Assignment 3

Long Zhang ------- gstCN0342

Q1.

Assignment #3. Long Zhang.

Q1: (a). $\because Q = I - (2vv^T)/(v^Tv)$ and known $I^T = I$

$\therefore Q^T = I^T - (2vv^T)^T/(v^Tv)^T$

$\because (vv^T)^T = (v^T)^T v^T$, $(v^Tv)^T = v^T (v^T)^T$

$\therefore Q^T = I - (2vv^T)/(v^Tv) = Q$.

$\therefore$ Q is symmetric

(b).

$$QQ^T = \left(I - 2\frac{vv^T}{v^Tv}\right)\left(I - \frac{vv^T}{v^Tv}\right)^T$$

$$= I - 4I\frac{vv^T}{v^Tv} + 4\frac{vv^T}{v^Tv}\cdot\frac{vv^T}{v^Tv}$$

$$= I - 4\frac{vv^T}{v^Tv} + 4\frac{v(v^Tv)v^T}{(v^Tv)^2}$$

$$= I - \frac{4vv^T}{v^Tv} + 4\frac{(v^Tv)vv^T}{(v^Tv)^2}$$

$$= I - 4\frac{vv^T}{v^Tv} + 4\frac{vv^T}{v^Tv}$$

$$= I.$$

therefore, Q is orthogonal.

(c.) known y is orthogonal to v $\Rightarrow v^Ty = 0$.

$$\Rightarrow (x - Qx)^T y = (x^T - x^TQ^T)y$$

$$= \left[x^T - x^T\left(I - 2\frac{vv^T}{v^Tv}\right)\right]y$$

$$= 2x^T\frac{vv^T}{v^Tv}y$$

$$= \frac{2}{v^Tv}x^T vv^T y$$

$\because v^Ty = 0$, therefore $\frac{2}{v^Tv}x^Tvv^Ty = 0$

$$\Rightarrow (x - Qx)^T y = 0$$

$\therefore$ $x - Qx$ is orthogonal to y.

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Q2.

Q2 Assignment #3. Long Zhang.

$$A = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 2 & 2 \end{bmatrix} \xrightarrow{\text{elemination}} A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{rank} = 2.$$

$\therefore$ dim $col(A) = 2 =$ dim $col(A^T)$

dim $Null(A) = 1$

$$col(A) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad col(A^T) = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$Null(A) = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \qquad Null(A^T) = \begin{bmatrix} -2 & -2 & 1 \end{bmatrix}$$

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Q3.A

Q_3. Assignment #3 Long Zhang.

(a). $\because \|A\|_p = \sup_{\|x\|=1} \left\{ \frac{\|Ax\|_p}{\|x\|_p} : x \in K^n \right\}$.

$\|cA\| = |c|\|A\|$ for any $c \in \mathbb{R}$, $\|A\| \geq 0$.

$\therefore \|Ax\|_p \leq \|A\|_p \|x\|_p$

$\therefore \|ABx\|_p \leq \|A\|_p \|Bx\|_p \leq \|A\|_p \|B\|_p \|x\|_p$

$\because \|AB\|_p = \max_{x \neq 0} \frac{\|ABx\|_p}{\|x\|_p}$ $\quad \|A\|_p \|B\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \cdot \max_{x \neq 0} \frac{\|Bx\|_p}{\|x\|_p}$.

$\therefore \cancel{\max_{x \neq 0} \frac{\|ABx\|_p}{\|x\|_p} \leq \max \frac{\|Ax\|_p}{\|x\|_p} \frac{\|Bx\|_p}{\|x\|_p} \leq \max \frac{\|Ax\|_p}{\|x\|_p} \max \frac{\|Bx\|_p}{\|x\|_p} \max \|x\|_p}$

$\therefore \cancel{\|AB\|_p \leq \|A\|_p \|B\|_p}$

$\therefore \|AB\|_p \leq \|A\|_p \|B\|_p$.

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Q3.B

Q3

(b). $A \cdot A = UDV^*$

$$\|Ax\|_2^2 = (Ax)^2 = X \cdot A \cdot A X = X \cdot V \cdot D V X.$$

$$= \|D^{\frac{1}{2}} V x\|_2^2 \le \|D^{\frac{1}{2}}\|_2^2 = \sigma_{max}^2.$$

$$\|A\|_2^2 \ge \|Ay\|_2^2 = yAAy$$

$$= \sigma_{max}^2 y \cdot y = \sigma_{max}^2 \|y\|_2^2$$

$$= \sigma_{max}^2.$$

$$\therefore \|A\|_2 = \sigma_{max}.$$

Q4.

Assignment 3
Long Zhang ------- gstCN0342

Q4 Assignment #3 Long Zhang.

(a). $K_2(A) = \|A\|_2 \|A^{-1}\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \cdot \max_{x \neq 0} \frac{\|A^{-1}x\|_2}{\|x\|_2}$

$$= \sigma_{max}(A) / \sigma_{min}(A)$$

$K_p(AB) = \|AB\| \, \|(AB)^{-1}\| = \|AB\| \, \|B^{-1}A^{-1}\|$

$K_p(A) K_p(B) = \|A\| \|A^{-1}\| \cdot \|B\| \|B^{-1}\|$

from Q3, we know: $\|AB\| \le \|A\| \|B\|$, $\|A\| \ge 0$

therefore: $\|AB\| \, \|B^{-1}A^{-1}\| \le \|A\| \|A^{-1}\| \, \|B\| \|B^{-1}\|$

$\therefore K_p(AB) \le K_p(A) K_p(B)$.

