Centroid

by

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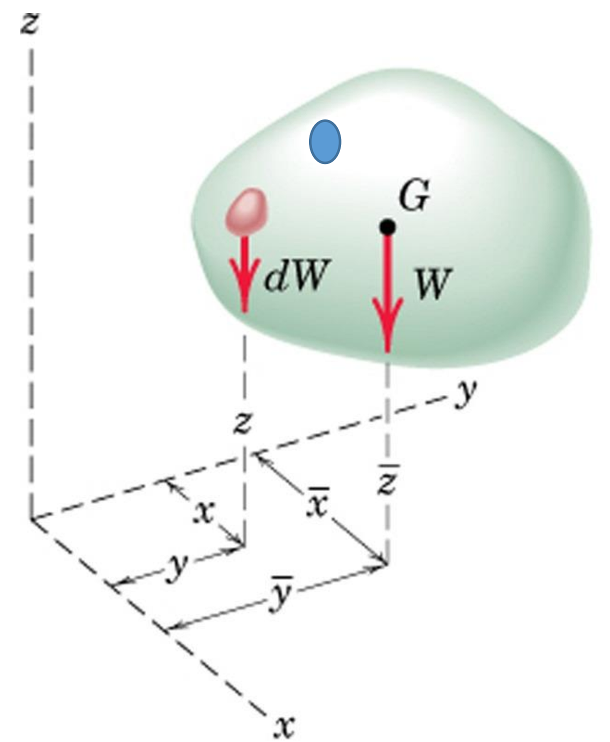
Centre of Gravity

Everybody is attracted towards the centre of the earth due gravity. The force of attraction is proportional to mass of the body. Everybody consists of innumerable particles, however the entire weight of a body is assumed to act through a single point and such a single point is called centre of gravity.

Every body has one and only centre of gravity.

- Locates the resultant weight of a system of particles
- Consider system of n particles fixed within a region of space
- The weights of the particles can be replaced by a single (equivalent) resultant weight having defined point G of application



Centre of Gravity

- Resultant weight = total weight of n particles

$$W_r = \sum W$$

- Sum of moments of weights of all the particles about x, y, z axes = moment of resultant weight about these axes
- Summing moments about the x axis,

$$\bar{x} W_r = \sum W_i \, x_i$$

$$\bar{x} = \frac{\sum W_i x_i}{W_r}$$

Centre of Gravity

- Summing moments about y axis,

$$\bar{y}W_r = \sum W_i\, y_i$$

$$\bar{y} = \frac{\sum W_i y_i}{W_r}$$

- Summing moments about the z axis,

$$\bar{z}W_r = \sum W_i\, z_i$$

$$\bar{z} = \frac{\sum W_i z_i}{W_r}$$

Centre of Mass

- A point where all of the mass could be concentrated
- It is the same as the center of gravity when the body is assumed to have uniform gravitational force

$$\bar{x} = \frac{\sum m_i x_i}{m}$$

$$\bar{y} = \frac{\sum m_i y_i}{m}$$

$$\bar{z} = \frac{\sum m_i z_i}{m}$$

$$m = \sum m_i$$

Centroids of Areas

In case of plane areas (bodies with negligible thickness) such as a triangle quadrilateral, circle etc., the total area is assumed to be concentrated at a single point and such a single point is called centroid of the plane area.

$$\bar{x} = \frac{\sum A_i x_i}{A}$$

$$\bar{y} = \frac{\sum A_i x_i}{A}$$

$$\bar{z} = \frac{\sum A_i x_i}{A}$$

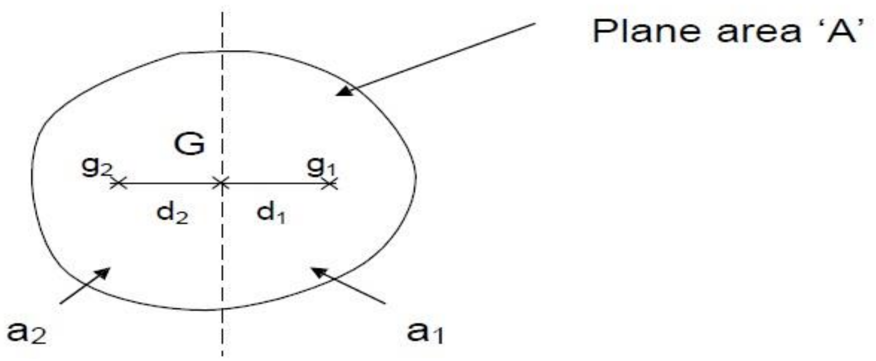
$$A = \sum A_i$$

Centroids

The term centre of gravity and centroid has the same meaning but the following differences.

1. Centre of gravity refer to bodies with mass and weight whereas, centroid refers to plane areas.

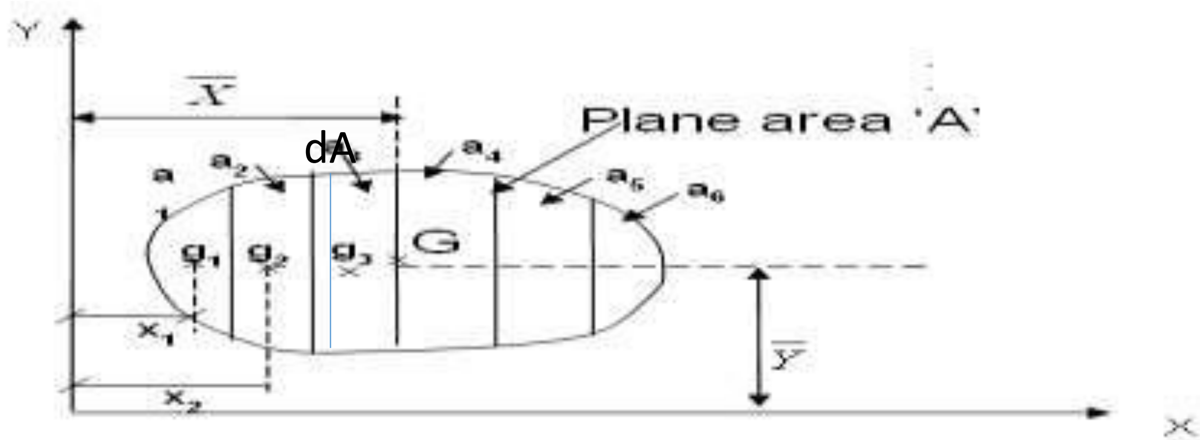
2. Centre of gravity is a point in a body through which the weight acts vertically downwards irrespective of the position, whereas the centroid is a point in a plane area such that the moment of areas about an axis through the centroid is zero



In the discussion on centroid, the area of any plane figure is assumed as a force equivalent to the centroid referring to the above figure G is said to be the centroid of the plane area A as long as

$$a1d1 – a2\ d2 = 0.$$

METHOD OF MOMENTS TO LOCATE THE CENTROID OF PLANE AREAS



Let us consider a plane area A lying in the XY plane.

