Digital Electronics
Boolean Algebra
Minimization Techniques

1st year of 4 year B.Tech.

Binary Logic and Gates :

Basic Logic Gates and Operations:

The basic logic operations are:

a) AND operation (X.Y)

b) OR operation (X+Y)

c) NOT operation ($\overline{X}$)

- **AND gate** is the hardware element that implement the AND operation.
- **OR gate** is the hardware element that implement the OR operation.
- **NOT gate** is the hardware element that implement the NOT operation.

AND	OR	NOT
$0 \cdot 0 = 0$ $0 \cdot 1 = 0$ $1 \cdot 0 = 0$ $1 \cdot 1 = 1$	$0 + 0 = 0$ $0 + 1 = 1$ $1 + 0 = 1$ $1 + 1 = 1$	$\overline{0} = 1$ $\overline{1} = 0$

Logic gates:

A logic gate is an electronic circuit that operates on one or more input signals to produce an output signal. A logic gate is also known building block of a digital circuit. Mostly, the logic gate consists of two inputs and one output. Gates produce the signals 1 or 0 if input requirements are satisfied. Digital computer uses different types of logical gates. Each gate has a specific function and graphical symbol. The function of the gate is expressed by means of an algebraic expression. The basic gates are described below:

- **AND Gate:**

The AND Gate contain two or more than to input values which produce only one output value. AND gate produces 1 output when all inputs are 1, otherwise the output will be 0. It can be explained with the help of two switches connected in series. In AND gate, current is flowing in the circuit only when both switches, A and B, are closed.

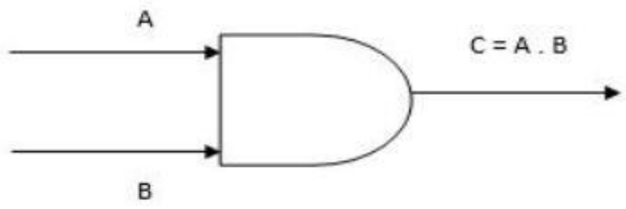
The switch contains two states which are ON or OFF. The ON means the logic 1 and the OFF means the logic 0. So, when both switches are ON, the output is 1 and when any of the switches are OFF, the output is 0.

The graphical symbol, logical circuit, algebraic expression and truth table of AND gate is shown below:

Truth table:

Input		Output
A	B	C = A . B
0	0	0
0	1	0
1	0	0
1	1	1

Graphical Symbol:



Algebraic Expression is, C = A . B

- **OR Gate:**

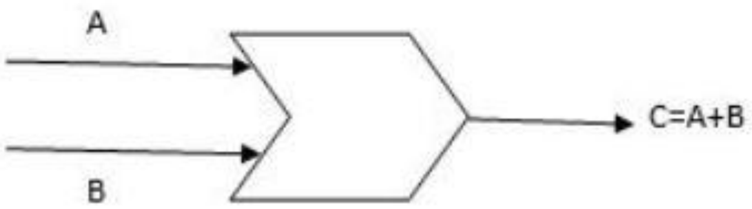
The OR Gate contains two or more than two input values which produce only one output value. OR gate produces 1 output, when one of the inputs is 1. If inputs are 0, then the output will be also 0. It can be explained by taking an example of two switches connected in parallel.

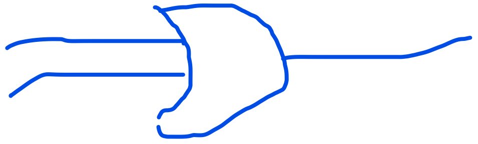
The graphical symbol, algebraic expression and truth table of OR gate is as shown below:

Truth table:

