# Exercise

1. A car moving at 90km/h was slowed down from the beginning of the curve at **A**, shown in figure-1. After travelling a distance of 100m, the speed reduced to 45km/h at **B**. If radius of the curve is 300m, determine the net acceleration of the car at point **A** and at point **B.** Assume retardation is constant.

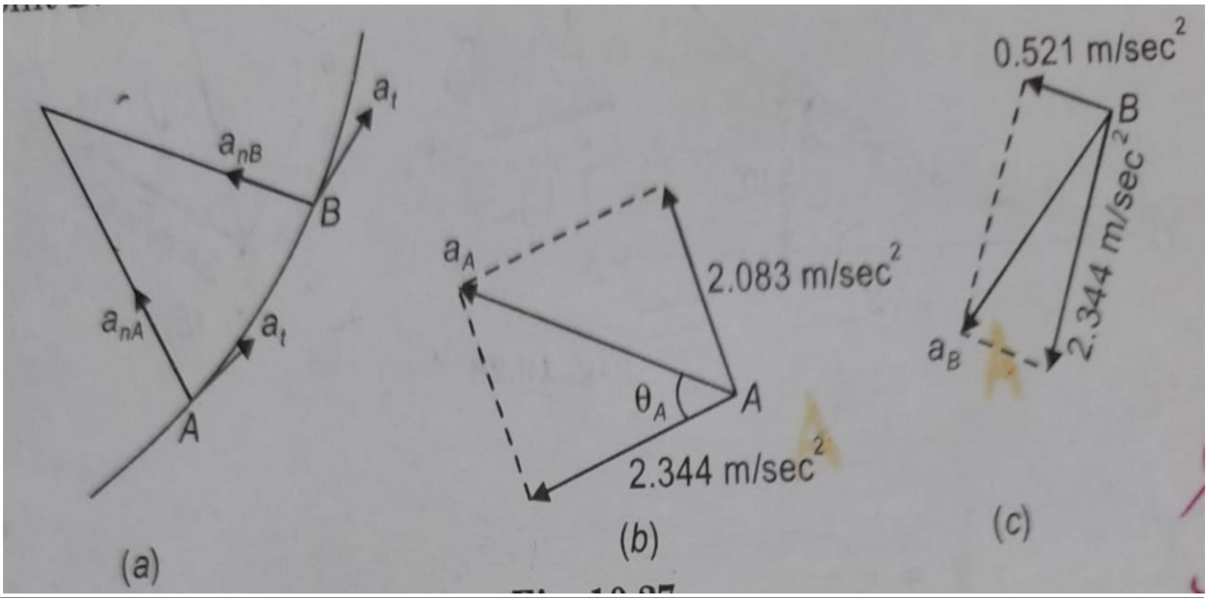



**Figure-1**

**Solution:**

Initial velocity = 90km/h = 25m/sec

Final velocity = 45km/h$=\dfrac{45\times1000}{60\times60}=12.5m/s$

$\therefore$ *From the equation* $v^2-u^2$=2as, we get

$$12.5^2-25^2=2a_t100$$

$a_t$= -2.344 m/s², constant throughout

At A $a_n=\dfrac{v^2}{\rho}=\dfrac{25^2}{300}=2.083$

$\therefore \quad a_A=\sqrt{(a_t^2+a_n^2)}=\sqrt{2.344^2+2.083^2}$

$=3.136\text{m/s}^2$

$\tan\theta_A=\dfrac{a_n}{a_t}=\dfrac{2.083}{2.344}$

Therefore $\theta_A=41.626^0$ as shown in figure-1

At B $a_n = \frac{v^2}{\rho} = \frac{12.5}{300} = 0.521 m/s^2$

$\therefore \qquad a_B = \sqrt{2.344^2 + 0.521^2}$

$\tan\theta_B = \frac{a_n}{a_t} = \frac{0.521}{2.344}$

$\theta_B = 12.531^0$, as shown in figure-1.

2. A car is moving down a sloping ground represented by the curve $x^2=240y$, where $x$ and $y$ are in meters. When the car is at position **A** as shown in figure-2, its velocity is 72km/h and the retardation is 2.4m/s$^2$. Determine the total acceleration at A.

