Generic two-state reaction-diffusion-advection equations

In[∘]:=
```
(* Define equations*)
sys1 = {
   D[u[x, t], t] -
    (Du * D[u[x, t], {x, 2}] + αu * D[u[x, t], x] + p1 * u[x, t] + p2 * v[x, t] + p3),
   D[v[x, t], t] -
    (Dv * D[v[x, t], {x, 2}] + αv * D[v[x, t], x] + p4 * u[x, t] + p5 * v[x, t] + p6)
  };
```

In[∘]:=
```
(* Obtain parameters in the above expression *)
params = Quiet[Complement[Variables[sys1], Join[{u[x, t], v[x, t]},
    Select[Variables[sys1], #[[0]][[1]] === u || #[[0]][[1]] === v &]]]]
```

Out[∘]=

$\{Du, Dv, p1, p2, p3, p4, p5, p6, \alpha u, \alpha v\}$

In[∘]:=
```
(* Define observation function *)
obsrep1 = u[x, t] → n[x, t] - v[x, t]
```

Out[∘]=

$u[x, t] \to n[x, t] - v[x, t]$

In[∘]:=
```
(* Substitute *)
sys2 = sys1 /.
  Table[D[obsrep1, {t, i[[1]]}, {x, i[[2]]}], {i, {{0, 0}, {1, 0}, {0, 1}, {0, 2}}}]
```

Out[∘]=

$$\{-p3 - p1\,(n[x,t] - v[x,t]) - p2\,v[x,t] + n^{(0,1)}[x,t] - v^{(0,1)}[x,t] - \alpha u\,\left(n^{(1,0)}[x,t] - v^{(1,0)}[x,t]\right) - Du\,\left(n^{(2,0)}[x,t] - v^{(2,0)}[x,t]\right),$$
$$-p6 - p4\,(n[x,t] - v[x,t]) - p5\,v[x,t] + v^{(0,1)}[x,t] - \alpha v\,v^{(1,0)}[x,t] - Dv\,v^{(2,0)}[x,t]\}$$

In[∘]:=
```
(* Solve sys1 for vxx and vt *)
sol1 = Solve[Table[expr == 0, {expr, sys2}], {D[v[x, t], {x, 2}], D[v[x, t], t]}][[1]]
```

Out[∘]=

$$\Big\{v^{(2,0)}[x,t] \to -\frac{1}{Du - Dv}\Big(-p3 - p6 - p1\,n[x,t] - p4\,n[x,t] + p1\,v[x,t] - p2\,v[x,t] + p4\,v[x,t] - p5\,v[x,t] + n^{(0,1)}[x,t] - \alpha u\,n^{(1,0)}[x,t] + \alpha u\,v^{(1,0)}[x,t] - \alpha v\,v^{(1,0)}[x,t] - Du\,n^{(2,0)}[x,t]\Big),$$
$$v^{(0,1)}[x,t] \to -\frac{1}{Du - Dv}\Big(-Dv\,p3 - Du\,p6 - Dv\,p1\,n[x,t] - Du\,p4\,n[x,t] + Dv\,p1\,v[x,t] - Dv\,p2\,v[x,t] + Du\,p4\,v[x,t] - Du\,p5\,v[x,t] + Dv\,n^{(0,1)}[x,t] - Dv\,\alpha u\,n^{(1,0)}[x,t] + Dv\,\alpha u\,v^{(1,0)}[x,t] - Du\,\alpha v\,v^{(1,0)}[x,t] - Du\,Dv\,n^{(2,0)}[x,t]\Big)\Big\}$$

In[ ]:=
```
(* Write out new expressions *)
sys3 = {
  D[v[x, t], {x, 2}] - (D[v[x, t], {x, 2}] /. sol1),
  D[v[x, t], t] - (D[v[x, t], t] /. sol1)
 }
```

