Energy and Momentum Methods

Introduction

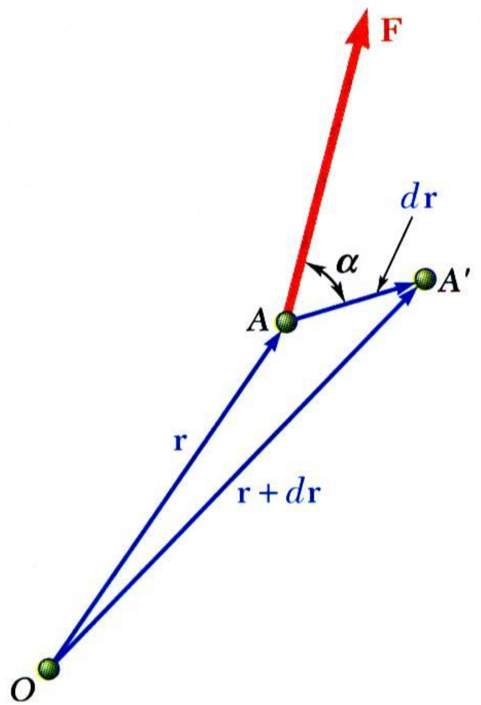
The problems dealing with the motion of particles were solved through the fundamental equation of motion, $\vec{F} = m\vec{a}$.

Method of work and energy: directly relates force, mass, velocity and displacement.

Method of impulse and momentum: directly relates force, mass, velocity, and time.

Work of a Force

$\vec{F} = F_x i + F_y j + F_z k$ $\quad\quad$ $\vec{r} = r_x i + dr_y j + r_z k$



- Differential vector $d\vec{r}$ is the *particle displacement*.

$dr = dxi + dyj + dyk$

- *Work of the force* is

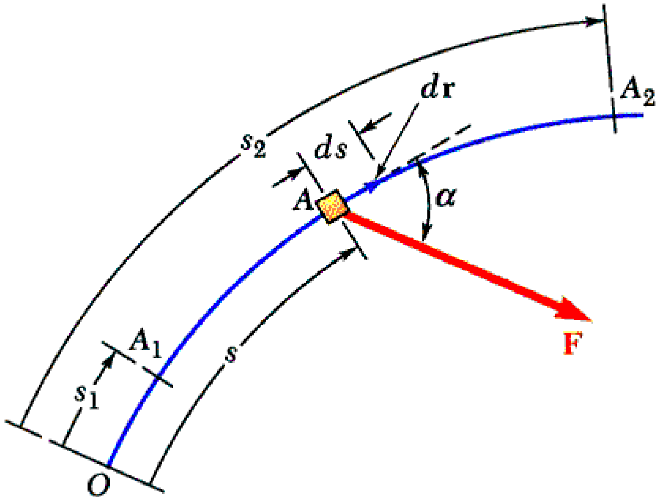
$$dU = \vec{F} \bullet d\vec{r}$$
$$= F\,ds\ \cos\alpha$$
$$= F_x dx + F_y dy + F_z dz$$

- Work is a scalar quantity, i.e., it has magnitude and sign but not direction.

- Dimensions of work are length $\times$ force. Units are

$$1\,\mathrm{J}\left(joule\right) = (1\,\mathrm{N})(1\,\mathrm{m}) \qquad 1\,\mathrm{ft}\cdot\mathrm{lb} = 1.356\,\mathrm{J}$$

Work of a Force

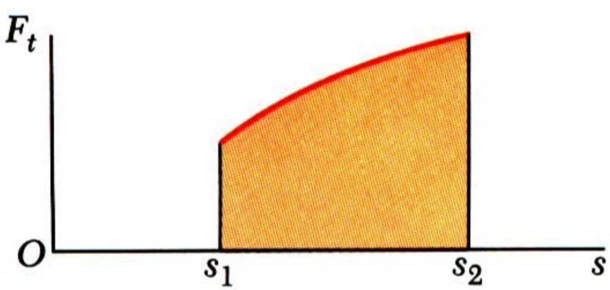


- Work of a force during a finite displacement,

$$U_{1\to 2} = \int_{A_1}^{A_2} \vec{F} \bullet d\vec{r}$$

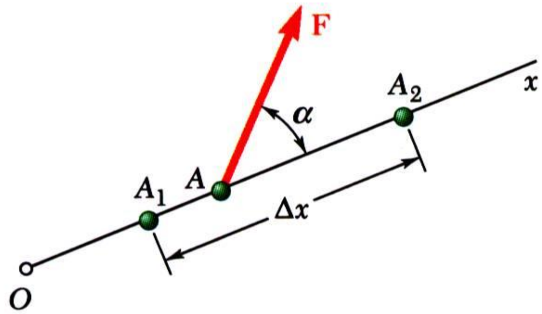
$$= \int_{s_1}^{s_2} (F\cos\alpha)\,ds = \int_{s_1}^{s_2} F_t\ ds$$

$$= \int_{A_1}^{A_2} \left(F_x dx + F_y dy + F_z dz\right)$$



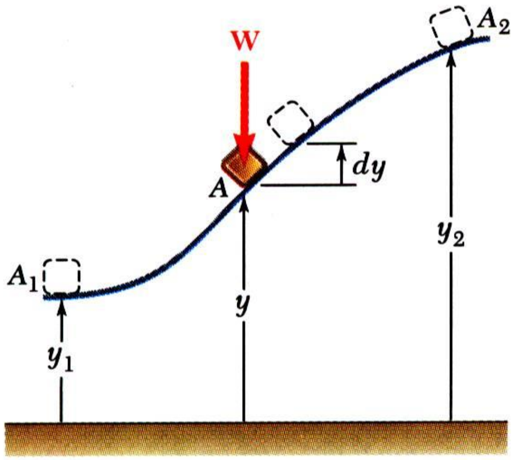
- Work is represented by the area under the curve of F_t plotted against s.

Work of a Force



- Work of a constant force in rectilinear motion,

$$U_{1\to 2} = (F\cos\alpha)\,\Delta x$$

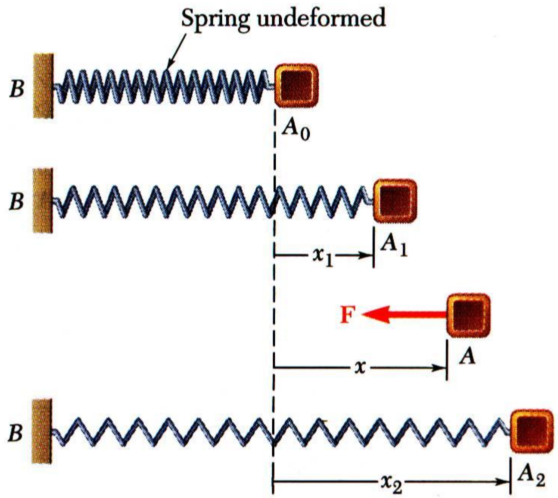


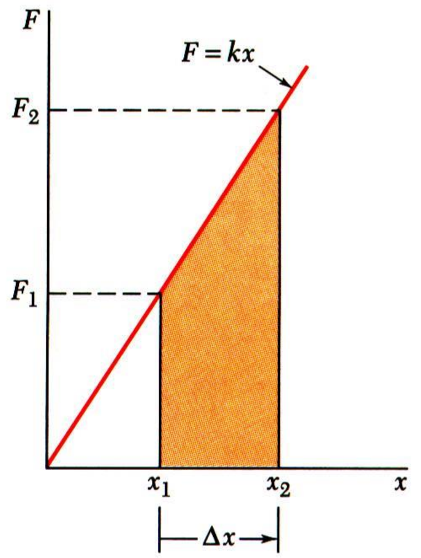
- Work of the force of gravity,

$$dU = F_x dx + F_y dy + F_z dz$$
$$= -W\ dy$$
$$U_{1\to 2} = -\int_{y_1}^{y_2} W\ dy$$
$$= -W(y_2 - y_1) = -W\ \Delta y$$

- Work *of the weight* is equal to product of weight *W* and vertical displacement d*y*.
- Work *of the weight* is positive when $\Delta y < 0$, i.e., when the weight moves down.

Work of a Force





- Magnitude of the force exerted by a spring is proportional to deflection,

$$F = kx$$

$$k = \text{spring constant } (\text{N/m or lb/in.})$$

- Work *of the force exerted by spring*,

$$dU = -F\,dx = -kx\,dx$$

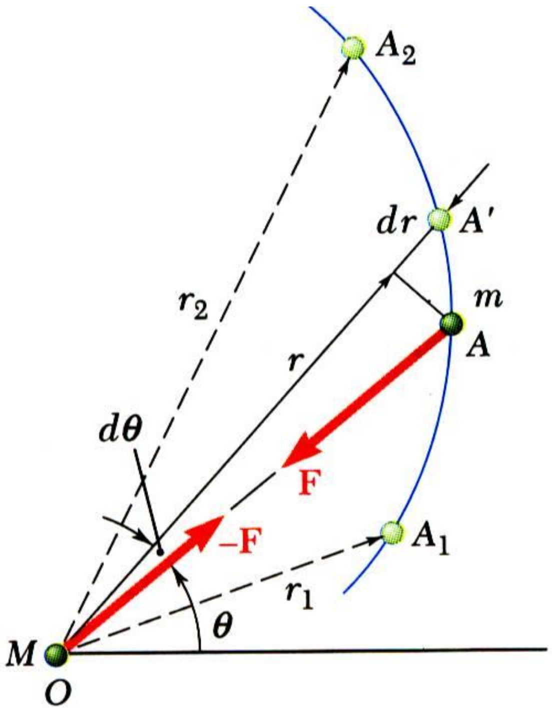
$$U_{1\to 2} = -\int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2$$

- Work *of the force exerted by spring* is positive when $x_2 < x_1$, i.e., when the spring is returning to its undeformed position.

- Work of the force exerted by the spring is equal to negative of area under curve of F plotted against x,

$$U_{1\to 2} = -\tfrac{1}{2}(F_1 + F_2)\,\Delta x$$

Work of a Force



Work of a gravitational force (assume particle M occupies fixed position O while particle m follows path shown),

$$dU = -F dr = -G\frac{Mm}{r^2}dr$$

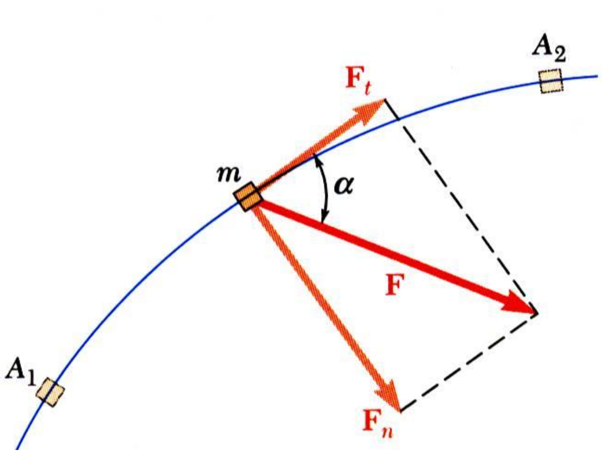
$$U_{1\to 2} = -\int_{r_1}^{r_2} G\frac{Mm}{r^2}dr = G\frac{Mm}{r_2} - G\frac{Mm}{r_1}$$

Work of a Force

Forces which *do not* do work ($ds = 0$ or $\cos \alpha = 0$):

- reaction at frictionless pin supporting rotating body,
- reaction at frictionless surface when body in contact moves along surface,
- reaction at a roller moving along its track, and
- weight of a body when its center of gravity moves horizontally.

