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Extremal graphs for weights

Béla Bollobás[a,c,*], Paul Erdős[a,b], Amites Sarkar[a,c]

[a] Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
[b] Mathematical Institute of the Hungarian Academy of Sciences, Reáltanoda St. 13-15, H-1053 Budapest, Hungary
[c] Trinity College, Cambridge CB2 1TQ, UK

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Abstract

Given a graph $G=(V,E)$ and $\alpha \in \mathbb{R}$, we write $w_\alpha(G)=\sum_{xy\in E} d_G(x)^\alpha d_G(y)^\alpha$, and study the function $w_\alpha(m)=\max\{w_\alpha(G)\colon e(G)=m\}$. Answering a question from Bollobás and Erdős (Graphs of external weights, to appear), we determine $w_1(m)$ for every m, and we also give bounds for the case $\alpha\neq 1$.

1. Introduction

The aim of this paper is to continue the work started by Bollobás and Erdős [1] on the α-weight of a graph with a given number of edges. For $\alpha\in\mathbb{R}$, the *α-weight* $w_\alpha(xy)$ of an edge xy of a graph G is defined as $w_\alpha(xy)=d(x)^\alpha d(y)^\alpha$, where $d(x)$ and $d(y)$ are the degrees of the vertices x and y. The *α-weight* $w_\alpha(G)$ of G is the sum of the α-weights of its edges.

In [1], Bollobás and Erdős studied the extremal α-weights of graphs with a given number of edges, with emphasis on the case $\alpha=-\frac{1}{2}$, when the weights are the so called *Randić weights*, as defined in [3]. They also proved that the Randić weight of a graph G of order n with no isolated vertices is at least $\sqrt{n-1}$, with equality if and only if $G\cong K_{1,n-1}$. Concerning the case $\alpha=1$, in [1] it was proved that if $m=\binom{k}{2}$ then the maximum 1-weight of a graph of size m is $m(k-1)^2$, with equality iff G is the union of K_k and isolated vertices. In [1] it was also conjectured that if $\binom{k}{2}<m\leqslant\binom{k+1}{2}$ then the maximum is attained on a graph of order $k+1$ which contains a complete graph of order k. One of our aims is to prove this conjecture. We do this in Section 2.

Our second main aim is to consider α-weights with $\alpha\neq 1$. What is the maximum α-weight of a graph with m edges, and what is the minimum? Rather trivially, for

* Corresponding author. E-mail: bollobas@msci.memphis.edu.

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$\alpha \leqslant 0$ the maximum is attained on m independent edges, and for $\alpha \geqslant 0$ the minimum is attained on m independent edges, so for $\alpha < 0$ and $\alpha > 0$ we are interested in graphs of minimum α-weight and maximum α-weight respectively. For positive values of α, considered in Section 3, it is convenient to distinguish three cases. When $0 \leqslant \alpha \leqslant 1$, as shown in [1], Hőlder's inequality together with our result for $\alpha = 1$ shows that among graphs of fixed size $\binom{k}{2}$, K_k has largest α-weight. When $\alpha > 1$, we have to work harder: complete graphs are no longer extremal, since it pays to have some edges of very high weight. Treating m as a large fixed parameter and letting α increase from 1 to $\frac{3}{2}$, the extremal graphs are close to the split graphs $K_t + \bar{K}_{m/t}$, where t rapidly decreases. Our result for $1 < \alpha < 2$, Theorem 6, only gives the correct leading term when α takes one of a discrete set of values. For $\alpha \geqslant 2$, it is not hard to show that $K_2 + \bar{K}_{m/2}$ is asymptotically best possible. Finally, in Section 4, we consider the case $\alpha < 0$. Here, repeated use of Causchy–Schwarz inequality shows that among graphs of size $\binom{k}{2}$, K_k has smallest α-weight (for $-1 \leqslant \alpha < 0$ this was already noted in [1]).

2. Graphs of extremal 1-weight

The aim of this section is to prove the following conjecture from [1].

Theorem 1. *Let k and r be positive integers with $0 < r \leqslant k$. Then all graphs G of size $m = \binom{k}{2} + r$ and minimal degree atleast one satisfy $w_1(G) \leqslant w_1(G_m)$, where the graph G_m consists of a complete graph of order k together with an additional vertex joined to r vertices of the complete graph, and has* 1-*weight*

$$w_1(G_m) = \binom{r}{2} k^2 + \binom{k-r}{2} (k-1)^2 + rk(k-r)(k-1) + r^2 k.$$

Before we are ready, we require three lemmas and the following generalisation of the notion of α-weight. For $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, the (ℓ, α)-*weight* of an edge xy of a graph G is

$$w_{(\ell,\alpha)}(xy) = (d_G(x) + \ell)^\alpha (d_G(y) + \ell)^\alpha,$$

and the (ℓ, α)-*weight* $w_{(\ell,\alpha)}(G)$ of a graph G is the sum of the (ℓ, α)-weights of its edges. Note that the $(0, \alpha)$-weight (of an edge or of a graph) is just the α-weight. From now on, we write d_x for $d_G(x)$.

