One Relator Groups and Regular Sectional Curvature

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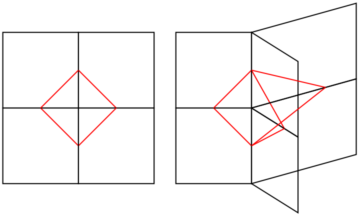
Overview

Angled 2-Complex

Definition

Let X be a 2 dimensional combinatorial CW-complex, the *link* at $v \in X^0$ is the graph $\mathrm{link}(v)$ corresponding to the “epsilon sphere” about v in X. There is a one-to-one correspondence between *corners* of 2-cells and edges in $\mathrm{link}(v)$. We say that X is angled if every *corner* c has been assigned a real number $\measuredangle c$ called the angle of c.



Curvature

Definition

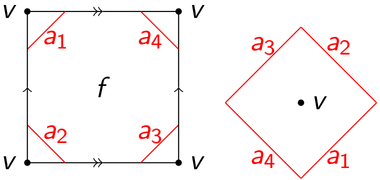
Let $|\partial f|$ denote the number of edges of a 2-cell f in X, the *curvature* of f is defined as

$$\kappa(f) = \left(\sum_{c \in Corners(f)} \measuredangle c \right) - (|\partial f| - 2)\pi \tag{1}$$

The curvature at a vertex v is defined as

$$\kappa(v) = 2\pi - \pi\chi(\text{link}(v)) - \sum_{c \in Corners(v)} \measuredangle c \tag{2}$$

Examples of Angled 2-Complexes



$$\kappa(f) = a_1 + a_2 + a_3 + a_4 - 2\pi$$

$$\begin{aligned}\kappa(v) &= 2\pi - \pi\chi(\text{link}(v)) - a_1 - a_2 - a_3 - a_4 \\ &= 2\pi - a_1 - a_2 - a_3 - a_4\end{aligned}$$

Combinatorial Gauss-Bonnet

Theorem

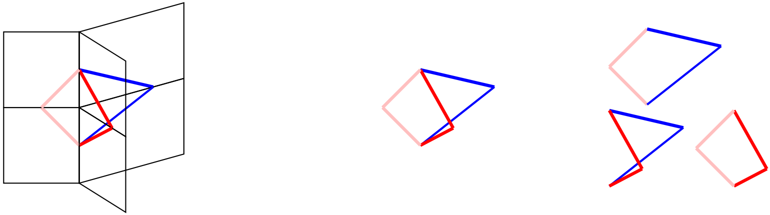
[W.Ballmann, S.Buyalo, 1996] If X is an angled 2-complex then the following equation holds

$$\sum_{f \in X^2} \kappa(f) + \sum_{v \in X^0} \kappa(v) = 2\pi\chi(X) \tag{3}$$

Sections

Definition

A *section* of X at a 0-cell x is a based immersion $(S, s) \to (X, x)$.
A *section* is called *regular* if $\mathrm{link}(s)$ is compact, connected and leafless.



Curvature of Sections

Definition

By pulling back the angles at the corners of 2-cells of X at x to corners of 2-cells of S at s we can define the *curvature of the section* $(S, s) \to (X, x)$ as $\kappa(s)$. We say that X has *sectional curvature* $\leq \alpha$ at x if all regular sections of X at x have curvature $\leq \alpha$.

We say that X has *sectional curvature* $\leq \alpha$ if each $x \in X^0$ has *sectional curvature* $\leq \alpha$ and $\kappa(f) \leq \alpha$ for each $f \in X^2$.

Motivation

Theorem

It was shown in [Wise, 2004] that the fundamental groups of angled complex with negative (non positive) regular sectional curvature have some nice properties. If the angles can also be incduced by a metric than the fundamental group has solvable generlized word problem.

Definition

A G is said to have solvable generalized word problem, if given any finitely generated subgroup H there is an algorithm that determines if a given $g \in G$ lies in H or not.

One-Relator Groups

Definition

A *one-relator group* is a group with a presentation of the form $\langle S \mid r \rangle$ where r is the single relation and $r \in F(S)$ where $F(S)$ is the free group on the generating set S

Example

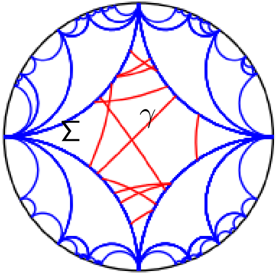
$\langle a, b \mid aba^{-1}b^{-1} \rangle, \langle a, b, c \mid acba^{-1}c^{-1}a \rangle$

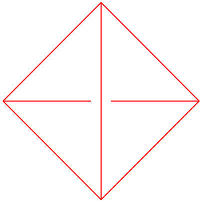
Definition

Every element of a free group $F(S)$ can be written uniquely as a *reduced* word in S. A word is *cyclically reduced* if every cyclic permutation of the word is *reduced*

2-Complex Construction

- Let G be a one-relator group with rank r.
- Take a surface Σ such that $\pi_1\Sigma$ is free of rank r.
- let γ be a curve of minimal intersection on Σ.
- Attach a 2-cell D along γ, such that $\partial D \to \gamma$ is an immersion.
- 0-cells are the intersections .





Conjecture

The probability that a one relator group arises as the fundamental group of a negatively (non positively) curved angled 2-complex tends to 1 as the length of the relator tends to ∞.

For the case of groups with 2 generators it can be shown that no presentation can arise as the fundamental group of an angled 2-complex of negative sectional curvature.

