Finite size scaling of conformal theories in the presence of a near-marginal operator

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The slowly evolving gauge coupling of strongly coupled gauge-fermion systems near the conformal boundary makes the numerical study of these models challenging. We show that finite size scaling techniques lead to inconsistent results if the near-marginal gauge coupling is neglected. When the corrections are included we find consistent results not only between different operators but between different gauge coupling values, even when combining data obtained using different lattice actions. Or results indicate that the SU(3) 12-fermion system is conformal with mass anomalous dimension $\gamma_m = 0.22(xx)$. We also consider the finite size scaling fits of the 8-fermion system where again we find consistent scaling, though in this case we are not able to distinguish conformal and chirally broken but volume squeezed behavior.

Strongly coupled gauge-fermion systems near the conformal window can be viable candidates to describe beyond-Standard Model physics. A walking gauge coupling with large anomalous dimension can give rise to an enhanced fermion condensate, while the slightly broken conformal symmetry results in a light dilaton that could play the role of the Higgs boson. While the non-perturbative properties of these systems are well suited to lattice studies, standard lattice methods are frequently not efficient to investigate the infrared properties of near-conformal systems. The problems are mainly due to the walking gauge coupling that makes this operator nearly marginal. In this paper we investigate finite size scaling, a well established method to predict critical scaling exponents, and show that it is essential to take into account the effect of the nearly marginal gauge coupling to obtain consistent results.

We concentrate on the SU(3) gauge system with 12 fundamental fermions, a controversial system. Several groups have studied the infrared properties of this model using different methods and different lattice actions, arriving at contradictory conclusions regarding its IR dynamics. (For a limited set of references see Refs. [1–15] .) In particular finite size scaling was considered in Ref. [4–6, 10, 11]. Inconsistencies of the scaling exponent as predicted by different operators lead some authors to strongly question the conformal behavior of this model.

We investigate this system at several gauge coupling values, and also analyze the published data of the Lattice Higgs (LHC) and Lattice KMI (LatKMI) collaborations [4, 11]. We develop a simple formalism that takes into account the effect of the nearly marginal gauge coupling in finite size scaling. We find consistent results for different operators when several gauge couplings are combined and even when different lattice actions are considered together, suggesting conformal infrared dynamics. This conclusion is further supported by the fact that at least some of the available data are in the strong coupling region relative to the apparent conformal fixed point. The predicted mass anomalous dimension, $\gamma_m = 0.20(xx)$, is consistent with results we obtained from the scaling of the Dirac operator spectral density as well. Preliminary results of our investigations have been reported in Ref. [16].

To further test our finite size scaling approach we analyze the available data for $N_f = 8$ fermion flavors from the LatKMI group [17] and the USBSM collaboration [18], combined with our own smaller data set. We find that all the data are consistent with conformal finite size scaling with scaling exponent $\gamma_m = 0.xx(xx)$. However this result could be consistent both with conformal and chirally broken infrared dynamics if all the data are in the weak coupling regime of a slowly walking system.

FINITE SIZE SCALING

Finite size scaling is a well understood technique in statistical physics. Its derivation is easiest using renormalization group analysis and has been reviewed recently in connection with infrared conformal systems [19–21]. Here we summarize only the steps relevant for the scaling of physical quantities "M_H" with mass (engineering) dimension $[M_H] = 1$.

For concreteness consider a system with one relevant operator, denoted by m, that has a scaling dimension $y_m > 0$. All other operators, denoted by g_i, are irrelevant with scaling exponents $y_i < 0$. Renormalization group arguments predict that in a finite spatial volume L^3, M_H depends only on specific combinations of the couplings, and can be written as

$$M_H = L^{-1} f\left(x, g_i m^{-y_i/y_m}\right), \tag{1}$$

where $x \equiv L m^{1/y_m}$. In the critical $m \to 0$ limit, $g_i m^{-y_i/y_m} \to 0$ and we find the familiar finite-size scaling formula

$$M_H = L^{-1} f_H(x), \tag{2}$$

where $f_H(x)$ is an arbitrary but unique scaling function that depends on the observable M_H. The exponent y_m is universal, characteristic of the corresponding fixed point.

