Euclidean plane and its relatives

A minimalist introduction

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References

Introduction

This book is meant to be rigorous, conservative, elementary and minimalist. At the same time it includes about the maximum what students can absorb in one semester.

Approximately third of the material used to be covered in high school, not any more.

The present book is based on the courses given by the author at the Pennsylvania State University as an introduction into Foundations of Geometry. The lectures were oriented to sophomore and senior university students. These students already had a calculus course. In particular, they are familiar with the real numbers and continuity. It makes it possible to cover the material faster and in a more rigorous way than it could be done in high school.

Prerequisite

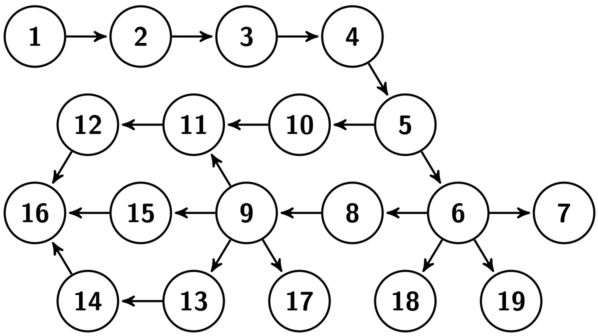
The students should be familiar with the following topics.

- Elementary set theory: $\in$, $\cup$, $\cap$, $\setminus$, $\subset$, $\times$.
- Real numbers: intervals, inequalities, algebraic identities.
- Limits, continuous functions and Intermediate value theorem.
- Standard functions: absolute value, natural logarithm, exponential function. Occasionally, trigonometric functions are used, but these parts can be ignored.
- Chapter 13 use matrix algebra of 2×2-matrices.
- To read Chapter 15, it is better to have some previous experience with *scalar product*, also known as *dot product*.
- To read Chapter 17, it is better to have some previous experience with complex numbers.

Overview

We use the so called *metric approach* introduced by Birkhoff. It means that we define the Euclidean plane as a *metric space* which satisfies a list of properties (*axioms*). This way we minimize the tedious parts which are unavoidable in the more classical Hilbert's approach. At the same time the students have a chance to learn basic geometry of metric spaces.

Here is dependency graph of the chapters.



In (1) we give all the definitions necessary to formulate the axioms; it includes metric space, lines, angle measure, continuous maps and congruent triangles.

Further we do Euclidean geometry: (2) Axioms and immediate corollaries, (3) Half-planes and continuity, (4) Congruent triangles, (5) Circles, motions, perpendicular lines, (6) Parallel lines and similar triangles — this is the first chapter where we use Axiom V, an equivalent of Euclid's parallel postulate. In (7) we give the most classical theorem of triangle geometry; this chapter is included mainly as an illustration.

In the following two chapters we discuss geometry of circles on the Euclidean plane: (8) Inscribed angles, (9) Inversion. It will be used to construct the model of the hyperbolic plane.

Further we discuss non-Euclidean geometry: (10) Neutral geometry — geometry without the parallel postulate, (11) Conformal disc model — this is a construction of the hyperbolic plane, an example of a neutral plane which is not Euclidean. In (12) we discuss geometry of the constructed hyperbolic plane — this is the highest point in the book.

In the reamining chapters we discuss some additional topics: (13) Affine geometry, (14) Projective geometry, (15) Spherical geometry,

(16) Projective model of the hyperbolic plane, (17) Complex coordinates, (18) Geometric constructions, (19) Area. The proofs in these chapters are not completely rigorous.

We encourage to use the visual assignments available at the author's website.

Disclaimer

It is impossible to find the original reference to most of the theorems discussed here, so I do not even try to. Most of the proofs discussed in the book already appeared in the Euclid's Elements.

Recommended books

- ◇ Kiselev's textbook [8] — a classical book for school students. Should help if you have trouble following this book.
- ◇ Moise's book, [13] — should be good for further study.
- ◇ Greenberg's book [7] — a historical tour in the axiomatic systems of various geometries.
- ◇ Prasolov's book [14] is perfect to master your problem-solving skills.
- ◇ Akopyan's book [1] — a collection of problems formulated in figures.
- ◇ Methodologically my lectures were very close to Sharygin's textbook [16]. This is the greatest textbook in geometry for school students, I recommend it to anyone who can read Russian.

Acknowlegments

Let me thank Matthew Chao, Alexander Lytchak, Alexei Novikov and Lukeria Petrunina for useful suggestions and correcting the misprints.

Chapter 1

Preliminaries

What is the axiomatic approach?

In the axiomatic approach, one defines the plane as anything which satisfies a given list of properties. These properties are called *axioms*. Axiomatic system for the theory is like the rules for a game. Once the axiom system is fixed, a statement is considered to be true if it follows from the axioms and nothing else is considered to be true.

The formulations of the first axioms were not rigorous at all. For example, Euclid described a *line* as *breadthless length* and *straight line* as a line which *lies evenly with the points on itself.* On the other hand, these formulations were sufficiently clear, so that one mathematician could understand the other.

The best way to understand an axiomatic system is to make one by yourself. Look around and choose a physical model of the Euclidean plane; imagine an infinite and perfect surface of a chalk board. Now try to collect the key observations about this model. Assume for now that we have intuitive understanding of such notions as *line* and *point.*

(i) We can measure distances between points.

(ii) We can draw a unique line which passes thru two given points.

(iii) We can measure angles.

(iv) If we rotate or shift we will not see the difference.

(v) If we change scale we will not see the difference.

These observations are good to start with. Further we will develop the language to reformulate them rigorously.

What is a model?

The Euclidean plane can be defined rigorously the following way:

Define a point *in the Euclidean plane as a pair of real numbers* (x, y) *and define the* distance *between the two points* (x_1, y_1) *and* (x_2, y_2) *by the following formula:*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

That is it! We gave a *numerical model* of Euclidean plane; it builds the Euclidean plane from the real numbers while the latter is assumed to be known.

Shortness is the main advantage of the model approach, but it is not intuitively clear why we define points and the distances this way.

On the other hand, the observations made in the previous section are intuitively obvious — this is the main advantage of the axiomatic approach.

An other advantage lies in the fact that the axiomatic approach is easily adjustable. For example, we may remove one axiom from the list, or exchange it to another axiom. We will do such modifications in Chapter 10 and further.

Metric spaces

The notion of metric space provides a rigorous way to say: *"we can measure distances between points"*. That is, instead of (i) on page 9, we can say *"Euclidean plane is a metric space"*.

1.1. Definition. *Let $\mathcal{X}$ be a nonempty set and d be a function which returns a real number $d(A, B)$ for any pair $A, B \in \mathcal{X}$. Then d is called* metric *on $\mathcal{X}$ if for any $A, B, C \in \mathcal{X}$, the following conditions are satisfied.*

(a) Positiveness:

$$d(A, B) \geqslant 0.$$

(b) $A = B$ if and only if

$$d(A, B) = 0.$$

(c) Symmetry:

$$d(A, B) = d(B, A).$$

(d) Triangle inequality:

$$d(A, C) \leqslant d(A, B) + d(B, C).$$

A metric space *is a set with a metric on it. More formally, a metric space is a pair* $(\mathcal{X}, d)$ *where* $\mathcal{X}$ *is a set and* d *is a metric on* $\mathcal{X}$.

The elements of $\mathcal{X}$ *are called* points *of the metric space. Given two points* $A, B \in \mathcal{X}$, *the value* $d(A, B)$ *is called* distance *from* A *to* B.

Examples

- *Discrete metric.* Let $\mathcal{X}$ be an arbitrary set. For any $A, B \in \mathcal{X}$, set $d(A, B) = 0$ if $A = B$ and $d(A, B) = 1$ otherwise. The metric d is called *discrete metric* on $\mathcal{X}$.
- *Real line.* Set of all real numbers ($\mathbb{R}$) with metric defined as

$$d(A, B) := |A - B|.$$

- *Metrics on the plane.* Let us denote by $\mathbb{R}^2$ the set of all pairs (x, y) of real numbers. Assume $A = (x_A, y_A)$ and $B = (x_B, y_B)$ are arbitrary points in $\mathbb{R}^2$. One can equip $\mathbb{R}^2$ with the following metrics:
  - *Euclidean metric,* denoted by d_2 and defined as

$$d_2(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}.$$

  - *Manhattan metric,* denoted by d_1 and defined as

$$d_1(A, B) = |x_A - x_B| + |y_A - y_B|.$$

  - *Maximum metric,* denoted by d_∞ and defined as

$$d_\infty(A, B) = \max\{|x_A - x_B|, |y_A - y_B|\}.$$

1.2. Exercise. *Prove that the following functions are metrics on* $\mathbb{R}^2$*: (a)* d_1*; (b)* d_2*; (c)* d_∞.

Shortcut for distance

Most of the time, we study only one metric on the space. Therefore, we will not need to name the metric function each time.

Given a metric space $\mathcal{X}$, the distance between points A and B will be further denoted by

$$AB \quad \text{or} \quad d_{\mathcal{X}}(A, B);$$

the latter is used only if we need to emphasize, that A and B are points of the metric space $\mathcal{X}$.

For example, the triangle inequality can be written as

$$AC \leqslant AB + BC.$$

For the multiplication, we will always use "$\cdot$", so AB should not be confused with $A{\cdot}B$.

Isometries, motions and lines

In this section, we define *lines* in a metric space. Once it is done the sentence *"We can draw a unique line which passes thru two given points."* becomes rigorous; see (ii) on page 9.

Recall that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is a *bijection*, if it gives an exact pairing of the elements of two sets. Equivalently, $f\colon \mathcal{X} \to \mathcal{Y}$ is a bijection, if it has an *inverse*; that is, a map $g\colon \mathcal{Y} \to \mathcal{X}$ such that $g(f(A)) = A$ for any $A \in \mathcal{X}$ and $f(g(B)) = B$ for any $B \in \mathcal{Y}$.

1.3. Definition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces and $d_{\mathcal{X}}$, $d_{\mathcal{Y}}$ be their metrics. The map*

$$f\colon \mathcal{X} \to \mathcal{Y}$$

is called distance-preserving *if*

$$d_{\mathcal{Y}}(f(A), f(B)) = d_{\mathcal{X}}(A, B)$$

for any $A, B \in \mathcal{X}$.

A bijective distance-preserving map is called an isometry*.*

Two metric spaces are called isometric *if there exists an isometry from one to the other.*

The isometry from a metric space to itself is also called a motion *of the space.*

1.4. Exercise. *Show that any distance-preserving map is* injective*; that is, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a distance-preserving map, then $f(A) \ne f(B)$ for any pair of distinct points $A, B \in \mathcal{X}$.*

1.5. Exercise. *Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a motion of the real line, then either (a) $f(x) = f(0) + x$ for any $x \in \mathbb{R}$, or (b) $f(x) = f(0) - x$ for any $x \in \mathbb{R}$.*

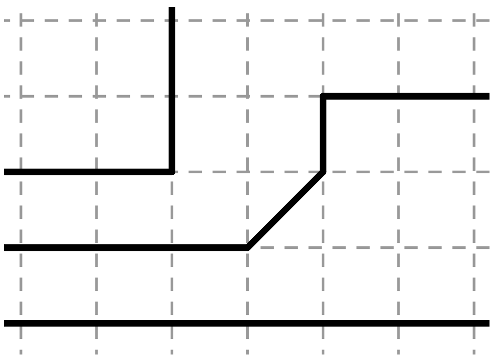
1.6. Exercise. *Prove that $(\mathbb{R}^2, d_1)$ is isometric to $(\mathbb{R}^2, d_\infty)$.*

1.7. Advanced exercise. *Describe all the motions of the Manhattan plane, defied on page 11.*

If $\mathcal{X}$ is a metric space and $\mathcal{Y}$ is a subset of $\mathcal{X}$, then a metric on $\mathcal{Y}$ can be obtained by restricting the metric from $\mathcal{X}$. In other words, the distance between two points of $\mathcal{Y}$ is defined to be the distance between these points in $= \mathcal{X}$. This way any subset of a metric space can be also considered as a metric space.

1.8. Definition. *A subset ℓ of metric space is called a* line, *if it is isometric to the real line.*

Note that a space with a discrete metric has no lines. The following picture shows examples of lines on the Manhattan plane $(\mathbb{R}^2, d_1)$.



1.9. Exercise. *Consider the graph $y = |x|$ in $= \mathbb{R}^2$. In which of the following spaces (a) $(\mathbb{R}^2, d_1)$, (b) $(\mathbb{R}^2, d_2)$, (c) $(\mathbb{R}^2, d_\infty)$ does it form a line? Why?*

1.10. Exercise. *How many points M are there on the line (AB) for which we have*

1. $AM = MB$?
2. $AM = 2{\cdot}MB$?

Half-lines and segments

Assume there is a line ℓ passing thru two distinct points P and $= Q$. In this case we might denote ℓ as (PQ). There might be more than one line thru P and Q, but if we write (PQ) we assume that we made a choice of such line.

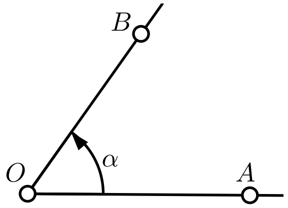
Let us denote by $[PQ)$ the *half-line* which starts at P and contains Q. Formally speaking, $[PQ)$ is a subset of (PQ) which corresponds to $[0, \infty)$ under an isometry $f\colon (PQ) \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.

1.11. Exercise. *Show that if $X \in [PQ)$, then $QX = |PX - PQ|$.*

The subset of line (PQ) between P and Q is called the *segment* between P and Q and denoted by $[PQ]$. Formally, the segment can be defined as the intersection of two half-lines: $[PQ] = [PQ) \cap [QP)$.

Angles

Our next goal is to introduce *angles* and *angle measures*; after that, the statement *"we can measure angles"* will become rigorous; see (iii) on page 9.



An ordered pair of half-lines which start at the same point is called an *angle*. The angle AOB (also denoted by $\angle AOB$) is the pair of half-lines $[OA)$ and $[OB)$. In this case the point O is called the *vertex* of the angle.

Intuitively, the angle measure tells how much one has to rotate the first half-line counterclockwise, so it gets the position of the second half-line of the angle. The full turn is assumed to be $2\cdot\pi$; it corresponds to the angle measure in radians.

The angle measure of $\angle AOB$ is denoted by $\measuredangle AOB$; it is a real number in the interval $(-\pi, \pi]$.

The notations $\angle AOB$ and $\measuredangle AOB$ look similar, they also have close but different meanings, which better not be confused. For example, the equality $\angle AOB = \angle A'O'B'$ means that $[OA) = [O'A')$ and $[OB) = [O'B')$; in particular, $O = O'$. On the other hand the equality $\measuredangle AOB = \measuredangle A'O'B'$ means only equality of two real numbers; in this case O may be distinct from O'.

Here is the first property of angle measure which will become a part of the axiom.

Given a half-line $[OA)$ and $\alpha \in (-\pi, \pi]$ there is a unique half-line $[OB)$ such that $\measuredangle AOB = \alpha$.

Reals modulo $2\cdot\pi$

Consider three half-lines starting from the same point, $[OA)$, $[OB)$ and $[OC)$. They make three angles AOB, BOC and AOC, so the value $\measuredangle AOC$ should coincide with the sum $\measuredangle AOB + \measuredangle BOC$ up to full rotation. This property will be expressed by the formula

$$\measuredangle AOB + \measuredangle BOC \equiv \measuredangle AOC,$$

where "$\equiv$" is a new notation which we are about to introduce. The last identity will become a part of the axiom.

We will write

$$\alpha \equiv \beta$$

or

$$\alpha \equiv \beta \pmod{2\cdot\pi}$$

if $\alpha = \beta + 2\cdot\pi\cdot n$ for some integer n. In this case we say

"α is equal to β modulo $2\cdot\pi$".

For example

$$-\pi \equiv \pi \equiv 3\cdot\pi \quad \text{and} \quad \tfrac{1}{2}\cdot\pi \equiv -\tfrac{3}{2}\cdot\pi.$$

The introduced relation "$\equiv$" behaves as an equality sing, but

$$\dots \equiv \alpha - 2\cdot\pi \equiv \alpha \equiv \alpha + 2\cdot\pi \equiv \alpha + 4\cdot\pi \equiv \dots;$$

that is, if the angle measures differ by full turn, then they are considered to be the same.

With "$\equiv$", we can do addition, subtraction and multiplication with integer numbers without getting into trouble. That is, if

$$\alpha \equiv \beta \quad \text{and} \quad \alpha' \equiv \beta',$$

then

$$\alpha + \alpha' \equiv \beta + \beta', \quad \alpha - \alpha' \equiv \beta - \beta' \quad \text{and} \quad n\cdot\alpha \equiv n\cdot\beta$$

for any integer n. But "$\equiv$" does not in general respect multiplication with non-integer numbers; for example

$$\pi \equiv -\pi \quad \text{but} \quad \tfrac{1}{2}\cdot\pi \not\equiv -\tfrac{1}{2}\cdot\pi.$$

1.12. Exercise. *Show that $2\cdot\alpha \equiv 0$ if and only if $\alpha \equiv 0$ or $\alpha \equiv \pi$.*

Continuity

The angle measure is also assumed to be continuous. Namely, the following property of angle measure will become a part of the axiom.

The function

$$\measuredangle\colon (A, O, B) \mapsto \measuredangle AOB$$

is continuous at any triple of points (A, O, B) *such that* $O \neq A$ *and* $O \neq B$ *and* $\measuredangle AOB \neq \pi$.

To explain this property, we need to extend the notion of *continuity* to the functions between metric spaces. The definition is a straightforward generalization of the standard definition for the real-to-real functions.

Further, let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces, and $d_\mathcal{X}$, $d_\mathcal{Y}$ be their metrics.

A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called continuous at point $A \in \mathcal{X}$ if for any $\varepsilon > 0$ there is $\delta > 0$, such that

$$d_\mathcal{X}(A, A') < \delta \quad \Rightarrow \quad d_\mathcal{Y}(f(A), f(A')) < \varepsilon.$$

One may define a continuous map of several variables the same way. Assume $f(A, B, C)$ is a function which returns a point in the space $\mathcal{Y}$ for a triple of points (A, B, C) in the space $\mathcal{X}$. The map f might be defined only for some triples in $\mathcal{X}$.

Assume $f(A, B, C)$ is defined. Then, we say that f is continuous at the triple (A, B, C) if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_\mathcal{Y}(f(A, B, C), f(A', B', C')) < \varepsilon.$$

if $d_\mathcal{X}(A, A') < \delta$, $d_\mathcal{X}(B, B') < \delta$ and $d_\mathcal{X}(C, C') < \delta$.

1.13. Exercise. *Let* $\mathcal{X}$ *be a metric space.*

(a) *Let* $A \in \mathcal{X}$ *be a fixed point. Show that the function*

$$f(B) := d_\mathcal{X}(A, B)$$

is continuous at any point B.

(b) *Show that* $d_\mathcal{X}(A, B)$ *is continuous at any pair* $A, B \in \mathcal{X}$.

1.14. Exercise. *Let* $\mathcal{X}$, $\mathcal{Y}$ *and* $\mathcal{Z}$ *be a metric spaces. Assume that the functions* $f\colon \mathcal{X} \to \mathcal{Y}$ *and* $g\colon \mathcal{Y} \to \mathcal{Z}$ *are continuous at any point, and* $h = g \circ f$ *is its composition; that is,* $h(A) = g(f(A))$ *for any* $A \in \mathcal{X}$. *Show that* $h\colon \mathcal{X} \to \mathcal{Z}$ *is continuous at any point.*

1.15. Exercise. *Show that any distance-preserving map is continuous.*

Congruent triangles

Our next goal is to give a rigorous meaning for (iv) on page 9. To do this, we introduce the notion of *congruent triangles* so instead of *"if we rotate or shift we will not see the difference"* we say that for triangles, the side-angle-side congruence holds; that is, two triangles are congruent if they have two pairs of equal sides and the same angle measure between these sides.

An *ordered* triple of distinct points in a metric space $\mathcal{X}$, say A, B, C, is called a *triangle* ABC (briefly $\triangle ABC$). Note that the triangles ABC and ACB are considered as different.

Two triangles $A'B'C'$ and ABC are called *congruent* (written as $\triangle A'B'C' \cong \triangle ABC$) if there is a motion $f\colon \mathcal{X} \to \mathcal{X}$ such that

$$A' = f(A), \quad B' = f(B) \quad \text{and} \quad C' = f(C).$$

Let $\mathcal{X}$ be a metric space, and $f, g\colon \mathcal{X} \to \mathcal{X}$ be two motions. Note that the inverse $f^{-1} : \mathcal{X} \to \mathcal{X}$, as well as the composition $f \circ g : \mathcal{X} \to \mathcal{X}$ are also motions.

It follows that "$\cong$" is an *equivalence relation*; that is, any triangle congruent to itself, and the following two conditions hold.

- ⋄ If $\triangle A'B'C' \cong \triangle ABC$, then $\triangle ABC \cong \triangle A'B'C'$.
- ⋄ If $\triangle A''B''C'' \cong \triangle A'B'C'$ and $\triangle A'B'C' \cong \triangle ABC$, then

$$\triangle A''B''C'' \cong \triangle ABC.$$

Note that if $\triangle A'B'C' \cong \triangle ABC$, then $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$.

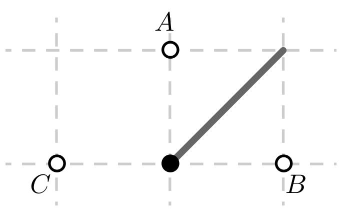
For a discrete metric, as well as some other metrics, the converse also holds. The following example shows that it does not hold in the Manhattan plane.

Example. Consider three points $A = (0, 1)$, $B = (1, 0)$ and $C = = (-1, 0)$ on the Manhattan plane $(\mathbb{R}^2, d_1)$. Note that

$$d_1(A, B) = d_1(A, C) = d_1(B, C) = 2.$$

On one hand,

$$\triangle ABC \cong \triangle ACB.$$



Indeed, it is easy to see that the map $(x, y) \mapsto (-x, y)$ is a motion of $(\mathbb{R}^2, d_1)$, which sends $A \mapsto A$, $B \mapsto C$ and $C \mapsto B$.

On the other hand,

$$\triangle ABC \not\cong \triangle BCA.$$

Indeed, arguing by contradiction, assume that f is a motion of $(\mathbb{R}^2, d_1)$ which sends $A \mapsto B$ and $B \mapsto C$. Note that a point M is a midpoint[1] of A and B if and only if $f(M)$ is a midpoint of B and C.

[1] M is a midpoint of A and B if $d_1(A, M) = d_1(B, M) = \frac{1}{2} \cdot d_1(A, B)$.

The set of midpoints for A and B is infinite, it contains all points (t,t) for $t \in [0,1]$ (it is the dark gray segment on the picture above). On the other hand, the midpoint for B and C is unique (it is the black point on the picture). Thus, the map f cannot be bijective, a contradiction.

Chapter 2

Axioms

The first system of axioms appears in the Euclid's "Elements", around 300 BC. Without any doubt, the Elements were the most successful and influential textbook ever written.

The systematic study of geometries as axiomatic systems was triggered by the discovery on non-Euclidean geometry in 19th century. The emerging this way branch of mathematics is called "Foundations of geometry".

The most popular system of axiom was proposed in 1899 by David Hilbert. This is also the first rigorous system by modern standards. It contains twenty axioms in five groups, six "primitive notions" and three "primitive terms"; these are not defined in terms of previously defined concepts.

We will use another system of axioms, which is very close to the system proposed by George Birkhoff in 1932.

This system is build upon metric spaces. In particular, we use the real numbers as a building block. By that reason this approach reminds a model-based introduction to Euclidean geometry. This approach makes possible to minimize the tedious parts which are unavoidable in the more conservative Hilbert's system.

The axioms use the notions of metric space, line, angle, triangle, equality modulo $2{\cdot}\pi$ ($\equiv$), and congruence of triangles ($\cong$). All this discussed in the preliminaries.

The axioms

I. The *Euclidean plane* is a metric space with at least two points.

II. There is one and only one line, that contains any two given distinct points P and Q in the Euclidean plane.

III. Any angle AOB in the Euclidean plane defines a real number in the interval $(-\pi, \pi]$. This number is called *angle measure of* $\angle AOB$ and denoted by $\measuredangle AOB$. It satisfies the following conditions:

   (a) Given a half-line $[OA)$ and $\alpha \in (-\pi, \pi]$, there is a unique half-line $[OB)$, such that $\measuredangle AOB = \alpha$.

   (b) For any points A, B and C, distinct from O we have

$$\measuredangle AOB + \measuredangle BOC \equiv \measuredangle AOC.$$

   (c) The function

$$\measuredangle\colon (A, O, B) \mapsto \measuredangle AOB$$

   is continuous at any triple of points (A, O, B), such that $O \neq A$ and $O \neq B$ and $\measuredangle AOB \neq \pi$.

IV. In the Euclidean plane, we have $\triangle ABC \cong \triangle A'B'C'$ if and only if

$$A'B' = AB, \quad A'C' = AC, \quad \text{and} \quad \measuredangle C'A'B' = \pm\measuredangle CAB.$$

V. If for two triangles ABC, $AB'C'$ in the Euclidean plane and for $k > 0$ we have

$$\begin{aligned} B' &\in [AB), & C' &\in [AC), \\ AB' &= k \cdot AB, & AC' &= k \cdot AC, \end{aligned}$$

then

$$B'C' = k \cdot BC, \quad \measuredangle ABC = \measuredangle AB'C', \quad \measuredangle ACB = \measuredangle AC'B'.$$

From now on, we can use no information about the Euclidean plane which does not follow from the five axioms above.

2.1. Exercise. *Show that the plane contains an infinite set of points.*

Lines and half-lines

2.2. Proposition.[✓] *Any two distinct lines intersect at most at one point.*

Proof. Assume two lines ℓ and m intersect at two distinct points P and Q. Applying Axiom II, we get $\ell = m$. □

2.3. Exercise. *Suppose $A' \in [OA)$ and $A' \neq O$. Show that*

$$[OA) = [OA').$$

2.4. Proposition.[✓] *Given $r \geqslant 0$ and a half-line $[OA)$ there is a unique $A' \in [OA)$ such that $OA' = r$.*

Proof. According to definition of half-line, there is an isometry

$$f\colon [OA) \to [0, \infty),$$

such that $f(O) = 0$. By the definition of isometry, $OA' = f(A')$ for any $A' \in [OA)$. Thus, $OA' = r$ if and only if $f(A') = r$.

Since isometry has to be bijective, the statement follows. □

Zero angle

2.5. Proposition.[✓] $\measuredangle AOA = 0$ *for any $A \neq O$.*

Proof. According to Axiom IIIb,

$$\measuredangle AOA + \measuredangle AOA \equiv \measuredangle AOA.$$

Subtract $\measuredangle AOA$ from both sides, we get $\measuredangle AOA \equiv 0$.

Since $-\pi < \measuredangle AOA \leqslant \pi$, we get $\measuredangle AOA = 0$. □

2.6. Exercise. *Assume $\measuredangle AOB = 0$. Show that $[OA) = [OB)$.*

2.7. Proposition.[✓] *For any A and B distinct from O, we have*

$$\measuredangle AOB \equiv -\measuredangle BOA.$$

Proof. According to Axiom IIIb,

$$\measuredangle AOB + \measuredangle BOA \equiv \measuredangle AOA$$

By Proposition 2.5, $\measuredangle AOA = 0$. Hence the result follows. □

[✓] Ignor this mark "✓" for a while; its meaning will be explained on page 77.

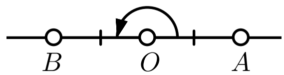
Straight angle

If $\measuredangle AOB = \pi$, we say that $\angle AOB$ is a *straight angle*. Note that by Proposition 2.7, if $\angle AOB$ is a straight, then so is $\angle BOA$.

We say that point O *lies between* points A and B, if $O \neq A$, $O \neq B$ and $O \in [AB]$.

2.8. Theorem.✓ *The angle AOB is straight if and only if O* lies *between A and B.*

Proof. By Proposition 2.4, we may assume that $OA = OB = 1$.



"If" part. Assume O lies between A and B. Set $\alpha = \measuredangle AOB$.

Applying Axiom IIIa, we get a half-line $[OA')$ such that $\alpha = \measuredangle BOA'$. By Proposition 2.4, we can assume that $OA' = 1$. According to Axiom IV,

$$\triangle AOB \cong \triangle BOA'.$$

Denote by f the corresponding motion of the plane; that is, f is a motion such that $f(A) = B$, $f(O) = O$ and $f(B) = A'$.

Then

$$(A'B) = f(AB) \ni f(O) = O.$$

Therefore, both lines (AB) and $(A'B)$ contain B and O. By Axiom II, $(AB) = (A'B)$.

By the definition of the line, (AB) contains exactly two points A and B on distance 1 from O. Since $OA' = 1$ and $A' \neq B$, we get $A = A'$.

By Axiom IIIb and Proposition 2.5, we get

$$\begin{aligned} 2 \cdot \alpha &= \measuredangle AOB + \measuredangle BOA' = \\ &= \measuredangle AOB + \measuredangle BOA \equiv \\ &\equiv \measuredangle AOA = \\ &= 0 \end{aligned}$$

Therefore, by Exercise 1.12, α is either 0 or π.

Since $[OA) \neq [OB)$, we have $\alpha \neq 0$, see Exercise 2.6. Therefore, $\alpha = \pi$.

"Only if" part. Suppose that $\measuredangle AOB = \pi$. Consider the line (OA) and choose a point B' on (OA) so that O lies between A and B'.

From above, we have $\measuredangle AOB' = \pi$. Applying Axiom IIIa, we get $[OB) = [OB')$. In particular, O lies between A and B. $\square$

A triangle ABC is called *degenerate* if A, B and C lie on one line. The following corollary is just a reformulation of Theorem 2.8.

2.9. Corollary.✓ *A triangle is degenerate if and only if one of its angles is equal to* π *or* 0.

2.10. Exercise. *Show that three distinct points A, O and B lie on one line if and only if*

$$2\cdot\measuredangle AOB \equiv 0.$$

2.11. Exercise. *Let A, B and C be three points distinct from O. Show that B, O and C lie on one line if and only if*

$$2\cdot\measuredangle AOB \equiv 2\cdot\measuredangle AOC.$$

Vertical angles

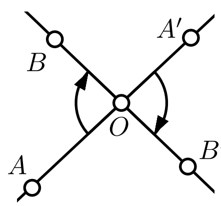
A pair of angles AOB and $A'OB'$ is called *vertical* if the point O lies between A and A' and between B and B' at the same time.

2.12. Proposition.✓ *The vertical angles have equal measures.*

Proof. Assume that the angles AOB and $A'OB'$ are vertical.

Note that the angles AOA' and BOB' are straight. Therefore, $\measuredangle AOA' = \measuredangle BOB' = \pi$.

It follows that



$$\begin{aligned}
0 &= \measuredangle AOA' - \measuredangle BOB' \equiv \\
&\equiv \measuredangle AOB + \measuredangle BOA' - \measuredangle BOA' - \measuredangle A'OB' \equiv \\
&\equiv \measuredangle AOB - \measuredangle A'OB'.
\end{aligned}$$

Hence the result follows. □

2.13. Exercise. *Assume O is the midpoint for both segments $[AB]$ and $[CD]$. Prove that $AC = BD$.*

Chapter 3

Half-planes

This chapter contains long proofs of intuitively evident statements. It is okay to skip it, but make sure you know definitions of positive/negative angles and that your intuition agrees with 3.8, 3.10, 3.11 and 3.16.

Sign of an angle

◇ The angle AOB is called *positive* if $0 < \measuredangle AOB < \pi$;

◇ The angle AOB is called *negative* if $\measuredangle AOB < 0$.

Note that according to the above definitions the straight angle as well as the zero angle are neither positive nor negative.

3.1. Exercise. *Show that $\angle AOB$ is positive if and only if $\angle BOA$ is negative.*

3.2. Exercise. *Let $\angle AOB$ be straight. Show that $\angle AOX$ is positive if and only if $\angle BOX$ is negative.*

3.3. Exercise. *Assume that the angles AOB and BOC are positive. Show that*

$$\measuredangle AOB + \measuredangle BOC + \measuredangle COA = 2\cdot\pi.$$

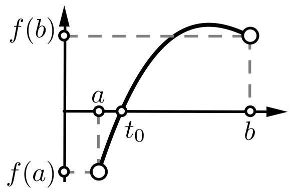
if $\angle COA$ is positive, and

$$\measuredangle AOB + \measuredangle BOC + \measuredangle COA = 0.$$

if $\angle COA$ is negative.

Intermediate value theorem

3.4. Intermediate value theorem. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Assume $f(a)$ and $f(b)$ have the opposite signs, then $f(t_0) = 0$ for some $t_0 \in [a,b]$.*



The Intermediate value theorem is assumed to be known; it should be covered in any calculus course. We will use the following corollary.

3.5. Corollary.✓ *Assume that for any $t \in$ $\in [0,1]$ we have three points in the plane O_t, A_t and B_t, such that*

(a) *Each function $t \mapsto O_t$, $t \mapsto A_t$ and $t \mapsto B_t$ is continuous.*
(b) *For for any $t \in [0,1]$, the points O_t, A_t and B_t do not lie on one line.*

Then $\angle A_0O_0B_0$ and $\angle A_1O_1B_1$ have the same sign.

Proof. Consider the function $f(t) = \measuredangle A_tO_tB_t$.

Since the points O_t, A_t and B_t do not lie on one line, Theorem 2.8 implies that $f(t) = \measuredangle A_tO_tB_t \neq 0$ nor π for any $t \in [0,1]$.

Therefore, by Axiom IIIc and Exercise 1.14, f is a continuous function.

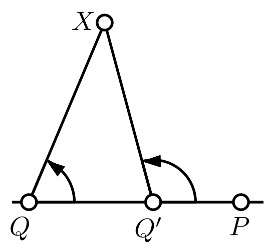
Further, by the Intermediate value theorem, $f(0)$ and $f(1)$ have the same sign; hence the result follows. $\square$

Same sign lemmas

3.6. Lemma.✓ *Assume $Q' \in [PQ)$ and $Q' \neq P$. Then for any $X \notin$ $\notin (PQ)$ the angles PQX and $PQ'X$ have the same sign.*

Proof. By Proposition 2.4, for any $t \in [0,1]$ there is a unique point $Q_t \in [PQ)$ such that

$$PQ_t = (1-t)\cdot PQ + t\cdot PQ'.$$



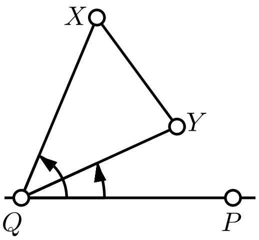
Note that the map $t \mapsto Q_t$ is continuous,

$$Q_0 = Q, \qquad Q_1 = Q'$$

and for any $t \in [0,1]$, we have $P \neq Q_t$.

Applying Corollary 3.5, for $P_t = P$, Q_t and $X_t = X$, we get that $\angle PQX$ has the same sign as $\angle PQ'X$. $\square$

3.7. Lemma.✓ *Assume $[XY]$ does not intersect (PQ), then the angles PQX and PQY have the same sign.*



The proof is nearly identical to the one above.

Proof. According to Proposition 2.4, for any $t \in [0,1]$ there is a point $X_t \in [XY]$, such that

$$XX_t = t{\cdot}XY.$$

Note that the map $t \mapsto X_t$ is continuous. Moreover, $X_0 = X$, $X_1 = Y$ and $X_t \notin (QP)$ for any $t \in [0,1]$.

Applying Corollary 3.5, for $P_t = P$, $Q_t = Q$ and X_t, we get that $\angle PQX$ has the same sign as $\angle PQY$. □

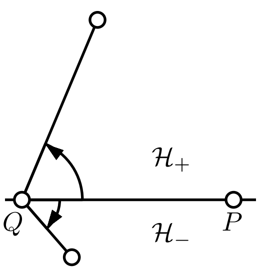
Half-planes

3.8. Proposition.✓ *The complement of a line (PQ) in the plane can be presented in a unique way as a union of two disjoint subsets called* half-planes *such that*

(a) Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if the angles PQX and PQY have the same sign.

(b) Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if $[XY]$ does not intersect (PQ).

We say that X and Y lie on *one side from* (PQ) if they lie in one of the half-planes of (PQ) and we say that P and Q lie on the *opposite sides from* ℓ if they lie in the different half-planes of ℓ.



Proof. Let us denote by $\mathcal{H}_+$ (correspondingly $\mathcal{H}_-$) the set of points X in the plane such that $\angle PQX$ is positive (correspondingly negative).

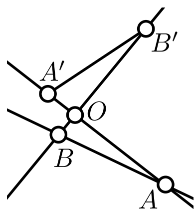
According to Theorem 2.8, $X \notin (PQ)$ if and only if $\measuredangle PQX \neq 0$ nor π. Therefore, $\mathcal{H}_+$ and $\mathcal{H}_-$ give the unique subdivision of the complement of (PQ) which satisfies the property *(a)*.

Now let us prove that the this subdivision depends only on the line (PQ); that is, if $(P'Q') = (PQ)$ and $X, Y \notin (PQ)$, then the angles PQX and PQY have the same sign if and only if the angles $P'Q'X$ and $P'Q'Y$ have the same sign.

Applying Exercise 3.2, we can assume that $P = P'$ and $Q' \in [PQ)$. It remains to apply Lemma 3.6.

(b). Assume $[XY]$ intersects (PQ). Since the subdivision depends only on the line (PQ), we can assume that $Q \in [XY]$. In this case, by Exercise 3.2, the angles PQX and PQY have opposite signs.

Now assume $[XY]$ does not intersect (PQ). In this case, by Lemma 3.7, $\angle PQX$ and $\angle PQY$ have the same sign. □



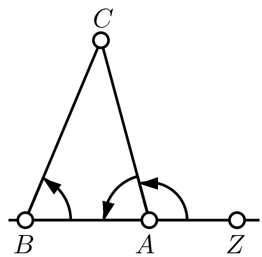
3.9. Exercise. *Assume that the angles AOB and $A'OB'$ are vertical. Show that the line (AB) does not intersect the segment $[A'B']$.*

Consider the triangle ABC. The segments $[AB]$, $[BC]$ and $[CA]$ are called *sides of the triangle.*

The following theorem is a corollary of Proposition 3.8.

3.10. Pasch's theorem.✓ *Assume line ℓ does not pass thru any vertex a triangle. Then it intersects either two or zero sides of the triangle.*

3.11. Signs of angles of a triangle.✓ *In any nondegenerate triangle ABC, the angles ABC, BCA and CAB have the same sign.*



Proof. Choose a point $Z \in (AB)$ so that A lies between B and Z.

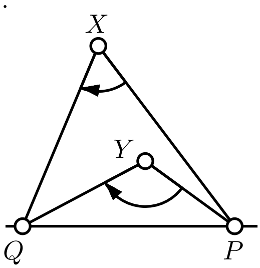
According to Lemma 3.6, the angles ZBC and ZAC have the same sign.

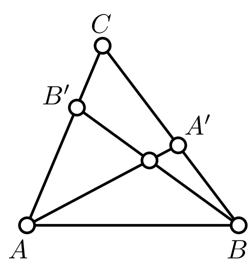
Note that $\measuredangle ABC = \measuredangle ZBC$ and

$$\measuredangle ZAC + \measuredangle CAB \equiv \pi.$$

Therefore, $\angle CAB$ has the same sign as $\angle ZAC$ which in turn has the same sign as $\measuredangle ABC = \measuredangle ZBC$.

Repeating the same argument for $\angle BCA$ and $\angle CAB$, we get the result. □





3.12. Exercise. *Show that two points $X, Y \notin (PQ)$ lie on the same side from (PQ) if and only if the angles PXQ and PYQ have the same sign.*

3.13. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle, $A' \in [BC]$ and $B' \in [AC]$. Show that the segments $[AA']$ and $[BB']$ intersect.*

3.14. Exercise. *Assume that the points X and Y lie on opposite sides from the line (PQ). Show that the half-line $[PX)$ does not interest $[QY)$.*

3.15. Advanced exercise. *Note that the following quantity*

$$\tilde{\measuredangle} ABC = \left[\begin{array}{lll} \pi & \text{if} & \measuredangle ABC = \pi \\ -\measuredangle ABC & \text{if} & \measuredangle ABC < \pi \end{array}\right.$$

can serve as the angle measure; that is, the axioms hold if one exchanges $\measuredangle$ to $\tilde{\measuredangle}$ everywhere.

Show that $\measuredangle$ and $\tilde{\measuredangle}$ are the only possible angle measures on the plane.

Show that without Axiom IIIc, this is no longer true.

Triangle with the given sides

Consider the triangle ABC. Set

$$a = BC, \qquad b = CA, \qquad c = AB.$$

Without loss of generality, we may assume that

$$a \leqslant b \leqslant c.$$

Then all three triangle inequalities for $\triangle ABC$ hold if and only if

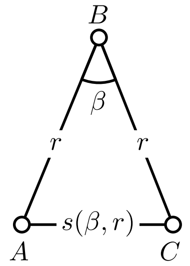
$$c \leqslant a + b.$$

The following theorem states that this is the only restriction on a, b and c.

3.16. Theorem.✓ *Assume that $0 < a \leqslant b \leqslant c \leqslant a + b$. Then there is a triangle ABC such that $a = BC$, $b = CA$ and $c = AB$.*

The proof requires some preparation; it is given in the end of section.

Assume $r > 0$ and $\pi > \beta > 0$. Consider the triangle ABC such that $AB = BC = r$ and $\measuredangle ABC = \beta$. The existence of such a triangle follows from Axiom IIIa and Proposition 2.4.



Note that according to Axiom IV, the values β and r define the triangle up to the congruence. In particular, the distance AC depends only on β and r. Set

$$s(\beta, r) := AC.$$

3.17. Proposition.✓ *Given $r > 0$ and $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < \beta < \delta$, then $s(r, \beta) < < \varepsilon$.*

Proof. Fix two points A and B such that $AB = r$.

Choose a point X such that $\measuredangle ABX$ is positive. Let $Y \in [AX)$ be the point such that $AY = \frac{\varepsilon}{8}$; it exists by Proposition 2.4.

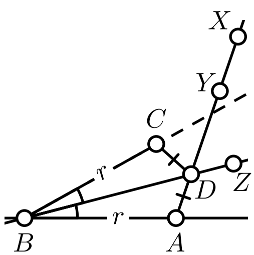
Note that X and Y lie on the same side from (AB); therefore, $\angle ABY$ is positive. Set $\delta = \measuredangle ABY$.

Assume $0 < \beta < \delta$, $\measuredangle ABC = \beta$ and $BC = r$.

Applying Axiom IIIa, we can choose a half-line $[BZ)$ such that $\measuredangle ABZ = \frac{1}{2}\cdot\beta$. Note that A and Y lie on the opposite sides from (BZ). Therefore, (BZ) intersects $[AY]$; denote by D the point of intersection.

Since $D \in (BZ)$, we get $\measuredangle ABD = \frac{\beta}{2}$ or $\frac{\beta}{2} - \pi$. The latter is impossible since D and Y lie on the same side from (AB). Therefore,

$$\measuredangle ABD = \measuredangle DBC = \tfrac{\beta}{2}.$$



By Axiom IV, $\triangle ABD \cong \triangle CBD$. In particular,

$$\begin{aligned} AC &\leqslant AD + DC = \\ &= 2\cdot AD \leqslant \\ &\leqslant 2\cdot AY = \\ &= \tfrac{\varepsilon}{4}. \end{aligned}$$

Hence the result follows. □

3.18. Corollary.✓ *Fix a real number $r > 0$ and two distinct points A and B. Then for any real number $\beta \in [0, \pi]$, there is a unique point C_β such that $BC_\beta = r$ and $\measuredangle ABC_\beta = \beta$. Moreover, the map $\beta \mapsto C_\beta$ is a continuous map from $[0, \pi]$ to the plane.*

Proof. The existence and uniqueness of C_β follows from Axiom IIIa and Proposition 2.4.

Note that if $\beta_1 \neq \beta_2$, then

$$C_{\beta_1}C_{\beta_2} = s(r, |\beta_1 - \beta_2|).$$

Therefore, Proposition 3.17 implies that the map $\beta \mapsto C_\beta$ is continuous. $\square$

Proof of Theorem 3.16. Fix the points A and B such that $AB = c$. Given $\beta \in [0, \pi]$, denote by C_β the point in the plane such that $BC_\beta = a$ and $\measuredangle ABC = \beta$.

According to Corollary 3.18, the map $\beta \mapsto C_\beta$ is continuous. Therefore, the function $b(\beta) = AC_\beta$ is continuous (formally, it follows from Exercise 1.13 and Exercise 1.14).

Note that $b(0) = c - a$ and $b(\pi) = c + a$. Since $c - a \leqslant b \leqslant c + a$, by the Intermediate value theorem (3.4) there is $\beta_0 \in [0, \pi]$ such that $b(\beta_0) = b$. Hence the result follows. $\square$

Chapter 4

Congruent triangles

Side-angle-side condition

Our next goal is to give conditions which guarantee congruence of two triangles. One of such conditions is Axiom IV; it is also called *side-angle-side congruence condition*, or briefly, *SAS congruence condition*.

Angle-side-angle condition

4.1. ASA condition.✓ *Assume that*

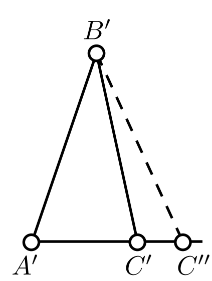
$$AB = A'B', \qquad \measuredangle ABC = \pm\measuredangle A'B'C', \qquad \measuredangle CAB = \pm\measuredangle C'A'B'$$

and $\triangle A'B'C'$ *is nondegenerate. Then*

$$\triangle ABC \cong \triangle A'B'C'.$$

Note that for degenerate triangles the statement does not hold. For example, consider one triangle with sides 1, 4, 5 and the other with sides 2, 3, 5.

Proof. According to Theorem 3.11, either



❶ $$\measuredangle ABC = \measuredangle A'B'C', \quad \measuredangle CAB = \measuredangle C'A'B'$$

or

❷ $$\measuredangle ABC = -\measuredangle A'B'C', \quad \measuredangle CAB = -\measuredangle C'A'B'.$$

Further we assume that ❶ holds; the case ❷ is analogous.

Let C'' be the point on the half-line $[A'C')$ such that $A'C'' = AC$.

By Axiom IV, $\triangle A'B'C'' \cong \triangle ABC$. Applying Axiom IV again, we get

$$\measuredangle A'B'C'' = \measuredangle ABC = \measuredangle A'B'C'.$$

By Axiom IIIa, $[B'C') = [BC'')$. Hence C'' lies on $(B'C')$ as well as on $(A'C')$.

Since $\triangle A'B'C'$ is not degenerate, $(A'C')$ is distinct from $(B'C')$. Applying Axiom II, we get $C'' = C'$.

Therefore, $\triangle A'B'C' = \triangle A'B'C'' \cong \triangle ABC$. □

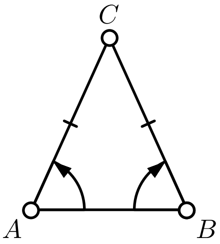
Isosceles triangles

A triangle with two equal sides is called *isosceles*; the remaining side is called the *base*.

4.2. Theorem.✓ *Assume $\triangle ABC$ is an isosceles triangle with the base $[AB]$. Then*

$$\measuredangle ABC \equiv -\measuredangle BAC.$$

Moreover, the converse holds if $\triangle ABC$ is nondegenerate.



The following proof is due to Pappus of Alexandria.

Proof. Note that

$$CA = CB, \quad CB = CA, \quad \measuredangle ACB \equiv -\measuredangle BCA.$$

Therefore, by Axiom IV,

$$\triangle CAB \cong \triangle CBA.$$

Applying the theorem on the signs of angles of triangles (3.11) and Axiom IV again, we get

$$\measuredangle CAB \equiv -\measuredangle CBA.$$

To prove the converse, we assume that $\measuredangle CAB \equiv -\measuredangle CBA$. By ASA condition 4.1, $\triangle CAB \cong \triangle CBA$. Therefore, $CA = CB$. □

A triangle with three equal sides is called *equilateral*.

