Moment of Inertia

by

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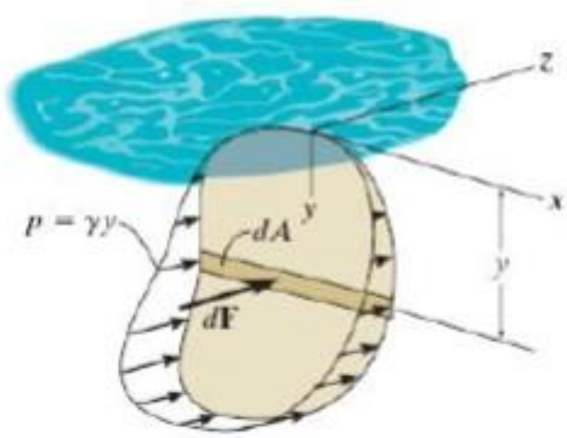
MOMENTS OF INERTIA FOR AREAS

- Centroid for an area is determined by the first moment of an area about an axis
- Second moment of an area is referred as the moment of inertia
- Moment of inertia of an area originates whenever one relates the normal stress σ or force per unit area



MOMENTS OF INERTIA FOR AREAS

First moment about x-axis=ydA

Second moment x-a

Moment of Inertia

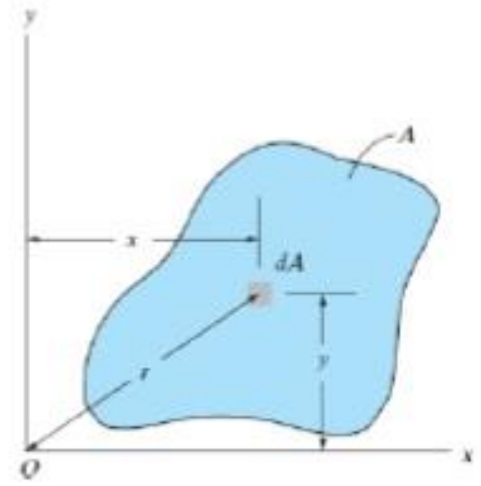
- Consider area A lying in the x-y plane
- Be definition, moments of inertia of the differential plane area dA about the x and y axes

$$dI_x = y^2 dA \quad dI_y = x^2 dA$$

- For entire area, moments of inertia are given by

$$I_x = \int_A y^2 dA$$

$$I_y = \int_A x^2 dA$$



MOMENTS OF INERTIA FOR AREAS

Moment of Inertia

- Formulate the second moment of dA about the pole O or z axis
- This is known as the polar axis

$$dJ_O = r^2 dA$$

where r is perpendicular from the pole (z axis) to the element dA

- Polar moment of inertia for entire area,

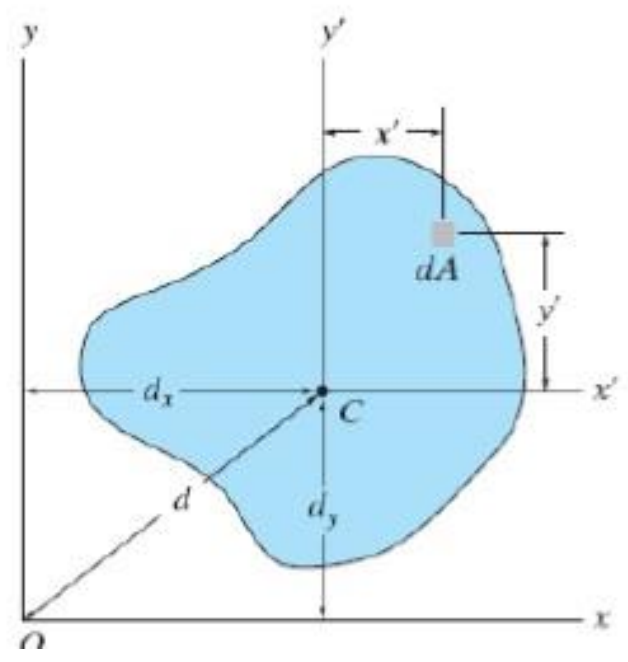
$$J_O = \int_A r^2 dA = I_x + I_y$$

Perpendicular Axis Theorem

- The moment of inertia (MI) of a plane area about an axis normal to the plane is equal to the sum of the moments of inertia about any two mutually perpendicular axis lying in the plane and passing through the given axis.
- That means the Moment of Inertia $I_z = I_x + I_y$

PARALLEL AXIS THEOREM FOR AN AREA

- For moment of inertia of an area known about an axis passing through its centroid, determine the moment of inertia of area about a corresponding parallel axis using the parallel axis theorem
- Consider moment of inertia of the shaded area
- A differential element dA is located at an arbitrary distance y′ from the centroidal x′ axis



PARALLEL AXIS THEOREM FOR AN AREA

- The fixed distance between the parallel x and x′ axes is defined as d_y
- For moment of inertia of dA about x axis

$$dI_x = \left(y' + d_y\right)^2 dA$$

- For entire area

$$I_x = \int_A \left(y' + d_y\right)^2 dA$$

$$= \int_A y'^2 \, dA + 2d_y \int_A y' \, dA + d_y^2 \int_A dA$$

- First integral represent the moment of inertia of the area about the centroidal axis

PARALLEL AXIS THEOREM FOR AN AREA

- Second integral = 0 since x′ passes through the area′s centroid C $\int y' dA = \bar{y}\int dA = 0; \quad \bar{y} = 0$
- Third integral represents the total area A