**Solution:**

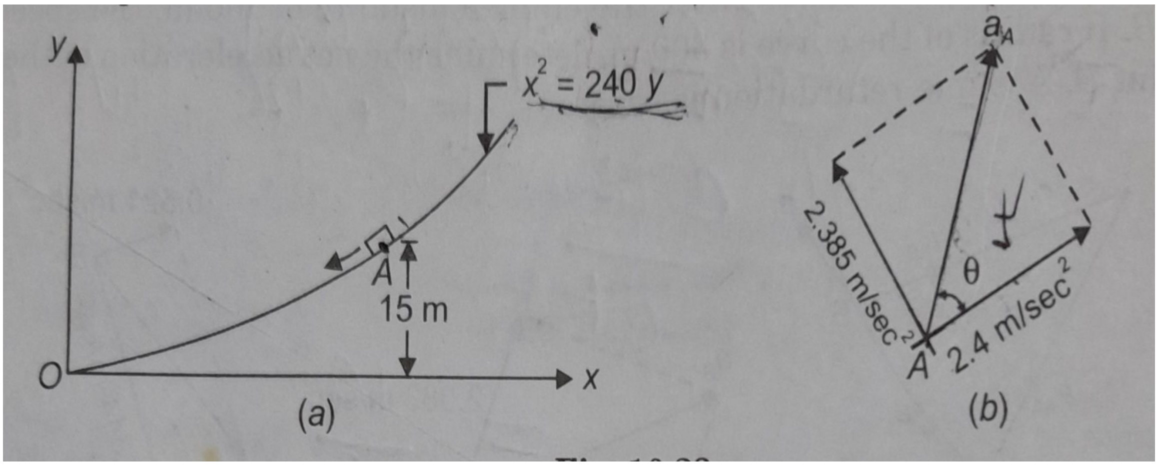



**Figure-2**

Now $y = \frac{x^2}{240}$

$$\frac{dy}{dx} = \frac{x}{120}$$

And $\frac{d^2y}{dx^2} = \frac{1}{120}$

$$\therefore \qquad \rho = \frac{[1 + (\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{[1 + (\frac{x}{120})^2]^{3/2}}{\frac{1}{120}}$$

At A $\quad y = 15\text{m}$

$x = \sqrt{240y} = \sqrt{240 \times 15} = 60\text{m}$

$$\therefore at\ A\ \rho = \frac{[1 + (\frac{60}{120})^2]^{3/2}}{\frac{1}{120}} = 167.705\ m$$

At A, $\quad v = 72\text{km/h} = \frac{72 \times 1000}{60 \times 60} = 20 m/s$

$\therefore$ At A $\quad a_n = \frac{v^2}{\rho} = \frac{(20)^2}{167.705} = 2.385\ m/s^2$

$a_t = -2.4\text{m/s}^2$

$a = \sqrt{a_t^2 + a_n^2} = 3.384\ \text{m/s}^2$

$\tan\theta = \frac{a_n}{a_t} = \frac{2.385}{2.4} = 44.82°$ as shown in figure-2

3. The motion of a particle in $x$-$y$ plane is given by $\mathbf{r} = t\ \mathbf{i} + (3t^2 - 4t)\mathbf{j}$, where the distance are in meter and time is in seconds.
   Determine the radius of curvature and the normal and tangential acceleration when the particle crosses the $x$-axis after the start of motion .

**Solution:**

Let $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} = t\mathbf{i} + (3t^2 - 4t)\mathbf{j}$

$\therefore \quad x = t$ and $y = 3x^2 - 4x = x(3x - 4)$

$y = 0$ at $x = 0$ and at $x = \frac{4}{3} m.$

Hence the point of interest A is at $x = 4/3$m.

$y = 3x^2 - 4x$

$\therefore \quad \frac{dy}{dx} = 6x - 4 \qquad \therefore [\frac{dy}{dx}]_{at\ A} = 6 \times \frac{4}{3} - 4 = 4$

$\frac{d^2y}{dx^2} = 6 \qquad \therefore [\frac{d^2y}{dx^2}]_{at\ A} = 6$

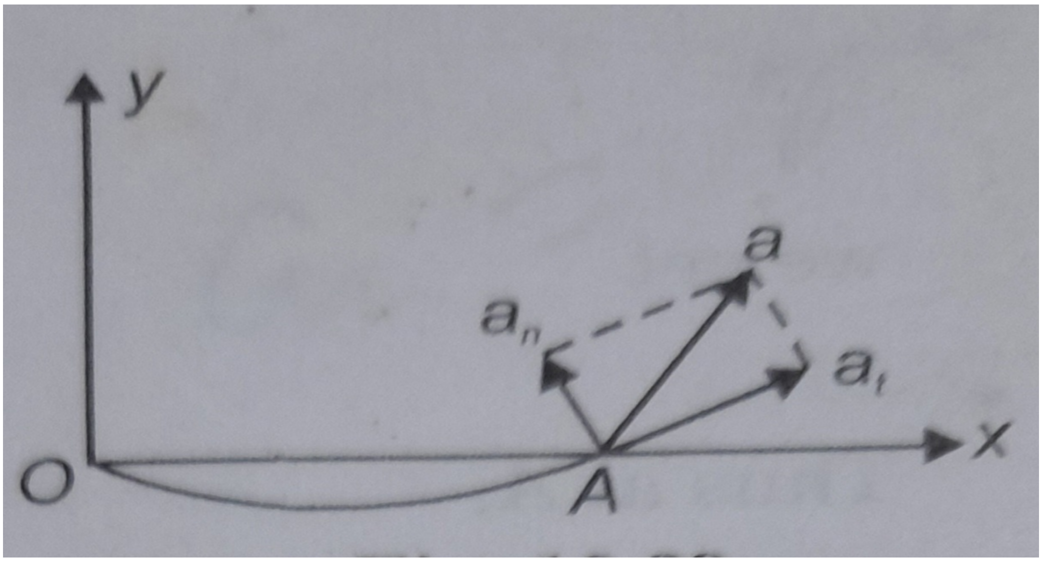



**Figure-3**

At A,

$\therefore \qquad \rho = \frac{[1+(\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{[1+4^2]^{3/2}}{6}$=11.682 m

