Euclidean Plane and its Relatives

A minimalistic introduction

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Introduction

The book is meant to be rigorous, conservative, elementary and minimalistic. At the same time it includes about the maximum what students can absorb in one semester.

Approximately third of the material used to be covered in high school, not any more.

The present book is based on the courses taught by the author at the Penn State as an introduction into Foundations of Geometry. The lectures were oriented to sophomore and senior university students. These students already had a calculus course. In particular they are familiar with the real numbers and continuity. It makes possible to cover the material faster and in a more rigorous way than it could be done in high school.

Prerequisite

The students has to be familiar with the following topics.

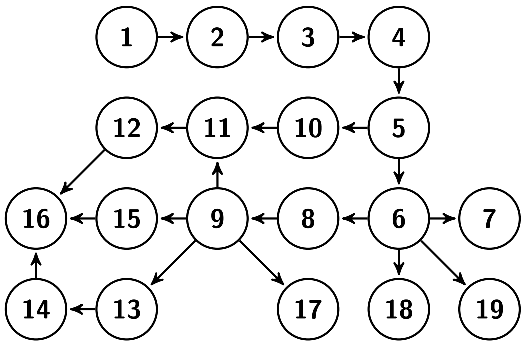
- Elementary set theory: $\in$, $\cup$, $\cap$, $\setminus$, $\subset$, $\times$.
- Real numbers: intervals, inequalities, algebraic identities.
- Limits, continuous functions and Intermediate value theorem.
- Standard functions: absolute value, natural logarithm, exponent. Occasionally, trigonometric functions are used, but these parts can be ignored.
- Chapter 13 use matrix algebra of $2{\times}2$-matrices.
- To read Chapter 15, it is better to have some previous experience with *scalar product*, also known as *dot product*.
- To read Chapter 17, it is better to have some previous experience with complex numbers.

Overview

We use so called *metric approach* introduced by Birkhoff. It means that we define Euclidean plane as a *metric space* which satisfies a list of properties. This way we minimize the tedious parts which are unavoidable in the more classical Hilbert's approach. At the same time the students have chance to learn basic geometry of metric spaces.

In the Chapter 1 we give all definitions necessary to formulate the axioms; it includes metric space, lines, angle measure, continuous maps and congruent triangles.

Euclidean geometry is discussed in the chapters 2–7. In the Chapter 2, we formulate the axioms and prove immediate corollaries. In the chapters 3–6 we develop Euclidean geometry to a dissent level. In Chapter 7 we give the most classical theorem of triangle geometry; this chapter included mainly as an illustration.



In the chapters 8–9 we discuss geometry of circles on the Euclidean plane. These two chapters will be used in the construction of the model of hyperbolic plane.

In the chapters 10–12 we discuss non-Euclidean geometry. In Chapter 10, we introduce the axioms of absolute geometry. In Chapter 11 we describe so called conformal disc model. This is a construction of hyperbolic plane, an example of absolute plane which is not Euclidean. In the Chapter 12 we discuss geometry of the constructed hyperbolic plane.

Chapters 13 and 14 discuss so called *incidence geometry*, it includes affine and projective geometries.

The chapters 15–19 contain additional topics: Spherical geometry, Projective model, Complex coordinates, Geometric constructions and Area correspondingly The proofs in these chapters are not completely rigorous.

Disclaimer

I am not doing history. It is impossible to find the original reference to most of the theorems discussed here, so I do not even try. Most of the proofs discussed in the book appeared already in the Euclid's Elements and the Elements are not the original source anyway.

Recommended books

◇ Kiselev's textbook [6] — a classical book for school students. Should help if you have trouble to follow my book.
◇ Moise's book, [11] — should be good for further study.
◇ Greenberg's book [5] — a historical tour through the axiomatic systems of various geometries.
◇ Prasolov's book [12] is perfect to master your problem-solving skills.
◇ Methodologically my lectures were very close to Sharygin's textbook [14]. This is the greatest textbook in geometry for school students, I recommend it to anyone who can read Russian.

Acknowlegments.

I would like to thank Matthew Chao for thoughtful reading and correcting dozens of mistakes.

Chapter 1

Preliminaries

What is the axiomatic approach?

In the axiomatic approach, one defines the plane as anything which satisfy a list of properties called *axioms*. Axiomatic system for the theory is like rules for the game. Once the axiom system is fixed, a statement considered to be true if it follows from the axioms and nothing else is considered to be true.

The formulations of the first axioms were not rigorous at all. For example, Euclid described a *line* as *breadthless length* and *straight line* as a line which *lies evenly with the points on itself*. On the other hand, these formulations were clear enough so that one mathematician could understand the other.

The best way to understand an axiomatic system is to make one by yourself. Look around and choose a physical model of the Euclidean plane, say imagine an infinite and perfect surface of chalk board. Now try to collect the key observations about this model. Assume for now that we have intuitive understanding of such notions as *line* and *point*.

(i) We can measure distances between points.

(ii) We can draw unique line which pass though two given points.

(iii) We can measure angles.

(iv) If we rotate or shift we will not see the difference.

(v) If we change scale we will not see the difference.

These observations are good enough to start with. Further we will develop the language to reformulate them rigorously.

What is a model?

Euclidean plane can be defined rigorously the following way.

Define a point *in the Euclidean plane is a pair of real numbers* (x, y) *and define the* distance *between two points* (x_1, y_1) *and* (x_2, y_2) *by the following formula.*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

That is it! We gave a *numerical model* of Euclidean plane; it builds the Euclidean plane from the real numbers while the later is assumed to be known.

Shortness is the main advantage of the model approach, but it is not intuitively clear why we define points and the distances this way.

On the other hand, the observations made in the previous section are intuitively obvious — this is the main advantage of the axiomatic approach.

An other advantage lies in the fact that the axiomatic approach is easily adjustable. For example we may remove one axiom from the list, or exchange it to an other axiom. We will do such modifications in Chapter 10 and further.

Metric spaces

The notion of metric space provides a rigorous way to say *"we can measure distances between points"*. That is, instead of (i) on page 9, we can say *"Euclidean plane is a metric space"*.

1.1. Definition. *Let $\mathcal{X}$ be a nonempty set and d be a function which returns a real number $d(A, B)$ for any pair $A, B \in \mathcal{X}$. Then d is called* metric *on $\mathcal{X}$ if for any $A, B, C \in \mathcal{X}$, the following conditions are satisfied.*

(a) Positiveness:

$$d(A, B) \geqslant 0.$$

(b) $A = B$ if and only if

$$d(A, B) = 0.$$

(c) Symmetry:

$$d(A, B) = d(B, A).$$

(d) Triangle inequality:

$$d(A, C) \leqslant d(A, B) + d(B, C).$$

A metric space *is a set with a metric on it. More formally, a metric space is a pair* $(\mathcal{X}, d)$ *where* $\mathcal{X}$ *is a set and* d *is a metric on* $\mathcal{X}$*.*

Elements of $\mathcal{X}$ *are called* points *of the metric space. Given two points* $A, B \in \mathcal{X}$ *the value* $d(A, B)$ *is called* distance *from* A *to* B*.*

Examples

- ◇ *Discrete metric.* Let $\mathcal{X}$ be an arbitrary set. For any $A, B \in \mathcal{X}$, set $d(A, B) = 0$ if $A = B$ and $d(A, B) = 1$ otherwise. The metric d is called *discrete metric* on $\mathcal{X}$.
- ◇ *Real line.* Set of all real numbers ($\mathbb{R}$) with metric defined as

$$d(A, B) \stackrel{\text{def}}{=\!=} |A - B|.$$

- ◇ *Metrics on the plane.* Let us denote by $\mathbb{R}^2$ the set of all pairs (x, y) of real numbers. Assume $A = (x_A, y_A)$ and $B = (x_B, y_B)$ are arbitrary points in $\mathbb{R}^2$. One can equip $\mathbb{R}^2$ with the following metrics.
  - ○ *Euclidean metric,* denoted as d_2 and defined as

$$d_2(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}.$$

  - ○ *Manhattan metric,* denoted as d_1 and defined as

$$d_1(A, B) = |x_A - x_B| + |y_A - y_B|.$$

  - ○ *Maximum metric,* denoted as d_∞ and defined as

$$d_\infty(A, B) = \max\{|x_A - x_B|, |y_A - y_B|\}.$$

1.2. Exercise. *Prove that the following functions are metrics on* $\mathbb{R}^2$*: (a)* d_1*; (b)* d_2*; (c)* d_∞*.*

Shortcut for distance

Most of the time we study only one metric on the space. Therefore we will not need to name the metric function each time.

Given a metric space $\mathcal{X}$, the distance between points A and B will be further denoted as

$$AB \quad \text{or} \quad d_{\mathcal{X}}(A, B);$$

the later is used only if we need to emphasize that A and B are points of the metric space $\mathcal{X}$.

For example, the triangle inequality can be written as

$$AC \leqslant AB + BC.$$

For the multiplication we will always use "$\cdot$", so AB should not be confused with $A{\cdot}B$.

Isometries, motions and lines

In this section we define *lines* in a metric space. Once it is done the sentence *"We can draw unique line which pass though two given points."* becomes rigorous; see (ii) on page 9.

Recall that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is a *bijection* if it gives an exact pairing of the elements of two sets. Equivalently, $f\colon \mathcal{X} \to \mathcal{Y}$ is a bijection if it has an *inverse*; that is, a map $g\colon \mathcal{Y} \to \mathcal{X}$ such that $g(f(A)) = A$ for any $A \in \mathcal{X}$ and $f(g(B)) = B$ for any $B \in \mathcal{Y}$.

1.3. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces and $d_{\mathcal{X}}$, $d_{\mathcal{Y}}$ be their metrics. A map*

$$f\colon \mathcal{X} \to \mathcal{Y}$$

is called distance-preserving *if*

$$d_{\mathcal{Y}}(f(A), f(B)) = d_{\mathcal{X}}(A, B)$$

for any $A, B \in \mathcal{X}$.

A bijective distance-preserving map is called an isometry*.*

Two metric spaces are called isometric *if there exists an isometry from one to the other.*

The isometry from a metric space to itself is also called motion *of the space.*

1.4. Exercise. *Show that any distance preserving map is* injective*; that is, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a distance preserving map then $f(A) \neq f(B)$ for any pair of distinct points $A, B \in \mathcal{X}$.*

1.5. Exercise. *Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a motion of the real line then either (a) $f(x) = f(0) + x$ for any $x \in \mathbb{R}$, or (b) $f(x) = f(0) - x$ for any $x \in \mathbb{R}$.*

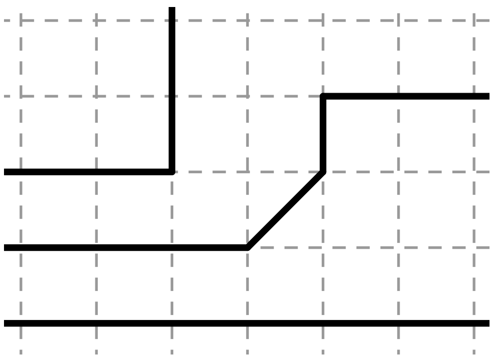
1.6. Exercise. *Prove that $(\mathbb{R}^2, d_1)$ is isometric to $(\mathbb{R}^2, d_\infty)$.*

1.7. Advanced exercise. *Describe all the motions of the Manhattan plane.*

If $\mathcal{X}$ is a metric space and $\mathcal{Y}$ is a subset of $\mathcal{X}$, then a metric on $\mathcal{Y}$ can be obtained by restricting the metric from $\mathcal{X}$. In other words, the distance between two points of $\mathcal{Y}$ is defined to be the distance between these points in $\mathcal{X}$. This way any subset of a metric space can be also considered as a metric space.

1.8. Definition. *A subset ℓ of metric space is called* line *if it is isometric to the real line.*

Note that a space with discrete metric has no lines. The following picture shows examples of lines on the Manhattan plane $(\mathbb{R}, d_1)$.



1.9. Exercise. *Consider graph $y = |x|$ in $\mathbb{R}^2$. In which of the following spaces (a) $(\mathbb{R}^2, d_1)$, (b) $(\mathbb{R}^2, d_2)$ (c) $(\mathbb{R}^2, d_\infty)$ it forms a line? Why?*

1.10. Exercise. *How many points M on the line (AB) for which we have*

1. $AM = MB$?
2. $AM = 2{\cdot}MB$?

Half-lines and segments

Assume there is a line ℓ passing through two distinct points P and Q. In this case we might denote ℓ as (PQ). There might be more than one line through P and Q, but if we write (PQ) we assume that we made a choice of such line.

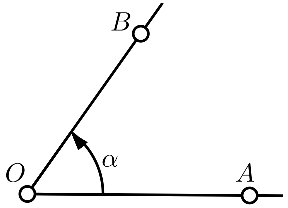
Let us denote by $[PQ)$ the half-line which starts at P and contains Q. Formally speaking, $[PQ)$ is a subset of (PQ) which corresponds to $[0, \infty)$ under an isometry $f\colon (PQ) \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.

1.11. Exercise. *Show that if $X \in [PQ)$ then $QX = |PX - PQ|$.*

The subset of line (PQ) between P and Q is called segment between P and Q and denoted as $[PQ]$. Formally, segment can defined as the intersection of two half-lines: $[PQ] = [PQ) \cap [QP)$.

Angles

Our next goal is to introduce *angles* and *angle measures*; after that the statement *"we can measure angles"* will become rigorous; see (iii) on page 9.



An ordered pair of half-lines which start at the same point is called *angle*. An angle formed by two half-lines $[OA)$ and $[OB)$ will be denoted as $\angle AOB$. In this case the point O is called *vertex* of the angle.

Intuitively, the angle measure tells how much one has to rotate the first half-line counterclockwise so it gets the position of the second half-line of the angle. The full turn is assumed to be $2\cdot\pi$; it corresponds to the angle measure in radians.

The angle measure of $\angle AOB$ is denoted as $\measuredangle AOB$; it is a real number in the interval $(-\pi, \pi]$.

The notations $\angle AOB$ and $\measuredangle AOB$ look similar, they also have close but different meanings, which better not to be confused. For example, the equality $\angle AOB = \angle A'O'B'$ means that $[OA) = [O'A')$ and $[OB) = [O'B')$, in particular $O = O'$. On the other hand the equality $\measuredangle AOB = \measuredangle A'O'B'$ means only equality of two real numbers; in this case O may be distinct from O'.

Here is the first property of angle measure which will become a part of the axiom.

Given a half-line $[OA)$ and $\alpha \in (-\pi, \pi]$ there is unique half-line $[OB)$ such that $\measuredangle AOB = \alpha$.

Reals modulo $2\cdot\pi$

Consider three half-lines starting from the same point, $[OA)$, $[OB)$ and $[OC)$. They make three angles $\angle AOB$, $\angle BOC$ and $\angle AOC$, so the angle measure $\measuredangle AOC$ should coincide with the sum $\measuredangle AOB + \measuredangle BOC$ up to full rotation. This property will be expressed by a formula

$$\measuredangle AOB + \measuredangle BOC \equiv \measuredangle AOC,$$

where "$\equiv$" is a new notation which we are about to introduce. The last identity will become a part of the axiom.

We will write

$$\alpha \equiv \beta \quad \text{or} \quad \alpha \equiv \beta \pmod{2\cdot\pi}$$

if $\alpha = \beta + 2\cdot\pi\cdot n$ for some integer n. In this case we say

"α is equal to β modulo $2\cdot\pi$".

For example

$$-\pi \equiv \pi \equiv 3\cdot\pi \quad \text{and} \quad \tfrac{1}{2}\cdot\pi \equiv -\tfrac{3}{2}\cdot\pi.$$

The introduced relation "$\equiv$" behaves roughly as equality, but the angle measures which differ by full turn

$$\dots, \alpha - 2\cdot\pi, \alpha, \alpha + 2\cdot\pi, \alpha + 4\cdot\pi, \dots$$

are considered to be the same.

With "$\equiv$", we can do addition subtraction and multiplication by integer number without getting into trouble. That is, if

$$\alpha \equiv \beta \quad \text{and} \quad \alpha' \equiv \beta'$$

then

$$\alpha + \alpha' \equiv \beta + \beta', \qquad \alpha - \alpha' \equiv \beta - \beta' \quad \text{and} \quad n\cdot\alpha \equiv n\cdot\beta$$

for any integer n. But "$\equiv$" does not in general respect multiplication by non-integer numbers; for example

$$\pi \equiv -\pi \quad \text{but} \quad \tfrac{1}{2}\cdot\pi \not\equiv -\tfrac{1}{2}\cdot\pi.$$

1.12. Exercise. *Show that $2\cdot\alpha \equiv 0$ if and only if $\alpha \equiv 0$ or $\alpha \equiv \pi$.*

Continuity

The angle measure is also assumed to be continuous. Namely, the following property of angle measure which will become a part of the axiom.

The function

$$\measuredangle\colon (A, O, B) \mapsto \measuredangle AOB$$

is continuous at any triple of points (A, O, B) such that $O \ne A$ and $O \ne B$ and $\measuredangle AOB \ne \pi$.

To explain this property we need to extend the notion of *continuity* to the functions between metric spaces. The definition is a straightforward generalization of the standard definition for the real-to-real functions.

Further $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces and $d_{\mathcal{X}}$, $d_{\mathcal{Y}}$ be their metrics.

A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called continuous at point $A \in \mathcal{X}$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_{\mathcal{X}}(A, A') < \delta \quad \Rightarrow \quad d_{\mathcal{Y}}(f(A), f(A')) < \varepsilon.$$

The same way one may define a continuous map of several variables. Say, assume $f(A, B, C)$ is a function which returns a point in the space $\mathcal{Y}$ for a triple of points (A, B, C) in the space $\mathcal{X}$. The map f might be defined only for some triples in $\mathcal{X}$.

Assume $f(A, B, C)$ is defined. Then we say that f continuous at the triple (A, B, C) if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$(f(A, B, C), f(A', B', C')) < \varepsilon.$$

if $d_{\mathcal{X}}(A, A') < \delta$, $d_{\mathcal{X}}(B, B') < \delta$ and $d_{\mathcal{X}}(C, C') < \delta$.

1.13. Exercise. *Let $\mathcal{X}$ be a metric space.*

(a) Let $A \in \mathcal{X}$ be a fixed point. Show that the function

$$f(B) \stackrel{\text{def}}{=\!=} d_{\mathcal{X}}(A, B)$$

is continuous at any point B.

(b) Show that $d_{\mathcal{X}}(A, B)$ is a continuous at any pair $A, B \in \mathcal{X}$.

1.14. Exercise. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be a metric spaces. Assume that the functions $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are continuous at any point and $h = g \circ f$ is its composition; that is, $h(A) = g(f(A))$ for any $A \in \mathcal{X}$. Show that $h\colon \mathcal{X} \to \mathcal{Z}$ is continuous at any point.*

1.15. Exercise. *Show that any distance preserving map is continuous.*

Congruent triangles

Our next goal is to give a rigorous meaning for (iv) on page 9. To do this, we introduce the notion of *congruent triangles* so instead of *"if we rotate or shift we will not see the difference"* we say that for triangles side-angle-side congruence holds; that is, if two triangles are congruent if they have two pairs of equal sides and the same angle measure between these sides.

An *ordered* triple of distinct points in a metric space $\mathcal{X}$, say A, B, C is called *triangle* and denoted as $\triangle ABC$. Note that the triangles $\triangle ABC$ and $\triangle ACB$ are considered as different.

Two triangles $\triangle A'B'C'$ and $\triangle ABC$ are called *congruent* (briefly $\triangle A'B'C' \cong \triangle ABC$) if there is a motion $f\colon \mathcal{X} \to \mathcal{X}$ such that

$$A' = f(A), \quad B' = f(B) \ \text{ and } \ C' = f(C).$$

Let $\mathcal{X}$ be a metric space and $f, g\colon \mathcal{X} \to \mathcal{X}$ be two motions. Note that the inverse $f^{-1} : \mathcal{X} \to \mathcal{X}$, as well as the composition $f \circ g : \mathcal{X} \to \mathcal{X}$ are also motions.

It follows that "$\cong$" is an equivalence relation; that is, any triangle congruent to itself and the following two conditions hold.

- ⋄ If $\triangle A'B'C' \cong \triangle ABC$ then $\triangle ABC \cong \triangle A'B'C'$.
- ⋄ If $\triangle A''B''C'' \cong \triangle A'B'C'$ and $\triangle A'B'C' \cong \triangle ABC$ then

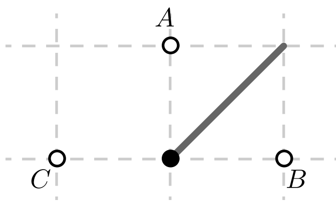
$$\triangle A''B''C'' \cong \triangle ABC.$$

Note that if $\triangle A'B'C' \cong \triangle ABC$ then $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$.

For discrete metric, as well some other metric spaces the converse also holds. The following example shows that it does not hold in the Manhattan plane.

Example. Consider three points $A = (0, 1)$, $B = (1, 0)$ and $C = = (-1, 0)$ on the Manhattan plane $(\mathbb{R}^2, d_1)$. Note that

$$d_1(A, B) = d_1(A, C) = d_1(B, C) = 2.$$



On one hand

$$\triangle ABC \cong \triangle ACB.$$

Indeed, it is easy to see that the map $(x, y) \mapsto (-x, y)$ is a motion of $(\mathbb{R}^2, d_1)$ which sends $A \mapsto A$, $B \mapsto C$ and $C \mapsto B$.

On the other hand

$$\triangle ABC \ncong \triangle BCA.$$

Indeed, assume there is a motion f of $(\mathbb{R}^2, d_1)$ which sends $A \mapsto B$ and $B \mapsto C$. Note that a point M is a midpoint[1] of A and B if and only if $f(M)$ is a midpoint of B and C. The set of midpoints for A and B is infinite, it contains all points (t, t) for $t \in [0, 1]$ (it is the dark gray segment on the picture). On the other hand the midpoint for B and C is unique (it is the black point on the picture). Thus the map f can not be bijective, a contradiction.

[1] M is a midpoint of A and B if $d_1(A, M) = d_1(B, M) = \frac{1}{2} \cdot d_1(A, B)$.

Chapter 2

The Axioms

The Axioms

Summarizing the above discussion, let us give an axiomatic system of the Euclidean plane.

I. The *Euclidean plane* is a metric space with at least two points.
II. There is one and only one line, that contains any two given distinct points P and Q in the Euclidean plane.
III. Any angle $\angle AOB$ in the Euclidean plane defines a real number in the interval $(-\pi, \pi]$. This number is called *angle measure of* $\angle AOB$ and denoted by $\measuredangle AOB$. It satisfies the following conditions:

   (a) Given a half-line $[OA)$ and $\alpha \in (-\pi, \pi]$ there is unique half-line $[OB)$ such that $\measuredangle AOB = \alpha$

   (b) For any points A, B and C distinct from O we have

   $$\measuredangle AOB + \measuredangle BOC \equiv \measuredangle AOC.$$

   (c) The function

   $$\measuredangle\colon (A, O, B) \mapsto \measuredangle AOB$$

   is continuous at any triple of points (A, O, B) such that $O \neq A$ and $O \neq B$ and $\measuredangle AOB \neq \pi$.

IV. In the Euclidean plane, we have $\triangle ABC \cong \triangle A'B'C'$ if and only if

$$A'B' = AB, \quad A'C' = AC, \quad \text{and} \quad \measuredangle C'A'B' = \pm\measuredangle CAB.$$

V. If for two triangles $\triangle ABC$, $\triangle AB'C'$ in the Euclidean plane and $k > 0$ we have

$$\begin{aligned} B' &\in [AB), & C' &\in [AC) \\ AB' &= k{\cdot}AB, & AC' &= k{\cdot}AC \end{aligned}$$

then

$$B'C' = k{\cdot}BC, \quad \measuredangle ABC = \measuredangle AB'C' \quad \text{and} \quad \measuredangle ACB = \measuredangle AC'B'.$$

This set of axioms is very close to the one given by Birkhoff in [4].

From now on, we can use no information about Euclidean plane which does not follow from the five axioms above.

Lines and half-lines

2.1. Proposition. *Any two distinct lines intersect at most at one point.*

Proof. Assume two lines ℓ and m intersect at two distinct points P and Q. Applying Axiom II, we get $\ell = m$. $\square$

2.2. Exercise. *Suppose $A' \in [OA)$ and $A' \neq O$ show that*

$$[OA) = [OA').$$

2.3. Proposition. *Given $r \geqslant 0$ and a half-line $[OA)$ there is unique $A' \in [OA)$ such that $OA' = r$.*

Proof. According to definition of half-line, there is an isometry

$$f\colon [OA) \to [0, \infty),$$

such that $f(O) = 0$. By the definition of isometry, $OA' = f(A')$ for any $A' \in [OA)$. Thus, $OA' = r$ if and only if $f(A') = r$.

Since isometry has to be bijective, the statement follows. $\square$

Zero angle

2.4. Proposition. *$\measuredangle AOA = 0$ for any $A \neq O$.*

Proof. According to Axiom IIIb,

$$\measuredangle AOA + \measuredangle AOA \equiv \measuredangle AOA.$$

Subtract $\measuredangle AOA$ from both sides, we get $\measuredangle AOA \equiv 0$.

Since $-\pi < \measuredangle AOA \leqslant \pi$, we get $\measuredangle AOA = 0$. □

2.5. Exercise. *Assume $\measuredangle AOB = 0$. Show that $[OA) = [OB)$.*

2.6. Proposition. *For any A and B distinct from O, we have*

$$\measuredangle AOB \equiv -\measuredangle BOA.$$

Proof. According to Axiom IIIb,

$$\measuredangle AOB + \measuredangle BOA \equiv \measuredangle AOA$$

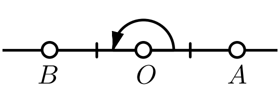
By Proposition 2.4, $\measuredangle AOA = 0$. Hence the result follows. □

Straight angle

If $\measuredangle AOB = \pi$, we say that $\angle AOB$ is a *straight angle*. Note that by Proposition 2.6, if $\angle AOB$ is a straight angle then so is $\angle BOA$.

We say that point O *lies between* points A and B if $O \neq A$, $O \neq B$ and $O \in [AB]$.

2.7. Theorem. *The angle $\angle AOB$ is straight if and only if O* lies between *A and B.*



Proof. By Proposition 2.3, we may assume that $OA = OB = 1$.

($\Leftarrow$). Assume O lies between A and B. Set $\alpha = \measuredangle AOB$.

Applying Axiom IIIa, we get a half-line $[OA')$ such that $\alpha = \measuredangle BOA'$. By Proposition 2.3, we can assume that $OA' = 1$. According to Axiom IV,

$$\triangle AOB \cong \triangle BOA'.$$

Denote by h the corresponding motion of the plane; that is, h is a motion such that $h(A) = B$, $h(O) = O$ and $h(B) = A'$.

Then $(A'B) = h(AB) \ni h(O) = O$. Therefore both lines (AB) and $(A'B)$, contain B and O. By Axiom II, $(AB) = (A'B)$.

By the definition of the line, (AB) contains exactly two points A and B on distance 1 from O. Since $OA' = 1$ and $A' \neq B$, we get $A = A'$.

By Axiom IIIb and Proposition 2.4, we get

$$\begin{aligned}2\cdot\alpha &= \measuredangle AOB + \measuredangle BOA' = \\ &= \measuredangle AOB + \measuredangle BOA \equiv \\ &\equiv \measuredangle AOA = \\ &= 0\end{aligned}$$

Therefore, by Exercise 1.12, α is either 0 or π.

Since $[OA) \neq [OB)$, we have $\alpha \neq 0$, see Exercise 2.5. Therefore $\alpha = \pi$.

$(\Rightarrow)$. Suppose that $\measuredangle AOB = \pi$. Consider line (OA) and choose point B' on (OA) so that O lies between A and B'.

From above, we have $\measuredangle AOB' = \pi$. Applying Axiom IIIa, we get $[OB) = [OB')$. In particular, O lies between A and B. $\square$

A triangle $\triangle ABC$ is called *degenerate* if A, B and C lie on one line. The following corollary is just a reformulation of Theorem 2.7.

2.8. Corollary. *A triangle is degenerate if and only if one of its angles is equal to π or 0.*

2.9. Exercise. *Show that three distinct points A, O and B lie on one line if and only if*

$$2\cdot\measuredangle AOB \equiv 0.$$

2.10. Exercise. *Let A, B and C be three points distinct from O. Show that B, O and C lie on one line if and only if*

$$2\cdot\measuredangle AOB \equiv 2\cdot\measuredangle AOC.$$

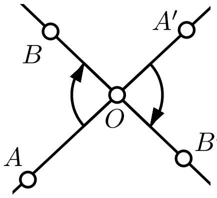
Vertical angles

A pair of angles $\angle AOB$ and $\angle A'OB'$ is called *vertical* if the point O lies between A and A' and between B and B' at the same time.

2.11. Proposition. *The vertical angles have equal measures.*

Proof. Assume that the angles $\angle AOB$ and $\angle A'OB'$ are vertical.

Note that the angles $\angle AOA'$ and $\angle BOB'$ are straight. Therefore $\measuredangle AOA' = \measuredangle BOB' = \pi$.



It follows that

$$\begin{aligned} 0 &= \measuredangle AOA' - \measuredangle BOB' \equiv \\ &\equiv \measuredangle AOB + \measuredangle BOA' - \measuredangle BOA' - \measuredangle A'OB' \equiv \\ &\equiv \measuredangle AOB - \measuredangle A'OB'. \end{aligned}$$

Hence the result follows. □

2.12. Exercise. *Assume O is the midpoint for both segments $[AB]$ and $[CD]$. Prove that $AC = BD$.*

Chapter 3

Half-planes

This chapter contains long proofs of intuitively evident statements. It is okay to skip it, but make sure you know definitions of positive/negative angles and that your intuition agrees with 3.8, 3.10, 3.11 and 3.16.

Sign of angle

◇ The angle $\angle AOB$ is called *positive* if $0 < \measuredangle AOB < \pi$;
◇ The angle $\angle AOB$ is called *negative* if $\measuredangle AOB < 0$.

Note that according to the above definitions the straight angle as well as zero angle are neither positive nor negative.

3.1. Exercise. *Show that $\angle AOB$ is positive if and only if $\angle BOA$ is negative.*

3.2. Exercise. *Let $\angle AOB$ be a straight angle. Show that $\angle AOX$ is positive if and only if $\angle BOX$ is negative.*

3.3. Exercise. *Assume that the angles $\angle AOB$ and $\angle BOC$ are positive. Show that*

$$\measuredangle AOB + \measuredangle BOC + \measuredangle COA = 2\cdot\pi.$$

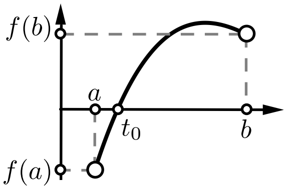
if $\angle COA$ is positive and

$$\measuredangle AOB + \measuredangle BOC + \measuredangle COA = 0.$$

if $\angle COA$ is negative.

Intermediate value theorem

3.4. Intermediate value theorem. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Assume $f(a)$ and $f(b)$ have the opposite signs then $f(t_0) = 0$ for some $t_0 \in [a,b]$.*



The Intermediate value theorem is assumed to be known; it should be covered in any calculus course. We will use the following corollary.

3.5. Corollary. *Assume that for any $t \in$ $\in [0,1]$ we have three points in the plane O_t, A_t and B_t such that*

(a) Each function $t \mapsto O_t$, $t \mapsto A_t$ and $t \mapsto B_t$ is continuous.
(b) For for any $t \in [0,1]$, the points O_t, A_t and B_t do not lie on one line.

Then the angles $\angle A_0O_0B_0$ and $\angle A_1O_1B_1$ have the same sign.

Proof. Consider the function $f(t) = \measuredangle A_tO_tB_t$.

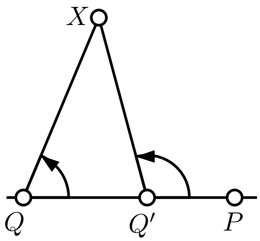
Since the points O_t, A_t and B_t do not lie on one line, Theorem 2.7 implies that $f(t) = \measuredangle A_tO_tB_t \neq 0$ nor π for any $t \in [0,1]$.

Therefore by Axiom IIIc and Exercise 1.14, f is a continuous function.

Further, by Intermediate value theorem, $f(0)$ and $f(1)$ have the same sign; hence the result follows. □

Same sign lemmas

3.6. Lemma. *Assume $Q' \in [PQ)$ and $Q' \neq P$. Then for any $X \notin$ $\notin (PQ)$ the angles $\angle PQX$ and $\angle PQ'X$ have the same sign.*



Proof. By Proposition 2.3, for any $t \in [0,1]$ there is unique point $Q_t \in [PQ)$ such that

$$PQ_t = (1-t)\cdot PQ + t\cdot PQ'.$$

Note that the map $t \mapsto Q_t$ is continuous,

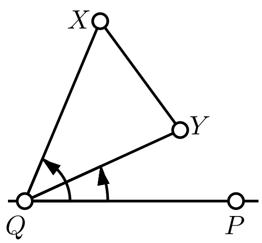
$$Q_0 = Q, \qquad Q_1 = Q'$$

and for any $t \in [0,1]$, we have $P \neq Q_t$.

Applying Corollary 3.5, for $P_t = P$, Q_t and $X_t = X$, we get that $\angle PQX$ has the same sign as $\angle PQ'X$. □

3.7. Lemma. *Assume $[XY]$ does not intersect (PQ) then the angles $\measuredangle PQX$ and $\measuredangle PQY$ have the same sign.*

The proof is nearly identical to the one above.



Proof. According to Proposition 2.3, for any $t \in [0,1]$ there is a point $X_t \in [XY]$ such that

$$XX_t = t{\cdot}XY.$$

Note that the map $t \mapsto X_t$ is continuous, $X_0 = X$ and $X_1 = Y$ and for any $t \in [0,1]$, we have $Y_t \notin (QP)$.

Applying Corollary 3.5, for $P_t = P$, $Q_t = Q$ and X_t, we get that $\measuredangle PQX$ has the same sign as $\measuredangle PQY$. □

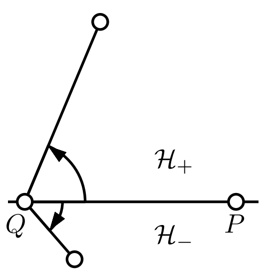
Half-planes

3.8. Proposition. *The complement of a line (PQ) in the plane can be presented in the unique way as a union of two disjoint subsets called* half-planes *such that*

(a) Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if the angles $\measuredangle PQX$ and $\measuredangle PQY$ have the same sign.

(b) Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if $[XY]$ does not intersect (PQ).

Further we say that X and Y lie on *one side from* (PQ) if they lie in one of the half-planes of (PQ) and we say that P and Q lie on the *opposite sides from* ℓ if they lie in the different half-planes of ℓ.



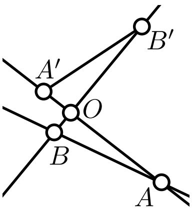
Proof. Let us denote by $\mathcal{H}_+$ (correspondingly $\mathcal{H}_-$) the set of points X in the plane such that $\measuredangle PQX$ is positive (correspondingly negative).

According to Theorem 2.7, $X \notin (PQ)$ if and only if $\measuredangle PQX \neq 0$ nor π. Therefore $\mathcal{H}_+$ and $\mathcal{H}_-$ give the unique subdivision of the complement of (PQ) which satisfies property (a).

Now let us prove that the this subdivision depends only on the line (PQ); that is, if $(P'Q') = (PQ)$ and $X, Y \notin (PQ)$ then the angles $\measuredangle PQX$ and $\measuredangle PQY$ have the same sign if and only if the angles $\measuredangle P'Q'X$ and $\measuredangle P'Q'Y$ have the same sign.

Applying Exercise 3.2, we can assume that $P = P'$ and $Q' \in [PQ)$. It remains to apply Lemma 3.6.

(b). Assume $[XY]$ intersects (PQ). Since the subdivision depends only on the line (PQ), we can assume that $Q \in [XY]$. In this case, by Exercise 3.2, the angles $\angle PQX$ and $\angle PQY$ have opposite signs.



Now assume $[XY]$ does not intersect (PQ). In this case, by Lemma 3.7, $\angle PQX$ and $\angle PQY$ have the same sign. □

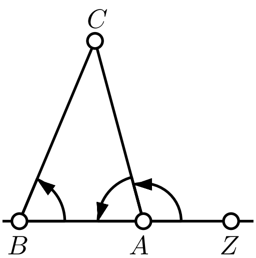
3.9. Exercise. *Assume that the angles $\angle AOB$ and $\angle A'OB'$ are vertical. Show that the line (AB) does not intersect the segment $[A'B']$.*

Consider triangle $\triangle ABC$. The segments $[AB]$, $[BC]$ and $[CA]$ are called *sides of the triangle.*

The following theorem is a corollary of Proposition 3.8.

3.10. Pasch's theorem. *Assume line ℓ does not pass through any vertex a triangle. Then it intersects either two or zero sides of the triangle.*

3.11. Signs of angles of triangle. *In any nondegenerate triangle $\triangle ABC$ the angles $\angle ABC$, $\angle BCA$ and $\angle CAB$ have the same sign.*



Proof. Choose a point $Z \in (AB)$ so that A lies between B and Z.

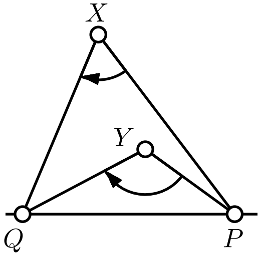
According to Lemma 3.6, the angles $\angle ZBC$ and $\angle ZAC$ have the same sign.

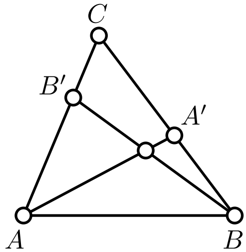
Note that $\measuredangle ABC = \measuredangle ZBC$ and

$$\measuredangle ZAC + \measuredangle CAB \equiv \pi.$$

Therefore $\angle CAB$ has the same sign as $\angle ZAC$ which in turn has the same sign as $\measuredangle ABC = \measuredangle ZBC$.

Repeating the same argument for $\angle BCA$ and $\angle CAB$, we get the result. □





3.12. Exercise. *Show that two points $X, Y \notin (PQ)$ lie on the same side from (PQ) if and only if the angles $\angle PXQ$ and $\angle PYQ$ have the same sign.*

3.13. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle, $A' \in [BC]$ and $B' \in [AC]$. Show that the segments $[AA']$ and $[BB']$ intersect.*

3.14. Exercise. *Assume that the points X and Y lie on the opposite sides from the line (PQ). Show that the half-line $[PX)$ does not interests $[QY)$.*

3.15. Advanced exercise. *Note that the following quantity*

$$\tilde{\measuredangle} ABC = \left[\begin{array}{lll} \pi & \textit{if} & \measuredangle ABC = \pi \\ -\measuredangle ABC & \textit{if} & \measuredangle ABC < \pi \end{array}\right.$$

can serve as the angle measure; that is, the axioms hold if one changes everywhere $\measuredangle$ to $\tilde{\measuredangle}$.

Show that $\measuredangle$ and $\tilde{\measuredangle}$ are the only possible angle measures on the plane.

Show that without Axiom IIIc, this is not longer true.

Triangle with the given sides

Consider a triangle $\triangle ABC$. Set

$$a = BC, \qquad b = CA, \qquad c = AB.$$

Without loss of generality we may assume that

$$a \leqslant b \leqslant c.$$

Then all three triangle inequalities for $\triangle ABC$ hold if and only if

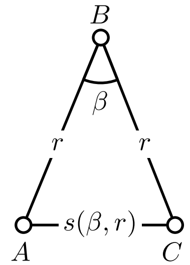
$$c \leqslant a + b.$$

The following theorem states that this is the only restriction on a, b and c.

3.16. Theorem. *Assume that $0 < a \leqslant b \leqslant c \leqslant a + b$. Then there is a triangle $\triangle ABC$ such that $a = BC$, $b = CA$ and $c = AB$.*

The proof requires some preparation.

Assume $r > 0$ and $\pi > \beta > 0$. Consider triangle $\triangle ABC$ such that $AB = BC = r$ and $\measuredangle ABC = \beta$. The existence of such triangle follow from Axiom IIIa and Proposition 2.3.



Note that according to Axiom IV, the values β and r define the triangle up to congruence. In particular the distance AC depends only on β and r. Set

$$s(\beta, r) \stackrel{\text{def}}{=\!=} AC.$$

3.17. Proposition. *Given $r > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < \beta < \delta$ then $s(r, \beta) < \varepsilon$.*

Proof. Fix two point A and B such that $AB = r$.

Choose a point X such that $\measuredangle ABX$ is positive. Let $Y \in [AX)$ be the point such that $AY = \frac{\varepsilon}{8}$; it exists by Proposition 2.3.

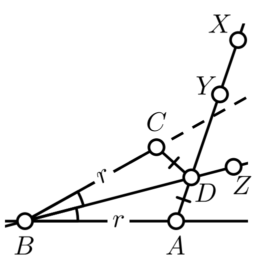
Note that X and Y lie on the same side from (AB); therefore $\angle ABY$ is positive. Set $\delta = \measuredangle ABY$.

Assume $0 < \beta < \delta$, $\measuredangle ABC = \beta$ and $BC = r$.

Applying Axiom IIIa, we can choose a half-line $[BZ)$ such that $\measuredangle ABZ = \frac{1}{2} \cdot \beta$. Note that A and Y lie on the opposite sides from (BZ). Therefore (BZ) intersects $[AY]$; denote by D the point of intersection.

Since $D \in (BZ)$, we get $\measuredangle ABD = \frac{\beta}{2}$ or $\frac{\beta}{2} - \pi$. The later is impossible since D and Y lie on the same side from (AB). Therefore

$$\measuredangle ABD = \measuredangle DBC = \tfrac{\beta}{2}.$$



By Axiom IV, $\triangle ABD \cong \triangle CBD$. In particular

$$\begin{aligned} AC &\leqslant AD + DC = \\ &= 2 \cdot AD \leqslant \\ &\leqslant 2 \cdot AY = \\ &= \tfrac{\varepsilon}{4}. \end{aligned}$$

Hence the result follows. □

3.18. Corollary. *Fix a real number $r > 0$ and two distinct points A and B. Then for any real number $\beta \in [0, \pi]$, there is unique point C_β such that $BC_\beta = r$ and $\measuredangle ABC_\beta = \beta$. Moreover, the map $\beta \mapsto C_\beta$ is a continuous map from $[0, \pi]$ to the plane.*

Proof. The existence and uniqueness of C_β follows from Axiom IIIa and Proposition 2.3.

Note that if $\beta_1 \neq \beta_2$ then

$$C_{\beta_1} C_{\beta_2} = s(r, |\beta_1 - \beta_2|).$$

Therefore Proposition 3.17 implies that the map $\beta \mapsto C_\beta$ is continuous. $\square$

Proof of Theorem 3.16. Fix points A and B such that $AB = c$. Given $\beta \in [0, \pi]$, denote by C_β the point in the plane such that $BC_\beta = a$ and $\measuredangle ABC = \beta$.

According to Corollary 3.18, the map $\beta \mapsto C_\beta$ is a continuous. Therefore function $b(\beta) = AC_\beta$ is continuous (formally it follows from Exercise 1.13 and Exercise 1.14).

Note that $b(0) = c - a$ and $b(\pi) = c + a$. Since $c - a \leqslant b \leqslant c + a$, by Intermediate value theorem (3.4) there is $\beta_0 \in [0, \pi]$ such that $b(\beta_0) = b$. Hence the result follows. $\square$

Chapter 4

Congruent triangles

Side-angle-side condition

Our next goal is to give conditions which guarantee congruence of two triangles. One of such conditions is Axiom IV, it is also called *side-angle-side congruence condition* or briefly *SAS congruence condition.*

Angle-side-angle condition

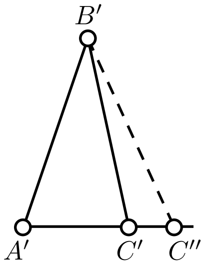
4.1. ASA condition. *Assume that*

$$AB = A'B', \qquad \measuredangle ABC = \pm\measuredangle A'B'C', \qquad \measuredangle CAB = \pm\measuredangle C'A'B'$$

and $\triangle A'B'C'$ *is nondegenerate. Then*

$$\triangle ABC \cong \triangle A'B'C'.$$

Note that for degenerate triangles the statement does not hold, say consider one triangle with sides 1, 4, 5 and the other with sides 2, 3, 5.



Proof. According to Theorem 3.11, either

❶
$$\begin{aligned}\measuredangle ABC &= \measuredangle A'B'C',\\ \measuredangle CAB &= \measuredangle C'A'B'\end{aligned}$$

or

❷
$$\begin{aligned}\measuredangle ABC &= -\measuredangle A'B'C',\\ \measuredangle CAB &= -\measuredangle C'A'B'.\end{aligned}$$

Further we assume that ❶ holds; the case ❷ is analogous.

Let C'' be the point on the half-line $[A'C')$ such that that $A'C'' = AC$.

By Axiom IV, $\triangle A'B'C'' \cong \triangle ABC$. Applying Axiom IV again, we get

$$\measuredangle A'B'C'' = \measuredangle ABC = \measuredangle A'B'C'.$$

By Axiom IIIa, $[B'C') = [BC'')$. Whence C'' lies on $(B'C')$ as well as on $(A'C')$.

Since $\triangle A'B'C'$ is not degenerate, $(A'C')$ is distinct from $(B'C')$. Applying Axiom II, we get $C'' = C'$.

Therefore $\triangle A'B'C' = \triangle A'B'C'' \cong \triangle ABC$. □

Isosceles triangles

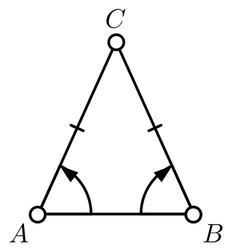
A triangle with two equal sides is called *isosceles*; the remaining side is called *base* of isosceles triangle.

4.2. Theorem. *Assume $\triangle ABC$ is an isosceles triangle with the base $[AB]$. Then*

$$\measuredangle ABC \equiv -\measuredangle BAC.$$

Moreover, the converse holds if $\triangle ABC$ is nondegenerate.

The following proof is due to Pappus of Alexandria.



