Differential Equation

① $(y'')^2 + (y')^3 + 3y = x^2$

Here order is 2

degree is 2

② $y''' + 3(y'')^2 + y' = e^x$

Order - 3

degree - 1

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Equation of First Order and First Degree

The simplest type of a differential equation of first order and first degree is the case in which the variables are separable. Such an equation is of the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ which can be written as

$$g(y)\,dy = f(x)\,dx$$

Hence $\int g(y)\,dy = \int f(x)\,dx + c$

Solve $\quad \frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

or, $\quad \frac{dy}{1+y^2} + \frac{dx}{1+x^2} = 0$

$$\Rightarrow \tan^{-1} y + \tan^{-1} x = c$$

Solve $\quad y' = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

Put $z = \frac{y}{x}$

Hence $zx = y$

$$x\frac{dz}{dx} + z = \frac{dy}{dx}$$

$$x\frac{dz}{dx} + \cancel{z} = \cancel{z} + \tan z$$

$$\Rightarrow \frac{dz}{\tan z} = \frac{dx}{x}$$

$$\Rightarrow \log \sin z = \log x + \log c$$

$$\Rightarrow \sin z = cx$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = cx$$

Homogeneous

A function $f(x,y)$ is called a homogeneous function of x and y of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ for all x, y

We consider a differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are homogeneous function of the same degree.

To solve this ~~thye~~ type of differential equation we put $y = vx$.

Solve $\frac{dy}{dx} = \frac{y^3 + 3x^2y}{x^3 + 3xy^2}$

Solve put $y = vx$

then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v^3 + 3v}{1 + 3v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^3 + 3v}{1 + 3v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^3 + 3v - v - 3v^3}{1 + 3v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v - 2v^2}{1 + 3v^2} = \frac{2v(1 - v)}{1 + 3v^2}$$

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$$\Rightarrow \frac{1+3v^2}{2v(1-v^2)}dv = 2\frac{dx}{x}$$

$$\Rightarrow 2\frac{dx}{x} = \frac{1}{v} - \frac{2}{1+v} + \frac{2}{1-v}$$

$$\Rightarrow 2\log x = \log v - 2\log(1+v) - 2\log(1-v)$$

$$\Rightarrow 2\log x = \log v\, 2\log(1+v)(1-v)$$

$$\Rightarrow \cancel{2\log x} = \cancel{\log v - 2\log}$$

$$\Rightarrow 2\log x = \log\frac{vc}{(1-v^2)^2}$$

$$\Rightarrow x^2 = \frac{cv}{(1-v)^2(1+v)^2}$$

$$\Rightarrow x^2 = \frac{c\frac{y}{x}}{\left(1-\frac{y}{x}\right)^2\left(1+\frac{y}{x}\right)^2}$$

$$\Rightarrow x^2 = \frac{cy}{x} \times \frac{x^4}{\cancel{(x^2-y^2)(x^2+y^2)}\,(x-y)^2(x+y)^2}$$

$$\Rightarrow (x^2-y^2)^2 = cxy$$

Solve ① $(x^2-y^2)\frac{dy}{dx} = 2xy$

② $x^2y\,dx - (x^3+y^3)dy = 0$

Non-homogeneous equation of first-degree in x and y.

We consider an equation of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$$

Case (i) If $ab_1 = ba_1$ then substitute $ax+by = v$ reduce the given equation to one in which the variables are separable.

Case (ii) If $ab_1 \neq ba_1$ then the substitution $x = X+h$ and $y = Y+k$ where h and k are such that $ah+bk+c=0$ and $a_1h+b_1k+c_1=0$ reduce the given equation to a homogeneous equation in X and Y which can be solved by using previous case. The final solution is got by replacing X and Y by $x-h$ and $y-k$ respectively.

Ex 1. Solve $\frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$

Put $3x-2y = v$

Hence $3 - 2\frac{dy}{dx} = \frac{dv}{dx}$

$$\frac{dv}{dx} = 3 - 2\left(\frac{2v+3}{v+1}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{-(v+3)}{v+1}$$

$$\Rightarrow dx = -\frac{v+1}{v+3}\,dv$$

$$= -\left(1 - \frac{2}{v+3}\right) dv$$

$x = -v + 2\log(v+3) + c'$ ⓪ ⑳

$\Rightarrow x = -(3x-2y) + 2\log(3x-2y+3) + c'$

$\Rightarrow 4x - 2y = 2\log(3x-2y+3) + c'$

$\Rightarrow 2x - y = \log(3x-2y+3) + c'$

Solve $\dfrac{dy}{dx} = \dfrac{3y-7x+7}{3x-7y-3}$

<u>Ex</u> Solve $\dfrac{dy}{dx} = \dfrac{x-y+1}{x+y-3}$

<u>Sol</u> $x = X+h, \quad y = Y+k$

$x-y+1 = 0$

$x+y-3 = 0$

Solving we get $(x,y) = (1,2)$

$x = X+1, \quad y = Y+2$

$$\frac{dY}{dX} = \frac{X+1-(Y+2)+1}{X+1+Y+2-3}$$

$$= \frac{X-Y}{X+Y}$$

$Y = vX$ Hence $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

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$$v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$ ⑥

or, $x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-2v-v^2}{1+v}$

or, $\frac{dv\,(1+v)}{1-2v-v^2} = \frac{dx}{x}$

or, $-\frac{1}{2}\log(1-2v-v^2) = \log x + \log c_1$

or, $\log(1-2v-v^2)^{-1/2} = \log c_1 x$

or, $(1-2v-v^2)^{-1/2} = c_1 x$

or, $1-2v-v^2 = c_2 x^{-2}$

or, $x^2(1-2v-v^2) = c_2$

or, $x^2\left(1-2\frac{y}{x}-\frac{y^2}{x^2}\right) = c_2$

or, $x^2 - 2xy - y^2 = c_2$

or, $(x-1)^2 - 2(x-1)(y-2) - (y-2)^2 = c_2$

or, $x^2 - 2xy - y^2 + 2x + 6y = c$

Solve $(6x-4y+1)\,dy = (3x-2y+1)\,dx$

Exact Differential equation

The differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if and only if

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$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Working Rule to solve exact differential equations

(a) Verify wheather the given equation $M\,dx + N\,dy = 0$ is exact i.e. verify

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(b) If exact, integrate M with respect to x keeping y as constant.

(c) Find out those terms in N which are free from x and integrate those terms with respect to y.

(d) The sum of these two expressions equated to an ordinary constant is the required general solution of the given exact equation.

Ex Verify wheather $(x^2-4xy-2y^2)\,dx + (y^2-4xy-2x^2)\,dy = 0$ is exact

Here $M = x^2 - 4xy - 2y^2$, $N = y^2 - 4xy - 2x^2$ (8)

$\therefore \frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x}$

Hence the given equation is exact.