Lemma 2. *Let k, ℓ and r be positive integers with $0 < r \leqslant k$. Let G be a graph of order n, without isolated vertices, having largest $(\ell, 1)$-weight among all graphs of size $m = \binom{k}{2} + r$. Then $\Delta(G) = n-1$.*

Proof. First observe that any two non-adjacent vertices in G have a common neighbour since otherwise by amalgamating the two vertices we could increase $w_{(\ell,1)}(G)$, while keeping $e(G)=m$. Let x be a vertex of maximal degree. Suppose, for a contradiction, that $d_x<n-1$ and let y be a vertex of maximal degree subject to the condition $xy\notin E(G)$. Let z be a common neighbour of x and y. Now let G' be G with the edge $yz\in E(G)$ replaced by the edge xy, and set $G_0=G-\{x,y,z\}$. Also, write S_x for the sum of the $(\ell,1)$-weights (in G) of edges incident with x, except for edge xz, S_y for the sum of the $(\ell,1)$-weights (in G) of edges incident with y, except for edge yz, and S_z for the sum of the $(\ell,1)$-weights (in G) of edges incident with z, except for edges xz and yz. Then

$$w_{(\ell,1)}(G)=\sum_{e\in E(G_0)} w_{(\ell,1)}(e)+S_x+S_y+S_z+(d_x+\ell)(d_z+\ell)+(d_y+\ell)(d_z+\ell)$$

and

$$\begin{aligned}w_{(\ell,1)}(G')=&\sum_{e\in E(G_0)} w_{(\ell,1)}(e)+S_x\frac{d_x+\ell+1}{d_x+\ell}+S_y+S_z\frac{d_z+\ell-1}{d_z+\ell}\\&+(d_x+\ell+1)(d_z+\ell-1)+(d_y+\ell)(d_y+\ell+1).\end{aligned}$$

As $w_{(\ell,1)}(G)$ is maximal for graphs of size m,

$$0\geqslant w_{(\ell,1)}(G')-w_{(\ell,1)}(G)=\frac{S_x}{d_x+\ell}-\frac{S_z}{d_z+\ell}+(d_x+1-d_z)(d_y+\ell-1). \tag{1}$$

Notice that $d_x+1-d_z>0$ and $d_y+\ell-1\geqslant d_y-1\geqslant 0$. Therefore we have

$$\frac{S_z}{d_z+\ell}\geqslant\frac{S_x}{d_x+\ell}. \tag{2}$$

Next, let $W=\Gamma_G(x)-(\Gamma_G(z)\cup\{z\})$. Note that $W\neq\emptyset$, since $d_x\geqslant d_z$ and $|W|=|\Gamma_G(z)-(\Gamma_G(x)\cup\{x\})|+d_x-d_z\geqslant 1$, as $y\in\Gamma_G(z)-(\Gamma_G(x)\cup\{x\})$. Let $w\in W$, and write T_x for the sum of the $(\ell,1)$-weights (in G) of edges incident with x, except for the edges wx and xz, and T_z for the sum of the $(\ell,1)$-weights (in G) of the edges incident with z, except for the edge xz. (We suppress the simple dependence of T_x and T_z on w.) Let G'' be G with the edge $xw\in E(G)$ replaced by the edge wz. Arguing as in (1), we find that

$$0\geqslant w_{(\ell,1)}(G'')-w_{(\ell,1)}(G)=\frac{T_z}{d_z+\ell}-\frac{T_x}{d_x+\ell}+(d_z+1-d_x)(d_w+\ell-1). \tag{3}$$

But

$$\begin{aligned}T_z&=S_z+(d_y+\ell)(d_z+\ell),\\T_x&=S_x-(d_x+\ell)(d_w+\ell),\end{aligned}$$

so (3) gives that

$$\begin{aligned}0&\geqslant\frac{S_z}{d_z+\ell}+(d_y+\ell)-\frac{S_x}{d_x+\ell}+(d_w+\ell)+(d_z+1-d_x)(d_w+\ell-1)\\&\geqslant(d_x+1-d_z)(d_y+\ell-1)+(d_y+\ell)+(d_w+\ell)+(d_z+1-d_x)(d_w+\ell-1)\end{aligned}$$

$$= (d_x - d_z)(d_y - d_w) + 2(d_y + d_w + 2\ell - 1)$$
$$> (d_x - d_z)(d_y - d_w).$$

Since $d_x \geqslant d_z$ we must have $d_x > d_z$ and $d_w > d_y$.