Particle Kinetic Energy: Principle of Work & Energy



- Consider a particle of mass m acted upon by force $\vec{F}$

$$F_t = ma_t = m\frac{dv}{dt}$$

$$= m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

$$F_t\, ds = mv\, dv$$

- Integrating from A_1 to A_2 ,

$$\int_{s_1}^{s_2} F_t ds = m\int_{v_1}^{v_2} v\, dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

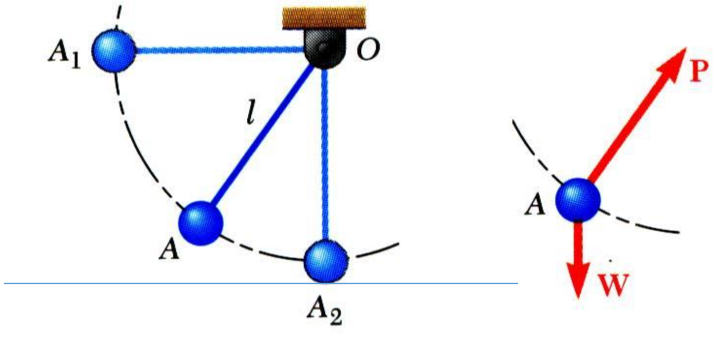
$$U_{1\to 2} = T_2 - T_1 \qquad T = \tfrac{1}{2}mv^2 = \textit{kinetic energy}$$

- *The work of the force* $\vec{F}$ *is equal to the change in kinetic energy of the particle.*

- Units of work and kinetic energy are the same:

$$T = \tfrac{1}{2}mv^2 = \mathrm{kg}\left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = \left(\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}\right)\mathrm{m} = \mathrm{N}\cdot\mathrm{m} = \mathrm{J}$$

Applications of the Principle of Work and Energy



- Wish to determine velocity of pendulum bob at A_2. Consider work & kinetic energy.

- Force $\vec{P}$ acts normal to path and does no work.

$$T_1 + U_{1\to 2} = T_2$$

$$0 + Wl = \frac{1}{2}\frac{W}{g}v_2^2$$

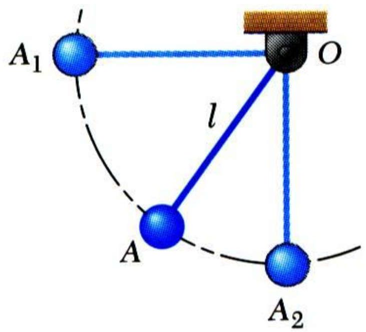
$$v_2 = \sqrt{2gl}$$

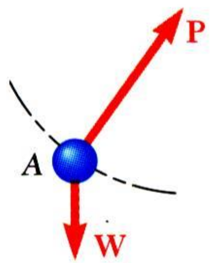
- Velocity found without determining expression for acceleration and integrating.

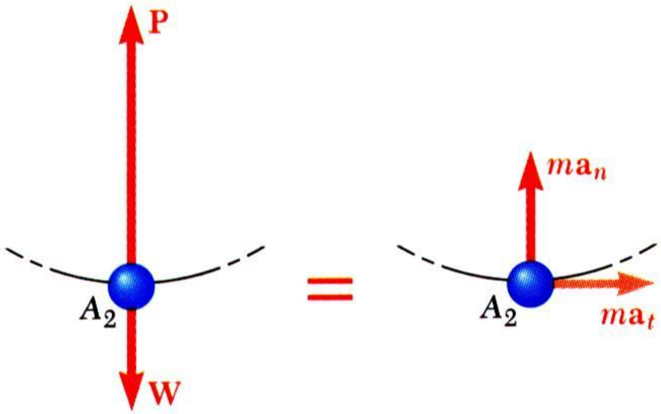
- All quantities are scalars and can be added directly.

- Forces which do no work are eliminated from the problem.

Applications of the Principle of Work and Energy







$$v_2 = \sqrt{2gl}$$

- Principle of work and energy cannot be applied to directly determine the acceleration of the pendulum bob.
- Calculating the tension in the cord requires supplementing the method of work and energy with an application of Newton’s second law.

  - As the bob passes through A_2 ,

$$\sum F_n = m\,a_n \qquad a_n = \frac{v^2}{r}$$

$$P - W = \frac{W}{g}\frac{v_2^2}{l}$$

$$P = W + \frac{W}{g}\frac{2gl}{l} = 3W$$

Power and Efficiency

- $Power = $ rate at which work is done.

$$= \frac{dU}{dt} = \frac{\vec{F} \bullet d\vec{r}}{dt}$$

$$= \vec{F} \bullet \vec{v}$$

- Dimensions of power are work/time or force*velocity. Units for power are

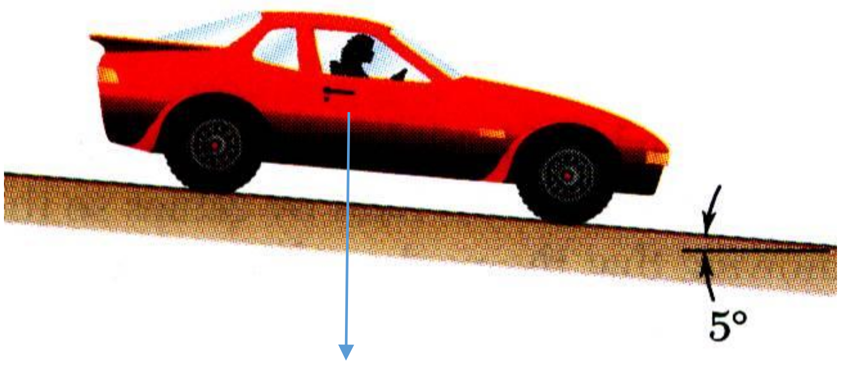
$$1\,\mathrm{W}\ (\text{watt}) = 1\frac{\mathrm{J}}{\mathrm{s}} = 1\,\mathrm{N} \cdot \frac{\mathrm{m}}{\mathrm{s}} \quad \text{or} \quad 1\,\mathrm{hp} = 550\,\frac{\mathrm{ft} \cdot \mathrm{lb}}{\mathrm{s}} = 746\,\mathrm{W}$$

- $\eta = $ efficiency

$$= \frac{\text{output work}}{\text{input work}}$$

$$= \frac{\text{power output}}{\text{power input}}$$

Problem#1



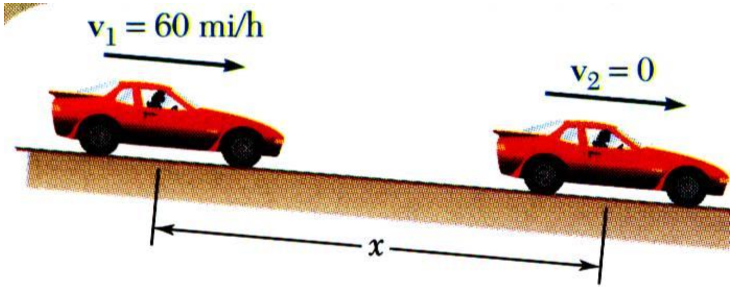
An automobile weighing 4000 N is driven down a 5° incline at a speed of 60 km/h when the brakes are applied causing a constant total breaking force of 1500 N.

Determine the distance traveled by the automobile as it comes to a stop.

SOLUTION:

- Evaluate the change in kinetic energy.
- Determine the distance required for the work to equal the kinetic energy change.

SOLUTION:

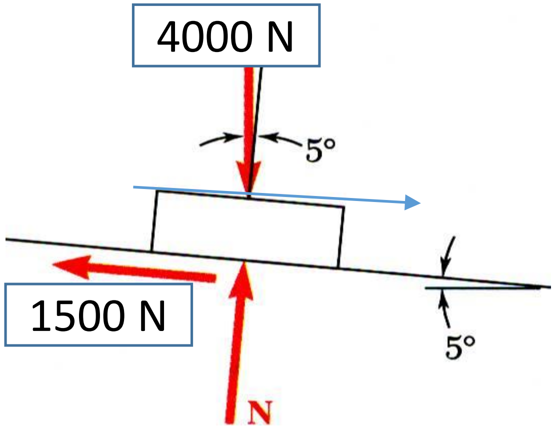


- Evaluate the change in kinetic energy.

$$v_1 = \left(60000\,\frac{\mathrm{m}}{\mathrm{h}}\right)\left(\frac{\mathrm{h}}{3600\,\mathrm{s}}\right) = 16.67\,\mathrm{m/s}$$

$$T_1 = \tfrac{1}{2} m v_1^2 = \tfrac{1}{2}(4000/9.81)(16.67)^2 = 65654.20\ \mathrm{N\text{-}m}$$

$$v_2 = 0 \qquad T_2 = 0$$



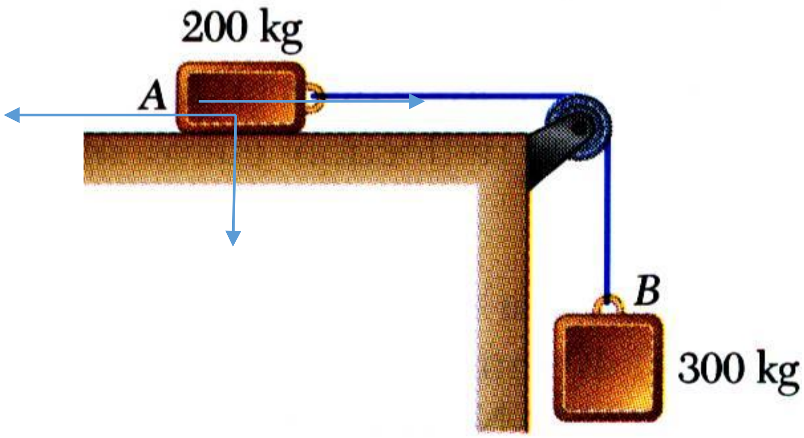
- Determine the distance required for the work to equal the kinetic energy change.

$$U_{1\to 2} = (-1500\,\mathrm{N})x + (4000\,\mathrm{N})(\sin 5°)x$$
$$= -(1151\,\mathrm{N})x$$

$$T_1 + U_{1\to 2} = T_2$$
$$65654.20\ \mathrm{N\text{-}m} - (1151\,\mathrm{N})x = 0$$

$$\boxed{x = 49.22\ \mathrm{m}}$$

Problem#2

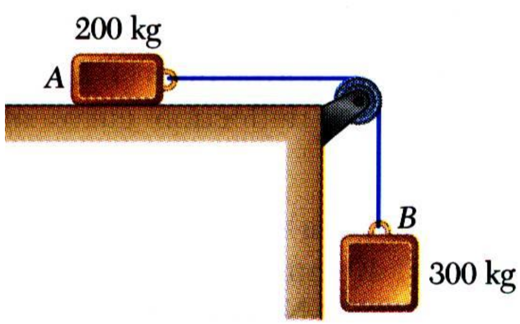


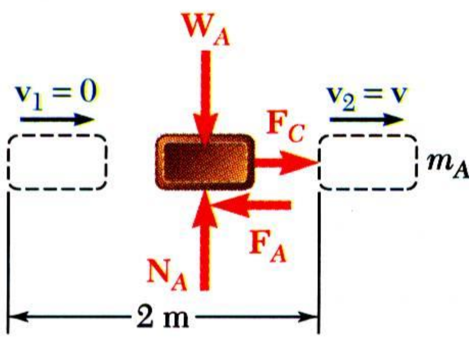
Two blocks are joined by an inextensible cable as shown. If the system is released from rest, determine the velocity of block *A* after it has moved 2 m. Assume that the coefficient of friction between block *A* and the plane is $\mu_k = 0.25$ and that the pulley is weightless and frictionless.

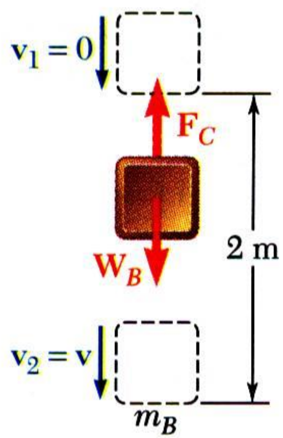
SOLUTION:

- Apply the principle of work and energy separately to blocks *A* and *B*.
- When the two relations are combined, the work of the cable forces cancel. Solve for the velocity.