Input		Output
A	B	C = A +B
0	0	0
0	1	1
1	0	1
1	1	1

Graphical Symbol





Algebraic Expression is, C = A+B

- **NOT Gate:**

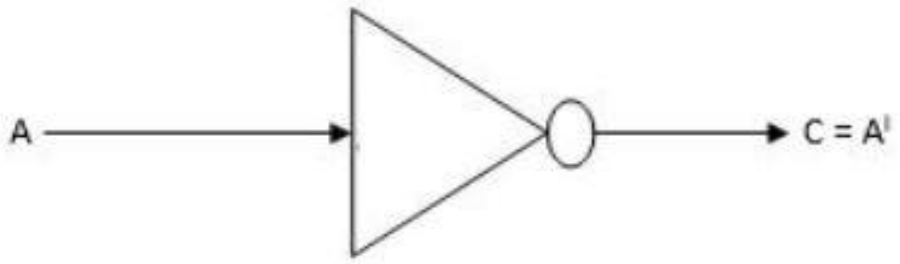
The NOT Gate contains only one input value which produces only one output value. This gate is also known as an inverter. So, this circuit inverts the logical sense of a binary signal. It produces the complemented function. If the input is 1, then this gate will produce 0 as output and vice-versa. The graphical symbol, algebraic expression and truth table of a NOT gate is given below.

Truth table:

A	A^{I}
0	1
1	0

Algebraic Expression is, $C = A^{I}$

Graphical Symbol:



- **NAND Gate:**

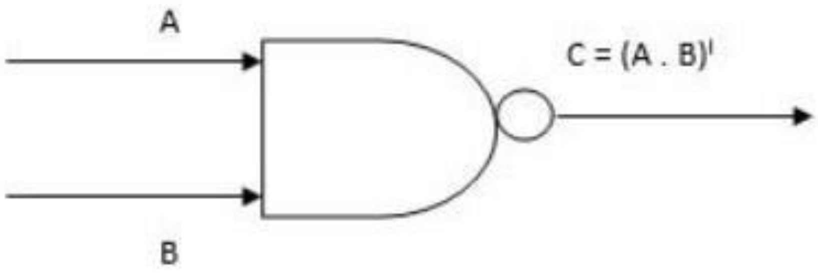
The NAND Gate contains two or more than two input values which produce only one output value. This gate is the combination of AND and NOT gates. This gate is a complement of AND function. This gate produces output 0, when all inputs are 1, otherwise, output will be 1.

The graphical symbol, algebraic expression and truth table of NAND gate is shown below:

Truth table:

Input		Output
A	B	C=(A . B)'
0	0	1
0	1	1
1	0	1
1	1	0

Graphical Symbol:



Algebraic Expression is, C = (A . B)'

- **NOR Gate:**

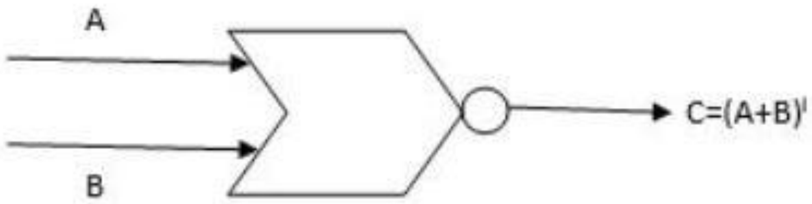
The NOR Gate contains two or more than two input values which produce only one output value. This gate is a combination of OR and NOT gate. This gate is the complement of the OR function. This gate produces 1 output, when all inputs are 0 otherwise output will 0.

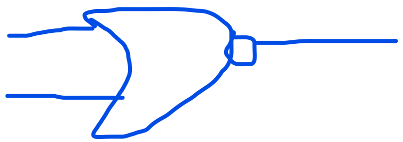
The graphical symbol, algebraic expression and truth table of NOR gate are given below:

Truth table:

Input		Output
A	B	$C = (A+B)'$
0	0	1
0	1	0
1	0	0
1	1	0

Graphical Symbol:





Algebraic Expression, $C = (A + B)'$

- **Exclusive OR (X-OR) Gate:**

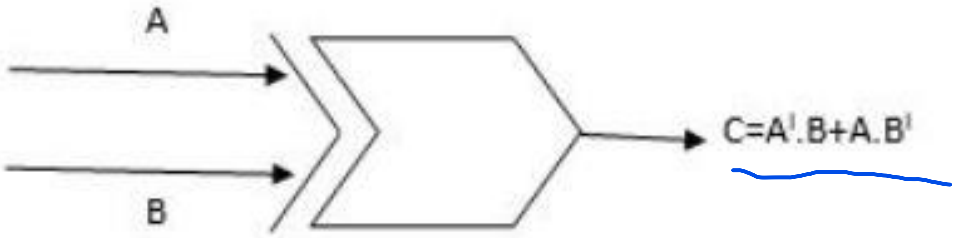
This gate contains two or more than two input values which produce only one output value. The graphical symbol of X-OR gate is similar to OR gate except for the additional curve line on the input side. This gate produces 1 as output, if any input is 1 and 0 if both inputs are either 1 or 0, otherwise its output is 0.

The graphical symbol, algebraic expression and truth table of X-OR gate is given below:

Truth table:

Input		Output
A	B	$C = A^{I} . B + A . B^{I}$
0	0	0
0	1	1
1	0	1
1	1	0

Graphical Symbol:

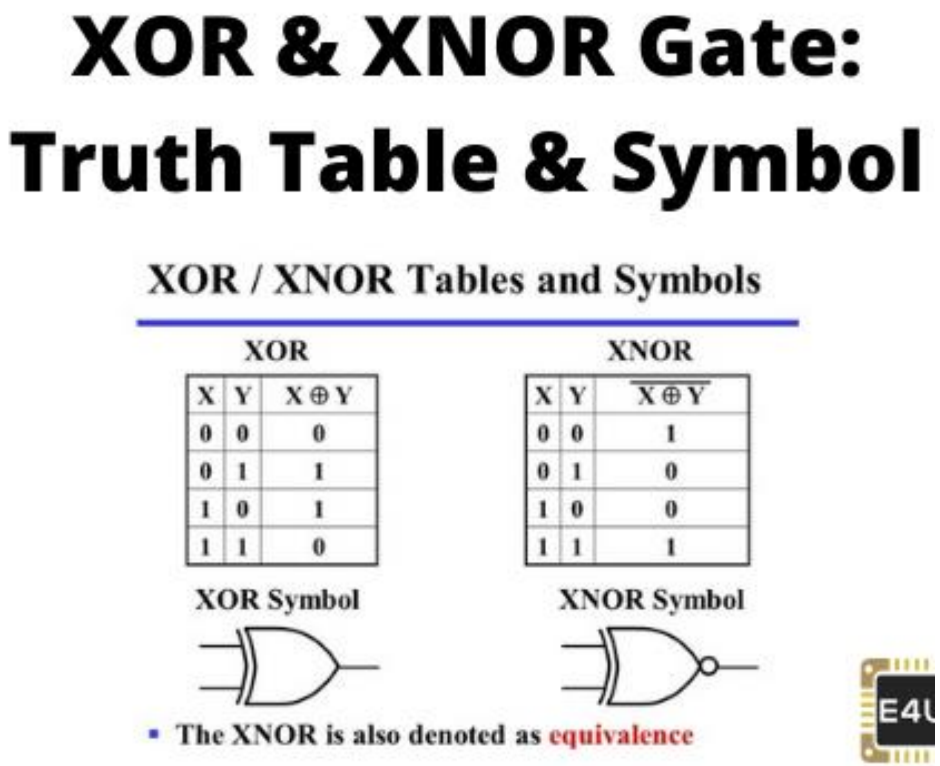


Algebraic Expression is, $C = A^{I} . B + A . B^{I}$

- **Exclusive NOR (X-NOR) Gate:**

This gate contains two or more than two input values which produce only one output value. The X-NOR is the complement of the X-OR, as indicated by the small circle in the graphical symbol. This gate produces 1 output, when all inputs are either 0 or 1, otherwise its output value is 0.

