Linear Algebra

A matrix A ~~over a field K~~ is a rectangular array of scalars usually presented in the following form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The rows of such a matrix A are the m horizontal lists of scalars

$$(a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, (a_{m1}, a_{m2}, \ldots a_{mn})$$

and the columns of A are the n vertical list of scalars

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that the element a_{ij}, called the ijth element, appear in row i and column j. We frequently denote such a matrix by simply writing $A = [a_{ij}]$

Matrix Addition and Scalar Multiplication ②

Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be two matrices with the same size, say $m\times n$ matrices.

The sum of of A and B written $A+B$, is the matrix obtained by adding corresponding elements from A and B. That is,

$$A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

The product of the matrix A by a scalar k, written $k.A$ or simply kA, is the matrix obtained by multiplying each element ~~k~~ of A by k. That is

$$kA=\begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & & & \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

We also observe that

$$-A=(-1)A, \quad A-B=A+(-B)$$

Matrix multiplication

The product of matrices A and B, written AB

Identity matrix

The unit matrix, denoted by I_n is the n-square matrix with 1's on the diagonal and 0's elsewise. The identity matrix I is similar to the scalar 1 in that, for any n-square matrix A,

$$AI = IA = A$$

Invertible Matrix

A square matrix A is said to be invertible or nonsingular if there exist a matrix B such that $AB = BA = I$

In case $|A| = 0$, the matrix A is not invertible.

Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ (4)

$|A| = 2\times5 - 4\times3 = -2$

$|A| \neq 0$, the matrix A is invertible

and $A^{-1} = -\frac{1}{2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$

Transpose of a matrix A

The transpose of a matrix A, written A^T, is the matrix obtained by writing the columns of A, in order as rows.

For example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Symmetric Matrices

A matrix A is symmetric if $A^T = A$

Equivalently $A = [a_{ij}]$ is symmetric if symmetric elements are equal-that is if each $a_{ij} = a_{ji}$.

A matrix A is said to be skew-symmetric if $A^T = -A$, or equivalently if each $a_{ij} = -a_{ji}$. Clearly the diagonal element of such a matrix must be zero, because $a_{ii} = -a_{ii}$ implies $a_{ii} = 0$

Ex Show that

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

are inverses.

$$AB = \begin{bmatrix} -11+0+12 & 2+0-2 & 2+0-2 \\ -22+4+18 & 4+0-3 & 4-1-3 \\ -44-4+48 & 8+0-8 & 8+1-8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AB = I$$

$\therefore$ A and B are inverses.

Rank of a matrix

Let A be a non-zero matrix of order $m \times n$. The rank of A is defined to be the greatest positive integer r such that A has at least one-zero minor of order r.

The rank of zero matrix A is defined to be 0.

The rank of A is also called the determinant rank of A.

If the rank of A be r, every minor of order $r+1$ is zero.

For a non-zero $m \times n$ matrix A, $0 < \text{rank} A \leq \min\{m, n\}$.

For a square matrix A of order n, rank of $A < n$, or $= n$ according as A is singular or non-singular.

1. Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{pmatrix}$

Since $\det A = 0$, rank of $A < 3$ and since there is a second order minor

$$\begin{vmatrix} 1 & 0 \\ 4 & -1 \end{vmatrix} \neq 0, \text{ rank of } A = 2$$

2. Let $A = \begin{pmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{pmatrix}$

Every minor of order 3 is zero, because the third row is a multiple of the first and therefore rank of $A < 3$.

There is a second order minor

$$\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} \neq 0 \text{ and so rank of } A = 2$$

Elementary Operations

An elementary operation ~~on~~ of a matrix A is

1. Interchange of two rows (or columns) of A

2. Multiplication of a row (or column) by a non-zero scalar c.
3. Addition of a scalar multiple of one row (or column) to another row (or column).

When applied to rows, the elementary operations are said to be elementary row operations. And when applied to columns they are said to be elementary column operations.

The interchange of the ith and jth row is denoted by R_{ij}

Multiplication of the ith row by a non-zero scalar c denoted by cR_i.

Addition of c times the jth row to the i-th row is denoted by $R_i + cR_j$.

Row equivalence, Column Equivalence

Let us consider the set S of all $m \times n$ matrices over a field F. A matrix B in S is said to be row equivalent (column equivalent) to a matrix A in S if B can be obtained by successive application of a finite number of elementary row operations (column operations) on A.

Def An $m \times n$ matrix A is called row reduced if

a) the first non-zero element in each non-zero row is 1 (called the leading 1)

b). in each column containing the leading 1 of some row, the leading 1 is the only non-zero element.

Def An $m \times n$ matrix A is said to be a row-reduced echelon matrix (or a row echelon matrix) if

a) A is row-reduced

b) there is an integer r $(0 \le r \le m)$ such that the first r rows of A are non-zero rows and the remaining rows (if there be any) are all zero rows

c) if the leading element of the ith non-zero row occurs in the kth column of A, then $k_1 < k_2 < \cdots < k_r$.

Examples of row-reduced echelon matrix are

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Def The rank of a matrix is the number of non-zero rows (numbers of pivots) in its row reduced echelon form.

$$\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_3-5R_1]{R_2-3R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_4-3R_2]{R_3-2R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow{R_{34}} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank is 3.

Determine the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 3 & 6 & 6 & 3 \end{pmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 3 & 3 \end{pmatrix} \xrightarrow{\frac{1}{6}R_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

$$\xrightarrow[R_3-3R_2]{R_1-R_2} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R \text{ (say)}$$

R is a row reduced echelon matrix and R has 2 non-zero rows. Therefore rank of R = 2

System of Linear Equation

A linear equation in unknowns $x_1, x_2, \dots x_n$ is an equation that can be put in the standard form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b \quad \text{--- (1)}$$

where $a_1, a_2, \dots a_n$ and b are constants. The constants a_k is called the coefficient of x_k and b is called the constant term of the equation. ②

A solution of the linear equation ① is a list of values for the unknowns or equivalently a vector u ~~of~~ in K^n, say

$x_1 = k_1,\ x_2 = k_2,\ \ldots,\ x_n = k_n$

such that ~~k~~ $a_1 k_1 + a_2 k_2 + \cdots + a_n k_n = b$

System of Linear Equations

A system of linear equations is a list of linear equations with the same unknowns. In particular, a system of m linear equations $L_1, L_2, \ldots L_m$ in n unknowns $x_1, x_2, \ldots x_n$ can be put in the standard form

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots& \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}\right\} \text{—— ②}$$

where a_{ij} and ~~b_j~~ b_i are constants. The number a_{ij} is the coefficient of the of the unknowns x_j in the equation L_i, and the number b_i is the constant of the equation L_i.

The system ② is called an $m \times n$ system. It is called a square system if $m = n$ — that is, if the number m of equations is equal to the number n of unknowns.