Proof. Note that

$$CA = CB, \;\; CB = CA, \;\; \measuredangle ACB \equiv -\measuredangle BCA.$$

Therefore by Axiom IV,

$$\triangle CAB \cong \triangle CBA.$$

Applying the theorem on the signs of angles of triangles (3.11) and Axiom IV again, we get

$$\measuredangle CAB \equiv -\measuredangle CBA.$$

To prove the converse, we assume $\measuredangle CAB \equiv -\measuredangle CBA$. By ASA condition 4.1, $\triangle CAB \cong \triangle CBA$. Therefore $CA = CB$. □

A triangle with three equal sides is called *equilateral.*

4.3. Exercise. *Let $\triangle ABC$ be an equilateral triangle. Show that*

$$\measuredangle ABC = \measuredangle BCA = \measuredangle CAB.$$

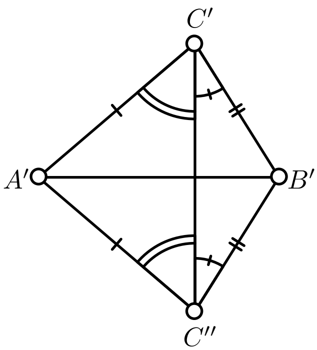
Side-side-side condition

4.4. SSS condition. *$\triangle ABC \cong \triangle A'B'C'$ if*

$$A'B' = AB, \quad B'C' = BC \quad \text{and} \quad C'A' = CA.$$

Proof. Choose C'' so that $A'C'' = A'C'$ and $\measuredangle B'A'C'' = \measuredangle BAC$. According to Axiom IV,

$$\triangle A'B'C'' \cong \triangle ABC.$$



It will suffice to prove that

$$\triangle A'B'C' \cong \triangle A'B'C''. \quad ❸$$

The condition ❸ trivially holds if $C'' = C'$. Thus it remains to consider the case $C'' \ne C'$.

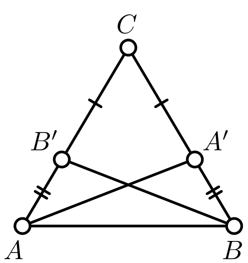
Clearly, the corresponding sides of $\triangle A'B'C'$ and $\triangle A'B'C''$ are equal. Whence the triangles $\triangle C'A'C''$ and $\triangle C'B'C''$ are isosceles. By Theorem 4.2, we have

$$\measuredangle A'C''C' \equiv -\measuredangle A'C'C'',$$
$$\measuredangle C'C''B' \equiv -\measuredangle C''C'B'.$$

By addition

$$\measuredangle A'C'B' \equiv -\measuredangle A'C''B'.$$

Applying Axiom IV again, we get ❸. □



4.5. Advanced exercise. *Let M be the midpoint of side $[AB]$ of a triangle $\triangle ABC$ and M' be the midpoint of side $[A'B']$ of a triangle $\triangle A'B'C'$. Assume $C'A' = CA$, $C'B' = CB$ and $C'M' = CM$. Prove that $\triangle A'B'C' \cong \triangle ABC$.*

4.6. Exercise. *Let $\triangle ABC$ be an isosceles triangle with the base $[AB]$ and the points $A' \in [BC]$ and $B' \in [AC]$ be such that $CA' = CB'$. Show that*

(a) *$\triangle AA'C \cong \triangle BB'C$;*
(b) *$\triangle ABB' \cong \triangle BAA'$.*

4.7. Exercise. *Show that if $AB + BC = AC$ then $B \in [AC]$.*

4.8. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle and let f be a motion of the plane such that*

$$f(A) = A, \quad f(B) = B \quad \textit{and} \quad f(C) = C.$$

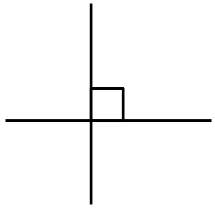
Show that f is the identity; that is, $f(X) = X$ for any point X on the plane.

Chapter 5

Perpendicular lines

Right, acute and obtuse angles

◇ If $|\measuredangle AOB| = \frac{\pi}{2}$, we say that the angle $\angle AOB$ is *right*;
◇ If $|\measuredangle AOB| < \frac{\pi}{2}$, we say that the angle $\angle AOB$ is *acute*;
◇ If $|\measuredangle AOB| > \frac{\pi}{2}$, we say that the angle $\angle AOB$ is *obtuse*.



On the diagrams, the right angles will be marked with a little square, as shown.

If $\angle AOB$ is right, we say also that $[OA)$ is *perpendicular* to $[OB)$; it will be written as $[OA) \perp [OB)$.

From Theorem 2.7, it follows that two lines (OA) and (OB) are appropriately called *perpendicular*, if $[OA) \perp [OB)$. In this case we also write $(OA) \perp (OB)$.

5.1. Exercise. *Assume point O lies between A and B and $X \neq O$. Show that $\angle XOA$ is acute if and only if $\angle XOB$ is obtuse.*

Perpendicular bisector

Assume M is the midpoint of the segment $[AB]$; that is, $M \in (AB)$ and $AM = MB$.

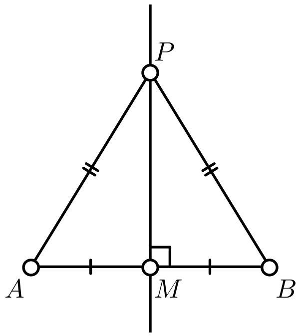
The line ℓ passing through M and perpendicular to (AB) is called *perpendicular bisector* to the segment $[AB]$.

5.2. Theorem. *Given distinct points A and B, all points equidistant from A and B and no others lie on the perpendicular bisector to $[AB]$.*

Proof. Let M be the midpoint of $[AB]$.

Assume $PA = PB$ and $P \neq M$. According to SSS (4.4), $\triangle AMP \cong \triangle BMP$. Whence

$$\measuredangle AMP = \pm\measuredangle BMP.$$



Since $A \neq B$, we have "$-$" in the above formula. Further,

$$\begin{aligned} \pi = \measuredangle AMB &\equiv \\ &\equiv \measuredangle AMP + \measuredangle PMB \equiv \\ &\equiv 2\cdot\measuredangle AMP. \end{aligned}$$

That is, $\measuredangle AMP = \pm\frac{\pi}{2}$ and therefore P lies on the perpendicular bisector.

To prove the converse, suppose $P \neq M$ is any point on the perpendicular bisector to $[AB]$. Then $\measuredangle AMP = \pm\frac{\pi}{2}$, $\measuredangle BMP = \pm\frac{\pi}{2}$ and $AM = BM$. Therefore $\triangle AMP \cong \triangle BMP$; in particular $AP = BP$. $\square$

5.3. Exercise. *Let ℓ be the perpendicular bisector to the segment $[AB]$ and X be an arbitrary point on the plane.*

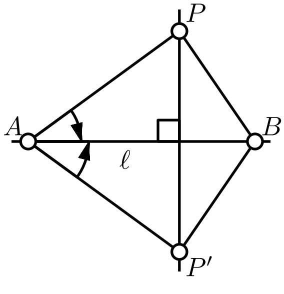
Show that $AX < BX$ if and only if X and A lie on the same side from ℓ.

5.4. Exercise. *Let $\triangle ABC$ be nondegenerate. Show that $AC > BC$ if and only if $|\measuredangle ABC| > |\measuredangle CAB|$.*

Uniqueness of perpendicular

5.5. Theorem. *There is one and only one line which pass through a given point P and perpendicular to a given line ℓ.*

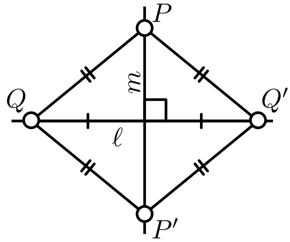
According to the above theorem, there is unique point $Q \in \ell$ such that $(QP) \perp \ell$. This point Q is called *foot point* of P on ℓ.



Proof. If $P \in \ell$ then both, existence and uniqueness, follow from Axiom III.

Existence for $P \notin \ell$. Let A, B be two distinct points of ℓ. Choose P' so that $AP' = AP$ and $\measuredangle P'AB \equiv -\measuredangle PAB$. According to Axiom IV, $\triangle AP'B \cong \triangle APB$. Therefore $AP = AP'$ and $BP = BP'$.

According to Theorem 5.2, A and B lie on perpendicular bisector to $[PP']$. In particular $(PP') \perp (AB) = \ell$.



Uniqueness for $P \notin \ell$. From above we can choose a point P' in such a way that ℓ forms the perpendicular bisector to $[PP']$.

Assume $m \perp \ell$ and $m \ni P$. Then m a perpendicular bisector to some segment $[QQ']$ of ℓ; in particular, $PQ = PQ'$.

Since ℓ is perpendicular bisector to $[PP']$, we get $PQ = P'Q$ and $PQ' = P'Q'$. Therefore

$$P'Q = PQ = PQ' = P'Q'.$$

By Theorem 5.2, P' lies on the perpendicular bisector to $[QQ']$ which is m. By Axiom II, $m = (PP')$. □

Reflection

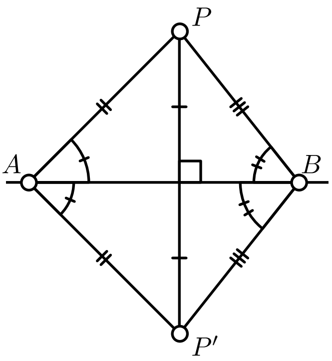
Assume a point P and a line (AB) are given. To find the *reflection* P' of P in (AB), one drops a perpendicular from P onto (AB), and continues it to the same distance on the other side.

According to Theorem 5.5, P' is uniquely determined by P.

Note that $P = P'$ if and only if $P \in (AB)$.

5.6. Proposition. *Assume P' is a reflection of the point P in the line (AB). Then $AP' = AP$ and if $A \neq P$ then $\measuredangle BAP' \equiv -\measuredangle BAP$.*

Proof. Note that if $P \in (AB)$ then $P = P'$ and by Corollary 2.8 $\measuredangle BAP = 0$ or π. Hence the statement follows.



If $P \notin (AB)$, then $P' \neq P$. By construction, the line (AB) is perpendicular bisector of $[PP']$. Therefore, according to Theorem 5.2, $AP' = AP$ and $BP' = BP$. In particular, $\triangle ABP' \cong \triangle ABP$. Therefore $\measuredangle BAP' = \pm\measuredangle BAP$.

Since $P' \neq P$ and $AP' = AP$, we get $\measuredangle BAP' \neq \measuredangle BAP$. That is, we are left with the case

$$\measuredangle BAP' = -\measuredangle BAP.$$

□

5.7. Corollary. *Reflection through the line is a motion of the plane. More over if $\triangle P'Q'R'$ is the reflection of $\triangle PQR$ then*

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR.$$

Proof. From the construction it follows that the composition of two reflections through the same line, say (AB), is the identity map. In particular reflection is a bijection.

Assume P', Q' and R' denote the reflections of the points P, Q and R through (AB). Let us first show that

➊ $$P'Q' = PQ \quad \text{and} \quad \measuredangle AP'Q' \equiv -\measuredangle APQ.$$

Without loss of generality we may assume that the points P and Q are distinct from A and B. By Proposition 5.6,

$$\begin{aligned} \measuredangle BAP' &\equiv -\measuredangle BAP, & \measuredangle BAQ' &\equiv -\measuredangle BAQ, \\ AP' &= AP, & AQ' &= AQ. \end{aligned}$$

It follows that $\measuredangle P'AQ' \equiv -\measuredangle PAQ$. Therefore $\triangle P'AQ' \cong \triangle PAQ$ and ➊ follows.

Repeating the same argument for P and R, we get

$$\measuredangle AP'R' \equiv -\measuredangle APR.$$

Subtracting the second identity in ➊, we get

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR.$$ □

5.8. Exercise. *Show that any motion of the plane can be presented as a composition of at most three reflections.*

Applying the exercise above and Corollary 5.7, we can divide the motions of the plane in two types, *direct* and *indirect motions*. The motion m is direct if

$$\measuredangle Q'P'R' = \measuredangle QPR$$

for any $\triangle PQR$ and $P' = m(P)$, $Q' = m(Q)$ and $R' = m(R)$; if instead we have

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR$$

for any $\triangle PQR$ then the motion m is called indirect.

5.9. Exercise. *Let X and Y be the reflections of P through the lines (AB) and (BC) correspondingly. Show that*

$$\measuredangle XBY \equiv 2\cdot\measuredangle ABC.$$

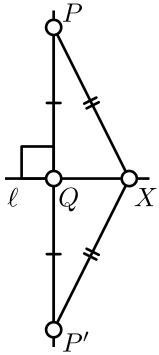
Perpendicular is shortest

5.10. Lemma. *Assume Q is the foot point of P on line ℓ. Then*

$$PX > PQ$$

for any point X on ℓ distinct from Q.

If P, Q and ℓ as above then PQ is called *distance from P to ℓ.*



Proof. If $P \in \ell$ then the result follows since $PQ = 0$. Further we assume that $P \notin \ell$.

Let P' be the reflection of P in ℓ. Note that Q is the midpoint of $[PP']$ and ℓ is perpendicular bisector of $[PP']$. Therefore

$$PX = P'X \quad \text{and} \quad PQ = P'Q = \tfrac{1}{2}\cdot PP'$$

Note that ℓ meets $[PP']$ at the point Q only. Therefore by the triangle inequality and Exercise 4.7,

$$PX + P'X > PP'.$$

Hence the result follows. □

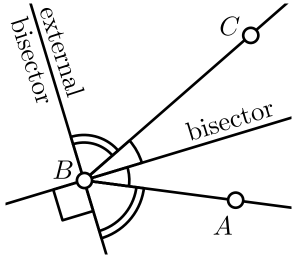
5.11. Exercise. *Assume $\angle ABC$ is right or obtuse. Show that*

$$BC > AB.$$

Angle bisectors

If $\measuredangle ABX \equiv -\measuredangle CBX$ then we say that line (BX) *bisects angle $\angle ABC$*, or line (BX) is the *bisector* of $\angle ABC$. If $\measuredangle ABX \equiv \pi - \measuredangle CBX$ then the line (BX) is called *external bisector* of $\angle ABC$.

Note that bisector and external bisector are uniquely defined by the angle.



Note that if $\measuredangle ABA' = \pi$; that is, if B lies between A and A', then bisector of $\angle ABC$ is the external bisector of $\angle A'BC$ and the other way around.

5.12. Exercise. *Show that for any angle, its bisector and external bisector are perpendicular.*

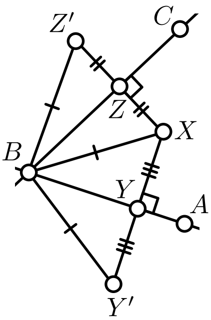
5.13. Lemma. *Assume $\measuredangle ABC \neq \pi$ nor 0. Given angle $\angle ABC$ and a point X, consider foot points Y and Z of X on (AB) and (BC).*

Then $XY = XZ$ if and only if X lies on the bisector or external bisector of $\angle ABC$.

Proof. Let Y' and Z' be the reflections of X through (AB) and (BC) correspondingly. By Proposition 5.6, $XB = Y'B = Z'B$.

Note that

$$XY' = 2{\cdot}XY \quad \text{and} \quad XZ' = 2{\cdot}XZ.$$



Applying SSS and then SAS congruence conditions, we get

$$\begin{gathered} XY = XZ \\ \Updownarrow \\ XY' = XZ' \\ \Updownarrow \\ \triangle BXY' \cong \triangle BXZ' \\ \Updownarrow \\ \measuredangle XBY' = \pm\measuredangle BXZ'. \end{gathered} \qquad ❷$$

According to Proposition 5.6,

$$\measuredangle XBA \equiv -\measuredangle Y'BA, \qquad\qquad \measuredangle XBC \equiv -\measuredangle Z'BC.$$

Therefore

$$2{\cdot}\measuredangle XBA \equiv \measuredangle XBY' \quad \text{and} \quad 2{\cdot}\measuredangle XBC \equiv -XBZ'.$$

That is, we can continue the chain of equivalence conditions ❷ the following way

$$\measuredangle XBY' \equiv \pm\measuredangle BXZ' \quad \Longleftrightarrow \quad 2{\cdot}\measuredangle XBA \equiv \pm 2{\cdot}\measuredangle XBC.$$

Since $(AB) \neq (BC)$, we have

$$2{\cdot}\measuredangle XBA \not\equiv 2{\cdot}\measuredangle XBC$$

(compare to Exercise 2.10). Therefore

$$XY = XZ \quad \Longleftrightarrow \quad 2{\cdot}\measuredangle XBA \equiv -2{\cdot}\measuredangle XBC.$$

The last identity means either

$$\measuredangle XBA + \measuredangle XBC \equiv 0$$

or

$$\measuredangle XBA + \measuredangle XBC \equiv \pi.$$

Hence the result follows. □

Circles

Given a positive real number r and a point O, the set Γ of all points on distant r from O is called *circle* with *radius* r and *center* O.

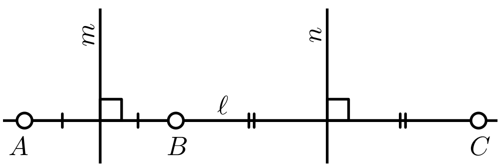
We say that a point P lies *inside* Γ if $OP < r$ and if $OP > r$, we say that P lies *outside* Γ.

5.14. Exercise. *Let Γ be a circle and $P \notin \Gamma$. Assume a line ℓ is passing through the point P intersects Γ at two distinct points X and Y. Show that P is inside Γ if and only if P lies between X and Y.*

A segment between two points on Γ is called *chord* of Γ. A chord passing through the center is called *diameter*.

5.15. Exercise. *Assume two distinct circles Γ and Γ' have a common chord $[AB]$. Show that the line between centers of Γ and Γ' forms a perpendicular bisector to $[AB]$.*

5.16. Lemma. *A line and a circle can have at most two points of intersection.*



Proof. Assume A, B and C are distinct points which lie on a line ℓ and a circle Γ with center O.

Then $OA = OB = OC$; in particular O lies on the perpendicular bisectors m and n to $[AB]$ and $[BC]$ correspondingly.

Note that the midpoints of $[AB]$ and $[BC]$ are distinct. Therefore m and n are distinct. The later contradicts the uniqueness of perpendicular (Theorem 5.5). $\square$

5.17. Exercise. *Show that two distinct circles can have at most two points of intersection.*

In consequence of the above lemma, a line ℓ and a circle Γ might have 2, 1 or 0 points of intersections. In the first case the line is called *secant line*, in the second case it is *tangent line*; if P is the only point of intersection of ℓ and Γ, we say that ℓ *is tangent to* Γ *at* P.

Similarly, according Exercise 5.17, two circles might have 2, 1 or 0 points of intersections. If P is the only point of intersection of circles Γ and Γ', we say that Γ *is tangent to* Γ *at* P.

5.18. Lemma. *Let ℓ be a line and Γ be a circle with center O. Assume P is a common point of ℓ and Γ. Then ℓ is tangent to Γ at P if and only if $(PO) \perp \ell$.*

Proof. Let Q be the foot point of O on ℓ.

Assume $P \neq Q$. Denote by P' the reflection of P through (OQ).

Note that $P' \in \ell$ and (OQ) is perpendicular bisector of $[PP']$. Therefore $OP = OP'$. Hence $P, P' \in \Gamma \cap \ell$; that is, ℓ is secant to Γ.

If $P = Q$ then according to Lemma 5.10, $OP < OX$ for any point $X \in \ell$ distinct from P. Hence P is the only point in the intersection $\Gamma \cap \ell$; that is, ℓ is tangent to Γ at P. □

5.19. Exercise. *Let Γ and Γ' be two distinct circles with centers at O and O' correspondingly. Assume Γ and Γ' intersect at point P. Show that Γ is tangent to Γ' if and only if O, O' and P lie on one line.*

5.20. Exercise. *Let Γ and Γ' be two distinct circles with centers at O and O' and radii r and r'.*

(a) Show that Γ is tangent to Γ' if and only if

$$OO' = r + r' \quad \text{or} \quad OO' = |r - r'|.$$

(b) Show that Γ intersects Γ' if and only if

$$|r - r'| \leqslant OO' \leqslant r + r'.$$

5.21. Exercise. *Assume three circles intersect at two points. Show that the centers of these circles lie on one line.*

Geometric constructions

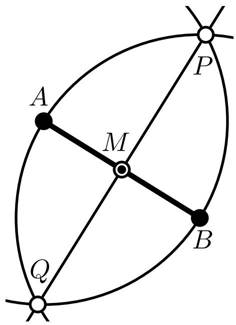
The *ruler-and-compass constructions* in the plane is the construction of points, lines, and circles using only an idealized ruler and compass. These construction problems provide a valuable source of exercises in geometry which we will use further in the book. In addition, Chapter 18 is devoted completely to the subject.

The idealized ruler can be used only to draw a line through given two points. The idealized compass can be used only to draw a circle with given center and radius. That is, given three points A, B and O we can draw the set of all points on distant AB from O. We may also mark new points in the plane as well as on the constructed lines, circles and their intersections (assuming that such points exist).

We can also look at the different set of instruments. For example, we may only use the ruler or we may invent a new instrument, say an instrument which produce midpoint for given two points.

As an example, let us consider the following problem:

5.22. Construction of midpoint. *Construct the midpoint of the given segment $[AB]$.*



Construction.

1. Construct the circle with center at A which is passing through B.
2. Construct the circle with center at B which is passing through A.
3. Mark both points of intersection of these circles, label them by P and Q.
4. Draw the line (PQ).
5. Mark the point of intersection of (PQ) and $[AB]$; this is the midpoint.

Typically, you need to proof that the construction produce what was expected. Here is a proof for the example above.

Proof. According to Theorem 5.2, (PQ) is the perpendicular bisector to $[AB]$. Therefore $M = (AB) \cap (PQ)$ is the midpoint of $[AB]$. $\square$

5.23. Exercise. *Make a ruler-and-compass construction of the line though the given point which is perpendicular to the given line.*

5.24. Exercise. *Make a ruler-and-compass construction of the center of the given circle.*

5.25. Exercise. *Make a ruler-and-compass construction of the lines tangent to the given circle which pass through the given point.*

5.26. Exercise. *Given two circles Γ_1 and Γ_2 and a segment $[AB]$ make a ruler-and-compass construction of a circle of radius AB which is tangent to each circle Γ_1 and Γ_2.*

Chapter 6

Parallel lines and similar triangles

Parallel lines

In consequence of Axiom II, any two distinct lines ℓ and m have either one point in common or none. In the first case they are *intersecting*; in the second case, ℓ and m are said to be *parallel* (briefly $\ell \parallel m$); in addition, a line is always regarded as parallel to itself.

6.1. Proposition. *Let ℓ, m and n be the lines in the plane. Assume that $n \perp m$ and $m \perp \ell$. Then $\ell \parallel n$.*

Proof. Assume contrary; that is, $\ell \nparallel n$. Then there is a point, say Z, of intersection of ℓ and n. Then by Theorem 5.5, $\ell = n$. In particular $\ell \parallel n$, a contradiction. $\square$

6.2. Theorem. *Given a point P and line ℓ in the Euclidean plane there is unique line m which pass though P and parallel to ℓ.*

The above theorem has two parts, existence and uniqueness. In the proof of uniqueness we will use Axiom V for the first time in this book.

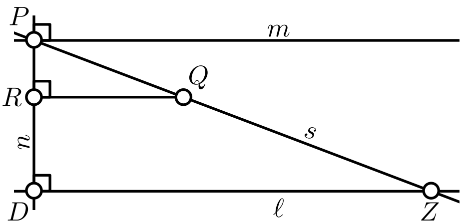
Proof; existence. Apply Theorem 5.5 two times, first to construct line m through P which is perpendicular to ℓ and second to construct line n through P which is perpendicular to m. Then apply Proposition 6.1.

Uniqueness. If $P \in \ell$ then $m = \ell$ by the definition of parallel lines. Further we assume $P \notin \ell$.

Let us construct lines $n \ni P$ and $m \ni P$ as in the proof of existence, so $m \parallel \ell$.

Assume there is yet an other line $s \ni P$ which is distinct from m and parallel to ℓ. Choose a point $Q \in s$ which lies with ℓ on the same side from m. Let R be the foot point of Q on n.

Let D be the point of intersection of n and ℓ. According to Proposition 6.1 $(QR) \parallel m$. Therefore Q, R and ℓ lie on the same side from m. In particular, $R \in [PD)$.



Choose $Z \in [PQ)$ such that

$$\frac{PZ}{PQ} = \frac{PD}{PR}.$$

Then by Axiom V, $(ZD) \perp (PD)$; that is, $Z \in \ell \cap s$, a contradiction. □

6.3. Corollary. *Assume ℓ, m and n are lines in the Euclidean plane such that $\ell \parallel m$ and $m \parallel n$. Then $\ell \parallel n$.*

Proof. Assume contrary; that is, $\ell \nparallel n$. Then there is a point $P \in \ell \cap n$. By Theorem 6.2, $n = \ell$, a contradiction. □

Note that from the definition, we have $\ell \parallel m$ if and only if $m \parallel \parallel \ell$. Therefore according to the above corollary "$\parallel$" is an equivalence relation.

6.4. Exercise. *Let k, ℓ, m and n be the lines in Euclidean plane. Assume that $k \perp \ell$ and $m \perp n$. Show that if $k \parallel m$ then $\ell \parallel n$.*

6.5. Exercise. *Make a ruler-and-compass construction of the line though the given point which is parallel to the given line.*

Similar triangles

Two triangles $\triangle A'B'C'$ and $\triangle ABC$ are *similar* (briefly $\triangle A'B'C' \sim \sim \triangle ABC$) if their sides are proportional, that is,

❶ $$A'B' = k{\cdot}AB, \quad B'C' = k{\cdot}BC \quad \text{and} \quad C'A' = k{\cdot}CA$$

for some $k > 0$ and

❷
$$\begin{aligned}\measuredangle A'B'C' &= \pm\measuredangle ABC,\\ \measuredangle B'C'A' &= \pm\measuredangle BCA,\\ \measuredangle C'A'B' &= \pm\measuredangle CAB.\end{aligned}$$

Remarks.

- ◇ According to 3.11, in the above three equalities the signs can be assumed to me the same.
- ◇ If $\triangle A'B'C' \sim \triangle ABC$ with $k = 1$ in ❶, then $\triangle A'B'C' \cong$ $\cong \triangle ABC$.
- ◇ Note that "$\sim$" is an *equivalence relation*. That is,
  - (i) $\triangle ABC \sim \triangle ABC$ for any $\triangle ABC$.
  - (ii) If $\triangle A'B'C' \sim \triangle ABC$ then
    $$\triangle ABC \sim \triangle A'B'C'$$
  - (iii) If $\triangle A''B''C'' \sim \triangle A'B'C'$ and $\triangle A'B'C' \sim \triangle ABC$ then
    $$\triangle A''B''C'' \sim \triangle ABC.$$

Using "$\sim$", the Axiom V can be formulated the following way.

6.6. Reformulation of Axiom V. *If for two triangles $\triangle ABC$, $\triangle AB'C'$ and $k > 0$ we have $B' \in [AB)$, $C' \in [AC)$, $AB' = k{\cdot}AB$ and $AC' = k{\cdot}AC$ then $\triangle ABC \sim \triangle AB'C'$.*

In other words, the Axiom V provides a condition which guarantee that two triangles are similar. Let us formulate yet three such conditions.

6.7. Similarity conditions. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ in the Euclidean plane are similar if one of the following conditions hold.*

(SAS) For some constant $k > 0$ we have

$$AB = k{\cdot}A'B',\ \ AC = k{\cdot}A'C'$$

$$\textit{and}\ \ \measuredangle BAC = \pm\measuredangle B'A'C'.$$

(AA) The triangle $\triangle A'B'C'$ is nondegenerate and

$$\measuredangle ABC = \pm\measuredangle A'B'C',\ \ \measuredangle BAC = \pm\measuredangle B'A'C'.$$

(SSS) For some constant $k > 0$ we have

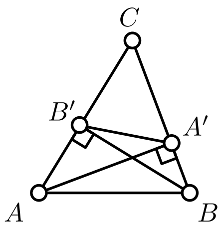
$$AB = k{\cdot}A'B',\ \ AC = k{\cdot}A'C',\ \ CB = k{\cdot}C'B'.$$

Each of these conditions is proved by applying the Axiom V with SAS, ASA and SSS congruence conditions correspondingly (see Axiom IV and the conditions 4.1, 4.4).

Proof. Set $k = \frac{AB}{A'B'}$. Choose points $B'' \in [A'B')$ and $C'' \in [A'C')$ so that $A'B'' = k \cdot A'B'$ and $A'C'' = k \cdot A'C'$. By Axiom V, $\triangle A'B'C' \sim \sim \triangle A'B''C''$.

Applying SAS, ASA or SSS congruence condition, depending on the case, we get $\triangle A'B''C'' \cong \triangle ABC$. Hence the result follows. □

A triangle with all acute angles is called *acute*.



6.8. Exercise. *Let $\triangle ABC$ be an acute triangle in the Euclidean plane. Denote by A' the foot point of A on (BC) and by B' the foot point of B on (AC). Prove that $\triangle A'B'C \sim \triangle ABC$.*

A bijection from plane to itself is called *angle preserving transformation*angle preserving if

$$\measuredangle ABC = \measuredangle A'B'C'$$

for any triangle $\triangle ABC$ and its image $\triangle A'B'C'$.

6.9. Exercise. *Show that any angle-preserving transformation of Euclidean plane multiplies all the distance by a fixed constant.*

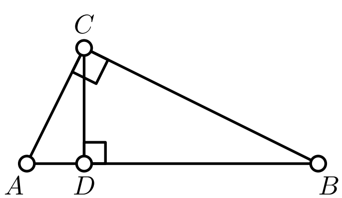
Pythagorean theorem

A triangle is called *right* if one of its angles is right. The side opposite the right angle is called the *hypotenuse*. The sides adjacent to the right angle are called *legs*.

6.10. Theorem. *Assume $\triangle ABC$ is a right triangle in the Euclidean plane with right angle at C. Then*

$$AC^2 + BC^2 = AB^2.$$

Proof. Let D be the foot point of C on (AB).



According to Lemma 5.10,

$$AD < AC < AB$$

and

$$BD < BC < AB.$$

Therefore D lies between A and B; in particular,

$$AD + BD = AB. \tag{❸}$$

Note that by AA similarity condition, we have

$$\triangle ADC \sim \triangle ACB \sim \triangle CDB.$$

In particular

$$\frac{AD}{AC} = \frac{AC}{AB} \quad \text{and} \quad \frac{BD}{BC} = \frac{BC}{BA}. \tag{❹}$$

Let us rewrite identities ❹ on an other way:

$$AC^2 = AB{\cdot}AD \qquad \text{and} \qquad BC^2 = AB{\cdot}BD.$$

Summing up above two identities and applying ❸, we get

$$AC^2 + BC^2 = AB{\cdot}(AD + BD) = AB^2.$$

□

6.11. Exercise. *Assume A, B, C and D are as in the proof above. Show that*

$$CD^2 = AD{\cdot}BD.$$

The following exercise is the converse to Pythagorean theorem.

6.12. Exercise. *Assume $\triangle ABC$ is a triangle in the Euclidean plane such that*

$$AC^2 + BC^2 = AB^2.$$

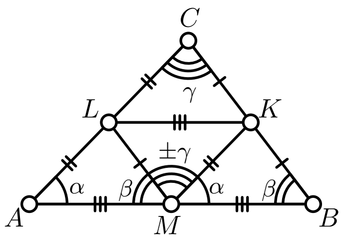
Prove that the angle at C is right.

Angles of triangle

6.13. Theorem. *In any triangle $\triangle ABC$ in the Euclidean plane, we have*

$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB \equiv \pi.$$

Proof. First note that if $\triangle ABC$ is degenerate then the equality follows from Lemma 2.7. Further we assume that $\triangle ABC$ is nondegenerate.



Set

$$\alpha = \measuredangle CAB,$$
$$\beta = \measuredangle ABC,$$
$$\gamma = \measuredangle BCA.$$

We need to prove that

❺ $$\alpha + \beta + \gamma \equiv \pi.$$

Let K, L, M be the midpoints of the sides $[BC]$, $[CA]$, $[AB]$ respectively. By Axiom V,

$$\triangle AML \sim \triangle ABC, \quad \triangle MBK \sim \triangle ABC, \quad \triangle LKC \sim \triangle ABC$$

and

$$LM = \tfrac{1}{2}{\cdot}BC, \qquad MK = \tfrac{1}{2}{\cdot}CA, \qquad KL = \tfrac{1}{2}{\cdot}AB.$$

According to SSS condition (6.7), $\triangle KLM \sim \triangle ABC$. Thus,

❻ $$\measuredangle MKL = \pm\alpha, \quad \measuredangle KLM = \pm\beta, \quad \measuredangle LMK = \pm\gamma.$$

According to 3.11, the "+" or "−" sign is to be the same throughout ❻.

If in ❻ we have "+" then ❺ follows since

$$\beta + \gamma + \alpha \equiv \measuredangle AML + \measuredangle LMK + \measuredangle KMB \equiv \measuredangle AMB \equiv \pi$$

It remains to show that we can not have "−" in ❻. In this case the same argument as above gives

$$\alpha + \beta - \gamma \equiv \pi.$$

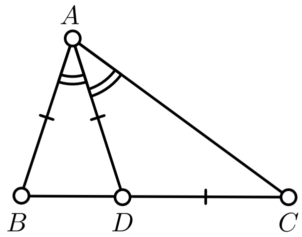
The same way we get

$$\alpha - \beta + \gamma \equiv \pi$$

Adding last two identities we get

$$2{\cdot}\alpha \equiv 0.$$

Equivalently $\alpha \equiv \pi$ or 0; that is, $\triangle ABC$ is degenerate, a contradiction. $\square$



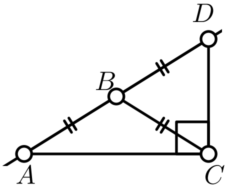
6.14. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle. Assume there is a point $D \in [BC]$ such that (AD) bisects $\angle BAC$ and $BA = AD = DC$. Find the angles of $\triangle ABC$.*

6.15. Exercise. *Show that*

$$|\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB| = \pi.$$

for any $\triangle ABC$ in the Euclidean plane.

6.16. Exercise. *Let $\triangle ABC$ be an isosceles nondegenerate triangle with base $[AC]$. Consider the point D on the extension of the side $[AB]$ such that $AB = BD$. Show that $\angle ACD$ is right.*



6.17. Exercise. *Let $\triangle ABC$ be an isosceles nondegenerate triangle with base $[AC]$. Assume that circle Γ pass through A, centered at a point on $[AB]$ and tangent to (BC) at the point X. Show that $\measuredangle CAX = \pm\frac{\pi}{4}$.*

Transversal property

If a line t intersects each line ℓ and m at one point then we say that t is a *transversal* to ℓ and m. On the diagram below, line (CB) is a transversal to (AB) and (CD).

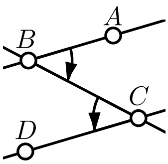
6.18. Transversal property. *In the Euclidean plane, $(AB) \parallel (CD)$ if and only if*

❼ $$2\cdot(\measuredangle ABC + \measuredangle BCD) \equiv 0.$$

Equivalently

$$\measuredangle ABC + \measuredangle BCD \equiv 0 \quad \text{or} \quad \measuredangle ABC + \measuredangle BCD \equiv \pi;$$

in the first case A and D lie on the opposite sides of (BC), in the second case A and D lie on the same sides of (BC).



Proof. If $(AB) \nparallel (CD)$ then there is $Z \in (AB) \cap (CD)$ and $\triangle BCZ$ is nondegenerate.

According to Theorem 6.13,

$$\measuredangle ZBC + \measuredangle BCZ \equiv \pi - \measuredangle CZB \not\equiv$$
$$\not\equiv 0 \text{ nor } \pi.$$

Note that $2 \cdot \measuredangle ZBC \equiv 2 \cdot \measuredangle ABC$ and $2 \cdot \measuredangle BCZ \equiv 2 \cdot \measuredangle BCD$. Therefore

$$2 \cdot (\measuredangle ABC + \measuredangle BCD) \equiv 2 \cdot \measuredangle ZBC + 2 \cdot \measuredangle BCZ \not\equiv 0;$$

that is, ❼ does not hold.

Note that if the points A, B and C are fixed, the identity ❼ uniquely defines the line (CD). Therefore by Theorem 6.2, if $(AB) \parallel \parallel (CD)$ then equality ❼ holds.

Applying Proposition 3.8, we get the last part of Transversal property. □

6.19. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle. Assume B' and C' are points on sides $[AB]$ and $[AC]$ such that $(B'C') \parallel (BC)$. Show that $\triangle ABC \sim \triangle AB'C'$.*

6.20. Exercise. *Trisect given segment with ruler and compass.*

Parallelograms

A *quadrilateral* is an ordered quadruple of distinct points in the plane. A quadrilateral formed by quadruple (A, B, C, D) will be denoted as $\square ABCD$.

Given a quadrilateral $\square ABCD$, the four segments $[AB]$, $[BC]$, $[CD]$ and $[DA]$ are called *sides of* $\square ABCD$; the remaining two segments $[AC]$ and $[BD]$ are called *diagonals of* $ABCD$.

6.21. Exercise. *Show for any quadrilateral $\square ABCD$ in the Euclidean plane we have*

$$\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB \equiv 0.$$

A quadrilateral $\square ABCD$ in the Euclidean plane is called *nondegenerate* if any three points from A, B, C, D do not lie on one line.

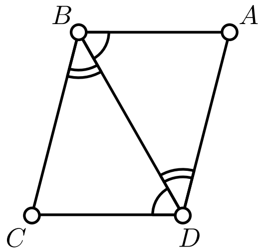
The nondegenerate quadrilateral $\square ABCD$ is called *parallelogram* if $(AB) \parallel (CD)$ and $(BC) \parallel (DA)$.

6.22. Lemma. *If $\square ABCD$ is a parallelogram then*

(a) $\measuredangle DAB = \measuredangle BCD$;
(b) $AB = CD$.

Proof. Since $(AB) \parallel (CD)$, the points C and D lie on the same side from (AB). Whence $\angle ABD$ and $\angle ABC$ have the same sign.

Analogously, $\angle CBD$ and $\angle CBA$ have the same sign.



Since $\measuredangle ABC \equiv -\measuredangle CBA$, we get that the angles $\angle DBA$ and $\angle DBC$ have opposite signs; that is, A and C lie on the opposite sides of (BD).

According to Transversal property (6.18),

$$\measuredangle BDC \equiv -\measuredangle DBA \quad \text{and} \quad \measuredangle DBC \equiv -\measuredangle BDA.$$

By ASA condition $\triangle ABD \cong \triangle CDB$. Which implies both statements in the lemma. $\square$

6.23. Exercise. *Assume $\square ABCD$ is a quadrilateral such that*

$$AB = CD = BC = DA.$$

Show that $\square ABCD$ is a parallelogram.

A quadrilateral as in the exercise above is called *rhombus.*

6.24. Exercise. *Show that diagonals of parallelogram intersect each other at the midpoint.*

A quadraliteral $\square ABCD$ is called *rectangle* if the angles $\angle ABC$, $\angle BCD$, $\angle CDA$ and $\angle DAB$ are right. Note that according to Transversal property (6.18), any rectangle is a parallelogram.

If in addition $AB = BC = CD = DA$ then the rectangle $\square ABCD$ is called *square.* In other words a square is a rectangle which is also a rhombus.

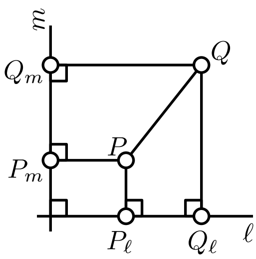
6.25. Exercise. *Show that parallelogram $\square ABCD$ is a rectangle if and only if $AC = BD$.*

6.26. Exercise. *Show that parallelogram $\square ABCD$ is a rhombus if and only if $(AC) \perp (BD)$.*

Assume $\ell \parallel m$ and $X, Y \in m$. Denote by X' and Y' the foot points of X and Y on ℓ. Note that $\square XYY'X'$ is a rectangle. By Lemma 6.22, $XX' = YY'$; that is the distance from any point $X \in m$ to ℓ is the same. This distance is called *distance between* ℓ and m.

Method of coordinates

The following exercise is important; it shows that our axiomatic definition agrees with the model definition described on page 11.



6.27. Exercise. *Let ℓ and m be perpendicular lines in the Euclidean plane. Given a points P denote by P_ℓ and P_m the foot points of P on ℓ and m correspondingly.*

(a) Show that for any $X \in \ell$ and $Y \in m$ there is unique point P such that $P_\ell = X$ and $P_m = Y$.

(b) Show that $PQ^2 = P_\ell Q_\ell^2 + P_m Q_m^2$ for any pair of points P and Q.

(c) Conclude that Euclidean plane is isometric to $(\mathbb{R}^2, d_2)$ defined on page 11.

Once this exercise solved, we can apply the method of coordinates to solve any problem in Euclidean plane geometry. This method is powerful, but it is often considered as a bad style.

6.28. Exercise. *Use the Exercise 6.27, to give an alternative proof of Theorem 3.16 in the Euclidean plane.*

That is, prove that given real numbers a, b and c such that

$$0 < a \leqslant b \leqslant c \leqslant a + c,$$

there is a triangle $\triangle ABC$ such that $a = BC$, $b = CA$ and $c = AB$.

6.29. Exercise. *Let (x_A, y_A) and (x_B, y_B) be the coordinates of distinct points A and B in the Euclidean plane. Show that the line (AB) is formed by points with coordinates (x, y) which satisfy the following equation*

$$(x - x_A)\cdot(y_B - y_A) = (y - y_A)\cdot(x_B - x_A).$$

Chapter 7

Triangle geometry

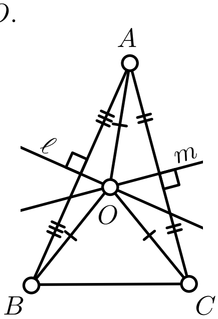
Triangle geometry studies of the properties of triangles, including associated centers and circles.

In this chapter we discuss the most basic results in triangle geometry, mostly to show that we develop enough machinery to prove things.

Circumcircle and circumcenter

7.1. Theorem. *Perpendicular bisectors to the sides of any nondegenerate triangle in the Euclidean plane intersect at one point.*

The point of the intersection of the perpendicular bisectors is called circumcenter. It is the center of the circumcircle of the triangle; that is, the circle which pass through all three vertices of the triangle. The circumcenter of the triangle is usually denoted by O.



Proof. Let $\triangle ABC$ be nondegenerate. Let ℓ and m be perpendicular bisectors to sides $[AB]$ and $[AC]$ correspondingly.

Assume ℓ and m intersect, let $O = \ell \cap n$.

Let us Theorem 5.2. Since $O \in \ell$, we have $OA = OB$ and since $O \in m$, we have $OA = OC$. It follows that $OB = OC$; that is, O lies on the perpendicular bisector to $[BC]$.

It remains to show that $\ell \nparallel m$; assume contrary. Since $\ell \perp (AB)$ and $m \perp (AC)$, we get $(AC) \parallel (AB)$ (see Exercise 6.4). Therefore by Theorem 5.5, $(AC) = (AB)$; that is, $\triangle ABC$ is degenerate, a contradiction. $\square$

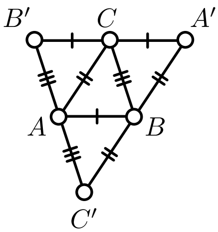
7.2. Exercise. *There is unique circle which pass through vertices of a given nondegenerate triangle in the Euclidean plane.*

Altitudes and orthocenter

An *altitude* of a triangle is a line through a vertex and perpendicular to the line containing the opposite side. The term *altitude* maybe also used for the distance from the vertex to its foot point on the line containing opposite side.

7.3. Theorem. *The three altitudes of any nondegenerate triangle in the Euclidean plane intersect in a single point.*

The point of intersection of altitudes is called *orthocenter*; it is usually denoted as H.



Proof. Let $\triangle ABC$ be nondegenerate.

Consider three lines $\ell \parallel (BC)$ through A, $m \parallel (CA)$ through B and $n \parallel (AB)$ through C. Since $\triangle ABC$ is nondegenerate, no pair of the lines ℓ, m and n is parallel. Set $A' = m \cap n$, $B' = n \cap \ell$ and $C' = \ell \cap m$.

Note that $\square ABA'C$, $\square BCB'A$ and $\square CBC'A$ are parallelograms. Applying Lemma 6.22 we get that $\triangle ABC$ is the median triangle of $\triangle A'B'C'$; that is, A, B and C are the midpoints of $[B'C']$, $[C'A']$ and $[A'B']$ correspondingly.

By Exercise 6.4, $(B'C') \parallel (BC)$, the altitudes from A is perpendicular to $[B'C']$ and from above it bisects $[B'C']$.

Whence the altitudes of $\triangle ABC$ are also perpendicular bisectors of the triangle $\triangle A'B'C'$. Applying Theorem 7.1, we get that altitudes of $\triangle ABC$ intersect at one point. $\square$

7.4. Exercise. *Assume H is the orthocenter of an acute triangle $\triangle ABC$ in the Euclidean plane. Show that A is orthocenter of $\triangle HBC$.*

Medians and centroid

A median of a triangle is a segment joining a vertex to the midpoint of the opposing side.

7.5. Theorem. *The three medians of any nondegenerate triangle in the Euclidean plane intersect in a single point. Moreover the point of intersection divides each median in ratio 2:1.*

The point of intersection of medians is called *centroid*; it is usually denoted by M.

Proof. Consider a nondegenerate triangle $\triangle ABC$. Let $[AA']$ and $[BB']$ be its medians.

According to Exercise 3.13, $[AA']$ and $[BB']$ are intersecting. Let us denote by M the point of intersection.

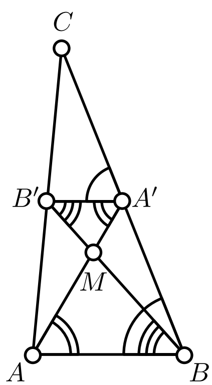
By SAS, $\triangle B'A'C \sim \triangle ABC$ and $A'B' = \frac{1}{2}\cdot AB$. In particular $\measuredangle ABC = \measuredangle B'A'C$.

Since A' lies between B and C, we get $\measuredangle BA'B' + \measuredangle B'A'C = \pi$. Therefore

$$\measuredangle B'A'B + \measuredangle A'BA = \pi.$$

By Transversal property (6.18), $(AB) \parallel (A'B')$.

Note that A' and A lie on the opposite sides from (BB'). Therefore by Transversal property (6.18) we get



$$\measuredangle B'A'M = \measuredangle BAM.$$

The same way we get,

$$\measuredangle A'B'M = \measuredangle ABM.$$

By AA condition, $\triangle ABM \sim \triangle A'B'M$.