Now, $\quad x=t \qquad$ and $y=3t^2-4t$

$\therefore \qquad v_x=\frac{dx}{dt} = 1 \qquad v_y = \frac{dy}{dt} = 6t - 4$

$$a_x = \frac{d^2x}{dt^2} = 0 \qquad a_y = \frac{d^2y}{dt^2} = 6$$

At A, $\quad x=\frac{4}{3}, t = x = \frac{4}{3}$

$\therefore \quad v_x$=1m/s $\quad v_y=6\times\frac{4}{3} - 4 = 4m/s$

$v=\sqrt{1^2 + 4^2} = 4.123m/s$

$a_x=0 \qquad a_y=6m/s^2$

$\therefore \quad a=\sqrt{a_x^2 + a_y^2}$=6m/s$^2$

At A $\quad a_n = \frac{v^2}{\rho} = \frac{(4.123)^2}{11.682} = 1.455\ m/s^2$

From the relation,

$$a=\sqrt{a_n^2 + a_t^2}$$

we get $6 = \sqrt{1.455^2 + a_t^2}$

$\therefore$ $a_t = 1.970 \text{ m/s}^2$

Thus at A, $\rho = 11.682\text{m}$

$a_n = 1.455 \text{ m/s}^2$

$a_t = 1.970 \text{ m/s}^2$

4. To anticipate the dip and hump in the road, the driver of car applies her brakes to produce a uniform deceleration. Her is 100kh/h at the bottom A of the dip and 50km/h at the top *C* of the hump, which is 120 m along the road from A .If the passenger experience a total acceleration of 3m/s$^2$ at A and if the radius of curvature of the hump at C is 150m, calculate(a) The radius of curvature ρ at A,(b) the acceleration at the inflection point B and (c) the total acceleration at C .

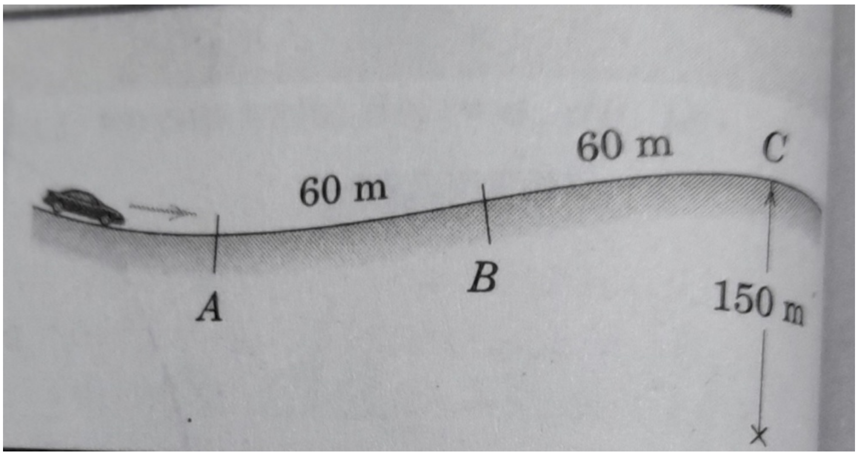



**Figure-4**

**SOLUTION:**

The dimension of the car are small compared with those of the path ,so we will treat the car as a particle. The velocities are

$v_A$ =(100km/h)(1h/3600s)(1000m/km) = 27.8m/s

$v_c = 50\frac{1000}{3600} = 13.89$ m/s

we find the constant deceleration along the path from

$[\int v dv = \int a_t \, ds]$ $\int_{v_a}^{v_c} v d\, v = a_t \int_0^s ds$

$a_t = \frac{1}{2s}(v_C^2 - v_A^2) = \frac{13.89^2 - 2\ \ .8^2}{2(120)} = -2.41 m/s^2$

