1. $P(A|B) = \dfrac{P(B|A) \cdot P(A)}{P(B)}$.

Buktikan pengyataan diatas.

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ dan $P(A \cap B) = P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$

Sehingga $P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$

$P(A|B) = \dfrac{P(B|A) \cdot P(A)}{P(B)}$.

2. Tentukan rumus $P(A|-B)$ dalam $P(A)$, $P(B)$, dan $P(A \cap B)$.

$P(A|-B) = \dfrac{P(-B|A)\, P(A)}{P(-B)}$; $P(-B) = 1 - P(B)$ dan $P(-B|A) = 1 - P(B|A)$

$P(B|A) = \dfrac{P(A \cap B)}{P(A)} \rightarrow 1 - P(B|A) = 1 - \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A) - P(A \cap B)}{P(A)}$

$P(A|-B) = \dfrac{\dfrac{P(A) - P(A \cap B)}{P(A)} \cdot P(A)}{1 - P(B)} = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$

3. Jika A dan B independen, buktikan
   a) A dan -B independen
   b) -A dan -B independen

A dan B independen jika $P(A|B) = P(A)$ dan $P(B|A) = B$.

a. A dan -B independen jika $P(A|-B) = P(A)$ dan $P(-B|A) = P(-B)$.

$$P(A|-B) = \frac{P(-B|A).P(A)}{P(-B)} = \frac{[1-P(B|A]. P(A)}{1-P(B)}$$; karena $P(B|A) = P(B)$,

maka $$\frac{[1-P(B|A]. P(A)}{1-P(B)} = \frac{[1-P(B)]\,P(A)}{1-P(B)} = P(A)$$

$$P(-B|A) = 1 - P(B|A) = 1 - \frac{P(B\cap A)}{P(A)} = \frac{P(A) - P(B|A).P(A)}{P(A)}$$

$$= \frac{\cancel{P(A)}\,[1-P(B|A)]}{\cancel{P(A)}} = 1 - P(B|A) = 1 - P(B) = P(-B).$$

Karena $P(A|-B) = P(A)$ dan $P(-B|A) = P(-B)$ maka A dan -B independen.

b. -A dan -B ~~tdk~~ Independen jika $P(-A|-B) = P(-A)$ dan $P(-B|-A) = P(-B)$

$P(-A|-B) = 1 - P(A|-B)$; dari 3a $P(A|-B) = P(A)$ sehingga
$1 - P(A|-B) = 1 - P(A) = P(-A)$.

$$P(-B|-A) = \frac{P(-A|-B).P(-B)}{P(-A)} = \frac{[1-P(A|-B)].P(-B)}{P(-A)}$$; karena $P(A|-B) = P(A)$

maka $$\frac{[1-P(A|-B)].P(-B)}{P(-A)} = \frac{[1-P(A)]\,P(-B)}{P(-A)} = \frac{P(-A)}{P(-A)}P(-B) = P(-B)$$

Karena $P(-A|-B) = P(-A)$ dan $P(-B|-A) = P(-B)$ maka -A dan -B adalah independen.

4.

	$X=\alpha$	$X=\beta$
$Y=\delta$	15	20
$Y=\eta$	75	30

a. $p(x=\alpha) = \frac{15+75}{140} = \frac{90}{140}$

b. $p(x \neq \beta) = 1 - p(x=\beta) = p(x=\alpha) = \frac{90}{140}$

c. $p(x=\beta \text{ atau } Y=\delta) = p(x=\beta) + p(Y=\delta) - p(x=\beta, Y=\delta)$

$= \frac{50}{140} + \frac{35}{140} - \frac{20}{140} = \frac{65}{140}$

d. $p(x=\alpha \text{ atau } Y=\eta) = p(x=\alpha) + p(Y=\eta) - p(x=\alpha, Y=\eta)$

$= \frac{90}{140} + \frac{105}{140} - \frac{75}{140} = \frac{120}{140}$

e. $p(x=\beta \mid Y=\eta) = \frac{p(x=\beta, Y=\eta)}{p(Y=\eta)} = \frac{\frac{30}{140}}{\frac{105}{140}} = \frac{30}{105}$

5. Peluang orang terkena kanker $= 0{,}01$, Peluang Tes positif jika orang terkena Kanker $= 0{,}98$.

Peluang tes negatif jika orang tidak kanker $= 0{,}97$.