Since $A'B' = \frac{1}{2}\cdot AB$, we have

$$\frac{A'M}{AM} = \frac{B'M}{BM} = \frac{1}{2}.$$

In particular M divides medians $[AA']$ and $[BB']$ in ratio 2:1.

Note that M is unique point on $[BB']$ such that

$$\frac{B'M}{BM} = \frac{1}{2}.$$

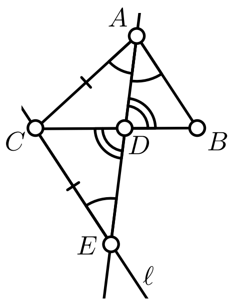
Repeating the same argument for vertices B and C we get that all medians $[CC']$ and $[BB']$ intersect in M. $\square$

7.6. Exercise. *Let $\square ABCD$ be a nondegenerate quadrilateral and X, Y, V, W be the midpoints of its sides $[AB]$, $[BC]$, $[CD]$ and $[DA]$. Show that $\square XYVW$ is a parallelogram.*

Bisector of triangle

7.7. Lemma. *Let $\triangle ABC$ be a nondegenerate triangle in the Euclidean plane. Assume that the bisector of $\angle BAC$ intersects $[BC]$ at the point D. Then*

❶ $$\frac{AB}{AC} = \frac{DB}{DC}.$$



Proof. Let ℓ be the line through C parallel to (AB). Note that $\ell \nparallel (AD)$; set $E = \ell \cap (AD)$.

Note that B and C lie on the opposite sides of (AD). Therefore by Transversal property (6.18),

❷ $$\measuredangle BAD = \measuredangle CED.$$

Further, note that the angles $\angle ADB$ and $\angle EDC$ are vertical; in particular, by 2.11

$$\measuredangle ADB = \measuredangle EDC.$$

By AA similarity condition, $\triangle ABD \sim \triangle ECD$. In particular,

❸ $$\frac{AB}{EC} = \frac{DB}{DC}.$$

Since (AD) bisects $\angle BAC$, we get $\measuredangle BAD = \measuredangle DAC$. Together with ❷, it implies that $\measuredangle CEA = \measuredangle EAC$. By Theorem 4.2, $\triangle ACE$ is isosceles; that is,

$$EC = AC.$$

Together with ❸, it implies ❶. □

7.8. Exercise. *Prove an analog of Lemma 7.7 for the external bisector.*

Incenter

7.9. Theorem. *The angle bisectors of any nondegenerate triangle intersect at one point.*

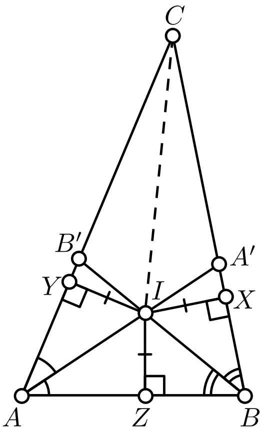
The point of intersection of bisectors is called *incenter*; it is usually denoted as I. The point I lies on the same distance from each side, it is the center of a circle tangent to each side of triangle. This circle is called *incircle* and its radius is called *inradius* of the triangle.

Proof. Let $\triangle ABC$ be a nondegenerate triangle.

Note that points B and C lie on the opposite sides from the bisector of $\angle BAC$. Hence this bisector intersects $[BC]$ at a point, say A'.

Analogously, there is $B' \in [AC]$ such the (BB') bisects $\angle ABC$.

Applying Pasch's theorem (3.10), twice for the triangles $\triangle AA'C$ and $\triangle BB'C$, we get that $[AA']$ and $[BB']$ intersect. Let us denote by I the point of intersection.



Let X, Y and Z be the foot points of I on (BC), (CA) and (AB) correspondingly. Applying Lemma 5.13, we get

$$IY = IZ = IX.$$

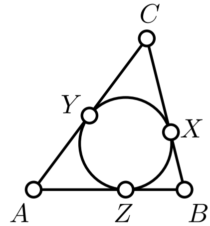
From the same lemma we get that I lies on a bisector or exterior bisector of $\angle BCA$.

The line (CI) intersects $[BB']$, the points B and B' lie on opposite sides of (CI). Therefore the angles $\angle ICB'$ and $\angle ICB$ have opposite signs. Note that $\angle ICA = \angle ICB'$. Therefore (CI) can not be exterior bisector of $\angle BCA$. Hence the result follows. $\square$

More exercises

7.10. Exercise. *Assume that bisector at one vertex of a nondegenerate triangle bisects the opposite side. Show that the triangle is isosceles.*

7.11. Exercise. *Assume that at one vertex of a nondegenerate triangle bisector coincides with the altitude. Show that the triangle is isosceles.*



7.12. Exercise. *Assume sides $[BC]$, $[CA]$ and $[AB]$ of $\triangle ABC$ are tangent to incircle at X, Y and Z correspondingly. Show that*

$$AY = AZ = \tfrac{1}{2}\cdot(AB + AC - BC).$$

By the definition, the *orthic triangle* is formed by the base points of its altitudes of the given triangle.

7.13. Exercise. *Prove that orthocenter of an acute triangle coincides with incenter of its orthic triangle.*

What should be an analog of this statement for an obtuse triangle?

Chapter 8

Inscribed angles

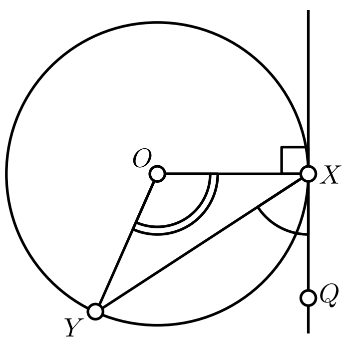
Angle between a tangent line and a chord

8.1. Theorem. *Let Γ be a circle with center O in the Euclidean plane. Assume line (XQ) is tangent to Γ at X and $[XY]$ is a chord of Γ. Then*

❶ $$2{\cdot}\measuredangle QXY \equiv \measuredangle XOY.$$

Equivalently,

$$\measuredangle QXY \equiv \tfrac{1}{2}{\cdot}\measuredangle XOY \quad \textit{or} \quad \measuredangle QXY \equiv \tfrac{1}{2}{\cdot}\measuredangle XOY + \pi.$$



Proof. Note that $\triangle XOY$ is isosceles. Therefore $\measuredangle YXO = \measuredangle OYX$.

Let us applying Theorem 6.13 to $\triangle XOY$. We get

$$\begin{aligned}\pi &\equiv \measuredangle YXO + \measuredangle OYX + \measuredangle XOY \equiv \\ &\equiv 2{\cdot}\measuredangle YXO + \measuredangle XOY.\end{aligned}$$

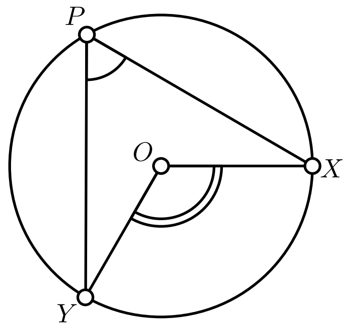
By Lemma 5.18, $(OX) \perp (XQ)$. Therefore

$$\measuredangle QXY + \measuredangle YXO \equiv \pm\tfrac{\pi}{2}.$$

Therefore

$$2{\cdot}\measuredangle QXY \equiv \pi - 2{\cdot}\measuredangle YXO \equiv \measuredangle XOY. \qquad \square$$

Inscribed angle



We say that triangle is *inscribed* in the circle Γ if all its vertices lie on Γ.

8.2. Theorem. *Let Γ be a circle with center O in the Euclidean plane, and X, Y be two distinct points on Γ. Then $\triangle XPY$ is inscribed in Γ if and only if*

❷ $$2\cdot\measuredangle XPY \equiv \measuredangle XOY.$$

Equivalently, if and only if

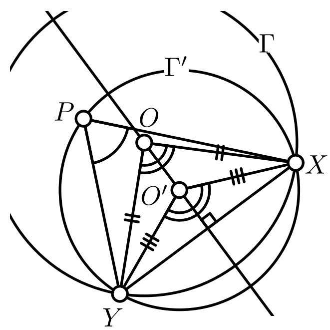
$$\measuredangle XPY \equiv \tfrac{1}{2}\cdot\measuredangle XOY \quad \textit{or} \quad \measuredangle XPY \equiv \tfrac{1}{2}\cdot\measuredangle XOY + \pi.$$

Proof. Choose a point Q such that $(PQ) \perp (OP)$. By Lemma 5.18, (PQ) is tangent to Γ.

According to Theorem 8.1,

$$2\cdot\measuredangle QPX \equiv \measuredangle POX,$$
$$2\cdot\measuredangle QPY \equiv \measuredangle POY.$$

Subtracting one identity from the other we get ❷.



Let us prove the converse. Assume that ❷ holds for some $P \notin \Gamma$. Note that $\measuredangle XOY \neq 0$ and therefore $\measuredangle XPY \neq 0$ nor π; that is, $\triangle PXY$ is nondegenerate.

Let Γ' be the circumcircle of $\triangle PXY$ and O' be its circumcenter; they exist by Exercise 7.2.

Note that $O' \neq O$. From above, we have

$$\measuredangle XOY \equiv 2\cdot\measuredangle XPY \equiv \measuredangle XO'Y.$$

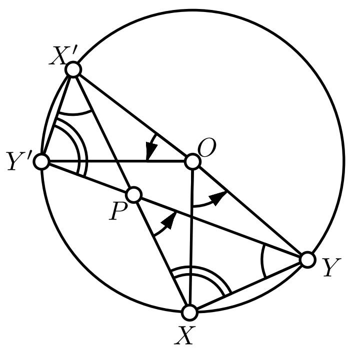
Note that $OX = OY$ and $O'X = O'Y$. By Theorem 5.2, (OO') is the perpendicular bisector to $[XY]$; equivalently X is the reflection of Y in (OO'). Applying Proposition 5.6, we get

$$\measuredangle XOO' \equiv -\measuredangle YOO', \qquad \measuredangle XO'O \equiv -\measuredangle YO'O.$$

Therefore

$$\begin{aligned} 2\cdot\measuredangle XOO' &\equiv \measuredangle XOO' + \measuredangle O'OY \equiv \\ &\equiv \measuredangle XOY \equiv \measuredangle XO'Y \equiv \\ &\equiv \measuredangle XO'O + \measuredangle OO'Y \\ &\equiv 2\cdot\measuredangle XO'O. \end{aligned}$$

By Transversal property 6.18, $(XO) \parallel (XO')$, a contradiction. □

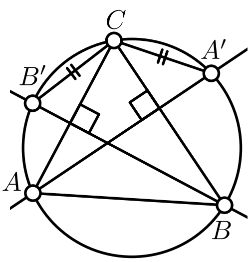


8.3. Exercise. *Let $[XX']$ and $[YY']$ be two chords of circle Γ with center O and radius r in the Euclidean plane. Assume (XX') and (YY') intersect at point P. Show that*

(a) $2 \cdot \measuredangle XPY = \measuredangle XOY + \measuredangle X'OY'$;
(b) $\triangle PXY \sim \triangle PY'X'$;
(c) $PX \cdot PX' = |OP^2 - r^2|$.

8.4. Exercise. *Assume that the chords $[XX']$, $[YY']$ and $[ZZ']$ of the circle Γ in the Euclidean plane intersect at one point. Show that*

$$XY' \cdot ZX' \cdot YZ' = X'Y \cdot Z'X \cdot Y'Z.$$

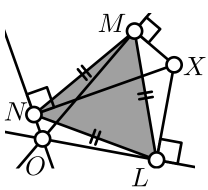


8.5. Exercise. *Let Γ be a circumcircle of $\triangle ABC$. $A' \neq A$ and $B' \neq B$ be the points of intersection of altitudes from A and B with Γ. Show that $\triangle A'B'C$ is isosceles.*

Recall that diameter of circle is its chord which pass through the center. Note that if $[XY]$ is diameter of circle with center O then $\measuredangle XOY = \pi$. Whence Theorem 8.2 implies the following.

8.6. Corollary. *Let Γ be a circle with diameter $[XY]$. Assume that point P is distinct from X and Y. Then $P \in \Gamma$ if and only if $\angle XPY$ is right.*

8.7. Exercise. *Given four points A, B, A' and B' construct a point Z such that both angles $\angle AZB$ and $\angle A'ZB'$ are right.*



8.8. Exercise. *Assume three lines ℓ, m and n intersect at point O and form six equal angles at O. Let X be a point distinct from O, denote by L, M and N be the footpoints of X on ℓ, m and n correspondingly. Show that $\triangle LMN$ is equilateral.*

8.9. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle in the Euclidean plane, A' and B' be foot points of altitudes from A and B. Show that A, B, A' and B' lie on one circle.*

What is the center of this circle?

8.10. Exercise. *Assume a line ℓ and a circle with center on ℓ are given. Make a ruler-only construction of the perpendicular to ℓ from the given point.*

Inscribed quadrilaterals

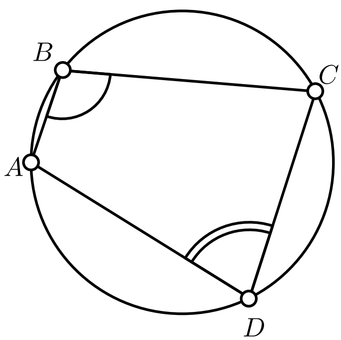
A quadrilateral $\square ABCD$ is called *inscribed* if all the points A, B, C and D lie on a circle or a line.

8.11. Theorem. *A quadrilateral $\square ABCD$ in the Euclidean plane is inscribed if and only if*

❸ $$2{\cdot}\measuredangle ABC + 2{\cdot}\measuredangle CDA \equiv 0.$$

Equivalently, if and only if

$$\measuredangle ABC + \measuredangle CDA \equiv \pi \quad \textit{or} \quad \measuredangle ABC \equiv -\measuredangle CDA.$$



Proof of Theorem 8.11. Assume $\triangle ABC$ is degenerate. By Corollary 2.8,

$$2{\cdot}\measuredangle ABC \equiv 0;$$

From the same corollary, we get

$$2{\cdot}\measuredangle CDA \equiv 0$$

if and only if $D \in (AB)$; hence the result follows.

It remains to consider the case if $\triangle ABC$ is nondegenerate.

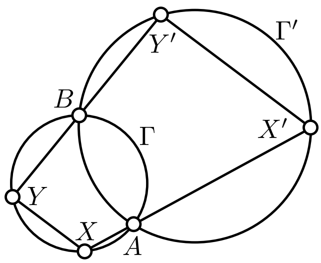
Denote by Γ the circumcircle of $\triangle ABC$ and let O be the center of Γ. According to Theorem 8.2,

❹ $$2{\cdot}\measuredangle ABC \equiv \measuredangle AOC.$$

From the same theorem, $D \in \Gamma$ if and only if

❺ $$2{\cdot}\measuredangle CDA \equiv \measuredangle COA.$$

Adding ❹ and ❺, we get the result. ☐



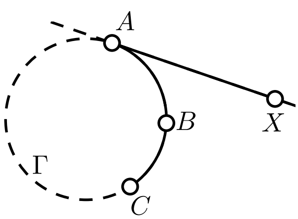
8.12. Exercise. *Let Γ and Γ' be two circles which intersect at two distinct points A and B. Assume $[XY]$ and $[X'Y']$ be the chords of Γ and Γ' correspondingly such that A lies between X and X' and B lies between Y and Y'. Show that $(XY)\parallel(X'Y')$.*

8.13. Exercise. *Let $[XY]$ and $[X'Y']$ be two parallel chords of a circle. Show that $XX'=YY'$.*

Arcs

A subset of a circle bounded by two points is called a circle arc.

More precisely, let Γ be a circle and A, B, C be distinct points on Γ. The subset which includes the points A, C as well as all the points on Γ which lie with B on the same side from (AC) is called *circle arc* ABC.



For the circle arc ABC, the points A and C are called *endpoints*. Note that there are two circle arcs of Γ with the given endpoints.

A half-line $[AX)$ is called *tangent* to arc ABC at A if the line (AX) is tangent to Γ and the points X and B lie on the same side from the line (AC).

If B lies on the line (AC), the arc ABC degenerates to one of two following a subsets of line (AC).

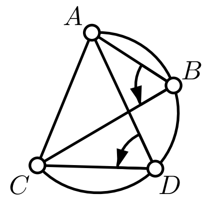
- ◇ If B lies between A and C then we define the arc ABC as the segment $[AC]$. In this case the half-line $[AC)$ is tangent to the arc ABC at A.
- ◇ If $B\in(AC)\backslash[AC]$ then we define the arc ABC as the line (AC) without all the points between A and C. If we choose points X and $Y\in(AC)$ such that the points X, A, C and Y appear in the same order on the line then the arc ABC is formed by two half-lines in $[AX)$ and $[CY)$. The half-line $[AX)$ is tangent to the arc ABC at A.
- ◇ In addition, any half-line $[AB)$ will be regarded as an arc. This degenerate arc has only one end point A and it assumed to be tangent to itself at A.

The circle arcs together with the degenerate arcs will be called *arcs*.

8.14. Proposition. *In the Euclidean plane, a point D lies on the arc ABC if and only if*

$$\measuredangle ADC = \measuredangle ABC$$

or D coincides with A or C.



Proof. Note that if A, B and C lie on one line then the statement is evident.

Assume Γ be the circle passing through A, B and C.

Assume D is distinct from A and C. According to Theorem 8.11, $D \in \Gamma$ if and only if

$$\measuredangle ADC = \measuredangle ABC \quad \text{or} \quad \measuredangle ADC \equiv \measuredangle ABC + \pi.$$

By Exercise 3.12, the first identity holds then B and D lie on one side of (AC); that is, D belongs to the arc ABC. If the second identity holds then the points B and D lie on the opposite sides from (AC), in this case D does not belong to the arc ABC. $\square$

8.15. Proposition. *In the Euclidean plane, a half-lines $[AX)$ is tangent to the arc ABC if and only if*

$$\measuredangle ABC + \measuredangle CAX \equiv \pi.$$

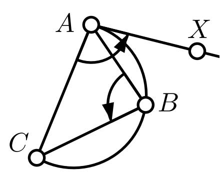
Proof. For a degenerate arc ABC the statement is evident. Further we assume the arc ABC is nondegenerate.

Applying theorems 8.1 and 8.2, we get

$$2\cdot\measuredangle ABC + 2\cdot\measuredangle CAX \equiv 0.$$

Therefore either

$$\measuredangle ABC + \measuredangle CAX \equiv \pi \quad \text{or} \quad \measuredangle ABC + \measuredangle CAX \equiv 0.$$



Since $[AX)$ is the tangent half-line to the arc ABC, X and B lie on the same side from (AC). Therefore the angles $\angle CAX$, $\angle CAB$ and $\angle ABC$ have the same sign. In particular $\measuredangle ABC + \measuredangle CAX \not\equiv 0$; that is, we are left with the case

$$\measuredangle ABC + \measuredangle CAX \equiv \pi.$$

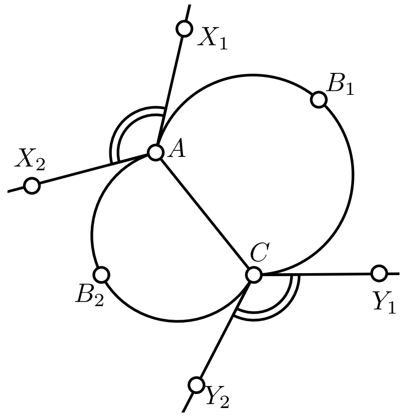
$\square$

8.16. Exercise. *Assume that in the Euclidean plane, the half-lines $[AX)$ and $[AY)$ are tangent to the arcs ABC and ACB correspondingly. Show that $\angle XAY$ is straight.*

8.17. Exercise. *Show that in the Euclidean plane, there is unique arc with endpoints at the given points A and C which is tangent at A to the given half line $[AX)$.*

8.18. Exercise. *Given two arcs AB_1C and AB_2C in the Euclidean plane, let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to arcs AB_1C and AB_2C at A and $[CY_1)$ and $[CY_2)$ be the half-lines tangent to arcs AB_1C and AB_2C at C. Show that*

$$\measuredangle X_1AX_2 \equiv -\measuredangle Y_1CY_2.$$



8.19. Exercise. *Given an acute triangle $\triangle ABC$ make a compass-and-ruler construction of the point Z such that*

$$\measuredangle AZB = \measuredangle BZC = \measuredangle CZA = \pm\tfrac{2}{3}\cdot\pi$$

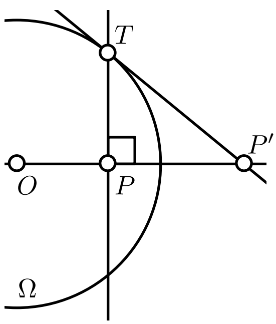
Chapter 9

Inversion

Let Ω be the circle with center O and radius r. The *inversion* of a point P in Ω is the point $P' \in [OP)$ such that

$$OP{\cdot}OP' = r^2.$$

In this case the circle Ω will be called the *circle of inversion* and its center O is called *center of inversion*.



The inversion of O is undefined.

Note that if P is inside Ω then P' is outside and the other way around. Further, $P = P'$ if and only if $P \in \Omega$.

Note that the inversion takes P' back to P.

9.1. Exercise. *Let P be a point inside of a circle Ω centered at O. Let T be a point where the perpendicular to (OP) from P intersects Ω. Let P' be the point where the tangent to Ω at T intersects (OP).*

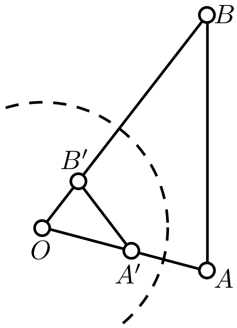
Show that P' is the inversion of P in the circle Ω.

9.2. Lemma. *Let Γ be a circle with center O in the Euclidean plane. Assume A' and B' are the inversions of A and B in Γ. Then*

$$\triangle OAB \sim \triangle OB'A'.$$

Moreover,

❶
$$\begin{aligned} \measuredangle AOB &\equiv -\measuredangle B'OA', \\ \measuredangle OBA &\equiv -\measuredangle OA'B', \\ \measuredangle BAO &\equiv -\measuredangle A'B'O. \end{aligned}$$



Proof. Let r be the radius of the circle of the inversion.

From the definition of inversion, we get

$$OA{\cdot}OA' = OB{\cdot}OB' = r^2.$$

Therefore

$$\frac{OA}{OB'} = \frac{OB}{OA'}.$$

Clearly

❷ $$\measuredangle AOB = \measuredangle A'OB' \equiv -\measuredangle B'OA'.$$

From SAS, we get

$$\triangle OAB \sim \triangle OB'A'.$$

Applying Theorem 3.11 and ❷, we get ❶. □

9.3. Exercise. *Let P' be the inversion of P in the circle Γ. Assume that $P \neq P'$. Show that the value $\frac{PX}{P'X}$ is the same for all $X \in \Gamma$.*

The converse to the above exercise also holds. Namely, given positive real number $k \neq 1$ and two distinct points P and P' in the Euclidean plane the locus of points X such that $\frac{PX}{P'X} = k$ forms a circle which is called *circle of Apollonius*. In this case P' is inverse of P in the circle of Apollonius.

9.4. Exercise. *Let A', B', C' be the images of A, B, C under the inversion in the incircle of $\triangle ABC$. Show that the incenter of $\triangle ABC$ is the orthocenter of $\triangle A'B'C'$.*

9.5. Exercise. *Make a ruler-and-compass construction of the inversion of the given point in the given circle.*

Cross-ratio

The following theorem gives some quantities expressed in distances or angles which do not change after inversion.

9.6. Theorem. *Let $\square ABCD$ and $\square A'B'C'D'$ be two quadrilaterals in the Euclidean plane such that the points A', B', C' and D' are the inversions of A, B, C, and D correspondingly.*

Then

(a)

$$\frac{AB{\cdot}CD}{BC{\cdot}DA} = \frac{A'B'{\cdot}C'D'}{B'C'{\cdot}D'A'}.$$

(b)

$$\measuredangle ABC + \measuredangle CDA \equiv -(\measuredangle A'B'C' + \measuredangle C'D'A').$$

(c) If the quadrilateral $\square ABCD$ is inscribed then so is $\square A'B'C'D'$.

Proof; (a). Let O be the center of the inversion. According to Lemma 9.2, $\triangle AOB \sim \triangle B'OA'$. Therefore

$$\frac{AB}{A'B'} = \frac{OA}{OB'}.$$

Analogously,

$$\frac{BC}{B'C'} = \frac{OC}{OB'}, \qquad \frac{CD}{C'D'} = \frac{OC}{OD'}, \qquad \frac{DA}{D'A'} = \frac{OA}{OD'}.$$

Therefore

$$\frac{AB}{A'B'} \cdot \frac{B'C'}{BC} \cdot \frac{CD}{C'D'} \cdot \frac{D'A'}{DA} = \frac{OA}{OB'} \cdot \frac{OB'}{OC} \cdot \frac{OC}{OD'} \cdot \frac{OD'}{OA} = 1.$$

Hence (a) follows.

(b). According to Lemma 9.2,

$$\begin{aligned} \measuredangle ABO &\equiv -\measuredangle B'A'O, & \measuredangle OBC &\equiv -\measuredangle OA'B', \\ \measuredangle CDO &\equiv -\measuredangle D'C'O, & \measuredangle ODA &\equiv -\measuredangle OA'D'. \end{aligned}$$

Summing these four identities we get

$$\measuredangle ABC + \measuredangle CDA \equiv -(\measuredangle D'C'B' + \measuredangle B'A'D').$$

Applying Axiom IIIb and Exercise 6.21, we get

$$\begin{aligned} \measuredangle A'B'C' + \measuredangle C'D'A' &\equiv -(\measuredangle B'C'D' + \measuredangle D'A'B') \equiv \\ &\equiv \measuredangle D'C'B' + \measuredangle B'A'D'. \end{aligned}$$

Hence (b) follows.

(c). Follows from (b) and Theorem 8.11. $\square$

Inversive plane and circlines

Let Ω be a circle with center O and radius r. Consider the inversion in Ω.

Recall that inversion of O is undefined. To deal with this problem it is useful to add to the plane an extra point; it will be called *the* *point at infinity* and we will denote it as ∞. We can assume that ∞ is inversion of O and the other way around.

The Euclidean plane with added a point at infinity is called *inversive plane.*

We will always assume that any line and half-line contains ∞.

It will be convenient to use notion of *circline*, which means *circle or line*; for instance we may say *if circline contains ∞ then it is a line* or *circline which does not contain ∞ is a circle.*

Note that according to Theorem 7.1, for any $\triangle ABC$ there is unique circline which pass through A, B and C.

9.7. Theorem. *In the inversive plane, inversion of a circline is a circline.*

Proof. Denote by O the center of inverse.

Let Γ be a circline. Choose three distinct points A, B and C on Γ. (If $\triangle ABC$ is nondegenerate then Γ is the circumcircle of $\triangle ABC$; if $\triangle ABC$ is degenerate then Γ is the line passing through A, B and C.)

Denote by A', B' and C' the inversions of A, B and C correspondingly. Let Γ' be the circline which pass though A', B' and C'. According to 7.1, Γ' is well defined.

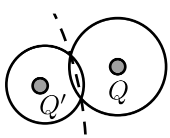
Assume D is a point of inversive plane which is distinct from A, C, O and ∞. Denote by D' the inversion of D.

By Theorem 9.6(c), $D' \in \Gamma'$ if and only if $D \in \Gamma$. Hence the result follows.

It remains to prove that $O \in \Gamma \Leftrightarrow \infty \in \Gamma'$ and $\infty \in \Gamma \Leftrightarrow O \in \Gamma'$. Since Γ is inversion of Γ' it is sufficient to prove only

$$\infty \in \Gamma \quad \Longleftrightarrow \quad O \in \Gamma'.$$

Since $\infty \in \Gamma$ we get that Γ is a line. Therefore for any $\varepsilon > 0$, the line Γ contains point P with $OP > r^2/\varepsilon$. For the inversion $P' \in \Gamma'$ of P, we have $OP' = r^2/OP < \varepsilon$. That is, the circline Γ' contains points arbitrary close to O. It follows that $O \in \Gamma'$. $\square$



9.8. Exercise. *Assume that if circle Γ' is the inversion of circle Γ in the Euclidean plane. Denote by Q the center of Γ and by Q' the inversion of Q.*

Show that Q' is not the center of Γ'.

Assume *circumtool* is a geometric construction tool which produce a circline passing through the given three points.

9.9. Exercise. *Show that with circumtool only, it is impossible to construct the center of given circle.*

9.10. Exercise. *Show that for any pair of tangent circles in the inversive plane there is an inversion which sends them to a pair of parallel lines.*

9.11. Theorem. *Consider the inversion of the inversive plane in the circle Ω with center O. Then*

(a) Line passing through O is inverted into itself.
(b) Line not passing through O is inverted into a circle which pass through O, and the other way around.
(c) A circle not passing through O is inverted into a circle not passing through O.

Proof. In the proof we use Theorem 9.7 without mentioning.

(a). Note that if line passing through O it contains both ∞ and O. Therefore its inversion also contains ∞ and O. In particular image is a line passing through O.

(b). Since any line ℓ pass through ∞, its image ℓ' has to contain O. If the line did not contain O then $\ell' \not\ni \infty$; that is ℓ' is not a line. Therefore ℓ' is a circle which pass through O.

(c). If circle Γ does not contain O then its image Γ' does not contain ∞. Therefore Γ' is a circle. Since $\Gamma \not\ni \infty$ we get $\Gamma' \not\ni O$. Hence the result follows. □

Ptolemy's identity

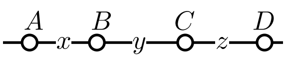
Here is one application of inversion, which we include as an illustration only; we will not use it further in the book.

9.12. Theorem. *Let $\square ABCD$ be an inscribed quadrilateral in the Euclidean plane. Assume that the points A, B, C and D appear on the circline in the same order. Then*

$$AB{\cdot}CD + BC{\cdot}DA = AC{\cdot}BD.$$

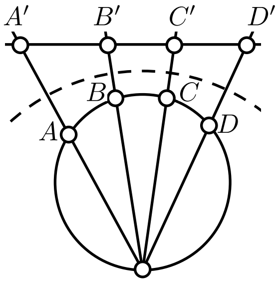
Proof. Assume the points A, B, C, D lie on one line in this order.

Set $x = AB$, $y = BC$, $z = CD$. Note that



$$x{\cdot}z + y{\cdot}(x + y + z) = (x + y){\cdot}(y + z).$$

Since $AC = x + y$, $BD = y + z$ and $DA = x + y + z$, it proves the identity.



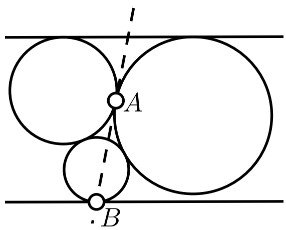
It remains to consider the case when quadrilateral $\square ABCD$ is inscribed in a circle, say Γ.

The identity can be rewritten as

$$\frac{AB\cdot DC}{BD\cdot CA}+\frac{BC\cdot AD}{CA\cdot DB}=1.$$

On the left hand side we have two cross-ratios. According to Theorem 9.6(a), the left hand side does not change if we apply an inversion to each point.

Consider an inversion in a circle centered at a point O which lie on Γ between A and D. By Theorem 9.11, this inversion maps Γ to a line. This reduces the problem to the case when A, B, C and D lie on one line, which was already considered. $\square$



In the proof above we noted that we rewrite Ptolemy identity in a form which is invariant with respect to inversion and then apply an inversion which makes it evident. The solution of the following exercise is based on the same idea; one has to apply an inversion with center at A.

9.13. Exercise. *Assume that three circles tangent to each other and to two parallel lines as shown on the picture. Show that the line passing through the point of tangency A and B on the diagram is also tangent to the two circles at A.*

Perpendicular circles

Assume two circles Γ and Ω intersect at two points say X and Y. Let ℓ and m be the tangent lines at X to Γ and Ω correspondingly. Analogously, ℓ' and m' be the tangent lines at Y to Γ and Ω.

From Exercise 8.18, we get that $\ell \perp m$ if and only if $\ell' \perp m'$.

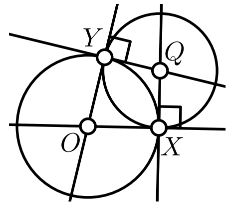
We say that circle Γ is *perpendicular* to circle Ω (briefly $\Gamma \perp \Omega$) if they intersect and the lines tangent to the circle at one point (and therefore both points) of intersection are perpendicular.

Similarly, we say that circle Γ is perpendicular to a line ℓ (briefly $\Gamma \perp \ell$) if $\Gamma \cap \ell \neq \varnothing$ and ℓ perpendicular to the tangent lines to Γ at one point (and therefore both points) of intersection. According to Lemma 5.18, it happens only if the line ℓ pass through the center of Γ.

Now we can talk about *perpendicular circlines.*

9.14. Theorem. *Assume* Γ *and* Ω *are distinct circles in the Euclidean plane. Then* $\Omega \perp \Gamma$ *if and only if the circle* Γ *coincides with its inversion in* Ω.

Proof. Denote by Γ' the inversion of Γ.



($\Rightarrow$) Let O be the center of Ω and Q be the center of Γ. Denote by X and Y the points of intersections of Γ and Ω. According to Lemma 5.18, $\Gamma \perp \Omega$ if and only if (OX) and (OY) are tangent to Γ.

Note that Γ' is also tangent to (OX) and (OY) at X and Y correspondingly. It follows that X and Y are the foot points of the center of Γ' on (OX) and (OY). Therefore both Γ' and Γ have the center Q. Finally, $\Gamma' = \Gamma$, since both circles pass through X.

($\Leftarrow$) Assume $\Gamma = \Gamma'$.

Since $\Gamma \neq \Omega$, there is a point P which lies on Γ, but not on Ω. Let P' be the inversion of P in Ω. Since $\Gamma = \Gamma'$, we have $P' \in \Gamma$, in particular the half-line $[OP)$ intersects Γ at two points. By Exercise 5.14, O lies outside of Γ.

As Γ has points inside and outside Ω, the circles Γ and Ω intersect. The later follows from Exercise 5.20(b).

Let X be a point of their intersection. We need to show (OX) is tangent to Γ; that is X is the only intersection point of (OX) and Γ.

Assume Z is an other point of intersection. Since O is outside of Γ, the point Z lies on the half-line $[OX)$.

Denote by Z' the inversion of Z in Ω. Clearly the three points Z, Z', X lie on Γ and (OX). The later contradicts Lemma 5.16. □

It is convenient to define *inversion in the line* as the reflection through this line. This way we can talk about *inversion in arbitrary circline.*

9.15. Corollary. *Let* Ω *and* Γ *be distinct circlines in the inversive plane. Then the inversion in* Ω *sends* Γ *to itself if and only if* $\Omega \perp \Gamma$.

Proof. By Theorem 9.14, it is sufficient to consider the case when Ω or Γ is a line.

Assume Ω is a line, so the inversion in Ω is reflection. In this case the statement follows from Corollary 5.7.

If Γ is a line then the statement follows from Theorem 9.11. □

9.16. Corollary. *Let* P *and* P' *be two distinct points in the Euclidean plane such that* P' *is the inversion of* P *in the circle* Ω. *Assume that a circline* Γ *pass through* P *and* P'. *Then* $\Gamma \perp \Omega$.

Proof. Without loss of generality we may assume that P is inside and P' is outside Ω. By Theorem 3.16, Γ intersects Ω. Denote by A a point of intersection.

Denote by Γ' the inversion of Γ. Since A is inversion of itself, the points A, P and P' lie on Γ'. By Exercise 7.2, $\Gamma' = \Gamma$ and by Theorem 9.14, $\Gamma \perp \Omega$. □

9.17. Corollary. *Let P and Q be two distinct points inside the circle Ω in the Euclidean plane. Then there is unique circline Γ perpendicular to Ω which pass through P and Q.*

Proof. Let P' be the inversion of point P in a circle Ω. According to Corollary 9.16, the circline passing through P and Q is perpendicular to Ω if and only if it pass though P'.

Note that P' lies outside of Ω. Therefore the points P, P' and Q are distinct.

According to Exercise 7.2, there is unique circline passing through P, Q and P'. Hence the result follows. □

9.18. Exercise. *Let Ω_1 and Ω_2 be two distinct circles in the Euclidean plane. Assume that the point P does not lie on Ω_1 nor on Ω_2. Show that there is unique circline passing through P which is perpendicular Ω_1 and Ω_2.*

9.19. Exercise. *Let P, Q, P' and Q' be points in the Euclidean plane. Assume P' and Q' are inversions of P and Q correspondingly. Show that the quadrilateral $\square PQP'Q'$ is inscribed.*

9.20. Exercise. *Let Ω_1 and Ω_2 be two perpendicular circles with centers at O_1 and O_2 correspondingly. Show that the inversion of O_1 in Ω_2 coincides with the inversion of O_2 in Ω_1.*

9.21. Exercise. *Three distinct circles Ω_1, Ω_2 and Ω_3 intersect at two points A and B. Assume that a circle $\Gamma \perp \Omega_1$ and $\Gamma \perp \Omega_2$, Show that $\Gamma \perp \Omega_3$.*

9.22. Exercise. *Assume you have two tools, first produce a circline which pass through given three points and the second produces inversion of given point in the given circle.*

Assume that a point P does not lie on two circles Ω_1, Ω_2. Using only the two tools above, construct a circline Γ which pass through P and perpendicular to both Ω_1 and Ω_2.

Angles after inversion

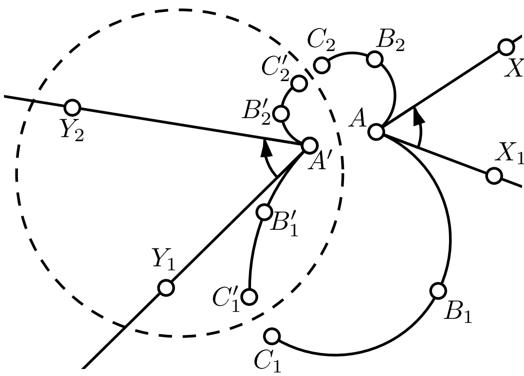
9.23. Proposition. *In the inversive plane, the inversion of an arc is an arc.*

Proof. Consider four distinct points A, B, C and D; let A', B', C' and D' be their inverses. We need to show that D lies on the arc ABC if and only if D' lies on the arc $A'B'C'$. According to Proposition 8.14, the later is equivalent to the following

$$\measuredangle ADC = \measuredangle ABC \quad \Longleftrightarrow \quad \measuredangle A'D'C' = \measuredangle A'B'C'$$

and it follows from Theorem 9.6(b). □

The following theorem roughly says that the angle between arcs changes sign after the inversion.



9.24. Theorem. *Let AB_1C_1, AB_2C_2 be two arcs in the inversive plane and $A'B_1'C_1'$, $A'B_2'C_2'$ be their inversions. Let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to AB_1C_1 and AB_2C_2 at A and $[A'Y_1)$ and $[A'Y_2)$ be the half-lines tangent to $A'B_1'C_1'$ and $A'B_2'C_2'$ at A'. Then*

$$\measuredangle X_1AX_2 \equiv -\measuredangle Y_1A'Y_2.$$

Proof. Applying to Proposition 8.15,

$$\begin{aligned}
\measuredangle X_1AX_2 &\equiv \measuredangle X_1AC_1 + \measuredangle C_1AC_2 + \measuredangle C_2AX_2 \equiv \\
&\equiv (\pi - \measuredangle C_1B_1A) + \measuredangle C_1AC_2 + (\pi - \measuredangle AB_2C_2) \equiv \\
&\equiv -(\measuredangle C_1B_1A + \measuredangle AB_2C_2 + \measuredangle C_2AC_1) \equiv \\
&\equiv -(\measuredangle C_1B_1A + \measuredangle AB_2C_1) - (\measuredangle C_1B_2C_2 + \measuredangle C_2AC_1).
\end{aligned}$$

The same way we get

$$\measuredangle Y_1A'Y_2 \equiv -(\measuredangle C_1'B_1'A' + \measuredangle A'B_2'C_1') - (\measuredangle C_1'B_2'C_2' + \measuredangle C_2'A'C_1').$$

By Theorem 9.6(b),

$$\begin{aligned}\measuredangle C_1B_1A + \measuredangle AB_2C_1 &\equiv -(\measuredangle C_1'B_1'A' + \measuredangle A'B_2'C_1'),\\ \measuredangle C_1B_2C_2 + \measuredangle C_2AC_1 &\equiv -(\measuredangle C_1'B_2'C_2' + \measuredangle C_2'A'C_1').\end{aligned}$$

Hence the result follows. □

9.25. Corollary. *Let P', Q' and Γ' be the inversions of the points P, Q and the circle Γ in the circle Ω of the Euclidean plane. Assume P is inversion of Q in Γ then P' is inversion of Q' in Γ'.*

Proof. If $P = Q$ then $P' = Q' \in \Gamma'$ therefore P' is inversion of Q' in Γ'.

It remains to consider the case $P \neq Q$. Let Δ_1 and Δ_2 be two distinct circles which intersect at P and Q. According to Corollary 9.16, $\Delta_1 \perp \Gamma$ and $\Delta_2 \perp \Gamma$.

Denote by Δ_1' and Δ_2' the inversions of Δ_1 and Δ_2 in Ω. Clearly Δ_1' and Δ_2' intersect at P' and Q'.

From Theorem 9.24, the later is equivalent to $\Delta_1' \perp \Gamma'$ and $\Delta_2' \perp$ $\perp \Gamma'$. By Corollary 9.15, the later implies that P' is the inversion of Q' in Γ'. □

Chapter 10

Absolute plane

Let us remove Axiom V from our axiomatic system, see pages 18–19. This way we define a new object called *absolute plane* or *neutral plane*. (In the absolute plane, the Axiom V may or may not hold.)

Clearly any theorem in absolute geometry holds in Euclidean geometry. In other words, Euclidean plane is an example of absolute plane. In the next chapter we will show that besides the Euclidean plane there are other examples of absolute plane.

Many theorems in Euclidean geometry hold in absolute geometry.

In this book, the Axiom V was used for the first time in the proof of uniqueness of parallel line in Theorem 6.2. Therefore all the statements before Theorem 6.2 also hold in absolute geometry.

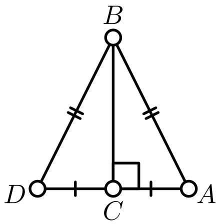
It makes all the discussed results about half-planes, signs of angles, congruence conditions, perpendicular lines and reflections true in absolute plane. If in the formulation of a statement above you do not see words "Euclidean plane" or "inversive plane", it means that the statement holds in absolute plane and the same proof works.

Let us give an example of theorem in absolute geometry, which admits a simpler proof in Euclidean geometry.

10.1. Theorem. *Assume that triangles $\triangle ABC$ and $\triangle A'B'C'$ have right angles at C and C' correspondingly, $AB = A'B'$ and $AC = A'C'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Euclidean proof. By Pythagorean theorem $BC = B'C'$. Then the statement follows from SSS congruence condition. □

Note that the proof of Pythagorean theorem used properties of similar triangles, which in turn used Axiom V. Whence the above proof is not working in absolute plane.



Absolute proof. Denote by D the reflection of A through (BC) and by D' the reflection of A' through $(B'C')$. Note that

$$AD = 2{\cdot}AC = 2{\cdot}A'C' = A'D',$$

$$BD = BA = B'A' = B'D'.$$

By SSS, we get $\triangle ABD \cong \triangle A'B'D'$.

The theorem follows since C is the midpoint of $[AD]$ and C' is the midpoint of $[A'D']$. $\square$

10.2. Exercise. *Give a proof of Exercise 7.10 which works in the absolute plane.*

10.3. Exercise. *Let $\square ABCD$ be an inscribed quadrilateral in the absolute plane. Show that*

$$\measuredangle ABC + \measuredangle CDA \equiv \measuredangle BCD + \measuredangle DAB.$$

Note that the Theorem 8.11 can not be applied in the above exercise; it use Theorems 8.1 and 8.2; which in turns use Theorem 6.13.

Two angles of triangle

In this section we will prove a weaker form of Theorem 6.13 which holds in absolute plane.

10.4. Proposition. *Let $\triangle ABC$ be nondegenerate triangle in the absolute plane. Then*

$$|\measuredangle CAB| + |\measuredangle ABC| < \pi.$$

Note according to 3.11, the angles $\angle ABC$, $\angle BCA$ and $\angle CAB$ have the same sign. Therefore in Euclidean plane the theorem follows immediately from Theorem 6.13. In absolute geometry we need to work more.

Proof. By 3.11, we may assume that $\angle CAB$ and $\angle ABC$ are positive.

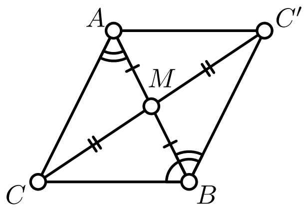
Let M be the midpoint of $[AB]$. Chose $C' \in (CM)$ distinct from C so that $C'M = CM$.

Note that the angles $\angle AMC$ and $\angle BMC'$ are vertical; in particular

$$\measuredangle AMC = \measuredangle BMC'.$$

By construction $AM = BM$ and $CM = C'M$. Therefore $\triangle AMC \cong \triangle BMC'$; in particular

$$\measuredangle CAB = \pm\measuredangle C'BA.$$



According to 3.11, the angles $\angle CAB$ and $\angle C'BA$ have the same sign as $\angle AMC$ and $\angle BMC'$. Therefore

$$\measuredangle CAB = \measuredangle C'BA.$$

In particular,

$$\begin{aligned}\measuredangle C'BC &\equiv \measuredangle C'BA + \measuredangle ABC \equiv \\ &\equiv \measuredangle CAB + \measuredangle ABC.\end{aligned}$$

Finally note that C' and A lie on the same side from (CB). Therefore the angles $\angle CAB$, $\angle ABC$ and $\angle C'BC$ are positive. By Exercise 3.3, the result follows. □

10.5. Exercise. *Assume A, B, C and D be points in absolute plane such that*

$$2{\cdot}\measuredangle ABC + 2{\cdot}\measuredangle BCD \equiv 0.$$

Show that $(AB) \parallel (CD)$.

Note that one can not extract the solution of the above exercise from the proof of Transversal property (6.18)

10.6. Exercise. *Prove* side-angle-angle congruence condition *in absolute plane.*

In other words, let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in absolute plane. Show that $\triangle ABC \cong \triangle A'B'C'$ if

$$AB = A'B', \quad \measuredangle ABC = \pm\measuredangle A'B'C' \quad \textit{and} \quad \measuredangle BCA = \pm\measuredangle B'C'A'.$$

Note that in the Euclidean plane, the above exercise follows from ASA and the theorem on sum of angles of triangle (6.13). However, Theorem 6.13 can not be used here since its proof use Axiom V. Later, in theorem Theorem 12.5, we will show that Theorem 6.13 does not hold in absolute plane.