The graphical symbol, algebraic expression and truth table of X-NOR gate is shown below:



Properties of Boolean Algebra

Commutative:	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative:	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Distributive:	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	$a + (b \cdot c) = (a + b) \cdot (a + c)$
Idempotent:	$a + a = a$	$a \cdot a = a$
Identity:	$a + 0 = a$	$a \cdot 1 = a$
Annihilate:	$a + 1 = 1$	$a \cdot 0 = 0$
Annihilate:	$a + \overline{a} = 1$	$a \cdot \overline{a} = 0$
DeMorgan:	$\overline{a + b} = \overline{a} \cdot \overline{b}$	$\overline{a \cdot b} = \overline{a} + \overline{b}$
Inverse:	$\overline{\overline{a}} = a$	
Absorption:	$a + ab = a$	$a.(a+b)=a$
	$a + \overline{a}b = a + b$	$a.(\overline{a} + b) = ab$
Consensus:	$ab + \overline{a}c + bc = ab + \overline{a}c$	$(a + b).(\overline{a} + c).(b + c) = (a + b).(\overline{a} + c)$

Proof of Boolean Laws:

(Distributive law)

$$
\begin{aligned}
A + BC &= A.1 + BC && [\text{ Since, } A.1 = A] \\
&= A(1 + B) + BC && [\text{Since, } B+1 = 1] \\
&= A.1 + AB + BC \\
&= A.(1 + C) + AB + BC \\
&= A\,(A + C) + B\,(A+C) && [\text{Since, } A.A = A.1 = A] \\
A + BC &= (A+B)(A+C)
\end{aligned}
$$

(Absorption Law):

$$A\,.\,(A+B) = A$$

Proof.

$$
\begin{aligned}
A.(A+B) &= A.A + A.B \\
&= A+AB && [\text{Since, } A.A = A] \\
&= A(1+B) \\
&= A.1
\end{aligned}
$$

$$A + (A\cdot B) = A$$

Proof.

$$
\begin{aligned}
A+(A.B) &= A.1 + AB && [\text{Since } A.1 = A] \\
&= A(1+B) && [\text{Since, } 1 + B = 1] \\
&= A.1 = A
\end{aligned}
$$

- $AB + \overline{A}C + BC = AB + \overline{A}C$ (Consensus Theorem)

Proof Steps	Justification
$AB + \overline{A}C + BC$	
$= AB + \overline{A}C + 1 \cdot BC$	Identity element
$= AB + \overline{A}C + (A + \overline{A}) \cdot BC$	Complement
$= AB + \overline{A}C + ABC + \overline{A}BC$	Distributive
$= AB + ABC + \overline{A}C + \overline{A}CB$	Commutative
$= AB \cdot 1 + ABC + \overline{A}C \cdot 1 + \overline{A}CB$	Identity element
$= AB\,(1+C) + \overline{A}C\,(1 + B)$	Distributive
$= AB \,.\, 1 + \overline{A}C \,.\, 1$	$1+X = 1$
$= AB + \overline{A}C$	Identity element

$$(A+B)(\bar{A}+C)(B+C) = (A+B)(\bar{A}+C)(B+C+0)$$

$$= (A+B)(\bar{A}+C)(B+C+A\bar{A})$$

$$= (A+B)(\bar{A}+C)(B+C+A)(B+C+\bar{A}) \qquad [\because A+BC = (A+B)(A+C)]$$

$$= (A+B)(A+B+C)(\bar{A}+C)(\bar{A}+C+B)$$

$$= (A+B)(\bar{A}+C) \qquad [\because A(A+B) = A]$$

Minimization Techniques of Boolean Algebra

- $Proof: AB + BC + \bar{B}C = AB + C$

$$\text{LHS} = AB + C(B + \bar{B})$$
$$= AB + C \cdot 1$$
$$= AB + C$$
$$(\text{LHS}) = (\text{RHS})$$

- Simplify: $AB + \overline{AC} + A\bar{B}C(AB + C)$

$= \ AB + \overline{AC} + A\bar{B}C \cdot AB + A\bar{B}C \cdot C$
$= \ AB + \overline{AC} + A\bar{B}C$ as $C \cdot C = C$
and $\bar{B} \cdot B = 0$

