $y = (2n+1)\dfrac{\pi}{2}, n \quad \mathbb{Z}$

(D) $x = 1$

Solution: By hypothesis

$$(\vec{a}\cdot\vec{c})\vec{b} \quad (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{b} = (4 \quad 2x \quad \sin y)\vec{b} + (x^2 \quad 1)\vec{c}$$

and $$|\vec{c}|^2\,\vec{a} = \vec{c}$$

Now

$$|\vec{c}|^2\,\vec{a} = \vec{c}$$
$$|\vec{c}|^2\,(\vec{a}\cdot\vec{c}) = |\vec{c}|^2$$
$$\vec{a}\cdot\vec{c} = 1$$

Therefore

$$(1+\vec{a}\cdot\vec{b})\vec{b} \quad (\vec{a}\cdot\vec{b})\vec{c} = (4 \quad 2x \quad \sin y)\vec{b} + (x^2 \quad 1)\vec{c}$$

This gives

$$1+\vec{a}\cdot\vec{b} = 4 \quad 2x \quad \sin y$$

and $$(\vec{a}\cdot\vec{b}) = x^2 \quad 1$$

Solving we get

$$1+1 \quad x^2 = 4 \quad 2x \quad \sin y$$
$$x^2 \quad 2x + 2 = \sin y$$
$$(x \quad 1)^2 + 1 = \sin y$$

So $1 \quad \sin y \quad 1 \quad x \quad 1 = 0$ and $\sin y = 1$. Therefore

$$x = 1$$

and $$y = (4n+1)\frac{\pi}{2}$$

Answers: (B), (D)

10. Let $ABCDEF$ be a regular hexagon (Figure 6.51) such that $\overline{AD} = x\overline{BC}$ and $\overline{CF} = y\overline{AB}$. Then

(A) $xy = \quad 4$ (B) $\overline{AF}\cdot\overline{AB} + \frac{1}{2}\overline{BC}^2 = 0$

(C) $xy = 4$ (D) $\overline{AB}\cdot\overline{AF} + \frac{1}{3}\overline{BC}^2 = 0$

Solution: Let "O" be the centre of the hexagon. Then

$$x\overline{BC} = \overline{AD} = 2\overline{AO} = 2\overline{BC}$$
$$x = 2$$
$$y\overline{AB} = \overline{CF} = 2\overline{CO} = \quad 2\overline{AB}$$
$$y = \quad 2$$

Hence

$$xy = \quad 4$$

(A) is correct. Now

$$\overline{AB}\cdot\overline{AF} = |\overline{AB}|^2 \cos 120° = \quad \frac{1}{2}(BC)^2$$

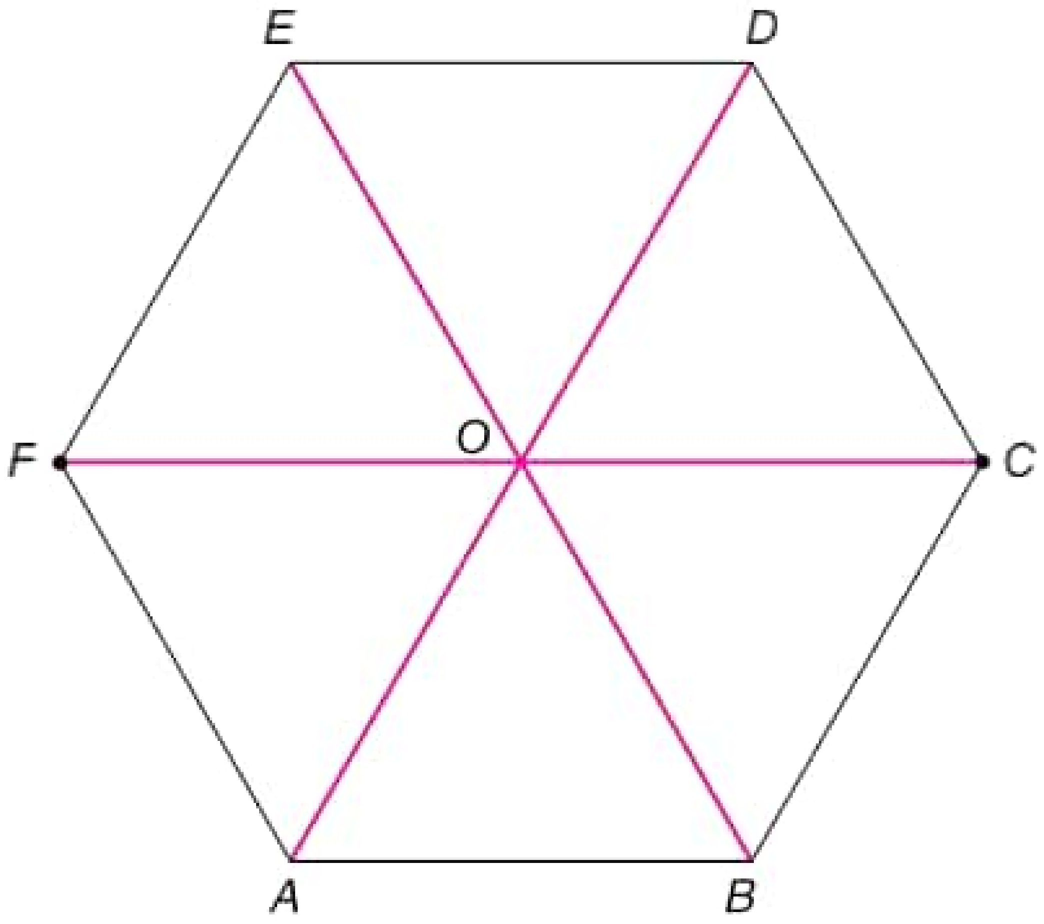

FIGURE 6.51 Multiple correct choice type question 10.

(B) is correct.

Answers: (A), (B)

11. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{\alpha} = \vec{a} \quad (\vec{a}\cdot\vec{b})\vec{b}$ and $\vec{\beta} = \vec{a} \quad \vec{b}$, then $|\vec{\beta}|$ is

(A) $|\vec{\alpha}|$ (B) $|\vec{\alpha}| + |\vec{\alpha}\cdot\vec{a}|$

(C) $|\vec{\alpha}| + |\vec{\alpha}\cdot\vec{b}|$ (D) $|\vec{\alpha}| + \vec{\alpha}\cdot(\vec{a}+\vec{b})$

Solution: Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = \cos\theta$$

Now

$$|\vec{\alpha}|^2 = |\vec{a} \quad (\vec{a}\cdot\vec{b})\vec{b}|^2$$
$$= |\vec{a}|^2 + (\vec{a}\cdot\vec{b})^2|\vec{b}|^2 \quad 2(\vec{a}\cdot\vec{b})^2$$
$$= 1 + \cos^2\theta \quad 2\cos^2\theta$$
$$= \sin^2\theta$$
$$= (|\vec{a}||\vec{b}|\sin\theta)^2 \quad (\because |\vec{a}| = |\vec{b}| = 1)$$
$$= |\vec{a} \quad \vec{b}|^2$$
$$= |\vec{\beta}|^2$$

Therefore

$$|\vec{\beta}| = |\vec{\alpha}|$$

Hence (A) is correct. Now

$$\vec{\alpha}\cdot\vec{b} = [\vec{a} \quad (\vec{a}\cdot\vec{b})\vec{b}]\cdot\vec{b}$$
$$= \vec{a}\cdot\vec{b} \quad (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{b})$$
$$= \vec{a}\cdot\vec{b} \quad \vec{a}\cdot\vec{b} \quad (\because |\vec{b}| = 1)$$
$$= 0$$

Since

$$|\vec{\alpha}|+|\vec{\alpha}\cdot\vec{b}|=|\vec{\alpha}|+0=|\vec{\alpha}|=|\vec{\beta}|$$

Therefore (C) is correct.

Answers: (A), (C)

12. The vector $(1/3)(2\vec{i}-2\vec{j}+\vec{k})$ is

(A) a unit vector

(B) makes an angle $\pi/3$ with the vector $2\vec{i}-4\vec{j}+3\vec{k}$

(C) parallel to the vector $-\vec{i}+\vec{j}-\frac{1}{2}\vec{k}$

(D) perpendicular to the vector $3\vec{i}+2\vec{j}-2\vec{k}$

Solution: Let

$$\vec{a}=\frac{1}{3}(2\vec{i}-2\vec{j}+\vec{k})$$

so that

$$|\vec{a}|^2=\frac{1}{9}(2^2+2^2+1)=1$$

Therefore

$$|\vec{a}|=1$$

(A) is correct.
Let $\vec{b}=2\vec{i}-4\vec{j}+3\vec{k}$. Then

$$\vec{a}\cdot\vec{b}=\frac{1}{3}(4+8+3)=5$$

Let $\theta=(\vec{a},\vec{b})$. Then

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{5}{1\cdot\sqrt{4+16+9}}=\frac{5}{\sqrt{29}}$$

Hence (B) is not correct.
Now

$$-\vec{i}+\vec{j}-\frac{1}{2}\vec{k}=-\frac{1}{2}(2\vec{i}-2\vec{j}+\vec{k})=-\frac{3}{2}\vec{a}$$

Therefore $\vec{a}$ is parallel $\vec{b}$. Hence (C) is correct.
Also

$$\vec{a}\cdot(3\vec{i}+2\vec{j}-2\vec{k})=\frac{1}{3}(6-4-2)=0$$

Therefore (D) is correct.

Answers: (A), (C), (D)

13. For three vectors $\vec{a},\vec{b},\vec{c},\vec{b}\cdot(\vec{a}\times\vec{c})$ is

(A) not equal to $\vec{a}\cdot(\vec{b}\times\vec{c})$

(B) not equal to $(\vec{b}\times\vec{c})\cdot\vec{a}$

(C) not equal to $(\vec{a}\times\vec{b})\cdot\vec{c}$

(D) not equal to $\vec{c}\cdot(\vec{b}\times\vec{a})$

Solution: It is known that

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b})$$

Therefore

$$\vec{b}\cdot(\vec{a}\times\vec{c})=\vec{c}\cdot(\vec{b}\times\vec{a})$$

Therefore (D) is not correct.

Answers: (A), (B), (C)

14. For any two vectors $\vec{a}$ and $\vec{b}$ which of the following are true?

(A) $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$

(B) $|1-\vec{a}\cdot\vec{b}|^2+|\vec{a}+\vec{b}+\vec{a}\times\vec{b}|^2=(1+|\vec{a}|^2)(1+|\vec{b}|^2)$

(C) $(\vec{a}\times\vec{b})\times\vec{a}=|\vec{a}|^2\vec{b}+(\vec{a}\cdot\vec{b})\vec{a}$

(D) $\vec{a}\times(\vec{b}\times\vec{a})=|\vec{a}|^2\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}$

Solution:

(A) We have

$$|\vec{a}\times\vec{b}|^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b}\end{vmatrix} \quad \text{(Theorem 6.48)}$$

$$=|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2$$

Therefore

$$|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$$

So (A) is correct.

(B) We have

$$|1-\vec{a}\cdot\vec{b}|^2+|\vec{a}+\vec{b}+\vec{a}\times\vec{b}|^2$$

$$=1-2(\vec{a}\cdot\vec{b})+(\vec{a}\cdot\vec{b})^2+|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})+|\vec{a}\times\vec{b}|^2$$

$$[\because\ \vec{a}\cdot(\vec{a}\times\vec{b})=\vec{b}\cdot(\vec{a}\times\vec{b})=0]$$

$$=1+|\vec{a}|^2|\vec{b}|^2\cos^2\theta+|\vec{a}|^2+|\vec{b}|^2+|\vec{a}|^2|\vec{b}|^2\sin^2\theta$$

$$=1+|\vec{a}|^2+|\vec{b}|^2+|\vec{a}|^2|\vec{b}|^2(\cos^2\theta+\sin^2\theta)$$

$$=(1+|\vec{a}|^2)(1+|\vec{b}|^2)$$

This implies (B) is correct.

(C) From Theorem 6.46 we have

$$(\vec{a}\times\vec{b})\times\vec{a}=(\vec{a}\cdot\vec{a})\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}$$

Hence (C) is not correct.

(D) Now

$$\vec{a}\times(\vec{b}\times\vec{a})=(\vec{a}\cdot\vec{a})\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}$$

Therefore (D) is correct.

Answers: (A), (B), (D)

15. Let $\overline{AB} = 3\vec{i} + 4\vec{k}$ and $\overline{AC} = 5\vec{i} - 2\vec{j} + 4\vec{k}$. Then

(A) length of the median through A of ΔABC is $\sqrt{33}$

(B) the length of the altitude from A to the base BC of ΔABC is $\sqrt{\frac{41}{2}}$

(C) the area of ΔABC is $\sqrt{41}$ sq. units

(D) the area of ΔABC is 10 sq. units

Solution:

(A) Suppose AD is the median. Then

$$\overline{AD} = \frac{\overline{AB} + \overline{AC}}{2} = \frac{8\vec{i} - 2\vec{j} + 8\vec{k}}{2} = 4\vec{i} - \vec{j} + 4\vec{k}$$

So

$$|\overline{AD}| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$$

Hence (A) is correct.

(B) We have

$$\overline{AB} + \overline{BC} = \overline{AC} \Rightarrow \overline{BC} = \overline{AC} - \overline{AB} = 2\vec{i} - 2\vec{j}$$

Suppose AM is the altitude from A to BC (Figure 6.52). Then

$$\sin B = \frac{AM}{AB}$$

$$\Rightarrow AM = (AB)\sin B = \frac{(BC)(AB)\sin B}{BC} = \frac{|\overline{BC} \times \overline{AB}|}{|\overline{BC}|} \qquad (6.67)$$

Now

$$\overline{BC} \times \overline{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 0 \\ 3 & 0 & 4 \end{vmatrix} = -8\vec{i} - 8\vec{j} + 6\vec{k}$$

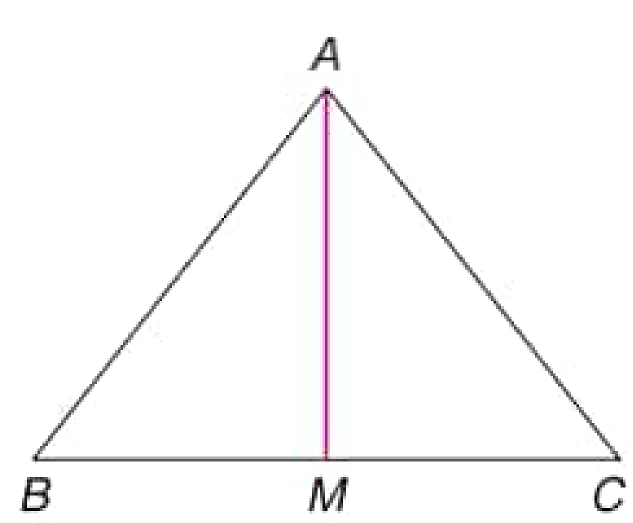

FIGURE 6.52 Multiple correct choice type question 15.

Therefore

$$|\overline{BC} \times \overline{AB}| = \sqrt{64 + 64 + 36} = \sqrt{164}$$

Also

$$|\overline{BC}| = \sqrt{8}$$

Now by Eq. (6.67)

$$AM = \frac{\sqrt{164}}{\sqrt{8}} = \sqrt{\frac{41}{2}}$$

Hence (B) is correct.

(C) We have

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 4 \\ 5 & -2 & 4 \end{vmatrix} = 8\vec{i} + 8\vec{j} + 6\vec{k}$$

Therefore by Theorem 6.29,

$$\text{Area of } \Delta ABC = \frac{1}{2}|\overline{BC} \times \overline{AB}| = \frac{1}{2}\sqrt{164} = \sqrt{41}$$

This means (C) is correct.