Now $\int M dx = \int (x^2 - 4xy - 2y^2) dx$

$= \frac{x^3}{3} - 2x^2y - 2y^2x$

In $\int N dy = \int (y^2 - 4xy - 2x^2) dy$

$= \frac{y^3}{3} - 2xy^2 - 2x^2y$

Complete integral is $\frac{x^3}{3} - 2x^2y - 2y^2x + \frac{y^3}{3} = c'$

or, $x^3 + y^3 - 6xy(x+y) = c$

Solve $\quad x dx + y dy - \left(\frac{x dy - y dx}{x^2+y^2} \right) = 0$

The equation can be written as

$\left(x + \frac{y}{x^2+y^2} \right) dx + \left(y - \frac{x}{x^2+y^2} \right) dy = 0$

$M = x + \frac{y}{x^2+y^2}$, $N = y - \frac{x}{x^2+y^2}$

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$$\frac{\partial M}{\partial y} = \frac{x^2y^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}$$

$$\int M\,dx = \frac{1}{2}x^2 + \tan^{-1}\left(\frac{x}{y}\right)$$

In N the term free from x is y whose integral with respect to y is $\frac{y^2}{2}$.

Hence the complete solution is

$$\frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1}\left(\frac{x}{y}\right) = c'$$

$$\text{or, } x^2+y^2+2\tan^{-1}\left(\frac{x}{y}\right) = c$$

Solve $x(x^2+y^2-a^2)\,dx + y(x^2-y^2-b^2)\,dy = 0$

Linear Differential Equation

If $Mdx+Ndy=0$ is not exact then any function μ which is such that $\mu(Mdx+Ndy)=0$ becomes exact is called an integrating factor (I.F) of the given differential equation $Mdx+Ndy=0$

Def A differential equation is said to be linear if the dependent variable and ~~the~~ its derivative appear only in the first de

Hence a linear differential equation of first-order is of the form

$$\frac{dy}{dx} + Py = Q$$

where P, Q are functions of x along alone.

Note : We observe that

$$d\left(y e^{\int P dx}\right) = \frac{dy}{dx} e^{\int P dx} + Py e^{\int P dx}$$

$$= e^{\int P dx}\left(\frac{dy}{dx} + Py\right)$$

∴ Multiplying the given equation by $e^{\int P dx}$ we get

$$d\left(y e^{\int P dx}\right) = e^{\int P dx} Q$$

$$\text{or, } y e^{\int P dx} = \int e^{\int P dx} Q \, dx + c$$

Thus $e^{\int P dx}$ is an I.F of the given equation and the general solution of equation ① is given by equation ②.

<u>Solve</u> $(1+y^2)\,dx + (x - \tan^{-1}y)\,dy = 0$ (11)

$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$ which is a linear first order differential equation in x.

$P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$

Now $\int P\,dy = \tan^{-1}y$

$\therefore x e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \tan^{-1}y \, dy$

Substituting $t = \tan^{-1}y$ in the right-hand side we get

$x e^{\tan^{-1}y} = \int t e^t \, dt = e^t (t-1) + c$

$= e^{\tan^{-1}y} (\tan^{-1}y - 1) + c$

or $x = \tan^{-1}y - 1 + c e^{-\tan^{-1}y}$

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Solve $\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x$, given $y=0$ (12)
when $x = \frac{\pi}{2}$

$$\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x$$

$$\int P\,dx = \int \cot x\,dx = \log\sin x$$

$$y\, e^{\log\sin x} = \int \sin x\,(4x\operatorname{cosec} x)\,dx$$

$$\text{or, } y\sin x = \frac{4x^2}{2} + c$$

$$\text{or, } y\sin x = 2x^2 + c$$

when $x = \frac{\pi}{2}$, $y=0$, $c = -\frac{\pi^2}{2}$

$$\therefore y\sin x = 2x^2 - \frac{\pi^2}{2}$$

Solve (1) $\frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}$

(2) $y' - y\tan x = e^x\sec x$

(3) $(x+1)y' - ny = e^x(x+1)^{n+1}$

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Bernoulli's Equation

The equation $\frac{dy}{dx} + Py = y^n Q$ where P, Q are functions of x ~~only~~ alone is called Bernoulli's equation.

When $n=0$ or $n=1$, it is already linear. For other values of n it can be reduced to a linear equation by the substitution $z = y^{1-n}$. With this substitution the given equation becomes

$$\frac{dz}{dx} + (1-n)Pz = (1-n)Q$$

which is linear and can be solved as previous.

<u>Ex</u> Solve $xy' + y = y^2 \log x$

$$y' + \frac{y}{x} = y^2 \frac{\log x}{x}$$

$$y^{-2}y' + \frac{y^{-1}}{x} = \frac{\log x}{x}$$

Put $z = y^{-1}$

Hence $y^{-2}y' = -z'$

The given equation is transformed to

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x}\log x$$

Here $P = -\frac{1}{x}$, $Q = -\frac{1}{x}\log x$

$$\therefore \int P\,dx = -\int \frac{dx}{x} = -\log x$$

$$e^{\int P\,dx} = e^{-\log x} = x^{-1} = \frac{1}{x}$$

$$z\left(\frac{1}{x}\right) = \int -\frac{1}{x}\log x \cdot \frac{1}{x}\,dx + c$$

$$= -\int \frac{1}{x^2}\log x\,dx + c$$

$$= \frac{1}{x}\log x - \int \frac{dx}{x^2} + c$$

$$= \left(\frac{1}{x}\right)\log x + \frac{1}{x} + c$$

$$= \frac{1}{x}(\log x + 1) + c$$

or, $\frac{1}{xy} = \frac{1}{x}(\log x + 1) + c$

or, $\frac{1}{y} = (\log x + 1) + cx$

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① $y' - 2y \tan x = y^2 \tan^2 x$

② $(1-x^2)y' - xy = x^2y^2$

③ $y' + y \cos x = y^n \sin 2x$

Equation of First-order and Higher-degree

Throughout this section we denote $\frac{dy}{dx}$ by p

A differential equation of first order and nth degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} - - - + P_{n-1} p + P_n = 0$$

where $P_1, P_2, - - P_n$ are functions of x and y.

Type A

Equation solvable for p

Suppose the left-hand side of equation (1) can be factorised into factors of the first-degree then equation ① becomes $(p-R_1)(p-R_2) - - (p-R_n) = 0$

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Obtain a solution $f_i(x,y,c)=0$ corresponding (16)
to the equation $p-R_i=0$ for $i=1,2,\dots n$.
Then the general solution of equation ① given by

$$f_1(x,y,c)\,f_2(x,y,c)\cdots f_n(x,y,c)=0$$

Solve Solve $p^2-9p+18=0$

The given equation can be written as $(p-6)(p-3)=0$

$$p=6,\quad p=3$$

$$\frac{dy}{dx}=6 \Rightarrow y=6x+c$$

$$\frac{dy}{dx}=3 \Rightarrow y=3x+c$$

The solution is $(y-6x-c)(y-3x-c)=0$

Solve $4p^2-8p+3=0$

$$\Rightarrow (2p-1)(2p-3)=0$$

$$\Rightarrow p=\frac{1}{2},\quad p=\frac{3}{2}$$

$$p=\frac{1}{2} \Rightarrow y=\frac{x}{2}+c'$$

$$p=\frac{3}{2} \Rightarrow y=\frac{3x}{2}+c''$$

$(2y-x-c)(2y-3x-c)=0$

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Type B

Equation solvable for y

In this case, the equation can be put in the form

$$y = f(x, p) \text{ ——— } ①$$

Differentiating with respect to x we get

$$p = \phi\left(x, p, \frac{dp}{dx}\right) \text{ ——— } ②$$

which is a first-order first-degree differential equation with variables p and x.