The system ② is said to be homogeneous if all the constant terms are zero — that is if $b_1 = 0, b_2 = 0, \cdots b_m = 0$. Otherwise the system is said to be nonhomogeneous.

A solution of the system ② is a list of values for the unknowns ~~or equivalently~~ which is a solution of each of the equations of the system. The set of all solutions of the system is called the solution set or the general solution of the system.

The system ② of linear equations is said to be consistent if it has one or more solutions, and it is said to be inconsistent if it has no solution.

Therefore the system of linear equations is either inconsistent or consistent. If it is inconsistent then it has no solution. If it is consistent then it has either unique solution or infinite number of solutions.

Argumented and Coefficient Matrices of a system

Consider the general system ② of m equations in n unknowns. Such a system has associated with it the following two matrices:

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ - & - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The matrix M is called augmented matrix of the system and the matrix A is called the coefficient matrix.

If Rank of M = Rank of A

Then the system is consistent

i) If the system is consistent then if the number of equations is equal to the number of unknowns then it has unique solution.

ii) If the number of equation is $< n$

That is there are more unknowns than equations. Then we can arbitrarily assign values to the $n-m$ free variables and solve them and obtained infinite no. of solution.

1. (a) Solve each of the following system.

a) $x_1 + x_2 - 2x_3 + 4x_4 = 5$

$2x_1 + 2x_2 - 3x_3 + x_4 = 3$

$3x_1 + 3x_2 - 4x_3 - 2x_4 = 1$

Reduce its augmented matrix M to echelon form and then to row equivalent form as follows:

$$M = \left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -10 & -9 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of M = Rank of A = 2

$\therefore$ The system is consistent.

Rewrite the row canonical form in terms of a system of linear equations to obtain the free variable form of the solution.

That is
$$\left.\begin{aligned} x_1+x_2 \quad -- \; -10x_4 &= -9 \\ x_3-7x_4 &= -7 \end{aligned}\right\} \text{ or } \begin{aligned} x_1 &= -9-x_2+10x_4 \\ x_3 &= -7+7x_4 \end{aligned}$$

(The zero row is omitted in the solution).

Observe that x_1 and x_3 are the pivot variables and x_2 and x_4 are free variables.

So any arbitrary value of x_2 and x_4 there exists infinite no. of solutions.

b)
$$x_1+x_2-2x_3+3x_4=4$$
$$2x_1+3x_2+3x_3-x_4=3$$
$$5x_1+7x_2+4x_3+x_4=5$$

$$M=\begin{bmatrix} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3-5R_1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{bmatrix}$$

$$\xrightarrow{R_3-2R_2}\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

Rank of M = 3 , Rank of A = 2

∴ The system is inconsistent
i.e. it has no solution.

c) $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{bmatrix}$

c) $$x+2y+z=3$$
$$2x+5y-z=-4$$
$$3x-2y-z=5$$

$$M=\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1}\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3+8R_2}\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{bmatrix}\xrightarrow[R_3 \div 28]{}\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2+3R_3}\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus Rank of M = Rank of $A = 3$

The system has the unique solution $x = 2$, $y = -1$, $z = 3$

2. Determine wheather or not each of the following homogeneous system has a non-zero solution.

(a)
$$x + y - z = 0$$
$$2x - 3y + z = 0$$
$$x - 4y + 2z = 0$$

(b)
$$x + y - z = 0$$
$$2x + 4y - z = 0$$
$$3x + 2y + 2z = 0$$

(c)
$$x_1 + 2x_2 - 3x_3 + 4x_4 = 0$$
$$2x_1 - 3x_2 + 5x_3 - 7x_4 = 0$$
$$5x_1 + 6x_2 - 9x_3 + 8x_4 = 0$$

a) Reduce the system to echelon form as follows.

$$x + y - z = 0$$
$$2y + z = 0$$
$$-y + z = 0$$

and then

$$x + y - z = 0$$
$$-5y + 3z = 0$$

The system has a nonzero solution, because there are only two equations in the three unknowns in echelon form. Here z is a free variable. Let us say set $z=5$,

$\therefore y=3,\ x=2$

Thus $(2,3,5)$ is a particular non-zero solution.

b) Reduce the system to echelon form as follows:

$$\begin{aligned} x+y-z&=0 \\ 2y+z&=0 \\ -y+5z&=0 \end{aligned} \quad \text{and} \quad \begin{aligned} x+y-z&=0 \\ 2y+z&=0 \\ 11z&=0 \end{aligned}$$

In echelon form, there are three eqns in three variables. Thus the system has only the zero solution.

c) The system must have a non-zero solution, because there are four unknowns but only three equations.

$$x + 2y - z = 3$$
$$y + 2z = 3$$
$$3z = 4$$

$$\therefore x = \frac{17}{3}, \quad y = -\frac{2}{3}, \quad z = \frac{4}{3}$$

Unique solution

b) $$M = \begin{bmatrix} 1 & -2 & 4 & 2 \\ 2 & -3 & 5 & 3 \\ 3 & -4 & 6 & 7 \end{bmatrix}$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & -6 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & +3 \end{bmatrix}$$

Rank of $M = 3$, Rank $A = 2$

$$3 \neq 2$$

The system has no solution.

$$x+2y-z=3$$
$$y+2z=3$$
$$3z=4$$

$$\therefore\ x=\frac{17}{3},\ y=-\frac{2}{3},\ z=\frac{4}{3}$$

Unique solution

b) $$M=\begin{bmatrix} 1 & -2 & 4 & 2 \\ 2 & -3 & 5 & 3 \\ 3 & -4 & 6 & 7 \end{bmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1}\begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & -6 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3-2R_2}\begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & +3 \end{bmatrix}$$

Rank of $M=3$, Rank $A=2$

$$3\neq 2$$

The system has no solution.

c) $M = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{bmatrix}$

$$\xrightarrow[R_3 - 5R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & -7 & 1 \\ 0 & 2 & -14 & 2 \end{bmatrix}$$

$$\xrightarrow[R_1 - R_2]{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank M = Rank A = 2

The system is consistent

$$x + 10z = 0$$

$$y - 7z = 1$$

Here z is only free variable.

$x = -10a$, $y = 1 + 7a$, $z = a$

Solve (14)

$$x_1 + 2x_2 - 3x_3 - 2x_4 + 4x_5 = 1$$
$$2x_1 + 5x_2 - 8x_3 - x_4 + 6x_5 = 4$$
$$x_1 + 4x_2 - 7x_3 + 5x_4 + 2x_5 = 8$$

Solve each of the following system of (1)

$$x+2y-z=3$$
$$x+3y+z=5$$
$$3x+8y+4z=17$$

(a)

$$x-2y+4z=2$$
$$2x-3y+5z=3$$
$$3x-4y+6z=7$$

(b)

$$x+y+3z=1$$
$$2x+3y-z=3$$
$$5x+7y+z=7$$

(c)

(a)
$$M=\begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & 8 & 4 & 17 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 7 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

Rank M = Rank A

So the system is consistent

Vector Space

Def Let V be a nonempty set with two operations:

i) Vector addition: This assigns to any $u, v \in V$ a sum $u+v$ in V.

ii) Scalar Multiplication: This assigns to any $u \in V$, $k \in K$ a product $ku \in V$.