Out[ ]=

$$\left\{\frac{1}{Du-Dv}\left(-p3-p6-p1\,n[x,t]-p4\,n[x,t]+p1\,v[x,t]-p2\,v[x,t]+p4\,v[x,t]-p5\,v[x,t]+n^{(0,1)}[x,t]-\alpha u\,n^{(1,0)}[x,t]+\alpha u\,v^{(1,0)}[x,t]-\alpha v\,v^{(1,0)}[x,t]-Du\,n^{(2,0)}[x,t]\right)+v^{(2,0)}[x,t],\ v^{(0,1)}[x,t]+\frac{1}{Du-Dv}\left(-Dv\,p3-Du\,p6-Dv\,p1\,n[x,t]-Du\,p4\,n[x,t]+Dv\,p1\,v[x,t]-Dv\,p2\,v[x,t]+Du\,p4\,v[x,t]-Du\,p5\,v[x,t]+Dv\,n^{(0,1)}[x,t]-Dv\,\alpha u\,n^{(1,0)}[x,t]+Dv\,\alpha u\,v^{(1,0)}[x,t]-Du\,\alpha v\,v^{(1,0)}[x,t]-Du\,Dv\,n^{(2,0)}[x,t]\right)\right\}$$

In[ ]:=
```
(* Expand sys12 to make it appropriately determined *)
sys4 = Join[sys3, {
    D[sys3[[1]], x],
    D[sys3[[1]], t],
    D[sys3[[2]], x],
    D[sys3[[2]], {x, 2}]
   }];
```

In[ ]:=
```
(* Variables we wish to keep and eliminate *)
nvars = Join[{n[x, t]}, Quiet[Select[Variables[sys4], #[[0]][[1]] === n &]]]
vvars = Join[{v[x, t]}, Quiet[Select[Variables[sys4], #[[0]][[1]] === v &]]]
```

Out[ ]=

$$\left\{n[x,t],\ n^{(0,1)}[x,t],\ n^{(0,2)}[x,t],\ n^{(1,0)}[x,t],\ n^{(1,1)}[x,t],\ n^{(2,0)}[x,t],\ n^{(2,1)}[x,t],\ n^{(3,0)}[x,t],\ n^{(4,0)}[x,t]\right\}$$

Out[ ]=

$$\left\{v[x,t],\ v^{(0,1)}[x,t],\ v^{(1,0)}[x,t],\ v^{(1,1)}[x,t],\ v^{(2,0)}[x,t],\ v^{(2,1)}[x,t],\ v^{(3,0)}[x,t]\right\}$$

In[ ]:=
```
(* Convert system to matrix form *)
A1 = Table[Coefficient[Expand[expr], var], {expr, sys4}, {var, vvars}];
B1 = Table[{0, 0, 0, 0, 0, 0, 0} /. CoefficientRules[expr, vvars], {expr, sys4}];
```