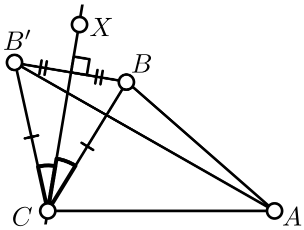
10.7. Exercise. *Assume that point D lies between the vertices A and B of triangle $\triangle ABC$ in the absolute plane. Show that*

$$CD < CA \quad \textit{or} \quad CD < CB.$$

Three angles of triangle

10.8. Proposition. *Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the absolute plane such that $AC = A'C'$ and $BC = B'C'$. Then*

$$AB < A'B' \quad \textit{if and only if} \quad |\measuredangle ACB| < |\measuredangle A'C'B'|.$$



Proof. Without loss of generality, we may assume that $A = A'$ and $C = C'$ and $\measuredangle ACB, \measuredangle ACB' \geqslant 0$. In this case we need to show that

$$AB < AB' \quad \Longleftrightarrow \quad \measuredangle ACB < \measuredangle ACB'.$$

Choose a point X so that

$$\measuredangle ACX = \tfrac{1}{2}\cdot(\measuredangle ACB + \measuredangle ACB').$$

Note that
- ⋄ (CX) bisects $\angle BCB'$
- ⋄ (CX) is the perpendicular bisector of $[BB']$.
- ⋄ A and B lie on the same side from (CX) if and only if

$$\measuredangle ACB < \measuredangle ACB'.$$

From Exercise 5.3, A and B lie on the same side from (CX) if and only if $AB < AB'$. Hence the result follows. □

10.9. Theorem. *Let $\triangle ABC$ be a triangle in the absolute plane. Then*

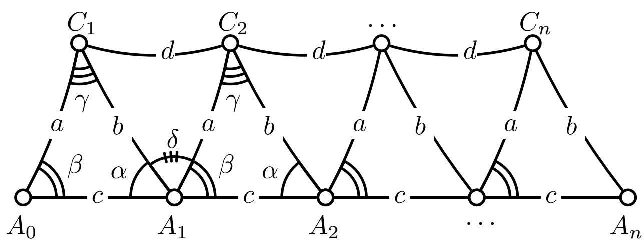
$$|\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB| \leqslant \pi.$$

The following proof is due to Legendre [8], earlier proofs were due to Saccheri [13] and Lambert [7].

Proof. Let $\triangle ABC$ be the given triangle. Set

$$\begin{aligned} a &= BC, & b &= CA, & c &= AB,\\ \alpha &= \measuredangle CAB, & \beta &= \measuredangle ABC, & \gamma &= \measuredangle BCA. \end{aligned}$$

Without loss of generality, we may assume that $\alpha, \beta, \gamma \geqslant 0$.



Fix a positive integer n. Consider points A_0, $A_1, \ldots, A_n$ on the half-line $[BA)$ so that $BA_i = i\cdot c$ for each i. (In particular, $A_0 = B$ and $A_1 = A$.) Let us construct the points C_1, $C_2, \ldots, C_n$, so that $\measuredangle A_i A_{i-1} C_i = \beta$ and $A_{i-1}C_i = a$ for each i.

This way we construct n congruent triangles

$$\begin{aligned}\triangle ABC = \triangle A_1A_0C_1 &\cong \\ \cong \triangle A_2A_1C_2 &\cong \\ \dots \\ \cong \triangle A_nA_{n-1}C_n&.\end{aligned}$$

Set $d = C_1C_2$ and $\delta = \measuredangle C_2A_1C_1$. Note that

❶ $$\alpha + \beta + \delta = \pi.$$

By Proposition 10.4, $\delta \geqslant 0$.

By construction

$$\triangle A_1C_1C_2 \cong \triangle A_2C_2C_3 \cong \ldots \cong \triangle A_{n-1}C_{n-1}C_n.$$

In particular, $C_iC_{i+1} = d$ for each i.

By repeated application of the triangle inequality, we get that

$$\begin{aligned}n\cdot c &= A_0A_n \leqslant \\ &\leqslant A_0C_1 + C_1C_2 + \dots + C_{n-1}C_n + C_nA_n = \\ &= a + (n-1)\cdot d + b.\end{aligned}$$

In particular,

$$c \leqslant d + \tfrac{1}{n}\cdot(a + b - d).$$

Since n is arbitrary positive integer, the later implies

$$c \leqslant d.$$

From Proposition 10.8 and SAS, the later is equivalent to

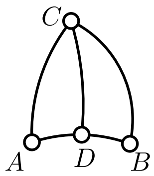
$$\gamma \leqslant \delta.$$

From ❶, the theorem follows. □

The *defect of triangle* $\triangle ABC$ is defined as

$$\operatorname{defect}(\triangle ABC) \stackrel{\text{def}}{=\!=} \pi - |\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB|.$$

Note that Theorem 10.9 sates that, defect of any triangle in absolute plane has to be nonnegative. According to Theorem 6.13, any triangle in Euclidean plane has zero defect.



10.10. Exercise. *Let $\triangle ABC$ be nondegenerate triangle in the absolute plane. Assume D lies between A and B. Show that*

$$\operatorname{defect}(\triangle ABC) = \operatorname{defect}(\triangle ADC) + \operatorname{defect}(\triangle DBC).$$

How to prove that something can not be proved?

Many attempts were made to prove that any theorem in Euclidean geometry holds in absolute geometry. The later is equivalent to the statement that Axiom V is a *theorem* in absolute geometry.

Many these attempts being accepted as proofs for long periods of time until the mistake was found.

There is a number of statements in the geometry of absolute plane which are equivalent to the Axiom V. It means that if we exchange the Axiom V to any of these statements then we will obtain an equivalent axiomatic system.

Here we give a short list of such statements. We are not going to prove the equivalence in the book.

10.11. Theorem. *An absolute plane is Euclidean if and only if one of the following equivalent conditions hold.*

(a) There is a line ℓ and a point $P \notin \ell$ such that there is only one line passing through P and parallel to ℓ.
(b) Every nondegenerate triangle can be circumscribed.
(c) There exists a pair of distinct lines which lie on a bounded distance from each other.
(d) There is a triangle with arbitrary large inradius.
(e) There is a nondegenerate triangle with zero defect.
(f) There exists a quadrilateral in which all angles are right.

It is hard to imagine an absolute plane, which does not satisfy some of the properties above. That is partly the reason why for the large number of false proofs; each used one of such statements by accident.

Let us formulate the negation of (a) above.

h-V. For any line ℓ and any point $P \notin \ell$ there are at least two lines which pass through P and parallel to ℓ.

By Theorem 6.2, an absolute plane which satisfies Axiom h-V is not Euclidean. Moreover, according to the Theorem 10.11 (which we do not prove) any non-Euclidean absolute plane Axiom h-V holds.

It opens a way to look for a proof by contradiction. Simply exchange Axiom V to Axiom h-V and start to prove theorems in the obtained axiomatic system. In the case if we arrive to a contradiction, we prove the Axiom V in absolute plane.

This idea was growing since 5th century; the most notable result were obtained by Saccheri in [13]. The more this new geometry was developed, it became more and more believable that there will be no contradiction; that is, the system of axioms I–IV and h-V is *consistent.*

This new type of geometry is now called *hyperbolic* or *Lobachevskian geometry.* In fact the following theorem holds.

10.12. Theorem. *The hyperbolic geometry is consistent if and only if so is the Euclidean geometry.*

The statement that hyperbolic geometry has no contradiction appears first in private letters of Bolyai, Gauss, Schweikart and Taurinus[1]. They all seem to be afraid to state it in public. Say, in 1818 Gauss writes to Gerling

> *...I am happy that you have the courage to express yourself as if you recognized the possibility that our parallels theory along with our entire geometry could be false. But the wasps whose nest you disturb will fly around your head....*

Lobachevsky came to the same conclusion independently. Unlike the others he had courage to state it in public and in print (see [9]). That cost him serious troubles.

It seems that Lobachevsky was also the first who had a proof of Theorem 10.12. Later Beltrami gave a cleaner proof of "if" part of the theorem. It was done by modeling points, lines, distances and angle measures of one geometry using some other objects in the other geometry. The same idea was used originally by Lobachevsky in the proof of the "only if" part of the theorem, see [10, 34].

The proof of Beltrami is the subject of the next chapter.

Arguably, the discovery of hyperbolic geometry was the second main discoveries of 19th century, trailing only the Mendel's laws.

Curvature

In a letter from 1824 Gauss writes:

> *The assumption that the sum of the three angles is less than π leads to a curious geometry, quite different from*

[1] The oldest surviving letters were the Gauss letter to Gerling 1816 and yet more convincing letter dated by 1818 of Schweikart sent to Gauss via Gerling.

> *ours but thoroughly consistent, which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of a determination of a constant, which cannot be designated a priori. The greater one takes this constant, the nearer one comes to Euclidean geometry, and when it is chosen indefinitely large the two coincide. The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. For example, the three angles of a triangle become as small as one wishes, if only the sides are taken large enough; yet the area of the triangle can never exceed a definite limit, regardless how great the sides are taken, nor indeed can it ever reach it.*

In the modern terminology the constant which Gauss mentions, can be expressed as $1/\sqrt{-k}$, where k denotes so called *curvature* of the absolute plane which we are about to introduce.

The identity in the Exercise 10.10 suggests that defect of triangle should be proportional to its area.[2]

In fact for any absolute plane there is a nonpositive real number k such that

$$k\cdot \operatorname{area}(\triangle ABC) + \operatorname{defect}(\triangle ABC) = 0$$

for any triangle $\triangle ABC$. This number k is called *curvature* of the plane.

For example, by Theorem 6.13, the Euclidean plane has zero curvature. By Theorem 10.9, curvature of any absolute plane is nonpositive.

It turns out that up to isometry, the absolute plane is characterized by its curvature; that is, two absolute planes are isometric if and only if they have the same curvature.

In the next chapter we will construct hyperbolic plane, this is an example of absolute plane with curvature $k = -1$.

Any absolute planes, distinct from Euclidean, can be obtained by rescaling metric on the hyperbolic plane. Indeed, if we rescale the metric by a positive factor c, the area changes by factor c^2, while defect stays the same. Therefore taking $c = \sqrt{-k}$, we can get the absolute plane given curvature $k < 0$. In other words, all the non-Euclidean absolute planes become identical if we use $r = 1/\sqrt{-k}$ as the unit of length.

In the Chapter 15, we briefly discuss the geometry of the unit

[2]The area in absolute plane discussed briefly in the end of Chapter 19, but instead the reader could also refer to intuitive understanding of area.

sphere. Although spheres are not absolute planes, the spherical geometry is a close relative of Euclidean and hyperbolic geometries.

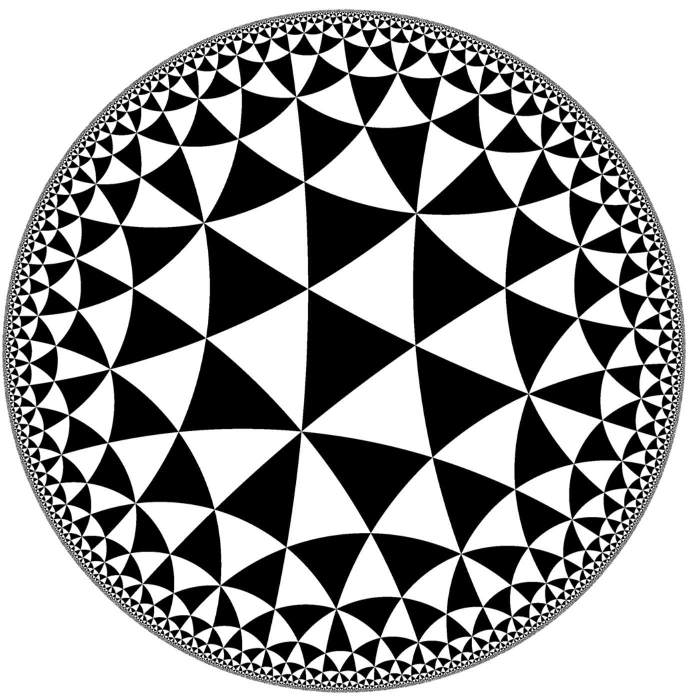
The nondegenerate spherical triangles have negative defect. Moreover if R is the radius of the sphere then

$$\tfrac{1}{R^2}\cdot\mathrm{area}(\triangle ABC)+\mathrm{defect}(\triangle ABC)=0$$

for any spherical triangle $\triangle ABC$. In other words, the sphere of radius R has positive curvature $k=\tfrac{1}{R^2}$.

Chapter 11

Hyperbolic plane



In this chapter we use inversive geometry to construct the model of hyperbolic plane — an example of absolute plane which is not Euclidean.

Namely, we construct so called *conformal disk model* of hyperbolic plane. This model was discovered by Beltrami in [3] and often called *Poincaré disc model.*

The figure above shows the conformal disk model of hyperbolic plane which is cut into congruent triangles with angles $\frac{\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{4}$.

Conformal disk model

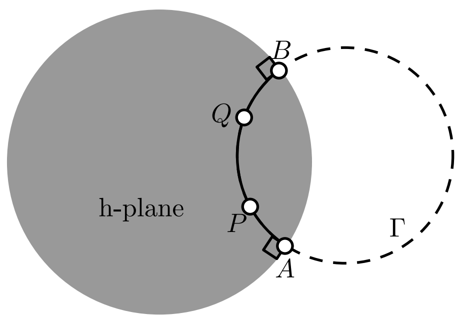
In this section we give new names for some objects in the Euclidean plane which will represent lines, angle measures, distances in the hyperbolic plane.

Hyperbolic plane. Let us fix a circle on the Euclidean plane and call it *absolute*. The set of points inside the absolute will be called *hyperbolic plane* (or *h-plane*).

Note that the points on absolute do *not* belong to the h-plane. The points in the h-plane will be also called *h-points*.

Often we will assume that the absolute is a unit circle.

Hyperbolic lines. The intersections of h-plane with circlines perpendicular to the absolute are called *hyperbolic lines* or *h-lines*.



By Corollary 9.17, there is unique h-line which pass through given two distinct h-points P and Q. This h-line will be denoted as $(PQ)_h$.

The arcs of hyperbolic lines will be called *hyperbolic segments* or *h-segments*. An h-segment with endpoints P and Q will be denoted as $[PQ]_h$.

The subset of h-line on one side from a point will be called *hyperbolic half-line* (or *h-half-line*). More precisely, h-half-line is an intersection of h-plane with arc which perpendicular to the absolute with only one endpoint in the h-plane. An h-half-line from P passing through Q will be denoted as $[PQ)_h$.

If Γ is the circle containing the h-line $(PQ)_h$ then the points of intersection of Γ with absolute are called *ideal points* of $(PQ)_h$. (Note that the ideal points of h-line do not belong to the h-line.)

Similarly a triangle in h-plane; that is, an ordered triple of h-points, say (P, Q, R), will be denoted by $\triangle PQR$.

So far $(PQ)_h$ is just a subset of h-plane; below we will introduce h-distance and later we will show that $(PQ)_h$ is a line for the h-distance in the sense of the Definition 1.8.

11.1. Exercise. *Show that the h-line is uniquely determined by its ideal points.*

11.2. Exercise. *Show that the h-line is uniquely determined by one of its ideal points and one h-point on it.*

11.3. Exercise. *Show that h-segment $[PQ]_h$ coincides with the Euclidean segment $[PQ]$ if and only if the line (PQ) pass through the center of the absolute.*

Hyperbolic distance. Let P and Q be distinct h-points. Denote by A and B be the ideal points of $(PQ)_h$. Without loss of generality, we may assume that on the Euclidean circle containing the h-line $(PQ)_h$, the points A, P, Q, B appear in the same order.

Consider function

$$\delta(P,Q) \stackrel{\text{def}}{=\!=} \frac{AQ{\cdot}BP}{QB{\cdot}PA}.$$

Note that right hand side is the cross-ratio, which appeared in Theorem 9.6. Set $\delta(P,P) = 1$ for any h-point P. Set

$$PQ_h \stackrel{\text{def}}{=\!=} \ln[\delta(P,Q)].$$

The proof that PQ_h is a metric on h-plane will be given below. For now it is just a function which returns a real value PQ_h for any pair of h-points P and Q.

11.4. Exercise. *Let O be the center of the absolute and the h-points O, X and Y lie on the one h-line in the same order. Assume $OX = XY$. Prove that $OX_h < XY_h$.*

Hyperbolic angles. Consider three h-points P, Q and R such that $P \neq Q$ and $R \neq Q$. The hyperbolic angle $\angle_h PQR$ is ordered pair of h-half-lines $[QP)_h$ and $[QR)_h$.

Let $[QX)$ and $[QY)$ be (Euclidean) half-lines which are tangent to $[QP]_h$ and $[QR]_h$ at Q. Then the *hyperbolic angle measure* (or *h-angle measure*) $\measuredangle_h PQR$ is defined as $\measuredangle XQY$.

11.5. Exercise. *Let ℓ be an h-line and P be an h-point which does not lie on ℓ. Show that there is unique h-line passing through P and perpendicular to ℓ.*

The plan

We defined all the *h-notions* needed in the formulation of the axioms I–IV and h-V. It remains to show that all these axioms hold; this will be done by the end of this chapter.

Once we are done with the proofs, we get that the model provides an example of absolute plane; in particular Exercise 11.5 can be proved the same way as Theorem 5.5.

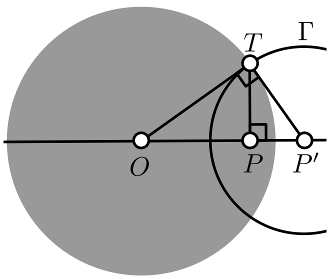
Most importantly we will prove the "if"-part of Theorem 10.12.

Indeed, any statement in hyperbolic geometry can be restated in the Euclidean plane using the introduced h-notions. Therefore if the system of axioms I–IV and h-V leads to a contradiction the so does the system axioms I–V.

Auxiliary statements

11.6. Lemma. *Consider h-plane with unit circle as the absolute. Let O be the center of absolute and P be an other h-point. Denote by P' the inversion of P in the absolute.*

Then the circle Γ with center P' and radius $1/\sqrt{1-OP^2}$ is perpendicular to the absolute. Moreover O is the inversion of P in Γ.



Proof. Follows from Exercise 9.20. □

Assume Γ is a circline which is perpendicular to the absolute. Consider the inversion $X \mapsto X'$ in Γ, or if Γ is a line, set $X \mapsto X'$ to be the reflection in Γ.

The following observation roughly says that the map $X \mapsto X'$ respects all the notions introduced in the previous section. Together with the lemma above, it implies that in any problem which formulated entirely *in h-terms* we can assume that a given h-point lies in the center of absolute.

11.7. Main observation. *The map $X \mapsto X'$ described above is a bijection of h-plane to itself. Moreover for any h-points P, Q, R such that $P \neq Q$ and $Q \neq R$ the following conditions hold*

(a) *The h-lines $(PQ)_h$, $[PQ)_h$ and $[PQ]_h$ are mapped to the h-lines $(P'Q')_h$, $[P'Q')_h$ and $[P'Q']_h$ correspondingly.*

(b) $\delta(P', Q') = \delta(P, Q)$ *and* $P'Q'_h = PQ_h$.

(c) $\measuredangle_h P'Q'R' \equiv -\measuredangle_h PQR$.

Proof. According to Theorem 9.14 the map sends the absolute to itself. Note that the points on Γ do not move, it follows that points inside of absolute remain inside after the mapping and the other way around.

Part (a) follows from 9.7 and 9.24.

Part (b) follows from Theorem 9.6.

Part (c) follows from Theorem 9.24. □

11.8. Exercise. *Let Γ be a circle which is perpendicular to the absolute and Q be an h-point lying on Γ. Assume P is an h-point and P' is its inversion in Γ. Show that $PQ_h = P'Q_h$.*

11.9. Exercise. *Consider the function*

$$\vartheta(P,Q) = 2\cdot\frac{PQ\cdot P'Q'}{PP'\cdot QQ'},$$

where P, Q are points in the h-plane and P', Q' are their inversions in the absolute. Show that

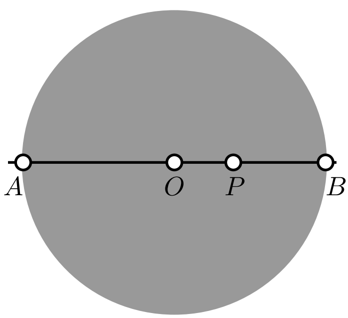
$$2 + 2\cdot\vartheta(P,Q) = \delta(P,Q) + \frac{1}{\delta(P,Q)}.$$

Conclude that

$$PQ_h = \ln\left[1 + \vartheta(P,Q) + \sqrt{\vartheta(P,Q)^2 + 2\cdot\vartheta(P,Q)}\right].$$

11.10. Lemma. *Assume that the absolute is a unit circle centered at O. Given a h-point P, set $x = OP$ and $y = OP_h$. Then*

$$y = \ln\frac{1+x}{1-x} \qquad \textit{and} \qquad x = \frac{e^y - 1}{e^y + 1}.$$



Proof. Note that h-line $(OP)_h$ lies in a diameter of absolute. Therefore if A and B are the ideal points as in the definition of h-distance. Therefore

$$OA = OB = 1,$$
$$PA = 1 + x,$$
$$PB = 1 - x.$$

In particular,

$$y = \ln\frac{AP\cdot BO}{PB\cdot OA} = \ln\frac{1+x}{1-x}.$$

Taking exponent of left and right hand side and applying obvious algebra manipulations we get

$$x = \frac{e^y - 1}{e^y + 1}.$$

□

11.11. Lemma. *Assume points P, Q and R appear on one h-line in the same order. Then*

$$PQ_h + QR_h = PR_h.$$

Proof. Note that

$$PQ_h + QR_h = PR_h$$

is equivalent to

$$\delta(P,Q)\cdot\delta(Q,R) = \delta(P,R). \tag{❶}$$

Let A and B be the ideal points of $(PQ)_h$. Without loss of generality we can assume that the points A, P, Q, R, B appear in the same order on the circline containing $(PQ)_h$. Then

$$\begin{aligned}
\delta(P,Q)\cdot\delta(Q,R) &= \frac{AQ\cdot BP}{QB\cdot PA}\cdot\frac{AR\cdot BQ}{RB\cdot QA} = \\
&= \frac{AR\cdot BP}{RB\cdot PA} = \\
&= \delta(P,R)
\end{aligned}$$

Hence ❶ follows. □

Let P be an h-point and $\rho > 0$. The set of all h-points Q such that $PQ_h = \rho$ is called *h-circle* with center P and *h-radius* ρ.

11.12. Lemma. *Any h-circle is formed by a Euclidean circle which lies completely in h-plane.*

More precisely for any h-point P and $\rho \geqslant 0$ there is a $\hat\rho \geqslant 0$ and a point $\hat P$ such that

$$PQ_h = \rho \quad\Longleftrightarrow\quad \hat{P}Q = \hat\rho$$

for any h-point Q.

Moreover, if O is the center of absolute then

1. *$\hat O = O$ for any ρ and*
2. *$\hat P \in (OP)$ for any $P \neq O$.*

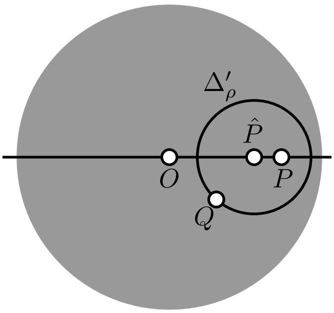
Proof. According to Lemma 11.10, $OQ_h = \rho$ if and only if

$$OQ = \hat\rho = \frac{e^\rho - 1}{e^\rho + 1}.$$

Therefore the locus of h-points Q such that $OQ_h = \rho$ is formed by the Euclidean circle, denote it by Δ_ρ.

If $P \neq O$, applying Lemma 11.6 and the Main observation (11.7) we get a circle Γ perpendicular to the absolute such that P is the inversion of O in Γ.

Let Δ'_ρ be the inversion of Δ_ρ in Γ. Since the inversion in Γ preserves the h-distance, $PQ_h = \rho$ if and only if $Q \in \Delta'_\rho$.



According to Theorem 9.7, Δ'_ρ is a Euclidean circle. Denote by $\hat{P}$ the Euclidean center and by $\hat{\rho}$ the Euclidean radius of Δ'_ρ.

Finally note that Δ'_ρ reflects to itself in (OP); that is, the center $\hat{P}$ lies on (OP). $\square$

11.13. Exercise. *Assume P, $\hat{P}$ and O be as in the Lemma 11.12 and $P \neq$ $\neq O$. Show that $\hat{P} \in [OP]$.*

Axiom I

Evidently h-plane contains at least two points. Therefore to show that Axiom I holds in h-plane we need to show that h-distance defined on page 86 is a metric on h-plane; that is, the conditions (a)–(d) in Definition 1.1 hold for h-distance.

The following claim says that h-distance meets the conditions (a) and (b).

11.14. Claim. *Given h-points P and Q, we have $PQ_h \geqslant 0$ and $PQ_h = 0$ if and only if $P = Q$.*

Proof. According to Lemma 11.6 and Main Observation (11.7), we may assume that Q is the center of absolute. In this case

$$\delta(Q,P) = \frac{1+QP}{1-QP} \geqslant 1$$

and the equality holds only if $P = Q$.

Therefore

$$QP_h = \ln[\delta(Q,P)] \geqslant 0.$$

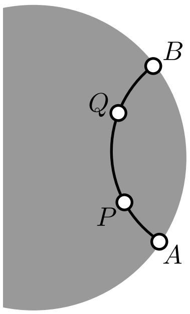
and the equality holds if and only if $P = Q$. $\square$

By the following claim says that h-distance meets the conditions 1.1(c).

11.15. Claim. *For any h-points P and Q, we have $PQ_h = QP_h$.*

Proof. Let A and B be ideal points of $(PQ)_h$ and A, P, Q, B appear on the circline containing $(PQ)_h$ in the same order.

Then



$$PQ_h = \ln \frac{AQ\cdot BP}{QB\cdot PA} = \\ = \ln \frac{BP\cdot AQ}{PA\cdot QB} = \\ = QP_h.$$

□

The following claim shows in particular that triangle inequality (which is condition 1.1(d)) holds for h-distance.

11.16. Claim. *Given a triple of h-points P, Q and R, we have*

$$PQ_h + QR_h \geqslant PR_h.$$

Moreover, the equality holds if and only if P, Q and R lie on one h-line in the same order.

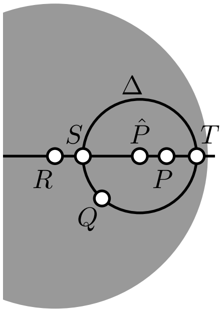
Proof. Without loss of generality, we may assume that $RP_h \geqslant PQ_h > 0$ and R is the center of absolute.

Denote by Δ the h-circle with center P and h-radius PQ_h. Let S and T be the points of intersection of (RP) and Δ.

Since $PQ_h \leqslant RP_h$, by Lemma 11.11 we can assume that the points R, S, P and T appear on the h-line in the same order.

Set $\rho = PQ_h$. According to Lemma 11.12, Δ is a Euclidean circle; denote by $\hat{P}$ its Euclidean center. Note that $\hat{P}$ is the (Euclidean) midpoint of $[ST]$.

By the Euclidean triangle inequality



❷ $$RT = R\hat{P} + \hat{P}Q \geqslant RQ$$

and the equality holds if and only if $T = Q$.

By Lemma 11.10,

$$RT_h = \ln \frac{1 + RT}{1 - RT}$$
$$RQ_h = \ln \frac{1 + RQ}{1 - RQ}.$$

Since the function $f(x) = \ln \frac{1+x}{1-x}$ is increasing for $x \in [0, 1)$, inequality ❷ implies

$$RT_h \geqslant RQ_h$$

and the equality holds if and only if only if $T = Q$.

Finally applying Lemma 11.11 again, we get

$$RT_h = RP_h + PQ_h.$$

Hence the claim follows. $\square$

Axiom II

Note that once the following claim is proved, Axiom II follows from Corollary 9.17.

11.17. Claim. *A subset of h-plane is an h-line if and only if it forms a line for h-distance in the sense of Definition 1.8.*

Proof. Let ℓ be an h-line. Applying the main observation we can assume that ℓ contains the center of absolute. In this case ℓ is formed by intersection of diameter of absolute and the h-plane. Let A and B be the endpoints of the diameter.

Consider the map $\iota\colon \ell \to \mathbb{R}$ defined as

$$\iota(X) = \ln \frac{AX}{XB}.$$

Note that $\iota\colon \ell \to \mathbb{R}$ is a bijection.

Further, if $X, Y \in \ell$ and the points A, X, Y and B appear on $[AB]$ in the same order then

$$\iota(Y) - \iota(X) = \ln \frac{AY}{YB} - \ln \frac{AX}{XB} = \ln \frac{AY \cdot BX}{YB \cdot XB} = XY_h;$$

that is, any h-line is a line for h-metric.

We proved that any h-line is a line for h-distance. The converse follows from Claim 11.16. $\square$

Axiom III

Note that the first part of Axiom III follows directly from the definition of h-angle measure defined on page 86. It remains to show that $\measuredangle_h$ satisfies the conditions IIIa, IIIb and IIIc on page 18.

The following two claims say that $\measuredangle_h$ satisfies IIIa and IIIb.

11.18. Claim. *Given an h-half-line $[OP)$ and $\alpha \in (-\pi, \pi]$ there is unique h-half-line $[OQ)$ such that $\measuredangle POQ = \alpha$.*

11.19. Claim. *For any h-points P, Q and R distinct from h-point O we have*

$$\measuredangle POQ + \measuredangle QOR \equiv \measuredangle POR.$$

Proof of 11.18 and 11.19. Applying the main observation, we may assume that O is the center of absolute. In this case, for any h-point $P \neq O$, $[OP)_h$ is the intersection of $[OP)$ with h-plane. Hence the claims 11.18 and 11.19 follow from the corresponding axioms of Euclidean plane. □

The following claim says that $\measuredangle_h$ satisfies IIIc.

11.20. Claim. *The function*

$$\measuredangle_h\colon (P, Q, R) \mapsto \measuredangle PQR$$

is continuous at any triple of points (P, Q, R) such that $Q \neq P$ and $Q \neq R$ and $\measuredangle_h PQR \neq \pi$.

Proof. Denote by O the center of the absolute. We can assume that Q is distinct from O.

Denote by Z the inversion of Q in the absolute and by Γ the circle perpendicular to the absolute which is centered at Q'. According to Lemma 11.6, the point O is the inversion of Q in Γ.

Denote by P' and R' the inversions in Γ of the points P and R correspondingly. Note that the point P' is completely determined by the points Q and P; moreover the map $(Q, P) \mapsto P'$ is continuous at any pair of points (Q, P) such that $Q \neq O$. The same is true for the map $(Q, R) \mapsto R'$

According to the Main Observation

$$\measuredangle_h PQR \equiv -\measuredangle_h P'OR'.$$

Since $\measuredangle_h P'OR' = \measuredangle P'OR'$ and the maps $(Q, P) \mapsto P'$, $(Q, R) \mapsto R'$ are continuous, the claim follows from the corresponding axiom of Euclidean plane. □

Axiom IV

The following claim says that Axiom IV holds holds in the h-plane.

11.21. Claim. *In the h-plane, we have $\triangle_h PQR \cong \triangle_h P'Q'R'$ if and only if*

$$Q'P'_h = QP_h, \quad Q'R'_h = QR_h, \quad \textit{and} \quad \measuredangle_h P'Q'R' = \pm\measuredangle PQR.$$

Proof. Applying the main observation, we can assume that both Q and Q' coincide with the center of absolute. In this case

$$\measuredangle P'QR' = \measuredangle_h P'QR' = \pm\measuredangle_h PQR = \pm\measuredangle PQR.$$

Since

$$QP_h = QP'_h \quad \text{and} \quad QR_h = QR'_h,$$

Lemma 11.10 implies that the same holds for the Euclidean distances; that is,

$$QP = QP' \quad \text{and} \quad QR = QR'.$$

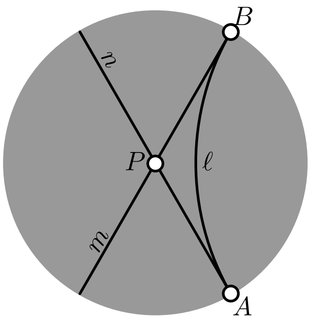
By SAS, there is a motion of Euclidean plane which sends Q to itself, P to P' and R to R'

Note that the center of absolute is fixed by the corresponding motion. It follows that this motion gives also a motion of h-plane; in particular the h-triangles $\triangle_h PQR$ and $\triangle_h P'QR'$ are h-congruent. □

Axiom h-V

Finally we need to check that the Axiom h-V on page 80 holds; that is, we need to prove the following claim.

11.22. Claim. *For any h-line ℓ and any h-point $P \notin \ell$ there are at least two h-lines which pass through P and have no points of intersection with ℓ.*



Instead o proof. Applying the main observation we can assume that P is the center of absolute.

The remaining part of proof can be guessed from the picture □

11.23. Exercise. *Show that in the h-plane there are 3 mutually parallel h-lines such that any pair of these three lines lies on one side from the remaining h-line.*

Chapter 12

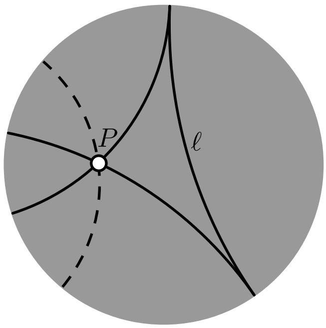
Geometry of h-plane

In this chapter we study the geometry of the plane described by conformal disc model. For briefness, this plane will be called *h-plane*. Note that we can work with this model directly from inside of Euclidean plane but we may also use the axioms of absolute geometry since according to the previous chapter they all hold in the h-plane.

Angle of parallelism

Let P be a point off an h-line ℓ. Drop a perpendicular $(PQ)_h$ from P to ℓ with foot point Q. Let φ be the least angle such that the h-line $(PZ)_h$ with $|\measuredangle_h QPZ| = \varphi$ does not intersect ℓ.

The angle φ is called *angle of parallelism* of P to ℓ. Clearly φ depends only on the h-distance $s = PQ_h$. Further $\varphi(s) \to \pi/2$ as $s \to 0$, and $\varphi(s) \to 0$ as $s \to \infty$. (In the Euclidean geometry the angle of parallelism is identically equal to $\pi/2$.)



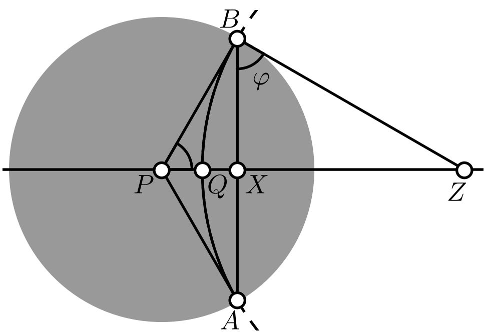
If ℓ, P and Z as above then the h-line $m = (PZ)_h$ is called *asymptotically parallel* to ℓ. In other words, two h-lines are asymptotically parallel if they share one ideal point. (In hyperbolic geometry the term *parallel lines* is often used for *asymptotically parallel lines*; we do not follow this convention.)

Given $P \notin \ell$ there are exactly two asymptotically parallel lines through P to ℓ; the remaining parallel lines to ℓ through P are called *ultra parallel*.

On the diagram, the two solid h-lines passing through P are asymptotically parallel to ℓ; the dotted h-line is ultra parallel to ℓ.

12.1. Proposition. *Let Q be the foot point of P on h-line ℓ. Denote by φ the angle of parallelism of P to ℓ. Then*

$$PQ_h = \tfrac{1}{2}\cdot \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}.$$



Proof. Applying a motion of h-plane if necessary, we may assume P is the center of absolute. Then the h-lines through P are formed by the intersections of Euclidean lines with the h-plane.

Let us denote by A and B the ideal points of ℓ. Without loss of generality we may assume that $\angle APB$ is positive. In this case

$$\varphi = \measuredangle QPB = \measuredangle APQ = \tfrac{1}{2}\cdot\measuredangle APB.$$

Let Z be the center of the circle Γ containing the h-line ℓ. Set X to be the point of intersection of the Euclidean segment $[AB]$ and (PQ).

Note that, $OX = \cos\varphi$ therefore by Lemma 11.10,

$$OX_h = \ln\tfrac{1+\cos\varphi}{1-\cos\varphi}.$$

Note that both angles $\angle PBZ$ and $\angle BXZ$ are right. Therefore, sine the $\angle PZB$ is shared, we get $\triangle ZBX \sim \triangle ZPB$. In particular

$$ZX\cdot ZP = ZB^2;$$

that is, X is the inversion of P in Γ.

The inversion in Γ is the reflection of h-plane through ℓ. Therefore

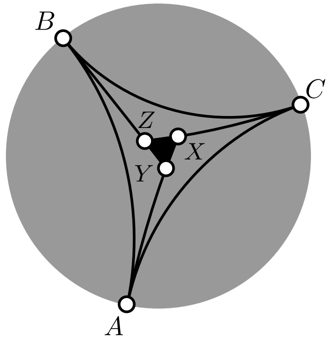
$$\begin{aligned} PQ_h &= QX_h = \\ &= \tfrac{1}{2}\cdot OX_h = \\ &= \tfrac{1}{2}\cdot\ln\tfrac{1+\cos\varphi}{1-\cos\varphi}. \end{aligned}$$

□

Inradius of h-triangle

12.2. Theorem. *Inradius of any h-triangle is less than* $\frac{1}{2}\cdot\ln 3$.

Proof. First note that any triangle in h-plane lies in an *ideal triangle*; that is, a region bounded by three pairwise asymptotically parallel lines.



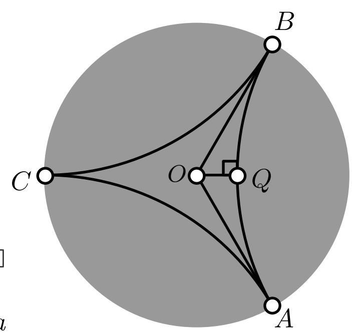
A proof can be seen in the picture. Consider arbitrary h-triangle $\triangle_h XYZ$. Denote by A, B and C the ideal points of the h-half-lines $[XY)_h$, $[YZ)_h$ and $[ZX)_h$.

It should be clear that inradius of the ideal triangle ABC is bigger than inradius of $\triangle_h XYZ$.

Applying an inverse if necessary, we can assume that h-incenter (O) of the ideal triangle is the center of absolute. Therefore, without loss of generality, we may assume

$$\measuredangle AOB = \measuredangle BOC = \measuredangle COA = \tfrac{2}{3}\cdot\pi.$$

It remains to find the inradius. Denote by Q the foot point of O on $(AB)_h$. Then OQ_h is the inradius. Note that the angle of parallelism of $(AB)_h$ at O is equal to $\frac{\pi}{3}$.



By Proposition 12.1,

$$\begin{aligned} OQ_h &= \tfrac{1}{2}\cdot\ln\frac{1+\cos\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \\ &= \tfrac{1}{2}\cdot\ln\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \\ &= \tfrac{1}{2}\cdot\ln 3. \end{aligned}$$

□

12.3. Exercise. *Let* $\square_h ABCD$ *be a quadrilateral in the h-plane such that the h-angles at* A, B *and* C *are right and* $AB_h = BC_h$. *Find the optimal upper bound for* AB_h.

Circles, horocycles and equidistants

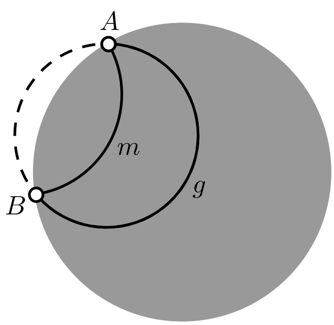
Note that according to Lemma 11.12, any h-circle is formed by a Euclidean circle which lies completely in the h-plane. Further any h-line is an intersection of the h-plane with the circle perpendicular to the absolute.

In this section we will describe the h-geometric meaning of the intersections of the other circles with the h-plane.

You will see that all these intersections formed by a *perfectly round shape* in the h-plane.

One may think of these curves as about trajectories of a car which drives in the plane with fixed position of the wheel. In the Euclidean plane, this way you either run along a circles or along a line.

In hyperbolic plane the picture is different. If you turn wheel far right, you will run along a circle. If you turn it less, at certain position of wheel, you will never come back, the path will be different from the line. If you turn the wheel further a bit, you start to run along a path which stays on the same distant from an h-line.



Equidistants of h-lines. Consider h-plane with absolute Ω. Assume a circle Γ intersects Ω in two distinct points A and B. Denote by g the intersection of Γ with the h-plane. Let us draw an h-line m with the ideal points A and B. According to Exercise 11.1, m is uniquely determined by its ideal points A and B.

Consider any h-line ℓ perpendicular to m; let Δ be the circle containing ℓ.

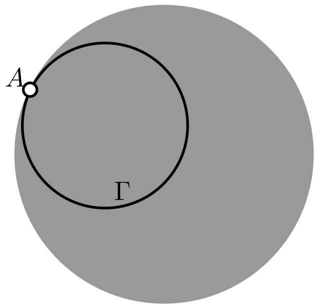
Note that $\Delta \perp \Gamma$. Indeed, according to Corollary 9.15, m and Ω inverted to themselves in Δ. It follows that A is the inversion of B in Δ. Finally, by Corollary 9.16, we get that $\Delta \perp \Gamma$.

Therefore inversion in Δ sends both m and g to themselves. So if $P', P \in g$ are inversions of each other in Δ then they lie on the same h-distance from m. Clearly we have plenty of choice for ℓ, which can be used to move points along g arbitrary keeping the distance to m.

It follows that g is formed by the set of points which lie on fixed h-distance and the same side from m.

Such curve g is called *equidistant* to h-line m. In Euclidean geometry the equidistant from a line is a line; apparently in hyperbolic geometry the picture is different.

Horocycles. If the circle Γ touches the absolute from inside at one point A then the complement $h = \Gamma \backslash \{A\}$ lies in the h-plane. This set is called *horocycle*. It also has perfectly round shape in the sense described above.

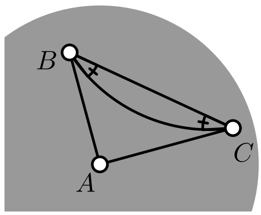


Horocycles are the boarder case between circles and equidistants to h-lines. A horocycle might be considered as a limit of circles which pass through fixed point which the centers running to infinity along a line. The same horocycle is a limit of equidistants which pass through fixed point to the h-lines running to infinity.

12.4. Exercise. *Find the leg of isosceles right h-triangle inscribed in a horocycle.*

Hyperbolic triangles

12.5. Theorem. *Any nondegenerate hyperbolic triangle has positive defect.*



Proof. Consider h-triangle $\triangle_h ABC$. According to Theorem 10.9,

❶ $$\operatorname{defect}(\triangle_h ABC) \geqslant 0.$$

It remains to show that in the case of equality the triangle $\triangle_h ABC$ degenerates.

Without loss of generality, we may assume that A is the center of absolute; in this case $\measuredangle_h CAB = \measuredangle CAB$. Yet we may assume that

$$\measuredangle_h CAB,\ \measuredangle_h ABC,\ \measuredangle_h BCA,\ \measuredangle ABC,\ \measuredangle BCA \geqslant 0.$$

Let D be an arbitrary point in $[CB]_h$ distinct from B and C. From Proposition 8.15

$$\measuredangle ABC - \measuredangle_h ABC \equiv \pi - \measuredangle CDB \equiv \measuredangle BCA - \measuredangle_h BCA.$$

From Exercise 6.15, we get

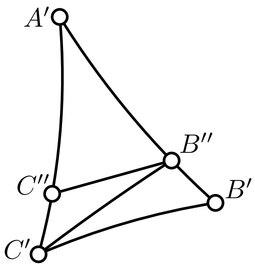
$$\operatorname{defect}(\triangle_h ABC) = 2\cdot(\pi - \measuredangle CDB).$$

Therefore if we have equality in ❶ then $\measuredangle CDB = \pi$. In particular the h-segment $[BC]_h$ coincides with Euclidean segment $[BC]$. By Exercise 11.3, the later can happen only if the h-line passes through the center of absolute; that is, if $\triangle_h ABC$ degenerates. $\square$

The following theorem states in particular that hyperbolic triangles are congruent if their corresponding angles are equal; in particular in hyperbolic geometry similar triangles have to be congruent.

12.6. AAA congruence condition. *Two nondegenerate triangles* $\triangle_h ABC$ *and* $\triangle_h A'B'C'$ *in the h-plane are congruent if* $\measuredangle_h ABC = \pm\measuredangle_h A'B'C'$, $\measuredangle_h BCA = \pm\measuredangle_h B'C'A'$ *and* $\measuredangle_h CAB = \pm\measuredangle_h C'A'B'$.

Proof. Note hat if $AB_h = A'B'_h$ then the theorem follows from ASA.



Assume contrary. Without loss of generality we may assume that $AB_h < A'B'_h$. Therefore we can choose the point $B'' \in [A'B']_h$ such that $A'B''_h = AB_h$.

Choose an h-half-line $[B''X)$ so that

$$\measuredangle_h A'B''X = \measuredangle_h A'B'C'.$$

According to Exercise 10.5, $(B''X)_h \parallel (B'C')_h$.

By Pasch's theorem (3.10), $(B''X)_h$ intersects $[A'C']_h$. Denote by C'' the point of intersection.

According to ASA, $\triangle_h ABC \cong \triangle_h A'B''C''$; in particular

❷ $$\mathrm{defect}(\triangle_h ABC) = \mathrm{defect}(\triangle_h A'B''C'').$$

Applying Exercise 10.10 twice, we get

❸ $$\begin{aligned}\mathrm{defect}(\triangle_h A'B'C') &= \mathrm{defect}(\triangle_h A'B''C'')+\\ &+\mathrm{defect}(\triangle_h B''C''C') + \mathrm{defect}(\triangle_h B''C'B').\end{aligned}$$

By Theorem 12.5, the defects has to be positive. Therefore

$$\mathrm{defect}(\triangle_h A'B'C') > \mathrm{defect}(\triangle_h ABC).$$

On the other hand,

$$\begin{aligned}\mathrm{defect}(\triangle_h A'B'C') &= |\measuredangle_h A'B'C'| + |\measuredangle_h B'C'A'| + |\measuredangle_h C'A'B'| =\\ &= |\measuredangle_h ABC| + |\measuredangle_h BCA| + |\measuredangle_h CAB| =\\ &= \mathrm{defect}(\triangle_h ABC),\end{aligned}$$

a contradiction. □

Recall that a bijection from plane to itself is called *angle preserving* if

$$\measuredangle ABC = \measuredangle A'B'C'$$

for any triangle $\triangle ABC$ and its image $\triangle A'B'C'$.