Then V is called a vector space (over the field K) if the following axioms hold for any vectors $u, v, w \in V$:

① $(u+v)+w = u+(v+w)$

② There is a vector in V, denoted by 0 and called the zero vector, such that, for any $u \in V$,

$$u+0 = 0+u = u$$

③ For each $u \in V$, there is a vector in V, denoted by $-u$, and called the negative of u such that

$$u+(-u) = (-u)+u = 0$$

④ $u+v = v+u$

⑤ $k(u+v) = ku + kv$, for any scalar $k \in K$ ③

⑥ $(a+b)u = au + bv$, for any scalars $a, b \in K$

⑦ $(ab)u = a(bu)$, for any scalars $a, b \in K$

⑧ $1u = u$ for the unit scalar $1 \in K$.

Th Let V be a vector space over a field K.

i) For any scalar $k \in K$ and $0 \in V$, $k0 = 0$.

ii) For $0 \in K$ and any ~~scalar~~ vector $u \in V$, $0u = 0$

iii) If $ku = 0$, where $k \in K$ and $u \in V$, then $k = 0$ or $u = 0$

iv) For any $k \in K$, and any $u \in V$, $(-k)u = k(-u) = -ku$

Polynomial space $P(t)$

Let $P(t)$ be the set of all polynomials of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_s t^s \quad (s = 1, 2, \cdots)$$

where the coefficients a_i belong to a field K. Then $P(t)$ is a vector space over K

using the following operations

i) Vector Addition: Here $p(t) + q(t)$ in $P(t)$ is the usual operation of addition of polynomials.

(ii) Scalar Multiplication: Here $kp(t)$ in $P(t)$ is the usual operation of the product of a scalar k and a polynomial $p(t)$.

The zero polynomial 0 is the zero vector in $P(t)$.

Linear ~~Conf~~ Combination

Let V be a vector space over a field K. A vector v in V is a linear combination of vectors $u_1, u_2, \ldots u_m$ in V if there exist scalars $a_1, a_2, \ldots a_m$ in K such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_mu_m$$

(4) Ex! Express $v = (1, -2, 5)$ in R^3 as a linear combination of the vectors

$$u_1 = (1, 1, 1), \quad u_2 = (1, 2, 3), \quad u_3 = (2, -1, 1)$$

$$v = xu_1 + yu_2 + zu_3$$

(4)

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$x + y + 2z = 1$

$x + 2y - z = -2$

$x + 3y + z = 5$

Reducing the system to echelon form

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{10}{3} \end{bmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & +4 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{10}{3} \end{bmatrix}$$

$$\xrightarrow[R_2+2R_3]{R_1-4R_3} \begin{bmatrix} 1 & 0 & 0 & -\frac{28}{3} \\ 0 & 1 & 0 & \frac{11}{3} \\ 0 & 0 & 1 & \frac{10}{3} \end{bmatrix}$$

$4 - \frac{40}{3}$

$-3 + \frac{20}{3}$

$$x + y + 2z = 1$$
$$y - 3z = -3$$
$$5z = 10$$

$$\Rightarrow z = 2, \; y = 3, \; x = -6$$

Thus $v = -6u_1 + 3u_2 + 2u_3$

Ex2 Express $v = (2, -5, 3)$ in R^3 as a linear combination of

$u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$, $u_3 = (1, -5, 7)$

$$v = xu_1 + yu_2 + zu_3$$

$$\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$$

$$x + 2y + z = 2$$
$$-3x - 4y - 5z = -5$$
$$2x - y + 7z = 3$$

Reducing the system to echelon form yields the system

$$x + 2y + z = 2, \quad 2y - 2z = 1, \quad 0 = 3$$

The system is inconsistent and so it has no solution .

Thus v can not be written as a linear combination of u_1, u_2, u_3.

Spanning sets

Let V be a vector space over ~~a field~~ K.
Vectors $u_1, u_2, \ldots u_m$ in V are said to span V or to form a spanning set of V if there v in V is a linear combination of the vectors $u_1, u_2, \ldots u_m$ — that is if there exist scalars $a_1, a_2, \ldots a_m$ in K such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m$$

Remark 1: Suppose $u_1, u_2, \ldots u_m$ span V. Then for any vector w, the set $w, u_1, u_2, \ldots u_m$ also spans V.

Remark 2: Suppose $u_1, u_2, \ldots u_m$ span V and suppose u_k is a linear combination of some of the other u's. Then the u's without u_k also span V.

Remark 3 Suppose $u_1, u_2, \ldots u_m$ span V and suppose one of the u's is the zero vector. Then the u's without the zero vector also span V.

Example 1. The following vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ is a spanning set set of $V = R^3$.

Let $v = (a, b, c)$ is any vector in R^3

Then $v = a e_1 + b e_2 + c e_3$

For example $(5, -6, 2) = -5 e_1 + (-6) e_2 + 2 e_3$

Subspaces.

Def Let V be a vector space over a field K and let W be a subset of V. Then W is a subspace of V if W is itself a vector space over K with respect to the operations of vectors addition and scalar multiplication on V.

Intersection of Subspaces

Let U and W be subspaces of a vector space V. We show that the intersection $U \cap W$ is also a subspace of V.

Linear Dependence and Independence

Let V be a vector space over a field K. We say that the vectors $v_1, v_2, \ldots v_m$ in V are linearly dependent if there exist scalars $a_1, a_2, \ldots a_m$ in K, not all of them 0, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0$$

Otherwise, we say that the vectors are linearly independent.

A set $S = \{v_1, v_2, \ldots v_m\}$ ~~are linearly independent~~ of vectors in V is linearly dependent or independent according to

whether the vectors $v_1, v_2, \dots v_m$ are linearly (1)
dependent or independent.

Note 1 Suppose 0 is one of the vectors $v_1, v_2, \dots v_m$ say $v_1 = 0$, then the vectors must be linearly dependent, because we have the following linear combination where the coefficients of $v_1 \neq 0$:

$$1v_1 + 0v_2 + \dots + 0v_m = 1\cdot 0 + 0 + \dots + 0 = 0$$

Note 2 Suppose v is a nonzero vector. Then v, by itself is linearly independent because $kv = 0$, $v \neq 0$ implies $k = 0$

Note 3 Suppose two of the vectors $v_1, v_2, \dots v_m$ are are equal or one is a scalar multiple of the other, say $v_1 = kv_2$. Then the vectors must be linearly dependent because we have the following linear combination where the coefficients of $v_1 \neq 0$:

$$v_1 - kv_2 + 0v_3 + \cdots + 0v_m = 0$$

~~Rem~~ Note 4 The vectors v_1 and v_2 are linearly dependent if and only if one of them is a multiple of the other.

