Linear dynamical systems with inputs & outputs

- inputs & outputs: interpretations
- impulse and step responses
- examples

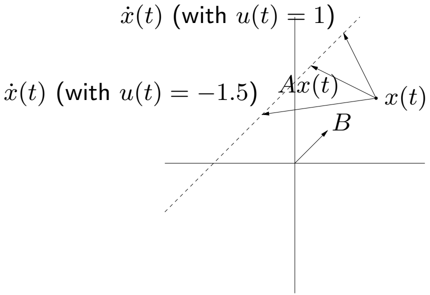
Inputs & outputs

recall continuous-time time-invariant LDS has form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

- Ax is called the *drift term* (of $\dot{x}$)
- Bu is called the input term (of $\dot{x}$)

picture, with $B \in \mathbb{R}^{2\times 1}$:



Interpretations

write $\dot{x} = Ax + b_1 u_1 + \cdots + b_m u_m$, where $B = \left[\begin{array}{ccc} b_1 & \cdots & b_m \end{array}\right]$

- state derivative is sum of autonomous term (Ax) and one term per input $(b_i u_i)$
- each input u_i gives another degree of freedom for $\dot{x}$ (assuming columns of B independent)

write $\dot{x} = Ax + Bu$ as $\dot{x}_i = \tilde{a}_i^\mathsf{T} x + \tilde{b}_i^\mathsf{T} u$, where $\tilde{a}_i^\mathsf{T}$, $\tilde{b}_i^\mathsf{T}$ are the rows of A, B

- ith state derivative is linear function of state x and input u

Response to input

- the solution to $\dot{x} = Ax + Bu$ is

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\, d\tau$$

- $e^{tA}x(0)$ is the unforced or autonomous response
- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix

Impulse response

impulse response $h(t) = Ce^{tA}B + D\delta(t)$

with $x(0) = 0$, $y = h * u$, *i.e.*,

$$y_i(t) = \sum_{j=1}^{m} \int_0^t h_{ij}(t - \tau)u_j(\tau)\, d\tau$$

interpretations:

- $h_{ij}(t)$ is impulse response from jth input to ith output
- $h_{ij}(t)$ gives $y_i(t)$ when $u(t) = e_j\delta(t)$
- $h_{ij}(\tau)$ shows how dependent output i is, on what input j was, τ seconds ago
- i indexes output; j indexes input; τ indexes time lag

Step response

the *step response* or *step matrix* is given by

$$s(t) = \int_0^t h(\tau)\, d\tau$$

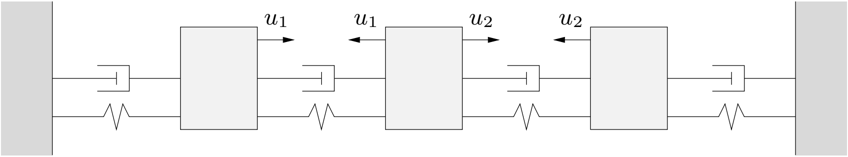
interpretations:

- $s_{ij}(t)$ is step response from jth input to ith output
- $s_{ij}(t)$ gives y_i when $u = e_j$ for $t \geq 0$

for invertible A, we have

$$s(t) = CA^{-1}\left(e^{tA} - I\right)B + D$$

Example 1



- unit masses, springs, dampers
- u_1 is tension between 1st & 2nd masses
- u_2 is tension between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

Example 1

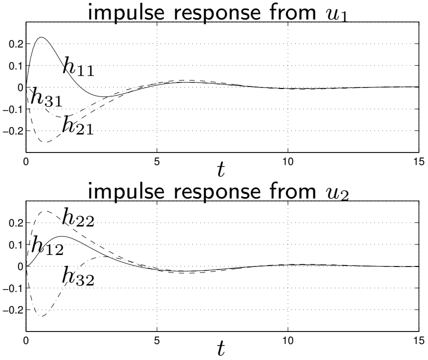
system is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

eigenvalues of A are

$$-1.71 \pm i0.71, \qquad -1.00 \pm i1.00, \qquad -0.29 \pm i0.71$$

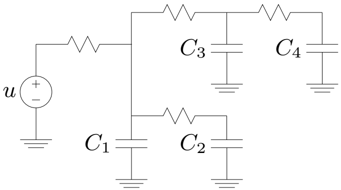
Example 1: Impulse response



roughly speaking:

- impulse at u_1 affects third mass less than other two
- impulse at u_2 affects first mass later than other two

Example 2: Circuit



- $u(t) \in \mathbb{R}$ is input (drive) voltage
- x_i is voltage across C_i
- output is state: $y = x$
- unit resistors, unit capacitors
- step response matrix shows delay to each node