4.3. Exercise. *Let $\triangle ABC$ be an equilateral triangle. Show that*

$$\measuredangle ABC = \measuredangle BCA = \measuredangle CAB.$$

Side-side-side condition

4.4. SSS condition.✓ *$\triangle ABC \cong \triangle A'B'C'$ if*

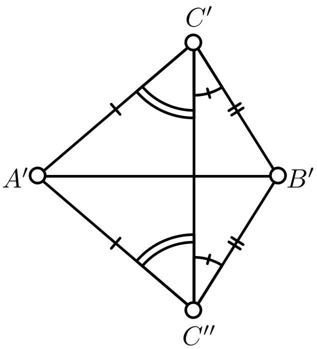
$$A'B' = AB, \quad B'C' = BC \quad \text{and} \quad C'A' = CA.$$

Proof. Choose C'' so that $A'C'' = A'C'$ and $\measuredangle B'A'C'' = \measuredangle BAC$. According to Axiom IV,

$$\triangle A'B'C'' \cong \triangle ABC.$$

It will suffice to prove that

❸ $$\triangle A'B'C' \cong \triangle A'B'C''.$$



The condition ❸ trivially holds if $C'' = C'$. Thus, it remains to consider the case $C'' \neq C'$.

Clearly, the corresponding sides of $\triangle A'B'C'$ and $\triangle A'B'C''$ are equal. Hence the triangles $\triangle C'A'C''$ and $\triangle C'B'C''$ are isosceles. By Theorem 4.2, we have

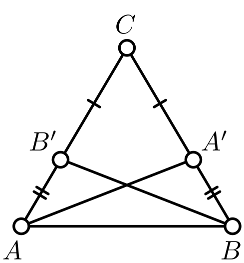
$$\measuredangle A'C''C' \equiv -\measuredangle A'C'C'',$$
$$\measuredangle C'C''B' \equiv -\measuredangle C''C'B'.$$

Adding them, we get

$$\measuredangle A'C''B' \equiv -\measuredangle A'C'B'.$$

Applying Axiom IV again, we get ❸. □

4.5. Advanced exercise. *Let M be the midpoint of the side $[AB]$ of $\triangle ABC$ and M' be the midpoint of the side $[A'B']$ of $\triangle A'B'C'$. Assume $C'A' = CA$, $C'B' = CB$ and $C'M' = CM$. Prove that $\triangle A'B'C' \cong \triangle ABC$.*



4.6. Exercise. *Let $\triangle ABC$ be an isosceles triangle with the base $[AB]$. Suppose that the points $A' \in [BC]$ and $B' \in [AC]$ are such that $CA' = CB'$. Show that*

(a) $\triangle AA'C \cong \triangle BB'C$;
(b) $\triangle ABB' \cong \triangle BAA'$.

4.7. Exercise. *Show that if $AB + BC = AC$ then $B \in [AC]$.*

4.8. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle and let f be a motion of the plane such that*

$$f(A) = A, \quad f(B) = B \quad \textit{and} \quad f(C) = C.$$

Show that f is the identity; that is, $f(X) = X$ for any point X on the plane.

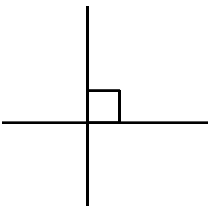
Chapter 5

Perpendicular lines

Right, acute and obtuse angles

⋄ If $|\measuredangle AOB| = \frac{\pi}{2}$, we say that $\angle AOB$ is *right*;
⋄ If $|\measuredangle AOB| < \frac{\pi}{2}$, we say that $\angle AOB$ is *acute*;
⋄ If $|\measuredangle AOB| > \frac{\pi}{2}$, we say that $\angle AOB$ is *obtuse*.

On the diagrams, the right angles will be marked with a little square, as shown.



If $\angle AOB$ is right, we say also that $[OA)$ is *perpendicular* to $[OB)$; it will be written as $[OA) \perp [OB)$.

From Theorem 2.8, it follows that two lines (OA) and (OB) are appropriately called *perpendicular*, if $[OA) \perp [OB)$. In this case we also write $(OA) \perp (OB)$.

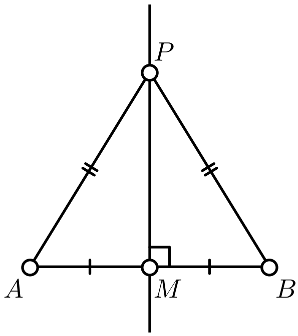
5.1. Exercise. *Assume point O lies between A and B and $X \neq O$. Show that $\angle XOA$ is acute if and only if $\angle XOB$ is obtuse.*

Perpendicular bisector

Assume M is the midpoint of the segment $[AB]$; that is, $M \in (AB)$ and $AM = MB$.

The line ℓ which passes thru M and perpendicular to (AB), is called the *perpendicular bisector* to the segment $[AB]$.

5.2. Theorem.✓ *Given distinct points A and B, all points equidistant from A and B and no others lie on the perpendicular bisector to $[AB]$.*



Proof. Let M be the midpoint of $[AB]$.

Assume $PA = PB$ and $P \neq M$. According to SSS (4.4), $\triangle AMP \cong \triangle BMP$. Hence

$$\measuredangle AMP = \pm\measuredangle BMP.$$

Since $A \neq B$, we have "$-$" in the above formula. Further,

$$\begin{aligned}\pi = \measuredangle AMB &\equiv \\ &\equiv \measuredangle AMP + \measuredangle PMB \equiv \\ &\equiv 2\cdot\measuredangle AMP.\end{aligned}$$

That is, $\measuredangle AMP = \pm\frac{\pi}{2}$. Therefore, P lies on the perpendicular bisector.

To prove the converse, suppose P is any point on the perpendicular bisector to $[AB]$ and $P \neq M$. Then $\measuredangle AMP = \pm\frac{\pi}{2}$, $\measuredangle BMP = \pm\frac{\pi}{2}$ and $AM = BM$. Therefore, $\triangle AMP \cong \triangle BMP$; in particular, $AP = BP$. □

5.3. Exercise. *Let ℓ be the perpendicular bisector to the segment $[AB]$ and X be an arbitrary point on the plane.*

Show that $AX < BX$ if and only if X and A lie on the same side from ℓ.

5.4. Exercise. *Let $\triangle ABC$ be nondegenerate. Show that $AC > BC$ if and only if $|\measuredangle ABC| > |\measuredangle CAB|$.*

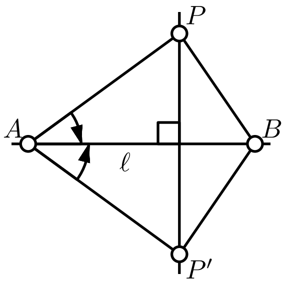
Uniqueness of a perpendicular

5.5. Theorem.✓ *There is one and only one line which passes thru a given point P and is perpendicular to a given line ℓ.*

According to the above theorem, there is a unique point $Q \in \ell$ such that $(QP) \perp \ell$. This point Q is called the *foot point* of P on ℓ.

Proof. If $P \in \ell$, then both, existence and uniqueness, follow from Axiom III.

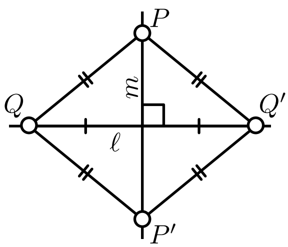
Existence for $P \notin \ell$. Let A and B be two distinct points of ℓ. Choose P' so that $AP' = AP$ and $\measuredangle P'AB \equiv -\measuredangle PAB$. According to Axiom IV, $\triangle AP'B \cong \triangle APB$. Therefore, $AP = AP'$ and $BP = BP'$.



According to Theorem 5.2, A and B lie on the perpendicular bisector to $[PP']$. In particular, $(PP') \perp (AB) = \ell$.

Uniqueness for $P \notin \ell$. From above we can choose a point P' in such a way that ℓ forms the perpendicular bisector to $[PP']$.

Assume $m \perp \ell$ and $m \ni P$. Then m is a perpendicular bisector to some segment $[QQ']$ of ℓ; in particular, $PQ = PQ'$.



Since ℓ is the perpendicular bisector to $[PP']$, we get $PQ = P'Q$ and $PQ' = P'Q'$. Therefore,

$$P'Q = PQ = PQ' = P'Q'.$$

By Theorem 5.2, P' lies on the perpendicular bisector to $[QQ']$, which is m. By Axiom II, $m = (PP')$. □

Reflection

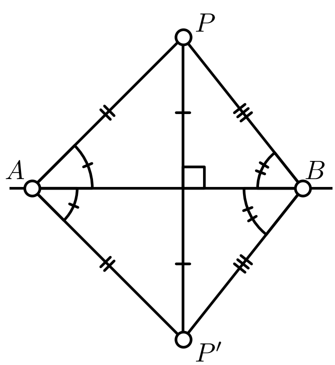
Assume the point P and the line (AB) are given. To find the *reflection* P' of P in (AB), one drops a perpendicular from P onto (AB), and continues it to the same distance on the other side.

According to Theorem 5.5, P' is uniquely determined by P.

Note that $P = P'$ if and only if $P \in (AB)$.

5.6. Proposition.✓ *Assume P' is a reflection of the point P in the line (AB). Then $AP' = AP$ and if $A \neq P$, then $\measuredangle BAP' \equiv -\measuredangle BAP$.*

Proof. Note that if $P \in (AB)$, then $P = P'$. By Corollary 2.9, $\measuredangle BAP = 0$ or π. Hence the statement follows.



If $P \notin (AB)$, then $P' \neq P$. By the construction of P', the line (AB) is perpendicular bisector of $[PP']$. Therefore, according to Theorem 5.2, $AP' = AP$ and $BP' = BP$. In particular, $\triangle ABP' \cong \triangle ABP$. Therefore, $\measuredangle BAP' = \pm\measuredangle BAP$.

Since $P' \neq P$ and $AP' = AP$, we get that $\measuredangle BAP' \neq \measuredangle BAP$. That is, we are left with the case

$$\measuredangle BAP' = -\measuredangle BAP. \qquad \square$$

5.7. Corollary.✓ *The reflection in a line is a motion of the plane. Moreover, if $\triangle P'Q'R'$ is the reflection of $\triangle PQR$, then*

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR.$$

Proof. From the construction, it follows that the composition of two reflections in the same line is the identity map. In particular, any reflection is a bijection.

Assume P', Q' and R' denote the reflections of the points P, Q and R in (AB). Let us first show that

$$❶ \qquad P'Q' = PQ \quad \text{and} \quad \measuredangle AP'Q' \equiv -\measuredangle APQ.$$

Without loss of generality, we may assume that the points P and Q are distinct from A and B. By Proposition 5.6,

$$\begin{aligned} \measuredangle BAP' &\equiv -\measuredangle BAP, & \measuredangle BAQ' &\equiv -\measuredangle BAQ, \\ AP' &= AP, & AQ' &= AQ. \end{aligned}$$

It follows that $\measuredangle P'AQ' \equiv -\measuredangle PAQ$. Therefore $\triangle P'AQ' \cong \triangle PAQ$ and ❶ follows.

Repeating the same argument for P and R, we get

$$\measuredangle AP'R' \equiv -\measuredangle APR.$$

Subtracting the second identity in ❶, we get

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR.$$ □

5.8. Exercise. *Show that any motion of the plane can be presented as a composition of at most three reflections.*

Applying the exercise above and Corollary 5.7, we can divide the motions of the plane in two types, *direct* and *indirect motions.* The motion f is direct if

$$\measuredangle Q'P'R' = \measuredangle QPR$$

for any $\triangle PQR$ and $P' = f(P)$, $Q' = f(Q)$ and $R' = f(R)$; if instead we have

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR$$

for any $\triangle PQR$, then the motion f is called indirect.

5.9. Exercise. *Let X and Y be the reflections of P in the lines (AB) and (BC) correspondingly. Show that*

$$\measuredangle XBY \equiv 2\cdot\measuredangle ABC.$$

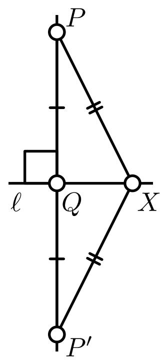
Perpendicular is shortest

5.10. Lemma.✓ *Assume Q is the foot point of P on the line ℓ. Then the inequality*

$$PX > PQ$$

holds for any point X on ℓ distinct from Q.

If P, Q and ℓ are as above, then PQ is called the *distance from P to ℓ.*



Proof. If $P \in \ell$, then the result follows since $PQ = 0$. Further we assume that $P \notin \ell$.

Let P' be the reflection of P in the line ℓ. Note that Q is the midpoint of $[PP']$ and ℓ is the perpendicular bisector of $[PP']$. Therefore

$$PX = P'X \quad \text{and} \quad PQ = P'Q = \tfrac{1}{2}\cdot PP'$$

Note that ℓ meets $[PP']$ only at the point Q. Therefore, by the triangle inequality and Exercise 4.7,

$$PX + P'X > PP'.$$

Hence the result follows. □

5.11. Exercise. *Assume $\angle ABC$ is right or obtuse. Show that*

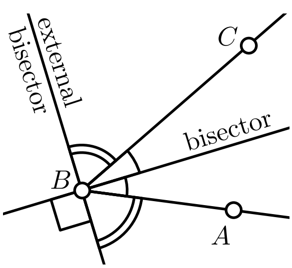
$$AC > AB.$$

Angle bisectors

If $\measuredangle ABX \equiv -\measuredangle CBX$, then we say that the line (BX) *bisects* $\angle ABC$, or line (BX) is the *bisector* of $\angle ABC$. If $\measuredangle ABX \equiv \pi - \measuredangle CBX$, then the line (BX) is called the *external bisector* of $\angle ABC$.

If $\measuredangle ABA' = \pi$; that is, if B lies between A and A', then bisector of $\angle ABC$ is the external bisector of $\angle A'BC$ and the other way around.

Note that the bisector and the external bisector are uniquely defined by the angle.

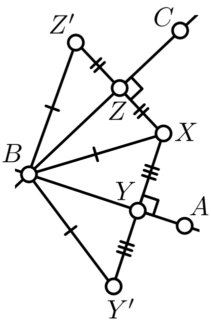


5.12. Exercise. *Show that for any angle, its bisector and external bisector are perpendicular.*

5.13. Lemma.✓ *Assume $\measuredangle ABC \neq \pi$ nor 0. Given the angle ABC and a point X, consider foot points Y and Z of X on (AB) and (BC).*

Then $XY = XZ$ if and only if X lies on the bisector or external bisector of $\angle ABC$.

Proof. Let Y' and Z' be the reflections of X in (AB) and (BC) correspondingly. By Proposition 5.6, $XB = Y'B = Z'B$.



Note that

$$XY' = 2{\cdot}XY \quad \text{and} \quad XZ' = 2{\cdot}XZ.$$

Applying SSS and then SAS, we get

$$\begin{gathered} XY = XZ \\ \Updownarrow \\ XY' = XZ' \\ \Updownarrow \\ \triangle BXY' \cong \triangle BXZ' \\ \Updownarrow \\ \measuredangle XBY' = \pm\measuredangle BXZ'. \end{gathered} \qquad ❷$$

According to Proposition 5.6,

$$\measuredangle XBA \equiv -\measuredangle Y'BA, \qquad \measuredangle XBC \equiv -\measuredangle Z'BC.$$

Therefore,

$$2{\cdot}\measuredangle XBA \equiv \measuredangle XBY' \quad \text{and} \quad 2{\cdot}\measuredangle XBC \equiv -XBZ'.$$

That is, we can continue the chain of equivalence conditions ❷ the following way:

$$\measuredangle XBY' \equiv \pm\measuredangle BXZ' \quad \Longleftrightarrow \quad 2{\cdot}\measuredangle XBA \equiv \pm 2{\cdot}\measuredangle XBC.$$

Since $(AB) \neq (BC)$, we have

$$2{\cdot}\measuredangle XBA \not\equiv 2{\cdot}\measuredangle XBC$$

(compare to Exercise 2.11). Therefore

$$XY = XZ \quad \Longleftrightarrow \quad 2{\cdot}\measuredangle XBA \equiv -2{\cdot}\measuredangle XBC.$$

The last identity means either

$$\measuredangle XBA + \measuredangle XBC \equiv 0$$

or

$$\measuredangle XBA + \measuredangle XBC \equiv \pi.$$

Hence the result follows. $\square$

Circles

Given a positive real number r and a point O, the set Γ of all points on distance r from O is called a *circle* with *radius* r and *center* O.

We say that a point P lies *inside* Γ if $OP < r$; if $OP > r$, we say that P lies *outside* Γ.

5.14. Exercise. *Let Γ be a circle and $P \notin \Gamma$. Assume a line ℓ is passing thru the point P and intersects Γ at two distinct points, X and Y. Show that P is inside Γ if and only if P lies between X and Y.*

A segment between two points on a circle is called a *chord* of the circle. A chord passing thru the center of the circle is called its *diameter.*

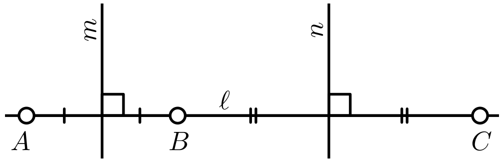
5.15. Exercise. *Assume two distinct circles Γ and Γ' have a common chord $[AB]$. Show that the line between centers of Γ and Γ' forms a perpendicular bisector to $[AB]$.*

5.16. Lemma.✓ *A line and a circle can have at most two points of intersection.*

Proof. Assume A, B and C are distinct points which lie on a line ℓ and a circle Γ with the center O.

Then $OA = OB = OC$; in particular, O lies on the perpendicular bisectors m and n to $[AB]$ and $[BC]$ correspondingly.

Note that the midpoints of $[AB]$ and $[BC]$ are distinct. Therefore, m and n are distinct. The latter contradicts the uniqueness of the perpendicular (Theorem 5.5). □



5.17. Exercise. *Show that two distinct circles can have at most two points of intersection.*

In consequence of the above lemma, a line ℓ and a circle Γ might have 2, 1 or 0 points of intersections. In the first case the line is called *secant line*, in the second case it is *tangent line*; if P is the only point of intersection of ℓ and Γ, we say that ℓ *is tangent to* Γ *at* P.

Similarly, according Exercise 5.17, two circles might have 2, 1 or 0 points of intersections. If P is the only point of intersection of circles Γ and Γ', we say that Γ *is tangent to* Γ *at* P.

5.18. Lemma.✓ *Let ℓ be a line and Γ be a circle with the center O. Assume P is a common point of ℓ and Γ. Then ℓ is tangent to Γ at P if and only if $(PO) \perp \ell$.*

Proof. Let Q be the foot point of O on ℓ.

Assume $P \neq Q$. Denote by P' the reflection of P in (OQ).

Note that $P' \in \ell$ and (OQ) is the perpendicular bisector of $[PP']$. Therefore, $OP = OP'$. Hence $P, P' \in \Gamma \cap \ell$; that is, ℓ is secant to Γ.

If $P = Q$, then according to Lemma 5.10, $OP < OX$ for any point $X \in \ell$ distinct from P. Hence P is the only point in the intersection $\Gamma \cap \ell$; that is, ℓ is tangent to Γ at P. □

5.19. Exercise. *Let Γ and Γ' be two distinct circles with centers at O and O' correspondingly. Assume Γ and Γ' intersect at point P. Show that Γ is tangent to Γ' if and only if O, O' and P lie on one line.*

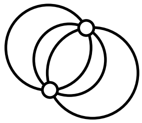
5.20. Exercise. *Let Γ and Γ' be two distinct circles with centers at O and O' and radii r and r'.*

(a) Show that Γ is tangent to Γ' if and only if

$$OO' = r + r' \quad \text{or} \quad OO' = |r - r'|.$$

(b) Show that Γ intersects Γ' if and only if

$$|r - r'| \leqslant OO' \leqslant r + r'.$$



5.21. Exercise. *Assume three circles intersect at two points as shown on the diagram. Prove that the centers of these circles lie on one line.*

Geometric constructions

The *ruler-and-compass constructions* in the plane is the construction of points, lines, and circles using only an idealized ruler and compass. These construction problems provide a valuable source of exercises in geometry, which we will use further in the book. In addition, Chapter 18 is devoted completely to the subject.

The idealized ruler can be used only to draw a line thru the given two points. The idealized compass can be used only to draw a circle with a given center and radius. That is, given three points A, B and O we can draw the set of all points on distant AB from O. We may also mark new points in the plane as well as on the constructed lines, circles and their intersections (assuming that such points exist).

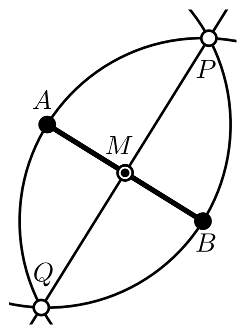
We can also look at the different set of construction tools. For example, we may only use the ruler or we may invent a new tool, say a tool which produces a midpoint for any given two points.

As an example, let us consider the following problem:

5.22. Construction of midpoint. *Construct the midpoint of the given segment* $[AB]$.

Construction.

1. Construct the circle with center at A which is passing thru B.
2. Construct the circle with center at B which is passing thru A.
3. Mark both points of intersection of these circles, label them with P and Q.
4. Draw the line (PQ).
5. Mark the point of intersection of (PQ) and $[AB]$; this is the midpoint.



Typically, you need to prove that the construction produces what was expected. Here is a proof for the example above.

Proof. According to Theorem 5.2, (PQ) is the perpendicular bisector to $[AB]$. Therefore, $M = (AB) \cap (PQ)$ is the midpoint of $[AB]$. □

5.23. Exercise. *Make a ruler-and-compass construction of a line thru a given point which is perpendicular to a given line.*

5.24. Exercise. *Make a ruler-and-compass construction of the center of a given circle.*

5.25. Exercise. *Make a ruler-and-compass construction of the lines tangent to a given circle which pass thru a given point.*

5.26. Exercise. *Given two circles* Γ_1 *and* Γ_2 *and a segment* $[AB]$ *make a ruler-and-compass construction of a circle with the radius* AB*, which is tangent to each circle* Γ_1 *and* Γ_2*.*

Chapter 6

Parallel lines and similar triangles

Parallel lines

In consequence of Axiom II, any two distinct lines ℓ and m have either one point in common or none. In the first case they are *intersecting*; in the second case, ℓ and m are said to be *parallel* (briefly $\ell \parallel m$); in addition, a line is always regarded as parallel to itself.

6.1. Proposition.✓ *Let ℓ, m and n be three lines. Assume that $n \perp m$ and $m \perp \ell$. Then $\ell \parallel n$.*

Proof. Assume the contrary; that is, $\ell \nparallel n$. Then there is a point, say Z, of intersection of ℓ and n. Then by Theorem 5.5, $\ell = n$.

Since any line is parallel to itself, we have $\ell \parallel n$, a contradiction. □

6.2. Theorem. *For any point P and any line ℓ there is a unique line m which passes thru P and is parallel to ℓ.*

The above theorem has two parts, existence and uniqueness. In the proof of uniqueness we will use Axiom V for the first time in this book.

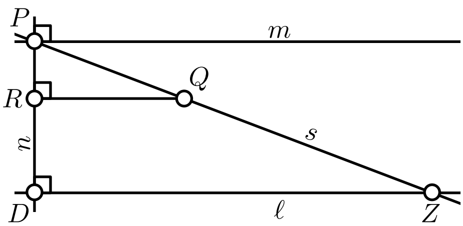
Proof; existence. Apply Theorem 5.5 two times, first to construct the line m thru P which is perpendicular to ℓ, and second to construct the line n thru P which is perpendicular to m. Then apply Proposition 6.1.

Uniqueness. If $P \in \ell$, then $m = \ell$ by the definition of parallel lines. Further we assume $P \notin \ell$.

Let us construct the lines $n \ni P$ and $m \ni P$ as in the proof of existence, so $m \parallel \ell$.

Assume there is yet another line $s \ni P$ which is distinct from m and parallel to ℓ. Choose a point $Q \in s$ which lies with ℓ on the same side from m. Let R be the foot point of Q on n.

Let D be the point of intersection of n and ℓ. According to Proposition 6.1 $(QR) \parallel m$. Therefore, Q, R and ℓ lie on the same side from m. In particular, $R \in [PD)$.



Choose $Z \in [PQ)$ such that

$$\frac{PZ}{PQ} = \frac{PD}{PR}.$$

Then by Axiom V, $(ZD) \perp (PD)$; that is, $Z \in \ell \cap s$, a contradiction. $\square$

6.3. Corollary. *Assume ℓ, m and n are lines such that $\ell \parallel m$ and $m \parallel n$. Then $\ell \parallel n$.*

Proof. Assume the contrary; that is, $\ell \nparallel n$. Then there is a point $P \in \ell \cap n$. By Theorem 6.2, $n = \ell$, a contradiction. $\square$

Note that from the definition, we have $\ell \parallel m$ if and only if $m \parallel \parallel \ell$. Therefore, according to the above corollary, "$\parallel$" is an *equivalence relation*. That is, for any lines ℓ, m and n the following conditions hold:

(i) $\ell \parallel \ell$;
(ii) if $\ell \parallel m$, then $m \parallel \ell$;
(iii) if $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.

6.4. Exercise. *Let k, ℓ, m and n be lines such that $k \perp \ell$ and $m \perp n$. Show that if $k \parallel m$, then $\ell \parallel n$.*

6.5. Exercise. *Make a ruler-and-compass construction of a line thru a given point which is parallel to a given line.*

Similar triangles

Two triangles $A'B'C'$ and ABC are called *similar* (briefly $\triangle A'B'C' \sim \sim \triangle ABC$) if their sides are proportional; that is,

❶ $$A'B' = k{\cdot}AB, \quad B'C' = k{\cdot}BC \quad \text{and} \quad C'A' = k{\cdot}CA$$

for some $k > 0$, and

❷ $$\begin{aligned} \measuredangle A'B'C' &= \pm\measuredangle ABC, \\ \measuredangle B'C'A' &= \pm\measuredangle BCA, \\ \measuredangle C'A'B' &= \pm\measuredangle CAB. \end{aligned}$$

Remarks.

- ◇ According to 3.11, in the above three equalities, the signs can be assumed to be the same.
- ◇ If $\triangle A'B'C' \sim \triangle ABC$ with $k = 1$ in ❶, then $\triangle A'B'C' \cong \cong \triangle ABC$.
- ◇ Note that "$\sim$" is an *equivalence relation*. That is,
  - (i) $\triangle ABC \sim \triangle ABC$ for any $\triangle ABC$.
  - (ii) If $\triangle A'B'C' \sim \triangle ABC$, then
  $$\triangle ABC \sim \triangle A'B'C'.$$
  - (iii) If $\triangle A''B''C'' \sim \triangle A'B'C'$ and $\triangle A'B'C' \sim \triangle ABC$, then
  $$\triangle A''B''C'' \sim \triangle ABC.$$

Using the notation "$\sim$", Axiom V can be formulated the following way.

6.6. Reformulation of Axiom V. *If for the two triangles $\triangle ABC$, $\triangle AB'C'$, and $k > 0$ we have $B' \in [AB)$, $C' \in [AC)$, $AB' = k{\cdot}AB$ and $AC' = k{\cdot}AC$, then $\triangle ABC \sim \triangle AB'C'$.*

In other words, the Axiom V provides a condition which guarantees that two triangles are similar. Let us formulate three more such *similarity conditions.*

6.7. Similarity conditions. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if one of the following conditions hold.*

(SAS) For some constant $k > 0$ we have

$$AB = k{\cdot}A'B', \quad AC = k{\cdot}A'C'$$

$$\text{and} \quad \measuredangle BAC = \pm\measuredangle B'A'C'.$$

(AA) The triangle $A'B'C'$ is nondegenerate and

$$\measuredangle ABC = \pm\measuredangle A'B'C', \quad \measuredangle BAC = \pm\measuredangle B'A'C'.$$

(SSS) For some constant $k > 0$ we have

$$AB = k{\cdot}A'B', \quad AC = k{\cdot}A'C', \quad CB = k{\cdot}C'B'.$$

Each of these conditions is proved by applying Axiom V with the SAS, ASA and SSS congruence conditions correspondingly (see Axiom IV and the conditions 4.1, 4.4).

Proof. Set $k = \frac{AB}{A'B'}$. Choose points $B'' \in [A'B')$ and $C'' \in [A'C')$, so that $A'B'' = k{\cdot}A'B'$ and $A'C'' = k{\cdot}A'C'$. By Axiom V, $\triangle A'B'C' \sim$ $\sim \triangle A'B''C''$.

Applying the SAS, ASA or SSS congruence condition, depending on the case, we get $\triangle A'B''C'' \cong \triangle ABC$. Hence the result follows. $\square$

A bijection from a plane to itself is called *angle preserving transformation* if

$$\measuredangle ABC = \measuredangle A'B'C'$$

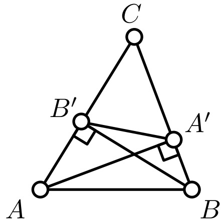
for any triangle ABC and its image $\triangle A'B'C'$.

6.8. Exercise. *Show that any angle-preserving transformation of the plane multiplies all the distance by a fixed constant.*

Method of similar triangles

The similarity of triangles provides a method for solving geometrical problems. To apply this method, we have to search for pairs of similar triangles and then use the proportionality of corresponding sides and/or equalities of corresponding angles.

The following exercise is a classical illustration of this method.



6.9. Exercise. *Let $\triangle ABC$ be an* acute *triangle; that is, all its angles are acute. Denote by A' the foot point of A on (BC) and by B' the foot point of B on (AC). Prove that $\triangle A'B'C \sim \triangle ABC$.*

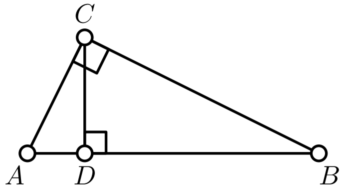
Pythagorean theorem

A triangle is called *right* if one of its angles is right. The side opposite the right angle is called the *hypotenuse*. The sides adjacent to the right angle are called *legs*.

6.10. Theorem. *Assume $\triangle ABC$ is a right triangle with the right angle at C. Then*

$$AC^2 + BC^2 = AB^2.$$

Proof. Let D be the foot point of C on (AB).



According to Lemma 5.10,

$$AD < AC < AB$$

and

$$BD < BC < AB.$$

Therefore, D lies between A and B; in particular,

❸ $$AD + BD = AB.$$

Note that by the AA similarity condition, we have

$$\triangle ADC \sim \triangle ACB \sim \triangle CDB.$$

In particular,

❹ $$\frac{AD}{AC} = \frac{AC}{AB} \quad \text{and} \quad \frac{BD}{BC} = \frac{BC}{BA}.$$

Let us rewrite the two identities in ❹:

$$AC^2 = AB{\cdot}AD \quad \text{and} \quad BC^2 = AB{\cdot}BD.$$

Summing up these two identities and applying ❸, we get

$$AC^2 + BC^2 = AB{\cdot}(AD + BD) = AB^2.$$ □

The idea in the proof above appears in the Elements, see [3, X.33], but the proof given there [3, I.47] is different; it uses area method discussed in Chapter 19.

6.11. Exercise. *Assume A, B, C and D are as in the proof above. Show that*

$$CD^2 = AD{\cdot}BD.$$

The following exercise is the converse to the Pythagorean theorem.

6.12. Exercise. *Assume $\triangle ABC$ is a triangle such that*

$$AC^2 + BC^2 = AB^2.$$

Prove that the angle at C is right.

Angles of triangles

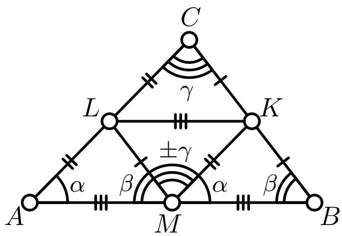
6.13. Theorem. *In any $\triangle ABC$, we have*

$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB \equiv \pi.$$

Proof. First note that if $\triangle ABC$ is degenerate, then the equality follows from Theorem 2.8. Further we assume that $\triangle ABC$ is nondegenerate.

Set

$$\alpha = \measuredangle CAB,$$
$$\beta = \measuredangle ABC,$$
$$\gamma = \measuredangle BCA.$$



We need to prove that

❺ $$\alpha + \beta + \gamma \equiv \pi.$$

Let K, L, M be the midpoints of the sides $[BC]$, $[CA]$, $[AB]$ respectively. By Axiom V,

$$\triangle AML \sim \triangle ABC, \quad \triangle MBK \sim \triangle ABC, \quad \triangle LKC \sim \triangle ABC$$

$\Delta$and

$$LM = \tfrac{1}{2}{\cdot}BC, \qquad MK = \tfrac{1}{2}{\cdot}CA, \qquad KL = \tfrac{1}{2}{\cdot}AB.$$

According to the SSS condition (6.7), $\triangle KLM \sim \triangle ABC$. Thus,

❻ $$\measuredangle MKL = \pm\alpha, \quad \measuredangle KLM = \pm\beta, \quad \measuredangle LMK = \pm\gamma.$$

According to 3.11, the "+" or "−" sign is to be the same in ❻.

If in ❻ we have "+", then ❺ follows since

$$\beta + \gamma + \alpha \equiv \measuredangle AML + \measuredangle LMK + \measuredangle KMB \equiv \measuredangle AMB \equiv \pi$$

It remains to show that we cannot have "−" in ❻. If this is the case, then the same argument as above gives

$$\alpha + \beta - \gamma \equiv \pi.$$

The same way we get

$$\alpha - \beta + \gamma \equiv \pi.$$

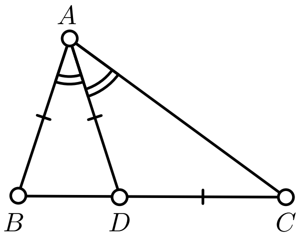
Adding the last two identities, we get

$$2{\cdot}\alpha \equiv 0.$$

Equivalently $\alpha \equiv \pi$ or 0; that is, $\triangle ABC$ is degenerate, a contradiction.

□

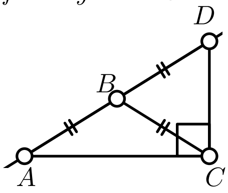
6.14. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle. Assume there is a point $D \in [BC]$ such that (AD) bisects $\angle BAC$ and $BA = AD = DC$. Find the angles of $\triangle ABC$.*



6.15. Exercise. *Show that*

$$|\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB| = \pi$$

for any $\triangle ABC$.



6.16. Exercise. *Let $\triangle ABC$ be an isosceles nondegenerate triangle with the base $[AC]$. Consider the point D on the extension of the side $[AB]$ such that $AB = BD$. Show that $\angle ACD$ is right.*

6.17. Exercise. *Let $\triangle ABC$ be an isosceles nondegenerate triangle with base $[AC]$. Assume that a circle is passing thru A, centered at a point on $[AB]$ and tangent to (BC) at the point X. Show that $\measuredangle CAX = \pm\frac{\pi}{4}$.*

Transversal property

If the line t intersects each line ℓ and m at one point, then we say that t is a *transversal* to ℓ and m. On the diagram below, line (CB) is a transversal to (AB) and (CD).

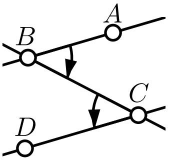
6.18. Transversal property. $(AB) \parallel (CD)$ *if and only if*

❼ $$2\cdot(\measuredangle ABC + \measuredangle BCD) \equiv 0.$$

Equivalently

$$\measuredangle ABC + \measuredangle BCD \equiv 0 \quad \textit{or} \quad \measuredangle ABC + \measuredangle BCD \equiv \pi;$$

in the first case A and D lie on the opposite sides of (BC), in the second case A and D lie on the same sides of (BC).



Proof. If $(AB) \nparallel (CD)$, then there is $Z \in (AB) \cap (CD)$ and $\triangle BCZ$ is nondegenerate.

According to Theorem 6.13,

$$\begin{aligned}\measuredangle ZBC + \measuredangle BCZ \equiv \pi - \measuredangle CZB \not\equiv \\ \not\equiv 0 \quad \text{nor} \quad \pi.\end{aligned}$$

Note that $2\cdot\measuredangle ZBC \equiv 2\cdot\measuredangle ABC$ and $2\cdot\measuredangle BCZ \equiv 2\cdot\measuredangle BCD$. Therefore,

$$2\cdot(\measuredangle ABC + \measuredangle BCD) \equiv 2\cdot\measuredangle ZBC + 2\cdot\measuredangle BCZ \not\equiv 0;$$

that is, ❼ does not hold.

Note that if the points A, B and C are fixed, the identity ❼ uniquely defines the line (CD). By Theorem 6.2, it follows that if $(AB) \parallel (CD)$, then the equality ❼ holds.

Applying Proposition 3.8, we get the last part of the transversal property. □

6.19. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle. Assume B' and C' are points on sides $[AB]$ and $[AC]$ such that $(B'C') \parallel (BC)$. Show that $\triangle ABC \sim \triangle AB'C'$.*

6.20. Exercise. *Trisect a given segment with a ruler and a compass.*

Parallelograms

A *quadrilateral* is an ordered quadruple of distinct points in the plane. The quadrilateral $ABCD$ will be also denoted by $\square ABCD$.

Given a quadrilateral $ABCD$, the four segments $[AB]$, $[BC]$, $[CD]$ and $[DA]$ are called *sides of* $\square ABCD$; the remaining two segments $[AC]$ and $[BD]$ are called *diagonals of* $\square ABCD$.

6.21. Exercise. *Show that for any quadrilateral $ABCD$, we have*

$$\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB \equiv 0.$$

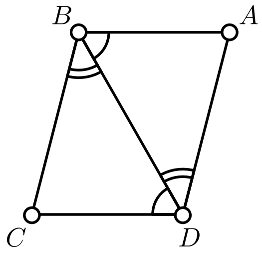
A quadrilateral $ABCD$ in the Euclidean plane is called *nondegenerate* if any three points from A, B, C, D do not lie on one line.

A nondegenerate quadrilateral $ABCD$ is called a *parallelogram* if $(AB) \parallel (CD)$ and $(BC) \parallel (DA)$.

6.22. Lemma. *If $\square ABCD$ is a parallelogram, then*

(a) $\measuredangle DAB = \measuredangle BCD$;

(b) $AB = CD$.



Proof. Since $(AB) \parallel (CD)$, the points C and D lie on the same side from (AB). Hence $\angle ABD$ and $\angle ABC$ have the same sign.

Analogously, $\angle CBD$ and $\angle CBA$ have the same sign.

Since $\measuredangle ABC \equiv -\measuredangle CBA$, we get that the angles DBA and DBC have opposite signs; that is, A and C lie on the opposite sides of (BD).

According to the transversal property (6.18),

$$\measuredangle BDC \equiv -\measuredangle DBA \quad \text{and} \quad \measuredangle DBC \equiv -\measuredangle BDA.$$

By the ASA condition $\triangle ABD \cong \triangle CDB$. The latter implies both statements in the lemma. □

6.23. Exercise. *Assume $ABCD$ is a quadrilateral such that*

$$AB = CD = BC = DA.$$

Show that $ABCD$ is a parallelogram.

A quadrilateral as in the exercise above is called a *rhombus*.

6.24. Exercise. *Show that diagonals of a parallelogram intersect each other at their midpoints.*

A quadraliteral $ABCD$ is called a *rectangle* if the angles ABC, BCD, CDA and DAB are right. Note that according to the transversal property 6.18, any rectangle is a parallelogram.

A rectangle with equal sides is called a *square*.

6.25. Exercise. *Show that the parallelogram $ABCD$ is a rectangle if and only if $AC = BD$.*

6.26. Exercise. *Show that the parallelogram $ABCD$ is a rhombus if and only if $(AC) \perp (BD)$.*

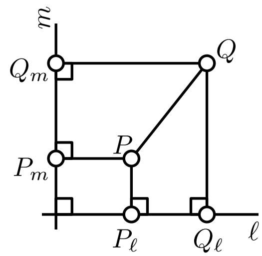
Assume $\ell \parallel m$, and $X, Y \in m$. Denote by X' and Y' the foot points of X and Y on ℓ. Note that $\square XYY'X'$ is a rectangle. By Lemma 6.22, $XX' = YY'$. That is, any point on m lies on the same distance from ℓ. This distance is called the *distance between ℓ and m*.

Method of coordinates

The following exercise is important; it shows that our axiomatic definition agrees with the model definition described on page 11.

6.27. Exercise. *Let ℓ and m be perpendicular lines in the Euclidean plane. Given a point P, denote by P_ℓ and P_m the foot points of P on ℓ and m correspondingly.*

(a) *Show that for any $X \in \ell$ and $Y \in m$ there is a unique point P such that $P_\ell = X$ and $P_m = Y$.*
(b) *Show that $PQ^2 = P_\ell Q_\ell^2 + P_m Q_m^2$ for any pair of points P and Q.*
(c) *Conclude that the plane is isometric to $(\mathbb{R}^2, d_2)$; see page 11.*



Once this exercise is solved, we can apply the method of coordinates to solve any problem in Euclidean plane geometry. This method is powerful, but it is often considered as a bad style.

6.28. Exercise. *Use the Exercise 6.27 to give an alternative proof of Theorem 3.16 in the Euclidean plane.*

That is, prove that given the real numbers a, b and c such that

$$0 < a \leqslant b \leqslant c \leqslant a + c,$$

there is a triangle ABC such that $a = BC$, $b = CA$ and $c = AB$.

6.29. Exercise. *Let (x_A, y_A) and (x_B, y_B) be the coordinates of distinct points A and B in the Euclidean plane. Show that the line (AB) is formed by points with coordinates (x, y) which satisfy the following equation*

$$(x - x_A)\cdot(y_B - y_A) = (y - y_A)\cdot(x_B - x_A).$$

Chapter 7

Triangle geometry

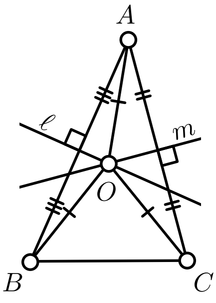
Triangle geometry is the study of the properties of triangles, including associated centers and circles.

We discuss the most basic results in triangle geometry, mostly to show that we have developed enough machinery to prove things.

Circumcircle and circumcenter

7.1. Theorem. *Perpendicular bisectors to the sides of any nondegenerate triangle intersect at one point.*

The point of intersection of the perpendicular bisectors is called *circumcenter*. It is the center of the *circumcircle* of the triangle; that is, the circle which passes thru all three vertices of the triangle. The circumcenter of the triangle is usually denoted by O.



Proof. Let $\triangle ABC$ be nondegenerate. Let ℓ and m be perpendicular bisectors to sides $[AB]$ and $[AC]$ correspondingly.

Assume ℓ and m intersect, let $O = \ell \cap m$.

Let us apply Theorem 5.2. Since $O \in \ell$, we have $OA = OB$ and since $O \in m$, we have $OA = OC$. It follows that $OB = OC$; that is, O lies on the perpendicular bisector to $[BC]$.

It remains to show that $\ell \nparallel m$; assume the contrary. Since $\ell \perp (AB)$ and $m \perp (AC)$, we get $(AC) \parallel (AB)$ (see Exercise 6.4). Therefore, by Theorem 5.5, $(AC) = (AB)$; that is, $\triangle ABC$ is degenerate, a contradiction. $\square$

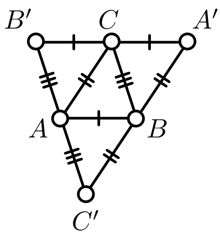
7.2. Exercise. *There is a unique circle which passes thru the vertexes of a given nondegenerate triangle in the Euclidean plane.*

Altitudes and orthocenter

An *altitude* of a triangle is a line thru a vertex and perpendicular to the line containing the opposite side. The term *altitude* maybe also used for the distance from the vertex to its foot point on the line containing opposite side.

7.3. Theorem. *The three altitudes of any nondegenerate triangle intersect in a single point.*

The point of intersection of altitudes is called *orthocenter*; it is usually denoted by H.



Proof. Let $\triangle ABC$ be nondegenerate.

Consider three lines $\ell \parallel (BC)$ thru A, $m \parallel$ $\parallel (CA)$ thru B and $n \parallel (AB)$ thru C. Since $\triangle ABC$ is nondegenerate, no pair of the lines ℓ, m and n is parallel. Set $A' = m \cap n$, $B' = n \cap \ell$ and $C' = \ell \cap m$.

Note that $\square ABA'C$, $\square BCB'A$ and $\square CBC'A$ are parallelograms. Applying Lemma 6.22 we get that $\triangle ABC$ is the median triangle of $\triangle A'B'C'$; that is, A, B and C are the midpoints of $[B'C']$, $[C'A']$ and $[A'B']$ correspondingly.

By Exercise 6.4, $(B'C') \parallel (BC)$, the altitude from A is perpendicular to $[B'C']$ and from above it bisects $[B'C']$.

Hence the altitudes of $\triangle ABC$ are also perpendicular bisectors of the triangle $A'B'C'$. Applying Theorem 7.1, we get that altitudes of $\triangle ABC$ intersect at one point. $\square$

7.4. Exercise. *Assume H is the orthocenter of an acute triangle ABC. Show that A is the orthocenter of $\triangle HBC$.*

Medians and centroid

A median of a triangle is the segment joining a vertex to the midpoint of the opposing side.

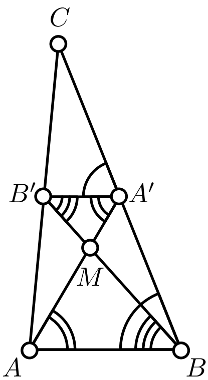
7.5. Theorem. *The three medians of any nondegenerate triangle intersect in a single point. Moreover, the point of intersection divides each median in the ratio 2:1.*

The point of intersection of medians is called the *centroid* of the triangle; it is usually denoted by M.

Proof. Consider a nondegenerate triangle ABC. Let $[AA']$ and $[BB']$ be its medians.

According to Exercise 3.13, $[AA']$ and $[BB']$ are intersecting. Let us denote the point of intersection by M.

By SAS, $\triangle B'A'C \sim \triangle ABC$ and $A'B' = \frac{1}{2}{\cdot}AB$. In particular, $\measuredangle ABC = \measuredangle B'A'C$.



Since A' lies between B and C, we get $\measuredangle BA'B' + \measuredangle B'A'C = \pi$. Therefore,

$$\measuredangle B'A'B + \measuredangle A'BA = \pi.$$

By the transversal property 6.18, $(AB) \parallel (A'B')$.

Note that A' and A lie on the opposite sides from (BB'). Therefore, by the transversal property 6.18, we get

$$\measuredangle B'A'M = \measuredangle BAM.$$

The same way we get,

$$\measuredangle A'B'M = \measuredangle ABM.$$

By AA condition, $\triangle ABM \sim \triangle A'B'M$.

Since $A'B' = \frac{1}{2}{\cdot}AB$, we have

$$\frac{A'M}{AM} = \frac{B'M}{BM} = \frac{1}{2}.$$

In particular, M divides medians $[AA']$ and $[BB']$ in ratio 2:1.