$= \ AB + \bar{A} + \bar{C} + A\bar{B}C$
$= \ A(B + \bar{B}C) + \bar{A} + \bar{C}$
$= A(B + C) + \bar{A} + \bar{C}$ $as\ A + A\ \bar{\mathrm{B}} = \mathrm{A} + \mathrm{B}$
$= AB + AC + \bar{A} + \vec{C}$
$= \bar{A} + AB + \bar{C} + AC$
$= \bar{A} + B + \bar{C} + A$
$= 1 + B + \bar{C} \ = \ 1$

- Simplify: $\bar{A}B + AB + \bar{A}\bar{B}$

$= \ (\bar{A} + A)B + \bar{A}\,\bar{B}$
$= 1 \cdot B + \bar{A}\,\bar{B}$
$= \ B + \bar{A}\,\bar{B}$
$= (\mathrm{B} + \bar{A})(B + \bar{B})$
$= B + \bar{A}$

- Simplify: $\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + A\bar{B}\bar{C} + AB\bar{C}$

$= \bar{A}\bar{C}(B + \bar{B}) + A\bar{C}(B + \bar{B})$

$= \bar{A}\bar{C} + A\bar{C}$

$= \ \bar{C}(\bar{A} + A)$

$= \ \bar{C}$

- Simplify $\overline{\overline{A\bar{B}+ABC}+A(B+A\bar{B})}$

$= \overline{\overline{A(\bar{B}+BC)}+A(B+A)}$

$= \overline{\overline{A(\bar{B}+C)}+A\cdot B+A\cdot A}$

$= \overline{\overline{A\bar{B}+AC}+AB+A}$

$= \overline{\overline{A\bar{B}+AC}+A(B+1)}$

$= \overline{\overline{A\bar{B}+AC}+A\cdot 1}$ $\quad as\ B+1=1$

$= \overline{(\overline{A\bar{B}}).(\overline{AC})+A}$ $\quad \text{applying De} - \text{Morgan's}$

$= \overline{(\bar{A}+B)\,.\,(\bar{A}+\bar{C})+A}$

$= \overline{(\bar{A}+B\bar{C})+A}$ $\quad since\ (A+B)(A+C)=A+BC$

$= \overline{\bar{A}+B\bar{C}+A}$

$= \overline{1+B\bar{C}}$ $\quad as\ \ A+\bar{A}=1$

$= \bar{1}$

$= 0$

Q:

- If $\bar{A}B + C\bar{D} = 0$ prove that $AB + \bar{C}(\bar{A} + \bar{D}) = AB + BD + \bar{B}\bar{D} + \bar{A}\bar{C}D$

LHS

$$
\begin{aligned}
AB + \bar{C}(\bar{A} + \bar{D}) + 0 &= AB + \bar{C}(\bar{A} + \bar{D}) + \bar{A}B + C\bar{D} \\
&= AB + \bar{A}\bar{C} + \bar{C}\bar{D} + \bar{A}B + C\bar{D} \\
&= B(A + \bar{A}) + \bar{D}(C + \bar{C}) + \bar{A}\bar{C} \\
&= B + \bar{D} + \bar{A}\bar{C}
\end{aligned}
$$

RHS

$$
\begin{aligned}
AB + BD + \bar{B}\bar{D} + \bar{A}\bar{C}D + 0 &= AB + BD + \bar{B}\bar{D} + \bar{A}\bar{C}D + \bar{A}B + C\bar{D} \\
&= B(A + \bar{A}) + BD + \bar{B}\bar{D} + \bar{A}\bar{C}D + C\bar{D} \\
&= B \cdot 1 + BD + \bar{B}\bar{D} + \bar{A}\bar{C}D + C\bar{D} \\
&= B(1 + D) + \bar{B}\bar{D} + \bar{A}\bar{C}D + C\bar{D} \\
&= B + \bar{B}\bar{D} + \bar{A}\bar{C}D + C\bar{D} \\
&= B + \bar{D} + \bar{A}\bar{C}D + C\bar{D} \\
&= B + \bar{D}(1 + C) + \bar{A}\bar{C}D \\
&= B + \bar{D} + \bar{A}\bar{C}D \\
&= B + \bar{D} + \bar{A}\bar{C} \qquad since \quad \bar{A} + AB = \bar{A} + B
\end{aligned}
$$

