Laplace Transform

Def Let $f(t)$ be a function defined on $[0,\infty)$. The Laplace Transform L is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

The Laplace Transform L acts on any function $f(t)$ for which the above integral exists.

$\int_0^\infty e^{-st} f(t)\,dt$ is a function of s and is denoted by $F(s)$. Thus

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Def When $f(t)$ is continuous and $L[f(t)] = F(s)$ we have $L^{-1}[F(s)] = f(t)$ and L^{-1} is the inverse laplace transform and $f(t)$ is the inverse Laplace transform of $F(s)$

Sufficient condition for existence of $L[f(t)]$

It is not true that the Laplace transform exists for all functions. For example $L\left[\frac{1}{t}\right]$ and $L\left[e^{t^2}\right]$ do not exist

Def A function f is said to be piecewise continuous on $[0, \infty)$ if it in any interval $0 \le a \le t \le b$, there are at most a finite number of points t_k, $k = 1, 2, \ldots n$ $(t_{k-1} < t_k)$, at which f has finite discontinuity and is continuous on each open interval $t_{k-1} < t < t_k$.

A function f is said to be of exponential order c, where $c > 0$, if there exist constants $M > 0$ and $T > 0$ such that $|f(t)| \le M e^{ct}$, for all $t > T$.

Ex 1. $f(t) = t$ is of exponential order and $c = 1$ and for $t > 0$, since $|t| \le e^t$.

2. $f(t) = t^2$ is of exponential order since

$$\lim_{t \to \infty} \frac{t^2}{e^{ct}} = \lim_{t \to \infty} \frac{2t}{c e^{ct}} = \lim_{t \to \infty} \frac{2}{c^2 e^{ct}} \quad \left[\text{by L'Hospital's Rule}\right]$$

$$= 0 \text{ if } c > 0$$

3. $f(t) = e^{t^2}$ is not of expontial order since

$$\lim_{t \to \infty} \frac{e^{t^2}}{e^{ct}} = \lim_{t \to \infty} e^{t^2 - ct} = \infty \text{ for any value of } c$$

Th Sufficient Conditions for existance of Laplace Transform ③

If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and is of exponential order c, then $L[f(t)]$ exists for $s > c$.

Note These conditions are not necessary for the existence of a Laplace transform.

For example the function $f(t) = t^{-1/2}$ is not piecewise continuous on $[0, \infty)$ but its Laplace transform exists.

Laplace Transform of some Standard Functions

1. $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

2. $L(1) = \frac{1}{s}$

2. $L(e^{at}) = \frac{1}{s-a}$ if $s-a>0$

$$L(e^{at}) = \int_0^\infty e^{at} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt = \lim_{m\to\infty} \int_0^m e^{-(s-a)t}\, dt$$

$$= \lim_{m \to \infty} \left[-\frac{e^{-(s-a)m}}{s-a} + \frac{1}{s-a} \right]$$ (4)

$$= \frac{1}{s-a} \text{ if } s-a > 0$$

Corollary 1. $L(e^{-at}) = \frac{1}{s+a}$ if $s+a > 0$

Corollary 2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$ and $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

Result 3. $L(\cos at) = \frac{s}{s^2+a^2}$

$$L(\cos at) = \text{Real part of } \int_0^\infty e^{-st} e^{iat}\, dt$$

$$= \text{Real part of } \left(\frac{1}{s-ai}\right)$$

$$= \text{Real part of } \frac{s+ai}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$

$$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

Result 4 $L(\sin at) = \frac{a}{s^2+a^2}$

Result 5 $L(\sinh at) = \frac{a}{s^2-a^2}$

$$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$$

$L(\cosh at) = \frac{s}{s^2-a^2}$

Solve Find the Laplace Transform of

$t^2 + \cos 2t \cos t + \sin^2 2t$

$L(t^2 + \cos 2t \cos t + \sin^2 2t)$

$= L(t^2) + L\left[\frac{1}{2}(\cos 3t + \cos t)\right] + L\left[\frac{1}{2}(1 - \cos 4t)\right]$

$= \frac{2}{s^3} + \frac{1}{2}\left[L(\cos 3t) + L(\cos t)\right]$

$+ \frac{1}{2}\left[L(1) - L(\cos 4t)\right]$

$= \frac{2}{s^3} + \frac{1}{2}\left[\frac{s}{s^2+9} + \frac{s}{s^2+1}\right] + \frac{1}{2}\left[\frac{1}{s} - \frac{4}{s^2+16}\right]$

Solve Find the Laplace Transform of

$$f(t) = \begin{cases} e^{-t} & 0 < t < 4 \\ 0 & t \geq 4 \end{cases}$$

$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$= \int_0^4 e^{-st} f(t)\, dt + \int_4^\infty e^{-st} f(t)\, dt$