$$I_x = \bar{I}_x + Ad_y^2$$

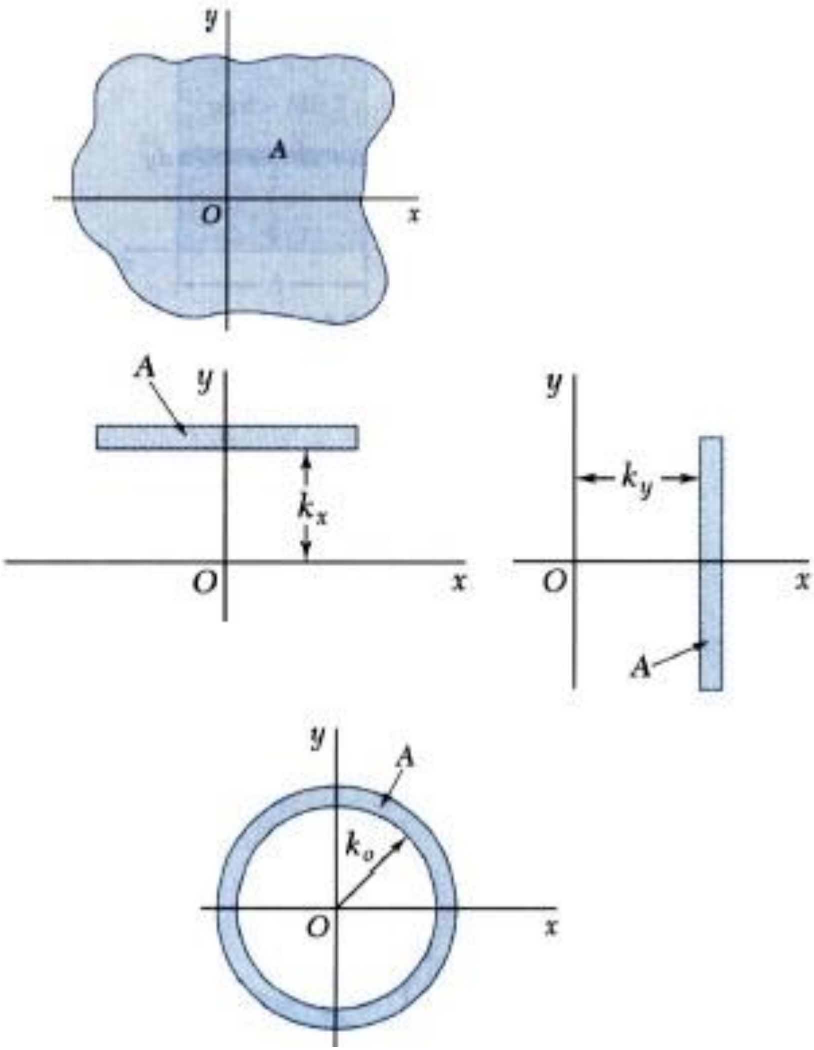
- Similarly

$$I_y = \bar{I}_y + Ad_x^2$$

- For polar moment of inertia about an axis perpendicular to the x-y plane and passing through pole O (z axis)

$$J_O = \bar{J}_C + Ad^2$$

Radius of Gyration of an Area



- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ *radius of gyration* with respect to the x axis

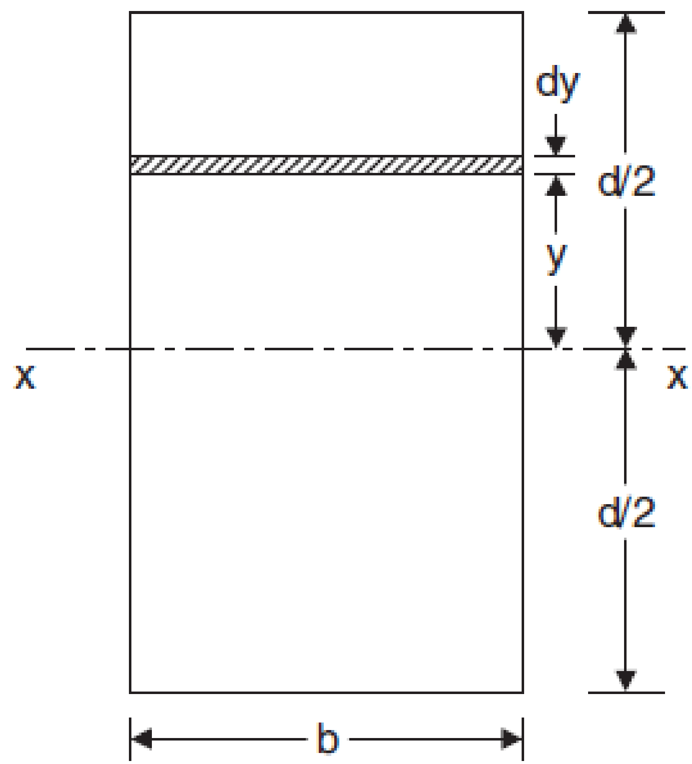
- Similarly,

$$I_y = k_y^2 A \quad k_y = \sqrt{\frac{I_y}{A}}$$

$$J_O = k_O^2 A \quad k_O = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_x^2 + k_y^2$$

Moment of Inertia of a Rectangle about the Centroidal Axis



Consider a rectangle of width *b* and depth *d*. Moment of inertia about the centroidal axis *x*-*x* parallel to the short side is required.

Moment of Inertia of a Rectangle about the Centroidal Axis

Consider an elemental strip of width *dy* at a distance *y* from the axis.

Moment of inertia of the elemental strip about the centroidal axis *xx* is:

$$= y^2 dA$$

$$= y^2 b\ dy$$

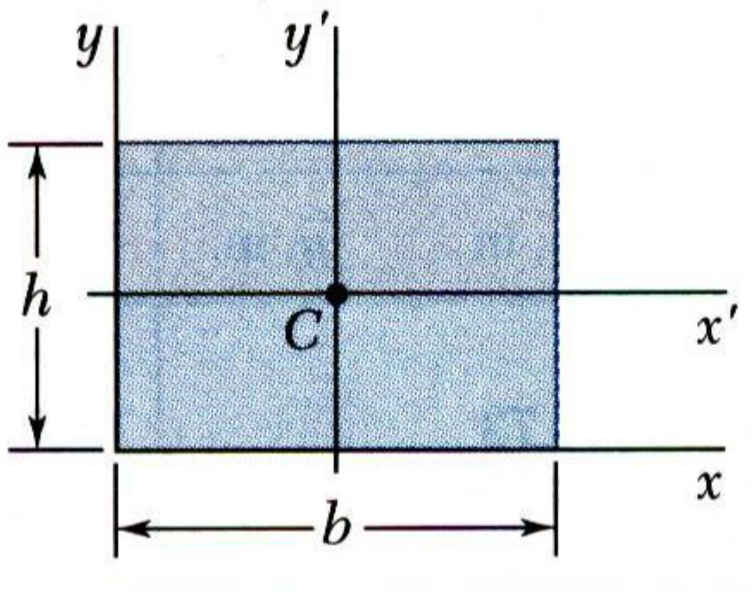
$$I_{xx} = \int_{-d/2}^{d/2} y^2 b\ dy$$

$$= b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$= b\left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$I_{xx} = \frac{bd^3}{12}$$