(b) $\because A \Delta x = \Delta b \Rightarrow \Delta x = A^{-1} \Delta b \Rightarrow \|b\| \le \|\Delta b\| \cdot \|A^{-1}\| = \frac{\|\Delta b\|}{\sigma_1(A)}$

$\because Ax = b \Rightarrow \|b\| \le \|A\| \|x\| = \sigma_n(A) \cdot \|x\| \Rightarrow \frac{1}{\|x\|} \le \frac{\sigma_n(A)}{\|b\|}$

$\therefore \|b\| \le \frac{\|\Delta b\|}{\sigma_1(A)}$, $\frac{1}{\|x\|} \le \frac{\sigma_n(A)}{\|b\|}$

$\therefore \frac{\|\Delta x\|}{\|x\|} \le K(A) \frac{\|\Delta b\|}{\|b\|}$

(c) After re-calculate x, the value is no longer 0.01 and 0., it's 0.01 and 0.01.

(d)

```
22matrix det=
 1.0
22matrix condition=
 40002.000075
PS F:\hw3>
```

Determinant A : 1

condition number: 40002.000075

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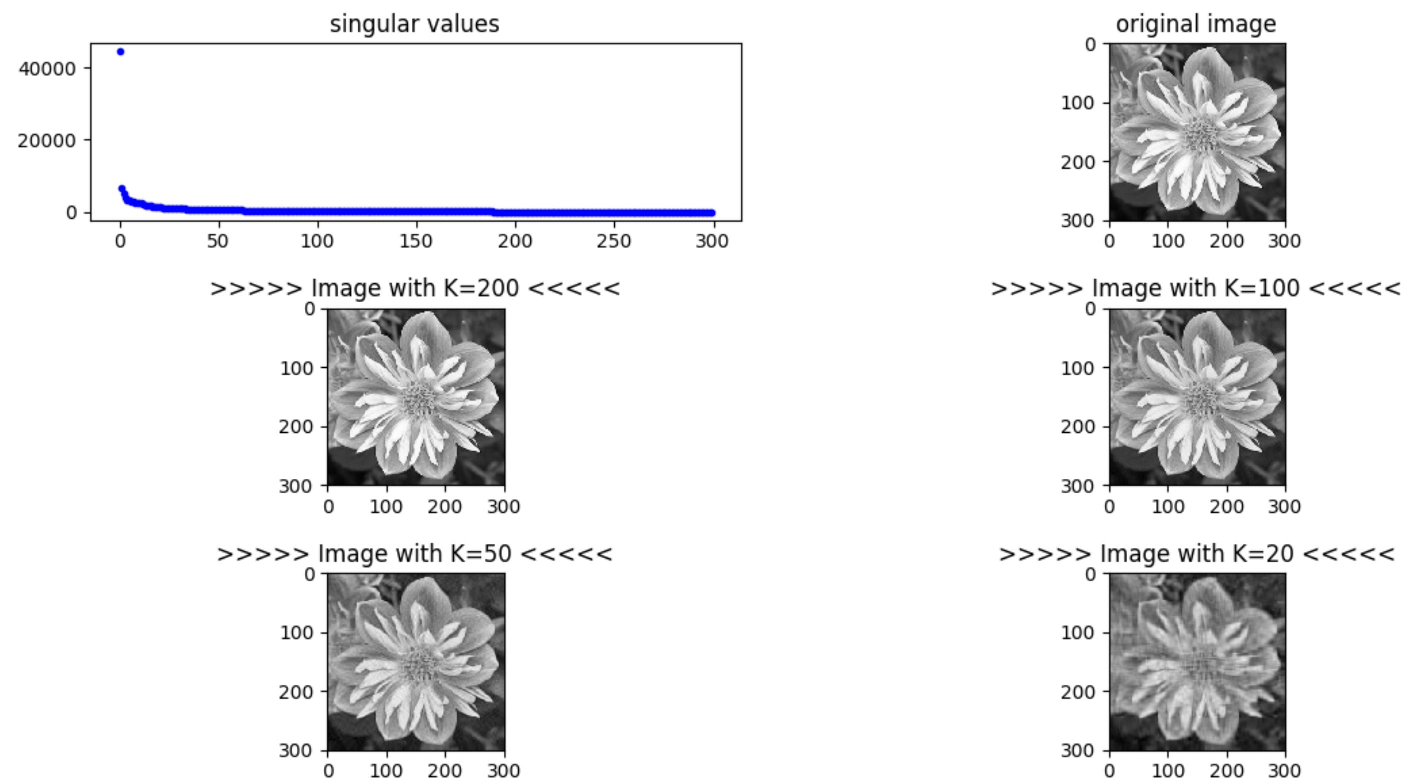
Q5.

Result:



As the K grows from 20 to 200 , the image become more clear, which means the compress rate is getting lower.

- **Is it worth transmitting when K = 200 ?**

Not really, when K = 200, we’re still keeping roughly 160000/300000 pixels. If we keep reduce K, we can get a better compress rate with a good quality.