Numerical Analysis

Let $y = f(x)$ be defined in $[a, b]$ and let $f(x)$ be specified by a given explicit formula. Then we can find the value or values of $f(x)$ corresponding to a fixed given value of x by simply substituting the value of x in the formula. But if $f(x)$ is not explicitly given by any formula, even then we can compute an approximate representive value of the function up to a desired degree of accuracy with the help of calculus of finite difference.

First Forward difference of the function $y = f(x)$ is denoted by

$$\Delta f(x) = f(x_0+h) - f(x_0) = y_1 - x_0 = \Delta y_0$$

$$\Delta f(x_0+h) = f(x_0+2h) - f(x_0+h) = y_2 - y_1 = \Delta y_1$$

$$\Delta f(x_0+2h) = f(x_0+3h) - f(x_0+2h) = y_3 - y_2 = \Delta y_2$$

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$$\Delta f(x_0 + \overline{n-1}\,h) = f(x_0+nh) - f(x_0 + \overline{n-1}\,h) = y_n - y_{n-1} = \Delta y_{n-1}$$

The differences of the first forward differences are called the second forward differences and are denoted by ②

$$\Delta^2 f(x_0) = \Delta^2 y_0$$
$$\Delta^2 f(x_0+h) = \Delta^2 y_1$$
$$- - - - - -$$
$$\Delta^2 f(x_0+\overline{n-1}\,h) = \Delta^2 y_{n-1}$$

where $\Delta^2 f(x_0) = \Delta f(x_0+h) - \Delta f(x_0)$

$$= f(x_0+2h) - f(x_0+h) - [f(x_0+h) - f(x_0)]$$

$$= f(x_0+2h) + f(x_0) - 2f(x_0+h)$$

$$= y_2 - 2y_1 + y_0 = \Delta^2 y_0$$

$$\Delta^2 f(x_0+h) = \Delta f(x_0+2h) - \Delta f(x_0+h)$$
$$= f(x_0+3h) - f(x_0+2h) - [f(x_0+2h) - f(x_0+h)]$$

$$= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h)$$

$$= y_3 - 2y_2 + y_1 = \Delta^2 y_1$$

$$\Delta^2 f(x_0+2h) = \Delta f(x_0+3h) - \Delta f(x_0+2h)$$

$$= f(x_0+4h) - f(x_0+3h) - [f(x_0+3h) - f(x_0+2h)]$$

$$= f(x_0+4h) - 2f(x_0+3h) + f(x_0+2h)$$

$$= y_4 - 2y_3 + y_2 = \Delta^2 y_2$$

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Similarly $\Delta^3 f(x_0) = \Delta^2 f(x_0+h) - \Delta^2 f(x_0)$

$$= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h) - [f(x_0+2h) - 2f(x_0+h) + f(x_0)]$$

$$= f(x_0+3h) - 3f(x_0+2h) + 3f(x_0+h) - f(x_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0 = \Delta^3 y_0$$

and so on.

Backward Differences

If $y_0 = f(x_0)$, $y_1 = f(x_1) = f(x_0+h)$,

$y_2 = f(x_2) = f(x_0+2h)$ - - - -

$y_{n-1} = f(x_{n-1}) = f(x_0+\overline{n-1}\,h)$, $y_n = f(x_n)$
$= f(x_0+nh)$

$\nabla f(x_0+h) = f(x_0+h) - f(x_0) = y_1 - y_0 = \nabla y_1$

$\nabla f(x_0+2h) = f(x_0+2h) - f(x_0+h) = y_2 - y_1 = \nabla y_2$

$\nabla f(x_0+3h) = f(x_0+3h) - f(x_0+2h) = y_3 - y_2 = \nabla y_3$

- - - - - - - - -

$\nabla f(x_0+nh) = f(x_0+nh) - f(x_0+\overline{n-1}\,h) = y_n - y_{n-1} = \nabla y_n$

The differences of first backward differences are called the second backward differences and are denoted by $\nabla^2 f(x_0+2h)$, $\nabla^2 f(x_0+3h)$, $\nabla^2 (x_0+4h)$, - - - etc.

where

$$\nabla^2 f(x_0+2h) = \nabla f(x_0+2h) - \nabla f(x_0+h)$$
$$= f(x_0+2h) - f(x_0+h)$$
$$- [f(x_0+h) - f(x_0)]$$
$$= f(x_0+2h) - 2f(x_0+h) + f(x_0)$$
$$= y_2 - 2y_1 + y_0 = \nabla^2 y_2$$

$$\nabla^2 f(x_0+3h) = \nabla f(x_0+3h) - \nabla f(x_0+2h)$$
$$= f(x_0+3h) - f(x_0+2h)$$
$$- [f(x_0+2h) - f(x_0+h)]$$
$$= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h)$$
$$= y_3 - 2y_2 + y_1 = \nabla^2 y_3$$

$$\nabla^2 f(x_0+4h) = \nabla f(x_0+4h) - \nabla f(x_0+3h)$$
$$= f(x_0+4h) - f(x_0+3h)$$
$$- [f(x_0+3h) - f(x_0+2h)]$$
$$= f(x_0+4h) - 2f(x_0+3h) + f(x_0+2h)$$
$$= y_4 - 2y_3 + y_2 = \nabla^2 y_4$$

and so on.

In general, we have

$$\nabla^n f(x) = \nabla^{n-1}(x) - \nabla^{n-1} f(x-h)$$

Prove that $\Delta . \nabla = \Delta - \nabla$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\nabla f(x) = f(x) - f(x-h)$$

$$\therefore \Delta . \nabla f(x) = \Delta \left[f(x) - f(x-h) \right]$$

$$= \Delta f(x) - \Delta f(x-h)$$

$$= \Delta f(x) - \left[f(x) - f(x-h) \right]$$

$$= \Delta f(x) - \nabla f(x)$$

$$= (\Delta - \nabla) f(x)$$

$$\therefore \Delta . \nabla = \Delta - \nabla$$

Shift Operator E and its relations with difference operator Δ.

The shifting operator E is defined by

$$E f(x) = f(x+h)$$

Again we know that

$$\Delta f(x) = f(x+h) - f(x)$$
$$= E f(x) - f(x)$$

or $(\Delta+1) f(x) = E f(x)$

Thus $\Delta+1 = E$ or $\Delta = E-1$

This relation are known as the relation between shift operator E and the difference operator Δ.

$$E^2 f(x) = E\left[E f(x)\right]$$
$$= E\left[f(x+h)\right]$$
$$= f(x+2h)$$

$$E^3 f(x) = E E^2 f(x)$$
$$= E f(x+2h) = f(x+3h)$$

$$E^n f(x) = f(x+nh)$$

Interpolation with Equal and Unequal Intervals

Suppose a function $y = f(x)$ is known for $(n+1)$ distinct values of x, say $x_0, x_1, x_2, \ldots x_{r-1}, x_r, x_{r+1}, \ldots, x_{n-1}, x_n$. There is no other information available about the function. That is the only information we have is

$$f(x_j) = y_j \quad (j = 0, 1, 2, \ldots n) \text{ —— } (1)$$

The problem of interpolation is to compute the value of $f(x)$, atleast approximate, for an argument, say x, not found in the table, i.e. for an argument other than $x_0, x_1, x_2, \ldots x_{r-1}, x_r, x_{r+1}, \ldots x_{n-1}, x_n$. The term <u>Interpolation</u> is used when x lies between the smallest and greatest of the interpolating points $x_0, x_1, x_2, \ldots x_{r-1}, x_r, x_{r+1}, \ldots x_{n-1}, x_n$ and the term <u>Extrapolation</u> is used when

when x lies slightly outside the range of the interpolating points. ~~since~~

Our problem will be to approximate $f(x)$ by a function $\phi(x)$, which is simple in nature and is such that

$$\phi(x_j) = f(x_j), \quad (j = 0, 1, 2, 3, \ldots n) \qquad (2)$$

This function $\phi(x)$ is known as the interpolation function and is used to compute the approximate value of the function $f(x)$ is at the desired values of x.