Suppose the equation ② can be solved to get a relation

$$\psi(x, p, c) = 0 \text{ ——— } ③$$

Then eliminating p from equations ① and ③, we get the required solution.

Solve $y - 2px = f(xp^2)$

Diff. w.r.to x, we get

$$p = 2p + 2x\frac{dp}{dx} + f'(xp^2)\left(p^2 + x2p\frac{dp}{dx}\right)$$

or, $\left(p + 2x\frac{dp}{dx}\right) + f'(xp^2)\,p\left(p + 2x\frac{dp}{dx}\right) = 0$

or, $\left(p + 2x\frac{dp}{dx}\right)\left[1 + pf'(xp^2)\right] = 0$

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$\therefore\ p + 2x\dfrac{dp}{dx} = 0$ or, $1 + p f'(xp^2) = 0$

Taking $p + 2x\dfrac{dp}{dx} = 0$

or, $2\dfrac{dp}{dx} + \dfrac{p}{x} = 0$

or, $2\log p + \log x = \log c$

or, $p^2 x = c$

Hence $p = \sqrt{\dfrac{c}{x}}$

Substituting p in the given equation

we get $y = 2\sqrt{\dfrac{c}{x}}\ x + f\left(x\dfrac{c}{x}\right)$

or, $y = 2\sqrt{cx} + f(c)$

Note: the factor $1 + p f'(xp^2)$ will lead to singular solution.

Solve $\quad 3x - y + \log p = 0$

or, $y = 3x + \log p$

Differentiating w.r.to x we get

$p = 3 + \dfrac{1}{p}\dfrac{dp}{dx}$

or, $p^2 - 3p = \dfrac{dp}{dx}$

or, $p(p-3) = \dfrac{dp}{dx}$

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$\therefore \; dx = \dfrac{dp}{p(p-3)} = \dfrac{1}{3}\left(\dfrac{1}{p-3} - \dfrac{1}{p}\right) dp$ (3)

$\Rightarrow 3x = \log(p-3) - \log p - \log c'$

$\Rightarrow 3x = \log\left(\dfrac{p-3}{pc}\right)$

$\Rightarrow \dfrac{p-3}{p} = c\,e^{3x}$

$\Rightarrow \dfrac{3}{p} = 1 - c\,e^{3x}$

$\Rightarrow p = \dfrac{3}{1-c\,e^{3x}}$

Substituting the value of p in the given equation we get the solution as

$$y = 3x + \log\left(\frac{3}{1-c\,e^{3x}}\right)$$

Solve $xp^2 - 2yp + x = 0$

$$y = \frac{x(p^2+1)}{2p}$$

Differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{p^2+1}{2p} + x\,\frac{2p\cdot 2p - (p^2+1)2}{4p^2}\,\frac{dp}{dx}$$

$$= \frac{p^2+1}{2p} + x\,\frac{p^2-1}{2p^2}\,\frac{dp}{dx}$$

$$\Rightarrow p = \frac{p(p^2+1) + x(p^2-1)p'}{2p^2}$$

$$2p^3 = p(p^2+1) + x(p^2-1)p'$$

or, $p^3 - p = xp'(p^2-1)$

or, $p(p^2-1) = xp'(p^2-1)$

or, $(p^2-1)(p - xp') = 0$

Taking $p'x = p$ we get $\frac{dp}{p} = \frac{dx}{x}$

$\therefore \log p = \log x + \log c$

or, $p = cx$

Substituting this in the given equation

$$xc^2x^2 - 2y \cdot cx + x = 0$$

or, $2cy = c^2x^2 + 1$

Note: the other factor will lead to singular solution

Type C

Equations solvable for x

In this case, the equation can be put in the form

$$x = f(y, p)$$

ifferentiating with respect to y we get

$$\frac{1}{p} = \psi\left(y, p, \frac{dp}{dy}\right)$$

which is a first-degree, first-order differential equation with variables p and y.

Suppose the equation (2) can be solved to get a relation

$$\phi(y, p, c) = 0$$

Then eliminating p from equation (1) and (3) we get the required solution.

Note: The factor which does not involve a derivative of p with respect to x or y will always lead to singular solution. Hence such a factor can be omitted.

Solve $\quad x = p + p^4$ —— (1)

Differentiating (1) with respect to y, we get

$$\frac{dx}{dy} = \frac{dp}{dy} + 4p^3\frac{dp}{dy}$$

or, $\frac{1}{p} = (1+4p^3)\frac{dp}{dy}$

or, $dy = (p+4p^4)\,dp$

or, $y = \frac{p^2}{2} + \frac{4p^5}{5} + c$ —— (2)

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Eliminating p from ① and ②, we get the required solution.

Solve $x = y + p^2$

Type D

Clairaut's equation

An equation of the form

$$y = px + f(p) \quad \text{——— ①}$$

is called Clairaut's equation.

Differentiating with respect to x we get

$$\cancel{p} = \cancel{p} + (x + f'(p)) \frac{dp}{dx}$$

$$\Rightarrow [x + f'(p)] \frac{dp}{dx} = 0$$

$$\therefore x + f'(p) = 0 \quad \text{or,} \quad \frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \Rightarrow p = c \quad (\text{a constant})$$

Now,

Hence the general solution of equation ① is

$$y = cx + f(c)$$

Remarks Using the equation $x + f'(p) = 0$ and

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the equation ① we obtain another solution ④ of the given differential equation. This solution is not included in the general solution equation ②. Such a solution is called a singular solution.

Solve $y = (x-a)p - p^2$

$$y = (x-a)p - p^2 = px - ap - p^2$$

$$y = px + f(p) \quad \text{where } f(p) = -ap - p^2$$

The general solution is

$$y = cx + f(c)$$

$$= cx + (-ca - c^2) \longrightarrow ①$$

To find the singular solution we differentiating the general solution ① w.r.to c

$$\therefore 0 = x - a - 2c$$

$$\therefore c = \frac{x-a}{2} \longrightarrow ②$$

Substituting equation ② in equation ①, we get the singular solution as

$$y = \left(\frac{x-a}{2}\right)x - a\left(\frac{x-a}{2}\right) - \left(\frac{x-a}{2}\right)^2$$

$$= \left(\frac{x-a}{2}\right)\left[x-a - \frac{x-a}{2}\right]$$

$$= \frac{(x-a)^2}{4}$$

$\therefore 4y = (x-a)^2$ which is the singular solution.