12.7. Exercise. *Show that any angle-preserving transformation of h-plane is a motion.*

Conformal interpretation

Let us give an other interpretation of the h-distance.

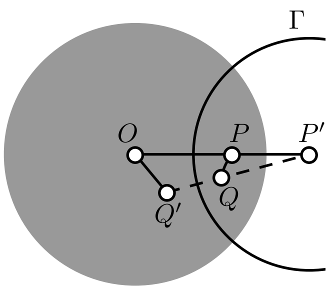
12.8. Lemma. *Consider h-plane with absolute formed by the unit circle centered at O. Fix a point P and let Q be an other point in the h-plane. Set $x = PQ$ and $y = PQ_h$ then*

$$\lim_{x\to 0}\tfrac{y}{x} = \frac{2}{1 - OP^2}.$$

The above formula tells that the h-distance from P to a near by point Q is nearly proportional to the Euclidean distance with the coefficient $\frac{2}{1-OP^2}$. The value $\lambda(P) = \frac{2}{1-OP^2}$ is called *conformal factor* of h-metric.

The value $\frac{1}{\lambda(P)} = \frac{1}{2}\cdot(1 - OP^2)$ can be interpenetrated as the *speed limit* at the given point P. In this case the h-distance is the minimal time needed to travel from one point of h-plane to the other point.

Proof. If $P = O$, then according to Lemma 11.10



$$\text{❹}\qquad \frac{y}{x} = \frac{\ln\frac{1+x}{1-x}}{x} \to 2$$

as $x \to 0$.

If $P \neq O$, denote by P' the inversion of P in the absolute. Denote by Γ the circle with center P' perpendicular to the absolute.

According to Main Observation 11.7 and Lemma 11.6 the inversion in Γ is a motion of h-plane which sends P to O. In particular, if we denote by Q' the inversion of Q in Γ then $OQ'_h = PQ_h$.

Set $x' = OQ'$ According to Lemma 9.2,

$$\frac{x'}{x} = \frac{OP'}{P'Q}.$$

Since P' is the inversion of P in the absolute, we have $PO\cdot OP' = 1$. Therefore

$$\frac{x'}{x} \to \frac{OP'}{P'P} = \frac{1}{1 - OP^2}$$

as $x \to 0$.

Together with ❹, it implies

$$\frac{y}{x} = \frac{y}{x'}\cdot\frac{x'}{x} \to \frac{2}{1 - OP^2}$$

as $x \to 0$. □

Here is an application of the lemma above.

12.9. Proposition. *The circumference of an h-circle of h-radius r is*

$$2\cdot\pi\cdot \operatorname{sh} r,$$

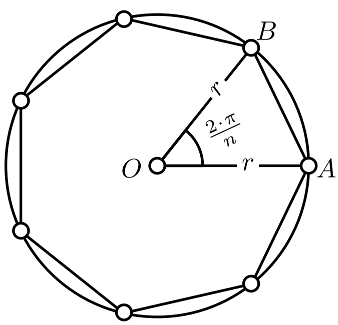
where $\operatorname{sh} r$ *denotes* hyperbolic sine *of r; that is,*

$$\operatorname{sh} r \stackrel{\text{def}}{=\!=} \frac{e^r - e^{-r}}{2}.$$

Before we proceed with the proof let us discuss the same problem in the Euclidean plane.

The circumference of the circle in the Euclidean plane can be defined as limit of perimeters of regular n-gons inscribed in the circle as $n \to \infty$.

Namely, let us fix $r > 0$. Given a positive integer n consider $\triangle AOB$ such that $\measuredangle AOB = \frac{2\cdot\pi}{n}$ and $OA = OB = r$. Set $x_n = AB$. Note that x_n is the side of regular n-gon inscribed in the circle of radius r. Therefore the perimeter of the n-gon is equal to $n\cdot x_n$.



The circumference of the circle with radius r might be defined as the limit

❺ $$\lim_{n\to\infty} n\cdot x_n = 2\cdot\pi\cdot r.$$

(This limit can be taken as the definition of π.)

In the following proof we repeat the same construction in the h-plane.

Proof. Without loss of generality we can assume that the center O of the circle is the center of absolute.

By Lemma 11.10, the h-circle with h-radius r is formed by the Euclidean circle with center O and radius

$$a = \frac{e^r - 1}{e^r + 1}.$$

Denote by x_n and y_n the Euclidean and hyperbolic side lengths of the regular n-gon inscribed in the circle.

Note that $x_n \to 0$ as $n \to \infty$. By Lemma 12.8,

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \frac{2}{1-a^2}.$$

Applying ❺, we get that the circumference of the h-circle can be found the following way

$$\begin{aligned}\lim_{n\to\infty} n\cdot y_n &= \frac{2}{1-a^2}\cdot \lim_{n\to\infty} n\cdot x_n = \\ &= \frac{4\cdot\pi\cdot a}{1-a^2} = \\ &= \frac{4\cdot\pi\cdot\left(\frac{e^r-1}{e^r+1}\right)}{1-\left(\frac{e^r-1}{e^r+1}\right)^2} = \\ &= 2\cdot\pi\cdot\frac{e^r-e^{-r}}{2} = \\ &= 2\cdot\pi\cdot\operatorname{sh} r.\end{aligned}$$

□

12.10. Exercise. *Denote by* $\operatorname{circum}_h(r)$ *the circumference of the h-circle of radius* r. *Show that*

$$\operatorname{circum}_h(r+1) > 2\cdot\operatorname{circum}_h(r)$$

for all $r > 0$.

Chapter 13

Affine geometry

Affine transformations

A bijection of Euclidean plane to itself is called *affine transformation* if it maps any line to a line.

We say that three points are *collinear* if they lie on one line. Note that affine transformation sends collinear points to collinear; the following exercise gives a converse.

13.1. Exercise. *Assume f is a bijection from Euclidean plane to itself which sends collinear points to collinear points. Show that f is an affine transformation. (In other words, show that f maps noncollinear points to noncollinear.)*

13.2. Exercise. *Show that affine transformation sends parallel lines to the parallel lines.*

Affine geometry studies so called *incidence structure* of Euclidean plane. The incidence structure is the data about which points lie on which lines and nothing else; we can not talk about distances, angles and so on. One may also say that affine geometry studies the properties of Euclidean plane which preserved under affine transformations.

Constructions with parallel tool and ruler

Let us consider geometric constructions with ruler and *parallel tool*; the later makes possible to draw a line through the given point parallel to the given line. By Exercisers 13.2, any construction with these two tools are invariant with respect to affine transformation. For example, to solve the following exercise, it is sufficient to prove that midpoint of given segment can be constructed with ruler and parallel tool.

13.3. Exercise. *Let M be the midpoint of segment $[AB]$ in the Euclidean plane. Assume that an affine transformation sends points A, B and M to A', B' and M' correspondingly. Show that M' is the midpoint of $[A'B']$.*

The following exercise will be used in the proof of Theorem 13.7.

13.4. Exercise. *Assume that in Euclidean plane we have 4 points with coordinates $(0,0)$, $(1,0)$, $(a,0)$ and $(b,0)$. Use ruler and parallel tool to construct the points with coordinates $(a{\cdot}b,0)$ and $(a+b,0)$.*

13.5. Exercise. *Use ruler and parallel tool to construct the center of the given circle.*

Matrix form

Since the lines are defined in terms of metric; any motion of Euclidean plane is also an affine transformation.

On the other hand, there are affine transformations of Euclidean plane which are not motions.

Consider Euclidean plane with coordinate system; let us use the column notation for the coordinates; that is, we will write $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ instead of (x,y).

As it follows from the theorem below, the so called *shear mapping* $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x+k\cdot y \\ y \end{smallmatrix}\right)$ is an affine transformation. The shear mapping can change the angle between vertical and horizontal lines almost arbitrary. The later can be used to prove impossibility of some constructions with ruler and parallel tool; here is one example.

13.6. Exercise. *Show that with ruler and parallel tool one can not construct a line perpendicular to the given line.*

13.7. Theorem. *A map β from the plane to itself is an affine transformation if and only if*

$$❶ \qquad \beta\colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$$

for some fixed invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and vector $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$.

In particular, any affine transformation of Euclidean plane is continuous.

In the proof of “only if” part, we will use the following algebraic lemma.

13.8. Algebraic lemma. *Assume $f\colon \mathbb{R}\to\mathbb{R}$ is a function such that*

$$\begin{aligned} f(1)&=1,\\ f(x+y)&=f(x)+f(y),\\ f(x\cdot y)&=f(x)\cdot f(y) \end{aligned}$$

for any $x,y\in\mathbb{R}$. Then $f(x)=x$ for any $x\in\mathbb{R}$.

Note that we do not assume that f is continuous.

The function f satisfying three conditions in the lemma is called *field automorphism*. Therefore the lemma states that the identity function is the only automorphism of the field of real numbers.

On the other hand, the conjugation $z\mapsto\bar z$ (see page 134) gives an example of nontrivial automorphism of complex numbers.

Proof. Since

$$f(0)+f(1)=f(0+1),$$

we get

$$f(0)+1=1;$$

that is,

❷ $$f(0)=0.$$

Further

$$0=f(0)=f(x)+f(-x).$$

Therefore

❸ $$f(-x)=-f(x)\ \text{for any}\ \ x\in\mathbb{R}.$$

Further

$$\begin{aligned} f(2)&=f(1)+f(1)=1+1=2\\ f(3)&=f(2)+f(1)=2+1=3\\ &\dots \end{aligned}$$

Together with ❸, the later implies that

$$f(n)=n\ \ \text{for any integer}\ \ n.$$

Since

$$f(m)=f(\tfrac{m}{n})\cdot f(n)$$

we get

$$f(\tfrac{m}{n})=\tfrac{m}{n}$$

for any rational number $\frac{m}{n}$.

Assume $a \geqslant 0$. Then the equation $x \cdot x = a$ has a real solution $x = \sqrt{a}$. Therefore $[f(\sqrt{a})]^2 = f(\sqrt{a}) \cdot f(\sqrt{a}) = f(a)$. Whence $f(a) \geqslant 0$. That is

❹ $$a \geqslant 0 \quad \Longrightarrow \quad f(a) \geqslant 0.$$

Applying ❸, we also get

❺ $$a \leqslant 0 \quad \Longrightarrow \quad f(a) \leqslant 0.$$

Finally, assume $f(a) \neq a$ for some $a \in \mathbb{R}$. Then there is a rational number $\frac{m}{n}$ which lies between a and $f(a)$; that is, the numbers $x = a - \frac{m}{n}$ and $y = f(a) - \frac{m}{n}$ have opposite signs. Since $f(\frac{m}{n}) = \frac{m}{n}$, we get $f(x) = y$. The later contradicts ❹ or ❺. $\square$

13.9. Lemma. *Assume γ is an affine transformation which fix three points $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ on the coordinate plane. Then γ is the identity map; that is, $\gamma\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ for any point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.*

Proof. Since affine transformation sends lines to lines, we get that each axes is mapped to itself.

According to Exercise 13.2, parallel lines are mapped to parallel lines. Therefore we get that horizontal lines mapped to horizontal lines and vertical lines mapped to vertical. In other words,

$$\gamma\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \begin{pmatrix} f(x) \\ h(y) \end{pmatrix}.$$

for some functions $f, h\colon \mathbb{R} \to \mathbb{R}$.

Note that $f(1) = h(1) = 1$ and according to Exercise 13.4, both f and h satisfies the other two conditions of Algebraic lemma 13.8. Applying the lemma, we get that f and h are identity functions and so is γ. $\square$

Proof of Theorem 13.7. Recall that matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible if

$$\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = a \cdot d - b \cdot c \neq 0;$$

in this case the matrix

$$\tfrac{1}{a \cdot d - b \cdot c} \cdot \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$$

is the inverse of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Assume that the map β is described by ❶. Note that

❻ $$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \tfrac{1}{a \cdot d - b \cdot c} \cdot \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x - v \\ y - w \end{smallmatrix}\right).$$

is inverse of β. In particular β is a bijection.

A line in the plane form the solutions of equations

❼ $$p{\cdot}x + q{\cdot}y + r = 0,$$

where $p \neq 0$ or $q \neq 0$. Find $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ from its β-image by formula ❻ and substitute the result in ❼. You will get the equation of the image of the line. The equation has the same type as ❼, with different constants, in particular it describes a line. Therefore β is an affine transformation.

To prove “only if” part, fix an affine transformation α. Set

$$\begin{aligned} \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \\ \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) - \alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \\ \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) - \alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right). \end{aligned}$$

Note that the points $\alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\alpha\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $\alpha\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ do not lie on one line. Therefore the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible.

For the affine transformation β defined by ❶ we have

$$\begin{aligned} \beta\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \\ \beta\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \\ \beta\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right). \end{aligned}$$

It remains to show that $\alpha = \beta$ or equivalently the composition $\gamma = \alpha \circ \beta^{-1}$ is the identity map.

Note that γ is an affine transformation which fix points $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. It remains to apply Lemma 13.9. $\square$

On inversive transformations

Recall that inversive plane is Euclidean plane with added a point at infinity, denoted as ∞. We assume that any line pass through ∞. The term *circline* stays for *circle or line*;

The *inversive transformation* is bijection from inversive plane to itself which sends circlines to circlines. Inversive geometry can be defined as geometry which *circline incidence structure* of inversive plane; that is we can say which points lie on which circlines.

13.10. Theorem. *A map from inversive plane to itself is an inversive transformation if and only if it can be presented as a composition of inversions and reflections.*

Proof. According to Theorem 9.7 any inversion is a inversive transformation. Therefore the same holds for composition of inversions.

To prove converse, fix an inversive transformation α.

Assume $\alpha(\infty) = \infty$. Recall that any circline passing through ∞ is a line. If follows that α maps lines to lines; that is, it is an affine transformation.

Further, α is not an arbitrary affine transformation, it maps circles to circles.

Composing α with a reflection, say ρ_1, we can assume that $\alpha' = \rho_1 \circ \alpha$ maps the unit circle with center at the origin to a concentric circle.

Composing the obtained map α' with a *homothety*

$$\chi\colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} k\cdot x \\ k\cdot y \end{smallmatrix}\right),$$

we can assume that $\alpha'' = \chi \circ \alpha'$ sends the unit circle to itself.

Composing the obtained map α'' a reflection ρ_2 in a line through the origin, we can assume that in addition $\alpha''' = \rho_2 \circ \alpha''$ the point $(1, 0)$ maps to itself.

By Exercise 13.5, α''' fixes the center of the circle; that is, it fixes the origin.

The obtained map α''' is an affine transformation. Applying Theorem 13.7, together with the properties of α'' described above we get

$$\alpha'''\colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$$

for an invertible matrix $\left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right)$. Since the point $(0, 1)$ maps to the unit circle we get

$$b^2 + d^2 = 1.$$

Since the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ maps to the unit circle we get

$$(b + d)^2 = 1.$$

It follows

$$\alpha'''\colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right);$$

that is, either α''' is the identity map or reflection the x-axis.

Note that the homothety χ is a composition of two inversions in concentric circles. Hence the result follows.

Now assume $P = \alpha(\infty) \neq \infty$. Consider an inversion β in a circle with center at P. Note that $\beta(P) = \infty$; therefore $\beta \circ \alpha(\infty) = \infty$. Since β is inversive, so is $\beta \circ \alpha$. From above we get that $\beta \circ \alpha$ is a composition of reflections and inversions therefore so is α. $\square$

13.11. Exercise. *Show that any reflection can be presented as a composition of three inverses.*

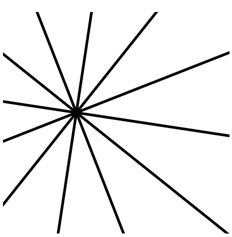
Note that exercise above together with Theorem 13.10, implies that any inversive map is a composition of inversions, no reflections are needed.

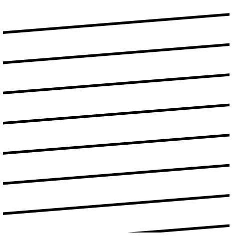
Chapter 14

Projective geometry

Real projective plane

In the Euclidean plane two distinct lines might have one or zero points of intersection (in the later case the lines are called parallel). Our aim is to extend Euclidean plane by ideal points so that any two distinct lines will have exactly one point of intersection.





A collection of lines in the Euclidean plane is called *concurrent* if they all intersect at single point or all of them pairwise parallel. A set of concurrent lines passing through each point in the plane is called *pencil*. There are two types of pencils first is the set of all lines passing through given point called the *center of the pencil* and second is the set of pairwise parallel lines.

Note that each point in Euclidean plane uniquely defines a pencil with center in it, but the parallel pencils have no center. Also any two lines completely determine the pencil containing both.

Let us add one ideal point for each parallel pencil, and assume that all these ideal points lie on one ideal line.

We obtain so called *real projective plane*. Each point in the real projective plane defined as a pencil of lines in Euclidean plane. We say that three points lie on one line the corresponding pencils contain a common line. (We assume that the ideal line belongs to each parallel pencil).

Euclidean space

Let us repeat the construction of metric d_2 (page 11) in the space.

We will denote by $\mathbb{R}^3$ the set of all triples (x, y, z) of real numbers. Assume $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ are arbitrary points. Define the metric on $\mathbb{R}^3$ the following way

$$AB \stackrel{\text{def}}{=\!=} \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

The obtained metric space is called *Euclidean space*.

Assume at least one of the real numbers a, b or c is distinct from zero. Then the subset of points $(x, y, z) \in \mathbb{R}^3$ described by equation

$$a{\cdot}x + b{\cdot}y + c{\cdot}z + d = 0$$

is called *plane*; here d is a real number.

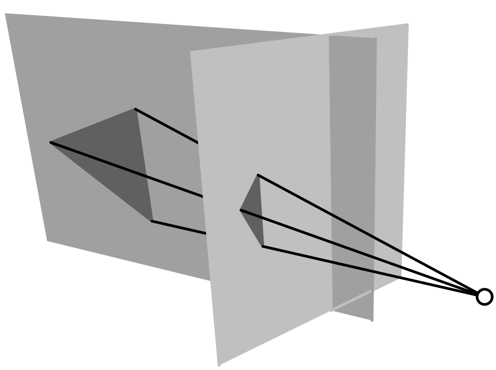
It is straightforward to show that any plane in Euclidean space is isometric to Euclidean plane. Further, any three points on the space lie on one plane. And any nonempty intersection of two distinct planes forms a line in each of these planes.

The later statements makes possible to generalize many notions and results from Euclidean plane geometry to Euclidean space by applying plane geometry in the planes of the space.

Perspective projection

Consider two planes Π and Π' in the Euclidean space. Let O be a point which does not belong neither to Π nor Π'.

Consider the *perspective projection from* Π *to* Π' *with center at* O. The projection of $P \in \Pi$ is defined as the point $P' \in \Pi'$ which lies on the line (OP).



Note that the perspective projection sends collinear points to collinear. Indeed, assume three points P, Q, R lie on one line ℓ in Π and P', Q', R' are their images in Π'. Then all the points P, Q, R, P', Q', R' lie in the plane, say Θ, containing O and ℓ. Therefore the points P', Q', R' lie in the line formed by intersection $\ell' = \Theta \cap \Pi'$.

The perspective projection is not a bijection between the planes;. Indeed, if the line (OP) is parallel to Π' (that is, if $(OP) \cap \Pi' = \varnothing$) then the perspective projection is not defined. Also if $(OP') \parallel \Pi$ for a point $P' \in \Pi'$ then point P' is not an image of the perspective projection.

A similar story happened with inversion. If inversion is considered for the Euclidean plane then is not defied at the center of inversion; it is also not the image of any point. To deal with this problem we passed to inversive plane which is Euclidean plane extended by one ideal point.

A similar strategy works for perspective projection $\Pi \to \Pi'$, but this time real projective plane is the right choice of extension. Denote by $\hat{\Pi}$ and $\hat{\Pi}'$ the corresponding real projective planes.

Note that there is a natural bijection between points in the real projective plane $\hat{\Pi}$ and all the lines passing through O. If $P \in \Pi$ then take the line (OP); if P is an ideal point of $\hat{\Pi}$, so defined by a parallel pencil of lines then take the line through O which is parallel to each lines in this pencil.

The same construction gives a bijection between points in the real projective plane $\hat{\Pi}'$ and all the lines passing through O. Composing these bijections we get a bijection $\hat{\Pi} \to \hat{\Pi}'$ which coincides with the perspective projection $P \mapsto P'$ where it is defined.

Note that the ideal line of $\hat{\Pi}$ maps to the line formed by intersection of Π' and the plane through O parallel to Π. Similarly the ideal line of $\hat{\Pi}'$ is the image of the line formed by intersection of Π and the plane through O parallel to Π'.

Strictly speaking this gives a transformation from one real projective plane to another, but if we identify the two planes, say by fixing a coordinate system in each, we get a projective transformation from the plane to itself.

14.1. Exercise. *Let O be the origin of (x, y, z)-coordinate space and the planes Π and Π' formed by the solutions of equations $x = 1$ and $y = 1$ correspondingly. The perspective projection from Π to Π' with center at O sends P to P'. Assume P has coordinates $(1, y, z)$, find the coordinates of P'.*

For which points $P \in \Pi$ the perspective projection is undefined? Which points $P' \in \Pi'$ are not images of points under perspective projection?

Projective transformations

A bijection from the real projective plane to itself which sends lines to lines is called *projective transformation.*

Projective and affine geometries study incidence structure of Euclidean and real projective plane correspondingly. One may also say that projective geometry studies the properties of real projective plane which preserved under projective transformations.

Note that any affine transformation defines a projective transformation on the corresponding real projective plane. We will call such projective transformations *affine*; these are projective transformations which send the ideal line to itself.

The perspective projection discussed in the previous section gives an example of projective transformation which is not affine.

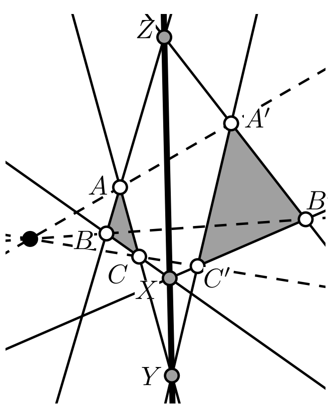
14.2. Theorem. *Any projective transformation can be obtained as a composition of an affine transformation and a perspective projection.*

Proof. Assume α is a projective transformation. If is α sends ideal line to itself then it has to be affine. Hence the theorem follows.

Assume α sends the ideal line to line ℓ, choose a perspective projection β which sends ℓ back to the ideal line. To do this we have to identify our plane with a plane Π in the space, then fix a point $O \notin \Pi$ and then choose a plane Π' which is parallel to the plane containing ℓ and O.

The composition $\beta \circ \alpha$ sends ideal line to itself, therefore it has to be affine. Hence the result follows. $\square$

Desargues' theorem



Loosely speaking, any statement in projective geometry can be formulated using only terms *collinear points*, *concurrent lines.*

Here is a classical example of a theorem in projective geometry.

14.3. Desargues' theorem. *Consider three concurrent lines (AA'), (BB') and (CC') in the real projective plane. Set*

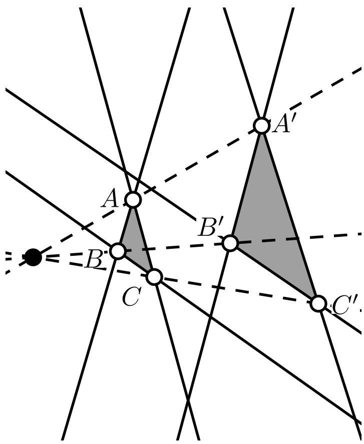
$$X = (BC) \cap (B'C'), \quad Y = (CA) \cap (C'A'), \quad Z = (AB) \cap (A'B')$$

Then the points X, Y and Z are collinear.

Proof. Without loss of generality, we may assume that the line (XY) is ideal. If not, apply a perspective projection which sends the line (XY) to the ideal line. That is, we can assume that $(BC) \parallel (B'C')$ and $(CA) \parallel (C'A')$ and we need to show that $(AB) \parallel (A'B')$.

Assume that the lines (AA'), (BB') and (CC') intersect at point O. Since $(BC) \parallel (B'C')$, Transversal property 6.18, implies that that $\measuredangle OBC = \measuredangle OB'C'$ and $\measuredangle OCB = \measuredangle OC'B'$. By AA similarity condition, $\triangle OBC \sim \triangle OB'C'$. In particular,

$$\frac{OB}{OB'} = \frac{OC}{OC'}.$$



The same way we get $\triangle OAC \sim$ $\sim \triangle OA'C'$ and

$$\frac{OA}{OA'} = \frac{OC}{OC'}.$$

Therefore

$$\frac{OA}{OA'} = \frac{OB}{OB'}.$$

By SAS similarity condition, we get $\triangle OAB \sim \triangle OA'B'$, in particular $\measuredangle OAB = \pm\measuredangle OA'B'$.

Note that $\measuredangle AOB = \measuredangle A'OB'$. Therefore

$$\measuredangle OAB = \measuredangle OA'B'.$$

By Transversal property 6.18, $(AB) \parallel (A'B')$.

The case $(AA') \parallel (BB') \parallel (CC')$ is done similarly. In this case the quadrilaterals $\square B'BCC'$ and $\square A'ACC'$ are parallelograms. Therefore

$$BB' = CC' = AA'.$$

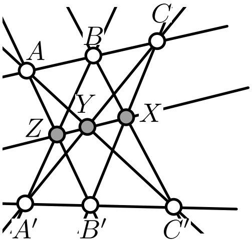
Whence $\square B'BAA'$ is a parallelogram and $(AB) \parallel (A'B')$. $\square$

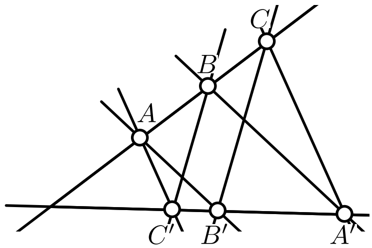
Here is an other classical theorem of of projective geometry.

14.4. Pappus's theorem. *Assume that two triples of points A, B, C, and A', B', C' are collinear. Set*

$$X = (BC') \cap (B'C), \quad Y = (CA') \cap (C'A), \quad Z = (AB') \cap (A'B).$$

Then the points X, Y, Z are collinear.





Pappus's theorem can be proved the same way as Desargues' theorem.

Idea of the proof. Applying a perspective projection, we can assume that X and Y lie on the ideal line. It remains to show that Z lies on the ideal line.

In other words, assuming that $(AB') \parallel (A'B)$ and $(AC') \parallel (A'C)$, we need to show that $(BC') \parallel (B'C)$.

14.5. Exercise. *Finish the proof of Pappus's theorem using the idea described above.*

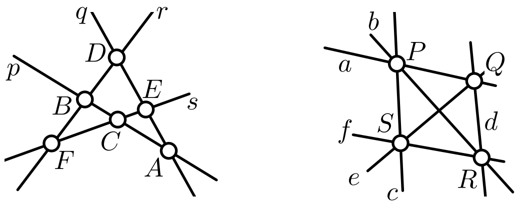
Duality

Let us fix a bijection between the set of lines and the set of points of the plane. Given a point P let us denote by lower case letter p the corresponding line. Also the other way around, given a line s we will denote upper case letter S the corresponding point.

The bijection between points and lines is called *duality*[1] if

$$P \in s \quad \Longleftrightarrow \quad p \ni S.$$

for any point P and line s.



Dual configurations.

Existence of duality in a plane says that the lines and the points in this plane have the same rights in terms of incidence.

[1]Usual definition of duality is more general; we consider a special case which is also called *polarity*.

14.6. Exercise. *Show that Euclidean plane does not admit a duality.*

14.7. Theorem. *The real projective plane admits a duality.*

Proof. Consider a plane Π and a point $O \notin \Pi$ in the Euclidean space; denote by $\hat{\Pi}$ the corresponding real projective plane.

Recall that there is a natural bijection between points in the real projective plane $\hat{\Pi}$ and all the lines passing through O. If $P \in \Pi$ then the corresponding line is (OP); if P is an ideal point of $\hat{\Pi}$, so P is defined as a parallel pencil of lines, take the line through O which is parallel to each lines in this pencil. Given $P \in \hat{\Pi}$, denote the obtained line $\dot{P}$.

Similarly there is a natural bijection between lines in $\hat{\Pi}$ and all the planes passing through O. If is s is a line in Π then take the plane containing O and s; if s is the ideal line of $\hat{\Pi}$, take the plane through O parallel to Π. Given a line s in $\hat{\Pi}$, denote the obtained plane as $\dot{s}$.

It is straightforward to check that $\dot{P} \subset \dot{s}$ if and only if $P \in s$; that is, the bijections $P \mapsto \dot{P}$ and $s \mapsto \dot{s}$ remember all the incidence structure of the real projective plane $\hat{\Pi}$.

It remains to construct a bijection $\dot{s} \mapsto \dot{S}$ between the set of planes and the set of lines passing through O such that

❶ $$\dot{r} \subset \dot{S} \quad \Longleftrightarrow \quad \dot{R} \supset \dot{s}$$

for any two lines line $\dot{r}$ and $\dot{s}$ passing through O.

Set $\dot{S}$ to be the plane through O which is perpendicular to $\dot{s}$. Note that both conditions ❶ are equivalent to $\dot{r} \perp \dot{s}$; hence the result follows. □

14.8. Exercise. *Consider Euclidean plane with (x, y)-coordinates, denote by O the origin. Given a point $P \neq O$ with coordinates (a, b) consider the line p formed by solution of the equation $a{\cdot}x + b{\cdot}y = 1$.*

Show that the correspondence P to p extends to duality of corresponding real projective plane.

Which line corresponds to O?

Which point of real projective plane corresponds to the line formed by solutions $a{\cdot}x + b{\cdot}y = 0$?

The existence of duality in the real projective planes makes possible to formulate an equivalent dual statement to any statement in projective geometry. Say, the dual statement for "the points X, Y and Z lie on one line ℓ" would be the "lines x, y and z intersect at one point L". Let us formulate the dual statement for Desargues' theorem 14.3.

14.9. Dual Desargues' theorem. *Consider the collinear points X, Y and Z. Assume that $X = (BC) \cap (B'C')$, $Y = (CA) \cap (C'A')$ and*

$Z = (AB) \cap (A'B')$. Then the lines (AA'), (BB') and (CC') are concurrent.

In this theorem the points X, Y and Z are dual to the lines (AA'), (BB') and (CC') in the original formulation, and the other way around.

Once Desargues' theorem is proved, applying duality (Theorem 14.7) we get Dual Desargues' theorem. Note that the Dual Desargues' theorem is the converse to the original Desargues' theorem 14.3. Thereofre Desargues' theorem and its dual can be packed in the following strong version of Desargues' theorem.

14.10. Strong Desargues' theorem. *Assume that $X = (BC) \cap \cap (B'C')$, $Y = (CA) \cap (C'A')$ and $Z = (AB) \cap (A'B')$. Then the lines (AA'), (BB') and (CC') are concurrent if and only if the points X, Y and Z are collinear.*

14.11. Exercise. *Formulate Dual Pappus's theorem, see 14.4.*

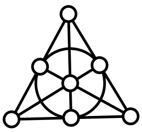
14.12. Exercise. *Given two parallel lines construct with ruler only the third parallel line through the given point.*

Axioms

Note that the real projective plane described above satisfies the following set of axioms.

I. Any two distinct points lie on a unique line.
II. Any two distinct lines pass through a unique point.
III. There exist at least four points of which no three are collinear.

Let us take these three axioms as a definition of *projective plane*; so the real projective plane discussed above becomes a particular example of projective plane.



There is an example of projective plane which contains exactly 3 points on each line. This is so called *Fano plane* which you can see on the diagram; it contains 7 points and 7 lines. This is an example of *finite projective plane*; that is, projective plane with finitely many points.

14.13. Exercise. *Show that any line in projective plane contains at least three points.*

Consider the following analog of Axiom III.

III$'$. There exist at least four lines of which no three are concurrent.

14.14. Exercise. *Show that Axiom III′ is equivalent to Axiom III.*

Exercise above show that in the axiomatic system of projective plane, lines and points have the same rights. In fact one can switch everywhere words “point” with “line”, “pass trough” with “lies on”, “collinear” with “concurrent” and we get an equivalent set of axioms.

14.15. Exercise. *Assume that one of the lines in a finite projective plane contains exactly $n + 1$ points.*

(a) Show that each line contains exactly $n + 1$ points.

(b) Show that the number of the points in the plane has to be

$$n^2 + n + 1.$$

(c) Show that in any finite projective plane the number of points coincides with the number of lines.

The number n in the above exercise is called *order* of finite projective plane. For example Fano plane has order 2. Here is one of the most famous open problem in finite geometry.

14.16. Conjecture. *The order of a finite projective plane is the power of a prime.*

Chapter 15

Spherical geometry

Spherical geometry is the geometry of the surface of the unit sphere. This type of geometry has practical applications in cartography, navigation and astronomy.

The spherical geometry is a close relative of Euclidean and hyperbolic geometries. Most of theorems of hyperbolic geometry have spherical analogs, but spherical geometry is easier to visualize.

We discuss few theorems in spherical geometry; the proofs are not completely rigorous.

Spheres in the space

Recall that Euclidean space is the set $\mathbb{R}^3$ of all triples (x, y, z) of real numbers such that the distance between a pair of points $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ is defined by the following formula

$$AB \stackrel{\text{def}}{=\!=} \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

The planes in the space defined as the set of solutions of equation

$$a{\cdot}x + b{\cdot}y + c{\cdot}z + d = 0$$

for real numbers a, b, c and d such that at least one of the numbers a, b or c is not zero. Equivalently plane can be defined as a subset of Euclidean space which is isometric to to Euclidean plane.

Sphere in the space is the direct analog of circle in the plane. Formally, *sphere* with center O and radius r is the set of points in the space which lie on the distance r from O.

Let A and B be two points on the unit sphere centered at O. The *spherical distance* from A to B (briefly AB_s) is defined as $|\measuredangle AOB|$.

In the spherical geometry, the role of lines play the *great circles*; that is, the intersection of the sphere with a plane passing through O.

Note that the great circles do not form lines in the sense of Definition 1.8. Also any two distinct great circles intersect at two antipodal points. In particular, the sphere does not satisfy the axioms of absolute plane.

Pythagorean theorem

Here is an analog of Pythagorean Theorems (6.10 and 16.8) in spherical geometry.

15.1. Theorem. *Let $\triangle_s ABC$ be a spherical triangle with right angle at C. Set $a = BC_s$, $b = CA_s$ and $c = AB_s$. Then*

$$\cos c = \cos a \cdot \cos b.$$

In the proof we will use the notion of scalar product which we are about to discuss.

Let A and B be two points in Euclidean space. Denote by $v_A = (x_A, y_A, z_A)$ and $v_B = (x_B, y_B, z_B)$ the position vectors of A and B correspondingly. The scalar product of two vectors v_A and v_B in $\mathbb{R}^3$ is defined as

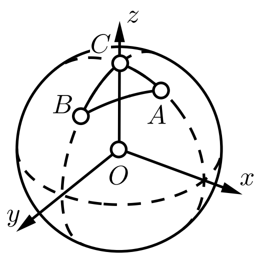
❶ $$\langle v_A, v_B \rangle \stackrel{\text{def}}{=\!=} x_A \cdot x_B + y_A \cdot y_B + z_A \cdot z_B.$$

Assume both vectors v_A and v_B are nonzero and φ is the angle measure between these two vectors. In this case the scalar product can be expressed the following way:

$$\langle v_A, v_B \rangle = |v_A| \cdot |v_B| \cdot \cos \varphi,$$

where

$$|v_A| = \sqrt{x_A^2 + y_A^2 + z_A^2}, \qquad |v_B| = \sqrt{x_B^2 + y_B^2 + z_B^2}.$$



Now, assume the points A and B lie on the unit sphere in $\mathbb{R}^3$ centered at the origin. In this case $|v_A| = |v_B| = 1$. By ❶ we get

❷ $$\cos AB_s = \langle v_A, v_B \rangle.$$

Proof. Since the angle at C is right, we can choose coordinates in $\mathbb{R}^3$ so that $v_C = (0, 0, 1)$,

v_A lies in xz-plane, so $v_A = (x_A, 0, z_A)$ and v_B lies in yz-plane, so $v_B = (0, y_B, z_B)$.

Applying, ❷, we get

$$z_A = \langle v_C, v_A\rangle = \cos b,$$
$$z_B = \langle v_C, v_B\rangle = \cos a.$$

Applying, ❷ again, we get

$$\begin{aligned}\cos c &= \langle v_A, v_B\rangle = \\ &= x_A{\cdot}0 + 0{\cdot}y_B + z_A{\cdot}z_B = \\ &= \cos b{\cdot}\cos a.\end{aligned}$$

□

15.2. Exercise. *Show that if $\triangle_s ABC$ be a spherical triangle with right angle at C and $AC_s = BC_s = \frac{\pi}{4}$ then $AB_s = \frac{\pi}{3}$.*

Try to find two solutions, with and without using the spherical Pythagorean theorem.

Inversion of the space

The inversion in the sphere defined the same way as we define inversion in the circle.

Formally, let Σ be the sphere with center O and radius r. The *inversion* in Σ of a point P is the point $P' \in [OP)$ such that

$$OP{\cdot}OP' = r^2.$$

In this case, the sphere Σ will be called the *sphere of inversion* and its center is called *center of inversion*.

We also add ∞ to the space and assume that the center of inversion is mapped to ∞ and the other way around. The space $\mathbb{R}^3$ with the point ∞ will be called *inversive space*.

The inversion of the space has many properties of the inversion of the plane. Most important for us is the analogs of theorems 9.6, 9.7, 9.24 which can be summarized as follows.

15.3. Theorem. *The inversion in the sphere has the following properties:*

(a) Inversion maps sphere or plane into sphere or plane.
(b) Inversion maps circle or line into circle or line.
(c) Inversion preserves cross-ratio; that is, if A', B', C' and D' be the inversions of the points A, B, C and D correspondingly then

$$\frac{AB{\cdot}CD}{BC{\cdot}DA} = \frac{A'B'{\cdot}C'D'}{B'C'{\cdot}D'A'}.$$

(d) *Inversion maps arcs into arcs.*

(e) *Inversion preserves the absolute value of the angle measure between tangent half-lines to the arcs.*

We do not present the proofs here, but they nearly repeat the corresponding proofs in plane geometry. To prove (a), you will need in addition the following lemma; its proof is left to the reader.

15.4. Lemma. *Let Σ be a subset of Euclidean space which contains at least two points. Fix a point O in the space.*

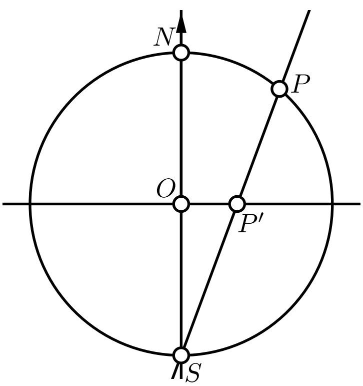
Then Σ is a sphere if and only if for any plane Π passing through O, the intersection $\Pi \cap \Sigma$ is either empty set, one point set or a circle.

The following observation helps to reduce part (b) to part (a).

15.5. Observation. *Any circle in the space can be presented as an intersection of two spheres.*

Stereographic projection

Consider the unit sphere Σ centered at the origin $(0,0,0)$. This sphere can be described by equation $x^2 + y^2 + z^2 = 1$.



The plane through P, O and S.

Denote by Π the xy-plane; it is defined by the equation $z = 0$. Clearly Π runs through the center of Σ.

Denote by $N = (0,0,1)$ the "North Pole" and by $S = (0,0,-1)$ be the "South Pole" of Σ; these are the points on the sphere which have extremal distances to Π. Denote by Ω the "equator" of Σ; it is the intersection $\Sigma \cap \Pi$.

For any point $P \neq S$ on Σ, consider the line (SP) in the space. This line intersects Π in exactly one point, say P'. We set in addition that $S' = \infty$.

The map $P \mapsto P'$ is called *stereographic projection from Σ to Π from the South Pole.* The inverse of this map $P' \mapsto P$ is called *stereographic projection from Π to Σ from the South Pole.*

The same way one can define *stereographic projections from the North Pole N.*

Note that $P = P'$ if and only if $P \in \Omega$.

Note that if Σ and Π as above. Then the stereographic projections $\Sigma \to \Pi$ and $\Pi \to \Sigma$ from S are the restrictions of the inversion in the sphere with center S and radius $\sqrt{2}$ to Σ and Π correspondingly.

From above and Theorem 15.3, it follows that the stereographic projection preserves the angles between arcs; more precisely *the absolute value of the angle measure* between arcs on the sphere.

This makes it particularly useful in cartography. A map of a big region of earth can not be done in the constant scale, but using stereographic projection, one can keep the angles between roads the same as on earth.

In the following exercises, we assume that Σ, Π, Ω, O, S and N are as above.

15.6. Exercise. *Show that the composition of stereographic projections from Π to Σ from S and from Σ to Π from N is the inversion of the plane Π in Ω.*

15.7. Exercise. *Show that stereographic projection $\Sigma \to \Pi$ sends the great circles to circlines on the plane which intersects Ω at two opposite points.*

15.8. Exercise. *Fix a point $P \in \Pi$ and let Q be yet an other point in Π. Denote by P' and Q' their stereographic projections to Σ. Set $x = PQ$ and $y = P'Q'_s$. Show that*

$$\lim_{x \to 0} \frac{y}{x} = \frac{2}{1 + OP^2}.$$

Compare with Lemma 12.8.

Central projection

Let Σ be the unit sphere centered at the origin which will be denoted as O. Denote by Π^+ the plane described by equation $z = 1$. This plane is parallel to xy-plane and it pass through the North Pole $N = (0, 0, 1)$ of Σ.

Recall that north hemisphere of Σ, is the subset of points $(x, y, z) \in$ $\in \Sigma$ such that $z > 0$. The north hemisphere will be denoted as Σ^+.

Given a point $P \in \Sigma^+$, consider half-line $[OP)$ and denote by P' the intersection of $[OP)$ and Π^+. Note that if $P = (x, y, z)$ then $P' = (\frac{x}{z}, \frac{y}{z}, 1)$. It follows that $P \mapsto P'$ is a bijection between Σ^+ and Π^+.

The described map $\Sigma^+ \to \Pi^+$ is called *central projection* of hemisphere Σ^+.

In spherical geometry, central projection is analogous to the projective model of hyperbolic plane which is discussed in Chapter 16.

Note that the central projection sends intersections of great circles with Σ^+ to the lines in Π^+. The later follows since great circles are formed by intersection of Σ with planes passing through the origin and the lines in Π^+ are formed by intersection of Π^+ with these planes.

15.9. Exercise. *Assume that N is the North Pole and $\triangle_s NBC$ has right angle at C and lies completely in the north hemisphere. Let $\triangle NB'C'$ be the image of $\triangle_s NBC$ under central projection.*

Observe that $\triangle NB'C'$ has right angle at C'.

Set

$$a = BC_s, \qquad b = CN_s, \qquad c = NB_s,$$
$$s = B'C', \qquad t = C'N, \qquad u = NB'.$$

Show that

$$s = \tfrac{\operatorname{tg} a}{\cos b}, \qquad t = \operatorname{tg} b, \qquad u = \operatorname{tg} c.$$

Use these identities together with the Euclidean Pythagorean theorem

$$u^2 = s^2 + t^2$$

for $\triangle NB'C'$ to prove spherical Pythagorean theorem

$$\cos c = \cos a \cdot \cos b$$

for $\triangle_s NBC$.

The following exercise is analogous to Exercise 16.4 in hyperbolic geometry.

15.10. Exercise. *Let $\triangle_s ABC$ be a nondegenerate spherical triangle. Assume that the plane Π^+ is parallel to the plane passing through A, B and C. Denote by A', B' and C' the central projections of A, B and C.*

(a) *Show that the midpoints of $[A'B']$, $[B'C']$ and $[C'A']$ are central projections of the midpoints of $[AB]_s$, $[BC]_s$ and $[CA]_s$ correspondingly.*

(b) *Use part (a) to show that medians of spherical triangle intersect at one point.*

Chapter 16

Projective model

The *projective model* is an other model of hyperbolic plane discovered by Beltrami; it is often called *Klein model*. The projective and conformal models are saying exactly the same thing but in two different languages. Some problems in hyperbolic geometry admit simpler proof using the projective model and others have simpler proof in the conformal model. Therefore it worth to know both.

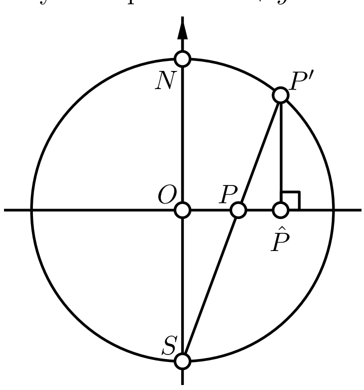
Special bijection of h-plane to itself

Consider the conformal disc model with absolute at the unit circle Ω centered at O. Choose a coordinate system (x, y) on the plane with origin at O, so the circle Ω is described by the equation $x^2 + y^2 = 1$.

Let us think of our plane Π as it lies in the Euclidean space as the xy-plane. Denote by Σ the unit sphere centered at O; it is described by the equation

$$x^2 + y^2 + z^2 = 1.$$

Set $S = (0, 0, -1)$ and $N = (0, 0, 1)$; these are the South and North Poles of Σ.



The plane through P, O and S.

Consider stereographic projection $\Pi \to \Sigma$ from S; given point $P \in \Pi$ denote its image in Σ as P'. Note that the h-plane is mapped to the North Hemisphere; that is, to the set of points (x, y, z) in Σ described by inequality $z > 0$.

For a point $P' \in \Sigma$ consider its foot point $\hat{P}$ on Π; this is the closest point on Π from P'.

The composition $P \mapsto \hat{P}$ of these two maps is a bijection of h-plane to itself.

Note that $P = \hat{P}$ if and only if $P \in \Omega$ or $P = O$.

