CSC420 – Assignment 1

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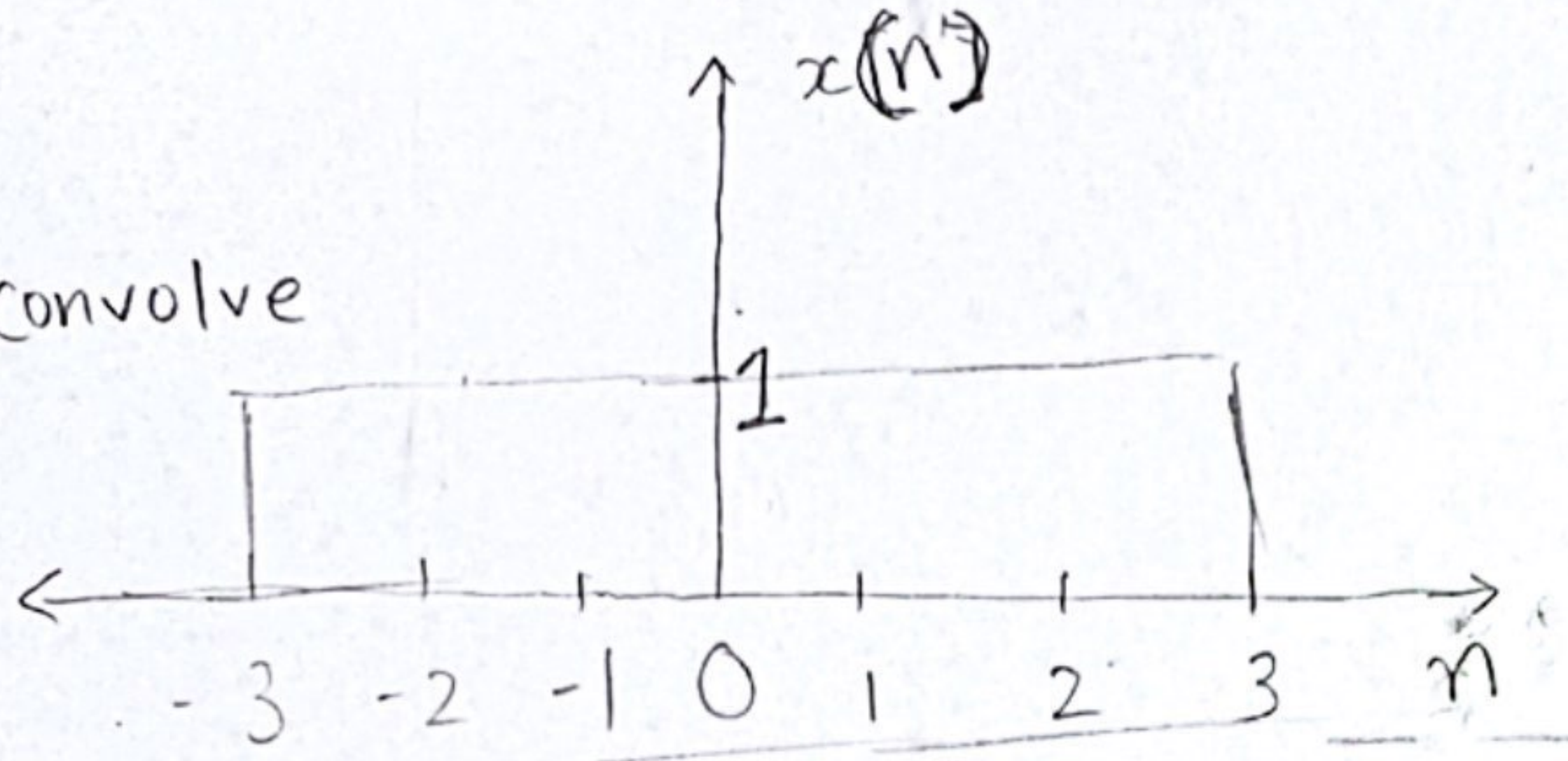
In solving the questions in this assignment, I worked together with my classmate Uttkarsh Berwal & 1005355018. I confirm that I have written the solutions/code/report in my own words

Part 1 :

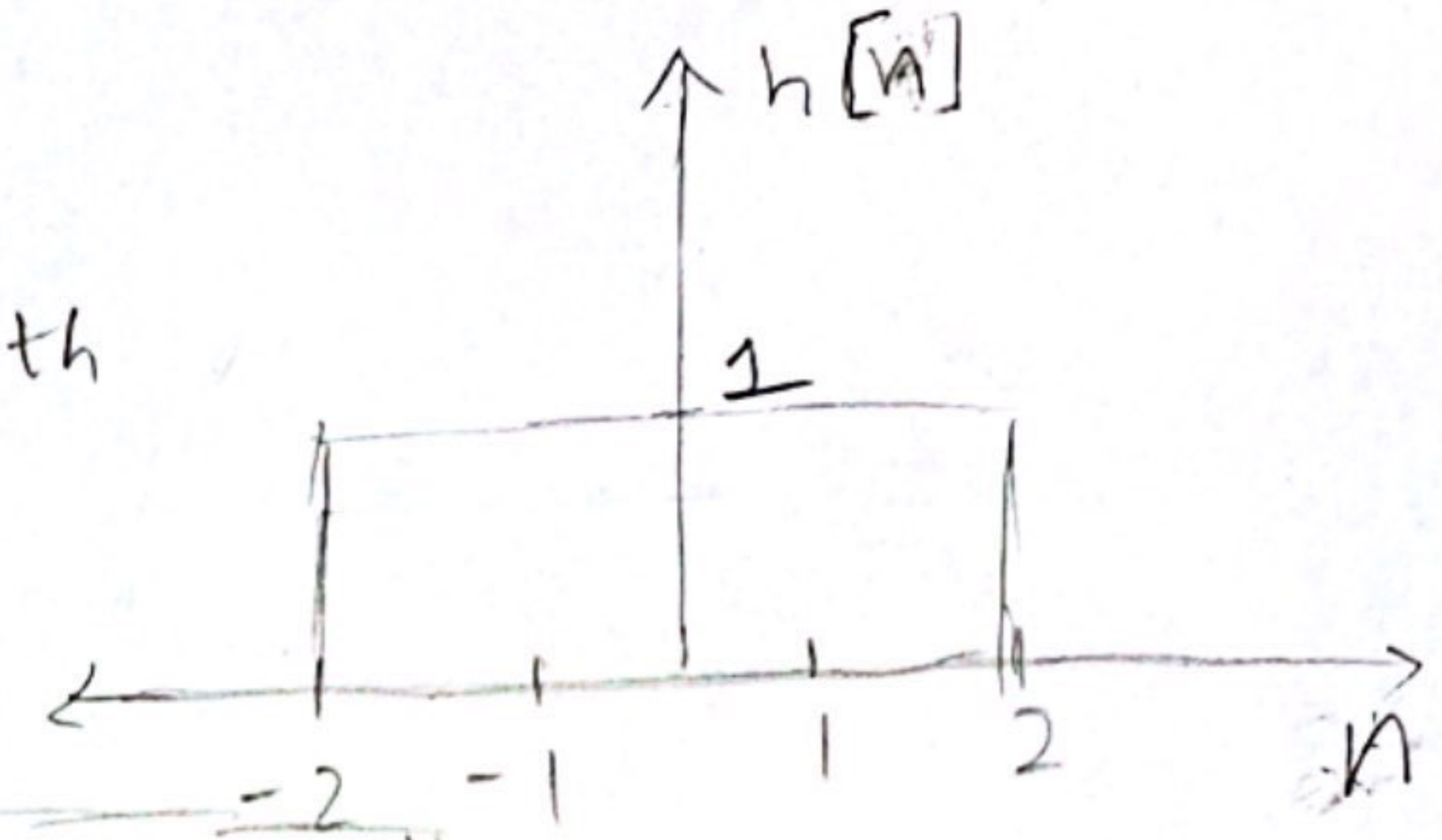
Q1 : Convolution .