Answers: (A), (B), (C)

16. If the lines

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-K}$$

and $$\frac{x-1}{K} = \frac{y-4}{2} = \frac{z-5}{1}$$

are coplanar, then K may be

(A) 0 (B) 3 (C) −2 (D) −3

Solution: Let the given lines be L_1 and L_2, respectively. L_1 is passing through the point (2, 3, 4) and having direction ratios 1, 1 and $-K$. L_2 is passing through the point (1, 4, 5) and having direction ratios K, 2, 1 (see Theorem 5.28, Chapter 5). Let $A = (2, 3, 4)$ and $B = (1, 4, 5)$. Also $\vec{a} = (1, 1, -K)$ and $\vec{b} = (K, 2, 1)$. The lines are coplanar if and only if vectors $\overline{AB}, \vec{a}$ and $\vec{b}$ are coplanar. Therefore by Corollary 6.16

$$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -K \\ K & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (1 + 2K) + (1 + K^2) - (2 - K) = 0$$

$$K^2 + 3K = 0$$
$$K = 0 \quad \text{or} \quad 3$$

Answers: (A), (D)

17. Consider the lines

$$L_1 : \frac{x \quad 3}{3} = \frac{y \quad 8}{1} = \frac{z \quad 3}{1}$$

$$L_2 : \frac{x+3}{3} = \frac{y+7}{2} = \frac{z \quad 6}{4}. \text{ Then}$$

(A) L_1 and L_2 are coplanar
(B) L_1 and L_2 are not coplanar and shortest distance between them is $2\sqrt{30}$
(C) L_1 and L_2 are skew lines
(D) the shortest distance between L_1 and L_2 is $3\sqrt{30}$

Solution: The line L_1 is passing through the point A (3, 8, 3) and having direction ratios 3, 1, 1. The line L_2 is passing through the point B (3, 7, 6) and having direction ratios –3, 2, 4. L_1 and L_2 are coplanar if and only if $\overline{AB}, \vec{a} = (3, \ 1, 1)$ and $\vec{b} = (\ 3, 2, 4)$ are coplanar. Therefore by Corollary 6.16, this would happen if and only if

$$\begin{vmatrix} 6 & 15 & 3 \\ 3 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 0$$

Now

$$\begin{vmatrix} 6 & 15 & 3 \\ 3 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 3[\ 2(\ 4 \quad 2) + 5(12+3) + (6 \quad 3)]$$
$$= 3[12 + 75 + 3] \quad 0$$

Therefore L_1 and L_2 are not coplanar and hence they are skew lines. Therefore (C) is correct.
The shortest distance between L_1 and L_2 is (using Theorem 6.44)

$$\frac{|\overline{AB} \times (\vec{a} \quad \vec{b})|}{|\vec{a} \quad \vec{b}|} \qquad (6.68)$$

Now

$$\overline{AB} = (\ 6, \ 15, 3) = 3(\ 2, \ 5, 1)$$

$$\vec{a} \quad \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix} = \ 6\vec{i} \quad 15\vec{j} + 3\vec{k}$$

Therefore

$$\overline{AB} \times (\vec{a} \quad \vec{b}) = 3(12 + 75 + 3) = 270$$
$$|\vec{a} \quad \vec{b}| = 3\sqrt{4 + 25 + 1} = 3\sqrt{30}$$

Hence by Eq. (6.68)

$$\text{Shortest distance} = \frac{270}{3\sqrt{30}}$$
$$= 3\sqrt{30}$$

So (D) is correct.

Answers: (C), (D)

18. Let $\vec{a}$ be a vector parallel to the line of intersection of the planes P_1 and P_2. Plane P_1 is parallel to the vectors $2\vec{j} + 3\vec{k}$ and $4\vec{j} \quad 3\vec{k}$. Plane P_2 is parallel to the vectors $\vec{j} \quad \vec{k}$ and $3\vec{i} + 3\vec{j}$. Then the angle between the vector $\vec{a}$ and a given vector $2\vec{i} + \vec{j} \quad 2\vec{k}$ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$ (C) $\frac{3\pi}{4}$ (D) $\frac{2\pi}{3}$

Solution: We have

$$\vec{n}_1 = \text{normal to } P_1 = (2\vec{j} + 3\vec{k}) \quad (4\vec{j} \quad 3\vec{k}) = \ 18\vec{i}$$

$$\vec{n}_2 = \text{normal to } P_2 = (\vec{j} \quad \vec{k}) \quad (3\vec{i} + 3\vec{j}) = 3(\vec{i} \quad \vec{j} \quad \vec{k})$$

Now $\vec{a}$ is parallel $\vec{n}_1 \quad \vec{n}_2$ angle between $\vec{a}$ and $2\vec{i} + \vec{j} \quad 2\vec{k}$ is same as angle between $\vec{n}_1 \quad \vec{n}_2$ and $2\vec{i} + \vec{j} \quad 2\vec{k}$. Then

$$\vec{n}_1 \quad \vec{n}_2 = \ 54 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & * \\ 1 & 1 & 1 \end{vmatrix} = 54(\ \vec{j} + \vec{k})$$

Let θ be the angle between $\vec{n}_1 \quad \vec{n}_2$ and $2\vec{i} + \vec{j} \quad 2\vec{k}$. Therefore

$$\cos\theta = \frac{54(\ 1 \quad 2)}{54\sqrt{2}(3)} = \ \frac{1}{\sqrt{2}}$$

This implies

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

Angle depends on the direction of $\vec{a}$ which is parallel to $\vec{n}_1 \quad \vec{n}_2$. If we take $\vec{a} = \vec{n}_2 \quad \vec{n}_1$, then

$$\theta = \frac{\pi}{4}$$

Answers: (B), (C)

19. $\vec{x}, \vec{y}$ and $\vec{z}$ are vectors of equal magnitudes $\sqrt{2}$ and each is inclined to the others at an angle of 60°. If $\vec{a} = \vec{x} \quad (\vec{y} \quad \vec{z}), \vec{b} = \vec{y} \quad (\vec{z} \quad \vec{x})$ and $\vec{c} = \vec{x} \quad \vec{y}$, then

(A) $\vec{x} \times \vec{y} = \vec{y} \times \vec{z} = \vec{z} \times \vec{x} = 1$
(B) $\vec{x} = \vec{a} \quad \vec{c}$
(C) $\vec{y} = \vec{b} \quad \vec{c}$
(D) $\vec{z} = \vec{b} + (\vec{a} \quad \vec{c})$

Solution: We have

$$\vec{x}\cdot\vec{y} = |\vec{x}||\vec{y}|\cos 60° = \sqrt{2} \quad \sqrt{2} \quad \frac{1}{2} = 1$$

Therefore (A) is correct.

$$\vec{a} = \vec{x} \quad (\vec{y} \quad \vec{z}) = (\vec{x}\cdot\vec{z})\vec{y} \quad (\vec{x}\cdot\vec{y})\vec{z} = \vec{y} \quad \vec{z} \tag{6.69}$$

$$\vec{b} = \vec{y} \quad (\vec{z} \quad \vec{x}) = (\vec{y}\cdot\vec{x})\vec{z} \quad (\vec{y}\cdot\vec{z})\vec{x} = \vec{z} \quad \vec{x} \tag{6.70}$$

Now by Eq. (6.69)

$$\begin{aligned}\vec{a} \quad \vec{c} &= (\vec{y} \quad \vec{z}) \quad (\vec{x} \quad \vec{y})\\ &= \vec{y} \quad (\vec{x} \quad \vec{y}) \quad \vec{z} \quad (\vec{x} \quad \vec{y})\\ &= (\vec{y}\cdot\vec{y})\vec{x} \quad (\vec{y}\cdot\vec{x})\vec{y} \quad [(\vec{z}\cdot\vec{y})\vec{x} \quad (\vec{z}\cdot\vec{x})\vec{y}]\\ &= 2\vec{x} \quad \vec{y} \quad \vec{x} + \vec{y} = \vec{x}\end{aligned}$$

Hence (B) is correct. Again

$$\begin{aligned}\vec{b} \quad \vec{c} &= (\vec{z} \quad \vec{x}) \quad (\vec{x} \quad \vec{y})\\ &= \vec{z} \quad (\vec{x} \quad \vec{y}) \quad \vec{x} \quad (\vec{x} \quad \vec{y})\\ &= (\vec{z}\cdot\vec{y})\vec{x} \quad (\vec{z}\cdot\vec{x})\vec{y} \quad [(\vec{x}\cdot\vec{y})\vec{x} \quad (\vec{x}\cdot\vec{x})\vec{y}]\\ &= \vec{x} \quad \vec{y} \quad \vec{x} + 2\vec{y} = \vec{y}\end{aligned}$$

Hence (C) is correct. From Eq. (6.70),

$$\vec{z} = \vec{b} + \vec{x} = \vec{b} + \vec{a} \quad \vec{c}$$

So (D) is correct.

Answers: (A), (B), (C), (D)

20. Let $P(3, 2, 6)$ be a point in the space and Q is a point on the line $\vec{r} = (\vec{i} \quad \vec{j} + 2\vec{k}) + \lambda(\quad 3\vec{i} + \vec{j} + 5\vec{k})$. Then

(A) the value of λ for which $\overline{PQ}$ is perpendicular to the vector $\vec{i} \quad 4\vec{j} + 3\vec{k}$ is 1/4

(B) the value of λ for which $\overline{PQ}$ has magnitude $\sqrt{29}$ is 0 or 34/35

(C) the value of λ for which $\overline{PQ}$ is equal to $\vec{i} \quad 4\vec{j} \quad 9\vec{k}$ is 1

(D) the value of λ for which $\overline{PQ}$ makes angle 90° with X-axis is 3/2

Solution: We have

$$Q = (1 \quad 3\lambda)\vec{i} + (\lambda \quad 1)\vec{j} + (2 + 5\lambda)\vec{k}$$

$$P = 3\vec{i} + 2\vec{j} + 6\vec{k}$$

Therefore

$$\overline{PQ} = (\quad 2 \quad 3\lambda)\vec{i} + (\lambda \quad 3)\vec{j} + (5\lambda \quad 4)\vec{k}$$

Now $\overline{PQ}$ is perpendicular to $\vec{i} \quad 4\vec{j} + 3\vec{k}$. This implies

$$(\quad 2 \quad 3\lambda)(1) + (\lambda \quad 3)(\quad 4) + (5\lambda \quad 4)3 = 0$$

$$8\lambda = 2 \qquad \lambda = \frac{1}{4}$$

Hence (A) is correct. Now

$$29 = |\overline{PQ}|^2 = (2 + 3\lambda)^2 + (\lambda \quad 3)^2 + (5\lambda \quad 4)^2$$

$$35\lambda^2 \quad 34\lambda = 0$$

$$\lambda = 0 \quad \text{or} \quad \frac{34}{35}$$

Hence (B) is correct. Again

$$\overline{PQ} = \vec{i} \quad 4\vec{j} \quad 9\vec{k}$$

$$2 \quad 3\lambda = 1, \lambda \quad 3 = \quad 4 \quad \text{and} \quad 5\lambda \quad 4 = \quad 9$$

Hence $\lambda = \quad 1$ and (C) is correct. $\overline{PQ}$ is perpendicular to X-axis means

$$\overline{PQ}\cdot\vec{i} = 0$$

$$2 \quad 3\lambda = 0$$

$$\lambda = \quad \frac{2}{3}$$

So (D) is not correct.

Answers: (A), (B), (C)

21. $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors such that

$$(\vec{a} \quad \vec{b})\cdot(\vec{c} \quad \vec{d}) = 1$$

and

$$\vec{a}\cdot\vec{c} = \frac{1}{2}$$

Then

(A) $\vec{a}, \vec{b}, \vec{c}$ are coplanar

(B) $\vec{b}, \vec{c}, \vec{d}$ are coplanar

(C) $\vec{b}, \vec{d}$ are non-parallel

(D) $\vec{a}, \vec{d}$ are parallel and $\vec{b}, \vec{c}$ are parallel

Solution: $(\vec{a} \quad \vec{b})\cdot(\vec{c} \quad \vec{d}) = 1$ the angle between $\vec{a} \quad \vec{b}$ and $\vec{c} \quad \vec{d}$ is 0. Therefore $\vec{a} \quad \vec{b}$ and $\vec{c} \quad \vec{d}$ are parallel vectors. So

$$(\vec{a} \quad \vec{b}) \quad (\vec{c} \quad \vec{d}) = \vec{0}$$

which implies

$$[\vec{a}\,\vec{b}\,\vec{d}]\vec{c} \quad [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = \vec{0} \tag{6.71}$$

and

$$[\vec{a}\,\vec{c}\,\vec{d}]\vec{b} \quad [\vec{b}\,\vec{c}\,\vec{d}]\vec{a} = 0 \tag{6.72}$$

If $\vec{a}, \vec{b}, \vec{c}$ are not coplanar, then from Eq. (6.71) it follows that $\vec{c}$ and $\vec{d}$ are parallel so that $\vec{c} \quad \vec{d} = \vec{0}$ which is not possible because $(\vec{a} \quad \vec{b})\times(\vec{c} \quad \vec{d}) = 1$. Therefore $\vec{a}, \vec{b}, \vec{c}$ are coplanar. Similarly from Eq. (6.72) $\vec{b}, \vec{c}, \vec{d}$ are coplanar. Hence (A) and (B) are correct.

Again

$$(\vec{a} \quad \vec{b})\times(\vec{c} \quad \vec{d}) = 1 \qquad |\vec{a} \quad \vec{b}||\vec{c} \quad \vec{d}|\cos\alpha = 1$$

where α is the angle between $\vec{a} \quad \vec{b}$ and $\vec{c} \quad \vec{d}$. So $\alpha = 0$. This means

$$|\vec{a} \quad \vec{b}||\vec{c} \quad \vec{d}| = 1$$

$$(|\vec{a}||\vec{b}|\sin\beta)(|\vec{c}||\vec{d}|\sin\gamma) = 1$$

where $\beta = (\vec{a}, \vec{b})$ and $\gamma = (\vec{c}, \vec{d})$. Hence

$$\sin\beta\sin\gamma = 1$$

$$\beta = \gamma = \frac{\pi}{2}$$

So $\vec{a}$ and $\vec{b}$ are at right angles and $\vec{c}$ and $\vec{d}$ are at right angles. Now suppose $\vec{b}$ and $\vec{d}$ are parallel. Since $\vec{a}$ is perpendicular to $\vec{b}, \vec{a}$ is perpendicular to $\vec{d}$. Now

$$1 = (\vec{a} \quad \vec{b})\times(\vec{c} \quad \vec{d}) = \vec{a}\times[\vec{b} \quad (\vec{c} \quad \vec{d})]$$

$$= \vec{a}\times[(\vec{i}\times\vec{d})\vec{c} \quad (\vec{b}\times\vec{c})\vec{d}]$$

$$= (\vec{a}\times\vec{c})(\vec{b}\times\vec{d}) \quad (\vec{a}\times\vec{d})(\vec{b}\times\vec{c})$$

$$= (\vec{a}\times\vec{c})(\vec{b}\times\vec{d}) \quad (\because \vec{a}\times\vec{d} = 0)$$

$$= \frac{1}{2}(\vec{b}\times\vec{d})$$

Hence $\vec{b}\times\vec{d} = 2$ which is not possible because $\vec{b}$ and $\vec{d}$ are unit vectors. So $\vec{b}$ and $\vec{d}$ cannot be parallel. Hence (C) is correct.

