Solutions to V.I. Arnold's Mathematical Methods of Classical Mechanics

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Preface

These solutions rose out of a personal goal I had set for myself to finish the most mathematically concrete treatment of Classical Mechanics to date, Mathematical Methods of Classical Mechanics by V. I. Arnold [1]. Being a graduate student working on galactic dynamics, I wanted to get a solid foundation on the bread and butter of my field. I was trained in physics without worrying too much about mathematical rigor and this exercise is my attempt at getting acquainted with the tools of differential geometry and its application to Hamiltonian systems. Further, a firm grasp of differential geometry is highly useful when learning general relativity and my goal after finishing Arnold's book is to tackle Wald's book on general relativity.

Working on galactic dynamics, almost all of the topics in this book are relevant to me except for the section on rigid body dynamics which I have skipped on my first reading. I may or may not get back to this chapter at a later time. Further, given that this is a completely solo effort made worse by my lack of experience with rigorous mathematical proofs, I am in no way claiming that my proofs are correct or as rigorous as they could be and I welcome corrections and suggestions from anyone taking their time to read this document. Working out these proofs as explicitly as possible helped me at least convince myself that a result is right, and as a physicist, this was my primary aim. This work is targeted mainly for physics students as a reference in case they are not accustomed with mathematical proofs and/or don't wish to spend their time trying to work out proofs, but would like to see what they look like.

I have tried my best to work things out as explicitly as possible using only the notations and results used in Arnold's book. In some sections one has to use results which are presented in further chapters. Since the book has a notoriously bad labeling system for problems, I have written out the problem statement along with the page number where the problem can be found in the book in parenthesis. Questions with self-explanatory answers that are provided in the text are not given different solutions. Notes or ideas that I found useful from different sources during this study are given as footnotes or in boxes where and when relevant.

Contents

Part I

Newtonian Mechanics

Chapter 1

Experimental Facts

1. (6) Show that every galilean transformation of the space $\mathbb{R}\times\mathbb{R}^3$ can be written in a unique way as the composition of a rotation, a translation, and a uniform motion $(g = g_1 \circ g_2 \circ g_3)$(thus the dimension of the galilean group is equal to $3+4+3 = 10$).

   Solution: We first consider a general linear transformation on $\mathbb{R}\times\mathbb{R}^3$ as the 4×4 matrix A and an affine transformation G given by $Ga = Aa+\lambda$ for any $a\in\mathbb{R}\times\mathbb{R}^3$ where λ is a constant vector. Since we require A to preserve the galilean structure, G has to preserve the time interval between two events $t(b-a)$, as well as the distance between two simultaneous events $\rho(a,a') = ||a-a'||$. To summarize, G has to satisfy

$$t(a-b) = t(Ga-Gb) \tag{1.1}$$
$$\rho(a,a') = \rho(Ga,Ga'). \tag{1.2}$$

   We now explicitly write out the transformation of an event $a\in\mathbb{R}\times\mathbb{R}^3$ under G.

$$Ga = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} + \begin{pmatrix} \lambda^0 \\ \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix} = \begin{pmatrix} A_{0i}a^i + \lambda_0 \\ A_{1i}a^i + \lambda^1 \\ A_{2i}a^i + \lambda^2 \\ A_{3i}a^i + \lambda^3 \end{pmatrix} \tag{1.3}$$

   where repeated indices are summed over $i = 0,..3$. The difference in the coordinate $i = 0$ is taken as the time interval map for $\mathbb{R}\times\mathbb{R}^3$, i.e., $t(a-b) = a^0 - b^0$. For the invariance of the time interval between two events a and b, we have

$$a^0 - b^0 = A_{00}(a^0-b^0) + A_{01}(a^1-b^1) + ... \tag{1.4}$$

   Since this has to hold for any two events, we are led to the conclusion that $A_{00} = 1$ and $A_{0i} = 0$, $i = 1,2,3$. Now to preserve distances for two simultaneous events a and a' $(a^0 = a'^0)$,

$$\sum_{i,j=1}^{3}(A_{ij}(a^j - a'^j))^2 = \sum_{i=1}^{3}(a^i - a'^i)^2 \tag{1.5}$$

   Thus the cofactor matrix $A^C = [A^C_{00}]$ obtained by removing the first row and first column must be an orthogonal matrix ($(A^C)^{\mathrm{T}}A^C = I_3$, where I_3 is the 3×3 identity matrix). Thus, a distance and time interval preserving transformation on $\mathbb{R}\times\mathbb{R}^3$ can be written down as

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & R_{11} & R_{12} & R_{13} \\ v_2 & R_{21} & R_{22} & R_{23} \\ v_3 & R_{31} & R_{31} & R_{33} \end{pmatrix} \tag{1.6}$$

   where $R_{i,j}$ are the elements of a 3×3 orthogonal matrix R, and v_1, v_2, v_3 represent the elements A_{10}, A_{20}, A_{30} of A respectively. To see that these terms represent a boost, we consider the action of a transformation A_{boost} with $R = I_3$

$$A_{\mathrm{boost}}a = \begin{pmatrix} a^0 \\ v_1a^0 + a^1 \\ v_2a^0 + a^2 \\ v_3a^0 + a^3 \end{pmatrix} \tag{1.7}$$

which represents the original point moving with a speed given by $\mathbf{v} = (v_1, v_2, v_3)$. Thus, we see that the final galilean transformation can be written down as

$$Ga = A_{\text{boost}} A_{\text{orthogonal}} a + \lambda \tag{1.8}$$

with

$$A_{\text{boost}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \qquad A_{\text{orthogonal}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{31} & R_{33} \end{pmatrix} \tag{1.9}$$

where the boost and rotation operations commute and can be well defined by the combined matrix A from (1.6). □

2. (6) Show that all galilean spaces are isomorphic to each other and, in particular, isomorphic to the coordinate space $\mathbb{R} \times \mathbb{R}^3$.

   Solution: Let E, E' be galilean spaces with underlying vector space $\mathbb{R}^4$, i.e., for any $a, b \in E$ or E', $a - b \in \mathbb{R}^4$. If we can find isomorphisms $\phi : E \to \mathbb{R} \times \mathbb{R}^3$ and $\phi' : E' \to \mathbb{R} \times \mathbb{R}^3$, then we can construct the isomorphism $\phi'^{-1} \circ \phi : E \to E'$ which is the required isomorphism between two arbitrary galilean spaces.

   We first construct a map $M : \mathbb{R}^4 \to \mathbb{R} \times \mathbb{R}^3$ that maps the time coordinate to the 0 index (thus the notation $\mathbb{R} \times \mathbb{R}^3$). The time map is a linear map $T : \mathbb{R}^4 \to \mathbb{R}$. Let $e_0, ...e_3$ be an arbitrary basis for $\mathbb{R}^4$. Then

$$v = v^i e_i \tag{1.10}$$
$$Tv = v^i (Te_i) \tag{1.11}$$

   We want to construct a basis e'_i where $T(e'_i) = \delta_{0i}$. Let the required transformation matrix be M, $e'_i = (M^{-1})^j_i e_j$. To satisfy the condition, $(M^{-1})^j_i T(e_j) = \delta_{0i} \implies T(e_j) = M^i_j \delta_{0i} = M^0_j$. For the remaining indices, we are free to choose any 3×4 matrix such that M is full-rank. Not all $T(e_i)$ can be zero, so assume that at least $T(e_3)$ is non-zero. We set

$$M = \begin{pmatrix} T(e_0) & T(e_1) & T(e_2) & T(e_3) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.12}$$

   Note that this choice is unimportant as even if all but one of the terms $T(e_i)$ with $i \neq 3$ is zero, we can set the cofactor matrix $[M^C_{0i}]$ to I_3 so that M thus defined is full rank. Now,

$$v'^i e'_i = v^k e_k \tag{1.13}$$
$$v'^i (M^{-1})^j_i e_j = v^k e_k \tag{1.14}$$
$$\implies v'^i (M^{-1})^j_i = v^j \tag{1.15}$$
$$\implies v'^i = M^i_j v^j \tag{1.16}$$

   Thus, we have a map $M : \mathbb{R}^4 \to \mathbb{R} \times \mathbb{R}^3$ such that $Tv = (Mv)^0$ i.e., we have separated the time coordinate $\mathbb{R}$ from the spacial coordinates $\mathbb{R}^3$ of v. Since M is full-rank, the map is also invertible.

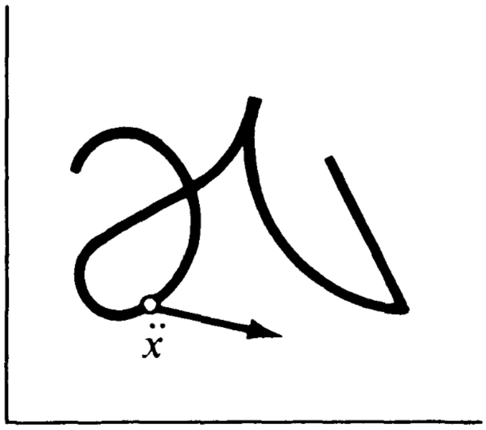


Figure 1.1

Now let $c \in E$ be fixed. Then we can write any $a \in E$ in the form $a = c + v$ where $v \in \mathbb{R}^4$ (this is effectively choosing an origin), define the map $P_c : E \to \mathbb{R}^4$ such that $P_c(a) = a - c = v \in \mathbb{R}^4$, which is clearly invertible ($P_c^{-1}(v) = c + v$). Now construct the composite map $\phi_c = M \circ P_c : E \to \mathbb{R} \times \mathbb{R}^3$. This is the required isomorphism. Note that this is not a canonical isomoprhism as one can make any choice for the "origin" $c \in E$. $\square$

3. (7) Is it possible for the trajectory of a differentiable motion on the plane to have the shape drawn in Figure 1.1? Is it possible for the acceleration vector to have the value shown?

   Solution: The trajectory shown is a perfectly reasonable motion. However, the acceleration vector shown is not possible. The velocity vector at every point on the curve points in the direction of the tangent to the curve. The direction of change of the tangent vector is given by the acceleration vector. Clearly, the tangent vector changes "inward" rather than "outward" as indicated by the arrow.

4. (10) Show that if a mechanical system consists of only one point, then its acceleration in an inertial coordinate system is equal to zero ("Newton's first law").

   Solution: If there is only one point in the system, $\ddot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}})$ is invariant under translations in time or space and under boosts with constant velocity. This implies

$$\mathbf{f}(t,\mathbf{x},\dot{\mathbf{x}}) = \mathbf{f}(t+s,\mathbf{x},\dot{\mathbf{x}}) \qquad \Longrightarrow \mathbf{f} = \mathbf{f}(\mathbf{x},\dot{\mathbf{x}})) \tag{1.17}$$
$$\mathbf{f}(\mathbf{x},\dot{\mathbf{x}})) = \mathbf{f}(\mathbf{x}+\mathbf{x_0},\dot{\mathbf{x}})) \qquad \Longrightarrow \mathbf{f} = \mathbf{f}(\dot{\mathbf{x}}) \tag{1.18}$$
$$\mathbf{f}(\dot{\mathbf{x}}) = \mathbf{f}(\dot{\mathbf{x}}+\mathbf{v_0}) \qquad \Longrightarrow \mathbf{f} = const. \tag{1.19}$$

   Invariance under orthogonal translations thus implies that $\mathbf{f} = 0$. $\square$

5. (10) A mechanical system consists of two points. At the initial moment their velocities (in some inertial coordinate system) are equal to zero. Show that the points will stay on the line which connected them at the initial moment.

   Solution: Let the point be 1 and 2. Choose a coordinate system such that at $t = 0$, $\mathbf{x}_1(0) = a\mathbf{u}_0$, $\mathbf{x}_2(0) = b\mathbf{u}_0$, and $\dot{\mathbf{x}}_1(0) = \dot{\mathbf{x}}_2(0) = 0$, where a and b are constants and u_0 is a vector parallel to the line joining 1 and 2. The forces on 1 and 2 are given by $\mathbf{f_i}(\mathbf{x_1} - \mathbf{x_2}, \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2.)$

Now, we note that if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are such that $\mathbf{v} = c\mathbf{w}$ for some $c \in \mathbb{R}$, a rotation R about the $\mathbf{v}$ leaves both vectors unchanged, i.e., $R\mathbf{v} = \mathbf{v}$, $R\mathbf{w} = \mathbf{w}$. Then

$$\mathbf{f}_i(R\mathbf{v}, R\mathbf{w}) = R\mathbf{f}_i(\mathbf{v}, \mathbf{w}) \tag{1.20}$$
$$= \mathbf{f}_i(\mathbf{v}, \mathbf{w}) \tag{1.21}$$
$$R\mathbf{f}_i(\mathbf{v}, \mathbf{w}) = \mathbf{f}_i(\mathbf{v}, \mathbf{w}) \tag{1.22}$$

Thus, if at any point the displacement and relative velocities are parallel to some vector $\mathbf{v}$, the force acting on the particles are also parallel to $\mathbf{v}$.

Back to our problem, define the function $F_i(x, y)\mathbf{u_0} \equiv \mathbf{f}_i(x\mathbf{u_0}, y\mathbf{u_0})$. Let $y_i(t)$ be the solution to the system $\ddot{y_i} = F_i(y_1 - y_2, \dot{y_1} - \dot{y_2})$, with initial conditions $y_1(0) = a$, $y_2(0) = b$, and $\dot{y_1}(0) = \dot{y_2}(0) = 0$. We can show that $\mathbf{x}_i(t) = \mathbf{u_0} y_i(t)$ is a solution to the original problem and thus the points always remain on the line parallel to u_0.

$$\ddot{\mathbf{x}}_i = \mathbf{u_0}\ddot{y}_i \tag{1.23}$$
$$= \mathbf{u_0}F_i(y_1 - y_2, \dot{y_1} - \dot{y_2}) \tag{1.24}$$
$$= \mathbf{f}_i(\mathbf{u_0}(y_1 - y_2), \mathbf{u_0}(\dot{y_1} - \dot{y_2})) \tag{1.25}$$
$$= \mathbf{f}_i(\mathbf{x_1} - \mathbf{x_2}, \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) \tag{1.26}$$

This solution is unique since the solution for the y_i's are unique. □

6. (10) A mechanical system consists of three points. At the initial moment their velocities (in some inertial coordinate system) are equal to zero. Show that the points always remain in the plane which contained them at the initial moment.

   Solution: Let the three points be 1, 2, 3. At $t = 0$, they lie on some plane τ with normal $\mathbf{N}$, and we are free to choose inertial coordinates that set the origin as $\mathbf{x}_1(0)$. Then $\mathbf{x}_2(0) - \mathbf{x}_1(0) = \mathbf{u}_0$, $\mathbf{x}_3(0) - \mathbf{x}_1(0) = \mathbf{v}_0$ are vectors that lie on τ and are perpendicular to $\mathbf{N}$.

   We now note that reflections are also distance preserving transformations and are also a valid galiliean transformations[1] (this is a problem in this chapter which we will argue now to proceed with this problem). Reflections are orthogonal transformations with $\det G = -1$. Invariance with respect to reflections means that there are no preferred orientations of coordinates in space. Thus we have the relation

$$\mathbf{F}(G\mathbf{x}, G\dot{\mathbf{x}}) = G\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) \tag{1.27}$$

   The forces that enter the equations of motion are functions of $\mathbf{x_i} - \mathbf{x_j}$ and $\dot{\mathbf{x}}_\mathbf{i} - \dot{\mathbf{x}}_\mathbf{j}$, i.e., $\mathbf{f}_i = \mathbf{f}_i(\mathbf{x_2} - \mathbf{x_1}, \mathbf{x_3} - \mathbf{x_1}, \dot{\mathbf{x_2}} - \dot{\mathbf{x_1}}, \dot{\mathbf{x_3}} - \dot{\mathbf{x_1}})$. Let $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4} \in Span(\mathbf{u_0}, \mathbf{v}_0)$, and G denote the reflection through the direction $\mathbf{N}$. Clearly, $G\mathbf{u}_0 = \mathbf{u}_0$ and $G\mathbf{v}_0 = \mathbf{v}_0$ and thus similar relations hold for the $\mathbf{w}_i$. Let $\mathbf{f}_i(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}) = \mathbf{f}_i^{||} + \mathbf{f}_i^{\perp}$ denote the components parallel and perpendicular to the plane τ (or perpendicular and parallel the normal $\mathbf{N}$) respectively. Now

$$G\mathbf{f}_i = G\mathbf{f}_i^{||} + G\mathbf{f}_i^{\perp} = \mathbf{f}_i^{||} - \mathbf{f}_i^{\perp}. \tag{1.28}$$

   But from galilean invariance we have

$$G\mathbf{f}_i(\mathbf{w}_1, ...) = \mathbf{f}_i(G\mathbf{w}_1, ...) = \mathbf{f}_i(\mathbf{w}_1, ...) = \mathbf{f}_i^{||} + \mathbf{f}_i^{\perp} \tag{1.29}$$

[1] Reflections are discrete transformations as opposed to the continuous translation, boost and rotation. The set of all galilean transformations barring rotations forms a Lie Group (see 8).