1 .a} $x[n] = \begin{cases} 1 & -3 \leq n \leq 3 \\ 0 & \text{otherwise} \end{cases}$ $h[n] = \begin{cases} 1 & -2 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$

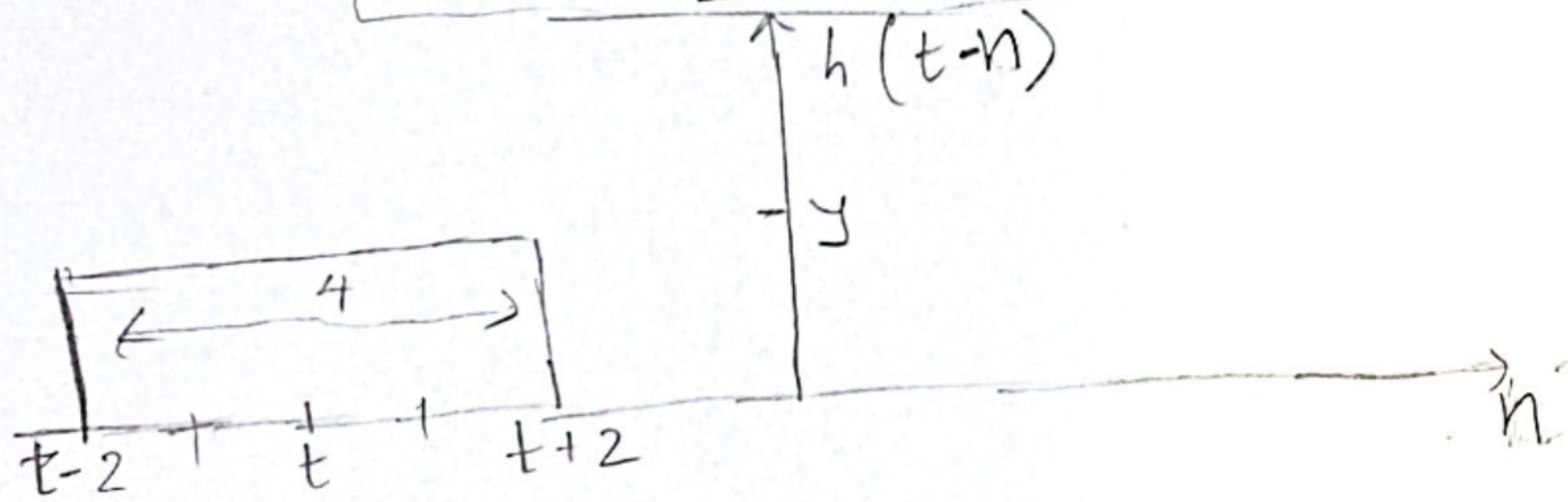
convolve



with



$$\boxed{y(t) = \int_{-\infty}^{\infty} x(n)\, h(n-z)\, dn}$$



Case 1 : $t < -5$ then $y(t) = 0$

Case 2 : $-5 \leq t < -1$, in this case the product of $x[n]$ and $h[t-n]$ be equal to 1 and the area will be increasing linearly till $t < -1$.

$$y(t) = \int_{-5}^{t} 1\, dn = t - (-5) = \boxed{t+5}$$

Case 3 : $-1 \le t \le 1$, Now in this case the product of $x(n)$ and $h(t-n)$ will be 1 but the area will be the same for all as the overall is in the complete area from $t-2$ to $t+2$.

$$y(t) = \int_{t-2}^{t+2} 1 \, dn = (t+2) - (t-2)$$
$$= t + 2 - t + 2 = \boxed{4}$$

Case 4 : $1 < t < 5$

For this case, again the product of $x(n)$ and $h(t-n)$ will be 4 but the area will be decreasing linearly as $h(t)$ overlap of $x(t)$ is decreasing as it keeps moving ahead.

$$y(t) = \int_{t-2}^{3} 1 \, dn = 3 - (t-2)$$
$$= 3 - t + 2 = \boxed{5-t}$$

Case 5 : $5 < t$ then $y(t) = 0$.