Therefore,

$$\begin{aligned}\frac{S_x}{d_x+\ell} &= \sum_{v\in\Gamma(x)\cap\Gamma(z)} (d_v+\ell) + \sum_{w\in W}(d_w+\ell)\\ &> \sum_{v\in\Gamma(x)\cap\Gamma(z)} (d_v+\ell) + (d_y+\ell)|\Gamma(x) - (\Gamma(z)\cup\{z\})|\\ &> \sum_{v\in\Gamma(x)\cap\Gamma(z)} (d_v+\ell) + (d_y+\ell)|\Gamma(z) - (\Gamma(x)\cup\{x\})|\\ &\geqslant \frac{S_z}{d_z+\ell},\end{aligned}$$

contradicting (2). □

In order to state the next lemma, we need another definition. The graph $G(d_1, d_2, \dots, d_N)$ has vertex set defined as the disjoint union

$$\bigcup_{0\leqslant j\leqslant N} I_j,$$

where $I_0 = \{v_1, v_2, \dots, v_N\}$, $|I_j| = d_j - d_{j+1}$ for $1 \leqslant j \leqslant N-1$ and $|I_N| = d_N - (N-1)$. For $1 \leqslant j \leqslant N$ we arrange that

$$\Gamma(v_j) = (I_0 - \{v_j\}) \cup \left(\bigcup_{j\leqslant k\leqslant N} I_k\right)$$

and

$$E\left(G\left[\bigcup_{1\leqslant j\leqslant N} I_j\right]\right) = \emptyset,$$

so that $d(v_j) = d_j$ for all j and

$$e(G(d_1, d_2, \dots, d_N)) = \sum_{i=1}^{N} d_i - \binom{N}{2}.$$

We will, of course, always have $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_N \geqslant N-1$. Each of these graphs, of order n, say, is the unique realization of a sequence corresponding to a vertex of the polytope K^n of degree sequences in E^n, as defined in [4]. Let F denote the family of graphs of the form $G(d_1, d_2, \dots, d_N)$ for $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_N \geqslant N-1$.

Lemma 3. *Let k, ℓ and r be positive integers with $0 < r \leqslant k$. If G is a graph of minimal degree at least one, having largest $(\ell, 1)$-weight among all graphs of size $m = \binom{k}{2} + r$, then $G \in F$.*

Proof. Suppose G is as in the hypotheses of the lemma and write $|G|=n$. We define a sequence $G=G_0,G_1,G_2,\ldots$ of graphs as follows. From Lemma 2 we know that $\Delta(G)=n-1$. Suppose that $d_G(x_1)=n-1$. The graph $G-\{x_1\}$ consists of a graph G_1 with no isolated vertices, together with a set J_1 of isolated vertices. If G_1 is the null graph, we are done. Otherwise we calculate

$$w_{(\ell,1)}(G)=(n+\ell-1)\sum_{x_1y\in E(G)}(d_y+\ell)+\sum_{yz\in E(G_1)}(d_y+\ell)(d_z+\ell) \tag{4}$$

$$=(n+\ell-1)(2m-(n-1)+(n-1)\ell)+w_{(\ell+1,1)}(G_1) \tag{5}$$

$$=(n+\ell-1)(2m+(\ell-1)(n-1))+w_{(\ell+1,1)}(G_1). \tag{6}$$

We claim that $\Delta(G_1)=|G_1|-1$. For if not we can use the proof of Lemma 2 to replace G_1 by a graph G_1' on the same vertex set as G_1 satisfying

$$e(G_1')=e(G_1)$$

and

$$w_{(\ell+1,1)}(G_1')>w_{(\ell+1,1)}(G),$$

and thereby produce a graph $G'=(V(G),E(G)\cup E(G_1')-E(G_1))$ with

$$e(G')=e(G)$$

and

$$w_{(\ell,1)}(G')>w_{(\ell,1)}(G).$$

Suppose that $d_{G_1}(x_2)=|G_1|-1$. Then the graph $G_1-\{x_2\}$ consists of a graph G_2 with no isolated vertices, together with a set J_2 of isolated vertices. If G_2 is the null graph then $G=G(d_G(x_1),d_G(x_2))$ and we are done. Otherwise we continue and find a sequence of vertices $\{x_3,x_4,\ldots\}$ and graphs $\{G_3,G_4,\ldots\}$. Eventually, the process terminates with a vertex $x_N\in V(G_{N-1})$ joined to a set J_N of isolated vertices. We then have $G=G(d_G(x_1),d_G(x_2),\ldots,d_G(x_N))\in F$. $\square$

For example, the only graphs in F of size 6 are $G(6),G(5,2),G(4,3)$ and $G(3,3,3)$ with $(\ell,1)$-weights $36+42\ell+6\ell^2, 39+36\ell+6\ell^2, 44+34\ell+6\ell^2$ and $54+36\ell+6\ell^2$, respectively. For $0\leqslant\ell\leqslant 2$, $G(3,3,3)$ has largest $(\ell,1)$-weight, while when $\ell=3$ we have

$$w_{(3,1)}(G(3,3,3))=w_{(3,1)}(G(6))>w_{(3,1)}(G(5,2))>w_{(3,1)}(G(4,3)),$$

and when $l\geqslant 4$, $G(6)$ has largest weight.

The final ingredient in the proof of our main theorem is a technical inequality concerning decreasing sequences of integers.