Solve $\cancel{x^2(y-px) = yp^2}$

$\sin px \cos y = \cos px \sin y + p$

or, $p = \sin(px - y)$

or, $px - y = \sin^{-1} p$

or, $y = px - \sin^{-1} p$

The general solution is $y = cx - \sin^{-1} c$

Solve $p = \log(px - y)$

or, $px - y = e^p$

or $y = px - e^p$

$\frac{dy}{dx} = \cancel{p} + x\frac{dp}{dx} - e^p\frac{dp}{dx}$

or, $\frac{dp}{dx}(x - e^p) = 0$

$\Rightarrow \frac{dp}{dx} = 0$

$\Rightarrow p = c$

$\therefore y = cx - e^c$

This is the general solution.

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$x = e^p \Rightarrow p = \log x$

$y = p e^p - e^p$
$= e^p(p-1)$

Eliminating p

$y = x(p-1)$
$= x(\log x - 1)$

This is the singular solution.

Linear equations of Higher Order

A linear equation of n^{th} order with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = X \quad \text{--- (1)}$$

where $a_1, a_2, a_3, \ldots$ are constants and X is a function of x. This equation can also be written in the form

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} \cdots + a_n) y = X \quad \text{--- (2)}$$

where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$ ... $D^n = \frac{d^n}{dx^n}$

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Consider $(D^n + a_1 D^{n-1} + a_2 D^{n-2} - - + a_n) y = 0$

The general solution of equation ② is given by

$Y = C_1 Y_1 + C_2 Y_2 + -- + C_n Y_n$

where $C_1, C_2, -- C_n$ are arbitrary constants and $Y_1, Y_2, -- Y_n$ are n independent solutions

Y is called the complementary function (C.F.) of the equation ①

Suppose u is a particular solution of eqn ①.

Then the general solution of equation ① is of the form $y = Y + u$ where Y is the complementary function and u is a particular integral (P.I.)

Thus $y = C.F + P.I$.

Method of Finding Complementary Function

Consider the differential equation

$y'' + ay' + by = 0$ — ①

Let $y = e^{mx}$ be a trial solution of equation (1)

of (1).

$$e^{mx}(m^2 + am + b) = 0$$

Since $e^{mx} \neq 0$

$$\therefore m^2 + am + b = 0 \quad \text{——— (2)}$$

Equ^n (2) is called the auxiliary equation (A.E.)

Case I If Roots of (2) are real and distinct

Then $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ is the general solution of (1).

Case II Roots of the (2) are real and unequal

$\therefore y = (c_1 + c_2 x) e^{mx}$ is the general solution of (1)

Case III Roots of A.E. are imaginary.

Let them be $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

The general solution of the equ^n (2) is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

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Results

1. If all the roots $m_1, m_2, \dots m_n$ are distinct and real then C.F. of ① is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots c_n e^{m_n x}$$

2. If k roots, say $m_1, m_2, \dots m_k$ are real and equal then the corresponding part of C.F. is given by

$$y = (c_1 + c_2 x + c_3 x^2 - \dots + c_k x^{k-1}) e^{m_1 x}$$

3. If $\alpha + i\beta$ is a complex root which occurs k times, the corresponding part of the C.F is given by

$$e^{\alpha x}\left\{\left[(c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1})\cos\beta x + (c_{k+1} + \dots + c_{2k} x^{k-1})\sin\beta x\right]\right\}$$

Type A

X is of the form e^{ax}

$D^n(e^{ax}) =$

(1) Solve $(D^2-5D+6)y=0$

The auxiliary equation is

$$m^2-5m+6=0$$

or, $m=2,3$

$$C.F = c_1e^{2x}+c_2e^{3x}$$

The general solution is $y=c_1e^{2x}+c_2e^{3x}$

(2) Solve $(D^3+D^2+D+1)y=0$

The auxiliary equation is

$$m^3+m^2+m+1=0$$

i.e. $(m+1)(m^2+1)=0$

or, $m=-1, \pm i$

The general solution is

$$y=c_1e^{-x}+c_2\cos x+c_3\sin x$$

(3) Solve $(D^2+D+1)^2y=0$

The auxiliary equn is $(m^2+m+1)^2=0$

$m = \frac{-1 \pm i\sqrt{3}}{2}$ (twice)

C.F $y = e^{-x/2}\left[(c_1+c_2x)\cos\frac{\sqrt{3}}{2}x + (c_3+c_4x)\sin\frac{\sqrt{3}}{2}x\right]$

Methods of finding Particular Integral

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} \cdots + a_n)y = X$

The equation can be written as

$f(D)\,y = X$

where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} \cdots + a_n$

Type A

X is of the form e^{ax}

since $D^n(e^{ax}) = a^n e^{ax}$

$\therefore \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ (provided $f(a) \neq 0$)

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Suppose $f(a) = 0$

$$\therefore \frac{1}{f(D)} e^{ax} = e^{ax} \frac{1}{f(D+a)} 1$$

// Solve $(D-2)^2 y = e^{2x}$

The auxiliary equ^n is

$$(m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

$$C.F = e^{2x}(c_1 + c_2 x)$$

$$P.I = \frac{1}{(D-2)^2} e$$

// Solve $(D^2+16) y = e^{-4x}$

$$P.I = \frac{1}{D^2+16} e^{-4x}$$

$$= \frac{e^{-4x}}{(-4)^2+16} = \frac{e^{-4x}}{32}$$

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Type B

X is of the form $\sin ax$ or $\cos ax$

Let $X = \sin ax$

We note that $\left[D^2(\sin ax)\right]^n = (-a^2)^n \sin ax$

$\therefore$ If $\phi(D^2)$ is a rational integral function of D^2

$$\phi(D^2)\sin ax = \phi(-a^2)\sin ax$$

$$\therefore \left[\frac{1}{\phi(D^2)}\right]\sin ax = \frac{1}{\phi(-a^2)}\sin ax \quad \text{If } \phi(-a^2) \neq 0$$

Thus the effect of $\frac{1}{\phi(D^2)}$ on $\sin ax$ is to replace D^2 by $-a^2$ if $\phi(-a^2) \neq 0$

Suppose $\phi(-a^2) = 0$. Hence D^2+a^2 is a factor of $\phi(D^2)$.

Let $\phi(D^2) = (D^2+a^2)\psi(D^2)$ and assume that $\psi(-a^2) \neq 0$

$$\text{Now } \left[\frac{1}{\phi(D^2)}\right]\sin ax = \frac{1}{(D^2+a^2)\psi(D^2)}\sin ax$$

$$= \frac{1}{\psi(-a^2)}\left(\frac{1}{D^2+a^2}\right)\sin ax$$

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Now $\left[\frac{1}{D^2+a^2}\right] \sin ax$

$= \frac{1}{D^2+a^2}$ (Imaginary part of e^{iax})

$=$ Imaginary part of $\frac{1}{D^2+a^2} e^{iax}$

Now $\left[\frac{1}{D^2+a^2}\right] e^{iax} = \frac{1}{(D+ia)(D-ia)} e^{iax}$

$$= \frac{1}{2ai}\left(\frac{1}{D-ia}\right) e^{iax}$$

$$= \frac{x e^{iax}}{2ai} = \frac{-ix e^{iax}}{2a}$$

$$= \frac{-ix}{2a}\left(\cos ax + i \sin ax\right)$$

$$\therefore \left[\frac{1}{\phi(D^2)}\right] \sin ax = \frac{1}{\phi(-a^2)}\left(\frac{-x\cos ax}{2a}\right)$$

__Note:__ The procedure is similar if $X = \cos ax$

__Type C__

X is of the form x^m (m positive integer)

Expand $\left[\frac{1}{f(D)}\right]$ in ascending powers of D as far as D^m and operate on x^m.