Example 2: Circuit

system is

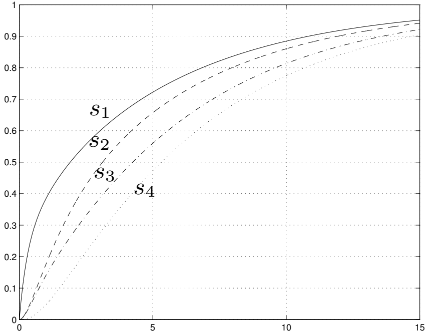
$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = x$$

eigenvalues of A are

$$-0.17, \qquad -0.66, \qquad -2.21, \qquad -3.96$$

Example 2: Circuit

step response matrix $s(t) \in \mathbb{R}^{4\times 1}$:



- shortest delay to x_1; longest delay to x_4
- delays consistent with slowest ($i.e.$, dominant) eigenvalue -0.17

DC or static gain matrix

- DC gain describes system under *static* conditions, $i.e.$, x, u, y constant:

$$0 = \dot{x} = Ax + Bu, \qquad y = Cx + Du$$

eliminate x to get $y = H_0 u$ where

$$H_0 = -CA^{-1}B + D$$

- if system is stable,

$$H_0 = \int_0^\infty h(t)\, dt = \lim_{t \to \infty} s(t)$$

if $u(t) \to u_\infty \in \mathbb{R}^m$, then $y(t) \to y_\infty \in \mathbb{R}^p$ where $y_\infty = H_0 u_\infty$

DC gain matrix

DC gain matrix for example 1 (springs):

$$H(0) = \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{bmatrix}$$

DC gain matrix for example 2 (RC circuit):

$$H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(do these make sense?)

Discretization with piecewise constant inputs

linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$

suppose $u_d : \mathbb{Z}_+ \to \mathbb{R}^m$ is a sequence, and

$$u(t) = u_d(k) \qquad \text{for} \quad kh \leq t < (k+1)h, \quad k = 0, 1, \ldots$$

define sequences

$$x_d(k) = x(kh), \qquad y_d(k) = y(kh), \qquad k = 0, 1, \ldots$$

- $h > 0$ is called the *sample interval* (for x and y) or *update interval* (for u)
- u is piecewise constant (called *zero-order-hold*)
- x_d, y_d are sampled versions of x, y

Discretization with piecewise constant inputs

$$
\begin{aligned}
x_d(k+1) &= x((k+1)h) \\
&= e^{hA}x(kh) + \int_0^h e^{\tau A} Bu((k+1)h - \tau)\, d\tau \\
&= e^{hA}x_d(k) + \left(\int_0^h e^{\tau A}\, d\tau\right) B\, u_d(k)
\end{aligned}
$$

x_d, u_d, and y_d satisfy discrete-time LDS equations

$$
x_d(k+1) = A_d x_d(k) + B_d u_d(k), \qquad y_d(k) = C_d x_d(k) + D_d u_d(k)
$$

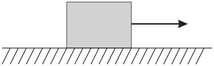
where

$$
A_d = e^{hA}, \qquad B_d = \left(\int_0^h e^{\tau A}\, d\tau\right) B, \qquad C_d = C, \qquad D_d = D
$$

called *discretized system*. If A is invertible, we can express integral as

$$
\int_0^h e^{\tau A}\, d\tau = A^{-1}\left(e^{hA} - I\right)
$$

Example: Force on mass



Newton's law gives continuous-time LDS

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

let's compute the discretization

$$\begin{aligned} A_d &= e^{Ah} \\ &= I + Ah + \frac{1}{2}A^2h^2 + \cdots \\ &= I + Ah \\ &= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Example: Force on mass

$$
\begin{aligned}
B_d &= \int_0^h e^{As} B \, ds \\
&= \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ds \\
&= \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix}
\end{aligned}
$$

so the discretization is

$$
\begin{aligned}
x_d(k+1) &= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x_d(k) + \begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix} u_d(k) \\
y_d(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_d(k)
\end{aligned}
$$

Stability of discretization

stability: if eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, then eigenvalues of A_d are $e^{h\lambda_1}, \ldots, e^{h\lambda_n}$

discretization preserves stability properties since

$$\Re\lambda_i < 0 \quad \Leftrightarrow \quad |e^{h\lambda_i}| < 1$$

for $h > 0$

Extensions and variations

- *offsets:* updates for u and sampling of x, y are offset in time
- *multirate:* u_i updated, y_i sampled at different intervals
  (usually integer multiples of a common interval h)

both very common in practice

Causality

interpretation of

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\, d\tau$$
$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau)\, d\tau + Du(t)$$

for $t \geq 0$:

current state ($x(t)$) and output ($y(t)$) depend on *past* input ($u(\tau)$ for $\tau \leq t$)