Note that M is a unique point on $[BB']$ such that

$$\frac{B'M}{BM} = \frac{1}{2}.$$

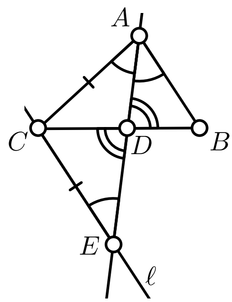
Repeating the same argument for vertices B and C we get that all medians $[CC']$ and $[BB']$ intersect at M. $\square$

7.6. Exercise. *Let $\square ABCD$ be a nondegenerate quadrilateral and X, Y, V, W be the midpoints of its sides $[AB]$, $[BC]$, $[CD]$ and $[DA]$. Show that $\square XYVW$ is a parallelogram.*

Bisector of triangle

7.7. Lemma. *Let $\triangle ABC$ be a nondegenerate triangle. Assume that the bisector of $\angle BAC$ intersects $[BC]$ at the point D. Then*

❶ $$\frac{AB}{AC} = \frac{DB}{DC}.$$



Proof. Let ℓ be the line passing thru C that is parallel to (AB). Note that $\ell \nparallel (AD)$; set

$$E = \ell \cap (AD).$$

Also note that B and C lie on the opposite sides of (AD). Therefore, by the transversal property (6.18),

❷ $$\measuredangle BAD = \measuredangle CED.$$

Further, note that the angles ADB and EDC are vertical; in particular, by 2.12

$$\measuredangle ADB = \measuredangle EDC.$$

By the AA similarity condition, $\triangle ABD \sim \triangle ECD$. In particular,

❸ $$\frac{AB}{EC} = \frac{DB}{DC}.$$

Since (AD) bisects $\angle BAC$, we get $\measuredangle BAD = \measuredangle DAC$. Together with ❷, it implies that $\measuredangle CEA = \measuredangle EAC$. By Theorem 4.2, $\triangle ACE$ is isosceles; that is,

$$EC = AC.$$

Together with ❸, it implies ❶. □

7.8. Exercise. *Prove an analog of Lemma 7.7 for the external bisector.*

Incenter

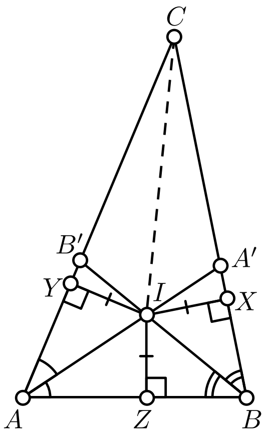
7.9. Theorem.✓ *The angle bisectors of any nondegenerate triangle intersect at one point.*

The point of intersection of bisectors is called the *incenter* of the triangle; it is usually denoted by I. The point I lies on the same distance from each side. In particular, it is the center of a circle tangent to each side of triangle. This circle is called the *incircle* and its radius is called the *inradius* of the triangle.

Proof. Let $\triangle ABC$ be a nondegenerate triangle.

Note that the points B and C lie on opposite sides of the bisector of $\angle BAC$. Hence this bisector intersects $[BC]$ at a point, say A'.

Analogously, there is $B' \in [AC]$ such that (BB') bisects $\angle ABC$.



Applying Pasch's theorem (3.10) twice for the triangles $AA'C$ and $BB'C$, we get that $[AA']$ and $[BB']$ intersect. Let us denote by I the point of intersection.

Let X, Y and Z be the foot points of I on (BC), (CA) and (AB) correspondingly. Applying Lemma 5.13, we get

$$IY = IZ = IX.$$

From the same lemma, we get that I lies on the bisector or on the exterior bisector of $\angle BCA$.

The line (CI) intersects $[BB']$, the points B and B' lie on opposite sides of (CI). Therefore, the angles ICB' and ICB have opposite signs. Note that $\angle ICA = \angle ICB'$. Therefore, (CI) cannot be the exterior bisector of $\angle BCA$. Hence the result follows. □

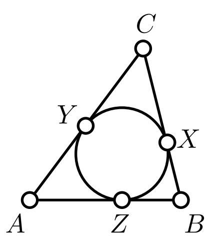
More exercises

7.10. Exercise. *Assume that an angle bisector of a nondegenerate triangle bisects the opposite side. Show that the triangle is isosceles.*

7.11. Exercise. *Assume that at one vertex of a nondegenerate triangle the bisector coincides with the altitude. Show that the triangle is isosceles.*

7.12. Exercise. *Assume sides $[BC]$, $[CA]$ and $[AB]$ of $\triangle ABC$ are tangent to the incircle at X, Y and Z correspondingly. Show that*

$$AY = AZ = \tfrac{1}{2}\cdot(AB + AC - BC).$$



By the definition, the vertexes of *orthic triangle* are the base points of the altitudes of the given triangle.

7.13. Exercise. *Prove that the orthocenter of an acute triangle coincides with the incenter of its orthic triangle.*

What should be an analog of this statement for an obtuse triangle?

Chapter 8

Inscribed angles

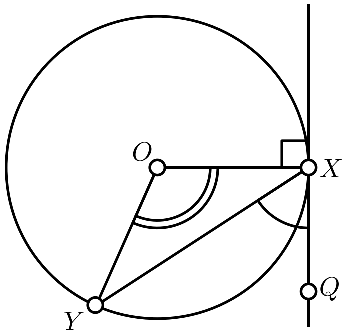
Angle between a tangent line and a chord

8.1. Theorem. *Let Γ be a circle with the center O. Assume the line (XQ) is tangent to Γ at X and $[XY]$ is a chord of Γ. Then*

❶ $$2\cdot\measuredangle QXY \equiv \measuredangle XOY.$$

Equivalently,

$$\measuredangle QXY \equiv \tfrac{1}{2}\cdot\measuredangle XOY \quad \textit{or} \quad \measuredangle QXY \equiv \tfrac{1}{2}\cdot\measuredangle XOY + \pi.$$



Proof. Note that $\triangle XOY$ is isosceles. Therefore, $\measuredangle YXO = \measuredangle OYX$.

Let us apply Theorem 6.13 to $\triangle XOY$. We get

$$\begin{aligned}\pi &\equiv \measuredangle YXO + \measuredangle OYX + \measuredangle XOY \equiv \\ &\equiv 2\cdot\measuredangle YXO + \measuredangle XOY.\end{aligned}$$

By Lemma 5.18, $(OX) \perp (XQ)$. Therefore,

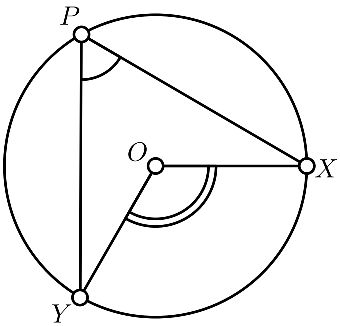
$$\measuredangle QXY + \measuredangle YXO \equiv \pm\tfrac{\pi}{2}.$$

Therefore,

$$2\cdot\measuredangle QXY \equiv \pi - 2\cdot\measuredangle YXO \equiv \measuredangle XOY.$$

□

Inscribed angle



We say that a triangle is *inscribed* in the circle Γ if all its vertices lie on Γ.

8.2. Theorem. *Let Γ be a circle with the center O, and X, Y be two distinct points on Γ. Then $\triangle XPY$ is inscribed in Γ if and only if*

❷ $$2\cdot\measuredangle XPY \equiv \measuredangle XOY.$$

Equivalently, if and only if

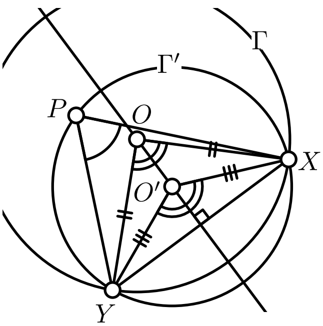
$$\measuredangle XPY \equiv \tfrac{1}{2}\cdot\measuredangle XOY \quad \textit{or} \quad \measuredangle XPY \equiv \tfrac{1}{2}\cdot\measuredangle XOY + \pi.$$

Proof; the "only if" part. Choose a point Q such that $(PQ) \perp (OP)$. By Lemma 5.18, (PQ) is tangent to Γ.

According to Theorem 8.1,

$$2\cdot\measuredangle QPX \equiv \measuredangle POX,$$
$$2\cdot\measuredangle QPY \equiv \measuredangle POY.$$

Subtracting one identity from the other, we get ❷.



"If" part. Assume that ❷ holds for some $P \notin \Gamma$. Note that $\measuredangle XOY \neq 0$. Therefore, $\measuredangle XPY \neq 0$ nor π; that is, $\triangle PXY$ is nondegenerate.

Let Γ' be the circumcircle of $\triangle PXY$ and O' be its circumcenter; they exist by Exercise 7.2.

Note that $O' \neq O$. From above, we have

$$\measuredangle XOY \equiv 2\cdot\measuredangle XPY \equiv \measuredangle XO'Y.$$

Note that $OX = OY$ and $O'X = O'Y$. By Theorem 5.2, (OO') is the perpendicular bisector to $[XY]$; equivalently X is the reflection of Y in (OO'). Applying Proposition 5.6, we get

$$\measuredangle XOO' \equiv -\measuredangle YOO', \qquad \measuredangle XO'O \equiv -\measuredangle YO'O.$$

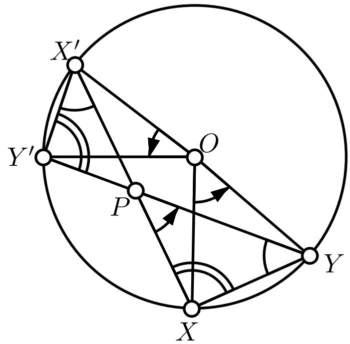
Therefore,

$$\begin{aligned} 2\cdot\measuredangle XOO' &\equiv \measuredangle XOO' + \measuredangle O'OY \equiv \\ &\equiv \measuredangle XOY \equiv \measuredangle XO'Y \equiv \\ &\equiv \measuredangle XO'O + \measuredangle OO'Y \\ &\equiv 2\cdot\measuredangle XO'O. \end{aligned}$$

By the transversal property 6.18, $(XO) \parallel (XO')$, a contradiction. $\square$

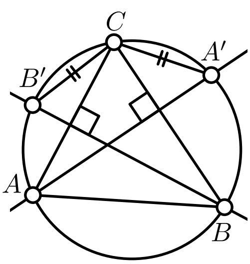
8.3. Exercise. *Let $[XX']$ and $[YY']$ be two chords of the circle Γ with the center O and the radius r. Assume (XX') and (YY') intersect at the point P. Show that*

(a) $2{\cdot}\measuredangle XPY = \measuredangle XOY + \measuredangle X'OY'$;
(b) $\triangle PXY \sim \triangle PY'X'$;
(c) $PX{\cdot}PX' = |OP^2 - r^2|$.



8.4. Exercise. *Three chords $[XX']$, $[YY']$ and $[ZZ']$ of the circle Γ intersect at one point. Show that*

$$XY'{\cdot}ZX'{\cdot}YZ' = X'Y{\cdot}Z'X{\cdot}Y'Z.$$



8.5. Exercise. *Let Γ be a circumcircle of $\triangle ABC$. Let us denote by A' and B' the second points of intersection of the altitudes from A and B with Γ. Show that $\triangle A'B'C$ is isosceles.*

Recall that the diameter of a circle is its chord which passes thru the center. Note that if $[XY]$ is the diameter of a circle with center O, then $\measuredangle XOY = \pi$. Hence Theorem 8.2 implies the following.

8.6. Corollary. *Let Γ be a circle with the diameter $[XY]$. Assume that the point P is distinct from X and Y. Then $P \in \Gamma$ if and only if $\angle XPY$ is right.*

8.7. Exercise. *Given four points A, B, A' and B', construct a point Z such that both angles AZB and $A'ZB'$ are right.*

8.8. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle, A' and B' be foot points of altitudes from A and B respectfully. Show that the four points A, B, A' and B' lie on one circle.*

What is the center of this circle?

8.9. Exercise. *Assume a line ℓ, a circle with its center on ℓ and a point $P \notin \ell$ are given. Make a ruler-only construction of the perpendicular to ℓ from P.*

Inscribed quadrilaterals

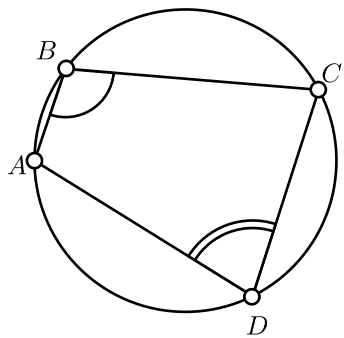
A quadrilateral $ABCD$ is called *inscribed* if all the points A, B, C and D lie on a circle or a line.

8.10. Theorem. *The quadrilateral $ABCD$ is inscribed if and only if*

❸ $$2{\cdot}\measuredangle ABC + 2{\cdot}\measuredangle CDA \equiv 0.$$

Equivalently, if and only if

$$\measuredangle ABC + \measuredangle CDA \equiv \pi \quad \textit{or} \quad \measuredangle ABC \equiv -\measuredangle CDA.$$



Proof of Theorem 8.10. Assume $\triangle ABC$ is degenerate. By Corollary 2.9,

$$2{\cdot}\measuredangle ABC \equiv 0;$$

From the same corollary, we get

$$2{\cdot}\measuredangle CDA \equiv 0$$

if and only if $D \in (AB)$; hence the result follows.

It remains to consider the case if $\triangle ABC$ is nondegenerate.

Denote by Γ the circumcircle of $\triangle ABC$ and let O be the center of Γ. According to Theorem 8.2,

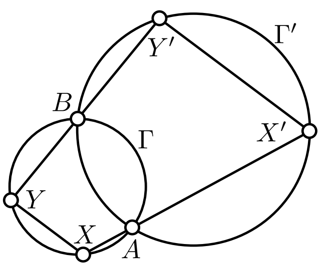
❹ $$2{\cdot}\measuredangle ABC \equiv \measuredangle AOC.$$

From the same theorem, $D \in \Gamma$ if and only if

❺ $$2{\cdot}\measuredangle CDA \equiv \measuredangle COA.$$

Adding ❹ and ❺, we get the result. □

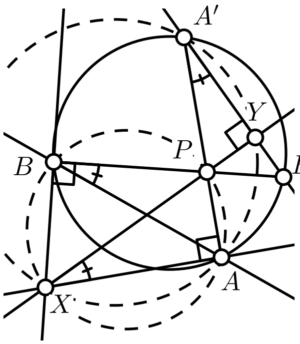
8.11. Exercise. *Let Γ and Γ' be two circles which intersect at two distinct points: A and B. Assume $[XY]$ and $[X'Y']$ are the chords of Γ and Γ' correspondingly, such that A lies between X and X' and B lies between Y and Y'. Show that $(XY) \parallel (X'Y')$.*



8.12. Exercise. *Let $[XY]$ and $[X'Y']$ be two parallel chords of a circle. Show that $XX' = YY'$.*

Method of additional circle

8.13. Problem. *Assume that two chords $[AA']$ and $[BB']$ intersect at the point P inside their circle. Let X be a point such that both angles XAA' and XBB' are right. Show that $(XP) \perp (A'B')$.*



Solution. Set $Y = (A'B') \cap (XP)$.

Both angles XAA' and XBB' are right; therefore

$$2\cdot\measuredangle XAA' \equiv 2\cdot\measuredangle XBB'.$$

By Theorem 8.10, $\square XAPB$ is inscribed. Applying this theorem again we get

$$2\cdot\measuredangle AXP \equiv 2\cdot\measuredangle ABP.$$

Since $\square ABA'B'$ is inscribed,

$$2\cdot\measuredangle ABB' \equiv 2\cdot\measuredangle AAB'.$$

It follows that

$$2\cdot\measuredangle AXY \equiv 2\cdot\measuredangle AA'Y.$$

By the same theorem $\square XAYA'$ is inscribed, and therefore,

$$2\cdot\measuredangle XAA' \equiv 2\cdot\measuredangle XYA'.$$

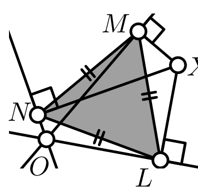
Since $\angle XAA'$ is right, so is $\angle XYA'$. That is $(XP) \perp (A'B')$. $\square$

The method used in the solution is called *method of additional circle*, since the circumcircles of the $\square XAPB$ and $\square XAPB$ above can be considered as *additional constructions.*

8.14. Exercise. *Find an inaccuracy in the solution of the problem 8.13 and try to fix it.*

Here is another problem of that type.

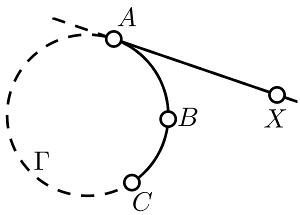
8.15. Exercise. *Assume three lines ℓ, m and n intersect at point O and form six equal angles at O. Let X be a point distinct from O. Denote by L, M and N the footpoints of X on ℓ, m and n correspondingly. Show that $\triangle LMN$ is equilateral.*



Arcs

A subset of a circle bounded by two points is called a circle arc.

More precisely, let Γ be a circle and A, B, C be distinct points on Γ. The subset which includes the points A, C as well as all the points on Γ which lie with B on the same side from (AC) is called *circle arc ABC*.



For the circle arc ABC, the points A and C are called *endpoints*. Note that there are two circle arcs of Γ with the given endpoints.

A half-line $[AX)$ is called *tangent* to the arc ABC at A if the line (AX) is tangent to Γ, and the points X and B lie on the same side from the line (AC).

If B lies on the line (AC), the arc ABC degenerates to one of two following a subsets of the line (AC).

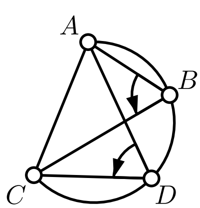
- ◇ If B lies between A and C, then we define the arc ABC as the segment $[AC]$. In this case the half-line $[AC)$ is tangent to the arc ABC at A.
- ◇ If $B \in (AC)\backslash[AC]$, then we define the arc ABC as the line (AC) without all the points between A and C. If we choose points X and $Y \in (AC)$ such that the points X, A, C and Y appear in the same order on the line, then the arc ABC is the union of two half-lines in $[AX)$ and $[CY)$. The half-line $[AX)$ is tangent to the arc ABC at A.
- ◇ In addition, any half-line $[AB)$ will be regarded as an arc. This degenerate arc has only one end point A and it assumed to be tangent to itself at A.

The circle arcs together with the degenerate arcs will be called *arcs*.

8.16. Proposition. *A point D lies on the arc ABC if and only if*

$$\measuredangle ADC = \measuredangle ABC$$

or D coincides with A or C.



Proof. Note that if A, B and C lie on one line, then the statement is evident.

Assume Γ be the circle passing thru A, B and C.

Assume D is distinct from A and C. According to Theorem 8.10, $D \in \Gamma$ if and only if

$$\measuredangle ADC = \measuredangle ABC \quad \text{or} \quad \measuredangle ADC \equiv \measuredangle ABC + \pi.$$

By Exercise 3.12, if the first identity holds, then B and D lie on one side of (AC); that is, D belongs to the arc ABC. If the second identity holds, then the points B and D lie on the opposite sides from (AC), in this case D does not belong to the arc ABC. □

8.17. Proposition. *The half-line $[AX)$ is tangent to the arc ABC if and only if*

$$\measuredangle ABC + \measuredangle CAX \equiv \pi.$$

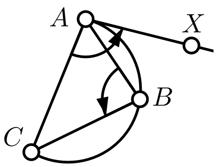
Proof. For a degenerate arc ABC, the statement is evident. Further we assume the arc ABC is nondegenerate.

Applying theorems 8.1 and 8.2, we get

$$2\cdot\measuredangle ABC + 2\cdot\measuredangle CAX \equiv 0.$$

Therefore, either

$$\measuredangle ABC + \measuredangle CAX \equiv \pi, \quad \text{or} \quad \measuredangle ABC + \measuredangle CAX \equiv 0.$$



Since $[AX)$ is the tangent half-line to the arc ABC, X and B lie on the same side from (AC). Therefore, the angles CAX, CAB and ABC have the same sign. In particular, $\measuredangle ABC + \measuredangle CAX \not\equiv 0$; that is, we are left with the case

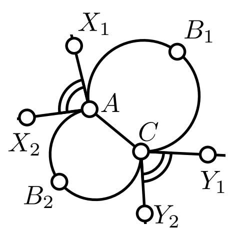
$$\measuredangle ABC + \measuredangle CAX \equiv \pi.$$ □

8.18. Exercise. *Assume that the half-lines $[AX)$ and $[AY)$ are tangent to the arcs ABC and ACB correspondingly. Show that $\angle XAY$ is straight.*

8.19. Exercise. *Show that there is a unique arc with endpoints at the given points A and C, which is tangent at A to the given half line $[AX)$.*

8.20. Exercise. *Given two arcs AB_1C and AB_2C, let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to the arcs AB_1C and AB_2C at A, and $[CY_1)$ and $[CY_2)$ be the half-lines tangent to the arcs AB_1C and AB_2C at C. Show that*

$$\measuredangle X_1AX_2 \equiv -\measuredangle Y_1CY_2.$$



8.21. Exercise. *Given an acute triangle ABC make a compass-and-ruler construction of the point Z such that*

$$\measuredangle AZB = \measuredangle BZC = \measuredangle CZA = \pm\tfrac{2}{3}\cdot\pi$$

Chapter 9

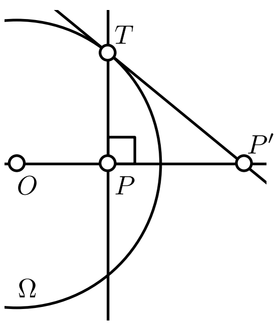
Inversion

Let Ω be the circle with center O and radius r. The *inversion* of a point P in Ω is the point $P' \in [OP)$ such that

$$OP{\cdot}OP' = r^2.$$

In this case the circle Ω will be called the *circle of inversion* and its center O is called the *center of inversion*.

The inverse of O is undefined.



Note that if P is inside Ω, then P' is outside and the other way around. Further, $P = P'$ if and only if $P \in \Omega$.

Note that the inversion takes P' back to P.

9.1. Exercise. *Let P be a point inside of the circle Ω centered at O. Let T be a point where the perpendicular to (OP) from P intersects Ω. Let P' be the point where the tangent line to Ω at T intersects (OP).*

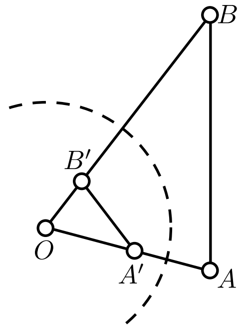
Show that P' is the inverse of P in the circle Ω.

9.2. Lemma. *Let Γ be a circle with the center O. Assume A' and B' are the inversions of A and B in Γ. Then*

$$\triangle OAB \sim \triangle OB'A'.$$

Moreover,

➊
$$\begin{aligned} \measuredangle AOB &\equiv -\measuredangle B'OA', \\ \measuredangle OBA &\equiv -\measuredangle OA'B', \\ \measuredangle BAO &\equiv -\measuredangle A'B'O. \end{aligned}$$



Proof. Let r be the radius of the circle of the inversion.

From the definition of inversion, we get

$$OA{\cdot}OA' = OB{\cdot}OB' = r^2.$$

Therefore,

$$\frac{OA}{OB'} = \frac{OB}{OA'}.$$

Clearly,

❷ $$\measuredangle AOB = \measuredangle A'OB' \equiv -\measuredangle B'OA'.$$

From SAS, we get

$$\triangle OAB \sim \triangle OB'A'.$$

Applying Theorem 3.11 and ❷, we get ❶. □

9.3. Exercise. *Let P' be the inverse of P in the circle Γ. Assume that $P \neq P'$. Show that the value $\frac{PX}{P'X}$ is the same for all $X \in \Gamma$.*

The converse to the exercise above also holds. Namely, given a positive real number $k \neq 1$ and two distinct points P and P' the locus of points X such that $\frac{PX}{P'X} = k$ forms a circle which is called the *circle of Apollonius*. In this case P' is inverse of P in the circle of Apollonius.

9.4. Exercise. *Let A', B' and C' be the images of A, B and C under the inversion in the incircle of $\triangle ABC$. Show that the incenter of $\triangle ABC$ is the orthocenter of $\triangle A'B'C'$.*

9.5. Exercise. *Make a ruler-and-compass construction of the inverse of a given point in a given circle.*

Cross-ratio

The following theorem gives some quantities expressed in distances or angles which do not change after inversion.

9.6. Theorem. *Let $ABCD$ and $A'B'C'D'$ be two quadrilaterals such that the points A', B', C' and D' are the inversions of A, B, C, and D correspondingly.*

Then

(a)

$$\frac{AB{\cdot}CD}{BC{\cdot}DA} = \frac{A'B'{\cdot}C'D'}{B'C'{\cdot}D'A'}.$$

(b)

$$\measuredangle ABC + \measuredangle CDA \equiv -(\measuredangle A'B'C' + \measuredangle C'D'A').$$

(c) If the quadrilateral $ABCD$ is inscribed, then so is $\square A'B'C'D'$.

Proof; (a). Let O be the center of the inversion. According to Lemma 9.2, $\triangle AOB \sim \triangle B'OA'$. Therefore,

$$\frac{AB}{A'B'} = \frac{OA}{OB'}.$$

Analogously,

$$\frac{BC}{B'C'} = \frac{OC}{OB'}, \qquad \frac{CD}{C'D'} = \frac{OC}{OD'}, \qquad \frac{DA}{D'A'} = \frac{OA}{OD'}.$$

Therefore,

$$\frac{AB}{A'B'} \cdot \frac{B'C'}{BC} \cdot \frac{CD}{C'D'} \cdot \frac{D'A'}{DA} = \frac{OA}{OB'} \cdot \frac{OB'}{OC} \cdot \frac{OC}{OD'} \cdot \frac{OD'}{OA} = 1.$$

Hence *(a)* follows.

(b). According to Lemma 9.2,

❸
$$\begin{aligned} \measuredangle ABO &\equiv -\measuredangle B'A'O, & \measuredangle OBC &\equiv -\measuredangle OC'B', \\ \measuredangle CDO &\equiv -\measuredangle D'C'O, & \measuredangle ODA &\equiv -\measuredangle OA'D'. \end{aligned}$$

By Axiom IIIb,

$$\begin{aligned} \measuredangle ABC &\equiv \measuredangle ABO + \measuredangle OBC, & \measuredangle D'C'B' &\equiv \measuredangle D'C'O + \measuredangle OC'B', \\ \measuredangle CDA &\equiv \measuredangle CDO + \measuredangle ODA, & \measuredangle B'A'D' &\equiv \measuredangle B'A'O + \measuredangle OA'D'. \end{aligned}$$

Therefore, summing the four identities in ❸, we get

$$\measuredangle ABC + \measuredangle CDA \equiv -(\measuredangle D'C'B' + \measuredangle B'A'D').$$

Applying Axiom IIIb and Exercise 6.21, we get

$$\begin{aligned} \measuredangle A'B'C' + \measuredangle C'D'A' &\equiv -(\measuredangle B'C'D' + \measuredangle D'A'B') \equiv \\ &\equiv \measuredangle D'C'B' + \measuredangle B'A'D'. \end{aligned}$$

Hence *(b)* follows.

(c). Follows from *(b)* and Theorem 8.10. $\square$

Inversive plane and circlines

Let Ω be a circle with the center O and the radius r. Consider the inversion in Ω.

Recall that inverse of O is undefined. To deal with this problem it is useful to add to the plane an extra point; it will be called the *point at infinity* and we will denote it as ∞. We can assume that ∞ is inverse of O and the other way around.

The Euclidean plane with an added point at infinity is called the *inversive plane*.

We will always assume that any line and half-line contains ∞.

It will be convenient to use the notion of *circline*, which means *circle or line*; for instance we may say *"if a circline contains ∞, then it is a line"* or *"a circline which does not contain ∞ is a circle"*.

Note that according to Theorem 7.1, for any $\triangle ABC$ there is a unique circline which passes thru A, B and C (if $\triangle ABC$ is degenerate, then this is a line and if not it is a circle).

9.7. Theorem. *In the inversive plane, inverse of a circline is a circline.*

Proof. Denote by O the center of the inversion.

Let Γ be a circline. Choose three distinct points A, B and C on Γ. (If $\triangle ABC$ is nondegenerate, then Γ is the circumcircle of $\triangle ABC$; if $\triangle ABC$ is degenerate, then Γ is the line passing thru A, B and C.)

Denote by A', B' and C' the inversions of A, B and C correspondingly. Let Γ' be the circline which passes thru A', B' and C'. According to 7.1, Γ' is well defined.

Assume D is a point of the inversive plane which is distinct from A, C, O and ∞. Denote by D' the inverse of D.

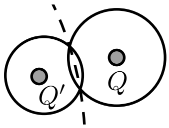
By Theorem 9.6c, $D' \in \Gamma'$ if and only if $D \in \Gamma$. Hence the result follows.

It remains to prove that $O \in \Gamma \Leftrightarrow \infty \in \Gamma'$ and $\infty \in \Gamma \Leftrightarrow O \in \Gamma'$. Since Γ is the inverse of Γ', it is sufficient to prove that

$$\infty \in \Gamma \quad \Longleftrightarrow \quad O \in \Gamma'.$$

Since $\infty \in \Gamma$, we get that Γ is a line. Therefore, for any $\varepsilon > 0$, the line Γ contains the point P with $OP > r^2/\varepsilon$. For the inversion $P' \in \Gamma'$ of P, we have $OP' = r^2/OP < \varepsilon$. That is, the circline Γ' contains points arbitrary close to O. It follows that $O \in \Gamma'$. $\square$

9.8. Exercise. *Assume that the circle Γ' is the inverse of the circle Γ. Denote by Q the center of Γ and by Q' the inverse of Q. Show that Q' is not the center of Γ'.*



Assume that a *circumtool* is a geometric construction tool which produces a circline passing thru any three given points.

9.9. Exercise. *Show that with only a circumtool, it is impossible to construct the center of a given circle.*

9.10. Exercise. *Show that for any pair of tangent circles in the inversive plane, there is an inversion which sends them to a pair of parallel lines.*

9.11. Theorem. *Consider the inversion of the inversive plane in the circle Ω with the center O. Then*

(a) A line passing thru O is inverted into itself.

(b) A line not passing thru O is inverted into a circle which passes thru O, and the other way around.

(c) A circle not passing thru O is inverted into a circle not passing thru O.

Proof. In the proof we use Theorem 9.7 without mentioning it.

(a). Note that if a line passes thru O, it contains both ∞ and O. Therefore, its inversion also contains ∞ and O. In particular, the image is a line passing thru O.

(b). Since any line ℓ passes thru ∞, its image ℓ' has to contain O. If the line does not contain O, then $\ell' \not\ni \infty$; that is, ℓ' is not a line. Therefore, ℓ' is a circle which passes thru O.

(c). If the circle Γ does not contain O, then its image Γ' does not contain ∞. Therefore, Γ' is a circle. Since $\Gamma \not\ni \infty$ we get $\Gamma' \not\ni O$. Hence the result follows. □

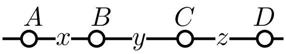
Method of inversion

Here is one application of inversion, which we include as an illustration only; we will not use it further in the book.

9.12. Ptolemy's identity. *Let $ABCD$ be an inscribed quadrilateral. Assume that the points A, B, C and D appear on the circline in the same order. Then*

$$AB{\cdot}CD + BC{\cdot}DA = AC{\cdot}BD.$$

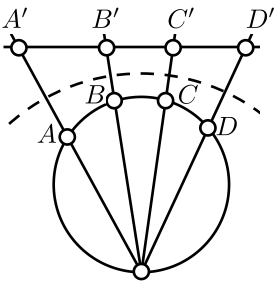
Proof. Assume the points A, B, C, D lie on one line in this order.



Set $x = AB$, $y = BC$, $z = CD$. Note that

$$x\cdot z + y\cdot(x + y + z) = (x + y)\cdot(y + z).$$

Since $AC = x + y$, $BD = y + z$ and $DA = x + y + z$, it proves the identity.



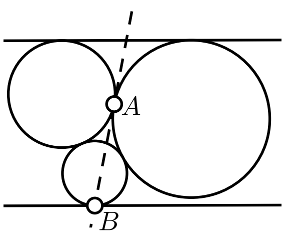
It remains to consider the case when the quadrilateral $ABCD$ is inscribed in a circle, say Γ.

The identity can be rewritten as

$$\frac{AB\cdot DC}{BD\cdot CA} + \frac{BC\cdot AD}{CA\cdot DB} = 1.$$

On the left hand side we have two cross-ratios. According to Theorem 9.6a, the left hand side does not change if we apply an inversion to each point.

Consider an inversion in a circle centered at a point O which lies on Γ between A and D. By Theorem 9.11, this inversion maps Γ to a line. This reduces the problem to the case when A, B, C and D lie on one line, which was already considered. □



In the proof above, we rewrite Ptolemy's identity in a form which is invariant with respect to inversion and then apply an inversion which makes the statement evident. The solution of the following exercise is based on the same idea; one has to apply an inversion with center at A.

9.13. Exercise. *Consider three circles tangent to each other and to two parallel lines as shown on the picture; denote by A and B the two points of tangency. Show that the line passing thru A and B is also tangent to the two circles at A.*

Perpendicular circles

Assume two circles Γ and Ω intersect at two points X and Y. Let ℓ and m be the tangent lines at X to Γ and Ω correspondingly. Analogously, ℓ' and m' be the tangent lines at Y to Γ and Ω.

From Exercise 8.20, we get that $\ell \perp m$ if and only if $\ell' \perp m'$.

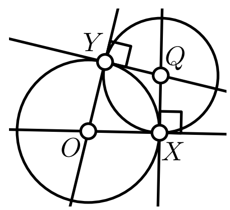
We say that the circle Γ is *perpendicular* to the circle Ω (briefly $\Gamma \perp \Omega$) if they intersect and the lines tangent to the circle at one point (and therefore, both points) of intersection are perpendicular.

Similarly, we say that the circle Γ is perpendicular to the line ℓ (briefly $\Gamma \perp \ell$) if $\Gamma \cap \ell \neq \varnothing$ and ℓ perpendicular to the tangent lines to Γ at one point (and therefore, both points) of intersection. According to Lemma 5.18, it happens only if the line ℓ passes thru the center of Γ.

Now we can talk about *perpendicular circlines.*

9.14. Theorem. *Assume Γ and Ω are distinct circles. Then $\Omega \perp \Gamma$ if and only if the circle Γ coincides with its inversion in Ω.*

Proof. Denote by Γ' the inverse of Γ.



"Only if" part. Let O be the center of Ω and Q be the center of Γ. Denote by X and Y the points of intersections of Γ and Ω. According to Lemma 5.18, $\Gamma \perp \Omega$ if and only if (OX) and (OY) are tangent to Γ.

Note that Γ' is also tangent to (OX) and (OY) at X and Y correspondingly. It follows that X and Y are the foot points of the center of Γ' on (OX) and (OY). Therefore, both Γ' and Γ have the center Q. Finally, $\Gamma' = \Gamma$, since both circles pass thru X.

"If" part. Assume $\Gamma = \Gamma'$.

Since $\Gamma \neq \Omega$, there is a point P which lies on Γ, but not on Ω. Let P' be the inverse of P in Ω. Since $\Gamma = \Gamma'$, we have $P' \in \Gamma$. In particular, the half-line $[OP)$ intersects Γ at two points. By Exercise 5.14, O lies outside of Γ.

As Γ has points inside and outside of Ω, the circles Γ and Ω intersect. The latter follows from Exercise 5.20*b*.

Let X be a point of their intersection. We need to show that (OX) is tangent to Γ; that is, X is the only intersection point of (OX) and Γ.

Assume Z is another point of intersection. Since O is outside of Γ, the point Z lies on the half-line $[OX)$.

Denote by Z' the inverse of Z in Ω. Clearly, the three points Z, Z', X lie on Γ and (OX). The latter contradicts Lemma 5.16. $\square$

It is convenient to define the *inversion in the line ℓ* as the reflection in ℓ. This way we can talk about *inversion in an arbitrary circline.*

9.15. Corollary. *Let Ω and Γ be distinct circlines in the inversive plane. Then the inversion in Ω sends Γ to itself if and only if $\Omega \perp \Gamma$.*

Proof. By Theorem 9.14, it is sufficient to consider the case when Ω or Γ is a line.

Assume Ω is a line, so the inversion in Ω is a reflection. In this case the statement follows from Corollary 5.7.

If Γ is a line, then the statement follows from Theorem 9.11. □

9.16. Corollary. *Let P and P' be two distinct points such that P' is the inverse of P in the circle Ω. Assume that the circline Γ passes thru P and P'. Then $\Gamma \perp \Omega$.*

Proof. Without loss of generality, we may assume that P is inside and P' is outside Ω. By Theorem 3.16, Γ intersects Ω. Denote by A a point of intersection.

Denote by Γ' the inverse of Γ. Since A is a self-inverse, the points A, P and P' lie on Γ'. By Exercise 7.2, $\Gamma' = \Gamma$ and by Theorem 9.14, $\Gamma \perp \Omega$. □

9.17. Corollary. *Let P and Q be two distinct points inside the circle Ω. Then there is a unique circline Γ perpendicular to Ω, which passes thru P and Q.*

Proof. Let P' be the inverse of the point P in the circle Ω. According to Corollary 9.16, the circline passing thru P and Q is perpendicular to Ω if and only if it passes thru P'.

Note that P' lies outside of Ω. Therefore, the points P, P' and Q are distinct.

According to Exercise 7.2, there is a unique circline passing thru P, Q and P'. Hence the result follows. □

9.18. Exercise. *Let Ω_1 and Ω_2 be two distinct circles in the Euclidean plane. Assume that the point P does not lie on Ω_1 nor on Ω_2. Show that there is a unique circline passing thru P which is perpendicular to Ω_1 and Ω_2.*

9.19. Exercise. *Let P, Q, P' and Q' be points in the Euclidean plane. Assume P' and Q' are inversions of P and Q correspondingly. Show that the quadrilateral $PQP'Q'$ is inscribed.*

9.20. Exercise. *Let Ω_1 and Ω_2 be two perpendicular circles with centers at O_1 and O_2 correspondingly. Show that the inverse of O_1 in Ω_2 coincides with the inverse of O_2 in Ω_1.*

9.21. Exercise. *Three distinct circles — Ω_1, Ω_2 and Ω_3, intersect at two points — A and B. Assume that a circle Γ is perpendicular to Ω_1 and Ω_2. Show that $\Gamma \perp \Omega_3$.*

9.22. Exercise. *Assume you have two construction tools: the circumtool which constructs a circline thru three given points, and a tool which constructs an inverse of a given point in a given circle.*

Assume that a point P does not lie on the two circles Ω_1, Ω_2. Using only the two given tools, construct a circline Γ which passes thru P, and perpendicular to both Ω_1 and Ω_2.

Angles after inversion

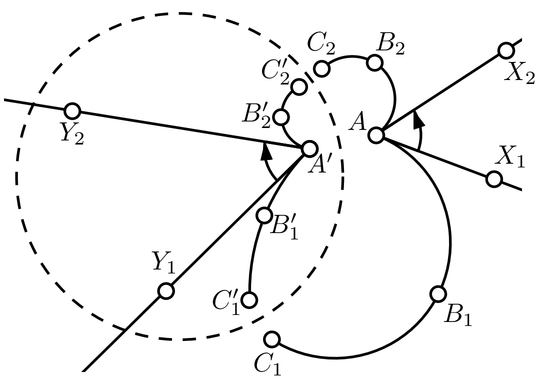
9.23. Proposition. *In the inversive plane, the inverse of an arc is an arc.*

Proof. Consider four distinct points A, B, C and D; let A', B', C' and D' be their inverses. We need to show that D lies on the arc ABC if and only if D' lies on the arc $A'B'C'$. According to Proposition 8.16, the latter is equivalent to the following:

$$\measuredangle ADC = \measuredangle ABC \quad \iff \quad \measuredangle A'D'C' = \measuredangle A'B'C'.$$

The latter follows from Theorem 9.6*b*. □

The following theorem states that the angle between arcs changes only its sign after the inversion.



9.24. Theorem. *Let AB_1C_1, AB_2C_2 be two arcs in the inversive plane, and the arcs $A'B_1'C_1'$, $A'B_2'C_2'$ be their inversions. Let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to AB_1C_1 and AB_2C_2 at A, and $[A'Y_1)$ and $[A'Y_2)$ be the half-lines tangent to $A'B_1'C_1'$ and $A'B_2'C_2'$ at A'. Then*

$$\measuredangle X_1AX_2 \equiv -\measuredangle Y_1A'Y_2.$$

Proof. Applying to Proposition 8.17,

$$\begin{aligned}\measuredangle X_1AX_2 &\equiv \measuredangle X_1AC_1 + \measuredangle C_1AC_2 + \measuredangle C_2AX_2 \equiv\\ &\equiv (\pi - \measuredangle C_1B_1A) + \measuredangle C_1AC_2 + (\pi - \measuredangle AB_2C_2) \equiv\\ &\equiv -(\measuredangle C_1B_1A + \measuredangle AB_2C_2 + \measuredangle C_2AC_1) \equiv\\ &\equiv -(\measuredangle C_1B_1A + \measuredangle AB_2C_1) - (\measuredangle C_1B_2C_2 + \measuredangle C_2AC_1).\end{aligned}$$

The same way we get

$$\measuredangle Y_1A'Y_2 \equiv -(\measuredangle C_1'B_1'A' + \measuredangle A'B_2'C_1') - (\measuredangle C_1'B_2'C_2' + \measuredangle C_2'A'C_1').$$

By Theorem 9.6b,

$$\begin{aligned}\measuredangle C_1B_1A + \measuredangle AB_2C_1 &\equiv -(\measuredangle C_1'B_1'A' + \measuredangle A'B_2'C_1'),\\ \measuredangle C_1B_2C_2 + \measuredangle C_2AC_1 &\equiv -(\measuredangle C_1'B_2'C_2' + \measuredangle C_2'A'C_1').\end{aligned}$$

Hence the result follows. □

9.25. Corollary. *Let P', Q' and Γ' be the inversions of the points P, Q and the circle Γ in the circle Ω. Assume P is the inverse of Q in Γ, then P' is the inverse of Q' in Γ'.*

Proof. If $P = Q$, then $P' = Q' \in \Gamma'$. Therefore, P' is the inverse of Q' in Γ'.

It remains to consider the case $P \neq Q$. Let Δ_1 and Δ_2 be two distinct circles which intersect at P and Q. According to Corollary 9.16, $\Delta_1 \perp \Gamma$ and $\Delta_2 \perp \Gamma$.

Denote by Δ_1' and Δ_2' the inversions of Δ_1 and Δ_2 in Ω. Clearly, Δ_1' and Δ_2' intersect at P' and Q'.

From Theorem 9.24, the latter is equivalent to $\Delta_1' \perp \Gamma'$ and $\Delta_2' \perp \perp \Gamma'$. By Corollary 9.15, P' is the inverse of Q' in Γ'. □

Chapter 10

Neutral plane

Let us remove Axiom V from our axiomatic system, see page 20. This way we define a new object called *neutral plane* or *absolute plane.* (In a neutral plane, the Axiom V may or may not hold.)

Clearly, any theorem in neutral geometry holds in Euclidean geometry. In other words, the Euclidean plane is an example of a neutral plane. In the next chapter we will construct an example of a neutral plane which is not Euclidean.

In this book, the Axiom V was used for the first time in the proof of uniqueness of parallel lines in Theorem 6.2. Therefore, all the statements before Theorem 6.2 also hold in neutral geometry.

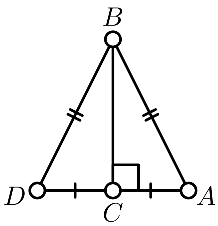
It makes all the discussed results about half-planes, signs of angles, congruence conditions, perpendicular lines and reflections true in neutral geometry. If a statement above marked with "✓", for example "**Theorem.**✓", then it holds in any neutral plane and the same proof works.

Let us give an example of a theorem in neutral geometry, which admits a simpler proof in Euclidean geometry.

10.1. Theorem. *Assume that triangles ABC and $A'B'C'$ have right angles at C and C' correspondingly, $AB = A'B'$ and $AC = A'C'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Euclidean proof. By the Pythagorean theorem $BC = B'C'$. Then the statement follows from the SSS congruence condition. □

Note that the proof of the Pythagorean theorem used properties of similar triangles, which in turn used Axiom V. Therefore this proof does not work in a neutral plane.



Neutral proof. Denote by D the reflection of A in (BC) and by D' the reflection of A' in $(B'C')$. Note that

$$AD = 2{\cdot}AC = 2{\cdot}A'C' = A'D',$$

$$BD = BA = B'A' = B'D'.$$

By SSS congruence condition 4.4, we get $\triangle ABD \cong \triangle A'B'D'$.

The theorem follows, since C is the midpoint of $[AD]$ and C' is the midpoint of $[A'D']$. □

10.2. Exercise. *Give a proof of Exercise 7.10 which works in the neutral plane.*

10.3. Exercise. *Let $ABCD$ be an inscribed quadrilateral in the neutral plane. Show that*

$$\measuredangle ABC + \measuredangle CDA \equiv \measuredangle BCD + \measuredangle DAB.$$

One cannot use the Theorem 8.10 to solve the exercise above, since it uses Theorems 8.1 and 8.2, which in turn uses Theorem 6.13.

Two angles of a triangle

In this section we will prove a weaker form of Theorem 6.13 which holds in any neutral plane.

10.4. Proposition. *Let $\triangle ABC$ be a nondegenerate triangle in the neutral plane. Then*

$$|\measuredangle CAB| + |\measuredangle ABC| < \pi.$$

Note that according to 3.11, the angles ABC, BCA and CAB have the same sign. Therefore, in the Euclidean plane the theorem follows immediately from Theorem 6.13. In neutral geometry, we need to work more.

Proof. By 3.11, we may assume that $\angle CAB$ and $\angle ABC$ are positive.

Let M be the midpoint of $[AB]$. Chose $C' \in (CM)$ distinct from C so that $C'M = CM$.

Note that the angles AMC and BMC' are vertical; in particular,

$$\measuredangle AMC = \measuredangle BMC'.$$

By construction, $AM = BM$ and $CM = C'M$. Therefore,

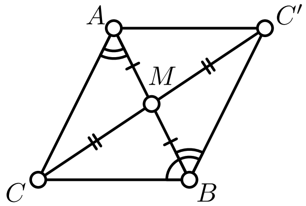
$$\triangle AMC \cong \triangle BMC';$$

in particular,

$$\measuredangle CAB = \pm\measuredangle C'BA.$$

According to 3.11, the angles CAB and $C'BA$ have the same sign as $\angle AMC$ and $\angle BMC'$. Therefore

$$\measuredangle CAB = \measuredangle C'BA.$$



In particular,

$$\measuredangle C'BC \equiv \measuredangle C'BA + \measuredangle ABC \equiv$$
$$\equiv \measuredangle CAB + \measuredangle ABC.$$

Finally, note that C' and A lie on the same side from (CB). Therefore, the angles CAB, ABC and $C'BC$ are positive. By Exercise 3.3, the result follows. □

10.5. Exercise. *Assume A, B, C and D are points in a neutral plane such that*

$$2\cdot\measuredangle ABC + 2\cdot\measuredangle BCD \equiv 0.$$

Show that $(AB) \parallel (CD)$.

Note that one cannot extract the solution of the above exercise from the proof of the transversal property (6.18)

10.6. Exercise. *Prove the* side-angle-angle congruence condition *holds in the neutral geometry.*

In other words, let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in a neutral plane. Show that $\triangle ABC \cong \triangle A'B'C'$ if

$$AB = A'B', \quad \measuredangle ABC = \pm\measuredangle A'B'C' \quad \textit{and} \quad \measuredangle BCA = \pm\measuredangle B'C'A'.$$

Note that in the Euclidean plane, the above exercise follows from ASA and the theorem on the sum of angles of a triangle (6.13). However, Theorem 6.13 cannot be used here, since its proof uses Axiom V. Later (Theorem 12.5) we will show that Theorem 6.13 does not hold in a neutral plane.

