TOPIC COVERED AT A GLANCE

1. FROOF FOR THE STATEMENT INFINTESIMAL ROTATION ,DEFINED AS A INFINITESIMALLY SMALL AMOUNT OF ANGLE OF ROTATION IS A VECTOR QUANTITY BUT FINITE ROTATION IS NOT A VECTOR QUANTITY BECAUSE IT DO NOT OBEY THE COMMUNICATIVE LAW OF ADDITION, ALTHOUGH POSSEESSING MAGNITUDE , DIRECTIONS AND SENSE.

2. DEFINITION OF ANGULAR VELOCITY AND ANGULAR ACCELERATION.

3. SCHEMATIC REPRESENTATION OF ANGULAR VELOCITY AND ACCELERATION.

4. REPRESENTATION OF DIFFERENT COORDINATE SYSTEMS.

5. SOME VECTOR OPERATIONS.

6. SOME VECTOR OPERATIONS IN DIFFERENT COORDINATE SYSTEMS.

7. LAWS OF MOTIONS IN LINEAR AND PLANE CURVILINEAR MOTION.

8. DERIVATIVES OF UNIT VECTORS.

Infinitesimal Rotation

Infinitesimal rotation $d\theta$, defined as a infinitesimally small amount of the angle of rotation, is a vector quantity. It is important to note here that finite rotation 'θ' is not a vector quantity. The reason is that the finite rotations θ_1, and θ_2, although possessing magnitudes, directions and sense, do not obey the commutative law of addition, i.e.,

$$\theta_1 + \theta_2 \neq \theta_2 + \theta_1 \tag{R2.41}$$

This has been demonstrated in Fig. R2.12 where a foot rule is shown subjected to rotations about the x and y axes respectively. An initial rotation $\theta_1 = \pi/2\,\mathbf{i}$ about the x-axis and a subsequent rotation $\theta_2 = \pi/2\,\mathbf{j}$ about the y-axis bring the sheet into positions (Aa) and (Aab). On the other hand, an initial rotation $\theta_2 = \pi/2\,\mathbf{j}$ about the y-axis and a subsequent rotation $\theta_1 = \pi/2\,\mathbf{i}$ about the x-axis result in positions (Ab) and (Aba) as shown. Obviously, the final positions (Aab) and (Aba) are not the

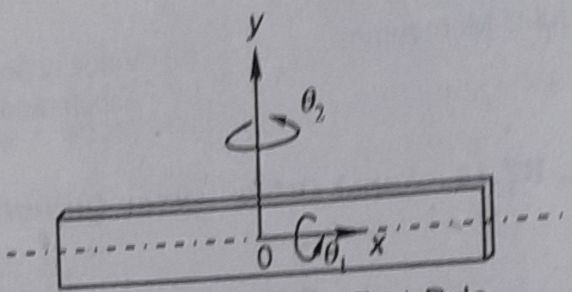
same showing that the commutative law for finite rotations fails; admission disqualifying them from being put under the category of vector quantities.

The case of addition of infinitesimal rotations $d\theta_1$ and $d\theta_2$ is also shown in Fig. R2.12. The position of the foot rule after an initial rotation $d\theta_1$ is shown as (Ba) whereas that after an initial rotation $d\theta_2$ is shown as (Bb). The final orientation of the foot rule after both $d\theta_1$ and $d\theta_2$ have been imparted, in either order, is approximately the same in the limiting case shown in the same figure. Hence,

$$\Delta\theta_1 + \Delta\theta_2 \approx \Delta\theta_2 + \Delta\theta_1 \tag{R2.42}$$