Type D

$X = e^{ax} v$ where v is any function of x.

We note that

$$\frac{1}{f(D)}\left(e^{ax} v\right) = e^{ax} \frac{1}{f(D+a)} v$$

Solve $(D^2-4)y = e^{2x} + e^{-4x}$

The auxiliary equⁿ is $m^2-4=0$

$\therefore m = \pm 2$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

$$\therefore \text{P.I} = \frac{1}{D^2-4}\left(e^{2x} + e^{-4x}\right)$$

$$= \frac{1}{D^2-4} e^{2x} + \frac{1}{D^2-4} e^{-4x}$$

$$= \frac{e^{2x}}{(2+2)(D-2)} + \frac{e^{-4x}}{4^2-4}$$

$$= \frac{1}{4}\frac{1}{D-2} e^{2x} + \frac{e^{-4x}}{12}$$

$$= \frac{1}{4} e^{2x} \frac{1}{D+2-2} 1 + \frac{e^{-4x}}{12}$$

$$= \frac{xe^{2x}}{4} + \frac{1}{12} e^{-4x}$$

The solution $y =$ C.F + P.I.

$$= c_1 e^{2x} + c_2 e^{-2x} + \frac{xe^{2x}}{4} + \frac{1}{12} e^{-4x}$$

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Solve $y'' + 4y' + 13y = 2e^{-x}$ given $y(0) = 0,\ y'(0) = -1$ (4)

The auxiliary equ^n is

$$m^2 + 4m + 13 = 0$$

$$\therefore m = -2 \pm 3i$$

$$\therefore C.F = e^{-2x}\left(c_1 \cos 3x + c_2 \sin 3x\right)$$

$$\therefore P.I = \frac{1}{D^2 + 4D + 13}\, 2e^{-x}$$

$$= \frac{2e^{-x}}{1 - 4 + 13} = \frac{1}{5} e^{-x}$$

The general solution is $y = C.F. + P.I.$

$$= e^{-2x}\left(c_1 \cos 3x + c_2 \sin 3x\right) + \frac{1}{5} e^{-x}$$

Now $y(0) = 0 \Rightarrow c_1 = -\frac{1}{5}$

$$y' = e^{-2x}\left[-3c_1 \sin 3x + 3c_2 \cos 3x\right] - 2e^{-2x}\left(c_1 \cos 3x + c_2 \sin 3x\right) - \frac{1}{5} e^{-x}$$

$$\therefore y'(0) = -1 \Rightarrow -1 = 3c_2 - 2c_1 - \frac{1}{5}$$

Hence $c_2 = -\frac{2}{5}$

$$\therefore y = -\frac{1}{5} e^{-2x}\left(\cos 3x + 2\sin 3x\right) + \frac{1}{5} e^{-x}$$

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Solve $(D^3+3D^2+3D+1)y = e^{-x}$

The auxiliary equ^n is

$m^3+3m^2+3m+1=0$

$\therefore (m+1)^3 = 0$

$\Rightarrow m = -1, -1, -1$

$\therefore C.F. = (c_1+c_2x+c_3x^2)e^{-x}$

$\therefore P.I. = \frac{1}{(D+1)^3} e^{-x}$

$= e^{-x} \frac{1}{(D-1+1)^3} 1$

$= e^{-x} \frac{1}{D^3} 1$

$= \frac{x^3 e^{-x}}{3!}$

$\therefore$ The general solution is

$y = (c_1+c_2x+c_3x^2)e^{-x} + \frac{x^3e^{-x}}{3!}$

Solve $(D^2+D+1)y = \sin 2x$

The auxiliary equation is

$m^2+m+1=0$

$\therefore m = \frac{-1 \pm i\sqrt{3}}{2}$

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$\therefore$ C.F. $= e^{-x/2}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$\therefore$ P.I. $= \dfrac{1}{D^2+D+1} \sin 2x$

$= \dfrac{1}{-4+D+1} \sin 2x$

$= \dfrac{1}{-3+D} \sin 2x$

$= \dfrac{D+3}{D^2-9} \sin 2x$

$= (D+3) \dfrac{\sin 2x}{-4-9}$

$= \dfrac{2\cos 2x}{13} - \dfrac{3}{13} \sin 2x$

$\therefore$ The general solution is

$y = $ C.F. + P.I.

Solve $\quad (D^2+1)y = x$

The auxiliary equ^n is

$m^2+1=0$

$\therefore m = \pm i$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

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$\therefore P.I = \frac{1}{D^2+1} x$

$= (1+D^2)^{-1} x$

$= (1-D^2) x$ (Using binomial expansion)

$= x$

$\therefore$ The general solution is $y = C.F. + P.I.$

$= c_1 \cos x + c_2 \sin x + x$

<u>Solve</u> $(D^2+9) y = \cos 3x$

The auxiliary eqn is $m^2+9=0$

$\therefore m = \pm 3i$

$\therefore C.F. = c_1 \cos 3x + c_2 \sin 3x$

$\therefore P.I. = \frac{1}{D^2+9} \cos 3x$

$=$ Real part of $\left(\frac{1}{D^2+9}\right) e^{i3x}$

$=$ Real part of $\frac{1}{(D+3i)(D-3i)} e^{i3x}$

$=$ Real part of $\frac{1}{6i} \frac{1}{D-3i} e^{i3x}$

$=$ Real part of $\frac{1}{6i(-3i)} \left(e^{i3x} \frac{1}{D+\cancel{3i-3i}} 1 \right.$

