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Invariance Principle and the Asymptotic Behavior of T-S Fuzzy Systems

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Abstract. In this paper, the asymptotic behavior of nonlinear systems is studied by means of a T-S fuzzy system, which exactly represents the nonlinear system in question, and the extended Invariance Principle. An important feature of the proposed approach is the exhibition of conditions to estimate the attracting invariant set in terms of LMIs.

Keywords. T-S Fuzzy systems, Extended LaSalle Principle, Linear Matrix Inequality

1 Introduction

The Takagi-Sugeno (T-S) fuzzy modelling approach can be used to represent a large class of nonlinear systems by means of a sum of averaged linear models [8]. This modelling approach facilitates the analysis and control design of nonlinear systems [2, 5, 11] by creating a middle ground between the linear and nonlinear dynamics [6]. The potential advantages of this approach are: (i) stability can be analyzed using linear matrix inequalities (LMIs) in a Lyapunov formulation, which can be efficiently solved by convex programming techniques, and (ii) a fuzzy dependent Lyapunov function is usually less conservative than a quadratic one [3]. Following this approach, several formulations have been developed to obtain less conservative conditions for stability, among which, fuzzy Lyapunov functions (FLFs) and polynomial Lyapunov functions (PLFs) have attracted a lot of attention [4].

The Invariance Principle and the Extension of LaSalle's Invariance Principle [1] provide information about the asymptotic behavior of trajectories with conditions that are usually less Preliminariesrestrictive than Lyapunov's method, in the sense that it allows for the derivative of the candidate Lyapunov-like function to be positive in a bounded region. This result has been applied to study stability and the asymptotic behavior of many classes of nonlinear systems, including the a class of switched T-S fuzzy systems [12] with promising results. The authors, however have not found similar application for non-switched T-S systems.

In this work, we propose the application of the extended Invariance Principle to study the asymptotic behavior of nonlinear systems by using a T-S fuzzy system model that exactly represents the nonlinear system in question. Particularly, conditions, in the form of LMIs, are developed to ensure the existence of a bounded positive invariant set that attracts trajectories of the nonlinear system.

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2 Preliminaries

Let us assume that the following nonlinear system:

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a complete $\mathcal{C}^1$ vector field, can be exactly represented [8], [10], [7] by the TS-fuzzy model: [9]

$$\dot{x} = \sum_{i \in \mathcal{R}} h_i A_i x, \quad \mathcal{R} = \{1, ..., r\} \tag{2}$$

in the following set of the state space

$$Z = \{x \in \mathbb{R}^n : |x_\nu| \leq \bar{x}_\nu, \forall \nu \in \mathcal{N}\} \tag{3}$$

where $\mathcal{R}$ is the set of indexes representing the r fuzzy rules, $\mathcal{N} = \{1, ..., n\}$, and $A_i \in \mathbb{R}^{n \times n}$ is a time invariant matrix. The topological boundary of Z is the set $\partial Z = \bigcup_{\nu \in \mathcal{N}} Z_\nu$, where $Z_\nu = \{x \in Z : x_\nu = \bar{x}_\nu\} \bigcup \{x \in Z : x_\nu = -\bar{x}_\nu\} \quad \forall \nu \in \mathcal{N}$. The membership functions $h_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\forall i \in \mathcal{R}$ are $\mathcal{C}^1$ and have the following convex properties:

$$\sum_{i \in \mathcal{R}} h_i(x) = 1, \quad h_i(x) \geq 0 \ \forall i \in \mathcal{R}, \ \forall x \in Z. \tag{4}$$

Our objective is to study the asymptotic behavior of (1) inside Z by studying the asymptotic behavior of (2) using the Extension of LaSalle's Invariance Principle [1].

