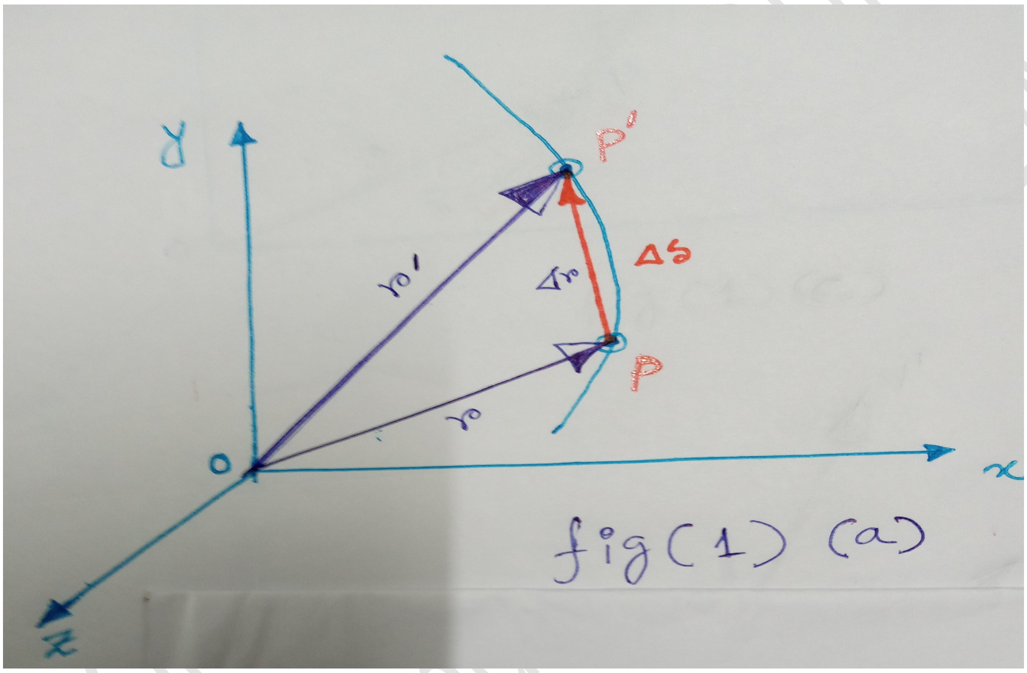
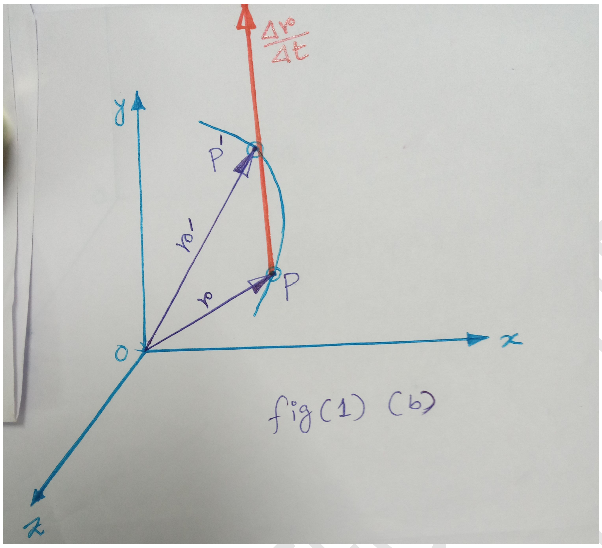
Expression for Position vector,Velocity and acceleration

When a particle moves along a curved , other than a straight line ,we understood that the particle is in curve linear motion. To define the position P occupied by the particle at an given instant of time t.Let us select a fixed reference system(no translation and no rotation with respect to time), as x,y,z axes as shown in figure (1) (a).