Equations (1.28) and (1.29) imply that $\mathbf{f}_i^{\perp} = 0$. Thus if all arguments $\mathbf{w}_i$ lie on a plane, the forces $\mathbf{f}_i$ on all the particles also lies on the same plane.

Now we define

$$\mathbf{f}_i(a_1\mathbf{u_0} + b_1\mathbf{v_0}, ..., a_4\mathbf{u_0} + b_4\mathbf{v_0}) \equiv \mathbf{u_0}F_i^0(a_1, b_1, ...a_4, b_4) + \mathbf{v_0}F_i^1(a_1, b_1, ..., a_4, b_4) \quad (1.30)$$

Let $y_i^j(t)$ be the solutions for $\ddot{y}_i^j = F_i^j(y_2^j - y_1^j, y_3^j - y_1^j, \dot{y}_2^j - \dot{y}_1^j, \dot{y}_3^j - \dot{y}_2^j)$, with initial conditions $y_1^0(0) = y_1^1(0) = 0$, $y_2^0(0) = 1, y_2^1(0) = 0$, $y_3^0(0) = 0, y_3^1(0) = 1$ and $\dot{y}_i^j(0) = 0$ for $i = 1, 2, 3$ and $j = 1, 2$. Consider solutions of the form $\mathbf{x}_i(t) = y_i^0(t)\mathbf{u}_0 + y_i^1(t)\mathbf{v}_0$. Similar to the previous problem, it can be shown that these are also valid solutions. Now the system of differential equations for the y_i^j are 6 second order equations with 12 initial conditions and thus possess a unique solution and thus, this solution for the $\mathbf{x}_i$ are unique. Thus we have shown that the trajectories of three particles stay on a plane if they started from rest in some inertial coordinate. □

7. (10) A mechanical system consists of two points. Show that for any initial conditions there exists an inertial coordinate system in which the two points remain in a fixed plane.

   Solution: Let the two points be 1 and 2. They have initial conditions $\mathbf{x}_1(0) = \mathbf{a}_1$, $\mathbf{x}_2(0) = \mathbf{a}_2$, $\dot{\mathbf{x}}_1(0) = \mathbf{u}_0$, and $\dot{\mathbf{x}}_2(0) = \mathbf{v}_0$. The equations of motion take the form

$$\ddot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_1 - \mathbf{x}_2, \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) \quad (1.31)$$

   Let $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{v} = \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2$). Consider the vector $\mathbf{L} = \mathbf{r} \times \mathbf{v}$. If the direction of $\mathbf{L}$ does not change with time, $\mathbf{r}$ and $\mathbf{v}$ lie on a plane perpendicular to $\mathbf{L}$ that moves at some speed parallel to $\mathbf{L}$. Now,

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}} \quad (1.32)$$
$$= \mathbf{r} \times \ddot{\mathbf{r}} \quad (1.33)$$

   At some instant, let G denote reflection about the plane on which the particles lie that is perpendicular to $\mathbf{L}$. Using the invariance of $\mathbf{r}$ and $\mathbf{v}$ on this reflection, we get $G\ddot{\mathbf{r}} = G\mathbf{f}_1(\mathbf{r}, \mathbf{v}) - G\mathbf{f}_2(\mathbf{r}, \mathbf{v}) = \mathbf{f}_1(G\mathbf{r}, G\mathbf{v}) - \mathbf{f}_2(G\mathbf{r}, G\mathbf{v}) = \mathbf{f}_1(\mathbf{r}, \mathbf{v}) - G\mathbf{f}_2(\mathbf{r}, \mathbf{v}) = \ddot{\mathbf{r}}$. Thus, we have shown that at every instant of the motion, the relative acceleration $\ddot{\mathbf{r}}$ is perpendicular to $\mathbf{L}$ and thus coplanar with $\mathbf{r}$ and $\mathbf{v}$. Thus the direction of $\mathbf{r} \times \ddot{\mathbf{r}}$ is parallel to that of $\mathbf{L}$ and the direction of $\mathbf{L}$ does not change with time. Further, one can also show that $G\ddot{\mathbf{x}}_i = G\mathbf{f}_i(\mathbf{r}, \mathbf{v}) = \mathbf{f}_i(G\mathbf{r}, G\mathbf{v}) = \mathbf{f}_i(\mathbf{r}, \mathbf{v}) = \mathbf{x}_i$ and thus the acceleration of the particles also lies on the plane perpendicular to $\mathbf{L}$ at every instant.

   We now find the inertial coordinates in which the particles appears to move on a plane. At every instant of motion, the component of the velocity of 1 parallel to $\mathbf{L}$ is given by

$$\dot{\mathbf{x}}_1^{||} = \dot{\mathbf{x}}_1 . \frac{(\mathbf{r} \times \mathbf{v})}{|(\mathbf{r} \times \mathbf{v}|} = -\dot{\mathbf{x}}_1 . \frac{(\mathbf{r} \times \dot{\mathbf{x}}_2)}{|(\mathbf{r} \times \mathbf{v}|} = \mathbf{r} . \frac{(\dot{\mathbf{x}}_1 \times \dot{\mathbf{x}}_2)}{|(\mathbf{r} \times \mathbf{v}|} \quad (1.34)$$

   where we have used the properties of the triple product. Similarly,

$$\dot{\mathbf{x}}_2^{||} = \dot{\mathbf{x}}_2 . \frac{(\mathbf{r} \times \mathbf{v})}{|(\mathbf{r} \times \mathbf{v}|} = \dot{\mathbf{x}}_2 . \frac{(\mathbf{r} \times \dot{\mathbf{x}}_1)}{|(\mathbf{r} \times \mathbf{v}|} = \mathbf{r} . \frac{(\dot{\mathbf{x}}_1 \times \dot{\mathbf{x}}_2)}{|(\mathbf{r} \times \mathbf{v}|} = \dot{\mathbf{x}}_1^{||} \quad (1.35)$$

Now, we have shown that at every instant of the motion, $\ddot{\mathbf{x}}_i$ is perpendicular to $\mathbf{L}$. This the components of the velocity parallel to $\mathbf{L}$ are constant and are set by the initial conditions. We define $\mathbf{v}_{\text{inertial}} \equiv \dot{\mathbf{x}}_1^{||} = \dot{\mathbf{x}}_2^{||}$ given by

$$\mathbf{v}_{\text{inertial}} = (\mathbf{a}_1 - \mathbf{a}_2).\frac{(\mathbf{u}_0 \times \mathbf{v}_0)}{|\mathbf{u}_0 \times \mathbf{v}_0|} \tag{1.36}$$

Thus, by carrying out a boost with velocity $\mathbf{v}_{\text{inertial}}$, the velocities of the particles parallel to $\mathbf{L}$ vanish and one sees the particles moving on a plane perpendicular to $\mathbf{L}$. $\square$

8. (11) Show that mechanics "through the looking glass" is identical to ours.

   Solution: As we have mentioned before, reflections are orthogonal transformations and thus are distance preserving maps. They also form a subset of galilean transformations. If a motion $\mathbf{x}_i(t)$ satisfies $\ddot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}, \dot{\mathbf{x}})$, then so does $G\mathbf{x}_i(t)$ and thus

$$G\ddot{\mathbf{x}} = G\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{F}(G\mathbf{x}, G\dot{\mathbf{x}}) \tag{1.37}$$

9. (11) Is the class of inertial systems unique?

   Solution: Given in text.

Chapter 2

Investigations of the Equations of Motion

1. (16) Show that through every phase point there is one and only one phase curve.

   Solution: The phase flow is given by the equations

$$\dot{x} = y \qquad \dot{y} = f(x). \tag{2.1}$$

   The equation for y can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x}y = f(x) \tag{2.2}$$

$$\implies \int_{y_0}^{y} y'\mathrm{d}y' = \int_{x_0}^{x} f(x')\mathrm{d}x \tag{2.3}$$

$$\implies \frac{y^2}{2} - F(x) = \frac{y_0^2}{2} - F(x_0) = const. \tag{2.4}$$

   where (x_0, y_0) are initial conditions, and $F(x)$ is the anti-derivative of $f(x)$, which is defined up to a constant. We note that due to the appearance of F on both sides of equation (2.4), the choice of this arbitrary constant is not important. The equation (2.4) is the equation of a 1-D curve in phase space that passes through the point (x_0, y_0). This is a well defined unique curve that is determined by only the initial conditions, which can be taken at any point on the curve. □

2. (18) Prove this (that the local maximum points ofthe potential energy are unstable, but the minimum points are stable equilibrium positions).

   Problem: The solution to this is straightforward. $f(x) = -\mathrm{d}U/\mathrm{d}x = U'(x)$, where U is the potential. If x_0 is an equilibrium point, $U'(x_0) = 0$. Thus the force at a point $x_0 + \epsilon$ is given by

$$f(x_0 + \epsilon) = -\epsilon U''(x_0) + \mathcal{O}(\epsilon^2). \tag{2.5}$$

   If the point x_0 is a minimum, $U''(x_0) > 0$ and thus the force tends to drive the body back to equilibrium (stable). On the other hand, if it is a maximum, $U''(x_0) < 0$ and the force tends to drive the body away from equilibrium(unstable).

3. (18) How many phase curves make up the separatrix (figure eight) curve, corresponding to the level E_2 ?

   Solution: Three: two 1-D curves, and a single unstable equilibrium point.

4. (18) Determine the duration of motion along the separatrix.

   Solution: As the particle approaches the equilibrium point, it's velocity starts to tend towards zero from energy conservation, while the force acting on it also tends to 0. To estimate the time, we use the result from the next problem which we will prove soon.

$$\Delta t = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2(E - U(x))}} \tag{2.6}$$

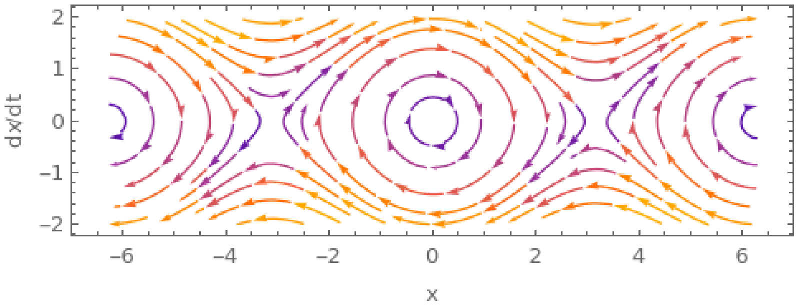


Figure 2.1: Phase curves of planar pendulum

Let $E = U(x_0)$, $x_1 = x_0 - \epsilon_0$, and $x_2 = x_0$. We can change the integration variable to ϵ and expanding the potential about x_0, we have $U(x_0-\epsilon) = U(x_0)+U''(x_0)\epsilon^2/2$. Thus, to leading order in ϵ we get

$$\Delta t \sim \int_{\epsilon_0}^{0} \frac{\mathrm{d}\epsilon}{\sqrt{-U''(x_0)\epsilon^2}} \tag{2.7}$$

which clearly diverges logarithmically. The term under the square root is positive as $U''(x_0) < 0$ due to x_0 being a maxima. Thus the time taken is infinite.

Alternatively as mentioned by Arnold, there is a unique phase curve (the 0 dimensional equilibrium point) at the phase point $(x_0, 0)$. Using the fact that there is only once phase curve passing through every point, we conclude that seperatrix trajectory always moves towards the equilibrium point but never reaches it and thus takes infinite time.

5. (18) Show that the time it takes to go from x_1 to x_2 (in one direction)is equal to

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2(E-U(x))}} \tag{2.8}$$

Solution:

$$E = T + U = \dot{x}^2/2 + U(x) \tag{2.9}$$

$$\implies \dot{x}^2 = 2(E - U(x)) \tag{2.10}$$

$$\implies \int_{t_1}^{t_2} \mathrm{d}t = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2(E-U(x))}} \tag{2.11}$$

□

6. (19) Draw the phase curves, given the potential energy graph in Figure 11 (of Arnold's Book). Solution: Given in text

7. (19) Draw the phase curves for the "equation of an ideal planar pendulum": $\ddot{x} = -\sin x$.

   Solution: The procedure for drawing these images is to plot the potential and work out the turning points corresponding to a given energy. The plot generated using the StreamPlot function in Mathematica is shown in figure 2.1

8. (19) Draw the phase curves for the "equation of a pendulum on a rotating axis": $\ddot{x} = -\sin x + M$ Solution: See figure 2.2.

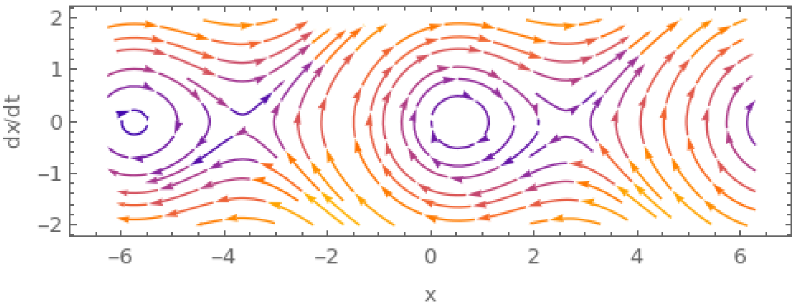


Figure 2.2: Phase curves of rotating pendulum

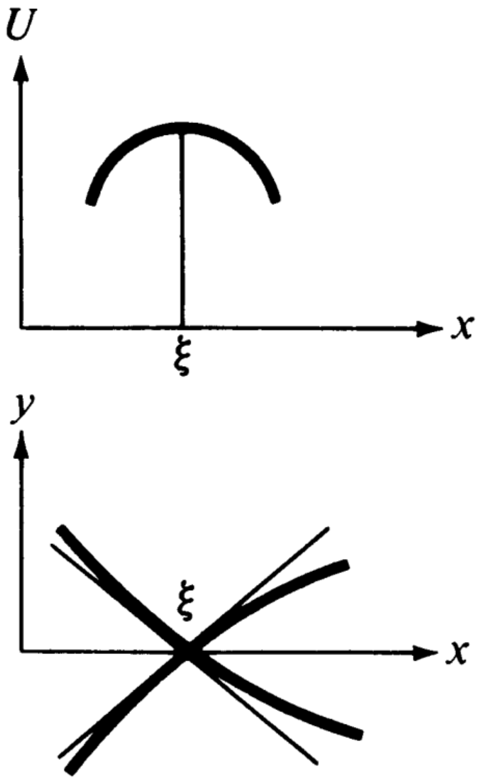


Figure 2.3: Critical energy level lines

9. (19) Find the tangent lines to the branches of the critical level corresponding to maximal potential energy $E = U(\xi)$ (Figure 2.3).

   Solution: From the first problem in this chapter, we have

$$\frac{y^2}{2} + U(x) = \frac{y_0^2}{2} + U(\xi) = U(\xi) \implies y = \pm\sqrt{2(U(\xi) - U(x))} \tag{2.12}$$

   Expanding $U(x)$ near the equilibrium point ξ, $U(x) = U(\xi) + (x-\xi)U'(\xi) + (x-\xi)^2U''(\xi)/2 + \mathcal{O}((x-\xi)^3)$, which finally gives

$$y = \pm(x-\xi)\sqrt{U''(\xi)} \tag{2.13}$$

10. (20) Let $S(E)$ be the area enclosed by the closed phase curve corresponding to the energy level E. Show that the period of motion along this curve is equal to $T = \mathrm{d}S/\mathrm{d}E$.