$= \int_0^4 e^{-st} e^{-t}\, dt + 0$

$= \int_0^4 e^{-(s+1)t}\, dt = -\left[\frac{e^{-(s+1)t}}{s+1}\right]_0^4$

$$= \frac{4\, e^{-4(s+1)}}{s+1}$$

Solve Find $L[f(t)]$ for the following

a) $f(t) = \begin{cases} 4 & 0 \le t < 2 \\ 0 & t \geq 2 \end{cases}$

b) $f(t) = t^2 e^{-2t}$

Solve $L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

$$= 4\int_0^2 e^{-st}\, dt + \int_2^\infty e^{-st} \cdot 0\, dt$$

$$= 4 \frac{e^{-st}}{-s} \Big|_0^2$$

$$= \frac{4}{s}\left[1 - e^{-2s}\right]$$

b) $L[f(t)] = \int_0^\infty e^{-st}\, t^2 e^{-2t}\, dt$

$$= \int_0^\infty t^2 e^{-(s+2)t}\, dt$$

$$= \left[\frac{t^2 e^{-(s+2)t}}{-(s+2)} - 2t \frac{e^{-(s+2)t}}{(s+2)^2} + 2 \frac{e^{-(s+2)t}}{-(s+2)^3} \right]_0^\infty$$

$$= \frac{2}{(s+2)^3}$$

Laplace Transform of Heaviside's unit step function

Def The unit step function $U(t-a)$ or $U_a(t)$ defined by

$$U(t-a) = \begin{cases} 0 & \text{if } 0 \le t < a \\ 1 & \text{if } t \ge a \end{cases}$$

Result $L[U(t-a)] = \frac{e^{-as}}{s}$

$$L[U(t-a)] = \int_0^\infty e^{-st} U(t-a)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} \cdot 1\,dt$$

$$= \int_a^\infty e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_a^\infty$$

$$= \frac{e^{-as}}{s} \quad (\because s > 0)$$

Corollary $L(U(t)) = \frac{1}{s}$

Dirac delta function or unit impulse function

The dirac delta function at $t=a$ denoted by $\delta(t-a)$ is defined as

$$\delta(t-a) = \lim_{h \to 0} \frac{1}{h}\left[U(t-a) - U(t-a-h)\right]$$

Note: Strictly speaking as $h \to 0$, the function $\delta(t-a)$ becomes infinite at $t=a$ so that $\delta(t-a)$ is not a well defined function. Actually it is a limiting mathematical operation and not a function as its name indicates. However we treat it as some type of quasi function which assumes the value zero for all other points $t \neq a$.

Also $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$ hence it is called a unit impulse function.

Remark In mechanical problems, the delta function is used to represent an impulse defined as the integral of a large force applied for a very short time.

Properties of Laplace Transform

Result 1: (Change of Scale)

If $L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt = \bar{F}(s)$, then

$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

Corollary $L^{-1}[f(\lambda s)] = \frac{1}{\lambda} f\left(\frac{t}{\lambda}\right)$

Result 2 First shifting or s shifting theorem

If $L[f(t)] = F(s)$ then

a) $L[e^{-at} f(t)] = F(s+a)$

b) $L[e^{at} f(t)] = F(s-a)$

Result 3 The second shifting theorem

If $L[f(t)] = F(s)$, then $L[f(t-a)\,U(t-a)]$
$= e^{-as} F(s)$

Corollary If $L^{-1}[F(s)] = f(t)$ then
$L^{-1}[e^{-as} F(s)] = f(t-a)\,U_a(t)$

Solve Find the Laplace transform of

a) $e^{t} t^{3/2}$ (b) $e^{t} \cosh 2t$ (c) $t^2 \cosh t$ (d) $e^{-t} \int_0^t \sin 2x \, dx$

Sol $L[e^{t} t^{3/2}] = L[t^{3/2}]_{s \to s-1}$ (by First shifting theorem)

$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

Corollary $L^{-1}[f(\lambda s)] = \frac{1}{\lambda} f\left(\frac{t}{\lambda}\right)$

Result 2 First shifting or s shifting theorem

If $L[f(t)] = F(s)$ then

a) $L[e^{-at} f(t)] = F(s+a)$

b) $L[e^{at} f(t)] = F(s-a)$

Result 3 The second Shifting theorem

If $L[f(t)] = F(s)$, then $L[f(t-a) U(t-a)]$
$= e^{-as} F(s)$

Corollary If $L^{-1}[F(s)] = f(t)$ then
$L^{-1}[e^{-as} F(s)] = f(t-a) U_a(t)$

Solve Find the Laplace transform of

a) $e^{t} t^{3/2}$ (b) $e^{t} \cosh 2t$ (c) $t^{2} \cosh t$ (d) $e^{-t} \int_0^t \sin 2x \, dx$

Sol $L[e^{t} t^{3/2}] = L[t^{3/2}]_{s \to s-1}$ (by First shifting theorem)

$$= \left[\frac{3\sqrt{\pi}}{4s^{5/2}}\right]_{s\to s-1}$$

$$= \frac{3\sqrt{\pi}}{4(s-1)^{5/2}}$$

b) $L\left[e^{t}\cosh 2t\right] = L\left[\cosh 2t\right]_{s\to s-1}$

$$= \left[\frac{s}{s^2-4}\right]_{s\to s-1}$$

$$= \frac{s-1}{(s-1)^2-4} = \frac{s-1}{s^2-2s-3}$$

c) $L\left[t^2\cosh t\right] = L\left[t^2\left(\frac{e^t+e^{-t}}{2}\right)\right]$

$$= \frac{1}{2}\left[L\left(e^{t}t^2\right) + L\left(e^{-t}t^2\right)\right]$$

$$= \frac{1}{2}\left[L\left(t^2\right)_{s\to s-1} + L\left(t^2\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\left(\frac{2}{s^3}\right)_{s\to s-1} + \left(\frac{2}{s^3}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^3} + \frac{2}{(s+1)^3}\right]$$