Problem#2







SOLUTION:

- Apply the principle of work and energy separately to blocks *A* and *B*.

$$W_A = (200\,\mathrm{kg})\left(9.81\,\mathrm{m/s^2}\right) = 1962\,\mathrm{N}$$

$$F_A = \mu_k N_A = \mu_k W_A = 0.25(1962\,\mathrm{N}) = 490\,\mathrm{N}$$

$$T_1 + U_{1\to 2} = T_2 :$$

$$0 + F_C(2\,\mathrm{m}) - F_A(2\,\mathrm{m}) = \tfrac{1}{2} m_A v^2$$

$$F_C(2\,\mathrm{m}) - (490\,\mathrm{N})(2\,\mathrm{m}) = \tfrac{1}{2}(200\,\mathrm{kg})v^2$$

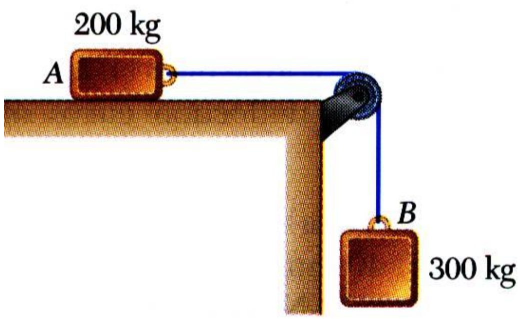
$$W_B = (300\,\mathrm{kg})\left(9.81\,\mathrm{m/s^2}\right) = 2940\,\mathrm{N}$$

$$T_1 + U_{1\to 2} = T_2 :$$

$$0 - F_c(2\,\mathrm{m}) + W_B(2\,\mathrm{m}) = \tfrac{1}{2} m_B v^2$$

$$-F_c(2\,\mathrm{m}) + (2940\,\mathrm{N})(2\,\mathrm{m}) = \tfrac{1}{2}(300\,\mathrm{kg})v^2$$

Problem#2



- When the two relations are combined, the work of the cable forces cancel. Solve for the velocity.

$$F_C(2\,\mathrm{m}) - (490\,\mathrm{N})(2\,\mathrm{m}) = \tfrac{1}{2}(200\,\mathrm{kg})v^2$$

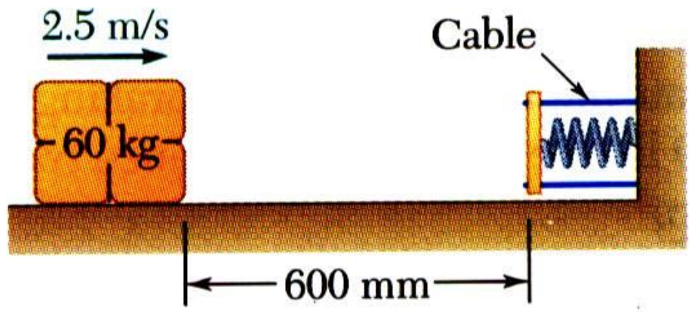
$$-F_c(2\,\mathrm{m}) + (2940\,\mathrm{N})(2\,\mathrm{m}) = \tfrac{1}{2}(300\,\mathrm{kg})v^2$$

$$(2940\,\mathrm{N})(2\,\mathrm{m}) - (490\,\mathrm{N})(2\,\mathrm{m}) = \tfrac{1}{2}(200\,\mathrm{kg} + 300\,\mathrm{kg})v^2$$

$$4900\,\mathrm{J} = \tfrac{1}{2}(500\,\mathrm{kg})v^2$$

$$\boxed{v = 4.43\,\mathrm{m/s}}$$

Problem#3



A spring is used to stop a 60 kg package which is sliding on a horizontal surface. The spring has a constant k = 20 kN/m and is held by cables so that it is initially compressed 120 mm. The package has a velocity of 2.5 m/s in the position shown and the maximum deflection of the spring is 40 mm.

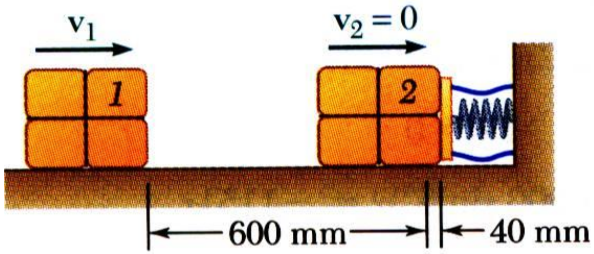
Determine *(a)* the coefficient of kinetic friction between the package and surface and *(b)* the velocity of the package as it passes again through the position shown.

SOLUTION:

- Apply the principle of work and energy between the initial position and the point at which the spring is fully compressed and the velocity is zero. The only unknown in the relation is the friction coefficient.

- Apply the principle of work and energy for the rebound of the package. The only unknown in the relation is the velocity at the final position.

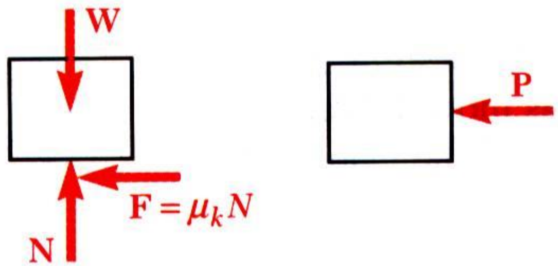
Problem#3



SOLUTION:

- Apply principle of work and energy between initial position and the point at which spring is fully compressed.

$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(60\,\text{kg})(2.5\,\text{m/s})^2 = 187.5\,\text{J} \qquad T_2 = 0$$

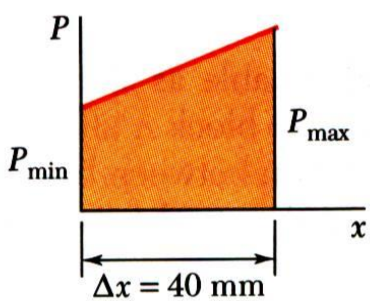


$$\begin{aligned}(U_{1\to 2})_f &= -\mu_k W\,x \\ &= -\mu_k(60\,\text{kg})(9.81\,\text{m/s}^2)(0.640\,\text{m}) = -(377\,\text{J})\mu_k\end{aligned}$$

$$P_{\min} = kx_0 = (20\,\text{kN/m})(0.120\,\text{m}) = 2400\,\text{N}$$

$$P_{\max} = k(x_0 + \Delta x) = (20\,\text{kN/m})(0.160\,\text{m}) = 3200\,\text{N}$$

$$\begin{aligned}(U_{1\to 2})_e &= -\tfrac{1}{2}(P_{\min} + P_{\max})\Delta x \\ &= -\tfrac{1}{2}(2400\,\text{N} + 3200\,\text{N})(0.040\,\text{m}) = -112.0\,\text{J}\end{aligned}$$

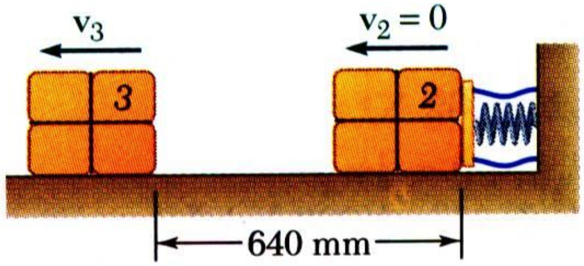


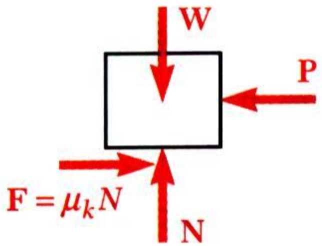
$$T_1 + U_{1\to 2} = T_2 :$$

$$187.5\,\text{J} - (377\,\text{J})\mu_k - 112\,\text{J} = 0$$

$$\boxed{\mu_k = 0.20}$$

Problem#3





- Apply the principle of work and energy for the rebound of the package.

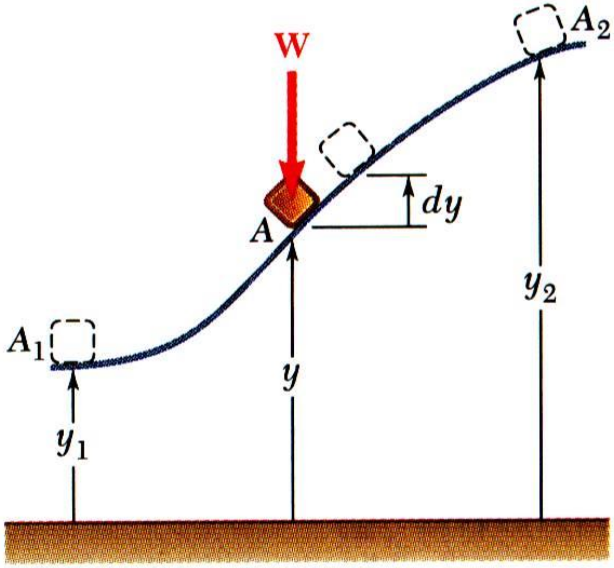
$$T_2 = 0 \qquad T_3 = \tfrac{1}{2} m v_3^2 = \tfrac{1}{2}(60\,\text{kg})v_3^2$$

$$U_{2\to 3} = \left(U_{2\to 3}\right)_f + \left(U_{2\to 3}\right)_e = -(377\,\text{J})\mu_k + 112\,\text{J}$$
$$= +36.5\,\text{J}$$

$$T_2 + U_{2\to 3} = T_3 :$$
$$0 + 36.5\,\text{J} = \tfrac{1}{2}(60\,\text{kg})v_3^2$$

$$\boxed{v_3 = 1.103\,\text{m/s}}$$

Potential Energy



- Work of the force of gravity $\vec{W}$,

$$U_{1\to 2} = W\,y_1 - W\,y_2$$

- Work is independent of path followed; depends only on the initial and final values of *Wy*.

$$V_g = Wy$$

$=$ *potential energy* of the body with respect to *force of gravity*.

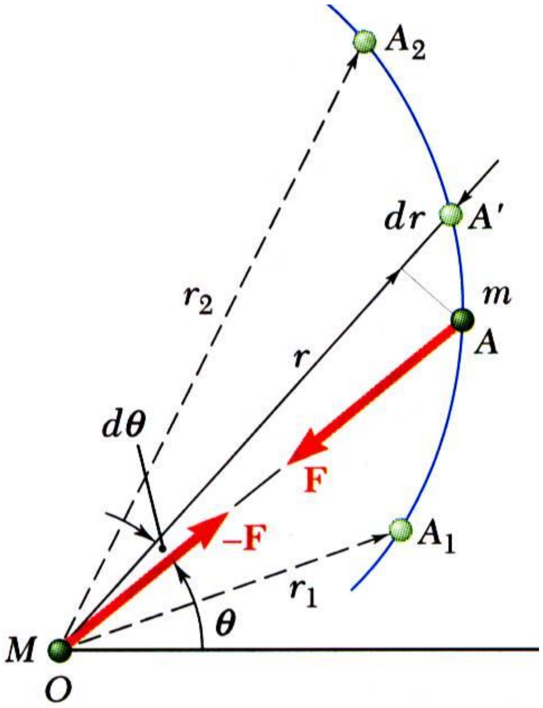
$$U_{1\to 2} = \left(V_g\right)_1 - \left(V_g\right)_2$$

- Choice of datum from which the elevation *y* is measured is arbitrary.

- Units of work and potential energy are the same:

$$V_g = Wy = \mathrm{N}\cdot\mathrm{m} = \mathrm{J}$$

Potential Energy



- Previous expression for potential energy of a body with respect to gravity is only valid when the weight of the body can be assumed constant.

- For a space vehicle, the variation of the force of gravity with distance from the center of the earth should be considered.