In[ ]:=
```
(* Row reduction (automatic) *)
op = RowReduce[Join[A1, IdentityMatrix[6], 2]][[ ;; , -6 ;;]];
FullSimplify[op.A1] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{(\text{Du}-\text{Dv})\left((\text{Du}-\text{Dv})(\text{p1}-\text{p2}+\text{p4}-\text{p5})-(\alpha\text{u}-\alpha\text{v})^2\right)}{(\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5}))(\text{p1}-\text{p2}+\text{p4}-\text{p5})^2} & \frac{-((\text{Du}-\text{Dv})(2\,\text{Dv}\,\text{p1}-2\,\text{Dv}\,\text{p2}+\text{Du}\,\text{p4}+\text{Dv}\,\text{p4}-(\text{Du}+\text{Dv})\,\text{p5})\,\alpha\text{u})+\cdots}{(\cdots} \\ 0 & 1 & 0 & 0 & 0 & \frac{(\text{Du}-\text{Dv})\left(\text{p1}-\text{p2}+\text{p4}-\text{p5}-\frac{(\alpha\text{u}-\alpha\text{v})(\text{p4}\,\alpha\text{u}-\text{p5}\,\alpha\text{u}+(\text{p1}-\text{p2})\,\alpha\text{v})}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}\right)}{(\text{p1}-\text{p2}+\text{p4}-\text{p5})^2} & \frac{(\alpha\text{u}-\alpha\text{v})\left(\text{Dv}^2(\text{p1}-\text{p2})^2+\text{Dv}(\text{p4}-\text{p5})\left(2\,\text{Du}\,(\cdots\right.\right.}{(\cdots} \\ 0 & 0 & 1 & 0 & 0 & -\frac{(\text{Du}-\text{Dv})(\alpha\text{u}-\alpha\text{v})}{(\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5}))(\text{p1}-\text{p2}+\text{p4}-\text{p5})} & \frac{\text{Dv}^2(-\text{p1}+\text{p2})+\text{Du}^2(\cdots}{\cdots} \\ 0 & 0 & 0 & 1 & 0 & \frac{(\text{Du}-\text{Dv})(\text{p4}\,\alpha\text{u}-\text{p5}\,\alpha\text{u}+(\text{p1}-\text{p2})\,\alpha\text{v})}{(\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5}))(\text{p1}-\text{p2}+\text{p4}-\text{p5})} & \frac{-\text{Dv}\,\text{p1}+\cdots}{\cdots} \\ 0 & 0 & 0 & 0 & 1 & \frac{\text{Du}-\text{Dv}}{\text{Dv}\,\text{p1}-\text{Dv}\,\text{p2}+\text{Du}\,\text{p4}-\text{Du}\,\text{p5}} & \\ 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

In[ ]:=
```
op // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{\text{Du}-\text{Dv}}{\text{Dv}\,\text{p1}-\text{Dv}\,\text{p2}+\text{Du}\,\text{p4}-\text{Du}\,\text{p5}} & \frac{(\text{Du}-\text{Dv})(-2\,\text{Dv}\,\text{p1}\,\alpha\text{u}+2\,\text{Dv}\,\text{p2}\,\alpha\text{u}-\text{Du}\,\text{p4}\,\alpha\text{u}-\text{Dv}\,\text{p4}\,\alpha\text{u}+\text{Du}\,\text{p5}\,\alpha\text{u}+\text{Dv}\,\text{p5}\,\alpha\text{u}+\text{Du}\cdots}{(\text{p1}-\text{p2}+\text{p4}-\text{p5})^2(\text{Dv}\,\text{p1}-\text{Dv}\,\text{p2}+\cdots} \\ 0 & 0 & \frac{(\text{Dv}\,\text{p1}-\text{Dv}\,\text{p2}+\text{Du}\,\text{p4}-\text{Du}\,\text{p5})}{(\text{p1}-\text{p2}+\text{p4}-\text{p5})^2} \\ 0 & 0 & \frac{\text{Du}-\text{Dv}}{\text{p1}-\text{p2}+\text{p4}-\text{p5}} \\ 0 & 0 & \frac{-\text{Dv}\,\text{p1}+\text{Dv}\,\text{p2}-\text{Du}\,\text{p4}+\cdots}{\text{p1}-\text{p2}+\text{p4}-\text{p5}} \\ 0 & 0 & 0 \\ 1 & \frac{-\text{p1}+\text{p2}-\text{p4}+\text{p5}}{\text{Dv}\,\text{p1}-\text{Dv}\,\text{p2}+\text{Du}\,\text{p4}-\text{Du}\,\text{p5}} & \frac{\left(\frac{\text{p1}}{\text{Du}-\text{Dv}}-\frac{\text{p2}}{\text{Du}-\text{Dv}}+\frac{\text{p4}}{\text{Du}-\text{Dv}}-\frac{\text{p5}}{\text{Du}-\text{Dv}}\right)^2\left(-\frac{\text{Dv}\,\text{p1}}{\text{Du}-\text{Dv}}+\frac{\text{Dv}\,\text{p2}}{\text{Du}-\text{Dv}}-\frac{\text{Du}\,\text{p4}}{\text{Du}-\text{Dv}}+\frac{\text{Du}\,\text{p5}}{\text{Du}-\text{Dv}}\right)\left(\frac{\alpha\text{u}}{\text{Du}-\text{Dv}}-\frac{\alpha\text{v}}{\text{Du}-\text{Dv}}\right)-\left(\frac{\text{p1}}{\text{Du}-\text{Dv}}-\frac{\text{p2}}{\text{Du}-\text{Dv}}+\frac{\text{p4}}{\text{Du}-\text{Dv}}-\frac{\text{p5}}{\text{Du}-\text{Dv}}\right)^2\left(-(\cdots\right.}{\left(\frac{\text{p1}}{\text{Du}-\text{Dv}}-\frac{\text{p2}}{\text{Du}-\text{Dv}}+\frac{\text{p4}}{\text{Du}-\text{Dv}}-\frac{\text{p5}}{\text{Du}-\text{Dv}}\right)^3\left(-\frac{\text{Dv}\,\text{p1}}{\text{Du}-\text{Dv}}+\cdots\right.} \end{pmatrix}$$