Answers: (A), (B), (C)

22. Let $\vec{a}$ and $\vec{b}$ be perpendicular unit vectors and $\vec{c}$ be a unit vector equally inclined to both $\vec{a}$ and $\vec{b}$ at an angle θ. If $\vec{c} = x\vec{a} + y\vec{b} + z(\vec{a} \quad \vec{b})$, then

(A) $x = y = \cos\theta$

(B) $x = \cos\theta, y = \sin\theta$

(C) $x = \cos\theta, z = \quad \cos 2\theta$

(D) $x = y = \cos\theta, z = \cos 2\theta$

Solution: By hypothesis $\vec{a}\times\vec{b} = 0, \vec{a}\times\vec{c} = \cos\theta = \vec{b}\times\vec{c}$. Now

$$\vec{a}\times\vec{c} = x(\vec{a}\times\vec{a}) + y(\vec{b}\times\vec{a}) + z[\vec{a}\ \vec{a}\ \vec{b}]$$

$$= x + 0 + 0$$

Therefore

$$x = \vec{a}\times\vec{c} = \cos\theta$$

Similarly

$$y = \vec{b}\times\vec{c} = \cos\theta$$

Again

$$1 = |\vec{c}|^2$$

$$= x^2|\vec{a}|^2 + y^2|\vec{b}|^2 + z^2|\vec{a} \quad \vec{b}|^2 + 2xy(\vec{a}\times\vec{b})$$

$$+ 2zx[\vec{a}\ \vec{a}\ \vec{b}] + 2yz[\vec{b}\ \vec{a}\ \vec{b}]$$

$$= x^2 + y^2 + z^2|\vec{a}|^2|\vec{b}|^2\sin^2 90^\circ$$

$$= \cos^2\theta + \cos^2\theta + z^2$$

Therefore

$$z^2 = \quad \cos 2\theta$$

Answers: (A), (C)

23. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$. If the angle between $\vec{b}$ and $\vec{c}$ is $\pi/6$, then $\vec{a} = \lambda(\vec{b} \quad \vec{c})$ where λ may be

(A) 4 (B) 4 (C) 2 (D) 2

Solution: We have

$$1 = |\lambda||\vec{b} \quad \vec{c}|$$

$$= |\lambda||\vec{b}||\vec{c}|\sin\frac{\pi}{6}$$

$$= |\lambda| \ \frac{1}{2}\div$$

Therefore

$$\lambda = \pm 2$$

Answers: (C), (D)

24. The position vectors of two points A and C are $9\vec{i}$ $\vec{j} + 2\vec{k}$ and $7\vec{i} \quad 2\vec{j} + 7\vec{k}$, respectively (Figure 6.53). The point of intersection of the vectors $\overline{AB} = 4\vec{i}$ $\vec{j} + 3\vec{k}$ and $\overline{CD} = 2\vec{i} \quad \vec{j} + 2\vec{k}$ is P. $\overline{PQ}$ is perpendicular to both $\overline{AB}$ and $\overline{CD}$ and $|\overline{PQ}| = 15$. Then the position vectors of the point Q is

(A) (4, 11, 11) (B) (6, 9, 9)

(C) (4, 11, 11) (D) (6, 9, 9)

Solution: Equation of the lines AB and CD are, respectively (see Theorem 5.27, Chapter 5)

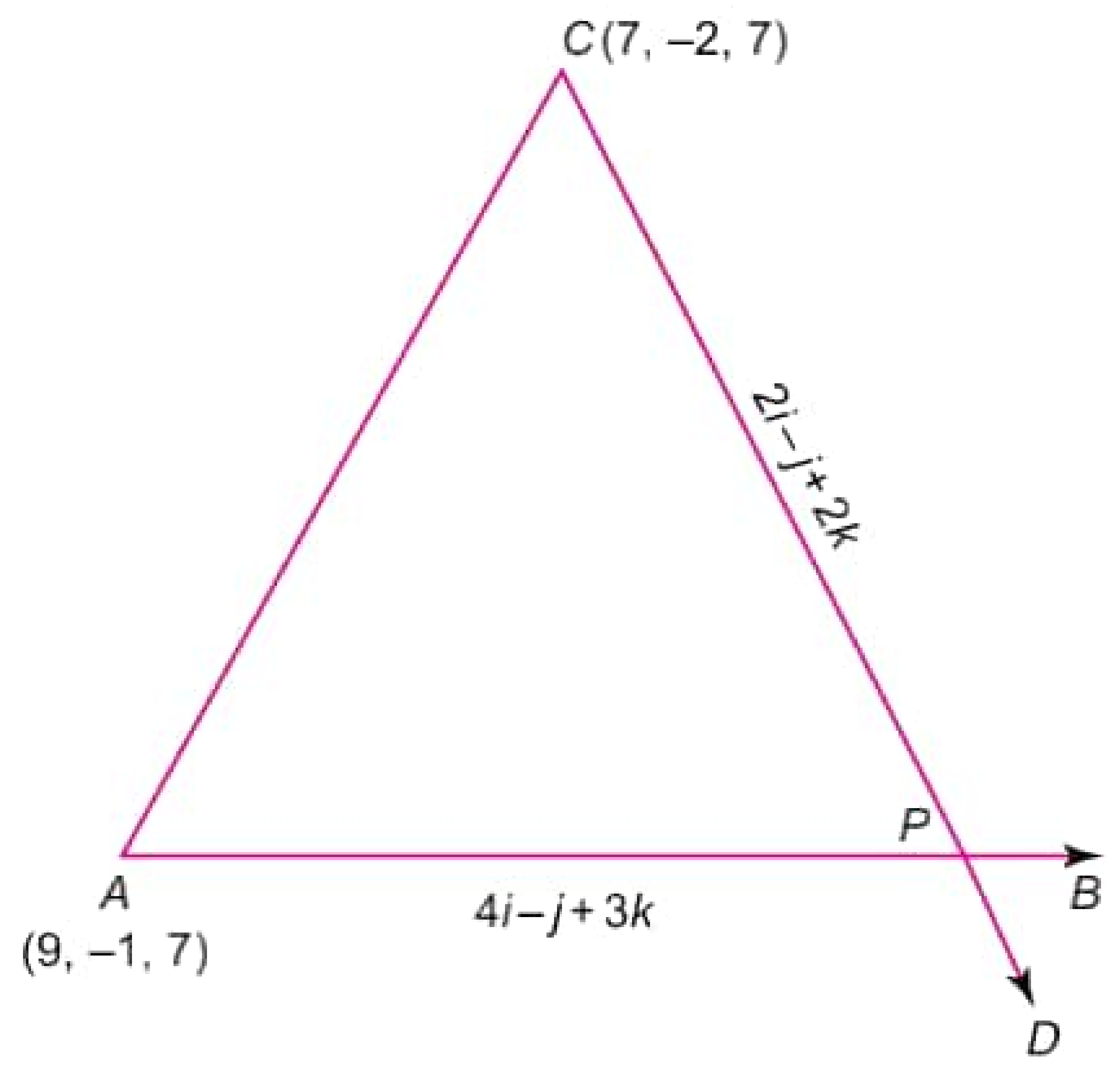

FIGURE 6.53 Multiple correct choice type question 24.

$$\vec{r} = (9\vec{i} - \vec{j} + 7) + t(4\vec{i} - \vec{j} + 3\vec{k}), t \in \mathbb{R}$$

and $$\vec{r} = (7\vec{i} - 2\vec{j} + 7\vec{k}) + s(2\vec{i} - \vec{j} + 2\vec{k}), s \in \mathbb{R}$$

Equating the corresponding coefficients in the above equations we get

$$9 + 4t = 7 + 2s$$

or $$4t - 2s = -2 \quad (6.73)$$

$$-1 - t = -2 - s$$

or $$t - s = 1 \quad (6.74)$$

$$7 + 3t = 7 + 2s$$

or $$3t - 2s = 0 \quad (6.75)$$

Now $t = -2$ and $s = -3$ satisfy Eqs. (6.73), (6.74) and (6.75). Therefore

$$P = (9\vec{i} - \vec{j} + 7\vec{k}) - 2(4\vec{i} - \vec{j} + 3\vec{k}) = \vec{i} + \vec{j} + \vec{k}$$

Now

$$\overline{AB} \times \overline{CD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 3 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \vec{i} - 2\vec{j} - 2\vec{k}$$

Since $\overline{PQ}$ is perpendicular to both $\overline{AB}$ and $\overline{CD}$, let

$$\overline{PQ} = \lambda(\overline{AB} \times \overline{CD}) = \lambda(\vec{i} - 2\vec{j} - 2\vec{k}), \lambda \in \mathbb{R}$$

Then

$$|\overline{PQ}| = 15$$

$$\Rightarrow 3|\lambda| = 15$$

$$\Rightarrow \lambda = \pm 5$$

Case 1: $\lambda = 5$. This implies

$$\overline{PQ} = 5\vec{i} - 10\vec{j} - 10\vec{k}$$

Hence

$$\overline{OQ} = \overline{PQ} + \overline{OP}$$

$$= (5\vec{i} - 10\vec{j} - 10\vec{k}) + (\vec{i} + \vec{j} + \vec{k})$$

$$= 6\vec{i} - 9\vec{j} - 9\vec{k}$$

So (B) is correct.

Case 2: $\lambda = -5$. This implies

$$\overline{PQ} = -5\vec{i} + 10\vec{j} + 10\vec{k}$$

Therefore

$$\overline{OQ} = \overline{PQ} + \overline{OP} = (-5\vec{i} + 10\vec{j} + 10\vec{k}) + (\vec{i} + \vec{j} + \vec{k})$$

$$= -4\vec{i} + 11\vec{j} + 11\vec{k}$$

So (C) is correct.

Answers: (B), (C)

25. In ΔABC, let I and O be the incentre and circumcentre, respectively. Then

(A) $a\overline{AI} + b\overline{BI} + c\overline{CI} = \vec{0}$ where a, b and c are the sides lengths of BC, CA and AB, respectively

(B) $\overline{OI} = \frac{1}{2s}(a\overline{OA} + b\overline{OB} + c\overline{OC})$ where $2s = a + b + c$

(C) if H is the orthocentre, then $\overline{OH} = \overline{OA} + \overline{OB} + \overline{OC}$

(D) $\overline{OG} = \frac{1}{3}(\overline{OA} + \overline{OB} + \overline{OC})$

Solution: Let AD be the internal bisector of $\underline{|A}$ meeting the side BC in D. Then

$$BD : DC = c : b$$

Take the circumcentre "O" as origin and let $\overline{OA} = \vec{\alpha}$, $\overline{OB} = \vec{\beta}$ and $\overline{OC} = \vec{\gamma}$. Then

$$\overline{OD} = \frac{b\vec{\beta} + c\vec{\gamma}}{b + c}$$

Now I divides AD in the ratio $(b + c) : a$. So

$$\overline{OI} = \frac{a\vec{\alpha} + (b + c)\left(\frac{b\vec{\beta} + c\vec{\gamma}}{b + c}\right)}{a + (b + c)}$$

$$= \frac{1}{2s}(a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma})$$

Hence (B) is correct. Also

$$\overline{AI} = \overline{OI} \quad \overline{OA} = \frac{1}{2s}(a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}) \quad \vec{\alpha}$$

$$= \frac{1}{2s}(b\vec{\beta} + c\vec{\gamma} \quad b\vec{\alpha} \quad c\vec{\alpha})$$

Similarly

$$\overline{BI} = \frac{1}{2s}(a\vec{\alpha} + c\vec{\gamma} \quad a\vec{\beta} \quad c\vec{\beta})$$

$$\overline{CI} = \frac{1}{2s}(a\vec{\alpha} + b\vec{\beta} \quad a\vec{\gamma} \quad b\vec{\gamma})$$

Therefore

$$a\overline{AI} + b\vec{\beta} + c\vec{\gamma} = \frac{1}{2s}[ab(\vec{\beta} \quad \vec{\alpha}) \quad ab(\vec{\beta} \quad \vec{\alpha}) + \cdots]$$

$$= \frac{1}{2s}(\vec{0}) = \vec{0}$$

So (A) is correct. For (C) see Example 5.3, Chapter 5, and (D) follows from the fact the centroid G divides the median through A in the ratio 2:1 reckoning from A.

Answers: (A), (B), (C), (D)

Matrix-Match Type Questions

1. $\vec{e}_1, \vec{e}_2$ and $\vec{e}_3$ are unit vectors such that

$$\vec{e}_1 + \vec{e}_2 + \vec{e}_3 = \vec{a}$$
$$\vec{e}_1 \quad (\vec{e}_2 \quad \vec{e}_3) = b$$
$$(\vec{e}_1 \quad \vec{e}_2) \quad \vec{e}_3 = \vec{c}$$
$$\vec{a} \cdot \vec{e}_1 = \frac{3}{2}$$
$$\vec{a} \cdot \vec{e}_2 = \frac{7}{4}$$
$$|\vec{a}| = 2$$

Match the items of Column I to those of Column II.

Column I	*Column II*
(A) $\vec{e}_1 \cdot \vec{e}_2$ is	(p) $\frac{2}{3}$
(B) $\vec{e}_2 \cdot \vec{e}_3$ is	(q) $\quad \frac{1}{4}$
(C) $\vec{e}_3 \cdot \vec{e}_1$ is	(r) $\frac{1}{2}$
	(s) 0
(D) $\vec{c} = \lambda \vec{e}_2$ where λ is	(t) $\frac{3}{4}$

Solution: We have

$$\vec{e}_1 \quad (\vec{e}_2 \quad \vec{e}_3) = \vec{b} \quad (\vec{e}_1 \cdot \vec{e}_3)\vec{e}_2 \quad (\vec{e}_1 \cdot \vec{e}_2)\vec{e}_2 = \vec{b} \qquad (6.76)$$

and

$$(\vec{e}_1 \quad \vec{e}_2) \quad \vec{e}_3 = \vec{c} \quad (\vec{e}_1 \cdot \vec{e}_3)\vec{e}_2 \quad (\vec{e}_2 \cdot \vec{e}_3)\vec{e}_1 = \vec{c} \qquad (6.77)$$

Now

$$\vec{e}_1 + \vec{e}_2 + \vec{e}_3 = \vec{a} \quad 1 + 1 + 1 + 2(\vec{e}_1 \cdot \vec{e}_2) + 2(\vec{e}_2 \cdot \vec{e}_3) + 2(\vec{e}_3 \cdot \vec{e}_1) = |\vec{a}|^2 = 4$$

Therefore

$$\vec{e}_1 \cdot \vec{e}_2 + \vec{e}_2 \cdot \vec{e}_3 + \vec{e}_3 \cdot \vec{e}_1 = \frac{1}{2} \qquad (6.78)$$

Again

$$\vec{a} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$$
$$\vec{a} \cdot \vec{e}_1 = \frac{3}{2}$$
$$\vec{a} \cdot \vec{e}_2 = \frac{7}{4}$$

These imply

$$\frac{3}{2} = \vec{a} \cdot \vec{e}_1 = 1 + \vec{e}_1 \cdot \vec{e}_2 + \vec{e}_1 \cdot \vec{e}_3$$

and

$$\frac{7}{4} = \vec{a} \cdot \vec{e}_2 = \vec{e}_1 \cdot \vec{e}_2 + 1 + \vec{e}_2 \cdot \vec{e}_3$$

Therefore

$$\vec{e}_1 \cdot \vec{e}_2 + \vec{e}_1 \cdot \vec{e}_3 = \frac{1}{2}$$

and

$$\vec{e}_1 \cdot \vec{e}_2 + \vec{e}_2 \cdot \vec{e}_3 = \frac{3}{4}$$

Using these values in Eq. (6.78) we have

$$\vec{e}_2 \cdot \vec{e}_3 = 0$$
$$\vec{e}_3 \cdot \vec{e}_1 = \quad \frac{1}{4}$$
$$\vec{e}_1 \cdot \vec{e}_2 = \frac{3}{4}$$

Using the values of $\vec{e}_1 \times \vec{e}_2 = 3/4, \vec{e}_2 \times \vec{e}_3 = 0$ and $\vec{e}_3 \times \vec{e}_1 = \ \ (1/4)$ in Eqs. (6.76) and (6.77) we have

$$\frac{1}{4}\vec{e}_2 \quad \frac{3}{4}\vec{e}_3 = \vec{b}$$

and
$$\vec{c} = \ \frac{1}{4}\vec{e}_2$$

Answer: (A) → (t), (B) → (s), (C) → (q), (D) → q

2. Match the items of Column I to those of Column II.

Column I	*Column II*
(A) If $\vec{a} = \vec{i} + \vec{j} + \vec{k}, \vec{a} \times \vec{b} = 1$ and $\vec{a} \quad \vec{b} = \vec{j} \quad \vec{k}$ then $\lvert\vec{b}\rvert$ is	(p) 0
(B) If $\vec{a} = \vec{i} \quad 3\vec{j} + \vec{k}, \vec{b} = 2\vec{i} + \lambda\vec{j} + \vec{k}$ and $\vec{c} = 3\vec{i} + \vec{j} \quad 2\vec{k}$ are coplanar, then λ equals	(q) 1
(C) If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then $[\vec{a} \quad \vec{b}\ \vec{b} + \vec{c}\ \vec{c} + \vec{a}]$ is equal to	(r) $\frac{1}{4}$
(D) $\vec{a}, \vec{b}$ are unit vector which include angle $\pi/6$. If $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, then $[\vec{a}\ \vec{b}\ \vec{c}]^2$ equals	(s) 4

Solution:

(A) We have

$$(\vec{a} \quad \vec{b}) \quad \vec{a} = (\vec{j} \quad \vec{k}) \quad (\vec{i} + \vec{j} + \vec{k})$$

$$(\vec{a} \times \vec{a})\vec{b} \quad (\vec{b} \times \vec{a})\vec{a} = 2\vec{i} \quad \vec{j} \quad \vec{k}$$

$$3\vec{b} \quad \vec{a} = 2\vec{i} \quad \vec{j} \quad \vec{k}$$

Therefore

$$3\vec{b} = 3\vec{i} \quad \text{or} \quad \vec{b} = \vec{i}$$

so that $|\vec{b}| = 1$.