To sum up,

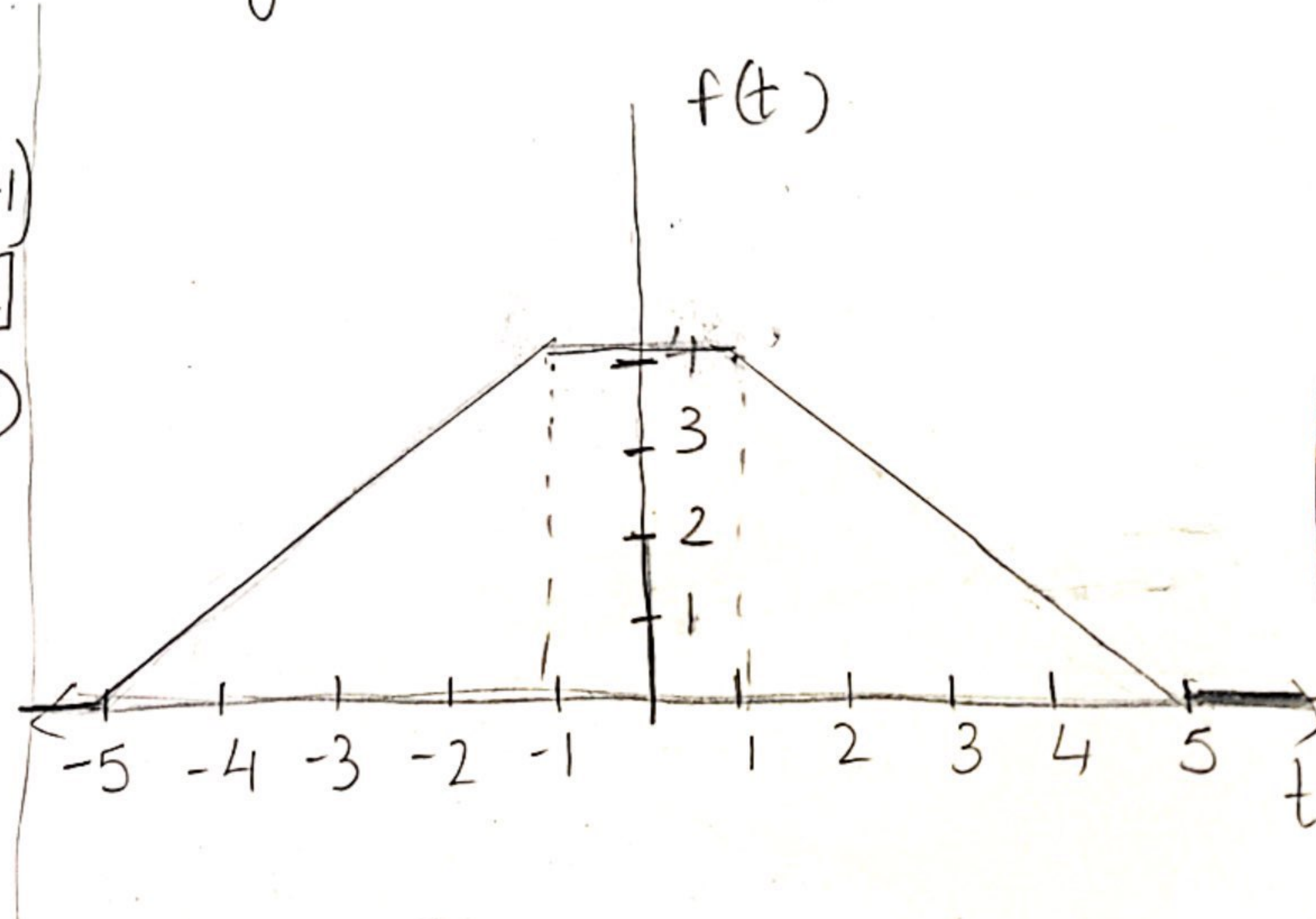
Case 1 : 0 for $t < -5$

Case 2 : $t+5$ for $t \in [-5, -1)$

Case 3 : 4 for $t \in [-1, 1]$

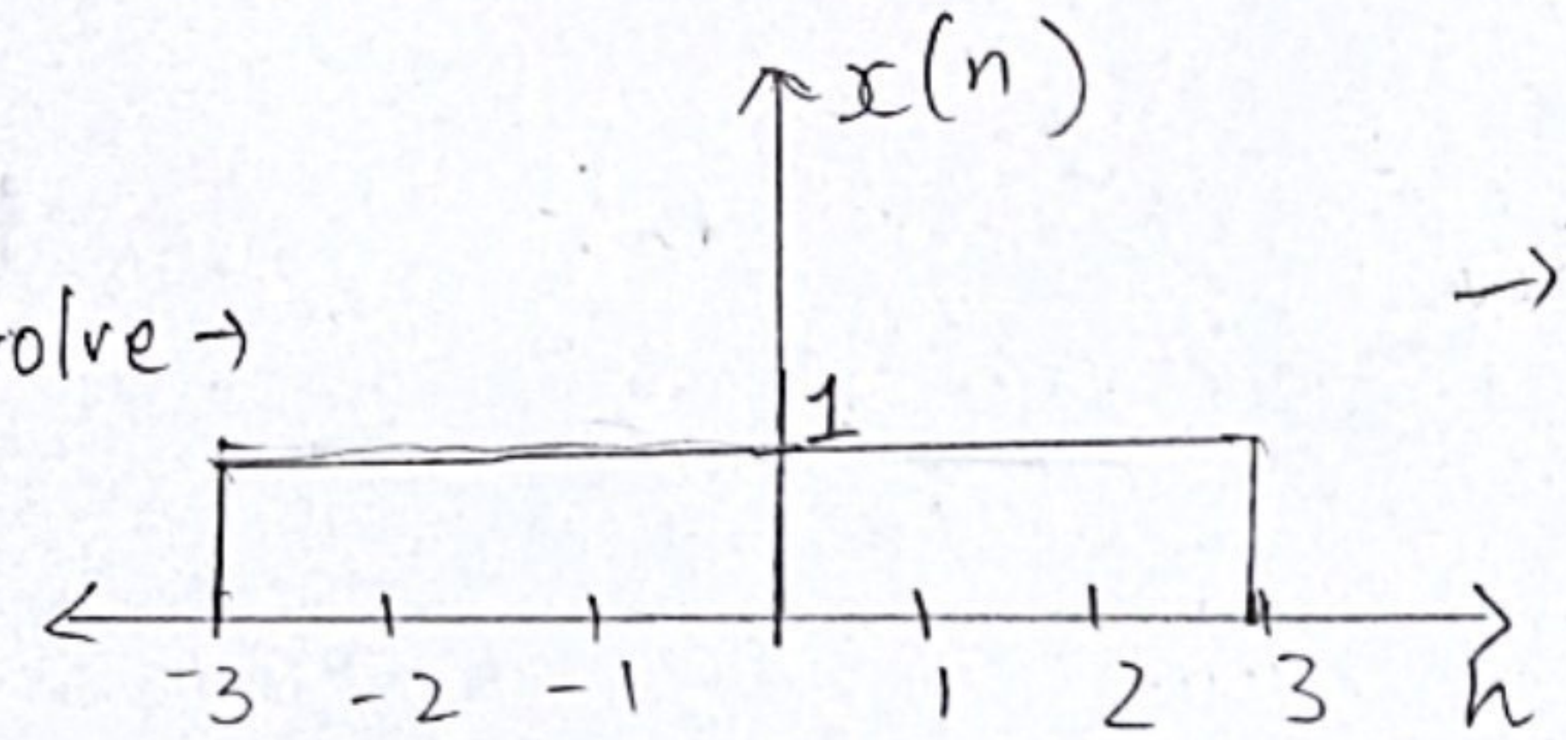
Case 4 : $5-t$ for $t \in (1, 5)$

Case 5 : 0 for $t > 5$.