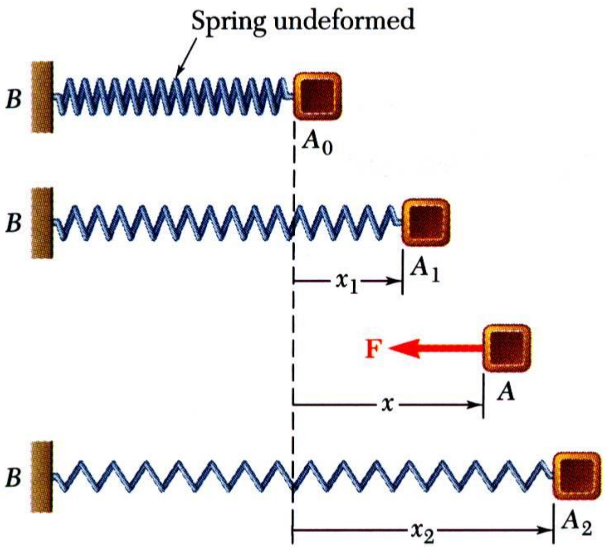
- Work of a gravitational force,

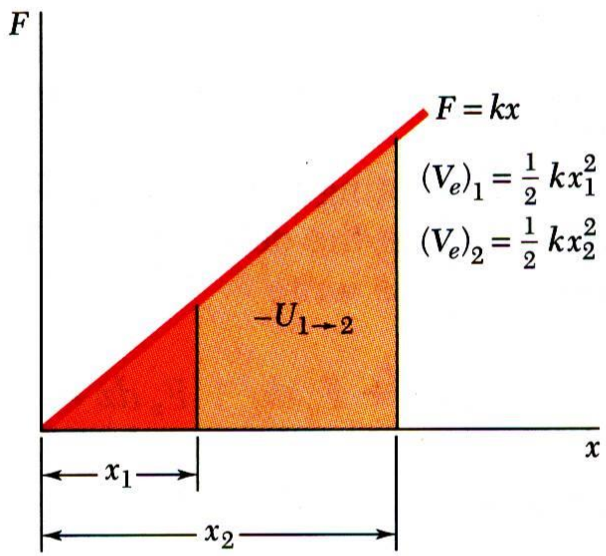
$$U_{1\to 2} = \frac{GMm}{r_2} - \frac{GMm}{r_1}$$

- Potential energy V_g when the variation in the force of gravity can not be neglected,

$$V_g = -\frac{GMm}{r} = -\frac{WR^2}{r}$$

Potential Energy





- Work of the force exerted by a spring depends only on the initial and final deflections of the spring,

$$U_{1\to 2} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2$$

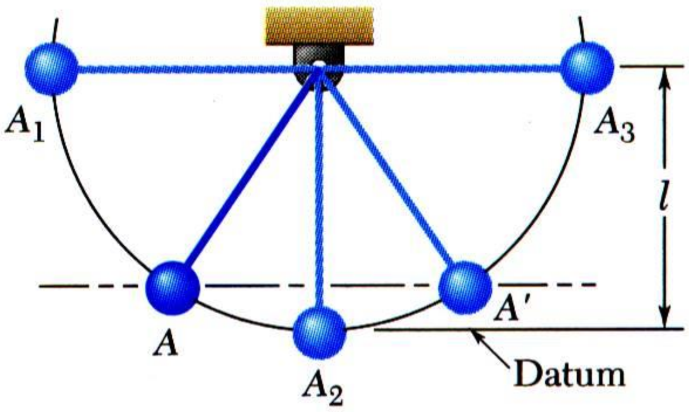
- The potential energy of the body with respect to the elastic force,

$$V_e = \tfrac{1}{2}kx^2$$

$$U_{1\to 2} = (V_e)_1 - (V_e)_2$$

- Note that the preceding expression for V_e is valid only if the deflection of the spring is measured from its undeformed position.

Conservation of Energy

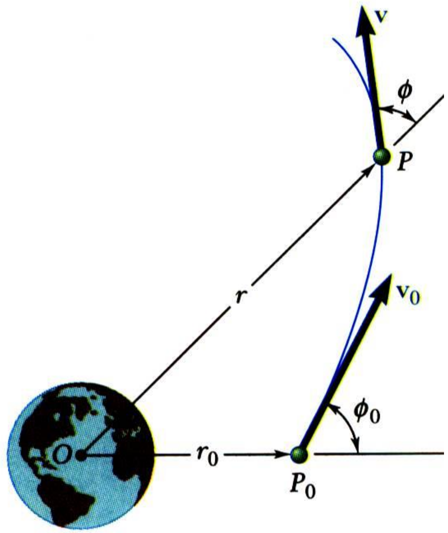


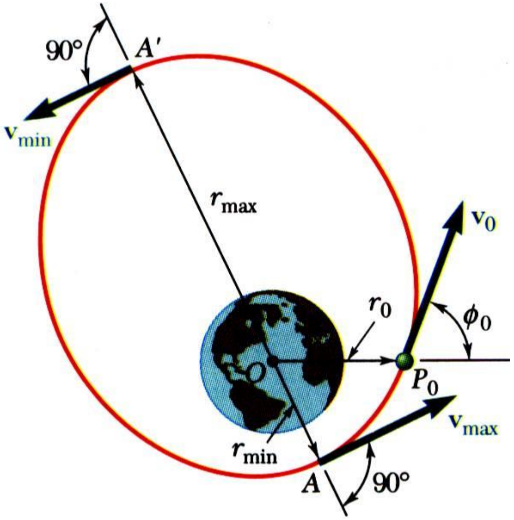
$$T_1 = 0 \quad V_1 = W\ell$$
$$T_1 + V_1 = W\ell$$

$$T_2 = \tfrac{1}{2} m v_2^2 = \frac{1}{2}\frac{W}{g}(2g\ell) = W\ell \quad V_2 = 0$$
$$T_2 + V_2 = W\ell$$

- Work of a conservative force,
  $$U_{1\to 2} = V_1 - V_2$$
- Concept of work and energy,
  $$U_{1\to 2} = T_2 - T_1$$
- Follows that
  $$T_1 + V_1 = T_2 + V_2$$
  $$E = T + V = \text{constant}$$
- When a particle moves under the action of conservative forces, the total mechanical energy is constant.
- Friction forces are not conservative. Total mechanical energy of a system involving friction decreases.
- Mechanical energy is dissipated by friction into thermal energy. Total energy is constant.

Motion Under a Conservative Central Force





- When a particle moves under a conservative central force, both the principle of conservation of angular momentum

$$r_0 m v_0 \sin\phi_0 = r m v \sin\phi$$

and the principle of conservation of energy

$$T_0 + V_0 = T + V$$

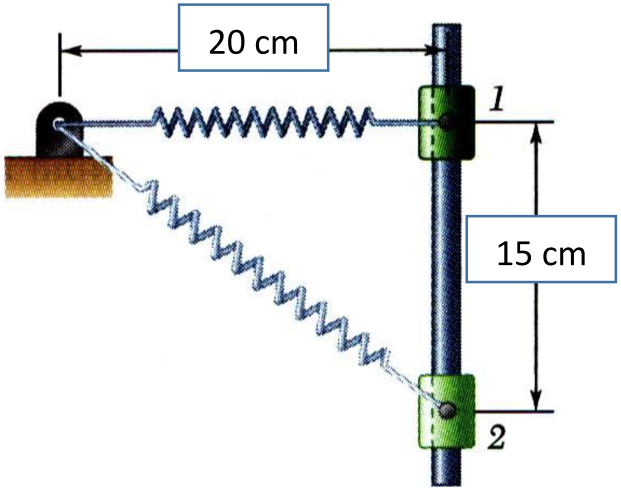
$$\tfrac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2} m v^2 - \frac{GMm}{r}$$

may be applied.

- Given r, the equations may be solved for v and φ.

- At minimum and maximum r, $\varphi = 90^{\circ}$. Given the launch conditions, the equations may be solved for r_{min}, r_{max}, v_{min}, and v_{max}.

Problem#4



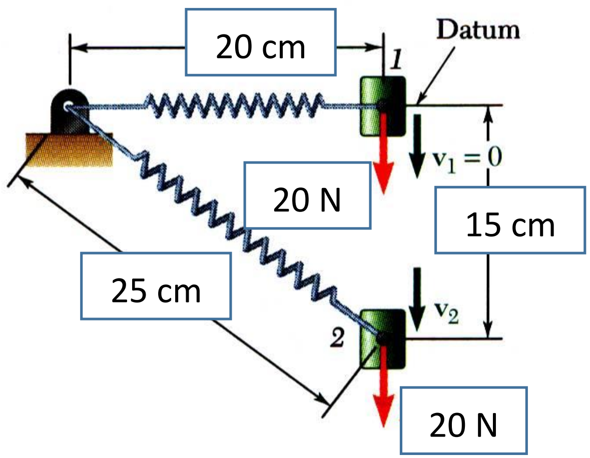
A 20 N collar slides without friction along a vertical rod as shown. The spring attached to the collar has an undeflected length of 10 cm. and a constant of 3 N/cm.

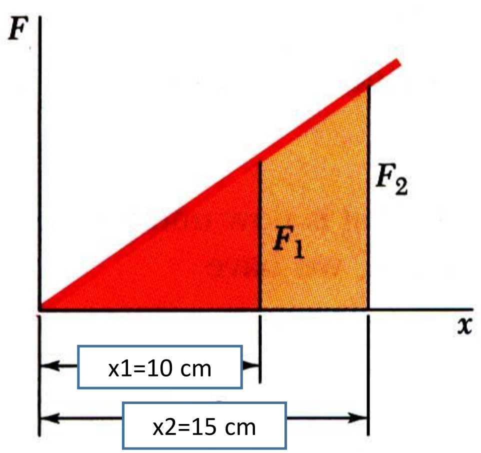
If the collar is released from rest at position 1, determine its velocity after it has moved 15 cm. to position 2.

SOLUTION:

- Apply the principle of conservation of energy between positions 1 and 2.
- The elastic and gravitational potential energies at 1 and 2 are evaluated from the given information. The initial kinetic energy is zero.
- Solve for the kinetic energy and velocity at 2.

Problem#4





SOLUTION:

- Apply the principle of conservation of energy between positions 1 and 2.

Position 1:

$$V_e = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(3\,\text{N/cm})(20\,\text{cm.} - 10\,\text{cm.})^2 = 150\,\text{N.cm}$$

$$V_1 = V_e + V_g = 15\,\text{N.cm} + 0 = 15\,\text{N.cm}$$

$$T_1 = 0$$

Position 2:

$$V_e = \tfrac{1}{2}kx_2^2 = \tfrac{1}{2}(3\,\text{N/cm.})(25\,\text{cm.} - 10\,\text{cm.})^2 = 337.5N.cm$$

$$V_g = Wy = (20\,\text{N})(-15\,\text{cm.}) = -300N.cm$$

$$V_2 = V_e + V_g = 337.5 - 300 = 37.5N.cm$$

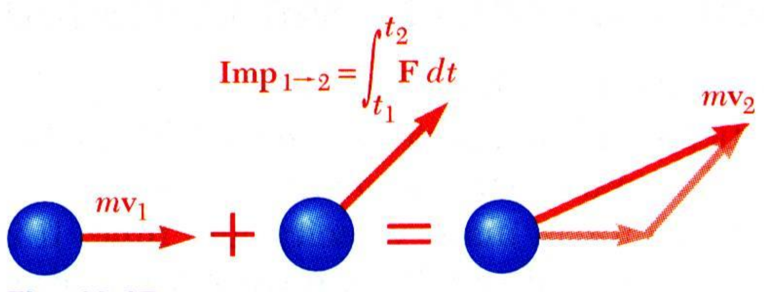
$$T_2 = \tfrac{1}{2}mv_2^2 = \frac{1}{2}\frac{20}{9.81}v_2^2 = 0.203v_2^2$$

Problem#4

Conservation of Energy:

$$T_1 + V_1 = T_2 + V_2$$

Principle of Impulse and Momentum



- From Newton's second law,

$$\vec{F} = \frac{d}{dt}(m\vec{v}) \qquad m\vec{v} = \text{linear momentum}$$

$$\vec{F}dt = d(m\vec{v})$$

$$\int_{t_1}^{t_2} \vec{F}dt = m\vec{v}_2 - m\vec{v}_1$$

$$\int_{t_1}^{t_2} \vec{F}dt = \mathbf{Imp}_{1\to2} = \text{impulse of the force } \vec{F}$$

$$m\vec{v}_1 + \mathbf{Imp}_{1\to2} = m\vec{v}_2$$

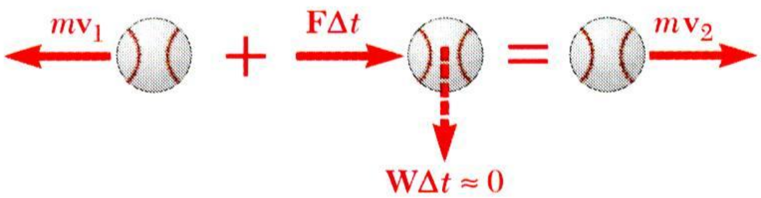
- Dimensions of the impulse of a force are
  *force*time.*

- Units for the impulse of a force are

$$\mathrm{N \cdot s} = \left(\mathrm{kg \cdot m/s^2}\right)\cdot \mathrm{s} = \mathrm{kg \cdot m/s}$$

- The final momentum of the particle can be obtained by adding vectorially its initial momentum and the impulse of the force during the time interval.

