1)

$$T\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right] = T\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+y_1 + x_2+y_2 + x_3+y_3 \\ 2x_1+2y_1 + x_2+y_2 + 2x_3+2y_3 \\ x_1+y_1 + 2x_2+2y_2 + x_3+y_3 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1+x_2+x_3) + (y_1+y_2+y_3) \\ (2x_1+x_2+2x_3) + (2y_1+y_2+2y_3) \\ (x_1+2x_2+x_3) + (y_1+2y_2+y_3) \end{bmatrix}$$

$$= T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + T\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$T\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} = \begin{bmatrix} cx_1 + cx_2 + cx_3 \\ 2cx_1 + cx_2 + 2cx_3 \\ cx_1 + 2cx_2 + cx_3 \end{bmatrix}$$

$$= c\begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_1 + x_2 + 2x_3 \\ x_1 + 2x_2 + x_3 \end{bmatrix}$$

$$= cT\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let, $x_3 = K$

$\therefore x_2 = 0$

and $x_1 = -x_2 - x_3 = -K$

$\therefore \text{Ker} = K \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So, dimension $= 1$

2)

$$\left[\begin{array}{ccc:c} 1 & 1 & a & 1 \\ 1 & a & 1 & 4 \\ a & 1 & 1 & b \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - aR_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 3 \\ 0 & 1-a & 1-a^2 & b-a \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc:c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 3 \\ 0 & 0 & 2-a-a^2 & 3+b-a \end{array}\right]$$

(i)

$$2 - a - a^2 \neq 0$$

$$\therefore a^2 + a - 2 \neq 0$$

$$\therefore a \neq \frac{-1 \pm \sqrt{1^2 - 4.1.(-2)}}{2.1} = -2, 1$$

$$\therefore a \neq -2, 1$$

(ii)

$$a = -2, 1$$

$$3 + b - a = 0$$

$$\therefore b = a - 3 = -5, -2$$

(iii) b is in the solution part of the Augumented matrix.

3) (i)

$$f(s) = \frac{s}{(s+1)(s^2-4s+13)}$$

let,

$$\frac{s}{(s+1)(s^2-4s+13)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-4s+13}$$

or, $s = A(s^2-4s+13) + (Bs+C)(s+1)$

or, $s = As^2 - 4As + 13A + Bs^2 + Bs + Cs + C$

or, $s = (A+B)s^2 + (-4A+B+C)s + (13A+C)$

$A+B=0 \quad \therefore B=-A$

$13A+C=0 \quad \therefore C=-13A$

$-4A+B+C=1$

or, $-4A - A - 13A = 1$

or, $-18A = 1$

$\therefore A = -\frac{1}{18} \qquad \therefore B = -\left(-\frac{1}{18}\right) = \frac{1}{18}$

$\therefore C = \frac{13}{18}$

$$f(s) = -\frac{1}{18}\left(\frac{1}{s+1}\right) + \frac{\frac{1}{18}s + \frac{13}{18}}{s^2-4s+13}$$

$$= -\frac{1}{18}\left(\frac{1}{s+1}\right) + \frac{1}{18}\left(\frac{s}{s^2-4s+13}\right) + \frac{13}{18}\left(\frac{1}{s^2-4s+13}\right)$$

$$= -\frac{1}{18}\left(\frac{1}{s+1}\right) + \frac{1}{18}\left(\frac{s-2+2}{(s-2)^2+3^2}\right) + \frac{13}{18}\left(\frac{1}{(s-2)^2+3^2}\right)$$

$$f(s) = -\frac{1}{18}\left(\frac{1}{s+1}\right) + \frac{1}{18}\left(\frac{s-2}{(s-2)^2+3^2}\right) + \frac{15}{18}\left(\frac{1}{(s-2)^2+3^2}\right)$$