Find the Laplace transform of

a) $t^2 u(t-2)$ (b) $e^{2t}(t-5)u(t-5)$

Sol

$$L[t^2 u(t-2)]$$

$$= L\left[\left((t-2)^2 + 4t - 4\right) u(t-2)\right]$$

$$= L\left[(t-2)^2 u(t-2)\right] + 4L\left[t\, u(t-2)\right] - 4L\left[u(t-2)\right]$$

$$= L\left[(t-2)^2 u(t-2)\right] + 4L\left[(t-2+2)u(t-2)\right] - 4L\left[u(t-2)\right]$$

$$= e^{-2s} L(t^2) + 4\cancel{e}L\left[(t-2)u(t-2)\right] + 4L\left[u(t-2)\right]$$

$$= e^{-2s} L(t^2) + 4e^{-2s} L[t] + \frac{4e^{-2s}}{s}$$

[by Second shifting theorem]

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$$

$$= \frac{e^{-2s}}{s^3}\left(2 + 4s + 4s^2\right)$$

b) $L\left[e^{2t}(t-5)\,u(t-5)\right]$

$= L\left[e^{2(t-5)}\,e^{10}\,(t-5)\,u(t-5)\right]$

$= e^{10}\,e^{-5s}\,L\left[e^{2t}\,t\right]$ (by Second shifting theorem)

$= e^{10}\,e^{-5s}\,L[t]_{s\to s-2}$

$= \dfrac{e^{10-5s}}{(s-2)^2}$

Find the Laplace transform of $\sin t\, u(t-\pi)$ where $u(t-\pi)$ is the unit step function

Solve $\sin t\,u(t-\pi) = \sin(\pi + t-\pi)\,u(t-\pi)$

$= -\sin(t-\pi)\,u(t-\pi)$

$L\left[\sin t\; u(t-\pi)\right]$

$= L\left[-\sin(t-\pi)\;u(t-\pi)\right]$

$= L\left[f(t-\pi)\,u(t-\pi)\right]$ where $f(t) = -\sin t$

$= e^{-\pi s}\,L\left[f(t)\right]$

$= e^{-\pi s}\,L(-\sin t) = -\dfrac{e^{-\pi s}}{s^2+1}$

Find the Laplace transform of $t^2 u(t-3)$ (13)

Sol $t^2 u(t-3) = [(t-3)+3]^2 u(t-3)$

$= [(t-3)^2 + 6(t-3) + 9]\, u(t-3)$

$= f(t-3)\, u(t-3)$ where $f(t) = t^2 + 6t + 9$

$\therefore L(t^2 u(t-3)) = L[f(t-3)\, u(t-3)]$

$= e^{-3s} L[f(t)]$ (by second shifting theorem)

$= e^{-3s} L[t^2 + 6t + 9]$

$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$

Find the inverse Laplace transform of

$$\frac{3(s^2-1)^2}{2s^5}$$

Sol $L^{-1}\left[\frac{3(s^2-1)^2}{2s^5}\right] = \frac{3}{2} L^{-1}\left[\frac{s^4 - 2s^2 + 1}{s^5}\right]$

$= \frac{3}{2}\left[L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s^5}\right)\right]$

$= \frac{3}{2}\left[1 - \frac{2t^2}{2!} + \frac{t^4}{4!}\right] = \frac{3}{2} - \frac{3t^2}{2} + \frac{t^4}{16}$

// Find the inverse Laplace transform of (14)

a) $\frac{1}{(s+3)^2+25}$ (b) $\frac{s}{(s+2)^2}$

Sol $L^{-1}\left[\frac{1}{(s+3)^2+25}\right] = e^{-3t} L^{-1}\left[\frac{1}{s^2+5^2}\right]$

$= \frac{e^{-3t}}{5} L^{-1}\left[\frac{5}{s^2+5^2}\right]$

$= \frac{e^{-3t}}{5} \sin 5t$

b) $L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right]$

$= L^{-1}\left[\frac{1}{s+2}\right] - 2 L^{-1}\left[\frac{1}{(s+2)^2}\right]$

$= e^{-2t} L^{-1}\left(\frac{1}{s}\right) - 2 e^{-2t} L^{-1}\left(\frac{1}{s^2}\right)$

$= e^{-2t} - 2t e^{-2t} = e^{-2t}(1-2t)$

// Find the L.T. of $\frac{s+1}{s^2+2s+2}$

$L^{-1}\left(\frac{s+1}{s^2+2s+2}\right) = L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)$