Lemma 4. *Let $d_1,d_2,\ldots,d_N$ be positive integers satisfying*

$$\sum_{i=1}^{N}d_i=cN+l,\quad l<N,\ d_1\geqslant d_2\geqslant\cdots\geqslant d_N\geqslant N-1. \tag{7}$$

Then

$$\sum_{i=1}^{N}(i-1)d_i^2 \leqslant \binom{N}{2}c^2 + \binom{l}{2}(2c+1), \tag{8}$$

obtained by setting

$$d_1 = d_2 = \cdots = d_l = c+1, \quad d_{l+1} = d_{l+2} = \cdots = d_N = c, \tag{9}$$

in other words making $(d_1, d_2, \ldots, d_N)$ *a balanced sequence.*

Proof. We use induction on N. If $N=2$ we have to maxmize d_2^2 subject to $d_1 \geqslant d_2$ with $d_1 + d_2$ fixed, so we should make d_1 and d_2 as equal as possible. Thus the induction starts.

Consider now a fixed $N \geqslant 3$ and assume that balanced sequences maximize f for smaller values of N. Take an optimal sequence $(d_1, d_2, \ldots, d_N)$ satisfying (7), and write $d_N = b = a - x$. Then

$$\sum_{i=1}^{N}(i-1)d_i^2 = \sum_{i=1}^{N-1}(i-1)d_i^2 + (N-1)b^2,$$

and so $\sum_{i=1}^{N-1}(i-1)d_i^2$ is maximal subject to the constraints $\sum_{i=1}^{N-1} d_i = cN + l - b$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{N-1}$. Therefore, by the induction hypothesis, $(d_1, d_2, \ldots, d_{N-1})$ is balanced, so that

$$d_1 = d_2 = \cdots = d_m = a+1, \qquad d_{m+1} = \cdots = d_{N-1} = a \tag{10}$$

with

$$Na + m - x = cN + l. \tag{11}$$

For notational simplicity, write

$$f(d_1, d_2, \ldots, d_N) = \sum_{i=1}^{N}(i-1)d_i^2.$$

In this notation, we must show that

$$f(a+1, a+1, \ldots, a+1, a, a, \ldots, a, a-x) \leqslant f(c+1, c+1, \ldots, c+1, c, c, \ldots, c).$$

Setting

$$\begin{aligned} F(N,a,m,x,c,l) = \binom{N}{2}c^2 + \binom{l}{2}(2c+1) - \binom{N}{2}a^2 \\ - \binom{m}{2}(2a+1) + (N-1)(2a-x)x, \end{aligned}$$

we need that

$$F(N,a,m,x,c,l) \geqslant 0, \tag{12}$$

provided the following four conditions hold:

$$c = a - \frac{x+l-m}{N}, \tag{13}$$

$$0 \leqslant m \leqslant N-2, \tag{14}$$

$$1 \leqslant l \leqslant N-1, \tag{15}$$

$$1 \leqslant x \leqslant a - N+1. \tag{16}$$

Here, (16) comes from the condition $b \geqslant N-1$, and we can suppose that $l>0$ since

$$f(c+1,c,c,\dots,c) = f(c,c,\dots,c),$$

while our proof will show that

$$f(a+1,a+1,\dots,a+1,a,a,\dots,a,a-x+1) \leqslant f(c+1,c,c,\dots,c),$$

which will give

$$f(a+1,a+1,\dots,a+1,a,a,\dots,a,a-x) < f(c,c,\dots,c).$$

Further, since a and c are integers, (13), (14), and (15) imply $c \leqslant a$.

The calculations involved in the proof of (12) are fairly lengthy, so we only outline them below.

It is convenient to deal with the cases $c=a$ and $c=a-1$ separately. When $c=a$, (13) implies $m=x+l$ and (12) reduces to an inequality $F_0(N,a,x,l) \leqslant 0$, where F_0 increases with l. When l is as large as possible, that is when $l=N-2-x$, this inequality is easily checked.

If $c=a-1$, (13) implies that $m=x+l-N$, and (12) becomes an inequality $F_1(N,a,x,l) \geqslant 0$. Differentiating F_1 with respect to l shows that F_1 is minimized when l and m are approximately equal and so we need only prove some simple inequalities in N,a and x.

In the following, then, we may assume $c \leqslant a-2$. Together with (13), this gives

$$x \geqslant 2N+m-1, \tag{17}$$

and, coupled with (16) and (17) implies that

$$a \leqslant 3N+m-l-1. \tag{18}$$

Differentiating (12), we find that $\partial F/\partial x$ decreases with x, so we need only check (12) when x is either as large as possible or as small as possible. As by (16) and (17),

$$2N+m-l \leqslant x \leqslant a-N+1,$$

we have to consider the cases $x=a-N+1$ and $x=2N+m-l$.

Case A. $x=a-N+1$: We can rewrite (12) as an inequality $F_2(N,a,m,l) \geqslant 0$, and $\partial F_2/\partial m$ decreases with m, so we must consider $F_2(N,a,m,l) \geqslant 0$ when m is either maximal or minimal subject to the constraints (14) and (18).