and in the limit,

$$d\theta_1 + d\theta_2 = d\theta_2 + d\theta_1$$



Initial Position of the Foot Rule

with a thickness

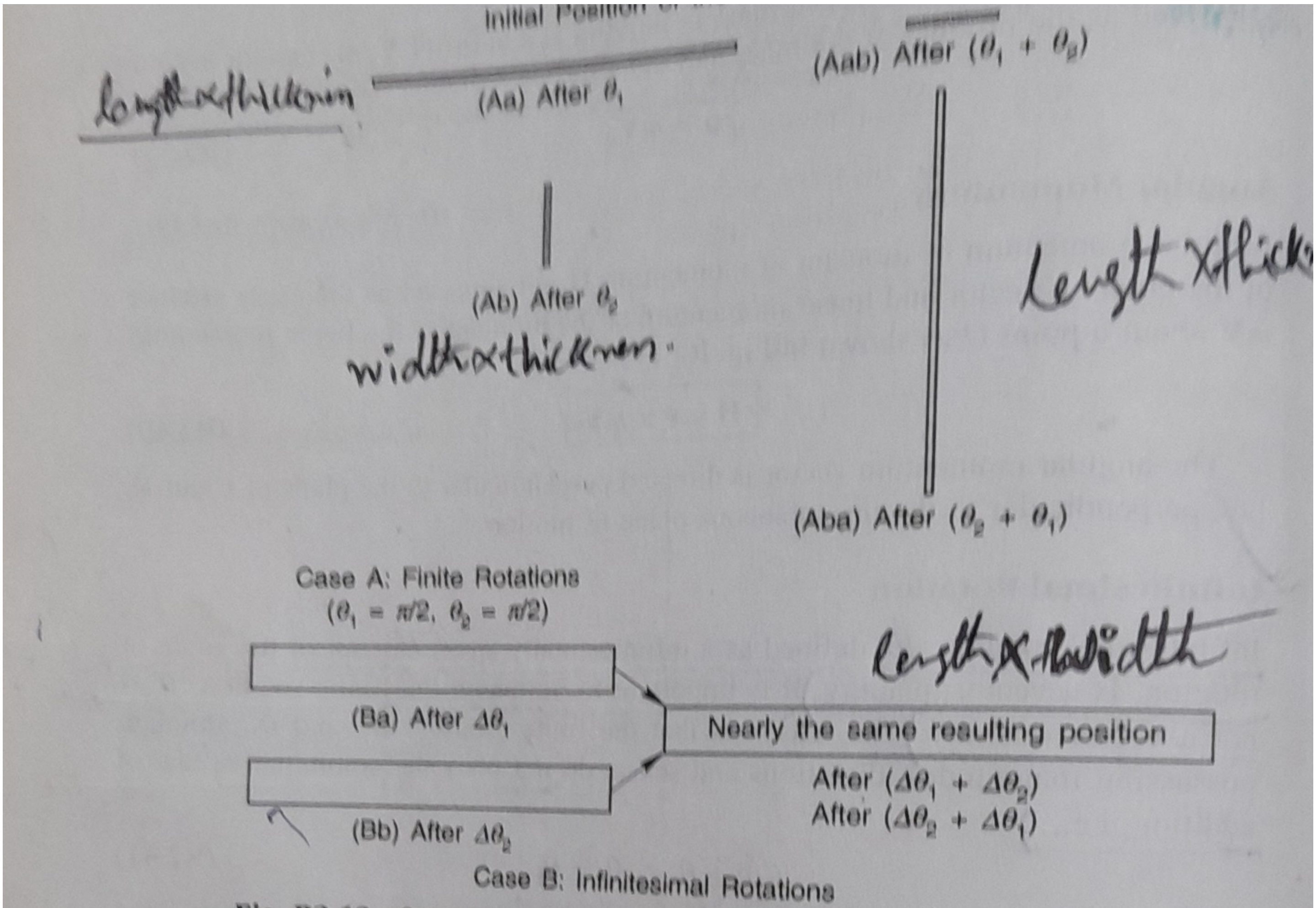


Fig. R2.12 ***Summation of Finite and Infinitesimal Rotations***

Angular Velocity

Angular velocity ω is defined as the rate of change of angular displacement or rotation of a body about an axis

$$\omega = \frac{d\theta}{dt} \quad (R2.43)$$

It follows from the vector nature of infinitesimal rotation $d\theta$ and the scalar nature of time interval dt that the angular velocity must be a vector quantity.