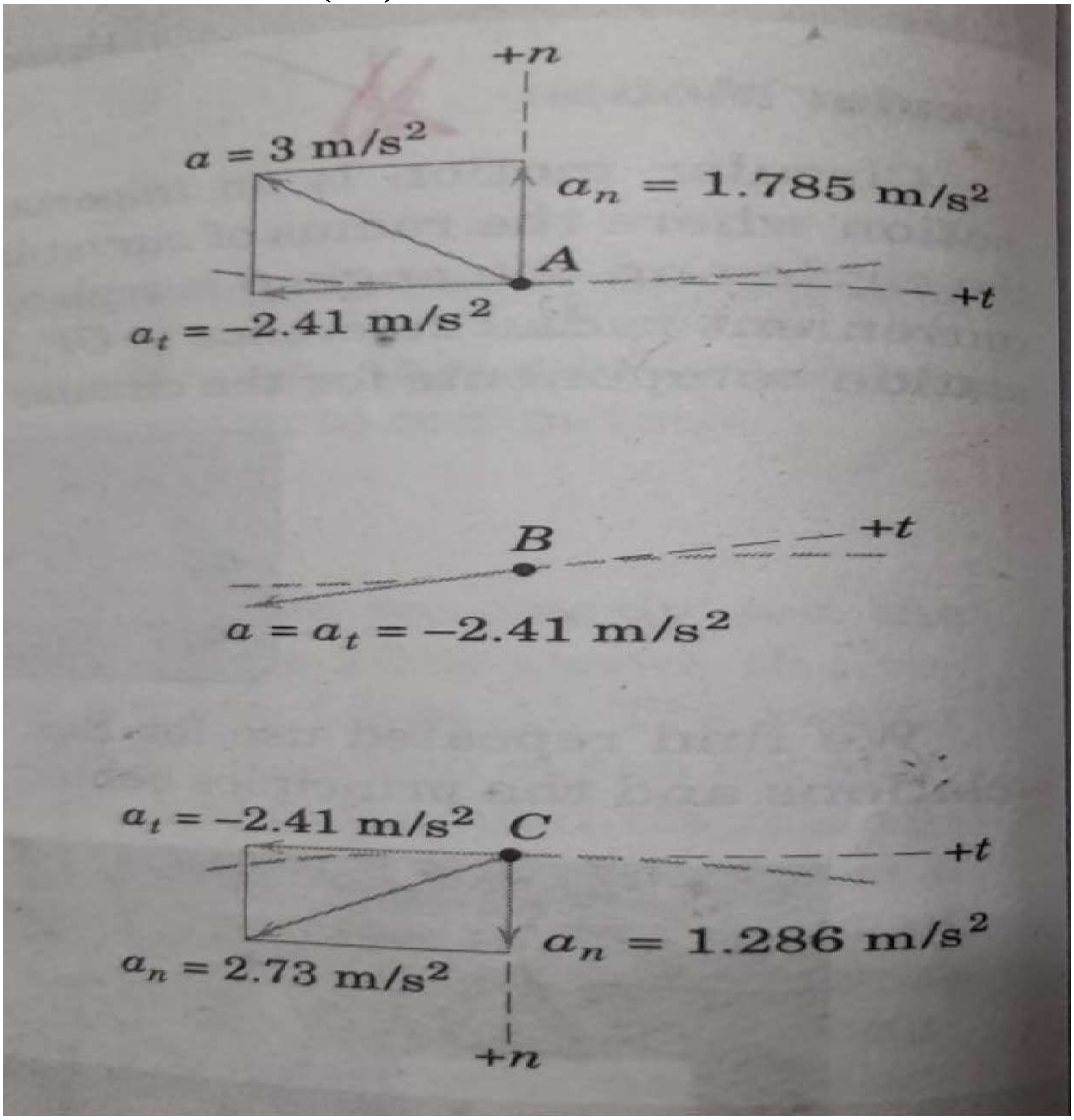



**Figure-5**

**(a)condition at A**

with the total acceleration given and $a_t$ determined, we can easily compute $a_n$ and hence ρ from

$[a^2 = a_n^2 + a_t^2]$ $\quad a_n^2 = 3^2 - (2.41)^2 = 3.19 \quad a_n = 1.785 m/s^2$

$[a_n = v^2/\rho] \quad \rho = {v^2}/{a_n} = \frac{(27.8)^2}{1.785} = 432m$

**(b) condition at B**

since the radius of curvature is infinite at the inflection at the inflection point $a_n = 0$ and $a$=$a_t$= -2.41m/s$^2$

**(c)condition at C.**

The normal acceleration becomes

[$a_n$=$v^2/\rho$]        $a_n$ = (13.89)$^2$/150 =1.286 m/s$^2$

With unit vector $e_n$ and $e_t$ in the $n$-and $t$- direction, the acceleration may be write

$$a = 1.286e_n - 2.41e_t \text{ m/s}^2$$

where the magnitude of a is

$[a = \sqrt{a_n^2 + a_t^2}\ ]$  $a = \sqrt{((1.286)^2 + (-2.41)^2)} = 2.73 m/s^2$ **Ans.**

The acceleration vectors representing the conditions at each of the three points are shown for clarification.

5. A particle is projected to move along a parabola $y^2$=4x . At a certain instant, when passing through a point P(4,4) its speed is 5 m/s and the rate of increase of it's speed in 3m/s$^2$ along the path.
   Express the velocity and acceleration of the particle in terms of rectangular coordinates.

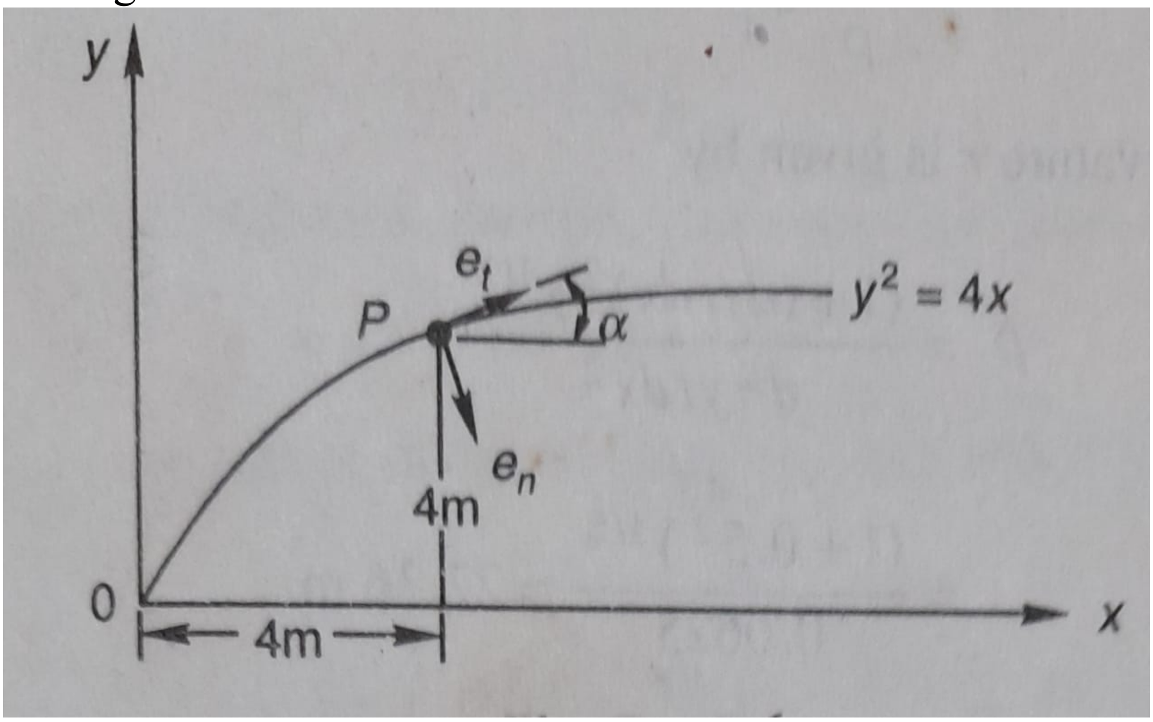



**Figure-6**

**Solution:**

Since the data relate to the path of the particle, the path coordinates may be used to advantage. The unit vectors are related as follows:

$e_t = cos \propto \boldsymbol{i} + sin \propto \boldsymbol{j}$**.......[1]**

$e_n = sin \propto \boldsymbol{i} - cos \propto \boldsymbol{j} \ldots.[\mathbf{2}]$

Where tan∝=$\frac{dy}{dx}$ $at\ P$

From the equation of the path $y^2$=4x

Differentiation with respect to x yields $2y\frac{dy}{dx} = 4$

And $\frac{dy}{dx} = \frac{2}{y} = \frac{1}{\sqrt{x}}$

Which at P is $\frac{1}{\sqrt{4}} = 0.5\ and\ \ \propto = \tan^{-1} 0.5 = 26.57° = 0.464\,rad$

Equation [1] and [2] at point P becomes

$e_t = 0.894\boldsymbol{i} + 0.447\boldsymbol{j}$

$e_n = 0.447\boldsymbol{i} - 0.894\boldsymbol{j}$

The velocity of the particle is given by

$$V = Ve_t = 5(0.894\boldsymbol{i} + 0.447\boldsymbol{j})$$

$$= 4.47\mathbf{i} + 2.235\mathbf{j}\ \text{m/s}$$

The tangential component of acceleration is

$a_t = (0.894\boldsymbol{i} + 0.447\boldsymbol{j}) = (2.68\boldsymbol{i} + 1.34\boldsymbol{j})\text{m/s}^2$

The normal component of acceleration is

$$a_n = \frac{V^2}{\rho} e_n$$

The radius of curvature *r* is given by

$$\therefore \qquad \rho = \frac{[1 + (\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{[1 + 0.5^2]^{3/2}}{0.0625} = 22.36\,m$$

Because $\frac{d^2y}{dx^2}$ =(-1/2$x^{-3/2}$)= 0.0625

The normal component of acceleration is

$a_n = \frac{V^2}{\rho} e_n = \frac{5^2}{22.36} \times (0.447\boldsymbol{i} - 0.894\boldsymbol{j}) = 0.5\mathbf{i}-\mathbf{j}$

The acceleration is , therefore, given by

$\boldsymbol{a}=\boldsymbol{a}_n+\boldsymbol{a}_t=2.68\mathbf{i}+1.34\mathbf{j}+0.5\mathbf{i}-\mathbf{j}=(3.18\mathbf{i} + 0.34\mathbf{j})m/s^2$ **Ans.**

6. A particle moves in the *xy* plane with a velocity of 30 m/s directed at an angle of $\tan^{-1}\frac{4}{3}$ as shown in figure-7 . It accelerated as $a_x$=-1.8 $m/s^2$ and $a_y$=-9$m/s^2$.Compute the radius of curvature of the path and the rate of change of speed along the path.

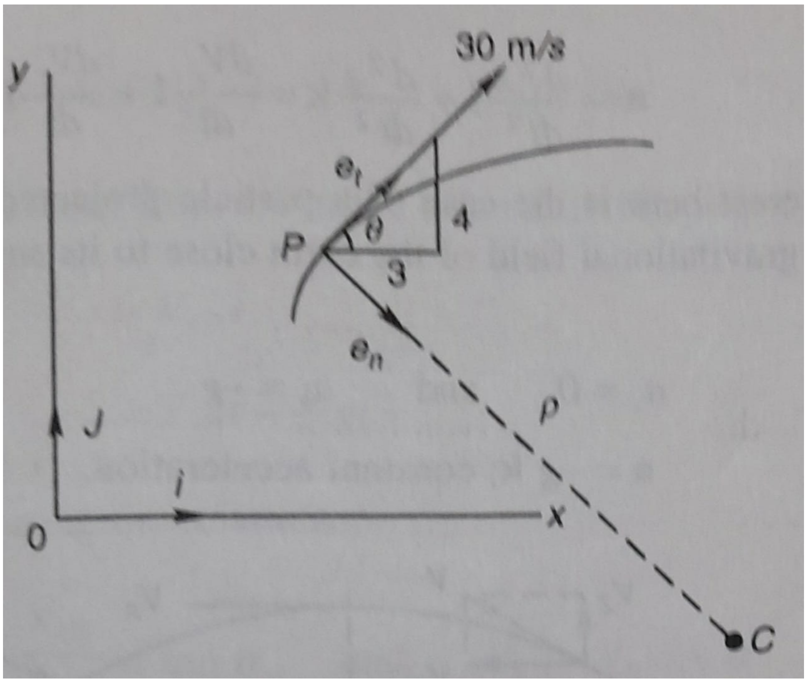



**Figure-7**

**Solution :**

The unit vector along the velocity vector is

$e_t = cos\theta\boldsymbol{i} + sin\theta\boldsymbol{j} = 0.6\boldsymbol{i} + 0.8\boldsymbol{j}$ .