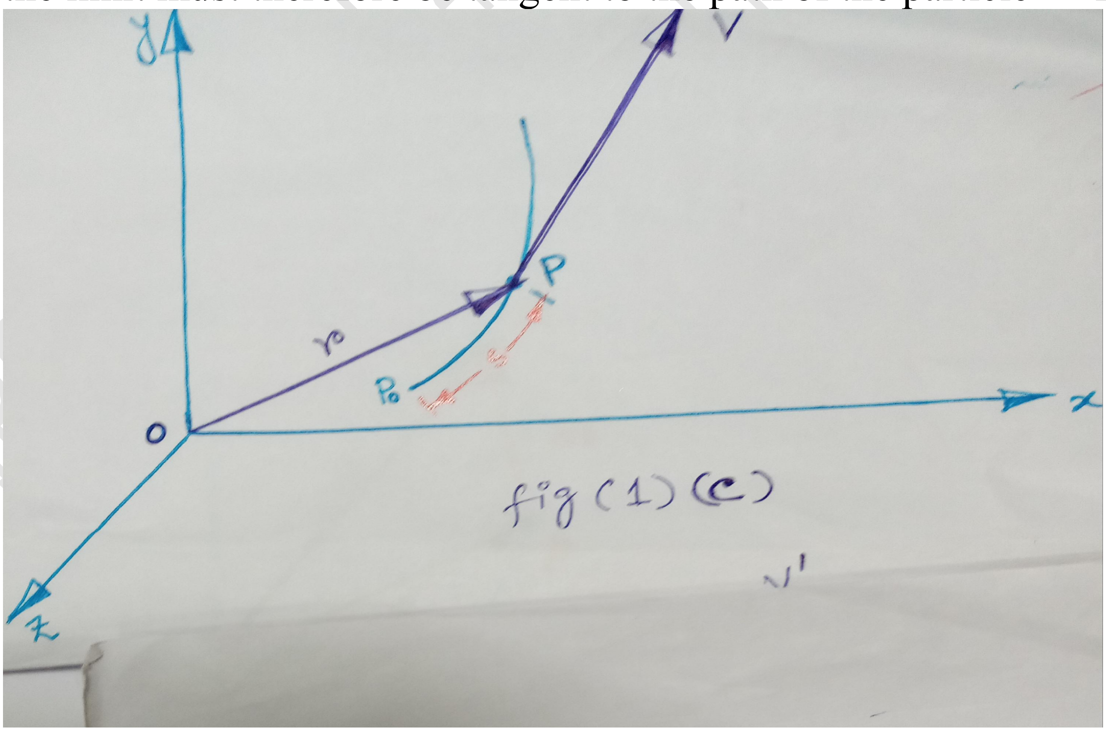
Let us draw the vector **r** ,joining the origin O and point **P**. Since the **r** is a position vector of the particle at the instant of time t, will represent the instantaneous position of particle at **P**,Similarly vector r' defines the position **P'** occupied by the same particle at a latter time $t+\Delta t$ .The vector Δ **r** joining **P** and **P'** represents the change in position vector during time interval Δ t since as we can easily check from figure (1) (a) . the vector r' is obtained by adding the vector **r** and Δ **r** according to triangle rule. We note that Δ **r** represents that a change in direction as well as a change in magnitude of the position vector **r**. The average velocity of the particle over the time interval Δ t is defined as the quotient of Δ **r** and Δ t.

Since Δ **r** is a vector and Δ t is scalar,the quotient $\Delta r/\Delta t$ is a vector attached at **P**, of the same direction as Δ **r** and of magnitude equal to the magnitude of $\Delta r/\Delta t$. figure (1) (b) .



The instantaneous velocity of the particle at time t is obtained by choosing shorter and shorter time interval Δt and, correspondingly shorter and shorter vector increments $\Delta \mathbf{r}$. The instantaneous velocity is thus represented by the vector $v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}$ ....eqn 1.1.

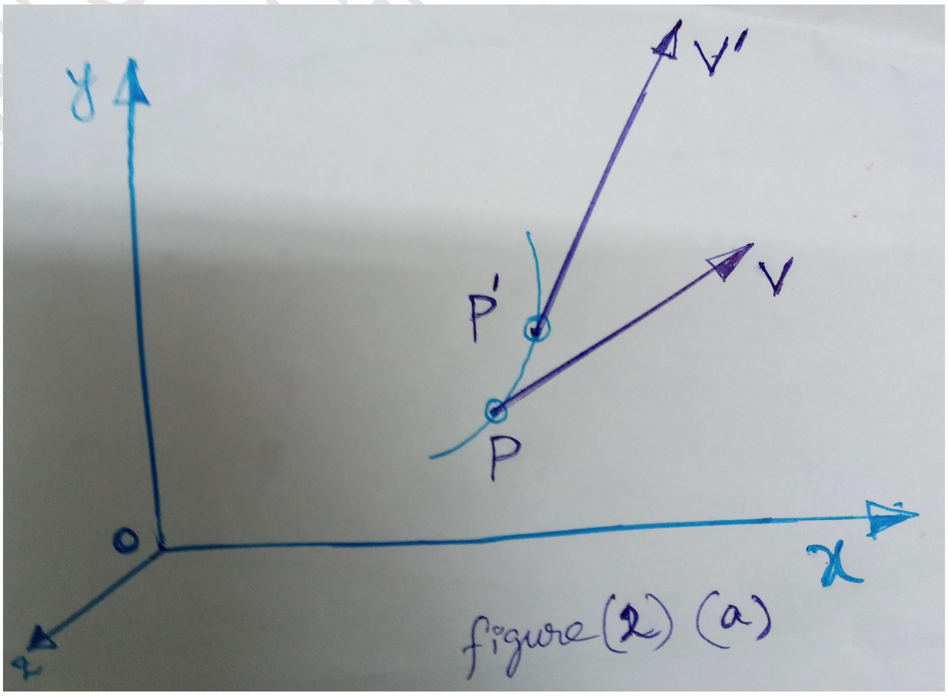
AS Δt and $\Delta \mathbf{r}$ becomes shorter . the points **P** and **P'** get closer ; the **v** obtained in the limit must therefore be tangent to the path of the particle figure (1) (c) .