Let G be the centroid of the plane area.

It is required to locate the position of centroid G $(\bar{X}, \bar{Y})$ with respect to the reference axis like Y- axis and X- axis

Let us divide the given area A into smaller elemental areas a1, a2, a3 …….. as shown in figure.

Let g1 ,g2, g3…… be the centroids of elemental areas a1, a2, a3 …….. etc.

METHOD OF MOMENTS TO LOCATE THE CENTROID OF PLANE AREAS

The moments of area about Y axis is

$$A.\bar{X} \qquad (1)$$

Let x1, x2, x3 etc. be the distance of the centroids g1 g2 g3 etc. from Y- axis

The sum of the moments of the elemental areas about Y axis is

$$a1 \,.\, x1 + a2 \,.\, x2 + a3 \,.\, x3 + \ldots\ldots \qquad (2)$$

Equating (1) and (2)

$$A.\bar{X} = a1 \,.\, x1 + a2 \,.\, x2 + a3 \,.\, x3 + \ldots\ldots$$

$$\bar{X} = \frac{a1 \,.\, x1 + a2 \,.\, x2 + a3 \,.\, x3 + \ldots\ldots}{A}$$

$$\bar{X} = \frac{\sum a_i x_i}{A} \text{ or } \bar{X} = \frac{\int x.dA}{A}$$

Where a or dA represents an elemental area in the area A , x is the distance of elemental area from Y axis.

METHOD OF MOMENTS TO LOCATE THE CENTROID OF PLANE AREAS

Similarly,

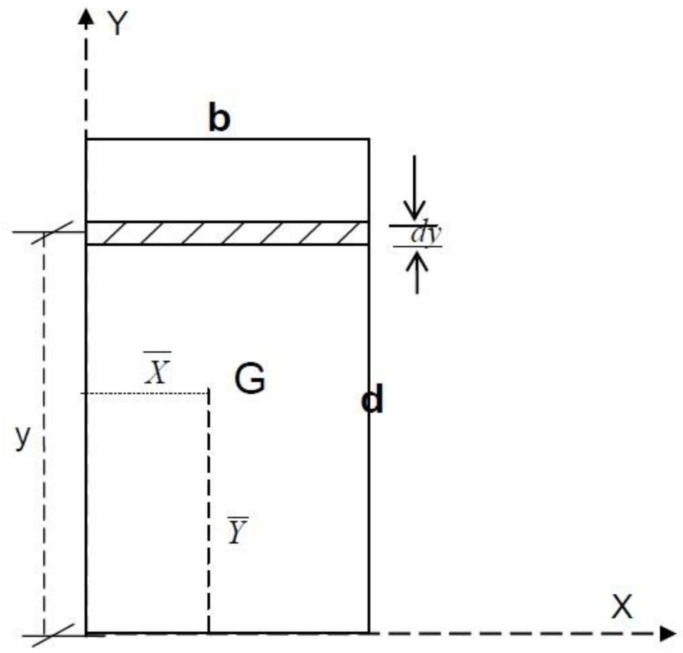
$$\bar{Y} = \frac{\text{a1 . y1 + a2 . y2 + a3 . y3 +......}}{A}$$

$$\bar{Y} = \frac{\sum a_i y_i}{A} \text{ or } \bar{Y} = \frac{\int y.dA}{A}$$

Where a or dA represents an elemental area in the area A , y is the distance of elemental area from X axis.

TO LOCATE THE CENTROID OF A RECTANGLE FROM

d



Let us consider a rectangle of breadth b and depth d. let g be the centroid of the rectangle.

Let us consider the X and Y axis as shown in the figure.

Let us consider an elemental area dA of breadth b and depth dy lying at a distance of y from the X axis.