2.1 Candidate Lyapunov-type Function

We define a candidate auxiliary scalar $\mathcal{C}^1$ function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$:

$$V(x) = x' \sum_{k \in G} h_k(x) P_k x, \tag{5}$$

with $P_k = P_k' \in \mathbb{R}^{n \times n}, \ \ \forall k \in G \subset \mathcal{R}$. Set G is a subset of $\mathcal{R}$, which can be conveniently chosen to include only a subset of the membership functions $h_k(x)$ in the candidate function. We will impose characteristics on matrices P_k with $k \in G$ such that we can use $V(x)$ to conclude about asymptotic behavior of (1). Taking the derivative of (5) yields:

$$\dot{V}(x) = \dot{x}' \sum_{k \in G} h_k P_k x + x' \sum_{k \in G} \dot{h}_k P_k x + x' \sum_{k \in G} h_k P_k \dot{x}. \tag{6}$$

Substituting (2) into (6) results in:

$$\dot{V}(x) = x' \sum_{j \in \mathcal{R}} h_j A_j' \sum_{k \in G} h_k P_k x + x' \sum_{k \in G} \dot{h}_k P_k x + x' \sum_{k \in G} h_k P_k \sum_{j \in \mathcal{R}} h_j A_j x. \tag{7}$$

Finally, rewriting the derivative as a quadratic form, we obtain:

$$\dot{V}(x) = x' \big[\sum_{j \in \mathcal{R}} \sum_{k \in G} h_k h_j (A_j' P_k + P_k A_j) + \sum_{k \in G} \dot{h}_k P_k \big] x. \tag{8}$$

2.2 Invariant Sets

Let us define the $L-$level set:

$$\Omega_L = \{x \in \mathbb{R}^n : V(x) < L\}. \tag{9}$$

We will first show that it is always possible to choose L such that $\Omega_L \subset Z$. This is important because (2) only exactly represents (1) inside Z and also because this guarantees that Ω_L is bounded. With that purpose, we know, for every $P_k, k \in G$ and $x \in \mathbb{R}^n$, that:

$$x^T P_k x \geq \lambda_{min}(P_k)||x||^2. \tag{10}$$

Multiplying (10) by h_k and summing over k yields:

$$\sum_{k \in G} h_k x^T P_k x = V(x) \geq \sum_{k \in G} h_k \lambda_{min}(P_k)||x||^2. \tag{11}$$

This inequality is valid $\forall x \in \mathbb{R}^n$, and, in particular, is valid on ∂Z, which is a compact set. This implies that V assumes a minimum value on ∂Z and the following is true:

$$V(y) \geq \min_{x \in \partial Z} \sum_{k \in G} h_k \lambda_{min}(P_k)||x||^2, \quad \forall y \in \partial Z. \tag{12}$$

The minimum of the sum is greater than the sum of the minimums, so

$$\min_{x \in \partial Z} \sum_{k \in G} h_k \lambda_{min}(P_k)||x||^2 \geq \sum_{k \in G} \min_{x \in \partial Z} \lambda_{min}(P_k) h_k ||x||^2. \tag{13}$$

If we restrict the norm of x to its smallest value at the border of Z, we can also write that V is bounded below by b, where

$$b = \sum_{k \in G} \min_{x \in \partial Z} \lambda_{min}(P_k) h_k \min_{\nu \in \mathcal{N}} \bar{x}_\nu^2. \tag{14}$$

Consequently, if we choose $L < b$, we guarantee that $\Omega_L \subset Z$.

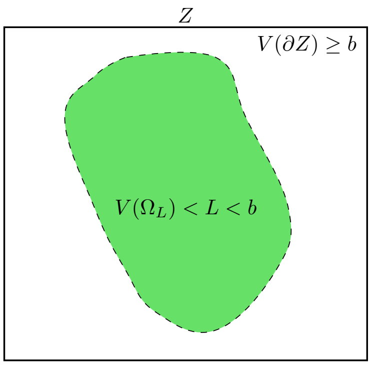

Figure 1: Diagram showing that $\Omega_L \subset Z$

Now, let C be the set with positive derivative of the candidate function V:

$$C = \{x \in \Omega_L : \dot{V}(x) > 0\} \tag{15}$$

According to (8),

$$C = \{x \in \Omega_L : x'\big[\sum_{j \in \mathcal{R}} \sum_{k \in G} h_k h_j (A_j' P_k + P_k A_j) + \sum_{k \in G} \dot{h}_k P_k\big] x > 0\} \tag{16}$$

3 Conditions for the existence of Invariant Sets

In this section, we will obtain LMI conditions for the existence of invariant sets for system (1) exploring the Extended Invariance Principle and the formulation (2). Initially, we analyze the derivative $\dot{V}$ of the candidate function V in order to establish a bounded region in which this derivative assumes positive values. With that in mind, initially, we establish conditions to ensure the first term of the expression in (16) is negative definite, that is

$$x' \sum_{k\in G} \sum_{j\in\mathcal{R}} h_k h_j \big(A_j' P_k + P_k A_j\big) x < 0. \tag{17}$$