Case A1. $x=a-N+1$ *and* $m=a-3N+l+1$: In this subcase, (13) yields $c=a-2$. Relation (12) becomes $F_3(N,a,l)\geqslant 0$ and differentiation with respect to l identifies the few cases to check.

Case A2. $x=a-N+1$ *and* $m=N-2$: We may suppose (since we are not in case A1) that $m=N-2\leqslant a-3N+l$. Relation (12) is now equivalent to a new inequality $F_4(N,a,l)\geqslant 0$, and this time F_4 increases with a. Therefore we need only look at the case when a is as small as possible, and from (18) this is precisely the case

$$x=3N-2-l,\quad m=N-2,\quad a=4N-3-l,\quad c=4N-5-l. \tag{19}$$

Once again this subcase is readily checked, completing the proof of case A2.

Case A3. $x=a-N+1$ *and* $m=0$: Inequality (12) becomes $F_5(N,a,l)\geqslant 0$, where F_5 also increases with a, so the only case to examine is that where a is minimal, which is the easily checked case

$$x=2N-l,\quad m=0,\quad a=3N-l-1,\quad c=3N-l-3. \tag{20}$$

This concludes case A3 and therefore case A.

Case B. $x=2N+m-l$: From (16) we obtain

$$a\geqslant 3N+m-l+1. \tag{21}$$

Moreover, $c=a-2$. We find that if F_6 is the function obtained by substituting $x=2N+m-l$ in F then $\partial F_6/\partial a>0$. Therefore we need only check the case when a is minimal, and from (21) this is the case $a=3N+m-l-1$. But then we also have $x=a-N+1$, and we are back in case A. This concludes the proof of (12), and therefore of (8). □

Proof of Theorem 1. Lemma 3 shows that we have only to maximize

$$w_1(G(d_1,d_2,\ldots,d_N))$$

given the constraints

$$d_1\geqslant d_2\geqslant\cdots\geqslant d_N\geqslant N-1 \tag{22}$$

and

$$\sum_{i=1}^{N} d_i=m+\binom{N}{2}. \tag{23}$$

An elementary calculation gives

$$w_1(G(d_1,\ldots,d_2,\ldots,d_N))=\left(\sum_{i=1}^{N} d_i\right)^2+\sum_{i=1}^{N}(i-1)d_i^2-N(N-1)\sum_{i=1}^{N} d_i. \tag{24}$$

First we fix N, thus also fixing $\sum_{i=1}^{N} d_i$. Lemma 4 shows that with these constraints, (24) is maximized by making the d_i as equal as possible. The remainder of the proof consists of comparing such balanced sequences, each one corresponding to a different value of N. The admissible values of N all satisfy $\binom{N}{2} \leqslant m$, from (22) and (23), and we will show that taking N maximal maximizes (24). The balanced sequence for this value of N corresponds to the graph G_m in the statement of the theorem.

Suppose that N is not maximal subject to $\binom{N}{2} \leqslant m$, let $(d_1, d_2, \ldots, d_N)$ be a balanced sequence satisfying (22) and (23). Then $d_N \geqslant N$, for otherwise

$$d_N = N - 1,$$

$$d_1 \leqslant N$$

and

$$m + \binom{N}{2} = \sum_{i=1}^{N} d_i < N^2 = \binom{N+1}{2} + \binom{N}{2},$$

so that

$$m < \binom{N+1}{2},$$

a contradiction. Thus $d_N \geqslant N$. Create a new sequence by adding $d_{N+1} = N$. Conditions (22) and (23) are still valid, and the right-hand side of (24) is unchanged. Therefore, we can increase the right-hand side of (24) by balancing our new sequence, and continue until N is maximal. This completes the proof of the theorem. □

Note that if we had needed to maximize the function

$$g(d_1, d_2, \ldots, d_N) = \sum_{i=1}^{N} i d_i^2$$

instead of f, where the d_i are subject to the constraints in Lemma 4, we would make N as small as possible instead of as large as possible.

3. Graphs of maximal α-weight for $\alpha > 0$

As mentioned in the introduction, we distinguish three cases, $0 \leqslant \alpha \leqslant 1$, $1 < \alpha < 2$ and $\alpha \geqslant 2$. The following result deals with the first of these. For $m \geqslant 1$, we define k and r by the expressions

$$m = \binom{k}{2} + r,$$

$$0 < r \leqslant k,$$

and write

$$w(m) = w_1(G_m) = \binom{r}{2} k^2 + \binom{k-r}{2} (k-1)^2 + rk(k-r)(k-1) + r^2 k,$$

so that $w(m)$ is the largest possible 1-weight of a graph of size m.

Theorem 5. *Let G be a graph of size m with no isolated vertices. Then*

$$w_\alpha(G) \leqslant m^{1-\alpha} w(m)^\alpha$$

for $0 \leqslant \alpha < 1$. For $\alpha \neq 0$, we have equality if and only if G is complete.