In[ ]:=
```
(* Polynomial equation in n *)
expr1 = Expand[FullSimplify[op.B1][[-1]]]
```

Out[ ]=

$$-\frac{\text{p3}\,\text{p4}}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{p3}\,\text{p5}}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{p1}\,\text{p6}}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}-\frac{\text{p2}\,\text{p6}}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}-\frac{\text{p2}\,\text{p4}\,n[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{p1}\,\text{p5}\,n[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}-\frac{\text{p1}\,n^{(0,1)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}-\frac{\text{p5}\,n^{(0,1)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{n^{(0,2)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{p5}\,\alpha\text{u}\,n^{(1,0)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{p1}\,\alpha\text{v}\,n^{(1,0)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}-\frac{\alpha\text{u}\,n^{(1,1)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}-\frac{\alpha\text{v}\,n^{(1,1)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{Dv}\,\text{p1}\,n^{(2,0)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{Du}\,\text{p5}\,n^{(2,0)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\alpha\text{u}\,\alpha\text{v}\,n^{(2,0)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}-\frac{\text{Du}\,n^{(2,1)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}-\frac{\text{Dv}\,n^{(2,1)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{Dv}\,\alpha\text{u}\,n^{(3,0)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{Du}\,\alpha\text{v}\,n^{(3,0)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}+\frac{\text{Du}\,\text{Dv}\,n^{(4,0)}[x,t]}{\text{Dv}(\text{p1}-\text{p2})+\text{Du}(\text{p4}-\text{p5})}$$

In[ ]:=
```
(* Get coefficients *)
coef1 = Values@CoefficientRules[expr1, nvars]
```

Out[ ]=

$$\left\{-\frac{p2\ p4}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}+\frac{p1\ p5}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5},\right.$$
$$-\frac{p1}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}-\frac{p5}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5},$$
$$\frac{1}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5},\ \frac{p5\ \alpha u}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}+\frac{p1\ \alpha v}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5},$$
$$-\frac{\alpha u}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}-\frac{\alpha v}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5},$$
$$\frac{Dv\ p1}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}+\frac{Du\ p5}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}+\frac{\alpha u\ \alpha v}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5},$$
$$-\frac{Du}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}-\frac{Dv}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5},$$
$$\frac{Dv\ \alpha u}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}+\frac{Du\ \alpha v}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5},$$
$$\frac{Du\ Dv}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5},\ -\frac{p3\ p4}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}+$$
$$\left.\frac{p3\ p5}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}+\frac{p1\ p6}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}-\frac{p2\ p6}{Dv\ p1-Dv\ p2+Du\ p4-Du\ p5}\right\}$$

In[ ]:=
```
(* Normalise so one coefficient is unity *)
coef3 = FullSimplify[coef1 / coef1[[3]]]
```

Out[ ]=

$\{-p2\ p4+p1\ p5,\ -p1-p5,\ 1,\ p5\ \alpha u+p1\ \alpha v,\ -\alpha u-\alpha v,\ Dv\ p1+Du\ p5+\alpha u\ \alpha v,$
$-Du-Dv,\ Dv\ \alpha u+Du\ \alpha v,\ Du\ Dv,\ p3\ (-p4+p5)+(p1-p2)\ p6\}$

Verify that a single polynomial expression is expected

In[ ]:=
```
sys5 = Join[
   Catenate[Table[Table[D[sys3[[1]], {x, i}, {t, k - i}], {i, 0, k}], {k, 0, 2}]],
   Catenate[Table[Table[D[sys3[[2]], {x, i}, {t, k - i}], {i, 0, k}], {k, 0, 3}]]
  ];
```

In[ ]:=
```
vvarsexp = Join[{v[x, t]}, Quiet[Select[Variables[sys5], #[[0]][[1]] === v &]]];
```

In[ ]:=
```
(* Convert system to matrix form *)
A2 = Table[Coefficient[Expand[expr], var], {expr, sys5}, {var, vvarsexp}];
B2 = Table[{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0} /.
    CoefficientRules[expr, vvarsexp], {expr, sys5}];
M2 = Transpose[Insert[Transpose[A2], B2, Length[vvarsexp] + 1]];
```

In[ ]:=
```
MatrixRank[M2] - MatrixRank[A2]
```

Out[ ]=

1