If a point is located with a position vector $\mathbf{r}$ with respect to an origin O on the axis of rotation, the linear velocity $\mathbf{V}$ of the point is given by

$$\mathbf{V} = \boldsymbol{\omega} \times \mathbf{r} \quad \text{(R2.44)}$$

as shown in Fig. R2.11(b).

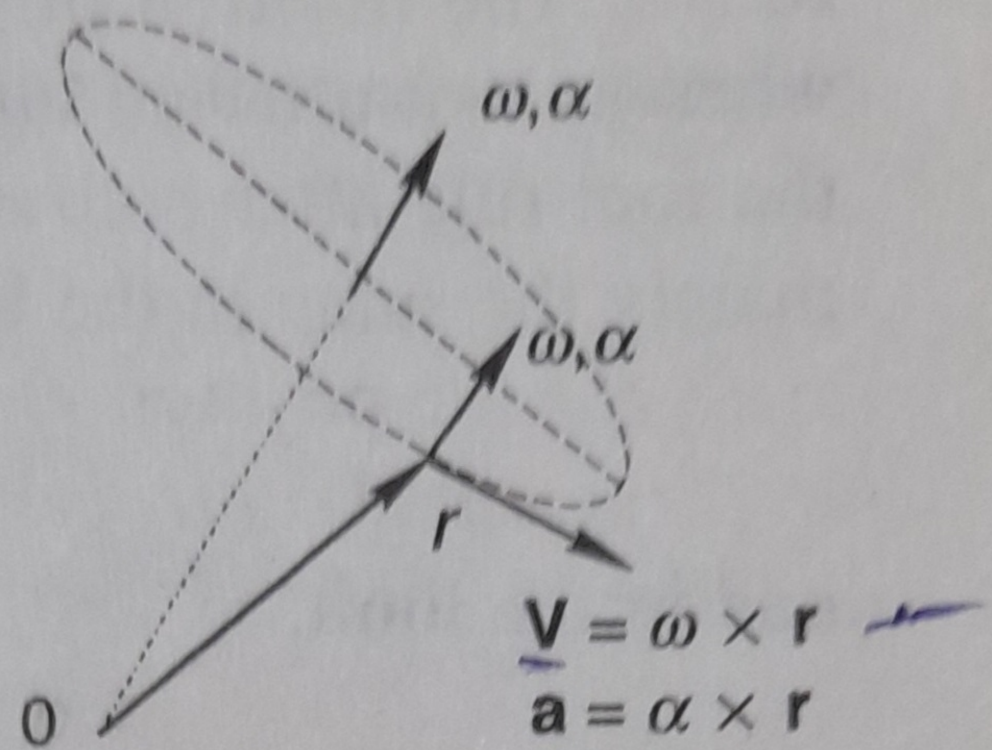
Angular Acceleration

Angular acceleration $\boldsymbol{\alpha}$ is defined as the rate of change of angular velocity, i.e.,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = \frac{d^2\boldsymbol{\theta}}{dt^2} \quad \text{(R2.45}$$

The linear acceleration $\mathbf{a}$ of a point with a positive vector $\mathbf{r}$ with respect to an origi O on the axis of rotation is given by