Answer: (A) → (q)

(B) $\vec{a}, \vec{b}, \vec{c}$ are coplanar. This implies

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & \lambda & 1 \\ 3 & 1 & 2 \end{vmatrix} = 0$$

$$2\lambda \quad 1 + 3(\ 4 \quad 3) + (2 \quad 3\lambda) = 0$$

$$5\lambda \quad 20 = 0$$

$$\lambda = \ 4$$

Answer: (B) → (s)

(C) We have from Quick Look 11 that

$$[\vec{a} \quad \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} [\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$= 0[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

Answer: (C) → (p)

(D) We have

$$|\vec{a} \quad \vec{b}| = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \quad \vec{b}) \times \vec{c} = |\vec{a} \quad \vec{b}||\vec{c}| \sin 90° = \frac{1}{2}$$

Therefore

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \frac{1}{4}$$

Answer: (D) → (r)

3. Match the items of Column I to those of Column II.

Column I	*Column II*
(A) Let $\vec{a} = 2\vec{i} \quad \vec{j} + \vec{k}, \vec{b} = \vec{i} + 2\vec{j} \quad \vec{k}$ and $\vec{c} = \vec{i} + \vec{j} \quad 2\vec{k}$. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ has magnitude $\sqrt{2/3}$ is	(p) $2\vec{i} \quad \vec{j} + 5\vec{k}$ (q) $\sqrt{5}(\vec{i} + \vec{j} \quad 2\vec{k})$
(B) Let $\vec{a} = 2\vec{i} + \vec{j} + \vec{k}, \vec{b} = \vec{i} + 2\vec{j} \quad \vec{k}$. Then a vector coplanar with $\vec{a}$ and $\vec{b}$ and having magnitude $2\sqrt{5}$ and perpendicular to $\vec{a}$ is	(r) $2\vec{i} + 3\vec{j} \quad 3\vec{k}$
(C) Let $\vec{a} = \vec{i} \quad \vec{j}, \vec{b} = \vec{j} \quad \vec{k}, \vec{c} = \vec{k} \quad \vec{i}$. If $\vec{d}$ is a vector of magnitude $\sqrt{30}$ such that $\vec{a} \times \vec{d} = 0 = [\vec{b}\ \vec{c}\ \vec{d}]$, then $\vec{d}$ may be	(s) $\sqrt{10}(\ \vec{j} + \vec{k})$
(D) $\vec{a} = \vec{i} + \vec{j} + \vec{k}, \vec{c} = \vec{j} \quad \vec{k}$. If $\vec{b}$ is a vector such that $\vec{a} \times \vec{b} = 3$ and $\vec{a} \quad \vec{b} = \vec{c}$ is	(t) $\frac{1}{3}(5\vec{i} + 2\vec{i} + 2\vec{j})$

Solution:

(A) Neither $\vec{b}$ nor $\vec{c}$ is a required vector. Let $\vec{d}$ be the required vector and

$$\vec{d} = \vec{b} + x\vec{c} = (1 + x)\vec{i} + (2 + x)\vec{j} \quad (1 + 2x)\vec{k}$$

Therefore

$$\frac{|\vec{d}\times\vec{a}|}{|\vec{a}|}=\sqrt{\frac{2}{3}}$$

$$(\vec{d}\times\vec{a})^2=4$$

$$[2(1+x)\quad(2+x)\quad(1+2x)]^2=4$$

$$x+1=\pm 2$$

$$\boldsymbol{x}=1,\quad 3$$

Answer: (A) → (p), (r)

(B) Let $\vec{c}$ be required vector and

$$\vec{c}=x\vec{a}+y\vec{b}=(2x+y)\vec{i}+(x+2y)\vec{j}+(x\quad y)\vec{k}$$

Then

$$\vec{c}\times\vec{a}=0\qquad 2(2x+y)+1(x+2y)+(x\quad y)=0$$

$$6x+3y=0$$

$$2x+y=0 \tag{6.79}$$

Also

$$|\vec{c}|=2\sqrt{5}\qquad (2x+y)^2+(x+2y)^2+(x\quad y)^2=20$$

$$9x^2+9x^2=20\quad[\text{by Eq. (6.79)}]$$

$$\boldsymbol{x}=\pm\frac{\sqrt{10}}{3}$$

Now $x=\sqrt{10}/3$ and $y=\quad 2\sqrt{10}/3$ imply

$$\vec{c}=\sqrt{10}(\quad\vec{j}+\vec{k})$$

Answer: (B) → (s)

(C) Let $\vec{d}=x\vec{i}+y\vec{j}+z\vec{k}$. Therefore

$$\vec{d}\times\vec{a}=0\qquad x\quad y=0 \tag{6.80}$$

$$[\vec{b}\;\vec{c}\;\vec{d}]=0\qquad \begin{vmatrix} x & y & z\\ 0 & 1 & 1\\ 1 & 0 & 1\end{vmatrix}=0$$

$$x+y+z=0 \tag{6.81}$$

From Eqs. (6.80) and (6.81) we have

$$y=x\quad\text{and}\quad z=\quad 2x \tag{6.82}$$

Now

$$|\vec{d}|=30\qquad x^2+y^2+z^2=30$$

$$6x^2=30\quad[\text{by Eq. (6.82)}]$$

$$x=\pm\sqrt{5}$$

Also

$$x=\sqrt{5}\qquad \vec{d}=\sqrt{5}\vec{i}+\sqrt{5}\vec{j}\quad 2\sqrt{5}\vec{k}=\sqrt{5}(\vec{i}+\vec{j}\quad 2\vec{k})$$

Answer: (C) → (q)

(D) We have

$$\vec{a}=\vec{i}+\vec{j}+\vec{k}$$

$$\vec{c}=\vec{j}\quad\vec{k}$$

$$\vec{a}\times\vec{b}=3$$

$$\vec{a}\quad\vec{b}=\vec{c}$$

Now

$$\vec{a}\quad\vec{b}=\vec{c}=\vec{j}\quad\vec{k}$$

$$(\vec{a}\quad\vec{b})\quad\vec{a}=(\vec{j}\quad\vec{k})\quad(\vec{i}+\vec{j}+\vec{k})$$

$$(\vec{a}\times\vec{a})\vec{b}\quad(\vec{b}\times\vec{a})\vec{a}=2\vec{i}\quad\vec{j}\quad\vec{k}$$

$$3\vec{b}\quad 3\vec{a}=2\vec{i}\quad\vec{j}\quad\vec{k}$$

$$3\vec{b}\quad 3(\vec{i}+\vec{j}+\vec{k})=2\vec{i}\quad\vec{j}\quad\vec{k}$$

$$\vec{b}=\frac{1}{3}(5\vec{i}+2\vec{j}+2\vec{k})$$

Answer: (D) → (t)

4. Let $\vec{a}=2\vec{i}+\vec{j}\quad\vec{k},\vec{b}=\vec{i}+2\vec{j}+3\vec{k},\vec{c}=\quad 3\vec{i}\quad\vec{j}+\vec{k}$ and suppose

$$\vec{l}=\frac{\vec{b}\quad\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

$$\vec{m}=\frac{\vec{c}\quad\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

$$\vec{n}=\frac{\vec{a}\quad\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

Let $\vec{d}$ be $\vec{i}+\vec{j}+\vec{k}$. If $\vec{d}=x\vec{l}+y\vec{m}+z\vec{n}$, then match the items of Column I to those of Column II.

Column I	*Column II*
(A) The value of $[\vec{a}\;\vec{b}\;\vec{c}]$ equals	(p) 5
(B) Value of x is	(q) 2
	(r) 1
(C) Value of y equals	(s) 1
(D) z is equal to	(t) 2

Solution: We have

$$\vec{d}=x\vec{l}+y\vec{m}+z\vec{n}=x\frac{(\vec{b}\quad\vec{c})}{[\vec{a}\;\vec{b}\;\vec{c}]}+y\frac{(\vec{c}\quad\vec{a})}{[\vec{a}\;\vec{b}\;\vec{c}]}+z\frac{(\vec{a}\quad\vec{b})}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

Now

$$[\vec{a}\;\vec{b}\;\vec{c}]=\begin{vmatrix} 2 & 1 & 1\\ 1 & 2 & 3\\ 3 & 1 & 1\end{vmatrix}$$

$$= 2(2+3) \quad 1(1+9) \quad 1(\;\; 1+6)$$
$$= \;\; 5$$

Also

$$2 = \vec{d}\times\vec{a} = x\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = x$$

$$x = 2$$

$$1 = \vec{d}\times\vec{b} = y\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = y$$

$$y = 1$$

$$1 = \vec{d}\times\vec{c} = z$$

$$z = \;\; 1$$

Answer: (A) → (p), (B) → (q), (C) → (r), (D) → (s)

5. Match the items of Column I to those of Column II.

Column I	***Column II***
(A) If $\vec{a}$ and $\vec{b}$ are non-zero vectors and $\lvert\vec{a}+\vec{b}\rvert = \lvert\vec{a}\rvert + \lvert\vec{b}\rvert$, then the angle between $\vec{a}$ and $\vec{b}$ is	(p) 0
	(q) $\text{Cos}^{1}\frac{1}{3}÷$
(B) If $\vec{a}$ and $\vec{b}$ non-zero vectors such that $\lvert\vec{a}+\vec{b}\rvert = \lvert\vec{a} \quad \vec{b}\rvert$, then angle θ between them is	(r) $\frac{\pi}{2}$
(C) In a regular tetrahedron, the angle between the opposite pair of edges is	(s) $\frac{\pi}{3}$
(D) In a cube, the angle between any two diagonals is	(t) $\text{Cos}^{1}\frac{1}{\sqrt{3}}÷$

Solution:

(A) Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then

$$\lvert\vec{a}+\vec{b}\rvert^2 = (\lvert\vec{a}\rvert+\lvert\vec{b}\rvert)^2 \quad 2\lvert\vec{a}\rvert\lvert\vec{b}\rvert\cos\theta = 2\lvert\vec{a}\rvert\lvert\vec{b}\rvert$$
$$\theta = 0$$

Answer: (A) → (p)

(B) We have

$$\lvert\vec{a}+\vec{b}\rvert^2 = \lvert\vec{a} \quad \vec{b}\rvert^2 \quad 2(\vec{a}\times\vec{b}) = \quad 2(\vec{a}\times\vec{b})$$
$$\vec{a}\times\vec{b} = 0$$

Answer: (B) → (r)

(C) Let $ABCD$ be a regular tetrahedron. Therefore, all the faces are equal triangles of equal edges. Take A as origin and let $\overline{AB} = \vec{b}$, $\overline{AC} = \vec{c}$ and $\overline{AD} = \vec{d}$. (AB, CD) is one pair of opposite edges and

$$\overline{AB}\times\overline{CD} = \vec{b}\times(\vec{d} \quad \vec{c})$$
$$= \vec{b}\times\vec{d} \quad \vec{b}\times\vec{c}$$
$$= \lvert\vec{b}\rvert\lvert\vec{d}\rvert\cos 60^\circ \quad \lvert\vec{c}\rvert\lvert\vec{d}\rvert\cos 60^\circ$$
$$= 0 \quad (\because \lvert\vec{b}\rvert = \lvert\vec{c}\rvert = \lvert\vec{d}\rvert)$$

Therefore angle between AB or CD is $\pi/2$.

Answer: (C) → (r)

(D) Let $\overline{OA} = \vec{a}$, $\overline{OB} = \vec{b}$ and $\overline{OC} = \vec{c}$ such that $\lvert\vec{a}\rvert = \lvert\vec{b}\rvert = \lvert\vec{c}\rvert$. Consider the cube with $\vec{a}, \vec{b}, \vec{c}$ as coterminous edges (student is advised to draw the diagram). We can see that $\vec{a}+\vec{b}+\vec{c}$ and $\vec{a}+\vec{b} \quad \vec{c}$ are a pair of diagonals. If θ is the angle between them, then

$$\cos\theta = \frac{\vec{a}\times\vec{a} + \vec{b}\times\vec{b} \quad \vec{c}\times\vec{c}}{\lvert\vec{a}+\vec{b}+\vec{c}\rvert\lvert\vec{a}+\vec{b} \quad \vec{c}\rvert} = \frac{1}{\sqrt{3} \quad \sqrt{3}} = \frac{1}{3}$$

Answer: (D) → (q)

Comprehension-Type Questions

1. Passage: If $\vec{a}$ and $\vec{b}$ are non-zero vectors and θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\theta = \text{Cos}^{1}\frac{\vec{a}\times\vec{b}}{\lvert\vec{a}\rvert\lvert\vec{b}\rvert}÷$$

Answer the following questions.

(i) If $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that each is perpendicular to sum of the other two and $\lvert\vec{a}\rvert = 2, \lvert\vec{b}\rvert = 3$ and $\lvert\vec{c}\rvert = 6$. Then the angle between $\vec{a}$ and $\vec{a}+\vec{b}+\vec{c}$ is

(A) $\text{Cos}^{1}\frac{2}{7}÷$ (B) $\frac{\pi}{4}$

(C) $\text{Cos}^{1}\frac{1}{7}÷$ (D) $\frac{\pi}{2}$

(ii) In the above problem, the angle between $\vec{a} \quad \vec{b}$ and $\vec{a}+\vec{b}+\vec{c}$ is

(A) $\text{Cos}^{1}\frac{3}{7}÷$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ (D) $\text{Cos}^{1}\frac{6}{7}÷$

(iii) If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular unit vectors, then the angle between the vectors $\vec{b} \quad \vec{c}+\vec{c} \quad \vec{a}+\vec{a} \quad \vec{b}$ and $\vec{a}+\vec{b}+\vec{c}$ is

(A) $\text{Cos}^{1}\frac{1}{7}÷$ (B) $\text{Cos}^{1}(1)$

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

Solution:

(i) By hypothesis $\vec{a}\cdot(\vec{b}+\vec{c})=0, \vec{b}\cdot(\vec{c}+\vec{a})=0$ and $\vec{c}\cdot(\vec{a}+\vec{b})=0$. Therefore

$$\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$$
$$\vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{a}=0$$
$$\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}=0$$

Adding all the three equations, we have

$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=0$$

and hence

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$$

Therefore

$$[\vec{a}\,\vec{b}\,\vec{c}]=|\vec{a}||\vec{b}||\vec{c}|=2\times3\times6=36$$

Now

$$|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\sum(\vec{b}\cdot\vec{c})$$
$$=4+9+36+0=49$$

So

$$|\vec{a}+\vec{b}+\vec{c}|=7$$

Now, angle between $\vec{a}+\vec{b}+\vec{c}$ and $\vec{a}$ is

$$\mathrm{Cos}^{-1}\left(\frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a})}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}\right)=\mathrm{Cos}^{-1}\left(\frac{\vec{a}\cdot\vec{a}}{7\times2}\right)$$
$$=\mathrm{Cos}^{-1}\left(\frac{4}{7\times2}\right)$$
$$=\mathrm{Cos}^{-1}\left(\frac{2}{7}\right)$$

Answer: (A)

(ii) We have

$$|\vec{a}\times\vec{b}|=2\times3=6$$

Now since $\vec{a},\vec{b},\vec{c}$ are at right angles,

$$(\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b}+\vec{c})=[\vec{a}\,\vec{b}\,\vec{c}]=|\vec{a}||\vec{b}||\vec{c}|$$
$$=2\times3\times6=36$$

Therefore angle between $\vec{a}\times\vec{b}$ and $\vec{a}+\vec{b}+\vec{c}$ is

$$\mathrm{Cos}^{-1}\left(\frac{(\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}\times\vec{b}||\vec{a}+\vec{b}+\vec{c}|}\right)=\mathrm{Cos}^{-1}\left(\frac{36}{6\times7}\right)=\mathrm{Cos}^{-1}\left(\frac{6}{7}\right)$$

Answer: (D)

(iii) We have

$$(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b}+\vec{c})=3[\vec{a}\,\vec{b}\,\vec{c}]=3$$

Now

$$|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|^2=|\vec{b}\times\vec{c}|^2+|\vec{c}\times\vec{a}|^2$$
$$+|\vec{a}\times\vec{b}|^2+2\sum(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a})$$
$$=1+1+1+2\sum(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a}) \qquad (6.83)$$

By Theorem 6.48

$$(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a})=\begin{vmatrix}\vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{a}\\ \vec{c}\cdot\vec{c} & \vec{c}\cdot\vec{a}\end{vmatrix}$$
$$=\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}=0$$

From Eq. (6.83)

$$|\vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}\quad \vec{a}\times\vec{b}|=\sqrt{3}$$

Also

$$|\vec{a}+\vec{b}+\vec{c}|=\sqrt{3}$$

Therefore angle between $\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$ and $\vec{a}+\vec{b}+\vec{c}$ is

$$\mathrm{Cos}^{-1}\left(\frac{3}{\sqrt{3}\times\sqrt{3}}\right)=\mathrm{Cos}^{-1}(1)=0$$

Note: You can take $\vec{a},\vec{b}$ and $\vec{c}$ as $\vec{i},\vec{j},\vec{k}$.