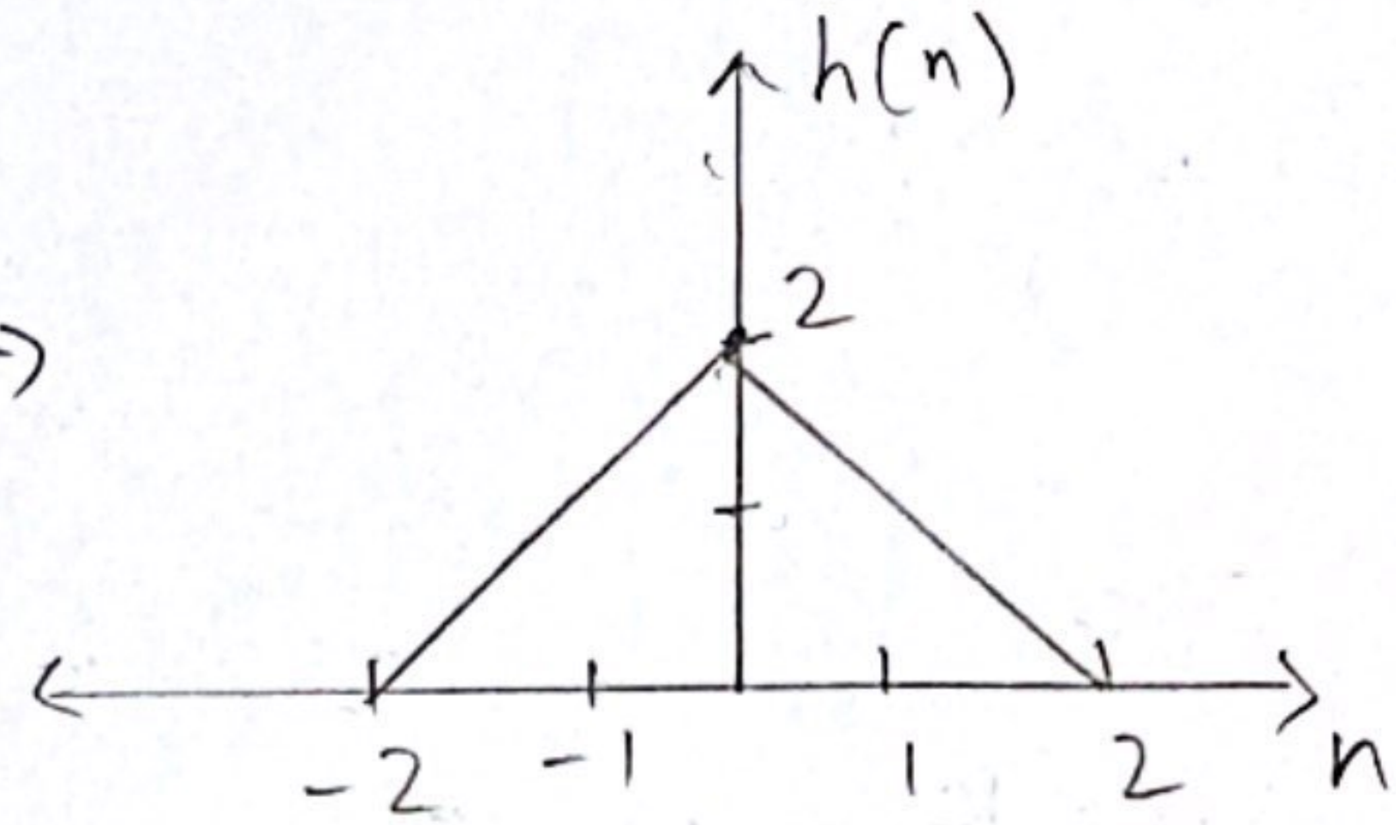


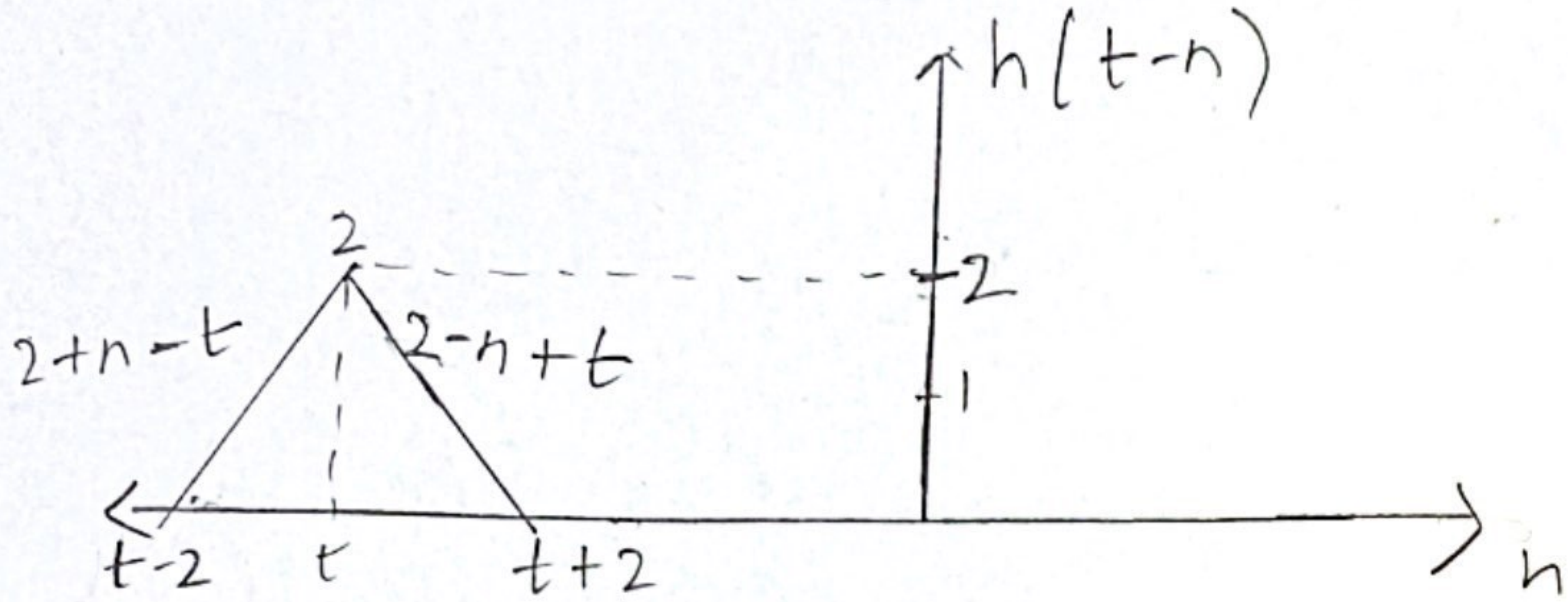
1.b} $x[n] = \begin{cases} 1 & -3 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$ $\quad h[n] = \begin{cases} 2-|n| & -2 \le n \le n \\ 0 & \text{otherwise.} \end{cases}$