Area Moments of Inertia: Standard MIs



$$I_{xx} = \frac{bh^3}{12} + bh(\frac{h}{2})^2$$

Moment of inertia about x-axis

$$I_{xx} = \frac{1}{3}bh^3$$

Moment of inertia about y-axis

$$I_{yy} = \frac{1}{3}hb^3$$

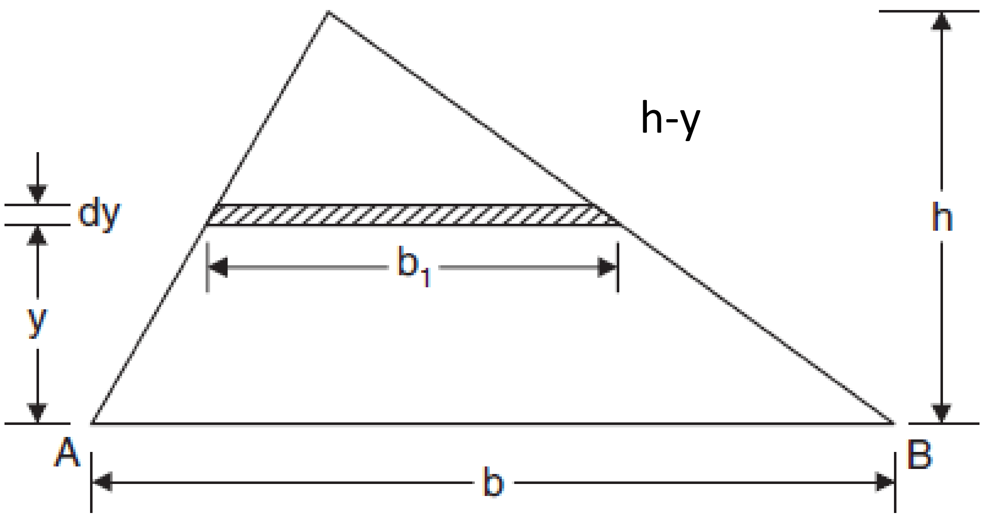
Moment of inertia about x'-axis

$$I_{x'x'} = \frac{1}{12}bh^3$$

Moment of inertia about y'-axis

$$I_{y'y'} = \frac{1}{12}hb^3$$

Moment of Inertia of a Triangle about its Base:



$$dA = \frac{h-y}{h} bdy$$

Moment of inertia of a triangle with base width *b* and height *h* is to be determined about the base *AB.*

Consider an elemental strip at a distance y from the base AB. Let dy be the thickness of the strip and dA its area. Width of this strip is given by:

$$b_1 = \frac{(h-y)}{h} \times b$$

Moment of Inertia of a Triangle about its Base:

Moment of inertia of this strip about *AB*

$$= y^2 dA$$

$$= y^2 \, b_1 \, dy$$

$$= y^2 \, \frac{(h-y)}{h} \times b \times dy$$

Moment of inertia of the triangle about *AB*,

$$I_{AB} = \int_0^h \frac{y^2 (h-y) b dy}{h}$$

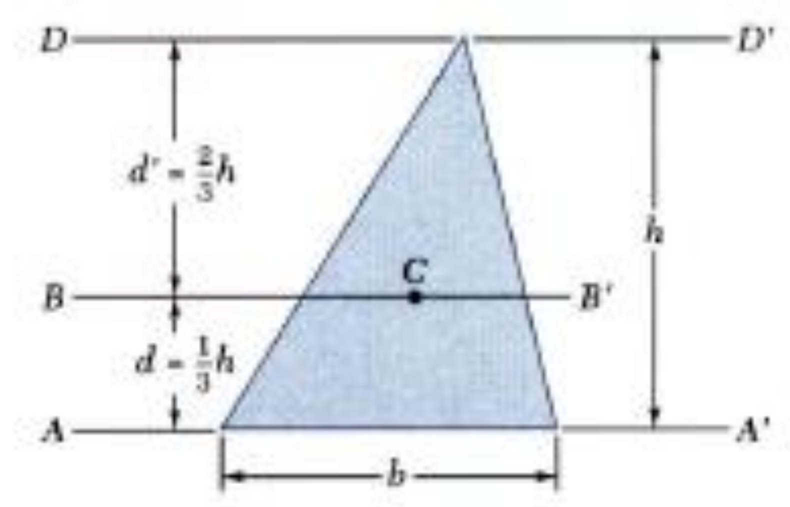
$$= \int_0^h \left(y^2 - \frac{y^3}{h} \right) b dy$$

$$= b \left[\frac{y^3}{3} - \frac{y^4}{4h} \right]_0^h$$

$$= b \left[\frac{h^3}{3} - \frac{h^4}{4h} \right]$$

$$I_{AB} = \frac{bh^3}{12}$$

Moment of inertia of a triangle with respect to a centroidal axis



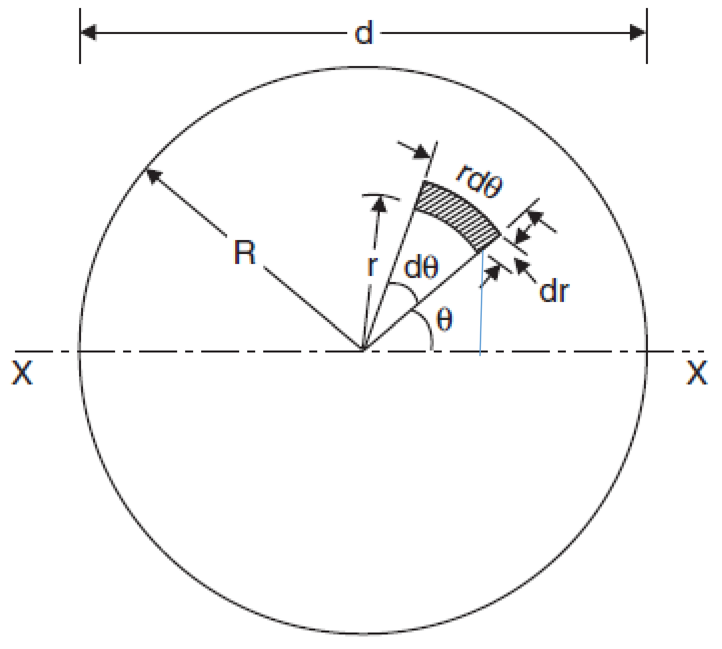
Moment of inertia of a triangle with respect to a centroidal axis,

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$

$$I_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh\left(\tfrac{1}{3}h\right)^2$$

$$= \tfrac{1}{36}bh^3$$

Moment of Inertia of a Circle about its Diametral Axis:



$$dA = dr.rd\theta$$

$$\frac{y}{r} = \sin\theta$$

Moment of inertia of a circle of radius *R* is required about it's diametral axis as shown in Figure.