    Solution: Let the motion have turning points x_1, x_2 i.e., $U(x_1) = U(x_2) = E$, with $x_1 < x_2$. Then the area under the phase space curve is given by

$$\begin{aligned} S(E) &= \int_{x_1}^{x_2} \mathrm{d}x|\dot{x}| + \int_{x_2}^{x_1} \mathrm{d}x(-|\dot{x}|) &(2.14)\\ &= 2\int_{x_1}^{x_2} \mathrm{d}x\sqrt{2(E-U(x))} &(2.15)\end{aligned}$$

    where the first and second terms in equation (2.14) represent the motion from x_1 to x_2 and the reverse motion with negative velocity from x_2 to x_1 respectively. Now

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}E} &= 2\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2(E-U(x))}} &(2.16)\\ &= T_{x_1\to x_2} + T_{x_2\to x_1} &(2.17)\end{aligned}$$

    which gives the total time period of the motion. We have used the result of problem 5. □

11. (20) Let E_0 be the value of the potential function at a minimum point ξ. Find the period T_0 of small oscillations in a neighborhood of the point ξ, where $T_0 = \lim_{E\to E_0} T(E)$.

    Solution: Taylor expand the potential $U(x)$ near the minimum point

$$U(x) = E_0 + \frac{(x-\xi)^2}{2}U''(\xi) + \mathcal{O}((x-\xi)^3) \tag{2.18}$$

    The equation of motion gives

$$\ddot{x} = -U'(x) = -(x-\xi)U''(\xi). \tag{2.19}$$

    Let $z = x - \xi$. Equation (2.19) can be written as $\ddot{z} = -zU''(\xi)$. This is the equation of a 1-D harmonic oscillator with frequency $\omega = \sqrt{U''(\xi)}$. Thus the time period of the motion is given by $T_0 = 2\pi/\sqrt{U''(\xi)}$

12. (20) Consider a periodic motion along the closed phase curve corresponding to the energy level E. Is it stable in the sense of Liapunov?

    Solution: An equilibrium point ξ is said to be Liapunov stable if $\forall\epsilon > 0, \exists\ \delta(\epsilon) > 0$ such that if $|x(0) - \xi| < \delta(\epsilon)$ then $|x(t) - \xi| < \epsilon\ \forall t > 0$. Simply put, we must be able to confine the motion to an arbitrarily small region of configuration space around ξ by starting the motion sufficiently close to ξ. However, for a given periodic motion with turning points $x_1 < \xi < x_2$, this is not possible for any $\epsilon < \min(\xi - x_1, x_2 - \xi)$, as $|x(t) - \xi|$ will exceed this value at some point of the periodic orbit. Thus the motion is not Liapunov stable.

13. (21) Show that the system with potential energy $U = -x^4$ does not define a phase flow.

Solution: In order for the motion in a potential to constitute a phase flow, it must be possible to extend the solution to the entire time axis. If the motion reaches infinity in a finite amount of time, then the group properties given in the text are destroyed and thus the the motion will not define a phase flow. We now calculate the time period of motion to infinity in this potential.

There are two cases: $E > 0$ and $E < 0$. For $E < 0$, let the particle have initial condition $x(0) = x_0$, $\dot{x}(0) = 0$. Its energy is given by $E = U(x_0)$. The time period for motion up to a point x_1 is given by (5)

$$T = \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2(E - U(x))}} \tag{2.20}$$

$$= \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2(x^4 - x_0^4)}} \tag{2.21}$$

$$\tag{2.22}$$

Let $y^4 = x^4/x_0^4 > 0$ and let $x_1 \to \infty$. Then

$$T_\infty = \frac{1}{\sqrt{2}x_0} \int_1^\infty \frac{\mathrm{d}y}{\sqrt{y^4 - 1}} = \frac{1}{2x_0} K\left(\frac{1}{\sqrt{2}}\right) \sim \frac{1.043}{x_0}, \tag{2.23}$$

where K is the complete elliptic integral of the first kind (see Gradshteyn and Ryzhik - Table of Integrals, Series, and Products 7th ed. [2] (GR) 3.166-17). For $E > 0$, let $x(0) = x_0$, $\dot{x}(0) = v_0$. Its energy is given by $E = mv_0^2/2 + U(x_0)$

$$T = \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2(mv_0^2/2 + x^4 - x_0^4)}}. \tag{2.24}$$

Let $z^4 = x^4/(mv_0^2/2 - x_0^4) > 0$ and let $x_1 \to \infty$. Then

$$T_\infty = \frac{1}{\sqrt{2(mv_0^2/2 - x_0^4)}} \int_{z_0}^\infty \frac{\mathrm{d}z}{\sqrt{z^4 + 1}} = \frac{1}{\sqrt{2(mv_0^2/2 - x_0^4)}} F\left(\alpha, \frac{1}{\sqrt{2}}\right), \tag{2.25}$$

where $z_0 = x_0^4/(mv_0^2/2 - x_0^4)$, F is the incomplete elliptic integral of first kind (GR-3.166-1) and $\alpha = (z_0^2 - 1)/(z_0^2 + 1)$. Thus, we see that in the quartic potential, the particle reaches infinity at a finite time, and thus one cannot define a one-parameter group of diffeomorphisms. □

14. (21) Show that if the potential energy is positive, then there is a phase flow.

Solution: Let $U(x) > 0$. For bound orbits, the motion is periodic and thus can be extended to the entire time axis. For unbound orbits, let at $t = 0$, $x(0) = x_0$, $\dot{x}(0) = 0$, so that the energy $E = U(x_0) > 0$. The time to reach some x_1 is given by

$$T = \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2(U(x_0) - U(x))}} > \int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{2U(x_0)}} = \frac{x_1 - x_0}{\sqrt{2U(x_0)}} \tag{2.26}$$

where we have used the fact that U is positive and the orbit is unbound. Thus we see that the time period is bounded below by a linear function of x, and thus, for every finite t, the motion in x is finite, and thus the motion can be extended to the whole time axis. Thus we can define a phase flow.

15. (21) Draw the image of the circle $x^2 + (y-1)^2 < 1/4$ under the action of the transformation of the phase flow for the equations (a) of the "inverse pendulum," $\ddot{x} = x$ and (b) of the "nonlinear pendulum," $\ddot{x} = -\sin x$.

Solution: The stable equilibrium points and seperatrix can easily be inferred by observing the potential. We will derive the general solutions for the trajectories here, which can be plotted using any graphing software (DESMOS, Mathematica, etc).

(a) The general solution to the ODE can be seen to be

$$x(t) = ae^t + be^{-t} \tag{2.27}$$
$$y(t) = ae^t - be^{-t}. \tag{2.28}$$

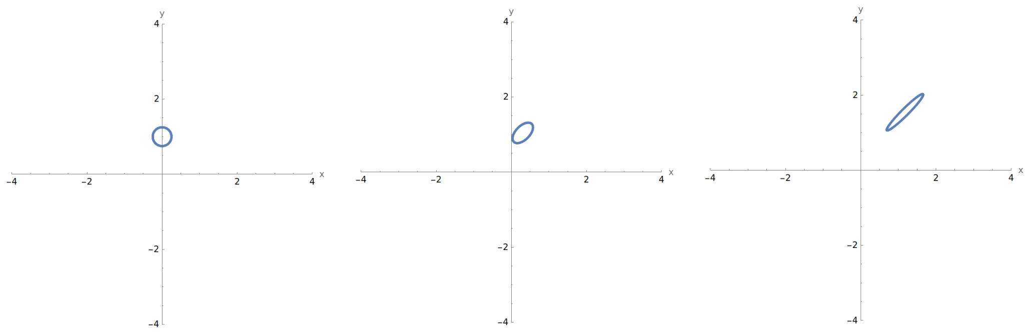
Using initial conditions $y(0) = y_0$, $x(0) = x_0$, we get

$$x(t) = \frac{x_0 + y_0}{2}e^t + \frac{x_0 - y_0}{2}e^{-t} \tag{2.29}$$
$$y(t) = \frac{x_0 + y_0}{2}e^t - \frac{x_0 - y_0}{2}e^{-t}. \tag{2.30}$$

This can be inverted easily by using (x, y) as the initial conditions and evolving the system to $-t$. This gives

$$x_0 = \frac{x + y}{2}e^{-t} + \frac{x - y}{2}e^t \tag{2.31}$$
$$y_0 = \frac{x + y}{2}e^{-t} - \frac{x - y}{2}e^t. \tag{2.32}$$

One can now plot the region in the (x, y) plane at time t corresponding to (x_0, y_0) lying in the original circle at $t = 0$. They are ellipses as seen in figure 2.4.



(a) $t = 0$ (b) $t = 0.3$ (c) $t = 1$

Figure 2.4: Phase plot for the inverse pendulum. The points lying on the boundary of the initial circle and subsequent motion are shown in blue.

(b) This system is a lot more involved and uses elliptic integrals. The results and definitions used here may be found in GR-8.1. The ODE for x can be written as

$$y\mathrm{d}y = -\sin x \mathrm{d}x \tag{2.33}$$

$$y^2 - y_0^2 = 2(\cos x - \cos x_0) \tag{2.34}$$

$$y = \sqrt{y_0^2 + 4(\sin^2(x_0/2) - \sin^2(x/2))} \tag{2.35}$$

$$\mathrm{d}t = \frac{\mathrm{d}x}{\sqrt{y_0^2 + 4(\sin^2(x_0/2) - \sin^2(x/2))}} \tag{2.36}$$

$$t = \int_{x_0}^{x} \frac{\mathrm{d}x}{2\sqrt{k_0^2 - \sin^2(x/2)}}, \tag{2.37}$$

where we have used standard trigonometric formulae and defined $k_0^2 \equiv y_0^2/4 + \sin^2(x_0/2)$. Define the variable z such that $\sin z = \sin(x/2)/k_0$. Carrying out the change of variables, it is easy to see that

$$t = \int_{z_0}^{z} \frac{\mathrm{d}z}{\sqrt{1 - k_0^2 \sin^2(z)}} \tag{2.38}$$

$$= F(z, k_0) - F(z_0, k_0), \tag{2.39}$$

where $\sin z_0 = \sin(x_0/2)/k_0$. To invert this equation, we use the definition of the Jacobi elliptical integrals (see GR-8.14)

$$u \equiv \int_0^{\mathrm{am}(u,k)} \frac{\mathrm{d}\alpha}{1 - k^2 \sin^2 \alpha} \tag{2.40}$$

$$\equiv \int_0^{\mathrm{sn}(u,k)} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{2.41}$$

We thus get,

$$z = \mathrm{am}(t + F(z_0, k_0), k_0) \tag{2.42}$$

$$\sin z = \mathrm{sn}(t + F(z_0, k_0), k_0) \tag{2.43}$$

$$x(t) = 2 \arcsin(k_0 \cdot \mathrm{sn}(t + F(z_0, k_0), k_0)). \tag{2.44}$$

Now, using equation (2.34), we get

$$y(t) = 2\sqrt{k_0^2 - k_0^2 \mathrm{sn}^2(t + F(z_0, k_0), k_0)} \tag{2.45}$$

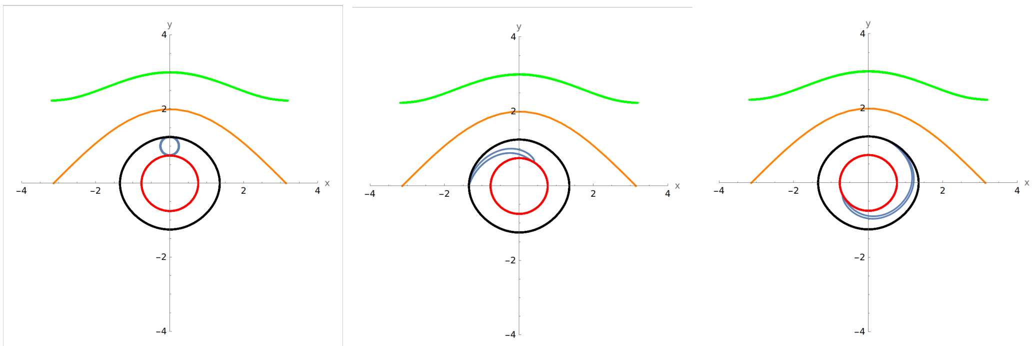
$$= 2k_0 \cdot \mathrm{cn}(t + F(z_0, k_0), k_0). \tag{2.46}$$

Similar to the previous part, one can obtain the initial conditions (x_0, y_0) as a function of $(x(t), y(t))$

$$x_0 = 2 \arcsin(k \cdot \mathrm{sn}(-t + F(\tilde{z}, k), k)) \tag{2.47}$$

$$y_0 = 2k \cdot \mathrm{cn}(-t + F(\tilde{z}, k), k), \tag{2.48}$$

where $\sin \tilde{z} = \sin(x/2)/k$. The circle condition may now be enforced on the initial condition which then leads to a constraint on the coordinates at later times. Qualitatively, all points trace an elliptical trajectory, with the point starting from $(0, 3/4)$ having the smallest size and the point starting from $(5/4, 0)$ being the largest, and all other points lying in-between. There is unstable equilibrium at $x = \pm\pi$ and a stable equilibrium at $x = 0$. Any point with energy higher than the maximum value of the potential, 1, is unbound. At $x = 0$, points starting with $y > 2$ are unbound. The two curves passing through $y = \pm 2$ and $x = 0$ form the seperatrix. As the motion proceeds, the circle is smeared more and more in phase space (phase mixing). The phase plot is shown in figure 2.5.



(a) $t = 0$ (b) $t = 26$ (c) $t = 50$

Figure 2.5: Phase plot for the non-linear pendulum. The inner and outer ellipses bounding ellipses are shown in red and black. The points lying on the boundary of the initial circle and subsequent motion are shown in blue. The seperatrix is shown in orange, and an unbound orbit is shown in green.

16. (22) Find an example of a system of the form $\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in E^2$, which is not conservative.

Solution: A system is conservative iff the force can be expressed as the gradient of a potential function. This requirement translates to

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}. \tag{2.49}$$

Consider the force $\mathbf{f}$ with $f_1 = x_2^2$ and $f_2 = x_1^2$. Clearly the above condition is violated and thus it is not conservative.

17. (24) Show that the phase curves are great circles of this sphere. (A great circle is the intersection of a sphere with a two-dimensional plane passing through its center.)

Solution: The solutions as functions of time are given in the text:

$$x_1 = c_1 \cos t + c_2 \sin t \qquad x_2 = c_3 \cos t + c_4 \sin t \tag{2.50}$$
$$y_1 = -c_1 \sin t + c_2 \cos t \qquad y_2 = -c_3 \sin t + c_4 \cos t \tag{2.51}$$

It is straightforward to eliminate $\cos t$ and $\sin t$ to obtain the expressions

$$\frac{x_1 c_1 + y_1 c_2}{c_1^2 + c_2^2} = \frac{x_2 c_3 + y_2 c_4}{c_3^2 + c_4^2} \tag{2.52}$$
$$\frac{x_1 c_2 - y_1 c_1}{c_1^2 + c_2^2} = \frac{x_2 c_4 - y_2 c_3}{c_3^2 + c_4^2} \tag{2.53}$$

These are the equations of two hyperplanes that pass through the origin $(asx1 = x2 = y1 = y2 = 0$ is a valid solution). Their intersection is a 2-plane which passes through the origin[1]. Thus on the constant energy surface π_{E_0}, the phase curves are great circles.

18. (24) Show that the set of phase curves on the surface π_{E_0} forms a two-dimensional sphere. The formula $w = (x_1 + iy_1)/(x_2 + iy_2)$ gives the "Hopf map" from the three-sphere π_{E_0} to the two-sphere (the complex plane of w completed by the point at infinity). Our phase curves are the pre-images of points under the Hopf map.