Note 5: If the set $\{v_1, \ldots, v_m\}$ is linearly independent, then any rearrangement of the vectors $\{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$ is also linearly independent.

Note 6 If a set S of vectors is linearly independent, then any subset of S is linearly independent. Alternatively, if S contains a linearly dependent subset, then S is linearly dependent.

Ex 1. Let $u = (1,1,0)$, $v = (1,3,2)$, $w = (4,9,5)$ (11)

Then ~~xx~~ u, v, w are linearly dependent

because

$$3u + 5v - 2w = (0,0,0) = 0$$

Ex 2 We show that the vectors $u = (1,2,3)$, $v = (2,5,7)$, $w = (1,3,5)$ are linearly independent.

$$xu + yv + zw = 0$$

$$\left.\begin{aligned} x + 2y + z &= 0 \\ 2x + 5y + 3z &= 0 \\ 3x + 7y + 5z &= 0 \end{aligned}\right\} \quad \begin{aligned} x + 2y + z &= 0 \\ y + z &= 0 \\ 2z &= 0 \end{aligned}$$

$$x = 0,\ y = 0,\ z = 0$$

$$xu + yv + zw = 0$$

$$\Rightarrow x = 0,\ y = 0,\ z = 0$$

$\therefore$ u, v, w are linearly ~~indend~~ independent.

Ex 3 Determine whether or not the vectors $u = (1,1,2)$, $v = (2,3,1)$, $w = (4,5,5)$ are linearly dependent

$$xu + yv + zw = 0$$

$$\left.\begin{aligned} x + 2y + 4z &= 0 \\ x + 3y + 5z &= 0 \\ 2x + y + 5z &= 0 \end{aligned}\right\} \quad \begin{aligned} x + 2y + 4z &= 0 \\ y + z &= 0 \end{aligned}$$

The echelon system has only two nonzero equations in three unknowns hence, it has a free variable and a non zero solution. Thus u, v, w are lineary dependent.

Ex 4 ~~(a)~~ Determine whether or not each of following ~~set~~ list of vectors in L.I.

(13)

a) $u_1 = (1,2,5)$, $u_2 = (1,3,1)$,

$u_3 = (2,5,7)$, $u_4 = (3,1,4)$

b) $u = (1,2,5)$, $v = (2,5,1)$, $w = (1,5,2)$

c) $u = (1,2,3)$, $v = (0,0,0)$, $w = (1,5,6)$

Ex Let $V = \mathbb{R}^3$ ①

Show that W is not a subspace of V, where

a) $W = \{(a,b,c) : a \geq 0\}$ b) $W = \{(a,b,c) : a^2+b^2+c^2 \leq 1\}$

Sol^n a) W consists of those vectors whose first entry is nonnegative. Thus $v = (1,2,3)$ belongs to W. Let $k = -3$. Then

$kv = (-3,-6,-9)$ does not belong to W, because -3 is negative. Thus W is not a subspace of V.

b) W consists of vectors whose length does not exceed 1. Hence $u = (1,0,0)$ and $v = (0,1,0)$ belong to W, but $u+v = (1,1,0)$ does not belong to W, because

$1^2+1^2+0^2 = 2 > 1$. Thus W is not a subspace of V.

Ex Let $V = P(t)$, the vector space of real polynomials. Determine whether or not W is a subspace of V.

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a) W consists of all polynomials with integral coefficients.

b) W consists of all polynomials with degree ≥ 6 and the zero polynomial.

Ans: a) No, because scalar multiple of polynomials in W do not always belong to W. For example

$$f(t) = 3 + 6t + 7t^2 \in W$$

$$\text{but } \frac{1}{2} f(t) = \frac{3}{2} + 3t + \frac{7}{2} t^2 \notin W$$

b) Yes, in each case W consists of the zero polynomial, and sums and scalar multiples of polynomials in W belong to W.

Linear span

Show that the vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$ $u_3 = (1,5,8)$ span R^3.

$\underline{\underline{\text{Sol}^n}}$ We need to show that an arbitrary vector $v = (a, b, c)$ in R^3 is a linear combination of u_1, u_2, u_3.

~~Seat~~ Set $v = xu_1 + yu_2 + zu_3$

That is $(a, b, c) = x(1,1,1) + y(1,2,3) + z(1,5,8)$

$$= (x+y+z,\ x+2y+5z,\ x+3y+8z)$$

$$\therefore \left.\begin{aligned} x+y+z &= a \\ x+2y+5z &= b \\ x+3y+8z &= c \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} x+y+z &= a \\ y+4z &= b-a \\ 2y+7z &= c-a \end{aligned}\right\}$$

or
$$\begin{aligned} x+y+z &= a \\ y+4z &= b-a \\ -z &= c-2b+a \end{aligned}$$

The above system is in echelon form and is consistent

$$x = -a+5b+3c,\ y = 3a-7b+4c,$$

$$z = a+2b-c \text{ is a solution.}$$

Thus u_1, u_2, u_3 span R^3.

Ex Find conditions on a, b, c so that (4)
$v = (a,b,c)$ in R^3 belongs to $W = span(u_1, u_2, u_3)$
where $u_1 = (1,2,0)$, $u_2 = (-1,1,2)$, $u_3 = (3,0,-4)$

$$(a,b,c) = x(1,2,0) + y(-1,1,2) + z(3,0,-4)$$

$$\left.\begin{aligned} x - y + 3z &= a \\ 2x + y &= b \\ 2y - 4z &= c \end{aligned}\right\} \text{ or } \left.\begin{aligned} x - y + 3z &= a \\ 3y - 6z &= b - 2a \\ 2y - 4z &= c \end{aligned}\right\}$$

$$\text{or} \quad \begin{aligned} x - y + 3z &= a \\ 3y - 6z &= b - 2a \\ 0 &= 4a - 2b + 3c \end{aligned}$$

The vector $v = (a,b,c)$ belongs to W iff the system is consistent and it is consistent iff $4a - 2b + 3c = 0$

$\therefore u_1, u_2, u_3$ do not span the whole space R^3.

Basis and Dimension

Defn: A set $S = \{u_1, u_2, \ldots u_n\}$ of vectors is a basis of V if it has the following two properties (1) S is linearly independent (2) S spans V.

Defn: A set $S = \{u_1, u_2, \ldots u_n\}$ of vectors is a basis of V if every $v \in V$ can be written uniquely as a linear combination of the basis vectors.