TO LOCATE THE CENTROID OF A RECTANGLE FROM

We know that

$$\bar{Y} = \frac{\int y.dA}{A}$$

Area of rectangle A=bd

Elemental area dA=bdy

$$\bar{Y} = \frac{\int_0^d y.bdy}{bd}$$

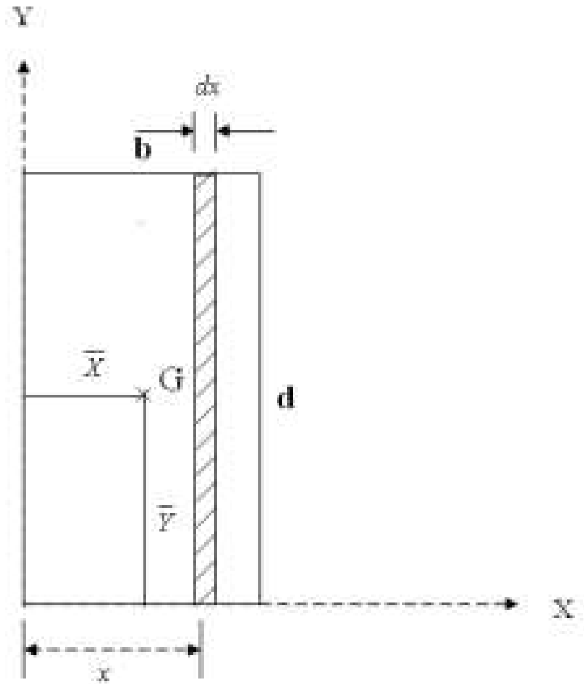
$$\bar{Y} = \frac{\int_0^d y.dy}{d}$$

$$\bar{Y} = \frac{1}{d}\left[\frac{y^2}{2}\right]_0^d$$

$$\bar{Y} = \frac{d}{2}$$

TO LOCATE THE CENTROID OF A RECTANGLE FROM

Similarly



We know that

$$\bar{X} = \frac{\int x.dA}{A}$$

Area of rectangle A=bd
Elemental area dA=ddx

TO LOCATE THE CENTROID OF A RECTANGLE FROM

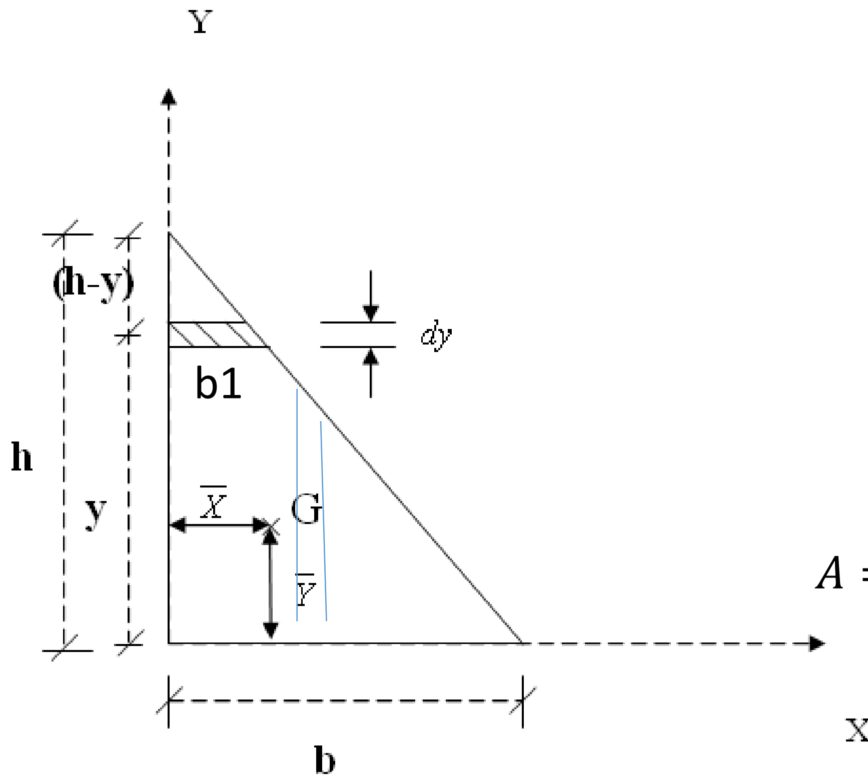
$$\bar{X} = \frac{\int_0^d x.ddx}{bd}$$

$$\bar{X} = \frac{\int_0^d x.dx}{b}$$

$$\bar{X} = \frac{1}{b}\left[\frac{x^2}{2}\right]_0^b$$

$$\bar{X} = \frac{b}{2}$$

Centroid of a triangle



$$\frac{h-y}{h}=\frac{b1}{b}$$

$$=> b1 = \frac{b(h-y)}{h}$$

$$A = \frac{1}{2}bh$$

Let us consider a right angled triangle with a base b and height h as shown in figure.

Let G be the centroid of the triangle.

Let us consider the X- axis and Y- axis as shown in figure.

Centroid of a triangle

Let us consider an elemental area dA of width b1 and thickness dy, lying at a distance y from X-axis.

We know that,

$$\bar{Y} = \frac{\int y.dA}{A}$$

Area of triangle

$$A = \frac{1}{2}\text{bh}$$

Elemental area $\quad \text{dA=b1.dy=}\frac{b(h-y)}{h}\text{dy}$

$$\bar{Y} = \frac{\int_0^h y.\frac{b(h-y)}{h}\,\mathrm{d}y}{\frac{1}{2}\,\text{bh}}$$

Centroid of a triangle

$$\bar{Y} = \frac{\int_0^h \frac{(h-y)}{h} \mathrm{y}\mathrm{d}y}{\frac{1}{2}\mathrm{h}}$$

$$\bar{Y} = \frac{2\int_0^h (\mathrm{hy}-y^2)\mathrm{d}y}{h^2}$$

$$\bar{Y} = \frac{2}{h^2}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h$$

$$\bar{Y} = \frac{2}{h^2}\left[\frac{h^3}{2} - \frac{h^3}{3}\right]_0^h$$

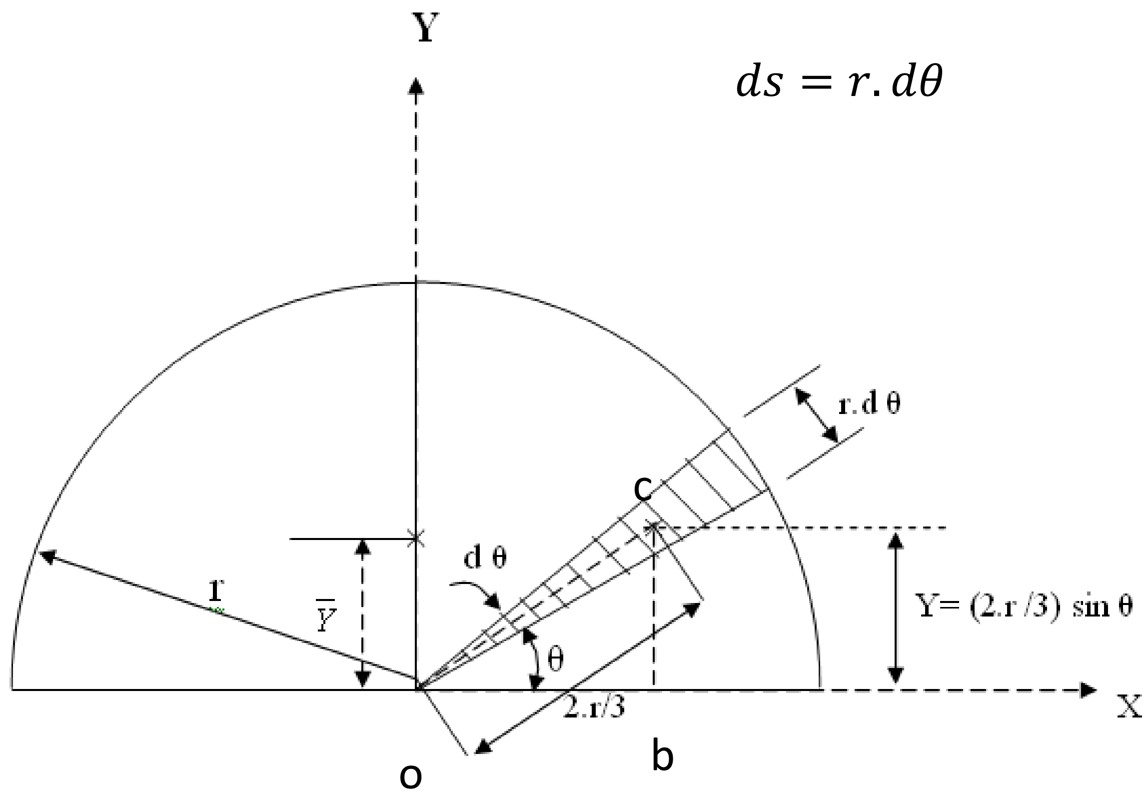
$$\bar{Y} = \frac{2}{h^2}\frac{h^3}{6}$$

$$\bar{Y} = \frac{h}{3}$$

Similarly,

$$\bar{X} = \frac{b}{3}$$

Centroid of a semi circle



$$ds = r.d\theta$$

$$\bar{X} = 0$$

Let us consider a semi-circle, with a radius =r.

Let O be the centre of the semi-circle .

let G be centroid of the semi-circle.

Let us consider the x and y axes as shown in figure.

Centroid of a semi circle

Let us consider an elemental area dA with centroid g as shown in figure.