Solution: A useful reference to topics such as the Hopf map is Differential Geometry and Lie Groups for Physicists by Fecko [3]. Define the complex variables $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. The constant energy 3-sphere π_{E_0} is defined by $E_0 = |z_1|^2 + |z_2|^2$. From the equations of motion given in the text, we can make the following identifications:

$$\dot{z_1} = \dot{x_1} + i\dot{y_1} \tag{2.54}$$
$$= y_1 - ix_1 \tag{2.55}$$
$$= -iz_1 \tag{2.56}$$

Thus we have $z_1 = z_{10}\exp(-it)$ and similarly, $z_2 = z_{20}\exp(-it)$. We now define the map $\psi : \mathbb{C}^2 \to \mathbb{C} \cup \{\infty\}$. The action of ψ on (z_1, z_2) is given by

$$\psi(z_1, z_2) = \begin{cases} z_1/z_2 \in \mathbb{C} & , z_2 \neq 0 \\ \infty & , z_2 = 0. \end{cases} \tag{2.57}$$

For points that lie on a given surface of constant $E = E_0$, we have shown that phase curves are given by $z_1 = z_{10}\exp(-it)$ and $z_2 = z_{20}\exp(-it)$ with the constraint that the curves lie on the three sphere $E_0 = |z_1|^2 + |z_2|^2 = |z_{10}|^2 + |z_{20}|^2$. We note that multiplying z_1 and z_2 by a constant phase factor represents the same phase curve that only starts from a different point at $t = 0$. Thus we are free to set $z_1 = |z_{10}|\exp(i\phi - it)$ and $z_2 = |z_{20}|\exp(-it)$. Combined with the constant energy constraint, we see that every phase curve on π_{E_0} can be represented using two numbers, the real number $a = |z_{10}|/|z_{20}|$ which along with E_0 gives us $|z_{10}|$ and $|z_{20}|$, and the phase ϕ. a is set to ∞ if $z_{20} = 0$. We denote the set of all phase curves on π_{E_0} as the set $\tilde{\pi}_{E_0}$ which has local coordinates (a, ϕ).The action of ψ on the coordinates of phase curves lying on π_{E_0} is thus

$$\psi(z_{10}\exp(-it), z_{20}\exp(-it)) = \begin{cases} z_{10}/z_{20} = |z_{10}/z_{20}|\exp(i\phi) \in \mathbb{C} & , z_{20} \neq 0 \\ \infty & , z_{20} = 0. \end{cases} \tag{2.58}$$

We can define a new map $\tilde{\psi} : \tilde{\pi}_{E_0} \to \mathbb{C} \cup \{\infty\}$ such that

$$\tilde{\psi}(a, \phi) = \begin{cases} a\exp(i\phi) \in \mathbb{C} & , a \neq 0 \\ \infty & , a = \infty. \end{cases} = w \tag{2.59}$$

We need to show that this mapping is injective and surjective. To show that the map is surjective, consider a point $w \in \mathbb{C}$. We need to find a phase curve that maps to w under $\tilde{\psi}$. We have $w = |z_{10}/z_{20}|\exp(i\phi)$ and thus $|w| = |z_{10}|/|z_{20}|$. From the constant energy surface condition, we get $E_0 = |z_{10}|^2 + |z_{20}|^2 = |z_{20}|^2(1 + |w|^2)$, or $|z_{20}|^2 = E_0/(1 + |w|^2)$. This gives $|z_{10}|^2 = |w|^2 E_0/(1 + |w|^2)$. We are now free

[1]The analog in 3D is the intersection of two 2-planes that pass through the origin giving rise to a line that passes through the origin.

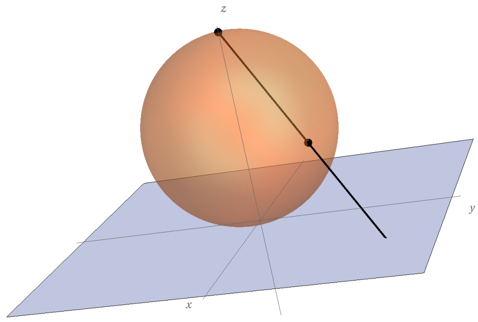


Figure 2.6: Stereographic projection of the 2-sphere on to $\mathbb{R}^2$.

to choose $\phi = \text{Arg}(w)$ and thus we have the required pre-image $z_1 = (|w|^2 E_0/(1 + |w|^2)) \exp(i\text{Arg}(w) - it)$ and $z_2 = (E_0/(1+|w|^2)) \exp(-it)$. For the point at infinity, one sets $|z_{20}| = 0$ and $|z_{10}|^2 = E_0$. To show that the map is injective, assume two phase curves (z_1, z_2) and (z_1', z_2') map to the same point w. Then we have

$$w = \frac{|z_{10}|}{|z_{20}|} \exp(i\phi) = \frac{|z_{10}'|}{|z_{20}'|} \exp(i\phi') \tag{2.60}$$

which gives $|w| = |z_{10}|/|z_{20}| = |z_{10}'|/|z_{20}'|$ and $\phi = \phi' \text{mod}(2\pi)$. Now, the constant energy constraint gives

$$E_0 = |z_{10}|^2 + |z_{20}|^2 = |z_{10}'|^2 + |z_{20}'|^2 \tag{2.61}$$

$$\implies |z_{20}|^2(1 + |w|^2) = |z_{20}'|^2(1 + |w|^2) \tag{2.62}$$

$$\implies |z_{20}|^2 = |z_{20}'|^2 \implies |z_{10}|^2 = |z_{10}'|^2 \tag{2.63}$$

Putting it all together, we see that $z_1' = |z_{10}'| \exp(i\phi' - it) = |z_{10}| \exp(i\phi + i2n\pi - it) = |z_{10}| \exp(i\phi - it) = z_1$, $z_2' = |z_{20}'| \exp(-it) = |z_{20}| \exp(-it) = z_2$, where n is some integer. Thus the two phase curves are identical and the mapping is injective.

We have shown that each phase curve on the three sphere of constant energy π_{E_0} gets mapped to a unique point in $\mathbb{C} \cup \{\infty\}$ under $\tilde{\psi}$ and thus the map $\tilde{\psi} : \tilde{\pi}_{E_0} \to \mathbb{C} \cup \{\infty\}$ is bijective. We will now find an isomorphism between $\mathbb{C} \cup \{\infty\}$ and $\mathbb{S}^2$ and establish that $\mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$. For this, we note that there exists a bijective map $Q : \mathbb{C} \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ such that $Q(x + iy) = (x, y) \in \mathbb{R}^2$ and $Q(\infty) = \infty$. We now define the stereographic projection $P : \mathbb{S}^2 \to \mathbb{R}^2 \cup \{\infty\}$. We place the sphere such that the south pole is at the origin of $\mathbb{R}^2$. As shown in figure 2.6, for any point on $\mathbb{S}^2$, we draw a line from the north pole through the point and find its intersection with the plane, which gives a point in $\mathbb{R}^2$ which in turn can be identified with a point in $\mathbb{C}$. It is easy to see that this mapping is bijective. The point corresponding to the north pole is mapped to the point at infinity. Thus we have established the required isomorphism between $\mathbb{C} \cup \{\infty\}$ and $\mathbb{S}^2$. Thus, there is an isomorphism between the set of phase curves on any given constant energy surface $\tilde{\pi}_{E_0}$ and the 2-sphere $\mathbb{S}^2$ given by $P^{-1} \circ Q \circ \tilde{\psi} : \tilde{\pi} \to \mathbb{S}^2$. $\square$

19. (24) Find the projection of the phase curves on the x_1, x_2 plane (i.e.,draw the orbits of the motion of a point)

    Solution: The orbit for a given E can be obtained by using $E = (x_1^2+y_1^2+x_2^2+y_2^2)/2 = (c_1^2 + c_2^2 + c_3^2 + c_4^2)/2$. We can solve for y_1 and y_2 from equations (2.52) and (2.53)

$$y_1 = \frac{(c_1^2 + c_2^2)x_2 + (c_1c_3 + c_2c_4)x_1}{c_1c_4 - c_2c_3} \tag{2.64}$$
$$y_2 = \frac{(c_3^2 + c_4^2)x_1 - (c_1c_3 + c_2c_4)x_2}{c_1c_4 - c_2c_3} \tag{2.65}$$

    Using the energy constraint, one gets the equation of the orbit

$$E = \frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{2} \tag{2.66}$$
$$= \frac{(c_1^2 + c_2^2 + c_3^2 + c_4^2)((c_3^2 + c_4^2)x_1^2 - 2(c_1c_3 + c_2c_4)x_1x_2 + (c_1^2 + c_2^2)x_2^2)}{2(c_2c_3 - c_1c_4)^2} \tag{2.67}$$
$$1 = \frac{((c_3^2 + c_4^2)x_1^2 - 2(c_1c_3 + c_2c_4)x_1x_2 + (c_1^2 + c_2^2)x_2^2)}{2(c_2c_3 - c_1c_4)^2} \tag{2.68}$$

    This is the equation of an ellipse which can easily be plotted given initial conditions.

20. (25) Show that this rectangle is inscribed in the ellipse $U \leq E$.

    Solution: From energy conservation, $E_1 + E_2 = E \geq (x_1^2 + \omega^2 x_2^2)/2 = U$. The trajectory is not allowed to leave this area as that would imply negative total kinetic energy. Thus the rectangle that contains the bounds on the individual x_1 and x_2 motions will lie to the interior of this ellipse. The maximum value of the potential is reached when motion in both x_1 and x_2 have 0 kinetic energy and they reach their extreme values of $x_1 = \pm\sqrt{2E_1}$ and $x_2 = \pm\sqrt{2E_2}/\omega$. But these points lie on the ellipse $U = E$. For finite values of kinetic energy, the motion will have smaller values of $|x_1|$ and $|x_2|$, and will thus be interior to this rectangle. Thus the rectangle is inscribed in the ellipse.

    In other words, the ellipse denotes the area where the total kinetic energy is positive, while the inscribed rectangle denotes the area where kinetic energies of both x_1 and x_2 motions are positive. The latter is a subset of the former, and thus lies in the interior.

21. (27) Show that this curve is a parabola.

    Solution: $x_1 = A_1 \sin(t + \phi_1)$, $x_2 = A_2 \sin(2t + \phi_2)$.

$$x_2 = A_2 \sin(2t + \phi_2) \tag{2.69}$$
$$= A_2 \sin(2(t + \phi_1) + \phi_2 - 2\phi_1) \tag{2.70}$$
$$= A_2(\sin(2(t + \phi_1)) \cos(\phi_2 - 2\phi_1) + \cos(2(t + \phi_2)) \sin(\phi_2 - 2\phi_1)) \tag{2.71}$$
$$= A_2(\cos(\phi_2 - 2\phi_1)\left(2\frac{x_1}{A_1}\sqrt{1 - \frac{x_1^2}{A_1^2}}\right) + \sin(\phi_2 - 2\phi_1)\left(1 - 2\frac{x_1^2}{A_1^2}\right)) \tag{2.72}$$

    Set the phase such that $\phi_2 - 2\phi_1 = \pi/2$. Then, we get $x_2 = A_2 - 2A_2x_1^2/A_1^2$ which is a parabola.

22. (28) Show that if $\omega = m/n$ then the Lissajous figure is a closed algebraic curve; but if ω is irrational, then the Lissajous figure fills the rectangle everywhere densely. What does the corresponding phase trajectory fill out?

Solution: $x_1 = A_1 \sin(t + \phi_1)$, $x_2 = A_2 \sin(\omega t + \phi_2)$. If the curve is to be closed, we require that after some time period Δt both curves return to the same point they started at, i.e., $\Delta t = 2\pi n$ and $\omega \Delta t = 2\pi m$, where n, m are integers. We see that closed orbits occur iff $\omega = m/n$ for some integers m and n. If ω is irrational, then by the above argument, the orbits are never closed. Since phase curves never cross and the motion is not closed, given enough time the trajectory of this motion in phase space will fill the region allowed by the energy constraints densely. This implies that the orbit visits all locations in configuration space permitted by energy conservation of both x_1 and x_2 motion, i.e., the rectangle with sides A_1 and A_2.

The region in phase space corresponding to the conservation of E_1 and E_2 is the intersection of the cylinders $E_1 = (x_1^2+y_1^2)/2$ and $E_2 = (x_2^2+y_2^2)/2$, which is inscribed in the 3-sphere $E = (x_1^2 + y_1^2 + x_2^2 + y_2^2)/2$

23. (29) Show that the vector field $F_1 = x_2$, $F_2 = -x_1$ is not conservative.

Solution: A field is said to be conservative iff it can be written as the gradient of a scalar potential function U (i.e., $F_1 = \partial U/\partial x_1$, $F_2 = \partial U/\partial x_2$). In 2D, a necessary condition for a force field to be conservative may be expressed using the commuting of partial derivatives as $\partial F_1/\partial x_2 = \partial F_2/\partial x_1$. For the given case we clearly see that this condition does not hold and thus the field is not conservative.

24. (29) Is the field in the plane minus the origin given by $F_1 = x_2/(x_1^2 + x_2^2)$, $F_2 = -x_1(x_1^2 + x_2^2)$ conservative? Show that a field is conservative if and only if its work along any closed contour is equal to zero.

We start by checking the partial derivative condition and it is easy to show that:

$$\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1} = \frac{(x_1 - x_2)(x_1 + x_2)}{(x_1 + x_2)^2}. \tag{2.73}$$

If we can find the required potential function we are done. Integrating $\partial U/\partial x_1 = F_1$ we get $U = \arctan(x_1/x_2) + C_1(x_2)$, where C_1 is an arbitrary function of only x_1. Similarly, integrating $\partial U/\partial x_2 = F_2$ gives $-\arctan(x_2/x_1) + C_2(x_1)$. Setting $C_1 = 0, C_2 = \pi/2$, and using $\arctan(\alpha) + \arctan(1/\alpha) = \pi/2$, we see that the potential $U = \arctan(x_1/x_2)$ is the necessary potential.

For the second part, if the field $\mathbf{F}$ is conservative, then

$$\int_\gamma (\mathbf{F}, \mathrm{d}\mathbf{S}) = -U(M_0) + U(M_0) = 0 \tag{2.74}$$

where γ is a closed path and M_0 is any point on this path. Conversely, if the work done by F on any closed path is 0, then consider the closed path C shown in figure 2.7. The path C is made up of two paths γ_1 and γ_2 and thus,

$$\int_C (\mathbf{F}, \mathrm{d}\mathbf{S}) = \int_{\gamma_1} (\mathbf{F}, \mathrm{d}\mathbf{S}) + \int_{\gamma_2} (\mathbf{F}, \mathrm{d}\mathbf{S}) \tag{2.75}$$

We now deform the curve C to a new curve C' such that γ_1 is changed to γ_1' and γ_2 is left intact as shown in figure 2.7. The shape of the curve γ_1' is completely arbitrary and we only need the end points to remain the same. We can again write

$$\int_{C'} (\mathbf{F}, \mathrm{d}\mathbf{S}) = \int_{\gamma_1'} (\mathbf{F}, \mathrm{d}\mathbf{S}) + \int_{\gamma_2} (\mathbf{F}, \mathrm{d}\mathbf{S}). \tag{2.76}$$

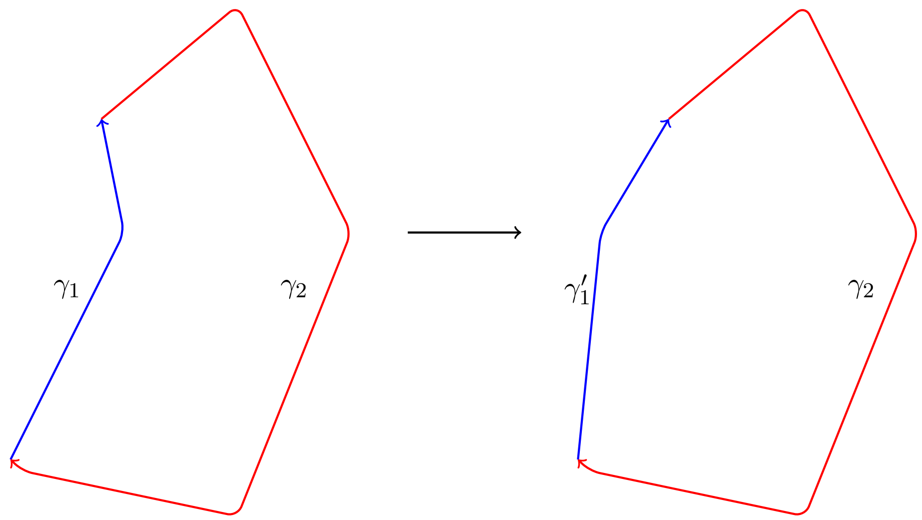


Figure 2.7: The closed path is made up of two paths. One path (blue) is deformed.

