Gauss-Newton method

- nonlinear least-squares
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Nonlinear least-squares

nonlinear least-squares (NLLS) problem: find $x \in \mathbb{R}^n$ that minimizes

$$\|r(x)\|^2 = \sum_{i=1}^{m} r_i(x)^2,$$

where $r : \mathbb{R}^n \to \mathbb{R}^m$

- $r(x)$ is a vector of 'residuals'
- reduces to (linear) least-squares if $r(x) = Ax - y$

Position estimation from ranges

estimate position $x \in \mathbb{R}^2$ from approximate distances to beacons at locations $b_1, \ldots, b_m \in \mathbb{R}^2$ *without* linearizing

- we measure $\rho_i = \|x - b_i\| + v_i$
  (v_i is range error, unknown but assumed small)
- NLLS estimate: choose $\hat{x}$ to minimize

$$\sum_{i=1}^{m} r_i(x)^2 = \sum_{i=1}^{m} (\rho_i - \|x - b_i\|)^2$$

Gauss-Newton method for NLLS

NLLS: find $x \in \mathbb{R}^n$ that minimizes $\|r(x)\|^2 = \sum_{i=1}^{m} r_i(x)^2$, where $r : \mathbb{R}^n \to \mathbb{R}^m$

- in general, very hard to solve exactly
- many good heuristics to compute *locally optimal* solution

Gauss-Newton method:

given starting guess for x
repeat
  linearize r near current guess
  new guess is linear LS solution, using linearized r
until convergence

Gauss-Newton method, more detail

- linearize r near current iterate $x^{(k)}$:

$$r(x) \approx r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)})$$

  where Dr is the Jacobian: $(Dr)_{ij} = \partial r_i / \partial x_j$

- write linearized approximation as

$$r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)}) = A^{(k)}x - b^{(k)}$$

$$A^{(k)} = Dr(x^{(k)}), \qquad b^{(k)} = Dr(x^{(k)})x^{(k)} - r(x^{(k)})$$

- at kth iteration, we approximate NLLS problem by linear LS problem:

$$\|r(x)\|^2 \approx \left\| A^{(k)}x - b^{(k)} \right\|^2$$

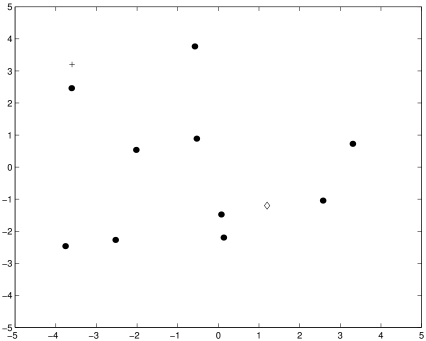
- next iterate solves this linearized LS problem:

$$x^{(k+1)} = \left(A^{(k)T} A^{(k)} \right)^{-1} A^{(k)T} b^{(k)}$$

- repeat until convergence (which *isn't* guaranteed)

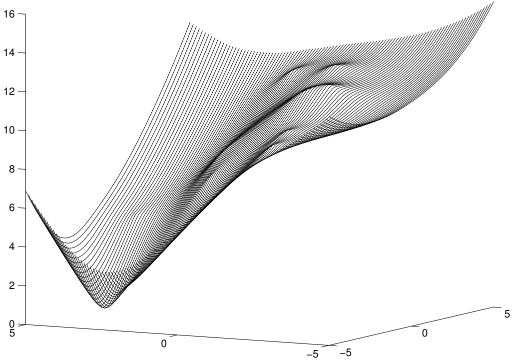
Gauss-Newton example

- 10 beacons
- $+$ true position $(-3.6, 3.2)$; $\diamondsuit$ initial guess $(1.2, -1.2)$
- range estimates accurate to ± 0.5



Example

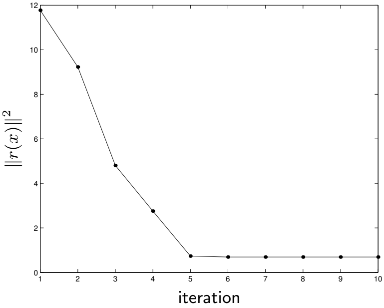
NLLS objective $\|r(x)\|^2$ versus x:



- for a linear LS problem, objective would be nice quadratic 'bowl'
- bumps in objective due to strong nonlinearity of r

Convergence

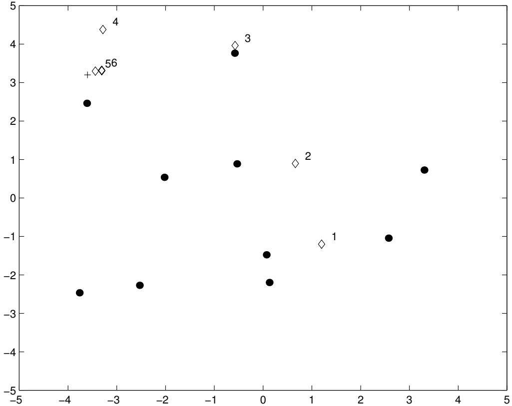
objective of Gauss-Newton iterates:



- $x^{(k)}$ converges to (in this case, global) minimum of $\|r(x)\|^2$
- convergence takes only five or so steps
- final estimate is $\hat{x} = (-3.3, 3.3)$
- estimation error is $\|\hat{x} - x\| = 0.31$, (smaller than range accuracy!)

Convergence

convergence of Gauss-Newton iterates:



Regularized Gauss-Newton

useful varation on Gauss-Newton: add regularization term

$$\|A^{(k)}x - b^{(k)}\|^2 + \mu\|x - x^{(k)}\|^2$$

so that next iterate is not too far from previous one (hence, linearized model still pretty accurate)