Impulsive Motion



- Force acting on a particle during a very short time interval that is large enough to cause a significant change in momentum is called an *impulsive force*.

- When impulsive forces act on a particle,

  $$m\vec{v}_1 + \sum \vec{F}\,\Delta t = m\vec{v}_2$$

- When a baseball is struck by a bat, contact occurs over a short time interval but force is large enough to change sense of ball motion.

- *Nonimpulsive forces* are forces for which $\vec{F}\Delta t$ is small and therefore, may be neglected.

Problem#6

An automobile weighing 4000 N is driven down a 5° incline at a speed of 60 km/h when the brakes are applied, causing a constant total braking force of 1500 N.

Determine the time required for the automobile to come to a stop.

SOLUTION:

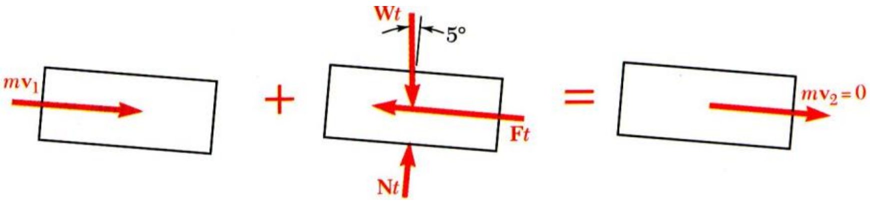
- Apply the principle of impulse and momentum. The impulse is equal to the product of the constant forces and the time interval.

SOLUTION:

- Apply the principle of impulse and momentum.

$$m\vec{v}_1 + \sum \mathbf{Imp}_{1\to 2} = m\vec{v}_2$$

$$v_1 = \left(60000\,\frac{\mathrm{m}}{\mathrm{h}}\right)\left(\frac{\mathrm{h}}{3600\,\mathrm{s}}\right) = 16.67\,\mathrm{m/s}$$



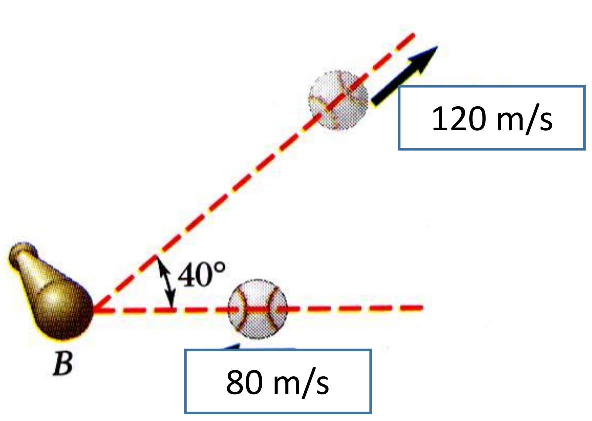
Taking components parallel to the incline,

$$mv_1 + (W\sin 5^\circ)t - Ft = 0$$

$$\left(\frac{4000}{9.81}\right)(16.67\,m/s) + (4000\sin 5^\circ)t - 1500t = 0$$

$$\boxed{t = 1.27\,\mathrm{s}}$$

Problem#7

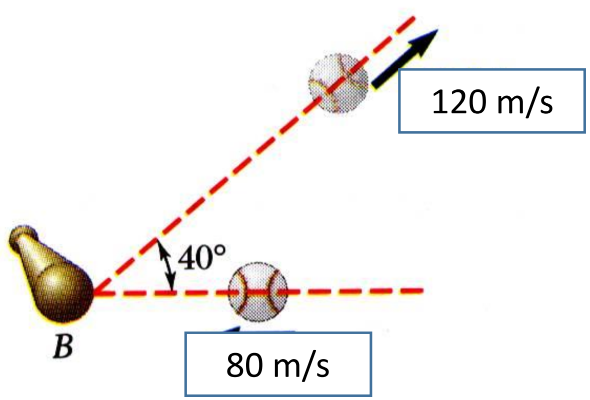


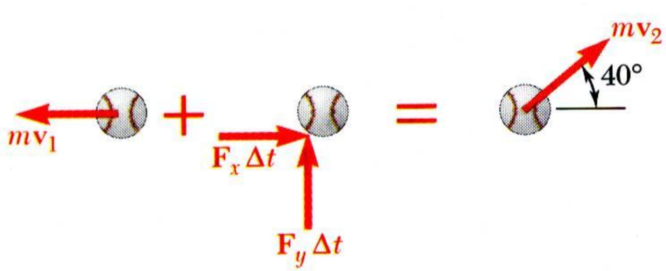
A 100 gm baseball is pitched with a velocity of 80 m/s. After the ball is hit by the bat, it has a velocity of 120 m/s in the direction shown. If the bat and ball are in contact for 0.015 s, determine the average impulsive force exerted on the ball during the impact.

<u>SOLUTION</u>:

- Apply the principle of impulse and momentum in terms of horizontal and vertical component equations.

Problem#7





<u>SOLUTION</u>:

- Apply the principle of impulse and momentum in terms of horizontal and vertical component equations.

$$m\vec{v}_1 + \mathbf{Imp}_{1\to 2} = m\vec{v}_2$$

x component equation:

$$-mv_1 + F_x\Delta t = mv_2 \cos 40^\circ$$

$$-0.1(80) + F_x(0.15) = 0.1(120\cos 40^\circ)$$

$$F_x = 114.61N$$

y component equation:

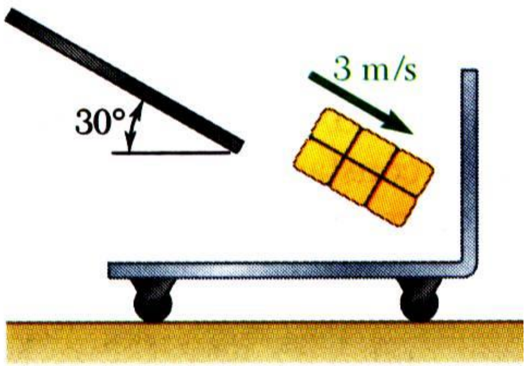
$$0 + F_y\Delta t = mv_2 \sin 40^\circ$$

$$F_y(0.15) = 0.1(120\cos 40^\circ)$$

$$F_y = 61.28N$$

$$\boxed{\vec{F} = (114.61)\vec{i} + (61.28)\vec{j}, \quad F = 129.96N}$$

Problem#8



A 10 kg package drops from a chute into a 25 kg cart with a velocity of 3 m/s. Knowing that the cart is initially at rest and can roll freely, determine *(a)* the final velocity of the cart, *(b)* the impulse exerted by the cart on the package, and *(c)* the fraction of the initial energy lost in the impact.

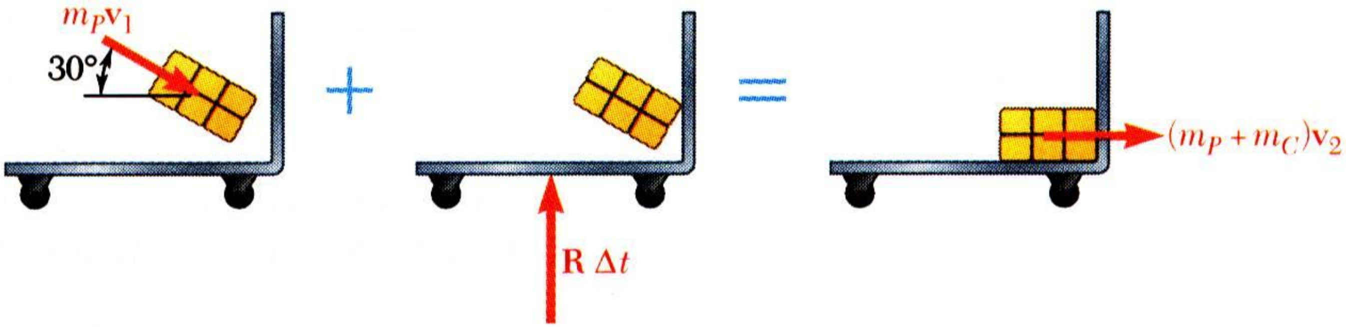
<u>SOLUTION</u>:

- Apply the principle of impulse and momentum to the package-cart system to determine the final velocity.
- Apply the same principle to the package alone to determine the impulse exerted on it from the change in its momentum.

Problem#8

<u>SOLUTION</u>:

- Apply the principle of conservation of momentum to the package-cart system to determine the final velocity.



$$m_p\vec{v}_1 + m_c\vec{v}c = (m_p + m_c)\vec{v}_2$$

$$\overrightarrow{Vc} = 0$$

x components:

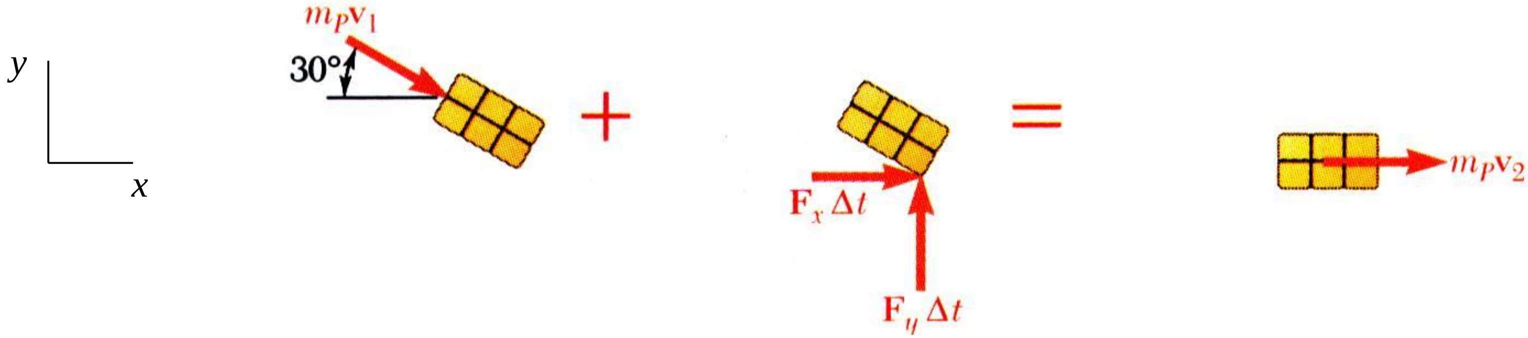
$$m_p v_1 \cos 30° + 0 = (m_p + m_c)v_2$$

$$(10\text{ kg})(3\text{ m/s})\cos 30° = (10\text{ kg} + 25\text{ kg})v_2$$

$$\boxed{v_2 = 0.742\text{ m/s}}$$

Problem#8

- Apply the same principle to the package alone to determine the impulse exerted on it from the change in its momentum.



$$m_p\vec{v}_1 + \sum \mathbf{Imp}_{1\to 2} = m_p\vec{v}_2$$

x components:

$$m_p v_1 \cos 30^\circ + F_x\Delta t = m_p v_2$$

$$(10\text{ kg})(3\text{ m/s})\cos 30^\circ + F_x\Delta t = (10\text{ kg})v_2 \qquad F_x\Delta t = -18.56\text{ N}\cdot\text{s}$$