The unit vector along the inward normal is

$e_n = 0.8\boldsymbol{i} - 0.6\boldsymbol{j}$

In terms of path coordinates, the acceleration is expressed as

$$a = f e_t + \frac{V^2}{\rho} e_n$$

Where f is the rate of change of speed along the path and ρ is the radius of curvature .

$(-1.8\boldsymbol{i} - 9\boldsymbol{j})= f(0.6\boldsymbol{i} + 0.8\boldsymbol{j}) + \frac{900}{\rho}(0.8\boldsymbol{i} - 0.6\boldsymbol{j})$

Which results in two equations.

$$0.6f + \frac{720}{\rho} = -1.8$$

And

$$0.8f - \frac{540}{\rho} = -9$$

Thus $f$= -8.3m/s and ρ = 227m.

7. Investigate the motion of the point *A, B and C* of a connecting rod (figure-8 )if the crankpin *A* is moving with uniform speed *v* along the circle of radius *r* and the point *B* is constrained to followed the *x* axis.

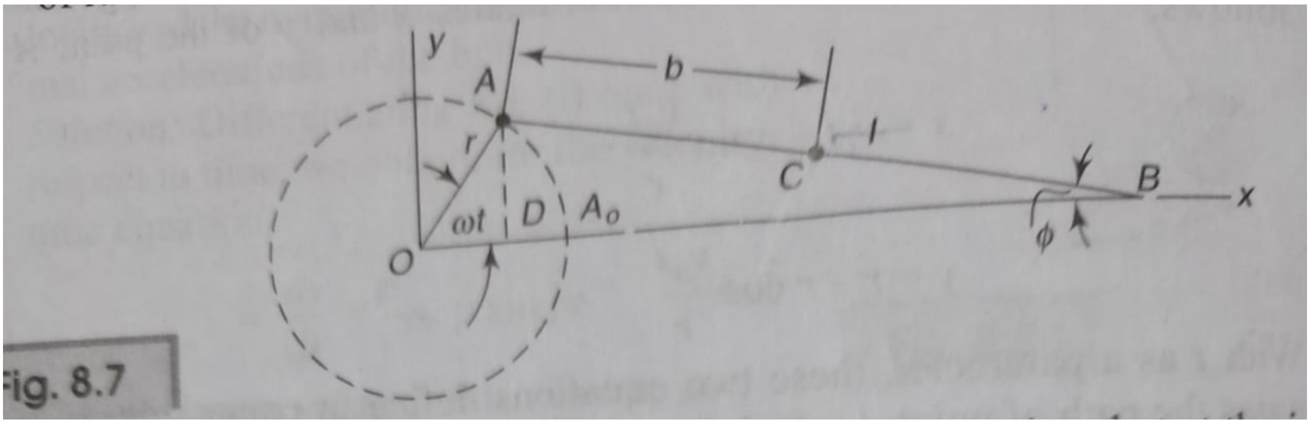



**Figure-8**

**Solution:**

We begin with the motion of point *A*, assuming that at the initial moment (t = 0) it has the position $A_0$ . Then denoting by ω the angle of the arc which the point *A* describe in unit time, we have ω =*v/r*, where *r* is the length of the crank. The angle $A_0OA$ is equal to ωt, and coordinates of the point A are

$x = r\cos\omega t$ ..............(1) and $y = r\sin\omega t$ ..............(2)

The coordinate x of the point B, which, of course , has rectilinear motion , is obtained by projecting on the x axis the length r of the crank and length l of the connecting rod .Then

$x = r\cos\omega t + l\cos\varphi$....[3]

Noting from figure-8 that

$r\sin\omega t = l\sin\varphi$....[4]

We obtain the following express for cosφ:

$$\cos\varphi = \sqrt{1 - \sin^2\varphi} = \sqrt{1 - \frac{r^2}{l^2}\sin^2\omega t}$$

And substitution in Eq(2) above, we have

$$x = r\cos\omega t + b\sqrt{1 - \frac{r^2}{l^2}\sin^2\omega t} \ldots\ldots\ldots\ldots [3]$$

It is interesting to note that, while the projection D of the crankpin on the *x* axis performs simple harmonic motion, the motion of point B is more complicated.

For any point C on the axis of the connecting rod at the distance *b* from the crankpin *A*, we obtain

$$x = r\cos\omega t + b\sqrt{1 - \frac{r^2}{l^2}\sin^2\omega t}\ ,\ y = \frac{l-b}{l} r\sin\omega t \ldots\ldots\ldots [4]$$

In the particular case where $r = l$, we have from Eq.(4)

$x = (r+b)\cos\omega t, \quad y = (r-b)\sin\omega t$............[5]

Eliminating *t* between Eq[5],we obtain for the path of C.

$$\frac{x^2}{(r+b)^2} + \frac{y^2}{(r-b)^2} = 1 \ldots\ldots\ldots\ldots\ldots [6]$$

Thus each point on the axis of the connecting rod describes an ellipse. This fact is utilized in a device called the ellipsograph for drawing ellipses.