Consider an element of sides rdθ and dr as shown in the figure. It's moment of inertia about the diametral axis x-x:

$$= y^2 dA$$
$$= (r \sin \theta)^2 r\, d\theta\, dr$$
$$= r^3 \sin^2 \theta\, d\theta\, dr$$

Moment of Inertia of a Circle about its Diametral Axis:

Moment of inertia of the circle about *x*-*x* is given by

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2 \theta \, d\theta \, dr$$

$$= \int_0^R \int_0^{2\pi} \frac{(1 - \cos 2\theta)}{2} \, d\theta \, dr$$

$$= \int_0^R \frac{r^3}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} dr$$

$$= \left[\frac{r^4}{8} \right]_0^R [2\pi - 0 + 0 - 0] = \frac{2\pi}{8} R^4$$

$$I_{xx} = \frac{\pi R^4}{4}$$

Moment of Inertia of a Circle about its Diametral Axis:

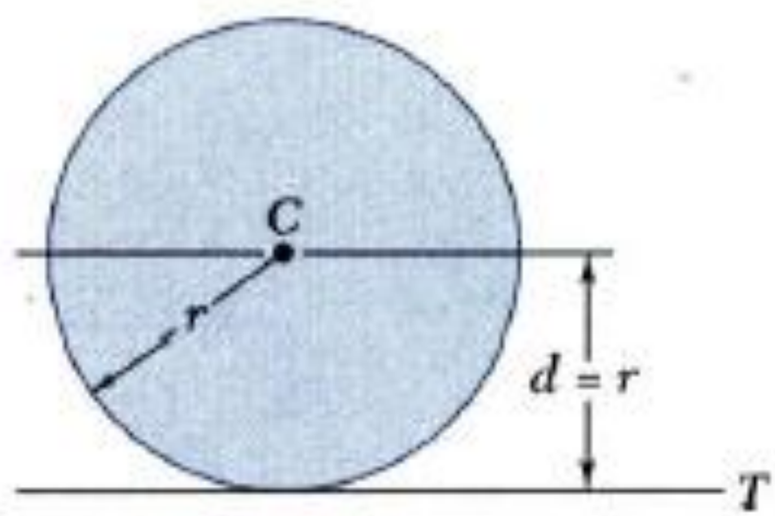
If *d* is the diameter of the circle, then

$$R = \frac{d}{2}$$

$$I_{xx} = \frac{\pi}{4}\left(\frac{d}{2}\right)^4$$

$$I_{xx} = \frac{\pi d^4}{64}$$

Area Moments of Inertia circular area with respect to a tangent to the circle

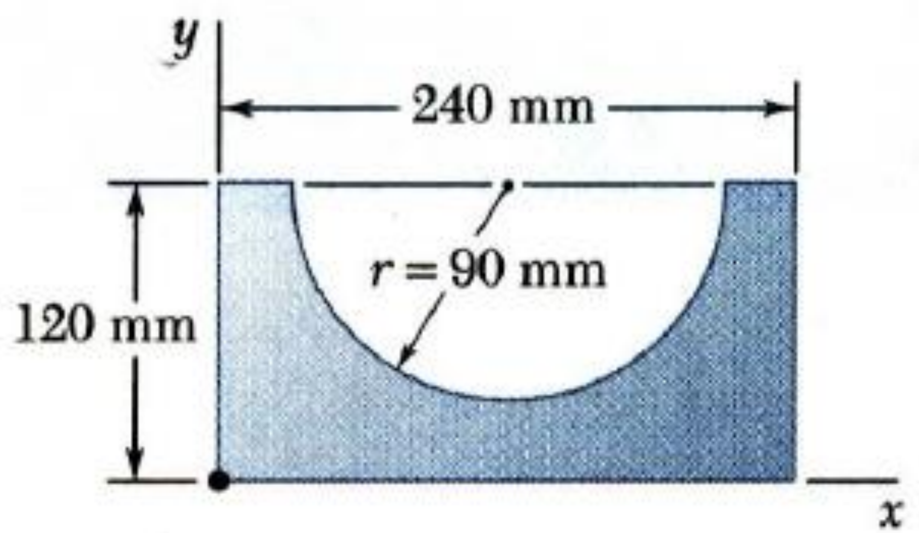


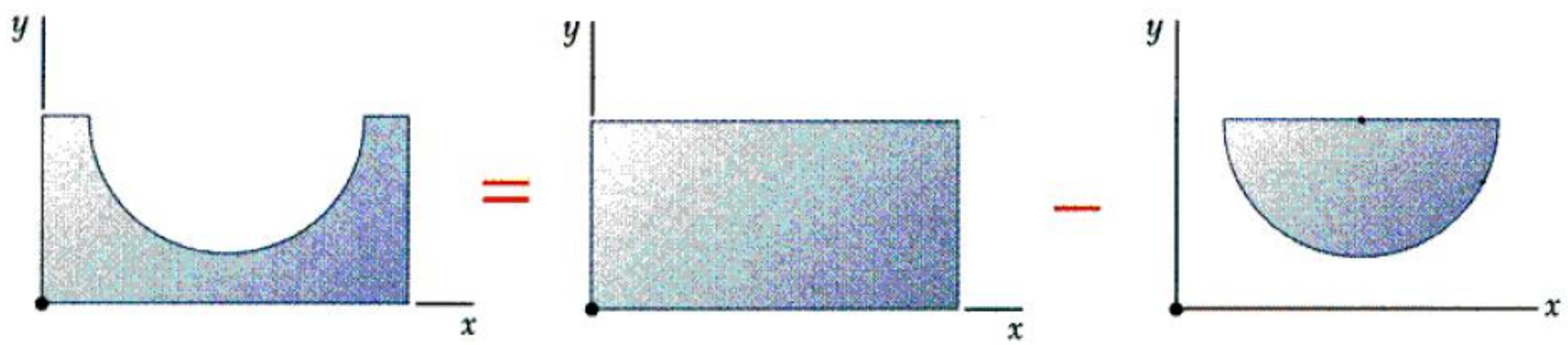
- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right)r^2$$

$$= \tfrac{5}{4}\pi r^4$$

Compute the moments of inertia of the bounding rectangle



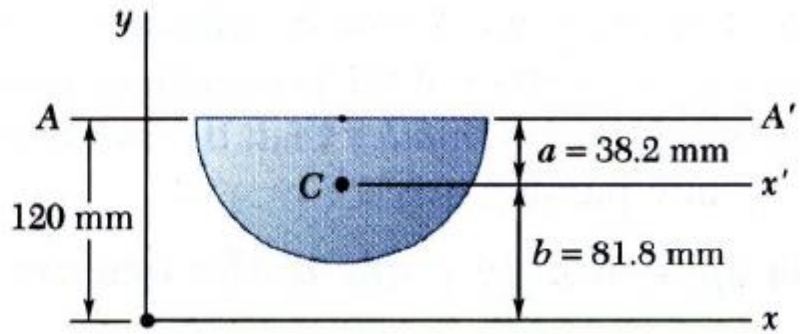


Compute the moments of inertia of the bounding rectangle

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the x axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120)^3 = 138.2\times10^6\,mm^4$$



