Solutions to V.A. Arnold's Mathematical Methods of Classical Mechanics

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Preface

These solutions rose out of a personal goal I had set for myself to finish the most mathematically concrete treatment of Classical Mechanics to date, Mathematical Methods of Classical Mechanics by V.A.Arnold [1]. Being a graduate student working on galactic dynamics, I wanted to get a solid foundation on the bread and butter of my field. I was trained in physics without worrying too much about mathematical rigor and this exercise is my attempt at getting acquainted with the tools of differential geometry and its application to Hamiltonian systems. Further, a firm grasp of differential geometry is highly useful when learning general relativity and my goal after finishing Arnold's book is to tackle Wald's book on general relativity.

Working on galactic dynamics, almost all of the topics in this book are relevant to me except for the section on rigid body dynamics which I have skipped on my first reading. I may or may not get back to this chapter at a later time. Further, given that this is a completely solo effort made worse by my lack of experience with rigorous mathematical proofs, I am in no way claiming that my proofs are correct or as rigorous as they could be and I welcome corrections and suggestions from anyone taking their time to read this document. Working out these proofs as explicitly as possible helped me at least convince myself that a result is right, and as a physicist, this was my primary aim. This work is targeted mainly for physics students as a reference in case they are not accustomed with mathematical proofs and/or don't wish to spend their time trying to work out proofs, but would like to see what they look like.

I have tried my best to work things out as explicitly as possible using only the notations and results used in Arnold's book. In some sections one has to use results which are presented in further chapters. Since the book has a notoriously bad labeling system for problems, I have written out the problem statement along with the page number where the problem can be found in the book in parenthesis. Questions with self-explanatory answers that are provided in the text are not given different solutions. Notes or ideas that I found useful from different sources during this study are given as footnotes or in boxes where and when relevant. There will also be a dedicated subsection proving the theorems of Cartan calculus.

Contents

Part I

Newtonian Mechanics

Chapter 1

Experimental Facts

1. (6) Show that every galilean transformation of the space $\mathbb{R}\times\mathbb{R}^3$ can be written in a unique way as the composition of a rotation, a translation, and a uniform motion ($g = g_1 \circ g_2 \circ g_3$)(thus the dimension of the galilean group is equal to $3+4+3 = 10$).

   Solution: We first consider a general linear transformation on $\mathbb{R}\times\mathbb{R}^3$ as the 4×4 matrix A and an affine transformation G given by $Ga = Aa + \lambda$ for any $a \in \mathbb{R}\times\mathbb{R}^3$ where λ is a constant vector. Since we require A to preserve the galilean structure, G has to preserve the time interval between two events $t(b-a)$, as well as the distance between two simultaneous events $\rho(a, a') = ||a-a'||$. To summarize, G has to satisfy

$$t(a-b) = t(Ga-Gb) \tag{1.1}$$
$$\rho(a, a') = \rho(Ga, Ga'). \tag{1.2}$$

   We now explicitly write out the transformation of an event $a \in \mathbb{R}\times\mathbb{R}^3$ under G.

$$Ga = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} + \begin{pmatrix} \lambda^0 \\ \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix} = \begin{pmatrix} A_{0i}a^i + \lambda_0 \\ A_{1i}a^i + \lambda^1 \\ A_{2i}a^i + \lambda^2 \\ A_{3i}a^i + \lambda^3 \end{pmatrix} \tag{1.3}$$

   where repeated indices are summed over $i = 0, ..3$. The difference in the coordinate $i = 0$ is taken as the time interval map for $\mathbb{R}\times\mathbb{R}^3$, i.e., $t(a-b) = a^0 - b^0$. For the invariance of the time interval between two events a and b, we have

$$a^0 - b^0 = A_{00}(a^0 - b^0) + A_{01}(a^1 - b^1) + ... \tag{1.4}$$

   Since this has to hold for any two events, we are led to the conclusion that $A_{00} = 1$ and $A_{0i} = 0$, $i = 1, 2, 3$. Now to preserve distances for two simultaneous events a and a' ($a^0 = a'^0$),

$$\sum_{i,j=1}^{3} (A_{ij}(a^j - a'^j))^2 = \sum_{i=1}^{3} (a^i - a'^i)^2 \tag{1.5}$$

   Thus the cofactor matrix $A^C = [A^C_{00}]$ obtained by removing the first row and first column must be an orthogonal matrix ($(A^C)^{\mathrm{T}}A^C = I_3$, where I_3 is the 3×3 identity matrix). Thus, a distance and time interval preserving transformation on $\mathbb{R}\times\mathbb{R}^3$ can be written down as

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & R_{11} & R_{12} & R_{13} \\ v_2 & R_{21} & R_{22} & R_{23} \\ v_3 & R_{31} & R_{31} & R_{33} \end{pmatrix} \tag{1.6}$$

   where $R_{i,j}$ are the elements of a 3×3 orthogonal matrix R, and v_1, v_2, v_3 represent the elements A_{10}, A_{20}, A_{30} of A respectively. To see that these terms represent a boost, we consider the action of a transformation A_{boost} with $R = I_3$

$$A_{\text{boost}}a = \begin{pmatrix} a^0 \\ v_1a^0 + a^1 \\ v_2a^0 + a^2 \\ v_3a^0 + a^3 \end{pmatrix} \tag{1.7}$$

which represents the original point moving with a speed given by $\mathbf{v} = (v_1, v_2, v_3)$. Thus, we see that the final galilean transformation can be written down as

$$Ga = A_{\text{boost}} A_{\text{orthogonal}} a + \lambda \tag{1.8}$$

with

$$A_{\text{boost}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \qquad A_{\text{orthogonal}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{31} & R_{33} \end{pmatrix} \tag{1.9}$$

where the boost and rotation operations commute and can be well defined by the combined matrix A from (1.6). □

2. (6) Show that all galilean spaces are isomorphic to each other and, in particular, isomorphic to the coordinate space $\mathbb{R} \times \mathbb{R}^3$.

   Solution: Let E, E' be galilean spaces with underlying vector space $\mathbb{R}^4$, i.e., for any $a, b \in E$ or E', $a - b \in \mathbb{R}^4$. If we can find isomorphisms $\phi : E \to \mathbb{R} \times \mathbb{R}^3$ and $\phi' : E' \to \mathbb{R} \times \mathbb{R}^3$, then we can construct the isomorphism $\phi'^{-1} \circ \phi : E \to E'$ which is the required isomorphism between two arbitrary galilean spaces.