A Lower Bound

Let D be the attaching disc, and let f be a face on the surface Σ, suppose $\kappa(f) \leq 0$ and $\kappa(v) \leq 0$ for any $v \in X^0$. The combinatorial Gauss-Bonnet theorem gives

$$
\begin{aligned}
2\pi\chi(X) &= \kappa(D) + \sum_{f \subset \Sigma, f \in X^2} \kappa(f) + \sum_{v \in X^0} \kappa(v) \\
&\leq \kappa(D)
\end{aligned}
$$

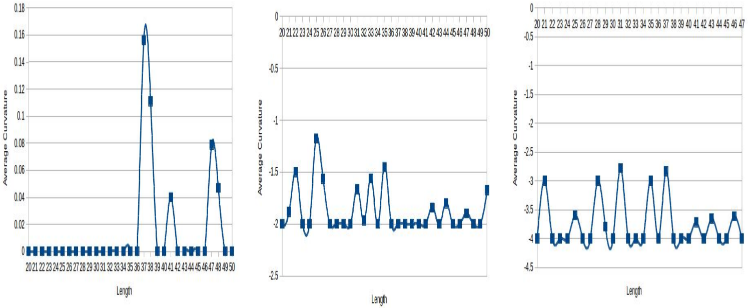
The 2 generator case is when $\chi(X) = 0$.

First Implementation for Generating Examples

1. Randomly generate cyclically reduced word.
2. Generate curve of minimal intersection for given word (Chris Arettines).
3. Find intersections and regions on surface.
4. Generate angle contraints due to regions and links
5. Use linear programming to minimize curvature of attaching disc subject to the constraints.
6. Return minimal value for attaching disc.
7. Check if value is less than or equal to 0.

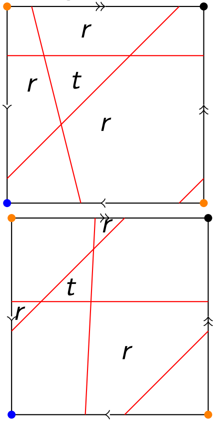
Some Plots

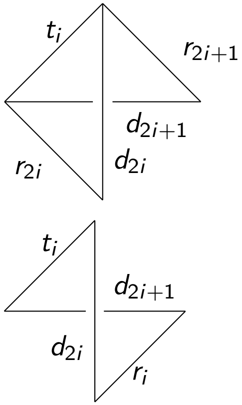
- A fixed number of examples were run for each word length, and the averages at each word length were plotted.
- The following plots seem to suggest that the curvature of attaching disc may be bounded near $2\pi\chi(X)$.
- Plots correspond to $\chi(X) = 0$, $\chi(X) = -1$, $\chi(X) = -2$



2 Examples On the Thrice Puncture Sphere

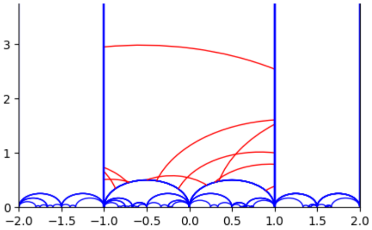
The first example was shown in [Hass, Scott, 1998] to be a configuration that can never arise as a geodesic.





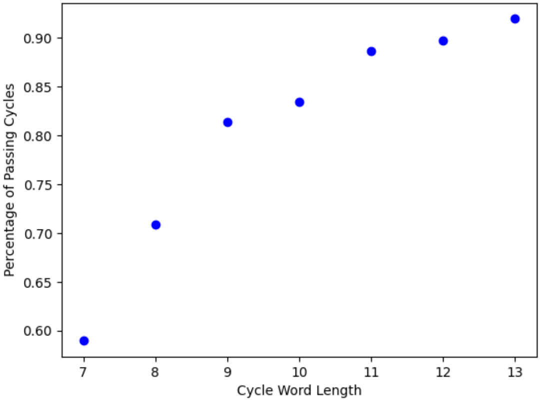
Second Implementation

- The second implementation is a work in progress.
- Similar to first implementation except curves are geodesics with respect to a hyperbolic metric.
- Examples can only be generated for punctured torus (plans to extend).
- Inequalities can be exported so that they can be imported into `polymake`.



Plot of Cycle results

This plot was generated by considering all words in a cyclic class.



Polytopes

- A different approach to the problem is to consider the polytope given by the inequalities.
- Some configurations lead to different polytope dimensions.
- This provides a new way to compare two examples.
- Curves from the same word cycle class may have polytopes of different dimension.
- For example the curves corresponding to the words BBabaabababaBA, BabaababaBAB lead to polytopes with dimension 5 and 8 respectively.

Polymatroids

Definition

A set function $f : \mathcal{P}(S) \to \mathbb{R}$, is called submodular if for any $T, U \subseteq S$, we have $f(U \cup T) + f(U \cap T) \leq f(U) + f(T)$. We say f is supermodular if $-f$ is submodular

Definition

Given a submodular function $f : \mathcal{P}(S) \to \mathbb{R}$, define the following polytope as the polymatroid polytope associated to f.

$$\mathrm{P}_f = \{x \in \mathbb{R}^S \mid x(U) \geq 0, x(U) \leq f(U) \text{ for each } U \subseteq S\}$$

An analogous definition can be made for contrapolymatroids associated to a supermodular function f.

Observations

- The polytopes that arise from the inequalities of the faces contain a polymatroid polytope.
- The polytopes that arise from the inequalities at the vertices contain a contra polymatroid polytope.
- This reduces the problem of nonpositive sectional curvature to a question about the intersection of these polymatroid polytopes.

$$2\pi - \pi\chi(\mathrm{link}(v)) \leq \sum_{c \in Corners(v)} \measuredangle c$$

$$(|\partial f| - 2)\pi \geq \left(\sum_{c \in Corners(f)} \measuredangle c \right)$$

Code repository

https://github.com/antonydellavecchia/one_relator_curvature

References

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