$\therefore L^{-1}(f(s)) =$

$$-\frac{1}{18}L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{18}L^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right) + \frac{15}{18}L^{-1}\left(\frac{1}{(s-2)^2+3^2}\right)$$

$$= -\frac{1}{18}e^{-t} + \frac{1}{18}e^{2t}\cos 3t + \frac{15}{18} \times \frac{1}{3}e^{2t}\sin 3t$$

$$= -\frac{e^{-t}}{18} + \frac{e^{2t}\cos 3t}{18} + \frac{5e^{2t}\sin 3t}{18}$$

$$= \frac{e^{2t}(5\sin 3t + \cos 3t)}{18} - \frac{e^{-t}}{18} \quad \text{(Ans)}$$

(ii) $$f(s) = \frac{(s+1)^2}{(s+2)^4} = \frac{(s+2)^2}{(s+2)^4} - \frac{2s}{(s+2)^4} - \frac{3}{(s+2)^4}$$

$$= \frac{1}{(s+2)^2} - \frac{2(s+2)}{(s+2)^4} + \frac{4}{(s+2)^4} - \frac{3}{(s+2)^4}$$

$$= \frac{1}{(s+2)^2} - \frac{2}{(s+2)^3} + \frac{1}{(s+2)^4}$$

$$L^{-1}(f(s)) =$$

$$L^{-1}\left(\frac{1}{(s+2)^2}\right) - L^{-1}\left(\frac{2}{(s+2)^3}\right) + L^{-1}\left(\frac{1}{(s+2)^4}\right)$$

$$= \frac{e^{-2t}t}{1} - \frac{2e^{-2t}t^2}{2} + \frac{e^{-2t}t^3}{6}$$

$$= \left(\frac{t^2}{6} - t + 1\right)te^{-2t}$$

4) (a) $y'' + 4y' + 4y = \sin x \qquad y(0)=2$

$y'(0)=0$

or, $L(y''+4y'+4y) = L(\sin x)$

or, $L(y'') + 4L(y') + 4L(y) = \frac{1}{s^2+1}$

or, $s^2L(y) - sy(0) - y'(0) +$

$4(sL(y) - y(0)) + 4L(y) = \frac{1}{s^2+1}$

or, $s^2L(y) - 2s - 0 + 4sL(y) - 8 + 4(L(y)) = \frac{1}{s^2+1}$

or, $L(y)(s^2+4s+4) - (2s+8) = \frac{1}{s^2+1}$

or, $L(y) = \dfrac{\frac{1}{s^2+1} + 2s + 8}{s^2+4s+4}$

or, $L(y) = \dfrac{1+2(s+4)(s^2+1)}{(s^2+1)(s+2)^2}$

or, $L^{-1}(L(y)) = L^{-1}\left(\dfrac{1+2(s+4)(s^2+1)}{(s^2+1)(s+2)^2}\right)$

or, $y = L^{-1}\left(\dfrac{1}{(s^2+1)(s+2)^2}\right) + 2L^{-1}\left(\dfrac{s+4}{(s+2)^2}\right)$

Let,

$$\frac{1}{(s^2+1)(s+2)^2} = \frac{As+B}{s^2+1} + \frac{C}{s+2} + \frac{D}{(s+2)^2}$$

or, $1 = (As+B)(s+2)^2 + C(s^2+1)(s+2) + D(s^2+1)$

or, $1 = (A+C)s^3 + (4A+B+2C+D)s^2 + (4A+4B+C)\,s + (4B+2C+D)$

$A+C=0 \qquad \therefore C=-A$

$4A+4B+C=0$

or, $3A+4B=0$

$\therefore B=-\frac{3}{4}A$

$4B+2C+D=1$

or, $-3A-2A+D=1$

$\therefore 1+5A=D$

$4A+B+2C+D=0$

or, $4A-\frac{3}{4}A+2(-A)+1+5A=0$

$\therefore A=-\frac{4}{25}$

And: $B = -\frac{3}{4} \times -\frac{4}{25} = \frac{3}{25}$

$$C = -\left(-\frac{4}{25}\right) = \frac{4}{25}$$

$$D = 1 + 5 \times \left(-\frac{4}{25}\right) = \frac{1}{5}$$

$$y = L^{-1}\left[\frac{-\frac{4}{25}S + \frac{3}{25}}{(S^2+1)} + \frac{\frac{4}{25}}{(S+2)} + \frac{\frac{1}{5}}{(S+2)^2}\right]$$