Convolve →



→ with →





$$\Rightarrow \quad y(t) = \int_{-\infty}^{\infty} x(n)\, h(n-z)\, dn$$

Case 1 : $t < -5$, $\quad y(t) = 0 \quad$ — (1) $\quad$ [because there is no overlap of $x(n)$ & $h(t-n)$]

Case 2 ; $-5 \le t < -3$, in this case the product of $x(n)$ & $h(t-n)$ will be $(1)(2-n+t)$

the area will be calculated between -3 to $(t+2)$ as thats when they overlap.

$$\therefore \; y(t) = \int_{-3}^{t+2} (2-n+t)\, dn$$

$$= 2n - \frac{n^2}{2} + tn \Big|_{-3}^{t+2} = \boxed{\frac{t^2 + 10t + 25}{2}} \quad \text{— (2)}$$

Case 3: $-3 \leq t < -1$, in this case both the slopes start to overlap as we move right.

The area of overlap region is between -3 to -1

$$y(t) = \int_{-3}^{t} 2+n-t \, dn + \int_{t}^{t+2} 2-n+t \, dn$$

$$= \boxed{\frac{t^2}{2} - t + \frac{7}{2}} \quad — (3)$$

Case 4: $-1 \leq t < 1$, in this case the whole triangle and both slopes of $h(t-n)$ is overlaped with $x(n)$.

$$y(t) = \int_{t-2}^{t} 2+n-t \cdot dn + \int_{t}^{t+2} 2-n+t \, dn = \boxed{4} \quad — (4)$$

Case 5: $1 \leq t < 3$

Similar to Case (3).

$$y(t) = \int_{t-2}^{t} 2+n-t \cdot dn + \int_{t}^{3} 2-n+t \, dn$$

$$= \boxed{\frac{t^2}{2} + t + \frac{7}{2}} \quad — (5)$$

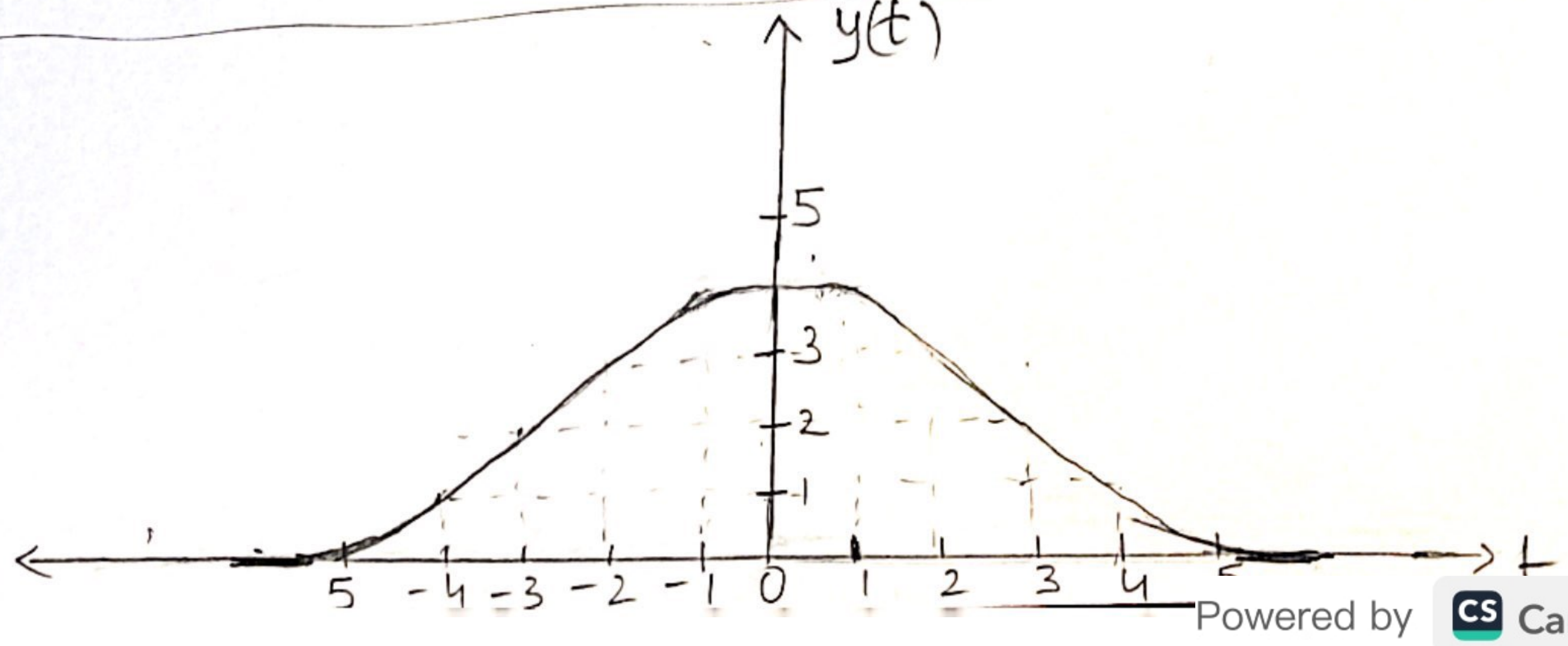
Case 6: $3 \leq t < 5$

Similar to Case (2).

$$y(t) = \int_{t-2}^{3} 2+n-t \, dn$$

$$= \boxed{\frac{1}{2}(t-5)^2} \quad — (6)$$

Case 7: $t > 5$, $y(t) = 0$ [since no overlap] — (7)



2a} Consider a discrete linear time-invariant system T with discrete input signal $x(n)$ and impulse response $h(n)$, $T[\delta(n)] = h(n)$, where:

impulse function $\delta(n) = \begin{cases} 1, & \text{if } n=0 \\ 0, & \text{else.} \end{cases}$

→ To Prove : $T[x(n)] = h(n) * x(n)$ [$*$ → convolution operation]

→ Using the given hint, first we will represent $x(n)$ as a function of $\delta(n)$

$$\therefore x(n) = \sum_{i=-\infty}^{\infty} x(i)\delta(n-i) \quad — ①$$

We get the above equation as when $i=n$, RHS would be $x(n)$ and the summation for all the other values of i will only be 0.