Neglecting the curvature, the elemental area becomes an isosceles triangle with base r.dθ and height r.

Let y be the distance of centroid g from x axis

Here $y = \frac{2r}{3} \sin\theta$

We know that,

$$\bar{Y} = \frac{\int y.dA}{A}$$

Total area:

$$A = \frac{\pi r^2}{2}$$

Centroid of a semi circle

Elemental area

$$dA = \frac{1}{2} r d\theta . r$$

$$dA = \frac{r^2}{2} d\theta$$

$$\bar{Y} = \frac{\int \frac{2r}{3} \sin\theta . \frac{r^2}{2} d\theta}{\frac{\pi r^2}{2}}$$

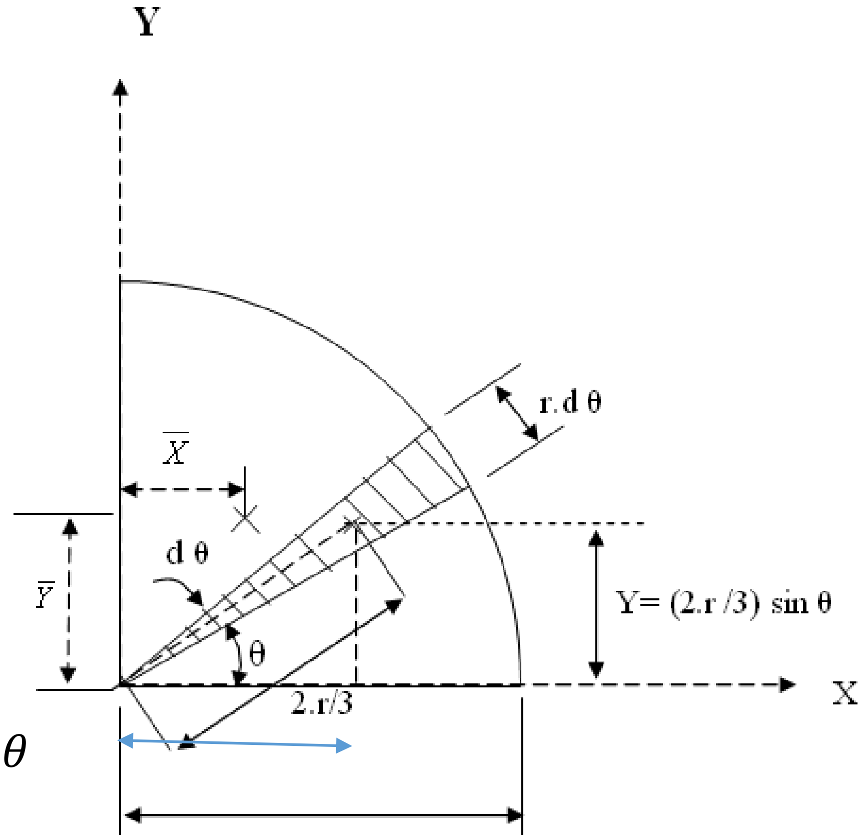
$$\bar{Y} = \frac{2r}{3\pi} \int_0^{\pi} \sin\theta \, d\theta$$

$$\bar{Y} = \frac{2r}{3\pi} [-\cos\theta]_0^{\pi}$$

$$\bar{Y} = \frac{2r}{3\pi} [1 + 1]$$

$$\bar{Y} = \frac{4r}{3\pi}$$

Centroid of a quarter circle



$$x = \frac{2r}{3} \cos \theta$$

Let us consider a quarter circle with radius r.

Let O be the centre and G be the centroid of the quarter circle.

Let us consider the x and y axes as shown in figure.

Centroid of a semi circle

Let us consider an elemental area dA with centroid g as shown in figure.

Neglecting the curvature, the elemental area becomes an isosceles triangle with base r.dθ and height r.

Let y be the distance of centroid g from x axis

Here $y = \frac{2r}{3}\sin\theta$

We know that,

$$\bar{Y} = \frac{\int y.dA}{A}$$

Total area:

$$A = \frac{\pi r^2}{4}$$

Centroid of a semi circle

Elemental area

$$dA = \frac{1}{2} r d\theta . r$$

$$dA = \frac{r^2}{2} d\theta$$

$$\bar{Y} = \frac{\int \frac{2r}{3} \sin\theta . \frac{r^2}{2} d\theta}{\frac{\pi r^2}{4}}$$

$$\bar{Y} = \frac{4r}{3\pi} \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta$$

$$\bar{Y} = \frac{4r}{3\pi} \lfloor -\cos\theta \rfloor_0^{\frac{\pi}{2}}$$

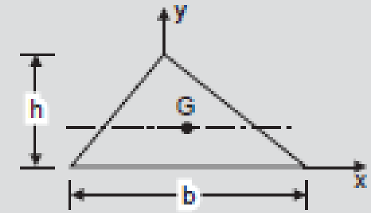
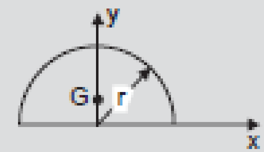
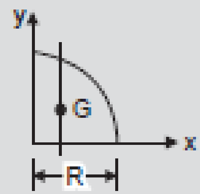
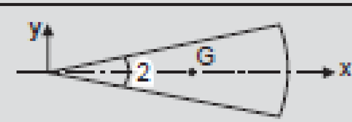
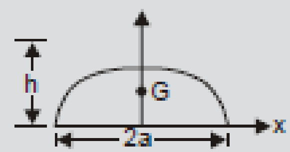
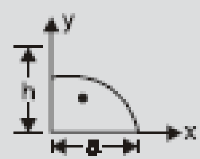
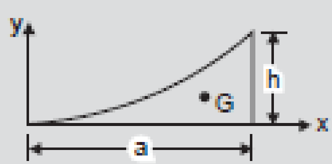
$$\bar{Y} = \frac{4r}{3\pi} [0 + 1]$$

$$\bar{Y} = \frac{4r}{3\pi}$$

Similarly,

$$\bar{X} = \frac{4r}{3\pi}$$

Centroid of Some Common Figures

Shape	*Figure*	$\bar{x}$	$\bar{y}$	*Area*
Triangle		—	$\frac{h}{3}$	$\frac{bh}{2}$
Semicircle		0	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{2}$
Quarter circle		$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{4}$
Sector of a circle		$\frac{2R}{3\alpha}\sin a$	0	αR^2
Parabola		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Semi parabola		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$

Centroid of Composite Sections

In engineering practice, use of sections which are built up of many simple sections is very common.

Such sections may be called as built-up sections or composite sections.

To locate the centroid of composite sections, one need not go for the method of integration.