If one of the irrelevant operators, let's say g_0, is nearly marginal with scaling exponent $y_0 \lesssim 0$, the term

$g_0 m^{-y_0/y_m}$ can remain significant and has to be included in the scaling analysis. This leads to the modified finite-size scaling formula

$$M_H = L^{-1} f\left(x, g_0 m^{\omega}\right), \tag{3}$$

where $\omega \equiv -y_0/y_m \gtrsim 0$. The scaling function $f\left(x, g_0 m^{\omega}\right)$ is analytic even at the fixed point, and can be expanded as

$$LM_H = F_H(x)\left\{1 + g_0 m^{\omega} G_H(x) + \mathcal{O}\left(g_0^2 m^{2\omega}\right)\right\}. \tag{4}$$

The first term is the usual finite-size scaling expression; the second term accounts for leading corrections to scaling due to the nearly-marginal g_0 coupling.

In the limit $x \to 0$, both $F_H(x)$ and $G_H(x)$ approach finite constants. In the infinite-volume limit, with small but fixed m, $F_H(x) \propto x$ while $G_H(x)$ remains finite. Our simulations cover a limited range $0.5 \lesssim x \lesssim 5$, over which we approximate $G_H(x)$ by a constant, $G_H(x) = c_G$, so

$$\frac{LM_H}{1 + c_G g_0 m^{\omega}} = F_H(x). \tag{5}$$

One can test the validity of this assumption by repeating the analyses using subsets of the data restricted to smaller ranges in x. Eq. 5 is very similar to the original Eq. 2 and can be fitted similarly. However, the analysis now involves three parameters: $c_0 \equiv c_G g_0$, y_0 and y_m.

FINITE SIZE SCALING FITS

In our numerical studies we use nHYP smeared staggered fermions with smearing parameters $(0.5, 0.5, 0.4)$ to ensure the numerical stability of simulations. Our gauge action contains fundamental and adjoint plaquette terms with $\beta_A/\beta_F = -0.25$ to avoid the potential scaling violation effects known to exist at positive adjoint plaquette coupling. In Ref. [7] we reported on the phase structure and other properties of this action.

In our previous studies we were able to run simulations in the $m = 0$ chiral limit with periodic spatial boundary conditions on volumes as large as $32^3 \times 64$ at gauge couplings $\beta_F \geq 2.8$, up to the single-site shift symmetry broken $\cancel{S^4}$ lattice phase [22]. In the present work we consider gauge couplings $\beta_F = 2.8$, 4.0, 4.5, 5.0, 5.5 and 6.0 on volumes $16^3 \times 32$, $20^3 \times 40$, $24^3 \times 48$ and $32^3 \times 64$.We choose the bare mass in the range $0.005 \leq m \leq 0.12$, requiring that the vector meson mass $M_V < 0.8$.

We start with a finite size scaling analysis using the usual form of Eq. 2, ignoring the effect of a potential near-marginal operator. We consider each operator and β_F data set independently and fit the $M_H L$ versus x with two independent quadratic polynomials, one at $x < x_0$ and the other at $x \geq x_0$. We minimize the χ^2 of this fit in terms of x_0 and y_m. The first row of Table I lists the relevant fit parameters, as well as the χ^2 per degree of freedom of the fit for the pseudo scalar at $\beta_F = 4.0$. (Figure 3 in Ref. [16] illustrated the "curve collapse" of this fit.) The left panel of Fig. 1 shows the results for y_m of similar analyses at other β_F values, as well as for the vector meson and f_π. The scaling exponents show significant variations between the three observables and as functions of β_F, suggesting that there is no consistent finite size scaling when using the form of Eq. 2.

Next we take into account the leading corrections according to Eq. 5. We are not able to constrain the value of the exponent y_0 from individual data sets so at this stage we fix $y_0 = -0.36$, the perturbative 2-loop value. Minimizing χ^2 in y_m, c_0 and x_0 results in a factor of two decrease as the second row of Table I shows.**Check this fit - we have two "2nd rows" now.** We obtain consistent results when fitting only the small- or large-x regions, justifying our approximation of constant $G(x) = c_G$.

Repeating this analysis at other gauge couplings, and for the other two quantities considered, leads to the results in the right panel of Fig. 1, showing consistency between all three operators in the whole β_F range investigated. Not surprisingly the errors are significantly larger with the corrected fit, especially for f_π where the data constrain the correction coefficient c_0 only weakly.

