Improper Integral

The definition of a definite integral $\int_a^b f(x)\,dx$ represents that (i) the limits a, b are finite (ii) that the integrand is bounded and integrable in $a \leq x \leq b$.

Hence when either (or both) of these assumptions are not satisfied, that is when a limit is infinite or the integrand becomes infinite in $a \leq x \leq b$, we need new definitions of such integrals called improper integrals.

Types of Improper integral

Improper integrals are of two main types

i) The interval increases without limit

ii) The integrands has a finite number of infinite discontinuities ②

Type I

Under type I we have three kinds of unbounded ranges over which integrals may be taken.

① Let $f(x)$ be be bounded and integrable in $a \leq x \leq B$ for every $B > a$. Then the symbol $\int_a^{\infty} f(x)\,dx$, called the improper integral, is said to converge or to exist if $\lim_{B \to \infty} \int_a^B f(x)\,dx$ exists finitely and we write

$$\int_a^{\infty} f(x)\,dx = \lim_{B \to \infty} \int_a^B f(x)\,dx$$

② If $f(x)$ be bounded and integrable in $A \leq x \leq b$ for every $A < b$ and $\lim_{A \to \infty} \int_A^b f(x)\,dx$ exists finitely, then we say that the improper integral $\int_{-\infty}^{b} f(x)dx$ exists and is convergent and we write:

$$\int_{-\infty}^{b} f(x)dx = \lim_{A \to -\infty} \int_{A}^{b} f(x)\,dx$$

If the limit tends to plus infinity or to minus infinity, then we say that the improper integral $\int_{-\infty}^{b}$ ~~$f(x)dx$ exists~~ and is is said to diverges. And if there is no limit, the integral is said to be oscillatory.

3) If $f(x)$ be bounded and integrable in $A \leq x \leq a$ for every $A < a$ and in $a \leq x \leq B$ for every $B > a$ and $\lim_{A \to -\infty} \int_{A}^{a} f(x)\,dx$ and $\lim_{B \to \infty} \int_{a}^{B} f(x)dx$ for $A < a < B$ exist finitely, then we say that $\int_{-\infty}^{\infty} f(x)dx$ is convergent and we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)\,dx$$

$$= \lim_{A \to -\infty} \int_{A}^{a} f(x)\,dx + \lim_{B \to \infty} \int_{a}^{B} f(x)dx$$

1. Does the improper integral $\int_0^\infty \frac{1}{1+x^2}\,dx$ exists?

An: $$\lim_{B\to\infty}\int_0^B \frac{1}{1+x^2}\,dx = \lim_{B\to\infty}\left[\tan^{-1}x\right]_0^B$$

$$= \lim_{B\to\infty}\left(\tan^{-1}B - \tan^{-1}0\right)$$

$$= \lim_{B\to\infty}\tan^{-1}B = \frac{\pi}{2}$$

Hence the integral $\int_0^\infty \frac{1}{1+x^2}\,dx$ does exist and is equal to $\frac{\pi}{2}$

2. Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$ if it converges

$$\lim_{B\to\infty}\int_1^B \frac{1}{x^2}\,dx = \lim_{B\to\infty}\left[-\frac{1}{x}\right]_1^B$$

$$= \lim_{B\to\infty}\left[-\frac{1}{B}+1\right] = 1$$

Hence $\int_1^\infty \frac{1}{x^2}\,dx$ converges and its value is 1

3. Evaluate $\int_a^\infty \sin x\,dx$ if it exists.

An: $$\lim_{B\to\infty}\int_a^B \sin x\,dx = \lim_{B\to\infty}\left[-\cos x\right]_a^B$$

$$= \lim_{B\to\infty}\left(\cos a - \cos B\right)$$ oscillates finitely. Therefore $\int_a^\infty \sin x\,dx$ is oscillatory.

4. Evaluate $\int_0^{\infty} e^{x}\,dx$ if it exist

<u>Sl</u> Now $\lim\limits_{B\to\infty} \int_0^B e^{x}\,dx = \lim\limits_{B\to\infty} \left(e^{B}-1\right)$

Since $e^{B}-1$ increases beyond all bounds as $B\to\infty$, this integral does not converge.