Th Let V be a vector space such that one basis has m elements and another basis has n elements then $m = n$.

A vector space V is said to be of finite dimension n or n dimensional, written $\dim V = n$

The vector space $\{0\}$ is defined to have dimension 0.

Suppose a vector space V does not have a finite basis. Then V is said to be of infinite dimension or to be infinite-dimensional.

Lemma Suppose $\{v_1, v_2, \dots v_n\}$ spans V and suppose $\{w_1, w_2, \dots w_m\}$ is linearly independent. Then $m \leq n$ and V is spanned by a set of the form $\{w_1, w_2, \dots w_m, v_{i_1}, v_{i_2}, \dots v_{i_{n-m}}\}$

Thus in particular, $n+1$ or more vectors in V are linearly dependent.

Theorems on Bases

Th Let V be a vector space of finite dimension n. Then

i) Any $n+1$ or more vectors in V are linearly dependent.

ii) Any linearly independent set $S = \{u_1, u_2, \dots u_n\}$ with n elements is a basis of V.

iii) Any spanning set $T = \{v_1, v_2, \ldots v_n\}$ of V with n elements is a basis of V.

Th Suppose S spans a vector space V.

Then

i) Any maximum number of linearly independent vectors in S form a basis of V.

ii) Suppose one deletes from S every vector that is a linear combination of preceding vectors in S. Then the seemaing vectors form a basis of V.

Th Let V be a vector space of finite dimension and let $S = \{u_1, u_2, \ldots u_r\}$ be a set of linearly independent vectors in V. Then S is a part of a basis of V, that is, S may be extended to a basis of V.

Th Let W be a subspace of a real vector space V of n dimensional. Then $\dim W \leq n$.

In particular, if $\dim W = n$, then $W = V$.

Example Suppose the vectors u, v, w are linearly independent. Show that the vectors $u+v$, $u-v$, $u-2v+w$ are also linearly independent.

Sol^n $x(u+v) + y(u-v) + z(u-2v+w) = 0$

or, $(x+y+z)u + (x-y-2z)v + zw = 0$

Because u, v, w are linearly independent

Hence
$$x+y+z=0$$
$$x-y-2z=0$$
$$z=0$$

The only solution is $x=0, y=0, z=0$

Thus $u+v$, $u-v$, $u-2v+w$ are linearly independent.

Determine whether or not each of the following is a basis of R^3.

a) $(1,1,1), (1,0,1)$ (b) $(1,2,3), (1,3,5), (1,0,1), (2,3,0)$ (c) $(1,1,1), (1,2,3), (2,-1,1)$

(d) $(1,1,2), (1,2,5), (5,3,4)$

(a) and (b) No, because a basis of R^3 must contains exactly three elements because $\dim R^3 = 3$

c) The three vectors form a basis iff. they are linearly independent. Thus the matrix whose rows are the given vectors, and row reduce the matrix to echelon form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

The echelon matrix has no zero rows. Hence the three vectors are linearly independent and so they do form a basis of R^3.

d) Form the matrix whose rows are the given vectors, and row reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix has a zero row hence the three vectors are linearly dependent and so they do not form a basis of R^3.

Example

Determine whether $(1,1,1,1)$, $(1,2,3,2)$, $(2,5,6,4)$, $(2,6,8,5)$ form a basis of R^4. If not find the dimension of the subspace they span.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix has a zero row.

Hence the four vectors are linearly

dependent and do not form a basis of R^4.
Because the echelon matrix has three nonzero rows, the four vectors span a ~~and~~ subspace of dimension 3.

Ex Find a basis and dimension of the subspace ~~of~~ W of R^4 where

a) $W = \{(a, b, c) : a+b+c=0\}$

b) $W = \{(a, b, c) : (a=b=c)\}$

a) Note that $W \neq R^3$, because for example $(1,2,3) \notin W$. Then $\dim W < 3$.
Note that $u_1 = (1,0,-1)$ and $u_2 = (0,1,-1)$ are two independent vectors in W. Thus $\dim W = 2$ and ~~so~~ u_1, u_2 form a basis of W.

b) The vector $u = (1,1,1) \in W$. Any vector $w \in W$ has the form $w = (k,k,k)$. Hence $w = ku$. Thus u spans W and $\dim W = 1$.

Ex Find the dimension and a basis of the solution space W of each homogeneous system.

a) $x + 2y + 2z - s + 3t = 0$

$x + 2y + 3z + s + t = 0$

$3x + 6y + 8z + s + 5t = 0$

Reduce the system to echelon form

$$\left.\begin{aligned} x + 2y + 2z - s + 3t &= 0 \\ z + 2s - 2t &= 0 \\ 2z + 4s - 4t &= 0 \end{aligned}\right\}$$

or, $\quad x + 2y + 2z - s + 3t = 0$

$\qquad\qquad z + 2s - 2t = 0$

The system in echelon form has two equations in five unknowns. Hence the system has $5 - 2 = 3$ free variable. which are y, s, t. Thus $\dim W = 3$.

We obtain a basis for W:

① Set $y = 1, s = 0, t = 0$, $\quad v_1 = (-2, 1, 0, 0, 0)$

② Set $y = 0, s = 1, t = 0$, $\quad v_2 = (5, 0, -2, 1, 0)$

③ Set $y = 0, s = 0, t = 1$, $\quad v_3 = (-7, 0, 2, 0, 1)$

The set $\{u_1, u_2, u_3\}$ is a basis of the solution space of W.

b) $x + 2y + z - 2t = 0$

$2x + 4y + 4z - 3t = 0$

$3x + 6y + 7z - 4t = 0$

Here the coefficient matrix A

$$= \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 4 & 4 & -3 \\ 3 & 6 & 7 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is

$$x + 2y + z - 2t = 0$$

$$2z + t = 0$$

The free variable are y, t $\dim W = 2$

i) set $y = 1$, $z = 0$, $u_1 = (-2, 1, 0, 0)$

(ii) set $y = 0$, $z = 2$, $u_2 = (6, 0, -1, 2)$

Then $\{u_1, u_2\}$ is a basis of W.

c) $x + y + 2z = 0$

$2x + 3y + 3z = 0$

$x + 3y + 5z = 0$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

Here no free variable. Thus $\dim W = 0$

Linear Mapping (Linear Transformation) ①

Def. Let V and U be vector spaces over the same field K. A mapping $F: V \to U$ is called a linear mapping or linear transformation if

① For any vectors $u, w \in V$,

$$F(u+w) = F(u) + F(w)$$

② For any scalar k and vector $v \in V$, $F(kv) = kF(v)$

Note $F(0)=0$. Thus every linear mapping takes the zero vector into the zero vector.