We strengthen the finite size scaling fit by combining data at different gauge couplings. If the gauge coupling is an irrelevant operator the scaling function $F_H(x)$ should be independent of β_F. Such a global fit allows us to determine both scaling dimensions y_m and y_0. The fit also depends on the scaling function $F_H(x)$ that is independent of β_F, and the coefficients c_0 that depend on the gauge coupling. In addition we need a new parameter s_m that rescales the bare mass $m \to s_m m$ to a common reference value. In the following we choose $s_m = 1$ at $\beta_F = 4.0$. s_m depends on the gauge coupling but is independent of the operator. The next entry in Table I lists the results of a fit to the pseudo scalar that combines $\beta_F = 4.0$, 4.5 and 5.0. The universal fit parameters y_m and y_0 are consistent with previous values. χ^2/dof of the combined fit is somewhat smaller but comparable to the $\beta_F = 4.0$ one. Including data sets at $\beta_F = 2.8$, 5.5 and 6.0 does not change the conclusion. As expected the scale factors s_m increase with decreasing β_F but the c_0 coefficients decrease, suggesting that the conformal fixed point with $g_0 \approx 0$ occurs at larger gauge coupling.

If the gauge coupling is an irrelevant operator, one should be able to combine different lattice actions, not only different gauge couplings. Both the LHC and LatKMI collaborations [4, 11] published their spectrum results which we combine with our data at all available β_f couplings. As the next entry of Table I shows such a combined fit with 191 independent degrees of freedom is possible and results in χ^2/dof=1.2. The left panel of Fig. 2 illustrates the curve collapse of this fit.

Until now we considered only the pseudo scalar mass.

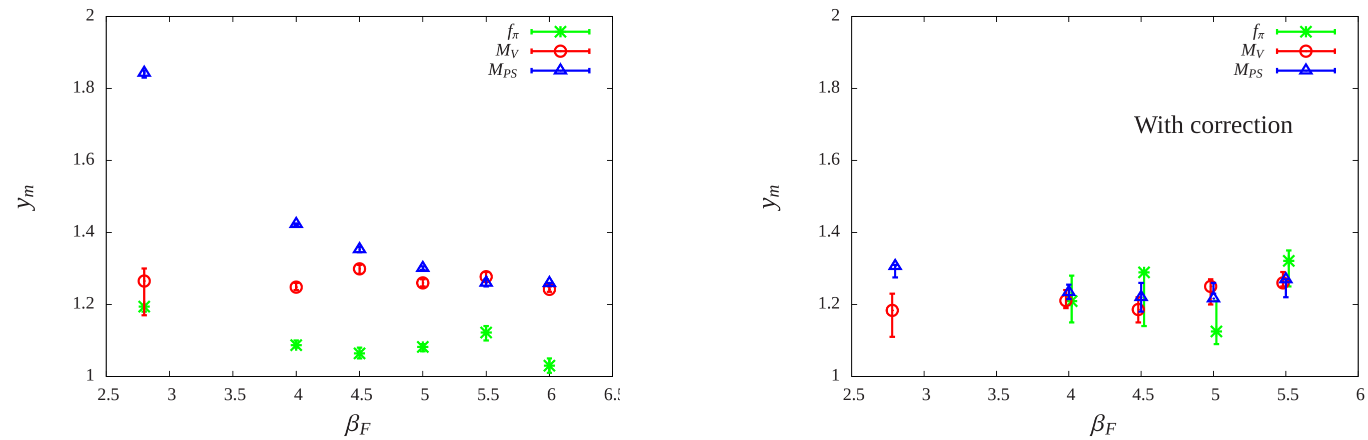

FIG. 1: The scaling dimension y_m predicted by finite size scaling, as a function of the gauge coupling β_F for the pseudoscalar (blue triangles), vector (red circles) and f_π (green $\times$s). Left: fits including only the relevant mass operator. Right: fits including both the relevant operator and leading irrelevant corrections with $y_0 = -0.36$ fixed at the two-loop value.**Redo these fits and cheek the errors!**

Operator	β	y_m	y_0	c_0	s_m	χ^2/dof[dof]	χ^2/dof[dof]
PS	4.0	1.4216(33)	-	0	-	5.9[42]	3.3 [29]
PS	4.0	1.178 (15)	-0.36	-0.790 (42)	-	1.9[42]	1.5 [28]
PS	4.0	1.24 (15)	-0.36	-0.65	-		1.3 [28]
PS	4.0	1.220(15)	-0.429(27)	-0.688(39)	1	3.6[117]	
	4.5			-0.486(49)	0.71		
	5.0			-0.379(56)	0.55		**1.2 [95]**
PS	2.8	1.1952(96)	-0.339(11)	-1.189(11)**1.62 ??**	3.0	7.6 [225]	
	4.0			-0.765(22)	1		
	4.5			-0.614(28)	0.71		
	5.0			-0.545(30)	0.55		
	5.5			-0.466(33)	0.44		
	6.0			-0.147 (46)	0.31		1.4[152]
	LHC[4]			-0.852 (19)	1.1		
	KMI 3.7[11]			-0.525(31)	0.43		
	KMI 4.0[11]			-0.xx(31)	0.xx		1.2 [191]
PS,V	4.0	1.xx	-0.xx	xx	1	xxx	
	4.5			xx	xx		
	5.0			xx	xx		
PS,V,f_π	4.0,4.5,5.0		as above	xx	1		
	LHC[4]			xx	xx		
	KMI[11]			xx	xx		