16.1. Exercise. *Show that the map $P \mapsto \hat{P}$ described above can be described the following way: set $\hat{O} = O$ and for any other point point P take $\hat{P} \in [OP)$ such that*

$$O\hat{P} = \frac{2 \cdot x}{1 + x^2},$$

where $x = OP$.

16.2. Lemma. *Let $(PQ)_h$ be an h-line with the ideal points A and B. Then $\hat{P}, \hat{Q} \in [AB]$.*

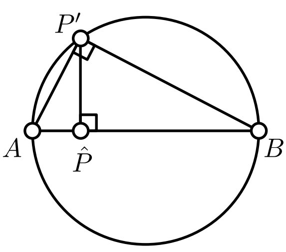
Moreover

❶ $$\frac{A\hat{Q} \cdot B\hat{P}}{\hat{Q}B \cdot \hat{P}A} = \left(\frac{AQ \cdot BP}{QB \cdot PA}\right)^2.$$

In particular, if A, P, Q, B appear on the line in the same order then

$$PQ_h = \tfrac{1}{2} \cdot \ln \frac{A\hat{Q} \cdot B\hat{P}}{\hat{Q}B \cdot \hat{P}A}.$$

Proof. Consider the stereographic projection $\Pi \to \Sigma$ from the South Pole. Denote by P' and Q' the images of P and Q.



The plane Λ.

According to Theorem 15.3(c),

❷ $$\frac{AQ \cdot BP}{QB \cdot PA} = \frac{AQ' \cdot BP'}{Q'B \cdot P'A}.$$

By Theorem 15.3(e), each circline in Π which is perpendicular to Ω is mapped to a circle in Σ which is still perpendicular to Ω. It follows that the stereographic projection sends $(PQ)_h$ to the intersection of the north hemisphere of Σ with a plane, say Λ, perpendicular to Π.

Consider the plane Λ. It contains points A, B, P', $\hat{P}$ and the circle $\Gamma = \Sigma \cap \Lambda$. (It also contains Q' and $\hat{Q}$ but we will not use these points for a while.)

Note that

- ◇ $A, B, P' \in \Gamma$,
- ◇ $[AB]$ is a diameter of Γ,
- ◇ $(AB) = \Pi \cap \Lambda$,
- ◇ $\hat{P} \in [AB]$
- ◇ $(P'\hat{P}) \perp (AB)$.

Since $[AB]$ is the diameter of Γ, by Corollary 8.6, the angle $\angle AP'B$ is right. Hence $\triangle A\hat{P}P' \sim \triangle AP'B \sim \triangle P'\hat{P}B$. In particular

$$\frac{AP'}{BP'} = \frac{A\hat{P}}{P'\hat{P}} = \frac{P'\hat{P}}{B\hat{P}}.$$

Therefore

❸ $$\frac{A\hat{P}}{B\hat{P}} = \left(\frac{AP'}{BP'}\right)^2.$$

The same way we get

❹ $$\frac{A\hat{Q}}{B\hat{Q}} = \left(\frac{AQ'}{BQ'}\right)^2.$$

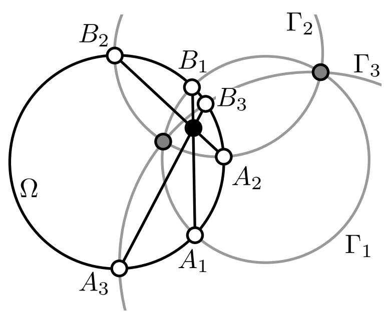
Finally note that ❷+❸+❹ imply ❶.

The last statement follows from ❶ and the definition of h-distance. Indeed,

$$\begin{aligned} PQ_h &\overset{\text{def}}{=\!=} \ln \frac{AQ\cdot BP}{QB\cdot PA} = \\ &= \ln \left(\frac{A\hat{Q}\cdot B\hat{P}}{\hat{Q}B\cdot \hat{P}A}\right)^{\frac{1}{2}} = \\ &= \tfrac{1}{2}\cdot \ln \frac{A\hat{Q}\cdot B\hat{P}}{\hat{Q}B\cdot \hat{P}A}. \end{aligned}$$

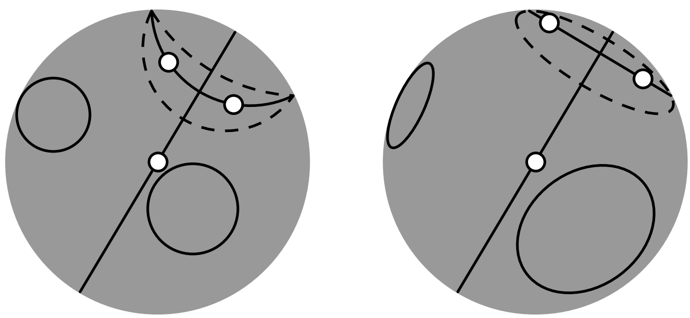
□

16.3. Exercise. *Let Γ_1, Γ_2 and Γ_3 be three circles perpendicular to the circle Ω. Let us denote by $[A_1B_1]$, $[A_2B_2]$ and $[A_3B_3]$ the common chords of Ω and Γ_1, Γ_2, Γ_3 correspondingly. Show that the chords $[A_1B_1]$, $[A_2B_2]$ and $[A_3B_3]$ intersect at one point inside Ω if and only if Γ_1, Γ_2 and Γ_3 intersect at two points.*



Projective model

The following picture illustrates the map $P \mapsto \hat{P}$ described in the previous section. If you take the picture on the left and apply the map $P \mapsto \hat{P}$, you get the picture on the right. The picture on the right gives a new way to look at the hyperbolic plane, which is called *projective model*. One may think of the map $P \mapsto \hat{P}$ as about translation from one language to the other.



Conformal model Projective model

In the projective model things look different; some become simpler, other things become more complicated.

Lines. The h-lines in the projective model are formed by chords. More precisely, they are formed by the intersections of chords of the absolute with the h-plane.

Circles and equidistants. The h-circles and equidistants in the projective model are formed by the certain type of ellipses and their arcs. It follows since the stereographic projection sends circles one the plane to the circles on the unit sphere and the foot point projection of circle back to plane is formed by ellipse. (One may define ellipse as the foot point projection of a circle which lies in the space to the plane.)

Distance. To find the h-distance between the points P and Q in the projective model, you have to find the points of intersection, say A and B, of the Euclidean line (PQ) with the absolute; then, by Lemma 16.2,

$$PQ_h = \tfrac{1}{2} \cdot \ln \frac{AQ \cdot BP}{QB \cdot PA},$$

assuming the points A, P, Q, R appear on the line in the same order.

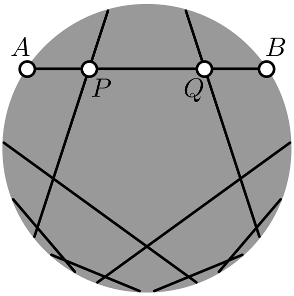
Angles. The angle measures in the projective model are very different from the Euclidean angles and it is hard to figure out by looking on the picture. For example all the intersecting h-lines on the picture are perpendicular. There are two useful exceptions

◇ If O is the center of absolute then

$$\measuredangle_h AOB = \measuredangle AOB.$$

◇ If O is the center of absolute and $\measuredangle OAB = \pm\frac{\pi}{2}$ then

$$\measuredangle_h OAB = \measuredangle OAB = \pm\tfrac{\pi}{2}.$$



To find the angle measure in the projective model, you may apply a motion of h-plane which moves the vertex of the angle to the center of absolute; once it is done the hyperbolic and Euclidean angles have the same measure.

Motions. The motions of h-plane in the conformal and projective models are relevant to inversive transformations and projective transformation in the same way. Namely:

◇ Any inversive transformations which preserve h-plane describe a motion of h-plane in the conformal model.

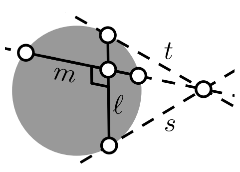
◇ Any projective transformation which preserve h-plane describes a motion in the projective model.

The following exercise is hyperbolic analog of Exercise 15.10. This is the first example of a statement which admits an easier proof using the projective model.

16.4. Exercise. *Let P and Q be the point in h-plane which lie on the same distance from the center of absolute. Observe that in the projective model, h-midpoint of $[PQ]_h$ coincides with the Euclidean midpoint of $[PQ]_h$.*

Conclude that if an h-triangle is inscribed in an h-circle then its medians intersect at one point.

Think how to prove the same for a general h-triangle.



16.5. Exercise. *Let ℓ and m are h-lines in the projective model. Denote by s and t the Euclidean lines tangent to the absolute at the ideal points of ℓ and denote by $\tilde{m}$ the Euclidean line which contains m. Show that if the lines s, t and $\tilde{m}$ meet at one point then ℓ and m are perpendicular h-lines in the projective model.*

16.6. Exercise. *Give a proof of Proposition 12.1 using projective model.*

16.7. Exercise. *Use projective model to find the inradius of the ideal triangle.*

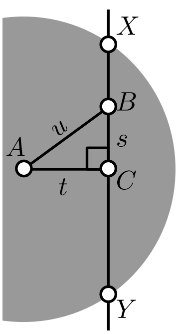
Hyperbolic Pythagorean theorem

16.8. Theorem. *Assume that $\triangle_h ACB$ is a triangle in h-plane with right angle at C. Set $a = BC_h$, $b = CB_h$ and $c = AB_h$. Then*

❺ $$\operatorname{ch} c = \operatorname{ch} a \cdot \operatorname{ch} b.$$

where ch *denotes* hyperbolic cosine*; that is, the function defined the following way*

$$\operatorname{ch} x \stackrel{\text{def}}{=\!=} \tfrac{e^x + e^{-x}}{2}.$$



Proof. We will use projective model of h-plane with a unit circle as the absolute.

We can assume that A is the center of absolute. Therefore both $\angle_h ACB$ and $\angle ACB$ are right.

Set $s = BC$, $t = CA$, $u = AB$. According to Euclidean Pythagorean theorem (6.10),

$$u^2 = s^2 + t^2.$$

Note that

$$b = \tfrac{1}{2} \cdot \ln \frac{1+t}{1-t};$$

therefore

❻ $$\operatorname{ch} b = \frac{\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} + \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}}}{2} = \\ = \frac{1}{\sqrt{1-t^2}}.$$

The same way we get

$$c = \tfrac{1}{2} \cdot \ln \frac{1+u}{1-u}$$

and

$$❼\qquad \operatorname{ch} c = \frac{\left(\frac{1+u}{1-u}\right)^{\frac12} + \left(\frac{1-u}{1+u}\right)^{\frac12}}{2} = \\ = \frac{1}{\sqrt{1-u^2}}.$$

Let X and Y are the ideal points of $(BC)_h$. Applying the Pythagorean theorem (6.10) again, we get

$$CX = CY = \sqrt{1-t^2}.$$

Therefore

$$a = \tfrac12 \cdot \ln \frac{\sqrt{1-t^2}+s}{\sqrt{1-t^2}-s}$$

and

$$❽\qquad \begin{aligned} \operatorname{ch} a &= \frac{\left(\frac{\sqrt{1-t^2}+s}{\sqrt{1-t^2}-s}\right)^{\frac12} + \left(\frac{\sqrt{1-t^2}-s}{\sqrt{1-t^2}+s}\right)^{\frac12}}{2} = \\ &= \frac{\sqrt{1-t^2}}{\sqrt{1-t^2-s^2}} \\ &= \frac{\sqrt{1-t^2}}{\sqrt{1-u^2}} \end{aligned}$$

Finally note that ❻+❼+❽ implies ❺. □

Bolyai's construction

Assume we need to construct a line asymptotically parallel to the given line through the given point. The initial configuration is given by three points, say P, A and B and we need to construct a line through P which is asymptotically parallel to $\ell = (AB)$.

Note that ideal points do not lie in the h-plane, so there is no way to use them in the construction.

The following construction was given by Bolyai. We assume that you know a compass-and-ruler construction of the perpendicular line to the given line through the given point; see the solution of Exercise 5.23.

16.9. Bolyai's construction.

1. *Construct the line m through P which perpendicular to ℓ. Denote by Q the foot point of P on ℓ.*
2. *Construct the line n through P which perpendicular to m.*

3. *Draw the circle Γ_1 with center Q through P and mark by R a point of intersection of Γ_1 with ℓ.*
4. *Construct the line k through R which perpendicular to n.*
5. *Draw the circle Γ_2 with center P through Q and mark by T a point of intersection of Γ_2 with k.*
6. *The line $(PT)_h$ is asymptotically parallel to ℓ.*

16.10. Exercise. *Explain what happens if one performs the Bolyai construction in the Euclidean plane.*

To prove that Bolyai's construction gives the asymptotically parallel line in h-plane, it is sufficient to show the following.

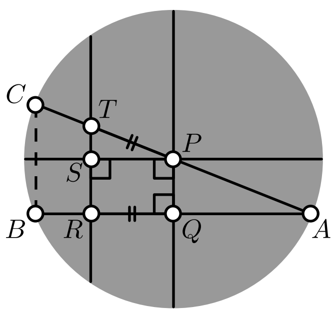
16.11. Proposition. *Assume P, Q, R, S, T be points in h-plane such that*

- ◇ $S \in (RT)_h$,
- ◇ $(PQ)_h \perp (QR)_h$,
- ◇ $(PS)_h \perp (PQ)_h$,
- ◇ $(RT)_h \perp (PS)_h$ *and*
- ◇ $(PT)_h$ *and* $(QR)_h$ *are asymptotically parallel.*

Then $QR_h = PT_h$.

Proof. We will use the projective model. Without loss of generality, we may assume that P is the center of absolute. As it was noted on page 129, in this case the corresponding Euclidean lines are also perpendicular; that is, $(PQ) \perp (QR)$, $(PS) \perp (PQ)$ and $(RT) \perp (PS)$.

Denote by A be the ideal point of $(QR)_h$ and $(PT)_h$. Denote by B and C the remaining ideal points of $(QR)_h$ and $(PT)_h$ correspondingly.



Note that the Euclidean lines (PQ), (TR) and (CB) are parallel.

Therefore

$$\triangle AQP \sim \triangle ART \sim \triangle ABC.$$

In particular,

$$\frac{AC}{AB} = \frac{AT}{AR} = \frac{AP}{AQ}.$$

It follows that

$$\frac{AT}{AR} = \frac{AP}{AQ} = \frac{BR}{CT} = \frac{BQ}{CP}.$$

In particular,

$$\frac{AT\cdot CP}{TC\cdot PA} = \frac{AR\cdot BQ}{RB\cdot QA};$$

hence $QR_h = PT_h$. □

Chapter 17

Complex coordinates

In this chapter we give an interpretation of inversive geometry using complex coordinates. The results of this chapter will not be used in this book, but they lead to deeper understanding of both concepts.

Complex numbers

Informally, a complex number is a number that can be put in the form

➊ $$z = x + i{\cdot}y,$$

where x and y are real numbers and $i^2 = -1$.

The set of complex numbers will be further denoted by $\mathbb{C}$. If x, y and z as in ➊, then x is called the real part and y the imaginary part of the complex number z. Briefly it is written as

$$x = \operatorname{Re} z \quad \text{and} \quad y = \operatorname{Im} z.$$

On the more formal level, a complex number is a pair of real numbers (x, y) with addition and multiplication described below. The formula $x + i{\cdot}y$ is only a convenient way to write the pair (x, y).

$$(x_1 + i{\cdot}y_1) + (x_2 + i{\cdot}y_2) \stackrel{\text{def}}{=\!=} (x_1 + x_2) + i{\cdot}(y_1 + y_2);$$

$$(x_1 + i{\cdot}y_1){\cdot}(x_2 + i{\cdot}y_2) \stackrel{\text{def}}{=\!=} (x_1{\cdot}x_2 - y_1{\cdot}y_2) + i{\cdot}(x_1{\cdot}y_2 + y_1{\cdot}x_2).$$

Complex coordinates

Recall that one can think of Euclidean plane as the set of all pairs of real numbers (x, y) equipped with the metric

$$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

where $A=(x_A,y_A)$ and $B=(x_B,y_B)$.

One can pack coordinates (x,y) of a point in the Euclidean plane, in one complex number $z=x+i{\cdot}y$. This way we get one-to-one correspondence between points of Euclidean plane and $\mathbb{C}$. Given a point $Z=(x,y)$, the complex number $z=x+i{\cdot}y$ is called *complex coordinate* of Z.

Note that if O, E and I are the points in the plane with complex coordinates 0, 1 and i then $\measuredangle EOI=\pm\frac{\pi}{2}$. Further we assume that $\measuredangle EOI=\frac{\pi}{2}$; if not, one has to change the direction of the y-coordinate.

Conjugation and absolute value

Let $z=x+i{\cdot}y$ and both x and y are real. Denote by Z the point in the plane with complex coordinate z.

If $y=0$, we say that the complex number z is *real* and if $x=0$ we say that z is *imaginary*. The set of points with real and imaginary complex coordinates form lines in the plane, which are called *real* and *imaginary* lines which will be denoted as $\mathbb{R}$ and $i{\cdot}\mathbb{R}$.

The complex number $\bar z=x-iy$ is called *complex conjugate* of z.

Note that the point $\bar Z$ with complex coordinate $\bar z$ is the reflection of Z in the real line.

It is straightforward to check that

$$x=\operatorname{Re}z=\frac{z+\bar z}{2},\quad y=\operatorname{Im}z=\frac{z-\bar z}{i{\cdot}2},\quad x^2+y^2=z{\cdot}\bar z. \tag{❷}$$

The last formula in ❷ makes possible to express the quotient $\frac{w}{z}$ of two complex numbers w and $z=x+i{\cdot}y$:

$$\frac{w}{z}=\tfrac{1}{z\cdot\bar z}{\cdot}w{\cdot}\bar z=\tfrac{1}{x^2+y^2}{\cdot}w{\cdot}\bar z.$$

Note that

$$\overline{z+w}=\bar z+\bar w,\quad \overline{z-w}=\bar z-\bar w,\quad \overline{z{\cdot}w}=\bar z{\cdot}\bar w,\quad \overline{z/w}=\bar z/\bar w;$$

that is, complex conjugation *respects* all the arithmetic operations.

The value

$$\begin{aligned}|z|&=\sqrt{x^2+y^2}=\\&=\sqrt{z{\cdot}\bar z}\end{aligned}$$

is called *absolute value* of z. If $|z|=1$ then z is called *unit complex number*.

Note that if Z and W are points in the Euclidean plane and z and w their complex coordinates then

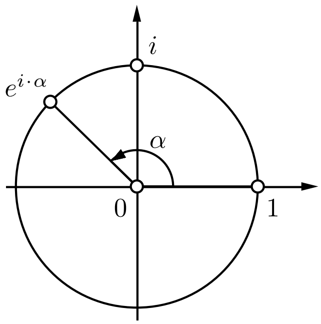
$$ZW=|z-w|.$$

Euler's formula

Let α be a real number. The following identity is called *Euler's formula.*

$$❸ \qquad e^{i\cdot\alpha} = \cos\alpha + i\cdot\sin\alpha.$$

In particular, $e^{i\cdot\pi} = -1$ and $e^{i\cdot\frac{\pi}{2}} = i$.



Geometrically Euler's formula means the following. Assume that O and E are the point with complex coordinates 0 and 1 correspondingly. Assume $OZ = 1$ and $\measuredangle EOZ \equiv \alpha$ then $e^{i\cdot\alpha}$ is the complex coordinate of Z. In particular, the complex coordinate of any point on the unit circle centered at O can be uniquely expressed as $e^{i\cdot\alpha}$ for some $\alpha \in (-\pi, \pi]$.

Why should you think that ❸ is true? The proof of Euler's identity depends on the way you define exponent. If you never had to take exponent of imaginary number, you may take the right hand side in ❸ as the definition of the $e^{i\cdot\alpha}$.

In this case formally nothing has to be proved, but it is better to check that $e^{i\cdot\alpha}$ the satisfies familiar identities. Mainly,

$$e^{i\cdot\alpha}\cdot e^{i\cdot\beta} = e^{i\cdot(\alpha+\beta)}.$$

The later can be proved using the following trigonometric formulas, which we assume to be known:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta\\ \sin(\alpha+\beta) &= \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta\end{aligned}$$

If you know power series for sine, cosine and exponent, the following might also convince that ❸ is the right definition.

$$\begin{aligned}e^{i\cdot\alpha} &= 1 + i\cdot\alpha + \frac{(i\cdot\alpha)^2}{2!} + \frac{(i\cdot\alpha)^3}{3!} + \frac{(i\cdot\alpha)^4}{4!} + \frac{(i\cdot\alpha)^5}{5!} + \dots = \\ &= 1 + i\cdot\alpha - \frac{\alpha^2}{2!} - i\cdot\frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + i\cdot\frac{\alpha^5}{5!} - \dots = \\ &= \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \dots\right) + i\cdot\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \dots\right) = \\ &= \cos\alpha + i\cdot\sin\alpha.\end{aligned}$$

Argument and polar coordinates

As above, assume that O and E denote the points with complex coordinates 0 and 1 correspondingly.

Let Z be the point distinct form O. Set $\rho = OZ$ and $\vartheta = \measuredangle EOZ$. The pair (ρ, ϑ) is called *polar coordinates* of Z.

If z is the complex coordinate of Z then then $\rho = |z|$. The value ϑ is called argument of z (briefly, $\vartheta = \arg z$). In this case

$$z = \rho{\cdot}e^{i\cdot\vartheta} = \rho{\cdot}(\cos\vartheta + i\cdot\sin\vartheta).$$

Note that

$$\arg(z{\cdot}w) \equiv \arg z + \arg w \quad \text{and} \quad \arg\tfrac{z}{w} \equiv \arg z - \arg w$$

if $z, w \neq 0$. In particular, if Z, V, W be points with complex coordinates z, v and w correspondingly then

❹
$$\begin{aligned}\measuredangle VZW &= \arg\left(\frac{w-z}{v-z}\right) \equiv \\ &\equiv \arg(w-z) - \arg(v-z)\end{aligned}$$

once the left hand side is defined.

17.1. Exercise. *Use the formula ❹ to show that*

$$\measuredangle ZVW + \measuredangle VWZ + \measuredangle WZV \equiv \pi$$

for any $\triangle ZVW$ in the Euclidean plane.

17.2. Exercise. *Assume that points V, W and Z have complex coordinates v, w and $v{\cdot}w$ correspondingly and the point O and E as above. Show that*

$$\triangle OEV \sim \triangle OWZ.$$

The following Theorem is a reformulation of Theorem 8.11 which use complex coordinates.

17.3. Theorem. *Let $\square UVWZ$ be a quadrilateral and u, v, w and z be the complex coordinates of its vertices. Then $\square UVWZ$ is inscribed if and only if the number*

$$\frac{(v-u){\cdot}(z-w)}{(v-w){\cdot}(z-u)}$$

is real.

The value $\frac{(v-u)\cdot(w-z)}{(v-w)\cdot(z-u)}$ is called *complex cross-ratio*, it will be discussed in more details below.

17.4. Exercise. *Observe that the complex number $z \neq 0$ is real if and only if $\arg z = 0$ or π; in other words, $2\cdot \arg z \equiv 0$.*

Use this observation to show that Theorem 17.3 is indeed a reformulation of Theorem 8.11.

Möbius transformations

17.5. Exercise. *Watch video "Möbius Transformations Revealed" by Douglas Arnold and Jonathan Rogness. (It is 3 minutes long and available on YouTube.)*

The complex plane $\mathbb{C}$ extended by one ideal number ∞ is called extended complex plane. It is denoted by $\hat{\mathbb{C}}$, so $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Möbius transformation of $\hat{\mathbb{C}}$ is a function of one complex variable z which can be written as

$$f(z) = \frac{a\cdot z + b}{c\cdot z + d},$$

where the coefficients a, b, c, d are complex numbers satisfying $a\cdot d - b\cdot c \neq 0$. (If $a\cdot d - b\cdot c = 0$ the function defined above is a constant and is not considered to be a Möbius transformation.)

In case $c \neq 0$, we assume that

$$f(-d/c) = \infty \quad \text{and} \quad f(\infty) = a/c;$$

and if $c = 0$ we assume

$$f(\infty) = \infty.$$

Elementary transformations

The following three types of Möbius transformations are called *elementary.*

1. $z \mapsto z + w$,
2. $z \mapsto w\cdot z$ for $w \neq 0$,
3. $z \mapsto \frac{1}{z}$.

The geometric interpretations. As before we will denote by O the point with complex coordinate 0.

The first map $z \mapsto z + w$, corresponds to so called *parallel translation* of Euclidean plane, its geometric meaning should be evident.

The second map is called *rotational homothety* with center at O. That is, the point O maps to itself and any other point Z maps to a point Z' such that $OZ' = |w|\cdot OZ$ and $\measuredangle ZOZ' = \arg w$.

The third map can be described as a composition of inversion in the unit circle centered at O and the reflection in $\mathbb{R}$ (the composition can be taken in any order). Indeed, $\arg z \equiv -\arg \frac{1}{z}$ therefore

$$\arg z = \arg(1/\bar z);$$

that is, if the points Z and Z' have complex coordinates z and $1/\bar z$ then $Z' \in [OZ)$. Clearly $OZ = |z|$ and $OZ' = |1/\bar z| = \frac{1}{|z|}$. Therefore Z' is inversion of Z in the unit circle centered at O. Finally the reflection of Z' in $\mathbb{R}$, has complex coordinate $\frac{1}{z} = \overline{(1/\bar z)}$.

17.6. Proposition. *A map $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation if and only if it can be expressed as a composition of elementary Möbius transformation.*

Proof; ($\Rightarrow$). Consider, the Möbius transformation

$$f(z) = \frac{a\cdot z + b}{c\cdot z + d}.$$

It is straightforward to check that

❺ $$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z),$$

where

- ◇ $f_1(z) = z + \frac{d}{c}$,
- ◇ $f_2(z) = \frac{1}{z}$,
- ◇ $f_3(z) = -\frac{a\cdot d - b\cdot c}{c^2}\cdot z$,
- ◇ $f_4(z) = z + \frac{a}{c}$

if $c \ne 0$ and

- ◇ $f_1(z) = \frac{a}{d}\cdot z$,
- ◇ $f_2(z) = z + \frac{b}{d}$,
- ◇ $f_3(z) = f_4(z) = z$

if $c = 0$.

($\Leftarrow$). We need to show that composing elementary transformations, we can only get Möbius transformations. Note that it is sufficient to check that composition of a Möbius transformations

$$f(z) = \frac{a\cdot z + b}{c\cdot z + d}.$$

with any elementary transformation is a Möbius transformations.

The later is done by means of direct calculations.

$$\begin{aligned}\frac{a\cdot(z+w)+b}{c\cdot(z+w)+d}&=\frac{a\cdot z+(b+a\cdot w)}{c\cdot z+(d+c\cdot w)},\\ \frac{a\cdot(w\cdot z)+b}{c\cdot(w\cdot z)+d}&=\frac{(a\cdot w)\cdot z+b}{(c\cdot w)\cdot z+d},\\ \frac{a\cdot\frac{1}{z}+b}{c\cdot\frac{1}{z}+d}&=\frac{b\cdot z+a}{d\cdot z+c}.\end{aligned}$$

□

17.7. Corollary. *The image of circline under Möbius transformation is a circline.*

Proof. By Proposition 17.6, it is sufficient to check that each elementary transformation sends circline to circline.

For the first and second elementary transformation the later is evident.

As it was noted above, the map $z\mapsto\frac{1}{z}$ is a composition of inversion and reflection. By Theorem 9.11, inversion sends circline to circline. Hence the result follows. □

17.8. Exercise. *Show that inverse of Möbius transformation is a Möbius transformation.*

17.9. Exercise. *Given distinct values $z_0, z_1, z_\infty\in\hat{\mathbb{C}}$, construct a Möbius transformation f such that $f(z_0)=0$, $f(z_1)=1$ and $f(z_\infty)=$ $=\infty$. Show that such transformation is unique.*

17.10. Exercise. *Show that any inversion is complex conjugate to a Möbius transformation.*

Complex cross-ratio

Given four distinct complex numbers u, v, w, z, the complex number

$$\frac{(u-w)\cdot(v-z)}{(v-w)\cdot(u-z)}$$

is called *complex cross-ratio*; it will be denoted as $(u,v;w,z)$.

If one of the numbers u,v,w,z, is ∞, then the complex cross-ratio has to be defined by taking the appropriate limit; in other words, we assume that $\frac{\infty}{\infty}=1$. For example,

$$(u,v;w,\infty)=\frac{(u-w)}{(v-w)}.$$

Assume that U, V, W and Z be the points with complex coordinates u, v, w and z correspondingly. Note that

$$\begin{aligned}\frac{UW\cdot VZ}{VW\cdot UZ} &= |(u,v;w,z)|,\\ \measuredangle WUZ + \measuredangle ZVW &= \arg\frac{u-w}{u-z} + \arg\frac{v-z}{v-w} \equiv\\ &\equiv \arg(u,v;w,z).\end{aligned}$$

It makes possible to reformulate Theorem 9.6 using the complex coordinates the following way.

17.11. Theorem. *Let $\square UWVZ$ and $\square U'W'V'Z'$ be two quadrilaterals such that the points U', W', V' and Z' are inversions of U, W, V, and Z correspondingly. Assume u, w, v, z, u', w', v' and z' be the complex coordinates of U, W, V, Z, U', W', V' and Z' correspondingly.*

Then

$$(u',v';w',z') = \overline{(u,v;w,z)}.$$

The following Exercise is a generalization of the Theorem above. It admits a short and simple solution which use Proposition 17.6.

17.12. Exercise. *Show that complex cross-ratios are invariant under Möbius transformations. That is, if a Möbius transformation maps four distinct complex numbers u, v, w, z to complex numbers u', v', w', z' respectively, then*

$$(u',v';w',z') = (u,v;w,z).$$

Schwarz–Pick theorem

The following theorem shows that the metric in the conformal disc model naturally appears in other branches of mathematics. We do not give a proof, but it can be found in any textbook on geometric complex analysis.

Let us denote by $\mathbb{D}$ the unit disc in the complex plane centered at 0; that is, a complex number z belongs to $\mathbb{D}$ if and only if $|z| < 1$.

Let us use the disc $\mathbb{D}$ as h-plane in the conformal disc model; the h-distance between $z, w \in \mathbb{D}$ will be denoted as $d_h(z,w)$.

A function $f\colon \mathbb{D} \to \mathbb{C}$ is called *holomorphic* if for every $z \in \mathbb{D}$ there is a complex number s such that

$$f(z+w) = f(z) + s\cdot w + o(|w|).$$

In other words, f is *complex-differentiable* at any $z \in \mathbb{D}$. The number s above is called derivative of f at z and denoted as $f'(z)$.

17.13. Schwarz–Pick theorem. *Assume $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic function. Then*

$$d_h(f(z), f(w)) \leqslant d_h(z, w)$$

for any $z, w \in \mathbb{D}$.

Moreover if equality holds for one pair of distinct numbers $z, w \in \mathbb{D}$ then it holds for any pair and $f\colon \mathbb{D} \to \mathbb{D}$ is a motion of the h-plane.

17.14. Exercise. *Show that Schwarz lemma stated below follows from Schwarz–Pick theorem.*

17.15. Schwarz lemma. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function and $f(0) = 0$. Then $|f(z)| \leqslant |z|$ for any $z \in \mathbb{D}$.*

Moreover if equality holds for some $z \neq 0$ then there is a unit complex number u such that $f(z) = u{\cdot}z$ for any $z \in \mathbb{D}$.

Chapter 18

Geometric constructions

Geometric constructions has great pedagogical value as an introduction to mathematical proofs. We were using construction problems everywhere starting from Chapter 5.

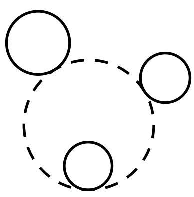
In this chapter we briefly discuss the classical results in geometric constructions.

Classical problems

In this section we list couple of classical construction problems; each known for more than thousand years. The solutions of the following two problems are quite nontrivial.

18.1. Brahmagupta's problem. *Construct an inscribed quadrilateral with given sides.*

18.2. Apollonius's problem. *Construct a circle which is tangent to three given circles in a plane.*



The following exercise is a simplified version of Apollonius's problem, which is still nontrivial.

18.3. Exercise. *Construct a circle which pass through the given point and tangent to two intersecting lines.*

The following three problems can not be solved in principle; that is, the needed compass-and-ruler construction does not exist.

Doubling the cube. *Construct the side of a new cube, which has volume twice bigger than the volume of the given cube.*

In other words, given a segment of length a one needs to construct a segment of length $\sqrt[3]{2}\cdot a$.

Squaring the circle. *Construct a square with the same area as the given circle.*

If r is the radius of the given circle, we need to construct a segment of length $\sqrt{\pi}\cdot r$.

Angle trisection. *Divide the given angle into three equal angles.*

In fact there is no compass-and-ruler construction which trisects angle with measure $\frac{\pi}{3}$. Existence of such construction would imply constructability of regular 9-gon which is prohibited by the following famous result.

18.4. Gauss–Wantzel theorem. *A regular n-gon is constructable with ruler and compass if and only if n is the product of a power of* 2 *and any number of distinct Fermat primes.*

The *Fermat prime* is a prime numbers of the form 2^k+1 for some integer k. Only five Fermat primes are known today:

$$3, 5, 17, 257, 65537.$$

For example,

- ◇ one can construct a regular 340-gon since $340 = 2^2\cdot 5\cdot 17$ and 5 as well as 17 are Fermat primes;
- ◇ one can not construct a regular 7-gon since 7 is not a Fermat prime;
- ◇ one can not construct a regular 9-gon; although $9 = 3\cdot 3$ is a product of two Fermat primes, these primes are not distinct.

The impossibility of these constructions was proved only in 19th century. The method used in the proofs is indicated in the next section.

Constructable numbers

In the classical compass-and-ruler constructions initial configuration can be completely described by a finite number of points; each line is defined by two points on it and each circle is described by its center and a point on it (equivalently, you may describe a circle by tree points on it).

The same way the result of construction can be described by a finite collection of points.

Choose a coordinate system, such that one of the initial points as the origin $(0,0)$ and yet an other initial point has coordinates $(1,0)$. In this coordinate system, the initial configuration of n points is described by $2\cdot n-4$ numbers, their coordinates, say $(x_3, y_3, x_4, y_4, \dots, x_n, y_n)$.

It turns out that the coordinates of any point constructed with compass and ruler can be written through the numbers $x_3, y_3, x_4, y_4, \dots$ $\dots, x_n, y_n$ using the four arithmetic operations "$+$", "$-$", "$\cdot$", "$/$" and the square root "$\sqrt{\ }$".

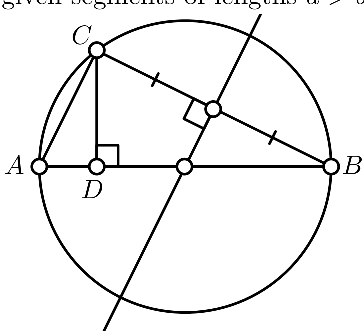
For example, assume we want to find the points $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ of intersection of a line passing through $A = (x_A, y_A)$ and $B = (x_B, y_B)$ and the circle with center $O = (x_O, y_O)$ which pass through a point $W = (x_W, y_W)$. Let us write the equations of these circle and line in the coordinates (x, y):

$$\left[\begin{array}{c}(x-x_O)^2+(y-y_O)^2=(x_W-x_O)^2+(y_W-y_O)^2\\ (x-x_A)\cdot(y_B-y_A)=(y-y_A)\cdot(x_B-x_A)\end{array}\right.$$

Expressing y from the second equation and substituting the result in the first one, gives us a quadratic equation in x, which can be solved using "$+$", "$-$", "$\cdot$", "$/$" and "$\sqrt{\ }$" only.

The same can be performed for the intersection of two circles. The intersection of two lines is even simpler; it described as a solution of two linear equations and can be expressed using only four arithmetic operations; the square root "$\sqrt{\ }$" is not needed.

On the other hand, it is easy to produce compass-and-ruler constructions which produce a segment of length $a+b$ and $a-b$ from two given segments of lengths $a > b$.



To perform "$\cdot$", "$/$" and "$\sqrt{\ }$" consider the following diagram. Let $[AB]$ be diameter of a circle; Consider point C on the circle and let D be the foot point of C on $[AB]$. Note that

$$\triangle ABC \sim \triangle ACD \sim \triangle BDC.$$

It follows that $AD\cdot DC = BD^2$.

Using this diagram, one should guess the compass-and-ruler constructions which produce a segments of lengths $\sqrt{a\cdot b}$ and $\frac{a^2}{b}$. Say to construct $\sqrt{a\cdot b}$ one can do the following: (1) construct points A, B and $D \in [AB]$ such that $AD = a$ and $BD = b$ (2) construct a circle Γ with diameter $[AB]$ (3) draw the perpendicular ℓ through D to (AB) (4) let C be an intersection of Γ and ℓ. Then $DC = \sqrt{a\cdot b}$.

Taking 1 for a or b above we can produce $\sqrt{a}$, a^2, $\frac{1}{b}$. Further combining these constructions we can produce $a\cdot b = (\sqrt{a\cdot b})^2$, $\frac{a}{b} = a\cdot\frac{1}{b}$.

In other words we produced a *compass-and-ruler calculator*, which can do "$+$", "$-$", "$\cdot$", "$/$" and "$\sqrt{\ }$".

The discussion above gives a rough sketch of the proof of the following theorem:

18.5. Theorem. *Assume that initial configuration of geometric construction is given by points $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (x_3, y_3), \dots$ $\dots, A_n = (x_n, y_n)$. Then a point $X = (x, y)$ can be constructed using a compass-and-ruler construction if and only if both coordinates x and y can be expressed from the integer numbers and $x_3, y_3, x_4, y_4, \dots, x_n, y_n$ using the arithmetic operations "$+$", "$-$", "$\cdot$", "$/$" and the square root "$\sqrt{\ }$".*

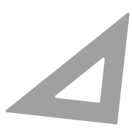
The numbers which can be expressed from the given numbers using the arithmetic operations and the square root "$\sqrt{\ }$" are called *constructable*; if the list of given numbers is not given then we can only use the integers. The theorem above translates any compass-and-ruler construction problem into purely algebraic language — we need to decide if the given number is constructable. For example:

- ◇ Impossibility of solution for doubling cube problem states that $\sqrt[3]{2}$ is not a constructable number. That is $\sqrt[3]{2}$ can not be expressed through integers using "$+$", "$-$", "$\cdot$", "$/$" and "$\sqrt{\ }$".
- ◇ Impossibility of solution for squaring the circle states that $\sqrt{\pi}$, or equivalently π, is not a constructable number.
- ◇ Gauss–Wantzel theorem tells for which integers n the number $\cos \frac{2\cdot\pi}{n}$ is constructable.

Some of these statements might look evident, but a rigorous proofs require some knowledge of field theory which is out of scope of this book.

In the next section we discuss a similar but simpler examples of impossible constructions with an unusual tool.

Constructions with set square



Set square is a construction tool which can produce a line through the given point which makes angle $\frac{\pi}{2}$ or $\pm\frac{\pi}{4}$ with the given line.

18.6. Exercise. *Trisect the given segment with ruler and set square.*

Consider ruler-and-set-square construction. Using the same idea as in the previous section, we can define *ruler-and-set-square constructable numbers* and prove the following analog of Theorem 18.5.

18.7. Theorem. *Assume that initial configuration of geometric construction is given by points* $A_1 = (0, 0)$*,* $A_2 = (1, 0)$*,* $A_3 = (x_3, y_3), \dots$ $\dots, A_n = (x_n, y_n)$*. Then a point* $X = (x, y)$ *can be constructed using a ruler-and-set-square construction if and only if both coordinates* x *and* y *can be expressed from the integer numbers and* $x_3, y_3, x_4, y_4, \dots, x_n, y_n$ *using the arithmetic operations "+", "−", "·", "/".*

We omit the proof of this theorem, but it can be build on the ideas described in the previous section. Let us show how to use this theorem to show impossibility of some constructions with ruler and set square.

Note that if all the coordinates $x_3, y_3, \dots, x_n, y_n$ are rational numbers then the theorem above implies that with ruler and set square one can only construct the points with rational coordinates. A point with both rational coordinates is called *rational*, and if at least one of the coordinates is irrational, then the point is called *irrational.*

18.8. Exercise. *Show that an equilateral triangle in Euclidean plane has at least one irrational point.*

Conclude that with ruler and set square one can not construct an equilateral triangle.

18.9. Exercise. *Make a ruler-and-set-square construction which verifies if the given triangle is equilateral. (We assume that we can "verify" if two constructed points coincide.)*

More impossible constructions

In this section we discuss yet an other source of impossible constructions.

Recall that a *circumtool* produces a circle passing through given three points or a line if all three points lie on one line. Let us restate Exercise 9.9.

Exercise. *Show that with circumtool only, it is impossible to construct the center of given circle* Γ*.*

Remark. In geometric constructions we allow to choose some free points, say any point on the plane, or a point on a constructed line, or a point which does not lie on a constructed line and so on.

In principle, when you make such a free choice it is possible to mark the center of Γ by an accident. Nether the less, we do not accept such coincidence as true construction; we say that a construction produce the center if it produce it for any free choices.

Solution. Arguing by contradiction, assume we have a construction of the center.

Apply to whole construction an inversion in a circle perpendicular to Γ. According to Corollary 9.15, the circle Γ maps to itself. Since inversion sends circline to circline, we get that all construction is mapped to an equivalent construction; that is, a constriction with different choice of free points.

According to Exercise 9.8, the inversion sends the center of Γ to an other point. That is, following the same construction, we can end up at a different point, a contradiction. $\square$

18.10. Exercise. *Show that there is no circumtool-only construction which verifies if the given point is the center of given circle. (We assume that we can only "verify" if two constructed points coincide.)*

A similar example of impossible constructions for ruler and parallel tool is given in Exercise 13.6.

Let us discuss yet an other example for ruler-only construction. Note that ruler-only constructions are invariant with respect to the projective transformations. In particular, to solve the following exercise, it is sufficient to construct a projective transformation which fix two points A and B and moves its midpoint.

18.11. Exercise. *Show that midpoint of given segment can not be constructed with ruler only.*

The following theorem is a stronger version of the exercise above.

18.12. Theorem. *The center of given circle can not be constructed with ruler only.*

Proof. To prove this theorem it is sufficient to construct a projective transformation which sends the given circle Γ to a circle, say Γ' but the center of Γ' is not the image of center of Γ.

Let Γ be a circle which lies in the plane Π in the Euclidean space.

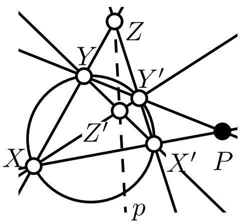
By Theorem 15.3, inversion of circle in a sphere is a circle or a line. Fix a sphere Σ with center O so that the inversion Γ' of Γ is a circle and the plane Π' containing Γ' is not parallel to Π; any sphere Σ in general position will do.

Denote by Z and Z' the centers of Γ and Γ'. Note that $Z' \notin (OZ)$. It follows that the perspective projection $\Pi \to \Pi'$ with center at O sends Γ to Γ', but Z' is not the image of Z. $\square$

Construction of polar

Assume Γ is a circle in the plane. Given point P consider two chords $[XX']$ and $[YY']$ of Γ such that $(XX')\cap(YY')=P$. Let $Z=(XY)\cap(X'Y')$ and $Z'=(XY')\cap(X'Y)$ and $p=(ZZ')$. If $P\in\Gamma$ we assume that p is the tangent to Γ at P.

18.13. Claim. *The line $p=(ZZ')$ does not depend on the choice of chords. Moreover $P\mapsto p$ is a duality (see page 115).*



The line p is called polar of P with respect to Γ. The same way the point P is called polar of the line p with respect to Γ.

We will not give a proof of this claim, but will try to use it in the constructions.

18.14. Exercise. *Let p be the polar line of point P with respect to the circle Γ. Assume that p intersects Γ at points V and W. Show that the lines (PV) and (PW) are tangent to Γ.*

Come up with a ruler-only construction of the tangent lines to the given circle through the given point.

18.15. Exercise. *Assume two concentric circles Γ and Γ' are given. Construct the common center of Γ and Γ' with ruler only.*

Chapter 19

Area

Any rigorous introduction to area is tedious. On the other hand, there is a more general notion called *Lebesgue measure*. Historically, the notion of area inspire the development of Lebesgue measure; but we will use Lebesgue measure to define of area.

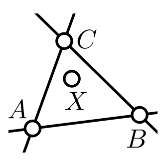
The construction of Lebesgue measure typically use the method of coordinates and it is included in any textbook in Real Analysis. The tedious part based of the properties of Lebesgue measure is packed in proof Theorem 19.7 and it is given without proof.

Solid triangles

We say that a point X lies *inside* a nondegenerate triangle $\triangle ABC$ if the following three condition hold:

◇ A and X lie on the same side from the line (BC);
◇ B and X lie on the same side from the line (CA);
◇ C and X lie on the same side from the line (AB).

The set of all points inside $\triangle ABC$ and on its sides $[AB]$, $[BC]$, $[CA]$ will be called *solid triangle* and denoted by $\blacktriangle ABC$.



19.1. Exercise. *Show that solid triangle is* convex*; that is if $X, Y \in \blacktriangle ABC$ then $[XY] \subset \blacktriangle ABC$.*

The notations $\triangle ABC$ and $\blacktriangle ABC$ look similar, they also have close but different meanings, which better not to be confused. Recall that $\triangle ABC$ is an ordered triple of distinct points (see page 17), while $\blacktriangle ABC$ is an infinite set of points.

In particular $\blacktriangle ABC = \blacktriangle BAC$; indeed any point which belong to the set $\blacktriangle ABC$ also belongs to the set $\blacktriangle BAC$ and the other way around. On the other hand, $\triangle ABC \neq \triangle BAC$ simply because the sequence of points (A, B, C) is distinct from the sequence (B, A, C).

Also, in general $\triangle ABC \ncong \triangle BAC$, but it is always true that $\blacktriangle ABC \cong \blacktriangle BAC$, where congruence of the sets $\blacktriangle ABC$ and $\blacktriangle BAC$ is understood the following way.

19.2. Definition. *Two sets $\mathcal{S}$ and $\mathcal{T}$ in the plane are called* congruent *(briefly $\mathcal{S} \cong \mathcal{T}$) if $\mathcal{T} = f(\mathcal{S})$ for some motion f of the plane.*

If $\triangle ABC$ is not degenerate and

$$\blacktriangle ABC \cong \blacktriangle A'B'C',$$

then after relabeling the vertices of $\triangle ABC$ we will have

$$\triangle ABC \cong \triangle A'B'C'.$$

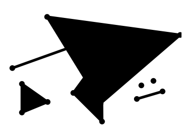
The existence of such relabeling follow from the exercise.