Proof. Fix G with $e(G)=m$ and $\delta(G) \geqslant 1$. The case $\alpha=0$ is trivial. Suppose first then that $0<\alpha<1$. Setting $p=1/\alpha$ and $q=1/(1-\alpha)$, Hőlder's inequality together with Theorem 1 shows that

$$\begin{aligned} w_\alpha(G) &= \sum_{xy\in E} (d_x d_y)^\alpha 1^{1-\alpha} \leqslant \left(\sum_{xy\in E} (d_x d_y)^{\alpha p} \right)^{1/p} \left(\sum_{xy\in E} 1 \right)^{1/q} \\ &= m^{1-\alpha} w(G)^\alpha \leqslant m^{1-\alpha} w(m)^\alpha, \end{aligned}$$

with equality iff $G \cong G_m$ and all edges have equal weight, so that $m=\binom{k}{2}$ and $G \cong K_k$. $\square$

We turn to the case $1<\alpha<2$. For convenience, we define

$$w_\alpha(m) = \max\{w_\alpha(G)\colon\ e(G)=m\}.$$

Here, when maximizing $w_\alpha(G)$ over graphs G of fixed size m, it is advantageous to have some vertices of very large degree (exactly how large depends on α). We therefore consider the split graphs $S(r,s)$ which are such that

$$V(S(r,s)) = \{v_1, v_2, \ldots, v_r, v_{r+1}, v_{r+2}, \ldots, v_{r+s}\},$$

$$E(S(r,s)) = E_1 \cup E_2,$$

where

$$E_1 = \{\{v_i, v_j\}\colon\ 1 \leqslant i < j \leqslant r\},$$

$$E_2 = \{\{v_i, v_j\}\colon\ 1 \leqslant i \leqslant r, r+1 \leqslant j \leqslant r+s\},$$

so that $S(r,s)$ is simply $K_{r,s}$ with the first class "filled in". Note further that the split graphs are a subfamily of F. It seems natural to guess that, assuming m has the appropriate divisibility properties, a graph of size m and maximum α-weight is close to $S(t,(m-\binom{t}{2})/t)$ for some t. We have

$$\begin{aligned} w_\alpha\left(S\left(t, \frac{m-\binom{t}{2}}{t}\right)\right) &= \binom{t}{2}\left(t-1+\frac{m-\binom{t}{2}}{t}\right)^{2\alpha} \\ &\quad + \left(m-\binom{t}{2}\right) t^\alpha \left(t-1+\frac{m-\binom{t}{2}}{t}\right)^\alpha \end{aligned}$$

$$= \binom{t}{2}\left(\frac{m}{t}\right)^{2\alpha}\left(1+\frac{\binom{t}{2}}{m}\right)^{2\alpha}$$
$$+\left(m-\binom{t}{2}\right)m_\alpha\left(1+\frac{\binom{t}{2}}{m}\right)^{\alpha}.$$

The first term is about $\frac{1}{2}m^{2\alpha}$ while the second is at most $4m^{1+\alpha}$. A quick differentiation shows that we should take t around $1+1/(2\alpha-2)$. To summarize, when $t=1+1/(2\alpha-2)$ is an integer, and when t divides $m-\binom{t}{2}$, $S(t,(m-\binom{t}{2})/t)$, with weight asymptotically equal to

$$\frac{(2\alpha-2)^{2\alpha-2}}{2(2\alpha-1)^{2\alpha-1}}m^{2\alpha}$$

is a good candidate for an extremal graph. For comparison, if in addition $m=\binom{k}{2}$, the complete graph of size m has α-weight asymptotically equal to $2^\alpha m^{1+\alpha}$.

The proof of the next theorem relies on the observation that only the terms in which d_x and d_y are both large contribute significantly to $w_\alpha(G)=\sum_{xy\in E(G)}d_x^\alpha d_y^\alpha$: a similar observation is made in [2].

Theorem 6. *For $\alpha>1$ we have*

$$w_\alpha(m)\leqslant\frac{(2\alpha-2)^{2\alpha-2}}{2(2\alpha-1)^{2\alpha-1}}m^{2\alpha}+O(m^{2\alpha-((\alpha-1)/(\alpha+1))}.$$

In particular,

$$w_\alpha(m)\sim\frac{(2\alpha-2)^{2\alpha-2}}{2(2\alpha-1)^{2\alpha-1}}m^{2\alpha}$$

when $1/(2\alpha-2)$ is an integer.