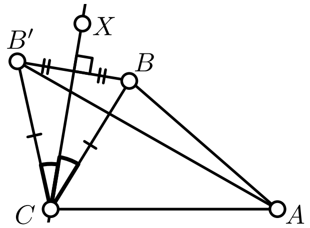
10.7. Exercise. *Assume that the point D lies between the vertices A and B of $\triangle ABC$ in a neutral plane. Show that*

$$CD < CA \quad \textit{or} \quad CD < CB.$$

Three angles of triangle

10.8. Proposition. *Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the neutral plane such that $AC = A'C'$ and $BC = B'C'$. Then*

$$AB < A'B' \quad \text{if and only if} \quad |\measuredangle ACB| < |\measuredangle A'C'B'|.$$



Proof. Without loss of generality, we may assume that $A = A'$ and $C = C'$ and $\measuredangle ACB, \measuredangle ACB' \geqslant 0$. In this case we need to show that

$$AB < AB' \iff \measuredangle ACB < \measuredangle ACB'.$$

Choose a point X so that

$$\measuredangle ACX = \tfrac{1}{2}\cdot(\measuredangle ACB + \measuredangle ACB').$$

Note that

- $\diamond$ (CX) bisects $\angle BCB'$.
- $\diamond$ (CX) is the perpendicular bisector of $[BB']$.
- $\diamond$ A and B lie on the same side from (CX) if and only if

$$\measuredangle ACB < \measuredangle ACB'.$$

From Exercise 5.3, A and B lie on the same side from (CX) if and only if $AB < AB'$. Hence the result follows. ☐

10.9. Theorem. *Let $\triangle ABC$ be a triangle in the neutral plane. Then*

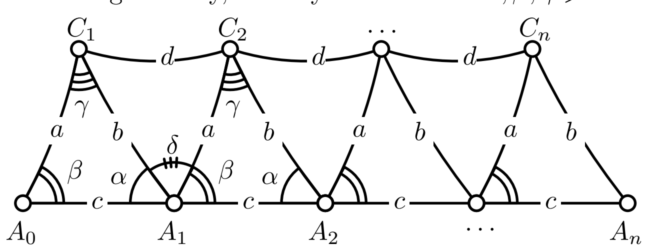
$$|\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB| \leqslant \pi.$$

The following proof is due to Legendre [10], earlier proofs were due to Saccheri [15] and Lambert [9].

Proof. Let $\triangle ABC$ be the given triangle. Set

$$\begin{aligned} a &= BC, & b &= CA, & c &= AB, \\ \alpha &= \measuredangle CAB, & \beta &= \measuredangle ABC, & \gamma &= \measuredangle BCA. \end{aligned}$$

Without loss of generality, we may assume that $\alpha, \beta, \gamma \geqslant 0$.



Fix a positive integer n. Consider the points $A_0, A_1, \dots, A_n$ on the half-line $[BA)$, such that $BA_i = i\cdot c$ for each i. (In particular, $A_0 = B$ and $A_1 = A$.) Let us construct the points $C_1, C_2, \dots, C_n$, so that $\measuredangle A_i A_{i-1} C_i = \beta$ and $A_{i-1}C_i = a$ for each i.

We have constructed n congruent triangles

$$\begin{aligned} \triangle ABC = \triangle A_1 A_0 C_1 &\cong \\ \cong \triangle A_2 A_1 C_2 &\cong \\ \dots \\ \cong \triangle A_n A_{n-1} C_n&. \end{aligned}$$

Set $d = C_1C_2$ and $\delta = \measuredangle C_2 A_1 C_1$. Note that

❶ $$\alpha + \beta + \delta = \pi.$$

By Proposition 10.4, $\delta \geqslant 0$.

By construction

$$\triangle A_1 C_1 C_2 \cong \triangle A_2 C_2 C_3 \cong \dots \cong \triangle A_{n-1} C_{n-1} C_n.$$

In particular, $C_i C_{i+1} = d$ for each i.

By repeated application of the triangle inequality, we get that

$$\begin{aligned} n\cdot c = A_0 A_n &\leqslant \\ &\leqslant A_0 C_1 + C_1 C_2 + \dots + C_{n-1} C_n + C_n A_n = \\ &= a + (n-1)\cdot d + b. \end{aligned}$$

In particular,

$$c \leqslant d + \tfrac{1}{n}\cdot(a + b - d).$$

Since n is arbitrary positive integer, the latter implies

$$c \leqslant d.$$

From Proposition 10.8 and SAS, the latter is equivalent to

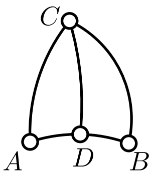
$$\gamma \leqslant \delta.$$

From ❶, the theorem follows. □

The *defect of triangle* $\triangle ABC$ is defined as

$$\operatorname{defect}(\triangle ABC) := \pi - |\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB|.$$

Note that Theorem 10.9 sates that, the defect of any triangle in a neutral plane has to be nonnegative. According to Theorem 6.13, any triangle in the Euclidean plane has zero defect.



10.10. Exercise. *Let $\triangle ABC$ be nondegenerate triangle in the neutral plane. Assume D lies between A and B. Show that*

$$\operatorname{defect}(\triangle ABC) = \operatorname{defect}(\triangle ADC) + \operatorname{defect}(\triangle DBC).$$

How to prove that something cannot be proved

Many attempts were made to prove that any theorem in Euclidean geometry holds in neutral geometry. The latter is equivalent to the statement that Axiom V is a *theorem* in neutral geometry.

Many of these attempts were accepted as proofs for long periods of time, until a mistake was found.

There is a number of statements in neutral geometry which are equivalent to the Axiom V. It means that if we exchange the Axiom V to any of these statements, then we will obtain an equivalent axiomatic system.

The following theorem provides a short list of such statements. We are not going to prove it in the book.

10.11. Theorem. *A neutral plane is Euclidean if and only if one of the following equivalent conditions holds.*

(a) There is a line ℓ and a point $P \notin \ell$ such that there is only one line passing thru P and parallel to ℓ.
(b) Every nondegenerate triangle can be circumscribed.
(c) There exists a pair of distinct lines which lie on a bounded distance from each other.
(d) There is a triangle with an arbitrary large inradius.
(e) There is a nondegenerate triangle with zero defect.
(f) There exists a quadrilateral in which all the angles are right.

It is hard to imagine a neutral plane, which does not satisfy some of the properties above. That is partly the reason for the large number of false proofs; each used one of such statements by accident.

Let us formulate the negation of *(a)* above as a new axiom; we label it h-V as a *hyperbolic version* of Axiom V on page 20.

h-V. For any line ℓ and any point $P \notin \ell$ there are at least two lines which pass thru P and parallel to ℓ.

By Theorem 6.2, a neutral plane which satisfies Axiom h-V is not Euclidean. Moreover, according to the Theorem 10.11 (which we do not prove) in any non-Euclidean neutral plane, Axiom h-V holds.

It opens a way to look for a proof by contradiction. Simply exchange Axiom V to Axiom h-V and start to prove theorems in the obtained axiomatic system. In the case if we arrive to a contradiction, we prove the Axiom V in neutral plane. This idea was growing since the 5th century; the most notable results were obtained by Saccheri in [15].

The system of axioms I–IV and h-V defines a new geometry which is now called *hyperbolic* or *Lobachevskian geometry*. The more this geometry was developed, it became more and more believable that there is no contradiction; that is, the system of axioms I–IV and h-V is *consistent*. In fact, the following theorem holds.

10.12. Theorem. *The hyperbolic geometry is consistent if and only if so is the Euclidean geometry.*

The claims that hyperbolic geometry has no contradiction can be found in private letters of Gauss, Schweikart and Taurinus.[1] They all seem to be afraid to state it in public. For instance, in 1818 Gauss writes to Gerling:

> *...I am happy that you have the courage to express yourself as if you recognized the possibility that our parallels theory along with our entire geometry could be false. But the wasps whose nest you disturb will fly around your head.*

Lobachevsky came to the same conclusion independently. Unlike the others, he had the courage to state it in public and in print (see [11]). That cost him serious troubles. Couple of years latter, also independently, Bolyai published his work (see [6]).

It seems that Lobachevsky was the first who had a proof of Theorem 10.12 altho its formulation required rigorous axiomatics, which were not developed at his time. Later, Beltrami gave a cleaner proof of the "if" part of the theorem. It was done by modeling points, lines, distances and angle measures of one geometry using some other objects in another geometry. The same idea was used earlier by Lobachevsky; in [12, 34] he modeled the Euclidean plane in the hyperbolic space.

The proof of Beltrami is the subject of the next chapter.

The discovery of hyperbolic geometry was one of the main scientific discoveries of the 19th century, on the same level are Mendel's laws and the law of multiple proportions.

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[1]The oldest surviving letters were the Gauss letter to Gerling in 1816 and yet more convincing letter dated 1818 of Schweikart sent to Gauss via Gerling.

Curvature

In a letter from 1824 Gauss writes:

> *The assumption that the sum of the three angles is less than π leads to a curious geometry, quite different from ours but thoroughly consistent, which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of a determination of a constant, which cannot be designated a priori. The greater one takes this constant, the nearer one comes to Euclidean geometry, and when it is chosen indefinitely large the two coincide. The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. For example, the three angles of a triangle become as small as one wishes, if only the sides are taken large enough; yet the area of the triangle can never exceed a definite limit, regardless how great the sides are taken, nor indeed can it ever reach it.*

In modern terminology, the constant which Gauss mentions, can be expressed as $1/\sqrt{-k}$, where $k \leqslant 0$, is the so called *curvature* of the neutral plane, which we are about to introduce.

The identity in Exercise 10.10 suggests that the defect of a triangle should be proportional to its area.[2]

In fact, for any neutral plane, there is a nonpositive real number k such that

$$k\cdot \operatorname{area}(\triangle ABC) + \operatorname{defect}(\triangle ABC) = 0$$

for any $\triangle ABC$. This number k is called the *curvature* of the plane.

For example, by Theorem 6.13, the Euclidean plane has zero curvature. By Theorem 10.9, the curvature of any neutral plane is nonpositive.

It turns out that up to isometry, the neutral plane is characterized by its curvature; that is, two neutral planes are isometric if and only if they have the same curvature.

In the next chapter, we will construct a *hyperbolic plane*; this is, an example of neutral plane with curvature $k = -1$.

Any neutral planes, distinct from Euclidean, can be obtained by rescaling the metric on the hyperbolic plane. Indeed, if we rescale the metric by a positive factor c, the area changes by factor c^2, while the defect stays the same. Therefore, taking $c = \sqrt{-k}$, we can get the neutral plane of the given curvature $k < 0$. In other words, all the

[2]The area in the neutral plane is discussed briefly in the end of Chapter 19, but the reader could also refer to an intuitive understanding of area measurement.

non-Euclidean neutral planes become identical if we use $r = 1/\sqrt{-k}$ as the unit of length.

In Chapter 15, we discuss the geometry of the unit sphere. Altho spheres are not neutral planes, the spherical geometry is a close relative of Euclidean and hyperbolic geometries.

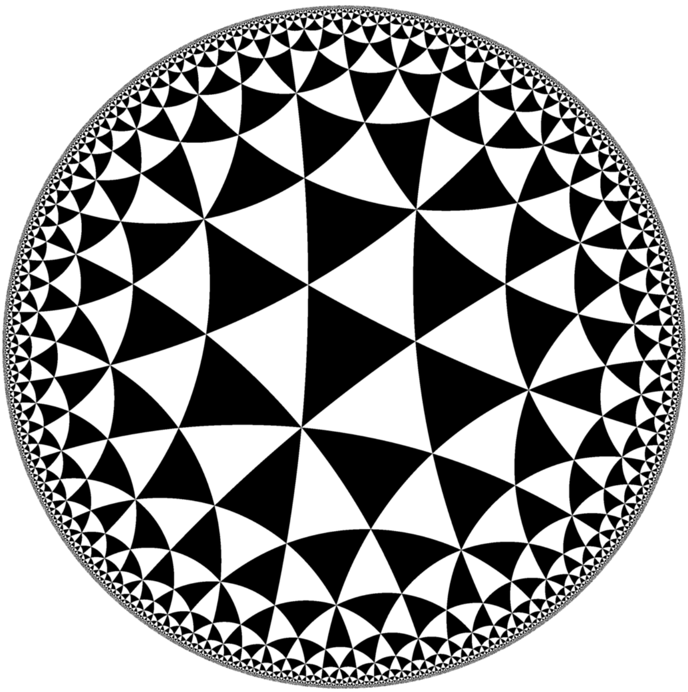
Nondegenerate spherical triangles have negative defects. Moreover, if R is the radius of the sphere, then

$$\tfrac{1}{R^2} \cdot \operatorname{area}(\triangle ABC) + \operatorname{defect}(\triangle ABC) = 0$$

for any spherical triangle ABC. In other words, the sphere of the radius R has the curvature $k = \frac{1}{R^2}$.

Chapter 11

Hyperbolic plane



In this chapter, we use inversive geometry to construct the model of hyperbolic plane — a neutral plane which is not Euclidean.

Namely, we construct the so called *conformal disk model* of the hyperbolic plane. This model was discovered by Beltrami in [4] and often called the *Poincaré disc model*.

The figure above shows the conformal disk model of the hyperbolic plane which is cut into congruent triangles with angles $\frac{\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{4}$.

Conformal disk model

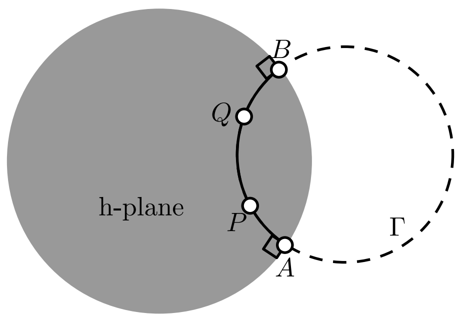
In this section, we give new names for some objects in the Euclidean plane which will represent lines, angle measures and distances in the hyperbolic plane.

Hyperbolic plane. Let us fix a circle on the Euclidean plane and call it *absolute*. The set of points inside the absolute will be called the *hyperbolic plane* (or *h-plane*).

Note that the points on the absolute do *not* belong to the h-plane. The points in the h-plane will be also called *h-points*.

Often we will assume that the absolute is a unit circle.

Hyperbolic lines. The intersections of the h-plane with circlines perpendicular to the absolute are called *hyperbolic lines* or *h-lines*.



By Corollary 9.17, there is a unique h-line which passes thru the given two distinct h-points P and Q. This h-line will be denoted by $(PQ)_h$.

The arcs of hyperbolic lines will be called *hyperbolic segments* or *h-segments*. An h-segment with endpoints P and Q will be denoted by $[PQ]_h$.

The subset of an h-line on one side from a point will be called a *hyperbolic half-line* (or *h-half-line*). More precisely, an h-half-line is an intersection of the h-plane with arc which perpendicular to the absolute with only one endpoint in the h-plane. An h-half-line starting at P and passing thru Q will be denoted by $[PQ)_h$.

If Γ is the circle containing the h-line $(PQ)_h$, then the points of intersection of Γ with the absolute are called *ideal points* of $(PQ)_h$. (Note that the ideal points of an h-line do not belong to the h-line.)

An ordered triple of h-points, say (P, Q, R) will be called *h-triange* PQR and denoted by $\triangle_h PQR$.

So far, $(PQ)_h$ is just a subset of the h-plane; below we will introduce h-distance and later we will show that $(PQ)_h$ is a line for the h-distance in the sense of the Definition 1.8.

11.1. Exercise. *Show that the h-line is uniquely determined by its ideal points.*

11.2. Exercise. *Show that the h-line is uniquely determined by one of its ideal points and one h-point on it.*

11.3. Exercise. *Show that the h-segment $[PQ]_h$ coincides with the Euclidean segment $[PQ]$ if and only if the line (PQ) pass thru the center of the absolute.*

Hyperbolic distance. Let P and Q be distinct h-points. Denote by A and B be the ideal points of $(PQ)_h$. Without loss of generality, we may assume that on the Euclidean circle containing the h-line $(PQ)_h$, the points A, P, Q, B appear in the same order.

Consider the function

$$\delta(P, Q) := \frac{AQ \cdot BP}{QB \cdot PA}.$$

Note that the right hand side is the cross-ratio, which appeared in Theorem 9.6. Set $\delta(P, P) = 1$ for any h-point P. Set

$$PQ_h := \ln[\delta(P, Q)].$$

The proof that PQ_h is a metric on the h-plane will be given below. For now it is just a function which returns a real value PQ_h for any pair of h-points P and Q.

11.4. Exercise. *Let O be the center of the absolute and the h-points O, X and Y lie on one h-line in the same order. Assume $OX = XY$. Prove that $OX_h < XY_h$.*

Hyperbolic angles. Consider three h-points P, Q and R such that $P \neq Q$ and $R \neq Q$. The *hyperbolic angle PQR* (briefly $\angle_h PQR$) is an ordered pair of h-half-lines $[QP)_h$ and $[QR)_h$.

Let $[QX)$ and $[QY)$ be (Euclidean) half-lines which are tangent to $[QP]_h$ and $[QR]_h$ at Q. Then the *hyperbolic angle measure* (or *h-angle measure*) of $\angle_h PQR$ denoted by $\measuredangle_h PQR$ and defined as $\measuredangle XQY$.

11.5. Exercise. *Let ℓ be an h-line and P be an h-point which does not lie on ℓ. Show that there is a unique h-line passing thru P and perpendicular to ℓ.*

Plan of the proof

We defined all the *h-notions* needed in the formulation of the axioms I–IV and h-V. It remains to show that all these axioms hold; this will be done by the end of this chapter.

Once we are done with the proofs, we get that the model provides an example of a neutral plane; in particular, Exercise 11.5 can be proved the same way as Theorem 5.5.

Most importantly we will prove the "if"-part of Theorem 10.12.

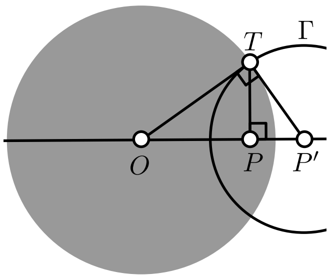
Indeed, any statement in hyperbolic geometry can be restated in the Euclidean plane using the introduced h-notions. Therefore, if the system of axioms I–IV and h-V leads to a contradiction, then so does the system axioms I–V.

Auxiliary statements

11.6. Lemma. *Consider an h-plane with a unit circle as the absolute. Let O be the center of the absolute and P be another h-point. Denote by P' the inverse of P in the absolute.*

Then the circle Γ with the center P' and radius $1/\sqrt{1-OP^2}$ is perpendicular to the absolute. Moreover, O is the inverse of P in Γ.

Proof. Follows from Exercise 9.20. □



Assume Γ is a circline which is perpendicular to the absolute. Consider the inversion $X \mapsto X'$ in Γ, or if Γ is a line, set $X \mapsto X'$ to be the reflection in Γ.

The following observation says that the map $X \mapsto X'$ respects all the notions introduced in the previous section. Together with the lemma above, it implies that in any problem which is formulated entirely *in h-terms* we can assume that a given h-point lies in the center of the absolute.

11.7. Main observation. *The map $X \mapsto X'$ described above is a bijection of the h-plane to itself. Moreover, for any h-points P, Q, R such that $P \ne Q$ and $Q \ne R$, the following conditions hold:*

(a) The h-line $(PQ)_h$, h-half-line $[PQ)_h$ and h-segment $[PQ]_h$ are mapped to $(P'Q')_h$, $[P'Q')_h$ and $[P'Q']_h$ correspondingly.

(b) $\delta(P',Q') = \delta(P,Q)$ and $P'Q'_h = PQ_h$.

(c) $\measuredangle_h P'Q'R' \equiv -\measuredangle_h PQR$.

It is instructive to compare this observation with Proposition 5.6.

Proof. According to Theorem 9.14, the map sends the absolute to itself. Note that the points on Γ do not move, it follows that points inside of the absolute remain inside after the mapping and the other way around.

Part *(a)* follows from 9.7 and 9.24.

Part *(b)* follows from Theorem 9.6.

Part *(c)* follows from Theorem 9.24. □

11.8. Exercise. *Let Γ be a circle which is perpendicular to the absolute and let Q be an h-point lying on Γ. Assume P is an h-point and P' is its inversion in Γ. Show that $PQ_h = P'Q_h$.*

11.9. Exercise. *Consider the function*

$$\theta(P,Q) = 2\cdot\frac{PQ\cdot P'Q'}{PP'\cdot QQ'},$$

where P, Q are points in the h-plane and P', Q' are their inversions in the absolute. Show that

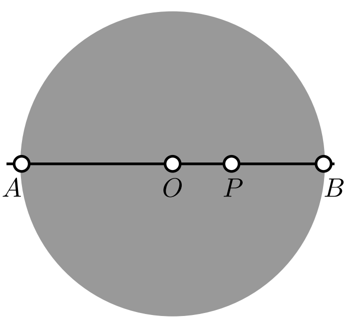
$$2 + 2\cdot\theta(P,Q) = \delta(P,Q) + \frac{1}{\delta(P,Q)}.$$

Conclude that

$$PQ_h = \ln\left[1 + \theta(P,Q) + \sqrt{\theta(P,Q)^2 + 2\cdot\theta(P,Q)}\right].$$

11.10. Lemma. *Assume that the absolute is a unit circle centered at O. Given an h-point P, set $x = OP$ and $y = OP_h$. Then*

$$y = \ln\frac{1+x}{1-x} \qquad \textit{and} \qquad x = \frac{e^y - 1}{e^y + 1}.$$



Proof. Note that the h-line $(OP)_h$ forms a diameter of the absolute. If A and B are the ideal points as in the definition of h-distance, then

$$OA = OB = 1,$$
$$PA = 1 + x,$$
$$PB = 1 - x.$$

In particular,

$$y = \ln\frac{AP\cdot BO}{PB\cdot OA} = \ln\frac{1+x}{1-x}.$$

Taking the exponential function of the left and the right hand side and applying obvious algebra manipulations, we get

$$x = \frac{e^y - 1}{e^y + 1}.$$

□

11.11. Lemma. *Assume the points P, Q and R appear on one h-line in the same order. Then*

$$PQ_h + QR_h = PR_h.$$

Proof. Note that

$$PQ_h + QR_h = PR_h$$

is equivalent to

❶ $$\delta(P,Q)\cdot\delta(Q,R) = \delta(P,R).$$

Let A and B be the ideal points of $(PQ)_h$. Without loss of generality, we can assume that the points A, P, Q, R and B appear in the same order on the circline containing $(PQ)_h$. Then

$$\begin{aligned}\delta(P,Q)\cdot\delta(Q,R) &= \frac{AQ\cdot BP}{QB\cdot PA}\cdot\frac{AR\cdot BQ}{RB\cdot QA} = \\ &= \frac{AR\cdot BP}{RB\cdot PA} = \\ &= \delta(P,R)\end{aligned}$$

Hence ❶ follows. □

Let P be an h-point and $\rho > 0$. The set of all h-points Q such that $PQ_h = \rho$ is called an *h-circle* with the center P and the *h-radius* ρ.

11.12. Lemma. *Any h-circle is a Euclidean circle which lies completely in the h-plane.*

More precisely for any h-point P and $\rho \geqslant 0$ there is a $\hat{\rho} \geqslant 0$ and a point $\hat{P}$ such that

$$PQ_h = \rho \quad\Longleftrightarrow\quad \hat{P}Q = \hat{\rho}$$

for any h-point Q.

Moreover, if O is the center of the absolute, then

1. *$\hat{O} = O$ for any ρ and*
2. *$\hat{P} \in (OP)$ for any $P \neq O$.*

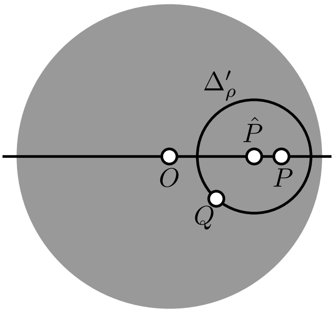
Proof. According to Lemma 11.10, $OQ_h = \rho$ if and only if

$$OQ = \hat{\rho} = \frac{e^\rho - 1}{e^\rho + 1}.$$

Therefore, the locus of h-points Q such that $OQ_h = \rho$ is a Euclidean circle, denote it by Δ_ρ.

If $P \neq O$, applying Lemma 11.6 and the main observation (11.7) we get a circle Γ perpendicular to the absolute such that P is the inverse of O in Γ.

Let Δ'_ρ be the inverse of Δ_ρ in Γ. Since the inversion in Γ preserves the h-distance, $PQ_h = \rho$ if and only if $Q \in \Delta'_\rho$.



According to Theorem 9.7, Δ'_ρ is a Euclidean circle. Denote by $\hat{P}$ the Euclidean center and by $\hat{\rho}$ the Euclidean radius of Δ'_ρ.

Finally, note that Δ'_ρ reflects to itself in (OP); that is, the center $\hat{P}$ lies on (OP). □

11.13. Exercise. *Assume P, $\hat{P}$ and O are as in the Lemma 11.12 and $P \neq$ $\neq O$. Show that $\hat{P} \in [OP]$.*

Axiom I

Evidently, the h-plane contains at least two points. Therefore, to show that Axiom I holds in the h-plane, we need to show that the h-distance defined on page 88 is a metric on h-plane; that is, the conditions *(a)*–*(d)* in Definition 1.1 hold for h-distance.

The following claim says that the h-distance meets the conditions *(a)* and *(b)*.

11.14. Claim. *Given the h-points P and Q, we have $PQ_h \geqslant 0$ and $PQ_h = 0$ if and only if $P = Q$.*

Proof. According to Lemma 11.6 and the main observation (11.7), we may assume that Q is the center of the absolute. In this case

$$\delta(Q,P) = \frac{1+QP}{1-QP} \geqslant 1$$

and therefore

$$QP_h = \ln[\delta(Q,P)] \geqslant 0.$$

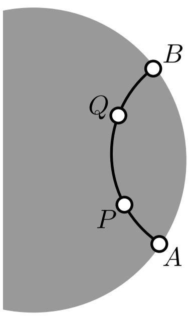
Moreover, the equalities holds if and only if $P = Q$. □

The following claim says that the h-distance meets the condition 1.1*c*.

11.15. Claim. *For any h-points P and Q, we have $PQ_h = QP_h$.*

Proof. Let A and B be ideal points of $(PQ)_h$ and A, P, Q, B appear on the circline containing $(PQ)_h$ in the same order.

Then



$$\begin{aligned} PQ_h &= \ln \frac{AQ \cdot BP}{QB \cdot PA} = \\ &= \ln \frac{BP \cdot AQ}{PA \cdot QB} = \\ &= QP_h. \end{aligned}$$

□

The following claim shows, in particular, that the triangle inequality (which is condition 1.1*d*) holds for *h*-distance.

11.16. Claim. *Given a triple of h-points P, Q and R, we have*

$$PQ_h + QR_h \geqslant PR_h.$$

Moreover, the equality holds if and only if P, Q and R lie on one h-line in the same order.

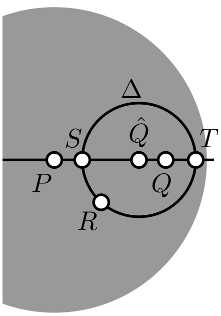
Proof. Without loss of generality, we may assume that P is the center of the absolute and $PQ_h \geqslant QR_h > 0$.

Denote by Δ the h-circle with the center Q and h-radius $\rho = QR_h$. Let S and T be the points of intersection of (PQ) and Δ.

By Lemma 11.11, $PQ_h \geqslant QR_h$. Therefore, we can assume that the points P, S, Q and T appear on the h-line in the same order.

According to Lemma 11.12, Δ is a Euclidean circle; denote by $\hat{Q}$ its Euclidean center. Note that $\hat{Q}$ is the (Euclidean) midpoint of $[ST]$.

By the Euclidean triangle inequality



❷ $$PT = P\hat{Q} + \hat{Q}R \geqslant PR$$

and the equality holds if and only if $T = R$.

By Lemma 11.10,

$$\begin{aligned} PT_h &= \ln \frac{1 + PT}{1 - PT}, \\ PR_h &= \ln \frac{1 + PR}{1 - PR}. \end{aligned}$$

Since the function $f(x) = \ln \frac{1+x}{1-x}$ is increasing for $x \in [0, 1)$, inequality ❷ implies

$$PT_h \geqslant PR_h$$

and the equality holds if and only if $T = R$.

Finally, applying Lemma 11.11 again, we get

$$PT_h = PQ_h + QR_h.$$

Hence the claim follows. □

Axiom II

Note that once the following claim is proved, Axiom II follows from Corollary 9.17.

11.17. Claim. *A subset of the h-plane is an h-line if and only if it forms a line for the h-distance in the sense of Definition 1.8.*

Proof. Let ℓ be an h-line. Applying the main observation (11.7) we can assume that ℓ contains the center of the absolute. In this case, ℓ is an intersection of a diameter of the absolute and the h-plane. Let A and B be the endpoints of the diameter.

Consider the map $\iota\colon \ell \to \mathbb{R}$ defined as

$$\iota(X) = \ln \frac{AX}{XB}.$$

Note that $\iota\colon \ell \to \mathbb{R}$ is a bijection.

Further, if $X, Y \in \ell$ and the points A, X, Y and B appear on $[AB]$ in the same order, then

$$\iota(Y) - \iota(X) = \ln \frac{AY}{YB} - \ln \frac{AX}{XB} = \ln \frac{AY\cdot BX}{YB\cdot XB} = XY_h.$$

We proved that any h-line is a line for h-distance. The converse follows from Claim 11.16. □

Axiom III

Note that the first part of Axiom III follows directly from the definition of the h-angle measure defined on page 88. It remains to show that $\measuredangle_h$ satisfies the conditions IIIa, IIIb and IIIc on page 20.

The following two claims say that $\measuredangle_h$ satisfies IIIa and IIIb.

11.18. Claim. *Given an h-half-line $[OP)_h$ and $\alpha \in (-\pi, \pi]$, there is a unique h-half-line $[OQ)_h$ such that $\measuredangle_h POQ = \alpha$.*

11.19. Claim. *For any h-points P, Q and R distinct from an h-point O, we have*

$$\measuredangle_h POQ + \measuredangle_h QOR \equiv \measuredangle_h POR.$$

Proof of 11.18 and 11.19. Applying the main observation, we may assume that O is the center of the absolute. In this case, for any h-point $P \neq O$, $[OP)_h$ is the intersection of $[OP)$ with h-plane. Hence the claims 11.18 and 11.19 follow from the corresponding axioms of the Euclidean plane. □

The following claim says that $\measuredangle_h$ satisfies IIIc.

11.20. Claim. *The function*

$$\measuredangle_h \colon (P, Q, R) \mapsto \measuredangle_h PQR$$

is continuous at any triple of points (P, Q, R) such that $Q \neq P$, $Q \neq R$ and $\measuredangle_h PQR \neq \pi$.

Proof. Denote by O the center of the absolute. We can assume that Q is distinct from O.

Denote by Z the inverse of Q in the absolute and by Γ the circle perpendicular to the absolute which is centered at Z. According to Lemma 11.6, the point O is the inverse of Q in Γ.

Denote by P' and R' the inversions in Γ of the points P and R correspondingly. Note that the point P' is completely determined by the points Q and P. Moreover, the map $(Q, P) \mapsto P'$ is continuous at any pair of points (Q, P) such that $Q \neq O$. The same is true for the map $(Q, R) \mapsto R'$

According to the main observation

$$\measuredangle_h PQR \equiv -\measuredangle_h P'OR'.$$

Since $\measuredangle_h P'OR' = \measuredangle P'OR'$ and the maps $(Q, P) \mapsto P'$, $(Q, R) \mapsto R'$ are continuous, the claim follows from the corresponding axiom of the Euclidean plane. □

Axiom IV

The following claim says that Axiom IV holds in the h-plane.

11.21. Claim. *In the h-plane, we have $\triangle_h PQR \cong \triangle_h P'Q'R'$ if and only if*

$$Q'P'_h = QP_h, \quad Q'R'_h = QR_h \quad \text{and} \quad \measuredangle_h P'Q'R' = \pm\measuredangle PQR.$$

Proof. Applying the main observation, we can assume that both Q and Q' coincide with the center of the absolute. In this case

$$\measuredangle P'QR' = \measuredangle_h P'QR' = \pm\measuredangle_h PQR = \pm\measuredangle PQR.$$

Since

$$QP_h = QP'_h \quad \text{and} \quad QR_h = QR'_h,$$

Lemma 11.10 implies that the same holds for the Euclidean distances; that is,

$$QP = QP' \quad \text{and} \quad QR = QR'.$$

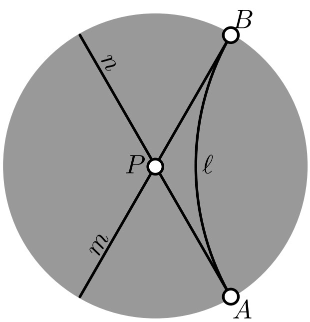
By SAS, there is a motion of the Euclidean plane which sends Q to itself, P to P' and R to R'

Note that the center of the absolute is fixed by the corresponding motion. It follows that this motion gives also a motion of the h-plane; in particular, the h-triangles $\triangle_h PQR$ and $\triangle_h P'QR'$ are h-congruent. □

Axiom h-V

Finally, we need to check that the Axiom h-V on page 82 holds; that is, we need to prove the following claim.

11.22. Claim. *For any h-line ℓ and any h-point $P \notin \ell$ there are at least two h-lines which pass thru P and have no points of intersection with ℓ.*



Instead of proof. Applying the main observation we can assume that P is the center of the absolute.

The remaining part of the proof can be guessed from the picture □

11.23. Exercise. *Show that in the h-plane there are 3 mutually parallel h-lines such that any pair of these three lines lies on one side from the remaining h-line.*

Chapter 12

Geometry of the h-plane

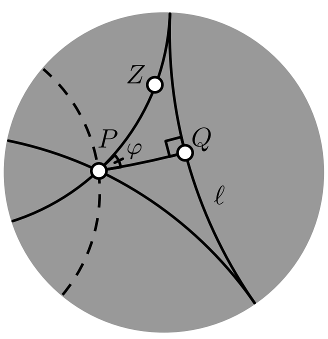
In this chapter, we study the geometry of the plane described by the conformal disc model. For briefness, this plane will be called the *h-plane*.

We can work with this model directly from inside of the Euclidean plane. We may also use the axioms of neutral geometry since they all hold in the h-plane; the latter proved in the previous chapter.

Angle of parallelism

Let P be a point off an h-line ℓ. Drop a perpendicular $(PQ)_h$ from P to ℓ; let Q be its foot point. Let φ be the smallest value such that the h-line $(PZ)_h$ with $|\measuredangle_h QPZ| = \varphi$ does not intersect ℓ.

The value φ is called the *angle of parallelism* of P to ℓ. Clearly, φ depends only on the h-distance $s = PQ_h$. Further, $\varphi(s) \to \pi/2$ as $s \to 0$, and $\varphi(s) \to 0$ as $s \to \infty$. (In the Euclidean geometry, the angle of parallelism is identically equal to $\pi/2$.)



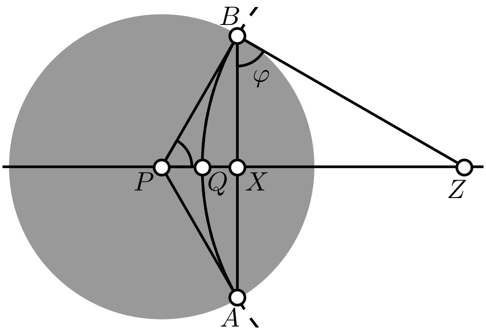
If ℓ, P and Z are as above, then the h-line $m = (PZ)_h$ is called *asymptotically parallel* to ℓ. In other words, two h-lines are asymptotically parallel if they share one ideal point. (In hyperbolic geometry, the term *parallel lines* is often used for *asymptotically parallel lines*; we do not follow this convention.)

Given $P \notin \ell$, there are exactly two asymptotically parallel lines thru P to ℓ; the remaining parallel lines are called *ultra parallel*.

On the diagram, the two solid h-lines passing thru P are asymptotically parallel to ℓ; the dashed h-line is ultra parallel to ℓ.

12.1. Proposition. *Let Q be the foot point of P on h-line ℓ. Denote by φ the angle of parallelism of P to ℓ. Then*

$$PQ_h = \tfrac{1}{2}\cdot \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}.$$



Proof. Applying a motion of the h-plane if necessary, we may assume P is the center of the absolute. Then the h-lines thru P are the intersections of Euclidean lines with the h-plane.

Let us denote by A and B the ideal points of ℓ. Without loss of generality, we may assume that $\angle APB$ is positive. In this case

$$\varphi = \measuredangle QPB = \measuredangle APQ = \tfrac{1}{2}\cdot\measuredangle APB.$$

Let Z be the center of the circle Γ containing the h-line ℓ. Set X to be the point of intersection of the Euclidean segment $[AB]$ and the line (PQ).

Note that, $PX = \cos\varphi$. Therefore, by Lemma 11.10,

$$PX_h = \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}.$$

Note that both angles PBZ and BXZ are right. Since the angle PZB is shared, $\triangle ZBX \sim \triangle ZPB$. In particular,

$$ZX\cdot ZP = ZB^2;$$

that is, X is the inverse of P in Γ.

The inversion in Γ is the reflection of the h-plane in ℓ. Therefore

$$\begin{aligned} PQ_h &= QX_h = \\ &= \tfrac{1}{2}\cdot PX_h = \\ &= \tfrac{1}{2}\cdot \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}. \end{aligned}$$

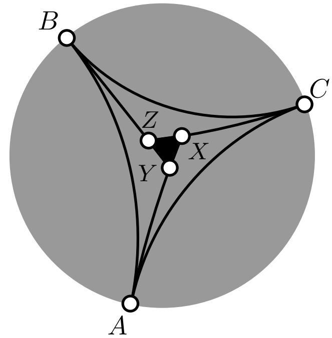
$\square$

Inradius of h-triangle

12.2. Theorem. *The inradius of any h-triangle is less than* $\frac{1}{2} \cdot \ln 3$.

Proof. Fix an h-triangle XYZ. Denote by A, B and C the ideal points of the h-half-lines $[XY)_h$, $[YZ)_h$ and $[ZX)_h$.

Consider the *ideal triangle* ABC; its *sides* are the three pairwise asymptotically parallel lines with ideal points A, B and C.



It should be clear that the inradius of the ideal triangle ABC is bigger than the inradius of $\triangle_h XYZ$.

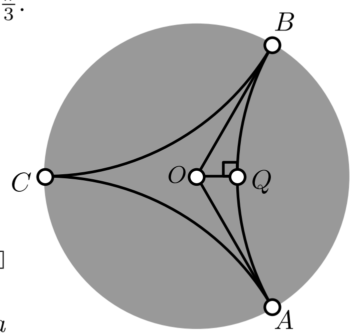
Applying an inversion if necessary, we can assume that the h-incenter (O) of the ideal triangle is the center of the absolute (see 11.7). Therefore, without loss of generality, we may assume

$$\measuredangle AOB = \measuredangle BOC = \measuredangle COA = \tfrac{2}{3} \cdot \pi.$$

It remains to find the inradius. Denote by Q the foot point of O on $(AB)_h$. Then OQ_h is the inradius. Note that the angle of parallelism of $(AB)_h$ at O is equal to $\frac{\pi}{3}$.

By Proposition 12.1,

$$\begin{aligned} OQ_h &= \tfrac{1}{2} \cdot \ln \frac{1+\cos\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \\ &= \tfrac{1}{2} \cdot \ln \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \\ &= \tfrac{1}{2} \cdot \ln 3. \end{aligned}$$



□

12.3. Exercise. *Let* $\square_h ABCD$ *be a quadrilateral in the h-plane such that the h-angles at* A, B *and* C *are right and* $AB_h = BC_h$. *Find the optimal upper bound for* AB_h.

Circles, horocycles and equidistants

Note that according to Lemma 11.12, any h-circle is a Euclidean circle which lies completely in the h-plane. Further, any h-line is an intersection of the h-plane with the circle perpendicular to the absolute.

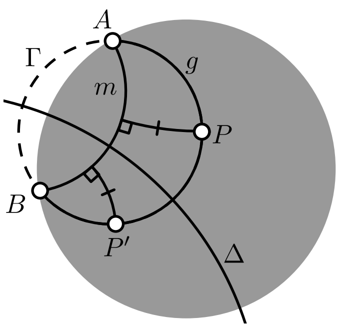
In this section we will describe the h-geometric meaning of the intersections of the other circles with the h-plane.

You will see that all these intersections have a *perfectly round shape* in the h-plane.

One may think of these curves as trajectories of a car which drives in the plane with a fixed position of the steering wheel.

In the Euclidean plane, this way you either run along a circle or along a line.

In the hyperbolic plane, the picture is different. If you turn the steering wheel to the far right, you will run along a circle. If you turn it less, at a certain position of the wheel, you will never come back to the same point, but the path will be different from the line. If you turn the wheel further a bit, you start to run along a path which stays at some fixed distance from an h-line.



Equidistants of h-lines. Consider the h-plane with the absolute Ω. Assume a circle Γ intersects Ω in two distinct points, A and B. Denote by g the intersection of Γ with the h-plane. Let us draw an h-line m with the ideal points A and B. According to Exercise 11.1, m is uniquely defined.

Consider any h-line ℓ perpendicular to m; let Δ be the circle containing ℓ.

Note that $\Delta \perp \Gamma$. Indeed, according to Corollary 9.15, m and Ω invert to themselves in Δ. It follows that A is the inverse of B in Δ. Finally, by Corollary 9.16, we get that $\Delta \perp \Gamma$.

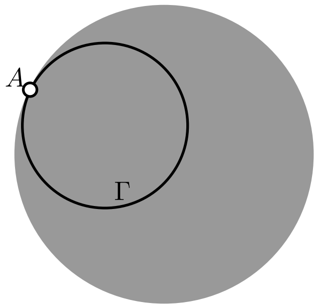
Therefore, inversion in Δ sends both m and g to themselves. For any two points $P', P \in g$ there is a choice of ℓ and Δ as above such that P' is the invese of P in Δ. By the main observation (11.7) the inversion in Δ is a motion of the h-plane. Therefore, all points of g lie on the same distance from m.

In other words, g is the set of points which lie on a fixed h-distance and on the same side from m.

Such a curve g is called *equidistant* to h-line m. In Euclidean geometry, the equidistant from a line is a line; apparently in hyperbolic geometry the picture is different.

Horocycles. If the circle Γ touches the absolute from inside at one point A, then the complement $h = \Gamma \backslash \{A\}$ lies in the h-plane. This set is called a *horocycle*. It also has a perfectly round shape in the sense described above.

Horocycles are the border case between circles and equidistants to h-lines. A horocycle might be considered as a limit of circles thru a fixed point with the centers running to infinity along a line. The same horocycle is a limit of equidistants which pass thru fixed point to the h-lines running to infinity.

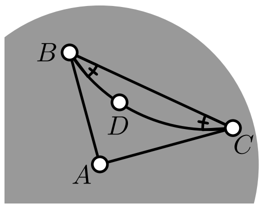


12.4. Exercise. *Find the leg of an isosceles right h-triangle inscribed in a horocycle.*

Hyperbolic triangles

12.5. Theorem. *Any nondegenerate hyperbolic triangle has a positive defect.*

Proof. Fix an h-triangle ABC. According to Theorem 10.9,



$$❶ \qquad \operatorname{defect}(\triangle_h ABC) \geqslant 0.$$

It remains to show that in the case of equality, $\triangle_h ABC$ degenerates.

Without loss of generality, we may assume that A is the center of the absolute; in this case $\measuredangle_h CAB = \measuredangle CAB$. Yet we may assume that

$$\measuredangle_h CAB, \quad \measuredangle_h ABC, \quad \measuredangle_h BCA, \quad \measuredangle ABC, \quad \measuredangle BCA \geqslant 0.$$

Let D be an arbitrary point in $[CB]_h$ distinct from B and C. From Proposition 8.17, we have

$$\measuredangle ABC - \measuredangle_h ABC \equiv \pi - \measuredangle CDB \equiv \measuredangle BCA - \measuredangle_h BCA.$$

From Exercise 6.15, we get

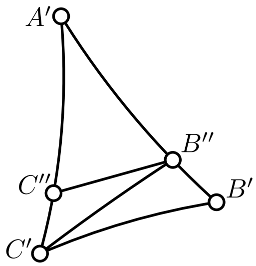
$$\operatorname{defect}(\triangle_h ABC) = 2\cdot(\pi - \measuredangle CDB).$$

Therefore, if we have equality in ❶, then $\measuredangle CDB = \pi$. In particular, the h-segment $[BC]_h$ coincides with the Euclidean segment $[BC]$. By Exercise 11.3, the latter can happen only if the h-line $(BC)_h$ passes thru the center of the absolute (A); that is, if $\triangle_h ABC$ degenerates. □

The following theorem states, in particular, that nondegenerate hyperbolic triangles are congruent if their corresponding angles are equal. In particular, in hyperbolic geometry, similar triangles have to be congruent.

12.6. AAA congruence condition. *Two nondegenerate triangles $\triangle_h ABC$ and $\triangle_h A'B'C'$ in the h-plane are congruent if $\measuredangle_h ABC = \pm\measuredangle_h A'B'C'$, $\measuredangle_h BCA = \pm\measuredangle_h B'C'A'$ and $\measuredangle_h CAB = \pm\measuredangle_h C'A'B'$.*

Proof. Note hat if $AB_h = A'B'_h$, then the theorem follows from ASA.



Assume the contrary. Without loss of generality, we may assume that $AB_h < A'B'_h$. Therefore, we can choose the point $B'' \in [A'B']_h$ such that $A'B''_h = AB_h$.

Choose an h-half-line $[B''X)$ so that

$$\measuredangle_h A'B''X = \measuredangle_h A'B'C'.$$

According to Exercise 10.5, $(B''X)_h \parallel (B'C')_h$.

By Pasch's theorem (3.10), $(B''X)_h$ intersects $[A'C']_h$. Denote by C'' the point of intersection.

According to ASA, $\triangle_h ABC \cong \triangle_h A'B''C''$; in particular,

$$\text{defect}(\triangle_h ABC) = \text{defect}(\triangle_h A'B''C''). \quad ❷$$

Applying Exercise 10.10 twice, we get

$$\begin{aligned}\text{defect}(\triangle_h A'B'C') &= \text{defect}(\triangle_h A'B''C'')+\\ &\quad + \text{defect}(\triangle_h B''C''C') + \text{defect}(\triangle_h B''C'B').\end{aligned} \quad ❸$$

By Theorem 12.5, all the defects have to be positive. Therefore

$$\text{defect}(\triangle_h A'B'C') > \text{defect}(\triangle_h ABC).$$

On the other hand,

$$\begin{aligned}\text{defect}(\triangle_h A'B'C') &= |\measuredangle_h A'B'C'| + |\measuredangle_h B'C'A'| + |\measuredangle_h C'A'B'| =\\ &= |\measuredangle_h ABC| + |\measuredangle_h BCA| + |\measuredangle_h CAB| =\\ &= \text{defect}(\triangle_h ABC),\end{aligned}$$

a contradiction. □

Recall that a bijection from a plane to itself is called *angle preserving* if

$$\measuredangle ABC = \measuredangle A'B'C'$$

for any $\triangle ABC$ and its image $\triangle A'B'C'$.

12.7. Exercise. *Show that any angle-preserving transformation of the h-plane is a motion.*

Conformal interpretation

Let us give another interpretation of the h-distance.

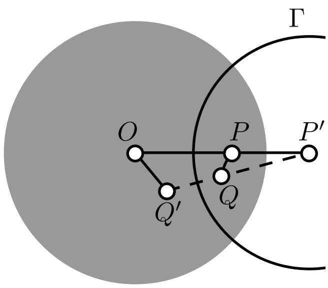
12.8. Lemma. *Consider the h-plane with the unit circle centered at O as the absolute. Fix a point P and let Q be another point in the h-plane. Set $x = PQ$ and $y = PQ_h$, then*

$$\lim_{x\to 0} \frac{y}{x} = \frac{2}{1 - OP^2}.$$

The above formula tells us that the h-distance from P to a nearby point Q is almost proportional to the Euclidean distance with the coefficient $\frac{2}{1-OP^2}$. The value $\lambda(P) = \frac{2}{1-OP^2}$ is called the *conformal factor* of the h-metric.

The value $\frac{1}{\lambda(P)} = \frac{1}{2}\cdot(1-OP^2)$ can be interpreted as the *speed limit* at the given point P. In this case the h-distance is the minimal time needed to travel from one point of the h-plane to another point.