8. Starting from rest, a motor boat travels around a circular path of $\rho$ =50m at a speed that increases with time, **v = (0.2t²) m/s**.Find the magnitude of boat`s velocity and acceleration at the instant **t=3s** using

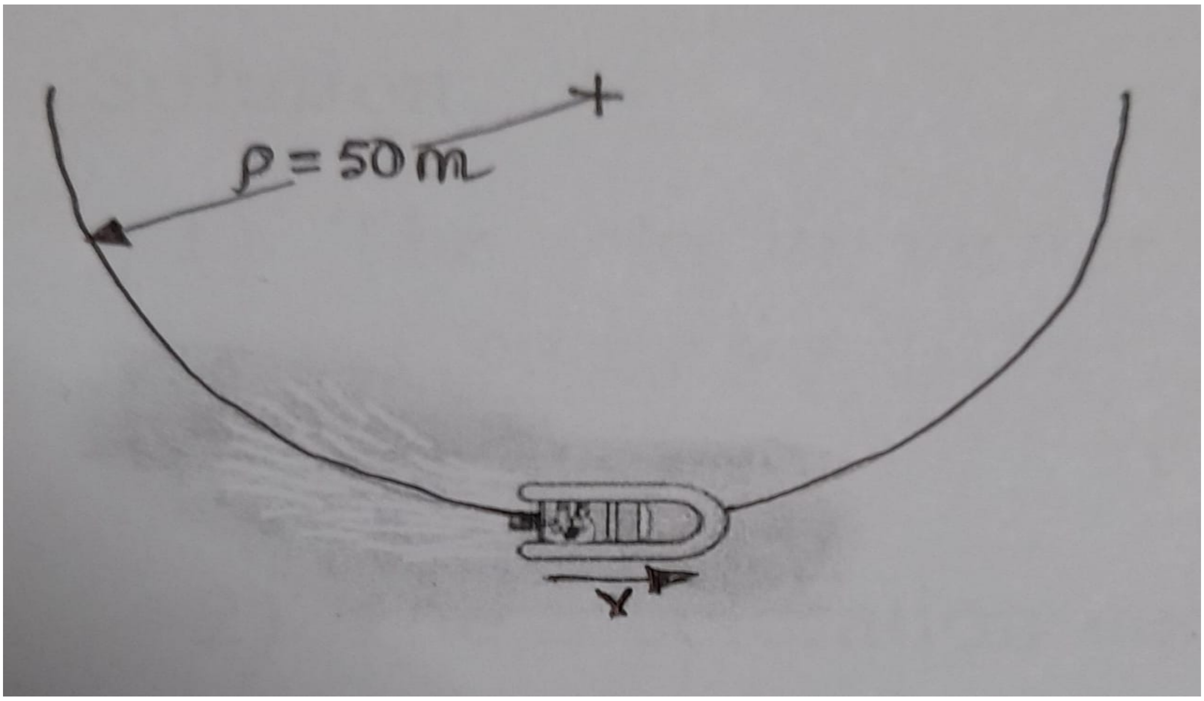



**Figure-9**

**Hint:** The boat starts from rest(**v=0** when **t=0**).

**1)** Calculate the velocity at **t=3s** using **v(t)**.

**2)** Calculate the tangential and normal component of acceleration and then the magnitude of the acceleration vector.

**Solution:**

1) The velocity vector is $\boldsymbol{v}=V\boldsymbol{u}_t$, where the magnitude is given by $\mathbf{v=(0.2t^2)}$m/s. At t = 3s:

$$V=0.2t^2=0.2(3)^2=1.8 m/s$$

2) The acceleration vector is $\boldsymbol{a} = \mathbf{a_t \boldsymbol{u}_t + a_n \boldsymbol{u}_n = v\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n}$ .

Tangential component: $a_t = v = d(.2t^2)/dt = 0.4t\ m/s^2$

At t = 3s: $\mathbf{a_t = 0.4t = 0.4(3) = 1.2\ m/s^2}$

Normal component :$a_n = v^2/\rho = (0.2t^2)^2/(\rho)\ m/s^2$

At t=3s: $a_n = [(0.2)(3^2)]^2/(50) = 0.0648\ m/s^2$

The magnitude of the acceleration is

$a=[(a_t) + (a_n)^2]^{0.5} = [(1.2)^2 + (0.0648)^2]^{0.5} = 1.20\ m/s^2$ **Ans.**

10. A jet plane travels along a vertical parabolic path defined by the equation $y=0.4x^2$ .At a point A, the jet has a speed of 200 m/s, which is increasing at the rate of 0.8 $m/s^2$. Find the magnitude of the plane's acceleration when it is at point A.