Proof. Let G be a graph of size m. Suppose that $V(G)=\{v_1,v_2,\ldots,v_n\}$, where $d(v_i)=d_i$, and that $d_1\geqslant d_2\geqslant\cdots\geqslant d_n>0$. Write

$$S=\{i\in[n]:\ d_i>m^\gamma\},$$
$$T=\{i\in[n]:\ d_i\leqslant m^\gamma\},$$
$$W=\{v_i:\ i\in S\},$$

for some $\frac{1}{2}<\gamma<1$, so that W is the set of vertices of large degree. Now

$$\begin{aligned} w_\alpha(G) &= \sum_{xy\in E(G)}d_x^\alpha d_y^\alpha\leqslant\sum_{1\leqslant i<j\leqslant n}d_i^\alpha d_j^\alpha\\ &=\sum_{1\leqslant i<j\leqslant n, j\in S}d_i^\alpha d_j^\alpha+\sum_{1\leqslant i<j\leqslant n, j\in T}d_i^\alpha d_j^\alpha\\ &\leqslant\sum_{1\leqslant i<j\leqslant n, j\in S}d_i^\alpha d_j^\alpha+\left(\sum_{i=1}^n d_i^\alpha\right)\left(\sum_{j\in T}d_j^\alpha\right)\end{aligned}$$

$$= \sum_{1\leqslant i<j\leqslant n, j\in S} d_i^\alpha d_j^\alpha + \left(\sum_{i=1}^n d_i d_i^{\alpha-1}\right)\left(\sum_{j\in T} d_j d_j^{\alpha-1}\right)$$

$$\leqslant \sum_{1\leqslant i<j\leqslant n, j\in S} d_i^\alpha d_j^\alpha + 2m^\alpha 2m^{1+\gamma(\alpha-1)}$$

$$= \sum_{1\leqslant i<j\leqslant n, j\in S} d_i^\alpha d_j^\alpha + 4m^{2+(1+\gamma)(\alpha-1)}.$$

There are less than $2m^{1-\gamma}$ vertices in W, and so they span less than $2m^{2-2\gamma}$ edges. Writing $\beta = 1 + 2m^{1-2\gamma}$, we have

$$\sum_{1\leqslant i<j\leqslant n, j\in S} d_i^\alpha d_j^\alpha \leqslant \frac{1}{2}\left(\sum_{1\leqslant i<j\leqslant n, j\in S} d_i^{2\alpha-1} d_j + d_i d_j^{2\alpha-1}\right)$$

$$= \frac{1}{2}\left(\sum_{i,j\in S, i\neq j} d_i^{2\alpha-1} d_j\right) = \frac{1}{2}\left(\sum_{i\in S} d_i \sum_{j\in S} d_j^{2\alpha-1} - \sum_{j\in S} d_j^{2\alpha}\right)$$

$$\leqslant \frac{1}{2}\left(\beta m \sum_{j\in S} d_j^{2\alpha-1} - \sum_{j\in S} d_j^{2\alpha}\right) = \frac{1}{2}\sum_{j\in S} d_j d_j^{2\alpha-2}(\beta m - d_j)$$

$$\leqslant \frac{1}{2}\sum_{j\in S} d_j (\beta m)^{2\alpha-1}\frac{(2\alpha-2)^{2\alpha-2}}{(2\alpha-1)^{2\alpha-1}} \leqslant \frac{(2\alpha-2)^{2\alpha-2}}{2(2\alpha-1)^{2\alpha-1}}(\beta m)^{2\alpha}$$

$$= \frac{(2\alpha-2)^{2\alpha-2}}{2(2\alpha-1)^{2\alpha-1}} m^{2\alpha} + O(m^{1+2\alpha-2\gamma}).$$

Putting the pieces together, we obtain

$$w_\alpha(G) \leqslant \frac{(2\alpha-2)^{2\alpha-2}}{2(2\alpha-1)^{2\alpha-1}} m^{2\alpha} + 4m^{2+(1+\gamma)(\alpha-1)} + O(m^{1+2\alpha-2\gamma}).$$

Finally, we choose γ so that

$$2 + (1+\gamma)(\alpha-1) = 1 + 2\alpha - 2\gamma,$$

giving $\gamma = \alpha/(1+\alpha)$ and the stated result. □

When $\alpha < \frac{3}{2}$, $1/(2\alpha-2)$ is never an integer, so that the bound in Theorem 6 is not realized by the split graph $S(2,(m-1)/2)$. However, due to the simple nature of $S(2,(m-1)/2)$, it is possible to show that, at least for $\alpha \geqslant 2$, we have

$$w_\alpha(m) \sim w_\alpha\left(S\left(2, \frac{m-1}{2}\right)\right) \sim \left(\frac{m}{2}\right)^{2\alpha}.$$

First we need a simple lemma.

Lemma 7. *Let $\alpha \geqslant 2$ and let $x_1, \dots, x_n$ be positive real numbers whose sum is unity. Then*

$$\sum_{1 \leqslant i < j \leqslant n} (x_i x_j)^\alpha \leqslant 4^{-\alpha},$$

with equality iff only two x_i are non-zero, and they are both $\frac{1}{2}$.