In general, we write $f(x) \simeq \phi(x)$ —— (3)

If we write $f(x) = \phi(x) + R_{n+1}(x)$ —— (4)

then we say that $R_{n+1}(x)$ is the remainder or error committed in replacing $f(x)$ by $\phi(x)$. This function $\phi(x)$ may be of different types. When $\phi(x)$ is a polynomial we call it a Parabolic or Polynomial Interpolation. (10)

Newton's Forward Interpolation Formula

Let a function $f(x)$ is known for $(n+1)$ distinct equispaced arguments namely

$x_0, x_1, x_2, \dots x_{r-1}, x_r, x_{r+1}, \dots x_{n-1}, x_n$

such that

$$x_r = x_0 + rh \quad (r = 0, 1, 2, \dots n) \quad \text{——} (1)$$

where h is the length of each space, and the corresponding entries are

$f(x_0) = y_0, f(x_1) = y_1, f(x_2) = y_2, \dots$

$f(x_r) = y_r, \dots f(x_{n-1}) = y_{n-1}, f(x_n) = y_n$

i.e. $f(x_j) = y_j \quad (j = 0, 1, 2, \dots n)$ —— (2)

Now from (1), we have

$$x_n - x_r = (x_0 + nh) - (x_0 + rh) = (n - r)h \quad \text{——} (3)$$

Now our object is to find a polynomial $p(x)$ of degree less than or equal to n.

$$p(x_j) = f(x_j) = y_j \quad (j = 0, 1, 2, --n) \qquad (4)$$

As $p(x)$ is a polynomial of degree n, we take $p(x)$ as

$$p(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1)$$
$$+ A_3(x-x_0)(x-x_1)(x-x_2)$$
$$+ --- + A_n(x-x_0)(x-x_1) --- (x-x_{n-1}) \qquad (5)$$

The constants A_j $(j = 0, 1, 2, 3, --n)$ are to be determined successively by using (4) as follows:

Substituting $x = x_0$, we get from (5) and (4)

$$p(x_0) = f(x_0) = y_0 = A_0$$

$$\therefore A_0 = f(x_0) = y_0$$

Substituting $x = x_1$ we have

$$p(x_2) = f(x_2) = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_0)(x_2 - x_1)$$

$$\therefore A_2 = \frac{y_2 - A_0 - A_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y_2 - y_0 - \frac{\Delta y_0}{h} \cdot 2h}{2h \cdot h}$$

$$= \frac{y_2 - y_0 - 2\Delta y_0}{2h^2} = \frac{y_2 - y_0 - 2(y_1 - y_0)}{2h^2}$$

$$= \frac{y_2 - 2y_1 + y_0}{2h^2}$$

$$= \frac{\Delta^2 y_0}{2!\, h^2} = \frac{\Delta^2 f(x_0)}{2!\, h^2}$$

Substituting $x = x_3$, we have from (5) and (4)

$$p(x_3) = f(x_3) = y_3$$

$$= A_0 + A_1(x_3 - x_0) + A_2(x_3 - x_0)(x_3 - x_1) + A_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)$$

$$A_3 = \frac{y_3 - A_0 - A_1(x_3 - x_0) - A_2(x_3 - x_0)(x_3 - x_1)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

$$= \frac{y_3 - y_0 - \frac{\Delta y_0}{h}.3h - \frac{\Delta^2 y_0}{2h^2}.3h.2h}{3h.2h.h}$$

$$= \frac{y_3 - y_0 - 3\Delta y_0 - 3\Delta^2 y_0}{3.2.1.h^3} = \frac{y_3 - y_0 - 3(y_1 - y_0) - 3(y_2 - 2y_1 + y_0)}{3!\,h^3}$$

$$= \frac{y_3 - 3y_2 + 3y_1 - y_0}{3!\,h^3} = \frac{\Delta^3 y_0}{3!\,h^3} = \frac{\Delta^3 f(x_0)}{3!\,h^3}$$

Similarly $A_r = \frac{\Delta^r y_0}{r!\,h^r} = \frac{\Delta^r f(x_0)}{r!\,h^r}$

Substituting A_r's in (5), we get

$$p(x) = y_0 + (x-x_0)\frac{\Delta y_0}{h} + (x-x_0)(x-x_1)\frac{\Delta^2 y_0}{2!\,h^2}$$
$$+ (x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 y_0}{3!\,h^3}$$
$$+ \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\frac{\Delta^n y_0}{n!\,h^n} \quad \text{——— (6)}$$

$$\therefore f(x) \simeq p(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{h}$$
$$+ (x-x_0)(x-x_1)\frac{\Delta^2 f(x_0)}{2!\,h^2} + \cdots$$

$$+ (x-x_0)(x-x_1)(x-x_2) \cdots (x-x_{n-2})(x-x_{n-1}) \frac{\Delta^n f(y_0)}{n!\, h^n} \quad \text{—— (7)}$$

We now transfer the formula (6) and (7) to a more convenient and useful form by introducing a dimensionless quantity u, called phase, given by

$$u = \frac{x-x_0}{h} \quad \text{or} \quad x = x_0 + hu \quad \text{—— (8)}$$

We have also from (1), $x_r = x_0 + rh$

$$\therefore x - x_r = (u-r)h \quad \text{—— (9)}$$

Using (9) ~~and~~ in (6) and (7), we get

$$p(x) = p(x_0 + hu)$$

$$= y_0 + u\,\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{2!} + u(u-1)(u-2)\frac{\Delta^3 y_0}{3!}$$

$$+ \cdots + u(u-1)(u-2)(u-3)\cdots(u-\overline{n-1})\frac{\Delta^n y_0}{n!} \quad \text{—— (10)}$$

or, $$f(x) \simeq p(x) = f(x_0) + u\,\Delta f(x_0) + u(u-1)\frac{\Delta^2 f(x_0)}{2!}$$

$$+ u(u-1)(u-2)\frac{\Delta^3 f(x_0)}{3!} + \cdots$$

$$+ u(u-1)(u-2)\cdots(u-n+1)\frac{\Delta^n f(x_0)}{n!} \quad \text{—— (11)}$$

This formula (10) or (11) is known as Newton's Forward Interpolation formula.

Newton's Backward Interpolation Formula

Let a function $f(x)$ is known for $(n+1)$ equispaced arguments, namely $x_n, x_{n-1}, x_{n-2}, \ldots x_{r+1}, x_r, x_{r-1}, \ldots x_2, x_1, x_0$ such that

$$x_r = x_0 + rh, \quad (r = 0, 1, 2, \ldots n) \quad \text{—} (1)$$

where h is the length of each space and the corresponding entries are

$$f(x_n) = y_n,\ f(x_{n-1}) = y_{n-1},\ f(x_{n-2}) = y_{n-2},$$

$$\ldots f(x_r) = y_r, \ldots, f(x_2) = y_2,\ f(x_1) = y_1$$

and $f(x_0) = y_0$,

i.e., $f(x_j) = y_j \quad (j = n, n-1, n-2, \ldots r+1, r, r-1, \ldots 2, 1, 0)$ — (2)

Thus from (1), we have

$$x_{n-r} - x_n = \{x_0 + (n-r)h\} - \{x_0 - nh\}$$

$$= -rh \quad \text{—} (3)$$

Here also, our purpose is to find a polynomial $p^B(x)$ of degree n, simple in nature and which replaces $f(x)$ on the set of interpolating points (arguments) x_j ($j = n, n-1, \ldots 2, 1, 0$) ②

i.e. $p^B(x_j) = f(x_j) = y_j$ $(j = n, n-1, n-2, \ldots r+1, r, r-1, \ldots 2, 1, 0)$ —— ④

We assume

$$p^B(x) = B_n + B_{n-1}(x-x_n) + B_{n-2}(x-x_n)(x-x_{n-1})$$
$$+ B_{n-3}(x-x_n)(x-x_{n-1})(x-x_{n-2})$$
$$+ \cdots + B_0 (x-x_n)(x-x_{n-1})(x-x_{n-2})$$
$$\cdots\cdots (x-x_2)(x-x_1) \quad \text{—— ⑤}$$

The constants B_j ($j = n, n-1, \ldots 3, 2, 1, 0$) will be determined successively by ④ as follows:

Substituting $x = x_n$, we get from ⑤ and ④

$$\cancel{p^B(x_j)} \quad p^B(x_n) = f(x_n) = y_n = B_n ;$$

$$\therefore \; B_n = y_n = f(x_n)$$

Substituting $x = x_{n-1}$, we get from ⑤ and ④

$$p^B(x_{n-1}) = f(x_{n-1}) = y_{n-1}$$
$$= B_n + B_{n-1}(x_{n-1} - x_n)$$
$$= B_n - h B_{n-1} \quad (\text{by ③})$$

$$\therefore B_{n-1} = \frac{B_n - y_{n-1}}{h} = \frac{B_n - y_{n-1}}{h}$$

$$= \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_n}{h} = \frac{\Delta y_{n-1}}{h}$$

$$= \frac{\nabla f(x_n)}{h} = \frac{\Delta f(x_{n-1})}{h}$$

Substituting $x = x_{n-2}$, we have from (5) and (4)

$$p^B(x_{n-2}) = f(x_{n-2}) = y_{n-2}$$

$$= B_n + B_{n-1}(x_{n-2} - x_n) + B_{n-2}(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$= B_n + B_{n-1}(-2h) + B_{n-2}(-2h)(-h) \quad (\text{by } (3))$$

$$\therefore B_{n-2} = \frac{y_{n-2} - B_n + 2h\,B_{n-1}}{2h^2} = \frac{y_{n-2} - y_n + 2h\frac{\nabla y_n}{h}}{2h^2}$$

$$= \frac{y_{n-2} - y_n + 2(y_n - y_{n-1})}{2h^2}$$

$$= \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2h^2} = \frac{\Delta^2 y_{n-2}}{2h^2}$$

$$= \frac{\nabla^2 f(x_n)}{2h^2} = \frac{\Delta^2 f(x_{n-2})}{2h^2}$$

Substituting $x = x_{n-3}$ we get from (5) and (4)

$$p^3(x_{n-3}) = f(x_{n-3}) = y_{n-3}$$

$$= B_n + (-3h)B_{n-1} + (-3h)(-2h)B_{n-2}$$
$$+ (-3h)(-2h)(-h)B_{n-3} \quad [\text{by } (3)]$$

$\therefore$ ~~B_{n-3} =~~

or, $$y_{n-3} = B_n + (-3h)B_{n-1} + (-3h)(-2h)B_{n-2}$$
$$+ (-3h)(-2h)(-h)B_{n-3}$$

$$\therefore B_{n-3} = \frac{B_n - 3h\,B_{n-1} + 6h^2 B_{n-2} - y_{n-3}}{6h^3}$$

$$= \frac{y_n - 3\nabla y_n + 3\nabla^2 y_n - y_{n-3}}{3!\,h^3}$$

$$= \frac{y_n - 3(y_n - y_{n-1}) + 3(y_n - 2y_{n-1} + y_{n-2}) - y_{n-3}}{3!\,h^3}$$

$$= \frac{y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3}}{3!\,h^3}$$

$$= \frac{\nabla^3 y_n}{3!\,h^3} = \frac{\Delta^3 y_{n-3}}{3!\,h^3} = \frac{\nabla^3 f(x_n)}{3!\,h^3} = \frac{\Delta^3 f(x_{n-3})}{3!\,h^3}$$

Similarly we have

$$B_{n-r} = \frac{\nabla^r y_n}{r!\, h^r} = \frac{\Delta^r f(x_{n-r})}{r!\, h^r}$$

Substituting B_{n-r}'s $(r = 0, 1, 2, --n)$ in 5,

we get

$$p^B(x) = y_n + (x-x_n)\frac{\nabla y_n}{h} + (x-x_n)(x-x_{n-1})\frac{\nabla^2 y_n}{2!\, h^2}$$

$$+ (x-x_n)(x-x_{n-1})(x-x_{n-2})\frac{\nabla^3 y_n}{3!\, h^3} + ----$$

$$---+ (x-x_n)(x-x_{n-1})(x-x_{n-2})---(x-x_2)(x-x_1)\frac{\nabla^n y_n}{n!\, h^n} \quad \text{—— (6)}$$

or,
$$p^B(x) = f(x_n) + (x-x_n)\frac{\Delta f(x_{n-1})}{h}$$

$$+ (x-x_n)(x-x_{n-1})\frac{\Delta^2 f(x_{n-2})}{2!\, h^2}$$

$$+ --- + (x-x_n)(x-x_{n-1})(x-x_{n-2})\frac{\Delta^3 f(x_{n-3})}{3!\, h^3}$$

$$+ ----+ (x-x_n)(x-x_{n-1})--(x-x_2)(x-x_1)\frac{\Delta^n f(x_0)}{n!\, h^n} \quad \text{—— (7)}$$

We now transfer the formula ⑥ and ⑥ ⑦ to a more convenient and useful form by introducing a non-dimensional quantity u, called phase, given by

$$u = \frac{x - x_n}{h}, \quad x = x_n + hu \quad \text{— ⑧}$$

Again we have from ③

$$x_{n-r} = x_n - rh$$

Thus $x - x_{n-r} = (u+r)h$ — ⑨

Using ⑨ in ⑥ and ⑦, we get

$$p^B(x) = p^B(x_n + hu)$$

$$= y_n + u\nabla y_n + u(u+1)\frac{\nabla^2 y_n}{2!} + \frac{u(u+1)(u+2)\nabla^3 y_n}{3!}$$

$$+ \cdots + u(u+1)(u+2)\cdots(u+n-1)\frac{\nabla^n y_n}{n!}$$

— ⑩

or, $f(x) = p^B(x)$

$$= f(x_n) + u\Delta f(x_{n-1}) + u(u+1)\frac{\Delta^2 f(x_{n-2})}{2!}$$

$$+ u(u+1)(u+2)\frac{\Delta^3 f(x_{n-3})}{3!}$$

$$+u(u+1)(u+2)\frac{\Delta^3 f(x_{n-3})}{3!}$$

$$+ - - - - + u(u+1)(u+2) - - - (u+n-1)\frac{\Delta^n f(x_0)}{n!} \quad \longrightarrow (11)$$

The formula (10) or (11) is known as Newton's Backward Interpolation Formula.

Solve

Given

x : 1	2	3	4	5	6	7	8
$f(x)$: 1	8	27	64	125	216	343	512

find (i) $f(1.5)$ (ii) $f(7.5)$

Sol^n The difference table is

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		6
7	343		42	
		169		
8	512			

(i) To compute $f(1.5)$, we use Newton's Forward Formula (10), as the point $x = 1.5$ is near the beginning of the table. Thus we have

$$f(x) = y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{2!} + \frac{u(u-1)(u-2)\Delta^3 y_3}{3!}$$

$$+ \cdots$$

where $u = \frac{x - x_0}{h}$

Here $x = 1.5$, $x_0 = 1$, $h = 1$

$$\therefore u = \frac{1.5-1}{1} = 0.5$$

$$f(1.5) = 1 + 0.5 \times 7 + (0.5)(-0.5)\frac{12}{2!}$$

$$+ (0.5)(-0.5)(-1.5) \times \frac{6}{3!}$$

$= 1 + 3.5 - 1.5 + 0.375$

$= 3.375$

(ii) For $f(7.5)$, as the point is near the end of the table, we use Newton's Backward formula (10).

$$f(x) = y_n + u \cdot \Delta y_{n-1} + \frac{u(u+1)}{2!}\Delta^2 y_{n-2} + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_{n-3} + \cdots$$

Here $x = 7.5$, $x_n = 8$ and $h = 1$

$$u = \frac{x - x_n}{h} = \frac{7.5 - 8}{1} = -0.5$$

$$\therefore f(7.5) = 512 + (-0.5)(169) + \frac{(-0.5)(0.5)}{2!} \times 42 + \frac{(-0.5)(0.5)(1.5)}{3!} \times 6$$

$= 512 - 84.5 - 5.25 - 0.375$

$= \cancel{41} \; 421.875$

Ex Given the following table

x	0	5	10	15	20
$f(x)$	1.0	1.6	3.8	8.2	15.4

Construct the difference table and compute $f(21)$ by Newton's Backward Formula.