   We first construct a map $M : \mathbb{R}^4 \to \mathbb{R} \times \mathbb{R}^3$ that maps the time coordinate to the 0 index (thus the notation $\mathbb{R} \times \mathbb{R}^3$). The time map is a linear map $T : \mathbb{R}^4 \to \mathbb{R}$. Let $e_0, ...e_3$ be an arbitrary basis for $\mathbb{R}^4$. Then

$$v = v^i e_i \tag{1.10}$$

$$Tv = v^i (Te_i) \tag{1.11}$$

   We want to construct a basis e'_i where $T(e'_i) = \delta_{0i}$. Let the required transformation matrix be M, $e'_i = (M^{-1})^j_i e_j$. To satisfy the condition, $(M^{-1})^j_i T(e_j) = \delta_{0i} \implies T(e_j) = M^i_j \delta_{0i} = M^0_j$. For the remaining indices, we are free to choose any 3×4 matrix such that M is full-rank. Not all $T(e_i)$ can be zero, so assume that at least $T(e_3)$ is non-zero. We set

$$M = \begin{pmatrix} T(e_0) & T(e_1) & T(e_2) & T(e_3) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.12}$$

   Note that this choice is unimportant as even if all but one of the terms $T(e_i)$ with $i \neq 3$ is zero, we can set the cofactor matrix $[M^C_{0i}]$ to I_3 so that M thus defined is full rank. Now,

$$v'^i e'_i = v^k e_k \tag{1.13}$$

$$v'^i (M^{-1})^j_i e_j = v^k e_k \tag{1.14}$$

$$\implies v'^i (M^{-1})^j_i = v^j \tag{1.15}$$

$$\implies v'^i = M^i_j v^j \tag{1.16}$$

   Thus, we have a map $M : \mathbb{R}^4 \to \mathbb{R} \times \mathbb{R}^3$ such that $Tv = (Mv)^0$ i.e., we have separated the time coordinate $\mathbb{R}$ from the spacial coordinates $\mathbb{R}^3$ of v. Since M is full-rank, the map is also invertible.

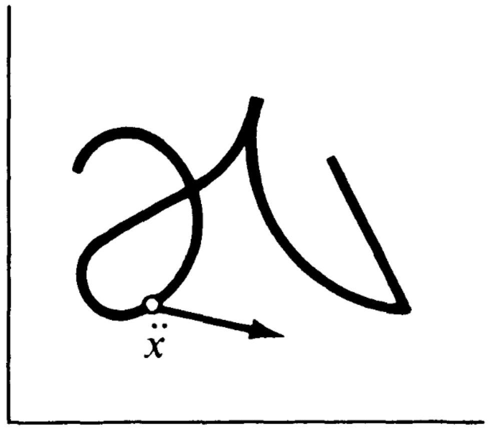


Figure 1.1

Now let $c \in E$ be fixed. Then we can write any $a \in E$ in the form $a = c + v$ where $v \in \mathbb{R}^4$ (this is effectively choosing an origin), define the map $P_c : E \to \mathbb{R}^4$ such that $P_c(a) = a - c = v \in \mathbb{R}^4$, which is clearly invertible ($P_c^{-1}(v) = c + v$). Now construct the composite map $\phi_c = M \circ P_c : E \to \mathbb{R} \times \mathbb{R}^3$. This is the required isomorphism. Note that this is not a canonical isomoprhism as one can make any choice for the "origin" $c \in E$. □

3. (7) Is it possible for the trajectory of a differentiable motion on the plane to have the shape drawn in Figure 1.1? Is it possible for the acceleration vector to have the value shown?

   Solution: The trajectory shown is a perfectly reasonable motion. However, the acceleration vector shown is not possible. The velocity vector at every point on the curve points in the direction of the tangent to the curve. The direction of change of the tangent vector is given by the acceleration vector. Clearly, the tangent vector changes "inward" rather than "outward" as indicated by the arrow.

4. (10) Show that if a mechanical system consists of only one point, then its acceleration in an inertial coordinate system is equal to zero ("Newton's first law").

   Solution: If there is only one point in the system, $\ddot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}})$ is invariant under translations in time or space and under boosts with constant velocity. This implies

$$\mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}(t + s, \mathbf{x}, \dot{\mathbf{x}}) \implies \mathbf{f} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}})) \tag{1.17}$$

$$\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}})) = \mathbf{f}(\mathbf{x} + \mathbf{x_0}, \dot{\mathbf{x}})) \implies \mathbf{f} = \mathbf{f}(\dot{\mathbf{x}}) \tag{1.18}$$

$$\mathbf{f}(\dot{\mathbf{x}}) = \mathbf{f}(\dot{\mathbf{x}} + \mathbf{v_0}) \implies \mathbf{f} = \mathit{const.} \tag{1.19}$$

   Invariance under orthogonal translations thus implies that $\mathbf{f} = 0$. □

5. (10) A mechanical system consists of two points. At the initial moment their velocities (in some inertial coordinate system) are equal to zero. Show that the points will stay on the line which connected them at the initial moment.

   Solution: Let the point be 1 and 2. Choose a coordinate system such that at $t = 0$, $\mathbf{x}_1(0) = a\mathbf{u}_0$, $\mathbf{x}_2(0) = b\mathbf{u}_0$, and $\dot{\mathbf{x}}_1(0) = \dot{\mathbf{x}}_2(0) = 0$, where a and b are constants and u_0 is a vector parallel to the line joining 1 and 2. The forces on 1 and 2 are given by $\mathbf{f_i}(\mathbf{x_1} - \mathbf{x_2}, \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2.)$

Now, we note that if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are such that $\mathbf{v} = c\mathbf{w}$ for some $c \in \mathbb{R}$, a rotation R about the $\mathbf{v}$ leaves both vectors unchanged, i.e., $R\mathbf{v} = \mathbf{v}$, $R\mathbf{w} = \mathbf{w}$. Then

$$\mathbf{f}_i(R\mathbf{v}, R\mathbf{w}) = R\mathbf{f}_i(\mathbf{v}, \mathbf{w}) \tag{1.20}$$

$$= \mathbf{f}_i(\mathbf{v}, \mathbf{w}) \tag{1.21}$$

$$R\mathbf{f}_i(\mathbf{v}, \mathbf{w}) = \mathbf{f}_i(\mathbf{v}, \mathbf{w}) \tag{1.22}$$

Thus, if at any point the displacement and relative velocities are parallel to some vector $\mathbf{v}$, the force acting on the particles are also parallel to $\mathbf{v}$.

Back to our problem, define the function $F_i(x, y)\mathbf{u_0} \equiv \mathbf{f}_i(x\mathbf{u_0}, y\mathbf{u_0})$. Let $y_i(t)$ be the solution to the system $\ddot{y_i} = F_i(y_1 - y_2, \dot{y_1} - \dot{y_2})$, with initial conditions $y_1(0) = a$, $y_2(0) = b$, and $\dot{y_1}(0) = \dot{y_2}(0) = 0$. We can show that $\mathbf{x}_i(t) = \mathbf{u_0}y_i(t)$ is a solution to the original problem and thus the points always remain on the line parallel to u_0.

$$\ddot{\mathbf{x}}_i = \mathbf{u_0}\ddot{y_i} \tag{1.23}$$

$$= \mathbf{u_0}F_i(y_1 - y_2, \dot{y_1} - \dot{y_2}) \tag{1.24}$$

$$= \mathbf{f}_i(\mathbf{u_0}(y_1 - y_2), \mathbf{u_0}(\dot{y_1} - \dot{y_2}) \tag{1.25}$$

$$= \mathbf{f}_i(\mathbf{x_1} - \mathbf{x_2}, \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) \tag{1.26}$$

This solution is unique since the solution for the y_i's are unique. □

6. (10) A mechanical system consists of three points. At the initial moment their velocities (in some inertial coordinate system) are equal to zero. Show that the points always remain in the plane which contained them at the initial moment.