→ Substitute ① in given $T[\delta(n)] = h(n)$:

$$T[x(n)] = T\left[\sum_{i=-\infty}^{\infty} x(i)\,\delta(n-i)\right]$$

$$= \sum_{i=-\infty}^{\infty} x(i)\, T[\delta(n-i)] \quad \text{[note: } x(i) \to \text{constant]}$$

Since its Time-Invariant $h(n-i) = T[\delta(n-i)]$

$$\therefore \sum_{i=-\infty}^{\infty} x(i)h(n-i)$$

$$= h(n) * x(n)$$

Hence Proved $T[x(n)] = h(n) * x(n)$

2b)

We will have to find out if convolution is linear and invariant or not, as Gaussian blurring is convolving with the Gaussian filter.

To Prove : Convolution is linear.

Proof : Let $g_1(x)$ and $g_2(x)$ be two convolutions.

$$g_1(x) = \int_{-\infty}^{\infty} f_1(\tau) h(x-\tau) d\tau$$

$$g_1(y) = \int_{-\infty}^{\infty} f_2(\tau) h(x-\tau) d\tau$$

which gives us :

$$= \int_{-\infty}^{\infty} (\alpha f_1(\tau) + \beta f_2(\tau)) h(x-\tau) d\tau$$

$$= \alpha \int_{-\infty}^{\infty} f(\tau) h(x-\tau) d\tau + \beta \int_{-\infty}^{\infty} f_2(\tau) h(x-z) d\tau$$

$$= \alpha g_1(x) + \beta g_2(x)$$

We proved the Convolution is linear and following the same path Gaussian blurring is Linear.

To Prove : Convolution is time-invariant.

Proof : $g(x) = \int_{-\infty}^{\infty} f(\tau) h(x-\tau) d\tau$

which gives us : $\int_{-\infty}^{\infty} f(\tau - t) h(x-\tau) d\tau$

$$= \int_{-\infty}^{\infty} f(u) h(x-t-u) du \quad [u = \tau - t]$$

$$= g(x-t)$$

We proved convolution is time invariant and following the same path Gaussian Blurring is Time Invariant.

2.c} $T[x(n)] = x(-n)$

$$T[a_1 x_1(n) + a_2 x_n(n)] = a_1 x_1(-n) + a_2 x_2(-n)$$
$$= a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

Hence Time reversal is Linear.

Now lets shift input by $-n_0$:

$$T[x(n-n_0)] = x(-n+n_0) \neq x(-n-n_0)$$

Since the output is shifted.

$\therefore$ Time reversal is NOT time invariant.

3} Vectors can be used to represent polynomials.
For example, 3rd degree polynomial $(a_3x^3+a_2x^2+a_1x+a_0)$ can be represented by vector $[a_3, a_2, a_1, a_0]$

To prove: If u and v are vectors of polynomial coefficients, prove that convolving them is equivalent to multiplying the two polynomials to represent eachother.

Proof: The convolution of $f(x)$ & $h(x)$ is

$$g(x) = \sum_{i=-k}^{k} f(i)h(x-i)$$

Lets assume that the values of vector are as follows

$\vec{u}$: $[u_0, u_1, u_2 \ldots u_n]$ & $\vec{v}$: $[v_0, v_1, v_2 \ldots v_n]$

Let the polynomial of $\vec{u}$ be $A = \sum_{i=0}^{n} a_i x^i$

Let the polynomial of $\vec{v}$ be $B = \sum_{j=0}^{n} b_j x^j$

$$A * B = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j X$$

In the above equation $a_i = 0$, where $i > n$ & $b_i = 0$ where $j > m$ because of proper zero padding.

$$\sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j x^{i+j} = \sum_{i=0}^{n+m} \sum_{j=0}^{n+m} a_i b_j x^{i+j} \quad \text{— (1)}$$

Assume $\underline{i+j=k}$, which also gives us $\underline{j=k-i}$

and substitute the above in — (1)

$$= \sum_{k=0}^{n+m} \sum_{i=0}^{k} a_i b_{k-i} X^k \approx \boxed{g_k = \sum_{i=0}^{k} u_i v_{k-i}}$$

The above is same as convolution of two vectors.

Hence proved convolving polynomial coefficients is equivalent to multiplication of two polynomials.

4}

To prove: Laplacian is infact rotation invariant.
$\Delta I = I_{rr} + I_{r'r'}$ where r & r' are orthogonal directions

Proof: First using the hint of Polar Coordinates
→ let the coordinates be (r, θ) for cartesian coordinate (x, y)

→ here $x = r\cos\theta$ and $y = r\sin\theta$.

$\therefore$ $u = r\cos(\theta + \alpha) = x\cos\alpha - y\sin\alpha$
$v = r\sin(\theta + \alpha) = x\sin\alpha + y\cos\alpha$

From the above equation we can express the image at any angle after rotating it.

The next step is to differentiate $I'(u, v)$ by applying Chain rule.

→ $$\frac{\partial I'}{\partial x} = \frac{\partial u}{\partial x} \cdot \frac{\partial I'}{\partial u} + \frac{\partial v}{\partial x} \cdot \frac{\partial I'}{\partial v}$$
$$= \cos\alpha \frac{\partial I'}{\partial u} + \sin\alpha \frac{\partial I'}{\partial v}$$

→ $$\frac{\partial^2 I'}{\partial x^2} = \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial u}\left(\frac{\partial I'}{\partial x}\right) + \frac{\partial v}{\partial x} \cdot \frac{\partial}{\partial v}\left(\frac{\partial I'}{\partial x}\right)$$
$$= \cos^2\alpha \frac{\partial^2 I'}{\partial u^2} + \sin^2\alpha \frac{\partial^2 I'}{\partial v^2} + 2\sin\alpha\cos\alpha \frac{\partial^2 I'}{\partial u \partial v} \quad \text{— (1)}$$

$$\rightarrow \frac{\partial I'}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial I'}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial I'}{\partial v} = -\sin\alpha \frac{\partial I'}{\partial u} + \cos\alpha \frac{\partial I'}{\partial v}$$

$$\rightarrow \frac{\partial^2 I'}{\partial^2 y^2} = \sin^2\alpha \frac{\partial^2 I'}{\partial u^2} + \cos^2\alpha \frac{\partial^2 I'}{\partial v^2} - 2\sin\alpha\cos\alpha \frac{\partial^2 I'}{\partial u \partial v} \quad \text{— } (2)$$