Proof. If $P = O$, then according to Lemma 11.10



❹ $$\frac{y}{x} = \frac{\ln\frac{1+x}{1-x}}{x} \to 2$$

as $x \to 0$.

If $P \neq O$, denote by P' the inverse of P in the absolute. Denote by Γ the circle with the center P' perpendicular to the absolute.

According to the main observation (11.7) and Lemma 11.6, the inversion in Γ is a motion of the h-plane which sends P to O. In particular, if we denote by Q' the inverse of Q in Γ, then $OQ'_h = PQ_h$.

Set $x' = OQ'$. According to Lemma 9.2,

$$\frac{x'}{x} = \frac{OP'}{P'Q}.$$

Since P' is the inverse of P in the absolute, we have $PO{\cdot}OP' = 1$. Therefore,

$$\frac{x'}{x} \to \frac{OP'}{P'P} = \frac{1}{1 - OP^2}$$

as $x \to 0$.

Together with ❹, it implies that

$$\frac{y}{x} = \frac{y}{x'}{\cdot}\frac{x'}{x} \to \frac{2}{1 - OP^2}$$

as $x \to 0$. □

Here is an application of the lemma above.

12.9. Proposition. *The circumference of an h-circle of the h-radius r is*

$$2\cdot\pi\cdot \operatorname{sh} r,$$

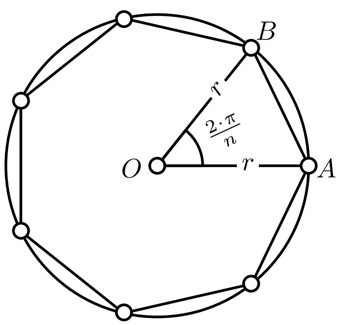
where $\operatorname{sh} r$ *denotes the* hyperbolic sine *of* r*; that is,*

$$\operatorname{sh} r := \frac{e^r - e^{-r}}{2}.$$

Before we proceed with the proof, let us discuss the same problem in the Euclidean plane.

The circumference of a circle in the Euclidean plane can be defined as the limit of perimeters of regular n-gons inscribed in the circle as $n \to \infty$.

Namely, let us fix $r > 0$. Given a positive integer n, consider $\triangle AOB$ such that $\measuredangle AOB = \frac{2\cdot\pi}{n}$ and $OA = OB = r$. Set $x_n = AB$. Note that x_n is the side of regular n-gon inscribed in the circle of radius r. Therefore, the perimeter of the n-gon is $n\cdot x_n$.



The circumference of the circle with the radius r might be defined as the limit

❺ $$\lim_{n\to\infty} n\cdot x_n = 2\cdot\pi\cdot r.$$

(This limit can be taken as the definition of π.)

In the following proof, we repeat the same construction in the h-plane.

Proof. Without loss of generality, we can assume that the center O of the circle is the center of the absolute.

By Lemma 11.10, the h-circle with the h-radius r is the Euclidean circle with the center O and the radius

$$a = \frac{e^r - 1}{e^r + 1}.$$

Denote by x_n and y_n the side lengths of the regular n-gons inscribed in the circle in the Euclidean and hyperbolic plane correspondingly.

Note that $x_n \to 0$ as $n \to \infty$. By Lemma 12.8,

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \frac{2}{1-a^2}.$$

Applying ❺, we get that the circumference of the h-circle can be found the following way:

$$\begin{aligned}\lim_{n\to\infty} n\cdot y_n &= \frac{2}{1-a^2}\cdot \lim_{n\to\infty} n\cdot x_n = \\ &= \frac{4\cdot\pi\cdot a}{1-a^2} = \\ &= \frac{4\cdot\pi\cdot\left(\frac{e^r-1}{e^r+1}\right)}{1-\left(\frac{e^r-1}{e^r+1}\right)^2} = \\ &= 2\cdot\pi\cdot\frac{e^r-e^{-r}}{2} = \\ &= 2\cdot\pi\cdot\operatorname{sh} r.\end{aligned}$$

□

12.10. Exercise. *Denote by* $\operatorname{circum}_h(r)$ *the circumference of the h-circle of the radius* r*. Show that*

$$\operatorname{circum}_h(r+1) > 2\cdot\operatorname{circum}_h(r)$$

for all $r > 0$*.*

Chapter 13

Affine geometry

Affine transformations

A bijection of Euclidean plane to itself is called *affine transformation* if it maps any line to a line.

We say that three points are *collinear* if they lie on one line. Note that affine transformation sends collinear points to collinear; the following exercise gives a converse.

13.1. Exercise. *Assume f is a bijection from Euclidean plane to itself which sends collinear points to collinear points. Show that f is an affine transformation. (In other words, show that f maps noncollinear points to noncollinear.)*

13.2. Exercise. *Show that affine transformation sends parallel lines to the parallel lines.*

Affine geometry studies the so called *incidence structure* of the Euclidean plane. The incidence structure says which points lie on which lines and nothing else; we cannot talk about distances, angle measures and so on. In other words, affine geometry studies the properties of the Euclidean plane which preserved under affine transformations.

Constructions with parallel tool and ruler

Let us consider geometric constructions with ruler and *parallel tool*; the latter makes possible to draw a line thru a given point parallel to a given line. By Exercisers 13.2, any construction with these two tools are invariant with respect to affine transformation. For example, to solve the following exercise, it is sufficient to prove that midpoint of given segment can be constructed with ruler and parallel tool.

13.3. Exercise. *Let M be the midpoint of segment $[AB]$ in the Euclidean plane. Assume that an affine transformation sends the points A, B and M to A', B' and M' correspondingly. Show that M' is the midpoint of $[A'B']$.*

The following exercise will be used in the proof of Theorem 13.7.

13.4. Exercise. *Assume that in Euclidean plane we have 4 points with the coordinates $(0,0)$, $(1,0)$, $(a,0)$ and $(b,0)$. Using a ruler and a parallel tool, construct the points with the coordinates $(a{\cdot}b,0)$ and $(a+b,0)$.*

13.5. Exercise. *Use ruler and parallel tool to construct the center of the given circle.*

Matrix form

Since the lines are defined in terms of metric; any motion of Euclidean plane is also an affine transformation.

On the other hand, there are affine transformations of Euclidean plane which are not motions.

Consider the Euclidean plane with coordinate system; let us use the column notation for the coordinates; that is, we will write $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ instead of (x,y).

As it follows from the theorem below, the so called *shear mapping* $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x+k\cdot y \\ y \end{smallmatrix}\right)$ is an affine transformation. The shear mapping can change the angle between vertical and horizontal lines almost arbitrary. The latter can be used to prove impossibility of some constructions with ruler and parallel tool; here is one example.

13.6. Exercise. *Show that with a ruler and a parallel tool one cannot construct a line perpendicular to a given line.*

13.7. Theorem. *A map β from the plane to itself is an affine transformation if and only if*

❶ $$\beta\colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} a\cdot x+b\cdot y+v \\ c\cdot x+d\cdot y+w \end{smallmatrix}\right)$$

for a fixed invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and a vector $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$.

In particular, any affine transformation of Euclidean plane is continuous.

In the proof of the "only if" part, we will use the following algebraic lemma.

13.8. Algebraic lemma. *Assume $f\colon \mathbb{R}\to\mathbb{R}$ is a function such that*

$$\begin{aligned} f(1)&=1,\\ f(x+y)&=f(x)+f(y),\\ f(x\cdot y)&=f(x)\cdot f(y) \end{aligned}$$

for any $x,y\in\mathbb{R}$. Then $f(x)=x$ for any $x\in\mathbb{R}$.

Note that we do not assume that f is continuous.

The function f satisfying three conditions in the lemma is called *field automorphism*. Therefore, the lemma states that the identity function is the only automorphism of the field of real numbers. For the field of complex numbers, the conjugation $z\mapsto\bar z$ (see page 138) gives an example of nontrivial automorphism.

Proof. Since

$$f(0)+f(1)=f(0+1),$$

we get

$$f(0)+1=1;$$

that is,

❷ $$f(0)=0.$$

Further

$$0=f(0)=f(x)+f(-x).$$

Therefore,

❸ $$f(-x)=-f(x)\quad\text{for any}\quad x\in\mathbb{R}.$$

Further

$$\begin{aligned} f(2)&=f(1)+f(1)=1+1=2;\\ f(3)&=f(2)+f(1)=2+1=3;\\ &\dots \end{aligned}$$

Together with ❸, the latter implies that

$$f(n)=n\quad\text{for any integer}\quad n.$$

Since

$$f(m)=f(\tfrac{m}{n})\cdot f(n)$$

we get

❹ $$f(\tfrac{m}{n}) = \tfrac{m}{n}$$

for any rational number $\frac{m}{n}$.

Assume $a \geqslant 0$. Then the equation $x \cdot x = a$ has a real solution $x = \sqrt{a}$. Therefore, $[f(\sqrt{a})]^2 = f(\sqrt{a}) \cdot f(\sqrt{a}) = f(a)$. Hence $f(a) \geqslant 0$. That is,

❺ $$a \geqslant 0 \quad \Longrightarrow \quad f(a) \geqslant 0.$$

Applying ❸, we also get

❻ $$a \leqslant 0 \quad \Longrightarrow \quad f(a) \leqslant 0.$$

Finally, assume $f(a) \neq a$ for some $a \in \mathbb{R}$. Then there is a rational number $\frac{m}{n}$ which lies between a and $f(a)$; that is, the numbers

$$x = a - \tfrac{m}{n} \quad \text{and} \quad y = f(a) - \tfrac{m}{n}$$

have opposite signs.

By ❹,

$$\begin{aligned} y + \tfrac{m}{n} &= f(a) = \\ &= f(x + \tfrac{m}{n}) = \\ &= f(x) + f(\tfrac{m}{n}) = \\ &= f(x) + \tfrac{m}{n}; \end{aligned}$$

that is, $f(x) = y$. The latter contradicts ❺ or ❻. □

13.9. Lemma. *Assume γ is an affine transformation which fix three points $\binom{0}{0}$, $\binom{1}{0}$ and $\binom{0}{1}$ on the coordinate plane. Then γ is the identity map; that is, $\gamma\binom{x}{y} = \binom{x}{y}$ for any point $\binom{x}{y}$.*

Proof. Since affine transformation sends lines to lines, we get that each axes is mapped to itself.

According to Exercise 13.2, parallel lines are mapped to parallel lines. Therefore, we get that horizontal lines mapped to horizontal lines and vertical lines mapped to vertical. In other words,

$$\gamma\binom{x}{y} = \begin{pmatrix} f(x) \\ h(y) \end{pmatrix}.$$

for some functions $f, h\colon \mathbb{R} \to \mathbb{R}$.

Note that $f(1) = h(1) = 1$ and according to Exercise 13.4, both f and h satisfies the other two conditions of Algebraic lemma 13.8.

Applying the lemma, we get that f and h are identity functions and so is γ. □

Proof of Theorem 13.7. Recall that matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible if

$$\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = a{\cdot}d - b{\cdot}c \neq 0;$$

in this case the matrix

$$\tfrac{1}{a\cdot d - b\cdot c} \cdot \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$$

is the inverse of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Assume that the map β is described by ❶. Note that

❼ $$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \tfrac{1}{a\cdot d - b\cdot c} \cdot \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x-v \\ y-w \end{smallmatrix}\right).$$

is inverse of β. In particular, β is a bijection.

Any line in the plane is given by equation

❽ $$p{\cdot}x + q{\cdot}y + r = 0,$$

where $p \neq 0$ or $q \neq 0$. Find $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ from its β-image by formula ❼ and substitute the result in ❽. You will get the equation of the image of the line. The equation has the same type as ❽, with different constants; in particular, it describes a line. Therefore, β is an affine transformation.

To prove the "only if" part, fix an affine transformation α. Set

$$\begin{aligned} \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \\ \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) - \alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \\ \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) - \alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right). \end{aligned}$$

Note that the points $\alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\alpha\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $\alpha\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ do not lie on one line. Therefore, the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible.

For the affine transformation β defined by ❶ we have

$$\begin{aligned} \beta\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \\ \beta\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \\ \beta\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) &= \alpha\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right). \end{aligned}$$

It remains to show that $\alpha = \beta$ or equivalently the composition $\gamma = \alpha \circ \beta^{-1}$ is the identity map.

Note that γ is an affine transformation which fix points $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. It remains to apply Lemma 13.9. □

On inversive transformations

Recall that inversive plane is Euclidean plane with added a point at infinity, denoted by ∞. We assume that every line passes thru ∞. Recall that the term *circline* stays for *circle or line*;

An *inversive transformation* is a bijection from inversive plane to itself which sends circlines to circlines. *Inversive geometry* studies the *circline incidence structure* of inversive plane; it says which points lie on which circlines.

13.10. Theorem. *A map from inversive plane to itself is an inversive transformation if and only if it can be presented as a composition of inversions and reflections.*

Proof. According to Theorem 9.7 any inversion is a inversive transformation. Therefore, the same holds for composition of inversions and reflection.

To prove the converse, fix an inversive transformation α.

Assume $\alpha(\infty) = \infty$. Recall that any circline passing thru ∞ is a line. If follows that α maps lines to lines; that is, it is an affine transformation.

Note that α is not an arbitrary affine transformation — it maps circles to circles.

Composing α with a reflection, say ρ_1, we can assume that $\alpha' = = \rho_1 \circ \alpha$ maps the unit circle with center at the origin to a concentric circle.

Composing the obtained map α' with a *homothety*

$$\chi\colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} k\cdot x \\ k\cdot y \end{smallmatrix}\right),$$

we can assume that $\alpha'' = \chi \circ \alpha'$ sends the unit circle to itself.

Composing the obtained map α'' with a reflection ρ_2 in a line thru the origin, we can assume that $\alpha''' = \rho_2 \circ \alpha''$ maps the point $(1,0)$ to itself.

By Exercise 13.5, α''' fixes the center of the circle; that is, it fixes the origin.

The obtained map α''' is an affine transformation. Applying Theorem 13.7, together with the properties of α'' described above we get

$$\alpha'''\colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$$

for an invertible matrix $\left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right)$. Since the point $(0,1)$ maps to the unit circle we get

$$b^2 + d^2 = 1.$$

Since the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ maps to the unit circle we get

$$(b+d)^2 = 1.$$

It follows

$$\alpha''' \colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right);$$

that is, either α''' is the identity map or reflection the x-axis.

Note that the homothety χ is a composition of two inversions in concentric circles. Therefore, α is a composition of inversions and reflections if and only are so is α', α'' and α'''.

In the remaining case $\alpha(\infty) \neq \infty$, set $P = \alpha(\infty)$. Consider an inversion β in a circle with center at P. Note that $\beta(P) = \infty$; therefore, $\beta \circ \alpha(\infty) = \infty$. Since β is inversive, so is $\beta \circ \alpha$. From above we get that $\beta \circ \alpha$ is a composition of reflections and inversions; therefore, so is α. □

13.11. Exercise. *Show that any reflection can be presented as a composition of three inversions.*

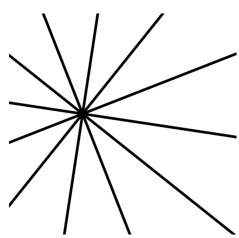
Note that exercise above together with Theorem 13.10, implies that any inversive map is a composition of inversions, no reflections are needed.

Chapter 14

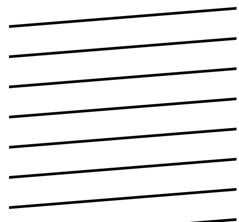
Projective geometry

Real projective plane

In the Euclidean plane, two distinct lines might have one or zero points of intersection (in the latter case the lines are called parallel). Our aim is to extend Euclidean plane by ideal points so that any two distinct lines will have exactly one point of intersection.



A collection of lines in the Euclidean plane is called *concurrent* if they all intersect at a single point or all of them pairwise parallel. A maximal set of concurrent lines in the plane is called *pencil*. There are two types of pencils: *central pensil* containing all lines passing thru a fixed point called the *center of the pencil* and *parallel pensil* containing pairwise parallel lines.



Each point in the Euclidean plane is uniquely defines a central pencil with the center in it. Note that any two lines completely determine the pencil containing both.

Let us add one *ideal point* for each parallel pencil, and assume that all these ideal points lie on one *ideal line*. We also assume that the ideal line belongs to each parallel pencil.

We obtain the so called *real projective plane*. It comes with an incidence structure — we say that three points lie on one line if the corresponding pencils contain a common line.

Projective geometry studies this incidence structure on the projective plane. Loosely speaking, any statement in projective geometry can be formulated using only terms *collinear points*, *concurrent lines*.

Euclidean space

Let us repeat the construction of metric d_2 (page 11) in the space.

We will denote by $\mathbb{R}^3$ the set of all triples (x, y, z) of real numbers. Assume $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ are arbitrary points. Define the metric on $\mathbb{R}^3$ the following way:

$$AB := \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

The obtained metric space is called *Euclidean space*.

Assume at least one of the real numbers a, b or c is distinct from zero. Then the subset of points $(x, y, z) \in \mathbb{R}^3$ described by equation

$$a{\cdot}x + b{\cdot}y + c{\cdot}z + d = 0$$

is called *plane*; here d is a real number.

It is straightforward to show the following:

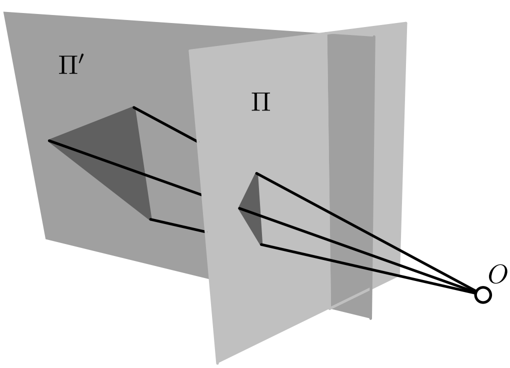
- ◇ Any plane in the Euclidean space is isometric to the Euclidean plane.
- ◇ Any three points in the space lie on a plane.
- ◇ An intersection of two distinct planes (if it is nonempty) is a line in each of these planes.

The latter statements makes possible to generalize many notions and results from Euclidean plane geometry to the Euclidean space by applying plane geometry in the planes of the space.

Perspective projection

Consider two planes Π and Π' in the Euclidean space. Let O be a point which does not belong neither to Π nor Π'.

Let us define the *perspective projection from* Π *to* Π' *with center at* O. The projection of the $P \in \Pi$ is defined as the point $P' \in \Pi'$ which lies on the line (OP).



Note that the perspective projection sends collinear points to collinear. Indeed, assume three points P, Q, R lie on one line ℓ in Π and P', Q', R' are their images in Π'. Let Θ be the plane containing O and ℓ. Then all the points P, Q, R, P', Q', R' lie on Θ. Therefore, the points P', Q', R' lie on the intersection line $\ell' = \Theta \cap \Pi'$.

The perspective projection is not a bijection between the planes;. Indeed, if the line (OP) is parallel to Π' (that is, if $(OP) \cap \Pi' = \varnothing$) then the perspective projection is not defined. Also, if $(OP') \parallel \Pi$ for a point $P' \in \Pi'$, then point P' is not an image of the perspective projection.

A similar story happened with inversion. An inversion is not defied at its center; morover, the center is not an inverse of any point. To deal with this problem we passed to inversive plane which is Euclidean plane extended by one ideal point.

A similar strategy works for perspective projection $\Pi \to \Pi'$, but this time real projective plane is the right choice of extension. Denote by $\hat{\Pi}$ and $\hat{\Pi}'$ the corresponding real projective planes.

Let us define a bijection between points in the real projective plane $\hat{\Pi}$ and the set Λ of all the lines passing thru O. If $P \in \Pi$, then take the line (OP); if P is an ideal point of $\hat{\Pi}$, so it is defined by a parallel pencil of lines, then take the line thru O parallel to the lines in this pencil.

The same construction gives a bijection between Λ and $\hat{\Pi}'$. Composing these two bijections $\hat{\Pi} \to \Lambda \to \hat{\Pi}'$, we get a bijection between $\hat{\Pi}$ and $\hat{\Pi}'$ which coincides with the perspective projection $P \mapsto P'$ where it is defined.

Note that the ideal line of $\hat{\Pi}$ maps to the intersection line of Π' and the plane thru O parallel to Π. Similarly the ideal line of $\hat{\Pi}'$ is the image of the intersection line of Π and the plane thru O parallel to Π'.

Strictly speaking we described a transformation from one real projective plane to another, but if we identify the two planes, say by fixing a coordinate system in each, we get a projective transformation from the plane to itself.

14.1. Exercise. *Let O be the origin of (x, y, z)-coordinate space and the planes Π and Π' are given by the equations $x = 1$ and $y = 1$ correspondingly. The perspective projection from Π to Π' with center at O sends P to P'. Assume P has coordinates $(1, y, z)$, find the coordinates of P'.*

For which points $P \in \Pi$ the perspective projection is undefined? Which points $P' \in \Pi'$ are not images of points under perspective projection?

Projective transformations

A bijection from the real projective plane to itself which sends lines to lines is called *projective transformation.*

Projective geometry studies the properties of real projective plane which preserved under projective transformations.

Note that any affine transformation defines a projective transformation on the corresponding real projective plane. We will call such projective transformations *affine*; these are projective transformations which send the ideal line to itself.

The perspective projection discussed in the previous section gives an example of projective transformation which is not affine.

14.2. Theorem. *Given a line ℓ in the real projective plane, there is a perspective projection which sends ℓ to the ideal line.*

Moreover, any projective transformation can be obtained as a composition of an affine transformation and a perspective projection.

Proof. Identify the projective plane with a plane Π in the space. Fix a point $O \notin \Pi$ and choose a plane Π' which is parallel to the plane containing ℓ and O. The corresponding perspective projection sends ℓ to the ideal line.

Assume α is a projective transformation. If is α sends ideal line to itself, then it has to be affine. Hence the theorem follows.

If α sends the ideal line to the line ℓ, choose a perspective projection β which sends ℓ back to the ideal line. The composition $\beta\circ\alpha$ sends ideal line to itself. Therefore, $\beta \circ \alpha$ is affine. Hence the result follows. □

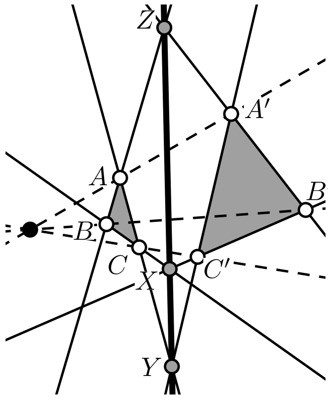
Moving points to infinity

Theorem 14.2 makes possible to take any line in the projective plane and declare it to be ideal. In other words, we can choose preferred affine plane by removing one line from the projective plane. This construction provides a method for solving problems in projective geometry which will be illustrated by the following classical example.

14.3. Desargues' theorem. *Consider three concurrent lines (AA'), (BB') and (CC') in the real projective plane. Set*

$$X = (BC) \cap (B'C'), \quad Y = (CA) \cap (C'A'), \quad Z = (AB) \cap (A'B').$$

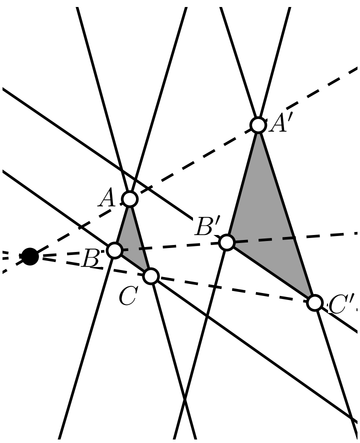
Then the points X, Y and Z are collinear.



Proof. Without loss of generality, we may assume that the line (XY) is ideal. If not, apply a perspective projection which sends the line (XY) to the ideal line. That is, we can assume that $(BC) \parallel (B'C')$ and $(CA) \parallel (C'A')$ and we need to show that $(AB) \parallel (A'B')$.

Assume that the lines (AA'), (BB') and (CC') intersect at point O. Since $(BC) \parallel (B'C')$, the transversal property (6.18) implies that $\measuredangle OBC = \measuredangle OB'C'$ and $\measuredangle OCB = \measuredangle OC'B'$. By the AA similarity condition, $\triangle OBC \sim \triangle OB'C'$. In particular,

$$\frac{OB}{OB'} = \frac{OC}{OC'}.$$



The same way we get $\triangle OAC \sim \triangle OA'C'$ and

$$\frac{OA}{OA'} = \frac{OC}{OC'}.$$

Therefore,

$$\frac{OA}{OA'} = \frac{OB}{OB'}.$$

By the SAS similarity condition, we get $\triangle OAB \sim \triangle OA'B'$; in particular, $\measuredangle OAB = \pm\measuredangle OA'B'$.

Note that $\measuredangle AOB = \measuredangle A'OB'$. Therefore,

$$\measuredangle OAB = \measuredangle OA'B'.$$

By the transversal property 6.18, $(AB) \parallel (A'B')$.

The case $(AA') \parallel (BB') \parallel (CC')$ is done similarly. In this case the quadrilaterals $B'BCC'$ and $A'ACC'$ are parallelograms. Therefore,

$$BB' = CC' = AA'.$$

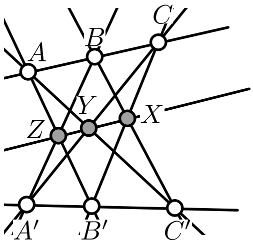
Hence $\square B'BAA'$ is a parallelogram and $(AB) \parallel (A'B')$. $\square$

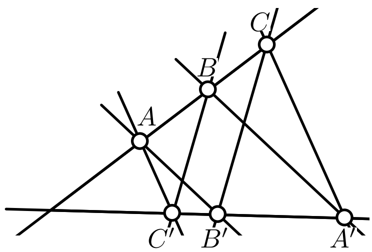
Here is another classical theorem of projective geometry.

14.4. Pappus's theorem. *Assume that two triples of points A, B, C, and A', B', C' are collinear. Set*

$$X = (BC') \cap (B'C), \quad Y = (CA') \cap (C'A), \quad Z = (AB') \cap (A'B).$$

Then the points X, Y, Z are collinear.





Pappus's theorem can be proved the same way as Desargues' theorem.

Idea of the proof. Applying a perspective projection, we can assume that X and Y lie on the ideal line. It remains to show that Z lies on the ideal line.

In other words, assuming that $(AB') \parallel (A'B)$ and $(AC') \parallel (A'C)$, we need to show that $(BC') \parallel (B'C)$.

14.5. Exercise. *Finish the proof of Pappus's theorem using the idea described above.*

Duality

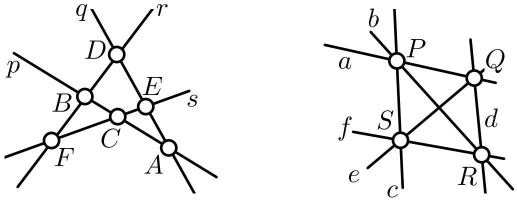
Assume we have a bijection between the set of lines and the set of points of the plane. Let us denote it the following way: given a point P let us denote by lower case letter p the corresponding line; and the other way around, given a line s we will denote upper case letter S the corresponding point.

The bijection between points and lines is called *duality*[1] if

$$P \in s \qquad \Longleftrightarrow \qquad p \ni S.$$

for any point P and line s.

[1]Usual definition of duality is more general; we consider a special case which is also called *polarity*.



Dual configurations.

Existence of duality in a plane says that the lines and the points in this plane have the same rights in terms of incidence.

14.6. Exercise. *Show that Euclidean plane does not admit a duality.*

14.7. Theorem. *The real projective plane admits a duality.*

Proof. Consider a plane Π and a point $O \notin \Pi$ in the Euclidean space; denote by $\hat{\Pi}$ the corresponding real projective plane.

Recall that there is a natural bijection between points in the real projective plane $\hat{\Pi}$ and all the lines passing thru O. If $P \in \Pi$, then the corresponding line is (OP); if P is an ideal point of $\hat{\Pi}$, so P is defined as a parallel pencil of lines, take the line thru O which is parallel to each lines in this pencil. Given $P \in \hat{\Pi}$, denote the obtained line $\dot{P}$.

Similarly there is a natural bijection between lines in $\hat{\Pi}$ and all the planes passing thru O. If s is a line in Π, then take the plane containing O and s; if s is the ideal line of $\hat{\Pi}$, take the plane thru O parallel to Π. Given a line s in $\hat{\Pi}$, denote the obtained plane as $\dot{s}$.

It is straightforward to check that $\dot{P} \subset \dot{s}$ if and only if $P \in s$; that is, the bijections $P \mapsto \dot{P}$ and $s \mapsto \dot{s}$ remember all the incidence structure of the real projective plane $\hat{\Pi}$.

It remains to construct a bijection $\dot{s} \mapsto \dot{S}$ between the set of planes and the set of lines passing thru O such that

❶ $$\dot{r} \subset \dot{S} \iff \dot{R} \supset \dot{s}$$

for any two lines line $\dot{r}$ and $\dot{s}$ passing thru O.

Set $\dot{S}$ to be the plane thru O which is perpendicular to $\dot{s}$. Note that both conditions ❶ are equivalent to $\dot{r} \perp \dot{s}$; hence the result follows. □

14.8. Exercise. *Consider the Euclidean plane with (x, y)-coordinates, denote by O the origin. Given a point $P \neq O$ with coordinates (a, b) consider the line p given by the equation $a{\cdot}x + b{\cdot}y = 1$.*

Show that the correspondence P to p extends to a duality of the real projective plane.

Which line corresponds to O?

Which point of the real projective plane corresponds to the line $a{\cdot}x + b{\cdot}y = 0$?

The existence of duality in the real projective planes makes possible to formulate an equivalent dual statement to any statement in projective geometry. For example, the dual statement for "the points X, Y and Z lie on one line ℓ" would be the "lines x, y and z intersect at one point L". Let us formulate the dual statement for Desargues' theorem 14.3.

14.9. Dual Desargues' theorem. *Consider the collinear points X, Y and Z. Assume that*

$$X = (BC) \cap (B'C'), \quad Y = (CA) \cap (C'A'), \quad Z = (AB) \cap (A'B').$$

Then the lines (AA'), (BB') and (CC') are concurrent.

In this theorem the points X, Y and Z are dual to the lines (AA'), (BB') and (CC') in the original formulation, and the other way around.

Once Desargues' theorem is proved, applying duality (14.7) we get Dual Desargues' theorem. Note that the Dual Desargues' theorem is the converse to the original Desargues' theorem 14.3. Thereofre Desargues' theorem and its dual can be packed in the following strong version of Desargues' theorem.

14.10. Strong Desargues' theorem. *Assume that*

$$X = (BC) \cap (B'C'), \quad Y = (CA) \cap (C'A'), \quad Z = (AB) \cap (A'B').$$

Then the lines (AA'), (BB') and (CC') are concurrent if and only if the points X, Y and Z are collinear.

14.11. Exercise. *Formulate the Dual Pappus's theorem (see 14.4).*

14.12. Exercise. *Given two parallel lines, construct with a ruler only the third parallel line thru the given point.*

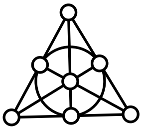
Axioms

Note that the real projective plane described above satisfies the following set of axioms.

p-I. Any two distinct points lie on a unique line.
p-II. Any two distinct lines pass thru a unique point.
p-III. There exist at least four points of which no three are collinear.

Let us take these three axioms as a definition of the *projective plane*; so the real projective plane discussed above becomes a particular example of projective plane.

There is an example of projective plane which contains exactly 3 points on each line. This is the so called *Fano plane* which you can see on the diagram; it contains 7 points and 7 lines. This is an example of *finite projective plane*; that is, projective plane with finitely many points.



14.13. Exercise. *Show that any line in projective plane contains at least three points.*

Consider the following analog of Axiom p-III.

p-III′. There exist at least four lines of which no three are concurrent.

14.14. Exercise. *Show that Axiom* p-III′ *is equivalent to Axiom* p-III. *That is,*

p-I, p-II and p-III imply p-III′,

and

p-I, p-II and p-III′ imply p-III.

The exercise above shows that in the axiomatic system of projective plane, lines and points have the same rights. In fact, one can switch everywhere words "point" with "line", "pass trough" with "lies on", "collinear" with "concurrent" and we get an equivalent set of axioms.

14.15. Exercise. *Assume that one of the lines in a finite projective plane contains exactly $n+1$ points.*

(a) Show that each line contains exactly $n+1$ points.
(b) Show that the number of the points in the plane has to be

$$n^2+n+1.$$

(c) Show that there is no projective plane with exactly 10 points.
(d) Show that in any finite projective plane the number of points coincides with the number of lines.

The number n in the above exercise is called *order* of finite projective plane. For example Fano plane has order 2. Here is one of the most famous open problem in finite geometry.

14.16. Conjecture. *The order of any finite projective plane is a power of a prime number.*

Chapter 15

Spherical geometry

Spherical geometry studies the surface of a unit sphere. This geometry has applications in cartography, navigation and astronomy.

The spherical geometry is a close relative of the Euclidean and hyperbolic geometries. Most of the theorems of hyperbolic geometry have spherical analogs, but spherical geometry is easier to visualize.

Euclidean space

Recall that Euclidean space is the set $\mathbb{R}^3$ of all triples (x,y,z) of real numbers such that the distance between a pair of points $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ is defined by the following formula:

$$AB := \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

The planes in the space are defined as the set of solutions of equation

$$a{\cdot}x + b{\cdot}y + c{\cdot}z + d = 0$$

for real numbers a, b, c and d such that at least one of the numbers a, b or c is not zero. Any plane in the Euclidean space is isometric to the Euclidean plane.

A sphere in the space is the direct analog of circle in the plane. Formally, *sphere* with center O and radius r is the set of points in the space which lie on the distance r from O.

Let A and B be two points on the unit sphere centered at O. The *spherical distance* from A to B (briefly AB_s) is defined as $|\measuredangle AOB|$.

In spherical geometry, the role of lines play the *great circles*; that is, the intersection of the sphere with a plane passing thru O.

Note that the great circles do not form lines in the sense of Definition 1.8. Also, any two distinct great circles intersect at two antipodal points. In particular, the sphere does not satisfy the axioms of the neutral plane.

Pythagorean theorem

Here is an analog of the Pythagorean theorems (6.10 and 16.8) in spherical geometry.

15.1. Theorem. *Let $\triangle_s ABC$ be a spherical triangle with a right angle at C. Set $a = BC_s$, $b = CA_s$ and $c = AB_s$. Then*

$$\cos c = \cos a \cdot \cos b.$$

In the proof, we will use the notion of the scalar product which we are about to discuss.

Let A and B be two points in the Euclidean space. Denote by $v_A = (x_A, y_A, z_A)$ and $v_B = (x_B, y_B, z_B)$ the position vectors of A and B correspondingly. The scalar product of the two vectors v_A and v_B in $\mathbb{R}^3$ is defined as

❶ $$\langle v_A, v_B\rangle := x_A{\cdot}x_B + y_A{\cdot}y_B + z_A{\cdot}z_B.$$

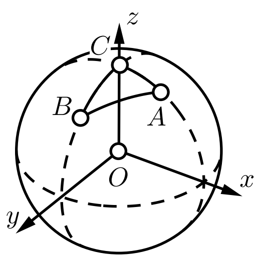
Assume both vectors v_A and v_B are nonzero; denote by φ is the angle measure between them. Then the scalar product can be expressed the following way:

❷ $$\langle v_A, v_B\rangle = |v_A|{\cdot}|v_B|{\cdot}\cos\varphi,$$

where

$$|v_A| = \sqrt{x_A^2 + y_A^2 + z_A^2}, \qquad |v_B| = \sqrt{x_B^2 + y_B^2 + z_B^2}.$$

Now, assume that the points A and B lie on the unit sphere Σ in $\mathbb{R}^3$ centered at the origin. In this case $|v_A| = |v_B| = 1$. By ❷ we get



❸ $$\cos AB_s = \langle v_A, v_B\rangle.$$

Proof. Since the angle at C is right, we can choose the coordinates in $\mathbb{R}^3$ so that $v_C = (0, 0, 1)$, v_A lies in the xz-plane, so $v_A = (x_A, 0, z_A)$, and v_B lies in yz-plane, so $v_B = (0, y_B, z_B)$.

Applying, ❸, we get

$$z_A = \langle v_C, v_A \rangle = \cos b,$$
$$z_B = \langle v_C, v_B \rangle = \cos a.$$

Applying, ❶ and ❸, we get

$$\begin{aligned}\cos c &= \langle v_A, v_B \rangle = \\ &= x_A{\cdot}0 + 0{\cdot}y_B + z_A{\cdot}z_B = \\ &= \cos b{\cdot}\cos a.\end{aligned}$$

□

15.2. Exercise. *Show that if $\triangle_s ABC$ is a spherical triangle with a right angle at C, and $AC_s = BC_s = \frac{\pi}{4}$, then $AB_s = \frac{\pi}{3}$.*

Inversion of the space

The inversion in a sphere is defined the same way as we define the inversion in a circle.

Formally, let Σ be the sphere with the center O and radius r. The *inversion* in Σ of a point P is the point $P' \in [OP)$ such that

$$OP{\cdot}OP' = r^2.$$

In this case, the sphere Σ will be called the *sphere of inversion* and its center is called the *center of inversion.*

We also add ∞ to the space and assume that the center of inversion is mapped to ∞ and the other way around. The space $\mathbb{R}^3$ with the point ∞ will be called *inversive space.*

The inversion of the space has many properties of the inversion of the plane. Most important for us are the analogs of theorems 9.6, 9.7, 9.24 which can be summarized as follows:

15.3. Theorem. *The inversion in the sphere has the following properties:*

(a) Inversion maps a sphere or a plane into a sphere or a plane.

(b) Inversion maps a circle or a line into a circle or a line.

(c) Inversion preserves the cross-ratio; that is, if A', B', C' and D' are the inversions of the points A, B, C and D correspondingly, then

$$\frac{AB{\cdot}CD}{BC{\cdot}DA} = \frac{A'B'{\cdot}C'D'}{B'C'{\cdot}D'A'}.$$

(d) Inversion maps arcs into arcs.

(e) Inversion preserves the absolute value of the angle measure between tangent half-lines to the arcs.

We do not present the proofs here, but they nearly repeat the corresponding proofs in plane geometry. To prove *(a)*, you will need in addition the following lemma; its proof is left to the reader.

15.4. Lemma. *Let Σ be a subset of the Euclidean space which contains at least two points. Fix a point O in the space.*

Then Σ is a sphere if and only if for any plane Π passing thru O, the intersection $\Pi \cap \Sigma$ is either empty set, one point set or a circle.

The following observation helps to reduce part *(b)* to part *(a)*.

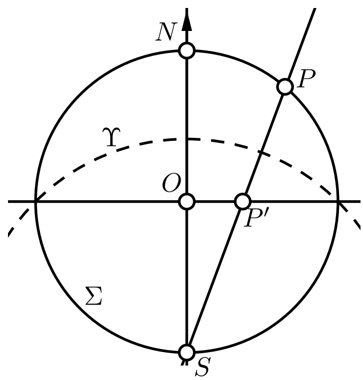
15.5. Observation. *Any circle in the space is an intersection of two spheres.*

Stereographic projection

Consider the unit sphere Σ centered at the origin $(0,0,0)$. This sphere can be described by the equation $x^2 + y^2 + z^2 = 1$.

Denote by Π the xy-plane; it is defined by the equation $z = 0$. Clearly, Π runs thru the center of Σ.

Denote by $N = (0,0,1)$ the "north pole" and by $S = (0,0,-1)$ be the "south pole" of Σ; these are the points on the sphere which have extremal distances to Π. Denote by Ω the "equator" of Σ; it is the intersection $\Sigma \cap \Pi$.



The plane thru P, O and S.

For any point $P \neq S$ on Σ, consider the line (SP) in the space. This line intersects Π in exactly one point, denoted by P'. Set $S' = \infty$.

The map $\xi_s \colon P \mapsto P'$ is called the *stereographic projection from Σ to Π with respect to the south pole.* The inverse of this map $\xi_s^{-1} \colon P' \mapsto P$ is called the *stereographic projection from Π to Σ with respect to the south pole.*

The same way, one can define the *stereographic projections ξ_n and ξ_n^{-1} with respect to the north pole N.*

Note that $P = P'$ if and only if $P \in \Omega$.

Note that if Σ and Π are as above, then the composition of the stereographic projections $\xi_s : \Sigma \to \Pi$ and $\xi_s^{-1} : \Pi \to \Sigma$ are the restrictions of the inversion in the sphere Υ with the center S and radius $\sqrt{2}$ to Σ and Π correspondingly.

From above and Theorem 15.3, it follows that the stereographic projection preserves the angles between arcs; more precisely *the absolute value of the angle measure* between arcs on the sphere.

This makes it particularly useful in cartography. A map of a big region of earth cannot be done in a constant scale, but using a stereographic projection, one can keep the angles between roads the same as on earth.

In the following exercises, we assume that Σ, Π, Υ, Ω, O, S and N are as above.

15.6. Exercise. *Show that $\xi_n \circ \xi_s^{-1}$, the composition of stereographic projections from Π to Σ from S, and from Σ to Π from N is the inverse of the plane Π in Ω.*

15.7. Exercise. *Show that a stereographic projection $\Sigma \to \Pi$ sends the great circles to circlines on the plane which intersects Ω at two opposite points.*

The following exercise is analogous to Lemma 12.8.

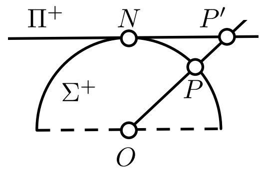
15.8. Exercise. *Fix a point $P \in \Pi$ and let Q be yet another point in Π. Denote by P' and Q' their stereographic projections to Σ. Set $x = PQ$ and $y = P'Q'_s$. Show that*

$$\lim_{x \to 0} \frac{y}{x} = \frac{2}{1 + OP^2}.$$

Central projection

The central projection is analogous to the projective model of hyperbolic plane which is discussed in Chapter 16.

Let Σ be the unit sphere centered at the origin which will be denoted by O. Denote by Π^+ the plane defined by the equation $z = 1$. This plane is parallel to the xy-plane and it passes thru the north pole $N = (0, 0, 1)$ of Σ.



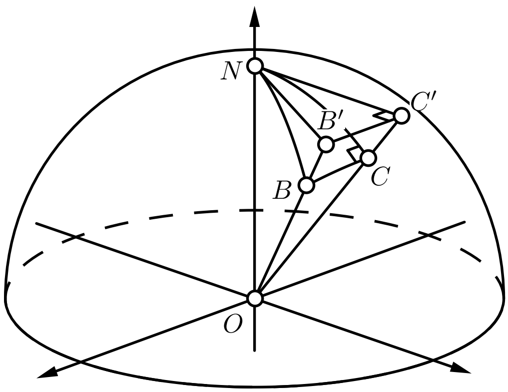
Recall that the northern hemisphere of Σ, is the subset of points $(x, y, z) \in \Sigma$ such that $z > 0$. The northern hemisphere will be denoted by Σ^+.

Given a point $P \in \Sigma^+$, consider the half-line $[OP)$ and denote by P' the intersection of $[OP)$ and Π^+. Note that if $P = (x, y, z)$, then $P' = (\frac{x}{z}, \frac{y}{z}, 1)$. It follows that the map $P \mapsto P'$ is a bijection between Σ^+ and Π^+.

The described map $\Sigma^+ \to \Pi^+$ is called the *central projection* of the hemisphere Σ^+.

Note that the central projection sends the intersections of the great circles with Σ^+ to the lines in Π^+. The latter follows since the great circles are intersections of Σ with planes passing thru the origin as well as the lines in Π^+ are the intersection of Π^+ with these planes.

15.9. Exercise. *Assume that N is the north pole, $\triangle_s NBC$ has right angle at C and lies completely in the north hemisphere. Let $\triangle NB'C'$ be the image of $\triangle_s NBC$ under the central projection.*



Observe that $\triangle NB'C'$ has a right angle at C'. Set

$$\begin{aligned} a &= BC_s, & b &= CN_s, & c &= NB_s, \\ s &= B'C', & t &= C'N, & u &= NB'. \end{aligned}$$

Show that

$$s = \tfrac{\operatorname{tg} a}{\cos b}, \qquad t = \operatorname{tg} b, \qquad u = \operatorname{tg} c.$$

Use these identities together with the Euclidean Pythagorean theorem $u^2 = s^2 + t^2$ for $\triangle NB'C'$ to prove the spherical Pythagorean theorem

$$\cos c = \cos a \cdot \cos b$$

for $\triangle_s NBC$.

The following exercise is analogous to Exercise 16.4 in hyperbolic geometry.

15.10. Exercise. *Let $\triangle_s ABC$ be a nondegenerate spherical triangle. Assume that the plane Π^+ is parallel to the plane passing thru A, B and C. Denote by A', B' and C' the central projections of A, B and C.*

(a) *Show that the midpoints of* $[A'B']$, $[B'C']$ *and* $[C'A']$ *are central projections of the midpoints of* $[AB]_s$, $[BC]_s$ *and* $[CA]_s$ *correspondingly.*

(b) *Use part (a) to show that the medians of a spherical triangle intersect at one point.*

Chapter 16

Projective model

The *projective model* is another model of hyperbolic plane discovered by Beltrami; it is often called *Klein model*. The projective and conformal models are saying exactly the same thing but in two different languages. Some problems in hyperbolic geometry admit simpler proof using the projective model and others have simpler proof in the conformal model. Therefore, it worth to know both.

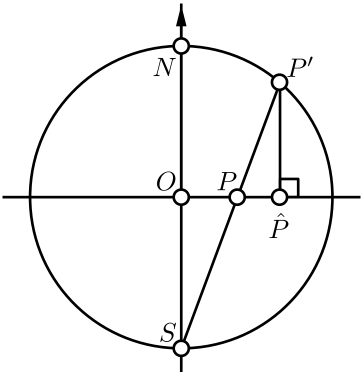
Special bijection of the h-plane to itself

Consider the conformal disc model with the absolute at the unit circle Ω centered at O. Choose a coordinate system (x, y) on the plane with the origin at O, so the circle Ω is described by the equation $x^2+y^2 = 1$.

Let us think that our plane is the coordinate xy-plane in the Euclidean space; denote it by Π. Let Σ be the unit sphere centered at O; it is described by the equation

$$x^2 + y^2 + z^2 = 1.$$

Set $S = (0, 0, -1)$ and $N = (0, 0, 1)$; these are the south and north poles of Σ.



The plane thru P, O and S.

Consider stereographic projection $\Pi \to \Sigma$ from S; given point $P \in \Pi$ denote its image in Σ by P'. Note that the h-plane is mapped to the *north hemisphere*; that is, to the set of points (x, y, z) in Σ described by the inequality $z > 0$.

For a point $P' \in \Sigma$ consider its foot point $\hat{P}$ on Π; this is the closest point on Π from P'.

The composition $P \mapsto P' \mapsto \hat{P}$ of these two maps is a bijection of the h-plane to itself.

Note that $P = \hat{P}$ if and only if $P \in \Omega$ or $P = O$.

16.1. Exercise. *Show that the map $P \mapsto \hat{P}$ described above can be described the following way: set $\hat{O} = O$ and for any other point P take $\hat{P} \in [OP)$ such that*

$$O\hat{P} = \frac{2{\cdot}x}{1+x^2},$$

where $x = OP$.

16.2. Lemma. *Let $(PQ)_h$ be an h-line with the ideal points A and B. Then $\hat{P}, \hat{Q} \in [AB]$.*

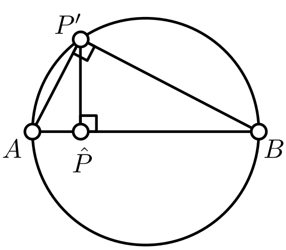
Moreover,

❶ $$\frac{A\hat{Q}{\cdot}B\hat{P}}{\hat{Q}B{\cdot}\hat{P}A} = \left(\frac{AQ{\cdot}BP}{QB{\cdot}PA}\right)^2.$$

In particular, if A, P, Q, B appear on the line in the same order, then

$$PQ_h = \tfrac{1}{2}\cdot\ln\frac{A\hat{Q}{\cdot}B\hat{P}}{\hat{Q}B{\cdot}\hat{P}A}.$$

Proof. Consider the stereographic projection $\Pi \to \Sigma$ from the south pole S. Denote by P' and Q' the images of P and Q.