$=$ Real part of $\frac{1}{18} e^{i3x} \frac{1}{D} 1$

$=$ Real part of $\frac{x e^{i3x}}{18}$

$\therefore P.I = \dfrac{x e^{3x}}{18}$

$\therefore$ The general solution is

$$Y = C.F + P.I$$

<u>Solve</u> $(D^2+3D+2)\,Y = x^2$

The auxiliary equ^n is $m^2+3m+2=0$

or, $m = -2, -1$

$$\therefore C.F = c_1 e^{-2x} + c_2 e^{-x}$$

$$\therefore P.I. = \frac{1}{(D+2)(D+1)}\,x^2$$

$$= \frac{1}{(D^2+3D+2)}\,x^2$$

$$= \frac{1}{2}\left[1+\frac{D^2+3D}{2}\right]^{-1} x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2+3D}{2}+\left(\frac{3D+D^2}{2}\right)^2\right] x^2$$

$$= \frac{1}{2}\left(1-\frac{3D+D^2}{2}+\frac{9D^2}{4}\right) x^2$$

$$= \frac{1}{2}\left(x^2-\frac{3}{\cancel{2}}.\cancel{2}x-\frac{1}{2}.2+\frac{9}{4}.2\right)$$

$$= \frac{1}{2}\left(x^2-3x-1+\frac{9}{2}\right)$$

$$= \frac{1}{2}\left(x^2-3x+\frac{7}{2}\right)$$

$\therefore$ The general solution is

$$y = C.F. + P.I$$

Solve Find the PI of $(D^2-4D+3)y = e^x \cos 2x$

$$\therefore P.I. = \frac{1}{D^2-4D+3} e^x \cos 2x$$

$$= e^x \frac{1}{(D+1)^2 - 4(D+1)+3} \cos 2x$$

$$= e^x \frac{1}{D^2+2D+1-4D-4+3} \cos 2x$$

$$= e^x \frac{1}{D^2-2D} \cos 2x$$

$$= e^x \frac{1}{-4-2D} \cos 2x$$

$$= -e^x \frac{1}{2(D+2)} \cos 2x$$

$$= -\frac{e^x}{2} \frac{D-2}{D^2-4} \cos 2x$$

$$= \frac{-e^x (D-2) \cos 2x}{2(-4-4)}$$

$$= \frac{e^x}{8\times 2}(-2\sin 2x - 2\cos 2x)$$

$$= -\frac{e^x}{8 \cancel{\times 2}} \cancel{2}(\sin 2x + \cos 2x) = \frac{-e^x}{8}(\sin 2x + \cos 2x)$$

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The general solution is $y = C.F + P.I.$

Solve $(D^2-2D+2)y = e^x \sin x$

The auxiliary equn is $m^2-2m+2=0$

or, $m = 1 \pm i$

$\therefore$ C.F. $= e^x (c_1 \cos x + c_2 \sin x)$

$$P.I. = \frac{1}{D^2-2D+2} e^x \sin x$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+2} \sin x$$

$$= e^x \frac{1}{D^2+1} \sin x$$

$$= e^x \frac{1}{D^2+1} \sin x$$

$$= e^x \text{ Imaginary part of } \left[\frac{1}{(D+i)(D-i)}\right] e^{ix}$$

$$= e^x \text{ Imaginary part of } \frac{1}{2i} \frac{1}{D-i} e^{ix}$$

$$= e^x \text{ Imaginary part of } \frac{1}{2i} e^{ix} \frac{1}{D+\cancel{i}-\cancel{i}} 1$$

$$= e^x \text{ Imaginary part of } \frac{1}{2i} e^{ix} x$$

$$= e^x \text{ Imaginary part of } \frac{2i}{-4} x (\cos x + i \sin x)$$

$$= e^x \frac{-1}{2} x \cos x$$

$= -\frac{1}{2} x e^x \cos x$

$\therefore$ The general solution is

$$y = C.F + P.I$$

Solve $(D^3 - 3D^2 + 3D - 1)y = x^2 e^x$

The auxiliary equation is

$$m^3 - 3m^2 + 3m + 1 = 0$$

or $(m+1)^3 = 0$

or $m = -1, -1, -1$

$\therefore C.F = e^{-x}(c_1 + c_2 x + c_3 x^2)$

$$\therefore P.I. = \frac{1}{(D-1)^3} x^2 e^x$$

$$= e^x \frac{1}{(D+\cancel{1}-\cancel{1})^3} x^2$$

$$= e^x \frac{1}{D^3} x^2$$

$$= e^x \frac{x^5}{60}$$

$\therefore$ The general solution is

$$y = C.F + P.I.$$

$$\therefore y = e^{-x}(c_1 + c_2 x + c_3 x^2) + \frac{x^5 e^x}{60}$$

Solve $(D^2-4D+4)y = 3x^2e^{2x}\sin 2x$ (12)

Sol The auxiliary equation is

$$m^2-4m+4=0$$

or, $(m-2)^2=0$

or, $m=2,2$

$\therefore$ C.F $=(C_1+C_2x)e^{2x}$

$\therefore$ P.I $=\dfrac{1}{(D-2)^2}\left(3x^2e^{2x}\sin 2x\right)$

$$=3e^{2x}\frac{1}{(D+\cancel{2}-\cancel{2})^2}x^2\sin 2x$$

$$=3e^{2x}\frac{1}{D^2}x^2\sin 2x$$

$=3e^{2x}\times$ Imaginary part of $\dfrac{1}{D^2}x^2e^{i2x}$

$=3e^{2x}$ Imaginary part of $e^{i2x}\dfrac{1}{(D+2i)^2}x^2$

$=3e^{2x}$ Imaginary part of $e^{i2x}\dfrac{1}{\cancel{D^2}-4\left(1+\frac{D}{2i}\right)^2}x^2$

$=3e^{2x}$ Imaginary part of $e^{i2x}\left(-\frac{1}{4}\right)\left(1-\frac{2D}{2i}-\frac{3D^2}{4}\right)x^2$

$=3e^{2x}$ Imaginary part of $e^{i2x}\left(-\frac{1}{4}\right)\left(x^2-\frac{2x}{i}-\frac{3}{4}\cdot 2\right)$

$=3e^{2x}$ Imaginary part of $e^{i2x}\left(-\frac{1}{4}x^2+\frac{2}{i}x+\frac{3}{8}\right)$

$=3e^{2x}$ Imaginary part of $e^{i2x}\left(-\frac{1}{4}x^2-2ix+\frac{3}{8}\right)$

$= 3e^{2x}$ Imaginary part of

$$\left(\cos 2x + i\sin 2x\right)\left(-\frac{1}{4}x^2 + \frac{3}{8} - 2ix\right)$$

$$= 3e^{2x}\left(-2x\cos 2x + \left(-\frac{1}{4}x^2 + \frac{3}{8}\right)\sin 2x\right)$$

$$= 3e^{2x}\left[\left(-\frac{x^2}{4} + \frac{3}{8}\right)\sin 2x - 2x\cos 2x\right]$$

The general solution is

$$y = C.F + P.I.$$

Homogeneous Linear Equations

Consider a differential equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_{n-1} x \frac{dy}{dx} + P_n y = X \quad \text{--- (1)}$$

Where $P_1, P_2, \ldots P_n$ are constants and X is a function of x. This equation is called a homogeneous linear equation.