More generally For any scalars $a_i \in K$ and any vectors $v_i \in V$

$$F(a_1v_1 + a_2v_2 + \cdots + a_mv_m) = a_1F(v_1) + a_2F(v_2) + \cdots + a_mF(v_m)$$

Example

1. Let $F: R^3 \to R^3$ be the mapping defined by $F(x,y,z) = (x,y,0)$. We show that F is linear.

Let $v = (a, b, c)$ and $w = (a', b', c')$

Then
$$F(v+w) = F(a+a', b+b', c+c') = (a+a', b+b', 0) = (a, b, 0) + (a', b', 0) = F(v) + F(w)$$

and for any scalar k,
$$F(kv) = F(ka, kb, kc) = (ka, kb, 0) = k(a, b, 0) = kF(v)$$

Thus F is linear.

Example 2 : Let $G: R^2 \to R^2$ defined by

$$G(x, y) = (x+1, y+2)$$

$$G(0) = G(0, 0) = (1, 2) \neq 0$$

Thus the zero vector is not mapped into the zero vector. Hence G is not linear

Kernel and Image of a linear Mapping

Def Let $F: V \to U$ be a linear mapping.
The kernel of F, written Ker F, is the set of elements of in V that map into the zero vector 0 in U.

That is $\text{Ker } F = \{v \in V : F(v) = 0\}$

The image (or range) of F, written in Im F, is the set of image points in U

That is $\text{Im } F = \{u \in U : \text{there exists } v \in V \text{ for which } F(v) = u\}$

Th Let $F: V \to U$ be a linear mapping.
Then the kernel of F is a subspace of V and the image of F is a subspace of U.

Rank and Nullity of Linear Mapping

Let $F: V \to U$ be a linear mapping. The rank of F is defined to be the dimension of its image and the nullity of F is defined to be the dimension of its kernel; namely,

$$\text{rank}(F) = \dim(\text{Im} F) \quad \text{and} \quad \text{Nullity}(F) = \dim(\text{Ker} F)$$

__Th__ Let V be of finite dimension, and let $F: V \to U$ be linear. Then

$$\dim V = \dim(\text{Ker} F) + \dim(\text{Im} F)$$
$$= \text{nullity}(F) + \text{rank}(F)$$

__Ex__ Let $F: R^4 \to R^3$ be the linear mapping defined by

$$F(x,y,z,t) = (x-y+z+t,\ 2x-2y+3z+4t,\ 3x-3y+4z+5t)$$

a) Find a basis and the dimension of the image of F.

First find the image of the usual basis vectors of R^4.

$F(1,0,0,0) = (1,2,3)$, $F(0,1,0,0) = (-1,-2,-3)$

$F(0,0,1,0) = (1,3,4)$, $F(0,0,0,1) = (1,4,5)$

The image vectors span Im F. Hence form the matrix M whose rows are these image vectors and row reduce to echelon form:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $(1,2,3), (0,1,1)$ form a basis of Im F.

Hence $\dim(\operatorname{Im} F) = 2$, $\operatorname{rank} F = 2$

b) Find a basis and the dimension of the kernel of the mapping F:

Set $F(v) = 0$ where $v = (x, y, z, t)$

$$F(x,y,z,t) = (x-y+z+t,\ 2x-2y+3z+4t,\ 3x-3y+4z+5t)$$

$$= (0,0,0)$$

Set corresponding components equal to each other to form the following homogeneous system whose solution space is Ker F:

$$\left.\begin{array}{l} x-y+z+t=0 \\ 2x-2y+3z+4t=0 \\ 3x-3y+4z+5t=0 \end{array}\right\} \text{ or } \left.\begin{array}{r} x-y+z+t=0 \\ z+2t=0 \\ z+2t=0 \end{array}\right\}$$

$$\text{or} \quad \begin{array}{r} x-y+z+t=0 \\ z+2t=0 \end{array}$$

The free variables are y and t,

Hence $\dim(\mathrm{Ker}\,F)=2$ or Nullity $(F)=2$

i) Set $y=1,\ t=0$ to obtain the solution $(-1,+1,0,0)$

(ii) set $y=0,\ t=1$ to obtain the solution $(1,0,-2,1)$

Thus $(-1,1,0,0)$ and $(1,0,-2,1)$ form a basis for Ker F.

As expected $\dim(\mathrm{Ker}\,F)+\dim(\mathrm{Im}\,F)=4=\dim(\mathbb{R}^4)$

Th The dimension of the solution space W of a homogeneous system $AX=0$ of linear equations is $s=n-r$, where n is the number

number of unknowns and r is the rank of the coefficient matrix A.

Singular and Nonsingular Linear mappings

Let $F: V \to U$ be a linear mapping.
Recall that $F(0) = 0$. F is said to be singular if the image of some nonzero vector v is 0 — that is if there exists $v \neq 0$ such that $F(v) = 0$. Thus $F: V \to U$ is nonsingular if the zero vector 0 is the only vector whose image under F is 0 or in other words; if $\ker F = \{0\}$.

Th Let $F: V \to U$ be a nonsingular linear mapping. Then the image of any linearly independent set is linearly independent.

Th A linear mapping $F: V \to U$ is one to one iff if F is nonsingular.

Ex Suppose the mapping $F: R^2 \to R^2$ is defined by $F(x,y) = (x+y, x)$. Show that F is linear.

Ans: We need to show that $F(v+w) = F(v) + F(w)$ and $F(kv) = kF(v)$, where u and v are any elements of R^2 and k is any scalar.

Let $v = (a, b)$, $w = (a', b')$

$v + w = (a+a', b+b')$

$kv = (ka, kb)$

$F(v) = (a+b, a)$, $F(w) = (a'+b', a')$

$F(v+w) = F(a+a', b+b')$

$= (a+a'+b+b', a+a')$

$= (a+b, a) + (a'+b', a')$

$= F(v) + F(w)$

and $F(kv) = F(ka, kb)$

$= (ka+kb, ka) = k(a+b, a)$

$= kF(v)$

Because v, w, k were arbitrary, F is linear.

Ex Suppose $F: R^2 \to R^2$ defined by $F(x,y,z) = (x+y+z, 2x-3y+4z)$. Show that F is linear.

Ex Show that the following mappings are not linear.

a) $F: R^2 \to R^2$ defined by $F(x,y) = (xy, x)$

b) $F: R^2 \to R^2$ defined by $F(x,y) = (x+3, 2y, x+y)$

c) $F: R^3 \to R^2$ defined by $F(x,y,z) = (|x|, y+z)$

a) Let $v = (1,2)$, $w = (3,4)$

$v + w = (4,6)$

$F(v) = (1 \cdot 2, 1) = (2,1)$ $\quad F(w) = (3 \times 4, 3) = (12,3)$

Hence $F(v+w) = (4 \cdot 6, 4) = (24,6)$

$\neq F(v) + F(w)$

b) Because $F(0,0) = (3,0,0) \neq (0,0,0)$

$\therefore F$ is not linear.

c) Let $v = (1, 2, 3)$, $k = -3$

$kv = (-3, -6, -9)$

$F(v) = (1, 5)$ and $kF(v) = -3(1, 5)$

$= (-3, -15)$

Thus $F(kv) = F(-3, -6, -9) = (+3, -15)$

$\neq kF(v)$

$\therefore$ F is not linear

<u>Ex</u> Let $F: R^2 \to R^2$ be the linear mapping for which $F(1, 2) = (2, 3)$, $F(0, 1) = (1, 4)$

Let (a, b) as a linear combination of $(1, 2)$ and $(0, 1)$.