$i.e.$, mapping from input to state and output is *causal* (with fixed *initial* state)

Idea of state

$x(t)$ is called *state* of system at time t since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs

Change of coordinates

start with LDS $\dot{x} = Ax + Bu$, $y = Cx + Du$

change coordinates in $\mathbb{R}^n$ to $\tilde{x}$, with $x = T\tilde{x}$

then

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

hence LDS can be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad y = \tilde{C}\tilde{x} + \tilde{D}u$$

where

$$\tilde{A} = T^{-1}AT, \qquad \tilde{B} = T^{-1}B, \qquad \tilde{C} = CT, \qquad \tilde{D} = D$$

TF is same (since u, y aren't affected):

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI - A)^{-1}B + D$$

Standard forms for LDS

can change coordinates to put A in various forms (diagonal, real modal, Jordan $\ldots$)

$e.g.$, to put LDS in *diagonal form*, find T s.t.

$$T^{-1}AT = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^\mathsf{T} \\ \vdots \\ \tilde{b}_n^\mathsf{T} \end{bmatrix}, \qquad CT = \begin{bmatrix} \tilde{c}_1 & \cdots & \tilde{c}_n \end{bmatrix}$$

Discrete-time systems

discrete-time LDS:

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

we have:

$$x(1) = Ax(0) + Bu(0),$$

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) \\ &= A^2x(0) + ABu(0) + Bu(1), \end{aligned}$$

and in general, for $t \in \mathbb{Z}_+$,

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau)$$

Discrete-time systems

Solution is

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} B u(\tau)$$

write this as

$$y(t) = CA^t x(0) + H * u$$

where $*$ is discrete-time convolution

$$y(t) = CA^t x(0) + \sum_{\tau=0}^{t} H(t-\tau) u(\tau)$$

and

$$H(t) = \begin{cases} D, & t = 0 \\ CA^{t-1}B, & t > 0 \end{cases}$$

is the impulse response

Block Toeplitz matrices

we have

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(t) \end{bmatrix} = \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & & & \ddots & \\ CA^{t-1}B & CA^{t-2}B & \dots & CB & D \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(t) \end{bmatrix} + \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^t \end{bmatrix} x(0)$$

- this matrix gives the output sequence $y(0), y(1), \dots$ in terms of the input sequence $u(0), u(1), \dots$ and the initial state $x(0)$
- *block Toeplitz* means blocks are constant along diagonals from top-left to bottom right
- we can use this to find controllers and estimators

Example: Point mass

unit point mass, with actuators applying force in directions

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has dynamics

$$x(k+1) = \begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{2}h^2 & 0 \\ h & 0 \\ 0 & \frac{1}{2}h^2 \\ 0 & h \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_2 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k)$$

here

- $x_1, x_2 =$ position, velocity in **x**-direction
  $x_3, x_4 =$ position, velocity in **y**-direction
- $h =$ sample time; we'll use $h = 1$.
- $u_i(k)$ force applied by actuator i at time k

Example: Point mass

we would like to drive it through the positions

$$y(20) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \qquad y(40) = \begin{bmatrix} 10 \\ -1 \end{bmatrix} \qquad y(70) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

at the above times

we have

$$y(t) = \sum_{\tau=0}^{t-1} CA^{t-1-\tau} Bu(\tau) + Du(t)$$

this gives the rows of

$$\begin{bmatrix} y(20) \\ y(40) \\ y(70) \end{bmatrix} = A_{\text{act}} \begin{bmatrix} u(0) \\ \vdots \\ u(70) \end{bmatrix}$$

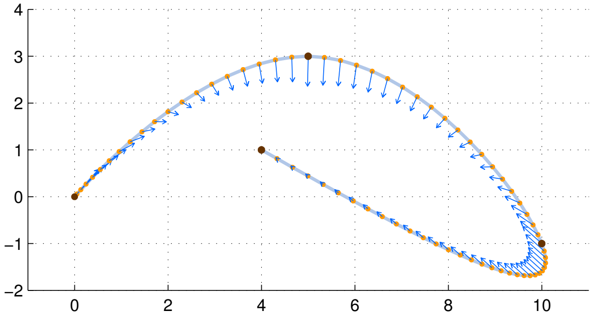
here A_{act} is 6×213.

Example: Point mass

let's find the minimum norm sequence of forces that meets the specifications

$$\begin{bmatrix} u(0) \\ \vdots \\ u(70) \end{bmatrix} = A_{\mathsf{act}}^{\dagger} \begin{bmatrix} 5 \\ 3 \\ 10 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

trajectory is



Example: Point mass

sequence of force inputs is