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Low rank SVD, orthogonal projection onto range of A

Say rank(A) = r < n. Then $A = U_r S_r V_r^T$. I know the orthogonal projection onto $Ran(A)$ should be $P = U_r U_r^T$ but I'm not sure how to show this.

$$P = A(A^T A)^{-1} A^T$$

If A had full column rank, then this would be easy since both V and S are invertible. But in this reduced rank case, S_r is nonsingular but V_r is rectangular and thus, not invertible. So I'm unsure how to simplify the $(A^T A)^{-1}$ term...

Also, any hint on how to show that the projection onto $N(A^T)$ is $P_N = \tilde{U}_r \tilde{U}_r^T$? (with $U = [U_r \tilde{U}_r]$). I thought it would just be $P_N = I - U_r U_r^T$

(linear-algebra) (matrices) (svd)

edited Oct 25 '13 at 7:55

asked Oct 25 '13 at 7:14
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1 Answer

In this economic version of the SVD of $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$, the $U_r \in \mathbb{R}^{m\times r}$ has orthonormal columns, S_r is square diagonal (and nonsingular) and $V_r \in \mathbb{R}^{r\times r}$ is square orthogonal matrix.

It is easy to show, that if $A = WT$, where $W \in \mathbb{R}^{m\times r}$ and $T \in \mathbb{R}^{r\times r}$ is nonsingular, then $\text{Range}(A) = \text{Range}(W)$. To see this, simply consider $x \in \text{Range}(A)$, that is, $x = Ay$ for some $y \in \mathbb{R}^r$. Then $x = Ay = WTy = Wz$ and thus $x \in \text{Range}(W)$. Also, if $x \in \text{Range}(W)$, that is $x = Wz$ for some z, then $x = WT(T^{-1}z) = WTx = Ax$, and hence $x \in \text{Range}(A)$. Since $\text{Range}(A) \subset \text{Range}(W)$ and $\text{Range}(W) \subset \text{Range}(A)$, we have $\text{Range}(A) = \text{Range}(W)$.

Now simply set $W = U_r$ and $T = S_r V_r^T$ to get that $\text{Range}(A) = \text{Range}(U)$. Note that S_r is nonsingular and V_r is orthogonal, so indeed, $S_r V_r^T$ is nonsingular as well.

The range of U_r is the range of A and similarly you can show that the range of $P = U_r U_r^T$ is the range of A. To see that it is the orthogonal projector, verify that $P^T = P$ and $P^2 = P$.

The projector P can be expressed as $P = A(A^T A)^{-1} A^T$ provided that the rank of A is n. As above, you can show this by verifying that the range of P defined in this way is the range of A and P is orthogonal projection: $P^T = P$ and $P^2 = P$. You can also use your SVD with $r = n$ to see that

$$\begin{aligned} P = A(A^T A)^{-1} A^T &= U_n S_n V_n^T (V_n S_n U_n^T U_n S_n V_n^T)^{-1} V_n S_n U_n^T = U_n S_n V_n^T (V_n S_n^2 V_n^T)^{-1} V_n S_n U_n^T \\ &= U_n S_n V_n^T V_n S_n^{-2} V_n^T V_n S_n U_n^T = U_n S_n S_n^{-2} S_n U_n^T = U_n U_n^T. \end{aligned}$$

For the rank deficient case $(r < n)$, P is given using the Moore-Pseudo pseudo-inverse $A^\dagger$ by $P = AA^\dagger$. (When $r = n$, $A^\dagger = (A^T A)^{-1} A^T$.) There are several (maybe many) ways how to characterize it: it can be, e.g., expressed using the SVD as $A^\dagger = V_n S_n^{-1} U_n^T$. Again, you can verify that $P = AA^\dagger = U_r S_r V_r^T V_r S_r^{-1} U_r^T = U_r U_r^T$.

For the last question, note that the nullspace of A^T is the orthogonal complement of the range of A. If P is the orthogonal projector to the range of A, then $Q = I - P$ is the orthogonal projector onto its orthogonal complement, the nullspace of A^T. So as you correctly think, $Q = I - U_r U_r^T$. However, the columns of $\tilde{U}_r$ form an orthonormal basis of $\text{Null}(A^T)$. Hence Q can also be expressed as $Q = \tilde{U}_r \tilde{U}_r^T$.

It is quite straightforward to verify that if the columns of the matrix $X \in \mathbb{R}^{m\times r}$ form a basis of some subspace of $\mathbb{R}^m$, the orthogonal projector onto this subspace (that is, $\text{Range}(X)$) is given by $X(X^T X)^{-1} X^T$. If X has orthonormal columns, (that is, $X^T X = I$), the projector is simply XX^T. Since $\tilde{U}_r$ is the orthonormal basis of the orthogonal complement of the range of A, $Q = \tilde{U}_r \tilde{U}_r^T$.

edited Oct 25 '13 at 12:34 answered Oct 25 '13 at 12:28

Algebraic Pavel
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