(B) <u>Type II</u>

(1) If $f(x)$ has an infinite discontinuity only at the left ~~ha~~ - hand end-point a, then by $\int_a^b f(x)\,dx$ we shall mean $\lim\limits_{\epsilon\to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx$

$$0<\epsilon<b-a$$

(2) If $f(x)$ has an infinite discontinuity only at b, by

$\int_a^b f(x)\,dx$ we shall mean $\lim\limits_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)\,dx$

$$0<\epsilon<b-a$$

(3) If $f(x)$ has an infinite disconitinuity at the point $x=c$ where $a<c<b$, then $\int_a^b f(x)\,dx$ we shall mean

$$\lim_{\varepsilon \to 0^+} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\delta \to 0^+} \int_{c+\delta}^{b} f(x)\,dx$$

If either of these limits fail to exist we say that the integral does not exist.

If however we make $\varepsilon = \delta$, and say that

$$\int_a^b f(x)\,dx \quad \text{means} \quad \lim_{\delta \to 0^+} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx \right]$$

We have what is called the Cauchy Principal value of $\int_a^b f(x)\,dx$ and write it as

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx \right]$$

It may sometimes happen that, the Cauchy Principal value integral exist when, according to the general definition ~~to the ge~~ the integral does not exist.

1. Prove that $\int_{-1}^{1} \frac{1}{x^3} dx$ exist in Cauchy Principal value sense but not in general sense.

Sol^n The integral is unbounded as $x \to 0$. Therefore we evaluate

$$\lim_{\varepsilon \to 0^+} \int_{-1}^{-\varepsilon} \frac{1}{x^3} dx + \lim_{\delta \to 0^+} \int_{\delta}^{1} \frac{1}{x^3} dx$$

$$= \lim_{\varepsilon \to 0^+} \left[-\frac{1}{2x^2} \right]_{-1}^{-\varepsilon} + \lim_{\delta \to 0^+} \left[-\frac{1}{2x^2} \right]_{\delta}^{1}$$

$$= \lim_{\varepsilon \to 0^+} \left[\frac{1}{2} - \frac{1}{2\varepsilon^2} \right] + \lim_{\delta \to 0^+} \left[-\frac{1}{2} + \frac{1}{2\delta^2} \right]$$

Since $\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon^2}$ and $\lim_{\delta \to 0^+} \frac{1}{2\delta^2}$ do not exist, the original integral does not exist. If however, we consider Cauchy Principal value, we are to find

$$\lim_{\varepsilon \to 0^+} \left[\int_{-1}^{-\varepsilon} \frac{1}{x^2} dx + \int_{\varepsilon}^{1} \frac{1}{x^2} dx \right]$$

$$= \lim_{\varepsilon \to 0^+} \left\{ \left(\frac{1}{2} - \frac{1}{2\varepsilon^2} \right) + \left(-\frac{1}{2} + \frac{1}{2\varepsilon^2} \right) \right\} = 0$$

since the terms involving ε cancel before the limit is taken.

Tests for Convergence : Type I

(A) Comparison test

Th If $f(x)$ be a non negative integrable function when $x \geq a$ and $\int_a^B f(x)\,dx$ is bounded above for every $B > a$, then $\int_a^\infty f(x)\,dx$ will converge; otherwise it will diverge to ∞.

Th If $f(x)$ and $g(x)$ be integrable functions when $x \geq a$ such that $0 \leq f(x) \leq g(x)$ then

i) $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.

ii) $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

B) <u>Limit test</u>

<u>Th.</u> Let $f(x)$ and $g(x)$ be integrable functions when $x \geq a$ and $g(x)$ be positive. Then if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lambda \neq 0$

the integrals $F = \int_a^\infty f(x)\,dx$ and $G = \int_a^\infty g(x)\,dx$ both converges absolutely or both diverge.

If $\frac{f}{g} \to 0$ and G converges, then F converges absolutely. If $\frac{f}{g} \to \pm\infty$ and G diverges, then F diverges.