$= e^{-t} L^{-1}\left(\frac{s}{s^2+1}\right)$

$= e^{-t} \cos t$

1) $L^{-1}\left[\frac{s}{a^2s^2+b^2}\right]$ ①

$= L^{-1}\left[\frac{s}{a^2\left[s^2+\left(\frac{b}{a}\right)^2\right]}\right]$

$= \frac{1}{a^2} L^{-1}\left[\frac{s}{s^2+\left(\frac{b}{a}\right)^2}\right]$

$= \frac{1}{a^2} \cos\left(\frac{bt}{a}\right)$

// Find $L^{-1}\left[\frac{cs+d}{(s+a)^2+b^2}\right]$

Sol^n $L^{-1}\left[\frac{cs+d}{(s+a)^2+b^2}\right]$

$= c L^{-1}\left[\frac{s}{(s+a)^2+b^2}\right] + d L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$

$= c L^{-1}\left[\frac{s+a-a}{(s+a)^2+b^2}\right] + d L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$

$= c L^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right] - ac L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$

$+ d L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$

$= c e^{-at} L^{-1}\left(\frac{s}{s^2+b^2}\right) - ac e^{-at} L^{-1}\left[\frac{1}{s^2+b^2}\right]$

$+ d e^{-at} L^{-1}\left[\frac{1}{s^2+b^2}\right]$

$= c e^{-at} L^{-1}\left(\frac{s}{s^2+b^2}\right) - \frac{ac}{b} e^{-at} L^{-1}\left[\frac{b}{s^2+b^2}\right] + \frac{d}{b} e^{-at} L^{-1}\left[\frac{b}{s^2+b^2}\right]$

$$= c e^{-at} \cos bt - ac e^{-at}\left(\frac{\sin bt}{b}\right) + d e^{-at}\left(\frac{\sin bt}{b}\right)$$

$$= c e^{-at} \cos bt + e^{-at}\left(\frac{\sin bt}{b}\right)(d - ac)$$

Solve Find $L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]$

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{1}{3}\left[\frac{1}{(s-1)^2} - \frac{1}{(s+2)^2}\right]$$

$$\therefore L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] = \frac{1}{3} L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3} L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{3} t e^{-t} - \frac{1}{3} e^{-2t} t$$

$$= \frac{1}{3} t\left(e^{-t} - e^{2t}\right)$$

Solve Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Let $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$

$$A(s+1)(s+2) + Bs(s+2) + Cs(s+1) = 1$$

$$B(-1)(-1+2) = 1$$

$$\Rightarrow B = \frac{1}{-1} = -1, \quad A = \frac{1}{2}, \quad C = \frac{1}{2}$$

$$\therefore \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

$$\therefore L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2}\cdot 1 - e^{-t}\cdot 1 + \frac{1}{2} e^{-2t}$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

Prob Find $L^{-1}\left[\frac{s^2-s+2}{s(s-3)(s+2)}\right]$

Sol^n $$\frac{s^2-s+2}{s(s-3)(s+2)} = -\frac{1}{3s} + \frac{8}{15(s-3)} + \frac{4}{5(s+2)}$$

$$\therefore L^{-1}\left[\frac{s^2-s+2}{s(s-3)(s+2)}\right]$$

$$= -\frac{1}{3} L^{-1}\left(\frac{1}{s}\right) + \frac{8}{15} L^{-1}\left(\frac{1}{s-3}\right) + \frac{4}{5} L^{-1}\left(\frac{1}{s+2}\right)$$

$$= -\frac{1}{3}\cdot 1 + \frac{8}{15} e^{+3t} + \frac{4}{5} e^{-2t}$$

$$= -\frac{1}{3} + \frac{8}{15} e^{3t} + \frac{4}{5} e^{-2t}$$

Prob Find $L^{-1}\left[\frac{5s+13}{s^2-6s+15}\right]$

$$L^{-1}\left[\frac{5s+13}{s^2-6s+15}\right) = L^{-1}\left[\frac{5s}{s^2-6s+15}\right] - 13 L^{-1}\left[\right.$$

$$= L^{-1}\left[\frac{5s-15+2}{(s-3)^2+6}\right]$$

$$= L^{-1}\left[\frac{5s-15}{(s-3)^2+6}\right] + 2L^{-1}\left[\frac{1}{(s+3)^2+6}\right]$$

$$= 5L^{-1}\left[\frac{s-3}{(s-3)^2+6}\right] + 2L^{-1}\left[\frac{1}{(s+3)^2+6}\right]$$

$$= 5e^{3t}L^{-1}\left[\frac{s}{s^2+6}\right] + 2e^{3t}L^{-1}\left[\frac{1}{s^2+6}\right]$$

$$= 5e^{3t}\cos(\sqrt{6}t) + \frac{2e^{3t}}{\sqrt{6}}\sin(\sqrt{6}t)$$

Solve Find the inverse Laplace Transform of $\dfrac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$

Solⁿ $L^{-1}\left[\dfrac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right]$

$$= L^{-1}\left[\frac{se^{-s/2}}{s^2+\pi^2}\right] + \pi L^{-1}\left[\frac{e^{-s}}{s^2+\pi^2}\right]$$

First we evaluate

$$L^{-1}\left[\frac{se^{-s/2}}{s^2+\pi^2}\right]$$

We know $L^{-1}\left[\dfrac{s}{s^2+\pi^2}\right] = \cos\pi t = f(t)$ (say)