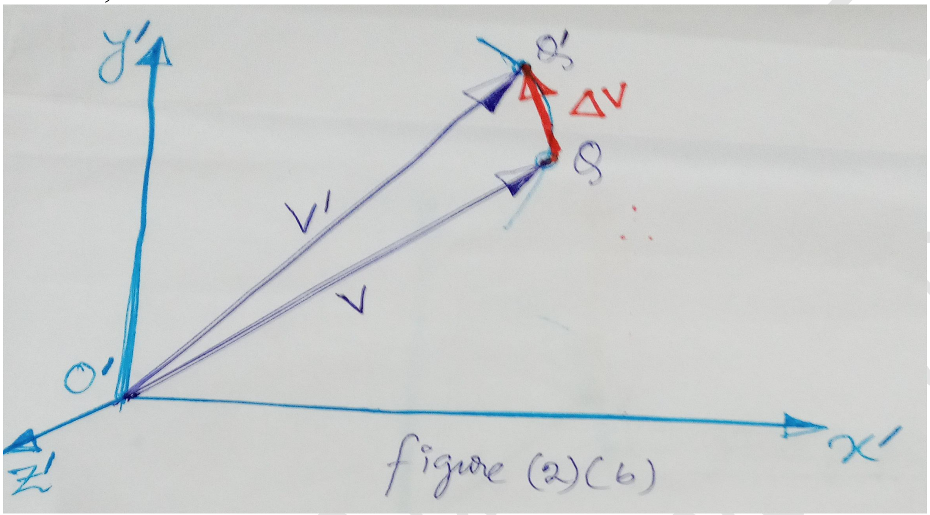
Since the position vector **r** depends upon the time t , we can refer to it as a *vector function* of the scalar variable t and denote it by **r**(t) . Extending the concept of derivative of a scalar function introduced in elementary calculus , We will refer to the limit of the quotient $\Delta \mathbf{r} / \Delta t$ as the derivative of the vector function **r**(t) . We write $v = \frac{d\mathbf{r}}{dt}$ …..eqn 1.2.

The magnitude v of the vector is called the speed of the particle . It can be obtained by substituting for the vector $\Delta \mathbf{r}$ in eqn 1.1 the magnitude of this vector represented by straight line segment PP'. But the length of the se gment PP' approaches the length Δs of the arc PP' as Δt decreases(figur e(1) (a)) and we can write $v = \lim_{\Delta t \to 0} \frac{PP'}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$, $v = \frac{ds}{dt}$ ……(1.3)

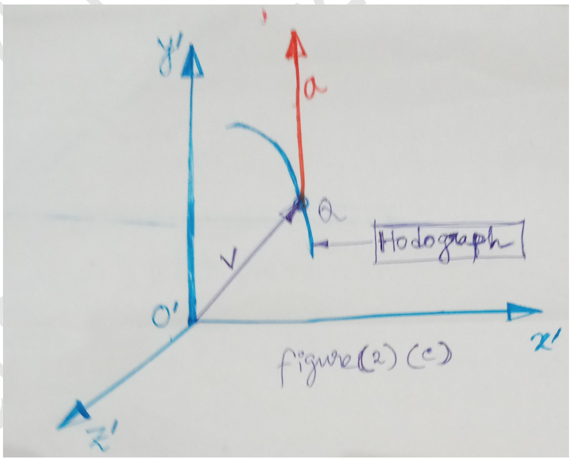
The speed v can thus be obtained by differentiating with respect to t the length s of the arc described by the particle. Consider the velocity **v** of the particle at time t and it's velocity **v'** at a latter time $t + \Delta t$ (figure 2) (a) .



Let us draw both vectors **v** and **v'** from the same origin O' (figure (2) (b)). The vect or Δv joining **Q** and **Q'** represents the change in the velocity of the particle during the time interval Δt,



Since, the vector **v'** can be obtained by adding the vectors **v** and Δv . We should note that Δv represents a change in the *direction* of the velocity as well as a change in speed v.

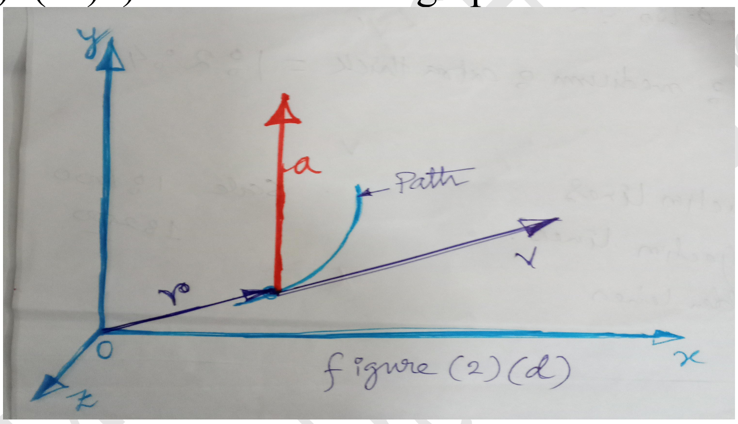


The *average acceleration* of the particle over the time interval Δt is defined as the q uotient of Δv and Δt. Science Δv is a vector and Δt is a scalar, the quotient $\Delta \mathbf{v}/\Delta t$ is a vector of the same direction as $\Delta \mathbf{V}$.

The *instantaneous acceleration* of the particle at time t is obtained by ch oosing smaller and smaller values for Δt and Δ**v** .The instantaneous acceler ation is thus represented by the vector $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$ ……...eqn1.4.