19.3. Exercise. *Let $\triangle ABC$ be nondegenerate and $X \in \blacktriangle ABC$. Show that X is a vertex of $\triangle ABC$ if and only if there is a line ℓ which intersects $\blacktriangle ABC$ at the single point X.*

Polygonal sets

Elementary set on the plane is a set of one of the following three types:

◇ one-point set;
◇ segment;
◇ solid triangle.



A set in the plane is called *polygonal* if it can be presented as a union of finite collection of elementary sets.

According to this definition, empty set $\varnothing$ is a polygonal set. Indeed, $\varnothing$ is a union of an empty collection of elementary sets.

A polygonal set is called *degenerate* if it can be presented as union of finite number of one-point sets and segments.

If X and Y lie on the opposite sides of the line (AB) then the union $\blacktriangle AXB \cup \blacktriangle BYA$ is a polygonal set which is called *solid quadrilateral* and denoted as $\blacksquare AXBY$. In particular, we can talk about *solid parallelograms*, *rectangles* and *squares* in the Euclidean plane.

Typically a polygonal set admits many presentation as union of finite collection of elementary sets. For example, if $\square AXBY$ is a parallelogram, then

$$\blacksquare AXBY = \blacktriangle AXB \cup \blacktriangle BYA = \blacktriangle XAY \cup \blacktriangle YBX.$$

19.4. Exercise. *Show that a solid square is not degenerate.*

19.5. Exercise. *Show that a circle is not a polygonal set.*

Definition of area

19.6. Claim. *For any two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$, the union $\mathcal{P}\cup\mathcal{Q}$ as well as the intersection $\mathcal{P}\cap\mathcal{Q}$ are also polygonal sets.*

A class of sets which closed with respect to union and intersection is called a *ring*. The claim above therefore states that polygonal sets in the plane form a ring.

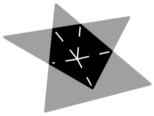
Semi-proof. Let us present $\mathcal{P}$ and $\mathcal{Q}$ as a union of finite collection of elementary sets $\mathcal{P}_1,\dots,\mathcal{P}_k$ and $\mathcal{Q}_1,\dots,\mathcal{Q}_n$ correspondingly.

Note that

$$\mathcal{P}\cup\mathcal{Q}=\mathcal{P}_1\cup\dots\cup\mathcal{P}_k\cup\mathcal{Q}_1\cup\dots\cup\mathcal{Q}_n.$$

Therefore $\mathcal{P}\cup\mathcal{Q}$ is polygonal.

Note that the union of all sets $\mathcal{P}_i\cap\mathcal{Q}_j$ forms $\mathcal{P}\cap\mathcal{Q}$.



Therefore in order to show that $\mathcal{P}\cap\mathcal{Q}$ is polygonal, it is sufficient to show that each $\mathcal{P}_i\cap\mathcal{Q}_j$ is polygonal for any pair i,j. The diagram should suggest an idea for the proof of the later statement. □

The following theorem defines the area as a function which returns a real number for any polygonal set and satisfying certain conditions. We omit the proof of this theorem. It follows from the construction of Lebesgue measure which can be found in any text book on Real Analysis.

19.7. Theorem. *For each polygonal set $\mathcal{P}$ in the Euclidean plane there is a real number s called* area of $\mathcal{P}$ *(briefly $s=\operatorname{area}\mathcal{P}$) such that*

$$\operatorname{area}\varnothing=0\quad\textit{and}\quad\operatorname{area}\mathcal{K}=1$$

where $\mathcal{K}$ the solid square with unit side and the conditions

$$\begin{gathered}\mathcal{P}\cong\mathcal{Q}\quad\Rightarrow\quad\operatorname{area}\mathcal{P}=\operatorname{area}\mathcal{Q};\\ \mathcal{P}\subset\mathcal{Q}\quad\Rightarrow\quad\operatorname{area}\mathcal{P}\leqslant\operatorname{area}\mathcal{Q};\\ \operatorname{area}\mathcal{P}+\operatorname{area}\mathcal{Q}=\operatorname{area}(\mathcal{P}\cup\mathcal{Q})+\operatorname{area}(\mathcal{P}\cap\mathcal{Q})\end{gathered}$$

hold for any two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$.

Moreover the area function

$$\mathcal{P} \mapsto \operatorname{area} \mathcal{P}$$

is uniquely defined by the above conditions.

Further you will see that based on this theorem, the concept of area can be painlessly developed to a dissent level without cheating.

It is also psossible to add Theorem 19.7 as an extra axiom, then the rest of the section becomes completely rigorous.

19.8. Proposition. *For any polygonal set $\mathcal{P}$ in the Euclidean plane, we have*

$$\operatorname{area} \mathcal{P} \geqslant 0.$$

Proof. Since $\varnothing \subset \mathcal{P}$, we get

$$\operatorname{area} \varnothing \leqslant \operatorname{area} \mathcal{P}.$$

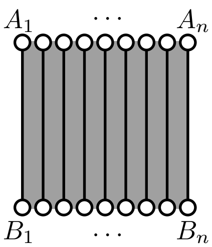
Since $\operatorname{area} \varnothing = 0$ the result follows. □

Vanishing area and subdivisions

19.9. Proposition. *Any one-point set as well as any segment in the Euclidean plane have vanishing area.*

Proof. Fix a line segment $[AB]$. Consider a sold square $\blacksquare ABCD$.

Note that given a positive integer n, there are n disjoint segments $[A_1B_1], \dots, [A_nB_n]$ in $\blacksquare ABCD$, such that each $[A_iB_i]$ is congruent to $[AB]$ in the sense of the Definition 19.2.



Applying the last identity in Theorem 19.7 few times, we get

$$\begin{aligned} n\cdot \operatorname{area}[AB] &= \operatorname{area}\left([A_1B_1] \cup \dots \cup [A_nB_n]\right) \leqslant \\ &\leqslant \operatorname{area}(\blacksquare ABCD) \end{aligned}$$

That is

$$\operatorname{area}[AB] \leqslant \tfrac{1}{n}\cdot \operatorname{area}(\blacksquare ABCD)$$

for any positive integer n. Therefore $\operatorname{area}[AB] \leqslant 0$.

On the other hand, by Proposition 19.8,

$$\operatorname{area}[AB] \geqslant 0.$$

Hence the result follows.

For any one-point set $\{A\}$ we have $\varnothing \subset \{A\} \subset [AB]$. Therefore

$$0 \leqslant \operatorname{area}\{A\} \leqslant \operatorname{area}[AB] = 0.$$

Whence $\operatorname{area}\{A\} = 0$. □

19.10. Corollary. *Any degenerate polygonal set in the Euclidean plane has vanishing area.*

Proof. Let $\mathcal{P}$ be a degenerate set, say

$$\mathcal{P} = [A_1B_1] \cup \dots \cup [A_nB_n] \cup \{C_1, \dots, C_k\}.$$

Applying Theorem 19.7 together with Proposition 19.8, we get

$$\begin{aligned}\operatorname{area}\mathcal{P} \leqslant \operatorname{area}[A_1B_1] + \dots + \operatorname{area}[A_nB_n] + \\ + \operatorname{area}\{C_1\} + \dots + \operatorname{area}\{C_k\}.\end{aligned}$$

By Proposition 19.9, the right hand side vanish. Hence the statement follows. □

We say that polygonal set $\mathcal{P}$ is *subdivided* into two polygonal sets $\mathcal{Q}_1$ and $\mathcal{Q}_2$ if $\mathcal{P} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ and the intersection $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is degenerate. (Recall that according to Claim 19.6, the set $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is polygonal.)

19.11. Proposition. *Assume polygonal sets $\mathcal{P}$ is subdivided into two polygonal set $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then*

$$\operatorname{area}\mathcal{P} = \operatorname{area}\mathcal{Q}_1 + \operatorname{area}\mathcal{Q}_2.$$

Proof. By Theorem 19.7,

$$\operatorname{area}\mathcal{P} = \operatorname{area}\mathcal{Q}_1 + \operatorname{area}\mathcal{Q}_2 - \operatorname{area}(\mathcal{Q}_1 \cap \mathcal{Q}_2).$$

Since $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is degenerate, by Corollary 19.10,

$$\operatorname{area}(\mathcal{Q}_1 \cap \mathcal{Q}_2) = 0.$$

Hence the result follows. □

Area of solid rectangles

19.12. Theorem. *The solid rectangle in the Euclidean plane with sides a and b has area $a{\cdot}b$.*

19.13. Algebraic lemma. *Assume that a function s returns a non-negative real number $s(a,b)$ for any pair of positive real numbers (a,b) and satisfies the following identities*

$$\begin{aligned} s(1,1) &= 1; \\ s(a,b+c) &= s(a,b)+s(a,c) \\ s(a+b,c) &= s(a,c)+s(b,c) \end{aligned}$$

for any $a,b,c>0$. Then

$$s(a,b)=a{\cdot}b$$

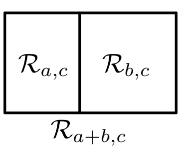
for any $a,b>0$.

The proof is similar to the proof of Lemma 13.8; we omit it.

Proof of Theorem 19.12. Denote by $\mathcal{R}_{a,b}$ the solid rectangle with sides a and b. Set

$$s(a,b)=\operatorname{area}\mathcal{R}_{a,b}.$$

By theorem 19.7, $s(1,1)=1$. That is, the first identity in the Algebraic Lemma holds.



Note that the rectangle $\mathcal{R}_{a+b,c}$ can be subdivided into two rectangle congruent to $\mathcal{R}_{a,c}$ and $\mathcal{R}_{b,c}$. Therefore by Proposition 19.11,

$$\operatorname{area}\mathcal{R}_{a+b,c}=\operatorname{area}\mathcal{R}_{a,c}+\operatorname{area}\mathcal{R}_{b,c}$$

That is, the second identity in the Algebraic Lemma holds. The proof of the third identity is analogues.

It remains to apply Algebraic lemma. $\square$

Area of solid parallelograms

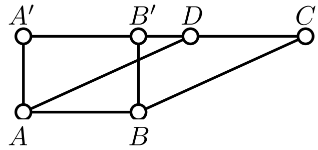
19.14. Proposition. *Let $\square ABCD$ be a parallelogram in the Euclidean plane, $a=AB$ and h be the distance between the lines (AB) and (CD). Then*

$$\operatorname{area}(\blacksquare ABCD)=a{\cdot}h.$$

Proof. Let us denote by A' and B' the foot points of A and B on the line (CD).

Note that $ABB'A'$ is a rectangle with sides a and h. By Proposition 19.12,

❶ $$\operatorname{area}(\blacksquare ABB'A') = h{\cdot}a.$$



Without loss of generality we may assume that $\blacksquare ABCA'$ contains $\blacksquare ABCD$ and $\blacksquare ABB'A'$.

In this case $\blacksquare ABB'D$ admits two subdivisions. First into $\blacksquare ABCD$ and $\blacktriangle AA'D$. Second into $\blacksquare ABB'A'$ and $\blacktriangle BB'C$.

By Proposition 19.11,

❷ $$\begin{aligned}\operatorname{area}(\blacksquare ABCD) + \operatorname{area}(\blacktriangle AA'D) &= \\ &= \operatorname{area}(\blacksquare ABB'A') + \operatorname{area}(\blacktriangle BB'C).\end{aligned}$$

Note that

❸ $$\triangle AA'D \cong \triangle BB'C.$$

Indeed, since $\square ABB'A'$ and $\square ABCD$ are parallelograms, by Lemma 6.22, we have $AA' = BB'$, $AD = BC$ and $DC = AB = A'B'$. It follows that $A'D = B'C$. Applying SSS congruence condition, we get ❸.

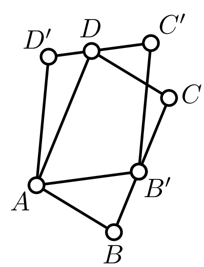
In particular,

❹ $$\operatorname{area}(\blacktriangle BB'C) = \operatorname{area}(\blacktriangle AA'D).$$

Subtracting ❹ from ❷, we get

$$\operatorname{area}(\blacksquare ABCD) = \operatorname{area}(\blacksquare ABB'D).$$

From ❶, the statement follows. □



19.15. Exercise. *Assume $\square ABCD$ and $\square AB'C'D'$ are two parallelograms such that $B' \in [BC]$ and $D \in$ $\in [C'D']$. Show that*

$$\operatorname{area}(\blacksquare ABCD) = \operatorname{area}(\blacksquare AB'C'D').$$

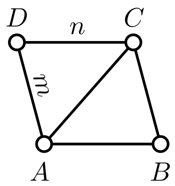
Area of solid triangles

19.16. Theorem. *Let $a = BC$ and h_A to be the altitude from A in $\triangle ABC$. Then*

$$\operatorname{area}(\blacktriangle ABC) = \tfrac{1}{2}\cdot a\cdot h_A.$$

Remark. Since $\triangle ABC$ completely determines the solid triangle $\blacktriangle ABC$, it is acceptable to write $\operatorname{area}(\triangle ABC)$ for $\operatorname{area}(\blacktriangle ABC)$.

Proof. Draw the line m through A which is parallel to (BC) and line n through C parallel to (AB). Note that $m \nparallel n$; set $D = m \cap n$. By construction, $\square ABCD$ is a parallelogram.



Note that $\blacksquare ABCD$ admits a subdivision into $\blacktriangle ABC$ and $\blacktriangle CDA$. Therefore

$$\operatorname{area}(\blacksquare ABCD) = \operatorname{area}(\blacktriangle ABC) + \operatorname{area}(\blacktriangle CDA)$$

Since $\square ABCD$ is a parallelogram, by Lemma 6.22 we have

$$AB = CD \quad\text{and}\quad BC = DA.$$

Therefore by SSS congruence condition, we have $\triangle ABC \cong \triangle CDA$. In particular

$$\operatorname{area}(\blacktriangle ABC) = \operatorname{area}(\blacktriangle CDA).$$

From above and Proposition 19.14, we get

$$\begin{aligned}\operatorname{area}(\blacktriangle ABC) &= \tfrac{1}{2}\cdot\operatorname{area}(\blacksquare ABCD) = \\ &= \tfrac{1}{2}\cdot h_A\cdot a\end{aligned}$$

□

19.17. Exercise. *Denote by h_A, h_B and h_C the altitudes of $\triangle ABC$ from vertices A, B and C correspondingly. Note that from Theorem 19.16, it follows that*

$$h_A\cdot BC = h_B\cdot CA = h_C\cdot AB.$$

Give a proof of this statement without using area.

19.18. Exercise. *Assume M lies inside the parallelogram $\square ABCD$; that is, M belongs to the solid parallelogram $\blacksquare ABCD$, but does not lie on its sides. Show that*

$$\operatorname{area}(\blacktriangle ABM) + \operatorname{area}(\blacktriangle CDM) = \tfrac{1}{2}\cdot\operatorname{area}(\blacksquare ABCD).$$

19.19. Exercise. *Assume that diagonals of a nondegenerate quadrilateral $\square ABCD$ intersect at point M. Show that*

$$\operatorname{area}(\blacktriangle ABM)\cdot \operatorname{area}(\blacktriangle CDM) = \operatorname{area}(\blacktriangle BCM)\cdot \operatorname{area}(\blacktriangle DAM).$$

19.20. Exercise. *Let r be the inradius of $\triangle ABC$ and p be its semiperimeter; that is $p = \frac{1}{2}\cdot(AB + BC + CA)$. Show that*

$$\operatorname{area}(\blacktriangle ABC) = p\cdot r.$$

19.21. Advanced exercise. *Show that for any affine transformation β there is a constant $k > 0$ such that the equality*

$$\operatorname{area}[\beta(\blacktriangle)] = k\cdot \operatorname{area}\blacktriangle.$$

holds for any solid triangle $\blacktriangle$.

Moreover, if β has the matrix form $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ then

$$k = \left|\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right| = |a\cdot d - b\cdot c|.$$

Area method

Slim proofs using area are often used to impress students. We will show couple of such examples in this section, but one should be aware that these proofs are not truly elementary — the price one pays to introduce the area function is quite high.

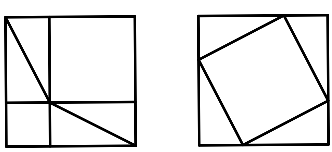
Proof of the Pythagorean theorem via the area method. We need to show that if a and b are legs and c is the hypotenuse of a right triangle then

$$a^2 + b^2 = c^2.$$

Denote by $\blacktriangle$ the right solid triangle with legs a and b and by $\blacksquare_x$ be the solid square with side x.

Let us construct two subdivisions of $\blacksquare_{a+b}$.

1. Subdivide $\blacksquare_{a+b}$ into two solid squares congruent to $\blacksquare_a$ and $\blacksquare_b$ and 4 solid triangles congruent to $\blacktriangle$, see the left diagram.



2. Subdivide $\blacksquare_{a+b}$ into one solid square congruent to $\blacksquare_c$ and 4 solid right triangles congruent to $\blacktriangle$, see the right diagram.

Applying Proposition 19.11 few times, we get.

$$\begin{aligned}\text{area}\,\blacksquare_{a+b} &= \text{area}\,\blacksquare_a + \text{area}\,\blacksquare_b + 4\cdot\,\text{area}\,\blacktriangle = \\ &= \text{area}\,\blacksquare_c + 4\cdot\,\text{area}\,\blacktriangle.\end{aligned}$$

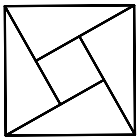
Therefore

$$\text{area}\,\blacksquare_a + \text{area}\,\blacksquare_b = \text{area}\,\blacksquare_c.$$

Since

$$\text{area}\,\blacksquare_x = x^2,$$

the statement follows. □



19.22. Exercise. *Build an other proof of Pythagorean theorem based on the diagram.*

(In the notations above it shows a subdivision of $\blacksquare_c$ *into* $\blacksquare_{a-b}$ *and 4 solid triangles congruent to* $\blacktriangle$*.)*

19.23. Exercise. *Show that the sum of distances from a point to the sides of an equilateral triangle is the same for all points inside the triangle.*

The following claim is simple but very useful.

19.24. Claim. *Assume that two triangles* $\triangle ABC$ *and* $\triangle A'B'C'$ *in the Euclidean plane have equal altitudes dropped from* A *and* A' *correspondingly. Then*

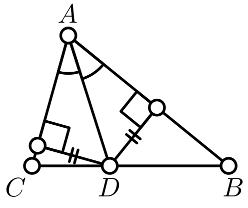
$$\frac{\blacktriangle A'B'C'}{\blacktriangle ABC} = \frac{B'C'}{BC}.$$

In particular, the same identity holds if $A = A'$ *and the bases* $[BC]$ *and* $[B'C']$ *lie on one line.*

Proof. Let h be the altitude. By Theorem 19.16,

$$\frac{\blacktriangle A'B'C'}{\blacktriangle ABC} = \frac{\frac{1}{2}\cdot h\cdot B'C'}{\frac{1}{2}\cdot h\cdot BC} = \frac{B'C'}{BC}.$$

□



Lemma 7.7 via Area method. We have to show that if $\triangle ABC$ is nondegenerate and the bisector of $\angle BAC$ intersects $[BC]$ at the point D. Then

$$\frac{AB}{AC} = \frac{DB}{DC}.$$

Applying Claim 19.24, we get

$$\frac{\text{area}(\blacktriangle ABD)}{\text{area}(\blacktriangle ACD)} = \frac{BD}{CD}.$$

By Lemma 5.13 the triangles $\triangle ABD$ and $\triangle ACD$ have equal altitudes from D. Applying Claim 19.24 again, we get

$$\frac{\operatorname{area}(\blacktriangle ABD)}{\operatorname{area}(\blacktriangle ACD)} = \frac{AB}{AC}.$$

Hence the result follows. □

19.25. Exercise. *Let X lies inside a nondegenerate triangle $\triangle ABC$. Show that X lies on the median from A if and only if*

$$\operatorname{area}(\blacktriangle ABX) = \operatorname{area}(\blacktriangle ACX).$$

19.26. Exercise. *Build a proof of Theorem 7.5 via area method based on the Exercise 19.25.*

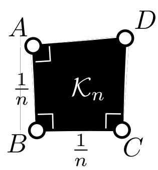
Namely, show that medians of nondegenerate triangle intersect at one point and the point of their intersection divides each median in the ration 1:2.

Area in the absolute planes and spheres

The theorem Theorem 19.7 will hold in the absolute planes and spheres if the solid unit square $\mathcal{K}$ exchanged to a fixed nondegenerate polygonal set. One has to make such change for good reason — hyperbolic plane and sphere have no unit squares.

The set $\mathcal{K}$ in this case plays role of the unit measure for the area and changing $\mathcal{K}$ will require conversion of area units.

According to the standard convention, the set $\mathcal{K}$ is taken so that on small scales area behaves like area in the Euclidean plane. Say if $\mathcal{K}_n$ denotes the solid quadrilateral $\blacksquare ABCD$ with right angles at A, B and C and $AB = BC = \frac{1}{n}$ then we may assume that



❺ $$n^2 \cdot \operatorname{area} \mathcal{K}_n \to 1 \quad \text{as} \quad n \to \infty.$$

This convention works equally well for spheres and absolute planes, including Euclidean plane. In spherical geometry equivalently we may assume that if r is the radius of the sphere then the area of whole sphere is $4 \cdot \pi \cdot r^2$.

Recall that *defect of triangle* $\triangle ABC$ is defined as

$$\operatorname{defect}(\triangle ABC) \stackrel{\text{def}}{=\!=} \pi - |\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB|.$$

It turns out that any absolute plane and sphere there is a real number k such that

❻ $$k\cdot \operatorname{area}(\triangle ABC) + \operatorname{defect}(\triangle ABC) = 0$$

for any triangle $\triangle ABC$.

This number k is called *curvature*; $k = 0$ for the Euclidean plane, $k = -1$ for the h-plane and $k = 1$ for the unit sphere and $k = \frac{1}{r^2}$ for the sphere of radius r. In particular it follows that any ideal triangle in h-plane has area π.

The identity ❻ suggest an alternative way to introuduce area function which works on spheres and all absolute planes except the Euclidean plane.

Quadrable sets

A set $\mathcal{S}$ in the plane is called *quadrable* if for any $\varepsilon > 0$ there are two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\mathcal{P} \subset \mathcal{S} \subset \mathcal{Q} \quad \text{and} \quad \operatorname{area}\mathcal{Q} - \operatorname{area}\mathcal{P} < \varepsilon.$$

If $\mathcal{S}$ is quadrable, its area can be defined as the (necessary unique) real number $\operatorname{area}\mathcal{S}$ such that the inequality

$$\operatorname{area}\mathcal{Q} \leqslant \operatorname{area}\mathcal{S} \leqslant \operatorname{area}\mathcal{P}$$

holds if $\mathcal{P}$ and $\mathcal{Q}$ are polygonal sets and $\mathcal{P} \subset \mathcal{S} \subset \mathcal{Q}$.

19.27. Exercise. *Let $\mathcal{D}$ be the unit disk; that is $\mathcal{D}$ is a set which contains the unit circle Γ and all the points inside Γ.*

Show that $\mathcal{D}$ is a quadrable set.

Since $\mathcal{D}$ is quadrable, the expression $\operatorname{area}\mathcal{D}$ makes sense. The constant π can be defined as $\pi = \operatorname{area}\mathcal{D}$.

It turns out that the class of quadrable sets is the largest class for which the area can be defined in such a way that it satisfies all the conditions in Theorem 19.7 incliding uniqueness.

There is a way to define area for all bounded sets which satisfies all the conditions in the Theorem 19.7 except uniqueness. (A set in the plane is called *bounded* if it lies inside a circle.)

In the hyperbolic plane and in the sphere sphere there is no similar construction. Read about Hausdorff–Banach–Tarski Paradox if you wonder why.

Hints

Chapter 1

Exercise 1.2. Among the conditions in Definition 1.1, only the triangle inequality require proof, the rest of conditions are evident. Let $A = (x_A, y_A)$, $B = (x_B, y_B)$ and $C = (x_C, y_C)$. Set

$$x_1 = x_B - x_A, \qquad y_1 = y_B - y_A,$$
$$x_2 = x_C - x_B, \qquad y_2 = y_C - y_B.$$

(a). The inequality

$$d_1(A, C) \leqslant d_1(A, B) + d_1(B, C)$$

can be written as

$$|x_1 + x_2| + |y_1 + y_2| \leqslant |x_1| + |y_1| + |x_2| + |y_2|.$$

The later follows since $|x_1 + x_2| \leqslant |x_1| + |x_2|$ and $|y_1 + y_2| \leqslant |y_1| + |y_2|$.

(b). The inequality

❶
$$d_2(A, C) \leqslant d_2(A, B) + d_2(B, C)$$

can be written as

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leqslant \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}.$$

Taking square of left and right hand sides, simplify take square again and again simplify. You should get the following inequality

$$0 \leqslant (x_1 \cdot y_2 - x_2 \cdot y_1)^2.$$

which is equivalent to ❶ and evidently true.

(c). The inequality

$$d_\infty(A, C) \leqslant d_\infty(A, B) + d_\infty(B, C)$$

can be written as

❷ $$\max\{|x_1+x_2|,|y_1+y_2|\} \leqslant \max\{|x_1|,|y_1|\}+\max\{|x_2|,|y_2|\}.$$

Without loss of generality we may assume that

$$\max\{|x_1+x_2|,|y_1+y_2|\} = |x_1+x_2|.$$

Further,

$$|x_1+x_2| \leqslant |x_1|+|x_2| \leqslant \max\{|x_1|,|y_1|\}+\max\{|x_2|,|y_2|\}.$$

Hence ❷ follows.

Exercise 1.4. If $A \neq B$ then $d_{\mathcal{X}}(A,B) > 0$. Since f is distance-preserving,

$$d_{\mathcal{Y}}(f(A),f(B)) = d_{\mathcal{X}}(A,B).$$

Therefore $d_{\mathcal{Y}}(f(A),f(B)) > 0$ and whence $f(A) \neq f(B)$.

Exercise 1.5. Set $f(0)=a$ and $f(1)=b$. Note that that $b=a+1$ or $a-1$. Moreover, $f(x)=a\pm x$ and at the same time, $f(x)=b\pm(x-1)$ for any x.

If $b=a+1$, it follows that $f(x)=a+x$ for any x.

The same way, if $b=a-1$, it follows that $f(x)=a-x$ for any x.

Exercise 1.6. Show that the map $(x,y)\mapsto(x+y,x-y)$ is an isometry $(\mathbb{R}^2,d_1)\to(\mathbb{R}^2,d_\infty)$. That is, you need to check that this map is bijective and distance preserving.

Exercise 1.7. First prove that *two points $A=(x_A,y_A)$ and $B=(x_B,y_B)$ on the Manhattan plane have unique midpoint if and only if $x_A=x_B$ or $y_A=y_B$*; compare with the example on page 17.

Then use above statement to prove that any motion of the Manhattan plane can be written in one of the following two ways

$$(x,y)\mapsto(\pm x+a,\pm y+b) \qquad \text{or} \qquad (x,y)\mapsto(\pm y+b,\pm x+a),$$

for some fixed real numbers a and b. (In each case we have 4 choices of signs, so for fixed pair (a,b) we have 8 distinct motions.)

Exercise 1.9. Assume three points A, B and C lie on one line. Note that in this case one of triangle inequalities with the points A, B and C becomes equality.

Set $A=(-1,1)$, $B=(0,0)$ and $C=(1,1)$. Show that for d_1 and d_2 all the triangle inequalities with the points A, B and C are strict. It follows that the graph is not a line.

For d_∞ show that $(x,|x|)\mapsto x$ gives the isometry of the graph to $\mathbb{R}$. Conclude that the graph is a line in $(\mathbb{R}^2,d_\infty)$.

Exercise 1.10. Applying the definition of line, the problems are reduced to the following.

Assume $a \neq b$, find the number of solutions for each of the following two equations

$$|x-a| = |x-b| \qquad \text{and} \qquad |x-a| = 2{\cdot}|x-b|.$$

Each can be solved by taking square of left and right hand sides. The numbers of solutions are 1 and 2 correspondingly.

Exercise 1.11. Fix an isometry $f\colon (PQ) \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) = q > 0$.

Assume that $f(X) = x$. By the definition of half-line $X \in [PQ)$ if and only if $x \geqslant 0$. The later holds if and only if

$$|x-a| = |x| - |a|.$$

Hence the result follows.

Exercise 1.12. The equation $2{\cdot}\alpha \equiv 0$ means that $2{\cdot}\alpha = 2{\cdot}k{\cdot}\pi$ for some integer k. Therefore $\alpha = k{\cdot}\pi$ for some integer k.

Equivalently $\alpha = 2{\cdot}n{\cdot}\pi$ or $\alpha = (2{\cdot}n+1){\cdot}\pi$ for some integer n. The first identity means that $\alpha \equiv 0$ and the second means $\alpha \equiv \pi$.

Exercise 1.13. *(a).* By triangle inequality

$$|f(A') - f(A)| \leqslant d(A', A).$$

Therefore we can take $\delta = \varepsilon$.

(b). By triangle inequality

$$\begin{aligned}|f(A',B') - f(A,B)| &\leqslant |f(A',B') - f(A,B')| + |f(A,B') - f(A,B)| \leqslant \\ &\leqslant d(A',A) + d(B',B)\end{aligned}$$

Therefore we can take $\delta = \frac{\varepsilon}{2}$.

Exercise 1.14. Fix $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ such that $f(A) = B$.

Fix $\varepsilon > 0$. Since g is continuous at B, there is $\delta_1 > 0$ such that

$$d_{\mathcal{Z}}(g(B'), g(B)) < \varepsilon \ \text{ if } \ d_{\mathcal{Y}}(B', B) < \delta_1.$$

Since f is continuous at A, there is $\delta_2 > 0$ such that

$$d_{\mathcal{Y}}(f(A'), f(A)) < \delta_1 \ \text{ if } \ d_{\mathcal{X}}(A', A) < \delta_2.$$

Since $f(A) = B$, we get

$$d_{\mathcal{Z}}(h(A'), h(A)) < \varepsilon \ \text{ if } \ d_{\mathcal{X}}(A', A) < \delta_2.$$

Hence the result follows.

Chapter 2

Exercise 2.2. By Axiom II, $(OA) = (OA')$. Therefore the statement boils down two the following.

Assume $f\colon \mathbb{R} \to \mathbb{R}$ is a motion of the line which sends $0 \to 0$ and one positive number to a positive number then f is an identity map.

The later follows from Exercise 1.5.

Exercise 2.5. By Proposition 2.4, $\measuredangle AOA = 0$. It remains to apply Axiom IIIa.

Exercise 2.9. Apply Proposition 2.4, Theorem 2.7 and Exercise 1.12.

Exercise 2.10. By Axiom IIIb,

$$2{\cdot}\measuredangle BOC \equiv 2{\cdot}\measuredangle AOC - 2{\cdot}\measuredangle AOB \equiv 0.$$

By Exercise 1.12, it implies that $\measuredangle BOC = 0$ or π. It remains to apply Exercise 2.5 and Theorem 2.7 correspondingly in these cases.

Exercise 2.12. Applying Proposition 2.11, we get $\measuredangle AOC = \measuredangle BOD$. It remains to apply Axiom IV.

Chapter 3

Exercise 3.1. Set $\alpha = \measuredangle AOB$ and $\beta = \measuredangle BOA$. Note that $\alpha = \pi$ if and only if $\beta = \pi$. Otherwise $\alpha = -\beta$. Hence the result follows.

Exercise 3.2. Set $\alpha = \measuredangle AOX$ and $\beta = \measuredangle BOX$. Since $\angle AOB$ is straight,

❶
$$\alpha - \beta \equiv \pi.$$

It follows that $\alpha = \pi \Leftrightarrow \beta = 0$ and $\alpha = 0 \Leftrightarrow \beta = \pi$.

In the remaining cases we have $|\alpha|, |\beta| < \pi$. If α and β have the same sign then $|\alpha - \beta| < \pi$ which contradicts ❶.

Exercise 3.3. Set $\alpha = \measuredangle BOC$, $\beta = \measuredangle COA$ and $\gamma = \measuredangle AOB$. By Axiom IIIb and Proposition 2.4

❷
$$\alpha + \beta + \gamma \equiv 0$$

Note that $0 < \alpha + \beta < 2{\cdot}\pi$ and $|\gamma| \leqslant \pi$. If $\gamma > 0$ then ❷ implies

$$\alpha + \beta + \gamma = 2{\cdot}\pi$$

and if $\gamma < 0$ then ❷ implies

$$\alpha + \beta + \gamma = 0.$$

Exercise 3.9. Note that O and A' lie on the same side from (AB). Analogously O and B' lie on the same side from (AB). Hence the result follows.

Exercise 3.12. Apply Theorem 3.11 for triangles $\triangle PQX$ and $\triangle PQY$ and then Proposition 3.8(a).

Exercise 3.13. Note that it is sufficient to consider the cases when $A' \neq B, C$ and $B' \neq A, C$.

Apply Pasch's theorem (3.10) twice; for $\triangle AA'C$ with line (BB') and for $\triangle BB'C$ with line (AA').

Exercise 3.14. Assume that Z is the point of intersection.

Note first $Z \neq P$ and $Z \neq Q$, therefore that $Z \notin (PQ)$.

Then show that Z and X lie on one side from (PQ). Repeat the argument to show that Z and Y lie on one side from (PQ). In particular X and Y lie on the same side from (PQ), a contradiction.

Chapter 4

Exercise 4.3. Apply Theorem 4.2 twice for different basis of $\triangle ABC$.

Exercise 4.5. Consider point D and D', so that M is the midpoint of $[AD]$ and M' is the midpoint of $[A'D']$. Show first that $\triangle ABD \cong \triangle A'B'D'$.

Exercise 4.6. To prove *(a)*, apply SAS. To prove *(b)*, use *(a)* and apply SSS.

Exercise 4.7. Choose $B' \in [AC]$ such that $AB = AB'$. Note that $BC = B'C$. By SSS, $\triangle ABC \cong AB'C$.

Exercise 4.8. Without loss of generality, we may assume that X is distinct from A, B and C. Set $f(X) = X'$; assume $X' \neq X$.

Note that $AX = AX'$, $BX = BX'$ and $CX = CX'$. By SSS we get $\measuredangle ABX = \pm\measuredangle ABX'$. Since $X \neq X'$, we get

$$\measuredangle ABX \equiv -\measuredangle ABX'.$$

The same way we get

$$\measuredangle CBX \equiv -\measuredangle CBX'.$$

Subtracting these two identities from each other, we get

$$\measuredangle ABC \equiv -\measuredangle ABC.$$

Therefore $\triangle ABC$ is degenerate, a contradiction.

Chapter 5

Exercise 5.1. By Axiom IIIb and Theorem 2.7

$$\measuredangle XOA - \measuredangle XOB \equiv \pi.$$

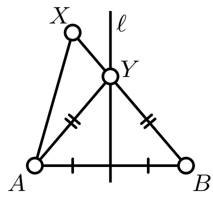
Since $|\measuredangle XOA|, |\measuredangle XOB| \leqslant \pi$, we get

$$|\measuredangle XOA| + |\measuredangle XOB| = \pi.$$

Hence the statement follows.

Exercise 5.3. Assume X and A lie on the same side from ℓ.

Note that A and B lie on the opposite sides of ℓ. Therefore, by Proposition 3.8, $[AX]$ does not intersect ℓ and $[BX]$ intersects ℓ, say at the point Y.



Note that $Y \notin [AX]$. By Exercise 4.7,

$$BX = AY + YX > AX.$$

This way we proved "if"-part. To prove "only if", it remains to switch A and B, repeat the above argument and yet apply Theorem 5.2.

Exercise 5.4. Apply Exercise 5.3, Theorem 4.1 and Exercise 3.3.

Exercise 5.8. Choose arbitrary nondegenerate triangle $\triangle ABC$. Denote by $\triangle \hat{A}\hat{B}\hat{C}$ its image after the motion.

If $A \neq \hat{A}$, apply the reflection in the perpendicular bisector of $[A\hat{A}]$. This reflection sends A to $\hat{A}$. Denote by B' and C' the reflections of B and C correspondingly.

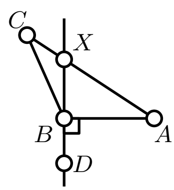
If $B' \neq \hat{B}$, apply the reflection in the perpendicular bisector of $[B'\hat{B}]$. This reflection sends B' to $\hat{B}$. Note that $\hat{A}\hat{B} = \hat{A}B'$; that is, $\hat{A}$ lies on the perpendicular bisector and therefore $\hat{A}$ reflects to itself. Denote by C'' the reflections of C'.

Finally if $C'' \neq \hat{C}$ apply the reflection in $(\hat{A}\hat{B})$. Note that $\hat{A}\hat{C} = \hat{A}C''$ and $\hat{B}\hat{C} = \hat{B}C''$; that is, (AB) is the perpendicular bisector of $[C''\hat{C}]$. Therefore this reflection sends C'' to $\hat{C}$.

Apply Exercise 4.8 to show that the composition of constructed reflections coincides with the given motion.

Exercise 5.9. Note that $\measuredangle XBA = \measuredangle ABP$, $\measuredangle PBC = \measuredangle CBY$. Therefore

$$\measuredangle XBY \equiv \measuredangle XBP + \measuredangle PBY \equiv 2{\cdot}(\measuredangle ABP + \measuredangle PBC) \equiv 2{\cdot}\measuredangle ABC.$$



Exercise 5.11. If $\angle ABC$ is right, the statement follows from Lemma 5.10. Therefore we can assume that $\angle ABC$ is obtuse.

Draw a line (BD) perpendicular to (BC). Since $\angle ABC$ is obtuse, the angles $\angle DBA$ and $\angle DBC$ have opposite signs.

By Proposition 3.8, the points A and C lie on the opposite sides from (AD). In particular $[AC]$ intersects (BD), say at point X.

Note that $AX < AC$ and by Lemma 5.10, $AB \leqslant AX$. Hence the result follows.

Exercise 5.12. Let (BX) and (BY) be internal and external bisectors of $\angle ABC$. Then

$$\begin{aligned}2\cdot\measuredangle XBY &\equiv 2\cdot\measuredangle XBA + 2\cdot\measuredangle ABY \equiv\\ &\equiv \measuredangle CBA + \pi + 2\cdot\measuredangle ABC \equiv\\ &\equiv \pi + \measuredangle CBC = \pi.\end{aligned}$$

Hence the result follows.

Exercise 5.14. Let O be the center of the circle. Note that we can assume that $O \neq P$.

Assume P lies between X and Y. By Exercise 5.1, we can assume that $\angle OPX$ is right or obtuse. By Exercise 5.11, $OP < OX$; that is, P lies inside Γ.

If P does not lie between X and Y, we can assume that X lies between P and Y. Since $OX = OY$, Exercise 5.11 implies that $\angle OXY$ is acute. Therefore $\angle OXP$ is obtuse. Applying Exercise 5.11 again we get $OP > OX$; that is, P lies outside Γ.

Exercise 5.15. Apply Theorem 5.2.

Exercise 5.17. Use Exercise 5.15 and the uniqueness of perpendicular (Theorem 5.5).

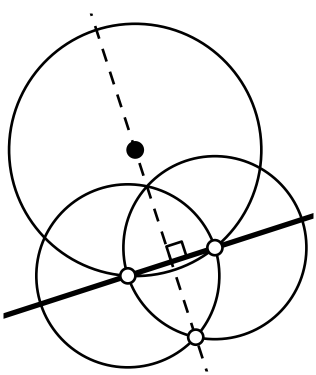
Exercise 5.19. Let P' be the reflection of P through (OO'). Note that P' lies on both circles and $P' \neq P$ if and only if $P \notin (OO')$.

Exercise 5.20. To prove (a), apply Exercise 5.19.

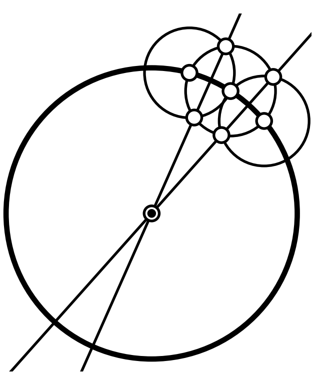
To prove (b), apply Theorem 3.16.

Exercise 5.21. Let A and B be the points of intersection. Note that the centers lie on the perpendicular bisector of the segment $[AB]$.

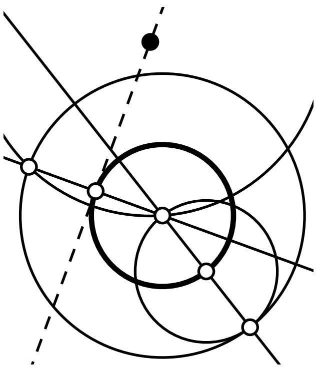
Exercise 5.23.



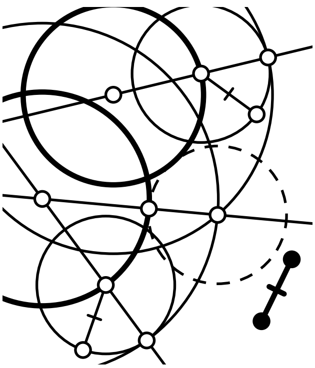
Exercise 5.24.



Exercise 5.25.



Exercise 5.26.



Chapter 6

Exercise 6.4. Show first that $k \perp n$.

Exercise 6.5. Repeat twice the construction in Exercise 5.23.

Exercise 6.8. First show that $\triangle AA'C \sim \triangle BB'C$.

Exercise 6.9. Applying AA similarity condition, we get that angle-preserving transformation multiplies the sides of any nondegenerate triangle by a number depending on triangle.

Note that for any two nondegenerate triangles which share one side this number is the same. Applying this observation to a chain of triangles leads to a solution.

Exercise 6.11. Apply that $\triangle ADC \sim \triangle CDB$.

Exercise 6.12. Apply Pythagorean theorem (6.10) and SSS congruence condition.

Exercise 6.14. Apply twice Theorem 4.2 and twice Theorem 6.13.

Exercise 6.15. If $\triangle ABC$ is degenerate then one of the angle measures is π and the other two are 0. Hence the result follows.

Assume $\triangle ABC$ is nondegenerate. Set $\alpha = \measuredangle CAB$, $\beta = \measuredangle ABC$ and $\gamma = \measuredangle BCA$.

By Theorem 3.11, we may assume that $0 < \alpha, \beta, \gamma < \pi$. Therefore

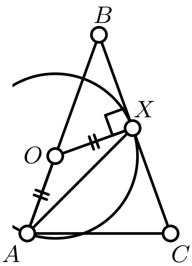
❶ $$0 < \alpha + \beta + \gamma < 3{\cdot}\pi.$$

By Theorem 6.13,

❷ $$\alpha + \beta + \gamma \equiv \pi.$$

From ❶ and ❷ the result follows.

Exercise 6.16. Apply twice Theorem 4.2 and once Theorem 6.13.



Exercise 6.17. Assume that O is the center of Γ.

Note that $\triangle AOX$ is isosceles and the angle $\angle OXC$ is right. Appling Theorems 6.13 and 4.2 and simplifyiing you should get

$$4\cdot\measuredangle CAX \equiv \pi.$$

Finally show that $\angle CAX$ has to be acute. It follows then that $\measuredangle CAX = \pm\frac{\pi}{4}$.

Exercise 6.19. By Transversal property 6.18,

$$\measuredangle B'BC \equiv \pi - \measuredangle C'B'B.$$

Since B' lies between A and B, we get $\measuredangle ABC = \measuredangle B'BC$ and $\measuredangle AB'C' + \measuredangle C'B'B \equiv \pi$. Whence $\measuredangle ABC = \measuredangle AB'C'$.

The same way we can prove that $\measuredangle BCA = \measuredangle B'C'A$. It remains to apply AA similarity condition.

Exercise 6.20. Assume we need to trisect segment $[AB]$. Construct a line $\ell \neq (AB)$ with four points A, C_1, C_2, C_3 such that C_1 and C_2 trisect $[AC_3]$. Draw the line (BC_3) and draw parallel lines through C_1 and C_2. The points of intersections of these two lines with (AB) trisect the segment $[AB]$.

Exercise 6.21. Apply Theorem 6.13 to $\triangle ABC$ and $\triangle BDA$.

Exercise 6.23. Since $\triangle ABC$ is isosceles, $\measuredangle CAB = \measuredangle BCA$.

By SSS, $\triangle ABC \cong \triangle CDA$. Therefore

$$\pm\measuredangle DCA = \measuredangle BCA = \measuredangle CAB.$$

Since $D \neq C$ and we get "$-$" in the last formula. In particular,

$$-\measuredangle DCA \equiv \measuredangle CAB.$$

It follows that $(AB) \parallel (CD)$.

Exercise 6.24. Apply Lemma 6.22, Transversal property (6.18) to the diagonals and a pair of opposite sides and then ASA-congruence condition (4.1).

Exercise 6.25. By Lemma Lemma 6.22 and SSS,

$$AC = BD \quad \Leftrightarrow \quad \measuredangle ABC = \pm\measuredangle BCD.$$

By Transversal property 6.18,

$$\measuredangle ABC + \measuredangle BCD \equiv \pi.$$

Therefore

$$AC = BD \quad \Leftrightarrow \quad \measuredangle ABC = \measuredangle BCD = \pm\tfrac{\pi}{2}.$$

Exercise 6.26. Consider a parallelogram $\square ABCD$. By Exercise 6.24, the diagolals $[AC]$ and $[BD]$ have common midpoint, say M.

Use SSS and Lemma 6.22 to show that

$$AB = CD \quad \Leftrightarrow \quad \triangle AMB \cong \triangle AMD \quad \Leftrightarrow \quad \measuredangle AMB = \pm\tfrac{\pi}{2}.$$

Exercise 6.27. *(a).* Use the uniqueness of parallel line (Theorem 6.2).

(b) Use lemma about parallelogram (Lemma 6.22) and Pythagorean theorem (6.10).

Exercise 6.28. Set $A = (0, 0)$, $B = (c, 0)$ and $C = (x, y)$. Clearly $AB = c$, $AC^2 = x^2 + y^2$ and $BC^2 = (c - x)^2 + y^2$.

It remains to show that the there is a pair of real numbers (x, y) which satisfy the following system of equations

$$\left[\begin{aligned} b^2 &= x^2 + y^2 \\ a^2 &= (c - x)^2 + y^2 \end{aligned}\right.$$

if $0 < a \leqslant b \leqslant c \leqslant a + c$. $\square$

Exercise 6.29. Without loss of generality, we can assume that $x_A \neq x_B$; otherwise switch x and y.

