Mathematics Assignment (MA 202) (Sem-2)

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(1.) Given $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Let, $v, w \in \mathbb{R}^2$ be any element in $\mathbb{R}^2$ and k be any scalar.

Let, $v = (a, b, c)$ and $u = (a', b', c')$

We need to show that $F(v+w) = f(v) + f(w)$ and $f(kv) = k f(v)$.

$\Rightarrow F(v+w) = f(a+a', b+b', c+c')$

$= (a+a'+b+b'+c+c', 2a+2a'+b+b'+2c+2c', a+a'+2b+2b'+c+c')$

$= (a+b+c, 2a+b+2c, a+2b, c) + (a'+b'+c', 2a'+b'+2c', a'+2b'+c')$

$= f(v) + f(w)$

Now, $f(k.v) = f(ka, kb, kc)$

$= (ka + kb + kc, 2ka + kb, 2kc, ka + 2kb + kc)$

$= k(a+b+c, 2a + b + 2c, a + 2b + c)$

$= k f(v)$

So, yes, it is linear.

Now, let, $f(v) = 0$ where, $v = (x_1, x_2, x_3)$ such that $v \in \mathbb{R}^3$.

$\Rightarrow f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3)$

$= (0, 0, 0)$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0$$
$$2x_1 + x_2 + 2x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$

or,

$$x_2 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$

$$\Rightarrow \quad x_2 = 0$$
$$x_1 + x_3 = 0$$

Here, x_3 is the only free variable.

So, $\dim(\ker f) = 1$

Now, let, $x_3 = c$

then, $x_1 = -c$ & $x_2 = 0$

$$\Rightarrow \quad \ker f = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix} = c\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \dim(\ker f) = 1.$$

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② Given,

$$x + y + az = 1$$
$$x + ay + z = 4$$
$$ax + y + z = b$$

Reducing the equation to its augmented matrix, M to echelon form and then to row reduced equivalent form, as follows:-

$$M = \left[\begin{array}{ccc:c} 1 & 1 & a & 1 \\ 1 & a & 1 & 4 \\ a & 1 & 1 & b \end{array}\right] \xrightarrow[R_3 - R_2]{R_1 - R_2} \left[\begin{array}{ccc:c} 0 & 1-a & a-1 & -3 \\ 1 & a & 1 & 4 \\ a-1 & 1-a & 0 & b-4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - aR_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 3 \\ 0 & 1-a & 1-a^2 & b-a \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc:c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 3 \\ 0 & 0 & 2-a-a^2 & b-a+3 \end{array}\right]$$

a) For unique solⁿ: $\text{Rank}(A) = \text{Rank}(M) = 3$

$$\Rightarrow \quad 2 - a - a^2 \neq 0$$
$$\Rightarrow \quad a^2 + a - 2 \neq 0$$
$$\Rightarrow \quad (a+2)(a-1) \neq 0 \longrightarrow a \neq -2, 1$$

So, the system has a unique sol^n $\forall$ values of a except $\{-2, 1\}$.

$\Rightarrow$ $a \in \mathbb{R} - \{-2, 1\}$.

b) <u>For more than one solution:</u>

$\Rightarrow$ $2 - a - a^2 = 0$ & $b - a + 3 = 0$

$\Rightarrow$ $a = \{-2, 1\}$

$\Rightarrow$ $b = a - 3 = \{-5, -2\}$.

So, for $(-2, -5)$ and $(1, -2)$, the system of equations has more than one solution.

c) The value of b does not affects whether the system has a unique sol^n, is because b is not a part of coefficient matrix.
b only appears in the right column part of the augmented matrix which never affects the solution being unique or not.

(3.) i) $f(s) = \dfrac{s}{(s+1)(s^2 - 4s + 13)}$

$\Rightarrow$ The ~~Laplace~~ inverse Laplace transform of function, $f(s)$ is given by :-

$$\Rightarrow L^{-1}\left[\frac{s}{(s+1)(s^2-4s+13)}\right] = L^{-1}\left[\frac{A}{(s+1)} + \frac{Bs+C}{(s^2-4s+13)}\right]$$

where, A, B & C are constants.

$$\Rightarrow \frac{s}{(s+1)(s^2-4s+13)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2-4s+13)}$$

$$\Rightarrow \frac{s}{(s+1)(s^2-4s+13)} = \frac{(As^2+Bs^2) + (-4A+B+C)s + (13A+C)}{(s+1)(s^2-4s+13)}$$

On comparing the coffecients, we get,

$$\left.\begin{aligned} A + B &= 0 \\ -4A + B + C &= 1 \\ 13A + C &= 0 \end{aligned}\right\} \quad \begin{aligned} A &= -B \\ A &= -\frac{C}{13} \end{aligned} \quad \Rightarrow \quad -4A - A - 13A = 1 \quad \left(A = -\frac{1}{18},\ B = +\frac{1}{18}\ \&\ C = +\frac{13}{18}\right)$$

$$\Rightarrow f(s) = \frac{-1}{18}\left(\frac{1}{s+1}\right) + \frac{\frac{1}{18}s + \frac{13}{18}}{(s^2-4s+13)}$$

$$= \frac{-1}{18}\left(\frac{1}{s+1}\right) + \frac{1}{18}\left(\frac{s}{s^2-4s+13}\right) + \frac{13}{18}\left(\frac{1}{s^2-4s+13}\right)$$

$$= \frac{-1}{18}\left(\frac{1}{s+1}\right) + \frac{1}{18}\left(\frac{s-2}{(s-2)^2+3^2}\right) + \frac{15}{18}\left(\frac{1}{(s-2)^2+3^2}\right)$$

