Homework 3
CS 210

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Question	Points	Score
1	10	
2	15	
3	10	
4	5	
5	10	
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9	5	
10	10	
Total	80	

Singular Value Decomposition

1. (T&B 4.1) Determine SVDs of the following matrices (by hand calculation):

   (a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

   Answer:

   Note: Answers follow the format $A = U\Sigma V^T$

   (a) The singular values are almost ready, we only have to deal with the negative sign in position $(2,2)$ through:

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   (b) Here, we only have to deal with the order of the singular values:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   (c) In this case we know the singular values should be 2 and 0 but we have to work on how to reorder the positions:

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(d) The key in this exercise is that matrix is rank 1, so we already know that the second singular value must be 0. So,

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}$$

We also know that range(A):$\left\{\alpha \begin{pmatrix} 1 \\ -0 \end{pmatrix} \forall \alpha \in \mathbb{R}\right\}$, so we have the first column of U:

$$U = \begin{pmatrix} 1 & u_{1,2} \\ 0 & u_{2,2} \end{pmatrix}$$

Similarly, we know that $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \in null(A)$ and $\left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\|_2 = \sqrt{2}$ and $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ must be part of the *null-space* of A. So,

$$V^T = \begin{pmatrix} v_{1,1} & v_{1,2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

From here, we can elaborate to complete matrices U and V^T as orthogonal matrices and then find σ_1. Finally, we obtain:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(e) Matrix A is also rank 1 and we can follow a similar procedure that in literal (d).

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

2. Let A be an $m \times n$ singular matrix of rank r with SVD

$$A = U\Sigma V^T = \left(\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{array} \right) \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \left(\begin{array}{c} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \vdots \\ \hline \mathbf{v}_n^T \end{array} \right)$$

$$= \begin{pmatrix} \hat{U} & \tilde{U} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \hat{V}^T \\ \tilde{V}^T \end{pmatrix}$$

where $\sigma_1 \geq \ldots \geq \sigma_r > 0$, $\hat{U}$ consists of the first r columns of U, $\tilde{U}$ consists of the remaining $m - r$ columns of U, $\hat{V}$ consists of the first r columns of V, and $\tilde{V}$ consists of the remaining $n - r$ columns of V. Give bases for the spaces range(A), null(A), range(A^T) and null(A^T) in terms of the components of the SVD of A, and a brief justification.

Answer:

- range(A): $span\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ or $\hat{U}$
- null(A): $span\{\mathbf{v}_{r+1}^T, \cdots, \mathbf{v}_n^T\}$ or $\tilde{V}^T$
- range(A^T): $span\{\mathbf{v}_1^T, \cdots, \mathbf{v}_r^T\}$ or $\hat{V}$
- null(A^T): $span\{\mathbf{u}_{r+1}, \cdots, \mathbf{u}_m\}$ or $\tilde{U}$

3. Use the SVD of A to show that for an $m \times n$ matrix of full column rank n, the matrix $A(A^TA)^{-1}A^T$ is an orthogonal projector onto range(A).

Answer:

I assume that $P = P^2$ and $P = P^T$ as it was already explained at class. To see that $P = A(A^TA)^{-1}A^T$ is an orthogonal projector onto range(A), we can use the reduced SVD ($A = \hat{U}\hat{\Sigma}V^T$) in the above equation:

$$P = A(A^TA)^{-1}A^T$$

$$P = (\hat{U}\hat{\Sigma}V^T)(V\hat{\Sigma}\hat{U}^T\hat{U}\hat{\Sigma}V^T)^{-1}V\hat{\Sigma}\hat{U}^T$$

As $\hat{U}^T\hat{U} = I$:

$$P = \hat{U}\hat{\Sigma}V^T(V\hat{\Sigma}\hat{\Sigma}V^T)^{-1}V\hat{\Sigma}\hat{U}^T$$

$$P = \hat{U}\hat{\Sigma}V^T(V\hat{\Sigma}^2V^T)^{-1}V\hat{\Sigma}\hat{U}^T$$

As V is squared and orthogonal, $V^{-1} = V^T$ holds:

$$P = \hat{U}\hat{\Sigma}V^TV\hat{\Sigma}^{-2}V^TV\hat{\Sigma}\hat{U}^T$$

As $V^TV = I$:

$$P = \hat{U}\hat{\Sigma}\hat{\Sigma}^{-2}\hat{\Sigma}\hat{U}^T$$

And finally:

$$P = \hat{U}\hat{U}^T$$

From question 2 we know that range(A): $span\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ or $\hat{U}$, so if P is in function of $\hat{U}$ it must be an orthogonal projector onto range(A).

Least Squares

4. Consider the least squares problem $\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2$. Which of the following statements are necessarily true?

   (a) If $\mathbf{x}$ is a solution to the least squares problem, then $A\mathbf{x} = \mathbf{b}$.

   (b) If $\mathbf{x}$ is a solution to the least squares problem, then the residual vector $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ is in the nullspace of A^T. ✓

   (c) The solution is unique.

   (d) A solution may not exist.

   (e) None of the above.