<u>Prob</u> Show that the improper integral $\int_a^\infty \frac{dx}{x^\mu}$ $(a>0)$ exists if $\mu > 1$ and does not exist if $\mu \leq 1$.

Proof $\int_a^B \frac{dx}{x^\mu} = \frac{1}{1-\mu}\left[B^{1-\mu} - a^{1-\mu}\right]$, when $\mu \neq 1$

$$\int_a^B \frac{dx}{x} = \log B - \log a \quad \text{when } \mu = 1$$

let $B \to \infty$, then $\int_a^\infty \frac{dx}{x^\mu} = \frac{a^{1-\mu}}{\mu-1}$ when $\mu > 1$

$= \infty$ when $\mu \leq 1$

Ex1 $\int_0^\infty \frac{dx}{e^x+1}$ converges ~~if~~ by Comparison test, since $0 \leq \frac{1}{e^x+1} \leq \frac{1}{e^x} = e^{-x}$ and

$\int_0^\infty e^{-x}dx$ converges.

Ex2 $\int_1^\infty \frac{dx}{\log x}$ diverges by comparison test since for $x \geq 2$, $\log x < x$, $\frac{1}{\log x} > \frac{1}{x}$

and $\int_1^\infty \frac{dx}{x}$ diverges.

$\therefore \int_1^\infty \frac{dx}{\log x}$ diverges.

Proof $\int_a^B \frac{dx}{x^\mu} = \frac{1}{1-\mu}\left[B^{1-\mu} - a^{1-\mu}\right]$, when $\mu \neq 1$

$\int_a^B \frac{dx}{x} = \log B - \log a$ when $\mu = 1$

let $B \to \infty$, then $\int_a^\infty \frac{dx}{x^\mu} = \frac{a^{1-\mu}}{\mu-1}$ when $\mu > 1$

$= \infty$ when $\mu \leq 1$

Ex1 $\int_0^\infty \frac{dx}{e^x+1}$ converges by comparison test, since $0 \leq \frac{1}{e^x+1} \leq \frac{1}{e^x} = e^{-x}$ and

$\int_0^\infty e^{-x}dx$ converges.

Ex2 $\int_1^\infty \frac{dx}{\log x}$ diverges by comparison test since for $x \geq 2$, $\log x < x$, $\frac{1}{\log x} > \frac{1}{x}$

and $\int_1^\infty \frac{dx}{x}$ diverges.

$\therefore \int_1^\infty \frac{dx}{\log x}$ diverges.

Ex $\int_0^\infty \frac{\cos x}{1+x^2}\,dx$ converges absolutely,

since $\int_0^\infty \left|\frac{\cos x}{1+x^2}\right| dx \le \int_0^\infty \frac{1}{1+x^2}\,dx$

and by μ-test $\lim_{x\to\infty}$

Th The μ-tests for Convergence

Let $f(x)$ be an integrable function when $x \geq a$. Then $F = \int_a^\infty f(x)\,dx$ converges absolutely if $\lim_{x\to\infty} x^\mu f(x) = \lambda$, $\mu > 1$

and F diverges if $\lim_{x\to\infty} x^\mu f(x) = \lambda\,(\neq 0)$ or $\pm\infty$, $\mu \leq 1$

Ex $\int_0^\infty \frac{\cos x}{1+x^2}\,dx$ converges absolutely

$$\int_0^\infty \left|\frac{\cos x}{1+x^2}\right| dx \leq \int_0^\infty \frac{1}{1+x^2}\,dx$$

and by μ-test $\lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} \frac{x^2}{1+x^2}$

$= 1$, $\mu = 2 > 1$

$\therefore \int_0^\infty \frac{1}{1+x^2}\,dx$ is convergs.

Hence $\int_0^\infty \frac{\cos x}{1+x^2}\,dx$ converges absolutely.

Ex Examine the convergence of

$$\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$$

$$\lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} \frac{x^2}{x\sqrt{1+x^2}} = 1, \quad \mu = 2 > 1$$

$\therefore \int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$ is convergent.