Hence, L.H.S = R.H.S

Sum of Products and Product of Sums (SOP) and (POS)

- Any Boolean expression can be in a standard, canonical, sum of product form(SOP) or product of sums form(POS)
- SOP: It is the logical sum of all the product terms, each of which contain all the variables in the expression either in complemented or uncomplemented form
- Each of these product terms are called minterms

SOP

X	Y	Z	Minterms Term	Minterms Designation
0	0	0	$\bar{X}\bar{Y}\bar{Z}$	mo
0	0	1	$\bar{X}\bar{Y}Z$	m1
0	1	0	$\bar{X}Y\bar{Z}$	m2
0	1	1	$\bar{X}YZ$	m3
1	0	0	$X\bar{Y}\bar{Z}$	m4
1	0	1	$X\bar{Y}Z$	m5
1	1	0	$XY\bar{Z}$	m6
1	1	1	XYZ	m7

SOP expressions of a Boolen function f(A,B,C) of three variables is expressed :

(A) $\bar{A}BC+\bar{A}BC+A\bar{B}\bar{C}+ABC$

(B) $\bar{A}\,\bar{B}\bar{C}+\bar{A}B\bar{C}+A\bar{B}C+AB\bar{C}$

(C) $AB\bar{C}+A\bar{B}\bar{C}+\bar{A}BC+\bar{A}\,\bar{B}\bar{C}$

(D) $\bar{A}\,\bar{B}\bar{C}+\bar{A}BC+AB\bar{C}+ABC$

- POS: It is the logical product of all the sum terms, each of which contain all the variables in the expression either in complemented or uncomplemented form
- Each of these sum terms are called maxterms

POS

Maxterms	
Term	Designation
$X+Y+Z$	M_0
$X+Y+\overline{Z}$	M_1
$X+\overline{Y}+Z$	M_2
$X+\overline{Y}+\overline{Z}$	M_3
$\overline{X}+Y+Z$	M_4
$\overline{X}+Y+\overline{Z}$	M_5
$\overline{X}+\overline{Y}+Z$	M_6
$\overline{X}+\overline{Y}+\overline{Z}$	M_7

The product of sum expression of a Boolean function f(A,B,C) of three variable is given by

$$F(A,B,C) = [A+B+\overline{C}][A+\overline{B}+\overline{C}][\overline{A}+B+C][\overline{A}+\overline{B}+\overline{C}]$$

- How to obtain a SOP form of a function:
    - Examine each term in the given logic function. Retain if it is a minterm
    - Check for variables that are missing in each product which is not a minterm
    - Multiply the product by $(X + \overline{X})$, for each variable X that is missing
    - Multiply all products and omit the redundant terms