Denote by ℓ the set of points with coordinates (x, y) satisfying

$$(x - x_A)\cdot(y_B - y_A) = (y - y_A)\cdot(x_B - x_A).$$

Note that $A, B \in \ell$

Show that the map $\ell \to \mathbb{R}$ defined as $(x, y) \mapsto \frac{AB}{|x_A - x_B|}\cdot x$ is an isometry; that is, ℓ is a line. It remains to apply Axiom II.

Chapter 7

Exercise 7.2. Apply Theorem 7.1 and Theorem 5.2.

Exercise 7.4. Note that $(AC) \perp (BH)$ and $(BC) \perp (AH)$ and apply Theorem 7.3.

Exercise 7.6. Use the idea from the proof of Theorem 7.5 to show that $(XY) \parallel (AC) \parallel (VW)$ and $(XV) \parallel (BD) \parallel (YW)$.

Exercise 7.8. If E is the point of intersection of (BC) with the external bisector of $\angle BAC$ then

$$\frac{AB}{AC} = \frac{EB}{EC}.$$

It can be proved along the same lines as Lemma 7.7.

Exercise 7.10. Apply Lemma 7.7. (Check Exercise 10.2 for yet an other solution.)

Exercise 7.11. Apply ASA for the two triangles which bisector cuts from the original triangle.

Exercise 7.12. Let I be the incenter. By SAS, we get $\triangle AIZ \cong \triangle AIY$ and therefore $AY = AZ$. The same way we get $BX = BZ$ and $CX = CY$. Hence the result follows.

Exercise 7.13. Let $\triangle ABC$ be the given acute triangle and $\triangle A'B'C'$ be its orthic triangle. Apply Exercise 6.8 to show that $\measuredangle A'B'C = \measuredangle AB'C'$. Conclude that (BB') is bisecting $\angle A'B'C'$.

If the triangle $\triangle ABC$ is obtuse then orthocenter coincides with one of the *excenters* of $\triangle ABC$; that is, the point of intersection of two external and one internal bisectors of $\triangle ABC$.

Chapter 8

Exercise 8.3. *(a).* Apply Theorem 8.2 for $\angle XX'Y$ and $\angle X'YY'$ and Theorem 6.13 for $\triangle PYX'$.

(b) Note first that the angles $\angle XPY$ and $\angle X'PY'$ are vertical. Therefore $\measuredangle XPY = \measuredangle X'PY'$.

Applying Theorem 8.2 we get

$$2{\cdot}\measuredangle Y'X'P \equiv 2{\cdot}\measuredangle PYX.$$

According to Theorem 3.11, $\angle Y'X'P$ and $\angle PYX$ have the same sign; therefore

$$\measuredangle Y'X'P = \measuredangle PYX.$$

It remains to apply the AA similarity condition.

(c) Apply (b) assuming $[YY']$ is the diameter of Γ.

Exercise 8.4. Apply Exercise 8.3(b) three times.

Exercise 8.5. Let X any Y be the foot points of altitudes from A and B. Denote by O the circumcenter.

By angle-angle condition, $\triangle AXC \sim \triangle BYC$. Thus

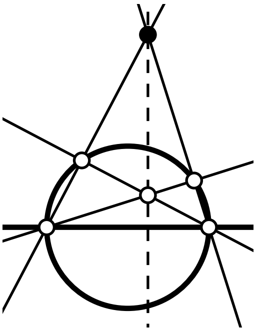
$$\measuredangle A'OC \equiv 2{\cdot}\measuredangle A'AC \equiv -2{\cdot}\measuredangle B'BC \equiv -\measuredangle B'OC.$$

By SAS, $\triangle A'OC \cong \triangle B'OC$. Therefore $A'C = B'C$.

Exercise 8.7. Construct the circles Γ and Γ' with diameters $[AB]$ and $[A'B']$. By Corollary 8.6, any point Z in the intersection $\Gamma \cap \Gamma'$ will do.

Exercise 8.8. By Corollary 8.6, the points L, M and N lie on the circle Γ with diameter $[OX]$. It remains to apply Theorem 8.2 for the circle Γ and two inscribed angles with vertex at O.

Exercise 8.9. Note that $\measuredangle AA'B = \pm\frac{\pi}{2}$ and $\measuredangle AB'B = \pm\frac{\pi}{2}$. Then apply Theorem 8.11 to $\square AA'BB'$.



If O is the center of the circle then

$$\measuredangle AOB \equiv 2{\cdot}\measuredangle AA'B \equiv \pi.$$

That is, O is the midpoint of $[AB]$.

Exercise 8.10. Guess the construction from the diagram.

To prove that it produce perpendicular line, apply Theorem 7.3 and Corollary 8.6.

Exercise 8.12. Apply Theorem 8.11 twice for $\square ABYX$ and $\square ABY'X'$ and use Transversal property (6.18).

Exercise 8.13. Apply Transversal property (6.18) and the theorem on inscribed angle (8.2).

Exercise 8.16. By Theorem 6.13,

$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB \equiv \pi.$$

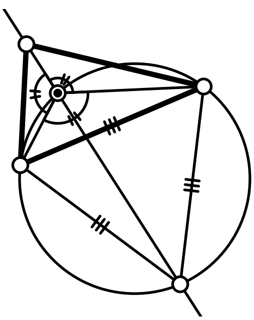
It remains to apply Proposition 8.15 twice.

Exercise 8.17. If $C \in (AX)$ then the arc is formed by $[AC]$ or two half-lines of (AX) with vertices at A and C.

Assume $C \notin (AX)$. Let ℓ be the line through A perpendicular to (AX) and m be the perpendicular bisector of $[AC]$. Note that $\ell \nparallel m$; set $O = \ell \cap m$. Note that the circle with center O passing though A is also passing through C and tangent to (AX).

Note that one the two arcs with endpoints A and C is tangent to $[AX)$.

The uniqueness follow from the propositions 8.14 and 8.15.



Exercise 8.18. Apply Proposition 8.15 twice.

Exercise 8.19. Guess the construction from the diagram. To show that it produce needed point, apply theorem on inscribed angle (8.2).

Chapter 9

Exercise 9.1. By Lemma 5.18, $\angle OTP'$ is right. Therefore $\triangle OPT \sim \triangle OTP'$ and in particular

$$OP{\cdot}OP' = OT^2.$$

Hence the result follows.

Exercise 9.3. Suppose O is the circle Γ and $X, Y \in \Gamma$; in particular, $OX = OY$.

Note that each X and Y is self-inverse. By Lemma 9.2,

$$\triangle OPX \sim \triangle OXP' \quad \text{and} \quad \triangle OPY \sim \triangle OYP'.$$

Therefore

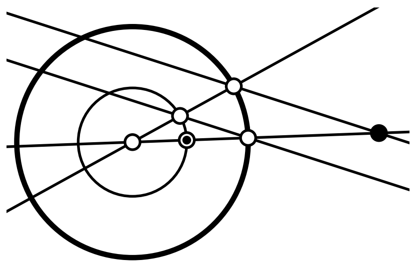
$$\frac{PX}{P'X} = \frac{OP}{OX} = \frac{OP}{OY} = \frac{PY}{P'Y}.$$

Hence the result follows.

Exercise 9.4. By Lemma 9.2,

$$\begin{aligned} &\measuredangle IA'B' \equiv -\measuredangle IBA, && \measuredangle IB'C' \equiv -\measuredangle ICB, && \measuredangle IC'A' \equiv -\measuredangle IAC, \\ &\measuredangle IB'A' \equiv -\measuredangle IAB, && \measuredangle IC'B' \equiv -\measuredangle IBC, && \measuredangle IA'C' \equiv -\measuredangle ICA. \end{aligned}$$

It remains to apply the theorem on the sum of angles of triangle (Theorem 6.13) to show that $(A'I) \perp (B'C')$, $(B'I) \perp (C'A')$ and $(C'I) \perp (B'A')$.



Exercise 9.5. Guess the construction from the diagram (the two nonintersecting lines on the diagram are parallel).

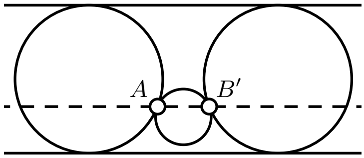
Exercise 9.8. Show first that for any $r > 0$ and any real numbers x, y distinct from 0, we have

$$\frac{r}{(x+y)/2} \neq \left(\frac{r}{x} + \frac{r}{y}\right)/2.$$

Denote by ℓ the line passing through Q, Q' and the center of the inversion. Note that for appropriately chosen isometry $\ell \to \mathbb{R}$, left hand side is the coordinate of the inversion of the center of Γ and right hand side is the coordinate of the of the center of inversion of Γ, assuming x and y are coordinates of the intersections $\ell \cap \Gamma$.

Exercise 9.9. The solution is given on page 147.

Exercise 9.10. Apply an inversion in a circle with the center at the only point of intersection of the circles; then use Theorem 9.11.



Exercise 9.13. Apply the inversion in a circle with center A. The point A will go to infinity, the two circles tangent at A will become parallel lines and the two parallel lines will become circles tangent at A.

Therefore you will see the picture as on the diagram and the statement becomes evident from its symmetry.

Exercise 9.18. Let P_1 and P_2 be the inversions of P in Ω_1 and Ω_2. Note that the points P, P_1 and P_2 are mutually distinct.

According to Theorem 7.1, there is unique circline Γ which pass through P, P_1 and P_2.

By Corollary 9.16, $\Gamma \perp \Omega_1$ and $\Gamma \perp \Omega_2$.

On the other hand if $\Gamma' \ni P$ and $\Gamma' \perp \Omega_1$, $\Gamma' \perp \Omega_2$ then by Theorem 9.14 we have $\Gamma' \ni P_1, P_2$. That is, $\Gamma' = \Gamma$.

Exercise 9.19. Apply Theorem 9.6(b), Exercise 6.21 and Theorem 8.2.

Exercise 9.20. Denote by T a point of intersection of Ω_1 and Ω_2. Let P be the foot point of T on (O_1O_2). Show first that

$$\triangle O_1PT \sim \triangle O_1TO_2 \sim \triangle TPO_2.$$

Conclude that P coincides with the inversions of O_1 in Ω_2 and of O_2 in Ω_1.

Exercise 9.21. Since $\Gamma \perp \Omega_1$ and $\Gamma \perp \Omega_2$, Corollary 9.15 implies that the circles Ω_1 and Ω_2 are inverted in Γ to themselves.

Therefore the points A and B are inversions of each other.

Since $\Omega_3 \ni A, B$, Corollary 9.16 implies that $\Omega_3 \perp \Gamma$.

Exercise 9.22. Follow the solution of Exercise 9.18.

Chapter 10

Exercise 10.2. Denote by D the midpoint of $[BC]$. Assume (AD) is the bisector of the angle at A.

Mark point $A' \in [AD)$ which is distinct from A and $AD = A'D$. Note that $\triangle CAD \cong \triangle BA'D$. In particular $\measuredangle BAA' = \measuredangle AA'B$. It remains to apply Theorem 4.2 for $\triangle ABA'$.

Exercise 10.5. Arguing by contradiction, assume

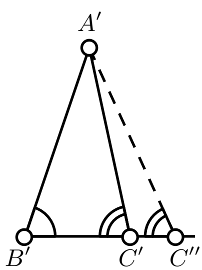
$$2\cdot(\measuredangle ABC + \measuredangle BCD) \equiv 0,$$

but $(AB) \nparallel (CD)$. Let Z be the point of intersection of (AB) and (CD).

Note that

$$2\cdot\measuredangle ABC \equiv 2\cdot\measuredangle ZBC, \qquad 2\cdot\measuredangle BCD \equiv 2\cdot\measuredangle BCZ.$$

Apply Proposition 10.4 to $\triangle ZBC$ and try to arrive to a contradiction.



Exercise 10.6. Let $C'' \in [B'C')$ be the point such that $B'C'' = BC$.

Note that by SAS, $\triangle ABC \cong \triangle A'B'C''$. Conclude that $\measuredangle B'C'A' = \measuredangle B'C''A'$.

Therefore it is sufficient to show that $C'' = C'$. If $C' \neq C''$ apply Proposition 10.4 to $\triangle A'C'C''$ and try to arrive to a contradiction.

(This proof was given in the Euclid's Elements [2, Book I, Proposition 26].)

Exercise 10.7. Use Exercise 5.4 and Proposition 10.4.

Alternatively, use the same argument as in the solution of Exercise 5.14.

Exercise 10.10. Note that $|\measuredangle ADC| + |\measuredangle CDB| = \pi$. Then apply the definition of defect.

Exercise 10.3. The statement is evident if A, B, C and D lie on one line.

In the remaining case, denote by O the center of the circumscribed circle. Apply theorem about isosceles triangle (4.2) to the triangles $\triangle AOB$, $\triangle BOC$, $\triangle COD$, $\triangle DOA$.

(Note that in the Euclidean plane the statement follows from Theorem 8.11 and Exercise 6.21, but one can not use these statements in the absolute plane.)

Chapter 11

Exercise 11.1. Let A and B be the ideal points of h-line ℓ. Note that the center of the Euclidean circle containing ℓ lies at the intersection of lines tangent to absolute at the ideal points of ℓ.

Exercise 11.2. Assume A is an ideal point of h-line ℓ and $P \in \ell$. Denote by P' the inversion of P in the absolute. By Corollary 9.15, ℓ lies in the intersection of h-plane and the (necessary unique) circline passing through P, A and P'.

Exercise 11.3. Denote by Ω the absolute and by O its center.

Let Γ be the circline containing $[PQ]_h$. Note that $[PQ]_h = [PQ]$ if and only if Γ is a line.

Denote by P' the inversion of P in Ω. Note that O, P and P' lie on one line.

By definition of h-line, $\Omega \perp \Gamma$. By Corollary 9.15, Γ pass through P and P'. Therefore Γ is a line if and only if it pass through O.

Exercise 11.4. Assume that the absolute is a unit circle.

Set $a = OX = OY$. Note that $0 < a < \frac{1}{2}$ and

$$OX_h = \ln \tfrac{1+a}{1-a}, \qquad XY_h = \ln \tfrac{(1+2\cdot a)\cdot(1-a)}{(1-2\cdot a)\cdot(1+a)}.$$

It remains to check that inequality

$$1 < \tfrac{1+a}{1-a} < \tfrac{(1+2\cdot a)\cdot(1-a)}{(1-2\cdot a)\cdot(1+a)}$$

holds if $0 < a < \frac{1}{2}$.

Exercise 11.5. Spell the meaning of terms “perpendicular” and “h-line” and then apply Exercise 9.18.

Exercise 11.8. Apply Main observation (11.7b).

Exercise 11.9. By Corollary 9.25 and Theorem 9.6, the quantity ϑ survives under inversion in a circle perpendicular to the absolute. Therefore by Main observation (11.7) we can assume that midpoint of $[PQ]_h$ to the center of absolute.

Set $a = OP = OQ$ then

$$\vartheta(P,Q) = \frac{8}{(\frac{1}{a}-a)^2}, \qquad\qquad \delta(P,Q) = \frac{(1+a)^2}{(1-a)^2}.$$

Hence the identity follows.

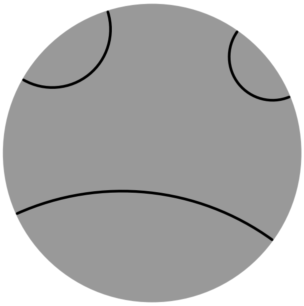
Exercise 11.13. Denote by X and Y the points of intersections of (OP) and Δ'_ρ. Consider an isometry $(OP) \to \mathbb{R}$ so that O corresponds to O, denote by x, y, p and $\hat{p}$ the real numbers corresponding to X, Y, P and $\hat{P}$.

We can assume that $p > 0$ and $x < y$.
Note that $\hat{p} = \frac{x+y}{2}$ and

$$\frac{(1+x)\cdot(1-p)}{(1-x)\cdot(1+p)} = \frac{(1+p)\cdot(1-y)}{(1-p)\cdot(1+y)}.$$

It remains to show that all this implies $0 < \hat{p} < p$.

Exercise 11.23. Look at the diagram and think.



Chapter 12

Exercise 12.3. Note that the angle of parallelism of B to $(CD)_h$ is bigger than $\frac{\pi}{4}$, and it converges to $\frac{\pi}{4}$ as $CD_h \to \infty$.

Applying Proposition 12.1, we get

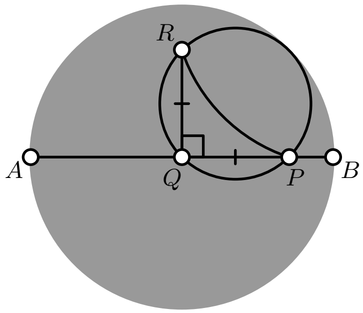
$$BC_h < \tfrac{1}{2}\cdot \ln \frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \ln\left(1+\sqrt{2}\right).$$

The right hand side is the limit of BC_h if $CD_h \to \infty$. Therefore $\ln\left(1+\sqrt{2}\right)$ is the optimal upper bound.

Exercise 12.4. Consider hyperbolic triangle $\triangle_h PQR$ with right angle at Q, such that $PQ = QR$ and the vertices P, Q and R lie on a horocycle.

Without loss of generality, we may assume that Q is the center of absolute. In this case $\measuredangle_h PQR = \measuredangle PQR = \pm\frac{\pi}{2}$.

The rest of proof should be easy to guess from the picture. The answer is $2\cdot\ln(1+\sqrt{2})$.



Exercise 12.7. In hyperbolic plane, the AAA-congruence condition (12.6) implies that any nondegenerate triangle is mapped to a congruent triangle. Hence the statement follows.

Exercise 12.10. Let us apply Proposition 12.9.

$$\begin{aligned}\operatorname{circum}_h(r+1) &= \pi\cdot(e^{r+1} - e^{-r-1}) = \pi\cdot e\cdot(e^{r} - e^{-r-2}) > \\ &> \pi\cdot e\cdot(e^{r} - e^{-r}) = e\cdot \operatorname{circum}_h(r) \geqslant 2\cdot \operatorname{circum}_h(r).\end{aligned}$$

Chapter 13

Exercise 13.1. Assume that triple of noncollinear points P, Q and R mapped to one line ℓ. Note that all three lines (PQ), (QR) and (RP) are mapped to ℓ. Therefore any line which connects two points on these three lines is mapped to ℓ.

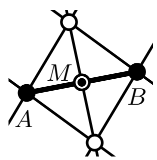
Note that any point in the plane lies on a line passing through two distinct points on these three lines. Therefore whole plane is mapped to ℓ. The later contradicts that the map is a bijection.

Exercise 13.2. Assume two distinct lines ℓ and m are mapped to intersecting lines ℓ' and m'. Denote by P' their point of intersection.

Let P be the inverse image of P'. By definition of affine map it has to lie on both ℓ and m; that is, ℓ and m are intersecting. Hence the result follows.

Exercise 13.3. According to the remark before the exercise, it is sufficient to construct the midpoint of $[AB]$ with ruler and parallel tool.

Guess the construction from the diagram.



Exercise 13.4. Let us denote by O, E, A and B the points with coordinates $(0,0)$, $(1,0)$, $(a,0)$ and $(b,0)$ correspondingly.

To construct point W with coordinates $(0,a+b)$, try to construct two parallelograms $\square ABPQ$ and $\square BWPQ$.

To construct Z with coordinates $(0,a\cdot b)$ choose a line $(OE') \neq (OE)$ and try construct the points $A' \in (OE')$ and $Z \in (OE)$ so that $\triangle OEE' \sim \triangle OAA'$ and $\triangle OE'B \sim \triangle OA'Z$.

Exercise 13.5. Draw two parallel chords $[XX']$ and $[YY']$. Set $Z = (XY) \cap (X'Y')$ and $Z' = (XY') \cap (X'Y)$. Note that (ZZ') pass through the center.

Repeat the same construction for an other pair of parallel chords. The center lies in the intersection of the obtained lines.

Exercise 13.6. Assume a construction produce two perpendicular lines. Apply a shear mapping which changes the angle between the lines.

Note that it transforms the construction to the same construction for an other free choices points. Therefore, this construction does not produce perpendicular lines in general (it might be the center only by coincidence).

Exercise 13.11. Denote by ℓ the line of the reflection. Choose a circle Γ with center not on ℓ. Let Ω be the inversion of ℓ in Γ; note that Ω is a circle.

Denote by ι_Γ and ι_Ω the inversions in Γ and Ω. Show that the composition $\iota_\Gamma\circ\iota_\Omega\circ\iota_\Gamma$ gives the reflection in ℓ.

Chapter 14

Exercise 14.1. Since O, P and P' lie on one line we have that coordinates of P' are proportional to the coordinates of P. The y coordinate of P' has to be equal to 1. Therefore P' has coordinates $(\frac{1}{y},1,\frac{z}{y})$.

Exercise 14.5. Assume that the lines (AB) and $(A'B')$ intersect at O. Since $(AB')\parallel(BA')$, we get $\triangle OAB'\sim\triangle OBA'$ and

$$\frac{OA}{OB}=\frac{OB'}{OA'}$$

Similarly since $(AC')\parallel(CA')$, we get

$$\frac{OA}{OC}=\frac{OC'}{OA'}.$$

Therefore

$$\frac{OB}{OC}=\frac{OC'}{OB'}.$$

Applying SAS similarity condition we get $\triangle OBC'\sim\triangle OCB'$. Therefore $(BC')\parallel(CB')$.

The case $(AB)\parallel(A'B')$ is similar.

Exercise 14.6. Assume contrary; let us use upper and lower cases of the same letter denote a point and its dual line.

Choose two parallel lines ℓ and m. Let L and M be their dual points. Set $s=(ML)$, then its dual point S have to lie on both ℓ and m; a contradiction.

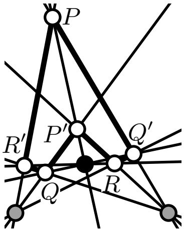
Exercise 14.8. Assume that point M has coordinates (a,b) and line s described by equation $p{\cdot}x+q{\cdot}y=1$. Then $M\in s$ is equivalent to $p{\cdot}a+q{\cdot}b=1$. The later is equivalent to $m\ni S$ if m is the line formed by solution of the equation $a{\cdot}x+b{\cdot}y=1$ and S is the point with coordinates (p,q).

To extend this bijection two whole plane assume that ideal line corresponds to the origin and the ideal point formed by pencil of lines $ax-by=c$ for different values of c corresponds to the line described by equation $ax+by=0$.

Exercise 14.11. Assume one set of concurrent lines a, b, c, and another set of concurrent lines a', b', c' are given. Set

$$\begin{aligned} P&=b\cap c', & Q&=c\cap a', & R&=a\cap b'\\ P'&=b'\cap c, & Q'&=c'\cap a, & R'&=a'\cap b\end{aligned}$$

Then the lines (PP'), (QQ') and (RR') are concurrent.



Exercise 14.12. Assume (AA') and (BB') are the given lines and C is the given point. Apply Dual Desargues' theorem 14.9 to construct C' so that (AA'), (BB') and (CC') are concurrent. Since $(AA') \parallel (BB')$ we get $(AA') \parallel (BB') \parallel (CC')$.

A similar solution can be build on the Dual Pappus's theorem, see the solution of Exercise 14.11.

Exercise 14.13. Let A, B, C and D be the point provided by Axiom III. Given a line ℓ we can assume that $A \notin \ell$, otherwise permute the labels on the points. Then by axioms I and II, the three lines (AB), (AC) and (AD) intersect ℓ at distinct points. In particular ℓ contains 3 points.

Exercise 14.14. Let A, B, C and D be the point provided by Axiom III. Show that the lines (AB), (BC), (CD) and (DA) satisfy Axiom III′. The proof of converse is similar.

Exercise 14.15. Let ℓ be a line with $n+1$ points on it.

By Axiom III, given any line m there is a point P which does not lie on ℓ nor on m.

By axioms I and II, there is a bijection between the lines passing through P and the points on ℓ. In particular there are exactly $n+1$ lines passing through P.

The same way there is bijection between the lines passing through P and the points on m. Hence (a) follows.

Fix a point X. By Axiom I, any point Y in the plane lies in unique line passing through X. From part (a), each such line contains X and yet n point. Hence the (b) follows.

To prove (c), show that the equation $n^2+n+1=10$ does not admit an integer solution and then apply part (b).

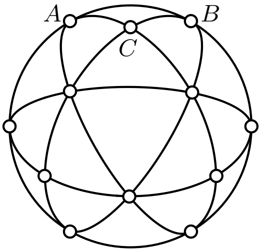
Chapter 15

Exercise 15.2. Applying Pythagorean theorem, we get

$$\cos AB_s = \cos AC_s \cdot \cos BC_s = \tfrac{1}{2}.$$

Therefore $AB_s = \frac{\pi}{3}$.

To see this without Pythagorean theorem, look at the tessellation of the sphere on the picture; it is made from 24 copies of $\triangle_s ABC$ and yet 8 equilateral triangles. From the symmetry of this tessellation, it follows that $[AB]_s$ occupies $\frac{1}{6}$ of the equator.



Exercise 15.6. Note that points on Ω do not move. Moreover, the points inside Ω are mapped outside of Ω and the other way around.

Further note that this maps sends circles to circles; moreover the perpendicular circles are mapped to perpendicular circles. In particular the circle perpendicular to Ω are mapped to itself.

Consider arbitrary point $P \notin \Omega$. Denote by P' the inversion of P in Ω. Choose two distinct circles which pass through P and P'. According to Corollary 9.16, $\Gamma_1 \perp \Omega$ and $\Gamma_2 \perp \Omega$.

Therefore the inversion in Ω sends Γ_1 to itself and the same holds for Γ_2.

The image of P has to lie on Γ_1 and Γ_2. Since the image has to be distinct from P, we get that it has to be P'.

Exercise 15.7. Apply Theorem 15.3b.

Exercise 15.8. Set $z = P'Q'$. Note that $\frac{y}{z} \to 1$ as $x \to 0$.

It remains to show that

$$\lim_{x \to 0} \frac{z}{x} = \frac{2}{1 + OP^2}.$$

Denote by O the center of the unit sphere Σ. Note that stereographic projection is the restriction of inversion in the sphere to the plane. Note that there is plane Π passing through O, P, Q, P' and Q'. In the plane Π the map $Q \mapsto Q'$ is an inversion in a circle $\Sigma \cap \Pi$.

This reduce the problem to Euclidean plane geometry. The remaining calculations in Π are similar to those in the proof of Lemma 12.8.

Exercise 15.9. Denote by O the center of the sphere.

To prove the identities

$$s = \tfrac{\operatorname{tg} a}{\cos b}, \qquad t = \operatorname{tg} b, \qquad u = \operatorname{tg} c$$

consider the sections by the planes containing the following triple of points (NBO), (BCO) and (CNO).

The remaining part of exercise is algebraic manipulations similar to the one given in the proof of Theorem 16.8.

Exercise 15.10. To prove (a), observe and use that $OA' = OB' = OC'$.

To prove (b), note that medians of spherical triangle $\triangle_s ABC$ map to the medians of Euclidean triangle $\triangle A'B'C'$. It remains to apply Theorem 7.5 for $\triangle A'B'C'$.

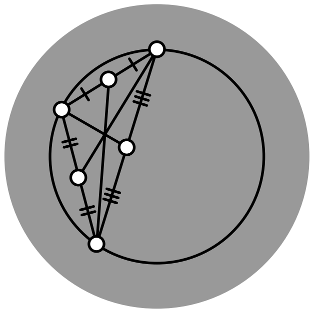
Chapter 16

Exercise 16.1. Let N, O, S, P, P' and $\hat{P}$ be as on the diagram on page 125.

Notice that $\triangle NOP \sim \triangle NP'S \sim \triangle P'\hat{P}P$ and $2 \cdot NO = NS$. It remains to do algebraic manipulations.

Exercise 16.3. Consider the bijection $P \mapsto \hat{P}$ of the h-plane with absolute Ω. Note that $\hat{P} \in [A_iB_i]$ if and only if $P \in \Gamma_i$.

Exercise 16.4. The observation follows since the reflection in the perpendicular bisector of $[PQ]$ is a motion of Euclidean plane and h-plane at the same time.

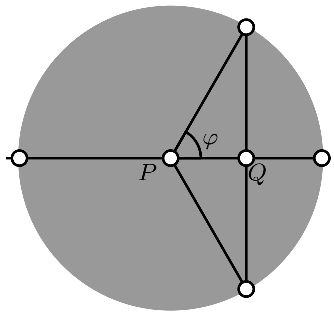


Without loss of generality we may assume that the center of circumcircle coincides with the center of absolute. In this case the h-median of the triangle coincide with the Euclidean medians. It remains to apply Theorem 7.5

Exercise 16.5. Let us denote by $\check\ell$ and $\check m$ the h-lines corresponding to ℓ and m in the conformal model. We need to show that $\check\ell \perp \check m$ as arcs in the Euclidean plane.

Note that the point Z where s and t meet is the center of the circle Γ containing $\check\ell$.

If $\tilde m$ passing through Z then the inversion in Γ exchanges the ideal points of $\check\ell$. In particular $\check\ell$ maps to itself. Hence the result follows.



Exercise 16.6. We can assume that P is the center of absolute. Then

$$PQ = \cos\varphi.$$

Therefore

$$PQ_h = \tfrac{1}{2}\cdot\ln\frac{1+\cos\varphi}{1-\cos\varphi}.$$

Exercise 16.7. Apply Exercise 16.6 for $\varphi = \frac{\pi}{3}$.

Exercise 16.10. In the Euclidean plane the circle Γ_2 is tangent to k; that is, the point T of intersection of Γ_2 and k is unique. It defines unique line (PT) parallel to ℓ.

Chapter 17

Exercise 17.1. Denote by z, v and w the complex coordinates of Z, V and W correspondingly. Then

$$\begin{aligned}\measuredangle ZVW + \measuredangle VWZ + \measuredangle WZV &\equiv \arg\tfrac{w-v}{z-v} + \arg\tfrac{z-w}{v-w} + \arg\tfrac{v-z}{w-z} \equiv\\ &\equiv \arg\tfrac{(w-v)\cdot(z-w)\cdot(v-z)}{(z-v)\cdot(v-w)\cdot(w-z)} \equiv\\ &\equiv \arg(-1) \equiv\\ &\equiv \pi.\end{aligned}$$

Exercise 17.2. Note and use that

$$\measuredangle EOV = \measuredangle WOZ = \arg v, \qquad \frac{OW}{OZ} = \frac{OZ}{OW} = |v|.$$

Exercise 17.4. Note that

$$\begin{aligned}\arg \frac{(v-u)\cdot(z-w)}{(v-w)\cdot(z-u)} &\equiv (\arg \frac{v-u}{z-u} + \arg \frac{z-w}{v-w} \equiv \\ &\equiv (\measuredangle ZUV + \measuredangle VWZ).\end{aligned}$$

The statement follows since value $\frac{(v-u)\cdot(z-w)}{(v-w)\cdot(z-u)}$ is real if and only if

$$2\cdot \arg \frac{(v-u)\cdot(z-w)}{(v-w)\cdot(z-u)} \equiv 0.$$

Exercise 17.8. Show that the inverse of each elementary transformation is elementary and use Proposition 17.6.

Exercise 17.9. The Möbius transformation

$$f(z) = \frac{(z_1 - z_\infty)\cdot(z - z_0)}{(z_1 - z_0)\cdot(z - z_\infty)}$$

meets the conditions.

To show uniqueness, assume there is an other Möbius transformation $g(z)$ which meets the conditions. Then the composition $h = g \circ f^{-1}$ is a Möbius transformation; set

$$h(z) = \frac{a\cdot z + b}{c\cdot z + d}.$$

Note that $h(\infty) = \infty$; therefore $c = 0$. Further $h(0) = 0$ implies $b = 0$. Finally, since $h(1) = 1$ we get $\frac{a}{d} = 1$. Therefore h is the *identity*; that is, $h(z) = z$ for any z. It follows that $g = f$.

Exercise 17.10. Assume z is the complex coordinate of point Z and z' is the coordinate of its inversion Z' in the circle with center at W with coordinate w and radius r.

By definition of inversion

$$\arg(z - w) = \arg(z' - w), \qquad |z - w|\cdot|z' - w| = r^2$$

It follows that

$$(\bar{z}' - \bar{w})\cdot(z - w) = r^2.$$

Equivalently,

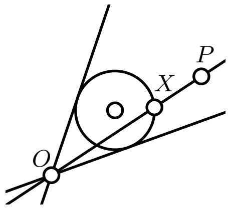
$$z' = \overline{\left(\frac{\bar{w}\cdot z + [r^2 - |w|^2]}{z - w}\right)}.$$

Exercise 17.12. Check the statement for each elementary transformation. Then apply Proposition 17.6.

Exercise 17.14. Apply Schwarz–Pick theorem for a function f such that $f(0) = 0$ and then apply Lemma 11.10.

Chapter 18

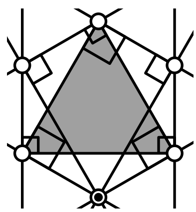
Exercise 18.3. Draw a line ℓ through the given point P and the point of intersection, say O of the lines. Construct a circle Γ tangent to both lines which crosses ℓ. Denote by X one of the points of intersections.



Consider the homothety with the center at O which sends X to P. The image of Γ is the needed circle.

Exercise 18.6. Note that with set square we can construct a line parallel to given line through the given point. It remains to modify the construction in Exercise 13.3.

Exercise 18.8. Assume that two vertices have rational coordinates, say (a_1, b_1) and (a_2, b_2). Use that the number $\sqrt{3}$ is irrational to show that the third vertex is an irrational point.



Exercise 18.9. Guess the construction from the diagram.

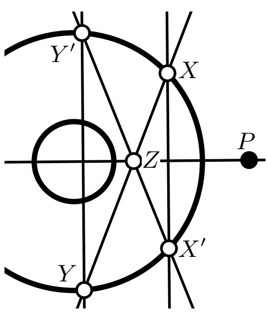
Exercise 18.10. Apply the same reasoning as in the solution of Exercise 9.9.

Exercise 18.11. Take the perspective projection as in Exercise 14.1. Choose $A = (1,1,1)$ and $B = (1,3,1)$ then $M = (1,2,1)$ is their midpoint.

Their images are $A' = (1,1,1)$, $B' = (\frac{1}{3}, 1, \frac{1}{3})$ and $M' = (\frac{1}{2}, 1, \frac{1}{2})$. Clearly M' is not the midpoint of $[A'B']$.

Exercise 18.14. The line v polar to V is tangent to Γ. Since $V \in p$, by Claim 18.13, we get $P \in v$; that is, $(PV) = v$. Hence the statement follows.

Exercise 18.15. Choose a point P outside of bigger circle. Construct the dual lines for given point for both circles. Note that these lines are parallel.



Assume the first line intersect the bigger circle at the points X and X'. the second line intersect the bigger circle at the points Y and Y'. Set $Z = (XY) \cap (X'Y')$. Note that (PZ) pass through the common center.

The center can be found as the intersection of the constructed line and an other line constructed the same way.

Chapter 19

Exercise 19.1. Assume contrary; that is there is a point $W \in [XY]$ such that $W \notin \blacktriangle ABC$.

Without loss of generality, we may assume that W and A lie on opposite sides from the line (BC).

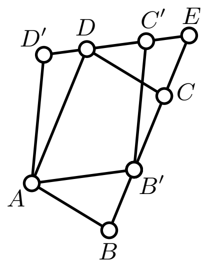
It imples that both segments $[WX]$ and $[WY]$ intersect (BC). By Axiom II, $W \in (BC)$, a contradiction.

Exercise 19.3. To prove the "only if" part consider the line passing through the vertex and parallel to the opposite side.

To prove "if" part, use Pasch's theorem 3.10.

Exercise 19.4. Assume contrary; that is a solid square can be presented as a union of finite collection of segments $[A_1B_1], \dots, [A_nB_n]$ and one-point sets $\{C_1\}, \dots, \{C_k\}$. Note that solid square contains infinte number of mutually nonparallel segments. Therefore we can choose a segment $[PQ]$ in the solid square which is not parallel to any segment $[A_1B_1], \dots, [A_nB_n]$.

It follows that $[PQ]$ has at most one common point with each the sets $[A_iB_i]$ and $\{C_i\}$. We arrive to a contradiction since $[PQ]$ contains infinte number of points.



Exercise 19.5. Note first that among elementary sets only one-point set are subsets of the a circle.

It remains to note that any circle contains infinite number of points.

Exercise 19.15. Denote by E the point of intersection of the lines (BC) and $(C'D')$.

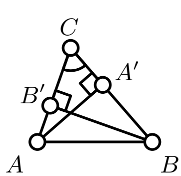
Use Proposition 19.14 to prove two identities

$$\operatorname{area}(\blacksquare AB'ED) = \operatorname{area}(\blacksquare ABCD), \quad \operatorname{area}(\blacksquare AB'ED) = \operatorname{area}(\blacksquare AB'C'D').$$

Hence the exercise follows.

Exercise 19.17. Without loss of generality, we may assume that the angles $\angle ABC$ and $\angle BCA$ are acute.

Denote by A' and B' the foot points of A and B on (BC) and (AC) correspondingly. Note that $h_A = AA'$ and $h_B = BB'$.



Note that $\triangle AA'C \sim \triangle BB'C$; indeed the the angle at C is shared and the angles at A' and B' are right. In particular

$$\frac{AA'}{BB'} = \frac{AC}{BC}$$

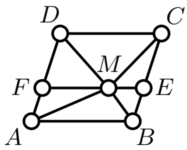
or, equivalently,

$$h_A \cdot BC = h_B \cdot AC.$$

The same argument, we get

$$h_C \cdot AB = h_B \cdot AC.$$

Hence the statement follows.



Exercise 19.18. Draw the line ℓ through M parallel to the sides $[AB]$ and $[CD]$; it subdivides $\blacksquare ABCD$ into two solid parallelograms, say $\blacksquare ABEF$ and $\blacksquare CDFE$. In particular,

$$\text{area}(\blacksquare ABCD) = \text{area}(\blacksquare ABEF) + \text{area}(\blacksquare CDFE).$$

By Proposition 19.14 and Theorem 19.16 we get

$$\text{area}(\blacksquare ABM) = \tfrac{1}{2}\cdot \text{area}(\blacksquare ABEF), \quad \text{area}(\blacksquare CDM) = \tfrac{1}{2}\cdot \text{area}(\blacksquare CDFE)$$

Hence the result follows.

Exercise 19.19. Denote by h_A and h_C the distances from A and C to the line (BD). According to Theorem 19.16,

$$\begin{aligned} \text{area}(\blacktriangle ABM) &= \tfrac{1}{2}\cdot h_A\cdot BM; & \text{area}(\blacktriangle BCM) &= \tfrac{1}{2}\cdot h_C\cdot BM;\\ \text{area}(\blacktriangle CDM) &= \tfrac{1}{2}\cdot h_C\cdot DM; & \text{area}(\blacktriangle ABM) &= \tfrac{1}{2}\cdot h_A\cdot DM.\end{aligned}$$

Therefore

$$\begin{aligned}\text{area}(\blacktriangle ABM)\cdot \text{area}(\blacktriangle CDM) &= \tfrac{1}{4}\cdot h_A\cdot h_C\cdot DM\cdot BM =\\ &= \text{area}(\blacktriangle BCM)\cdot \text{area}(\blacktriangle DAM).\end{aligned}$$

Exercise 19.20. Let I be the incenter of $\triangle ABC$. Note that $\blacktriangle ABC$ can be subdivided into $\blacktriangle IAB$, $\blacktriangle IBC$ and $\blacktriangle ICA$.

It remains to applying Theorem 19.16 to each of these triangles and summing up the results.

Exercise 19.22. Assuming $a > b$, we subdivided $\blacksquare_c$ into $\blacksquare_{a-b}$ and for triangles congruent to $\blacktriangle$. Therefore

❶ $$\text{area}\,\blacksquare_c = \text{area}\,\blacksquare_{a-b} + 4\cdot \text{area}\,\blacktriangle.$$

According to Theorem 19.16, $\text{area}\,\blacktriangle = \tfrac{1}{2}\cdot a\cdot b$. Therefore the identity ❶ can be written as

$$c^2 = (a-b)^2 + 2\cdot a\cdot b.$$

Simplifying, we get Pythagorean theorem.

The cases $a = b$ is yet simpler. The case $b > a$ can be done the same way.

Exercise 19.23. If X is a point inside of triangle $\triangle ABC$ then $\blacktriangle ABC$ is subdivided into $\blacktriangle ABX$, $\blacktriangle BCX$ and $\blacktriangle CAX$. Therefore

$$\text{area}(\blacktriangle ABX) + \text{area}(\blacktriangle BCX) + \text{area}(\blacktriangle CAX) = \text{area}(\blacktriangle ABC).$$

Set $a = AB = BC = CA$. Denote by h_1, h_2 and h_3 are the distances from X to the sides $[AB]$, $[BC]$ and $[CA]$. Then by Theorem 19.16,

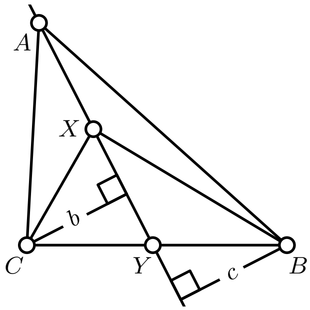
$$\begin{aligned}\text{area}(\blacktriangle ABX) &= \tfrac{1}{2}\cdot h_1\cdot a,\\ \text{area}(\blacktriangle BCX) &= \tfrac{1}{2}\cdot h_2\cdot a,\\ \text{area}(\blacktriangle CAX) &= \tfrac{1}{2}\cdot h_3\cdot a.\end{aligned}$$

Therefore

$$h_1 + h_2 + h_3 = 2\cdot\frac{\operatorname{area}(\blacktriangle ABC)}{a}.$$

Exercise 19.25. Let X be a point inside $\triangle ABC$. Denote by Y the point of intersection of (AX) and $[BC]$.

Denote by b and c the distances from B and C to the line (AX).



By Theorem 19.16, we get that the following equivalences

$$\begin{gathered}\operatorname{area}(\blacktriangle AXB) = \operatorname{area}(\blacktriangle AXC),\\ \Updownarrow\\ b = c,\\ \Updownarrow\\ \operatorname{area}(\blacktriangle AYB) = \operatorname{area}(\blacktriangle AYC),\\ \Updownarrow\\ BY = CY.\end{gathered}$$

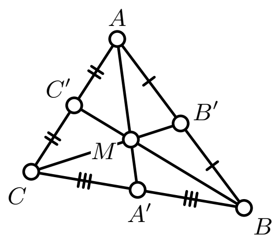
Exercise 19.26. Denote by M the intersection of two medians $[AA']$ and $[BB']$.

From Exercise 19.25 we have

$$\operatorname{area}(\blacktriangle ABM) = \operatorname{area}(\blacktriangle ACM), \qquad \operatorname{area}(\blacktriangle ABM) = \operatorname{area}(\blacktriangle CBM).$$

Therefore

$$\operatorname{area}(\blacktriangle BCM) = \operatorname{area}(\blacktriangle ACM).$$



According to Exercise 19.25, M lies on the median $[CC']$. That is, medians $[AA']$, $[BB']$ and $[CC']$ intersect at one point M.

By Theorem 19.16, we get

$$\begin{aligned}\operatorname{area}(\blacktriangle C'AM) &= \tfrac{1}{2}\cdot\operatorname{area}(\blacktriangle BAM)\\ &= \tfrac{1}{2}\cdot\operatorname{area}(\blacktriangle CAM)\end{aligned}$$

Applying Claim 19.24, we get

$$\frac{MC'}{MC} = \frac{\operatorname{area}(\blacktriangle C'AM)}{\operatorname{area}(\blacktriangle CAM)} = \tfrac{1}{2}.$$

Exercise 19.27. Let $\mathcal{P}_n$ and $\mathcal{Q}_n$ be the solid regular n-gons so that Γ is inscribed in $\mathcal{Q}_n$ and circumscribed around $\mathcal{P}_n$. Clearly

$$\mathcal{P}_n \subset \mathcal{D} \subset \mathcal{Q}_n.$$

Show that $\frac{\operatorname{area}\mathcal{P}_n}{\operatorname{area}\mathcal{Q}_n} = (\cos\frac{\pi}{n})^2$, in particular $\frac{\operatorname{area}\mathcal{P}_n}{\operatorname{area}\mathcal{Q}_n} \to 1$ as $n \to \infty$.

Next show that $\operatorname{area}\mathcal{Q}_n < 100$, say for all $n \geqslant 100$.

These two statements imply that $(\operatorname{area}\mathcal{Q}_n - \operatorname{area}\mathcal{P}_n) \to 0$. Hence the result follows.

Index

Used resources

[1] Aleksandrov, A. D. Minimal foundations of geometry. Siberian Math. J. 35 (1994), no. 6, 1057–1069 .

[2] *Euclid's Elements.*

[3] Eugenio Beltrami, *Teoria fondamentale degli spazii di curvatura costante,* Annali. di Mat., ser II, 2 (1868), 232–255.

[4] Birkhoff, George David, A Set of postulates for plane geometry, based on scale and protractors, *Annals of Mathematics* 33 (1932), 329–345.

[5] Marvin J. Greenberg, *Euclidean and Non-Euclidean Geometries: Development and History,* 4th ed., New York: W. H. Freeman, 2007.

[6] Kiselev, A. P., *Kiselev's Geometry. Book I. Planimetry,* Adapted from Russian by Alexander Givental.

[7] Lambert, Johann Heinrich, *Theorie der Parallellinien,* F. Engel, P. Stäckel (Eds.) (1786) Leipzig.

[8] Legendre, Adrien-Marie, *Eléments de géométrie*, 1794.

[9] Лобачевский, Никлай Иванович, О началах геометрии, *Казанский вестник,* вып. 25–28 (1829–1830 гг.).

[10] Lobachevsky, N. I., *Geometrische Untersuchungen zur Theorie der Parallellinien.* Berlin: F. Fincke, 1840.

[11] Moise, Edwin, *Elementary Geometry From An Advanced Standpoint,* 3rd ed. Boston: Addison-Wesley. 1990.

[12] Prasolov, Viktor, *Problems in Plane and Solid Geometry*, translated and edited by Dimitry Leites. 2006.

[13] Saccheri, Giovanni Girolamo, *Euclides ab omni nævo vindicatus*, 1733.

[14] Шарыгин, Игорь Фёдорович, *Геометрия 7–9*, М.: Дрофа, 1997.