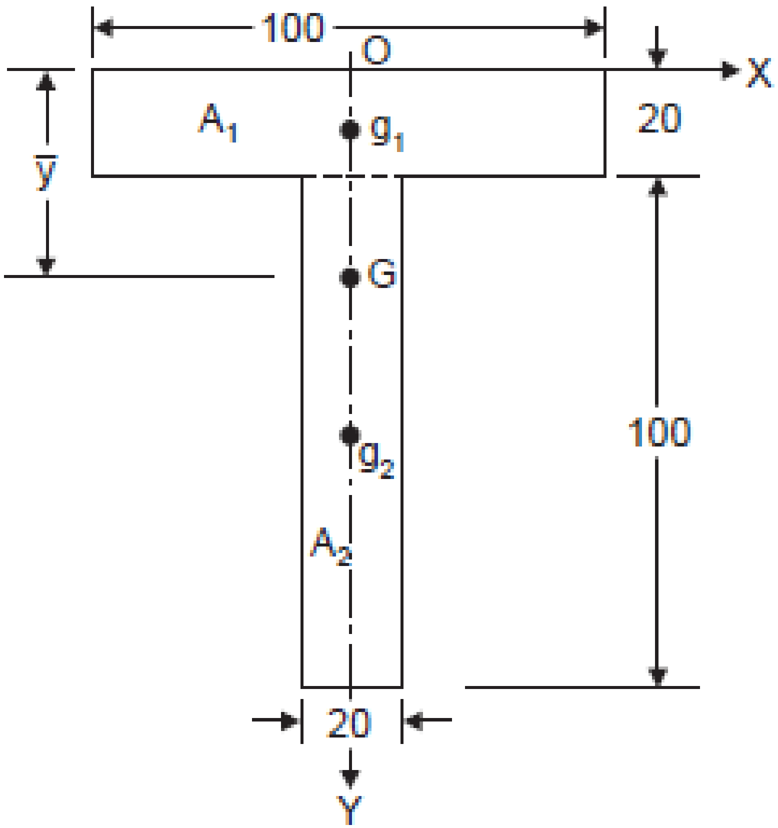
The given composite section can be split into suitable simple figures and then the centroid of each simple figure can be found by inspection or using the standard formulae.

Assuming the area of the simple figure as concentrated at its centroid, its moment about an axis can be found by multiplying the area with distance of its centroid from the reference axis.

After determining moment of each area about reference axis, the distance of centroid from the axis is obtained by dividing total moment of area by total area of the composite section.

PROBLEMS:

Locate the centroid of the T-section shown in figure



All dimensions in mm

Locate the centroid of the T-section shown in figure

Solution. Selecting the axis as shown in Fig. we can say due to symmetry centroid lies on y axis, *i.e.* $\bar{x} = 0$. Now the given T-section may be divided into two rectangles A_1 and A_2 each of size 100×20 and 20×100. The centroid of A_1 and A_2 are $g_1(0, 10)$ and $g_2(0, 70)$ respectively.

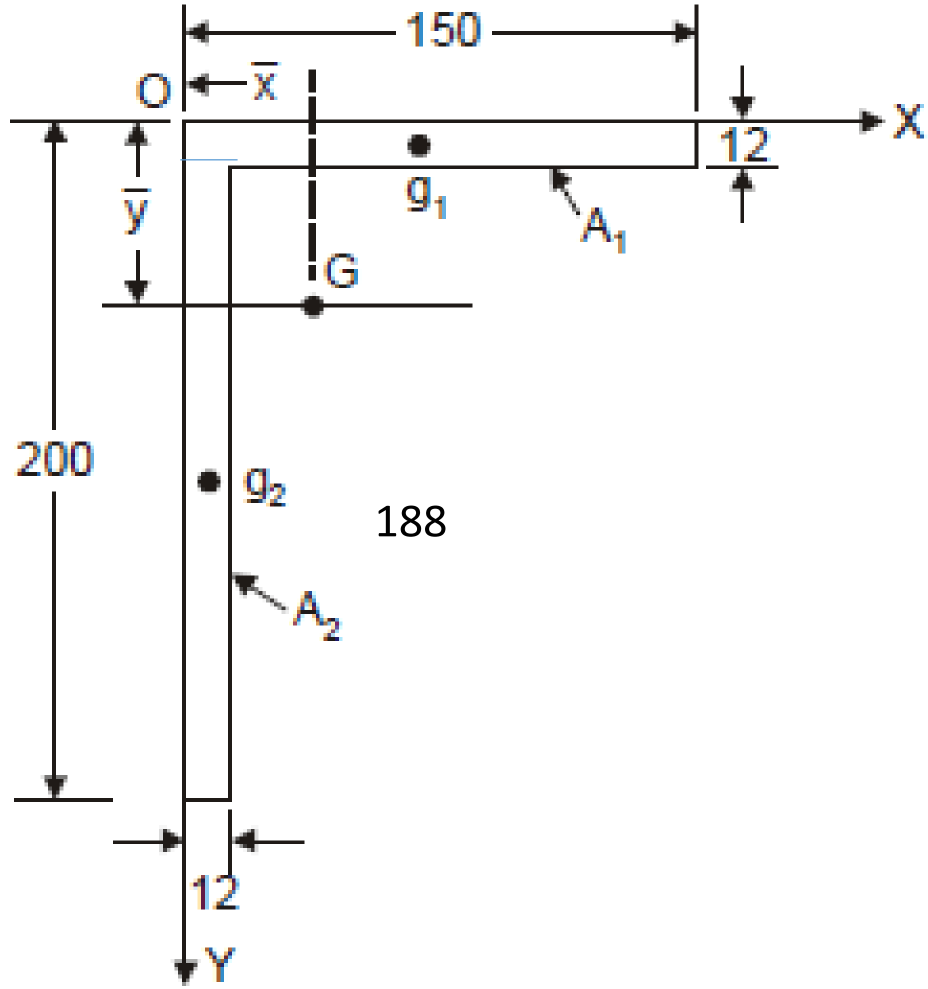
$\therefore$ The distance of centroid from top is given by:

$$\bar{y} = \frac{100 \times 20 \times 10 + 20 \times 100 \times 70}{100 \times 20 + 20 \times 100}$$

$$= 40 \text{ mm}$$

Hence, *centroid of T-section is on the symmetric axis at a distance 40 mm from the top.* **Ans.**