$\therefore$ By Heaviside Shift theorem we have

$$L^{-1}\left[\frac{e^{-s/2}s}{s^2+\pi^2}\right]$$

$$= f(t-1/2)\,U\left(t-\tfrac{1}{2}\right)$$

$$= \cos\pi(t-1/2)\;U(t-1/2)$$

$$= \cos\left(\frac{\pi}{2}-\pi t\right)\;U(t-1/2)$$

$$= \sin\pi t\;U(t-1/2)$$

We now find $L^{-1}\left[\frac{e^{-s}}{s^2+\pi^2}\right]$

We have $L^{-1}\left[\frac{1}{s^2+\pi^2}\right] = \frac{\sin\pi t}{\pi} = f(t)$ (say)

$\therefore$ By Heaviside shift theorem we have

$$L^{-1}\left[\frac{e^{-s}}{s^2+\pi^2}\right] = f(t-1)\,U(t-1)$$

$$= \frac{\sin[\pi(t-1)]}{\pi}\,U(t-1)$$

$$= -\frac{\sin\pi t}{\pi}\,U(t-1)$$

From ①

$$L^{-1}\left[\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right] = \sin\pi t\;U(t-1/2) - \frac{\sin\pi t}{\pi}\,U(t-1)$$

Derivative of Laplace Tramsform

If $L[f(t)] = F(s)$ then $L[tf(t)] = -\frac{d}{ds}[F(s)]$

Note In general we can p.t.

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\left(L[f(t)]\right)$$

Laplace Tramsform of derivative

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

Note In general

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Laplace Tramsform of integrals

Let $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Corollary $\int_0^t f(t)\,dt = L^{-1}\left[\frac{F(s)}{s}\right]$

Solve Find the Laplace Transform of ⑦

a) $te^{-t}\cos t$ (b) $t^2\cosh at$ (c) $t^2 e^{-at}$

(a) $L(te^{-t}\cos t) = -\frac{d}{ds} L(e^{-t}\cos t)$

$$= -\frac{d}{dt} F(s+1)$$

Where $F(s) = L(\cos t) = \frac{s}{s^2+1}$

Now $F(s+1) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$

$$\therefore L(te^{-t}\cos t) = -\frac{d}{ds}\left(\frac{s+1}{s^2+2s+2}\right)$$

$$= \frac{s^2+2s}{(s^2+2s+2)^2}$$

(b) $L(t^2\cosh at)$

$$=(-1)^2 \frac{d^2}{ds^2} L(\cosh at)$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2-a^2}\right) = -\frac{d}{ds}\left(\frac{a^2+s^2}{(s^2-a^2)^2}\right)$$

$$= \frac{2s(s^2+a^2)}{(s^2-a^2)^3}$$

(c) $L(t^2 e^{-at}) = (-1)^2 \frac{d^2}{ds^2}\left[L(e^{-at})\right]$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s+a}\right)$$

$$= \frac{d}{ds}\left(-\frac{1}{(s+a)^2}\right)$$

$$= \frac{2}{(s+a)^3}$$

9. Find the L.T of (a) $t\sin t$ (b) $\frac{\sin at}{t}$

a) $L(t\sin t) = (-1)\frac{d}{ds}L(\sin t)$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{2s}{(s^2+1)^2}$$

b) $L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\,ds$

$$= a\int_s^\infty \frac{ds}{s^2+a^2}$$

$$= \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

$$= \tan^{-1}\left(\frac{a}{s}\right)$$

Convolution Theorem

Let $f(t)$ and $g(t)$ be two functions defined for $t>0$. The convolution $f*g$ of f and g is defined as

$$(f*g)(t) = \int_0^t f(u)\,g(t-u)\,du$$

Convolution Theorem

If $f(t)$ and $g(t)$ are two functions defined for $t>0$ then

$$L\left[(f*g)(t)\right] = L[f(t)]\, L[g(t)]$$

(i.e) the Laplace transform of convolution of $f(t)$ and $g(t)$ is same as the product of the ~~con~~ Laplace transform of $f(t)$ and $g(t)$

Ex Find $L^{-1}\left[\frac{1}{s(s^2+1)}\right]$

We use convolution theorem we find

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s(s^2+1)}\right]$$

$$= L^{-1}\left(\frac{1}{s}\right) * L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= 1 * \sin t$$

$$= \int_0^t \sin(t-u)\,du = \cos(t-u)\Big|_0^t$$

$$= 1 - \cos t$$

Ex Using convolution theorem find

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]$$

Sol
$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s+2}\right]$$

$$= e^{-t} * e^{-2t}$$

$$= \int_0^t e^{-u} e^{-2(t-u)} du$$

$$= \int_0^t e^{u-2t} du$$

$$= e^{u-2t} \Big|_0^t$$

$$= e^{-t} - e^{-2t}$$

Ex Solve for y the integral equation

$$y(t) = t^2 + \int_0^t y(u) \sin(t-u)\, du$$

Sol^n
$$y(t) = t^2 + \int_0^t y(u) \sin(t-u)\, du$$

$$= t^2 + y(t) * \sin t$$

Applying Laplace transform to the given equation we get

$$L(y) = L(t^2) + L[y * \sin t]$$

$= L(t^2) + L(y) \cdot L(\sin t)$ (by convolution theorem)