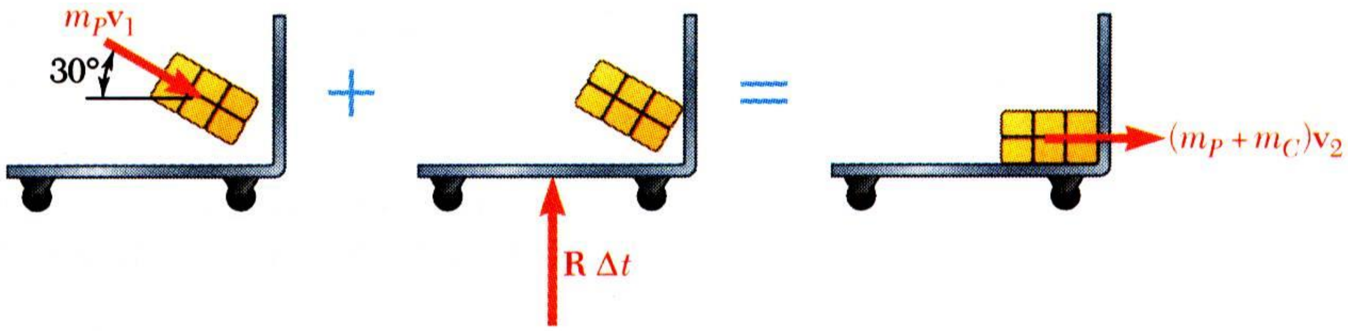
y components:

$$-m_p v_1 \sin 30^\circ + F_y\Delta t = 0$$

$$-(10\text{ kg})(3\text{ m/s})\sin 30^\circ + F_y\Delta t = 0 \qquad F_y\Delta t = 15\text{ N}\cdot\text{s}$$

$$\boxed{\sum \mathbf{Imp}_{1\to 2} = \vec{F}\Delta t = (-18.56\text{ N}\cdot\text{s})\vec{i} + (15\text{ N}\cdot\text{s})\vec{j} \qquad F\Delta t = 23.9\text{ N}\cdot\text{s}}$$

Problem#8



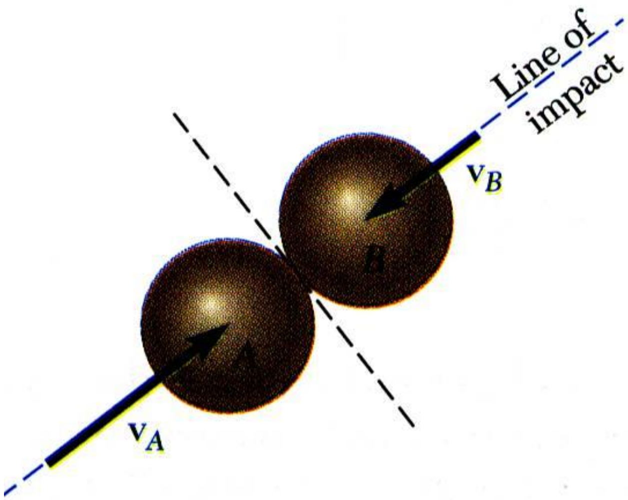
To determine the fraction of energy lost,

$$T_1 = \tfrac{1}{2} m_p v_1^2 = \tfrac{1}{2}(10\text{ kg})(3\text{ m/s})^2 = 45\text{ J}$$

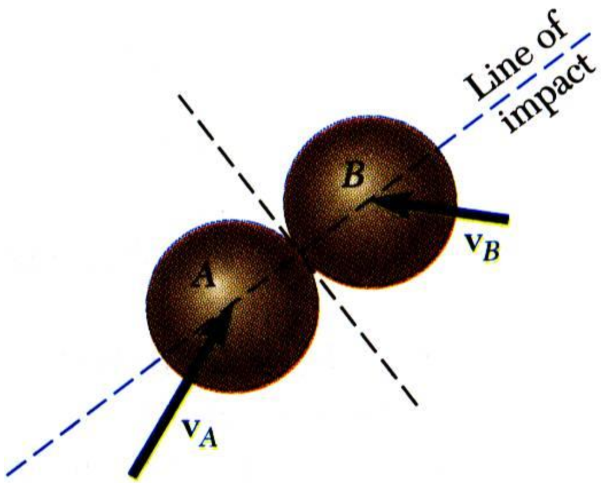
$$T2 = \tfrac{1}{2}(m_p + m_c)v_2^2 = \tfrac{1}{2}(10\text{ kg} + 25\text{ kg})(0.742\text{ m/s})^2 = 9.63\text{ J}$$

$$\boxed{\frac{T_1 - T_2}{T_1} = \frac{45\text{ J} - 9.63\text{ J}}{45\text{ J}} = 0.786}$$

Impact



Direct Central Impact



Oblique Central Impact

- *Impact:* Collision between two bodies which occurs during a small time interval and during which the bodies exert large forces on each other.

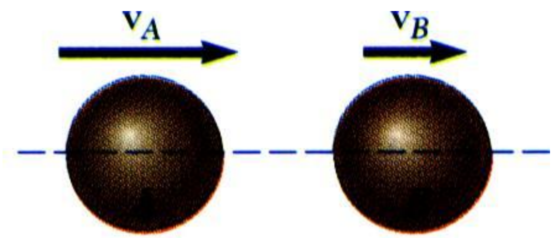
- *Line of Impact:* Common normal to the surfaces in contact during impact.

- *Central Impact:* Impact for which the mass centers of the two bodies lie on the line of impact; otherwise, it is an *eccentric impact*..

- *Direct Impact:* Impact for which the velocities of the two bodies are directed along the line of impact.

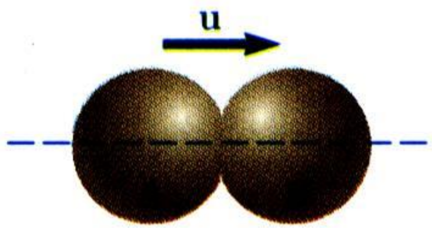
- *Oblique Impact:* Impact for which one or both of the bodies move along a line other than the line of impact.

Direct Central Impact

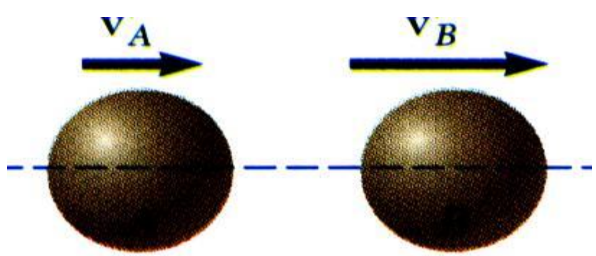


- Bodies moving in the same straight line, $v_A > v_B$.

- Upon impact the bodies undergo a *period of deformation*, at the end of which, they are in contact and moving at a common velocity.



- A *period of restitution* follows during which the bodies either regain their original shape or remain permanently deformed.

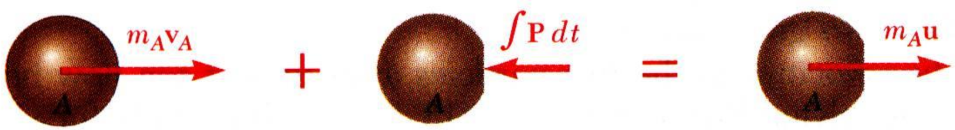


- Wish to determine the final velocities of the two bodies. The total momentum of the two body system is preserved,

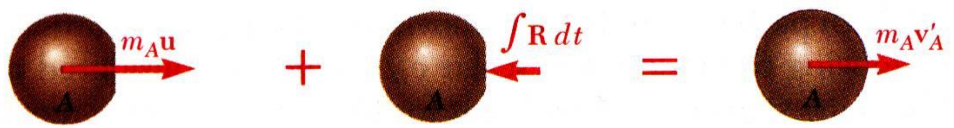
$$m_A v_A + m_B v_B = m_B v'_A + m_B v'_B$$

- A second relation between the final velocities is required.

Direct Central Impact



- Period of deformation: $m_A v_A - \int P dt = m_A u$



- Period of restitution: $m_A u - \int R dt = m_A v'_A$

$$e = \textit{coefficient of restitution}$$

$$= \frac{\int R dt}{\int P dt} = \frac{u - v'_A}{v_A - u}$$

$$0 \le e \le 1$$

- A similar analysis of particle *B* yields

$$e = \frac{v'_B - u}{u - v_B}$$

- Combining the relations leads to the desired second relation between the final velocities.

$$v'_B - v'_A = e(v_A - v_B)$$

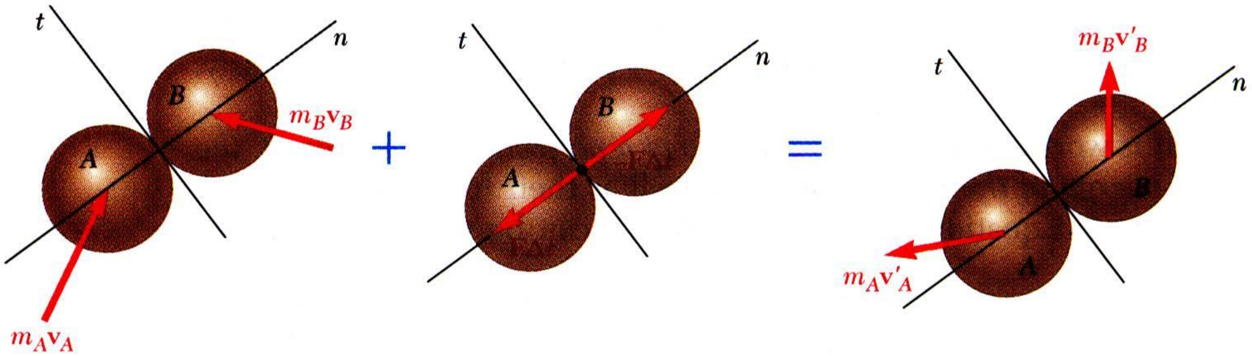
- *Perfectly plastic impact, e* = 0: $v'_B = v'_A = v'$

$$m_A v_A + m_B v_B = (m_A + m_B) v'$$

- *Perfectly elastic impact, e* = 1:
  Total energy and total momentum conserved.

$$v'_B - v'_A = v_A - v_B$$

Oblique Central Impact



- Final velocities are unknown in magnitude and direction. Four equations are required.

- No tangential impulse component; tangential component of momentum for each particle is conserved.

$$(v_A)_t = (v'_A)_t \qquad (v_B)_t = (v'_B)_t$$

- Normal component of total momentum of the two particles is conserved.

$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n$$

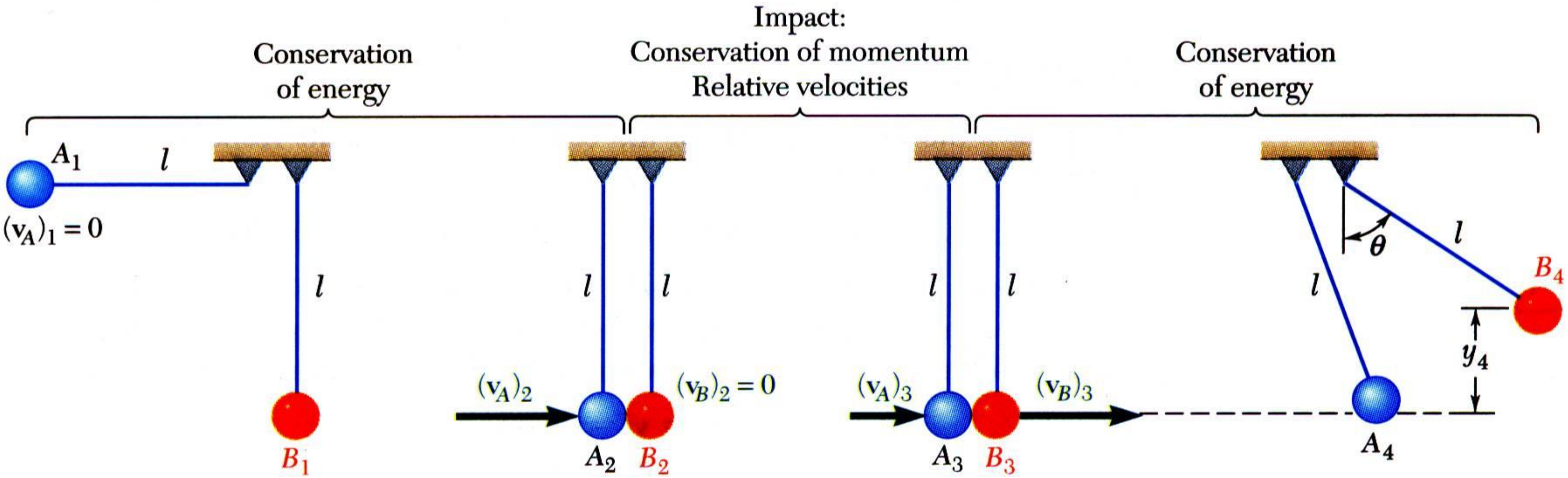
- Normal components of relative velocities before and after impact are related by the coefficient of restitution.

$$(v'_B)_n - (v'_A)_n = e\left[(v_A)_n - (v_B)_n\right]$$

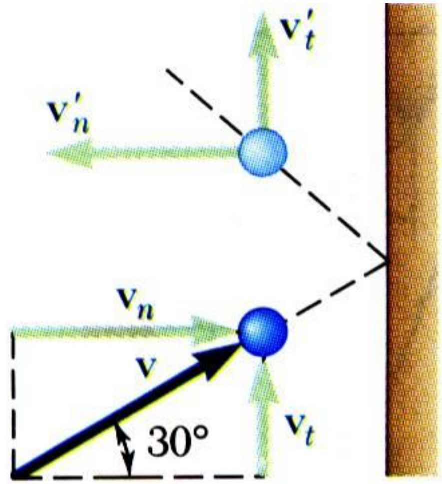
Problems Involving Energy and Momentum

- Three methods for the analysis of kinetics problems:
  - Direct application of Newton's second law
  - Method of work and energy
  - Method of impulse and momentum

- Select the method best suited for the problem or part of a problem under consideration.