Now,

$$L^{-1}[f(s)] = \frac{-1}{18} L^{-1}\left[\frac{1}{(s+1)}\right] + \frac{1}{18} L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + \frac{15}{18} L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]$$

$$= \frac{-e^{-t}}{18} + \frac{e^{2t}\cos 3t}{18} + \frac{15}{18}\times\frac{1}{3} e^{2t}\sin 3t$$

$$= \frac{-e^{-t}}{18} + \frac{e^{2t}\cos 3t}{18} + \frac{5}{18} e^{2t}\sin 3t$$

$$= \frac{1}{18}\left[e^{2t}(\cos 3t + 5\sin 3t) - e^{-t}\right].$$

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ii) $f(s) = \dfrac{(s+1)^2}{(s+2)^4}$

$$\Rightarrow f(s) = \frac{(s+2)^2 - 2s - 3}{(s+2)^4} = \frac{(s+2)^2 - 2(s+2) + 1}{(s+2)^4}$$

$$= \frac{1}{(s+2)^2} - \frac{2}{(s+2)^3} + \frac{1}{(s+2)^4}$$

$$\Rightarrow L^{-1}[f(s)] = L^{-1}\left[\frac{1}{(s+2)^2}\right] - 2L^{-1}\left[\frac{1}{(s+2)^3}\right] + L^{-1}\left[\frac{1}{(s+2)^4}\right]$$

$$= \frac{e^{-2t}\cdot t}{1!} - \frac{2e^{-2t}t^2}{2!} + \frac{e^{-2t}t^3}{3!}$$

$$= e^{-2t}t - e^{-2t}t^2 + \frac{e^{-2t}t^3}{6}$$

$$= \left(\frac{t^2}{6} - t + 1\right)\cdot e^{-2t}\cdot t$$

4. a) $y'' + 4y' + 4y = \sin x$, Given : $y(0) = 2$ & $y'(0) = 0$

Now, Laplace transform of the above differential equation gives :-

$\Rightarrow L[y'' + 4y' + 4y] = L[\sin x]$

$\Rightarrow L[y''] + 4L[y'] + 4L[y] = L[\sin x]$

$\Rightarrow \{s^2 L[y] - s\,y(0) - y'(0)\} + 4\{s L(y) - y(0)\} + 4L[y] = \frac{1}{s^2+1}$

$\Rightarrow s^2 L(y) - 2s + 4s L(y) - 8 + 4L(y) = \frac{1}{s^2+1}$

$\Rightarrow L(y)\{s^2+4s+4\} + (-2s-8) = \frac{1}{s^2+1}$

$\Rightarrow L(y)\{s^2+4s+4\} = \frac{1}{(s^2+1)} + 2s + 8$

$\Rightarrow L(y)\{s^2+4s+4\} = \frac{1+2s^3+8s^2+2s+8}{(s^2+1)} = \frac{1+2(s+4)(s^2+1)}{(s^2+1)}$

$\Rightarrow L(y) = \frac{1+2(s+4)(s^2+1)}{(s^2+1)(s+2)^2}$

On taking the inverse Laplace transform on both sides, we get,

$\Rightarrow L^{-1}(L(y)) = L^{-1}\left(\frac{1+2(s+4)(s^2+1)}{(s^2+1)(s+2)^2}\right)$

$\Rightarrow y = L^{-1}\left[\frac{1}{(s^2+1)(s+2)^2}\right] + 2L^{-1}\left[\frac{(s+4)}{(s+2)^2}\right]$ —— ①

Now, let, $\frac{1}{(s^2+1)(s+2)^2} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{Cs+D}{(s^2+1)}$

$\Rightarrow \frac{1}{(s^2+1)(s+2)^2} = \frac{A(s+2)(s^2+1) + B(s^2+1) + (Cs+D)(s+2)^2}{(s+2)^2(s^2+1)}$

$\Rightarrow (s^3+2s^2+s+2)A + (s^2+1)B + (s^3+4s^2+4s)C + (s^2+4s+4)D = 1$ {On comparing}

$\Rightarrow (A+C)s^3 + (2A+B+4C+D)s^2 + (A+4C+4D)s + (2A+B+4D) = 1$

$\Rightarrow A+C=0 \rightarrow \underline{A=-C}$

$\Rightarrow A+4C+4D=0 \rightarrow 3C+4D=0 \rightarrow \underline{D=\frac{-3C}{4}}$