The difference table is

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1.0			
		0.6		
5	1.6		1.6	
		2.2		0.6
10	3.8		2.2	
		4.4		0.6
15	8.2		2.8	
20	15.4	7.2		

Newton's Backward formula (10) is

$$f(x) = y_n + u\,\Delta y_{n-1} + \frac{u(u+1)}{2!}\Delta^2 y_{n-2} + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_{n-3} + \cdots$$

Here $x = 21$, $x_n = 20$, $h = 5$

$\therefore u = \frac{x - x_n}{h} = \frac{21-20}{5} = 0.2$

$$\therefore f(21) = 15.4 + (0.2)\times 7.2$$

$$+ \frac{0.2\times 1.2}{2}\times 2.8 + \frac{0.2\times 1.2\times 2.2}{6}\times 0.6$$

$$= 15.4 + 1.44 + 0.336 + 0.0528$$

$$= 17.2288 \simeq 17.2$$

Ex Calculate from the following table find the value of y when $x = 1.6$

x	1.0	1.5	2.0	2.5	3.0
~~$f(x)$~~ y	0.11246	0.14032	0.16800	0.19547	0.22270

The difference table is

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1.0	0.11246			
		0.02786		
			−0.00018	
1.5	0.14032			−0.00003
		0.02768		
2.0	0.16800		−0.00021	
		0.02747		−0.00003
2.5	0.19547		−0.00024	
3.0	0.22270	0.02723		

As the pt, is near the beginning of the table we use Newton's Forward Formula (10)

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Here $x = 1.6$, $x_0 = 1.0$, $h = 0.5$,

$$\therefore u = \frac{1.6-1}{0.5} = 1.2$$

$$\therefore f(1.6) = 0.11246 + 1.2 \times 0.02786 + \frac{1.2 \times 0.2}{2}(-0.00018) + \frac{1.2 \times 0.2 \times (-0.8)}{3!}(-0.00003)$$

$$= 0.11246 + 0.033432 - 0.0000216 + 0.000001$$

$$= 0.1458714 \simeq 0.14587$$

Note $x = 1.6$ is nearer to $x = 1.5$ than $x = 1.0$ taking $x_0 = 1.5$, we may get a better result than $x_0 = 1.0$

Taking $x = 1.6$, $x_0 = 1.5$, $u = \frac{1.6 - 1.5}{0.5} = 0.2$

$$f(1.6) = 0.14032 + 0.2 \times 0.02768 + \frac{0.2 \times (0.2-1)}{2}(-0.00021) + \frac{0.2(0.2-1)(0.2-2)}{6}(-0.0003)$$

$$= 0.14032 + 0.005536 + 0.0000168 - 0.00000144$$

$$= 0.1458714 \simeq 0.14587$$

Ex find $f(1.02)$

x	1.00	1.10	1.20	1.30
$f(x)$	0.8415	0.8912	0.9320	0.9636

The difference table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1.00	0.8415			
		0.0497		
1.10	0.8912		-0.0089	
		0.0408		-0.0003
1.20	0.9320		-0092	
		0.0316		
1.30	0.9636			

$$u = \frac{1.02 - 1.0}{0.1} = 0.2$$

$$f(1.02) = 0.8415 + 0.2\,(0.0497)$$

$$+ \frac{0.2\,(0.2-1)}{2}\,(-0.0089)$$

$$+ \frac{0.2\,(0.2-1)(0.2-2)}{6}\,(-0.0007)$$

$$= 0.8415 + 0.00994 + 0.000712 - 0$$

$$= 0.8521376 \simeq 0.8521$$

Ex $f(0.5)$, $f(2.8)$

x	0	1	2	3
$f(x)$	1	2	11	34

Ans: ① $u = \frac{0.5 - 0}{1} = 0.5$

Ans: 0.88

② $u = \frac{2.8 - 3}{1} = -0.2$

$f(2.8) = \cancel{28}\ 27.992$

Method of Bisection

It is an iterative method and is based on a well known theorem which states that if $f(x)$ be a continuous function in a closed interval $[a, b]$ and $f(a)f(b)<0$, then there exist at least one real root of the function of equation $f(x)=0$, between a and b.

If further $f'(x)$ exists and $f'(x)$ maintains same sign in $[a, b]$, i.e. $f(x)$ strictly monotone, then there is only one real root of $f(x)=0$, in $[a, b]$. The method of Bisection is nothing but a repeated applications of the above theorem.

We shall determine a sufficiently small interval $[a_0, b_0]$ by Graphical or Tabulation method, in which $f(a_0)f(b_0)<0$ and $f'(x)$ maintains same in $[a_0, b_0]$, so that there is only one real root of $f(x) = 0$. Now we shall find a sequence $\{x_n\}$, each member of which is a successive better approximation of a root say, α of $f(x) = 0$, in $[a_0, b_0]$ as follows. Let the interval $[a_0, b_0]$ be divided in two equal parts by x_1, i.e. $x_1 = \frac{a_0 + b_0}{2}$ and $f(x_1)$ is calculated. If $f(x_1) = 0$, then x_1 is an exact root of $f(x) = 0$. If $f(x_1) \neq 0$, then either $f(a_0) f(x_1) < 0$ or $f(x_1) . f(b_0) < 0$. If $f(a_0) f(x_1) < 0$, then the root α lies in $[a_0, x_1]$, otherwise α lies in $[x_1, b_0]$. For convenience we assume that α lies in $[x_1, b_0]$ and we re-name the interval as $[a_1, b_1]$ so that $b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$. Now we take $x_2 = \frac{a_1 + b_1}{2}$ and $f(x_2)$ is computed, then either $f(a_1)f(x_2) < 0$ or $f(x_2)f(b_1) < 0$, provided $f(x_2)$ is computed, then either $f(a_1)f(x_2) < 0$ or $f(x_2)f(b_1) < 0$, provided $f(x_2) = 0$ where x_2 is the exact root of $f(x) = 0$. We assume here that $f(a_1)f(x_2) < 0$, then the root α of $f(x) = 0$ lies in $[a_1, x_2]$ and we call it as $[a_2, b_2]$, where $b_2 - a_2 = \frac{1}{2}(b_1 - a_1)$ $= \frac{1}{2^2}(b_0 - a_0)$. Proceeding in this manner we find $x_{n+1} = \frac{a_n + b_n}{2}$ which is the $(n+1)$th approximation of the root α of $f(x) = 0$ and lies in the interval $[a_n, b_n]$ where $b_n - a_n$ $= \frac{1}{2^n}(b_0 - a_0)$ and $a_0 \leq a_n < b_n \leq b_0$ for all n. If ε_{n+1} be the error in approximation α by x_{n+1} then $\varepsilon_{n+1} = |\alpha - x_{n+1}| < b_n - a_n$ $< \frac{b_0 - a_0}{2^n} \to 0$ as $n \to \alpha$. Thus, this iterative process surely converges To get a root of $f(x)=0$ correct to p significant figures, we are to go up to qth iteration so that x_q and x_{q+1} have the same p significant figures.

Computation Scheme

1. Find an interval $[a_0, b_0]$ where $f(a_0)f(b_0) < 0$ and $f'(x)$ maintains same sign.

2. Write, n (number of iteration), a_n, b_n, $x_{n+1}\left(=\frac{a_n+b_n}{2}\right)$ and $f(x_{n+1})$ horizontally.

3. Insert +ve or −ve sign with a_n, as a_n(+ve) or a_n(−ve) according as $f(a_0)>0$ or $f(a_0)<0$ and −ve or +ve sign with b_n, as b_n(−ve) or b_n(+ve) according as $f(b_0)<0$ or $f(b_0)>0$.

4. In $(r+1)$th iteration, write $x_{r+1}\left(=\frac{a_r+b_r}{2}\right)$ in the column of a_n(+ve), if $f(x_{r+1})>0$ keeping b_r fixed in the column of b_n(−ve).