$= \frac{2}{s^3} + L(y) \cdot \frac{1}{s^2+1}$

$\therefore L(y)\left[1 - \frac{1}{s^2+1}\right] = \frac{2}{s^3}$

$\Rightarrow L(y)\frac{s^2}{s^2+1} = \frac{2}{s^3}$

$\therefore L(y) = \frac{2(s^2+1)}{s^5} = \frac{2}{s^3} + \frac{2}{s^5}$

$\therefore y = 2L^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{1}{s^5}\right]$

$= t^2 + \frac{t^4}{12}$

Solve for $f(t)$ using Laplace transform

$f'(t) = t + \int_0^t f(t-u)\cos u \, du$ given $f(0) = 4$

Sol^n $f'(t) = t + \int_0^t f(t-u)\cos u \, du$

$= t + (\cos t * f(t))$

Applying Laplace transform on both sides

$L[f'(t)] = L(t) + L[\cos t * f(t)]$

$= L(t) + L(\cos t)\, L[f(t)]$

④

$$sL[f(t)] - f(0) = \frac{1}{s^2} + \frac{s}{s^2+1} L(f(t))$$

$$\therefore L[f(t)]\left[s - \frac{s}{s^2+1}\right] = 4 + \frac{1}{s^2}$$

$$\text{i.e. } L[f(t)]\left(\frac{s^3}{s^2+1}\right) = \frac{4s^2+1}{s^2}$$

$$\text{or, } L[f(t)] = \frac{(4s^2+1)(s^2+1)}{s^5}$$

$$= \frac{4}{s} + \frac{5}{s^3} + \frac{1}{s^5}$$

Applying inverse Laplace tramform on both sides

$$f(t) = 4L^{-1}\left[\frac{1}{s}\right] + \frac{5}{2}L^{-1}\left[\frac{2}{s^3}\right] + \frac{1}{4!}L^{-1}\left[\frac{4!}{s^5}\right]$$

$$= 4.1 + \frac{5}{2}t^2 + \frac{t^4}{24}$$

Solution

$$y + \int_0^t y(u)\,du = e^{-t}$$

$$\therefore y + y(t) * 1 = e^{-t} \quad \text{(by the definition of convolution)}$$

Applying Laplace tramform on both sides we get

$$L(y) + L[y(t) * 1] = L[e^{-t}]$$

$$\therefore L(y) + L(y)\left(\frac{1}{s}\right) = \frac{1}{s+1}$$

i.e. $L(y)\left[1+\frac{1}{s}\right]=\frac{1}{s+1}$

i.e. $L(y)\,\frac{s+1}{s}=\frac{1}{s+1}$

i.e. $L(y)=\frac{s}{(s+1)^2}$

Applying inverse Laplace transform on both sides

$$y=L^{-1}\left[\frac{s}{(s+1)^2}\right]$$

$$=L^{-1}\left[\frac{s+1-1}{(s+1)^2}\right]$$

$$=L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$=e^{-t}-e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$=e^{-t}-te^{-t}$$

$$=e^{-t}(1-t)$$

Ex Use Laplace transform to solve $y'-y=e^t$ given that $y(0)=1$

Sol^n $y'-y=e^t$

Applying Laplace transform on both sides we get $L(y')-L(y)=L(e^t)$

$$\left[sL(y)-y(0)\right]-L(y)=\frac{1}{s-1}$$

$$(S-1)L(y) = \frac{1}{S-1} + 1$$

$$\therefore L(y) = \frac{S}{(S-1)^2}$$

$$\therefore y = L^{-1}\left[\frac{S}{(S-1)^2}\right]$$

$$= L^{-1}\left[\frac{S-1+1}{(S-1)^2}\right]$$

$$= L^{-1}\left[\frac{1}{S+1}\right] + L^{-1}\left[\frac{1}{(S-1)^2}\right]$$

$$= e^{+t} + t e^{+t}$$

$$= e^{t}(1+t)$$

Ex Using Laplace tramform solve

$y'' + 4y' + 13y = 2e^{-t}$ given $y(0) = 0$ and $y'(0) = -1$

Sol^n $y'' + 4y' + 13y = 2e^{-t}$

$$L[y'' + 4y' + 13y] = L[2e^{-t}]$$

$$\therefore L[y''] + 4L[y'] + 13L[y] = 2L[e^{-t}]$$

$$\therefore \{s^2L(y) - sy(0) - y'(0)\} + 4\{sL[y] - y(0)\} + 13L(y) = \frac{2}{s+1}$$

$$\therefore s^2L(y) + 1 - 4sL(y) + 13L(y) = \frac{2}{s+1}$$

[using $y(0) = 0$, $y'(0) = -1$]