From (1) & (2) we can find $\frac{\partial^2 I'}{\partial x^2} + \frac{\partial^2 I'}{\partial y^2}$

$$\frac{\partial^2 I'}{\partial x^2} + \frac{\partial^2 I'}{\partial y^2} = (\cos^2\alpha + \sin^2\alpha)\frac{\partial^2 I'}{\partial u} + (\cos^2\alpha + \sin^2\alpha)\frac{\partial^2 I'}{\partial v}$$
$$+ 2\sin\alpha\cancel{\cos\alpha}\frac{\partial^2 I'}{\partial u \partial v} - 2\sin\alpha\cancel{\cos\alpha}\frac{\partial^2 I'}{\partial u \partial v}$$

[here $\cos^2\alpha + \sin^2\alpha = 1$]

$$\frac{\partial^2 I'}{\partial x^2} + \frac{\partial^2 I'}{\partial y^2} = \frac{\partial^2 I'}{\partial u} + \frac{\partial^2 I'}{\partial v}$$

$\therefore$ The above gives us,

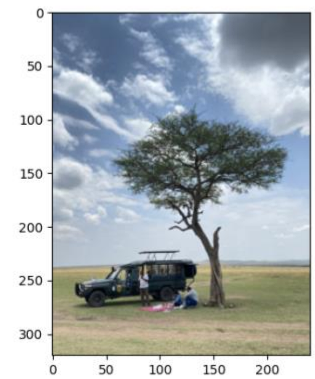
$$I_{xx} + I_{yy} = I'_{uu} + I'_{vv}$$
$$= I_{rr} + I_{r'r'}$$

$$\Delta I = \Delta I'$$

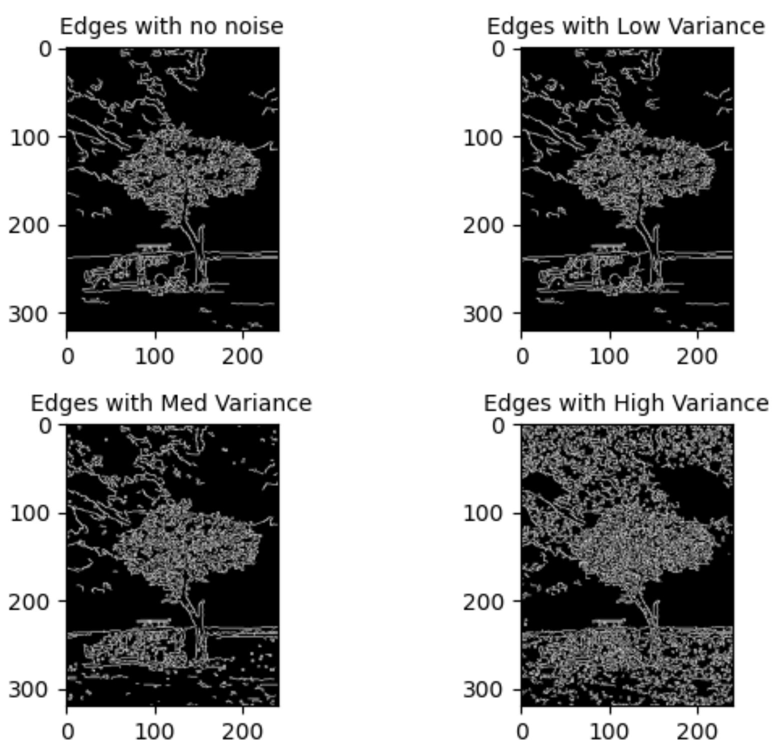
Hence proved Laplacian is infact rotation invariant.