$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2\,mm$$

$$b = 120 - a = 81.8\,mm$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$

$$= 12.72\times10^3\,mm^2$$

Compute the moments of inertia of the bounding rectangle

Half-circle:

moment of inertia with respect to AA',

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76\times10^6\,\text{mm}^4$$

Moment of inertia with respect to x',

$$\bar{I}_{x'} = I_{AA'} - Aa^2 = \left(25.76\times10^6\right) - \left(12.72\times10^3\right)(38.2)^2$$
$$= 7.20\times10^6\,mm^4$$

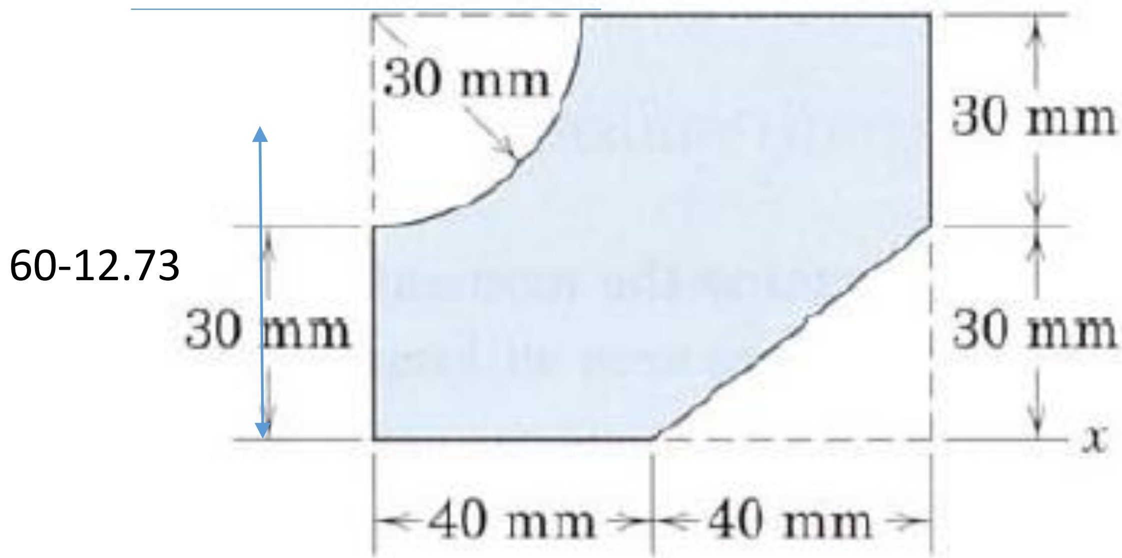
moment of inertia with respect to x,

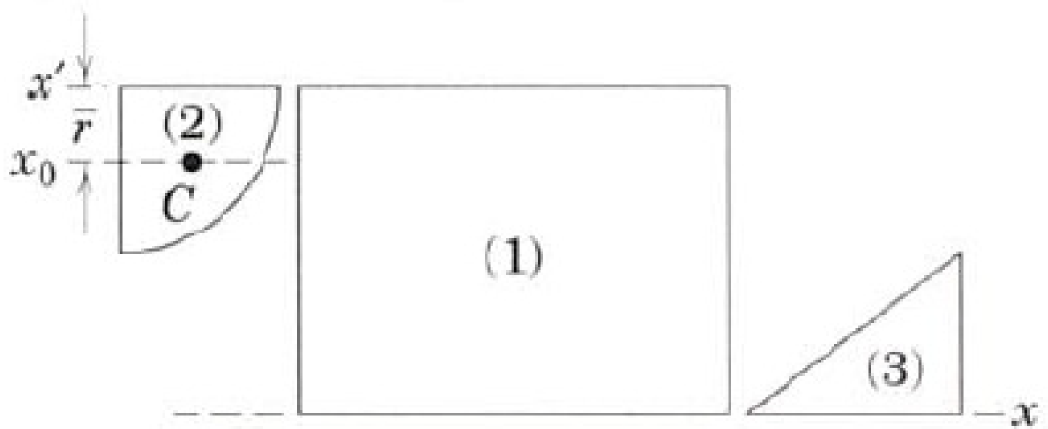
$$I_x = \bar{I}_{x'} + Ab^2 = 7.20\times10^6 + \left(12.72\times10^3\right)(81.8)^2$$
$$= 92.3\times10^6\,mm^4$$

$$I_x \quad = \quad 138.2\times10^6\,\text{mm}^4 \quad - \quad 92.3\times10^6\,\text{mm}^4$$

$$\boxed{I_x = 45.9\times10^6\,\text{mm}^4}$$

Compute the moments of inertia of shaded Area





Consider area (1)

$$I_x = \frac{1}{3}bh^3 = \frac{1}{3} \times 80 \times 60^3 = 5.76 \times 10^6\ mm^4$$

Consider area (2)

$$I_{x'} = \frac{1}{4}\left(\frac{\pi r^4}{4}\right) = \frac{\pi}{16}(30)^4 = 0.1590 \times 10^6\ mm^4$$

$$\bar{I_x} = 0.1590 \times 10^6 - \frac{\pi}{4}(30)^2 \times (12.73)^2 = 0.0445 \times 10^6\ mm^4$$

$$I_x = 0.0445 \times 10^6 + \frac{\pi}{4}(30)^2(60 - 12.73) = 1.624 \times 10^6\ mm^4$$

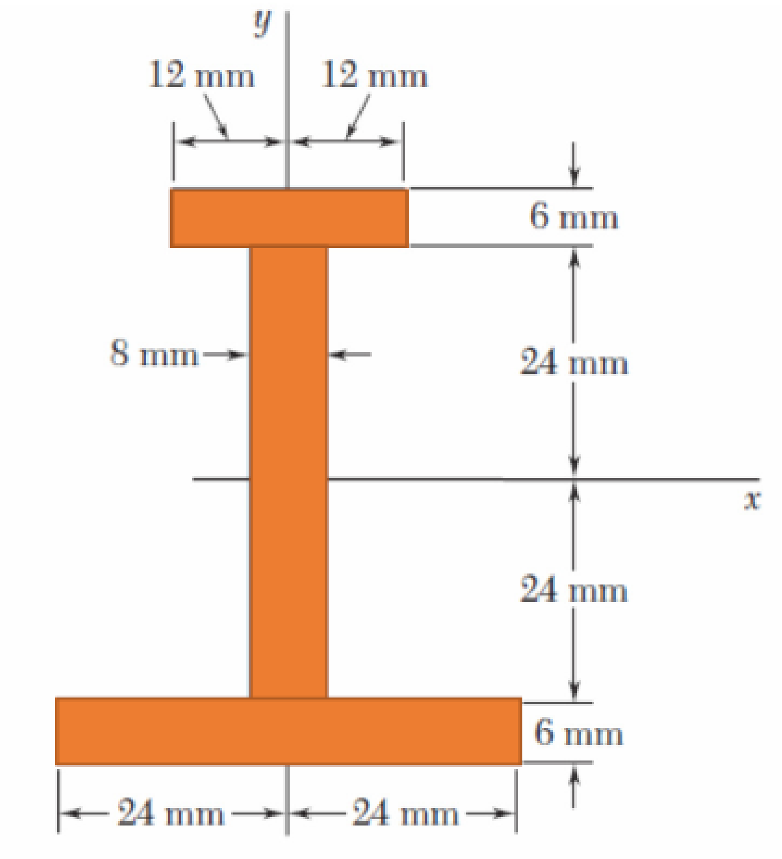
Consider area (3)