Otherwise, write $x_{r+1}\left(=\frac{a_r+b_r}{2}\right)$ in the column of b_n (-ve), if $f(x_{r+1})<0$, keeping a_r fixed in the column of a_n (+ve)

Solved

Find the positive root of the equation $x^3-3x+1.06=0$ by the method of bisection, correct to three decimal places.

Sol^n Let $f(x)=x^3-3x+1.06$

Now $f(0)=1.06>0$

$f(1)=-0.94<0$,

$f(2)=3.06>0$

Then, one positive root α lies in $(0,1)$ and other β lies in $(1,2)$

i) Computation of α $(0<\alpha<1)$

n	a_n (+ve)	b_n (−ve)	$x_{n+1} = \left(\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
0	0	1	0.5	−0.32
1	0	0.5	0.25	0.33
2	0.25	0.5	0.375	−0.012
3	0.25	0.375	0.312	0.154
4	0.312	0.375	0.343	0.071
5	0.343	0.375	0.359	0.029
6	0.359	0.375	0.367	0.008
7	0.367	0.375	0.371	−0.002
8	0.367	0.371	0.369	0.003
9	0.369	0.371	0.370	0.0006
10	0.370	0.371	0.3705	−0.0006
11	0.370	0.3705	0.37025	0.000006
12	0.37025	0.3705	0.370375	−0.0003
13	0.37025	0.370375	0.370312	

$\alpha = 0.370$, correct to three decimal places

(ii) Computation of β $(1 < \beta < 2)$. Here

$f(1) = -0.94$, $f(2) = 3.06$

n	a_n (−ve)	b_n (+ve)	$x_{n+1}\left(=\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
0	1	2	1.5	−0.06
1	1.5	2	1.75	1.17
2	1.5	1.75	1.62	0.45
3	1.5	1.62	1.56	0.17
4	1.5	1.56	1.53	0.05
5	1.5	1.53	1.515	−0.007
6	1.515	1.53	1.5225	0.0217
7	1.515	1.5225	1.5188	0.0067
8	1.515	1.5188	1.5169	−0.0003
9	1.5169	1.5188	1.51785	0.0034
10	1.5169	1.51785	1.517375	0.0015
11	1.5169	1.517375	1.517138	0.0006
12	1.5169	1.517138	1.517019	0.0001

$\therefore \beta = 1.517$, correct to three decimal places. Here a_n, b_n and x_{n+1} are equal upto three decimal places at the 11th step.

~~Solved~~ Solve the equation $x^3 - 9x + 1 = 0$ (8)

for the root lying between 2 and 3 correct to 3-significant figures.

Sol^n Let $f(x) = x^3 - 9x + 1$

$f(2) = -9$, $f(3) = 1$

$\therefore f(2) \cdot f(3) < 0$

n	a_n (-ve)	b_n (+ve)	$x_{n+1} \left(= \frac{a_n + b_n}{2}\right)$	$f(x_{n+1})$
0	2	3	2.5	-5.8
1	2.5	3	2.75	-2.9
2	2.75	3	2.88	-1.03
3	2.88	3	2.94	-0.05
4	2.94	3	2.97	0.47
5	2.94	2.97	2.955	0.21
6	2.94	2.955	2.9475	0.08
7	2.94	2.9475	2.9438	0.017
8	2.94	2.9438	2.9419	-0.016
9	2.9419	2.9438	2.9428	-0.003

In th 8th step, a_n, b_n and x_{n+1} are equal up to three significant figures.

$\therefore 2.94$ is the root, up to three significant figures.

Ex Compute one positive root of $2x-3\sin x-5=0$, by the bisection method correct to three significant figures.

Sol^n Let $f(x)=2x-3\sin x-5$

Here $f(0)=-5$, $f(1)=-5.5$, $f(2)=-3.7$

$f(3)=0.57$

$\therefore f(2)\cdot f(3)<0$. Then only one root lies between 2 and 3 since $f'(x)=2-3\cos x>0$ for $x\in[2,3]$

n	a_n (-ve)	b_n (+ve)	$x_{n+1}\left(=\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
0	2.0	3.0	2.5	-1.79
1	2.5	3.0	2.75	-0.64
2	2.75	3.0	2.875	-0.04
3	2.875	3.0	2.938	0.27
4	2.875	2.938	2.906	0.11
5	2.875	2.906	2.8905	0.036
6	2.875	2.8905	2.8828	-0.0021
7	2.8828	2.8905	2.8866	0.0165
8	2.8828	2.8866	2.8847	0.0072
9	2.8828	2.8847	2.8838	0.0028
10	2.8828	2.8838	2.8833	0.0003

In 9th step, a_n, b_n and x_{n+1} are equal upto three significant figures

$\therefore$ 2.88 is the root correct to three significant figures

<u>Ex</u> Find one root of $10^x + \sin x + 2x = 0$ by the bisection method three significant figure

$f(x) = 10^x + \sin x + 2x$

$f(0) = 1$, $f(1) = 12.8$, $f(-1) = -2.74$

$f(-1) f(0) < 0$. Also $f'(x) = 10^x \log x + \cos x + 2 > 0$

for $x \in [-1, 0]$

Ans: 0.207

Newton-Raphson Method

This is also an iterative method and is used to find isolated roots of an equation $f(x)=0$. The object of this method is to correct the approximate root of the equ^n successively to its exact value α. Initially, a crude approximation on small interval $[a_0, b_0]$ is found out in which only one root α(say) of $f(x)=0$ lies

Let $x = x_0 \; (a_0 \le x_0 < b_0)$ is an approximation of the root α of the equation $f(x)=0$. Let h be a small correction on x_0, then $x_1 = x_0+h$ is the correct root.

$$\therefore \; f(x_1) = 0 \Rightarrow f(x_0+h) = 0$$

Therefore by Taylor series expansion, we get

$$f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

As h is small, neglecting the second and higher power of h, we get

$$h = -\frac{f(x_0)}{f'(x_0)} \quad \text{—— (1)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{—— (2)}$$

Further, if h_1 be the correction on x_1, then $x_2 = x_1 + h_1$ is the correct root of $f(x) = 0$

$$\therefore f(x_2) = f(x_1 + h_1) = 0$$

Thus $f(x_1) + h_1 f'(x_1) + \frac{h_1^2}{2!} f''(x_1) + \cdots = 0$

Neglecting the second and higher power of h_1, we get

$$h_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$\therefore x_2 = x_1 + h_1 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{—— (3)}$$

Proceeding in this way, we get the $(n+1)$th corrected root as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{—— (4)}$$

The formula (4) generates a sequence of successive corrections on an approximate root x_0 to get the correct root α of $f(x)=0$, provided the ~~sq~~ sequence is convergent.

The formula (4) is known as the iteration formula for Newton-Raphson Method.

Geometrical Interpretation of Newton-Raphson Method.

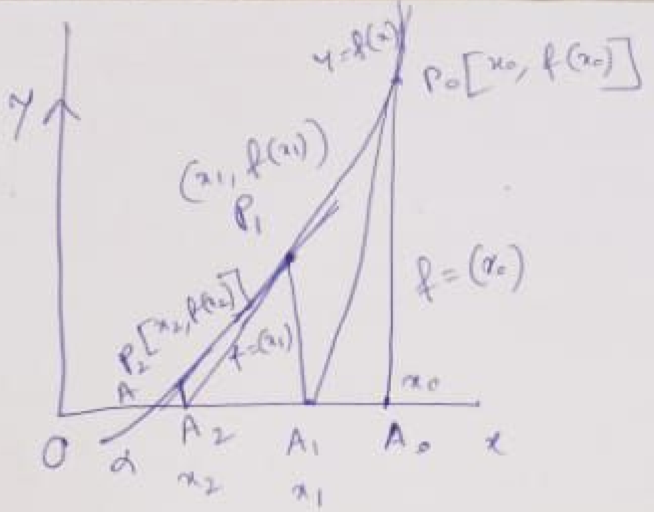
The curve $y=f(x)$ w.r.to. ox and oy as axes. Let the tangent at $P_0[x_0, f(x_0)]$ meet the x-axis, A_1, where $OA_1 = x_1$ and the tangent at $P_1[x_1, f(x_1)]$ meet the x-axis at A_2, where $OA_2 = x_2$, etc.