However, since the work done by the force on any closed path is 0, these two quantities are 0, and we are lead to the conclusion that

$$\int_{\gamma_1} (\mathbf{F}, \mathrm{d}\mathbf{S}) = \int_{\gamma_1'} (\mathbf{F}, \mathrm{d}\mathbf{S}), \tag{2.77}$$

and thus the work done by $\mathbf{F}$ along any path depends only on the end points and not the shape of the path. But from the theorem proved in the text, this means $\mathbf{F}$ is a conservative force. □

25. (30) Show that all vectors of a central field lie on rays through 0, and that the magnitude of the vector field at a point depends only on the distance from the point to the center of the field.

Solution: Let the force field be $\mathbf{F}(\mathbf{x})$ on E^2. The set of motions that leave the origin invariant are reflection about planes passing through the origin and rotations about the origin. From invariance under rotations about the origin G, we get $G\mathbf{F} = \mathbf{F}(G\mathbf{x}) \implies |\mathbf{F}| = |\mathbf{F}(G\mathbf{x})|$, where we have used the fact that G is an orthogonal matrix. Thus the magnitude of $\mathbf{F}$ does not depend on the direction of x and depends only on the magnitude. Thus $\mathbf{F}(\mathbf{x}) = F(x)\hat{\mathbf{n}}(\mathbf{x})$, where F is the magnitude of $\mathbf{F}$ and $\hat{\mathbf{n}}$ is the unit vector which may still depend on the direction of $\mathbf{x}$.

Now, let (x_1, x_2) be coordinates on E^2. We consider a point on the x_1 axis:

$$\mathbf{F}(x_1, 0) = F(x_1)(\alpha(x_1, x_2)\hat{x_1} + \beta(x_1, x_2)\hat{x_2}) \tag{2.78}$$

where $\alpha^2 + \beta^2 = 1$ sets the direction of the unit vector $\hat{\mathbf{n}}$. Let G_1 be rotation by $180°$. Applying G_1 to $\mathbf{F}(x_1, 0)$ gives

$$G_1\mathbf{F}(x_1, 0) = F(x_1)G_1(\alpha(x_1, x_2)\hat{x_1} + \beta(x_1, x_2)\hat{x_2}) \tag{2.79}$$
$$\mathbf{F}(-x_1, 0) = F(x_1)(-\alpha(x_1, x_2)\hat{x_1} - \beta(x_1, x_2)\hat{x_2}). \tag{2.80}$$

Now let R_1 be reflection about the x_2 axis. Applying R_1 to $\mathbf{F}(x_1, 0)$ gives

$$R_1\mathbf{F}(x_1, 0) = F(x_1)R_1(\alpha(x_1, x_2)\hat{x_1} + \beta(x_1, x_2)\hat{x_2}) \tag{2.81}$$
$$\mathbf{F}(-x_1, 0) = F(x_1)(-\alpha(x_1, x_2)\hat{x_1} + \beta(x_1, x_2)\hat{x_2}). \tag{2.82}$$

Combining equations (2.80) and (2.82) gives $\beta(x_1, x_2) = 0$ and thus, the force on the x_1 axis is pointed in the direction of the x_1 axis. But our choice of the (x_1, x_2) axes was completely arbitrary. For any point of interest we can construct the axes such that the point lies on the x_1 axis. Thus, we are lead to the conclusion that the direction of $\mathbf{F}$ at any point is along the line joining the origin and that point. □

26. (30) Compute the potential energy of a Newtonian field.

    Solution: $\mathbf{F} = -k(\mathbf{r}/|r|^3)$. Since the force is conservative, the potential function U satisfies

$$U(r_0) - U(r) = \int_{r_0}^{r} \mathrm{d}r' F(r') \tag{2.83}$$

$$= -\int_{r_0}^{r} \mathrm{d}r' \frac{k}{r'^2} \tag{2.84}$$

$$= \frac{k}{r} - \frac{k}{r_0}. \tag{2.85}$$

    Upto an arbitrary constant, $U(r) = -k/r$.

27. (36) For which values of α is motion along a circular orbit in the field with potential energy $U = r^\alpha$, $-2 < \alpha < \infty$, Liapunov stable?

    Solution: For a circular orbit, $\dot{r} = 0$, and thus $E = V(r_0)$ where V is the effective potential $V(r) = U(r) + M^2/2r^2$ and r_0 is the radius of the circulat orbit. The equilibrium point r_0 is the minimum of the effective potential and thus by differentiating the definition of $v = V(r)$ we get $M^2 = r_0^3 U'(r_0) = \alpha r_0^{2+\alpha}$. Now, the equation of motion is given by

$$\ddot{r} = \frac{M^2}{r^3} - U'(r). \tag{2.86}$$

    Let the orbit be slightly perturbed from the circular orbit : $r = r_0 + \delta r$ where $\delta r/r_0 \ll 1$. Plugging this into the equation of motion we get (r_0 is a constant)

$$\delta\ddot{r} = \frac{M^2}{r_0^3}\left(1 + \frac{\delta r}{r_0}\right)^{-3} - \alpha r_0^{\alpha-1}\left(1 + \frac{\delta r}{r_0}\right)^{\alpha-1} \tag{2.87}$$

$$= -\frac{3M^2\delta r}{r_0^4} - \alpha(\alpha-1)\frac{\delta r}{r_0} r_0^{\alpha-1} + \mathcal{O}((\delta r/r_0)^2) \tag{2.88}$$

$$= -3\alpha r_0^{\alpha-1}\frac{\delta r}{r_0} - \alpha(\alpha-1)\frac{\delta r}{r_0} r_0^{\alpha-1} \tag{2.89}$$

$$= -\alpha(\alpha+2)\delta r r_0^{\alpha-2} \tag{2.90}$$

    where we have ignored higher order terms and used the relation between r_0 and M^2. This is a harmonic oscillator ODE which can easily be solved to get

$$\delta r = \delta r_0 \exp(\pm r_0^{\frac{\alpha-2}{2}}\sqrt{-\alpha(\alpha+2)}t) = \delta r_0 \exp(\pm i r_0^{\frac{\alpha-2}{2}}\sqrt{\alpha(\alpha+2)}t), \tag{2.91}$$

    where δr_0 is a constant. We see that the system has either exponentially growing modes ($-2 < \alpha < 0$), or oscillatory modes ($\alpha > 0$). The oscillatory modes are radially stable. However, one also needs to make sure that the tangential time periods of the original and perturbed orbits are identical to make sure that the orbits remain close to one another. The tangential time period of the unperturbed circular orbit is $T_0 = 2\pi r_0/v_{t0} = 2\pi r_0^2/M = 2\pi/\sqrt{\alpha r_0^{\alpha-2}}$. The time period of tangential motion of the perturbed orbit can be estimated from

$$\dot{\theta} = \frac{M}{(r_0+\delta_r)^2} = \frac{M}{r_0^2} - 2\frac{M\delta r}{r_0^3} + \mathcal{O}(\delta r^2). \tag{2.92}$$

The angle Φ covered by the perturbed orbit in a time T_0 given by

$$\begin{aligned}
\Phi &= \int_0^{T_0} \mathrm{d}t \left(\frac{M}{r_0^2} - 2\frac{M\delta r}{r_0^3}\right) &(2.93)\\
&= 2\pi - 2\frac{M\delta r_0}{r_0^3}\int_0^{T_0} \mathrm{d}t \exp(\pm i r_0^{\frac{\alpha-2}{2}}\sqrt{\alpha(\alpha+2)}t) &(2.94)\\
&= 2\pi \mp 2\frac{M\delta r_0}{i r_0^{\frac{\alpha+4}{2}}\sqrt{\alpha(\alpha+2)}}(\exp(\pm i r_0^{\frac{\alpha-2}{2}}\sqrt{\alpha(\alpha+2)}T_0) - 1) &(2.95)\\
&= 2\pi \mp 2\frac{M\delta r_0}{i r_0^{\frac{\alpha+4}{2}}\sqrt{\alpha(\alpha+2)}}(\exp(\pm i r_0^{\frac{\alpha-2}{2}}\sqrt{(\alpha+2)}2\pi) - 1). &(2.96)
\end{aligned}$$

Note that the term in the exponential depends on the radius r_0 and only in the case $\alpha = 2$ does it become independent of r_0. For $\alpha = 2$, the term in the exponent becomes

$$\pm i r_0^{\frac{\alpha-2}{2}}\sqrt{(\alpha+2)}2\pi = \pm i4\pi \tag{2.97}$$

and thus the second term in equation (2.96) vanishes and $\Phi = 2\pi$. We see that only in the case $\alpha = 2$ does the term in the exponent remain an integer multiple of 2π for circular orbits of any radii. Thus when the unperturbed orbit completes an 2π revolution, so does the perturbed orbit, and thus the two orbits can remain arbitrarily close to each other. We also note that in this case, the time period of circular motion $T_0 = 2\pi/\sqrt{\alpha r_0^{\alpha-2}}$ also becomes independent of radius. Thus $\alpha = 2$ is the only Liapunov stable case.

28. (36) Find the angle Φ for an orbit close to the circle of radius r.

    Solution: For a circular orbit with radius r_0, $M^2 = r_0^3 U'(r_0)$. We perturb the orbit but adding a small amount of energy ϵ. Thus we have

$$E = \frac{M^2}{2r_0^2} + U(r_0) + \epsilon = \frac{M^2}{2r^2} + U(r) + \frac{\dot{r}^2}{2}. \tag{2.98}$$

The turning points of the perturbed orbit may be obtained by setting the kinetic energy to 0 and solving for r.

$$\begin{aligned}
\frac{M^2}{2r_0^2} + U(r_0) + \epsilon &= \frac{M^2}{2r_p^2} + U(r_p) &(2.99)\\
\implies \frac{M^2}{2r_0^2}\left(\frac{r_0^2}{r_p^2} - 1\right) &+ U(r_p) - U(r) = \epsilon &(2.100)
\end{aligned}$$

Now, assume that ϵ is small enough that $r_p = r_0 - \delta r_p$, with $\delta r_p \ll r_0$. Then, expanding up to second order we get

$$\frac{M^2}{2r_0^2}\left(2\frac{\delta r_p}{r_0} + 3\frac{\delta r_p^2}{r_0^2}\right) - \delta r_p U'(r_0) + \frac{\delta r_p^2}{2}U''(r_0) + \mathcal{O}(\delta r^3) = \epsilon \tag{2.101}$$

We see that the first order term vanishes for circular orbits, so ϵ is a second order term

$$\epsilon = \left(\frac{3M^2}{2r_0^4} + \frac{U''(r_0)}{2}\right)\delta r_p^2 \tag{2.102}$$

The same calculation can be carried out for $r_a = r_0 + \delta r_a$, and one sees that $\delta r_a = \delta r_p = \delta r_0$. Now computing Φ, we get

$$\Phi = \int_{r_p}^{r_p} \mathrm{d}r \frac{M/r^2}{\sqrt{2(\epsilon + M^2/2r_0^2 + U(r_0) - M^2/2r^2 - U(r))}} \tag{2.103}$$

Changing variables to $r = r_0 + \delta r$ and retaining only the highest non zero order, we get

$$\Phi = \int_{-\delta r_0}^{\delta r_0} \mathrm{d}\delta r \frac{(M/r_0^2)(1 - 2\delta r/r_0)}{\sqrt{2\left(\frac{3M^2}{2r_0^4} + \frac{U''(r_0)}{2}\right)(\delta r_0^2 - \delta r^2)}} \tag{2.104}$$

$$= \int_{-\delta r_0}^{\delta r_0} \mathrm{d}\delta r \frac{(M/r_0^2)}{\sqrt{2\left(\frac{3M^2}{2r_0^4} + \frac{U''(r_0)}{2}\right)(\delta r_0^2 - \delta r^2)}} \tag{2.105}$$

$$= \frac{(M/r_0^2)}{\sqrt{\frac{3M^2}{r_0^4} + U''(r_0)}} \arcsin(\delta r/\delta r_0)|_{-\delta r_0}^{\delta r_0} \tag{2.106}$$

$$= \frac{\pi M/r_0^2}{\sqrt{\frac{3M^2}{r_0^4} + U''(r_0)}}. \tag{2.107}$$

Now, using the fact that $M^2 = r_0^3 U'(r_0)$, we can rewrite this as

$$\Phi = \pi\sqrt{\frac{U'(r_0)}{3U'(r_0) + r_0 U''(r_0)}} \tag{2.108}$$

29. (36) Examine the shape of an orbit in the case when the total energy is equal to the value of the effective energy V a local maximum point.

    Solution: The local maximum of V is an unstable equilibrium point r_0 for the motion in r and thus for orbits that start at r_0 with energy $E = V(r_0)$ are circular. For orbits that approach r_0 with $E = V(r_0)$, the body never reaches r_0 (see problem 4 in chapter 1). Thus the after a large duration, the body spirals around r_0 with decreasing r and $0 < (r - r_0)/r_0 \ll 1$ if it started from $r > r_0$ and with increasingly r and $0 < (r_0 - r)/r_0 \ll 1$ if it started from $r < r_0$.

30. (37) Show that the angle Φ between the pericenter and apocenter is equal to the semiperiod of an oscillation in the one-dimensional system with potential energy $W(x) = U(M/x) + (x^2/2)$.

    Solution:

$$\Phi = \int_{r_p}^{r_a} \mathrm{d}r \frac{M/r^2}{\sqrt{2(E - U(r)) - M^2/r^2}} \tag{2.109}$$

Using the hint given, the substitution $x = M/r$, $\mathrm{d}x = -Mx^2\mathrm{d}r$

$$\Phi = \int_{x_{max}}^{x_{min}} \frac{-\mathrm{d}x}{\sqrt{2(E - U(M/x) - x^2/2)}} = \int_{x_{min}}^{x_{max}} \frac{\mathrm{d}x}{\sqrt{2(E - W(x))}}, \tag{2.110}$$

where $x_{min} = M/r_a$, $x_{max} = M/r_p$, and $W(x) = U(M/x) + (x^2/2)$.

31. (37) Find the angle Φ for an orbit close to the circle of radius r.

    Solution: See problem 28.

32. (37) For which values of U is the magnitude of Φ_{cir} independent of the radius r?

    Solution: We need to solve for U from the equation $\mathrm{d}\Phi_{circ}/\mathrm{d}r = 0$. This is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{U'}{3U' + rU''} = 0 \tag{2.111}$$

$$\implies U''(3U' + rU'') - U'(3U'' + U'' + rU''') = 0 \tag{2.112}$$

Let $U' = y$. Then, we get the ODE

$$ry''y + y'y - ry'^2 = 0. \tag{2.113}$$

Dividing by y^2, we see that

$$\frac{ry''}{y} + \frac{y'}{y} - \frac{ry'^2}{y^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{y'}{y}\right) = 0. \tag{2.114}$$

Solving, we get $ry' = py$ where p is a constant. Solving this linear ODE for y then gives $y = qr^p$, where q is another arbitrary constant. Now we defined $U' = y$. Finally, solving for U, we get see that if $p \neq -1$, $U = ar^\alpha$, $\alpha \neq 0$, where we have just defined arbitrary variables $\alpha = p+1$ and $a = q/p$ instead of p and q. If $p = -1$, then we have $U = b \ln r$. We also have to make sure that the quantity under the square root in the definition of Φ_{circ} is positive. This condition gives

$$\frac{qr^p}{3qr^p + qpr^p} > 0 \tag{2.115}$$

which gives $p > -3$ or $\alpha > -2$. Thus the allowed forms of U are $U = ar^\alpha$, $\alpha > -2$ and $U = b\ln r$.

33. (37) Let $U(r) \to \infty$ for $U = ar^\alpha$ as $r \to \infty$. Find $\lim_{E\to\infty} \Phi(E, M)$.