$$\therefore L(y)\left(s^2+4s+13\right) = \frac{2}{s+1} - 1 = \frac{1-s}{1+s}$$

$$\therefore L(y) = \frac{1-s}{(1+s)(s^2+4s+13)}$$

$$\therefore y = L^{-1}\left[\frac{1-s}{(1+s)(s^2+4s+13)}\right]$$

Now, by partial fractions

$$\frac{1-s}{(1+s)(s^2+4s+13)} = \frac{1}{5(s+1)} - \frac{s}{5(s^2+4s+13)} - \frac{8}{5(s^2+4s+13)}$$

$$\therefore y = \frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4s+13}\right] - \frac{8}{5}L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+2-2}{(s+2)^2+3^2}\right] - \frac{8}{5}L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] + \frac{2}{5}L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$- \frac{8}{5}L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t + \frac{2}{5\times 3}e^{-2t}\sin 3t - \frac{8}{5\times 3}e^{-2t}\sin 3t$$

$$\therefore y = \frac{e^{-t}}{5} - \frac{1}{5}e^{-2t}\cos 3t + \frac{2}{15}e^{-2t}\sin 3t - \frac{8}{15}e^{-2t}\sin 3t$$

$$= \frac{e^{-t}}{5} - \frac{1}{5}e^{-2t}\left(\cos 3t - \frac{2}{3}\sin 3t + \frac{8}{3}\sin 3t\right)$$

$$= \frac{e^{-t}}{5} - \frac{1}{5}e^{-2t}\left(\cos 3t + 2\sin 3t\right)$$

Ex Using Laplace transform solve

$x'' - 2x' + x = e^{2t}$, $x(0) = 0$, $x'(0) = -1$

Sol^n $x'' - 2x' + x = e^{2t}$

$\therefore L[x'' - 2x' + x] = L[e^{2t}]$

$\therefore L[x''] - 2L[x'] + L[x] = L[e^{2t}]$

$\therefore \{s^2 L(x) - s\,x(0) - x'(0)\} - 2\{sL(x) - x(0)\} + L(x) = \frac{1}{s-2}$

$\therefore s^2 L(x) + 1 - 2sL(x) + L(x) = \frac{1}{s-2}$ (using $x(0) = 0$, $x'(0) = 1$)

$\therefore L(x)\,[s^2 - 2s + 1] = \frac{1}{s-2} - 1 = \frac{3-s}{s-2}$

$\therefore L(x) = \frac{3-s}{(s-2)(s^2-2s+1)}$

$= \frac{3-s}{(s-1)^2(s-2)}$

$\therefore x = L^{-1}\left[\frac{3-s}{(s-1)^2(s-2)}\right]$

$\frac{3-s}{(s-1)^2(s-2)} = -\frac{1}{s-1} - \frac{2}{(s-1)^2} + \frac{1}{s-2}$

$\therefore x = L^{-1}\left[-\frac{1}{s-1}\right] - L^{-1}\left[\frac{2}{(s+1)^2}\right] + L^{-1}\left[\frac{1}{s-2}\right]$

$= -L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{s-2}\right]$

$x = -e^{-t} - 2te^{t} + e^{2t}$

Ex Solve by Laplace Transform method
$y''(t) + y'(t) = U(t-1)$ given $y(0)=0$, $y'(0)=1$

Sol^n $y''(t) + y'(t) = U(t-1)$

$\therefore L[y''(t)] + L[y'(t)] = L[U(t-1)]$

$\therefore [s^2 L(y) - s\,y(0) - y'(0)] + [s L(y) - y(0)] = \frac{e^{-s}}{s}$

Using $y(0)=0$, $y'(0)=1$ we get

$[s^2 L(y) - 1] + sL(y) = \frac{e^{-s}}{s}$

$\therefore (s^2+s)L(y) = \frac{e^{-s}}{s} + 1$

$\therefore L(y) = \frac{1}{s^2+s} + \frac{e^{-s}}{s(s^2+s)}$

$= \frac{1}{s(s+1)} + \frac{e^{-s}}{s^2(s+1)}$

$\therefore y = L^{-1}\left[\frac{1}{s(s+1)}\right] + L^{-1}\left[\frac{e^{-s}}{s^2(s+1)}\right]$

$= L^{-1}\left[\frac{s+1-s}{s(s+1)}\right] + L^{-1}\left[\frac{e^{-s}}{s^2(s+1)}\right]$

$= L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + U(t-1)L^{-1}\left[\frac{1}{s^2(s+1)}\right]$

$= 1 - e^{-t} + u(t-1)\, L^{-1}\left[-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right]$

$= 1 - e^{-t} + u(t-1)\left[-1 + t + e^{-t}\right]$

<u>Ex</u> Solve $ty'' + 2y' + ty = \sin t$, $y(0) = 1$

Taking L.T on both sides we get

$L[ty''] + 2L[y'] + L[ty] = L[\sin t]$

or, $-\frac{d}{ds}[L(y'')] + 2L(y') - \frac{d}{ds}L(y) = \frac{1}{s^2+1}$

or $-\frac{d}{ds}\left[s^2 L(y(t)) - s\,y(0) - y'(0)\right] + 2\left[s L(y) - y(0)\right]$