   Solution: Let the three points be 1, 2, 3. At $t = 0$, they lie on some plane τ with normal $\mathbf{N}$, and we are free to choose inertial coordinates that set the origin as $\mathbf{x}_1(0)$. Then $\mathbf{x}_2(0) - \mathbf{x}_1(0) = \mathbf{u}_0$, $\mathbf{x}_3(0) - \mathbf{x}_1(0) = \mathbf{v}_0$ are vectors that lie on τ and are perpendicular to $\mathbf{N}$.

   We now note that reflections are also distance preserving transformations and are also a valid galiliean transformations[1] (this is a problem in this chapter which we will argue now to proceed with this problem). Reflections are orthogonal transformations with $\det G = -1$. Invariance with respect to reflections means that there are no preferred orientations of coordinates in space. Thus we have the relation

$$\mathbf{F}(G\mathbf{x}, G\dot{\mathbf{x}}) = G\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) \tag{1.27}$$

   The forces that enter the equations of motion are functions of $\mathbf{x_i} - \mathbf{x_j}$ and $\dot{\mathbf{x}}_\mathbf{i} - \dot{\mathbf{x}}_\mathbf{j}$, i.e., $\mathbf{f}_i = \mathbf{f}_i(\mathbf{x_2} - \mathbf{x_1}, \mathbf{x_3} - \mathbf{x_1}, \dot{\mathbf{x_2}} - \dot{\mathbf{x_1}}, \dot{\mathbf{x_3}} - \dot{\mathbf{x_1}})$. Let $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4} \in Span(\mathbf{u_0}, \mathbf{v}_0)$, and G denote the reflection through the direction $\mathbf{N}$. Clearly, $G\mathbf{u}_0 = \mathbf{u}_0$ and $G\mathbf{v}_0 = \mathbf{v}_0$ and thus similar relations hold for the $\mathbf{w}_i$. Let $\mathbf{f}_i(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}) = \mathbf{f}_i^{||} + \mathbf{f}_i^{\perp}$ denote the components parallel and perpendicular to the plane τ (or perpendicular and parallel the normal $\mathbf{N}$) respectively. Now

$$G\mathbf{f}_i = G\mathbf{f}_i^{||} + G\mathbf{f}_i^{\perp} = \mathbf{f}_i^{||} - \mathbf{f}_i^{\perp}. \tag{1.28}$$

   But from galilean invariance we have

$$G\mathbf{f}_i(\mathbf{w}_1, ...) = \mathbf{f}_i(G\mathbf{w}_1, ...) = \mathbf{f}_i(\mathbf{w}_1, ...) = \mathbf{f}_i^{||} + \mathbf{f}_i^{\perp} \tag{1.29}$$

[1] Reflections are discrete transformations as opposed to the continuous translation, boost and rotation. The set of all galilean transformations barring rotations forms a Lie Group (see 8).

Equations (1.28) and (1.29) imply that $\mathbf{f}_i^{\perp} = 0$. Thus if all arguments $\mathbf{w}_i$ lie on a plane, the forces $\mathbf{f}_i$ on all the particles also lies on the same plane.

Now we define

$$\mathbf{f}_i(a_1\mathbf{u_0} + b_1\mathbf{v_0}, ..., a_4\mathbf{u_0} + b_4\mathbf{v_0}) \equiv \\ \mathbf{u_0}F_i^0(a_1, b_1, ...a_4, b_4) + \mathbf{v_0}F_i^1(a_1, b_1, ..., a_4, b_4) \quad (1.30)$$

Let $y_i^j(t)$ be the solutions for $\ddot{y}_i^j = F_i^j(y_2^j - y_1^j, y_3^j - y_1^j, \dot{y}_2^j - \dot{y}_1^j, \dot{y}_3^j - \dot{y}_2^j)$, with initial conditions $y_1^0(0) = y_1^1(0) = 0$, $y_2^0(0) = 1, y_2^1(0) = 0$, $y_3^0(0) = 0, y_3^1(0) = 1$ and $\dot{y}_i^j(0) = 0$ for $i = 1, 2, 3$ and $j = 1, 2$. Consider solutions of the form $\mathbf{x}_i(t) = y_i^0(t)\mathbf{u}_0 + y_i^1(t)\mathbf{v}_0$. Similar to the previous problem, it can be shown that these are also valid solutions. Now the system of differential equations for the y_i^j are 6 second order equations with 12 initial conditions and thus possess a unique solution and thus, this solution for the $\mathbf{x}_i$ are unique. Thus we have shown that the trajectories of three particles stay on a plane if they started from rest in some inertial coordinate. □

7. (10) A mechanical system consists of two points. Show that for any initial conditions there exists an inertial coordinate system in which the two points remain in a fixed plane.

Solution: Let the two points be 1 and 2. They have initial conditions $\mathbf{x}_1(0) = \mathbf{a}_1$, $\mathbf{x}_2(0) = \mathbf{a}_2$, $\dot{\mathbf{x}}_1(0) = \mathbf{u}_0$, and $\dot{\mathbf{x}}_2(0) = \mathbf{v}_0$. The equations of motion take the form

$$\ddot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_1 - \mathbf{x}_2, \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) \quad (1.31)$$

Let $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{v} = \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2$). Consider the vector $\mathbf{L} = \mathbf{r} \times \mathbf{v}$. If the direction of $\mathbf{L}$ does not change with time, $\mathbf{r}$ and $\mathbf{v}$ lie on a plane perpendicular to $\mathbf{L}$ that moves at some speed parallel to $\mathbf{L}$. Now,

$$\begin{aligned} \dot{\mathbf{L}} &= \dot{\mathbf{r}} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}} && (1.32) \\ &= \mathbf{r} \times \ddot{\mathbf{r}} && (1.33) \end{aligned}$$

At some instant, let G denote reflection about the plane on which the particles lie that is perpendicular to $\mathbf{L}$. Using the invariance of $\mathbf{r}$ and $\mathbf{v}$ on this reflection, we get $G\ddot{\mathbf{r}} = G\mathbf{f}_1(\mathbf{r}, \mathbf{v}) - G\mathbf{f}_2(\mathbf{r}, \mathbf{v}) = \mathbf{f}_1(G\mathbf{r}, G\mathbf{v}) - \mathbf{f}_2(G\mathbf{r}, G\mathbf{v}) = \mathbf{f}_1(\mathbf{r}, \mathbf{v}) - G\mathbf{f}_2(\mathbf{r}, \mathbf{v}) = \ddot{\mathbf{r}}$. Thus, we have shown that at every instant of the motion, the relative acceleration $\ddot{\mathbf{r}}$ is perpendicular to $\mathbf{L}$ and thus coplanar with $\mathbf{r}$ and $\mathbf{v}$. Thus the direction of $\mathbf{r} \times \ddot{\mathbf{r}}$ is parallel to that of $\mathbf{L}$ and the direction of $\mathbf{L}$ does not change with time. Further, one can also show that $G\ddot{\mathbf{x}}_i = G\mathbf{f}_i(\mathbf{r}, \mathbf{v}) = \mathbf{f}_i(G\mathbf{r}, G\mathbf{v}) = \mathbf{f}_i(\mathbf{r}, \mathbf{v}) = \mathbf{x}_i$ and thus the acceleration of the particles also lies on the plane perpendicular to $\mathbf{L}$ at every instant.