To guarantee this, we add and subtract the following canceling terms to the previous inequality:

$$\begin{aligned} x' \sum_{k\in G} \sum_{j\in\mathcal{R}} \Big[& h_k h_j (A_j' P_k + P_k A_j) + h_k h_j (L_k A_j + A_j' L_k') \\ & - h_k h_j (A_j' L_k' + L_k A_j) + A_j' h_k h_j (R_k' + R_k) A_j \\ & - A_j' h_k h_j (R_k + R_k') A_j \Big] x < 0, \end{aligned} \tag{18}$$

where $L_k, R_k \in \mathbb{R}^{n\times n}$. Rearranging the terms of the previous inequality and writing it in matrix form, we have

$$\sum_{j\in\mathcal{R}} h_j \begin{bmatrix} x' & x'A_j' \end{bmatrix} \sum_{k\in G} h_k \begin{bmatrix} L_k A_j + A_j' L_k' & * \\ P_k - L_k' + R_k A_j & -R_k - R_k' \end{bmatrix} \begin{bmatrix} x \\ A_j x \end{bmatrix} < 0. \tag{19}$$

Let us define:

$$\Upsilon_{kj} = \begin{bmatrix} L_k A_j + A_j' L_k' & * \\ P_k - L_k' + R_k A_j & -R_k - R_k' \end{bmatrix}, \tag{20}$$

so (19) can be compactly written as:

$$\sum_{j\in\mathcal{R}} h_j \begin{bmatrix} x' & x'A_j' \end{bmatrix} \sum_{k\in G} h_k \Upsilon_{kj} \begin{bmatrix} x \\ A_j x \end{bmatrix} < 0. \tag{21}$$

For (21) to hold, it is sufficient that

$$\Upsilon_{kj} < 0 \quad \forall k \in G, j \in \mathcal{R}. \tag{22}$$

If we impose (22), then the second term of the expression in (16) will be responsible for generating regions with positive derivative of V. More precisely, if we define:

$$\mathcal{D} = \{x \in \Omega_L : x' \sum_{k\in G} \dot{h}_k P_k x > 0\}, \tag{23}$$

then $C \subset \mathcal{D}$. If in addition, we assume that

$$\sup_{x\in\mathcal{D}} V(x) = l < L, \tag{24}$$

then, we have $\mathcal{D} \subset \Omega_l \subset \Omega_L \subset \Omega_b \subset Z$.

3.1 Main Results

In this section, we propose a theorem establishing conditions to ensure the existence of a bounded attracting set for system (1) by exploring the following extended invariance principle.

Lemma 3.1. *(Extended Invariance Principle) Consider system* (1) *and let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a* C^1 *function and* $c : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function, such that*

$$-\dot{V}(x) \geq c(x), \quad \forall x \in \mathbb{R}^n. \tag{25}$$

Let $L \in \mathbb{R}$ *be a constant such that* $\Omega_L = \{x \in \mathbb{R}^n : V(x) < L\}$ *is bounded. Let* $A := \{x \in \Omega_L : c(x) < 0\}$*, suppose that* $\sup_{x\in A} V(x) = l < L$ *and define* $\bar{\Omega}_l = \{x \in \mathbb{R}^n : V(x) \leq l\}$ *and* $E := \{x \in \Omega_L : c(x) = 0\} \bigcup \bar{\Omega}_l$*. Let* B *be the largest invariant set of the nonlinear system* (1) *contained in* E*. Then every solution of* (1) *starting in* Ω_L *converges to the invariant set* B*, as* $t \to \infty$*. Moreover, if* $x_0 \in \bar{\Omega}_l$ *then* $\varphi(t, x_0) \in \bar{\Omega}_l$ *for every* $t \leq 0$ *and* $\varphi(t, x_0)$ *tends to the largest invariant set of* (1) *contained in* $\bar{\Omega}_l$*.*

Proof. See [1]. □

Now, we are in a position to enunciate the main result of this paper:

Theorem 3.1. *Consider the nonlinear system* (1)*, which can be exactly represented by the T-S Fuzzy system* (2) *inside the bounded set* Z*. Let* L *such that* Ω_L *is bounded and* $\Omega_L \subset Z$*. Let* $\mathcal{D}$ *be defined as* (23) *and assume* (24)*. If there exist* P_k, L_k, R_k *such that* (22) *is satisfied, then every solution of* (1) *starting in* Ω_L*, converges to the largest invariant set of* (1) *inside* $E := \{x \in \Omega_L : \dot{V}(x) = 0\} \bigcup \Omega_l$*.*

Proof. Choosing $L < b$ as defined in (14), it is guaranteed that $\Omega_L \subset Z$ and, therefore, is bounded. Choosing also, $V(x)$ as in (5) and $c(x) = -x' \sum_{k\in G} \dot{h}_k P_k x$, we can directly apply Lemma 3.1. If (22) is satisfied, then the only possibility for $\dot{V}(x) > 0$ inside Ω_L is $\mathcal{D} \subset \Omega_l$. The result follows. □

Example 3.1. *Consider the following nonlinear system:*

$$\dot{x} = \begin{bmatrix} -10x_1 + \frac{30}{25}(x_1^2 + x_2^2 - 25)x_2 \\ -20x_2 \end{bmatrix}. \tag{26}$$

Using the sector nonlinearity approach [8], we obtain a T-S Fuzzy Model that exactly represents (26) *in the set* $Z = \{x \in \mathbb{R}^n : x_1 \in [-5, 5],\ x_2 \in [-5, 5]\}$*:*

$$\dot{x} = \sum_{i\in\mathcal{R}} h_i A_i x, \quad \mathcal{R} = \{1, 2\} \tag{27}$$

$$A_1 = \begin{bmatrix} -10 & 30 \\ 0 & -20 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -10 & -30 \\ 0 & -20 \end{bmatrix} \tag{28}$$

$$h_1(x) = \frac{x_1^2 + x_2^2}{50}, \quad h_2(x) = 1 - \frac{x_1^2 + x_2^2}{50} \tag{29}$$

Solving the LMIs defined in (22) *using convex programming techniques, the following results were obtained*

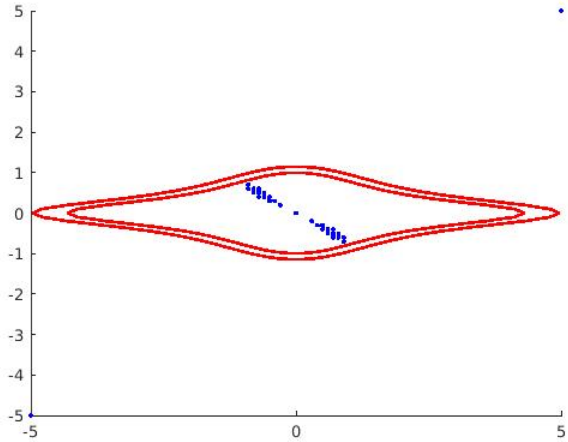

Figure 2: Sets Ω_L, Ω_l and $\mathcal{D}$

$$P_1 = \begin{bmatrix} 0.0289 & -0.0000 \\ -0.0000 & 10.0652 \end{bmatrix} \qquad L_1 = \begin{bmatrix} 0.0194 & -0.0000 \\ -0.0000 & 0.0652 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 0.0030 & -0.0000 \\ -0.0000 & 0.5000 \end{bmatrix} \tag{30}$$

Using $G = \{1\}$, $b = 0.35$ and $l = 0.2$. The feasibility of the LMI conditions guarantees that the solutions of the nonlinear system (26) starting inside Ω_L, with $L < 0.35$ converge to the largest invariant set inside $E \coloneqq \{x \in \Omega_L : \dot{V}(x) = 0\} \bigcup \Omega_l$. Figure 2 illustrates the relevant sets. The outer red line represents Ω_L, the inner red line represents Ω_l and the blue dots represent set $\mathcal{D}$ as defined in (23).

4 Final Considerations

In this work, we proposed a new method for analyzing the asymptotic behavior of nonlinear systems using a T-S fuzzy model and the extended Invariance Principle. In this result, we allowed $\dot{V}(x)$ to assume positive values in a bounded set and proposed methods to obtain estimates of the invariant sets by means of LMIs. In future results, we aim to improve the estimates of the invariant sets and also apply this result to more complex systems, that is, with a higher number of nonlinearities.

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