$$I_x = \frac{1}{12} bh^3 = \frac{1}{12} \times 40 \times 30^3 = 0.09 \times 10^6 \; mm^4$$

$$I_x = 5.76 \times 10^6 - 1.624 \times 10^6 - 0.09 \times 10^6 = \mathbf{4.05 \times 10^6 \; mm^4}$$

$$A = 60 \times 80 - \frac{1}{4}\pi(30)^2 - \frac{1}{2} 40 \times 30 = 3490 mm^2$$

Determine the moment of inertia and the radius of gyration of the area shown in the figure



$$\overline{I1_x} = \frac{1}{12}bd^3 = \frac{1}{12} \times 24 \times 6^3 = 432\ mm^4$$

$$\overline{I2_x} = \frac{1}{12}bd^3 = \frac{1}{12} \times 8 \times 48^3 = 73728\ mm^4$$

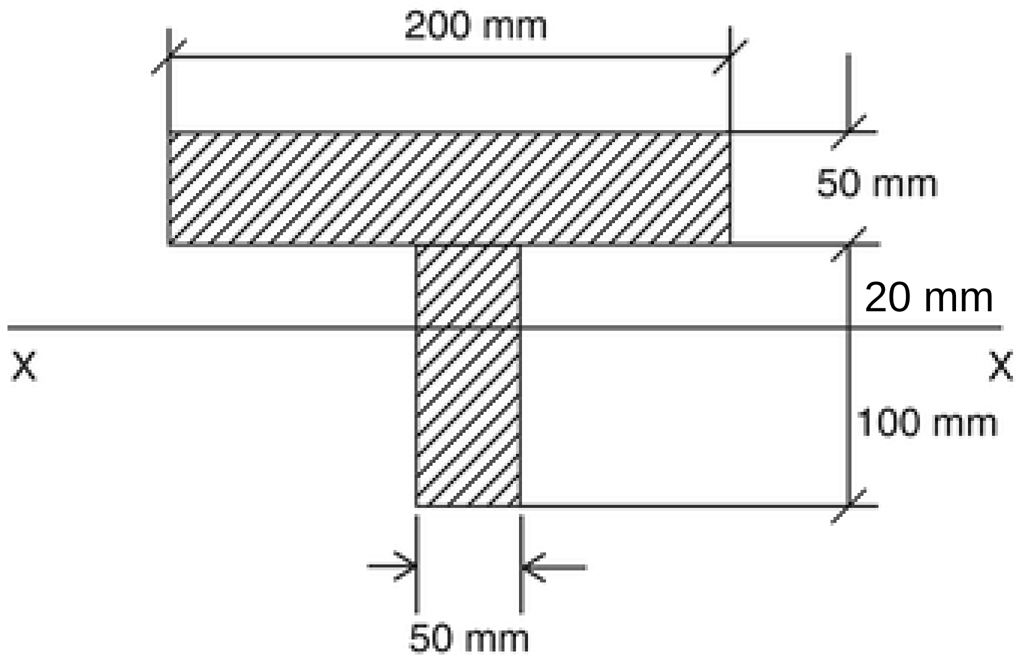
$$\overline{I3_x} = \frac{1}{12}bd^3 = \frac{1}{12} \times 48 \times 6^3 = 864\ mm^4$$

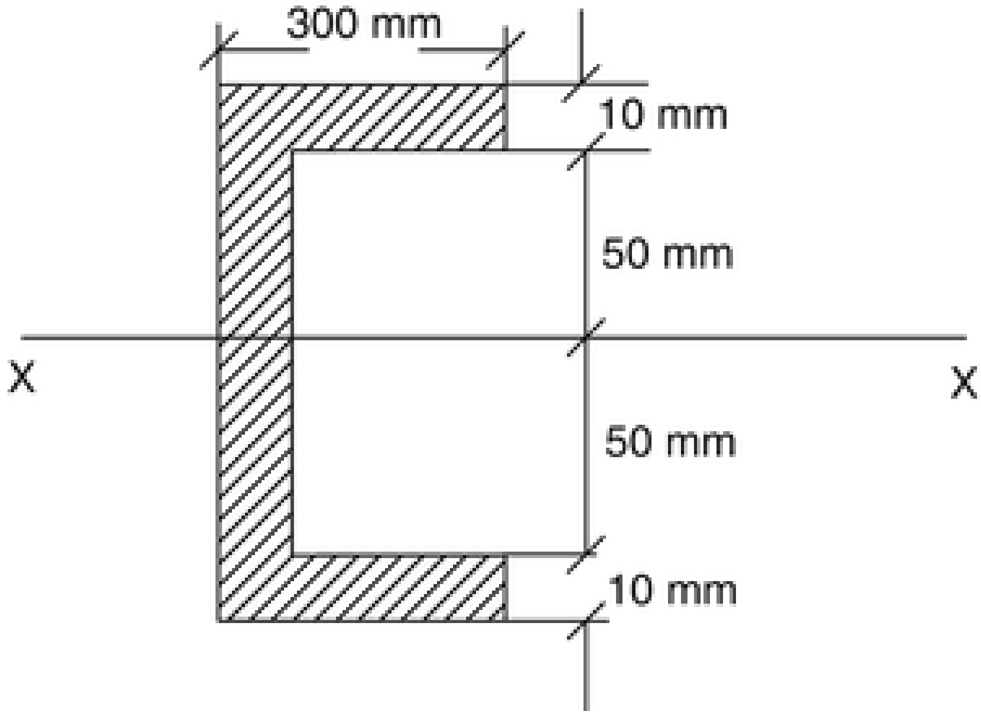
$$I1_x = \overline{I1_x} + Ah^2 = 432 + 24 \times 6 \times (24 + 3)^2 = 105408\ mm^4$$