TABLE I: Results of the finite size scaling analysis in the 12 flavor system. The pseudo scalar and vector masses, and f_π are analyzed at various β_F couplings with the nHYP action, combined with the published data of the LHC and Lat-KMI collaborations [4, 11]. c_0 denotes the amplitude of the gauge coupling in the fit and s_m is the scale factor of the bare mass relative to the $\beta_F = 4.0$ nHYP data. The last column lists the χ^2/dof of the fit. When several data sets are combined this number refers to all data at and above the line it appears.

The vector meson and f_π can be analyzed similarly. One can even combine the three operators. The fit now depends on the universal scaling dimensions y_m and y_0, three scaling functions $F_H(x)$ for the three operators, the scale factors s_m at each gauge coupling value and the c_0 coefficients that depend both on the operator and β_F. **We might fix sm from the pion and reduce parameters other ways as well. This is still to be done! Add Nf=8; interpret and conclude**

CONCLUSION

This is from the Pos We have demonstrated that apparent inconsistencies in finite size scaling analyses of the $N_f = 12$ system can be resolved by considering the effect of the leading irrelevant gauge coupling, at least for f_π and the pseudoscalar and vector meson masses. We find that all three quantities, when considered independently,

FIG. 2: Left panel: The best curve collapse fit for the pseudoscalar mass combining all available gauge couplings, including the published data of Ref. [4, 11]. The fit parameters are listed in Table I. Right panel: Similar fit combining the pseudo scalar and vector meson masses and f_π.

prefer an anomalous dimension $\gamma_m^\star = y_m^\star - 1 \approx 0.25$. By performing a combined fit to all data used in this work, we hope to strengthen our conclusion and obtain a robust prediction for $\gamma_m^\star$. It will also be important to consider other quantities, such as the string tension and mass of the lightest baryon, but at present we do not have these data available to analyze.

We expect that systems near the conformal boundary will generically possess a nearly-marginal gauge coupling. The initial results presented here suggest that this may have important effects that will need to be investigated in future studies of strongly-coupled many-flavor systems.

Finite size scaling techniques provide an effective tool to investigate models governed by a fixed point with only one relevant operator, especially if the irrelevant operators are strongly irrelevant, i.e., their scaling dimensions are much below zero. If this condition is not met, either very large volumes have to be used, or corrections to scaling have to be taken into account. Both perturbation theory and non-perturbative step scaling function calculations predict that in the 12-flavor systems the gauge coupling has very small scaling exponent, $-0.3 \lesssim y_0 \lesssim -0.1$ [1, 23]. In this paper we consider the possibility that some of the inconsistencies found in earlier investigations are due to this nearly-marginal gauge coupling.

In order to investigate the effects of a nearly-marginal irrelevant gauge operator, it is essential to study the system at many gauge coupling values. In this work we cover a wide range from a strong coupling near the onset of the “$\cancel{S^4}$” lattice phase [7] to as weak coupling as our lattice volumes allow. We find that finite size scaling using only the leading relevant operator predicts scaling exponents that depend both on the physical quantity considered as well as on the bare gauge coupling, as shown in the left panel of Fig. 1. When we include the corrections to scaling due to the nearly-marginal gauge coupling, our preliminary analysis predicts scaling exponents that are, within errors, independent of the gauge coupling and consistent for the pseudoscalar meson, vector meson, and f_π, as shown in the right panel of Fig. 1.

While we cannot prove that all physical quantities will scale consistently once corrections to scaling are taken into account – especially because these corrections might be more important to some quantities than to others – our results resolve some of the existing controversies of the 12-flavor system and reinforce the IR-conformal interpretation suggested by our earlier studies of the bare step scaling function [3], phase transitions [22] and Dirac eigenvalues [8]. Our finite size scaling results prefer a fairly small anomalous dimension, $\gamma_m^\star = y_m^\star - 1 \approx 0.25$. The statistical errors on $\gamma_m^\star$ are about 10%, with similar systematic uncertainties for the three quantities considered. At this point we cannot give a more precise error estimate, but note that this value is consistent with our findings for $\gamma_m^\star$ from the Dirac operator spectral density [8].

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