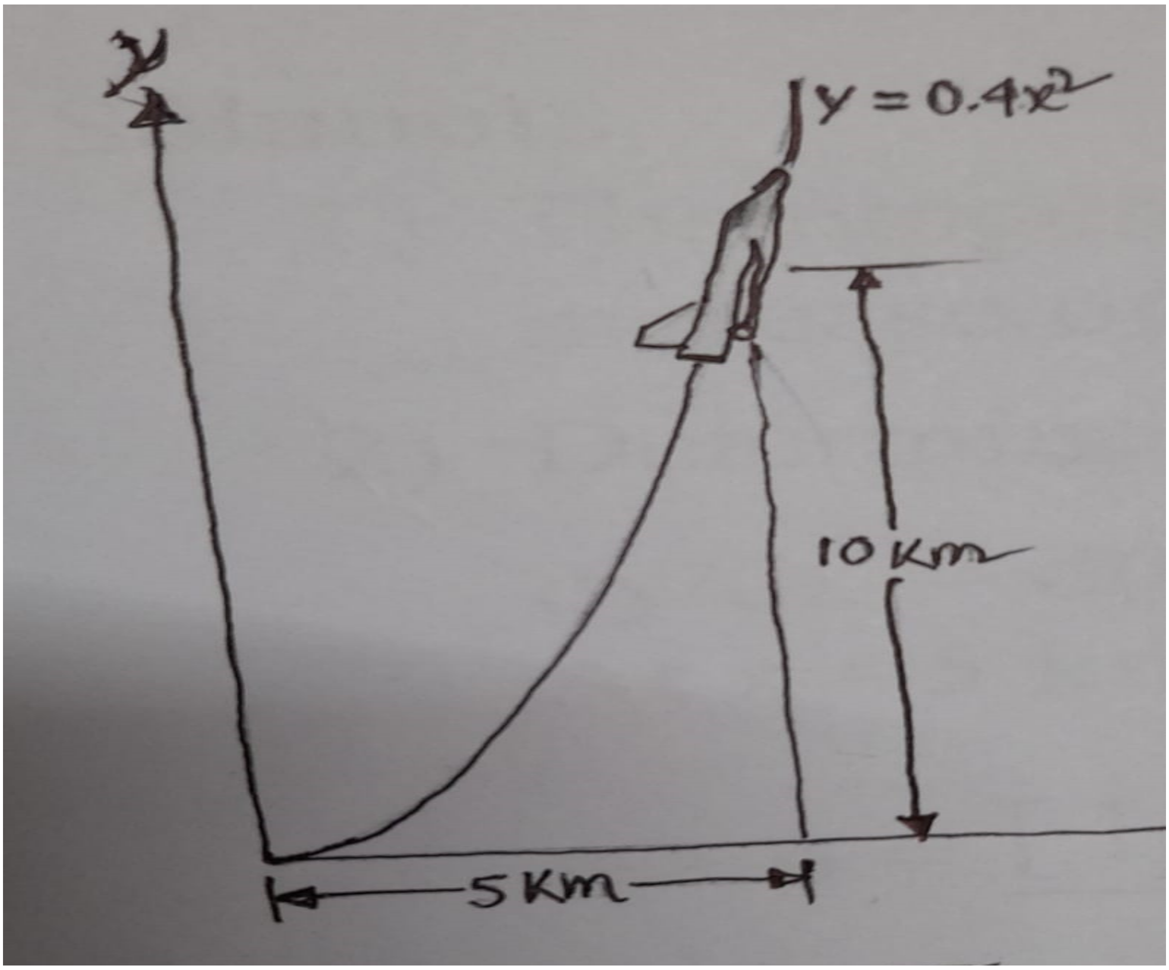



**Figure-10**

[Hints]

- The change in speed of the plane (0.8 $m/s^2$) is the tangential component of the total acceleration.
- Calculate the radius of curvature of the path at A.
- Calculate the normal component of acceleration.
- Determine the magnitude of the acceleration vector.

**Solution:**

The tangential component of acceleration is the rate of increase of the plane's speed, So $a_t = 0.8 m/s^2$ .

Determine the radius of curvature at point A(x=5km):

$$\frac{dy}{dx} = \frac{d(0.4x^2)}{dx} = 0.8x,$$

$$\frac{dy}{dx}_{atx=5\ km} = \frac{d(0.4x^2)}{dx} = 0.8x = 0.8(5) = 4$$

$$\frac{d^2y}{dx^2} = 0.8$$

$$\therefore \qquad \rho = \frac{[1 + (\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{[1 + (4)^2]^{3/2}}{0.85} = 87.62\ km$$

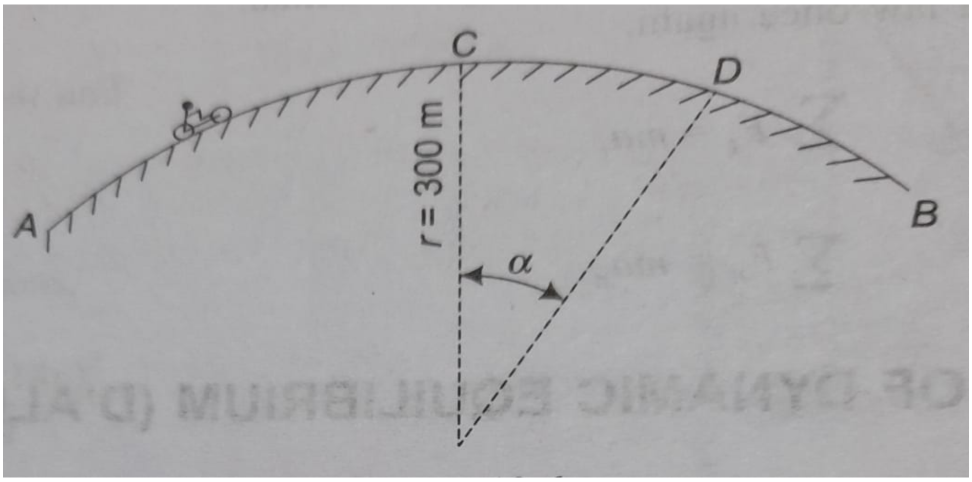
The normal component of acceleration is

$a_n = v^2/\rho = (200)^2/(87.62\times10^3) = 0.457\ m/s^2$

The magnitude of the acceleration vector is

$a=[(a_t) + (a_n)^2]^{0.5} = [(0.8)^2 + (0.457)^2]^{0.5} = 0.921\ m/s^2$ **Ans.**

11. A motor cycle and rider having a total weight W= 2225N travels in a vertical plane following a curve AB of radius r=300m at a speed of 72 km/h . Compute the thrust exerted by the road as it passes over the crest C on the curve as shown in figure-11.




**<u>Figure-11</u>**

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