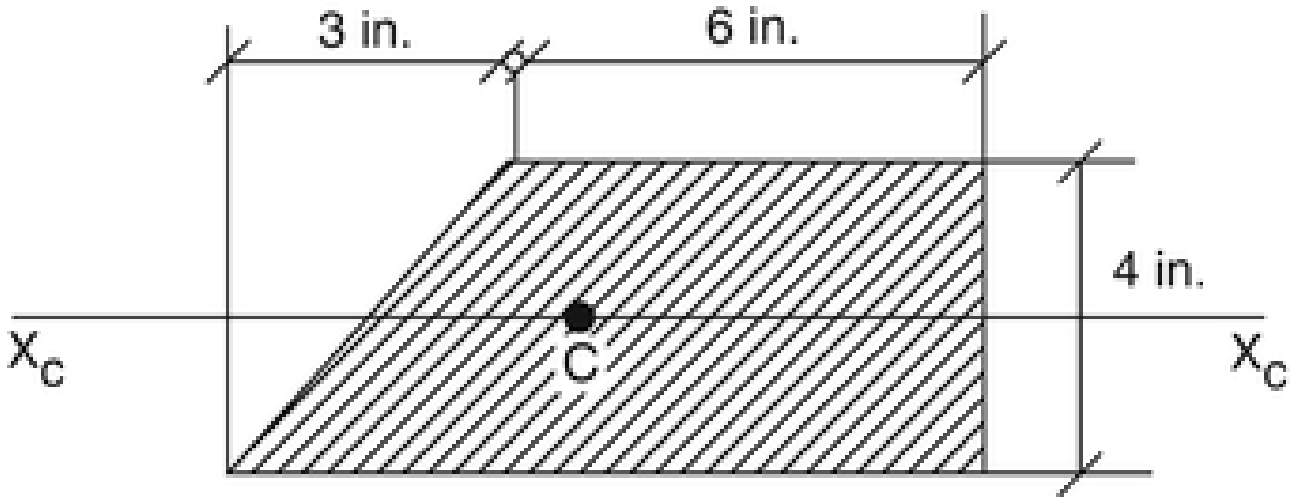
$$I3_x = \overline{I3_x} + Ah^2 = 864 + 48 \times 6 \times (24 + 3)^2 = 210816\ mm^4$$

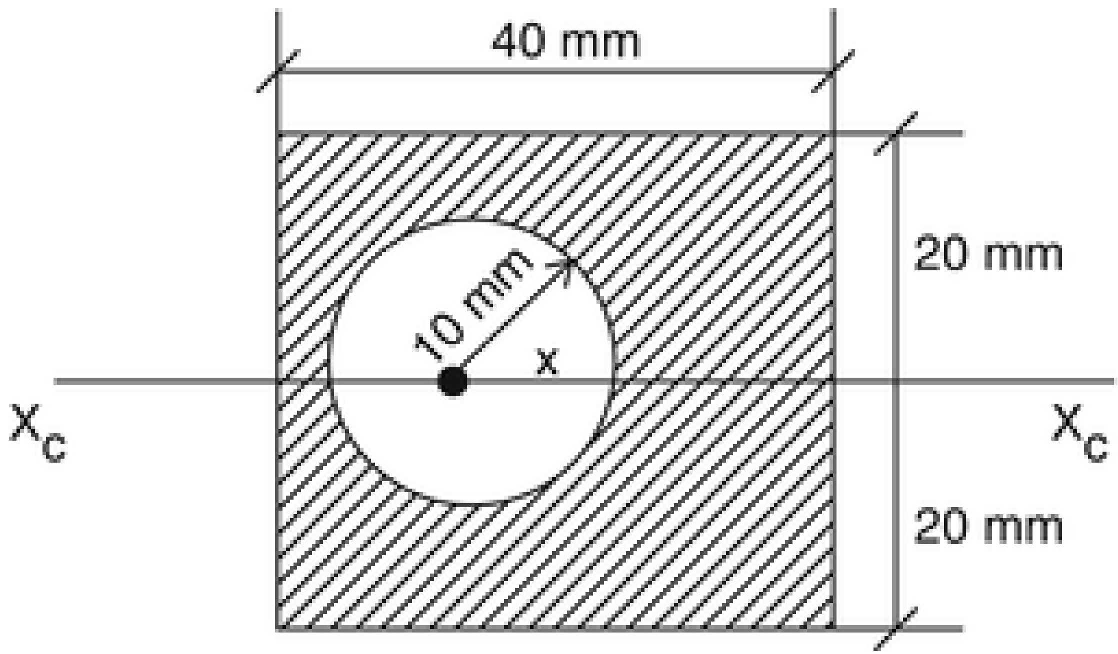
$$I_x = 105408 + 73728 + 210816 = 390 \times 10^3 mm^4$$

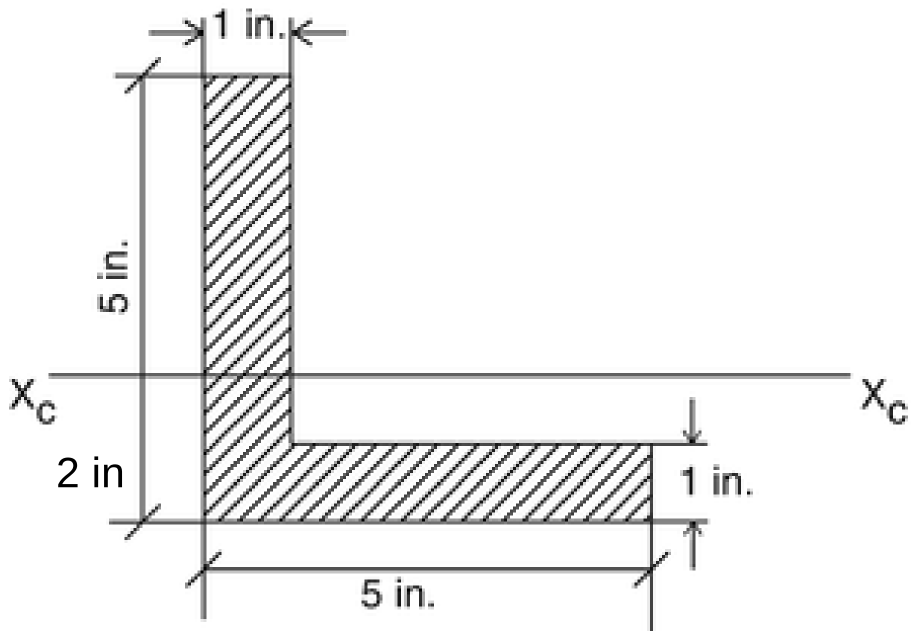
$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390 \times 10^3}{816}} = \mathbf{21.9\ mm}$$


200 mm
50 mm
20 mm
X
X
100 mm
50 mm


300 mm
10 mm
50 mm
X
X
50 mm
10 mm


3 in.
6 in.
4 in.
Xc
C
Xc

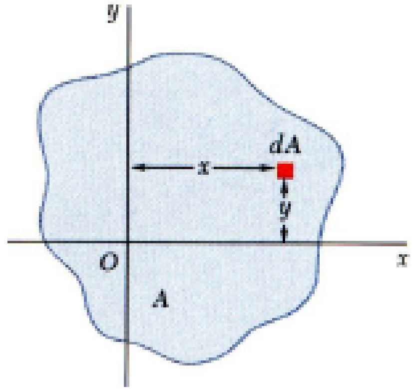

40 mm
20 mm
10 mm
x
X_c
X_c
20 mm



1 in.
5 in.
X
C
X
C
2 in
1 in.
5 in.