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} \quad \text{(R2.46}$$



(b) Velocity and Angular Velocity
Acceleration and Angular Acceleration

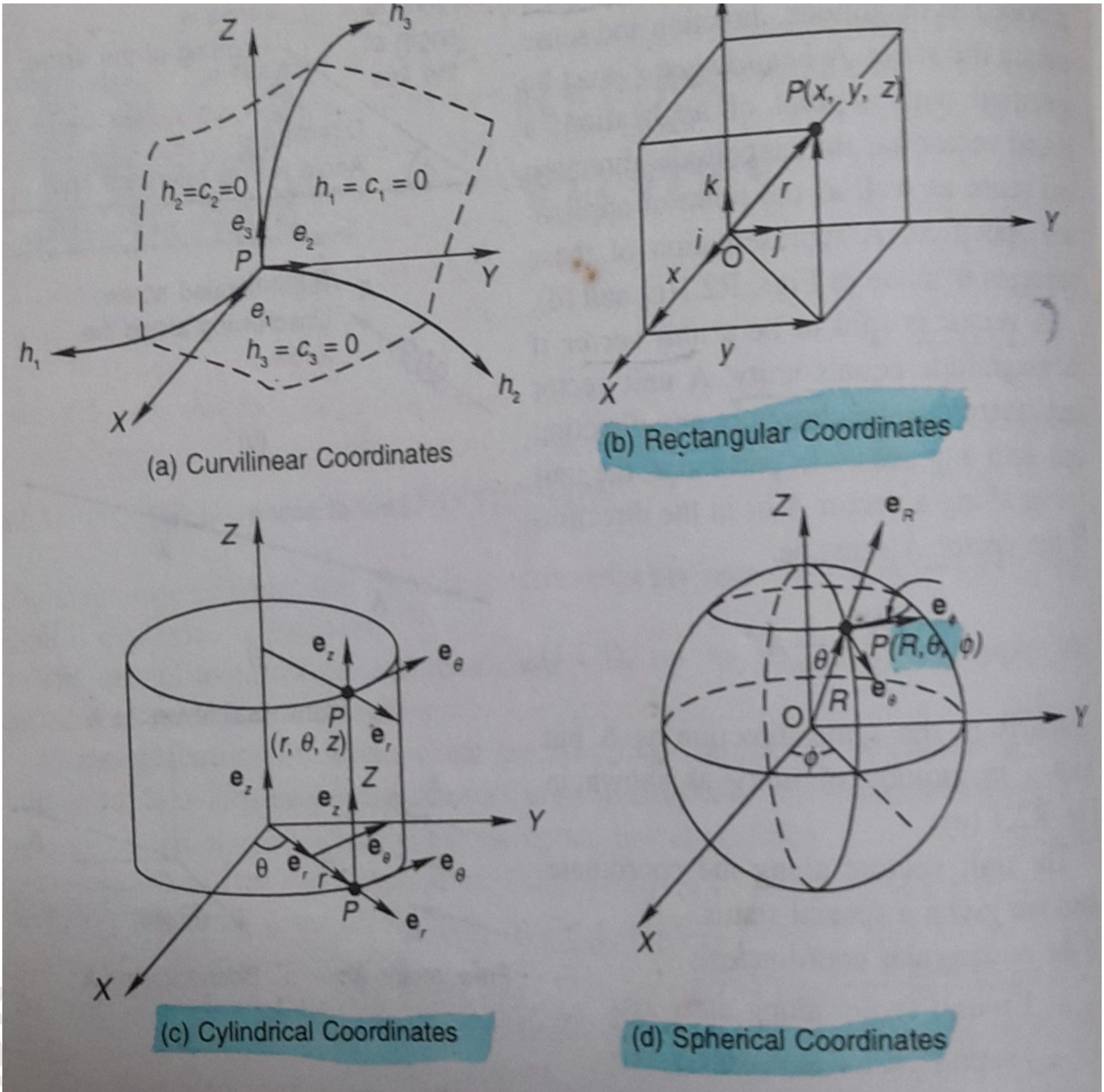


Fig. R2.2 *Coordinate Systems*

DIFFERENT COORDINATE SYSTEMS

R2.7 SOME VECTOR OPERATIONS

Some of the useful results in vector algebra and calculus commonly referred to by the engineering students are summarized in Table R2.1.

$$\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=\mathbf{e}_r\cdot\mathbf{e}_r=\mathbf{e}_\theta\cdot\mathbf{e}_\theta=\ldots=1$$

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{j}=\mathbf{e}_r\cdot\mathbf{e}_\theta=\mathbf{e}_\theta\cdot\mathbf{e}_\phi=\ldots=0$$

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{e}_r\times\mathbf{e}_r=\mathbf{e}_\theta\times\mathbf{e}_\theta=\ldots=0$$

$$\mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i},\ \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

$$\mathbf{e}_r\times\mathbf{e}_\theta=\mathbf{e}_z,\ \mathbf{e}_\theta\times\mathbf{e}_z=\mathbf{e}_r,\ \mathbf{e}_z\times\mathbf{e}_r=\mathbf{e}_\theta$$

$$\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A},\ (\mathbf{A}\times\mathbf{B})\times\mathbf{C}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{A}$$

$$\frac{d\mathbf{A}}{du}=\frac{dA_x}{du}\mathbf{i}+\frac{dA_y}{du}\mathbf{j}+\frac{dA_z}{du}\mathbf{k}$$