Answer: (B)

2. Passage: If $\vec{a},\vec{b},\vec{c}$ are non-coplanar vectors and $\vec{r}$ is any vector, then there exist unique set of scalars $\{x, y, z\}$ such that $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}$. Also three vectors $\vec{a},\vec{b},\vec{c}$ are non-coplanar if and only if $[\vec{a}\,\vec{b}\,\vec{c}]\neq0$. Let

$$\vec{a}=\vec{i}+\vec{j}-\vec{k}$$
$$\vec{b}=-\vec{i}+2\vec{j}+2\vec{k}$$
$$\vec{c}=-\vec{i}+2\vec{j}-\vec{k}$$

Answer the following three questions.

(i) If $\vec{d}=2\vec{i}+3\vec{j}+\vec{k}$ and $\vec{d}=x\vec{a}+y\vec{b}+z\vec{c}$, then

(A) $x=-\frac{7}{9}, y=\frac{11}{9}, z=-\frac{8}{9}$

(B) $x=\frac{7}{9}, y=\frac{11}{9}, z=-\frac{8}{9}$

(C) $x=\frac{7}{3}, y=\frac{11}{9}, z=-\frac{8}{9}$

(D) $x=\frac{11}{3}, y=-\frac{7}{3}, z=-\frac{8}{9}$

(ii) If the same $\vec{d}$ as in (i) is equal to $x(\vec{b}\times\vec{c})+y(\vec{c}\times\vec{a})+z(\vec{a}\times\vec{b})$, then

(A) $x=-\frac{4}{9}, y=-\frac{6}{9}, z=-\frac{1}{3}$

(B) $x=\frac{2}{9}, y=-\frac{6}{9}, z=-\frac{1}{9}$

(C) $x=-\frac{2}{9}, y=\frac{6}{9}, z=-\frac{1}{9}$

(D) $x=\frac{2}{9}, y=\frac{6}{9}, z=-\frac{1}{9}$

(iii) If the same

$$\vec{d}=x\left(\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right)+y\left(\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right)+z\left(\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right)$$

then

(A) $x=-2, y=-6, z=-1$

(B) $x=-2, y=6, z=1$

(C) $x=4, y=6, z=3$

(D) $x=2, y=3, z=1$

Solution:

(i) We have

$$2\vec{i}+3\vec{j}+\vec{k}=x\vec{a}+y\vec{b}+z\vec{c}$$
$$=(x-y-z)\vec{i}+(x+2y+2z)\vec{j}+(-x+2y-z)\vec{k}$$

Therefore

$$x-y-z=2 \quad (6.84)$$
$$x+2y+2z=3 \quad (6.85)$$
$$-x+2y-z=1 \quad (6.86)$$

Solving Eqs. (6.84), (6.85) and (6.86) we get that

$$x=\frac{7}{3}, y=\frac{11}{9}, z=-\frac{8}{9}$$

Answer: (C)

(ii) We have

$$[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}1 & 1 & -1\\ -1 & 2 & 2\\ -1 & 2 & -1\end{vmatrix}=-9$$

$$\vec{d}=2\vec{i}+3\vec{j}+\vec{k}=x(\vec{b}\times\vec{c})+y(\vec{c}\times\vec{a})+z(\vec{a}\times\vec{b})$$

$$4=\vec{d}\cdot\vec{a}=x[\vec{a}\,\vec{b}\,\vec{c}]=x(-9)\Rightarrow x=-\frac{4}{9}$$

$$6=\vec{d}\cdot\vec{b}=y[\vec{b}\,\vec{c}\,\vec{a}]=y(-9)\Rightarrow y=-\frac{2}{3}$$

$$3=\vec{d}\cdot\vec{c}=z[\vec{a}\,\vec{b}\,\vec{c}]z=(-9)\Rightarrow z=-\frac{1}{3}$$

Answer: (A)

(iii) We have

$$2\vec{i}+3\vec{j}+\vec{k}=\vec{d}=x\left(\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right)+y\left(\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right)+z\left(\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right)$$

Therefore

$$4=\vec{d}\cdot\vec{a}=x\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}=x$$

$$6=\vec{d}\cdot\vec{b}=y$$

$$3=\vec{d}\cdot\vec{c}=z$$

Answer: (C)

3. Passage: The equation of a straight line passing through the points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r}=(1-t)\vec{a}+t\vec{b}, t\in\mathbb{R}$. It is given that in $\Delta ABC, \underline{|A}=90°$ and the vertices B and C lie on the line joining the points $(-1,3,2)$ and $(1,1,3)$ and $\underline{|ABC}=30°$. If $A=(-1,2,3)$ (see Figure 6.54), then answer the following three questions.

(i) The vertex B may be

(A) $\left(\frac{2\sqrt{3}-1}{3}, \frac{7-2\sqrt{3}}{3}, \frac{7+\sqrt{3}}{3}\right)$

(B) $\left(\frac{2-\sqrt{3}}{3}, \frac{7-2\sqrt{3}}{3}, \frac{7+\sqrt{3}}{3}\right)$

(C) $\left(\frac{2\sqrt{3}+1}{3}, \frac{7-2\sqrt{3}}{3}, \frac{7+\sqrt{3}}{3}\right)$

(D) $\left(\frac{2\sqrt{3}-1}{3}, \frac{7+\sqrt{3}}{3}, \frac{7-2\sqrt{3}}{3}\right)$

(ii) The vertex C may be

(A) $\left(\frac{-3-2\sqrt{3}}{9}, \frac{2\sqrt{3}-3}{9}, \frac{21-\sqrt{3}}{9}\right)$

(B) $\left(\frac{2\sqrt{3}-3}{9}, \frac{21-2\sqrt{3}}{9}, \frac{21+\sqrt{3}}{9}\right)$

(C) $\left(\frac{2\sqrt{3}-3}{9}, \frac{-3-2\sqrt{3}}{9}, \frac{21+\sqrt{3}}{9}\right)$

(D) $\left(\frac{-3-2\sqrt{3}}{9}, \frac{21+\sqrt{3}}{9}, \frac{21-\sqrt{3}}{9}\right)$

(iii) If G is the centroid of the triangle, then $\overline{GA}+\overline{GB}+\overline{GC}$ equals

(A) $\overline{AB}+\overline{BC}+\overline{CA}$ (B) $\overline{AB}\times\overline{AC}$

(C) $\overline{BC}\times\overline{BA}$ (D) $\overline{CA}\times\overline{CB}$

Solution: Equation of the line BC is

$$\vec{r} = (1-t)(-1,3,2) + t(1,1,3)$$
$$= (2t-1, 3-2t, 2+t)$$

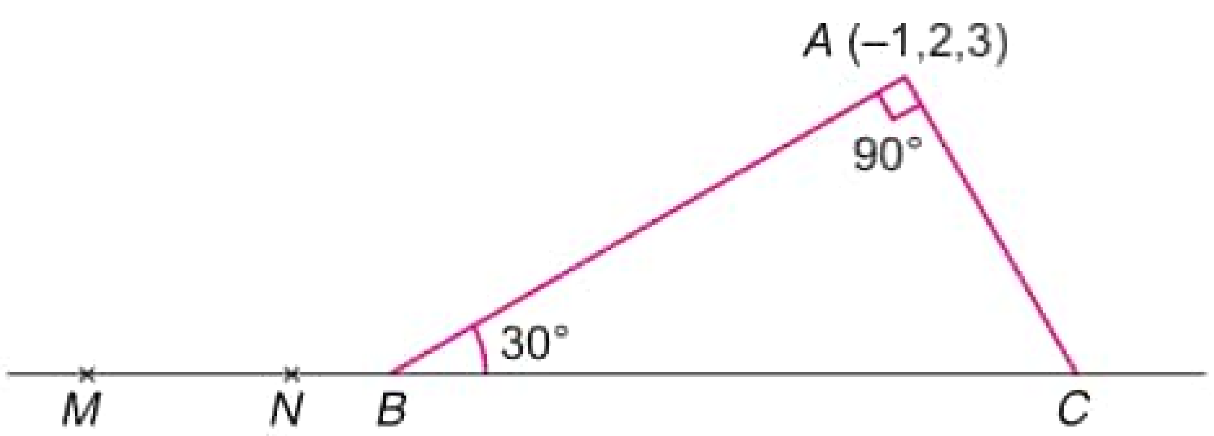

FIGURE 6.54 Comprehension-type question 3.

Therefore

$$\overline{BA} = (-2t, 2t-1, 1-t)$$
$$\overline{MN} = (2,-2,1)$$

Now

$$\frac{\sqrt{3}}{2} = \cos 30° = \frac{|-4t-4t+2+1-t|}{3\sqrt{4t^2+4t^2+1-4t+1+t^2-2t}}$$
$$= \frac{|9t-3|}{3\sqrt{9t^2-6t+2}}$$

Therefore

$$3(9t^2-6t+2) = 4(3t-1)^2$$
$$9t^2-6t-2=0$$
$$t = \frac{6 \pm 6\sqrt{3}}{18} = \frac{1 \pm \sqrt{3}}{3}$$

Let $\overline{CA} = (-2t_1, 2t_1-1, 1-t_1)$ and $\overline{BA} = (-2t, 2t-1, 1-t)$. Then

$$\overline{BA} \cdot \overline{CA} = 0 \Rightarrow 9tt_1 - 3t - 3t_1 + 2 = 0$$
$$\Rightarrow 3(3t-1)t_1 = 3t-2$$
$$\Rightarrow t_1 = \frac{3t-2}{3(3t-1)} = \frac{1}{3}\left[1 - \frac{1}{3t-1}\right]$$

Therefore

$$t = \frac{1+\sqrt{3}}{3} \Rightarrow t_1 = \frac{\sqrt{3}-1}{3\sqrt{3}}$$
$$t = \frac{1-\sqrt{3}}{3} \Rightarrow t_1 = \frac{\sqrt{3}+1}{3\sqrt{3}}$$

So

$$B = \left(\frac{2\sqrt{3}-1}{3}, \frac{7-2\sqrt{3}}{3}, \frac{7+\sqrt{3}}{3}\right)$$

and
$$C = \left(\frac{-2-\sqrt{3}}{3\sqrt{3}}, \frac{2+7\sqrt{3}}{3\sqrt{3}}, \frac{7\sqrt{3}-1}{3\sqrt{3}}\right)$$

OR

$$B = \left(\frac{-1-2\sqrt{3}}{3}, \frac{7+2\sqrt{3}}{3}, \frac{7-\sqrt{3}}{3}\right)$$

and
$$C = \left(\frac{2-\sqrt{3}}{3\sqrt{3}}, \frac{-2+7\sqrt{3}}{3\sqrt{3}}, \frac{7\sqrt{3}+1}{3\sqrt{3}}\right)$$

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the vertices A, B and C respectively. Then

$$G = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$
$$\overline{GA} = \vec{a} - \frac{\vec{a}+\vec{b}+\vec{c}}{3} = \frac{2\vec{a}-\vec{b}-\vec{c}}{3}$$

Similarly

$$\overline{GB} = \frac{2\vec{b}-\vec{c}-\vec{a}}{3}$$

and
$$\overline{GC} = \frac{2\vec{c}-\vec{a}-\vec{b}}{3}$$

Hence

$$\overline{GA} + \overline{GB} + \overline{GC} = \vec{0} = \overline{AB} + \overline{BC} + \overline{CA}$$

Answer: (i) → (A), (ii) → (B), (iii) → (A)

4. **Passage:** Two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if $\vec{b} = \lambda\vec{a}, \lambda \in \mathbb{R}$. Non-zero vectors $\vec{a}$ and $\vec{b}$ are at right angles if and only if $\vec{a}\cdot\vec{b} = 0$. If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, then every vector $\vec{r}$ in the plane determined by $\vec{a}$ and $\vec{b}$ can be written as $\vec{r} = x\vec{a} + y\vec{b}$ in one and only one way. Answer the following questions.

(i) If $\vec{b} = x\vec{i} + y\vec{j} + z\vec{k}$ such that $|\vec{b}| = 10$, $xyz < 0$ and is collinear with the vector $\vec{a} = 2\sqrt{2}\vec{i} - \vec{j} + 4\vec{k}$ then $\vec{b}$ is

(A) $(-4\sqrt{2}, 2, 8)$ (B) $(4\sqrt{2}, -2, 8)$
(C) $(-4\sqrt{2}, -2, -8)$ (D) $(4\sqrt{2}, 2, -8)$

(ii) Let $\vec{a} = (2,3,-1)$ and $\vec{b} = (1,-2,3)$. $\vec{c}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ and satisfies $\vec{c}\cdot\vec{d} = -6$ where $\vec{d} = 2\vec{i} - \vec{j} + \vec{k}$. Then $\vec{c}$ is equal to

(A) $(-3, 3, 3)$ (B) $(2, 2, -2)$
(C) $(2, 1, -1)$ (D) $(0, 1, 1)$

(iii) Let $\vec{a} = (-1,1,1)$ and $\vec{b} = (2,0,1)$. If $\vec{c}$ is vector coplanar with the vectors $\vec{a}$ and $\vec{b}$, is perpendicular to $\vec{b}$ and satisfies the condition $\vec{a}\cdot\vec{c} = 7$, then $\vec{c}$ equals

(A) $\left(-\frac{3}{2}, \frac{5}{2}, 3\right)$ (B) $(1, 0, -2)$

(C) (3, 0, 6) (D) $\frac{3}{2}, 0, \ 3\div$

Solution:

(i) Let

$$x\vec{i} + y\vec{j} + z\vec{k} = \vec{b} = \lambda\vec{a} = \lambda(2\sqrt{2}, \ \ 1, 4)$$

Therefore

$$x = 2\lambda\sqrt{2}, y = \ \ \lambda, z = 4\lambda$$

Now

$$|\vec{b}| = 10 \quad 8\lambda^2 + \lambda^2 + 16\lambda^2 = 100$$

$$\lambda = \pm 2$$

So

$$\lambda = 2 \quad x = 4\sqrt{2}, y = \ \ 2, z = 8$$

$$\lambda = \ \ 2 \quad x = \ \ 4\sqrt{2}, y = 2, z = \ \ 8$$

We have

$$xyz < 0 \quad x = 4\sqrt{2}, y = \ \ 2, z = 8$$

Answer: (B)

(ii) $\vec{a} = (2, 3, \ \ 1)$ and $\vec{b} = (1, \ \ 2, 3)$. Then

$$\vec{a} \quad \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 7\vec{i} \quad 7\vec{j} \quad 7\vec{k} = 7(\vec{i} \quad \vec{j} \quad \vec{k})$$

Since $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, let

$$\vec{c} = \lambda(\vec{a} \quad \vec{b}) = 7\lambda(\vec{i} \quad \vec{j} \quad \vec{k})$$

then

$$\vec{c} \times \vec{d} = \ \ 6 \quad 7\lambda(\vec{i} \quad \vec{j} \quad \vec{k}) \times (2\vec{i} \quad \vec{j} + \vec{k}) = \ \ 6$$

$$7\lambda(2 + 1 \quad 1) = \ \ 6$$

$$\lambda = \frac{3}{7}$$

Therefore

$$\vec{c} = 7\lambda(\vec{i} \quad \vec{j} \quad \vec{k}) = \ \ 3(\vec{i} \quad \vec{j} \quad \vec{k}) = (\ \ 3, 3, 3)$$

Answer: (A)

(iii) $\vec{a} = (\ \ 1, 1, 1)$ and $\vec{b} = (2, 0, 1)$. Let

$$\vec{c} = x\vec{a} + y\vec{b}$$

$$= (\ \ x + 2y, x, x + y) \tag{6.87}$$

Now

$\vec{c}$ is perpendicular to $\vec{b}$ $\quad \vec{c} \times \vec{b} = 0$

$$2x + 4y + x + y = 0$$

$$5y \quad x = 0 \tag{6.88}$$

Again

$$\vec{a} \times \vec{c} = 7 \quad x \quad 2y + x + x + y = 7$$

$$3x \quad y = 7 \tag{6.89}$$

Solving Eqs. (6.88) and (6.89) we have

$$x = \frac{5}{2}, y = \frac{1}{2}$$

Substituting these values in Eq. (9.87) we get

$$\vec{c} = \ \ \frac{5}{2} + \frac{2}{2}, \frac{5}{2}, \frac{5}{2} + \frac{1}{2}\div = \ \ \frac{3}{2}, \frac{5}{2}, 3\div$$

Answer: (A)

5. Passage: For any three vectors $\vec{a}, \vec{b}, \vec{c}$, the following hold good:

$$(\vec{a} \quad \vec{b}) \quad \vec{c} = (\vec{a} \times \vec{c})\vec{b} \quad (\vec{a} \times \vec{b})\vec{c}$$

$$\vec{a} \quad (\vec{b} \quad \vec{c}) = (\vec{a} \times \vec{c})\vec{b} \quad (\vec{a} \times \vec{b})\vec{c}$$

Let $\vec{a}$ and $\vec{b}$ be mutually perpendicular unit vectors. Answer the following three questions.