Equation (1) can be transformed to a linear equation with constant coefficients by the substitution $z = \log x$ i.e $x = e^z$

$\therefore$ We have $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\therefore xDy = \theta y, \text{ where } D = \frac{d}{dx}, \quad \theta = \frac{d}{dz}$$

Differentiating with respect to x again we get $$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\therefore x^2D^2y = (\theta^2 - \theta)y = \theta(\theta - 1)y$$

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Similarly $x^3D^3y = \theta(\theta-1)(\theta-2)y$

$\vdots$

$$x^nD^ny = \theta(\theta-1)(\theta-2)\cdots(\theta-n+1)\,y$$

Equation (1) can be transformed as

$$\left[\theta(\theta-1)(\theta-2)\cdots(\theta-n+1) + P_1\theta(\theta-1)\cdots(\theta-n+2) + \cdots + P_{n-1}\theta + P_n\right] y = Z \quad \text{(2)}$$

where Z is a function of z obtained from x by putting $x = e^z$

Hence equ^n (2) is a linear equation with constant coefficients and the complementary function can be found by the methods described in previous.

Solve $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} - 5y = \sin(\log x)$

Sol

Let $z = \log x$ and $\theta = \dfrac{d}{dz}$

$\therefore$ The given equation reduces to

$$\left[\theta(\theta-1) - 3\theta - 5\right] y = \sin z$$

i.e. $(\theta^2 - 4\theta - 5)\,y = \sin z$

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The auxiliary equation is

$$m^2-4m-5=0$$

or, $(m-5)(m+1)=0$

or, $m=5,-1$

$$\therefore C.F. = c_1e^{5z}+c_2e^{-z}$$

$$= c_1e^{5\log x}+c_2e^{-\log x}$$

$$= c_1x^5+c_2x^{-1}$$

$$\therefore P.I. = \left[\frac{1}{\theta^2-4\theta-5}\right]\sin z$$

$$= \frac{1}{-1-4\theta-5}\sin z$$

$$= \frac{1}{-2(3+2\theta)}\sin z$$

$$= \frac{2\theta-3}{-2(4\theta^2-9)}\sin z$$

$$= \frac{2\theta-3}{-2(-4-9)}\sin z$$

$$= \frac{2\theta-3}{26}\sin z$$

$$= \frac{2\cos z-3\sin z}{26}$$

$$= \frac{1}{13}\cos(\log x)-\frac{3}{26}\sin(\log x)$$

The general solution $y = C.F + P.I.$

Solve $x^2y'' - xy' + 4y = \cos(\log x) + x\sin(\log x)$

Sol Put $z = \log x$ and $\theta = \frac{d}{dz}$

$\therefore$ The given equation reduces to

$$[\theta(\theta-1) - \theta + 4]y = \cos z + e^z \sin z$$

or, $(\theta^2 - 2\theta + 4)y = \cos z + e^z \sin z$

Let $(\theta^2 - 2\theta + 4)y = 0$

$\therefore$ The auxiliary equation is

$m^2 - 2m + 4 = 0$

or, $m = 1 \pm i\sqrt{3}$

Hence $C.F. = e^z (c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z)$

$= x[c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x)]$

P.I.

$$= \frac{1}{\theta^2 - 2\theta + 4} \cos z + \frac{1}{\theta^2 - 2\theta + 4} e^z \sin z$$

$$= \frac{1}{-1^2 - 2\theta + 4} \cos z + e^z \frac{1}{(\theta+1)^2 - 2(\theta+1) + 4} \sin z$$

$$= \frac{1}{3 - 2\theta} \cos z + e^z \frac{1}{\theta^2 + 3} \sin z$$

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$$= \frac{1}{3-2\theta}\cos z + e^{z}\frac{1}{\theta^2+3}\sin z$$

$$= \frac{3+2\theta}{9-4\theta^2}\cos z + e^{z}\frac{\sin z}{-1^2+3}$$

$$= \frac{3+2\theta}{9+4}\cos z + e^{z}\frac{\sin z}{2}$$

$$= \frac{1}{13}(3\cos z + 2\sin z) + \frac{1}{2}e^{z}\sin z$$

$$= \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{1}{2}x\sin(\log x)$$

$\therefore$ The solution is $y = C.F + P.I.$

Solve $(x^2D^2 + 2xD + 4)y = x^2 + 2\log x$

Put $z = \log x$ and $\theta = \frac{d}{dz}$

Then the given equation reduces to

$$[\theta(\theta-1) + 2\theta + 4]\, y = e^{2z} + 2z$$

i.e. $(\theta^2 + \theta + 4)y = e^{2z} + 2z$

The auxiliary equ^n is

$$m^2 + m + 4 = 0$$

$$\therefore m = \frac{-1 \pm \sqrt{1-16}}{2} = \frac{-1 \pm i\sqrt{15}}{2}$$

$$\therefore C.F. = e^{-z/2}\left[c_1\cos\frac{\sqrt{15}}{2}z + c_2\sin\frac{\sqrt{15}}{2}z\right]$$

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$$= x^{-1/2}\left[c_1 \cos\left(\frac{\sqrt{15}}{2}\right)\log x + c_2 \sin\left(\frac{\sqrt{15}}{2}\right)\log x\right]$$

$$\therefore P.I. = \left[\frac{1}{\theta^2+\theta+4}\right]\left(e^{2z}+2z\right)$$

$$= P.I._1 + P.I._2$$

$P.I_1$

$$= \frac{1}{\theta^2+\theta+4} e^{2z} = \frac{e^{2z}}{10} = \frac{x^2}{10}$$

$$P.I._2 = \frac{1}{\theta^2+\theta+4} 2z$$

$$= \frac{1}{4\left(1+\frac{\theta^2+\theta}{4}\right)} 2z$$

$$= \frac{1}{2}\left(1+\frac{\theta^2+\theta}{4}\right)^{-1} z$$

$$= \frac{1}{2}\left(1-\frac{\theta^2+\theta}{4}\right) z$$

$$= \cancel{\frac{1}{2}\left(1-\frac{1}{4}\right)} = \frac{1}{2}z - \frac{1}{8}$$

$$= \frac{1}{2}\log x - \frac{1}{8}$$

$\therefore$ The general solution is

$$y = C.F + P.I$$

Solve: Solve the equation $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = x\cos(\log x)$.