If $F=f(x,y)$ where $x=x(t)$, $y=y(t)$

$$\frac{dF}{dt}=\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}$$

SOME VECTOR OPERATIONS IN DIFFERENT COORDINATES

Table R2.1 VECTOR OPERATIONS IN DIFFERENT COORDINATE SYSTEMS

	Cartesian	*Cylindrical*	*Spherical*
Vector $\mathbf{A}$	$A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}$	$A_r\mathbf{e}_r+A_\theta\mathbf{e}_\theta+A_z\mathbf{e}_z$	$A_R\mathbf{e}_R+A_\theta\mathbf{e}_\theta+A_\phi\mathbf{e}_\phi$
$A=\lvert\mathbf{A}\rvert$	$\left(A_x^2+A_y^2+A_z^2\right)^{1/2}$	$\left(A_r^2+A_\theta^2+A_z^2\right)^{1/2}$	$\left(A_R^2+A_\theta^2+A_\phi^2\right)^{1/2}$
$\mathbf{A}\cdot\mathbf{B}=AB\cos\alpha$	$A_xB_x+A_yB_y+A_zB_z$	$A_rB_r+A_\theta B_\theta+A_zB_z$	$A_RB_R+A_\theta B_\theta+A_\phi B_\phi$
$\mathbf{A}\times\mathbf{B}=AB\sin\alpha\,\mathbf{n}$	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}$	$\begin{vmatrix}\mathbf{e}_r&\mathbf{e}_\theta&\mathbf{e}_z\\A_r&A_\theta&A_z\\B_r&B_\theta&B_z\end{vmatrix}$	$\begin{vmatrix}\mathbf{e}_R&\mathbf{e}_\theta&\mathbf{e}_\phi\\A_R&A_\theta&A_\phi\\B_R&B_\theta&B_\phi\end{vmatrix}$
$(\mathbf{ABC})=\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$	$\begin{vmatrix}A_x&A_y&A_z\\B_x&B_y&B_z\\C_x&C_y&C_z\end{vmatrix}$	$\begin{vmatrix}A_r&A_\theta&A_z\\B_r&B_\theta&B_z\\C_r&C_\theta&C_z\end{vmatrix}$	$\begin{vmatrix}A_R&A_\theta&A_\phi\\B_R&B_\theta&B_\phi\\C_R&C_\theta&C_\phi\end{vmatrix}$
Operator $\nabla\equiv$	$\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$	$\mathbf{e}_r\frac{\partial}{\partial r}+\mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\mathbf{e}_z\frac{\partial}{\partial z}$	$\mathbf{e}_R\frac{\partial}{\partial R}+\mathbf{e}_\theta\frac{1}{R}\frac{\partial}{\partial\theta}+\mathbf{e}_\phi\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}$
grad $p=\nabla p\equiv$	$\frac{\partial p}{\partial x}\mathbf{i}+\frac{\partial p}{\partial y}\mathbf{j}+\frac{\partial p}{\partial z}\mathbf{k}$	$\frac{\partial p}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial p}{\partial\theta}\mathbf{e}_\theta+\frac{\partial p}{\partial z}\mathbf{e}_z$	$\frac{\partial p}{\partial R}\mathbf{e}_R+\frac{1}{R}\frac{\partial p}{\partial\theta}\mathbf{e}_\theta+\frac{1}{R\sin\theta}\frac{\partial p}{\partial\phi}\mathbf{e}_\phi$
div $\cdot\,\mathbf{A}=\nabla\cdot\mathbf{A}$	$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$	$\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{\partial A_z}{\partial z}$	$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2A_R)+\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,A_\theta)+\frac{1}{R\sin\theta}\frac{\partial A\phi}{\partial\phi}$
curl $\mathbf{A}=\nabla\times\mathbf{A}$	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\A_x&A_y&A_z\end{vmatrix}$	$\frac{1}{r}\begin{vmatrix}\mathbf{e}_r&r\mathbf{e}_\theta&\mathbf{e}_z\\\frac{\partial}{\partial r}&\frac{\partial}{\partial\theta}&\frac{\partial}{\partial z}\\A_r&rA_\theta&A_z\end{vmatrix}$	$\frac{1}{R^2\sin\theta}\begin{vmatrix}\mathbf{e}_R&R\mathbf{e}_\theta&R\sin\theta\,\mathbf{e}_\phi\\\frac{\partial}{\partial R}&\frac{\partial}{\partial\theta}&\frac{\partial}{\partial\phi}\\A_R&A_\theta&R\sin\theta\,A_\phi\end{vmatrix}$