(i) If $\vec{d}$ is any vector such that $\vec{d} \quad \vec{b} = \vec{a} \quad \vec{d}$, then $(\vec{d} \quad \vec{b}) \quad \vec{b}$ equal

(A) $\vec{a}$ (B) $\ \vec{d}$ (C) $\vec{d}$ (D) $2\vec{b}$

(ii) If $\vec{d} \quad \vec{b} = \vec{a} \quad \vec{d}$, then $\vec{d}$ equals

(A) $\frac{(\vec{a} \quad \vec{b}) \quad \vec{a}}{2}$ (B) $\frac{\vec{a} + (\vec{a} \quad \vec{b})}{2}$

(C) $\vec{a} \quad \vec{b}$ (D) $\frac{\vec{a} \quad (\vec{a} \quad \vec{b})}{2}$

(iii) If $\vec{d} \quad \vec{b} = \vec{a} \quad \vec{d}$ which of the following statements is false?

(A) Vectors $\vec{a}, \vec{d}$ and $\vec{d} \quad \vec{b}$ are linearly dependent

(B) $[\vec{d}\ \vec{a}\ \vec{d} \quad \vec{b}] \quad 0$

(C) $\vec{d} \times \vec{b} = 0$ and $|\vec{d}| = \frac{1}{\sqrt{2}}$

(D) $[\vec{d}\ \vec{a}\ \vec{d} \quad \vec{b}] = 0$

Solution: By hypothesis $\vec{d}\quad \vec{b}=\vec{a}\quad \vec{d}$ and $\vec{a}\times\vec{b}=0$. Therefore

$$0=(\vec{d}\quad \vec{b})\times\vec{b}=\vec{a}\times\vec{b}\quad \vec{d}\times\vec{b}=0\quad \vec{d}\times\vec{b}$$

$$\vec{d}\times\vec{b}=0$$

Now

$$\vec{d}\quad \vec{b}=\vec{a}\quad \vec{d}\qquad (\vec{d}\quad \vec{b})\quad \vec{b}=\vec{a}\quad \vec{b}\quad \vec{d}\quad \vec{b}$$

$$(\vec{d}\times\vec{b})\vec{b}\quad (\vec{b}\times\vec{b})\vec{d}=\vec{a}\quad \vec{b}\quad \vec{d}\quad \vec{b} \qquad (6.90)$$

$$\vec{d}=\vec{a}\quad \vec{b}\quad \vec{d}\quad \vec{b}=\vec{a}\quad \vec{b}\quad (\vec{a}\quad \vec{d})$$

$$(\because\ \vec{d}\quad \vec{b}=\vec{a}\quad \vec{d})$$

$$\vec{d}=\frac{\vec{a}\quad (\vec{a}\quad \vec{b})}{2}$$

$$(\vec{d}\quad \vec{b})\quad \vec{b}=\vec{a}\quad \vec{b}\quad \vec{d}\quad \vec{b}$$

$$=\vec{a}\quad \vec{b}\quad \vec{d}\quad (\vec{a}\quad \vec{b})\quad \text{[by Eq. (6.90)]}$$

$$=\quad \vec{d}$$

Also

$$\vec{d}=\frac{\vec{a}\quad (\vec{a}\quad \vec{b})}{2}$$

Again

$$[\vec{d}\ \vec{a}\ \vec{d}\quad \vec{b}]=(\vec{d}\quad \vec{a})\times(\vec{d}\quad \vec{b})$$

$$=\begin{vmatrix} \vec{d}\times\vec{d} & \vec{d}\times\vec{b} \\ \vec{a}\times\vec{d} & \vec{a}\times\vec{b} \end{vmatrix}$$

$$=\begin{vmatrix} \vec{d}\times\vec{d} & 0 \\ \vec{a}\times\vec{d} & 0 \end{vmatrix}=0$$

Therefore (B) is false.

Answer: (i) → (b), (ii) → (D), (iii) → (B)

Assertion–Reasoning Type Questions

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both Statements I and II are true and Statement II is a correct explanation for Statement I
(B) Both Statements I and II are true but Statement II is not a correct explanation for Statement I
(C) Statement I is true and Statement II is false
(D) Statement I is false and Statement II is true

1. Statement I: If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, then $\vec{b}\quad \vec{c}$, $\vec{c}\quad \vec{a}$ and $\vec{a}\quad \vec{b}$ are also non-coplanar.

Statement II: Vectors $\vec{a},\vec{b},\vec{c}$ are coplanar if and only if $[\vec{a}\ \vec{b}\ \vec{c}]=0$.

Solution: According to Theorem 6.34 Statement II is correct. Now

$$[\vec{b}\quad \vec{c}\ \vec{c}\quad \vec{a}\ \vec{a}\quad \vec{b}]=[\vec{a}\ \vec{b}\ \vec{c}]^2\quad 0$$

because by Statement I $\vec{a},\vec{b},\vec{c}$ are non-coplanar.

Answer: (A)

2. Statement I: If $\vec{a},\vec{b},\vec{c}$ are coplanar vectors, then

$$\begin{vmatrix} \vec{a}\times\vec{a} & \vec{a}\times\vec{b} & \vec{a} \\ \vec{b}\times\vec{a} & \vec{b}\times\vec{b} & \vec{b} \\ \vec{c}\times\vec{a} & \vec{c}\times\vec{b} & \vec{c} \end{vmatrix}=\vec{0}$$

Statement II: $[\vec{a}\ \vec{b}\ \vec{c}](\vec{a}\quad \vec{b})=\begin{vmatrix} \vec{a}\times\vec{a} & \vec{a}\times\vec{b} & \vec{a} \\ \vec{b}\times\vec{a} & \vec{b}\times\vec{b} & \vec{b} \\ \vec{c}\times\vec{a} & \vec{c}\times\vec{b} & \vec{c} \end{vmatrix}$

Solution: Let

$$\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$$

$$\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$$

$$\vec{c}=c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$$

Then

$$[\vec{a}\ \vec{b}\ \vec{c}](\vec{a}\quad \vec{b})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

If A and B are two matrices of $3\quad 3$ order, then we know that det (AB) = (det A) (det B). Therefore

$$[\vec{a}\ \vec{b}\ \vec{c}](\vec{a}\quad \vec{b})=\begin{vmatrix} \vec{a}\times\vec{a} & \vec{a}\times\vec{b} & \vec{a} \\ \vec{b}\times\vec{a} & \vec{b}\times\vec{b} & \vec{b} \\ \vec{c}\times\vec{a} & \vec{c}\times\vec{b} & \vec{c} \end{vmatrix}$$

$\vec{a},\vec{b},\vec{c}$ are coplanar $\quad [\vec{a}\ \vec{b}\ \vec{c}]=0$. So

$$[\vec{a}\ \vec{b}\ \vec{c}](\vec{a}\quad \vec{b})=\vec{0}$$

Answer: (A)

3. Statement I: The shortest distance between the lines

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

and $$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$$

is $2\sqrt{19}$.

Statement II: The shortest distance between two skew lines $\vec{r}=\vec{a}+t\vec{b}$ and $\vec{r}=\vec{c}+s\vec{d}$ is

$$\frac{|(\vec{a}-\vec{c})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$$

Solution: According to Theorem 6.44, Statement II is correct.
In the given lines $\vec{a}=(3,5,7)$, $\vec{b}=(1,-2,1)$, $\vec{c}=(-1,-1,-1)$ and $\vec{d}=(7,-6,1)$. Now

$$\vec{a}-\vec{c}=(4,6,8)$$

and $$\vec{b}\times\vec{d}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 1\\ 7 & -6 & 1\end{vmatrix}=4\vec{i}+6\vec{j}+8\vec{k}$$

Therefore the shortest distance is

$$\frac{|(\vec{a}-\vec{c})\cdot(4\vec{i}+6\vec{j}+8\vec{k})|}{2\sqrt{4+9+16}}=\frac{|16+36+64|}{2\sqrt{29}}$$

$$=\frac{58}{\sqrt{29}}=2\sqrt{29}$$

Hence Statement I is false.

Answer: (D)

4. Statement I: In any ΔABC, $\cos 2A+\cos 2B+\cos 2C\geq -\frac{3}{2}$.

Statement II: If O is the circum centre of ΔABC, then $|\overline{OA}+\overline{OB}+\overline{OC}|\geq 0$.

Solution: Statement II is clearly true. It is known that the angles $(\overline{OB},\overline{OC})=2A$, $(\overline{OC},\overline{OA})=2B$ and $(\overline{OA},\overline{OB})=2C$. Now

$$0\leq|\overline{OA}+\overline{OB}+\overline{OC}|^2=|\overline{OA}|^2+|\overline{OB}|^2+|\overline{OC}|^2+2\sum(\overline{OB}\cdot\overline{OC})$$

$$=3R^2+2R^2(\cos 2A+\cos 2B+\cos 2C)$$

where $OA=OB=OC=R$. Therefore

$$0\leq 3+2(\cos 2A+\cos 2B+\cos 2C)$$

Hence

$$-\frac{3}{2}\leq\cos 2A+\cos 2B+\cos 2C$$

Answer: (A)

5. $\overline{PQ},\overline{QR},\overline{RS},\overline{ST},\overline{TU}$ and $\overline{UP}$ represent the sides of a regular hexagon (Figure 6.55).

Statement I: $\overline{PQ}\times(\overline{RS}+\overline{ST})\neq\vec{0}$

Statement II: $\overline{PQ}\times\overline{RS}=\vec{0}$ and $\overline{PQ}\times\overline{ST}\neq\vec{0}$.

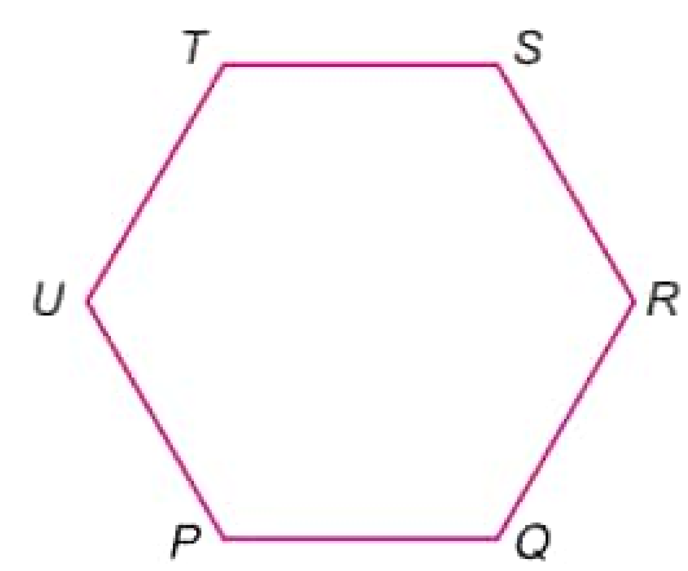

FIGURE 6.55 Assertion–reasoning type question 5.

Solution: We have

$$\overline{PQ}\times(\overline{RS}+\overline{ST})=\overline{PQ}\times\overline{RT}\neq 0$$

because $\overline{PQ}$ and $\overline{RT}$ are not parallel.
Statement I is true. Now

$$\overline{PQ}\times\overline{RS}\neq\vec{0}$$

because they are not parallel vectors. Also $\overline{PQ}\times\overline{ST}=\vec{0}$ because $\overline{PQ}$ and $\overline{ST}$ are parallel. Hence Statement II is false.

Answer: (C)

6. Consider the planes $3x-6y-2z=15$ and $2x+y-2z=5$.

Statement I: Let $\vec{n}_1=(3,-6,-2)$ and $\vec{n}_2=(2,1,-2)$ bee normals to the planes. Then $\vec{n}_1\times\vec{n}_2$ is parallel to the line of intersection of the planes.

Statement II: The vector $14\vec{i}+2\vec{j}+15\vec{k}$ is parallel to the line of intersection of the planes.

Solution: We have

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -6 & -2\\ 2 & 1 & -2\end{vmatrix}=14\vec{i}+2\vec{j}+15\vec{k}$$

Answer: (B)

Integer Answer Type Questions

1. Let $\vec{a} = (8, -10, 13)$. If $\vec{b} = (x, y, z)$ is a vector of magnitude $\sqrt{37}$ and collinear with $\vec{a}$ and making an acute angle with Z-axis, then the integral part of $x + y + z$ is ______.

Solution: We have

$$\vec{b} = \lambda\vec{a} \Rightarrow x = 8\lambda, y = -10\lambda, z = 13\lambda$$

Now

$$37 = |\vec{b}|^2 = x^2 + y^2 + z^2 = (333)\lambda^2$$

Therefore

$$\lambda^2 = \frac{1}{9} \quad \text{or} \quad \lambda = \pm\frac{1}{3}$$

But it makes acute angle with Z-axis. This implies

$$\lambda = \frac{1}{3}$$

Therefore

$$x = \frac{8}{3}, y = -\frac{10}{3} \quad \text{and} \quad z = \frac{13}{3}$$

So

$$x + y + z = \frac{11}{3} = 3.66\ldots$$

$$[x + y + z] = 3$$

Answer: 3

2. Let $\vec{a} = \vec{i} - \vec{j} + 3\vec{k}, \vec{b} = 3\vec{i} - 5\vec{j} + 6\vec{k}$. Then integer part of the magnitude of the projection of $2\vec{a} - \vec{b}$ on to the vector $\vec{a} + \vec{b}$ is ______.

Solution:

$$\vec{a} + \vec{b} = 4\vec{i} - 6\vec{j} + 9\vec{k}$$

$$2\vec{a} - \vec{b} = -\vec{i} + 3\vec{j}$$

$$(\vec{a} + \vec{b})\cdot(2\vec{a} - \vec{b}) = -4 - 18 = -22$$

$$|\vec{a} + \vec{b}| = \sqrt{133}$$

According to Quick Look 2, the magnitude of the projection of $2\vec{a} - \vec{b}$ on $(\vec{a} + \vec{b})$ is

$$\frac{|(\vec{a} + \vec{b})\cdot(2\vec{a} - \vec{b})|}{|\vec{a} + \vec{b}|} = \frac{22}{\sqrt{133}}$$

whose integer part is 1.

Answer: 1

3. $\overline{OA} = \vec{a}, \overline{OB} = 10\vec{a} + 2\vec{b}$ and $\overline{OC} = \vec{b}$ where O, A and C are non-collinear points. Let p be the area of the quadrilateral $OABC$ and q be the area of the parallelogram with OA and OC as adjacent sides. The $p = Kq$ where K is ______.

Solution: See single correct choice type question 57.

Answer: 6

4. P (1, −1, 2), Q (2, 0, −1) and R (0, 2, 1) are three points, then $[\overline{QR}\times\overline{RP}\ \overline{RP}\times\overline{PQ}\ \overline{PQ}\times\overline{QR}]$ is equal to ______.

Solution: We have

$$\overline{PQ} = \vec{i} + \vec{j} - 3\vec{k}$$

$$\overline{QR} = -2\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\overline{RP} = \vec{i} - 3\vec{j} + \vec{k}$$

Now

$$[\overline{PQ}\ \overline{QR}\ \overline{RP}] = \begin{vmatrix} 1 & 1 & -3 \\ -2 & 2 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 0$$

This is also evident from $\overline{PQ} + \overline{QR} + \overline{RP} = \vec{0}$. Now

$$[\overline{QR}\times\overline{RP}\ \overline{RP}\times\overline{PQ}\ \overline{PQ}\times\overline{QR}] = [\overline{PQ}\ \overline{QR}\ \overline{RP}]^2 = 0$$

Answer: 0

5. In a quadrilateral $ABCD$, $\overline{AB} = \vec{a}$, $\overline{AD} = \vec{d}$ and $\overline{AC} = m\vec{a} + n\vec{d}$ where m and n are positive integers. If the area of $ABCD$ is $5|\vec{a}\times\vec{d}|$, then the possible number of pairs (m, n) is ______.