The plane Λ.

According to Theorem 15.3c,

❷ $$\frac{AQ{\cdot}BP}{QB{\cdot}PA} = \frac{AQ'{\cdot}BP'}{Q'B{\cdot}P'A}.$$

By Theorem 15.3e, each circline in Π which is perpendicular to Ω is mapped to a circle in Σ which is still perpendicular to Ω. It follows that the stereographic projection sends $(PQ)_h$ to the intersection of the north hemisphere of Σ with a plane, say Λ, perpendicular to Π.

Consider the plane Λ. It contains points A, B, P', $\hat{P}$ and the circle $\Gamma = \Sigma \cap \Lambda$. (It also contains Q' and $\hat{Q}$ but we will not use these points for a while.)

Note that

- $A, B, P' \in \Gamma$,
- $[AB]$ is a diameter of Γ,
- $(AB) = \Pi \cap \Lambda$,
- $\hat{P} \in [AB]$
- $(P'\hat{P}) \perp (AB)$.

Since $[AB]$ is the diameter of Γ, by Corollary 8.6, the angle $AP'B$ is right. Hence $\triangle A\hat{P}P' \sim \triangle AP'B \sim \triangle P'\hat{P}B$. In particular

$$\frac{AP'}{BP'} = \frac{A\hat{P}}{P'\hat{P}} = \frac{P'\hat{P}}{B\hat{P}}.$$

Therefore

❸ $$\frac{A\hat{P}}{B\hat{P}} = \left(\frac{AP'}{BP'}\right)^2.$$

The same way we get

❹ $$\frac{A\hat{Q}}{B\hat{Q}} = \left(\frac{AQ'}{BQ'}\right)^2.$$

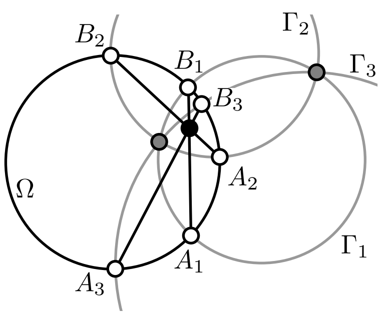
Finally, note that ❷+❸+❹ imply ❶.

The last statement follows from ❶ and the definition of h-distance. Indeed,

$$\begin{aligned} PQ_h &:= \ln \frac{AQ\cdot BP}{QB\cdot PA} = \\ &= \ln \left(\frac{A\hat{Q}\cdot B\hat{P}}{\hat{Q}B\cdot \hat{P}A}\right)^{\frac{1}{2}} = \\ &= \tfrac{1}{2}\cdot \ln \frac{A\hat{Q}\cdot B\hat{P}}{\hat{Q}B\cdot \hat{P}A}. \end{aligned}$$

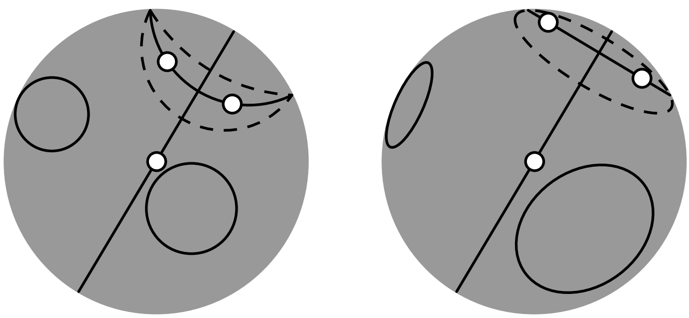
□

16.3. Exercise. *Let Γ_1, Γ_2 and Γ_3 be three circles perpendicular to the circle Ω. Let us denote by $[A_1B_1]$, $[A_2B_2]$ and $[A_3B_3]$ the common chords of Ω and Γ_1, Γ_2, Γ_3 correspondingly. Show that the chords $[A_1B_1]$, $[A_2B_2]$ and $[A_3B_3]$ intersect at one point inside Ω if and only if Γ_1, Γ_2 and Γ_3 intersect at two points.*



Projective model

The following picture illustrates the map $P \mapsto \hat{P}$ described in the previous section — if you take the picture on the left and apply the map $P \mapsto \hat{P}$, you get the picture on the right. The pictures are *conformal* and *projective model* of the hyperbolic plane correspondingly. The map $P \mapsto \hat{P}$ is a “translation” from one to another.



Conformal model Projective model

In the projective model things look different; some become simpler, other things become more complicated.

Lines. The h-lines in the projective model are chords of the absolute; more precisely, chords with the endpoints removed.

Circles and equidistants. The h-circles and equidistants in the projective model are certain type of ellipses and their open arcs.

It follows since the stereographic projection sends circles on the plane to circles on the unit sphere and the foot point projection of circle back to the plane is an ellipse. (One may define ellipse as a foot point projection of a circle.)

Distance. To find the h-distance between the points P and Q in the projective model, you have to find the points of intersection, say A and B, of the Euclidean line (PQ) with the absolute; then, by Lemma 16.2,

$$PQ_h = \tfrac{1}{2} \cdot \ln \frac{AQ \cdot BP}{QB \cdot PA},$$

assuming the points A, P, Q, R appear on the line in the same order.

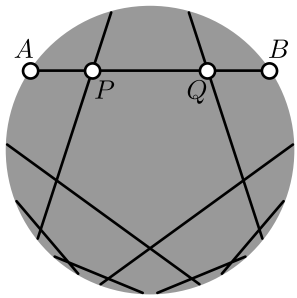
Angles. The angle measures in the projective model are very different from the Euclidean angles and it is hard to figure out by looking on the picture. For example all the intersecting h-lines on the picture are perpendicular. There are two useful exceptions

◇ If O is the center of the absolute, then

$$\measuredangle_h AOB = \measuredangle AOB.$$

◇ If O is the center of the absolute and $\measuredangle OAB = \pm\frac{\pi}{2}$, then

$$\measuredangle_h OAB = \measuredangle OAB = \pm\tfrac{\pi}{2}.$$



To find the angle measure in the projective model, you may apply a motion of the h-plane which moves the vertex of the angle to the center of the absolute; once it is done the hyperbolic and Euclidean angles have the same measure.

Motions. The motions of the h-plane in the conformal and projective models are relevant to inversive transformations and projective transformation in the same way. Namely:

◇ Any inversive transformations which preserve the h-plane describe a motion of the h-plane in the conformal model.

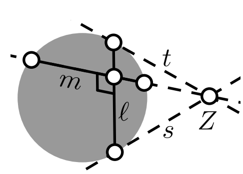
◇ Any projective transformation which preserve h-plane describes a motion in the projective model.

The following exercise is hyperbolic analog of Exercise 15.10. This is the first example of a statement which admits an easier proof using the projective model.

16.4. Exercise. *Let P and Q be the point in h-plane which lie on the same distance from the center of the absolute. Observe that in the projective model, h-midpoint of $[PQ]_h$ coincides with the Euclidean midpoint of $[PQ]_h$.*

Conclude that if an h-triangle is inscribed in an h-circle, then its medians intersect at one point.

Recall that an h-triangle might be also inscribed in a horocycle or an equidistant. Think how to prove the statement in this case.



16.5. Exercise. *Let ℓ and m are h-lines in the projective model. Denote by s and t the Euclidean lines tangent to the absolute at the ideal points of ℓ and denote by $\bar{m}$ the Euclidean line which contains m. Show that if the lines s, t and $\bar{m}$ meet at one point, then ℓ and m are perpendicular h-lines in the projective model.*

16.6. Exercise. *Give a proof of Proposition 12.1 using projective model.*

16.7. Exercise. *Use projective model to find the inradius of the ideal triangle.*

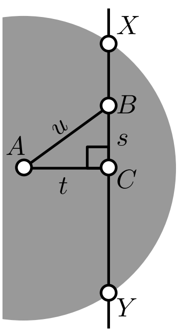
Hyperbolic Pythagorean theorem

16.8. Theorem. *Assume that $\triangle_h ACB$ is a triangle in h-plane with right angle at C. Set $a = BC_h$, $b = CB_h$ and $c = AB_h$. Then*

❺ $$\operatorname{ch} c = \operatorname{ch} a \cdot \operatorname{ch} b.$$

where ch *denotes* hyperbolic cosine*; that is, the function defined by*

$$\operatorname{ch} x := \tfrac{e^x + e^{-x}}{2}.$$



Proof. We will use projective model of the h-plane with a unit circle as the absolute.

We can assume that A is the center of the absolute. Therefore, both $\angle_h ACB$ and $\angle ACB$ are right.

Set $s = BC$, $t = CA$, $u = AB$. According to the Euclidean Pythagorean theorem (6.10),

$$u^2 = s^2 + t^2.$$

Note that

$$b = \tfrac{1}{2} \cdot \ln \frac{1+t}{1-t};$$

therefore

❻ $$\begin{aligned} \operatorname{ch} b &= \frac{\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} + \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}}}{2} = \\ &= \frac{1}{\sqrt{1-t^2}}. \end{aligned}$$

The same way we get

$$c = \tfrac{1}{2} \cdot \ln \frac{1+u}{1-u}$$

and

$$\text{❼} \qquad \operatorname{ch} c = \frac{\left(\frac{1+u}{1-u}\right)^{\frac12} + \left(\frac{1-u}{1+u}\right)^{\frac12}}{2} = \\ = \frac{1}{\sqrt{1-u^2}}.$$

Let X and Y are the ideal points of $(BC)_h$. Applying the Pythagorean theorem (6.10) again, we get

$$CX = CY = \sqrt{1-t^2}.$$

Therefore,

$$a = \tfrac12 \cdot \ln \frac{\sqrt{1-t^2}+s}{\sqrt{1-t^2}-s},$$

and

$$\text{❽} \qquad \begin{aligned} \operatorname{ch} a &= \frac{\left(\frac{\sqrt{1-t^2}+s}{\sqrt{1-t^2}-s}\right)^{\frac12} + \left(\frac{\sqrt{1-t^2}-s}{\sqrt{1-t^2}+s}\right)^{\frac12}}{2} = \\ &= \frac{\sqrt{1-t^2}}{\sqrt{1-t^2-s^2}} = \\ &= \frac{\sqrt{1-t^2}}{\sqrt{1-u^2}}. \end{aligned}$$

Finally, note that ❻+❼+❽ imply ❺. □

Bolyai's construction

Assume we need to construct a line thru P asymptotically parallel to the given line $\ell = (AB)$ in the h-plane.

Note that ideal points do not lie in the h-plane, so there is no way to use them in the construction.

In the following construction we assume that you know a compass-and-ruler construction of the perpendicular line; see Exercise 5.23.

16.9. Bolyai's construction.

1. *Drop a perpendicular from P to ℓ; denote it by m. Let Q be the foot point of P on ℓ.*
2. *Drop a perpendicular from P to m; denote it by n.*
3. *Mark by R a point on ℓ distinct from Q.*
4. *Drop a perpendicular from R to n; denote it by k.*
5. *Draw the circle Γ_2 with center P and the radius QR. Mark by T a point of intersection of Γ_2 with k.*

6. The line $(PT)_h$ is asymptotically parallel to ℓ.

16.10. Exercise. *Explain what happens if one performs the Bolyai construction in the Euclidean plane.*

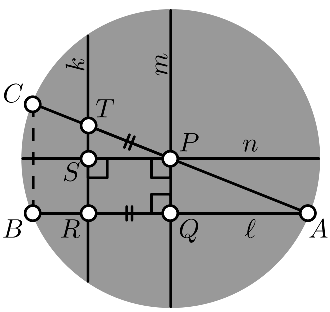
To prove that Bolyai's construction gives the asymptotically parallel line in the h-plane, it is sufficient to show the following.

16.11. Proposition. *Assume P, Q, R, S, T be points in h-plane such that*
- ◇ $S \in (RT)_h$,
- ◇ $(PQ)_h \perp (QR)_h$,
- ◇ $(PS)_h \perp (PQ)_h$,
- ◇ $(RT)_h \perp (PS)_h$ *and*
- ◇ $(PT)_h$ *and* $(QR)_h$ *are asymptotically parallel.*

Then $QR_h = PT_h$.

Proof. We will use the projective model. Without loss of generality, we may assume that P is the center of the absolute. As it was noted on page 132, in this case the corresponding Euclidean lines are also perpendicular; that is, $(PQ) \perp (QR)$, $(PS) \perp (PQ)$ and $(RT) \perp (PS)$.

Denote by A be the ideal point of $(QR)_h$ and $(PT)_h$. Denote by B and C the remaining ideal points of $(QR)_h$ and $(PT)_h$ correspondingly.



Note that the Euclidean lines (PQ), (TR) and (CB) are parallel.

Therefore,

$$\triangle AQP \sim \triangle ART \sim \triangle ABC.$$

In particular,

$$\frac{AC}{AB} = \frac{AT}{AR} = \frac{AP}{AQ}.$$

It follows that

$$\frac{AT}{AR} = \frac{AP}{AQ} = \frac{BR}{CT} = \frac{BQ}{CP}.$$

In particular,

$$\frac{AT \cdot CP}{TC \cdot PA} = \frac{AR \cdot BQ}{RB \cdot QA};$$

hence $QR_h = PT_h$. □

Chapter 17

Complex coordinates

In this chapter, we give an interpretation of inversive geometry using complex coordinates. The results of this chapter will not be used in this book, but they lead to deeper understanding of both concepts.

Complex numbers

Informally, a complex number is a number that can be put in the form

❶ $$z = x + i{\cdot}y,$$

where x and y are real numbers and $i^2 = -1$.

The set of complex numbers will be further denoted by $\mathbb{C}$. If x, y and z are as in ❶, then x is called the real part and y the imaginary part of the complex number z. Briefly it is written as

$$x = \operatorname{Re} z \quad \text{and} \quad y = \operatorname{Im} z.$$

On the more formal level, a complex number is a pair of real numbers (x, y) with the addition and multiplication described below. The formula $x + i{\cdot}y$ is only a convenient way to write the pair (x, y).

$$\begin{aligned}(x_1 + i{\cdot}y_1) + (x_2 + i{\cdot}y_2) &:= (x_1 + x_2) + i{\cdot}(y_1 + y_2);\\ (x_1 + i{\cdot}y_1){\cdot}(x_2 + i{\cdot}y_2) &:= (x_1{\cdot}x_2 - y_1{\cdot}y_2) + i{\cdot}(x_1{\cdot}y_2 + y_1{\cdot}x_2).\end{aligned}$$

Complex coordinates

Recall that one can think of the Euclidean plane as the set of all pairs of real numbers (x, y) equipped with the metric

$$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2},$$

where $A=(x_A,y_A)$ and $B=(x_B,y_B)$.

One can pack the coordinates (x,y) of a point in the Euclidean plane in one complex number $z=x+i{\cdot}y$. This way we get one-to-one correspondence between points of the Euclidean plane and $\mathbb{C}$. Given a point $Z=(x,y)$, the complex number $z=x+i{\cdot}y$ is called the *complex coordinate* of Z.

Note that if O, E and I are points in the plane with complex coordinates 0, 1 and i, then $\measuredangle EOI=\pm\tfrac{\pi}{2}$. Further, we assume that $\measuredangle EOI=\tfrac{\pi}{2}$; if not, one has to change the direction of the y-coordinate.

Conjugation and absolute value

Let $z=x+i{\cdot}y$ and both x and y be real. Denote by Z the point in the plane with the complex coordinate z.

If $y=0$, we say that the complex number z is *real* and if $x=0$ we say that z is *imaginary*. The set of points with real and imaginary complex coordinates form lines in the plane, which are called *real* and *imaginary* lines; they will be denoted as $\mathbb{R}$ and $i{\cdot}\mathbb{R}$.

The complex number $\bar z=x-iy$ is called the *complex conjugate* of z.

Note that the point $\bar Z$ with the complex coordinate $\bar z$ is the reflection of Z in the real line.

It is straightforward to check that

❷ $$x=\operatorname{Re}z=\frac{z+\bar z}{2},\quad y=\operatorname{Im}z=\frac{z-\bar z}{i{\cdot}2},\quad x^2+y^2=z{\cdot}\bar z.$$

The last formula in ❷ makes it possible to express the quotient $\frac{w}{z}$ of two complex numbers w and $z=x+i{\cdot}y$:

$$\frac{w}{z}=\tfrac{1}{z\cdot\bar z}{\cdot}w{\cdot}\bar z=\tfrac{1}{x^2+y^2}{\cdot}w{\cdot}\bar z.$$

Note that

$$\overline{z+w}=\bar z+\bar w,\qquad \overline{z-w}=\bar z-\bar w,\qquad \overline{z{\cdot}w}=\bar z{\cdot}\bar w,\qquad \overline{z/w}=\bar z/\bar w.$$

That is, the complex conjugation *respects* all the arithmetic operations.

The value

$$\begin{aligned}|z|&=\sqrt{x^2+y^2}=\\&=\sqrt{z{\cdot}\bar z}\end{aligned}$$

is called the *absolute value* of z. If $|z|=1$, then z is called a *unit complex number*.

Note that if Z and W are points in the Euclidean plane, z and w are their complex coordinates, then

$$ZW = |z - w|.$$

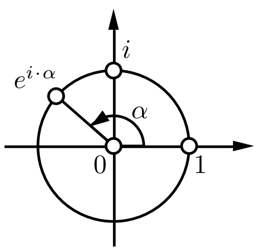
Euler's formula

Let α be a real number. The following identity is called *Euler's formula.*

❸ $$e^{i\cdot\alpha} = \cos\alpha + i\cdot\sin\alpha.$$

In particular, $e^{i\cdot\pi} = -1$ and $e^{i\cdot\frac{\pi}{2}} = i$.

Geometrically, Euler's formula means the following: Assume that O and E are the points with complex coordinates 0 and 1 correspondingly. Assume $OZ = 1$ and $\measuredangle EOZ \equiv \equiv \alpha$, then $e^{i\cdot\alpha}$ is the complex coordinate of Z. In particular, the complex coordinate of any point on the unit circle centered at O can be uniquely expressed as $e^{i\cdot\alpha}$ for some $\alpha \in (-\pi, \pi]$.



Why should you think that ❸ is true? The proof of Euler's identity depends on the way you define the exponential function. If you never had to take the exponential function of an imaginary number, you may take the right hand side in ❸ as the definition of the $e^{i\cdot\alpha}$.

In this case, formally nothing has to be proved, but it is better to check that $e^{i\cdot\alpha}$ satisfies familiar identities. Mainly,

$$e^{i\cdot\alpha}\cdot e^{i\cdot\beta} = e^{i\cdot(\alpha+\beta)}.$$

The latter can be proved using the following trigonometric formulas, which we assume to be known:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta\\ \sin(\alpha+\beta) &= \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta\end{aligned}$$

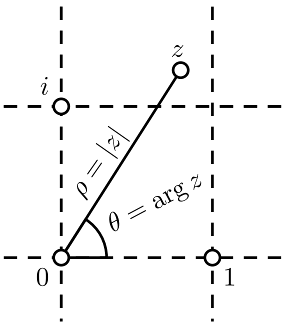
If you know the power series for the sine, cosine and exponential function, the following might convince that the identity ❸ holds:

$$\begin{aligned}
e^{i\cdot\alpha} &= 1 + i\cdot\alpha + \frac{(i\cdot\alpha)^2}{2!} + \frac{(i\cdot\alpha)^3}{3!} + \frac{(i\cdot\alpha)^4}{4!} + \frac{(i\cdot\alpha)^5}{5!} + \dots = \\
&= 1 + i\cdot\alpha - \frac{\alpha^2}{2!} - i\cdot\frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + i\cdot\frac{\alpha^5}{5!} - \dots = \\
&= \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \dots\right) + i\cdot\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \dots\right) = \\
&= \cos\alpha + i\cdot\sin\alpha.
\end{aligned}$$

Argument and polar coordinates

As before, we assume that O and E are the points with complex coordinates 0 and 1 correspondingly.

Let Z be the a point distinct form O. Set $\rho = OZ$ and $\theta = \measuredangle EOZ$. The pair (ρ, θ) is called the *polar coordinates* of Z.



If z is the complex coordinate of Z, then $\rho = |z|$. The value θ is called the argument of z (briefly, $\theta = \arg z$). In this case,

$$z = \rho\cdot e^{i\cdot\theta} = \rho\cdot(\cos\theta + i\cdot\sin\theta).$$

Note that

$$\arg(z\cdot w) \equiv \arg z + \arg w$$

and

$$\arg\tfrac{z}{w} \equiv \arg z - \arg w$$

if $z, w \neq 0$. In particular, if Z, V, W are points with complex coordinates z, v and w correspondingly, then

❹
$$\begin{aligned}
\measuredangle VZW &= \arg\left(\frac{w-z}{v-z}\right) \equiv \\
&\equiv \arg(w-z) - \arg(v-z)
\end{aligned}$$

if $\measuredangle VZW$ is defined.

17.1. Exercise. *Use formula ❹ to show that*

$$\measuredangle ZVW + \measuredangle VWZ + \measuredangle WZV \equiv \pi$$

for any $\triangle ZVW$ in the Euclidean plane.

17.2. Exercise. *Assume that points V, W and Z have complex coordinates v, w and $z = v{\cdot}w$ correspondingly and the point O and E are as above. Show that*

$$\triangle OEV \sim \triangle OWZ.$$

The following Theorem is a reformulation of Theorem 8.10 which uses complex coordinates.

17.3. Theorem. *Let $\square UVWZ$ be a quadrilateral and u, v, w and z be the complex coordinates of its vertices. Then $\square UVWZ$ is inscribed if and only if the number*

$$\frac{(v-u){\cdot}(z-w)}{(v-w){\cdot}(z-u)}$$

is real.

The value $\frac{(v-u)\cdot(w-z)}{(v-w)\cdot(z-u)}$ is called the *complex cross-ratio*; it will be discussed in more details below.

17.4. Exercise. *Observe that the complex number $z \neq 0$ is real if and only if $\arg z = 0$ or π; in other words, $2{\cdot}\arg z \equiv 0$.*

Use this observation to show that Theorem 17.3 is indeed a reformulation of Theorem 8.10.

Möbius transformations

17.5. Exercise. *Watch video "Möbius Transformations Revealed" by Douglas Arnold and Jonathan Rogness. (It is 3 minutes long and available on YouTube.)*

The complex plane $\mathbb{C}$ extended by one ideal number ∞ is called the *extended complex plane*. It is denoted by $\hat{\mathbb{C}}$, so $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

A *Möbius transformation* of $\hat{\mathbb{C}}$ is a function of one complex variable z which can be written as

$$f(z) = \frac{a{\cdot}z + b}{c{\cdot}z + d},$$

where the coefficients a, b, c, d are complex numbers satisfying $a{\cdot}d - b{\cdot}c \neq 0$. (If $a{\cdot}d - b{\cdot}c = 0$ the function defined above is a constant and is not considered to be a Möbius transformation.)

In case $c \neq 0$, we assume that

$$f(-d/c) = \infty \quad \text{and} \quad f(\infty) = a/c;$$

and if $c = 0$ we assume

$$f(\infty) = \infty.$$

Elementary transformations

The following three types of Möbius transformations are called *elementary*.

1. $z \mapsto z + w$,
2. $z \mapsto w \cdot z$ for $w \neq 0$,
3. $z \mapsto \frac{1}{z}$.

The geometric interpretations. As before we will denote by O the point with the complex coordinate 0.

The first map $z \mapsto z + w$, corresponds to the so called *parallel translation* of the Euclidean plane, its geometric meaning should be evident.

The second map is called the *rotational homothety* with the center at O. That is, the point O maps to itself and any other point Z maps to a point Z' such that $OZ' = |w| \cdot OZ$ and $\measuredangle ZOZ' = \arg w$.

The third map can be described as a composition of the inversion in the unit circle centered at O and the reflection in $\mathbb{R}$ (the composition can be taken in any order). Indeed, $\arg z \equiv -\arg \frac{1}{z}$. Therefore,

$$\arg z = \arg(1/\bar{z});$$

that is, if the points Z and Z' have complex coordinates z and $1/\bar{z}$, then $Z' \in [OZ)$. Clearly, $OZ = |z|$ and $OZ' = |1/\bar{z}| = \frac{1}{|z|}$. Therefore, Z' is the inverse of Z in the unit circle centered at O. Finally, the reflection of Z' in $\mathbb{R}$, has complex coordinate $\frac{1}{z} = \overline{(1/\bar{z})}$.

17.6. Proposition. *The map $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation if and only if it can be expressed as a composition of elementary Möbius transformations.*

Proof; the "only if" part. Fix a Möbius transformation

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d}.$$

Assume $c \neq 0$. Then

$$\begin{aligned} f(z) &= \frac{a \cdot z + b}{c \cdot z + d} = \\ &= \frac{a}{c} - \frac{a \cdot d - b \cdot c}{c \cdot (c \cdot z + d)} = \\ &= \frac{a}{c} - \frac{a \cdot d - b \cdot c}{c^2} \cdot \frac{1}{z + \frac{d}{c}}. \end{aligned}$$

That is,

❺
$$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z),$$

where f_1, f_2, f_3 and f_4 are elementary transformations of the following form:

- ◇ $f_1(z) = z + \frac{d}{c}$,
- ◇ $f_2(z) = \frac{1}{z}$,
- ◇ $f_3(z) = -\frac{a\cdot d - b\cdot c}{c^2}\cdot z$,
- ◇ $f_4(z) = z + \frac{a}{c}$.

If $c = 0$, then
$$f(z) = \frac{a\cdot z + b}{d}.$$
In this case
$$f(z) = f_2 \circ f_1(z),$$
where

- ◇ $f_1(z) = \frac{a}{d}\cdot z$,
- ◇ $f_2(z) = z + \frac{b}{d}$.

"If" part. We need to show that by composing elementary transformations, we can only get Möbius transformations. Note that it is sufficient to check that the composition of a Möbius transformations

$$f(z) = \frac{a\cdot z + b}{c\cdot z + d}.$$

with any elementary transformation $z \mapsto z + w$, $z \mapsto w\cdot z$ and $z \mapsto \frac{1}{z}$ is a Möbius transformations.

The latter is done by means of direct calculations.

$$\begin{aligned}
\frac{a\cdot(z+w)+b}{c\cdot(z+w)+d} &= \frac{a\cdot z + (b + a\cdot w)}{c\cdot z + (d + c\cdot w)},\\
\frac{a\cdot(w\cdot z)+b}{c\cdot(w\cdot z)+d} &= \frac{(a\cdot w)\cdot z + b}{(c\cdot w)\cdot z + d},\\
\frac{a\cdot\frac{1}{z}+b}{c\cdot\frac{1}{z}+d} &= \frac{b\cdot z + a}{d\cdot z + c}.
\end{aligned}$$
□

17.7. Corollary. *The image of a circline under a Möbius transformation is a circline.*

Proof. By Proposition 17.6, it is sufficient to check that each elementary transformation sends a circline to a circline.

For the first and second elementary transformation, the latter is evident.

As it was noted above, the map $z \mapsto \frac{1}{z}$ is a composition of inversion and reflection. By Theorem 9.11, the inversion sends a circline to a circline. Hence the result follows. $\square$

17.8. Exercise. *Show that the inverse of a Möbius transformation is a Möbius transformation.*

17.9. Exercise. *Given distinct values $z_0, z_1, z_\infty \in \hat{\mathbb{C}}$, construct a Möbius transformation f such that $f(z_0) = 0$, $f(z_1) = 1$ and $f(z_\infty) = \infty$. Show that such a transformation is unique.*

17.10. Exercise. *Show that any inversion is a composition of the complex conjugation and a Möbius transformation.*

Complex cross-ratio

Given four distinct complex numbers u, v, w and z, the complex number

$$\frac{(u-w)\cdot(v-z)}{(v-w)\cdot(u-z)}$$

is called the *complex cross-ratio*; it will be denoted by $(u, v; w, z)$.

If one of the numbers u, v, w, z is ∞, then the complex cross-ratio has to be defined by taking the appropriate limit; in other words, we assume that $\frac{\infty}{\infty} = 1$. For example,

$$(u, v; w, \infty) = \frac{(u-w)}{(v-w)}.$$

Assume that U, V, W and Z are the points with complex coordinates u, v, w and z correspondingly. Note that

$$\begin{aligned}
\frac{UW\cdot VZ}{VW\cdot UZ} &= |(u, v; w, z)|,\\
\measuredangle WUZ + \measuredangle ZVW &= \arg\frac{u-w}{u-z} + \arg\frac{v-z}{v-w} \equiv\\
&\equiv \arg(u, v; w, z).
\end{aligned}$$

It makes it possible to reformulate Theorem 9.6 using the complex coordinates the following way.

17.11. Theorem. *Let $UWVZ$ and $U'W'V'Z'$ be two quadrilaterals such that the points U', W', V' and Z' are inversions of U, W, V, and*

Z correspondingly. Assume u, w, v, z, u', w', v' and z' are the complex coordinates of U, W, V, Z, U', W', V' and Z' correspondingly.

Then

$$(u', v'; w', z') = \overline{(u, v; w, z)}.$$

The following exercise is a generalization of the Theorem above. It admits a short and simple solution which uses Proposition 17.6.

17.12. Exercise. *Show that complex cross-ratios are* invariant *under Möbius transformations.*

That is, if a Möbius transformation maps four distinct complex numbers u, v, w, z to complex numbers u', v', w', z' respectively, then

$$(u', v'; w', z') = (u, v; w, z).$$

Schwarz–Pick theorem

The following theorem shows that the metric in the conformal disc model naturally appears in other branches of mathematics. We do not give a proof, but it can be found in any textbook on geometric complex analysis.

Let us denote by $\mathbb{D}$ the unit disc in the complex plane centered at 0; that is, a complex number z belongs to $\mathbb{D}$ if and only if $|z| < 1$.

Let us use the disc $\mathbb{D}$ as a h-plane in the conformal disc model; the h-distance between $z, w \in \mathbb{D}$ will be denoted by $d_h(z, w)$.

A function $f\colon \mathbb{D} \to \mathbb{C}$ is called *holomorphic* if for every $z \in \mathbb{D}$ there is a complex number s such that

$$f(z + w) = f(z) + s \cdot w + o(|w|).$$

In other words, f is *complex-differentiable* at any $z \in \mathbb{D}$. The number s above is called the derivative of f at z and is denoted by $f'(z)$.

17.13. Schwarz–Pick theorem. *Assume $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic function. Then*

$$d_h(f(z), f(w)) \leqslant d_h(z, w)$$

for any $z, w \in \mathbb{D}$.

If the equality holds for one pair of distinct numbers $z, w \in \mathbb{D}$, then it holds for any pair. Moreover, $f\colon \mathbb{D} \to \mathbb{D}$ is a motion of the h-plane.

17.14. Exercise. *Show that the Schwarz lemma stated below follows from Schwarz–Pick theorem.*

17.15. Schwarz lemma. *Let* $f\colon \mathbb{D}\to\mathbb{D}$ *be a holomorphic function and* $f(0)=0$. *Then* $|f(z)|\leqslant|z|$ *for any* $z\in\mathbb{D}$.

Moreover, if equality holds for some $z\neq 0$, *then there is a unit complex number* u *such that* $f(z)=u\cdot z$ *for any* $z\in\mathbb{D}$.

Chapter 18

Geometric constructions

Geometric constructions have great pedagogical value as an introduction to mathematical proofs. We were using construction problems everywhere starting from Chapter 5.

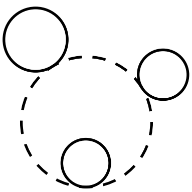
In this chapter we briefly discuss the classical results in geometric constructions.

Classical problems

In this section we list a couple of classical construction problems; each known for more than a thousand years. The solutions of the following two problems are quite nontrivial.

18.1. Problem of Brahmagupta. *Construct an inscribed quadrilateral with given sides.*

18.2. Problem of Apollonius. *Construct a circle which is tangent to three given circles.*



The following exercise is a simplified version of the problem of Apollonius, which is still nontrivial.

18.3. Exercise. *Construct a circle which passes thru a given point and is tangent to two intersecting lines.*

The following three problems cannot be solved in principle; that is, the needed compass-and-ruler construction does not exist.

Doubling the cube. *Construct the side of a new cube, which has the volume twice as big as the volume of a given cube.*

In other words, given a segment of the length a, one needs to construct a segment of length $\sqrt[3]{2}\cdot a$.

Squaring the circle. *Construct a square with the same area as a given circle.*

If r is the radius of the given circle, we need to construct a segment of length $\sqrt{\pi}\cdot r$.

Angle trisection. *Divide the given angle into three equal angles.*

In fact, there is no compass-and-ruler construction which trisects angle with measure $\frac{\pi}{3}$. Existence of such a construction would imply constructability of a regular 9-gon which is prohibited by the following famous result.

18.4. Gauss–Wantzel theorem. *A regular n-gon can be constructed with ruler and compass if and only if n is the product of a power of 2 and any number of distinct Fermat primes.*

A *Fermat prime* is a prime number of the form 2^k+1 for some integer k. Only five Fermat primes are known today:

$$3, 5, 17, 257, 65537.$$

For example,

- ◇ one can construct a regular 340-gon since $340 = 2^2\cdot 5\cdot 17$ and 5 as well as 17 are Fermat primes;
- ◇ one cannot construct a regular 7-gon since 7 is not a Fermat prime;
- ◇ one cannot construct a regular 9-gon; altho $9 = 3\cdot 3$ is a product of two Fermat primes, these primes are not distinct.

The impossibility of these constructions was proved only in 19th century. The method used in the proofs is indicated in the next section.

Constructible numbers

In the classical compass-and-ruler constructions initial configuration can be completely described by a finite number of points; each line is defined by two points on it and each circle is described by its center and a point on it (equivalently, you may describe a circle by three points on it).

The same way the result of construction can be described by a finite collection of points.

Choose a coordinate system, such that one of the initial points is the origin $(0, 0)$ and yet another initial point has the coordinates $(1, 0)$.

In this coordinate system, the initial configuration of n points is described by $2\cdot n - 4$ numbers — their coordinates $x_3, y_3, \dots, x_n, y_n$.

It turns out that the coordinates of any point constructed with a compass and ruler can be written thru the numbers $x_3, y_3, \dots, x_n, y_n$ using the four arithmetic operations "$+$", "$-$", "$\cdot$", "$/$" and the square root "$\sqrt{}$".

For example, assume we want to find the points $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ of the intersections of a line passing thru $A = (x_A, y_A)$ and $B = (x_B, y_B)$ and the circle with center $O = (x_O, y_O)$ which passes thru the point $W = (x_W, y_W)$. Let us write the equations of the circle and the line in the coordinates (x, y):

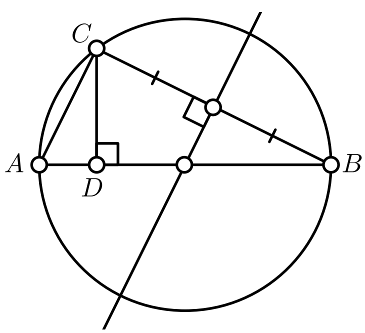
$$\begin{cases} (x - x_O)^2 + (y - y_O)^2 = (x_W - x_O)^2 + (y_W - y_O)^2, \\ \quad (x - x_A)\cdot(y_B - y_A) = (y - y_A)\cdot(x_B - x_A). \end{cases}$$

Expressing y from the second equation and substituting the result in the first one, gives us a quadratic equation in x, which can be solved using "$+$", "$-$", "$\cdot$", "$/$" and "$\sqrt{}$" only.

The same can be performed for the intersection of two circles. The intersection of two lines is even simpler; it is described as a solution of two linear equations and can be expressed using only four arithmetic operations; the square root "$\sqrt{}$" is not needed.

On the other hand, it is easy to produce compass-and-ruler constructions which produce segments of the lengths $a + b$ and $a - b$ from two given segments of lengths $a > b$.

To perform "$\cdot$", "$/$" and "$\sqrt{}$" consider the following diagram: let $[AB]$ be a diameter of a circle; fix a point C on the circle and let D be the foot point of C on $[AB]$. Note that



$$\triangle ABC \sim \triangle ACD \sim \triangle BDC.$$

It follows that $AD\cdot DC = BD^2$.

Using this diagram, one should guess the compass-and-ruler constructions which produce segments of lengths $\sqrt{a\cdot b}$ and $\frac{a^2}{b}$. To construct $\sqrt{a\cdot b}$, do the following: (1) construct points A, B and $D \in [AB]$ such that $AD = a$ and $BD = b$; (2) construct the circle Γ on the diameter $[AB]$; (3) draw the line ℓ thru D perpendicular to (AB); (4) let C be an intersection of Γ and ℓ. Then $DC = \sqrt{a\cdot b}$.

Taking 1 for a or b above, we can produce $\sqrt{a}$, a^2, $\frac{1}{b}$. Combining these constructions we can produce $a\cdot b = (\sqrt{a\cdot b})^2$, $\frac{a}{b} = a\cdot\frac{1}{b}$. In other words we produced a *compass-and-ruler calculator*, which can do "+", "−", "·", "/" and "$\sqrt{\ }$".

The discussion above gives a sketch of the proof of the following theorem:

18.5. Theorem. *Assume that the initial configuration of geometric construction is given by the points $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = = (x_3, y_3), \dots, A_n = (x_n, y_n)$. Then a point $X = (x, y)$ can be constructed using a compass-and-ruler construction if and only if both coordinates x and y can be expressed from the integer numbers and $x_3, y_3, x_4, y_4, \dots, x_n, y_n$ using the arithmetic operations "+", "−", "·", "/" and the square root "$\sqrt{\ }$".*

The numbers which can be expressed from the given numbers using the arithmetic operations and the square root "$\sqrt{\ }$" are called *constructible*; if the list of given numbers is not given, then we can only use the integers.

The theorem above translates any compass-and-ruler construction problem into a purely algebraic language. For example:

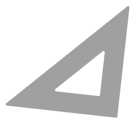
- ◇ The impossibility of a solution for doubling the cube problem states that $\sqrt[3]{2}$ is not a constructible number. That is $\sqrt[3]{2}$ cannot be expressed thru integers using "+", "−", "·", "/" and "$\sqrt{\ }$".
- ◇ The impossibility of a solution for squaring the circle states that $\sqrt{\pi}$, or equivalently π, is not a constructible number.
- ◇ The Gauss–Wantzel theorem says for which integers n the number $\cos\frac{2\cdot\pi}{n}$ is constructible.

Some of these statements might look evident, but rigorous proofs require some knowledge of abstract algebra (namely, field theory) which is out of the scope of this book.

In the next section, we discuss similar but simpler examples of impossible constructions with an unusual tool.

Constructions with the set square

A set square is a construction tool which can produce a line thru a given point which makes the angles $\frac{\pi}{2}$ or $\pm\frac{\pi}{4}$ to a given line.



18.6. Exercise. *Trisect a given segment with a ruler and a set square.*

Let us consider ruler-and-set-square constructions. Using the same idea as in the previous section, we can define *ruler-and-set-square constructible numbers* and prove the following analog of Theorem 18.5.

18.7. Theorem. *Assume that the initial configuration of a geometric construction is given by the points* $A_1 = (0,0)$*,* $A_2 = (1,0)$*,* $A_3 = (x_3, y_3), \dots, A_n = (x_n, y_n)$*. Then a point* $X = (x, y)$ *can be constructed using a ruler-and-set-square construction if and only if both coordinates* x *and* y *can be expressed from the integer numbers and* x_3*,* y_3*,* x_4*,* $y_4, \dots, x_n$*,* y_n *using the arithmetic operations "$+$", "$-$", "$\cdot$", "$/$".*

We omit the proof of this theorem, but it can be build on the ideas described in the previous section. Let us show how to use this theorem to show the impossibility of some constructions with a ruler and set a square.

Note that if all the coordinates $x_3, y_3, \dots, x_n, y_n$ are rational numbers, then the theorem above implies that with a ruler and a set square, one can only construct the points with rational coordinates. A point with both rational coordinates is called *rational*, and if at least one of the coordinates is irrational, then the point is called *irrational*.

18.8. Exercise. *Show that an equilateral triangle in the Euclidean plane has at least one irrational point.*

Conclude that with a ruler and a set square, one cannot construct an equilateral triangle.

18.9. Exercise. *Make a ruler-and-set-square construction which verifies if the given triangle is equilateral. (We assume that we can "verify" if two constructed points coincide.)*

More impossible constructions

In this section we discuss yet another source of impossible constructions.

Recall that a *circumtool* produces a circle passing thru any given three points or a line if all three points lie on one line. Let us restate Exercise 9.9.

Exercise. *Show that with a circumtool only, it is impossible to construct the center of a given circle* Γ*.*

Remark. In geometric constructions, we allow to choose some free points, say any point on the plane, or a point on a constructed line, or a point which does not lie on a constructed line and so on.

In principle, when you make such a free choice it is possible to mark the center of Γ by accident. Nevertheless, we do not accept such a coincidence as true construction; we say that a construction produces the center if it produces it for any free choices.

Solution. Arguing by contradiction, assume we have a construction of the center.

Apply an inversion in a circle perpendicular to Γ to the whole construction. According to Corollary 9.15, the circle Γ maps to itself. Since the inversion sends a circline to a circline, we get that the whole construction is mapped to an equivalent construction; that is, a constriction with a different choice of free points.

According to Exercise 9.8, the inversion sends the center of Γ to another point. That is, following the same construction, we can end up at a different point, a contradiction. $\square$

18.10. Exercise. *Show that there is no circumtool-only construction which verifies if the given point is the center of a given circle. (We assume that we can only "verify" if two constructed points coincide.)*

A similar example of impossible constructions for a ruler and a parallel tool is given in Exercise 13.6.

Let us discuss yet another example for a ruler-only construction. Note that ruler-only constructions are invariant with respect to the projective transformations. In particular, to solve the following exercise, it is sufficient to construct a projective transformation which fixes two points A and B and moves its midpoint.

18.11. Exercise. *Show that the midpoint of a given segment cannot be constructed with only a ruler.*

The following theorem is a stronger version of the exercise above.

18.12. Theorem. *The center of a given circle cannot be constructed with only a ruler.*

Sketch of the proof. It is sufficient to construct a projective transformation which sends the given circle Γ to a circle Γ' such that the center of Γ' is not the image of the center of Γ.

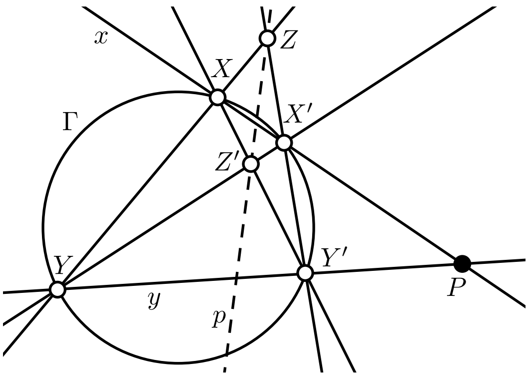
Let Γ be a circle which lies in the plane Π in the Euclidean space.

By Theorem 15.3, the inverse of a circle in a sphere is a circle or a line. Fix a sphere Σ with the center O so that the inversion Γ' of Γ is a circle and the plane Π' containing Γ' is not parallel to Π; any sphere Σ in a general position will do.

Denote by Z and Z' the centers of Γ and Γ'. Note that $Z' \notin (OZ)$. It follows that the perspective projection $\Pi \to \Pi'$ with center at O sends Γ to Γ', but Z' is not the image of Z. $\square$

Construction of a polar

Assume Γ is a circle in the plane and $P \notin \Gamma$. Draw two lines x and y thru P which intersect Γ at two pairs of points X, X' and Y, Y'. Let $Z = (XY) \cap (X'Y')$ and $Z' = (XY') \cap (X'Y)$. Consider the line $p = (ZZ')$.



The following claim will be used in the constructions without a proof.

18.13. Claim. *The constructed line $p = (ZZ')$ does not depend on the choice of the lines x and y. Moreover, $P \mapsto p$ is a duality (see page 118).*

The line p is called the polar of P with respect to Γ. The same way the point P is called the polar of the line p with respect to Γ.

18.14. Exercise. *Let p be the polar line of point P with respect to the circle Γ. Assume that p intersects Γ at points V and W. Show that the lines (PV) and (PW) are tangent to Γ.*

Come up with a ruler-only construction of the tangent lines to the given circle Γ thru the given point $P \notin \Gamma$.

18.15. Exercise. *Assume two concentric circles Γ and Γ' are given. Construct the common center of Γ and Γ' with a ruler only.*

Chapter 19

Area

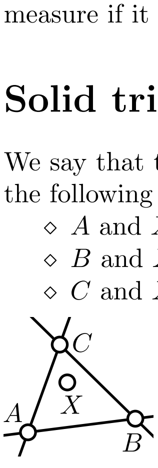
The area functional will be defined by Theorem 19.7. This theorem is given without proof, but it follows immediately from the properties of *Lebesgue measure* on the plane. The construction of Lebesgue measure typically use the method of coordinates and it is included in any textbook in Real Analysis. Based on this theorem, we develop the concept of area with no cheating.

We choose this approach since any rigorous introduction to area is tedious. We do not want to cheat and at the same time we do not want to waste your time; soon or later you will have to learn Lebesgue measure if it is not done already.

Solid triangles

We say that the point X lies *inside* a nondegenerate triangle ABC if the following three condition hold:

- ⋄ A and X lie on the same side from the line (BC);
- ⋄ B and X lie on the same side from the line (CA);
- ⋄ C and X lie on the same side from the line (AB).



The set of all points inside $\triangle ABC$ and on its sides $[AB]$, $[BC]$, $[CA]$ will be called *solid triangle ABC* and denoted by $\blacktriangle ABC$.

19.1. Exercise. *Show that any solid triangle is* convex*; that is, if $X, Y \in \blacktriangle ABC$, then $[XY] \subset \blacktriangle ABC$.*

The notations $\triangle ABC$ and $\blacktriangle ABC$ look similar, they also have close but different meanings, which better not to confuse. Recall that $\triangle ABC$ is an ordered triple of distinct points (see page 17), while $\blacktriangle ABC$ is an infinite set of points.

In particular, $\blacktriangle ABC = \blacktriangle BAC$ for any triangle ABC. Indeed, any point which belong to the set $\blacktriangle ABC$ also belongs to the set $\blacktriangle BAC$ and the other way around. On the other hand, $\triangle ABC \neq \triangle BAC$ simply because the sequence of points (A, B, C) is distinct from the sequence (B, A, C).

In general $\triangle ABC \ncong \triangle BAC$, but $\blacktriangle ABC \cong \blacktriangle BAC$, where congruence of the sets $\blacktriangle ABC$ and $\blacktriangle BAC$ is understood the following way.

19.2. Definition. *Two sets $\mathcal{S}$ and $\mathcal{T}$ in the plane are called* congruent *(briefly $\mathcal{S} \cong \mathcal{T}$) if $\mathcal{T} = f(\mathcal{S})$ for some motion f of the plane.*

If $\triangle ABC$ is not degenerate and

$$\blacktriangle ABC \cong \blacktriangle A'B'C',$$

then after relabeling the vertices of $\triangle ABC$ we will have

$$\triangle ABC \cong \triangle A'B'C'.$$

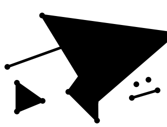
The existence of such relabeling follow from the exercise.

19.3. Exercise. *Let $\triangle ABC$ be nondegenerate and $X \in \blacktriangle ABC$. Show that X is a vertex of $\triangle ABC$ if and only if there is a line ℓ which intersects $\blacktriangle ABC$ at the single point X.*

Polygonal sets

Elementary set on the plane is a set of one of the following three types:

- ◇ one-point set;
- ◇ segment;
- ◇ solid triangle.



A set in the plane is called *polygonal* if it can be presented as a union of finite collection of elementary sets.

According to this definition, empty set $\varnothing$ is a polygonal set. Indeed, $\varnothing$ is a union of an empty collection of elementary sets.