    Solution: Using the definition of x and $W(x)$ in equation (2.110), we use the hint given in the text and define $y = x/x_{max}$. We further define $W_*(y) = W(x(y))/x_{max}^2$ so that

$$W_*(y) = \frac{y^2}{2} + \frac{1}{x_{max}^2}U\left(\frac{M}{yx_{max}}\right). \tag{2.116}$$

We further note that by definition of the turning points, $E = V(r_p) = W(x_{max}) = x_{max}^2 W_*(1)$. Using this, equation (2.110) becomes,

$$\Phi = \int_{y_{min}}^{1} \frac{\mathrm{d}y}{\sqrt{2(W_*(1) - W_*(y))}}, \tag{2.117}$$

where $y_{min} = x_{min}/x_{max}$. Now, as $E \to \infty$, $r_p \to 0$ and $r_a \to \infty$. This can be argued as follows. The turning points are the roots r_t of the equation $E = V(r_t)$. When $E \to \infty$, we see that $V(r) \to \infty$ for the cases $r \to 0$ (from the term $M^2/2r^2$), and $r \to \infty$ (from $U(r)$). This implies, $x_{min} \to 0$ and $x_{max} \to \infty$. Thus $y_{min} \to 0$ and

$$\lim_{x_{max}\to\infty} W_*(y) = \lim_{x_{max}\to\infty} \frac{y^2}{2} + \frac{1}{x_{max}^2} U\left(\frac{M}{yx_{max}}\right) = \lim_{x_{max}\to\infty} \frac{y^2}{2} + \frac{1}{x_{max}^2} U(0) = \frac{y^2}{2} \tag{2.118}$$

Thus, we have

$$\Phi = \int_0^1 \frac{\mathrm{d}y}{\sqrt{1-y^2}} = \pi/2 \tag{2.119}$$

34. (38) Let $U(r) = -kr^{-\beta}$, $0 < \beta < 2$. Find $\Phi_0 = \lim_{E\to -0} \Phi$.

    Solution: In the limit $E \to -0$,

$$\Phi \to \Phi_0 = \int_{r_p}^{r_a} \mathrm{d}r \frac{M/r^2}{\sqrt{-2U(r) - M^2/r^2}} \tag{2.120}$$

$$= \int_{r_p}^{r_a} \mathrm{d}r \frac{M/r^2}{\sqrt{2kr^{-\beta} - M^2/r^2}} \tag{2.121}$$

We see that $r_a \to \infty$, and $r_p = (\sqrt{2k}/M)^{2/(\beta-2)}$ by equating the effective potential to 0. Let $y = r_p/r$. Then we get

$$\Phi_0 = \int_0^1 \frac{\mathrm{d}y}{r_p\sqrt{r_p^{\beta-2}(r_p/y)^{-\beta} - (r_p/y)^{-2}}} = \int_0^1 \frac{\mathrm{d}y}{\sqrt{y^\beta - y^2}} = \int_0^1 \mathrm{d}y y^{-\beta/2}(1-y^{2-\beta})^{-1/2} \tag{2.122}$$

Define a new variable $z = y^{2-\beta}$ so that $\mathrm{d}y = z^{(\beta-1)/(2-\beta)}\mathrm{d}z/(2-\beta)$. We thus get

$$\Phi_0 = \int_0^1 \mathrm{d}z \frac{z^{(\beta-1)/(2-\beta)}}{2-\beta} z^{-\beta/2(2-\beta)}(1-z) \tag{2.123}$$

$$= \int_0^1 \mathrm{d}z \frac{z^{(\beta-1)/(2-\beta)}}{2-\beta} z^{-\beta/2(2-\beta)}(1-z)^{-1/2} \tag{2.124}$$

$$= \int_0^1 \mathrm{d}z \frac{z^{-1/2}}{2-\beta}(1-z)^{-1/2} \tag{2.125}$$

$$= \frac{1}{2-\beta}\mathrm{Beta}(1/2, 1/2) \tag{2.126}$$

$$= \frac{\pi}{2-\beta} \tag{2.127}$$

35. (38) Find all central fields in which bounded orbits exist and are all closed.

    Solution: The solution is given in the book. I will try to elaborate a little further. The angle Φ_{circ} evaluated in problem 28 is the angle covered in one full radial oscillation by an orbit that deviates very slightly from a circular orbit. For all these nearly circular orbits to be closed, we **necessarily** require $\Phi_{circ} = 2\pi m/n$ and it should not depend on the radius. Thus we require the potentials to have the forms from problem 32. For these potentials, $\Phi_{circ} = \pi/\sqrt{\alpha+2}$ as given in the text with $\alpha = 0$ for the logarithmic case.

    For the case $\alpha > 0$, bounded orbits only exist for $a > 0$. We have shown in problem 33 that if as $r \to \infty$ $U \to \infty$, then as $E \to \infty$ $\Phi \to \pi/2$ **irrespective of the value of** M. Equating $\pi/\sqrt{\alpha+2} = \pi/2$, we get $\alpha = 2$. This means that for $\alpha > 0$, only for the special case $\alpha = 2$ does it hold that the angle Φ remains constant and equal to a rational multiple of 2π for **ALL** bounded orbits from the lowest possible energy (circular orbit), to $E \to \infty$.

Now for the case $\alpha < 0$, it can be seen that bound orbits exist only for $a < 0$. We have shown that in problem 34 that in the limit $E \to -0$ which is the highest energy possible for a bound orbit, $\Phi \to \Phi_0 = \pi/(2+\alpha)$ **irrespective of the value of** M. Equating $(2+\alpha) = \sqrt{\alpha+2}$ we get $\alpha = -1$ and $\Phi = \pi$. In words, for $\alpha < 0$, $\alpha = -1$ is the only case in which the angle Φ of a closed orbit remains constant and equal to a rational multiple of 2π for **ALL** bounded orbits from the lowest possible energy (circular orbit), to $E \to -0$. Thus all bounded orbits are closed only for the cases $U = ar^2$, $a > 0$ and $U = -b/r$, $b > 0$. □

36. (40) At the entry of a satellite into a circular orbit at a distance 300 km from the earth the direction of its velocity deviates from the intended direction by 1° towards the earth. How is the perigee changed?

    Solution: The radius of the intended circular orbit of the satellite is $r_0 = 300\text{km} + R_e$ where $R_e \sim 6400\text{km}$ is the radius of the Earth. As given in the text, for small eccentricities, the orbit is very close to circular and deviations are second order in eccentricity. The new orbit is the original one tilted by 1° about the point of entry. Thus, we see that the change in perigee, which should simply have been r_0 for a circular orbit is given by

$$\delta r_p = r_0 \frac{1^\circ}{180^\circ}\pi \sim 118\text{km} \tag{2.128}$$

37. (41) How does the height of the perigee change if the actual velocity is 1 m/sec less than intended?

    Solution: We will assume unit mass for the satellite as it does not affect the result. For the gravity of the earth, the constant $k = GM_e$ where $G = 6.67 \times 10^{-11}\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$ is the gravitational constant and $M_e = 5.972 \times 10^{24}\text{kg}$ is the mass of the earth. For a circular orbit of radius a_0, it is easy to show using $\dot{r} = 0$ to get $E = V(a_0)$ along with $M = a_0 v$ and $V'(a_0) = 0$ that the velocity is given by $v_0 = \sqrt{GM_e/a_0}$. For $a_0 = 6700\text{km}$ it can be shown that $v_0 = 7.7 \times 10^8\text{m}\,\text{s}^{-1}$.

    Now, from the results in the text, $|E| = GM_e/2a$, where a is the semi-major axis for any orbit in the potential. In our case, when the velocity is reduced at the initial point, the satellite is at the apogee of a new elliptical orbit with semimajor axis a. The apogee distance is the radius of the original circular orbit $a(1+e) = a_0$. At the apogee, the energy of the satellite is also given by $|E| = GM_e/a_0 - v^2/2$. Using the two results for the energy we can write

$$\frac{GM_e}{2a} = \frac{GM_e}{a_0} - \frac{v^2}{2} \tag{2.129}$$

$$\frac{GM_e(1+e)}{2a_0} = \frac{GM_e}{a_0} - \frac{v^2}{2} \tag{2.130}$$

$$\implies v^2 = \frac{GM_e(1-e)}{a_0} = v_0^2(1-e). \tag{2.131}$$

    Now, $v = v_0 + \Delta v$ where $\Delta v = -1\text{ms}^{-1}$ and $|\Delta v| \ll v_0$. To linear order in $\Delta v/v_0$ we get $e = -2\Delta v/v_0 = 2.6 \times 10^{-4}$. The old perigee was $r_{p0} = a_0$. The new perigee is given by $r_p = a(1-e) = a_0(1-e)/(1+e) \approx a_0(1-2e)$. Thus the change in perigee is

$$\Delta r_p = r_p - r_{p0} = a_0(1-2e) - a_0 = -2ea_0 = 3484\text{m} \tag{2.132}$$

38. (41) The first cosmic velocity is the velocity of motion on a circular orbit of radius close to the radius of the earth. Find the magnitude of the first cosmic velocity v_1 and show that $v_2 = \sqrt{2}v_1$.

Solution: In this problem, v_2 is the escape velocity. For a circular orbit with radius $R_e \sim 6400$km, as we have done in the previous problem, using $\dot{r} = 0$ to get $E = V(R_e)$ along with $M = R_e v_1$and $V'(r_0) = 0$ gives $v_1 = \sqrt{GM_e/R_e} \approx 7.9\mathrm{km\,s^{-1}}$.

To escape the Earth, the minimum velocity is one such that the object reaches infinity with 0 velocity i.e., the total energy of the body must be zero. Writing the energy conservation condition we get

$$E = \frac{v_2^2}{2} - \frac{GM_e}{R_e} = 0 \tag{2.133}$$

which gives us $v_2 = \sqrt{2GM_e/R_e} = \sqrt{2}v_1$.

39. (41) During his walk in outer space, the cosmonaut A. Leonov threw the lens cap of his movie camera towards the earth. Describe the motion of the lens cap with respect to the spaceship, taking the velocity of the throw as $10\mathrm{m\,s^{-1}}$.

Solution: Let the radius of his circular orbit around the Earth be r_0. Let the trajectory of the lens cap be given by $\mathbf{r}(t)$ where we use polar coordinates (r, θ) with center at the earth. Ignoring the gravity of the ship and writing out the components of the equations of motion in the r and θ directions we get

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM_e}{r^2} \tag{2.134}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0. \tag{2.135}$$

Using a perturbative approach, we can write the motion of the lens cap as $r = r_0 + r_1(t)$, $\theta(t) = \theta_0(t) + \theta_1(t)$ where the subscript 0 quantities denote the motion of the spaceship. The zeroth and first order equations in the subscript 1 quantities give

$$\ddot{r_0} = 0 \tag{2.136}$$

$$\dot{\theta_0}^2 = \frac{GM_e}{r_0^3} \tag{2.137}$$

$$\ddot{r_1} - r_1\dot{\theta_0}^2 - 2\dot{\theta_0}\dot{\theta_1}r_0 = 2\frac{GM_e}{r_0^3}r_1 \tag{2.138}$$

$$2\dot{r_1}\dot{\theta_0} + r_0\ddot{\theta_1} = 0. \tag{2.139}$$

We denote the constant value of $\dot{\theta}$ as $1/T_0 = \sqrt{GM_e/r_0^3}$. Define dimensionless quantities $\tilde{r_1} = r_1/r_0$ and $\tilde{t} = t/T_0$. Using these definitions, and using the definition of T_0 , we get

$$\frac{\mathrm{d}^2\tilde{r_1}}{\mathrm{d}\tilde{t}^2} = 3\tilde{r_1} + 2\frac{\mathrm{d}\theta_1}{\mathrm{d}\tilde{t}} \tag{2.140}$$

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}\tilde{t}^2} + 2\frac{\mathrm{d}\tilde{r_1}}{\mathrm{d}\tilde{t}} = 0. \tag{2.141}$$

Differentiating (2.140) with respect to $\tilde{t}$ and using (2.141) gives

$$\frac{\mathrm{d}^3\tilde{r_1}}{\mathrm{d}\tilde{t}^3} = 3\frac{\mathrm{d}\tilde{r_1}}{\mathrm{d}\tilde{t}} - 4\frac{\mathrm{d}\tilde{r_1}}{\mathrm{d}\tilde{t}} = -\frac{\mathrm{d}\tilde{r_1}}{\mathrm{d}\tilde{t}} \tag{2.142}$$

$$\implies \frac{\mathrm{d}\tilde{r_1}}{\mathrm{d}\tilde{t}} = A\cos\tilde{t} + B\sin\tilde{t} \tag{2.143}$$

$$\implies \tilde{r_1} = A\sin\tilde{t} - B\cos\tilde{t} + C \tag{2.144}$$

where A, B, and C are arbitrary constants set by the initial conditions. From equation (2.143), we also get

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}\tilde{t}^2} = -2(A\cos\tilde{t} + B\sin\tilde{t}) \tag{2.145}$$

$$\implies \frac{\mathrm{d}\theta_1}{\mathrm{d}\tilde{t}} = -2(A\sin\tilde{t} - B\cos\tilde{t}) + C_1 \tag{2.146}$$

$$\implies \theta_1 = 2(A\cos\tilde{t} + B\sin\tilde{t}) + C_1\tilde{t} + D \tag{2.147}$$

where C_1 and D are also arbitrary constants. Now, at $t = 0$, we are given $\tilde{r_1} = 0$, $\mathrm{d}\tilde{r_1}/\mathrm{d}\tilde{t} = -10\mathrm{m\,s^{-1}}T_0/r_0 = \tilde{v_0}$, $\theta_1 = 0$, and $\mathrm{d}\theta_1/\mathrm{d}\tilde{t} = 0$. This gives $B = C$, $A = v_0$, $D = -2A = -2v_0$, and $2B = -C_1$. Thus, we have

$$\tilde{r_1} = v_0\sin\tilde{t} + B(1-\cos\tilde{t}) \tag{2.148}$$

$$\theta_1 = -2v_0(1-\cos\tilde{t}) + 2B(\sin\tilde{t} - \tilde{t}). \tag{2.149}$$

It appears as if we have an undefined constant in the problem. However, this is due to us differentiating equation (2.140) to solve the coupled second order differential equations. Substituting our solutions for $\tilde{r_1}$ and θ_1 back into equation (2.140) gives

$$\begin{aligned} -v_0\sin\tilde{t} + B\cos\tilde{t} &= 3(v_0\sin\tilde{t} + B(1-\cos\tilde{t})) + 2(-2v_0\sin\tilde{t} + 2B(\cos\tilde{t}-1)) \\ &= -v_0\sin\tilde{t} + B(\cos\tilde{t} - 1) \end{aligned} \tag{2.150}$$

which sets B to 0. Thus our final solutions are $\tilde{r_1} = v_0\sin\tilde{t}$ and $\theta_1 = -2v_0(1-\cos\tilde{t})$. Set up a local coordinate system (x, y) at the position of the space ship as shown in figure 2.8. Up to first order in the perturbation quantities, we have

$$x = (r_0 + r_1)\sin\theta_1 \tag{2.151}$$

$$y = r_0 - (r_0 + r_1)\cos\theta_1 \tag{2.152}$$

Taking $r_0 = R_e \approx 6400\mathrm{km}$, we have $v_0 = -10/\sqrt{GM/R_e} \approx -1/800$. The maximum value of x is at $\tilde{t} = 2n\pi + \pi$, where it attains the value $x_{max} = (r_0 + r_1)\sin(-4v_0) \approx 4 \times 6400/800 = 32\mathrm{km}$ and $y \approx -r_1 = 0$ to first order in the small quantites. The maximum value of $|y|$ is at $\tilde{t} = n\pi + \pi/2$ where it attains the value $y_{max} = r_0 - (r_0 + v_0r_0)\cos(-2v_0) \approx -v_0r_0 = 6400/800 = 8\mathrm{km}$ and $x = (r_0 + r_1)\sin(-2v_0) \approx 2 \times 6400/800 = 16\mathrm{km}$. Thus the lens cap traces an elliptical path with semi axes 16km and 8km, with the minor axes along the line joining the spaceship and the Earth. The cap starts on the side of the spaceship facing the earth, and returns to the opposite side. The time period of this motion is $2\pi \times T_0 = 2\pi \times 6400\mathrm{km}/8\mathrm{kms^{-1}} = 5027\mathrm{s} \approx 1.4\mathrm{hr}$. The trajectory is shown in figure 2.8.