Solution:

$$\begin{aligned}\text{Area of } ABCD &= \frac{1}{2}|\overline{AC}\times\overline{BD}| \\ &= \frac{1}{2}|(m\vec{a} + n\vec{d})\times(\vec{d} - \vec{a})| \\ &= \frac{1}{2}|(m + n)(\vec{a}\times\vec{d})| \\ &= \frac{m + n}{2}(\vec{a}\times\vec{d}) \\ &= 5|\vec{a}\times\vec{d}|\end{aligned}$$

This implies

$$\frac{m+n}{2} = 5$$

Therefore $m + n = 10$. Hence m takes values from 1 to 9 so that n takes values from 9 to 1. Hence number of pairs = 9.

Answer: 9

6. If $(\vec{b} \quad \vec{c}) \quad (\vec{c} \quad \vec{a}) = 3\vec{c}$, then the value of $[\vec{b} \quad \vec{c}\,\vec{c} \quad \vec{a} \quad \vec{a} \quad \vec{b}]$ is ______.

Solution: We have

$$3\vec{c} = (\vec{b} \quad \vec{c}) \quad (\vec{c} \quad \vec{a}) = [\vec{b}\,\vec{c}\,\vec{a}]\vec{c} \quad 0$$

Therefore

$$[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = 3$$

$$[\vec{b} \quad \vec{c}\,\vec{c} \quad \vec{a}\,\vec{a} \quad \vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 = 3^2$$

Answer: 9

7. If the point of intersection of the lines

$$\frac{x}{1} = \frac{y \quad 2}{2} = \frac{z+3}{3}$$

and
$$\frac{x \quad 2}{2} = \frac{y \quad 6}{2} = \frac{z \quad 3}{4}$$

is (x, y, z), then $y + x$ is ______.

Solution: Let L_1 and L_2 be the given lines, respectively. P is a point on L_1 and Q is a point on L_2. Therefore

$$P = (t, 2 + 2t, \quad 3 + 3t)$$

and
$$Q = (2 + 2s, 6 + 2s, 3 + 4s)$$

where $t, s \quad \mathbb{R}$. Now $P = Q$ implies

$$t = 2 + 2s \quad \text{or} \quad t \quad 2s = 2 \qquad (6.91)$$

$$2 + 2t = 6 + 2s \quad \text{or} \quad 2t \quad 2s = 4 \qquad (6.92)$$

and
$$3 + 3t = 3 + 4s \quad \text{or} \quad 3t \quad 4s = 6 \qquad (6.93)$$

From Eqs. (6.91) and (6.92) $t = +2, s = 0$ which also satisfy Eq. (6.93). Therefore the point of intersection is

$$(2, 6, 3) = (x, y, z)$$

So

$$y + x = 8$$

Answer: 8

8. If $A_1, A_2, \ldots A_8$ are vertices of a regular octagon, then

$${}^{7}_{j=1}(\overrightarrow{OA}_j \quad \overrightarrow{OA}_{j+1}) = K(\overrightarrow{OA}_1 \quad \overrightarrow{OA}_2)$$

where the value of K is ______.

Solution: Each side subtends an angle of $2\pi/8 = \pi/4$ at the centre of the octagon. Let "O" be the centre of the octagon and r the radius of the circumcircle of the octagon. Therefore

$$\overrightarrow{OA}_1 \quad \overrightarrow{OA}_2 = \quad r^2 \sin\frac{\pi}{4} \div \vec{n}$$

where $\vec{n}$ is the vector perpendicular to the plane of the polygon such that from the side of $\vec{n}$, the points $A_1, A_2, A_3, \ldots, A_n$ are in counterclock sense. Hence

$$\overrightarrow{OA}_2 \quad \overrightarrow{OA}_3 = \quad r^2 \sin\frac{\pi}{4} \div \vec{n}$$

$$\overrightarrow{OA}_3 \quad \overrightarrow{OA}_4 = \quad r^2 \sin\frac{\pi}{4} \div \vec{n}, \ldots \text{etc.}$$

Therefore

$${}^{7}_{j=1}(\overrightarrow{OA}_j \quad \overrightarrow{OA}_{j+1}) = 7 \quad r^2 \sin\frac{\pi}{4} \div \vec{n}$$

$$= 7(\overrightarrow{OA}_1 \quad \overrightarrow{OA}_2)$$

Answer: 7

9. $\overrightarrow{OA} = 2\vec{i} \quad 2\vec{j}, \overrightarrow{OB} = \vec{i} + \vec{j} \quad \vec{k}$ and $\overrightarrow{OC} = 3\vec{i} \quad \vec{k}$ are edges of a parallelopiped. Then the volume of the parallelopiped in cubic units is ______.

Solution:

$$[\overrightarrow{OA}\,\overrightarrow{OB}\,\overrightarrow{OC}] = \begin{vmatrix} 2 & 2 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= 2(\quad 1 + 0) + 2(\quad 1 + 3)$$

$$= \quad 2 + 4$$

$$= 2$$

$$\text{Volume of } OABC = |[\overrightarrow{OA}\,\overrightarrow{OB}\,\overrightarrow{OC}]| = 2$$

Answer: 2

10. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$. Let V_1 be the volume of the parallelopiped with edges $\vec{a}$, $\vec{b}$ and $\vec{c}$ and V_2 be the volume of the parallelopiped with coterminous edges $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$. Then $V_2 = KV_1$ where K equals ______.

Solution: We have

$$V_1 = |[\vec a\ \vec b\ \vec c]|$$

$$V_2 = |[\vec a + \vec b\ \ \vec b + \vec c\ \ \vec c + \vec a]| = |2[\vec a\ \vec b\ \vec c]| = 2V_1$$

Answer: 2

11. Let $\vec a, \vec b, \vec c$ be non-coplanar vectors and $A = 2\vec a + 3\vec b \quad \vec c, B = \vec a \quad 2\vec b + 3\vec c, C = 3\vec a + 4\vec b \quad 2\vec c$ and $D = \vec a \quad 6\vec b + 6\vec c$ be four points. Then $[\overline{AB}\ \overline{AC}\ \overline{AD}]$ is equal to ______.

Solution: We have

$$\overline{AB} = \quad \vec a \quad 5\vec b + 4\vec c$$

$$\overline{AC} = \vec a + \vec b \quad \vec c$$

$$\overline{AD} = \quad \vec a \quad 9\vec b + 7\vec c$$

Therefore using Quick Look 11 [part (1)] we have

$$[\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix} 1 & 5 & 4 \\ 1 & 1 & 1 \\ 1 & 9 & 7 \end{vmatrix} [\vec a\ \vec b\ \vec c]$$

$$= (2 + 30 \quad 32)[\vec a\ \vec b\ \vec c] = 0$$

Answer: 0

12. Let $ABCD$ be a parallelogram (Figure 6.56) in which AB is parallel to CD and AD is parallel to BC. K is a point on the side AD such that $AK = (1/5)(AD)$. L is a point on the diagonal AC such that $AL = (1/6)(AC)$. Then the area of the parallelogram for which $\overline{LB}$ and $\overline{LK}$ are adjacent sides is ______.

Solution: Take A as origin and let $\overline{AB} = \vec b$ and $\overline{AD} = \vec d$ so that $\overline{AC} = \vec b + \vec d$. By hypothesis

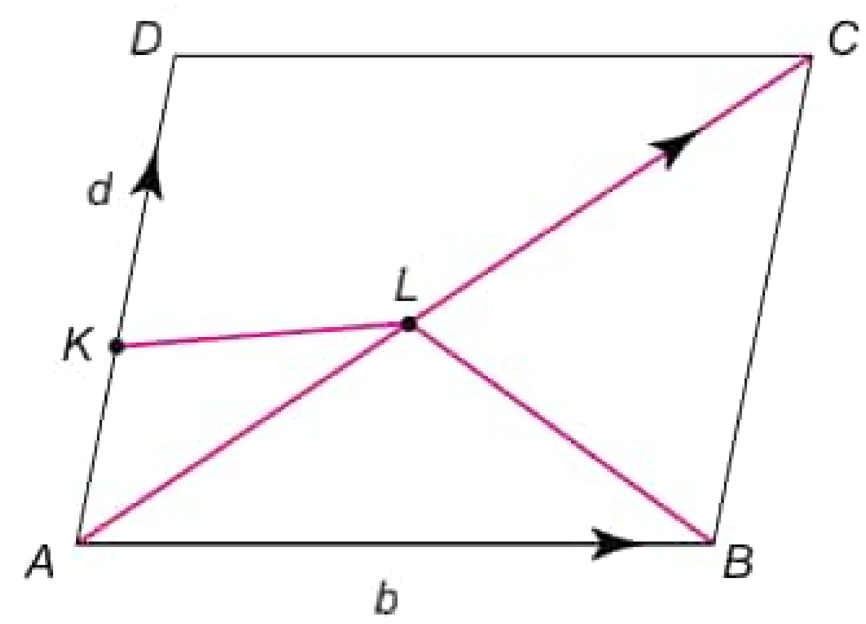

FIGURE 6.56 Integer type question 12.

$$\overline{AK} = \frac{1}{5}\vec d$$

$$\overline{AL} = \frac{1}{6}(\vec b + \vec d)$$

$$\overline{KL} = \frac{1}{6}(\vec b + \vec d) \quad \frac{1}{5}\vec d$$

$$= \frac{1}{30}(5\vec b \quad \vec d)$$

$$\overline{BL} = \frac{1}{6}(\vec b + \vec d) \quad \vec b$$

$$= \frac{1}{6}(\vec d \quad 5\vec b)$$

$$= \quad \frac{1}{6}(5\vec b \quad \vec d)$$

$$= (\quad 5)\left(\frac{1}{30}\right)(5\vec b \quad \vec d)$$

$$= (\quad 5)\overline{KL}$$

Therefore the vectors $\overline{LK}$ and $\overline{LB}$ are collinear vectors. Hence the area of the parallelogram for which $\overline{LK}$ and $\overline{LB}$ are adjacent sides is zero.

Answer: 0

SUMMARY

6.1 ***Scalar product (or dot product):*** Let $\vec a$ and $\vec b$ two vectors. Then we define $\vec a \cdot \vec b = \vec 0$ if either of them is zero vector. If $\vec a \quad \vec 0, \vec b \quad \vec 0$ and $\theta = (\vec a, \vec b)$ is the angle between $\vec a$ and $\vec b$, then $\vec a \cdot \vec b = |\vec a||\vec b|\cos\theta$.

QUICK LOOK

(i) $\vec a \cdot \vec b = \vec b \cdot \vec a$. But $(\vec a \cdot \vec b) \cdot \vec c$ has no meaning.

(ii) $0 \quad (\vec a, \vec b) < 90° \quad \cos(\vec a, \vec b) > 0 \quad \vec a \cdot \vec b > 0$.

(iii) $(\vec a, \vec b) = 90° \quad \vec a \cdot \vec b = 0$, when $\vec a \quad \vec 0 \quad \vec b$.

(iv) $(\vec a, \vec b) > 90° \quad \vec a \cdot \vec b < 0$.

6.2 ***Scalar and vector components:*** Let $\vec a$ and $\vec b$ be non-zero vectors and O, A and B be points in the space such that $\vec a = \overline{OA}$ and $\vec b = \overline{OB}$. Let Q be the foot of the perpendicular from B onto the support $\overline{OA}$ of $\vec a$. Then OQ is called the scalar component (or simply the component or the projection) of $\vec b$ on $\vec a$ if $(\vec a, \vec b) \quad 90°$, and $-OQ$ if $(\vec a, \vec b) > 90°$. $\overline{OQ}$ is called the *vector component* (or the orthogonal projection) of $\vec b$ on $\vec a$.

6.3 ***Orthogonal projection:*** The orthogonal projection of $\vec b$ on $\vec a$ is

$$\left(\frac{\vec b \cdot \vec a}{|\vec a|^2}\right)\vec a = \left(\vec b \cdot \frac{\vec a}{|\vec a|}\right)\frac{\vec a}{|\vec a|} = (\vec b \cdot \vec e)\vec e$$

where $\vec e = \vec a / |\vec a|$ is the unit vector in the direction of $\vec a$.

QUICK LOOK

(i) The orthogonal projection of $\vec{b}$ on $\lambda\vec{a}$ ($\lambda \neq 0$) is same as the orthogonal projection of $\vec{b}$ on $\vec{a}$.
(ii) The magnitude of the orthogonal projection of $\vec{b}$ on $\vec{a}$ is $\dfrac{|\vec{b}\cdot\vec{a}|}{|\vec{a}|}$.

6.4 ***Notation:*** $\vec{a}\cdot\vec{a}$ is denoted by $\vec{a}^2$ and note that $\vec{a}^2 = |\vec{a}|^2$.

6.5 ***For any two vectors $\vec{a}, \vec{b}$:***

(i) $(\vec{a}+\vec{b})^2 = \vec{a}^2 + 2(\vec{a}\cdot\vec{b}) + \vec{b}^2$

(ii) $(\vec{a}-\vec{b})^2 = \vec{a}^2 - 2(\vec{a}\cdot\vec{b}) + \vec{b}^2$

For three vectors $\vec{a}, \vec{b}, \vec{c}$,

$$(\vec{a}+\vec{b}+\vec{c})^2 = \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

6.6 ***Magnitude of sum and difference:*** If $\vec{a}$ and $\vec{b}$ are two vectors, then

(i) $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$. Equality holds $\Leftrightarrow$ either of $\vec{a}, \vec{b}$ is a zero vector or $\vec{a}, \vec{b}$ are like vectors.

(ii) $||\vec{a}| - |\vec{b}|| \le |\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$. Equality holds if and only if either $\vec{a}, \vec{b}$ is $\vec{0}$ or $\vec{a}, \vec{b}$ are like vectors.

QUICK LOOK

$|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$. Equality holds $\Leftrightarrow$ either of $\vec{a}, \vec{b}$ is $\vec{0}$ or they are collinear vectors.

6.7 ***Formulae:***

(i) $\vec{a} = (\vec{a}\cdot\vec{i})\vec{i} + (\vec{a}\cdot\vec{j})\vec{j} + (\vec{a}\cdot\vec{k})\vec{k}$

(ii) If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ and in particular $|\vec{a}|^2 = \vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2$

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If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, then $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

6.8 ***Expression for the angle between two vectors:*** Let $\vec{a}$ and $\vec{b}$ be non-zero vectors and $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ and $\theta = (\vec{a}, \vec{b})$. Then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1\vec{i} + a_2\vec{j} + a_3\vec{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

6.9 ***Cauchy–Schwartz inequality:*** For any triads of real numbers (a_1, a_2, a_3) and (b_1, b_2, b_3),

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

and the equality holds if and only if a_1, a_2, a_3 are proportional to b_1, b_2, b_3. In vector language,

$$|\vec{a}\cdot\vec{b}|^2 \le |\vec{a}|^2|\vec{b}|^2$$

where the equality holds if and only if $\vec{a}$ and $\vec{b}$ are collinear vectors.

6.10 ***Vector equation of a plane (Normal form):*** The equation of the plane, whose unit normal drawn from the origin is $\vec{n}$ and whose distance from the origin is a, is given by

$$\vec{r}\cdot\vec{n} = a$$

In particular, if the plane passes through origin, its equation is $\vec{r}\cdot\vec{n} = 0$.

6.11 ***Equation of the plane in Cartesian form:*** If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{n} = l\vec{i} + m\vec{j} + n\vec{k}$, then the equation of the plane is $lx + my + n = a$.

6.12 ***Cross-product:*** Let $\vec{a}$ and $\vec{b}$ be any vectors. We define

$$\vec{a}\times\vec{b} = \begin{cases} \vec{0} \text{ if either of } \vec{a}, \vec{b} \text{ is } \vec{0} \text{ or } \vec{a}, \vec{b} \text{ are collinear} \\ |\vec{a}||\vec{b}|\sin\theta\,\vec{n} \text{ otherwise} \end{cases}$$

where $\theta = (\vec{a}, \vec{b})$ and $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $(\vec{a}, \vec{b}, \vec{n})$ is a triad of right-handed system. $\vec{a}\times\vec{b}$ is called vector product or cross-product of $\vec{a}$ and $\vec{b}$.

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(i) $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow$ either of $\vec{a}, \vec{b}$ is $\vec{0}$ or $\vec{a}, \vec{b}$ are collinear vectors.