Area Moments of Inertia

Products of Inertia: for problems involving unsymmetrical cross-sections and in calculation of MI about rotated axes. It may be +ve, -ve, or zero

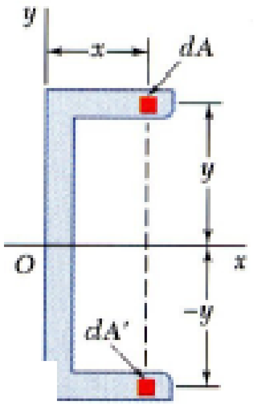


- Product of Inertia of area A w.r.t. x-y axes:

$$I_{xy} = \int xy \, dA$$

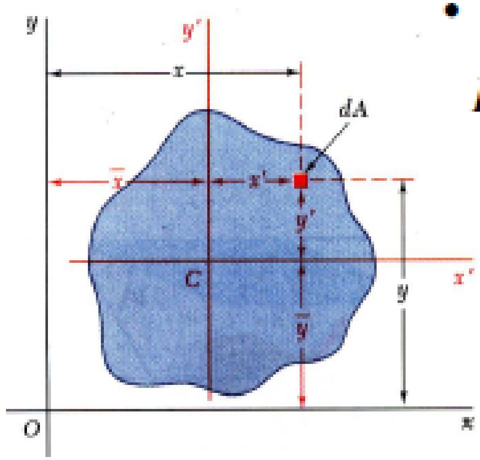
x and y are the coordinates of the element of area $dA = xy$

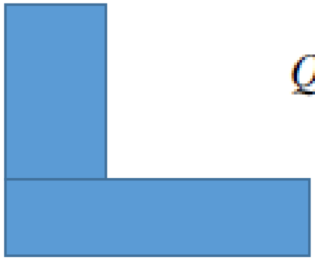
- When the x axis, the y axis, or both are an axis of symmetry, the product of inertia is zero.



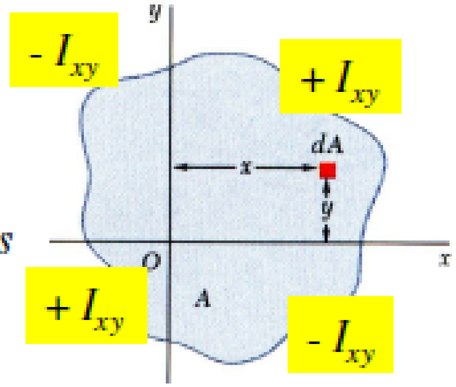
- Parallel axis theorem for products of inertia:

$$I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A$$



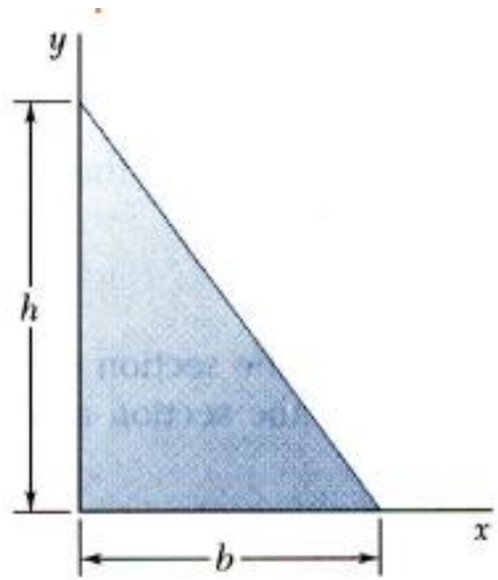


Quadrants



Area Moments of Inertia

Product of Inertia

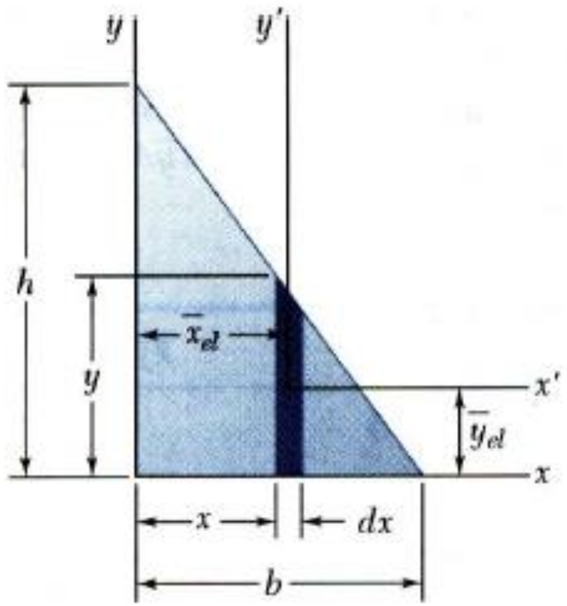


Determine the product of inertia of the right triangle *(a)* with respect to the *x* and *y* axes and *(b)* with respect to centroidal axes parallel to the *x* and *y* axes

SOLUTION:

• Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

• Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.



- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

$$y = h\left(1-\frac{x}{b}\right) \qquad dA = y\,dx = h\left(1-\frac{x}{b}\right)dx$$

$$\bar{x}_{el} = x \qquad \bar{y}_{el} = \tfrac{1}{2}y = \tfrac{1}{2}h\left(1-\frac{x}{b}\right)$$

Integrating dI_x from $x = 0$ to $x = b$,

$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el}\bar{y}_{el}\,dA = \int_0^b x\left(\tfrac{1}{2}\right)h^2\left(1-\frac{x}{b}\right)^2 dx$$

$$= h^2\int_0^b\left(\frac{x}{2}-\frac{x^2}{b}+\frac{x^3}{2b^2}\right)dx = h^2\left[\frac{x^2}{4}-\frac{x^3}{3b}+\frac{x^4}{8b^2}\right]_0^b$$

$$\boxed{I_{xy} = \tfrac{1}{24}b^2h^2}$$