Find the centroid of the unequal angle 200 ×150 × 12 mm, shown in Figure



All dimensions in mm

Find the centroid of the unequal angle 200 ×150 × 12 mm, shown in Figure

Solution. The given composite figure can be divided into two rectangles:

$$A_1 = 150 \times 12 = 1800 \text{ mm}^2$$

$$A_2 = (200 - 12) \times 12 = 2256 \text{ mm}^2$$

Total area $\quad A = A_1 + A_2 = 4056 \text{ mm}^2$

Selecting the reference axis x and y as shown in Fig. . The centroid of A_1 is g_1 (75, 6) and that of A_2 is:

$$g_2\left[6, 12+\frac{1}{2}(200-12)\right]$$

i.e., $\quad g_2$ (6, 106)

$$\therefore \quad \bar{x} = \frac{\text{Movement about } y \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1x_1 + A_2x_2}{A}$$

$$= \frac{1800 \times 75 + 2256 \times 6}{4056} = 36.62 \text{ mm}$$

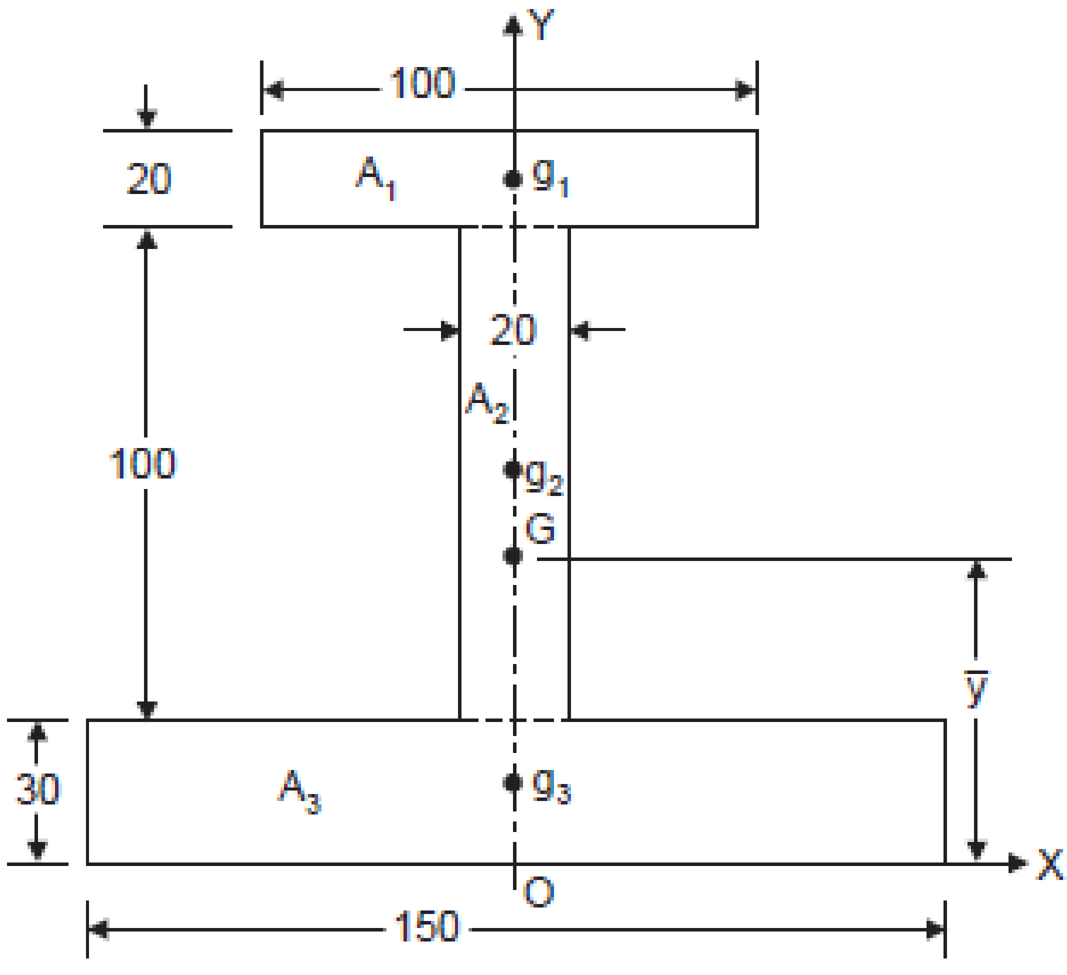
$$\bar{y} = \frac{\text{Movement about } x \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1y_1 + A_2y_2}{A}$$

$$= \frac{1800 \times 6 + 2256 \times 106}{4056} = 61.62 \text{ mm}$$

Thus, the centroid is at $\bar{x}$ = 36.62 mm and $\bar{y}$ = 61.62 mm as shown in the figure **Ans.**

Locate the centroid of the I-section shown in Figure



All dimensions in mm

Locate the centroid of the I-section shown in Figure

Solution. Selecting the co-ordinate system as shown in Fig. due to symmetry centroid must lie on y axis,

i.e., $\bar{x} = 0$

Now, the composite section may be split into three rectangles

$$A_1 = 100 \times 20 = 2000 \text{ mm}^2$$

Centroid of A_1 from the origin is:

$$y_1 = 30 + 100 + \frac{20}{2} = 140 \text{ mm}$$

Similarly $A_2 = 100 \times 20 = 2000 \text{ mm}^2$

$$y_2 = 30 + \frac{100}{2} = 80 \text{ mm}$$

$$A_3 = 150 \times 30 = 4500 \text{ mm}^2, \text{ and}$$

$$y_3 = \frac{30}{2} = 15 \text{ mm}$$

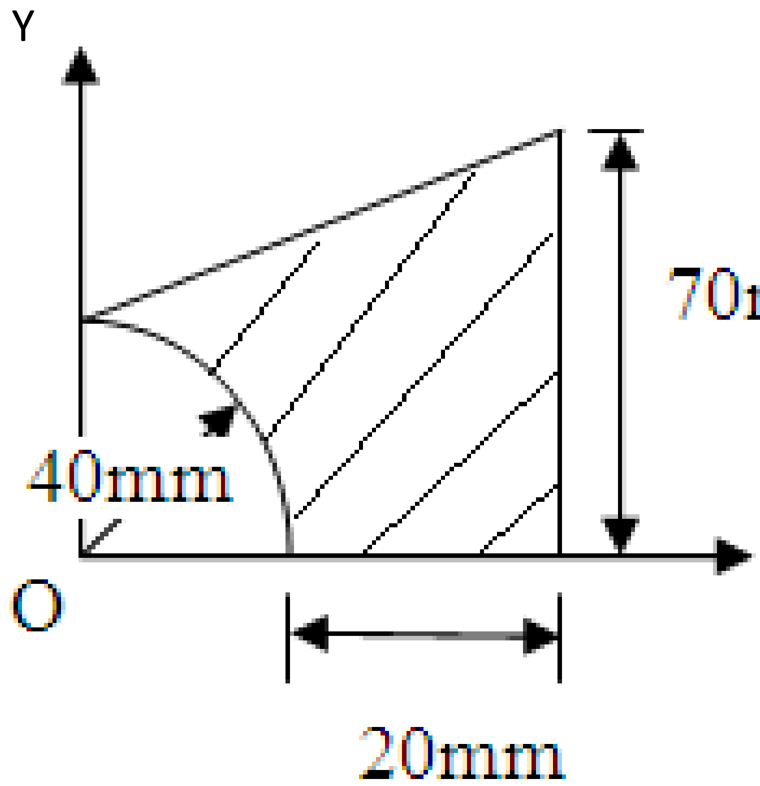
$$\therefore \quad \bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A}$$

$$= \frac{2000 + 140 + 2000 \times 80 + 4500 \times 15}{2000 + 2000 + 4500}$$

$$= 59.71 \text{ mm}$$

Thus, *the centroid is on the symmetric axis at a distance 59.71 mm from the bottom* as shown in Fig. **Ans.**

Determine the centroid of the lamina shown in figure.