Noting that the velocity **v** is a vector function **v**(t) of the time t , We can refer limit of the quotient $\Delta \mathbf{v} / \Delta t$ as the derivative of **v** with respect to t . We write $a = \frac{d\,\mathrm{v}}{dt}$ eqn1.5

We observe that the acceleration **a** is tangent to the curve described by the tip Q of the vector **v** when the latter is drawn from a fixed origin O' (figure (2) (c)) and that in general the acceleration is not tangent to the path of the particle (figure (2) (d)). The curve described by the tip of **v** and shown in figure (2) (c)) is called hodograph of the motion.

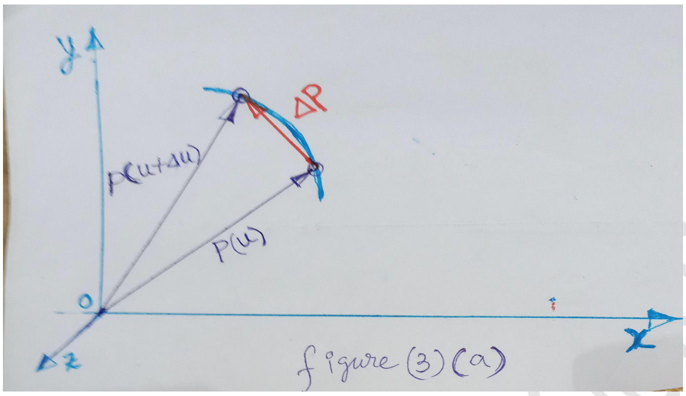


DERIVATIVES OF VECTOR FUNCTION

We saw that velocity **v** of a particle in curve linear motion can be represented by the derivative of the vector function **r**(t) characterizing the position of the particle. Similarly, the acceleration **a** of the particle can be represented by the derivative of the vector function **v**(t). In this section, We will give a formal definition of the derivative of a vector function and establish a few rules governing the differentiation of sums and the products of vector functions.

Let **P**(u) be a vector function of the scalar variable u. By that we mean that the scalar u completely defines the magnitude and direction of the vector **P.**

If the vector **P** is drawn from a fixed origin O and the Scalar u is allowed to vary , the tip of **P** will describe a given curve in space.Consider the vector **P** corresponding , respectively,to the value u and u+ Δ u of the scalar variable (figure(3)(a)).

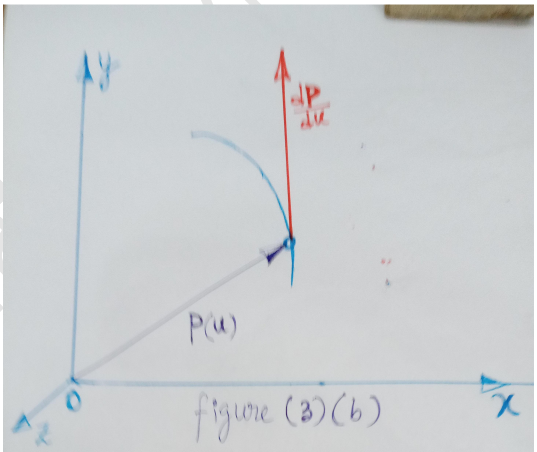


Let Δ **P** by the vector joining the tips of the two given vectors ; We write Δ **P**= **P**(u +Δ u)-**P**(u)

Dividing through by Δ u and letting Δ u approach zero, *We define the derivative of the vector function* **P**(u) :

$$\frac{dP}{du} = \lim_{\Delta u \to 0} \frac{\Delta P}{\Delta u} = \lim_{\Delta u \to 0} \frac{P(u + \Delta u) - P(u)}{\Delta u} \text{ ....eqn 1.6}$$

As Δ u approaches to zero, the line of action of Δ **P** becomes tangent to the curve of (figure (3) (a)). Thus, the derivative dP/du of the vector function **P**(u) *is tangent to the curve describing by the tip of* **P**(u) (figure (3) (b)).



The standard rules for the differentiation of the sums and products of scalar functions can be extended to vector functions. Consider first the sum of vector functions **P**(u) and **Q**(u) of the same scalar variable u. According to the definition , the derivative of the vector **P** +**Q** is

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$$\frac{d(\mathrm{P}+\mathrm{Q})}{du} = \lim_{\Delta u \to 0} \frac{\Delta(\mathrm{P}+\mathrm{Q})}{\Delta u} = \lim_{\Delta u \to 0}\left(\frac{\Delta \mathrm{P}}{\Delta u}+\frac{\Delta \mathrm{Q}}{\Delta u}\right)$$

Or since the limit of sum is equal to the sum of the limits of it's terms,

$$\frac{d(\mathrm{P}+\mathrm{Q})}{du} = \lim_{\Delta u \to 0} \frac{\Delta \mathrm{P}}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta \mathrm{Q}}{\Delta u}$$

$$\frac{d(\mathrm{P}+\mathrm{Q})}{du} = \frac{d\mathrm{P}}{du} + \frac{d\mathrm{Q}}{du} \quad \text{……………………eqn 1.7}$$

The product of *scalar function* f(u) *and a vector function* **P**(u) of the same scalar variable u will now be considered, The derivative of the vector *f***P** is

$$\frac{d(f\mathrm{P})}{du} = \lim_{\Delta u \to 0} \frac{(f+\Delta f)(\mathrm{P}+\Delta \mathrm{P}) - f\mathrm{P}}{\Delta u} = \lim_{\Delta u \to 0}\left(\frac{\Delta f}{\Delta u}\mathrm{P} + f\frac{\Delta \mathrm{P}}{\Delta u}\right)$$