We now find the inertial coordinates in which the particles appears to move on a plane. At every instant of motion, the component of the velocity of 1 parallel to $\mathbf{L}$ is given by

$$\dot{\mathbf{x}}_1^{||} = \dot{\mathbf{x}}_1.\frac{(\mathbf{r} \times \mathbf{v})}{|(\mathbf{r} \times \mathbf{v}|} = -\dot{\mathbf{x}}_1.\frac{(\mathbf{r} \times \dot{\mathbf{x}}_2)}{|(\mathbf{r} \times \mathbf{v}|} = \mathbf{r}.\frac{(\dot{\mathbf{x}}_1 \times \dot{\mathbf{x}}_2)}{|(\mathbf{r} \times \mathbf{v}|} \quad (1.34)$$

where we have used the properties of the triple product. Similarly,

$$\dot{\mathbf{x}}_2^{||} = \dot{\mathbf{x}}_2.\frac{(\mathbf{r} \times \mathbf{v})}{|(\mathbf{r} \times \mathbf{v}|} = \dot{\mathbf{x}}_2.\frac{(\mathbf{r} \times \dot{\mathbf{x}}_1)}{|(\mathbf{r} \times \mathbf{v}|} = \mathbf{r}.\frac{(\dot{\mathbf{x}}_1 \times \dot{\mathbf{x}}_2)}{|(\mathbf{r} \times \mathbf{v}|} = \dot{\mathbf{x}}_1^{||} \quad (1.35)$$

Now, we have shown that at every instant of the motion, $\ddot{\mathbf{x}}_i$ is perpendicular to $\mathbf{L}$. This the components of the velocity parallel to $\mathbf{L}$ are constant and are set by the initial conditions. We define $\mathbf{v}_{\text{inertial}} \equiv \dot{\mathbf{x}}_1^{||} = \dot{\mathbf{x}}_2^{||}$ given by

$$\mathbf{v}_{\text{inertial}} = (\mathbf{a}_1 - \mathbf{a}_2).\frac{(\mathbf{u}_0 \times \mathbf{v}_0)}{|\mathbf{u}_0 \times \mathbf{v}_0|} \tag{1.36}$$

Thus, by carrying out a boost with velocity $\mathbf{v}_{\text{inertial}}$, the velocities of the particles parallel to $\mathbf{L}$ vanish and one sees the particles moving on a plane perpendicular to $\mathbf{L}$.

□

8. (11) Show that mechanics "through the looking glass" is identical to ours.

   Solution: As we have mentioned before, reflections are orthogonal transformations and thus are distance preserving maps. They also form a subset of galilean transformations. If a motion $\mathbf{x}_i(t)$ satisfies $\ddot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}, \dot{\mathbf{x}})$, then so does $G\mathbf{x}_i(t)$ and thus

$$G\ddot{\mathbf{x}} = G\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{F}(G\mathbf{x}, G\dot{\mathbf{x}}) \tag{1.37}$$

9. (11) Is the class of inertial systems unique?

   Solution: Given in text.

CHAPTER 2

Investigations of the Equations of Motion

1. (16) Show that through every phase point there is one and only one phase curve.

   Solution: The phase flow is given by the equations

$$\dot{x} = y \qquad \dot{y} = f(x). \tag{2.1}$$

   The equation for y can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} y = f(x) \tag{2.2}$$
$$\implies \int_{y_0}^{y} y' \mathrm{d}y' = \int_{x_0}^{x} f(x') \mathrm{d}x \tag{2.3}$$
$$\implies \frac{y^2}{2} - F(x) = \frac{y_0^2}{2} - F(x_0) = const. \tag{2.4}$$

   where (x_0, y_0) are initial conditions, and $F(x)$ is the anti-derivative of $f(x)$, which is defined up to a constant. We note that due to the appearance of F on both sides of equation (2.4), the choice of this arbitrary constant is not important. The equation (2.4) is the equation of a 1-D curve in phase space that passes through the point (x_0, y_0). This is a well defined unique curve that is determined by only the initial conditions, which can be taken at any point on the curve. □

2. (18) Prove this (that the local maximum points ofthe potential energy are unstable, but the minimum points are stable equilibrium positions).

   Problem: The solution to this is straightforward. $f(x) = -\mathrm{d}U/\mathrm{d}x = U'(x)$, where U is the potential. If x_0 is an equilibrium point, $U'(x_0) = 0$. Thus the force at a point $x_0 + \epsilon$ is given by

$$f(x_0 + \epsilon) = -\epsilon U''(x_0) + \mathcal{O}(\epsilon^2). \tag{2.5}$$

   If the point x_0 is a minimum, $U''(x_0) > 0$ and thus the force tends to drive the body back to equilibrium (stable). On the other hand, if it is a maximum, $U''(x_0) < 0$ and the force tends to drive the body away from equilibrium(unstable).

3. (18) How many phase curves make up the separatrix (figure eight) curve, corresponding to the level E_2 ?

   Solution: Three: two 1-D curves, and a single unstable equilibrium point.

4. (18) Determine the duration of motion along the separatrix.

   Solution: As the particle approaches the equilibrium point, it's velocity starts to tend towards zero from energy conservation, while the force acting on it also tends to 0. To estimate the time, we use the result from the next problem which we will prove soon.

$$\Delta t = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2(E - U(x))}} \tag{2.6}$$

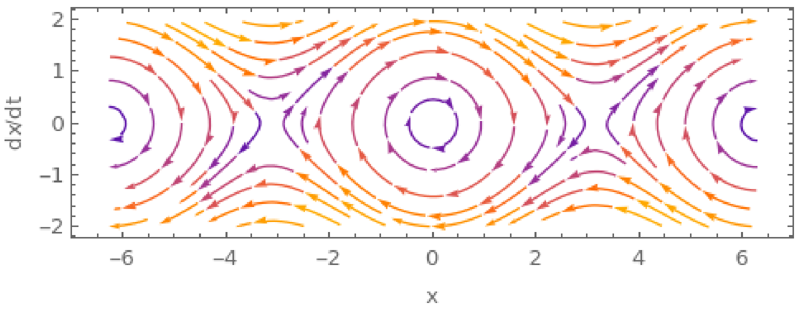


Figure 2.1: Phase curves of planar pendulum

Let $E = U(x_0)$, $x_1 = x_0 - \epsilon_0$, and $x_2 = x_0$. We can change the integration variable to ϵ and expanding the potential about x_0, we have $U(x_0-\epsilon) = U(x_0)+U''(x_0)\epsilon^2/2$. Thus, to leading order in ϵ we get

$$\Delta t \sim \int_{\epsilon_0}^{0} \frac{\mathrm{d}\epsilon}{\sqrt{-U''(x_0)\epsilon^2}} \tag{2.7}$$

which clearly diverges logarithmically. The term under the square root is positive as $U''(x_0) < 0$ due to x_0 being a maxima. Thus the time taken is infinite.