Ex $\int_a^\infty \frac{\sin^2 x}{x^2}\,dx \quad (a>0)$ converges

Since $0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$

$\int_a^\infty \frac{1}{x^2}\,dx$ is converges by μ-test

$$\lim_{x\to\infty} x^2 \times \frac{1}{x^2} = 1, \quad \mu = 2 > 1$$

$\therefore$ By Comparison test $\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ converges

Ex $\int_0^\infty e^{-x^2} dx$ converges by μ-test

Since $\lim_{x\to\infty} x^2 \times e^{-x^2} = \lim_{x\to\infty} \frac{x^2}{e^{x^2}} = \lim_{x\to\infty} \frac{\cancel{2}x}{e^{x^2}\cdot\cancel{2}x}$

$= 0 \ , \mu = 2 > 1$

Ex $\int_1^\infty e^{-x} x^n dx$ converges

$\lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} \frac{x^{n+2}}{e^x} \to 0 \ , \mu = 2 > 1$

Ex $\int_1^\infty \frac{\log x}{x^2} dx$ converges for as $x \to \infty$

$\lim_{x\to\infty} x^{1/2} \frac{\log x}{x^2}$

$= \lim_{x\to\infty} \frac{\log x}{x^{3/2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{3}{2} x^{+1/2}}$

$= \frac{2}{3} \lim_{x\to\infty} \frac{1}{x \cdot x^{1/2}} = \to 0$

Ex $\int_0^\infty \frac{x^{3/2}}{3x^2+5} dx \Rightarrow \lim_{x\to\infty} x^{1/2} f(x)$

$= \frac{1}{3}$ for $\mu = \frac{1}{2} < 1$

diverges.

Test for Convergence Type II

Comparison test

If $f(x)$ be a non-negative integrable function in $a \leq x \leq b$ and a be the only point of infinite discontinuity of $f(x)$ in a finite interval $[a, b]$ and $\int_{a+\varepsilon}^{b} f(x)\,dx$ is bounded ~~over~~ above for $0 < \varepsilon < b-a$, the integral $\int_{a}^{b} f(x)\,dx$ will converge otherwise it will diverge to ∞.

Limit Test

Let $f(x)$ and $g(x)$ be integrable functions when $a < x \leq b$ and $g(x)$ be positive, then if

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lambda \neq 0$$

the integrals $F = \int_{a}^{b} f(x)\,dx$ and $G = \int_{a}^{b} g(x)\,dx$ both converge absolutely or both diverge.

If $\frac{f}{g} \to 0$ and G Converges, then F converges absolutely. If $\frac{f}{g} \to \pm\infty$ and G diverges, then F diverges.

Comparison integral

Show that the integral $\int_a^b \frac{dx}{(x-a)^\mu}$ exist if $\mu < 1$ and does not exist if $\mu \geq 1$.

Proof When $\mu \neq 1$, we have

$$\int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^\mu} = \frac{1}{1-\mu}\left\{(b-a)^{1-\mu} - \varepsilon^{1-\mu}\right\}$$

and when $\mu = 1$

$$\int_{a+\varepsilon}^{b} \frac{dx}{x-a} = \log(b-a) - \log\varepsilon$$

On letting $\varepsilon \to 0^+$, we obtain

$$\int_a^b \frac{dx}{(x-a)^\mu} = \frac{(b-a)^{1-\mu}}{1-\mu} \quad \text{when } 0 < \mu < 1$$

$$= \infty \quad , \quad \text{when } \mu \geq 1$$

when $\mu \leq 0$, the integral is proper.

The μ-test for convergence

Let $f(x)$ be an integrable function in an arbitrary interval $(a+\varepsilon, b)$ where $0 < \varepsilon < b-a$.