Or recalling the properties of the limits of the sums and products,

$$\frac{d(f\mathrm{P})}{du} = \frac{df}{du}\mathrm{P} + f\frac{d\mathrm{P}}{du} \quad \text{………………………………………………….eqn 1.8}$$

The derivatives of the scalar product and the vector product of two vector functions **P**(u) and **Q**(u) can be obtained in a similar way . We have

$$\frac{d(\mathrm{P}.\mathrm{Q})}{du} = \frac{d\mathrm{P}}{du}.\mathrm{Q} + \mathrm{P}.\frac{d\mathrm{Q}}{du} \quad \text{……………………………………………….eqn 1.9}$$

$$\frac{d(\mathrm{P}\times\mathrm{Q})}{du} = \frac{d\mathrm{P}}{du}\times\mathrm{Q} + \mathrm{P}\times\frac{d\mathrm{Q}}{du} \quad \text{………………………………eqn 1.10}$$

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The properties established above can be used to determine the rectangular components of the derivative of a vector function **P**(u) . Resolving **P** into components along fixed rectangular axes x,y,z , we write

$\mathbf{P}=P_x\mathbf{i}+P_y\mathbf{j}+P_z\mathbf{k}$……………………………………………eqn 1.11

Where Px ,Py ,Pz are the rectangular scalar components of the vector **P** and **i**,**j**,**k** the unit vectors corresponding, respectively, to the x,y,and z axes . By the derivative of **P** is equal to the sum of the derivatives of the term in the right hand member . Since each of these terms is the product of a scalar and vector function , we should use (eqn 1.8) . But the unit vectors **i**,**j**,**k** have a constant magnitude (equal to 1) and fixed directions. The derivatives are therefore zero,and we write

$$\frac{dP}{du} = \frac{dP_x}{du}i + \frac{dP_y}{du}j + \frac{dP_z}{du}k \quad \text{……………………………………………eqn 1.12}$$

Noting that coefficient of the unit vectors are, by definition , the scalar component of the vector dP/du ,we conclude that *the rectangular scalar component of the derivatives* dP/du *of the vector function* **P**(u) are obtained by differentiating the corresponding scalar component of **P**.

RATE OF CHANGE OF A VECTOR

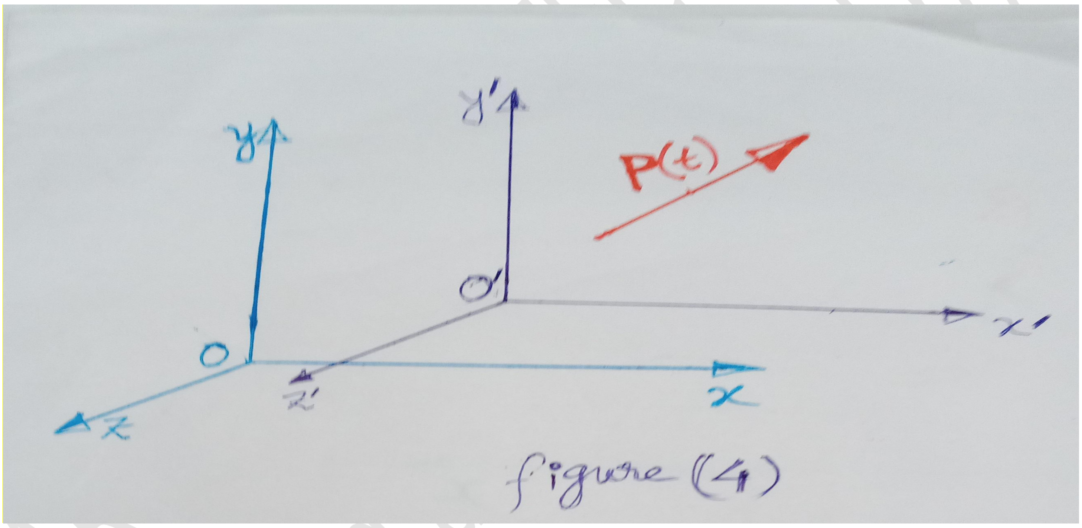
When the vector **P** is a function of time t , it's derivative dP/dt represents the *rate of change* of **P** with respect to the frames Oxyz. Resolving **P** into rectangular components , we have by (eqn 1.12) ,

$$\frac{dP}{dt} = \frac{dP_x}{dt}i + \frac{dP_y}{dt}j + \frac{dP_z}{dt}k$$

Or using dots to indicate differentiation with respect to t,

$$\dot{P} = \dot{P}_x i + \dot{P}_y j + \dot{P}_z k \quad \text{…………………………………eqn 1.13}$$

As the rate of change of a vector as observed from a moving frame of reference is, in general , different from it's rate of change as observed from a fixed frame of reference. However , if the moving frame O'x'y'z' is in translation , i.e , if it's axes remains parallel to the corresponding axes of the fixed frame Oxyz (Figure (4.)) The same unit vectors **i , j , k** are used in both frames, and at any given instant the vector **P** has the same components Px, Py ,Pz in both frames. It follows from (eqn 11.13) that the rate of change P is the same swith respect to the frames Oxyz and O'x'y'z'. We state, therefore : The rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation. This properties will greatly simplify our work, Since we will be concerned mainly with frames in translation.