Thus

$$P_0A_0 = A_1A_0 \tan \angle P_0A_1A_0 = A_1A_0 f'(x_0)$$

$$\text{or, } f(x_0) = (x_0 - x_1) f'(x_0)$$

$$\text{or, } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



Thus it is clear that the successive approximations of the root, i.e. $x_1, x_2, x_3, \ldots$ $\ldots x_{n+1}$ are obtained respectively by the points at which the tangents at $x_0, x_1, x_2, \ldots$ $\ldots x_n$ to the curve $y = f(x)$ meet the x-axis.

Remark Newton-Raphson Method fails when, $f'(x) = 0$ or very small in the neighbourhood of the root.

Remark If the initial approximation is very close to the root, then the convergence in Newton-Raphson Method is faster than the iteration Method.

Remark The initial approximation must be taken very close to the root, otherwise, the ~~iate~~ iterations may diverge.

Solve

Find the root of $x^3 - 8x - 4 = 0$, which between 3 and 4, by Newton-Raphson method, correct to four decimal places.

Let $f(x) = x^3 - 8x - 4$

$f(3) = -1 < 0$ and $f(4) = 23 > 0$

Thus $f(x) = 0$, has a root between 3 and 4

$\therefore f'(x) = 3x^2 - 8 \quad f'(3) = 19$

We take $x_0 = 3$ and the successive approximations are computed in the table as follows.

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	3	-1.0	19.0	0.05	3.05
1	3.05	-0.027	19.9075	0.0014	3.0514
2	3.0514	0.000513	19.9332	-0.0000257	3.051374
3	3.051374	-0.000005	19.932649	0.00000025	3.0513742

Thus 3.0514 is the root of the given equ^n, correct up to four decimal places

Solve Find a positive root of $x^3+2x-2=0$ by Newton-Raphson Method correct to two significant figures.

Solve

$$f(x) = x^3+2x-2$$

$$f(0) = -2, \quad f(0.5) = -0.75$$

$$f(0.7) = -0.11, \quad f(0.8) = 0.24$$

Here $f'(x) = 2x+2$ and $f'(0.7) = 3.4$

Thus $f(x)=0$ has a root between 0.7 and 0.8.

We take $x_0 = 0.7$

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	0.7	-0.11	3.4	0.03	0.73
1	0.73	-0.0071	3.46	0.00205	0.73205
2	0.73205	-0.0000028	3.4641	0.0000008	0.732051

$\therefore$ 0.73 is the root of the equ^n correct to two significant figures

Solve Find a positive root of $x + \ln x - 2 = 0$, by Newton-Raphson Method correct to six significant figures

Sol^n Let $f(x) = x + \ln x - 2$

$$\therefore f(1) = -1, \; f(1.5) = -0.09$$

$$f(2) = 0.69$$

$\therefore f(x) = 0$ has a root between 1.5 and 2.0

Now $f'(x) = 1 + \frac{1}{x}$ and $f'(1.5) = 1.67$

Taking $x_0 = 1.5$, the successive approximations are computed in the table as follows

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	1.5	−0.09	1.62	0.054	1.554
1	1.554	−0.00517	1.6435	0.003146	1.557146
2	1.557146	0.0000145	1.6422	−0.0000004	1.5571453
3	1.5571453	−0.0000005	1.6422	0.0000003	1.5571455
4	1.5571455	0.000000	1.6422	0.0000000	1.5571455

$\therefore$ 1.55714 is the root of the eqn, correct to ~~si~~ six significant figures.

__Solve__ Find by Newton-Raphson Method the root of $3x - \cos x - 1 = 0$

Let $f(x) = 3x - \cos x - 1$

$\therefore f(0) = -2,\ f(0.5) = -0.37$

$f(0.7) = 0.34,$

Thus, one real root of $f(x) = 0$ between 0.5 and 0.7.

Now $f'(x) = 3 + \sin x,\ f'(0.5) = 3.48$

Taking $x_0 = 0.5$, the successive approximation of the root are computed in the following table.

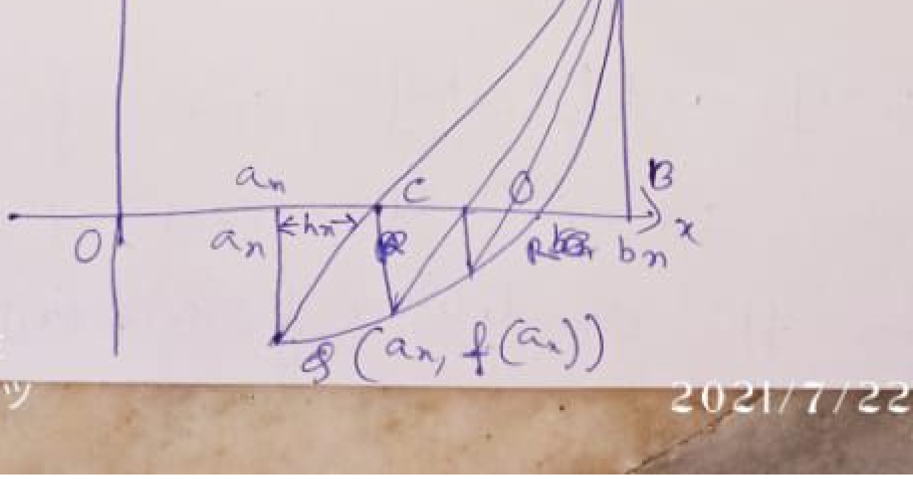
n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	0.5	−0.37	3.48	0.1063	0.6063
1	0.6063	−0.00286	3.56983	0.000801	0.607101
2	0.607101	−0.00000231	3.570489	0.00000064	0.6071016
3	0.6071016	−0.00000017	3.570489	0.00000003	0.6071019

∴ 0.60710 is the root of $f(x) = 0$ correct up to five decimal places

Regular Falsi Method or Method of False Position

In this method we first find a sufficiently small interval $[a_0, b_0]$ such that $f(a_0) f(b_0) < 0$.

This method is based on the assumption that the graph $y = f(x)$ in small interval $[a_0, b_0]$ can be represented by ~~the method joining~~ chord joining $(a_0, f(a_0))$ and $(b_0, f(b_0))$. Therefore, the point $x = a_1 = a_0 + h_0$ at which the chord meets the x-axis gives us an approximate value of the root α of the equation $f(x) = 0$. Thus, we obtain two intervals $[a_0, x_1]$ and $[x_1, b_0]$ one of which must contain the root α, depending upon the conditions $f(a_0) f(x_1) < 0$ or or $f(a_1)f(b_0)<0$. If $f(x_1)f(b_0)<0$, then α lies in the interval $[x_1, b_0]$ which we rename as $[a_1, b_1]$. Again we consider that the graph of $y=f(x)$ in $[a_1, b_1]$ as the chord joining $(a_1, f(a_1))$ and $(b_1, f(b_1))$, then the point of intersection of the chord with the x-axis (say) $x_2 = a_1 + h_1$ gives us an approximate value of the root α and x_2 is called the second approximation of the root α. Proceeding in this way shall get a sequence of $\{x_1, x_2, \ldots x_n\}$ each member of which is the successive approximation of an exact root α of the equation $f(x)=0$.



Solve: Compute the root of the equation $2x - \log_{10}x - 7 = 0$, by Regular Falsi method, which between 3 and 4. correct to three ~~significant~~ decimal places.