Then $F = \int_a^b f(x)\,dx$ converges absolutely

if $\lim_{x \to a^+} (x-a)^\mu f(x) = \lambda$, for $0 < \mu < 1$

and F diverges if $\lim_{x \to a^+} (x-a)^\mu f(x) = \lambda \; (\neq 0)$

or $\pm\infty$ for $\mu \geq 1$

Ex $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ converges

$$\lim_{x \to 0^+} (x-0)^\mu f(x) = \lim_{x \to 0^+} \cancel{(x-0)^{1/2}} \frac{1}{(1+x)\cancel{\sqrt{x}}}$$

$$= 1 \qquad \mu = \frac{1}{2} < 1$$

Ex ~~$\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$ converges since~~

$$\cancel{\lim_{x \to 0^+} (x-0)^{3/4} \frac{\log x}{\sqrt{x}} = \lim_{x \to 0^+} \frac{\log x}{x^{1/2 - 3/4}} = \lim_{x \to 0^+} \frac{\log x}{x^{-1/4}}}$$

$\int_1 \frac{\sqrt{x}}{\log x}$ diverges

$$\lim_{x \to 1^+} (x-1) \frac{\sqrt{x}}{\log x} = \lim_{x \to 1^+} \frac{\frac{3}{2} x^{1/2} - \frac{1}{2} x^{-1/2}}{\frac{1}{x}} = 1 \; \boxed{\text{L. Hospital Rule}}$$

Ex $\int_{1/2}^{1} \frac{dx}{\sqrt{x(1-x)}}$ converges

$$\lim_{x \to 1^-} (1-x)^{1/2} f(x) = \lim_{x \to 1^-} (1-x)^{1/2} \frac{1}{\sqrt{x(1-x)}} = 1 \quad \mu < 1$$

Convergence of Gamma and Beta Function

i) Gamma function

Let us discuss the convergence of

$$\int_0^{\infty} e^{-x} x^{n-1} \, dx$$

We write $f(x) = e^{-x} x^{n-1}$

$$I_1 = \int_0^1 e^{-x} x^{n-1} \, dx \; , \quad I_2 = \int_1^{\infty} e^{-x} x^{n-1} \, dx$$

The part I_1 is proper when $n \geq 1$, improper but absolutely convergent when $0 < n < 1$;

for as $x \to 0^+$, $\left[e^{-x} x^{n-1} \to \infty \text{ as } x \to 0^+ \right]$ by μ-test.

$$x^{1-n} f(x) = x^{1-n} e^{-x} x^{n-1} = e^{-x} \to 1$$

For $0 < \mu = 1-n < 1$, i.e. for $0 < n < 1$.

The part I_2 also converges absolutely for all values of n by μ-test, for as $x \to \infty$,

$$x^2 f(x) = x^2 e^{-x} x^{n-1} = e^{-x} x^{n+1} \to 0$$

Thus converges for $n > 0$. This is called gamma function denoted by $\Gamma(n)$.

Hence $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0$

(2) Beta function

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx$$

This is a proper integral when $m, n > 1$ but is improper at the lower limit when $m < 1$, at the upper limit when $n < 1$. We therefore, split it into two parts $I_1 + I_2$

where

$$I_1 = \int_0^{1/2} x^{m-1} (1-x)^{n-1} dx$$

$$I_2 = \int_{1/2}^1 x^{m-1} (1-x)^{n-1} dx$$

Now I_1 converges for $0 < m < 1$, diverges when $m \leq 0$, for as $x \to 0^+$, by μ-test

$$x^{1-m} f(x) = x^{1-m} x^{m-1} (1-x)^{n-1} = (1-x)^{n-1} \to 1$$

for $\mu = 1-m$ and for convergence $0 < \mu = 1-m < 1$

that is $0 < m < 1$. Also

$$x f(x) = x \cdot x^{m-1} (1-x)^{n-1}$$
$$= x^m (1-x)^{n-1} \to \begin{cases} 1 & \text{when } m = 0 \\ \infty & \text{when } m < 0 \end{cases}$$

where $f(x) = x^{m-1} (1-x)^{n-1}$

Next if we make the change of variable $x = 1-y$, the second ~~integrable~~ integral reduces to the first with m and n interchanged. Hence we may draw the same conclusion as before with n in place of m. Thus converges for $m, n > 0$. This is called Beta function denoted by $B(m,n)$

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx, \text{ for } m, n > 0$$

Ex $\int_0^{\infty} \frac{\cos x}{\sqrt{1+x^3}}\, dx$ converges absolutely by μ-test.