Proof. We use induction on n. For $n=2$ the result is immediate. Suppose $n \geqslant 3$ and $x_n = \min x_i$. If $x_n = 0$, we are done by the induction hypothesis. Otherwise,

$$\begin{aligned}\sum_{1 \leqslant i < j \leqslant n} (x_i x_j)^\alpha &= \sum_{1 \leqslant i < j \leqslant n-1} (x_i x_j)^\alpha + x_n^\alpha \sum_{i=1}^{n-1} x_i^\alpha \\ &\leqslant 4^{-\alpha}(1-x_n)^{2\alpha} + x_n^\alpha (1-x_n)^\alpha \\ &= 4^{-\alpha}\{(1-x_n)^{2\alpha} + (4x_n(1-x_n))^\alpha\} \\ &\leqslant 4^{-\alpha}\{(1-x_n)^4 + (4x_n(1-x_n))^2\} \\ &< 4^{-\alpha},\end{aligned}$$

using the induction hypothesis, the fact that $1 - x_n$ and $4x_n(1-x_n)$ are both at most unity, *and* (crucially) the inequality $x_n \leqslant \frac{1}{3}$. □

Theorem 8. *For $\alpha \geqslant 2$ fixed and $m \to \infty$, we have*

$$w_\alpha(m) = \left(\frac{m}{2}\right)^{2\alpha} + O(m^{2\alpha - ((\alpha-1)/(\alpha+1))}).$$

Proof. The split graph $S(2, \frac{(m-1)}{2})$ has α-weight given by

$$w_\alpha\left(S\left(2, \frac{m-1}{2}\right)\right) = \left(\frac{m+1}{2}\right)^{2\alpha} + (m-1)(m+1)^\alpha,$$

so we need only show that

$$w_\alpha(m) \leqslant \left(\frac{m}{2}\right)^{2\alpha} + O(m^{2\alpha - ((\alpha-1)/(\alpha+1))}).$$

To this end, if $e(G) = m$ and $\frac{1}{2} < \gamma < 1$, the proof of Theorem 6 gives

$$w_\alpha(G) \leqslant \sum_{1 \leqslant i < j \leqslant n, j \in S} d_i^\alpha d_j^\alpha + 4m^{2+(1+\gamma)(\alpha-1)},$$

and this together with Lemma 7 implies that

$$w_\alpha(G) \leqslant \left(\frac{\beta m}{2}\right)^{2\alpha} + 4m^{2+(1+\gamma)(\alpha-1)}.$$

Choosing $\gamma = \alpha/(1+\alpha)$ as before, the theorem follows. □

It would be interesting to investigate the case where m is not too large and $\alpha = 1 + \varepsilon$, for small positive ε. Further, it is possible that one can prove an exact result for $\alpha \geqslant 2$.

4. Graphs of minimal α-weight for $\alpha < 0$

All we use in this section is Theorem 5 (which relies on Theorem 1) and the Cauchy–Schwarz inequality.

Theorem 9. *Let G be a graph of size m with no isolated vertices. Then*

$$w_\alpha(G) \geqslant m^{1-\alpha} w(m)^\alpha$$

for $\alpha < 0$, with equality if and only if G is complete.

Proof. Write

$$(0, \infty) = \bigcup_{j \geqslant 0} A_j,$$

where

$$A_j = [1 - 2^{j+1}, 1 - 2^j).$$

We proceed by induction on j. For $\alpha \in A_0$, we may write

$$\begin{aligned} w_\alpha(G) w_{-\alpha}(G) &= \sum_{xy \in E} (d_x d_y)^\alpha \sum_{xy \in E} (d_x d_y)^{-\alpha} \\ &\geqslant \left\{ \sum_{xy \in E} (d_x d_y)^{\alpha/2} (d_x d_y)^{-\alpha/2} \right\}^2 = m^2 \end{aligned}$$

by the Cauchy–Schwarz inequality, so that using Theorem 5

$$w_\alpha(G) \geqslant \frac{m^2}{w_{-\alpha}(G)} \geqslant m^{1-\alpha} w(m)^\alpha,$$

with equality if and only if G is complete. Assume next that $w_\alpha(G) \geqslant m^{1-\alpha} w(m)^\alpha$ for $\alpha \in A_j$, with equality iff G is complete. Take $\alpha \in A_{j+1}$. Then, again by the Cauchy–Schwarz inequality,

$$\begin{aligned} w_\alpha(G) w_1(G) &= \sum_{xy \in E} (d_x d_y)^{-\alpha} \sum_{xy \in E} d_x d_y \\ &\geqslant \left\{ \sum_{xy \in E} (d_x d_y)^{\alpha/2} (d_x d_y)^{1/2} \right\}^2 = w_{(1+\alpha)/2}(G)^2. \end{aligned}$$

Now

$$\alpha \in A_{j+1} \Leftrightarrow \frac{1+\alpha}{2} \in A_j,$$

so that

$$w_\alpha(G) w_1(G) \geqslant w_{(1+\alpha)/2}(G)^2 \geqslant m^{1-\alpha} w(m)^{1+\alpha}$$

by induction, and so

$$w_\alpha(G) \geqslant m^{1-\alpha} w(m)^\alpha,$$

with equality iff G is complete, completing the induction step. □

As mentioned in Section 1, the case $-1 \leqslant \alpha < 0$ appears in [1].

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