Alternatively as mentioned by Arnold, there is a unique phase curve (the 0 dimensional equilibrium point) at the phase point $(x_0, 0)$. Using the fact that there is only once phase curve passing through every point, we conclude that seperatrix trajectory always moves towards the equilibrium point but never reaches it and thus takes infinite time.

5. (18) Show that the time it takes to go from x_1 to x_2 (in one direction)is equal to

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2(E-U(x))}} \tag{2.8}$$

Solution:

$$E = T + U = \dot{x}^2/2 + U(x) \tag{2.9}$$

$$\implies \dot{x}^2 = 2(E - U(x)) \tag{2.10}$$

$$\implies \int_{t_1}^{t_2} \mathrm{d}t = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2(E-U(x))}} \tag{2.11}$$

□

6. (19) Draw the phase curves, given the potential energy graph in Figure 11 (of Arnold's Book). Solution: Given in text

7. (19) Draw the phase curves for the "equation of an ideal planar pendulum": $\ddot{x} = -\sin x$.

   Solution: The procedure for drawing these images is to plot the potential and work out the turning points corresponding to a given energy. The plot generated using the StreamPlot function in Mathematica is shown in figure 2.1

8. (19) Draw the phase curves for the "equation of a pendulum on a rotating axis": $\ddot{x} = -\sin x + M$ Solution: See figure 2.2.

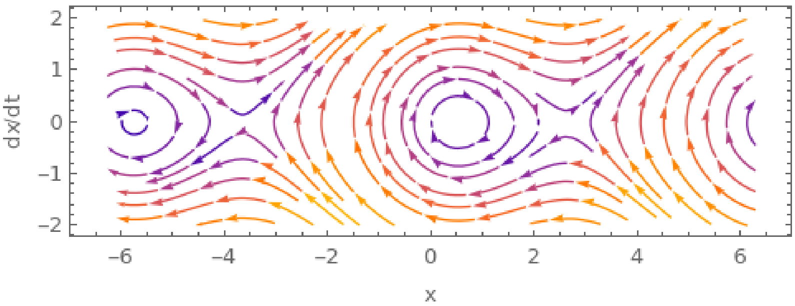


Figure 2.2: Phase curves of rotating pendulum

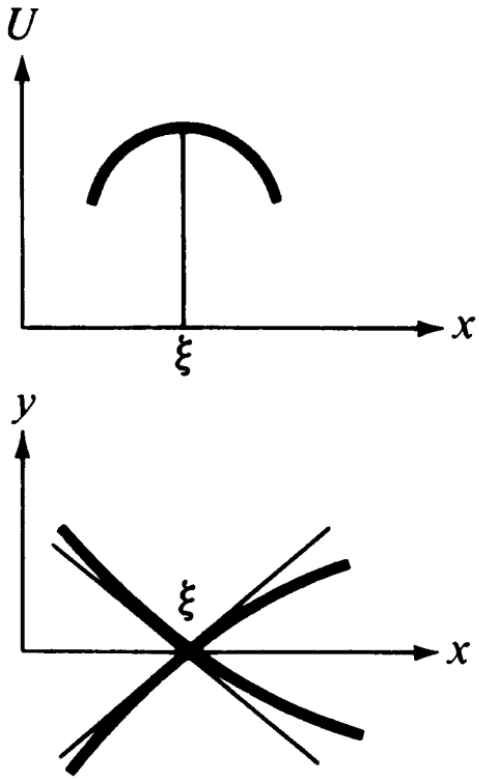


Figure 2.3: Critical energy level lines

9. (19) Find the tangent lines to the branches of the critical level corresponding to maximal potential energy $E = U(\xi)$ (Figure 2.3).

   Solution: From the first problem in this chapter, we have

$$\frac{y^2}{2} + U(x) = \frac{y_0^2}{2} + U(\xi) = U(\xi) \implies y = \pm\sqrt{2(U(\xi) - U(x))} \tag{2.12}$$

   Expanding $U(x)$ near the equilibrium point ξ, $U(x) = U(\xi) + (x-\xi)U'(\xi) + (x-\xi)^2 U''(\xi)/2 + \mathcal{O}((x-\xi)^3)$, which finally gives

$$y = \pm(x-\xi)\sqrt{U''(\xi)} \tag{2.13}$$

10. (20) Let $S(E)$ be the area enclosed by the closed phase curve corresponding to the energy level E. Show that the period of motion along this curve is equal to $T = \mathrm{d}S/\mathrm{d}E$.

   Solution: Let the motion have turning points x_1, x_2 i.e., $U(x_1) = U(x_2) = E$, with $x_1 < x_2$. Then the area under the phase space curve is given by

$$S(E) = \int_{x_1}^{x_2} \mathrm{d}x|\dot{x}| + \int_{x_2}^{x_1} \mathrm{d}x(-|\dot{x}|) \tag{2.14}$$

$$= 2\int_{x_1}^{x_2} \mathrm{d}x\sqrt{2(E - U(x))} \tag{2.15}$$

   where the first and second terms in equation (2.14) represent the motion from x_1 to x_2 and the reverse motion with negative velocity from x_2 to x_1 respectively. Now

$$\frac{\mathrm{d}S}{\mathrm{d}E} = 2\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2(E - U(x))}} \tag{2.16}$$

$$= T_{x_1\to x_2} + T_{x_2\to x_1} \tag{2.17}$$

   which gives the total time period of the motion. We have used the result of problem 5. □

11. (20) Let E_0 be the value of the potential function at a minimum point ξ. Find the period T_0 of small oscillations in a neighborhood of the point ξ, where $T_0 = \lim_{E\to E_0} T(E)$.

   Solution: Taylor expand the potential $U(x)$ near the minimum point

$$U(x) = E_0 + \frac{(x-\xi)^2}{2}U''(\xi) + \mathcal{O}((x-\xi)^3) \tag{2.18}$$

   The equation of motion gives

$$\ddot{x} = -U'(x) = -(x-\xi)U''(\xi). \tag{2.19}$$

   Let $z = x - \xi$. Equation (2.19) can be written as $\ddot{z} = -zU''(\xi)$. This is the equation of a 1-D harmonic oscillator with frequency $\omega = \sqrt{U''(\xi)}$. Thus the time period of the motion is given by $T_0 = 2\pi/\sqrt{U''(\xi)}$

12. (20) Consider a periodic motion along the closed phase curve corresponding to the energy level E. Is it stable in the sense of Liapunov?

   Solution: An equilibrium point ξ is said to be Liapunov stable if $\forall \epsilon > 0, \exists\ \delta(\epsilon) > 0$ such that if $|x(0) - \xi| < \delta(\epsilon)$ then $|x(t) - \xi| < \epsilon\ \forall t > 0$. Simply put, we must be able to confine the motion to an arbitrarily small region of configuration space around ξ by starting the motion sufficiently close to ξ. However, for a given periodic motion with turning points $x_1 < \xi < x_2$, this is not possible for any $\epsilon < \min(\xi - x_1, x_2 - \xi)$, as $|x(t) - \xi|$ will exceed this value at some point of the periodic orbit. Thus the motion is not Liapunov stable.

13. (21) Show that the system with potential energy $U = -x^4$ does not define a phase flow.