$$\lim_{x\to\infty} x^{5/4} \frac{\cos x}{\sqrt{1+x^3}} = \lim_{x\to\infty} \frac{\cos x \, x^{5/4}}{x^{3/2}\sqrt{1+\frac{1}{x^3}}}$$

$$= \lim_{x\to\infty} x^{5/4-3/2} \frac{\cos x}{\sqrt{1+\frac{1}{x^3}}}$$

$$= \lim_{x\to\infty} x^{\frac{5-6}{4}} \frac{\cos x}{\sqrt{1+\frac{1}{x^3}}}$$

$$= \lim_{x\to\infty} \frac{\cos x}{x^{1/4}\sqrt{1+\frac{1}{x^3}}} = 0 \qquad \mu = \frac{5}{4} > 1$$

Since absolute converges $\Rightarrow$ its ordinary convergence.

Ex $\int_0^{\pi} \frac{\sin x}{x^3}\, dx$ diverges by μ-test.

$$\lim_{x\to 0^+} x^2 \frac{\sin x}{x^3} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \quad \text{for } \mu = 2 > 1$$

Ex $\int_1^{\infty} \frac{\log x}{x+a}\, dx$ $(a>0)$ diverges by μ-test

$$\lim_{x\to\infty} x \frac{\log x}{x+a} = \lim_{x\to\infty} \frac{\log \cancel{x}}{\cancel{x}} \cdot \frac{\cancel{x}}{x+a} = \lim_{x\to\infty} \frac{\log x}{x} \cdot \frac{x^2}{x+a}$$

$$= \lim_{x\to\infty} \cancel{\frac{\frac{1}{x}}{1}} = \; = \infty \quad \text{for } \mu = 1$$

Ex

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx \text{ converges by } \mu\text{-test}$$

$$\lim_{x\to 0^+} \int_0^1 \frac{1-\cos x}{x^2}\,dx + \int_1^\infty \frac{1-\cos x}{x^2}$$

$$\lim_{x\to 0^+} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

and at the upper limit by μ-test

$$\lim_{x\to\infty} x^{3/2}\,\frac{1-\cos x}{x^2} = \lim_{x\to\infty} \frac{1-\cos x}{\sqrt{x}} = 0\,, \quad \mu=\frac{3}{2}>1$$

By μ-test it is convergent.

Ex Discuss the convergence of

$$\int_0^{\pi/2} \log \sin x\,dx$$

Sol^n The only singularity is at $x=0$.

Also
$$\log \sin x = \log\left(x\,\frac{\sin x}{x}\right)$$
$$= \log x + \log \frac{\sin x}{x}$$

By μ-test

$$\lim_{x\to 0^+} x^\mu \log \sin x = \lim_{x\to 0^+}\left(x^\mu \log x + x^\mu \log \frac{\sin x}{x}\right)$$

$$=0$$

(since $\lim_{x\to 0^+} x^{\mu}\log x = 0$, if $\mu > 0$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$)

for $\mu > 0$. See also that μ cannot be taken to be ≥ 1. Thus $0 < \mu < 1$. Hence the integral converges.

Ex $\int_1^{\infty} \frac{\sin x}{x^p}\, dx$ converges absolutely for $p > 1$,

$$\int_p^{\infty} \left|\frac{\sin x}{x^p}\right| dx \leq \int_1^{\infty} \frac{1}{x^p}\, dx \quad \text{for } x \geq 1$$

and $\int_1^{\infty} \frac{dx}{x^p}$ converges whenever $p > 1$.

Relation 1 For any $a > 0$,

$$\int_0^{\infty} e^{-at}\, t^{x-1}\, dt = \frac{\Gamma(x)}{a^x}, \quad x > 0$$

Proof Put $at = u$,

$$\int_{\varepsilon}^{B} e^{-at}\, t^{x-1}\, dt = \int_{a\varepsilon}^{aB} e^{-u} \frac{u^{x-1}}{a^{x-1}} \frac{du}{a}$$

As $\varepsilon \to 0^+$, $B \to \infty$

$$\int_0^\infty e^{-at}\, t^{x-1}\, dt = \frac{1}{a^x}\int_0^\infty e^{-u}\, u^{x-1}\, du = \frac{\Gamma(x)}{a^x}$$