(ii) $|\vec{a}\times\vec{b}| \le |\vec{a}||\vec{b}|$ equality holds if and only if $\vec{a}, \vec{b}$ are orthogonal to each other.

(iii) $(\vec{a}, \vec{b}, \vec{n})$ is a right-handed system $\Rightarrow$ $(\vec{b}, \vec{a}, -\vec{n})$ is a right-handed system so that

$$\vec{b}\times\vec{a} = |\vec{b}||\vec{a}|\sin\theta(-\vec{n}) = -(\vec{a}\times\vec{b})$$

6.13 ***About $(\vec{i}, \vec{j}, \vec{k})$:*** Let $(\vec{i}, \vec{j}, \vec{k})$ be a triad of mutually perpendicular unit vectors forming right-handed system. Then

(i) $\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0}$.

(ii) $\vec{i}\times\vec{j} = \vec{k}, \vec{j}\times\vec{k} = \vec{i}, \vec{k}\times\vec{i} = \vec{j}$.

6.14 ***Formulae:***

(i) If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then

$$\vec{a} \quad \vec{b} = (a_2b_3 \quad a_3b_2)\vec{i} \quad (a_1b_3 \quad a_3b_1)\vec{j} + (a_1b_2 \quad a_2b_1)\vec{k}$$

which is denoted by

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and can be expanded as in the case of usual determinant expansion.

(ii) If $\vec{a} = a_1\vec{\alpha} + a_2\vec{\beta} + a_3\vec{\gamma}$ and $\vec{b} = b_1\vec{\alpha} + b_2\vec{\beta} + b_3\vec{\gamma}$ where $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ are non-coplanar vectors, then

$$\vec{a} \quad \vec{b} = \begin{vmatrix} \vec{\beta} \quad \vec{\alpha} & \vec{\gamma} \quad \vec{\alpha} & \vec{\alpha} \quad \vec{\beta} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

6.15 If θ is the angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ then

$$\sin\theta = \frac{|\vec{a} \quad \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{\quad (a_1b_2 \quad a_2b_1)^2}}{\sqrt{\quad a_1^2}\sqrt{\quad b_1^2}}$$

where $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

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(i) $|\vec{a} \quad \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 \quad (\vec{a} \cdot \vec{b})^2$.

(ii) Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and hence perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$ is

$$\pm\frac{\vec{a} \quad \vec{b}}{|\vec{a} \quad \vec{b}|}$$

6.16 ***Important advise:*** To find the angle between two vectors, always use dot product because the cross-product gives $\sin\theta$ and $\sin\theta > 0$ for $0 < \theta < 180°$.

6.17 ***Vector area:*** Let D be a plane region bounded by closed curve ζ. On ζ_2 mark three points P_1, P_2 and P_3. Let $\vec{n}$ be unit vector perpendicular to the plane of the region D such that from the side of $\vec{n}$, the points P_1, P_2 and P_3 be in counterclock sense. If is the area of the plane region D, then $\vec{n}$ is called the vector area of the region D. From the side of $\vec{n}$ if P_1, P_2 and P_3 are clockwise, then $(\quad \vec{n})$ is the vector area of the region D.

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(i) Vector area of the region $D = \quad (\pm\vec{n})$ where $\vec{n}$ is unit normal to the plane region D and is its area.

(ii) |Vector area| = Area.

6.18 ***Vector area of a quadrilateral and a parallelogram:***

(i) $\vec{a} \quad \vec{b}$ is the vector area of the parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides and $(1/2)(\vec{a} \quad \vec{b})$ is the area of the triangle for which $\vec{a}$ and $\vec{b}$ are two of the sides.

(ii) The vector area of a quadrilateral $ABCD$ in terms of its diagonal vectors is $(1/2)(\overline{AC} \quad \overline{BD})$ which is also valid, if $ABCD$ is a parallelogram.

6.19 **(i)** The vector area of $\quad ABC$ is

$$\frac{1}{2}\overline{AB} \quad \overline{AC} = \frac{1}{2}\overline{CB} \quad \overline{BA} = \frac{1}{2}\overline{CA} \quad \overline{CB}$$

(ii) If $\vec{a}, \vec{b}$ and $\vec{c}$ are the vertices of a triangle, then its vector area is

$$\frac{1}{2}(\vec{b} \quad \vec{c} + \vec{c} \quad \vec{a} + \vec{a} \quad \vec{b})$$

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In both 6.18 and 6.19 the corresponding modulus of vector area will give area.

6.20 ***Formula:*** If $\vec{a}$ is any vector, then

$$|\vec{a} \quad \vec{i}|^2 + |\vec{a} \quad \vec{j}|^2 + |\vec{a} \quad \vec{k}|^2 = 2|\vec{a}|^2$$

6.21 ***Scalar triple product:*** If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then $(\vec{a} \quad \vec{b}) \cdot \vec{c}$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\,\vec{b}\,\vec{c}]$.

6.22 ***Note:***

(i) $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.

(ii) Four points A, B, C and D are coplanar the vectors $\overline{AB}, \overline{AC}$ and $\overline{AD}$ are coplanar $[\overline{AB}\,\overline{AC}\,\overline{AD}] = 0$.

6.23 ***Geometrical interpretation of $[\vec{a}\,\vec{b}\,\vec{c}]$:*** Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors and V be the volume of the parallelopiped with $\vec{a}$, $\vec{b}$, $\vec{c}$ as coterminus edges.

$[\vec a, \vec b, \vec c] = V$ or $-V$ according as $(\vec a, \vec b, \vec c)$ is a right-handed system or left-handed system. In fact $|[\vec a, \vec b, \vec c]| = V$. The volume of the tetrahedron whose coterminus edges are $\vec a, \vec b$ and $\vec c$

$$\frac{1}{6}|[\vec a\ \vec b\ \vec c]|$$

6.24 ***Interchange of cross and dot:*** Since the triads $(\vec a, \vec b, \vec c)$, $(\vec b, \vec c, \vec a)$ and $(\vec a, \vec b, \vec c)$ are all either right-handed systems or all left-handed systems, it follows that

$$[\vec a\ \vec b\ \vec c] = [\vec b\ \vec c\ \vec a] = [\vec c\ \vec a\ \vec b] = +V \text{ or } -V$$

Hence

$$(\vec a \quad \vec b)\times\vec c = [\vec a\ \vec b\ \vec c] = [\vec b\ \vec c\ \vec a] = (\vec b \quad \vec c)\times\vec a = \vec a\times(\vec b \quad \vec c)$$

since dot is commutative.

6.25 ***Formula for $[\vec a, \vec b, \vec c]$:***

(i) Let $\vec a = a_1\vec i + a_2\vec j + a_3\vec k$, $\vec b = b_1\vec i + b_2\vec j + b_3\vec k$ and $\vec c = c_1\vec i + c_2\vec j + c_3\vec k$ then

$$[\vec a, \vec b, \vec c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(ii) If $\vec a = a_1\vec l + a_2\vec m + a_3\vec n$, $\vec b = b_1\vec l + b_2\vec m + b_3\vec n$ and $\vec c = c_1\vec l + c_2\vec m + c_3\vec n$ where $\vec l, \vec m$ and $\vec n$ are non-coplanar vectors, then

$$[\vec a, \vec b, \vec c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\vec l\ \vec m\ \vec n]$$

6.26 ***Important note:*** If $\vec a, \vec b, \vec c$ are expressed in $\vec i, \vec j$ and $\vec k$ or any three non-coplanar vectors $\vec l, \vec m$ and $\vec n$, the necessary and sufficient condition for $\vec a, \vec b, \vec c$ to be coplanar is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

6.27 ***Two important formulae:*** Let $\vec a = a_1\vec i + a_2\vec j + a_3\vec k$, $\vec b = b_1\vec i + b_2\vec j + b_3\vec k$, $\vec c = c_1\vec i + c_2\vec j + c_3\vec k$, $\vec l = l_1\vec i + l_2\vec j + l_3\vec k$, $\vec m = m_1\vec i + m_2\vec j + m_3\vec k$ and $\vec n = n_1\vec i + n_2\vec j + n_3\vec k$ then

(i) (a) $$[\vec l\ \vec m\ \vec n](\vec a \quad \vec b) = \begin{vmatrix} \vec l\times\vec a & \vec l\times\vec b & \vec l \\ \vec m\times\vec a & \vec m\times\vec b & \vec m \\ \vec n\times\vec a & \vec n\times\vec b & \vec n \end{vmatrix}$$

(b) $$[\vec a\ \vec b\ \vec c](\vec a \quad \vec b) = \begin{vmatrix} \vec a\times\vec a & \vec a\times\vec b & \vec a \\ \vec b\times\vec a & \vec b\times\vec b & \vec b \\ \vec c\times\vec a & \vec c\times\vec b & \vec c \end{vmatrix}$$

(ii) $$[\vec l\ \vec m\ \vec n][\vec a\ \vec b\ \vec c] = \begin{vmatrix} \vec l\times\vec a & \vec l\times\vec b & \vec l\times\vec c \\ \vec m\times\vec a & \vec m\times\vec b & \vec m\times\vec c \\ \vec n\times\vec a & \vec n\times\vec b & \vec n\times\vec c \end{vmatrix}$$

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In 6.27 (i) (b), if $\vec a, \vec b, \vec c$ are coplanar, then

$$\begin{vmatrix} \vec a\times\vec a & \vec a\times\vec b & \vec a \\ \vec b\times\vec a & \vec b\times\vec b & \vec b \\ \vec c\times\vec a & \vec c\times\vec b & \vec c \end{vmatrix} = \vec 0$$

In (ii) if either of the triad $(\vec l, \vec m, \vec n)$ or $(\vec a, \vec b, \vec c)$ is a set of coplanar vectors, then

$$\begin{vmatrix} \vec l\times\vec a & \vec l\times\vec b & \vec l\times\vec c \\ \vec m\times\vec a & \vec m\times\vec b & \vec m\times\vec c \\ \vec n\times\vec a & \vec n\times\vec b & \vec n\times\vec c \end{vmatrix} = 0$$

6.28 ***Volume of parallelopiped and tetrahedron:*** Let $\vec a = a_1\vec i + a_2\vec j + a_3\vec k$, $\vec b = b_1\vec i + b_2\vec j + b_3\vec k$, $\vec c = c_1\vec i + c_2\vec j + c_3\vec k$. Then

(i) Volume of the parallelopiped with $\vec a, \vec b, \vec c$ as coterminus edges is the absolute value of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(ii) The volume of the tetrahedron with $\vec a, \vec b$ and $\vec c$ as coterminus edges is the absolute value of

$$\frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(iii) Volume of the prism is half of the volume of the parallelopiped.

6.29 ***Vector product of three vectors:*** For any three vectors $\vec a, \vec b$ and $\vec c$

(i) $(\vec a \quad \vec b) \quad \vec c = (\vec a\times\vec c)\vec b \quad (\vec b\times\vec c)\vec a$

(ii) $\vec a \quad (\vec b \quad \vec c) = (\vec a\times\vec c)\vec b \quad (\vec a\times\vec b)\vec c$

6.30 Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors such that $\vec{a}$ and $\vec{b}$ are non-collinear and $\vec{b}$ is perpendicular to neither $\vec{a}$ nor $\vec{c}$. Then

$$(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$$

if and only if the vectors $\vec{a}$ and $\vec{c}$ are collinear vectors.

6.31 ***Vector product and dot product of four vectors:*** Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be four vectors. Then

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{c}\,\vec{d}]\vec{b}-[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$

and also

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$

From this we have

$$[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}=[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{d}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}$$

and hence if $[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$ (i.e. $\vec{a},\vec{b},\vec{c}$ are non coplanar) then

$$\vec{d}=\frac{[\vec{b}\,\vec{c}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{a}+\frac{[\vec{c}\,\vec{a}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{b}+\frac{[\vec{a}\,\vec{b}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{c}$$

Therefore as per the space representation theorem (Theorem 5.20, Chapter 5), if $\vec{d}=x\vec{a}+y\vec{b}+z\vec{c}$, then

$$x=\frac{[\vec{b}\,\vec{c}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]},\quad y=\frac{[\vec{c}\,\vec{a}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\quad\text{and}\quad z=\frac{[\vec{a}\,\vec{b}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

(ii) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}$

6.32 ***For any three vectors $\vec{a},\vec{b},\vec{c}$:***

(i) $[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}]=2[\vec{a}\,\vec{b}\,\vec{c}]$

(ii) $[\vec{b}\times\vec{c}\;\;\vec{c}\times\vec{a}\;\;\vec{a}\times\vec{b}]=[\vec{a}\,\vec{b}\,\vec{c}]^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}$

6.33 ***Reciprocal systems:*** Let $\vec{a},\vec{b},\vec{c}$ be three non-coplanar vectors. Define

$$\overline{a'}=\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$\overline{b'}=\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$\overline{c'}=\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Then $(\overline{a'},\overline{b'},\overline{c'})$ is called reciprocal system of $(\vec{a},\vec{b},\vec{c})$. In these systems the pairs $\vec{a},\overline{a'};\vec{b},\overline{b'};\vec{c},\overline{c'}$ are called corresponding vectors and the other pairs are called non-corresponding vectors.

6.34 ***Properties of reciprocal systems:***

(i) $\vec{a}\cdot\overline{a'}=\vec{b}\cdot\overline{b'}=\vec{c}\cdot\overline{c'}=1$ and $\vec{a}\cdot\overline{b'}=\vec{a}\cdot\overline{c'}=\vec{b}\cdot\overline{a'}=\vec{b}\cdot\overline{a'}=\vec{b}\cdot\overline{c'}=\vec{c}\cdot\overline{a'}=\vec{c}\cdot\overline{b'}=0$

(ii) $[\vec{a}\,\vec{b}\,\vec{c}][\overline{a'}\,\overline{b'}\,\overline{c'}]=1$

(iii) If $(\overline{a''},\overline{b''},\overline{c''})$ is the reciprocal system of $(\overline{a'},\overline{b'},\overline{c'})$ then $\overline{a''}=\vec{a},\overline{b''}=\vec{b}$ and $\overline{c''}=c$.

Remark: Without using the name reciprocal systems, questions have been asked in the past JEE examinations.

Note: If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, then $\overline{a'},\overline{b'},\overline{c'}$ (defined above) are also non-coplanar and hence if $\vec{r}$ is any vector, such that $\vec{r}=x\overline{a'}+y\overline{b'}+z\overline{c'}$, then $x=\vec{r}\cdot\vec{a}$, $y=\vec{r}\cdot\vec{b}$ and $z=\vec{r}\cdot\vec{c}$.

6.35 ***Vector equation of a plane using a scalar triple product:***

(i) The equation of the plane passing through a point $A(\vec{a})$ and parallel to two non-collinear vectors $\vec{b}$ and $\vec{c}$ is

$$[\vec{r}\,\vec{b}\,\vec{c}]=[\vec{a}\,\vec{b}\,\vec{c}]$$

(ii) The equation of the plane passing through two points $A(\vec{a}), B(\vec{b})$ and parallel to a vector $\vec{c}$ is

$$[\vec{r}\,\vec{b}\,\vec{c}]+[\vec{r}\,\vec{c}\,\vec{a}]=[\vec{a}\,\vec{b}\,\vec{c}]$$

(iii) The equation of the plane passing three non-collinear points $A(\vec{a}), B(\vec{b})$ and $C(\vec{c})$ is

$$[\vec{r}\,\vec{b}\,\vec{c}]+[\vec{r}\,\vec{c}\,\vec{a}]+[\vec{r}\,\vec{a}\,\vec{b}]=[\vec{a}\,\vec{b}\,\vec{c}]$$

(iv) The equation of the plane containing the line $\vec{r}=\vec{a}+t\vec{b}$ and perpendicular plane $\vec{r}\cdot\vec{c}=q$ is

$$[\vec{r}\,\vec{b}\,\vec{c}]=[\vec{a}\,\vec{b}\,\vec{c}]$$

Finally we conclude the summary with the concept of skew lines and shortest distance between two skew lines.

6.36 ***Skew lines:*** Any two non-coplanar lines are called skew lines. That is two lines are said to be skew lines, if there is no plane containing both.

6.37 ***Shortest distance between two skew lines:*** Two lines $\vec{r}=\vec{a}+t\vec{b}$ and $\vec{r}=\vec{c}+s\vec{d}$ where $s,t\in\mathbb{R}$ are skew lines if and only if $[\vec{a}-\vec{c}\;\vec{b}\;\vec{d}]\neq 0$. If these lines are skew, then the shortest between them is

$$\frac{|(\vec{a}-\vec{c})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$$