$(a, b) = x(1, 2) + y(0, 1)$

$= (x, 2x + y)$

$\therefore x = a$, $b = 2x + y$, $y = -2a + b$

$F(a, b) = xF(1, 1) + yF(0, 1) = a(2, 3) + (-2a + b)(1, 4)$

$= (b, -5a+4b)$

Kernal and Image of the linear mapping

Ex 2 Let $F: R^4 \to R^3$ be the linear mapping defined by

$$F(x,y,z,t) = (x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$$

Find a basis and the dimension of

a) the image of F (b) the kernel of F.

a) Find the images of the usual basis of R^4

$F(1,0,0,0) = (1,1,1)$, $F(0,0,1,0) = (1,2,3)$

$F(0,1,0,0) = (-1,0,1)$, $F(0,0,0,1) = (1,-1,-3)$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $(1,1,1)$ and $(0,1,2)$ form a basis for Im F.

Hence $\dim(\operatorname{Im} F) = 2$

b) Set $F(v) = 0$ where $v = (x, y, z, t)$

$$F(x, y, z, t) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$$
$$= (0, 0, 0)$$

$$\left.\begin{aligned} x - y + z + t &= 0 \\ x \quad + 2z - t &= 0 \\ x + y + 3z - 3t &= 0 \end{aligned}\right\} \text{ or } \quad \left.\begin{aligned} x - y + z + t &= 0 \\ y + z - 2t &= 0 \\ 2y + 2z - 4t &= 0 \end{aligned}\right\} \text{ or}$$

$$\begin{aligned} x - y + z + t &= 0 \\ y + z - 2t &= 0 \end{aligned}$$

The free variables are z and t

Hence $\dim(\ker F) = 2$

i) set $z = 1,\ t = 0$, & the solution is $(2, 1, -1, 0)$

ii) Set $z = 0,\ t = 1$, the solution is $(1, 2, 0, 1)$

Thus $(2, 1, -1, 0)$, $(1, 2, 0, 1)$ form a basis of $\ker F$.

$$\dim(\ker F) + \dim(\operatorname{Im} F) = 2 + 2 = 4$$
$$= \dim R^4$$

Ex Let $G: R^3 \to R^3$ be the linear mapping defined by

$$G(x, y, z) = (x+2y-z,\ y+z,\ x+y-2z)$$

Find a basis and the dimension of (a) the image of G (b) the kernel of G.

a) Find the images of the usual basis of R^3

$$G(1,0,0) = (1,0,1),\quad G(0,1,0) = (2,1,1)$$

$$G(0,0,1) = (-1,1,-2)$$

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $(1,0,1)$ and $(0,1,-1)$ form a basis for $\mathrm{Im}\, G$. Hence $\dim(\mathrm{Im}\, G) = 2$

b) Set $G(v) = 0$ where $v = (x, y, z)$

$$G(x, y, z) = (x+2y-z,\ y+z,\ x+y-2z)$$
$$= (0, 0, 0)$$

Set corresponding

$$\left.\begin{array}{r} x+2y-z=0 \\ y+z=0 \\ x+y-2z=0 \end{array}\right\} \quad \begin{array}{r} x+2y-z=0 \\ y+z=0 \end{array}$$

$\therefore \dim(\ker G) = 1$, set $z=1$, $y=-1$, $x=3$

Then $(3,-1,1)$ is a basis of $\ker G$.

Linear Mappings and Matrices

Matrix Representation of a Linear Operator

Let T be a linear transformation from a vector space V into itself and suppose $S = \{u_1, u_2, \ldots u_n\}$ is a basis of V. Now $T(u_1), T(u_2), \ldots T(u_n)$ are vectors in V, and so each is a linear combination of the vectors in the basis S; say

$$T(u_1) = a_{11} u_1 + a_{12} u_2 + \cdots + a_{1n} u_n$$

$$T(u_2) = a_{21} u_1 + a_{22} u_2 + \cdots + a_{2n} u_n$$

$$\cdots \cdots \cdots$$

$$T(u_n) = a_{n1} u_1 + a_{n2} u_2 + \cdots + a_{nn} u_n$$

Def^n: The transpose of the above matrix of coefficients, denoted by $m_S(T)$ or $[T]_S$, is called the matrix representation of T relative to the basis S or simply the matrix of T in the basis S.

Using the coordinate vector notation, the matrix representation of T may be written in the form

$$m_S(T) = [T]_S = \big[[T(u_1)]_S, \cdots [T(u_n)]_S \big].$$

Prob: Let $F: R^2 \to R^2$ be the linear operator defined by $F(x,y) = (2x+3y,\ 4x-5y)$

Find the matrix representation of F relative to the basis $S = \{u_1, u_2\}$

$= \{(1,2),\ (2,5)\}$.

$F(u_1) = F(1,2) = (8,-6)$

$= x(1,2) + y(2,5)$

$\therefore x + 2y = 8$

$2x + 5y = -6$

Solve the system to obtain $x = 52, y = -22$

$\therefore F(u_1) = 52u_1 - 22u_2$

$F(u_2) = F(2,5) = (19,-17)$

$= x(1,2) + y(2,5)$

$x+2y = 19$
$2x+5y = -17$

$\therefore x = 129, \ y = -55$

$F(u_2) = 129u_1 - 55u_2$

Now write the coordinates of $F(u_1)$ and $F(u_2)$ as columns we obtain the matrix

$$[F]_S = \begin{bmatrix} 52 & 129 \\ -22 & -55 \end{bmatrix}$$

Prob: Let $F: R^2 \to R^2$ be defined by $F(x,y) = (2x+3y, 4x-5y)$. Find the matrix representation $[F]_S$ of F relative to the basis $S = \{u_1, u_2\} = \{(1,-2), (2,-5)\}$

An: $F(u_1) = F(1,-2) = (-4, 14)$

$= x(1,-2) + y(2,-5)$

$= 8u_1 + 6u_2$

$F(u_2) = F(2,-5) = (-11, 33)$

$= 11u_1 - 11u_2$

$$[F]_S = \begin{bmatrix} 8 & 11 \\ -6 & -11 \end{bmatrix}$$

Prob Let $F: R^3 \to R^2$ be the linear mapping defined by $F(x,y,z) = (3x+2y-4z,\ x-5y+3z)$.