A polygonal set is called *degenerate* if it can be presented as union of finite number of one-point sets and segments.

If X and Y lie on the opposite sides of the line (AB), then the union $\blacktriangle AXB \cup \blacktriangle BYA$ is a polygonal set which is called *solid quadrilateral* $AXBY$ and denoted by $\blacksquare AXBY$. In particular, we can talk about *solid parallelograms*, *rectangles* and *squares*.

Typically a polygonal set admits many presentation as union of a finite collection of elementary sets. For example, if $\square AXBY$ is a parallelogram, then

$$\blacksquare AXBY = \blacktriangle AXB \cup \blacktriangle BYA = \blacktriangle XAY \cup \blacktriangle YBX.$$

19.4. Exercise. *Show that a solid square is not degenerate.*

19.5. Exercise. *Show that a circle is not a polygonal set.*

Definition of area

19.6. Claim. *For any two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$, the union $\mathcal{P} \cup \mathcal{Q}$ as well as the intersection $\mathcal{P} \cap \mathcal{Q}$ are also polygonal sets.*

A class of sets which closed with respect to union and intersection is called a *ring of sets*. The claim above, therefore, states that polygonal sets in the plane form a ring of sets.

Informal proof. Let us present $\mathcal{P}$ and $\mathcal{Q}$ as a union of finite collection of elementary sets $\mathcal{P}_1, \dots, \mathcal{P}_k$ and $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ correspondingly.

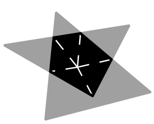
Note that

$$\mathcal{P} \cup \mathcal{Q} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k \cup \mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_n.$$

Therefore, $\mathcal{P} \cup \mathcal{Q}$ is polygonal.

Note that the union of all sets $\mathcal{P}_i \cap \mathcal{Q}_j$ is $\mathcal{P} \cap \mathcal{Q}$.

Therefore, in order to show that $\mathcal{P} \cap \mathcal{Q}$ is polygonal, it is sufficient to show that each $\mathcal{P}_i \cap \mathcal{Q}_j$ is polygonal for any pair i, j.



The diagram should suggest an idea for the proof of the latter statement in case if $\mathcal{P}_i$ and $\mathcal{Q}_j$ are solid triangles. The other cases are simpler; a formal proof can be built on Exercise 19.1 $\square$

The following theorem defines the area as a function which returns a real number for any polygonal set and satisfying certain conditions. We omit the proof of this theorem. It follows from the construction of Lebesgue measure which can be found in any text book on Real Analysis.

19.7. Theorem. *For each polygonal set $\mathcal{P}$ in the Euclidean plane there is a real number s called* area of $\mathcal{P}$ *(briefly $s = \operatorname{area}\mathcal{P}$) such that*

(a) $\operatorname{area}\varnothing = 0$ *and* $\operatorname{area}\mathcal{K} = 1$ *where $\mathcal{K}$ a solid square with unit side;*

(b) the conditions

$$\mathcal{P} \cong \mathcal{Q} \quad \Rightarrow \quad \operatorname{area}\mathcal{P} = \operatorname{area}\mathcal{Q};$$
$$\mathcal{P} \subset \mathcal{Q} \quad \Rightarrow \quad \operatorname{area}\mathcal{P} \leqslant \operatorname{area}\mathcal{Q};$$
$$\operatorname{area}\mathcal{P} + \operatorname{area}\mathcal{Q} = \operatorname{area}(\mathcal{P}\cup\mathcal{Q}) + \operatorname{area}(\mathcal{P}\cap\mathcal{Q})$$

hold for any two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$.
Moreover, the area function

$$\mathcal{P} \mapsto \operatorname{area}\mathcal{P}$$

is uniquely defined by the above conditions.

19.8. Proposition. *For any polygonal set $\mathcal{P}$ in the Euclidean plane, we have*

$$\operatorname{area}\mathcal{P} \geqslant 0.$$

Proof. Since $\varnothing \subset \mathcal{P}$, we get

$$\operatorname{area}\varnothing \leqslant \operatorname{area}\mathcal{P}.$$

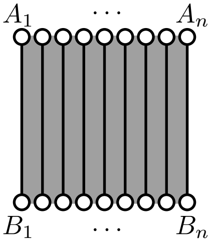
Since $\operatorname{area}\varnothing = 0$ the result follows. □

Vanishing area and subdivisions

19.9. Proposition. *Any one-point set as well as any segment in the Euclidean plane have vanishing area.*

Proof. Fix a line segment $[AB]$. Consider a sold square $\blacksquare ABCD$.

Note that given a positive integer n, there are n disjoint segments $[A_1B_1], \dots, [A_nB_n]$ in $\blacksquare ABCD$, such that each $[A_iB_i]$ is congruent to $[AB]$ in the sense of the Definition 19.2.



Applying the last identity in Theorem 19.7 few times, we get

$$\begin{aligned} n\cdot \operatorname{area}[AB] &= \operatorname{area}\left([A_1B_1]\cup\dots\cup[A_nB_n]\right) \leqslant \\ &\leqslant \operatorname{area}(\blacksquare ABCD) \end{aligned}$$

That is,

$$\operatorname{area}[AB] \leqslant \tfrac{1}{n}\cdot \operatorname{area}(\blacksquare ABCD)$$

for any positive integer n. Therefore, $\operatorname{area}[AB] \leqslant 0$.

On the other hand, by Proposition 19.8,

$$\operatorname{area}[AB] \geqslant 0.$$

Hence the result follows.

For any one-point set $\{A\}$ we have $\varnothing \subset \{A\} \subset [AB]$. Therefore,

$$0 \leqslant \operatorname{area}\{A\} \leqslant \operatorname{area}[AB] = 0.$$

Hence $\operatorname{area}\{A\} = 0$. ☐

19.10. Corollary. *Any degenerate polygonal set in the Euclidean plane has vanishing area.*

Proof. Let $\mathcal{P}$ be a degenerate set, say

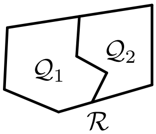
$$\mathcal{P} = [A_1B_1] \cup \dots \cup [A_nB_n] \cup \{C_1, \dots, C_k\}.$$

Applying Theorem 19.7 together with Proposition 19.8, we get

$$\begin{aligned}\operatorname{area}\mathcal{P} \leqslant \operatorname{area}[A_1B_1] + \dots + \operatorname{area}[A_nB_n] + \\ + \operatorname{area}\{C_1\} + \dots + \operatorname{area}\{C_k\}.\end{aligned}$$

By Proposition 19.9, the right hand side vanish.

On the other hand, by Proposition 19.8, $\operatorname{area}\mathcal{P} \geqslant 0$; hence the statement follows. ☐



We say that polygonal set $\mathcal{P}$ is *subdivided* into two polygonal sets $\mathcal{Q}_1$ and $\mathcal{Q}_2$ if $\mathcal{P} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ and the intersection $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is degenerate. (Recall that according to Claim 19.6, the set $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is polygonal.)

19.11. Proposition. *Assume polygonal sets $\mathcal{P}$ is subdivided into two polygonal set $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then*

$$\operatorname{area}\mathcal{P} = \operatorname{area}\mathcal{Q}_1 + \operatorname{area}\mathcal{Q}_2.$$

Proof. By Theorem 19.7,

$$\operatorname{area}\mathcal{P} = \operatorname{area}\mathcal{Q}_1 + \operatorname{area}\mathcal{Q}_2 - \operatorname{area}(\mathcal{Q}_1 \cap \mathcal{Q}_2).$$

Since $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is degenerate, by Corollary 19.10,

$$\operatorname{area}(\mathcal{Q}_1 \cap \mathcal{Q}_2) = 0.$$

Hence the result follows. ☐

Area of solid rectangles

19.12. Theorem. *The solid rectangle in the Euclidean plane with sides a and b has area $a{\cdot}b$.*

19.13. Algebraic lemma. *Assume that a function s returns a nonnegative real number $s(a,b)$ for any pair of positive real numbers (a,b) and satisfies the following identities*

$$\begin{aligned} s(1,1)&=1;\\ s(a,b+c)&=s(a,b)+s(a,c)\\ s(a+b,c)&=s(a,c)+s(b,c)\end{aligned}$$

for any $a,b,c>0$. Then

$$s(a,b)=a{\cdot}b$$

for any $a,b>0$.

The proof is similar to the proof of Lemma 13.8.

Proof. Note that if $a>a'$ and $b>b'$ then

$$s(a,b)\geqslant s(a',b'). \qquad ❶$$

Indeed, since s returns nonnegative numbers, we get

$$\begin{aligned} s(a,b)&=s(a',b)+s(a-a',b)\geqslant\\ &\geqslant s(a',b)=\\ &\geqslant s(a',b')+s(a',b-b')\geqslant\\ &\geqslant s(a',b').\end{aligned}$$

Applying the second and third identity few times we get that

$$s(a,m{\cdot}b)=s(m{\cdot}a,b)=m{\cdot}s(a,b)$$

for any positive integer m. Therefore

$$\begin{aligned} s(\tfrac{k}{l},\tfrac{m}{n})&=k{\cdot}s(\tfrac{1}{l},\tfrac{m}{n})=\\ &=k{\cdot}m{\cdot}s(\tfrac{1}{l},\tfrac{1}{n})=\\ &=k{\cdot}m{\cdot}\tfrac{1}{l}{\cdot}s(1,\tfrac{1}{n})=\\ &=k{\cdot}m{\cdot}\tfrac{1}{l}{\cdot}\tfrac{1}{n}{\cdot}s(1,1)=\\ &=\tfrac{k}{l}{\cdot}\tfrac{m}{n}\end{aligned}$$

for any positive integers k, l, m and n. That is, the needed identity holds for any pair of rational numbers $a = \frac{k}{l}$ and $b = \frac{m}{n}$.

Arguing by contradiction, assume $s(a, b) \neq a{\cdot}b$ for some pair of positive real numbers (a, b). We will consider two cases: $s(a, b) > a{\cdot}b$ and $s(a, b) < a{\cdot}b$.

If $s(a, b) > a{\cdot}b$, we can choose a positive integer n such that

❷ $$s(a, b) > (a + \tfrac{1}{n}){\cdot}(b + \tfrac{1}{n}).$$

Set $k = \lfloor a{\cdot}n \rfloor + 1$ and $m = \lfloor b{\cdot}n \rfloor + 1$; equivalently, k and m are positive integers such that

$$a < \tfrac{k}{n} \leqslant a + \tfrac{1}{n} \quad \text{and} \quad b < \tfrac{m}{n} \leqslant b + \tfrac{1}{n}.$$

By ❶, we get

$$\begin{aligned} s(a, b) &\leqslant s(\tfrac{k}{n}, \tfrac{m}{n}) = \\ &= \tfrac{k}{n} \cdot \tfrac{m}{n} \leqslant \\ &\leqslant (a + \tfrac{1}{n})(b + \tfrac{1}{n}), \end{aligned}$$

which contradicts ❷.

The case $s(a, b) < a{\cdot}b$ is similar. Fix a positive integer n such that $a > \frac{1}{n}$, $b > \frac{1}{n}$ and

❸ $$s(a, b) < (a - \tfrac{1}{n}){\cdot}(b - \tfrac{1}{n}).$$

Set $k = \lceil a{\cdot}n \rceil - 1$ and $m = \lceil b{\cdot}n \rceil - 1$; that is,

$$a > \tfrac{k}{n} \geqslant a - \tfrac{1}{n} \quad \text{and} \quad b > \tfrac{m}{n} \geqslant b - \tfrac{1}{n}.$$

Applying ❶ again, we get

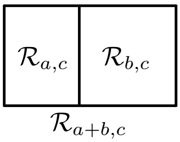
$$\begin{aligned} s(a, b) &\geqslant s(\tfrac{k}{n}, \tfrac{m}{n}) = \\ &= \tfrac{k}{n} \cdot \tfrac{m}{n} \geqslant \\ &\geqslant (a - \tfrac{1}{n})(b - \tfrac{1}{n}), \end{aligned}$$

which contradicts ❸. □

Proof of Theorem 19.12. Denote by $\mathcal{R}_{a,b}$ the solid rectangle with sides a and b. Set

$$s(a, b) = \operatorname{area} \mathcal{R}_{a,b}.$$

By theorem 19.7, $s(1, 1) = 1$. That is, the first identity in the Algebraic Lemma holds.



Note that the rectangle $\mathcal{R}_{a+b,c}$ can be subdivided into two rectangle congruent to $\mathcal{R}_{a,c}$ and $\mathcal{R}_{b,c}$. Therefore, by Proposition 19.11,

$$\operatorname{area}\mathcal{R}_{a+b,c} = \operatorname{area}\mathcal{R}_{a,c} + \operatorname{area}\mathcal{R}_{b,c}$$

That is, the second identity in the algebraic lemma holds. The proof of the third identity is analogues.

It remains to apply the algebraic lemma. $\square$

Area of solid parallelograms

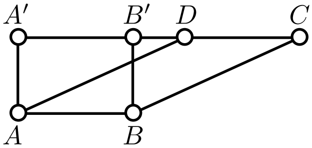
19.14. Proposition. *Let $\square ABCD$ be a parallelogram in the Euclidean plane, $a = AB$ and h be the distance between the lines (AB) and (CD). Then*

$$\operatorname{area}(\blacksquare ABCD) = a{\cdot}h.$$

Proof. Let us denote by A' and B' the foot points of A and B on the line (CD).

Note that $ABB'A'$ is a rectangle with sides a and h. By Proposition 19.12,

$$\operatorname{area}(\blacksquare ABB'A') = h{\cdot}a. \qquad ❹$$



Without loss of generality, we may assume that $\blacksquare ABCA'$ contains $\blacksquare ABCD$ and $\blacksquare ABB'A'$.

In this case $\blacksquare ABB'D$ admits two subdivisions. First into $\blacksquare ABCD$ and $\blacktriangle AA'D$. Second into $\blacksquare ABB'A'$ and $\blacktriangle BB'C$.

By Proposition 19.11,

$$\begin{aligned}\operatorname{area}(\blacksquare ABCD) + \operatorname{area}(\blacktriangle AA'D) &= \\ = \operatorname{area}(\blacksquare ABB'A') &+ \operatorname{area}(\blacktriangle BB'C).\end{aligned} \qquad ❺$$

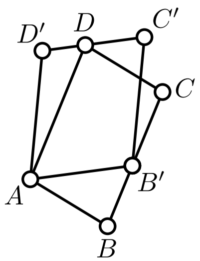
Note that

$$\triangle AA'D \cong \triangle BB'C. \qquad ❻$$

Indeed, since both $\square ABB'A'$ and $\square ABCD$ are parallelograms, by Lemma 6.22, we have $AA' = BB'$, $AD = BC$ and $DC = AB = A'B'$. It follows that $A'D = B'C$. Applying the SSS congruence condition, we get ❻.

In particular,

$$\operatorname{area}(\blacktriangle BB'C) = \operatorname{area}(\blacktriangle AA'D). \qquad ❼$$



Subtracting ❼ from ❺, we get

$$\operatorname{area}(\blacksquare ABCD) = \operatorname{area}(\blacksquare ABB'D).$$

From ❹, the statement follows. $\square$

19.15. Exercise. *Assume $\square ABCD$ and $\square AB'C'D'$ are two parallelograms such that $B' \in [BC]$ and $D \in$ $\in [C'D']$. Show that*

$$\operatorname{area}(\blacksquare ABCD) = \operatorname{area}(\blacksquare AB'C'D').$$

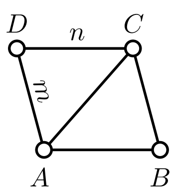
Area of solid triangles

19.16. Theorem. *Let $a = BC$ and h_A to be the altitude from A in $\triangle ABC$. Then*

$$\operatorname{area}(\blacktriangle ABC) = \tfrac{1}{2} \cdot a \cdot h_A.$$

Remark. It is acceptable to write $\operatorname{area}(\triangle ABC)$ for $\operatorname{area}(\blacktriangle ABC)$, since $\triangle ABC$ completely determines the solid triangle $\blacktriangle ABC$.

Proof. Draw the line m thru A which is parallel to (BC) and line n thru C parallel to (AB). Note that $m \nparallel n$; set $D = m \cap n$. By construction, $\square ABCD$ is a parallelogram.



Note that $\blacksquare ABCD$ admits a subdivision into $\blacktriangle ABC$ and $\blacktriangle CDA$. Therefore,

$$\operatorname{area}(\blacksquare ABCD) = \operatorname{area}(\blacktriangle ABC) + \operatorname{area}(\blacktriangle CDA)$$

Since $\square ABCD$ is a parallelogram, Lemma 6.22 implies that

$$AB = CD \quad \text{and} \quad BC = DA.$$

Therefore, by the SSS congruence condition, we have $\triangle ABC \cong$ $\cong \triangle CDA$. In particular

$$\operatorname{area}(\blacktriangle ABC) = \operatorname{area}(\blacktriangle CDA).$$

From above and Proposition 19.14, we get

$$\begin{aligned}\operatorname{area}(\blacktriangle ABC) &= \tfrac{1}{2} \cdot \operatorname{area}(\blacksquare ABCD) = \\ &= \tfrac{1}{2} \cdot h_A \cdot a\end{aligned}$$

$\square$

19.17. Exercise. *Denote by h_A, h_B and h_C the altitudes of $\triangle ABC$ from vertices A, B and C correspondingly. Note that from Theorem 19.16, it follows that*

$$h_A{\cdot}BC = h_B{\cdot}CA = h_C{\cdot}AB.$$

Give a proof of this statement without using area.

19.18. Exercise. *Assume M lies inside the parallelogram $ABCD$; that is, M belongs to the solid parallelogram $\blacksquare ABCD$, but does not lie on its sides. Show that*

$$\operatorname{area}(\blacktriangle ABM) + \operatorname{area}(\blacktriangle CDM) = \tfrac{1}{2}\cdot \operatorname{area}(\blacksquare ABCD).$$

19.19. Exercise. *Assume that diagonals of a nondegenerate quadrilateral $ABCD$ intersect at point M. Show that*

$$\operatorname{area}(\blacktriangle ABM)\cdot \operatorname{area}(\blacktriangle CDM) = \operatorname{area}(\blacktriangle BCM)\cdot \operatorname{area}(\blacktriangle DAM).$$

19.20. Exercise. *Let r be the inradius of $\triangle ABC$ and p be its* semiperimeter*; that is, $p = \frac{1}{2}{\cdot}(AB + BC + CA)$. Show that*

$$\operatorname{area}(\blacktriangle ABC) = p{\cdot}r.$$

19.21. Advanced exercise. *Show that for any affine transformation β there is a constant $k > 0$ such that the equality*

$$\operatorname{area}[\beta(\blacktriangle)] = k\cdot \operatorname{area}\blacktriangle.$$

holds for any solid triangle $\blacktriangle$.

Moreover, if β has the matrix form $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$, then

$$k = \left|\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right| = |a{\cdot}d - b{\cdot}c|.$$

Area method

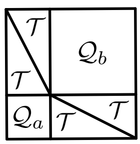
In this section we will give examples of slim proofs using area. Note that these proofs are not truly elementary since the price one pays to introduce the area function is high.

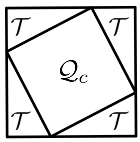
We start with the proof of the Pythagorean theorem. In the Elements of Euclid, Pythagorean theorem was formulated as equality ❽ below and the proof used a similar technique.

Proof. We need to show that if a and b are legs and c is the hypotenuse of a right triangle, then
$$a^2 + b^2 = c^2.$$

Denote by $\mathcal{T}$ the right solid triangle with legs a and b and by $\mathcal{Q}_x$ be the solid square with side x.

Let us construct two subdivisions of $\mathcal{Q}_{a+b}$.





1. Subdivide $\mathcal{Q}_{a+b}$ into two solid squares congruent to $\mathcal{Q}_a$ and $\mathcal{Q}_b$ and 4 solid triangles congruent to $\mathcal{T}$, see the left diagram.

2. Subdivide $\mathcal{Q}_{a+b}$ into one solid square congruent to $\mathcal{Q}_c$ and 4 solid right triangles congruent to $\mathcal{T}$, see the right diagram.

Applying Proposition 19.11 few times, we get.

$$\begin{aligned}\operatorname{area}\mathcal{Q}_{a+b} &= \operatorname{area}\mathcal{Q}_a + \operatorname{area}\mathcal{Q}_b + 4\cdot\operatorname{area}\mathcal{T} = \\ &= \operatorname{area}\mathcal{Q}_c + 4\cdot\operatorname{area}\mathcal{T}.\end{aligned}$$

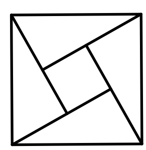
Therefore,

❽ $$\operatorname{area}\mathcal{Q}_a + \operatorname{area}\mathcal{Q}_b = \operatorname{area}\mathcal{Q}_c.$$

By Theorem 19.12,
$$\operatorname{area}\mathcal{Q}_x = x^2,$$
for any $x > 0$. Hence the statement follows. □

19.22. Exercise. *Build another proof of Pythagorean theorem based on the diagram.*

(In the notations above it shows a subdivision of $\mathcal{Q}_c$ into $\mathcal{Q}_{a-b}$ and four copies of $\mathcal{T}$ if $a > b$.)



19.23. Exercise. *Show that the sum of distances from a point to the sides of an equilateral triangle is the same for all points inside the triangle.*

Let us prove of Lemma 7.7 using the area method. That is, we need to show that if $\triangle ABC$ is nondegenerate and the bisector of $\angle BAC$ intersects $[BC]$ at the point D. Then
$$\frac{AB}{AC} = \frac{DB}{DC}.$$

In the proof we will use the following claim.

19.24. Claim. *Assume that two triangles ABC and $A'B'C'$ in the Euclidean plane have equal altitudes dropped from A and A' correspondingly. Then*

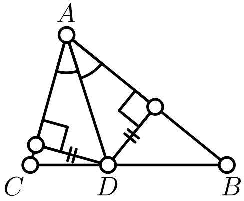
$$\frac{\operatorname{area}(\blacktriangle A'B'C')}{\operatorname{area}(\blacktriangle ABC)}=\frac{B'C'}{BC}.$$

In particular, the same identity holds if $A=A'$ and the bases $[BC]$ and $[B'C']$ lie on one line.

Proof. Let h be the altitude. By Theorem 19.16,

$$\frac{\operatorname{area}(\blacktriangle A'B'C')}{\operatorname{area}(\blacktriangle ABC)}=\frac{\frac12\cdot h\cdot B'C'}{\frac12\cdot h\cdot BC}=\frac{B'C'}{BC}.$$

□



Proof of Lemma 7.7. Applying Claim 19.24, we get

$$\frac{\operatorname{area}(\blacktriangle ABD)}{\operatorname{area}(\blacktriangle ACD)}=\frac{BD}{CD}.$$

By Lemma 5.13 the triangles ABD and ACD have equal altitudes from D. Applying Claim 19.24 again, we get

$$\frac{\operatorname{area}(\blacktriangle ABD)}{\operatorname{area}(\blacktriangle ACD)}=\frac{AB}{AC}.$$

Hence the result follows. □

19.25. Exercise. *Assume that the point X lies inside a nondegenerate triangle ABC. Show that X lies on the median from A if and only if*

$$\operatorname{area}(\blacktriangle ABX)=\operatorname{area}(\blacktriangle ACX).$$

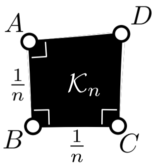
19.26. Exercise. *Build a proof of Theorem 7.5 based on the Exercise 19.25.*

Namely, show that medians of nondegenerate triangle intersect at one point and the point of their intersection divides each median in the ratio 1:2.

Area in the neutral planes and spheres

An analog of Theorem 19.7 holds in the neutral planes and spheres. In the formulation of this theorem, the solid unit square $\mathcal{K}$ has to be exchanged to a fixed nondegenerate polygonal set. One has to make such change for good reason — hyperbolic plane and sphere have no unit squares.

The set $\mathcal{K}$ in this case plays role of the unit measure for the area and changing $\mathcal{K}$ will require conversion of area units.



According to the standard convention, the set $\mathcal{K}$ is taken so that on small scales area behaves like area in the Euclidean plane. Say, if $\mathcal{K}_n$ denotes the solid quadrilateral $\blacksquare ABCD$ with right angles at A, B and C such that $AB = BC = \frac{1}{n}$, then we may assume that

$$n^2\cdot \operatorname{area}\mathcal{K}_n \to 1 \quad \text{as} \quad n \to \infty. \tag{9}$$

This convention works equally well for spheres and neutral planes, including Euclidean plane. In spherical geometry equivalently we may assume that if r is the radius of the sphere, then the area of whole sphere is $4\cdot\pi\cdot r^2$.

Recall that *defect of triangle* $\triangle ABC$ is defined as

$$\operatorname{defect}(\triangle ABC) := \pi - |\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB|.$$

It turns out that any neutral plane or sphere there is a real number k such that

$$k\cdot \operatorname{area}(\blacktriangle ABC) + \operatorname{defect}(\triangle ABC) = 0 \tag{10}$$

for any $\triangle ABC$.

This number k is called *curvature*; $k = 0$ for the Euclidean plane, $k = -1$ for the h-plane and $k = 1$ for the unit sphere and $k = \frac{1}{r^2}$ for the sphere of radius r.

In particular, it follows that any ideal triangle in h-plane has area π. Similarly in the unit sphere the area of equilateral triangle with right angles has area $\frac{\pi}{2}$; since whole sphere can be subdivided in 8 such triangles, we get that the area of unit sphere is $4\cdot\pi$.

The identity ❿ suggest a simpler way to introduce area function which works on spheres and all neutral planes, but does not work for the Euclidean plane.

Quadrable sets

A set $\mathcal{S}$ in the plane is called *quadrable* if for any $\varepsilon > 0$ there are two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\mathcal{P} \subset \mathcal{S} \subset \mathcal{Q} \quad \text{and} \quad \operatorname{area}\mathcal{Q} - \operatorname{area}\mathcal{P} < \varepsilon.$$

If $\mathcal{S}$ is quadrable, its area can be defined as the necessary unique real number $s = \operatorname{area}\mathcal{S}$ such that the inequality

$$\operatorname{area}\mathcal{Q} \leqslant s \leqslant \operatorname{area}\mathcal{P}$$

holds for any polygonal sets $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P} \subset \mathcal{S} \subset \mathcal{Q}$.

19.27. Exercise. *Let $\mathcal{D}$ be the unit disk; that is, $\mathcal{D}$ is a set which contains the unit circle Γ and all the points inside Γ.*

Show that $\mathcal{D}$ is a quadrable set.

Since $\mathcal{D}$ is quadrable, the expression $\operatorname{area}\mathcal{D}$ makes sense and the constant π can be defined as $\pi = \operatorname{area}\mathcal{D}$.

It turns out that the class of quadrable sets is the largest class for which the area can be defined in such a way that it satisfies all the conditions in Theorem 19.7 incliding uniqueness.

There is a way to define area for all bounded sets which satisfies all the conditions in the Theorem 19.7 excluding uniqueness. (A set in the plane is called *bounded* if it lies inside some circle.)

In the hyperbolic plane and in the sphere there is no similar construction. Read about Hausdorff–Banach–Tarski Paradox if you wonder why.

Hints

Chapter 1

Exercise 1.2. Only the triangle inequality requires a proof — the rest of conditions in Definition 1.1 are evident. Let $A=(x_A,y_A)$, $B=(x_B,y_B)$ and $C=(x_C,y_C)$. Set

$$\begin{aligned} x_1 &= x_B - x_A, & y_1 &= y_B - y_A,\\ x_2 &= x_C - x_B, & y_2 &= y_C - y_B. \end{aligned}$$

(a). The inequality

$$d_1(A,C) \leqslant d_1(A,B) + d_1(B,C)$$

can be written as

$$|x_1+x_2| + |y_1+y_2| \leqslant |x_1| + |y_1| + |x_2| + |y_2|.$$

The latter follows since $|x_1+x_2| \leqslant |x_1|+|x_2|$ and $|y_1+y_2| \leqslant |y_1|+|y_2|$.

(b). The inequality

❶ $$d_2(A,C) \leqslant d_2(A,B) + d_2(B,C)$$

can be written as

$$\sqrt{\left(x_1+x_2\right)^2 + \left(y_1+y_2\right)^2} \leqslant \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}.$$

Take the square of the left and the right hand sides, simplify, take the square again and simplify again. You should get the following inequality

$$0 \leqslant (x_1\cdot y_2 - x_2\cdot y_1)^2.$$

which is equivalent to ❶ and evidently true.

(c). The inequality

$$d_\infty(A,C) \leqslant d_\infty(A,B) + d_\infty(B,C)$$

can be written as

$$❷ \qquad \max\{|x_1 + x_2|, |y_1 + y_2|\} \leqslant \max\{|x_1|, |y_1|\} + \max\{|x_2|, |y_2|\}.$$

Without loss of generality, we may assume that

$$\max\{|x_1 + x_2|, |y_1 + y_2|\} = |x_1 + x_2|.$$

Further,

$$|x_1 + x_2| \leqslant |x_1| + |x_2| \leqslant \max\{|x_1|, |y_1|\} + \max\{|x_2|, |y_2|\}.$$

Hence ❷ follows.

Exercise 1.4. If $A \neq B$, then $d_{\mathcal{X}}(A, B) > 0$. Since f is distance-preserving,

$$d_{\mathcal{Y}}(f(A), f(B)) = d_{\mathcal{X}}(A, B).$$

Therefore, $d_{\mathcal{Y}}(f(A), f(B)) > 0$ and hence $f(A) \neq f(B)$.

Exercise 1.5. Set $f(0) = a$ and $f(1) = b$. Note that $b = a + 1$ or $a - 1$. Moreover, $f(x) = a \pm x$ and at the same time, $f(x) = b \pm (x - 1)$ for any x.

If $b = a + 1$, it follows that $f(x) = a + x$ for any x.

The same way, if $b = a - 1$, it follows that $f(x) = a - x$ for any x.

Exercise 1.6. Show that the map $(x, y) \mapsto (x + y, x - y)$ is an isometry $(\mathbb{R}^2, d_1) \to (\mathbb{R}^2, d_\infty)$. That is, you need to check if this map is bijective and distance-preserving.

Exercise 1.7. First prove that *two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ on the Manhattan plane have a unique midpoint if and only if $x_A = x_B$ or $y_A = y_B$*; compare with the example on page 17.

Then use above statement to prove that any motion of the Manhattan plane can be written in one of the following two ways

$$(x, y) \mapsto (\pm x + a, \pm y + b) \qquad \text{or} \qquad (x, y) \mapsto (\pm y + b, \pm x + a),$$

for some fixed real numbers a and b. (In each case we have 4 choices of signs, so for fixed pair (a, b) we have 8 distinct motions.)

Exercise 1.9. Assume three points A, B and C lie on one line. Note that in this case one of the triangle inequalities with the points A, B and C becomes equality.

Set $A = (-1, 1)$, $B = (0, 0)$ and $C = (1, 1)$. Show that for d_1 and d_2 all the triangle inequalities with the points A, B and C are strict. It follows that the graph is not a line.

For d_∞ show that $(x, |x|) \mapsto x$ gives the isometry of the graph to $\mathbb{R}$. Conclude that the graph is a line in $(\mathbb{R}^2, d_\infty)$.

Exercise 1.10. Applying the definition of the line, the problems are reduced to the following.

Assume that $a \neq b$, find the number of solutions for each of the following two equations

$$|x-a| = |x-b| \qquad \text{and} \qquad |x-a| = 2{\cdot}|x-b|.$$

Each can be solved by taking the square of the left and the right hand sides. The numbers of solutions are 1 and 2 correspondingly.

Exercise 1.11. Fix an isometry $f\colon (PQ) \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) = q > 0$.

Assume that $f(X) = x$. By the definition of the half-line $X \in [PQ)$ if and only if $x \geqslant 0$. The latter holds if and only if

$$|x-a| = |x| - |a|.$$

Hence the result follows.

Exercise 1.12. The equation $2{\cdot}\alpha \equiv 0$ means that $2{\cdot}\alpha = 2{\cdot}k{\cdot}\pi$ for some integer k. Therefore, $\alpha = k{\cdot}\pi$ for some integer k.

Equivalently, $\alpha = 2{\cdot}n{\cdot}\pi$ or $\alpha = (2{\cdot}n{+}1){\cdot}\pi$ for some integer n. The first identity means that $\alpha \equiv 0$ and the second means $\alpha \equiv \pi$.

Exercise 1.13. *(a).* By the triangle inequality,

$$|f(A') - f(A)| \leqslant d(A', A).$$

Therefore, we can take $\delta = \varepsilon$.

(b). By the triangle inequality,

$$\begin{aligned}|f(A',B') - f(A,B)| &\leqslant |f(A',B') - f(A,B')| + |f(A,B') - f(A,B)| \leqslant \\ &\leqslant d(A',A) + d(B',B)\end{aligned}$$

Therefore, we can take $\delta = \frac{\varepsilon}{2}$.

Exercise 1.14. Fix $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ such that $f(A) = B$.

Fix $\varepsilon > 0$. Since g is continuous at B, there is a positive value δ_1 such that

$$d_{\mathcal{Z}}(g(B'), g(B)) < \varepsilon \quad \text{if} \quad d_{\mathcal{Y}}(B', B) < \delta_1.$$

Since f is continuous at A, there is $\delta_2 > 0$ such that

$$d_{\mathcal{Y}}(f(A'), f(A)) < \delta_1 \quad \text{if} \quad d_{\mathcal{X}}(A', A) < \delta_2.$$

Since $f(A) = B$, we get

$$d_{\mathcal{Z}}(h(A'), h(A)) < \varepsilon \quad \text{if} \quad d_{\mathcal{X}}(A', A) < \delta_2.$$

Hence the result follows.

Chapter 2

Exercise 2.1. By Axiom I, there are at least two points in the plane. Therefore, by Axioms II, the plane contains a line. It remains to note that line is an infinite set of points.

Exercise 2.3. By Axiom II, $(OA) = (OA')$. Therefore, the statement boils down two the following.

Assume $f\colon \mathbb{R} \to \mathbb{R}$ is a motion of the line which sends $0 \to 0$ and one positive number to a positive number, then f is an identity map.

The latter follows from Exercise 1.5.

Exercise 2.6. By Proposition 2.5, $\measuredangle AOA = 0$. It remains to apply Axiom IIIa.

Exercise 2.10. Apply Proposition 2.5, Theorem 2.8 and Exercise 1.12.

Exercise 2.11. By Axiom IIIb,

$$2{\cdot}\measuredangle BOC \equiv 2{\cdot}\measuredangle AOC - 2{\cdot}\measuredangle AOB \equiv 0.$$

By Exercise 1.12, it implies that $\measuredangle BOC = 0$ or π. It remains to apply Exercise 2.6 and Theorem 2.8 correspondingly in these two cases.

Exercise 2.13. Applying Proposition 2.12, we get $\measuredangle AOC = \measuredangle BOD$. It remains to apply Axiom IV.

Chapter 3

Exercise 3.1. Set $\alpha = \measuredangle AOB$ and $\beta = \measuredangle BOA$. Note that $\alpha = \pi$ if and only if $\beta = \pi$. Otherwise $\alpha = -\beta$. Hence the result follows.

Exercise 3.2. Set $\alpha = \measuredangle AOX$ and $\beta = \measuredangle BOX$. Since $\angle AOB$ is straight,

❶ $$\alpha - \beta \equiv \pi.$$

It follows that $\alpha = \pi \Leftrightarrow \beta = 0$ and $\alpha = 0 \Leftrightarrow \beta = \pi$.

In the remaining cases we have $|\alpha|, |\beta| < \pi$. If α and β have the same sign, then $|\alpha - \beta| < \pi$ which contradicts ❶.

Exercise 3.3. Set $\alpha = \measuredangle BOC$, $\beta = \measuredangle COA$ and $\gamma = \measuredangle AOB$. By Axiom IIIb and Proposition 2.5

❷ $$\alpha + \beta + \gamma \equiv 0$$

Note that $0 < \alpha + \beta < 2{\cdot}\pi$ and $|\gamma| \leqslant \pi$. If $\gamma > 0$, then ❷ implies

$$\alpha + \beta + \gamma = 2{\cdot}\pi$$

and if $\gamma < 0$, then ❷ implies

$$\alpha + \beta + \gamma = 0.$$

Exercise 3.9. Note that O and A' lie on the same side from (AB). Analogously O and B' lie on the same side from (AB). Hence the result follows.

Exercise 3.12. Apply Theorem 3.11 for $\triangle PQX$ and $\triangle PQY$ and then apply Proposition 3.8*a*.

Exercise 3.13. Note that it is sufficient to consider the cases when $A' \neq \neq B, C$ and $B' \neq A, C$.

Apply Pasch's theorem (3.10) twice; for $\triangle AA'C$ with (BB') and for $\triangle BB'C$ with (AA').

Exercise 3.14. Assume that Z is the point of intersection.

Note that $Z \neq P$ and $Z \neq Q$. Therefore, $Z \notin (PQ)$.

Show that Z and X lie on one side from (PQ). Repeat the argument to show that Z and Y lie on one side from (PQ). It follows that X and Y lie on the same side from (PQ), a contradiction.

Chapter 4

Exercise 4.3. Apply Theorem 4.2 twice.

Exercise 4.5. Consider the points D and D', such that M is the midpoint of $[AD]$ and M' is the midpoint of $[A'D']$. Show first that $\triangle ABD \cong \triangle A'B'D'$.

Exercise 4.6. *(a)* Apply SAS.

(b) Use *(a)* and apply SSS.

Exercise 4.7. Choose $B' \in [AC]$ such that $AB = AB'$. Note that $BC = = B'C$. By SSS, $\triangle ABC \cong AB'C$.

Exercise 4.8. Without loss of generality, we may assume that X is distinct from A, B and C. Set $f(X) = X'$; assume $X' \neq X$.

Note that $AX = AX'$, $BX = BX'$ and $CX = CX'$. By SSS we get $\measuredangle ABX = \pm\measuredangle ABX'$. Since $X \neq X'$, we get

$$\measuredangle ABX \equiv -\measuredangle ABX'.$$

The same way we get

$$\measuredangle CBX \equiv -\measuredangle CBX'.$$

Subtracting these two identities from each other, we get

$$\measuredangle ABC \equiv -\measuredangle ABC.$$

Conclude that $\measuredangle ABC = 0$ or π. That is, $\triangle ABC$ is degenerate, a contradiction.

Chapter 5

Exercise 5.1. By Axiom IIIb and Theorem 2.8

$$\measuredangle XOA - \measuredangle XOB \equiv \pi.$$

Since $|\measuredangle XOA|, |\measuredangle XOB| \leqslant \pi$, we get

$$|\measuredangle XOA| + |\measuredangle XOB| = \pi.$$

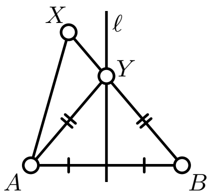
Hence the statement follows.

Exercise 5.3. Assume X and A lie on the same side from ℓ.

Note that A and B lie on the opposite sides of ℓ. Therefore, by Proposition 3.8, $[AX]$ does not intersect ℓ and $[BX]$ intersects ℓ, say at the point Y.

Note that $Y \notin [AX]$. By Exercise 4.7,

$$BX = AY + YX > AX.$$



This way we proved the "if" part. To prove the "only if" part, it remains to switch A and B, repeat the above argument and yet apply Theorem 5.2.

Exercise 5.4. Apply Exercise 5.3, Theorem 4.1 and Exercise 3.3.

Exercise 5.8. Choose arbitrary nondegenerate triangle ABC. Denote by $\triangle\hat{A}\hat{B}\hat{C}$ its image after the motion.

If $A \neq \hat{A}$, apply the reflection in the perpendicular bisector of $[A\hat{A}]$. This reflection sends A to $\hat{A}$. Denote by B' and C' the reflections of B and C correspondingly.

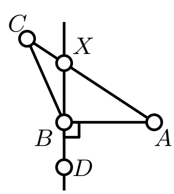
If $B' \neq \hat{B}$, apply the reflection in the perpendicular bisector of $[B'\hat{B}]$. This reflection sends B' to $\hat{B}$. Note that $\hat{A}\hat{B} = \hat{A}B'$; that is, $\hat{A}$ lies on the perpendicular bisector. Therefore, $\hat{A}$ reflects to itself. Denote by C'' the reflection of C'.

Finally, if $C'' \neq \hat{C}$, apply the reflection in $(\hat{A}\hat{B})$. Note that $\hat{A}\hat{C} = \hat{A}C''$ and $\hat{B}\hat{C} = \hat{B}C''$; that is, (AB) is the perpendicular bisector of $[C''\hat{C}]$. Therefore, this reflection sends C'' to $\hat{C}$.

Apply Exercise 4.8 to show that the composition of constructed reflections coincides with the given motion.

Exercise 5.9. Note that $\measuredangle XBA = \measuredangle ABP$, $\measuredangle PBC = \measuredangle CBY$. Therefore,

$$\measuredangle XBY \equiv \measuredangle XBP + \measuredangle PBY \equiv 2{\cdot}(\measuredangle ABP + \measuredangle PBC) \equiv 2{\cdot}\measuredangle ABC.$$



Exercise 5.11. If $\angle ABC$ is right, the statement follows from Lemma 5.10. Therefore, we can assume that $\angle ABC$ is obtuse.

Draw a line (BD) perpendicular to (BC). Since $\angle ABC$ is obtuse, the angles DBA and DBC have opposite signs.

By Proposition 3.8, A and C lie on the opposite sides from (AD). In particular, $[AC]$ intersects (BD) at a point; denote it by X.

Note that $AX < AC$ and by Lemma 5.10, $AB \leqslant AX$. Hence the result follows.

Exercise 5.12. Let (BX) and (BY) be internal and external bisectors of $\angle ABC$. Then

$$\begin{aligned} 2\cdot\measuredangle XBY &\equiv 2\cdot\measuredangle XBA + 2\cdot\measuredangle ABY \equiv \\ &\equiv \measuredangle CBA + \pi + 2\cdot\measuredangle ABC \equiv \\ &\equiv \pi + \measuredangle CBC = \pi. \end{aligned}$$

Hence the result follows.

Exercise 5.14. Let O be the center of the circle. Note that we can assume that $O \neq P$.

Assume P lies between X and Y. By Exercise 5.1, we can assume that $\angle OPX$ is right or obtuse. By Exercise 5.11, $OP < OX$; that is, P lies inside Γ.

If P does not lie between X and Y, we can assume that X lies between P and Y. Since $OX = OY$, Exercise 5.11 implies that $\angle OXY$ is acute. Therefore, $\angle OXP$ is obtuse. Applying Exercise 5.11 again we get $OP > OX$; that is, P lies outside Γ.

Exercise 5.15. Apply Theorem 5.2.

Exercise 5.17. Use Exercise 5.15 and Theorem 5.5.

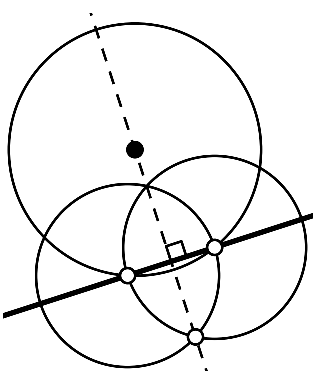
Exercise 5.19. Let P' be the reflection of P in (OO'). Note that P' lies on both circles and $P' \neq P$ if and only if $P \notin (OO')$.

Exercise 5.20. *(a)* Apply Exercise 5.19.

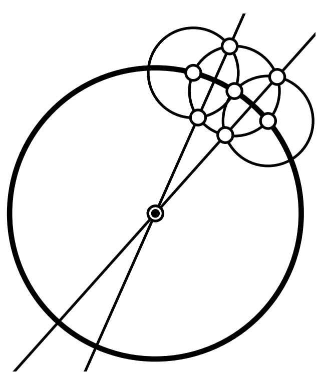
(b) Apply Theorem 3.16.

Exercise 5.21. Let A and B be the points of intersection. Note that the centers lie on the perpendicular bisector of the segment $[AB]$.

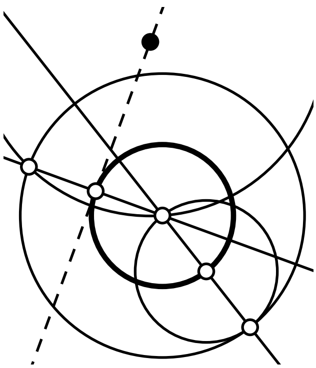
Exercise 5.23.



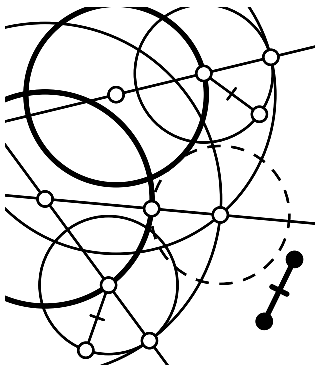
Exercise 5.24.



Exercise 5.25.



Exercise 5.26.



Chapter 6

Exercise 6.4. Show first that $k \perp n$.

Exercise 6.5. Repeat twice the construction in Exercise 5.23.

Exercise 6.8. By the AA similarity condition, the transformation multiplies the sides of any nondegenerate triangle by some number which may depend on the triangle.

Note that for any two nondegenerate triangles which share one side this number is the same. Applying this observation to a chain of triangles leads to a solution.

Exercise 6.9. First show that $\triangle AA'C \sim \triangle BB'C$.

Exercise 6.11. Apply that $\triangle ADC \sim \triangle CDB$.

Exercise 6.12. Apply Pythagorean theorem (6.10) and the SSS congruence condition.

Exercise 6.14. Apply twice Theorem 4.2 and twice Theorem 6.13.

Exercise 6.15. If $\triangle ABC$ is degenerate, then one of the angle measures is π and the other two are 0. Hence the result follows.

Assume $\triangle ABC$ is nondegenerate. Set $\alpha = \measuredangle CAB$, $\beta = \measuredangle ABC$ and $\gamma = \measuredangle BCA$.

By Theorem 3.11, we may assume that $0 < \alpha, \beta, \gamma < \pi$. Therefore,

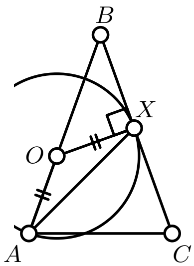
❶ $$0 < \alpha + \beta + \gamma < 3 \cdot \pi.$$

By Theorem 6.13,

❷ $$\alpha + \beta + \gamma \equiv \pi.$$

From ❶ and ❷ the result follows.

Exercise 6.16. Apply twice Theorem 4.2 and once Theorem 6.13.



Exercise 6.17. Denote by O the center of the circle.

Note that $\triangle AOX$ is isosceles and $\angle OXC$ is right. Appling Theorems 6.13 and 4.2 and simplifyiing you should get

$$4\cdot\measuredangle CAX \equiv \pi.$$

Show that $\angle CAX$ has to be acute. It follows then that $\measuredangle CAX = \pm\frac{\pi}{4}$.

Exercise 6.19. By the transversal property 6.18,

$$\measuredangle B'BC \equiv \pi - \measuredangle C'B'B.$$

Since B' lies between A and B, we get $\measuredangle ABC = \measuredangle B'BC$ and $\measuredangle AB'C' + \measuredangle C'B'B \equiv \pi$. Hence $\measuredangle ABC = \measuredangle AB'C'$.

The same way we can prove that $\measuredangle BCA = \measuredangle B'C'A$. It remains to apply the AA similarity condition.

Exercise 6.20. Assume we need to trisect segment $[AB]$. Construct a line $\ell \neq (AB)$ with four points A, C_1, C_2, C_3 such that C_1 and C_2 trisect $[AC_3]$. Draw the line (BC_3) and draw parallel lines thru C_1 and C_2. The points of intersections of these two lines with (AB) trisect the segment $[AB]$.

Exercise 6.21. Apply Theorem 6.13 to $\triangle ABC$ and $\triangle BDA$.

Exercise 6.23. Since $\triangle ABC$ is isosceles, $\measuredangle CAB = \measuredangle BCA$.