Solution: In order for the motion in a potential to constitute a phase flow, it must be possible to extend the solution to the entire time axis. If the motion reaches infinity in a finite amount of time, then the group properties given in the text are destroyed and thus the the motion will not define a phase flow. We now calculate the time period of motion to infinity in this potential.

There are two cases: $E > 0$ and $E < 0$. For $E < 0$, let the particle have initial condition $x(0) = x_0$, $\dot{x}(0) = 0$. Its energy is given by $E = U(x_0)$. The time period for motion up to a point x_1 is given by (5)

$$T = \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2(E - U(x))}} \tag{2.20}$$

$$= \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2(x^4 - x_0^4)}} \tag{2.21}$$

$$\tag{2.22}$$

Let $y^4 = x^4/x_0^4 > 0$ and let $x_1 \to \infty$. Then

$$T_\infty = \frac{1}{\sqrt{2}x_0} \int_1^\infty \frac{\mathrm{d}y}{\sqrt{y^4 - 1}} = \frac{1}{2x_0} K\left(\frac{1}{\sqrt{2}}\right) \sim \frac{1.043}{x_0}, \tag{2.23}$$

where K is the complete elliptic integral of the first kind (see Gradshteyn and Ryzhik - Table of Integrals, Series, and Products 7th ed. [2] (GR) 3.166-17). For $E > 0$, let $x(0) = x_0$, $\dot{x}(0) = v_0$. Its energy is given by $E = mv_0^2/2 + U(x_0)$

$$T = \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2(mv_0^2/2 + x^4 - x_0^4)}}. \tag{2.24}$$

Let $z^4 = x^4/(mv_0^2/2 - x_0^4) > 0$ and let $x_1 \to \infty$. Then

$$T_\infty = \frac{1}{\sqrt{2(mv_0^2/2 - x_0^4)}} \int_{z_0}^\infty \frac{\mathrm{d}z}{\sqrt{z^4 + 1}} = \frac{1}{\sqrt{2(mv_0^2/2 - x_0^4)}} F\left(\alpha, \frac{1}{\sqrt{2}}\right), \tag{2.25}$$

where $z_0 = x_0^4/(mv_0^2/2 - x_0^4)$, F is the incomplete elliptic integral of first kind (GR-3.166-1) and $\alpha = (z_0^2 - 1)/(z_0^2 + 1)$. Thus, we see that in the quartic potential, the particle reaches infinity at a finite time, and thus one cannot define a one-parameter group of diffeomorphisms. □

14. (21) Show that if the potential energy is positive, then there is a phase flow.

Solution: Let $U(x) > 0$. For bound orbits, the motion is periodic and thus can be extended to the entire time axis. For unbound orbits, let at $t = 0$, $x(0) = x_0$, $\dot{x}(0) = 0$, so that the energy $E = U(x_0) > 0$. The time to reach some x_1 is given by

$$T = \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2(U(x_0) - U(x))}} > \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2U(x_0)}} = \frac{x_1 - x_0}{\sqrt{2U(x_0)}} \tag{2.26}$$

where we have used the fact that U is positive and the orbit is unbound. Thus we see that the time period is bounded below by a linear function of x, and thus, for every finite t, the motion in x is finite, and thus the motion can be extended to the whole time axis. Thus we can define a phase flow.

15. (21) Draw the image of the circle $x^2 + (y-1)^2 < 1/4$ under the action of the transformation of the phase flow for the equations (a) of the "inverse pendulum," $\ddot{x} = x$ and (b) of the "nonlinear pendulum," $\ddot{x} = -\sin x$.

Solution: The stable equilibrium points and seperatrix can easily be inferred by observing the potential. We will derive the general solutions for the trajectories here, which can be plotted using any graphing software (DESMOS, Mathematica, etc).

(a) The general solution to the ODE can be seen to be

$$x(t) = ae^t + be^{-t} \tag{2.27}$$
$$y(t) = ae^t - be^{-t}. \tag{2.28}$$

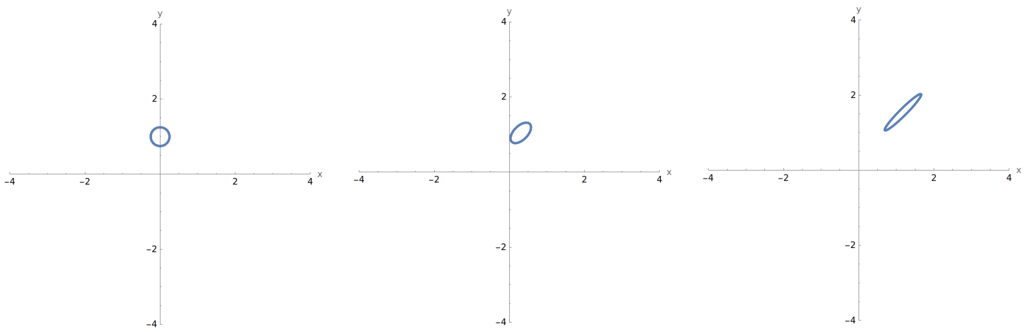
Using initial conditions $y(0) = y_0$, $x(0) = x_0$, we get

$$x(t) = \frac{x_0 + y_0}{2}e^t + \frac{x_0 - y_0}{2}e^{-t} \tag{2.29}$$
$$y(t) = \frac{x_0 + y_0}{2}e^t - \frac{x_0 - y_0}{2}e^{-t}. \tag{2.30}$$

This can be inverted easily by using (x, y) as the initial conditions and evolving the system to $-t$. This gives

$$x_0 = \frac{x + y}{2}e^{-t} + \frac{x - y}{2}e^t \tag{2.31}$$
$$y_0 = \frac{x + y}{2}e^{-t} - \frac{x - y}{2}e^t. \tag{2.32}$$

One can now plot the region in the (x, y) plane at time t corresponding to (x_0, y_0) lying in the original circle at $t = 0$. They are ellipses as seen in figure 2.4.



(a) $t = 0$ (b) $t = 0.3$ (c) $t = 1$

Figure 2.4: Phase plot for the inverse pendulum. The points lying on the boundary of the initial circle and subsequent motion are shown in blue.

(b) This system is a lot more involved and uses elliptic integrals. The results and definitions used here may be found in GR-8.1. The ODE for x can be written as

$$y \mathrm{d}y = -\sin x \mathrm{d}x \tag{2.33}$$

$$y^2 - y_0^2 = 2(\cos x - \cos x_0) \tag{2.34}$$

$$y = \sqrt{y_0^2 + 4(\sin^2(x_0/2) - \sin^2(x/2))} \tag{2.35}$$

$$\mathrm{d}t = \frac{\mathrm{d}x}{\sqrt{y_0^2 + 4(\sin^2(x_0/2) - \sin^2(x/2))}} \tag{2.36}$$

$$t = \int_{x_0}^{x} \frac{\mathrm{d}x}{2\sqrt{k_0^2 - \sin^2(x/2)}}, \tag{2.37}$$

where we have used standard trigonometric formulae and defined $k_0^2 \equiv y_0^2/4 + \sin^2(x_0/2)$. Define the variable z such that $\sin z = \sin(x/2)/k_0$. Carrying out the change of variables, it is easy to see that

$$t = \int_{z_0}^{z} \frac{\mathrm{d}z}{\sqrt{1 - k_0^2 \sin^2(z)}} \tag{2.38}$$

$$= F(z, k_0) - F(z_0, k_0), \tag{2.39}$$

where $\sin z_0 = \sin(x_0/2)/k_0$. To invert this equation, we use the definition of the Jacobi elliptical integrals (see GR-8.14)

$$u \equiv \int_0^{\mathrm{am}(u,k)} \frac{\mathrm{d}\alpha}{1 - k^2 \sin^2 \alpha} \tag{2.40}$$

$$\equiv \int_0^{\mathrm{sn}(u,k)} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{2.41}$$