RECTANGULAR COMPONENTS OF VELOCITY AND ACCELERATION

When the position of a particle P is defined at any instant by it's rectangular coordinates x,y and z , it is convenient to resolve the velocity **v** and acceleration **a** of the particle into rectangular components (Figure (4) (a)).

Resolving the position vector **r** of the particle into rectangular components, we write **r**= x**i**+y**j**+z**k**……….eqn 1.14

Where the coordinates of x,y,z are function of t .Differentiating twice , we obtain .

$$v = \frac{d\mathbf{r}}{dt} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}$$ ……………………………. eqn 1.15 And

$$\mathbf{a} = \frac{d\mathrm{v}}{dt} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}$$ **…………………………eqn1.16**

TANGENTIAL AND NORMAL COMPONENTS

The velocity of the particle is a vector tangent to the path of the particle but that, in general , the acceleration is not tangent to the path. It is some time convenient to resolve the acceleration into components directed, respectively , along the tangent and normal to the path of the particle.

PLANE MOTION OF THE PARTICLE

First let us consider a particle which moves along a curve contained in the plane of the figure . Let P be the position of a particle of given instant. We attach at P a unit vector $\mathbf{e}_t$ tangent to the path of the particle and pointing in the direction of motion (Figure ((5)(a)).

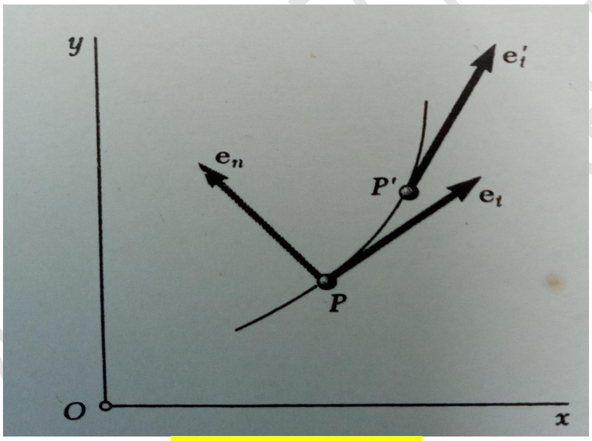


Figure-((5)(a))

Let $\mathbf{e}_t$' be the unit vector corresponding to the position P' of the particle at a latter instant . Drawing both vectors from the same origin O' , We define the vector $\Delta \mathbf{e}_t = \mathbf{e}_t' - \mathbf{e}_t$ (Figure-((5)(b)) .

Science $\mathbf{e}_t$ and $\mathbf{e}_t$' are of unit length there tips lie on a circle of radius 1. Denoting by $\Delta\theta$ the angle formed by $\mathbf{e}_t$ and $\mathbf{e}_t$' ,We find that the magnitude of $\Delta \mathbf{e}_t$ is $2\sin(\Delta\theta/2)$.

Considering now the vector $\Delta \mathbf{e}_t/\Delta\theta$,we note that as $\Delta\theta$ approaches zero , This vector be comes tangent to the unit circle of (figure-(5)(b)), i.e, perpendicular to $\mathbf{e}_t$, and that it magnitude approaches.

$$\lim_{\Delta\theta \to 0} \frac{2Sin(\Delta\theta/2)}{(\Delta\theta)} = \lim_{\Delta\theta \to 0} \frac{Sin(\Delta\theta/2)}{(\Delta\theta/2)} = 1$$

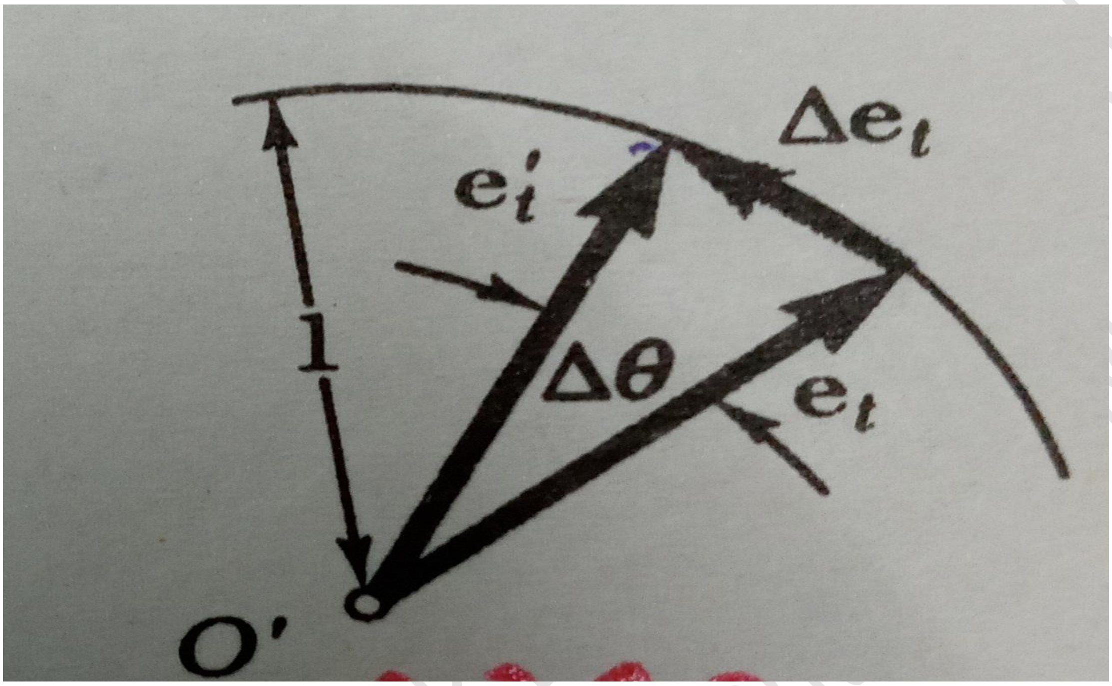


Figure-((5)(b))