$\Rightarrow 2A+B+4D=1 \rightarrow -2C+B-3C=1 \rightarrow \underline{B=1+5C}$

Now, $2A + B + 4C + D = 0$

$\Rightarrow -2C + (1+5C) + 4C - \frac{3C}{4} = 0$

$\Rightarrow C = -\frac{4}{25}$

$\Rightarrow A = \frac{4}{25}, \quad B = 1 - \frac{4}{5} = \frac{1}{5}, \quad C = -\frac{4}{25}, \quad D = \frac{3}{25}$

Putting these values in ①, we get,

$$\Rightarrow y = \frac{4}{25} L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{5} L^{-1}\left[\frac{1}{(s+2)^2}\right] + L^{-1}\left[\frac{-4s+3}{25(s^2+1)}\right] + 2L^{-1}\left[\frac{1}{s+2}\right] + 2L^{-1}\left[\frac{2}{(s+2)^2}\right]$$

$$\Rightarrow y = \frac{4}{25} L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{5} L^{-1}\left[\frac{1}{(s+2)^2}\right] - \frac{4}{25} L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{3}{25} L^{-1}\left[\frac{1}{s^2+1}\right] + 2L^{-1}\left[\frac{1}{s+2}\right] + 4L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$\Rightarrow y = \frac{4}{25} e^{-2t} + \frac{1}{5} e^{-2t} t - \frac{4}{25} \cos t + \frac{3}{25} \sin t + 2e^{-2t} + 4e^{-2t}.t$$

$$\Rightarrow y = \frac{1}{25}(3\sin t - 4\cos t) + e^{-2t}\left[\frac{4}{25} + 2\right] + e^{-2t} t\left[\frac{1}{5} + 4\right]$$

$$\Rightarrow y = \frac{1}{25}(3\sin t - 4\cos t) + \frac{54}{25} e^{-2t} + \frac{21}{5} e^{-2t}.t$$

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b) $y'' - 2y' + y = xe^x$, given: $y(0) = 0$ & $y'(0) = 0$

Taking Laplace transform of the above equation on both sides, we get,

$$\Rightarrow L[y'' - 2y' + y] = L[xe^x]$$

$$\Rightarrow \{s^2 L(y) - s y(0) - y'(0)\} - 2\{s L(y) - y(0)\} + L(y) = L[xe^x]$$

$$\Rightarrow s^2 L(y) - 2s L(y) + L(y) = L[xe^x]$$

$$\Rightarrow L(y) = \frac{L[xe^x]}{(s^2 - 2s + 1)} = \frac{1}{(s-1)^2 (s-1)^2} = \frac{1}{(s-1)^4}$$

Taking inverse Laplace transform of the above equation, we get,

$$\Rightarrow L^{-1}[L(y)] = L^{-1}\left[\frac{1}{(s-1)^4}\right]$$

$$\Rightarrow y = e^t L^{-1}\left[\frac{1}{s^4}\right] = e^t . \frac{t^3}{3!} = \frac{t^3 e^t}{6}$$

(5.) a) $y(t) = t + \frac{1}{6}\int_0^t (t-u)^3 y(u)\, du$

~~Taking~~ Using convolution theorem, we get ;

$$\int_0^t (t-u)^3 \cdot y(u)\, du = (t)^3 \cdot y(t)$$

On Putting this value in the main equation and taking Laplace transform on both sides, we get,

$$\Rightarrow L[y(t)] = L\left[t + \frac{1}{6} t^3 \cdot y(t)\right]$$

$$\Rightarrow L[y(t)] = L[t] + \frac{1}{6} L[y(t) \cdot t^3]$$

$$\Rightarrow L[y(t)] = \frac{1}{s^2} + \frac{1}{6} L[y(t)] \cdot L[t^3]$$

$$\Rightarrow L[y(t)] = \frac{1}{s^2} + \frac{1}{6} \cdot \frac{6}{s^4} \cdot L[y(t)]$$

$$\Rightarrow L[y(t)]\left(1 - \frac{1}{s^4}\right) = \frac{1}{s^2}$$

$$\Rightarrow L[y(t)] = \frac{s^2}{s^4 - 1}$$

$\Rightarrow$ On taking inverse Laplace transform of the above equation from both sides, we get :

$$\Rightarrow L^{-1}[L[y(t)]] = L^{-1}\left[\frac{s^2}{s^4 - 1}\right]$$

$$\Rightarrow y(t) = \frac{1}{2} L^{-1}\left[\frac{s^2+1+s^2-1}{(s^2+1)(s^2-1)}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{(s^2+1)}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{(s^2-1)}\right]$$

$$\Rightarrow y(t) = \frac{1}{2}\left[\sin t + \frac{1}{2} L^{-1}\left(\frac{2}{(s+1)(s-1)}\right)\right]$$

$$\Rightarrow y(t) = \frac{1}{2}\left[\sin t + \frac{1}{2}\left(\frac{1}{(s-1)} - \frac{1}{(s+1)}\right)\right]$$

$$\Rightarrow y(t) = \frac{\sin t}{2} + \frac{1}{4}\left[e^{-t} - e^{t}\right]$$