By SSS, $\triangle ABC \cong \triangle CDA$. Therefore,

$$\pm\measuredangle DCA = \measuredangle BCA = \measuredangle CAB.$$

Since $D \neq C$ and we get "$-$" in the last formula. In particular,

$$-\measuredangle DCA \equiv \measuredangle CAB.$$

It follows that $(AB) \parallel (CD)$.

Exercise 6.24. Apply Lemma 6.22 together with the transversal property (6.18) to the diagonals and a pair of opposite sides. After that use the ASA-congruence condition (4.1).

Exercise 6.25. By Lemma 6.22 and SSS,

$$AC = BD \quad\iff\quad \measuredangle ABC = \pm\measuredangle BCD.$$

By the transversal property (6.18),

$$\measuredangle ABC + \measuredangle BCD \equiv \pi.$$

Therefore,

$$AC = BD \quad\iff\quad \measuredangle ABC = \measuredangle BCD = \pm\tfrac{\pi}{2}.$$

Exercise 6.26. Fix a parallelogram $ABCD$. By Exercise 6.24, its diagolals $[AC]$ and $[BD]$ have common midpoint, say M.

Use SSS and Lemma 6.22 to show that

$$AB = CD \quad \Longleftrightarrow \quad \triangle AMB \cong \triangle AMD \quad \Longleftrightarrow \quad \measuredangle AMB = \pm\tfrac{\pi}{2}.$$

Exercise 6.27. *(a).* Use the uniqueness of parallel line (Theorem 6.2).

(b) Use Lemma 6.22 and the Pythagorean theorem (6.10).

Exercise 6.28. Set $A = (0,0)$, $B = (c,0)$ and $C = (x,y)$. Clearly, $AB = c$, $AC^2 = x^2 + y^2$ and $BC^2 = (c-x)^2 + y^2$.

It remains to show that the there is a pair of real numbers (x,y) which satisfy the following system of equations

$$\begin{cases} b^2 = x^2 + y^2 \\ a^2 = (c-x)^2 + y^2 \end{cases}$$

if $0 < a \leqslant b \leqslant c \leqslant a + c$. □

Exercise 6.29. Without loss of generality, we can assume that $x_A \ne x_B$; otherwise switch x and y.

Denote by ℓ the set of points with coordinates (x,y) satisfying

$$(x - x_A)\cdot(y_B - y_A) = (y - y_A)\cdot(x_B - x_A).$$

Note that $A, B \in \ell$

Show that the map $\ell \to \mathbb{R}$ defined as $(x,y) \mapsto \frac{AB}{|x_A - x_B|}\cdot x$ is an isometry; that is, ℓ is a line. It remains to apply Axiom II.

Chapter 7

Exercise 7.2. Apply Theorem 7.1 and Theorem 5.2.

Exercise 7.4. Note that $(AC) \perp (BH)$ and $(BC) \perp (AH)$ and apply Theorem 7.3.

Exercise 7.6. Use the idea from the proof of Theorem 7.5 to show that $(XY) \parallel (AC) \parallel (VW)$ and $(XV) \parallel (BD) \parallel (YW)$.

Exercise 7.8. If E is the point of intersection of (BC) with the external bisector of $\angle BAC$, then

$$\frac{AB}{AC} = \frac{EB}{EC}.$$

It can be proved along the same lines as Lemma 7.7.

Exercise 7.10. Apply Lemma 7.7. See also the solution of Exercise 10.2.

Exercise 7.11. Apply ASA for the two triangles which the bisector cuts from the original triangle.

Exercise 7.12. Let I be the incenter. By SAS, we get $\triangle AIZ \cong \triangle AIY$. Therefore, $AY = AZ$. The same way we get $BX = BZ$ and $CX = CY$. Hence the result follows.

Exercise 7.13. Let $\triangle ABC$ be the given acute triangle and $\triangle A'B'C'$ be its orthic triangle. Apply Exercise 6.9 to show that $\measuredangle A'B'C = \measuredangle AB'C'$. Conclude that (BB') is bisecting $\angle A'B'C'$.

If $\triangle ABC$ is obtuse, then orthocenter coincides with one of the *excenters* of $\triangle ABC$; that is, the point of intersection of two external and one internal bisectors of $\triangle ABC$.

Chapter 8

Exercise 8.3. *(a).* Apply Theorem 8.2 for $\angle XX'Y$ and $\angle X'YY'$ and Theorem 6.13 for $\triangle PYX'$.

(b) Note first that the angles XPY and $X'PY'$ are vertical. Therefore, $\measuredangle XPY = \measuredangle X'PY'$.

Applying Theorem 8.2 we get

$$2\cdot\measuredangle Y'X'P \equiv 2\cdot\measuredangle PYX.$$

According to Theorem 3.11, $\angle Y'X'P$ and $\angle PYX$ have the same sign; therefore

$$\measuredangle Y'X'P = \measuredangle PYX.$$

It remains to apply the AA similarity condition.

(c) Apply *(b)* assuming $[YY']$ is the diameter of Γ.

Exercise 8.4. Apply Exercise 8.3b three times.

Exercise 8.5. Let X any Y be the foot points of altitudes from A and B. Denote by O the circumcenter.

By AA condition, $\triangle AXC \sim \triangle BYC$. Thus

$$\measuredangle A'OC \equiv 2\cdot\measuredangle A'AC \equiv -2\cdot\measuredangle B'BC \equiv -\measuredangle B'OC.$$

By SAS, $\triangle A'OC \cong \triangle B'OC$. Therefore, $A'C = B'C$.

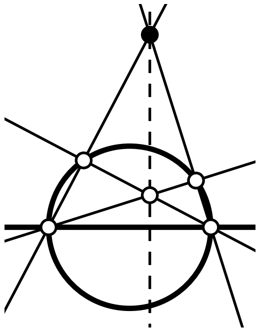
Exercise 8.7. Construct the circles Γ and Γ' with diameters $[AB]$ and $[A'B']$. By Corollary 8.6, any point Z in the intersection $\Gamma \cap \Gamma'$ will do.

Exercise 8.8. Note that $\measuredangle AA'B = \pm\frac{\pi}{2}$ and $\measuredangle AB'B = \pm\frac{\pi}{2}$. Then apply Theorem 8.10 to $\square AA'BB'$.

If O is the center of the circle, then

$$\measuredangle AOB \equiv 2\cdot\measuredangle AA'B \equiv \pi.$$

That is, O is the midpoint of $[AB]$.



Exercise 8.9. Guess the construction from the diagram. To prove it, apply Theorem 7.3 and Corollary 8.6.

Exercise 8.11. Apply Theorem 8.10 twice for $\square ABYX$ and $\square ABY'X'$ and use the transversal property (6.18).

Exercise 8.12. Apply the transversal property (6.18) and the theorem on inscribed angle (8.2).

Exercise 8.14. In the proof we assume that the lines $(A'B')$ and (XP) are not parallel, otherwise the first line in the proof does not make sense.

In addition, we use without mentioning the following three identities:

$$\begin{aligned} 2\cdot\measuredangle AXP &\equiv 2\cdot\measuredangle AXY, \\ 2\cdot\measuredangle ABP &\equiv 2\cdot\measuredangle ABB', \\ 2\cdot\measuredangle AA'B' &\equiv 2\cdot\measuredangle AA'Y. \end{aligned}$$

Exercise 8.15. By Corollary 8.6, the points L, M and N lie on the circle Γ with diameter $[OX]$. It remains to apply Theorem 8.2 for the circle Γ and two inscribed angles with vertex at O.

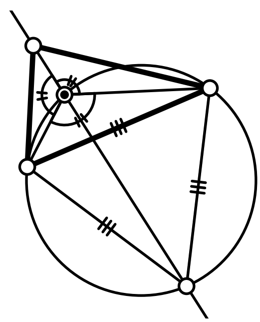
Exercise 8.18. By Theorem 6.13,

$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB \equiv \pi.$$

It remains to apply Proposition 8.17 twice.

Exercise 8.19. If $C \in (AX)$, then the arc is the line segment $[AC]$ or the union of two half-lines in (AX) with vertices at A and C.

Assume $C \notin (AX)$. Let ℓ be the line thru A perpendicular to (AX) and m be the perpendicular bisector of $[AC]$.



Note that $\ell \nparallel m$; set $O = \ell \cap m$. Note that the circle with center O passing thru A is also passing thru C and tangent to (AX).

Note that one the two arcs with endpoints A and C is tangent to $[AX)$.

The uniqueness follow from the propositions 8.16 and 8.17.

Exercise 8.20. Apply Proposition 8.17 twice.

Exercise 8.21. Guess the construction from the diagram. To show that it produce needed point, apply theorem on inscribed angle (8.2).

Chapter 9

Exercise 9.1. By Lemma 5.18, $\angle OTP'$ is right. Therefore, $\triangle OPT \sim$ $\sim \triangle OTP'$ and in particular

$$OP{\cdot}OP' = OT^2.$$

Hence the result follows.

Exercise 9.3. Denote by O the center of Γ. Suppose $X, Y \in \Gamma$; in particular, $OX = OY$.

Note that the invesion sends X and Y to itself. By Lemma 9.2,

$$\triangle OPX \sim \triangle OXP' \quad \text{and} \quad \triangle OPY \sim \triangle OYP'.$$

Therefore,

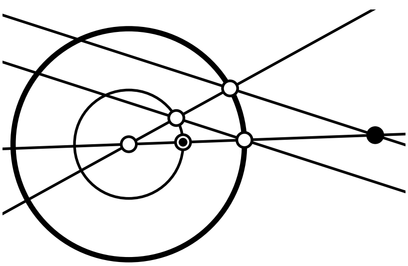
$$\frac{PX}{P'X} = \frac{OP}{OX} = \frac{OP}{OY} = \frac{PY}{P'Y}.$$

Hence the result follows.

Exercise 9.4. By Lemma 9.2,

$$\begin{aligned} &\measuredangle IA'B' \equiv -\measuredangle IBA, && \measuredangle IB'C' \equiv -\measuredangle ICB, && \measuredangle IC'A' \equiv -\measuredangle IAC,\\ &\measuredangle IB'A' \equiv -\measuredangle IAB, && \measuredangle IC'B' \equiv -\measuredangle IBC, && \measuredangle IA'C' \equiv -\measuredangle ICA. \end{aligned}$$

It remains to apply the theorem on the sum of angles of triangle (6.13) to show that $(A'I) \perp (B'C')$, $(B'I) \perp (C'A')$ and $(C'I) \perp (B'A')$.



Exercise 9.5. Guess the construction from the diagram (the two nonintersecting lines on the diagram are parallel).

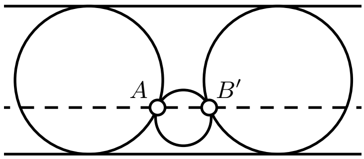
Exercise 9.8. Show first that for any $r > 0$ and any real numbers x, y distinct from 0, we have

$$\frac{r}{(x+y)/2} \neq \left(\frac{r}{x} + \frac{r}{y}\right)/2.$$

Denote by ℓ the line passing thru Q, Q' and the center of the inversion O. Choose an isometry $\ell \to \mathbb{R}$ which sends O to 0; assume x and y denote the values for the two points of intersection $\ell \cap \Gamma$. Then the left hand side above is the coordinate of Q' and the right hand side is the coordinate of the center of Γ'.

Exercise 9.9. A solution is given on page 152.

Exercise 9.10. Apply an inversion in a circle with the center at the only point of intersection of the circles; then use Theorem 9.11.



Exercise 9.13. Apply the inversion in a circle with center A. The point A will go to infinity, the two circles tangent at A will become parallel lines and the two parallel lines will become circles tangent at A; see the diagram.

It remains to show that the dashed line AB' is parallel to the other two lines.

Exercise 9.18. Let P_1 and P_2 be the inversions of P in Ω_1 and Ω_2. Note that the points P, P_1 and P_2 are mutually distinct.

According to Theorem 7.1, there is a unique circline Γ which pass thru P, P_1 and P_2.

By Corollary 9.16, $\Gamma \perp \Omega_1$ and $\Gamma \perp \Omega_2$.

On the other hand, if $\Gamma' \ni P$ and $\Gamma' \perp \Omega_1$, $\Gamma' \perp \Omega_2$, then by Theorem 9.14 we have $\Gamma' \ni P_1, P_2$. That is, $\Gamma' = \Gamma$.

Exercise 9.19. Apply Theorem 9.6*b*, Exercise 6.21 and Theorem 8.2.

Exercise 9.20. Denote by T a point of intersection of Ω_1 and Ω_2. Let P be the foot point of T on (O_1O_2). Show first that

$$\triangle O_1PT \sim \triangle O_1TO_2 \sim \triangle TPO_2.$$

Conclude that P coincides with the inversions of O_1 in Ω_2 and of O_2 in Ω_1.

Exercise 9.21. Since $\Gamma \perp \Omega_1$ and $\Gamma \perp \Omega_2$, Corollary 9.15 implies that the circles Ω_1 and Ω_2 are inverted in Γ to themselves.

Therefore, the points A and B are inversions of each other.

Since $\Omega_3 \ni A, B$, Corollary 9.16 implies that $\Omega_3 \perp \Gamma$.

Exercise 9.22. Follow the solution of Exercise 9.18.

Chapter 10

Exercise 10.2. Denote by D the midpoint of $[BC]$. Assume (AD) is the bisector of the angle at A.

Let $A' \in [AD)$ be the point distinct from A such that $AD = A'D$. Note that $\triangle CAD \cong \triangle BA'D$. In particular, $\measuredangle BAA' = \measuredangle AA'B$. It remains to apply Theorem 4.2 for $\triangle ABA'$.

Exercise 10.3. The statement is evident if A, B, C and D lie on one line.

In the remaining case, denote by O the circumcenter. Apply theorem about isosceles triangle (4.2) to the triangles AOB, BOC, COD, DOA.

(Note that in the Euclidean plane the statement follows from Theorem 8.10 and Exercise 6.21, but one cannot use these statements in the neutral plane.)

Exercise 10.5. Arguing by contradiction, assume

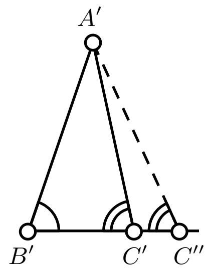
$$2\cdot(\measuredangle ABC + \measuredangle BCD) \equiv 0,$$

but $(AB) \nparallel (CD)$. Let Z be the point of intersection of (AB) and (CD).

Note that

$$2\cdot\measuredangle ABC \equiv 2\cdot\measuredangle ZBC, \qquad\qquad 2\cdot\measuredangle BCD \equiv 2\cdot\measuredangle BCZ.$$

Apply Proposition 10.4 to $\triangle ZBC$ and try to arrive to a contradiction.



Exercise 10.6. Let $C'' \in [B'C')$ be the point such that $B'C'' = BC$.

Note that by SAS, $\triangle ABC \cong \triangle A'B'C''$. Conclude that $\measuredangle B'C'A' = \measuredangle B'C''A'$.

Therefore, it is sufficient to show that $C'' = C'$. If $C' \ne C''$ apply Proposition 10.4 to $\triangle A'C'C''$ and try to arrive to a contradiction.

(This proof was given in the Euclid's Elements [3, Book I, Proposition 26].)

Exercise 10.7. Use Exercise 5.4 and Proposition 10.4.

Alternatively, use the same argument as in the solution of Exercise 5.14.

Exercise 10.10. Note that $|\measuredangle ADC| + |\measuredangle CDB| = \pi$. Then apply the definition of the defect.

Chapter 11

Exercise 11.1. Let A and B be the ideal points of h-line ℓ. Note that the center of the Euclidean circle containing ℓ lies at the intersection of the lines tangent to the absolute at the ideal points of ℓ.

Exercise 11.2. Assume A is an ideal point of h-line ℓ and $P \in \ell$. Denote by P' the inverse of P in the absolute. By Corollary 9.15, ℓ lies in the intersection of the h-plane and the (necessary unique) circline passing thru P, A and P'.

Exercise 11.3. Denote by Ω the absolute and by O its center.

Let Γ be the circline containing $[PQ]_h$. Note that $[PQ]_h = [PQ]$ if and only if Γ is a line.

Denote by P' the inverse of P in Ω. Note that O, P and P' lie on one line.

By definition of h-line, $\Omega \perp \Gamma$. By Corollary 9.15, Γ pass thru P and P'. Therefore, Γ is a line if and only if it pass thru O.

Exercise 11.4. Assume that the absolute is a unit circle.

Set $a = OX = OY$. Note that $0 < a < \frac{1}{2}$ and

$$OX_h = \ln\tfrac{1+a}{1-a}, \qquad\qquad XY_h = \ln\tfrac{(1+2\cdot a)\cdot(1-a)}{(1-2\cdot a)\cdot(1+a)}.$$

It remains to check that inequality

$$1 < \tfrac{1+a}{1-a} < \tfrac{(1+2\cdot a)\cdot(1-a)}{(1-2\cdot a)\cdot(1+a)}$$

holds if $0 < a < \frac{1}{2}$.

Exercise 11.5. Spell the meaning of terms "perpendicular" and "h-line" and then apply Exercise 9.18.

Exercise 11.8. Apply the main observation (11.7b).

Exercise 11.9. By Corollary 9.25 and Theorem 9.6, the quantity θ survives under an inversion in a circle perpendicular to the absolute. Therefore, by the main observation (11.7) we can assume that the midpoint of $[PQ]_h$ is the center of the absolute.

Set $a = OP = OQ$, then

$$\theta(P,Q) = \frac{8}{(\frac{1}{a} - a)^2}, \qquad \delta(P,Q) = \frac{(1+a)^2}{(1-a)^2}.$$

Hence the identity follows.

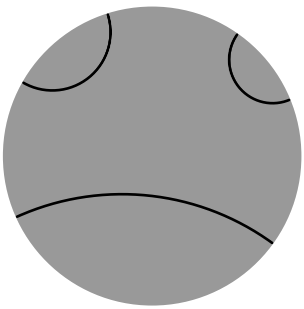
Exercise 11.13. Denote by X and Y the points of intersections of (OP) and Δ'_ρ. Consider an isometry $(OP) \to \mathbb{R}$ so that O corresponds to 0, denote by x, y, p and $\hat{p}$ the real numbers corresponding to X, Y, P and $\hat{P}$.

We can assume that $p > 0$ and $x < y$. Note that $\hat{p} = \frac{x+y}{2}$ and

$$\frac{(1+x)\cdot(1-p)}{(1-x)\cdot(1+p)} = \frac{(1+p)\cdot(1-y)}{(1-p)\cdot(1+y)}.$$

It remains to show that all this implies $0 < \hat{p} < p$.

Exercise 11.23. Look at the diagram and think.



Chapter 12

Exercise 12.3. Note that the angle of parallelism of B to $(CD)_h$ is bigger than $\frac{\pi}{4}$, and it converges to $\frac{\pi}{4}$ as $CD_h \to \infty$.

Applying Proposition 12.1, we get

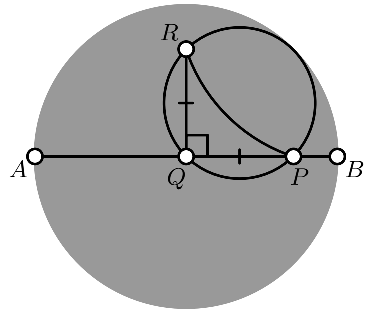
$$BC_h < \tfrac{1}{2}\cdot \ln \frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \ln\left(1+\sqrt{2}\right).$$

The right hand side is the limit of BC_h if $CD_h \to \infty$. Therefore, $\ln\left(1+\sqrt{2}\right)$ is the optimal upper bound.

Exercise 12.4. Consider hyperbolic triangle PQR with right angle at Q, such that $PQ_h = QR_h$ and the vertices P, Q and R lie on a horocycle.

Without loss of generality, we may assume that Q is the center of the absolute. In this case $\measuredangle_h PQR = \measuredangle PQR = \pm\frac{\pi}{2}$ and $PQ = QR$

The rest of proof should be easy to guess from the picture. The answer is $2\cdot\ln(1+\sqrt{2})$.



Exercise 12.7. In hyperbolic plane, the AAA-congruence condition (12.6) implies that any nondegenerate triangle is mapped to a congruent triangle. Hence the statement follows.

Exercise 12.10. Let us apply Proposition 12.9.

$$\begin{aligned}\operatorname{circum}_h(r+1) &= \pi\cdot(e^{r+1} - e^{-r-1}) = \pi\cdot e\cdot(e^{r} - e^{-r-2}) > \\ &> \pi\cdot e\cdot(e^{r} - e^{-r}) = e\cdot\operatorname{circum}_h(r) \geqslant 2\cdot\operatorname{circum}_h(r).\end{aligned}$$

Chapter 13

Exercise 13.1. Assume that triple of noncollinear points P, Q and R mapped to one line ℓ. Note that all three lines (PQ), (QR) and (RP) are mapped to ℓ. Therefore, any line which connects two points on these three lines is mapped to ℓ.

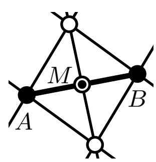
Note that any point in the plane lies on a line passing thru two distinct points on these three lines. Therefore, whole plane is mapped to ℓ. The latter contradicts that the map is a bijection.

Exercise 13.2. Assume the two distinct lines ℓ and m are mapped to intersecting lines ℓ' and m'. Denote by P' their point of intersection.

Let P be the inverse image of P'. By definition of affine map it has to lie on both ℓ and m; that is, ℓ and m are intersecting. Hence the result follows.

Exercise 13.3. According to the remark before the exercise, it is sufficient to construct the midpoint of $[AB]$ with ruler and parallel tool.

Guess the construction from the diagram.



Exercise 13.4. Let us denote by O, E, A and B the points with the coordinates $(0,0)$, $(1,0)$, $(a,0)$ and $(b,0)$ correspondingly.

To construct a point W with the coordinates $(0, a+b)$, try to construct two parallelograms $ABPQ$ and $BWPQ$.

To construct Z with coordinates $(0, a\cdot b)$ choose a line $(OE') \neq (OE)$ and try construct the points $A' \in (OE')$ and $Z \in (OE)$ so that $\triangle OEE' \sim \sim \triangle OAA'$ and $\triangle OE'B \sim \triangle OA'Z$.

Exercise 13.5. Draw two parallel chords $[XX']$ and $[YY']$. Set $Z = (XY) \cap (X'Y')$ and $Z' = (XY') \cap (X'Y)$. Note that (ZZ') passes thru the center.

Repeat the same construction for another pair of parallel chords. The center lies in the intersection of the obtained lines.

Exercise 13.6. Assume a construction produce two perpendicular lines. Apply a shear mapping which changes the angle between the lines.

Note that it transforms the construction to the same construction for another free choices points. Therefore, this construction does not produce perpendicular lines in general (it might be the center only by coincidence).

Exercise 13.11. Fix a line ℓ. Choose a circle Γ with center not on ℓ. Let Ω be the inverse of ℓ in Γ; note that Ω is a circle.

Denote by ι_Γ and ι_Ω the inversions in Γ and Ω. Show that the composition $\iota_\Gamma \circ \iota_\Omega \circ \iota_\Gamma$ is the reflection in ℓ.

Chapter 14

Exercise 14.1. Since O, P and P' lie on one line we have that coordinates of P' are proportional to the coordinates of P. The y coordinate of P' has to be equal to 1. Therefore, P' has coordinates $(\frac{1}{y}, 1, \frac{z}{y})$.

Exercise 14.5. Assume that the lines (AB) and $(A'B')$ intersect at O. Since $(AB') \parallel (BA')$, we get $\triangle OAB' \sim \triangle OBA'$ and

$$\frac{OA}{OB} = \frac{OB'}{OA'}$$

Similarly since $(AC') \parallel (CA')$, we get

$$\frac{OA}{OC} = \frac{OC'}{OA'}.$$

Therefore

$$\frac{OB}{OC} = \frac{OC'}{OB'}.$$

Applying the SAS similarity condition we get $\triangle OBC' \sim \triangle OCB'$. Therefore, $(BC') \parallel (CB')$.

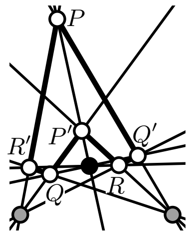
The case $(AB) \parallel (A'B')$ is similar.

Exercise 14.6. Assume the contrary. Choose two parallel lines ℓ and m. Let L and M be their dual points. Set $s = (ML)$, then its dual point S have to lie on both ℓ and m; a contradiction.

Exercise 14.8. Assume $M = (a, b)$ and the line s is given by the equation $p{\cdot}x + q{\cdot}y = 1$. Then $M \in s$ is equivalent to $p{\cdot}a + q{\cdot}b = 1$.

The latter is equivalent to $m \ni S$ where m is the line given by the equation $a{\cdot}x + b{\cdot}y = 1$ and $S = (p, q)$.

To extend this bijection to whole projective plane assume that (1) the ideal line corresponds to the origin and (2) the ideal point given by the pencil of the lines $ax - by = c$ for different values of c corresponds to the line given by the equation $ax + by = 0$.



Exercise 14.11. Assume one set of concurrent lines a, b, c, and another set of concurrent lines a', b', c' are given. Set

$$\begin{aligned} P &= b \cap c', & Q &= c \cap a', & R &= a \cap b' \\ P' &= b' \cap c, & Q' &= c' \cap a, & R' &= a' \cap b \end{aligned}$$

Then the lines (PP'), (QQ') and (RR') are concurrent.

Exercise 14.12. Assume (AA') and (BB') are the given lines and C is the given point. Apply Dual Desargues' theorem 14.9 to construct C' so that (AA'), (BB') and (CC') are concurrent. Since $(AA') \parallel (BB')$ we get $(AA') \parallel (BB') \parallel (CC')$.

A similar solution can be build on the Dual Pappus's theorem, see Exercise 14.11.

Exercise 14.13. Let A, B, C and D be the point provided by Axiom p-III. Given a line ℓ we can assume that $A \notin \ell$, otherwise permute the labels of the points. Then by axioms p-I and p-II, the three lines (AB), (AC) and (AD) intersect ℓ at distinct points. In particular, ℓ contains 3 points.

Exercise 14.14. Let A, B, C and D be the point provided by Axiom p-III. Show that the lines (AB), (BC), (CD) and (DA) satisfy Axiom p-III$'$. The proof of the converse is similar.

Exercise 14.15. Let ℓ be a line with $n+1$ points on it.

By Axiom p-III, given any line m there is a point P which does not lie on ℓ nor on m.

By axioms p-I and p-II, there is a bijection between the lines passing thru P and the points on ℓ. In particular, there are exactly $n+1$ lines passing thru P.

The same way there is bijection between the lines passing thru P and the points on m. Hence *(a)* follows.

Fix a point X. By Axiom p-I, any point Y in the plane lies in a unique line passing thru X. From part *(a)*, each such line contains X and yet n point. Hence *(b)* follows.

To solve *(c)*, show that the equation $n^2 + n + 1 = 10$ does not admit an integer solution and then apply part *(b)*.

To solve *(d)*, count the number of lines crossing a given line using the part *(a)* and apply *(b)*.

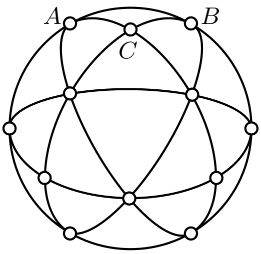
Chapter 15

Exercise 15.2. Applying Pythagorean theorem, we get

$$\cos AB_s = \cos AC_s \cdot \cos BC_s = \tfrac{1}{2}.$$

Therefore, $AB_s = \frac{\pi}{3}$.

Alternatively, look at the tessellation of the sphere on the picture; it is made from 24 copies of $\triangle_s ABC$ and yet 8 equilateral triangles. From the symmetry of this tessellation, it follows that $[AB]_s$ occupies $\frac{1}{6}$ of the equator.



Exercise 15.6. Note that points on Ω do not move. Moreover, the points inside Ω are mapped outside of Ω and the other way around.

Further, note that this map sends circles to circles; moreover, the perpendicular circles are mapped to perpendicular circles. In particular, the circle perpendicular to Ω are mapped to itself.

Consider arbitrary point $P \notin \Omega$. Denote by P' the inverse of P in Ω. Choose two distinct circles which pass thru P and P'. According to Corollary 9.16, $\Gamma_1 \perp \Omega$ and $\Gamma_2 \perp \Omega$.

Therefore, the inversion in Ω sends Γ_1 to itself and the same holds for Γ_2.

The image of P has to lie on Γ_1 and Γ_2. Since the image of P is distinct from P, we get that it has to be P'.

Exercise 15.7. Apply Theorem 15.3*b*.

Exercise 15.8. Set $z = P'Q'$. Note that $\frac{y}{z} \to 1$ as $x \to 0$.

It remains to show that

$$\lim_{x \to 0} \frac{z}{x} = \frac{2}{1 + OP^2}.$$

Denote by O the center of the unit sphere Σ. Recall that stereographic projection is the inversion in the sphere Υ restricted to the plane. Note that there is plane Π passing thru O, P, Q, P' and Q'. In the plane Π the map $Q \mapsto Q'$ is an inversion in a circle $\Sigma \cap \Pi$.

This reduce the problem to Euclidean plane geometry. The remaining calculations in Π are similar to those in the proof of Lemma 12.8.

Exercise 15.9. Denote by O the center of the sphere.

To prove the identities

$$s = \tfrac{\operatorname{tg} a}{\cos b}, \qquad t = \operatorname{tg} b, \qquad u = \operatorname{tg} c$$

consider the sections by the planes containing the following triple of points (NBO), (BCO) and (CNO).

The remaining part of exercise is algebraic manipulations similar to the one given in the proof of Theorem 16.8.

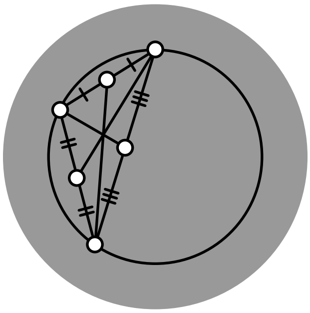
Exercise 15.10. *(a).* Observe and use that $OA' = OB' = OC'$.

(b). Mote that medians of spherical triangle ABC map to the medians of Euclidean triangle $A'B'C'$. It remains to apply Theorem 7.5 for $\triangle A'B'C'$.

Chapter 16

Exercise 16.1. Let N, O, S, P, P' and $\hat{P}$ be as on the diagram on page 129.

Notice that $\triangle NOP \sim \triangle NP'S \sim \triangle P'\hat{P}P$ and $2{\cdot}NO = NS$. It remains to do algebraic manipulations.



Exercise 16.3. Consider the bijection $P \mapsto \hat{P}$ of the h-plane with absolute Ω. Note that $\hat{P} \in [A_iB_i]$ if and only if $P \in \Gamma_i$.

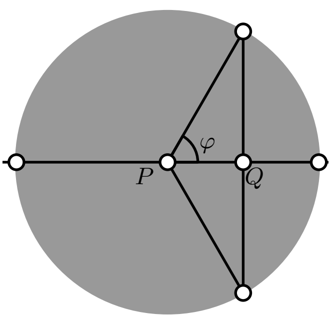
Exercise 16.4. The observation follows since the reflection in the perpendicular bisector of $[PQ]$ is a motion of the Euclidean plane, and a motion of the h-plane as well.

Without loss of generality, we may assume that the center of circumcircle coincides with the center of the absolute. In this case the h-median of the triangle coincide with the Euclidean medians. It remains to apply Theorem 7.5

Exercise 16.5. Let us denote by $\hat{\ell}$ and $\hat{m}$ the h-lines in the conformal model which correspond to ℓ and m. We need to show that $\hat{\ell} \perp \hat{m}$ as arcs in the Euclidean plane.

The point Z where s and t meet is the center of the circle Γ containing $\hat{\ell}$.

If $\bar{m}$ passing thru Z, then the inversion in Γ exchanges the ideal points of $\hat{\ell}$. In particular, $\hat{\ell}$ maps to itself. Hence the result follows.



Exercise 16.6. We can assume that P is the center of the absolute. Then

$$PQ = \cos\varphi.$$

Therefore

$$PQ_h = \tfrac{1}{2}\cdot \ln \frac{1+\cos\varphi}{1-\cos\varphi}.$$

Exercise 16.7. Apply Exercise 16.6 for $\varphi = \frac{\pi}{3}$.

Exercise 16.10. In the Euclidean plane, the circle Γ_2 is tangent to k; that is, the point T of intersection of Γ_2 and k is unique. It defines a unique line (PT) parallel to ℓ.

Chapter 17

Exercise 17.1. Denote by z, v and w the complex coordinates of Z, V and W correspondingly. Then

$$\begin{aligned}\measuredangle ZVW + \measuredangle VWZ + \measuredangle WZV &\equiv \arg \tfrac{w-v}{z-v} + \arg \tfrac{z-w}{v-w} + \arg \tfrac{v-z}{w-z} \equiv \\ &\equiv \arg \tfrac{(w-v)\cdot(z-w)\cdot(v-z)}{(z-v)\cdot(v-w)\cdot(w-z)} \equiv \\ &\equiv \arg(-1) \equiv \\ &\equiv \pi.\end{aligned}$$

Exercise 17.2. Note and use that

$$\measuredangle EOV = \measuredangle WOZ = \arg v, \qquad \frac{OW}{OZ} = \frac{OZ}{OW} = |v|.$$

Exercise 17.4. Note that

$$\begin{aligned}\arg \frac{(v-u)\cdot(z-w)}{(v-w)\cdot(z-u)} &\equiv \arg \frac{v-u}{z-u} + \arg \frac{z-w}{v-w} \equiv \\ &\equiv \measuredangle ZUV + \measuredangle VWZ.\end{aligned}$$

The statement follows since value $\frac{(v-u)\cdot(z-w)}{(v-w)\cdot(z-u)}$ is real if and only if

$$2\cdot \arg \frac{(v-u)\cdot(z-w)}{(v-w)\cdot(z-u)} \equiv 0.$$

Exercise 17.8. Show that the inverse of each elementary transformation is elementary and use Proposition 17.6.

Exercise 17.9. The Möbius transformation

$$f(z) = \frac{(z_1 - z_\infty)\cdot(z - z_0)}{(z_1 - z_0)\cdot(z - z_\infty)}$$

meets the conditions.

To show uniqueness, assume there is another Möbius transformation $g(z)$ which meets the conditions. Then the composition $h = g \circ f^{-1}$ is a Möbius transformation; set

$$h(z) = \frac{a\cdot z + b}{c\cdot z + d}.$$

Note that $h(\infty) = \infty$; therefore, $c = 0$. Further, $h(0) = 0$ implies $b = 0$. Finally, since $h(1) = 1$ we get $\frac{a}{d} = 1$. Therefore, h is the *identity*; that is, $h(z) = z$ for any z. It follows that $g = f$.

Exercise 17.10. Let Z' be the inverse of the point Z. Assume that the circle of the inversion has center W and radius r. Denote by z, z' and w the complex coordinate of the points Z, Z' and W correspondingly.

By definition of inversion

$$\arg(z - w) = \arg(z' - w), \qquad\qquad |z - w|\cdot|z' - w| = r^2$$

It follows that

$$(\bar{z}' - \bar{w})\cdot(z - w) = r^2.$$

Equivalently,

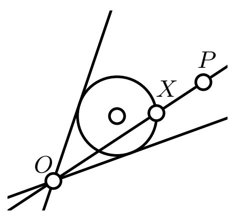
$$z' = \overline{\left(\frac{\bar{w}\cdot z + [r^2 - |w|^2]}{z - w}\right)}.$$

Exercise 17.12. Check the statement for each elementary transformation. Then apply Proposition 17.6.

Exercise 17.14. Apply Schwarz–Pick theorem for a function f such that $f(0) = 0$ and then apply Lemma 11.10.

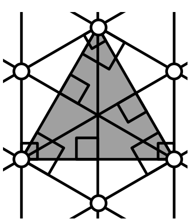
Chapter 18

Exercise 18.3. Let O be the point of intersection of the lines. Draw a line ℓ thru the given point P and O. Construct a circle Γ tangent to both lines which crosses ℓ. Denote by X one of the points of intersections.



Consider the homothety with the center at O which sends X to P. The image of Γ is the needed circle.

Exercise 18.6. Note that with set square we can construct a line parallel to given line thru the given point. It remains to modify the construction in Exercise 13.3.



Exercise 18.8. Assume that two vertices have rational coordinates, say (a_1, b_1) and (a_2, b_2). Find the coordinates of the third vertex. Use that the number $\sqrt{3}$ is irrational to show that the third vertex is an irrational point.

Exercise 18.9. Guess the construction from the diagram. Prove that it verifies that the triangle is equilateral.

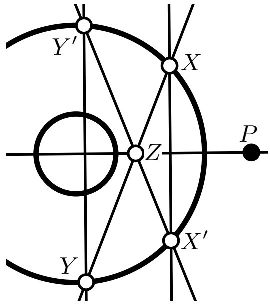
Exercise 18.10. Apply the same argument as in Exercise 9.9.

Exercise 18.11. Consider the perspective projection as in Exercise 14.1. Let $A = (1, 1, 1)$, $B = (1, 3, 1)$ and $M = (1, 2, 1)$. Note that M is the midpoint of $[AB]$.

Their images are $A' = (1, 1, 1)$, $B' = (\tfrac{1}{3}, 1, \tfrac{1}{3})$ and $M' = (\tfrac{1}{2}, 1, \tfrac{1}{2})$. Clearly, M' is not the midpoint of $[A'B']$.

Exercise 18.14. The line v polar to V is tangent to Γ. Since $V \in p$, by Claim 18.13, we get $P \in v$; that is, $(PV) = v$. Hence the statement follows.

Exercise 18.15. Choose a point P outside of bigger circle. Construct the dual lines for given point for both circles. Note that these lines are parallel.



Assume that the lines intersect the bigger circle at two pairs of points X, X' and Y, Y'. Set $Z = (XY) \cap (X'Y')$. Note that the line (PZ) passes thru the common center.

The center can be found as the intersection of the constructed line and another line constructed the same way.

Chapter 19

Exercise 19.1. Assume the contrary; that is, there is a point $W \in [XY]$ such that $W \notin \blacktriangle ABC$.

Without loss of generality, we may assume that W and A lie on opposite sides from the line (BC).

It imples that both segments $[WX]$ and $[WY]$ intersect (BC). By Axiom II, $W \in (BC)$, a contradiction.

Exercise 19.3. To prove the "only if" part, consider the line passing thru the vertex which is parallel to the opposite side.

To prove the "if" part, use Pasch's theorem (3.10).

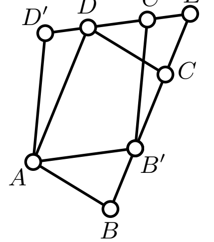
Exercise 19.4. Assume the contrary; that is, a solid square $\mathcal{Q}$ can be presented as a union of finite collection of segments $[A_1B_1], \dots, [A_nB_n]$ and one-point sets $\{C_1\}, \dots, \{C_k\}$.

Note $\mathcal{Q}$ contains infinite number of mutually nonparallel segments. Therefore, we can choose a segment $[PQ]$ in $\mathcal{Q}$ which is not parallel to any of the segments $[A_1B_1], \dots, [A_nB_n]$.

It follows that $[PQ]$ has at most one common point with each the sets $[A_iB_i]$ and $\{C_i\}$. We arrive to a contradiction since $[PQ]$ contains infinte number of points.

Exercise 19.5. Note first that among elementary sets only one-point set are subsets of the a circle.

It remains to note that any circle contains infinite number of points.



Exercise 19.15. Denote by E the point of intersection of the lines (BC) and $(C'D')$.

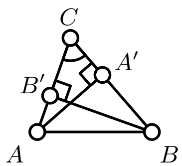
Use Proposition 19.14 to prove two identities

$$\operatorname{area}(\blacksquare AB'ED) = \operatorname{area}(\blacksquare ABCD),$$
$$\operatorname{area}(\blacksquare AB'ED) = \operatorname{area}(\blacksquare AB'C'D').$$

Hence the statement follows.

Exercise 19.17. Without loss of generality, we may assume that the angles ABC and BCA are acute.

Denote by A' and B' the foot points of A and B on (BC) and (AC) correspondingly. Note that $h_A = AA'$ and $h_B = BB'$.



Note that $\triangle AA'C \sim \triangle BB'C$; indeed the angle at C is shared and the angles at A' and B' are right. In particular

$$\frac{AA'}{BB'} = \frac{AC}{BC}$$

or, equivalently,

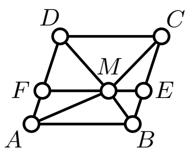
$$h_A \cdot BC = h_B \cdot AC.$$

Along the same lines, we get

$$h_C \cdot AB = h_B \cdot AC.$$

Hence the statement follows.

Exercise 19.18. Draw the line ℓ thru M parallel to the sides $[AB]$ and $[CD]$; it subdivides $\blacksquare ABCD$ into two solid parallelograms denoted by $\blacksquare ABEF$ and $\blacksquare CDFE$. In particular,



$$\operatorname{area}(\blacksquare ABCD) = \operatorname{area}(\blacksquare ABEF) + \operatorname{area}(\blacksquare CDFE).$$

By Proposition 19.14 and Theorem 19.16 we get

$$\begin{aligned}\operatorname{area}(\blacktriangle ABM) &= \tfrac{1}{2}\cdot \operatorname{area}(\blacksquare ABEF),\\ \operatorname{area}(\blacktriangle CDM) &= \tfrac{1}{2}\cdot \operatorname{area}(\blacksquare CDFE)\end{aligned}$$

Hence the result follows.

Exercise 19.19. Denote by h_A and h_C the distances from A and C to the line (BD). According to Theorem 19.16,

$$\begin{aligned}\operatorname{area}(\blacktriangle ABM) &= \tfrac{1}{2}\cdot h_A \cdot BM; & \operatorname{area}(\blacktriangle BCM) &= \tfrac{1}{2}\cdot h_C \cdot BM;\\ \operatorname{area}(\blacktriangle CDM) &= \tfrac{1}{2}\cdot h_C \cdot DM; & \operatorname{area}(\blacktriangle ABM) &= \tfrac{1}{2}\cdot h_A \cdot DM.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{area}(\blacktriangle ABM)\cdot \operatorname{area}(\blacktriangle CDM) &= \tfrac{1}{4}\cdot h_A \cdot h_C \cdot DM \cdot BM =\\ &= \operatorname{area}(\blacktriangle BCM)\cdot \operatorname{area}(\blacktriangle DAM).\end{aligned}$$

Exercise 19.20. Let I be the incenter of $\triangle ABC$. Note that $\blacktriangle ABC$ can be subdivided into $\blacktriangle IAB$, $\blacktriangle IBC$ and $\blacktriangle ICA$.

It remains to applying Theorem 19.16 to each of these triangles and sum up the results.

Exercise 19.22. Assuming $a > b$, we subdivided $\mathcal{Q}_c$ into $\mathcal{Q}_{a-b}$ and four triangles congruent to $\mathcal{T}$. Therefore

$$❶ \qquad \operatorname{area}\mathcal{Q}_c = \operatorname{area}\mathcal{Q}_{a-b} + 4\cdot \operatorname{area}\mathcal{T}.$$

According to Theorem 19.16, $\operatorname{area}\mathcal{T} = \tfrac{1}{2}\cdot a\cdot b$. Therefore, the identity ❶ can be written as

$$c^2 = (a-b)^2 + 2\cdot a\cdot b.$$

Simplifying, we get the Pythagorean theorem.

The cases $a = b$ is yet simpler. The case $b > a$ can be done the same way.

Exercise 19.23. If X is a point inside of $\triangle ABC$, then $\blacktriangle ABC$ is subdivided into $\blacktriangle ABX$, $\blacktriangle BCX$ and $\blacktriangle CAX$. Therefore

$$\operatorname{area}(\blacktriangle ABX) + \operatorname{area}(\blacktriangle BCX) + \operatorname{area}(\blacktriangle CAX) = \operatorname{area}(\blacktriangle ABC).$$

Set $a = AB = BC = CA$. Denote by h_1, h_2 and h_3 are the distances from X to the sides $[AB]$, $[BC]$ and $[CA]$. Then by Theorem 19.16,

$$\begin{aligned}\operatorname{area}(\blacktriangle ABX) &= \tfrac{1}{2}\cdot h_1\cdot a,\\ \operatorname{area}(\blacktriangle BCX) &= \tfrac{1}{2}\cdot h_2\cdot a,\\ \operatorname{area}(\blacktriangle CAX) &= \tfrac{1}{2}\cdot h_3\cdot a.\end{aligned}$$

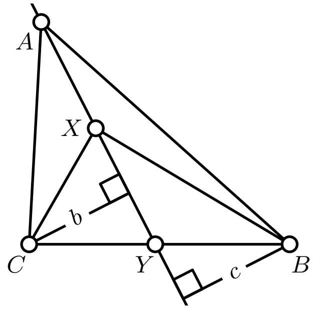
Therefore,

$$h_1 + h_2 + h_3 = 2\cdot\frac{\operatorname{area}(\blacktriangle ABC)}{a}.$$

Exercise 19.25. Let X be a point inside $\triangle ABC$. Denote by Y the point of intersection of (AX) and $[BC]$.

Denote by b and c the distances from B and C to the line (AX).

By Theorem 19.16, we get the following equivalences



$$\begin{gathered}\operatorname{area}(\blacktriangle AXB) = \operatorname{area}(\blacktriangle AXC),\\ \Updownarrow\\ b = c,\\ \Updownarrow\\ \operatorname{area}(\blacktriangle AYB) = \operatorname{area}(\blacktriangle AYC),\\ \Updownarrow\\ BY = CY.\end{gathered}$$

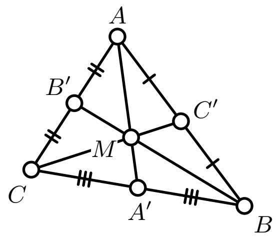
Exercise 19.26. Denote by M the intersection of two medians $[AA']$ and $[BB']$.

From Exercise 19.25 we have

$$\operatorname{area}(\blacktriangle ABM) = \operatorname{area}(\blacktriangle ACM), \qquad \operatorname{area}(\blacktriangle ABM) = \operatorname{area}(\blacktriangle CBM).$$

Therefore,

$$\operatorname{area}(\blacktriangle BCM) = \operatorname{area}(\blacktriangle ACM).$$



According to Exercise 19.25, M lies on the median $[CC']$. That is, medians $[AA']$, $[BB']$ and $[CC']$ intersect at one point M.

By Theorem 19.16, we get

$$\begin{aligned}\operatorname{area}(\blacktriangle C'AM) &= \tfrac{1}{2}\cdot \operatorname{area}(\blacktriangle BAM)\\ &= \tfrac{1}{2}\cdot \operatorname{area}(\blacktriangle CAM)\end{aligned}$$

Applying Claim 19.24, we get

$$\frac{MC'}{MC} = \frac{\operatorname{area}(\blacktriangle C'AM)}{\operatorname{area}(\blacktriangle CAM)} = \tfrac{1}{2}.$$

Exercise 19.27. Let $\mathcal{P}_n$ and $\mathcal{Q}_n$ be the solid regular n-gons so that Γ is inscribed in $\mathcal{Q}_n$ and circumscribed around $\mathcal{P}_n$. Clearly,

$$\mathcal{P}_n \subset \mathcal{D} \subset \mathcal{Q}_n.$$

Show that $\frac{\operatorname{area}\mathcal{P}_n}{\operatorname{area}\mathcal{Q}_n} = (\cos\frac{\pi}{n})^2$; in particular,

$$\frac{\operatorname{area}\mathcal{P}_n}{\operatorname{area}\mathcal{Q}_n} \to 1 \quad \text{as} \quad n\to\infty.$$

Next show that $\operatorname{area}\mathcal{Q}_n < 100$, say for all $n \geqslant 100$.

These two statements imply that $(\operatorname{area}\mathcal{Q}_n - \operatorname{area}\mathcal{P}_n) \to 0$. Hence the result follows.

Index

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