Thus, the vector obtained in the limit is a unit vector along the normal to the path of the particle,in the direction towards which $\mathbf{e}_t$, turns. Denoting this vector by $\mathbf{e}_n$, We write

$$\mathbf{e}_n = \lim_{\Delta\theta\to 0} \frac{\Delta \mathrm{e}_t}{\Delta\theta}$$

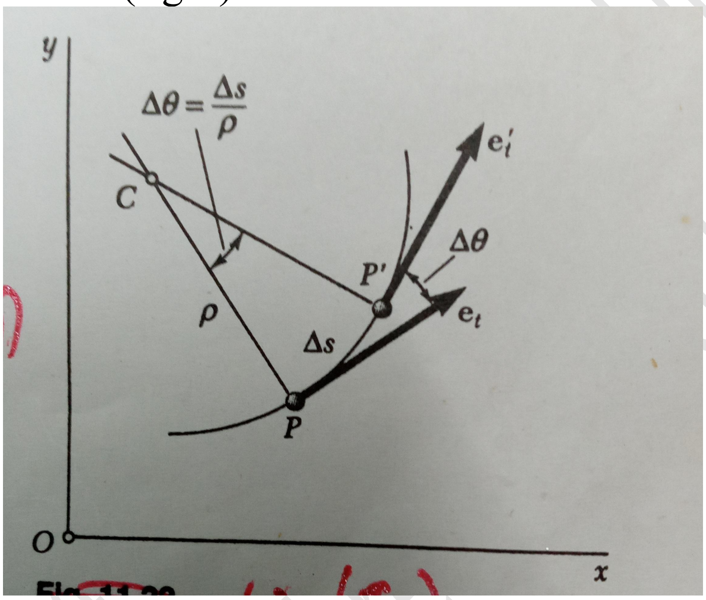
$$\mathbf{e}_n = \frac{d\mathrm{e}_t}{d\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{eqn } 1.17$$

Since the velocity **v** of the particle is tangent to the path , it can be expressed as the product of the scalar v and the unit vector $\mathbf{e}_t$. We have
$\mathbf{v}=v\mathbf{e}_t$ ……………………………………………………..eqn1.18
To obtain acceleration of the particle,(eqn 1.18) will be differentiated with respect to t . Applying the rule for differentiation of the product of a scalar and vector function we write $a = \frac{d\,\mathrm{v}}{dt} = \frac{dv}{dt}\mathrm{et} + v\frac{d\mathrm{et}}{dt}$ ……eqn1.19

But, $\dfrac{d\,e_t}{dt} = \dfrac{d\,e_t}{d\,\theta}\dfrac{d\,\theta}{ds}\dfrac{ds}{dt}$ Recalling that $\dfrac{ds}{dt} = v$, from eqn 1.3

that $de_t/d\theta = \mathbf{e_n}$, and from elementary calculus that $\dfrac{d\,\theta}{ds} = 1/\rho$,Where ρ is the radius

of curvature of the path at P (fig-6)



(Figure-(6))