Problem#9

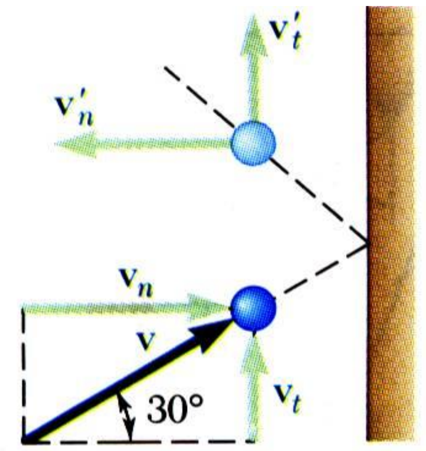


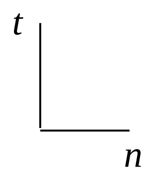
A ball is thrown against a frictionless, vertical wall. Immediately before the ball strikes the wall, its velocity has a magnitude v and forms angle of 30° with the horizontal. Knowing that $e = 0.90$, determine the magnitude and direction of the velocity of the ball as it rebounds from the wall.

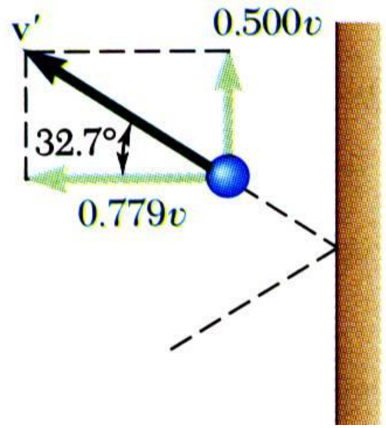
<u>SOLUTION</u>:

- Resolve ball velocity into components normal and tangential to wall.
- Impulse exerted by the wall is normal to the wall. Component of ball momentum tangential to wall is conserved.
- Assume that the wall has infinite mass so that wall velocity before and after impact is zero. Apply coefficient of restitution relation to find change in normal relative velocity between wall and ball, i.e., the normal ball velocity.

Problem#9







SOLUTION:

- Resolve ball velocity into components parallel and perpendicular to wall.

$$v_n = v\cos 30° = 0.866v \qquad v_t = v\sin 30° = 0.500v$$

- Component of ball momentum tangential to wall is conserved.

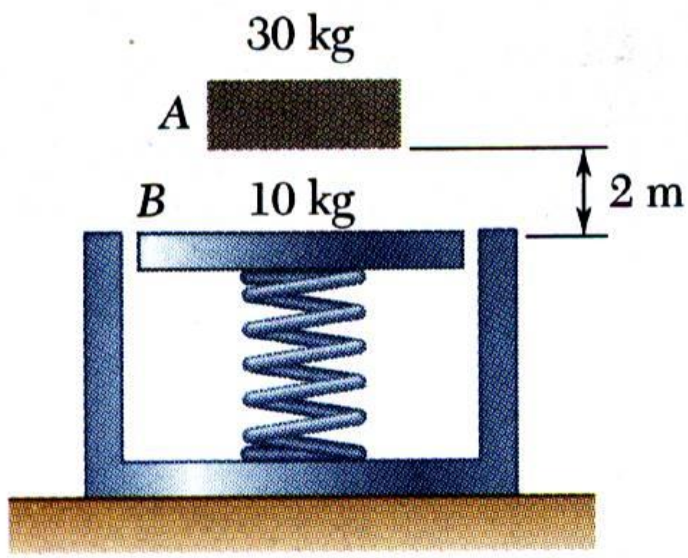
$$v'_t = v_t = 0.500v$$

- Apply coefficient of restitution relation with zero wall velocity.

$$0 - v'_n = e(v_n - 0)$$
$$v'_n = -0.9(0.866v) = -0.779v$$

$$\boxed{\begin{array}{l} \vec{v}' = -0.779v\,\vec{\lambda}_n + 0.500v\,\vec{\lambda}_t \\ v' = 0.926v \quad \tan^{-1}\left(\dfrac{0.779}{0.500}\right) = 32.7° \end{array}}$$

Problem#10

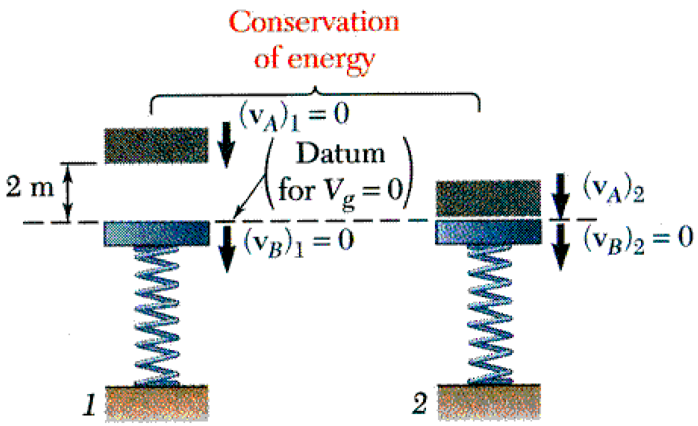


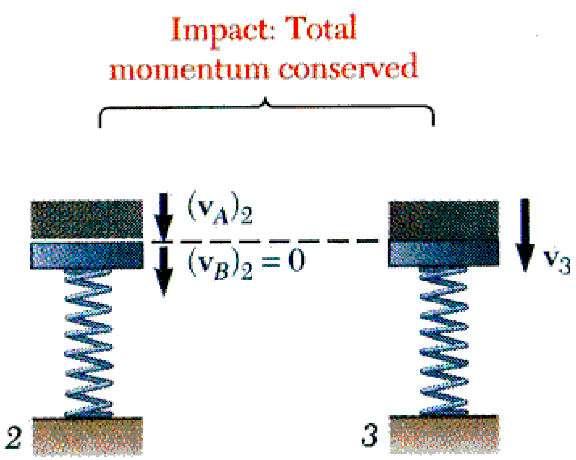
A 30 kg block is dropped from a height of 2 m onto the 10 kg pan of a spring scale. Assuming the impact to be perfectly plastic, determine the maximum deflection of the pan. The constant of the spring is k = 20 kN/m.

SOLUTION:

- Apply the principle of conservation of energy to determine the velocity of the block at the instant of impact.
- Since the impact is perfectly plastic, the block and pan move together at the same velocity after impact. Determine that velocity from the requirement that the total momentum of the block and pan is conserved.
- Apply the principle of conservation of energy to determine the maximum deflection of the spring.

Problem#10





SOLUTION:

- Apply principle of conservation of energy to determine velocity of the block at instant of impact.

$$T_1 = 0 \qquad V_1 = W_A y = (30)(9.81)(2) = 588\text{ J}$$

$$T_2 = \tfrac{1}{2} m_A (v_A)_2^2 = \tfrac{1}{2}(30)(v_A)_2^2 \qquad V_2 = 0$$

$$T_1 + V_1 = T_2 + V_2$$

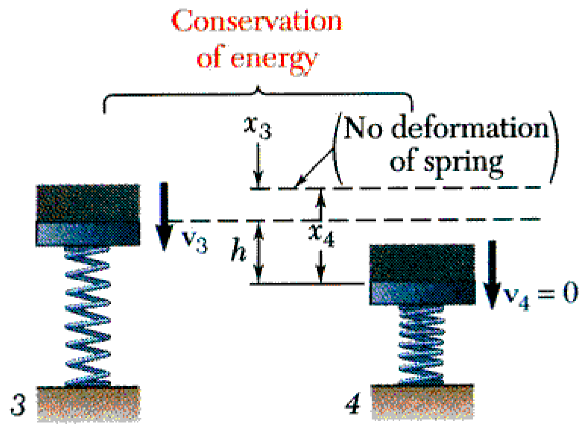
$$0 + 588\text{ J} = \tfrac{1}{2}(30)(v_A)_2^2 + 0 \qquad (v_A)_2 = 6.26\text{ m/s}$$

- Determine velocity after impact from requirement that total momentum of the block and pan is conserved.

$$m_A (v_A)_2 + m_B (v_B)_2 = (m_A + m_B) v_3$$

$$(30)(6.26) + 0 = (30 + 10) v_3 \qquad v_3 = 4.70\text{ m/s}$$

Problem#10



Initial spring deflection due to pan weight:

$$x_3 = \frac{W_B}{k} = \frac{(10)(9.81)}{20 \times 10^3} = 4.91 \times 10^{-3}\,\text{m}$$

- Apply the principle of conservation of energy to determine the maximum deflection of the spring.

$$T_3 = \tfrac{1}{2}(m_A + m_B)v_3^2 = \tfrac{1}{2}(30 + 10)(4.7)^2 = 442\ \text{J}$$

$$V_3 = V_g + V_e$$

$$= 0 + \tfrac{1}{2}kx_3^2 = \tfrac{1}{2}\left(20 \times 10^3\right)\left(4.91 \times 10^{-3}\right)^2 = 0.241\ \text{J}$$

$$T_4 = 0$$

$$V_4 = V_g + V_e = (W_A + W_B)(-h) + \tfrac{1}{2}kx_4^2$$

$$= -392(x_4 - x_3) + \tfrac{1}{2}\left(20 \times 10^3\right)x_4^2$$

$$= -392\left(x_4 - 4.91 \times 10^{-3}\right) + \tfrac{1}{2}\left(20 \times 10^3\right)x_4^2$$

$$T_3 + V_3 = T_4 + V_4$$

$$442 + 0.241 = 0 - 392\left(x_4 - 4.91 \times 10^{-3}\right) + \tfrac{1}{2}\left(20 \times 10^3\right)x_4^2$$

$$x_4 = 0.230\ \text{m}$$

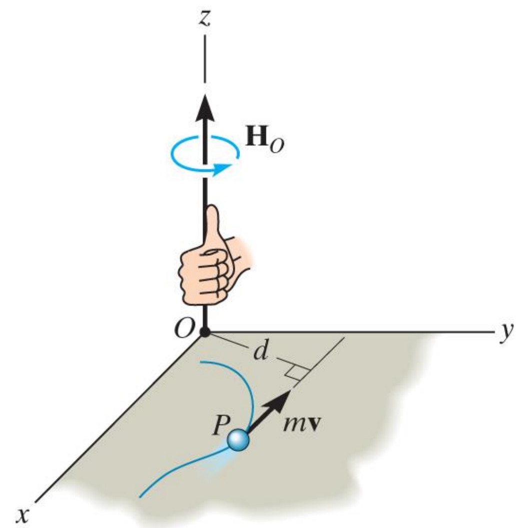
$$h = x_4 - x_3 = 0.230\ \text{m} - 4.91 \times 10^{-3}\ \text{m} \qquad \boxed{h = 0.225\ \text{m}}$$

Angular Momentum

The angular momentum $\overrightarrow{H_0}$ of a particle about point O is defined as the ”moment” of the particle’s linear momentum about O. It is sometimes referred to as the moment of the momentum.

