作业三

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第一题

(a) 证明：根据 Gauss 消元法的步骤，有

$$a_{ji}^{(1)} = a_{ji} - \frac{a_{j1}}{a_{11}} a_{1i} \tag{1}$$

和

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \tag{2}$$

利用对称矩阵的性质 $a_{ij} = a_{ji}$，即可验证

$$a_{ij}^{(1)} = a_{ji}^{(1)} \tag{3}$$

(b) 没看明白题意

(c) MATLAB 程序显示如下：

```
clear, clc, clf
A = [4,-2,4,2;-2,10,-2,-7;4,-2,8,4;2,-7,4,7];
b = [8;2;16;6]';
showAb(A,b);
X = cholesky(A,b);
fprintf("X:\n");
disp(X);
```

```
%Cholesky•
 function [X]=cholesky(A,b)
[N,N]=size(A);
X=zeros(N,1);
Y=zeros(N,1);

for i=1:N
A(i,i)=sqrt(A(i,i)-A(i,1:i-1)*A(i,1:i-1)');
if A(i,i)==0 %¼
```

```
break
end

for j=i+1:N
A(j,i)=(A(j,i)-A(j,1:i-1)*A(i,1:i-1)')/A(i,i);
end
end
fprintf("A:\n");
disp(A);

%j̆´ •¨
 for j=1:N
Y(j)=(b(j)-A(j,1:j-1)*Y(1:j-1))/A(j,j);
end

%
A = A';
for k=N:-1:1
X(k)=(Y(k)-A(k,k+1:N)*X(k+1:N))/A(k,k);
end
```

输出结果为:

第二题

(a) 证明:

由题目有

$$X^{(k+1)} = (I - \omega A)X^{(k)} + Ib \tag{4}$$

设矩阵 A 的特征值为 λ_i，则矩阵 $G = I - \omega A$ 的特征值为

$$1 - \omega\lambda_i \tag{5}$$

若 Richardson 迭代方法收敛，则

$$\rho(1 - \omega\lambda_i) = \max|1 - \omega\lambda_i| < 1 \tag{6}$$

即

$$\omega < \frac{2}{\lambda_i}\ , \quad for\ \forall\lambda_i \tag{7}$$

其中 $0 < \lambda_1 <= \lambda_i <= \lambda_n$。
因此

$$\omega < 2/\lambda_n \tag{8}$$

(b) 由

$$\begin{aligned}\rho(1-\omega\lambda_i) &= \max|1-\omega\lambda_i| \\ &= \max(1-\omega\lambda_1, \omega\lambda_n - 1) \\ &= \begin{cases} 1-\omega\lambda_1 & \omega < \frac{2}{\lambda_1+\lambda_n} \\ \frac{\lambda_n-\lambda_1}{\lambda_n+\lambda_1} & \omega = \frac{2}{\lambda_1+\lambda_n} \\ \omega\lambda_n - 1 & \omega > \frac{2}{\lambda_1+\lambda_n} \end{cases}\end{aligned} \tag{9}$$

显然，ω 的最佳取值为

$$\omega_b = \frac{2}{\lambda_1+\lambda_n} \tag{10}$$

(c) MATLAB 程序显示如下：

```
clear, clc, clf
b = rand(5,1);
D = orth(rand(5,5));
B = diag([1,2,3,4,5]);
A = D\B*D;
x0 = A\b;
E = eig(A);
Emax = max(E);
Emin = min(E);
step = (2/Emax-0.002)/1000;
for omega = 0.01:step:(2/Emax -0.01)
    G = eye(5)-omega*A;
    x = zeros(5,1);
    k = 0;
    while norm(x-x0)>1e-5
        xnext = G*x + omega*b;
        e = norm(x-x0);
        x = xnext;
        enext = norm(x-x0);
        k = k+1;
    end
```

```
    roh = enext/e;
    plot(omega,roh,".");hold on
end
```

输出结果为:

第三题

(a) 由于 Gauss 积分可用 n 个采样点保证积分 $2n-1$ 阶的代数精度:

$$I(f)=\sum_{i=0}^{n}\omega_i f(x_i)=\int_{-1}^{1}f(x)\mathrm{d}x \tag{11}$$

当 $n=6$ 时，代数精度为 11，依次取 $f(x)$ 为 $1,x^2,x^4,x^6,x^8,x^{10}$，再计及节点和积分权重关于原点的对称性，有

$$\begin{cases}\sum_{i=0}^{5}\omega_i-2=0\\ \sum_{i=0}^{5}\omega_i x_i^2-2/3=0\\ \sum_{i=0}^{5}\omega_i x_i^4-2/5=0\\ \sum_{i=0}^{5}\omega_i x_i^6-2/7=0\\ \sum_{i=0}^{5}\omega_i x_i^8-2/9=0\\ \sum_{i=0}^{5}\omega_i x_i^{10}-2/11=0\\ x_0+x_5=0\\ x_1+x_4=0\\ x_2+x_3=0\\ \omega_0-\omega_5=0\\ \omega_1-\omega_4=0\\ \omega_2-\omega_3=0\end{cases} \tag{12}$$

(b) 由 Jacobian 定义可以直接写出：

$$
\begin{pmatrix}
0 & & \cdots & & & 0 & 1 & & \cdots & & & 1 \\
2\omega_0 x_0 & & \cdots & & & 2\omega_5 x_5 & x_0^2 & & \cdots & & & x_5^2 \\
4\omega_0 x_0^3 & & \cdots & & & 4\omega_5 x_5^3 & x_0^4 & & \cdots & & & x_5^4 \\
6\omega_0 x_0^5 & & \cdots & & & 6\omega_5 x_5^5 & x_0^6 & & \cdots & & & x_5^6 \\
8\omega_0 x_0^7 & & \cdots & & & 4\omega_5 x_5^7 & x_0^8 & & \cdots & & & x_5^8 \\
10\omega_0 x_0^9 & & \cdots & & & 4\omega_5 x_5^9 & x_0^{10} & & \cdots & & & x_5^{10} \\
1 & 0 & 0 & 0 & 0 & 1 & & & & & & \\
0 & 1 & 0 & 0 & 1 & 0 & & & & 0 & & \\
0 & 0 & 1 & 1 & 0 & 0 & & & & & & \\
 & & & & & & 1 & 0 & 0 & 0 & 0 & -1 \\
 & & & 0 & & & 0 & 1 & 0 & 0 & -1 & 0 \\
 & & & & & & 0 & 0 & 1 & -1 & 0 & 0
\end{pmatrix}
$$