Solve: $z = \log x$ and $\theta = \frac{d}{dz}$

Soln Put $z = \log x$ and $\theta = \frac{d}{dz}$

Then the given equation becomes

$$\left[\theta(\theta-1) + 3\theta + 5\right] y = e^{z} \cos z$$

$$\text{or } (\theta^2 + 2\theta + 5) y = e^{z} \cos z$$

The auxiliary eqn is $m^2 + 2m + 5 = 0$

$$\therefore m = -1 \pm 2i$$

$$\therefore C.F. = e^{-z}(c_1 \cos 2z + c_2 \sin 2z)$$

$$= x^{-1}\left(c_1 \cos(2\log x) + c_2 \sin(2\log x)\right)$$

$$P.I. = \frac{1}{\theta^2 + 2\theta + 5}(e^{z} \cos z)$$

$$= e^{z} \frac{1}{(\theta+1)^2 + 2(\theta+1) + 5} \cos z$$

$$= e^{z} \frac{1}{\theta^2 + 4\theta + 8} \cos z$$

$$= e^{z} \frac{1}{-1 + 4\theta + 8} \cos z$$

$$= e^{z} \frac{1}{7 + 4\theta} \cos z$$

$$= e^{z} \frac{4\theta - 7}{16\theta^2 - 49} \cos z$$

$$= e^{z} \frac{4\theta - 7}{-65} \cos z$$

$$= \frac{e^{z}}{-65}\left[4 \sin z - 7 \cos z\right]$$

$$= -\frac{4x}{65} \sin(\log x) + \frac{7x}{65} \cos(\log x)$$

$\therefore$ general solution is

$$y = C.F + P.I.$$

Solve Solve the $(2x+1)^2 y'' - 2(2x+1)y' - 12y = 6x$

Put $2x+1 = z$

Hence $\frac{dz}{dx} = 2$ and $x = \frac{z-1}{2}$

Now $y' = \frac{dy}{dz}\frac{dz}{dx} = 2\frac{dy}{dz}$

and $y'' = 2\frac{d^2y}{dz^2}\frac{dz}{dx} = 4\frac{d^2y}{dz^2}$

Hence the given equation reduces to a linear homogeneous equation

$$4z^2\frac{d^2y}{dz^2} - 4z\frac{dy}{dz} - 12y = 6\left(\frac{z-1}{2}\right) \quad \text{--- (1)}$$

Now put $u = \log z$ and $\theta = \frac{d}{du}$ and hence the equⁿ (1) reduces to

$$([4\theta(\theta-1) - 4\theta + 2)]y = 3(e^u - 1)$$

$$\Rightarrow (4\theta^2 - 8\theta - 12)y = 3(e^u - 1)$$

$$\Rightarrow (\theta^2 - 2\theta - 3)y = \frac{3}{4}(e^u - 1)$$

The auxiliary equⁿ is $m^2 - 2m - 3 = 0$

$\therefore (m-3)(m+1) = 0$ Hence $m = 3, -1$

$$\therefore C.F. = c_1 e^{3u} + c_2 e^{-u} = c_1 e^{3\log z} + c_2 e^{-\log z}$$
$$= c_1 z^3 + c_2 z^{-1} = c_1 (2x+1)^3 + c_2 (2x+1)^{-1}$$

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$\therefore$ P.I. $= \frac{3}{4}\left[\frac{1}{\theta^2-2\theta-3}\right](e^u-1)$

$= \frac{3}{4}\left(\frac{e^u}{-4}+\frac{1}{3}\right) = \frac{3}{4}\left(-\frac{z}{4}+\frac{1}{3}\right)$

$= \frac{3}{4}\left(\frac{2x+1}{-4}+\frac{1}{3}\right) = -\frac{3}{8}x+\frac{1}{16}$

$\therefore$ Hence the solution is $y =$ C.F. + P.I.

The Method of Variation of Parameters

The method of variation of parameters gives a method to determine a particular solution of the equation

$$y''+Py'+Qy = X \quad \text{——— (1)}$$

when the complementary function (C.F.) is known.

Let $y = au + bv$ where a, b are constants, be the known C.F. of equation (1).

We replace the constants a and b by unknown functions ϕ and ψ respectively which are functions of x and determine ϕ and ψ in such a way that

$y = \phi u + \psi v$ —(2) is a P.I. of equation (1).

We have $y' = (\phi u' + \psi v') + (\phi' u + \psi' v)$

Choose ϕ and ψ such that

$$\phi' u + \psi' v = 0 \quad \text{——— (3)}$$

Then $y' = \phi u' + \psi v'$ ——— (4)

$$y'' = \phi u'' + u'\phi' + \psi' v' + \psi v'' \quad \text{——— (5)}$$

Substituting eqn (2), (4) and (5) in eqn (1) and rearranging the terms we get

$$\phi(u'' + Pu' + Qu) + \psi(v'' + Pv' + Qv) + (\phi' u' + \psi' v') = X$$

$$\therefore \phi' u' + \psi' v' = X \quad \text{——— (6)}$$

(since u, v satisfy $y'' + Py' + Qy = 0$)

solving for ϕ' and ψ' from equations (3) and (6) we get

$$\phi' = \frac{-vX}{uv' - u'v}, \quad v' = \frac{uX}{uv' - u'v}$$

Integrating the above equations we can find ϕ and ψ.

$\therefore$ The general solution is $y = \phi u + \psi v$

Solve Using the method of variation of parameters solve $(D^2+1)\,y = 0$ (11)

Sol The auxiliary eqn is $m^2+1=0$

$$m = \pm i$$

$\therefore$ The $C.F = a\cos x + b\sin x$

Let $u = \cos x$, $v = \sin x$

$\therefore$ The general solution is given by

$$y = \phi u + \psi v$$

$$= \phi\cos x + \psi\sin x \qquad \text{--- (1)}$$

where ϕ and ψ are functions of x determined by the equations

$$\phi' u + \psi' v = 0 \quad \text{and} \quad \phi' u' + \psi' v' = x$$

Now $\phi' u + \psi' v = 0 \Rightarrow \phi'\cos x + \psi'\sin x = 0$ --- (2)

$\phi' u' + \psi' v' = x \Rightarrow -\phi'\sin x + \psi'\cos x = x$ --- (3)

(2) $\times\sin x$ + (3) $\times\cos x \Rightarrow \psi' = x\cos x$

Substituting in eqn (2) we get $\phi' = -x\sin x$

Now $\phi = -\int x\sin x\,dx = \int x\,d(\cos x)$

$$= x\cos x - \int \cos x\,dx$$

$$= x\cos x - \sin x + c_1$$

$\psi = \int x \cos x \, dx$

$= x \sin x + \cos x + c_2$

$\therefore$ The required solution (1) is given by

$y = \phi u + \psi v$

$= (x \cos x - \sin x + c_1) \cos x + (x \sin x + \cos x + c_2) \sin x$

$= x + c_1 \cos x + c_2 \sin x$

Solve $\quad \frac{d^2 y}{dx^2} + 4y = \tan 2x$

$m^2 + 4 = 0$

$m = \pm 2i$

The C.F $= a \cos 2x + b \sin 2x$

$u = \cos 2x, \quad v = \sin 2x$

The general solution is given by

$y = \phi u + \psi v$

$= \phi \cos 2x + \psi \sin 2x$

where ϕ and ψ are function of x determined by the equ^{n}s

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$\psi = \int x \cos x \, dx$

$= x \sin x + \cos x + c_2$

$\therefore$ The required solution ① is given by

$y = \phi u + \psi v$

$= (x \cos x - \sin x + c_1) \cos x + (x \sin x + \cos x + c_2) \sin x$

$= x + c_1 \cos x + c_2 \sin x$

Solve $\quad \dfrac{d^2y}{dx^2} + 4y = \tan 2x$

$m^2 + 4 = 0$

$m = \pm 2i$

The C.F $= a \cos 2x + b \sin 2x$

$u = \cos 2x, \quad v = \sin 2x$

The general solution is given by

$y = \phi u + \psi v$

$= \phi \cos 2x + \psi \sin 2x$

where ϕ and ψ are function of x determined by the equ^{ns}