40. (43) Investigate motion in a central field in n-dimensional euclidean space.

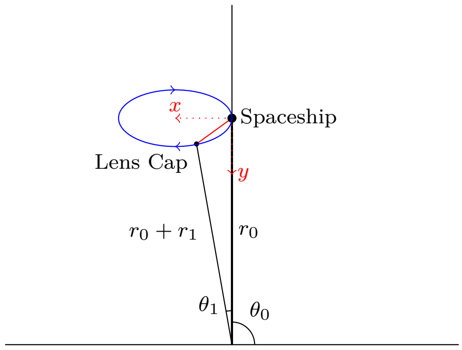


Figure 2.8: The local coordinate system for the cosmonaut are shown in red. The path of the lens cap is shown in blue

Solution: For a central field in n dimension, the equation of motion is given by

$$\ddot{\mathbf{r}} = -\frac{\partial U(r)}{\partial \mathbf{r}}, \tag{2.153}$$

where the potential U only depends on the magnitude of $\mathbf{r}$. Define the rank 2 tensor

$$M_{i_3 i_4} \equiv \epsilon_{i_1 i_2 i_3 i_4} r_{i_1} \dot{r}_{i_2} \tag{2.154}$$

where $\epsilon_{i_1 i_2 i_3 i_4}$ is the totally antisymmetric tensor defined as

$$\epsilon_{i_1 i_2 i_3 i_4} = \begin{cases} 1, & \text{if } \{i_1, i_2, i_3, i_4\} \text{ is an even permutaion of } Asc\{i_1, i_2, i_3, i_4\} \\ -1, & \text{if } \{i_1, i_2, i_3, i_4\} \text{ is an odd permutaion of } Asc\{i_1, i_2, i_3, i_4\} \\ 0, & \text{if atleast one } i_j = i_k, j \neq k \end{cases} \tag{2.155}$$

where $Asc\{i_1, i_2, i_3, i_4\}$ represents the numbers $\{i_1, i_2, i_3, i_4\}$ sorted in ascending order. Taking the time derivative of M, we get

$$\dot{M}_{i_3 i_4} = \epsilon_{i_1 i_2 i_3 i_4} \dot{r}_{i_1} \dot{r}_{i_2} + \epsilon_{i_1 i_2 i_3 i_4} r_{i_1} \ddot{r}_{i_2} \tag{2.156}$$

$$= 0 - \epsilon_{i_1 i_2 i_3 i_4} r_{i_1} \frac{\partial U(r)}{\partial r_{i_2}} \tag{2.157}$$

$$= -\epsilon_{i_1 i_2 i_3 i_4} r_{i_1} U'(r) \frac{\partial r}{\partial r_{i_2}} \tag{2.158}$$

$$= -\epsilon_{i_1 i_2 i_3 i_4} r_{i_1} U'(r) \frac{r_{i_2}}{r} \tag{2.159}$$

$$= 0, \tag{2.160}$$

where we have used the fact that the product of a totally antisymmetric tensor with a symmetric tensor is 0 along with the equation of motion in a central potential. Thus we have shown that in the central potential the quantity $M_{i_3 i_4}$ is conserved. Further, for the motion in a central potential with a given M, it is then straightforward to show that $(\mathbf{r}, M) = (\dot{\mathbf{r}}, M) = (\ddot{\mathbf{r}}, M) = 0$ at all times. Thus, given initial conditions $(\mathbf{r}_0, \dot{\mathbf{r}}_0)$, we have $M_{i_3 i_4} = \epsilon_{i_1 i_2 i_3 i_4} r_{0 i_1} \dot{r}_{0 i_2}$ and $\mathbf{r}, \dot{\mathbf{r}} \in Ker(M) = Span(\mathbf{r}_0, \dot{\mathbf{r}}_0)$, which is a plane if $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ are linearly independent or a line if they are dependent.

Another way to see this is to use the rotation invariance of the central potential and orient the coordinates such that $\mathbf{r}_0 = (r_0, 0, 0, ..)$, $\dot{\mathbf{r}}_0 = (v_{10}, v_{20}, 0, 0, ...)$. The force in the $3, 4, ..n$ coordinates are initially 0 and the initial position and velocity are also 0. Thus, since the equations of motions are second order, the only solutions are $r_j = \dot{r}_j = 0$, $j = 3, 4, ...n$. The motion is confined to the $1-2$ plane which is the same as $Span(\mathbf{r}_0, \dot{\mathbf{r}}_0)$.

41. (43) Show that if a field is axially symmetric and conservative, then its potential energy has the form $U = U(r, z)$, where r, ϕ, and z are cylindrical coordinates.

    Solution: Let the vector field be invariant under the group of rotations that leave the points on the z axis invariant. Such a rotation can be parameterized by the rotation angle $\Delta\phi$. Let the vector field be $\mathbf{F}(r, \phi, z)$ in cylindrical polar coordinates. Then from the invariance under the group of rotations under consideration, we have

$$G_{\Delta\phi}\mathbf{F}(r, \phi, z) = \mathbf{F}(G_{\Delta\phi}(r, \phi, z)) = \mathbf{F}(r, \phi + \Delta\phi, z), \tag{2.161}$$

    and thus the magnitudes are independent of ϕ. If the field is conservative, then there exists a scalar potential function $U(r, z, \phi)$ such that $\mathbf{F} = -\partial U/\partial\mathbf{r}$. Let $\mathbf{F} = F_r(r, z)\hat{r} + F_\phi(r, z)\hat{\phi} + F_z(r, z)\hat{z}$ which gives $\partial U/\partial\phi = F_\phi(r, z)$ and so on for r and z. The $\hat{\phi}$ equation in particular on solving gives $U = F_\phi(r, z)\phi + H(r, z)$ where H is an arbitrary function of r and z. Using this in the $\hat{r}$ equation, one gets $F_r(r, z) = \phi\partial F_\phi(r, z)/\partial r + \partial H_\phi(r, z)/\partial r$. But this violates the fact that F_r is independent of ϕ unless $F_\phi = 0$. The same conclusion can be drawn from the $\hat{z}$ equation. Thus, we conclude that $\partial U/\partial\phi = 0$ and $U = U(r, z)$. □

    Alternatively one can also show that $F_\phi = 0$ by requiring that the work around a closed path, say a circle of fixed radius, must be 0.

42. (46) Show that the center of mass is well defined, i.e., does not depend on the choice of the origin of reference for radius vectors.

    Solution: The center of mass is defined as

$$\mathbf{r}_{CM} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}. \tag{2.162}$$

    Suppose we change the origin to a new coordinate system $\mathbf{r}' = \mathbf{r} - \mathbf{a}$. If the center of mass is a well defined quantity, then we should have $\mathbf{r}'_{CM} = \mathbf{r}_{CM} - \mathbf{a}$. Evaluating the quantity on the LHS gives

$$\mathbf{r}'_{CM} = \frac{\sum m_i \mathbf{r}'_i}{\sum m_i} = \frac{\sum m_i(\mathbf{r}_i - \mathbf{a})}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} - \mathbf{a}\frac{\sum m_i}{\sum m_i} = \mathbf{r}_{CM} - \mathbf{a} \tag{2.163}$$

□

43. (50) Determine the major semi-axis of the ellipse which the center of the earth describes around the common center of mass of the earth and the moon. Where is this center of mass, inside the earth or outside? (The mass of the moon is 1/81 times the mass of the earth.)

    Solution: The center of mass is given by

$$\mathbf{r}_0 = \frac{M_e \mathbf{r}_e + m\mathbf{r}_m}{M_e + m}, \tag{2.164}$$

    where M_e is the mass of the Earth, m is the mass of the moon, $\mathbf{r}_e$ is the position of the earth, and $\mathbf{r}_m$ is the position of the moon. The relative position vector is given by $\mathbf{r} = \mathbf{r}_e - \mathbf{r}_m$. In this scenario, the potential is given by $U(r) = -GM_e m/r$, and thus we have

$$\ddot{\mathbf{r}} = -\frac{M_e + m}{M_e m}\frac{GM_e m}{r^3}\mathbf{r} = -\frac{G(M_e + m)}{r^3}\mathbf{r}. \tag{2.165}$$

This is the Kepler problem with $k = G(M_e + m)$. The vector r thus traces out an elliptical trajectory with semi-major axis a related to the time-period T by Kepler's third law

$$T^2 = \frac{4\pi^2}{k}a^3 = \frac{4\pi^2}{G(M + m_e)}a^3 \tag{2.166}$$

$$\implies a = \left(\frac{G(M_e + m)}{4\pi^2}T^2\right)^{1/3}. \tag{2.167}$$

Now, for the Earth-Moon system, $T \approx 28$days and given $m/M_e = 1/81$. We want to know the major semi-axes of the ellipse formed by the center of the Earth about the center of mass. We have from the definition of the center of mass,

$$\mathbf{r}_e - \mathbf{r}_0 = \frac{m\mathbf{r}}{M_e + m}. \tag{2.168}$$

Thus the motion of the Earth (also the moon) about the center of mass is just the motion of $\mathbf{r}$ scaled scaled by a factor. Thus the semi-major axes of this motion is given by

$$a_{e0} = \frac{ma}{M_e + m} = \frac{1}{1 + 81}M_e^{1/3}\left(\frac{G(1 + 1/81)}{4\pi^2}T^2\right)^{1/3} \tag{2.169}$$

$$= \frac{1}{82}(6 \times 10^{24})^{1/3}\left(\frac{82}{81}\frac{6.67 \times 10^{-11}}{4\pi^2}(28 \times 24 \times 3600)^2\right)^{1/3} \text{km} \tag{2.170}$$

$$\approx 4776\text{km}. \tag{2.171}$$

The radius of the Earth is 6400km and thus the center of mass is inside the Earth.

44. If the radius of a planet is α times the radius of the earth and its mass β times that of the earth, find the ratio of the acceleration of the force of gravity and the first and second cosmic velocities to the corresponding quantities for the earth.

    Solution: For a planet of radius R and mass M, the acceleration due to gravity is $g = GM/R^2$ and the first and second cosmic velocities are proportional to $\sqrt{GM/R}$. Thus the ratios for two different planets 1 and 2 are given by

$$\frac{g_1}{g_2} = \frac{M_1}{M_2}\frac{R_2^2}{R_1^2} = \beta\alpha^{-2} \tag{2.172}$$

$$\frac{v_1}{v_2} = \sqrt{\frac{M_1}{M_2}\frac{R_2}{R_1}} = \sqrt{\beta/\alpha} \tag{2.173}$$

45. A desert animal has to cover great distances between sources of water. How does the maximal time the animal can run depend on the size L of the animal?

    Solution: Given in text.

46. How does the running velocity of an animal on level ground and uphill depend on the size L of the animal?

    Solution: Given in text.

47. How does the height of an animal's jump depend on its size?

    Solution: Given in text.

Part II

Lagrangian Mechanics

Chapter 3

Variational principles

1. (58) Cite examples where there are many extremals connecting two given points, and others where there are none at all.

   Solution: Consider opposite points on a 2-sphere of radius r. There are infinitely many paths of shortest distance $d = \pi r$ connecting the points.

   Consider the motion of a particle in a potential with a barrier. There are no extremals connecting points on opposite sides of the barrier if the energy of the particle is lower than the barrier height.

2. (59) Find the differential equation for the family of all straight lines in the plane in polar coordinates.

   Solution: First, lets obtain the differential equation using the general form of a line in cartesian coordinates: $y = mx + c \implies r\sin\phi = mr\cos\phi + c$. Differentiating once with respect to t, we get

$$\dot{r}\sin\phi + r\dot{\phi}\cos\phi = m(\dot{r}\cos\phi - r\dot{\phi}\sin\phi). \tag{3.1}$$

   Solving for m and differentiating again gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}\sin\phi + r\dot{\phi}\cos\phi}{\dot{r}\cos\phi - r\dot{\phi}\sin\phi}\right) = 0 \tag{3.2}$$

   which can be simplified to give

$$r\dot{r}\ddot{\phi} + 2\dot{\phi}\dot{r}^2 + \dot{\phi}^3 r^2 - r\ddot{r}\dot{\phi} = 0. \tag{3.3}$$

   We can obtain this expression using the Euler-Lagrange(EL) equations as well. The Lagrangian for the length of a curve joining two points in flat space is given by $L = \sqrt{\dot{r}^2 + r^2\dot{\phi}^2}$.We apply the EL equations to L:

$$\frac{\partial L}{\partial r} = \frac{r\dot{\phi}^2}{\sqrt{\dot{r}^2 + r^2\dot{\phi}^2}} \qquad \frac{\partial L'}{\partial \dot{r}} = \frac{\dot{r}}{\sqrt{\dot{r}^2 + r^2\dot{\phi}^2}} \tag{3.4}$$

$$\frac{\ddot{r}}{\sqrt{\dot{r}^2 + r^2\dot{\phi}^2}} - \frac{\dot{r}(\dot{r}\ddot{r} + r\dot{r}\dot{\phi}^2 + r^2\dot{\phi}\ddot{\phi})}{(\dot{r}^2 + r^2\dot{\phi}^2)^{3/2}} = \frac{r\dot{\phi}^2}{\sqrt{\dot{r}^2 + r^2\dot{\phi}^2}} \tag{3.5}$$

$$\implies \dot{\phi}(\dot{\phi}\ddot{r}r^2 - 2\dot{\phi}r\dot{r}^2 - \dot{r}r^2\ddot{\phi} - r^3\dot{\phi}^3) = 0 \tag{3.6}$$

$$\implies r\dot{r}\ddot{\phi} + 2\dot{\phi}\dot{r}^2 + \dot{\phi}^3 r^2 - r\ddot{r}\dot{\phi} = 0. \tag{3.7}$$

   Thus we obtain the same equation. It is straightforward to check that the ϕ EL equation gives the same result.

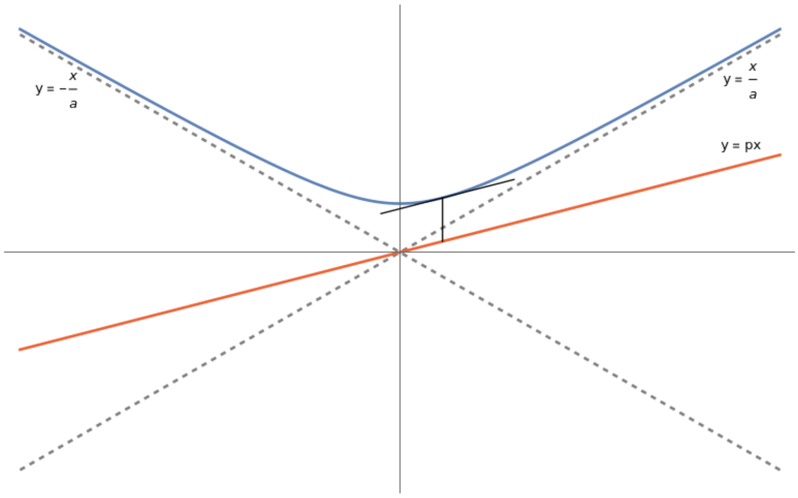


Figure 3.1: Hyperbola with the maximum distance for some p shown. Asymptotes are shown as dashed lines

3. (60) Show that this extremum is a minimum.

   Solution: We are asked to show that the extremum from solving the EL equation for the Lagrangian of a free particle in 3D is a minimum. Let q_{i0}, $i = 1, 2, 3$ denote the straight line path between two points t_0 and t_1 which is the solution from the EL equations. Consider a nearby path $q_i = q_{i0} + h_i$ where $h_i(t_0) = h_i t_1 = 0$ between the same end points. The action for this path is given by

$$S' = \int_{t_0}^{t_1} (\dot{q_1}^2 + \dot{q_2}^2 + \dot{q_3}^2)\mathrm{d}t \tag{3.8}$$

$$= \int_{t_0}^{t_1} (\dot{q}_{10}^2 + \dot{q}_{20}^2 + \dot{q}_{30}^2)\mathrm{d}t + 2\int_{t_0}^{t_1} (\dot{q}_{10}\dot{h}_1 + \dot{q}_{20}\dot{h}_2 + \dot{q}_{30}\dot{h}_3) + \int_{t_0}^{t_1} (\dot{h}_1^2 + \dot{h}_2^2 + \dot{h}_3^2)\mathrm{d}t. \tag{3.9}$$

   However, the EL equations for $L = m\sum \dot{q}_i^2/2$ show that the extremal path satisfies $\dot{q}_{i0} = \text{const}$. The second term in equation (3.9) vanishes since h_i vanish at the end points. We thus get

$$S' = S + \int_{t_0}^{t_1} (\dot{h}_1^2 + \dot{h}_2^2 + \dot{h}_3^2)\mathrm{d}t > S \tag{3.10}$$

   for any deviation h_i. Thus, the extremal is a minimum. □

4. (62) Show that the domain of g can be a point, a closed interval, or a ray if f is defined on the whole x axis. Prove that if f is defined on a closed interval, then g is defined on the whole p axis.