We have $\dfrac{de_t}{dt} = \dfrac{v}{\rho} e_n$ ……………………………………eqn 1.20

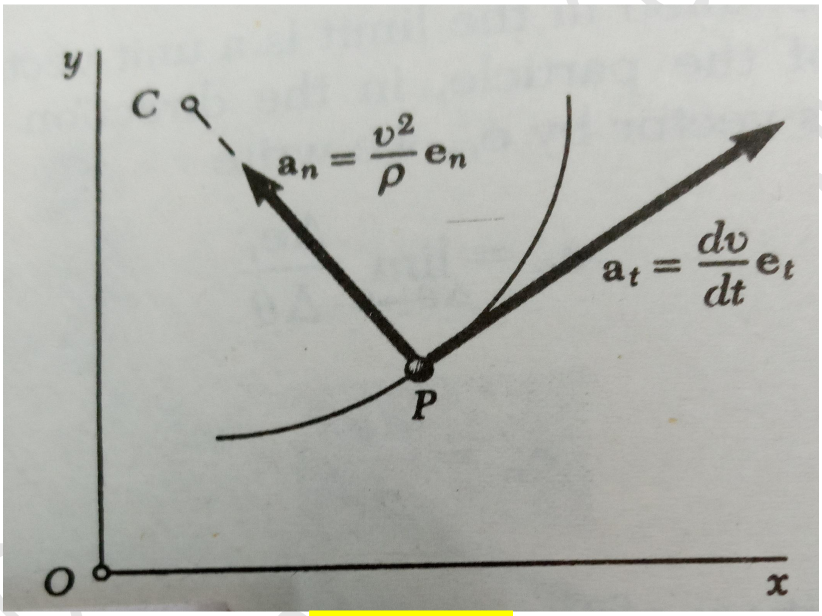
Substituting in eqn 1.19 , We obtain

$\mathbf{a} = \dfrac{dv}{dt} e_t + \dfrac{v^2}{\rho} e_n$ ……………………………………eqn 1.21

Thus the scalar component of acceleration are

$a_t = \dfrac{dv}{dt}$ and $a_n = \dfrac{v^2}{\rho}$ ……………………………………eqn1.22

The relations obtained express that the tangential component of the acceleration is equal to the rate of change of the speed of the particle , while the normal component is equal to the square of the speed divided by the radius of curvature of the path at P. If the speed of the particle increases , $\mathbf{a}_t$ is positive and the vector component at points in the direction of motion. If the speed of the particle decreases , $\mathbf{a}_t$ is negative and $\mathbf{a}_t$ points against the direction of motion. The vector component $\mathbf{a}_n$, on the other hand , is always directed towards the centre of curvature C of the path (figure-(7)).



(Figure-(7))

We conclude from the above that the tangential component of the acceleration reflects a change in the speed of the particle , while it's normal component reflects a change in the direction of motion of the particle .The acceleration of the particle will be zero only if both it's components are zero.Thus, the acceleration of a particle moving with constant speed along a curve will not be zero unless the particle happens to pass through a point of inflection of the curve (where the radius of the curve is infinite)or unless the curve is a straight line.The facts that the normal component of acceleration depends upon the radius of curvature of the path followed by the particle is taken into account in the design of structures or mechanism as widely different as airplane wings,railroad tracks and cams. In order to avoid sudden changes in acceleration of the air particles flowing past a wing, wing profiles are designed without any sudden change in curvature.

Similar care is taken in designing railroad curves to avoid sudden changes in acceleration of the cars(which would be hard on the equipment and unpleasant for the passengers).A straight section of track, for instance, is never directly followed by a circular section. Special transition sections are used to help pass smoothly from the infinite radius of curvature of the straight section to the finite radius of the circular track.Likewise in the design of high speed cams, abrupt changes in acceleration are avoided by using transition curves which produce a continuous change in acceleration.

EXPRESSION FOR RADIUS OF CURVATURE(ρ) FOR PLANE CURVILINEAR MOTION IF THE EQUATION OF MOTION OF THE PARTICLE IS EXPRESSED AS y=f(x) .

Let us consider the motion of a particle is described as plane curvilinear motion with the help of a fixed coordinate axis Oxy. Consider the motion of a single particle moving along a curved path (as described in figure below). Let point A on the curve be the position of a moving particle at a given instant.Let us assume a unit vector e_t tangent to the path at point A and e_t' be the unit vector corresponding to point B. The equation of motion of the particle is expressed as y=f(x) . Now we can derive the expression for radius of curvature at a specific point on the curve will be

$$\rho = \frac{[1+(\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}}$$

Referring to the figure-8, It may be shown that Angle AME=dθ

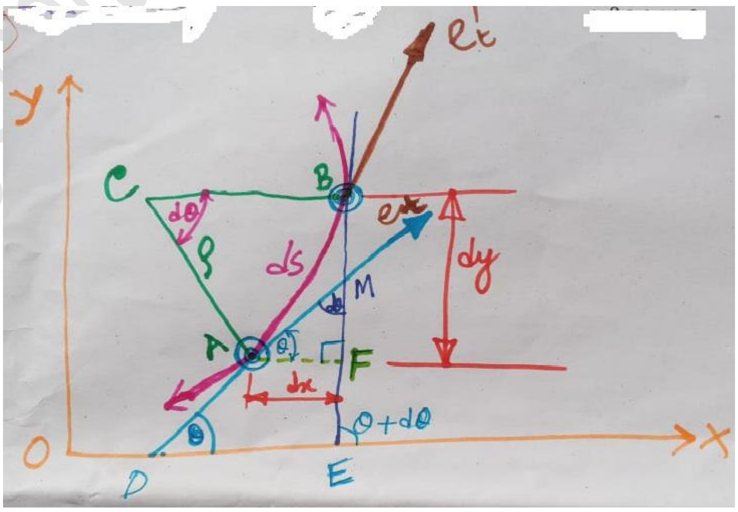


Figure-8

Angle ACB=Angle AME=$d\theta$ and ds=$\rho\, d\theta$

$$\frac{1}{\rho} = \frac{d\theta}{ds}$$

Since ds is an elemental length treating ABF as a triangle we may write as

$$\frac{ds}{dx} = \sec\theta$$

$$\frac{dy}{dx} = \tan\theta$$

$$\frac{d^2y}{dx^2} = Sec^2\theta \; \frac{d\theta}{ds} \; \frac{ds}{dx}$$

$$\frac{d^2y}{dx^2} = Sec^2\theta \; \frac{1}{\rho} \; \sec\theta$$

$$\frac{d^2y}{dx^2} = \frac{1}{\rho} \; Sec^3\theta$$

Rearranging we obtain,

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Where ρ = Radius of curvature

(Hence proved)

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