We thus get,

$$z = \mathrm{am}(t + F(z_0, k_0), k_0) \tag{2.42}$$

$$\sin z = \mathrm{sn}(t + F(z_0, k_0), k_0) \tag{2.43}$$

$$x(t) = 2 \arcsin(k_0 \cdot \mathrm{sn}(t + F(z_0, k_0), k_0)). \tag{2.44}$$

Now, using equation (2.34), we get

$$y(t) = 2\sqrt{k_0^2 - k_0^2 \mathrm{sn}^2(t + F(z_0, k_0), k_0)} \tag{2.45}$$

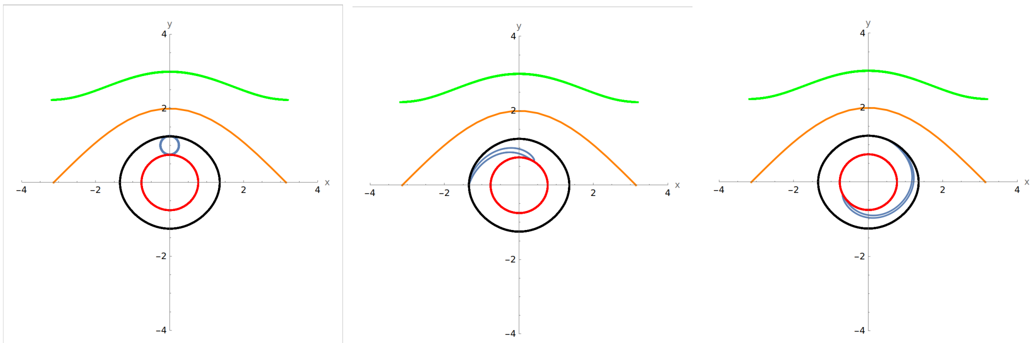
$$= 2k_0 \cdot \mathrm{cn}(t + F(z_0, k_0), k_0). \tag{2.46}$$

Similar to the previous part, one can obtain the initial conditions (x_0, y_0) as a function of $(x(t), y(t))$

$$x_0 = 2 \arcsin(k \cdot \mathrm{sn}(-t + F(\tilde{z}, k), k)) \tag{2.47}$$

$$y_0 = 2k \cdot \mathrm{cn}(-t + F(\tilde{z}, k), k), \tag{2.48}$$

where $\sin \tilde{z} = \sin(x/2)/k$. The circle condition may now be enforced on the initial condition which then leads to a constraint on the coordinates at later times. Qualitatively, all points trace an elliptical trajectory, with the point starting from $(0, 3/4)$ having the smallest size and the point starting from $(5/4, 0)$ being the largest, and all other points lying in-between. There is unstable equilibrium at $x = \pm\pi$ and a stable equilibrium at $x = 0$. Any point with energy higher than the maximum value of the potential, 1, is unbound. At $x = 0$, points starting with $y > 2$ are unbound. The two curves passing through $y = \pm 2$ and $x = 0$ form the seperatrix. As the motion proceeds, the circle is smeared more and more in phase space (phase mixing). The phase plot is shown in figure 2.5.



(a) $t = 0$ (b) $t = 26$ (c) $t = 50$

Figure 2.5: Phase plot for the non-linear pendulum. The inner and outer ellipses bounding ellipses are shown in red and black. The points lying on the boundary of the initial circle and subsequent motion are shown in blue. The seperatrix is shown in orange, and an unbound orbit is shown in green.

16. (22) Find an example of a system of the form $\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in E^2$, which is not conservative.

    Solution: A system is conservative iff the force can be expressed as the gradient of a potential function. This requirement translates to

    $$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}. \tag{2.49}$$

    Consider the force $\mathbf{f}$ with $f_1 = x_2^2$ and $f_2 = x_1^2$. Clearly the above condition is violated and thus it is not conservative.

17. (24) Show that the phase curves are great circles of this sphere. (A great circle is the intersection of a sphere with a two-dimensional plane passing through its center.)

    Solution: The solutions as functions of time are given in the text:

    $$x_1 = c_1 \cos t + c_2 \sin t \qquad x_2 = c_3 \cos t + c_4 \sin t \tag{2.50}$$
    $$y_1 = -c_1 \sin t + c_2 \cos t \qquad y_2 = -c_3 \sin t + c_4 \cos t \tag{2.51}$$

    It is straightforward to eliminate $\cos t$ and $\sin t$ to obtain the expressions

    $$\frac{x_1 c_1 + y_1 c_2}{c_1^2 + c_2^2} = \frac{x_2 c_3 + y_2 c_4}{c_3^2 + c_4^2} \tag{2.52}$$
    $$\frac{x_1 c_2 - y_1 c_1}{c_1^2 + c_2^2} = \frac{x_2 c_4 - y_2 c_3}{c_3^2 + c_4^2} \tag{2.53}$$

    These are the equations of two hyperplanes that pass through the origin $(asx1 = x2 = y1 = y2 = 0$ is a valid solution). Their intersection is a 2-plane which passes through the origin[1]. Thus on the constant energy surface π_{E_0}, the phase curves are great circles.

18. (24) Show that the set of phase curves on the surface π_{E_0} forms a two-dimensional sphere. The formula $w = (x_1 + iy_1)/(x_2 + iy_2)$ gives the "Hopf map" from the three-sphere π_{E_0} to the two-sphere (the complex plane of w completed by the point at infinity). Our phase curves are the pre-images of points under the Hopf map.