Soln: Let $f(x) = 2x - \log_{10}x - 7$

Here $f(3) = -1.48$, $f(4) = 0.40$

Therefore one root of $f(x) = 0$ between 3 and 4. Now we compute the successive approximations of the root as under:

n	$a_n(-)$	$b_n(+)$	$f(a_n)$	$f(b_n)$	h_n^*	x_{n+1}^{**}	$f(x_{n+1})$
0	3.0	4.0	-1.48	0.40	0.79	3.79	
1	3.0	3.79	-1.48	0.0014	0.789	3.789	
2	3.789	3.79	-0.00052	0.0014	0.000271	3.789271	
3	3.789271	3.79	-0.0000014	0.0014	0.0000007	3.7892717	

$f(x_{n+1})$

$0.0014 > 0$

$-0.00052 < 0$

$-0.0000014 < 0$

$-0.0000012 < 0$

(4)

$$* \; h_n = \frac{|f(a_n)|\,(b_n - a_n)}{|f(a_n)| + |f(b_n)|}$$

$$** \; x_{n+1} = a_n + h_n$$

$$= a_n + \frac{|f(a_n)|\,(b_n - a_n)}{|f(a_n)| + |f(b_n)|}$$

Solve

Find a root of $f(x) = 0$ correct to ~~three~~ four decimal places where $f(x) = 3x - \cos x - 1$

Sol^n $f(x) = 3x - \cos x - 1$

$f(0) = -1$, $f(1) = 1.46$

n	a_n	b_n	$f(a_n)$	$f(b_n)$	h_n^{*}	x_{n+1}^{**}	$f(x_{n+1})$
0	0.0	1.0	−1	1.46	0.41	0.41	$-0.67 < 0$
1	0.41	1.0	−0.67	1.46	0.18	0.59	$-0.061 < 0$
2	0.59	1.0	−0.061	1.46	0.0164	0.6064	$-0.0025 < 0$
3	0.6064	1.0	−0.0025	1.46	0.00067	0.60707	$-0.000113 < 0$
4	0.60707	1.0	−0.000113	1.46	0.0000304	0.6071004	$-0.0000045 < 0$
5	0.6071004	1.0	−0.0000045	1.46	0.0000012	0.6071016	$-0.00000017 < 0$

$$* \; h_n = \frac{|f(a_n)|\,(b_n - a_n)}{|f(a_n)| + |f(b_n)|}, \quad ** \; x_{n+1} = a_n + h_n$$

Solve Using Regula-Falsi Method find a root of $x^3+2x-2=0$ correct upto three significant figures

Solve

$$\text{Let } f(x) = x^3+2x-2$$

$$\text{Here } f(0) = -2,\ f(1) = 1$$

n	a_n	b_n	$f(a_n)$	$f(b_n)$	h_n^*	x_{n+1}^{**}	$f(x_{n+1})$
0	0.0	1.0	−2.0	1.0	0.67	0.67	$-0.36<0$
1	0.67	1.0	−0.36	1.0	0.087	0.757	$-0.052<0$
2	0.757	1.0	−0.052	1.0	0.012	0.769	$-0.00724<0$
3	0.769	1.0	−0.00724	1.0	0.00106	0.77066	$-0.00097<0$
4	0.77066	1.0	−0.00097	1.0	0.000222	0.770882	$-0.00013<0$

$$* \quad h_n = \frac{|f(a_n)|\,(b_n-a_n)}{|f(a_n)|+|f(b_n)|} \qquad ** \quad x_{n+1} = a_n+h_n$$

Solve Compute a root of $x\ln x=1$, by Regula-Falsi Method, correct to three decimal places.

An: Let $f(x) = x\ln x-1$. Here

$$f(1) = -1,\ f(2) = 0.39$$

n	$a_n(-)$	$b_n(+)$	$f(a_n)$	$f(b_n)$	h_n^*	x_{n+1}^{**}	$f(x_{n+1})$
0	1.0	2.0	−1.0	0.39	0.72	1.72	−0.06760
1	1.72	2.0	−0.067	0.39	0.0411	1.7611	−0.0033360
2	1.7611	2.0	−0.00333	0.39	0.002022	1.763122	−0.0001586
3	1.763122	2.0	−0.000158	0.39	0.000096	1.763218	−0.00007560

$$* \; h_n = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|}$$

$$** \; x_{n+1} = a_n + h_n$$

Q. Find a root of the equation $\sin x + \cos x = 1$, by Regular-Falsé method correct to four significant figures

Solⁿ $f(x) = \sin x + \cos x - 1$

Here $f(0) = 0$

$f(0.5) = 0.36$

$f(1.0) = 0.38$

$f(1.5) = 0.07$

$f(2.0) = -0.51$

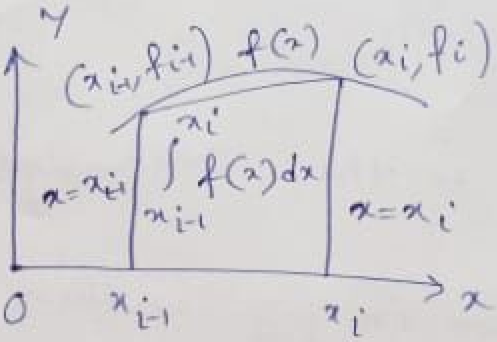
Thus, $f(x) = 0$ has a root between 1.5 and 2.0.

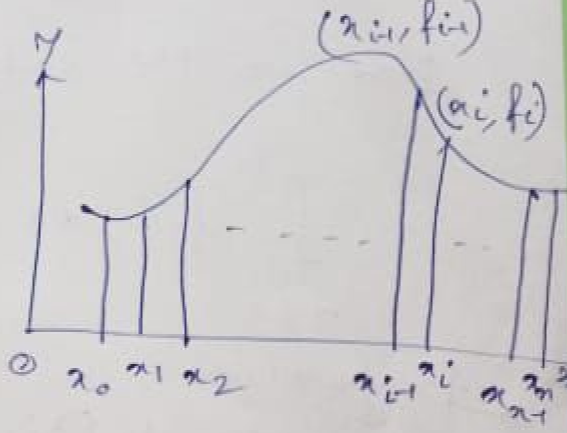
Trapezoidal Rule for integration

Trapezoidal Rule ($n=1$)

For the sub-interval $[x_0, x_1]$, we get

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1)$$





The Trapezoidal formula for the integration

$\int_{x_0}^{x_n} f(x)\,dx$ for the given curve $y=f(x)$ in $[x_0, x_n]$ which contains $[x_{i-1}, x_i]$ for $i=1,2,\dots n$ can be obtained.

$$\int_{x_0}^{x_n} f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$\simeq \sum_{i=1}^{n} \left[\frac{1}{2}(f_{i-1} - f_i) \times (x_{i-1} - x_i) \right]$$

$$= \frac{h}{2}\left[f_0 + 2\sum_{i=1}^{n-1} f_i + f_n \right]$$ [Assuming $h = x_{i-1} - x_i$]

Geometrical interpretation of Trapezoidal Rule (2)

The curve $y = f(x)$ is approximated by a set of $(n+1)$ points (x_i, f_i) for $i = 0, 1, 2, \dots n$ as shown in figure.

Now in the integration

$$\int_{x_{i-1}}^{x_i} f(x)\,dx$$ is the area of the trapezium as shown in fig. within the sub-intervals $[x_{i-1}, x_i]$ i.e. bounded by the line segments joining the pts (x_{i-1}, f_{i-1}) and (x_i, f_i), $x = x_{i-1}$, $x = x_i$, and the x-axis, (i.e. $y = 0$)

The area of this trapezium

$= \frac{1}{2} \times$ (sum of parallel sides) $\times$ (Distance between them)

$$= \frac{1}{2}(f_{i-1} + f_i) \times (x_{i-1} - x_i)$$

Therefore
$$\int_{x_0}^{x_n} f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx$$
$$= \sum_{i=1}^{n} \left[\frac{1}{2}(f_{i-1} + f_i) \times (x_{i-1} - x_i)\right]$$

Simpson $\frac{1}{3}$rd Rule $(n=2)$

For the sub-interval $[x_0, x_2]$ we get

$$\int_{x_0}^{x_2} f(x)\,dx \simeq h\int_0^2 \left(f_0 + \Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0\right)du$$

$$= h\left(2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0\right)$$

$$= h\left(2f_0 + 2(f_1 - f_0) + \frac{1}{3}(f_2 - 2f_1 + f_0)\right)$$

$$= \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)$$

Similarly, for the sub-interval $[x_{i-2}, x_i]$ we get

$$\int_{x_{i-2}}^{x_i} f(x)\,dx \simeq \frac{h}{3}\left(f_{i-2} + 4f_{i-1} + f_i\right)$$

Now, the Simpson $\frac{1}{3}$rd formula for the integration $\int_{x_0}^{x_n} f(x)\,dx$ for the given curve $y = f(x)$ in $[x_0, x_n]$ which contains $[x_{i-2}, x_{i-1}]$ and $[x_{i-1}, x_i]$ for $i = 2, \ldots N$ can be obtained.