5. (Heath 3.3) Set up the linear least squares system $A\mathbf{x} \approx \mathbf{b}$ for fitting the model function $f(t, \mathbf{x}) = x_1t + x_2e^t$ to the three data points $(1, 2), (2, 3), (3, 5)$. Is the least squares solution unique? Why or why not?

Answer:

We can set a linear systems from the above information:

$$\mathbf{x}_1 + \mathrm{e}\mathbf{x}_2 = 2$$

$$2\mathbf{x}_1 + \mathrm{e}^2\mathbf{x}_2 = 3$$

$$3\mathbf{x}_1 + \mathrm{e}^3\mathbf{x}_2 = 5$$

and in the $A\mathbf{x} = \mathbf{b}$ format:

$$\begin{pmatrix} 1 & \mathrm{e} \\ 2 & \mathrm{e}^2 \\ 3 & \mathrm{e}^3 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

It is clear that columns of A are linearly independent, so it is *full rank* and it has an unique solution.

6. (Heath 3.5) Let $\mathbf{x}$ be the solution to the linear least squares problem $A\mathbf{x} \approx \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

Let $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ be the corresponding residual vector. Which of the following three vectors is a possible value for $\mathbf{r}$? Why?

(a) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ ✓

Answer:

If $\mathbf{x}$ is the solution, then $\mathbf{r}$ must be $\perp$ to A and $A^T\mathbf{r} = 0$ holds. Just option (c) satisfies $A^T\mathbf{r} = 0$.

Orthogonal and Householder Matrices

7. (Heath 3.23) Which of the following matrices are orthogonal?

   (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ✓

   (b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ✓

   (c) $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$

   (d) $\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$ ✓

8. (Heath 3.24) Which of the following properties does an orthogonal $n \times n$ matrix necessarily have? (Circle all that apply.)

   (a) It is nonsingular. ✓

   (b) It preserves the Euclidean vector norm when multiplied times a vector. ✓

   (c) Its transpose is its inverse. ✓

(d) Its columns are orthonormal. ✓
(e) It is symmetric.
(f) It is diagonal.
(g) Its Euclidean matrix norm is 1. ✓
(h) Its Euclidean condition number is 1. ✓

9. A Householder matrix H

(a) has condition number 1.
(b) has the property $||H||_2 = 1$.
(c) is uniquely defined by $H\mathbf{x} = \mathbf{b}$ for two vector $\mathbf{x}$ and $\mathbf{b}$ such that $||\mathbf{x}||_2 = ||\mathbf{b}||_2$.
(d) Both (a) and (b).
(e) All of the above. ✓

10. Show that a $n \times n$ Householder matrix $H = I - 2\mathbf{v}\mathbf{v}^T/\mathbf{v}^T\mathbf{v}$ has an eigenvalue of 1 with multiplicity $n-1$ and an eigenvalue of -1 with multiplicity 1.

Answer:

Using the formula of H, we have:

$$H\mathbf{a} = \left(I - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)\mathbf{a}$$

$$H\mathbf{a} = \left(\mathbf{a} - 2\frac{\mathbf{v}(\mathbf{v}^T\mathbf{a})}{\mathbf{v}^T\mathbf{v}}\right)$$

and

$$\left(\mathbf{a} - 2\frac{\mathbf{v}(\mathbf{v}^T\mathbf{a})}{\mathbf{v}^T\mathbf{v}}\right) = \lambda\mathbf{a}$$

We also know that a Householder matrix has eigenvalues ± 1, so:

$$\left(\mathbf{a} - 2\frac{\mathbf{v}(\mathbf{v}^T\mathbf{a})}{\mathbf{v}^T\mathbf{v}}\right) = -\mathbf{a}$$

or

$$\left(\mathbf{a} - 2\frac{\mathbf{v}(\mathbf{v}^T\mathbf{a})}{\mathbf{v}^T\mathbf{v}}\right) = \mathbf{a}$$

So, we have to evaluate when

$$\mathbf{a} - 2\frac{\mathbf{v}(\mathbf{v}^T\mathbf{a})}{\mathbf{v}^T\mathbf{v}} \tag{1}$$

becomes positive and when it becomes negative. Following this, just in the case that $\mathbf{a} = \mathbf{v}$, we have:

$$\left(\mathbf{v} - 2\frac{\mathbf{v}(\mathbf{v}^T\mathbf{v})}{\mathbf{v}^T\mathbf{v}}\right) = \mathbf{v} - 2\mathbf{v} = -\mathbf{v}$$

On the other hand, if $\mathbf{a} = \mathbf{u}$ and $\mathbf{u} \perp \mathbf{v}$, we have

$$\left(\mathbf{u} - 2\frac{\mathbf{v}(\mathbf{v}^T\mathbf{u})}{\mathbf{v}^T\mathbf{v}}\right) = \mathbf{u}$$

since $\mathbf{v}^T\mathbf{u} = 0$ because they are perpendicular.

Therefore, we have just one case where equation 1 becomes negative and this explains the eigenvalue of -1 with multiplicity 1. However, there are $n-1$ cases where equation 1 becomes positive since there are $n-1$ independent vectors orthogonal to $\mathbf{v}$ (eigenvalue of 1 with multiplicity $n-1$).