    Solution: A useful reference to topics such as the Hopf map is Differential Geometry and Lie Groups for Physicists by Fecko [3]. Define the complex variables $z_1 = x_1+iy_1$ and $z_2 = x_2 + iy_2$. The constant energy 3-sphere π_{E_0} is defined by $E_0 = |z_1|^2 + |z_2|^2$. From the equations of motion given in the text, we can make the following identifications:

$$\dot{z_1} = \dot{x_1} + i\dot{y_1} \tag{2.54}$$
$$= y_1 - ix_1 \tag{2.55}$$
$$= -iz_1 \tag{2.56}$$

Thus we have $z_1 = z_{10}\exp(-it)$ and similarly, $z_2 = z_{20}\exp(-it)$. We now define the map $\psi : \mathbb{C}^2 \to \mathbb{C} \cup \{\infty\}$. The action of ψ on (z_1, z_2) is given by

$$\psi(z_1, z_2) = \begin{cases} z_1/z_2 \in \mathbb{C} & , z_2 \neq 0 \\ \infty & , z_2 = 0. \end{cases} \tag{2.57}$$

For points that lie on a given surface of constant $E = E_0$, we have shown that phase curves are given by $z_1 = z_{10}\exp(-it)$ and $z_2 = z_{20}\exp(-it)$ with the constraint that the curves lie on the three sphere $E_0 = |z_1|^2 + |z_2|^2 = |z_{10}|^2 + |z_{20}|^2$. We note that multiplying z_1 and z_2 by a constant phase factor represents the same phase curve that only starts from a different point at $t = 0$. Thus we are free to set $z_1 = |z_{10}|\exp(i\phi - it)$ and $z_2 = |z_{20}|\exp(-it)$. Combined with the constant energy constraint, we see that every phase curve on π_{E_0} can be represented using two numbers, the real number $a = |z_{10}|/|z_{20}|$ which along with E_0 gives us $|z_{10}|$ and $|z_{20}|$, and the phase ϕ. a is set to ∞ if $z_{20} = 0$. We denote the set of all phase curves on π_{E_0} as the set $\tilde{\pi}_{E_0}$ which has local coordinates (a, ϕ).The action of ψ on the coordinates of phase curves lying on π_{E_0} is thus

$$\psi(z_{10}\exp(-it), z_{20}\exp(-it)) = \begin{cases} z_{10}/z_{20} = |z_{10}/z_{20}|\exp(i\phi) \in \mathbb{C} & , z_{20} \neq 0 \\ \infty & , z_{20} = 0. \end{cases} \tag{2.58}$$

We can define a new map $\tilde{\psi} : \tilde{\pi}_{E_0} \to \mathbb{C} \cup \{\infty\}$ such that

$$\tilde{\psi}(a, \phi) = \begin{cases} a\exp(i\phi) \in \mathbb{C} & , a \neq 0 \\ \infty & , a = \infty. \end{cases} = w \tag{2.59}$$

We need to show that this mapping is injective and surjective. This is a map from To show that the map is surjective, consider a point $w \in \mathbb{C}$. We need to find a phase curve that maps to w under π. We have $w = |z_{10}/z_{20}|\exp(i\phi)$ and thus $|w| = |z_{10}|/|z_{20}|$. From the constant energy surface condition, we get $E_0 = |z_{10}|^2 + |z_{20}|^2 = |z_{20}|^2(1 + |w|^2)$, or $|z_{20}|^2 = E_0/(1 + |w|^2)$. This gives $|z_{10}|^2 = |w|^2 E_0/(1 + |w|^2)$.

[1]The analog in 3D is the intersection of two 2-planes that pass through the origin giving rise to a line that passes through the origin.

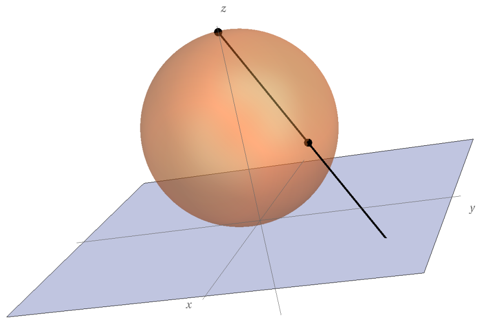


Figure 2.6: Stereographic projection of the 2-sphere on to $\mathbb{R}^2$.

We are now free to choose $\phi = \text{Arg}(w)$ and thus we have the required pre-image $z_1 = (|w|^2 E_0/(1+|w|^2))\exp(i\text{Arg}(w) - it)$ and $z_2 = (E_0/(1+|w|^2))\exp(-it)$. For the point at infinity, one sets $|z_{20}| = 0$ and $|z_{10}|^2 = E_0$. To show that the map is injective, assume two phase curves (z_1, z_2) and (z_1', z_2') map to the same point w. Then we have

$$w = \frac{|z_{10}|}{|z_{20}|}\exp(i\phi) = \frac{|z_{10}'|}{|z_{20}'|}\exp(i\phi') \tag{2.60}$$

which gives $|w| = |z_{10}|/|z_{20}| = |z_{10}'|/|z_{20}'|$ and $\phi = \phi' \text{mod}(2\pi)$. Now, the constant energy constraint gives

$$E_0 = |z_{10}|^2 + |z_{20}|^2 = |z_{10}'|^2 + |z_{20}'|^2 \tag{2.61}$$

$$\implies |z_{20}|^2(1+|w|^2) = |z_{20}'|^2(1+|w|^2) \tag{2.62}$$

$$\implies |z_{20}|^2 = |z_{20}'|^2 \implies |z_{10}|^2 = |z_{10}'|^2 \tag{2.63}$$

Putting it all together, we see that $z_1' = |z_{10}'|\exp(i\phi' - it) = |z_{10}|\exp(i\phi + i2n\pi - it) = |z_{10}|\exp(i\phi - it) = z_1$, $z_2' = |z_{20}'|\exp(-it) = |z_{20}|\exp(-it) = z_2$, where n is some integer. Thus the two phase curves are identical and the mapping is injective.

We have shown that each phase curve on the three sphere of constant energy π_{E_0} gets mapped to a unique point in $\mathbb{C}\cup\{\infty\}$ under $\tilde{\psi}$ and thus the map $\tilde{\psi} : \tilde{\pi}_{E_0} \to \mathbb{C}\cup\{\infty\}$ is bijective. We will now find an isomorphism between $\mathbb{C}\cup\{\infty\}$ and $\mathbb{S}^2$ and establish that $\mathbb{C}\cup\{\infty\} \cong \mathbb{S}^2$. For this, we note that there exists a bijective map $Q : \mathbb{C} \to \mathbb{R}^2$ such that $Q(x+iy) = (x, y) \in \mathbb{R}^2$. We now define the stereographic projection $P : \mathbb{S}^2 \to \mathbb{R}^2$. We place the sphere such that the south pole is at the origin of $\mathbb{R}^2$. As shown in figure 2.6, for any point on $\mathbb{S}^2$, we draw a line from the north pole through the point and find its intersection with the plane, which gives a point in $\mathbb{R}^2$ which in turn can be identified with a point in $\mathbb{C}$. It is easy to see that this mapping is bijective. The point corresponding to the north pole is mapped to the point at infinity. Thus we have established the required isomorphism between $\mathbb{C}\cup\{\infty\}$ and $\mathbb{S}^2$. Thus, there is an isomorphism between the set of phase curves on any given constant energy surface $\tilde{\pi}_{E_0}$ and the 2-sphere $\mathbb{S}^2$ given by $\tilde{\psi} \circ Q \circ P^{-1}$. $\square$

19. (24)Find the projection of the phase curves on the x_1, x_2 plane (i.e.,draw the orbits of the motion of a point)

Part II

Lagrangian Mechanics

Chapter 3

Variational principles

Chapter 4

Lagrangian mechanics on manifolds

Chapter 5

Oscillations

Chapter 6

Rigid Bodies

Part III

Hamiltonian Mechanics

Chapter 7

Differential Forms

Chapter 8

Symplectic manifolds

Chapter 9

Canonical formalism

Chapter 10

Introduction to perturbation theory

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[3] Marián Fecko. *Differential Geometry and Lie Groups for Physicists.* Cambridge University Press, 2006.