a) Find the matrix of F in the following bases of R^3 and R^2

$$S = \{w_1, w_2, w_3\} = \{(1,1,1), (1,1,0), (1,0,0)\}$$

and $S' = \{u_1, u_2\} = \{(1,3), (2,5)\}$

Sol^n

$$(a, b) = (-5a+2b)u_1 + (3a-b)u_2$$

$$F(w_1) = F(1,1,1) = (1,-1) = -7u_1 + 4u_2$$

$$F(w_2) = F(1,1,0) = (5,-4) = -33u_1 + 19u_2$$

$$F(w_3) = F(1,0,0) = (3,1) = -13u_1 + 8u_2$$

$$[F]_{S,S'} = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

Prob: Consider the linear transformation T on R^2 defined by $T(x,y) = (2x-3y,\ x+4y)$ and the following bases of R^2:

$$E = \{e_1, e_2\} = \{(1,0), (0,1)\}$$ and

$$S = \{u_1, u_2\} = \{(1,3), (2,5)\}$$

a) Find the matrix A representing T relative to the basis E and S (5)

b) Find the matrix B representing T relative to the bases S and E.

a) $(a, b) = \cancel{(-5a+2b)}u_1 \; xu_1 + yu_2$

$= (-5a+2b)u_1 + (3a-b)u_2$

$T(e_1) = T(1,0) = (2,1) = -8u_1 + 5u_2$

$T(e_2) = T(0,1) = (-3,4) = 23u_1 + 13u_2$

$$A = \begin{bmatrix} -8 & 23 \\ 5 & -13 \end{bmatrix}$$

b) $T(u_1) = T(1,3) = (-7,13) = -7e_1 + 13e_2$

$T(u_2) = T(2,5) = (-11,22) = -11e_1 + 22e_2$

$$B = \begin{bmatrix} -7 & -11 \\ 13 & 22 \end{bmatrix}$$

Prob Let $F: R^3 \to R^2$ be defined by $F(x,y,z) = (2x+y-z,\ 3x-2y+4z)$. Find the matrix A representing F relative to the bases

$S = \{(1,1,1), (1,1,0), (1,0,0)\}$ and $S' = \{(1,3), (1,4)\}$ (6)

Characteristic Polynomial, Cayley-Hamilton Theorem

Let $A = [a_{ij}]$ be an n-square matrix. The matrix $M = A - tI_n$, where I_n is the n-square identity matrix and t is an indeterminate, may be obtained by subtracting t down the diagonal of A. The negative of M is the matrix $tI_n - A$, and its determinant

$$\Delta(t) = \det(tI_n - A) = (-1)^n \det(A - tI_n)$$

which is a polynomial in t of degree n and is called the characteristic polynomial of A.

Cayley Hamilton

Every matrix A is a root of its characteristic polynomial.

Ex Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$. Its characteristic polynomial is

$$\Delta(t) = |tI - A| = \begin{vmatrix} t-1 & -3 \\ -4 & t-5 \end{vmatrix} = (t-1)(t-5) - 12$$

$$= t^2 - 6t - 7$$

The Cayley-Hamilton theorem, A is a root of $\Delta(t)$.

$$\Delta(A) = A^2 - 6A - 7I$$

$$= \begin{bmatrix} 13 & 18 \\ 24 & 37 \end{bmatrix} + \begin{bmatrix} -6 & -18 \\ -24 & -30 \end{bmatrix} + \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Ex Find the characteristic polynomial of A

$$= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 3 & 9 \end{bmatrix}$$

$$\Delta(t) = t^3 - 13t^2 + 31t - 17$$

Eigenvalue

Let A be any square matrix. A scalar λ is called an eigenvalue of A if there exists a nonzero vector v such that

~~or~~ $Av = \lambda v$

Th An n-square matrix A is similar to a diagonal matrix D iff A has n linearly independent eigenvectors. In this case, the diagonal elements of D are the ~~characteristic~~ corresponding eigen values and $D = P^{-1}AP$, where P is the ~~eo~~ matrix whose columns are the eigenvectors.

Q. Let $A = \begin{bmatrix} 3 & -1 \\ 2 & -6 \end{bmatrix}$

a) Find all eigenvalues and eigenvectors.

b) Find matrices P and D such that P is nonsingular and $D = P^{-1}AP$ is diagonal.

Ans; a) First find the characteristic polynomial of A

$$\Delta(t) = t^2 - tr(A) + |A|$$
$$= t^2 + 3t - 10 = (t-2)(t+5)$$

The roots $\lambda = 2, \lambda = -5$

Eigen values are 2, 5.

i) Substract $\lambda = 2$ dawn the diagonal of A to obtain the matrix $M = A - 2I$, where the corresponding homogeneous system $MX = 0$ yields the eigenvectors corresponding to $\lambda = 2$. We have

$$M = \begin{bmatrix} 1 & -4 \\ 2 & -8 \end{bmatrix} \quad \text{corresponding to} \quad \left. \begin{array}{l} x - 4y = 0 \\ 2x - 8y = 0 \end{array} \right\}$$

or, $x - 4y = 0$

The system has only one free variable, and $v_1 = (4, 1)$ is a nonzero solution.

Thus, $v_1 = (4, 1)$ is an eigenvector belonging to $\lambda = 2$

ii) Substract $\lambda = -5$ &

$$M = \begin{bmatrix} 8 & -4 \\ 2 & -1 \end{bmatrix} \quad \left. \begin{array}{l} 8x - 4y = 0 \\ 2x - y = 0 \end{array} \right\} \text{ or } 2x - y = 0$$

The system has only one free variable and $v_1 = (1, 2)$ is a nonzero solution.

Thus $v_2 = (1, 2)$ is an eigenvector belonging to $\lambda = 5$

b) Let P be the matrix whose columns are (10)
v_1 and v_2. Then

$$P = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad D = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$$

Note that D is the diagonal matrix whose entries are the eigenvalues of A corresponding to the eigenvectors appearing in P.

Prob Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

a) Find all eigenvalues and eigenvectors

b) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal. and P^{-1}

The eigen values are $\lambda = 1, \lambda = 4$

$$\left.\begin{aligned} x + 2y &= 0 \\ x + 2y &= 0 \end{aligned}\right\} \quad \text{or} \quad x + 2y = 0$$

$(2, -1)$, $v_1 = (2, -1)$ is an eigenvector belonging to 1.

ii) Subtract $\lambda = 4$

$$M = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \quad \left.\begin{matrix} -2x+2y=0 \\ x-y=0 \end{matrix}\right\} x-y=0$$

$x=1, y=1$, $v_2 = (1,1)$ is an eigen vector belonging $\lambda = 4$.

b) Let P be the matrix whose columns are v_1 and v_2. Then

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \qquad D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

where $P^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$