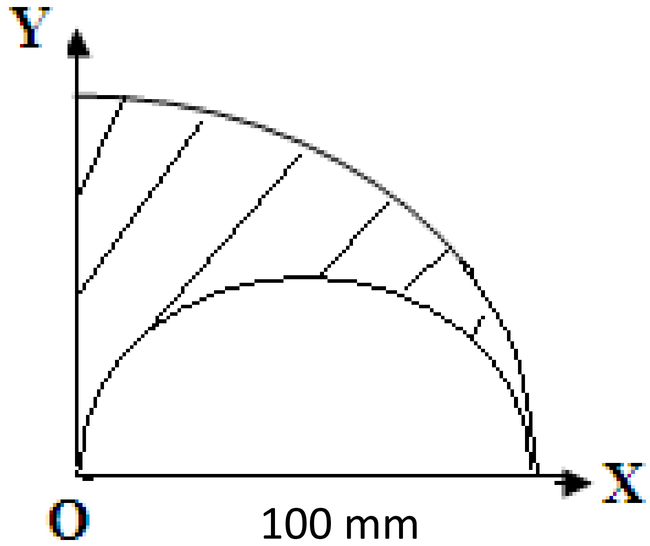
Determine the centroid of the lamina shown in figure.

Solution

Component	Area (mm^2)	X (mm)	Y (mm)	aX	Ay
Quarter circle	-1256.64	16.97	16.97	-21325.2	-21325.2
Triangle	900	40	50	36000	45000
Rectangle	2400	30	20	72000	48000
	$\sum a = 2043.36$			$\sum aX =$ 86674.82	$\sum aY =$ 71674.82

$\underline{X} = 42.42$ mm; $Y = \underline{\ }35.08$ mm

Find the centroid of the shaded area shown in fig, obtained by cutting a semicircle of diameter 100mm from the quadrant of a circle of radius 100mm.



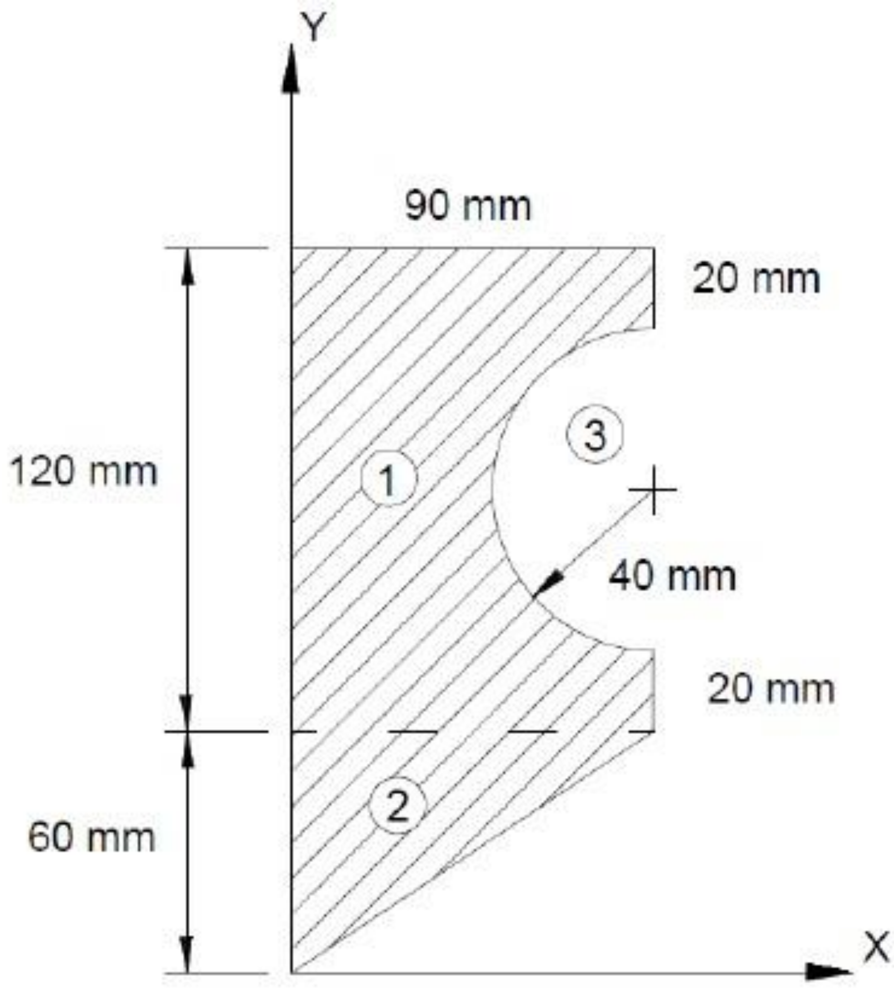
Find the centroid of the shaded area shown in fig, obtained by cutting a semicircle of diameter 100mm from the quadrant of a circle of radius 100mm.

Solution

Component	**Area (mm^2)**	**X (mm)**	**Y (mm)**	**aX**	**aY**
Quarter circle	7853.98	42.44	42.44	333322.9	333322.9
Semi circle	-3926.99	50	21.22	-196350	-83330.7
	$\sum$a= 3926.99			$\sum$ aX = 136973.4	$\sum$ aY = 249992.2

X = 34.88 mm; Y = 63.66 mm

Given: Area shown.
Find: Centroid (x , y), using a vertical element.