Scalar Formulation

Consider a particle of mass m moving in the x-y plane with a velocity $\vec{V}$.



Scalar Formulation

The magnitude of the angular momentum of this particle about point O has a magnitude equal to $dm|\bar{V}|$where d is the moment arm or the perpendicular distance from O to the line of action of $m\bar{V}$.

The direction can be found using the right hand rule Curling your right hand in the direction of $m\bar{V}$ from the reference O.

The extended thump points in the direction of $\overrightarrow{H_0}$.

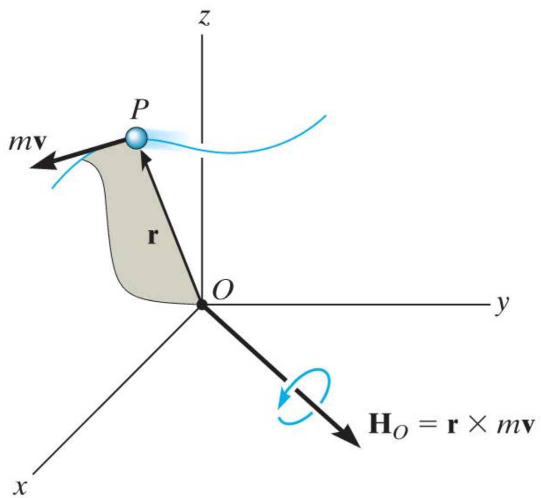
Common units for $\overrightarrow{H_0}$ are kg · m2/s or slug · ft2/s.

Vector Formulation:

If the particle is moving along a space curve and $\vec{r}$ is a position vector drawn from point O to the particle P then

$$\overrightarrow{H_0} = \vec{r} \times m\vec{V}$$

Vector Formulation:

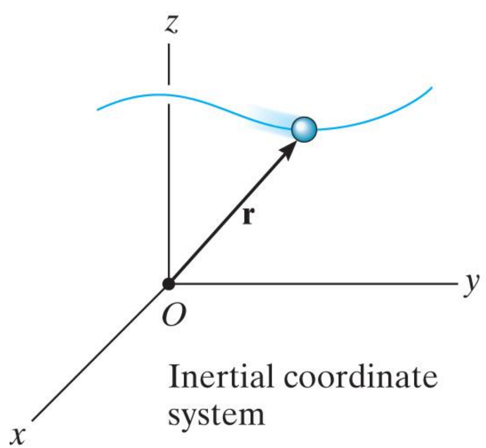


As shown in the figure $\overrightarrow{H_0}$ is perpendicular to the plane made of $\vec{r}$ and $m\vec{V}$.

In the cartesian coordinate system where all vectors are expressed in terms of the unit vector along the x, y and z axes the angular momentum is determined by evaluating the determinant:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ mV_x & mV_y & mV_z \end{vmatrix}$$

Relation between moment of a force and angular momentum:



(a)

Consider a particle of constant mass m moving in an inertial frame as shown in the figure. Newton's second applied to the particle states that

$$\sum \vec{F} = \frac{d\left(m\vec{V}\right)}{dt} = m\vec{a}$$

Relation between moment of a force and angular momentum:

The moment of the forces about point O $\sum \overrightarrow{M_0}$ can be expressed using Newton's second law as

$$\sum \overrightarrow{M_0} = \vec{r} \times \sum \vec{F} = \vec{r} \times m\dot{\vec{V}} = \frac{d}{dt}\left(\vec{r} \times m\vec{V}\right) = \dot{\overrightarrow{H_0}}$$

This equation states that the resultant moment about point O of all the forces acting on the particle is equal to the time rate of change of the particle's angular momentum about point O.

Angular Impulse and Momentum Principles

$\sum \overrightarrow{M_0} = \dot{\overrightarrow{H_0}}$ this equation can be rewritten as follows

$$\sum \overrightarrow{M_0} = \dot{\overrightarrow{H_0}} = \frac{d\overrightarrow{H_0}}{dt}$$

Now integrating above equation

$$\int \sum \overrightarrow{M_0} dt = \int d\overrightarrow{H_0}$$

we have assuming that at tine t = t1, $\overrightarrow{H_0} = \left(\overrightarrow{H_0}\right)_1$ and at time t = t2, $\overrightarrow{H_0} = \left(\overrightarrow{H_0}\right)_2$

$$\sum \int_{t1}^{t2} \overrightarrow{M_0} dt = \left(\overrightarrow{H_0}\right)_2 - \left(\overrightarrow{H_0}\right)_1$$

This equation is referred to as the principle of angular impulse and momentum.

Angular Impulse and Momentum Principles

The initial and final angular momenta $\overrightarrow{H_0} = \left(\overrightarrow{H_0}\right)_1$ and $\overrightarrow{H_0} = \left(\overrightarrow{H_0}\right)_2$ are defined as the moment of tth linear momentum of the particle at the instant t1 and t2 respectively.

The second term on the left side $\sum \int_{t1}^{t2} \overrightarrow{M_0} dt$ is called the angular impulse.

It is determined by integrating with respect to time, the moments of all the forces acting on the particle over the time period t1 to t2

$$angular\ impluse = \sum \int_{t1}^{t2} \overrightarrow{M_0} dt = \int_{t1}^{t2} \left(\vec{r} \times \vec{F}\right) dt$$

Conservation of Angular Momentum:

When the angular impulses acting on a particle are all zero during the time t1 to t2 we have

$$\left(\overrightarrow{H_0}\right)_1 = \left(\overrightarrow{H_0}\right)_2$$

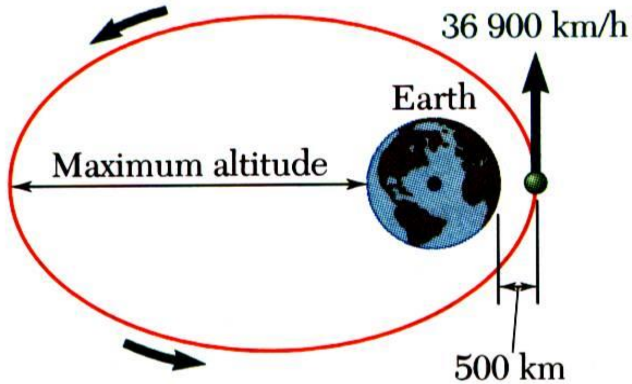
which is known as conservation of angular momentum.

If no external impulse is applied to the particle, both linear and angular momentum will be conserved.

Conservation of angular momentum for a system of particles is given by

$$\sum\left(\overrightarrow{H_0}\right)_1 = \sum\left(\overrightarrow{H_0}\right)_2$$

Problem#5



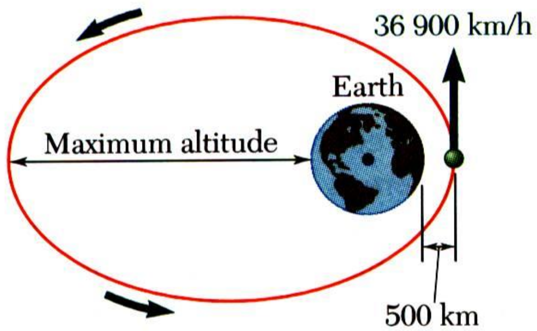
A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36900 km/h from an altitude of 500 km.

Determine *(a)* the maximum altitude reached by the satellite, and *(b)* the maximum allowable error in the direction of launching if the satellite is to come no closer than 200 km to the surface of the earth

SOLUTION:

- For motion under a conservative central force, the principles of conservation of energy and conservation of angular momentum may be applied simultaneously.
- Apply the principles to the points of minimum and maximum altitude to determine the maximum altitude.
- Apply the principles to the orbit insertion point and the point of minimum altitude to determine maximum allowable orbit insertion angle error.

SOLUTION



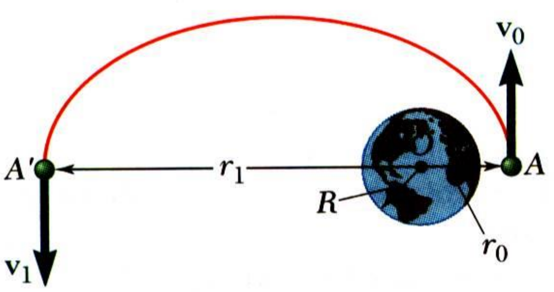
- Apply the principles of conservation of energy and conservation of angular momentum to the points of minimum and maximum altitude to determine the maximum altitude.

Conservation of energy:

$$T_A + V_A = T_{A'} + V_{A'} \qquad \tfrac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2} m v_1^2 - \frac{GMm}{r_1}$$

Conservation of angular momentum:

$$r_0 m v_0 = r_1 m v_1 \qquad v_1 = v_0 \frac{r_0}{r_1}$$



Combining,

$$\tfrac{1}{2} v_0^2 \left(1 - \frac{r_0^2}{r_1^2}\right) = \frac{GM}{r_0}\left(1 - \frac{r_0}{r_1}\right) \qquad 1 + \frac{r_0}{r_1} = \frac{2GM}{r_0 v_0^2}$$

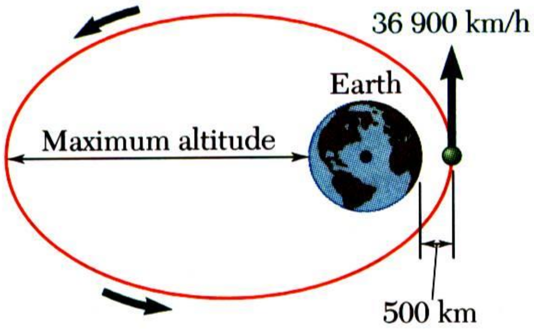
$$r_0 = 6370\ \text{km} + 500\ \text{km} = 6870\ \text{km}$$

$$v_0 = 36900\ \text{km/h} = 10.25 \times 10^6\ \text{m/s}$$

$$GM = gR^2 = \left(9.81\ \text{m/s}^2\right)\left(6.37 \times 10^6\ \text{m}\right)^2 = 398 \times 10^{12}\ \text{m}^3/\text{s}^2$$

$$\boxed{r_1 = 60.4 \times 10^6\ \text{m} = 60400\ \text{km}}$$

SOLUTION



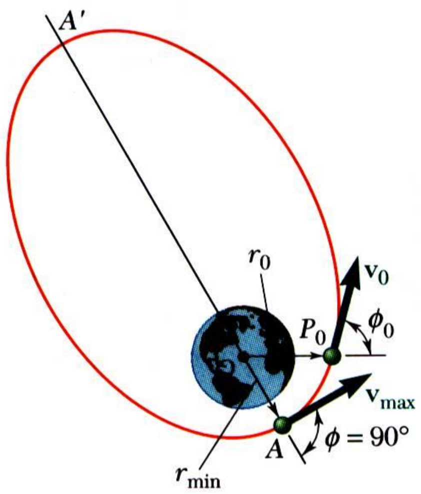
- Apply the principles to the orbit insertion point and the point of minimum altitude to determine maximum allowable orbit insertion angle error.

Conservation of energy:

$$T_0 + V_0 = T_A + V_A \qquad \tfrac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2} m v_{max}^2 - \frac{GMm}{r_{min}}$$

Conservation of angular momentum:

$$r_0 m v_0 \sin\phi_0 = r_{min} m v_{max} \qquad v_{max} = v_0 \sin\phi_0 \frac{r_0}{r_{min}}$$



Combining and solving for sin φ_0,

$$\sin\phi_0 = 0.9801$$

$$\varphi_0 = 90° \pm 11.5°$$

$$\boxed{\text{allowable error} = \pm 11.5°}$$