(Contd.)

Table R2.1 (*Contd*) **VECTOR OPERATIONS IN DIFFERENT COORDINATE SYSTEMS**

	Cartesian	Cylindrical	Spherical
Operator $\nabla^2 \equiv$ (Laplacian)	$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$	$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$
Operator $\frac{D}{Dt} \equiv (U\cdot\nabla)+\frac{\partial}{\partial t}$	$u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}$	$u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}+\frac{\partial}{\partial t}$	$u_R\frac{\partial}{\partial R}+\frac{u_\theta}{R}\frac{\partial}{\partial\theta}+\frac{u_\phi}{R\sin\theta}\frac{\partial}{\partial\phi}+\frac{\partial}{\partial t}$

Laws of Motions for Linear vs Plane curvilinear motiom

Table 5.1 Linear vs Angular Motion

	Linear Motion	*Angular Motion*	*Remarks*
Displacement	s	θ	$ds = r\,d\theta$
Velocity	$v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$	$V = r\omega$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$	$a = r\alpha$
Initial velocity	u	ω_0	
Expressions relating the displacement, velocity, accelera-tion and time	$V = u + at$ $s = ut + \frac{1}{2}at^2$ $V^2 - u^2 = 2as$ $s = \int V\,dt + C$ $V = \int a\,dt + K$	$\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega - \omega_0 = 2\alpha\theta$ $\theta = \int \omega\,dt + C'$ $\omega = \int \alpha\,dt + K'$	valid only for constant accele-ration

DERIVATIVES OF UNIT VECTOR

$e_\theta \equiv e_t$

(b) Derivatives of Unit Vectors

Derivatives of the unit vectors **i**, **j** and **k** along the ~~fixed~~ space coordinates x, y and z respectively must be zero with respect to any scalar.

$$\frac{d\mathbf{i}}{dt} = 0 = \frac{d\mathbf{j}}{dt} = \frac{d\mathbf{k}}{dt} \ldots$$

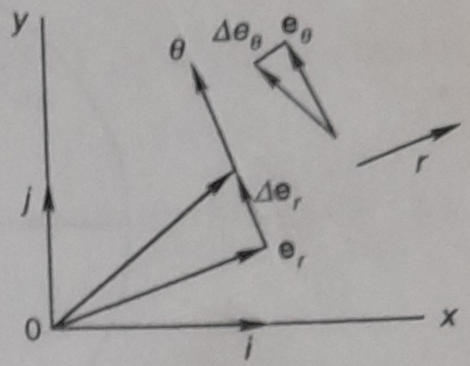


Fig. R2.8(b) ***Change in Unit Vectors***

Let us consider the derivatives of the unit vectors in radial and tangential direction, i.e., of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as shown in Fig. R2.8(b)

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$\frac{d\mathbf{e}_r}{dt} = (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\frac{d\theta}{dt}$$

Since

$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

and

$$\omega = \frac{d\theta}{dt}$$

$$\boxed{\frac{d\mathbf{e}_r}{dt} = \omega\,\mathbf{e}_\theta}$$

Similarly,

$$\frac{d\mathbf{e}_\theta}{dt} = (-\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j})\frac{d\theta}{dt}$$

$$= -\omega\,\mathbf{e}_r$$