$P(K) = 0{,}01 \rightarrow P(-K) = 0{,}99$;

$P(P \mid K) = 0{,}98 \rightarrow P(-P \mid K) = 0{,}02$

$P(-P \mid -K) = 0{,}97 \rightarrow P(P \mid -K) = 0{,}03$

Peluang orang terkena kanker jika hasil tes positif adalah $P(K \mid P)$.

$$P(K \mid P) = \frac{P(P \mid K) \cdot P(K)}{P(P)} = \frac{0{,}98 \cdot 0{,}01}{P(P)}$$

	K	-K
P	$P \cap K$	$P \cap -K$
-P	$K \cap P$	$-P \cap -K$

$$\begin{aligned} P(P) &= P(P \cap K) + P(P \cap -K) \\ &= P(P \mid K) \cdot P(K) + P(P \mid -K) \cdot P(-K) \\ &= 0{,}98 \cdot 0{,}01 + 0{,}03 \cdot 0{,}99 \end{aligned}$$

sehingga $P(K \mid P) = \dfrac{0{,}98 \cdot 0{,}01}{0{,}98 \cdot 0{,}01 + 0{,}03 \cdot 0{,}99} = 0{,}24$

6. $P(x_1 \dots x_n | \sigma, \mu) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right)$

tentukan μ_{map}.

~~$p(x_1 \dots x_n | \sigma, \mu) \Rightarrow$~~ $\mu_{map} = \arg\max P(x_1 \dots x_n | \mu, \sigma) \cdot p(\mu, \sigma)$.

$$p(\mu) = \frac{1}{\sigma_p\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\mu-\mu_p}{\sigma_p}\right)^2\right)$$

$$p(x_1 \dots x_n | \mu) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right).$$

$$p(\mu, \sigma | x_1 \dots x_n) = \frac{p(x_1 \dots x_n | \mu, \sigma) \cdot p(\mu, \sigma)}{p(x_1 \dots x_n)} \approx p(x_1 \dots x_n | \mu, \sigma) \cdot p(\mu, \sigma)$$

$= \frac{1}{\sigma_p\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\mu-\mu_p}{\sigma_p}\right)^2\right) \cdot \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right)$; untuk mencari μ_{max}, persamaan diatas diturunkan dan hasil turunannya sama dengan nol.

$$\ln(P(\mu, \sigma | x_1 \dots x_n)) = \ln\left(\frac{1}{\sigma_p\sqrt{2\pi}}\right) + \ln \exp\left(-\frac{1}{2}\left(\frac{\mu-\mu_p}{\sigma_p}\right)^2\right) + \ln\left[\prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right)\right]$$

$$= \ln\left(\frac{1}{\sigma_p\sqrt{2\pi}}\right) + -\frac{1}{2}\left(\frac{\mu-\mu_p}{\sigma_p}\right)^2 + \ln\left[\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n\right] + \ln\left[\prod_{i=1}^{n}\left(-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right)\right].$$

$$= \ln\left(\frac{1}{\sigma_p\sqrt{2\pi}}\right) + -\frac{1}{2}\left(\frac{\mu-\mu_p}{\sigma_p}\right)^2 + n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \sum_{i=1}^{n} -\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2;$$

$\frac{d}{d\mu}\left[\ln P(\mu, \sigma | x_1 \dots x_n)\right] = 0$, sehingga.

$$\frac{d}{d\mu}\ln\left(\frac{1}{\sigma_p\sqrt{2\pi}}\right) + \frac{d}{d\mu} -\frac{1}{2}\left(\frac{\mu-\mu_p}{\sigma_p}\right)^2 + \frac{d}{d\mu} n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \frac{d}{d\mu}\sum_{i=1}^{n} -\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2 = 0$$

$$\Leftrightarrow 0 + -\frac{(\mu-\mu_p)}{\sigma_p^2} + 0 + \sum_{i=1}^{n} \frac{d}{d\mu}\left(-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right) = 0$$

$$\Leftrightarrow -\left(\frac{\mu-\mu_p}{\sigma_p^2}\right) + \sum_{i=1}^{n} -\frac{(x_i-\mu)}{\sigma^2} = 0 \Leftrightarrow \frac{\sum_{i=1}^{n}\mu - \sum_{i=1}^{n} x_i}{\sigma^2} = \frac{\mu-\mu_p}{\sigma_p^2} \Leftrightarrow \frac{n\mu - \sum_{i=1}^{n} x_i}{\sigma^2} = \frac{\mu-\mu_p}{\sigma_p^2}.$$

$$\Leftrightarrow \frac{\mu_p}{\sigma_p^2} + \frac{\sum_{i=1}^{n} x_i}{\sigma^2} = \frac{\mu}{\sigma_p^2} + \frac{n\mu}{\sigma^2} \Leftrightarrow \frac{\sigma^2\mu_p + \left(\sum_{i=1}^{n} x_i\right)\sigma_p^2}{\cancel{\sigma_p^2\sigma^2}} = \frac{\sigma^2\mu + \sigma_p^2 n\mu}{\cancel{\sigma_p^2\sigma^2}} \Leftrightarrow$$

$$\sigma^2\mu_p + \sigma_p^2\sum_{i=1}^{n} x_i = \mu(\sigma^2 + \sigma_p^2 n). \Leftrightarrow \mu = \frac{\sigma^2\mu_p + \sigma_p^2\sum_{n=1}^{n} x_i}{\sigma^2 + \sigma_p^2 n}$$