$$+ 2L^{-1}\left[\frac{(S+2)}{(S+2)^2} + \frac{2}{(S+2)^2}\right]$$

$$= -\frac{4}{25}L^{-1}\left[\frac{S}{S^2+1^2}\right] + \frac{3}{25}L^{-1}\left[\frac{1}{S^2+1^2}\right]$$

$$+ \frac{4}{25}L^{-1}\left[\frac{1}{S+2}\right] + \frac{1}{5}L^{-1}\left[\frac{1}{(S+2)^2}\right]$$

$$+ 2L^{-1}\left[\frac{1}{S+2}\right] + 4L^{-1}\left[\frac{1}{(S+2)^2}\right]$$

$$= -\frac{4}{25}\cos t + \frac{3}{25}\sin t + \frac{4}{25}e^{-2t} + 2e^{-2t} + \frac{1}{5}e^{-2t}t + 4e^{-2t}t$$

$$= \frac{3}{25}\sin t - \frac{4}{25}\cos t + \frac{54}{25}e^{-2t} + \frac{21}{5}te^{-2t}$$

(b) $y'' - 2y' + y = xe^x$ $\quad y(0) = 0$

$y'(0) = 0$

or, $L(y'' - 2y' + y) = L(xe^x)$

or, $L(y'') - 2L(y') + L(y) = \frac{1}{(s-1)^2}$

or, $s^2L(y) - s\,y(0) - y'(0) - 2[sL(y) - y(0)] + L(y) = \frac{1}{(s-1)^2}$

or, $s^2L(y) - s.0 - 0 - 2sL(y) + 2.0 + L(y) = \frac{1}{(s-1)^2}$

or, $L(y)(s^2 - 2s + 1) = \frac{1}{(s-1)^2}$

or, $L(y) = \frac{1}{(s-1)^4}$

or, $L^{-1}(L(y)) = L^{-1}\left(\frac{1}{(s-1)^4}\right)$

$$\therefore\ y = L^{-1}\left(\frac{1}{(s-1)^4}\right)$$

$$= e^t L^{-1}\left(\frac{1}{s^4}\right) = e^t\,\frac{t^3}{3!}$$

$$= \frac{t^3 e^t}{6}.$$

5) (a)

$$L(y(t)) = L\left[t + \frac{1}{6}(y(t) * t^3)\right]$$

or, $L(y(t)) = L(t) + \frac{1}{6} L(y(t) * t^3)$

or, $L(y(t)) = \frac{1}{s^2} + \frac{1}{6} L(y(t)) L(t^3)$

or, $L(y(t)) = \frac{1}{s^2} + \frac{1}{6}\frac{6}{s^4} L(y(t))$

or, $L(y(t)) = \frac{s^2}{s^4 - 1}$

or, $L^{-1}(L(y(t))) = L^{-1}\left(\frac{s^2}{s^4-1}\right)$

or, $y(t) = L^{-1}\frac{1}{2}\left(\frac{s^2+1+s^2-1}{(s^2+1)(s^2-1)}\right)$

$$= \frac{1}{2}\left(L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2-1}\right)\right)$$

$$= \frac{1}{2}\left(\sin t + \frac{1}{2}L^{-1}\left(\frac{1+1}{(s+1)(s-1)}\right)\right)$$

$$= \frac{1}{2}\sin t + \frac{1}{2}\left(\frac{1}{2}\left(L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{1}{s-1}\right)\right)\right)$$

$$= \frac{1}{2}\sin t + \frac{1}{4}\left(e^{t} + e^{-t}\right)$$