Relation 2 $\Gamma(x+1) = x\,\Gamma(x)$, $x>0$

$$\int_\varepsilon^B \overset{u}{e^{-t}}\, \overbrace{t^{x-1}\, dt}^{dv} = e^{-t}\,\frac{t^x}{x}\Big|_\varepsilon^B + \frac{1}{x}\int_\varepsilon^B e^{-t}\, t^x\, dt$$

As $B \to \infty$ and $\varepsilon \to 0^+$, the integration part vanishes at both limits and therefore

$$\int_0^\infty e^{-t}\, t^{x-1}\, dt = \frac{1}{x}\int_0^\infty e^{-t}\, t^x\, dt$$

$$\Gamma(x) = \frac{1}{x}\,\Gamma(x+1) \quad \text{or} \quad \Gamma(x+1) = x\,\Gamma(x), \; x>0$$

Relation 3 $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt$$

$$= \lim_{B\to\infty} \int_0^B -e^{-t}\, dt$$

$$= \lim_{B\to\infty} \left(1 - e^{-B}\right) = 1$$

Relation 4 $\Gamma(x+1) = x!$ x being a positive integer.

$$\begin{aligned}\Gamma(x+1) &= x\Gamma(x) \\ &= x(x-1)\Gamma(x-1) \\ &= \text{- - - -} \\ &= x(x-1)(x-2)\text{- -}\ 3.2.1\,\Gamma(1) \\ &= x!\end{aligned}$$

The Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x,y>0$$

$$B(x,y) = \int_0^{1/2} t^{x-1}(1-t)^{y-1}\,dt + \int_{1/2}^{1} t^{x-1}(1-t)^{y-1}\,dt$$

Relation 1 $B(x,y) = B(y,x)$, for $x,y>0$

Proof Put $t = 1-u$

$$\int_\varepsilon^1 t^{x-1}(1-t)^{y-1}\,dt = \int_\delta^{1-\varepsilon} u^{y-1}(1-u)^{x-1}\,du$$

and on letting $\varepsilon \to 0^+$, $\delta \to 0^+$, we have

$$B(x,y) = B(y,x)$$

Relation 2 $B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta, \; x,y>0$

Proof Put $t = \sin^2\theta$, then

$$\int_\epsilon^{1-\delta} t^{x-1}(1-t)^{y-1}\,dt$$

$$= \int_\epsilon^{1-\delta} t^{x-1}(1-t)^{y-1}\,dt$$

$$= \int_{\sin^{-1}\sqrt{\epsilon}}^{\sin^{-1}\sqrt{1-\delta}} \sin^{2x-2}\theta \cos^{2y-2}\theta \cdot 2\sin\theta\cos\theta\, d\theta$$

and on letting $\epsilon \to 0^+, \delta \to 0^+$, we obtain

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = B(x,y)$$

$$= 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta, \; x,y>0$$

Relation 3 $B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

Put $x = \frac{1}{2} = y$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = \pi$$

Relation between Beta and Gamma Function

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \; x,y>0$$

Relation 4 $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi$$

But $\Gamma(1) = 1$

$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

II $\Gamma(m)\,\Gamma(1-m) = \pi \operatorname{cosec} m\pi, \quad 0 < m < 1$

Ex $\Gamma(4) = 3.2.1.\Gamma(1) = 6$

Ex

$$\int_0^{\pi/2} \sin^4 x \cos^4 x \, dx$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)} = \frac{1}{2}\,\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4.3.2.1.\Gamma(1)}$$

$$= \frac{3\pi}{258}$$

Ex Show that $\cancel{\int_0^1 \sqrt{1-x^4}\,dx}$ $\int_0^{\pi/2} \sqrt{\tan x}\,dx = \frac{\pi}{\sqrt{2}}$

$$\int_0^{\pi/2} \sqrt{\tan x}\,dx = \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x \, dx$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(1 - \frac{1}{4}\right)}{1} = \frac{1}{2} \pi \operatorname{cosec} \frac{\pi}{4}$$

$$= \frac{\pi}{\sqrt{2}}$$