

Linear functions

Linear equations

consider system of linear equations

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\vdots$$

$$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

can be written in matrix form as $y = Ax$, where

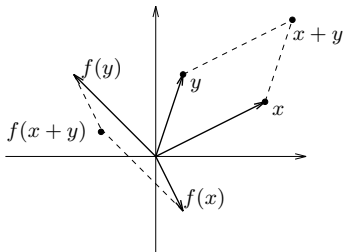
$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Linear functions

a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *linear* if

- ▶ $f(x + y) = f(x) + f(y), \forall x, y \in \mathbb{R}^n$
- ▶ $f(\alpha x) = \alpha f(x), \forall x \in \mathbb{R}^n \forall \alpha \in \mathbb{R}$

i.e., *superposition* holds



Matrix multiplication function

- ▶ consider function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $f(x) = Ax$, where $A \in \mathbb{R}^{m \times n}$
- ▶ matrix multiplication function f is linear
- ▶ **converse** is true: **any** linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be written as $f(x) = Ax$ for some $A \in \mathbb{R}^{m \times n}$
- ▶ representation via matrix multiplication is unique: for any linear function f there is only one matrix A for which $f(x) = Ax$ for all x
- ▶ $y = Ax$ is a concrete representation of a generic linear function

Interpretations of $y = Ax$

- ▶ y is measurement or observation; x is unknown to be determined
- ▶ x is 'input' or 'action'; y is 'output' or 'result'
- ▶ $y = Ax$ defines a function or transformation that maps $x \in \mathbb{R}^n$ into $y \in \mathbb{R}^m$

Interpretation of a_{ij}

$$y_i = \sum_{j=1}^n a_{ij} x_j$$

a_{ij} is *gain factor* from j th input (x_j) to i th output (y_i)

- ▶ i th *row* of A concerns i th *output*
- ▶ j th *column* of A concerns j th *input*
- ▶ $a_{27} = 0$ means 2nd output (y_2) doesn't depend on 7th input (x_7)
- ▶ $|a_{31}| \gg |a_{3j}|$ for $j \neq 1$ means y_3 depends mainly on x_1
- ▶ $|a_{52}| \gg |a_{i2}|$ for $i \neq 5$ means x_2 affects mainly y_5
- ▶ A is lower triangular, i.e., $a_{ij} = 0$ for $i < j$, means y_i only depends on x_1, \dots, x_i
- ▶ A is diagonal, i.e., $a_{ij} = 0$ for $i \neq j$, means i th output depends only on i th input

more generally, **sparsity pattern** of A , i.e., list of zero/nonzero entries of A , shows which x_j affect which y_i