   Solution: Note here that the term "distance" used in this problem refers to the value of $px - f(x)$ and not the absolute magnitude of this quantity.

   - Consider the straight line $f(x) = mx + c$ that is defined on the whole x axis. For $p = m$, there is no unique point farthest from the line, but all the points on $y = px$ are a distance c from $f(x)$. Thus $g(m) = c$. For $p > m$ and $p < m$, there is no greatest distance point from $f(x)$ and thus $g(p)$ is not defined. Hence, $\mathrm{Dom}(g) = \{m\}$.

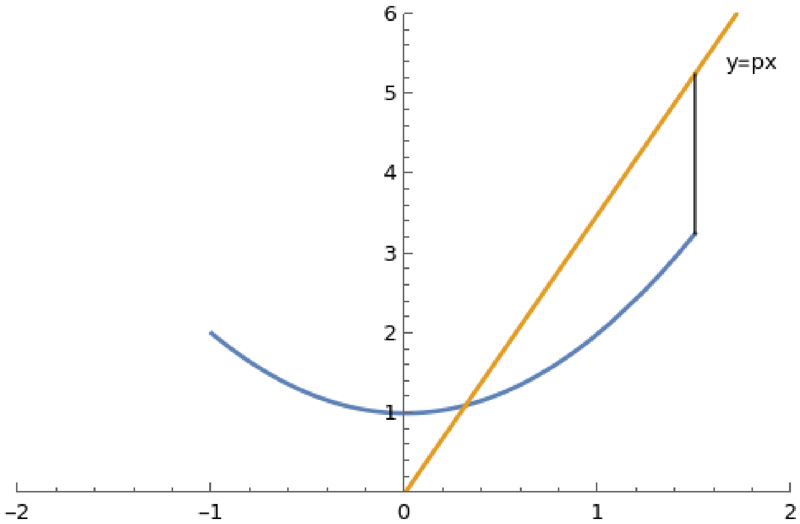


Figure 3.2: A curve defined on a closed interval.

- Consider a curve $f(x)$ defined on the entire x-axis such that the range of $f'(x)$ is bounded. $f(x) = \sqrt{x^2/a^2+1}$ which represents one such curve, with $\text{Range}(f') = (-1/a, 1/a)$ as shown in figure 3.1. As we can see, the the function $px - f(x)$ has a unique maximum value only for $p \in [-1/a, 1/a]$ where one can set $g(1/a) = g(-1/a) = 0$. Hence $\text{Dom}(g) = [-1/a, 1/a]$.
- Consider a function defined on the entire x axis such that $f'(x) > 0$ and unbounded. $f(x) = e^x$ is such a function. It is easy to see that $\forall p > 0$, $\exists x$ such that $f'(x) = p$. Thus there is a unique maximum value for $px - f(x)$. Hence $\text{Dom}(g) = [0, \infty)$.

Let $f(x)$ be a convex function defined on a closed interval. Let R denote $\text{Range}(f')$. For $p \in R$, one can define $g(p)$ in the usual way. For other values of p, the maximum value of $px - f(x)$ occurs at one of the end points of the domain of f as shown in figure 3.2. If the closed interval is $[a, b]$, for $p > f'(b)$, $g(p) = pb - f(b)$ and for $p < f'(a)$, $g(p) = pa - f(a)$. Thus, $g(p)$ can be defined for any value of $p \in \mathbb{R}$.

5. (64) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Let $\mathbb{R}^{n*}$ denote the dual vector space. Show that the formulas above completely define the mapping $g : \mathbb{R}^{n*} \to \mathbb{R}$ (under the condition that the linear form $\mathrm{d}f|_{\mathbf{x}}$ ranges over all of $\mathbb{R}^{n*}$ when $\mathbf{x}$ ranges over $\mathbb{R}^n$).

   We want to show that the quantity $F(\mathbf{p}, \mathbf{x}(\mathbf{p}))$ and thus map g is well defined. Suppose $\mathbf{x}(\mathbf{p})$ is not a unique quantity and for a given $\mathbf{p}$ there exists a subset $A \subset \mathbb{R}^n$ such that $\mathbf{p} = \mathrm{d}f_{\mathbf{x}}\ \forall\ \mathbf{x} \in A$. We want to show then that $F(\mathbf{p}, \mathbf{x}) = (\mathbf{p}, \mathbf{x}) - f(\mathbf{x})$ has the same value for all $\mathbf{x} \in A$, and thus $g(\mathbf{p})$ has a unique value. We first note that for a given $\mathbf{p}$, the function $F(\mathbf{p}, \mathbf{x})$ satisfies $\partial^2 F/\partial x_i \partial x_j = -\partial^2 f/\partial x_i \partial x_j$ and is thus negative semi-definite since $\partial^2 f/\partial x_i \partial x_j$ is positive semi definite. Let $\mathbf{x}_1$, $\mathbf{x}_2 \in A$. From mean value theorem,

$$F(\mathbf{x}_2) = F(\mathbf{x}_1) + \nabla_i F(\mathbf{x_1})(\mathbf{x}_2 - \mathbf{x}_1)_i + \frac{1}{2}(\mathbf{x}_2 - \mathbf{x}_1)_i . \nabla_i \nabla_j F(\mathbf{x_1} t + (1-t)\mathbf{x}_2) . (\mathbf{x}_2 - \mathbf{x}_1)_j . \tag{3.11}$$

   where we have dropped the $\mathbf{p}$ dependence on F for the time being and $0 \le t \le 1$.

Now, since F is negative semi-definite, we have

$$F(\mathbf{x}_2) \geq F(\mathbf{x}_1) + \nabla_i F(\mathbf{x_1})(\mathbf{x}_2 - \mathbf{x}_1)_i = F(\mathbf{x}_1) \tag{3.12}$$

where we have used the fact that $\mathbf{x}_1$ is a local maxima of F. One can use similar reasoning to also show that $F(\mathbf{x}_1) \geq F(\mathbf{x}_2)$ and thus conclude that $F(\mathbf{x}_1) = F(\mathbf{x}_2)$ and hence the map $g : \mathbb{R}^{n*} \to \mathbb{R}$ defined by the Legendre transform of a function with domain $\mathbb{R}^n$ is completely defined. □

6. (65) Let f be the quadratic form $f(\mathbf{x}) = \sum f_{ij} x_i x_j$. Show that its Legendre transform is again a quadratic form $g(\mathbf{p}) = \sum g_{ij} p_i p_j$, and that the values of both forms at the corresponding points coincide

$$f(\mathbf{x}(\mathbf{p})) = g(\mathbf{p}) \qquad g(\mathbf{p}(\mathbf{x})) = f(\mathbf{x})$$

Solution: Since we are working with quadratic forms, note that we are free assume f_{ij} and g_{ij} are symmetric matrices. We drop summation signs, and repeated indices are to be understood as summed over. To find $\mathbf{x}(\mathbf{p})$, we solve

$$p_i = \frac{\partial f}{\partial x_i} = 2 f_{ij} x_j \tag{3.13}$$

$$\implies x_j(\mathbf{p}) = \frac{1}{2}(f^{-1})_{ji} p_i \tag{3.14}$$

Thus, we get $g(\mathbf{p})$ as

$$g(\mathbf{p}) = \frac{1}{2} p_j (f^{-1})_{ji} p_i - f_{ij} \frac{1}{2}(f^{-1})_{ik} p_k \frac{1}{2}(f^{-1})_{jl} p_l \tag{3.15}$$

$$= \frac{1}{2} p_j (f^{-1})_{ji} p_i - \frac{1}{4}(f^{-1})_{ik} p_k \delta_{il} p_l \tag{3.16}$$

$$= \frac{1}{4} p_j (f^{-1})_{ji} p_i. \tag{3.17}$$

Hence the Legendre transform is a quadratic form $g(\mathbf{p}) = g_{ij} p_i p_j$, with $g_{ij} = (f^{-1})_{ij}/4$. We also compute

$$f(\mathbf{x}(\mathbf{p})) = f_{ij} \frac{1}{2}(f^{-1})_{ik} p_k \frac{1}{2}(f^{-1})_{jl} p_l = \frac{1}{4} p_j (f^{-1})_{ji} p_i = g(\mathbf{p}). \tag{3.18}$$

$\mathbf{p}(\mathbf{x})$ is given by $p_i(\mathbf{x}) = (g^{-1})_{ij} x_j / 2 = 2 f_{ij} x_j$ from the relations we have derived. Thus,

$$g(\mathbf{p}(\mathbf{x})) = \frac{1}{4}(f^{-1})_{ij} 2 f_{ik} x_k 2 f_{jl} x_l = f_{ij} x_i x_j = f(\mathbf{x}). \tag{3.19}$$

□

7. (68) Show that $\{g^t\}$ is a group.

Solution: The group property of g^t hinges crucially on the fact that the solution of Hamilton's equations, which are first order time ODEs depend only on the initial conditions, and are time reversible if $H = H(\mathbf{p}, \mathbf{q})$. Now we verify the group properties of g^t.

- The identity element is g^0, $g^0(\mathbf{p}(0), \mathbf{q}(0)) = (\mathbf{p}(0), \mathbf{q}(0))$
- The inverse of g^t is given by g^{-t}, which is the phase flow with time reversed.

- Closure property: $g^{t_1} \circ g^{t_2} = g^{t_1+t_2} \in \{g^t\}$. This follows from the fact that the end state of the flow depends only on the initial values of $\mathbf{p}$ and $\mathbf{q}$ from the properties of first order differential equations. Thus the phase point obtained by evolving the system from some initial condition to time t_1 and then evolving the system to a further time t_2 starting from t_1 is the same as evolving the system for a time $t_1 + t_2$.
- Associative property: $(g^{t_1} \circ g^{t_2}) \circ g^{t_3} = g^{t_1+t_2} \circ g^{t_3} = g^{t_1+t_2+t_3} = g^{t_1} \circ g^{t_2+t_3} = g^{t_1} \circ (g^{t_2} \circ g^{t_3})$

Note these assertions are made under the assumption that the motion is bound and well defined for all values of t. In case the particle goes off to infinity in finite time, it is not possible to define a phase flow.

8. (70) Prove Liouville's fromula $W = W_0 e^{\int \mathrm{tr} A \mathrm{d}t'}$ for the Wronskian determinant of the linear system $\dot{\mathbf{x}} = A(t)\mathbf{x}$.

Solution: For n dimension, by appropriate linear transformations, the system $\dot{\mathbf{x}} = A(t)\mathbf{x}$ can be decoupled to n homogeneous n-th order ordinary differential equations for each x_i. Thus there are n solution sets for $\mathbf{x}$. Let Ψ be the matrix of linearly independent solutions of $\mathbf{x}$, i.e, each column of Ψ is a solution $\mathbf{x}$. This matrix satisfies $\dot{\Psi} = A\Psi$.

Lemma. $\mathrm{d}(\det\Psi)/\mathrm{d}t = \mathrm{tr}(\dot{\Psi}\Psi^{-1})\det\Psi$

Proof.

$$\frac{\mathrm{d}(\det\Psi)}{\mathrm{d}t} = \lim_{h\to 0} \frac{\det\Psi(t+h) - \det\Psi(t)}{h} \tag{3.20}$$

$$\lim_{h\to 0} \frac{\det\Psi(t+h)\det\Psi^{-1}(t) - 1}{h}\det\Psi(t) \tag{3.21}$$

$$= \lim_{h\to 0} \frac{\det(\Psi(t+h)\Psi^{-1}(t)) - 1}{h}\det\Psi(t) \tag{3.22}$$

$$= \lim_{h\to 0} \frac{\det(E + h\dot{\Psi}(t)\Psi^{-1}(t) + \mathcal{O}(h^2)) - 1}{h}\det\Psi(t) \tag{3.23}$$

$$\tag{3.24}$$

Now, one can show that $\det(E + hM) = 1 + h\mathrm{tr}(M) + \mathcal{O}(h^2)$, by writing out the determinant explicitly. Thus we get

$$\frac{\mathrm{d}(\det\Psi)}{\mathrm{d}t} = \lim_{h\to 0} \frac{1 + h\mathrm{tr}(\dot{\Psi}(t)\Psi^{-1}(t)) - 1}{h}\det\Psi(t) \tag{3.25}$$

$$= \mathrm{tr}(\dot{\Psi}(t)\Psi^{-1}(t))\det\Psi(t) \tag{3.26}$$

□

Using the Lemma in our case, we get

$$\frac{\mathrm{d}(\det\Psi)}{\mathrm{d}t} = \mathrm{tr}(A\Psi\Psi^{-1})\det\Psi \tag{3.27}$$

$$= \mathrm{tr}(A)\det\Psi. \tag{3.28}$$

Now the Wronskian is given by $W \equiv \det\Psi$ and thus we have

$$\dot{W} = \mathrm{tr}(A)W \tag{3.29}$$

$$\implies W = W_0 \exp\left(\int \mathrm{tr}(A)\mathrm{d}t'\right) \tag{3.30}$$

□

9. (70) Show that in a Hamiltonian system it is impossible to have asymptotically stable equilibrium positions and asymptotically stable limit cycles in the phase space.

Solution: We state here a few definitons.

Definition. *A point x_e of a system is said to be* ***Lyapunov stable*** *if* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that if* $|x(0) - x_e| < \delta$, $|x(t) - x_e| < \epsilon \ \forall t > 0$.

In other words, the point x_e is Lyapunov stable if there is some starting point close enough to the point such that the point stays arbitrarily close to x_e at all later times.

Definition. *A point x_e of a system is said to be* ***asymptotically stable*** *if it is Lyapunov stable, and* $\exists \delta > 0$ *such that if* $|x(0) - x_e| < \delta$, $\lim_{t\to\infty} x(t) = x_e$.

Hamiltonian flows in phase space can have Lyapunov stable equilibrium points at points where $U'(q_e) = 0$ and $U''(q_e) > 0$ with $p = 0$. For example, if one starts the motion of a 1-D harmonic oscillator with $k = 1$ from $(0, \epsilon)$, the subsequent motion satisfies $|(p(t), x(t)) - (0, 0)| \leq \epsilon$, where $\epsilon/\sqrt{2}$ is the total energy, and thus the origin is a Lyapunov stable equilibrium point. However, we see that it is not asymptotically stable.

To show that this holds generally, let $x_e = (0, q_e)$ be an asymptotically stable equilibrium point in phase space. Then $\exists \delta > 0$ such that if $|x(0) - x_e| < \delta$, then $\lim_{t\to\infty} x(t) = x_e$. Consider the set of motions with $x(0)$ lying within the ball in phase space with radius δ centered at $(0, q_e)$. The volume occupied by the phase points of these motions is $v(0) = 4\pi\delta^3/3$. If the point is asymptotically stable, $\lim_{t\to\infty} x(t) = x_e$ for all points under consideration, and thus $\lim_{t\to\infty} v = 0$. This contradicts Liouville's theorem.

A similar argument holds for limit cycles. If there is an asymptotically stable limit cycle, consider the set of trajectories starting out sufficiently close to it. These trajectories take up a volume $v(0) \neq 0$ initially. However as $t \to \infty$, all these trajectories coincide with the limit cycle and thus $v \to 0$. This is again a contradiction of Liouville's theorem. □

Chapter 4

Lagrangian mechanics on manifolds

Chapter 5

Oscillations

Chapter 6

Rigid Bodies

Part III

Hamiltonian Mechanics

Chapter 7

Differential Forms

Chapter 8

Symplectic manifolds

Chapter 9

Canonical formalism

Chapter 10

Introduction to perturbation theory

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