Solution

Part	Area, A_i	$\bar{x}_i$	$\bar{y}_i$	$\bar{x}_i A_i$	$\bar{y}_i A_i$
1	10,800	45.0	120.0	486,000	1,296,000
2	2,700	30.0	40.0	81,000	108,000
3	- 2,510	73.0	120.0	- 183,000	- 301,000
Totals	10,990			384,000	1,103,000

$A_3 = \pi r^2/2 = \pi (40)^2/2 = 2{,}510$

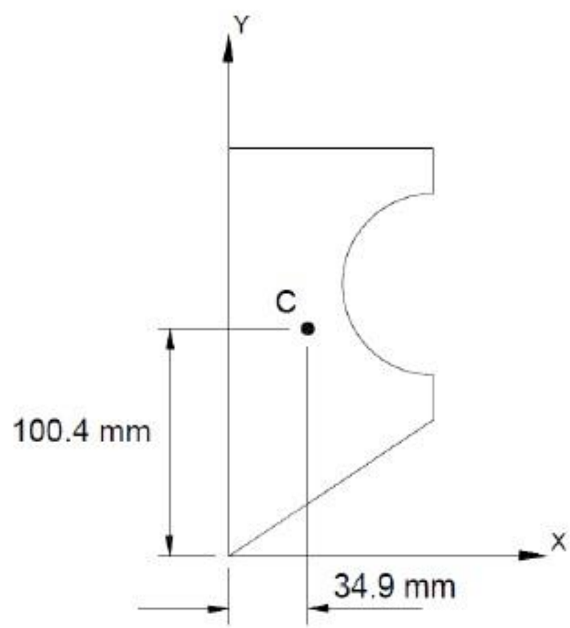
$x_3 = 90 - 4r/3\pi = 90 - 4(40)/3\pi = 73.0$

$\bar{x} = \Sigma \bar{x}_i A_i / \Sigma A_i = 384{,}000/10{,}990$

$\bar{x} = 34.9$ mm

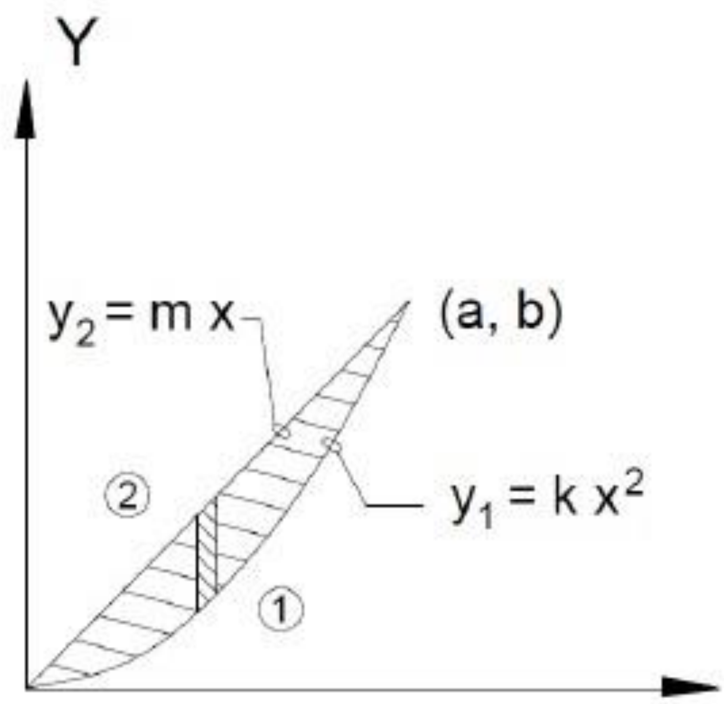
$\bar{y} = \Sigma \bar{y}_i A_i / \Sigma A_i = 1{,}103{,}000/10{,}990$

$\bar{y} = 100.4$ mm



Given: Area shown.

Find: Centroid ($\bar{x}$, $\bar{y}$), using a vertical element.



Solution

Applicable equations:

$A = \int dA$ $\qquad$ $dA = (y_2 - y_1)\, dx$

$Q_y = \bar{x} A = \int \bar{x}_{el}\, dA$ $\qquad$ $\bar{x}_{el} = x$

$Q_x = \bar{y} A = \int \bar{y}_{el}\, dA$ $\qquad$ $\bar{y}_{el} = \frac{1}{2}(y_2 + y_1)$

Define values of "m" and "k" in terms of "a" and "b".

For the line, when $x = a$ and $y = b$: $\quad y = mx \quad b = ma \quad m = b/a$

For the parabola, when $x = a$ and $y = b$: $\quad y = kx^2 \quad b = ka^2 \quad k = b/a^2$

To define values of "y" in terms of "x" to allow the integration with respect to x, write equations for the boundaries of the area (i.e. the line and the parabola).

- The equation for the line is $\quad y = (b/a)\, x$
- The equation for the parabola is $\quad y = (b/a^2)\, x^2$
- Now define dA and $\bar{y}_{el}$ in terms of "x".

$$dA = (y_2 - y_1)\, dx = [(b/a)\, x - (b/a^2)\, x^2]\, dx$$

$$\bar{y}_{el} = \frac{1}{2}(y_2 + y_1) = \frac{1}{2}[(b/a)\, x + (b/a^2)\, x^2]$$

Now solve for area A and the first moments of area Q_x and Q_y.

$$A = \int dA = \int x\, dy = \int_0^a [(b/a)\, x - (b/a^2)\, x^2]\, dx$$

$$= [(b/a)(x^2/2) - (b/a^2)(x^3/3)]\, \Big|_0^a = (b/a)(a^2/2) - (b/a^2)(a^3/3)$$

$$= ba/2 - ba/3$$

$$A = ba/6$$

$$\bar{x} A = \int \bar{x}_{el}\, dA = \int_0^a x\, [(b/a)\, x - (b/a^2)\, x^2]\, dx$$

$$= \int_0^a [(b/a)\, x^2 - (b/a^2)\, x^3]\, dx = [(b/a)(x^3/3) - (b/a^2)(x^4/4)]\, \Big|_0^a$$

$$= (b/a)(a^3/3) - (b/a^2)(a^4/4) = ba^2/3 - ba^2/4$$

$$\bar{x} A = ba^2/12$$

$$\bar{y} A = \int \bar{y}_{el}\, dA = \int_0^a \tfrac{1}{2}\, [(b/a)\, x + (b/a^2)\, x^2]\, [(b/a)\, x - (b/a^2)\, x^2]\, dx$$

$$= \tfrac{1}{2} \int_0^a [(b/a)^2\, x^2 - (b/a^2)^2\, x^4]\, dx = \tfrac{1}{2}\, [(b^2/a^2)(x^3/3) - (b^2/a^4)(x^5/5)]\, \Big|_0^a$$

$$= \tfrac{1}{2}\, [(b^2/a^2)(a^3/3) - (b^2/a^4)(a^5/5)] = \tfrac{1}{2}\, (b^2a/3 - b^2a/5)$$

$$\bar{y} A = b^2a/15$$

Finally, determine the location of the centroid.

$$\bar{x} = \bar{x}A/A = (ba^2/12)/(ba/6) = a/2$$

$$\bar{y} = \bar{y}A/A = (b^2a/15)/(ba/6) = 2b/5$$