Q5) Canny Edge Detector Robustness -

Image used -

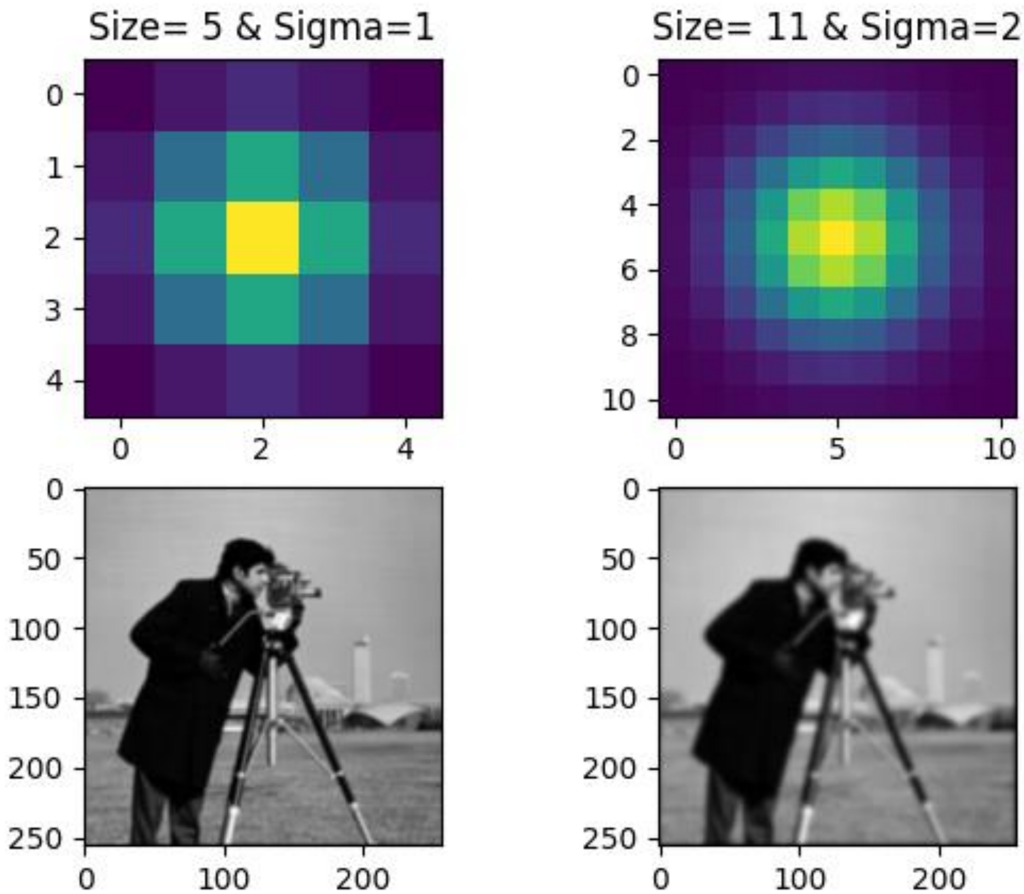


Output –

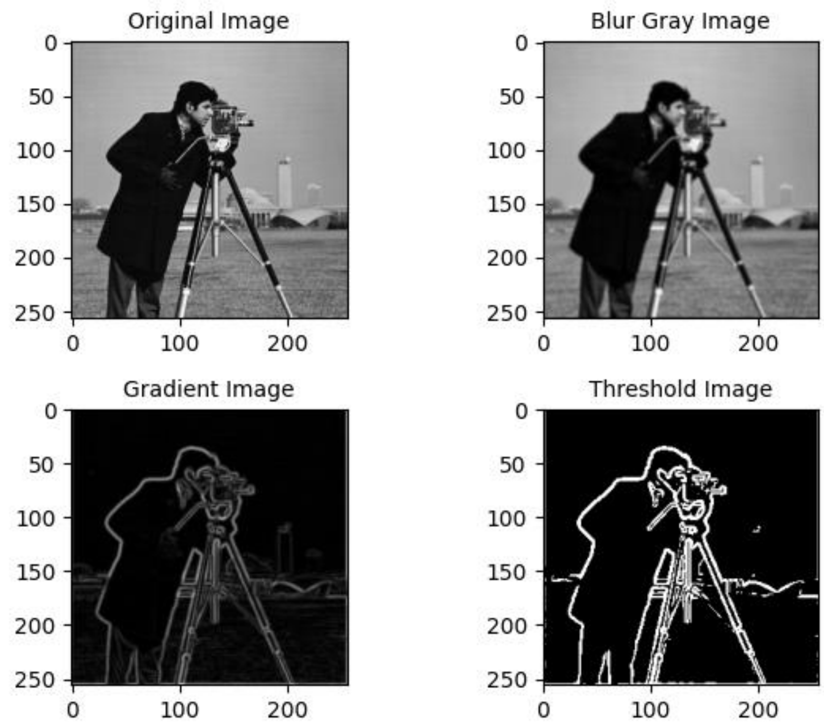


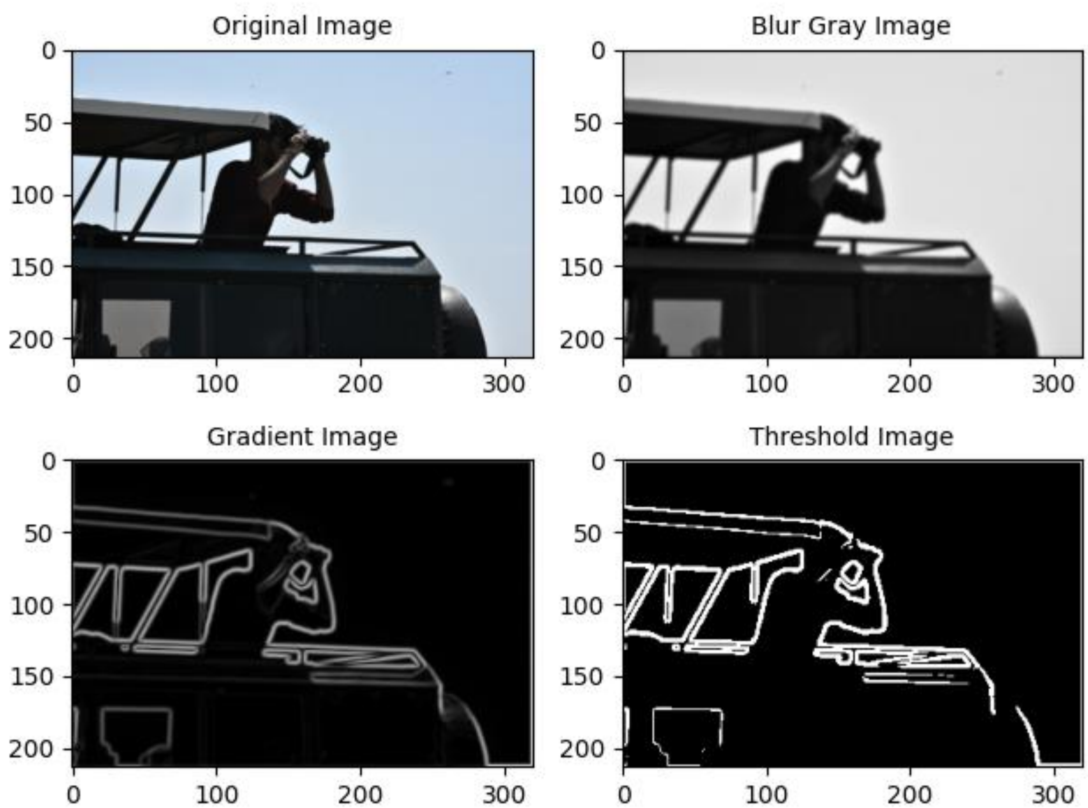
In Canny Edge Detector low variance gaussian noise does not make a noticeable difference, we can notice small change in median variance gaussian noise. But the edges would become more difficult to detect with an increase in noise variance, as seen with high variance noise. High variance noise will lead to less detection of image depending on the intensity of the image near the edges. To conclude, increase in variance leads to bad and inaccurate performance of the edge detector.

Q6 – Step 1



Q6 – Step 4


Original Image
Blur Gray Image
Gradient Image
Threshold Image


Original Image
Blur Gray Image
Gradient Image
Threshold Image

Process – We took an image and first converted it gray. Calculated the Gaussian matrix of size 5 and Sigma 1. Convolved the gray image with kernel (Gaussian matrix). Used the convolved image to find the gradient magnitude with given value of x and y. Last step was to take the output from before to find the threshold image.

Strengths – For my image, the edges are very well detected in gradient image because we ran a gaussian blur on the original image, the threshold result is also good. All main edges are well detected and accurate. This type of image with only one focus and no background will run accurately.

Weakness – For the given image, we can notice some details and edges getting missed. Mainly the background building and tower is missed and only the man with camera is in focus. To conclude, we miss the background edges.