$- \frac{d}{ds}L(y) = \frac{1}{s^2+1}$ $\left[\text{let } L(y) = F(s)\right]$

or, $-\frac{d}{ds}\left[s^2 F(s)\right] + y(0) + 2sF(s) - 2 \times y(0)$

$- F'(s) = \frac{1}{s^2+1}$

or, $-2sF(s) - s^2 F'(s) + 1 + 2sF(s) - 2 \cdot 1$

$- F'(s) = \frac{1}{s^2+1}$

or, $-F'(s)(s^2+1) - 1 = \frac{1}{s^2+1}$

or, $F'(s) = \left(-1 - \frac{1}{s^2+1}\right)\frac{1}{s^2+1}$

or, $F'(s) = -\frac{1}{s^2+1} - \frac{1}{(s^2+1)^2}$

Taking inverse L.T. on both sides we get

$$-ty = -\sin t - \frac{1}{2}(\sin t - t\cos t)$$

$\left[F(s) = \frac{1}{s}L(y)\right]$

$$\therefore y = -\frac{3}{2}\frac{\sin t}{t} - \frac{1}{2}\cos t$$

$F'(s) = \frac{d}{ds}F(s)$

$L^{-1}[F'(s)]$

$= -tf(t)$

Ex Solve the initial value problem

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0 \quad \text{with } y = 3 \text{ and } \frac{dy}{dx} = 7 \text{ at } x = 0$$

Using L.T.

$$L[y''] - 4L[y'] + 3L(y) = 0$$

or, $\left[s^2\bar{y} - sy(0) - y'(0)\right] - 4\left[s\bar{y} - y(0)\right] + 3\bar{y} = 0$

or, $\bar{y}\left[s^2 - 4s + 3\right] - 3s + 5 = 0$

or, $\bar{y} = \dfrac{3s-5}{s^2-4s+3}$

or, $\bar{y} = \dfrac{3(s-2)+1}{(s-2)^2-1}$

or, $y = L^{-1}\left[\dfrac{3(s-2)+1}{(s-2)^2-1}\right]$

$= e^{2t}L^{-1}\left(\dfrac{3s+1}{s^2-1}\right) = e^{2t}(3\cosh t + \sinh t)$

Solve $y'' + 2y' + 5y = 8\sin t + 4\cos t$ with $y(0) = 1$, $y\left(\frac{\pi}{4}\right) = \sqrt{2}$ (12)

Ans: Taking L.T. on both sides of the given equation we get

$L[y''] + 2L[y'] + 5 \cancel{y[t]}\, L[y] = 8L(\sin t) + 4L(\cos t)$

or, $s^2 L(y(t)) - s\,y(0) - y'(0)$

$+ 2sL(y(t)) - 2y(0) + 5L(y(t))$

$= \frac{8}{s^2+1} + \frac{4s}{s^2+1}$

or, $L[y(t)]\left[s^2+2s+5\right] - s - 2 + c = \frac{8+4s}{s^2+1}$ where $c = y'(0)$

or, $L[y(t)]\left[s^2+2s+5\right] = \frac{8+4s}{s^2+1} + s + 2 - c$

or $L[y(t)]\left[s^2+2s+5\right] = \frac{8+4s}{s^2+1} + s + 2 - c$

or $L[y(t)] = \frac{8+4s}{(s^2+1)(s^2+2s+5)} + \frac{s+2-c}{s^2+2s+5}$ —— (1)

Consider $\frac{8+4s}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}$

$$\therefore 8+4s = (As+B)(s^2+2s+5) + (Cs+D)(s^2+1)$$

$$\therefore \cancel{8+4s} = 8 = 5B+D \quad \text{—— (2)}$$

$$4 = 5A+2B+C \quad \text{—— (3)}$$

$$0 = 2A+B+D \quad \text{—— (4)}$$

$$0 = A+C \Rightarrow A=-C \quad \text{—— (5)}$$

Using this in (3) and (4) we get

$$-2C+B=2$$

$$-2C+B+D=0$$

Solving we get $D=-2$

$\therefore$ From (2) $B=2$

Using this values in (4) we get $A=0$ and hence from (5) $C=0$

$\therefore$ (1) becomes

$$L[y(t)] = 2\left[\frac{1}{s^2+1} - \frac{1}{s^2+2s+5}\right] + \frac{s+2-e}{s^2+2s+5}$$

$$= 2\left[\frac{1}{s^2+1} - \frac{1}{(s+1)^2+4}\right] + \frac{s+1-e+1}{(s+1)^2+4}$$

$$\therefore y(t) = 2\sin t - e^{-t} L^{-1}\left[\frac{2}{s^2+2^2}\right] + e^{-t} L^{-1}\left[\frac{s-e+1}{s^2+4}\right]$$

(14)

$= 2\sin t - e^{-t}\sin 2t + e^{-t}\left(\cos 2t - \frac{c}{2}\sin 2t + \frac{1}{2}\sin 2t\right)$ —⑥

Using $y\left(\frac{\pi}{4}\right) = \sqrt{2}$ in ⑥ we get

$$\sqrt{2} = \frac{2}{\sqrt{2}} - e^{-\pi/4} + e^{-\pi/4}\left(-\frac{c}{2} + \frac{1}{2}\right)$$

$\Rightarrow c = -1$

Using this in ⑥ we get

$$y = 2\sin t + e^{-t}\cos 2t$$