- Obtain the canonical sum of product form of the function

$$Y(A, B) = A + B$$

$$= A(B + \bar{B}) + B(A + \bar{A})$$

$$= AB + A\bar{B} + \bar{A}B + BA$$

$$= AB + A\bar{B} + \bar{A}B$$

- Obtain the canonical sum of product form of the function

$$
\begin{aligned}
Y &= A + BC + \overline{AB} \\
&= A + BC + \overline{A} + \overline{B} \\
&= A(B + \overline{B})(C + \overline{C}) + BC(A + \overline{A}) + \overline{A}(B + \overline{B})(C + \overline{C}) \\
&\quad + \overline{B}(A + \overline{A})(C + \overline{C}). \\
&= A(BC + B\overline{C} + \overline{B}C + \overline{B}\,\overline{C}) + ABC + \overline{A}(BC + B\overline{C}) \\
&\quad + \overline{B}C + \overline{B}C + \overline{B}(AC + A\overline{C} + \overline{A}C + \overline{A}C + \overline{A}C) \\
&= ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\,\overline{C} + ABC + \overline{A}BC \\
&\quad + \overline{A}B\overline{C} + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C} \\
&\quad + A\,\overline{B}\,C + A\overline{B}C + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,C + \overline{A}\,\overline{B}\,\overline{C} \\
&= ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\,\overline{C} + \overline{A}BC \\
&\quad + \overline{A}B\overline{C} + \overline{A}BC + \overline{A}\,\overline{B}\,\overline{C}.
\end{aligned}
$$

- How to obtain a POS form of a function:
  - Examine each term in the given logic function. Retain if it is a maxterm
  - Check for variables that are missing in each product which is not a maxterm
  - Add $(X.\overline{X})$ for each variable X that is missing
  - Expand the expression using distributive property and eliminate redundant terms

Obtain the canonical POS of the expression

$$Y = (A + \bar{B})(B + C)(A + \bar{C})$$

$$= (\mathrm{A} + \bar{B} + C\bar{C})(A\bar{A} + B + C)(A + B\bar{B} + C)$$

$$= (A + \bar{B} + C)(A + \bar{B} + \bar{C})(A + B + C)(\bar{A} + B + C)(A + B + C)(A + \overline{B} + C)$$

$$= (A + \bar{B} + C)(A + \bar{B} + \bar{C})(A + B + C)(\bar{A} + B + C)(A + B + C)$$

- Express the function $Y = A + \bar{B}C$ in (a) canonical SOP (b) canonical POS form

(a) $Y = A + \bar{B}C$

$$= A(B + \bar{B})(C + \bar{C}) + (A + \bar{A})\bar{B}C$$

$$= (AB + A\bar{B})(C + \bar{C}) + (A\bar{B}C + \bar{A}\bar{B}C)$$

$$= ABC + A\bar{B}C + AB\bar{C} + A\bar{B}\bar{C} + A\bar{B}C + \bar{A}\bar{B}C$$

$$= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C} + \bar{A}\bar{B}C$$

$$= m_7 + m_6 + m_5 + m_4 + m_1$$

$Therefore,\ Y = \sum(1,4,5,6,7)$

(b) $Y = A + \bar{B}C$

$$= (A + \bar{B})(A + C)$$ [Distributive Law]

$$= (A + \bar{B} + C\bar{C})(A + B\bar{B} + C)$$

$$= (A + \bar{B} + C)(A + \bar{B} + \bar{C})(A + B + C)(A + \bar{B} + C)$$

$$= (A + \bar{B} + C)(A + \bar{B} + \bar{C})(A + B + C)$$

$$= M_2 M_3 M_0$$

Therefore, $Y = \Pi(0,2,3)$

Deriving SOP from Truth Table

A	B	C	Min term
0	0	0	$m_0 = \bar{A}\,\bar{B}\,\bar{C}$
0	0	1	$m_1 = \bar{A}\,\bar{B}\,C$
0	1	0	$m_2 = \bar{A}\,B\,\bar{C}$
0	1	1	$m_3 = \bar{A}\,B\,C$
1	0	0	$m_4 = A\,\bar{B}\,\bar{C}$
1	0	1	$m_5 = A\,\bar{B}\,C$
1	1	0	$m_6 = A\,B\,\bar{C}$
1	1	1	$m_7 = A\,B\,C$

Deriving POS from Truth Table

A	B	C	Max term
0	0	0	$M_0 = A + B + C$
0	0	1	$M_1 = A + B + \bar{C}$
0	1	0	$M_2 = A + \bar{B} + C$
0	1	1	$M_3 = A + \bar{B} + \bar{C}$
1	0	0	$M_4 = \bar{A} + B + C$
1	0	1	$M_5 = \bar{A} + B + \bar{C}$
1	1	0	$M_6 = \bar{A} + \bar{B} + C$
1	1	1	$M_7 = \bar{A} + \bar{B} + \bar{C}$