Here is the theorem that we are proving.

Theorem. The following properties hold:

1. If B and C are inverses of A then $B=C$. Thus we can speak about **the** inverse of a matrix A, A^{-1}.
2. If A is invertible and k is a non-zero scalar then kA is invertible and $(kA)^{-1}=1/k\ A^{-1}$.
3. If A and B are invertible then AB is invertible and

   $(AB)^{-1}=B^{-1}\ A^{-1}$

   that is the inverse of the product is the product of inverses in the opposite order. In particular

   $(A^n)^{-1}=(A^{-1})^n$.

4. $(A^T)^{-1}=(A^{-1})^T$, the inverse of the transpose is the transpose of the inverse.
5. If A is invertible then $(A^{-1})^{-1}=A$.

Proof. 1. Indeed if $AB=\mathbf{I}$, $CA=\mathbf{I}$ then

$$B=\mathbf{I}*B=(CA)B=C(AB)=C*\mathbf{I}=C.$$

3. We need to prove that if A and B are invertible square matrices then $B^{-1}A^{-1}$ is the inverse of AB. Let us denote $B^{-1}A^{-1}$ by C (we always have to denote the things we are working with). Then by definition of the inverse we need to show that $(AB)C=C(AB)=\mathbf{I}$. Substituting $B^{-1}A^{-1}$ for C we get:

$$(AB)(B^{-1}A^{-1})=ABB^{-1}A^{-1}=A(BB^{-1})A^{-1}=A\mathbf{I}A^{-1}=AA^{-1}=\mathbf{I}.$$

We used the associativity of the product of matrices, the definition of an inverse and the fact that $\mathbf{I}A=A\mathbf{I}=A$ for every matrix A.

Other properties were left as exercises.