Therefore we can write

$$\int_{x_0}^{x_n} f(x)\,dx = \sum_{i=1}^{n/2} \int_{x_{2i-2}}^{x_{2i}} f(x)\,dx$$

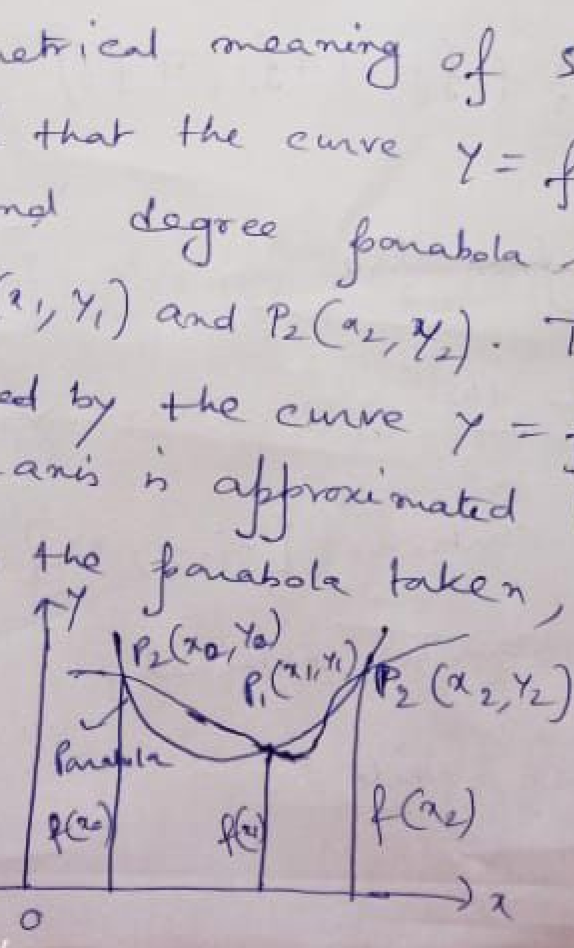
$$\approx \sum_{i=1}^{n/2} \left[\frac{h}{3}\left(f_{2i-2} + 4f_{2i-1} + f_{2i}\right)\right]$$

$$= \frac{h}{3}\left[f_0 + 4\sum_{i=1}^{n/2} f_{2i-1} + 2\sum_{i=1}^{n/2-1} f_{2i} + f_n\right]$$

[Assuming $h = x_{i-1} - x_i$, $i = 1, 2, \ldots, n$]

Geometrical Interpretation of Simpson's One-third Rule

The geometrical meaning of Simpson's One-third Rule is that the curve $y = f(x)$ is replaced by the second degree parabola through $P_0(x_0, y_0)$, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Therefore the area bounded by the curve $y = f(x)$, $x = x_0$, $x = x_2$ and the x-axis is approximated to the area bounded by the parabola taken, $x = x_0$, $x = x_2$ and x-axis.



(5)

Q. Evaluate $\int_0^1 (4x-3x^2)dx$ by taking 10 intervals by (i) Trapezoidal Rule (ii) Simpson's one third Rule Compute the exact value and find the absolute error and relative error.

Ans: Here $f(x) = 4x-3x^2$, $a=0$, $b=1$, $n=10$,

$\therefore h = \frac{1-0}{10} = 0.1$

x_i	$y_i = f(x_i)$	y_i	y_i	y_i
$i=0(1)10$	$i=0(1)10$	$(i=0,10)$	$(i=1,3,5,7,9)$	$(i=2,4,6,8)$
$x_0=0.0$	$y_0=0.0$	0.00	---	--
$x_1=0.1$	$y_1=0.37$	---	0.37	---
$x_2=0.2$	$y_2=0.68$		---	0.68
$x_3=0.3$	$y_3=0.93$		0.93	---
$x_4=0.4$	$y_4=1.12$		---	1.12
$x_5=0.5$	$y_5=1.25$		1.25	----
$x_6=0.6$	$y_6=1.32$		---	1.32
$x_7=0.7$	$y_7=1.33$		1.33	----
$x_8=0.8$	$y_8=1.28$		---	1.28
$x_9=0.9$	$y_9=1.17$		1.17	---
$x_{10}=1.0$	$y_{10}=1.00$	1.00	---	---

$\Sigma y_i = 1.00$, $\Sigma y_i = 5.05$ $\Sigma y_i = 4.40$

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(i) Now the Trapezoidal Rule is

$$I_T^c = \frac{h}{2}\left[y_0 + y_{10} + 2(y_1 + y_2 + y_3 \cdots y_n)\right]$$

$$= \frac{h}{2}\left[1.00 + 2\times(5.05 + 4.40)\right]$$

$$= 0.995$$

The exact value $= \int_0^1 (4x - 3x^2)\,dx$

$$= 2x^2 - x^3 \Big|_0^1$$

$$= 1.0000$$

$\therefore$ Absolute error = Exact value − Approx value

$$= 1.0000 - 0.995 = \cancel{0.005}$$

$$= \cancel{0.0005}\ 0.005$$

Relative error $= \dfrac{\text{Absolute error}}{\text{Exact value}} = \dfrac{0.005}{1.000} = 0.005$

(ii) Simpson's One-third Rule is

$$I_S = \frac{h}{3}\left[y_0 + y_{10} + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8 + \cancel{y_{10}})\right]$$

$$= \frac{0.1}{3}\left[1.00 + 4\times 5.05 + 2\times 4.40\right]$$

$$= \frac{0.1}{3}\times 30 = 1.000 \text{ (approx)}$$

Absolute error = Exact value − Appox value

$= 1 - 1 = 0$

Relative error $= \dfrac{\text{Absolute error}}{\text{Exact value}} = \dfrac{0}{1} = 0$

Ex Calculate the value of $\int_0^1 \frac{x}{1+x}\,dx$ correct up to three significant figures, taking six intervals by (i) Simpson's one-third Rule (ii) Trapezoidal Rule.

Ans: Here $f(x) = \dfrac{x}{1+x}$

x_i $i=0$ to 6	$y_i = f(x_i)$ $i=0$ to 6	y_i $i=(0,6)$	y_i $(i=1,3,5)$	y_i $(i=2,4)$
$x_0 = 0$	0.00000	0.0000	— —	— —
$x_1 = \frac{1}{6}$	0.14286	— —	0.14286	— —
$x_2 = \frac{2}{6}$	0.25000	——	— — —	0.25000
$x_3 = \frac{3}{6}$	0.33333	——	0.33333	— — — —
$x_4 = \frac{4}{6}$	0.40000	——	— — —	0.40000
$x_5 = \frac{5}{6}$	0.45454	——	0.45454	— — —
$x_6 = \frac{6}{6}$	0.50000	0.5000	— — — —	— — —

(i) Simpson's One-third Rule is

$$I_S = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{1}{18}\left[0.50000 + 4\times 0.93073 + 2\times 0.65000\right]$$

$$= 0.30683$$

$$I_S^C = 0.307$$

(ii) The Trapezoidal Rule is

$$I_T^C = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[0.50000 + 2(0.93073 + 0.65000)\right]$$

$$= 0.30512$$

$$I_T = 0.305$$

<u>Ex</u> Calculate the value of $\int_{1.2}^{1.6}\left(x+\frac{1}{x}\right)dx$, correct upto two significant figures, taking four intervals by (i) Simpson's One-third Rule (ii) Trapezoidal Rule

Ans: Here $f(x) = x + \frac{1}{x}$

$a = 1.2,\ b = 1.6,\ n = 4,\ h = \frac{b-a}{n} = 0.1$

x_i ($i = 0$ to 4)	$f(x_i)$ ($i = 0$ to 4)	y_i ($i = 0, 4$)	y_i ($i = 1, 3$)	y_i $i = 2$
$x_0 = 1.2$	2.0333	2.0333	---	--
$x_1 = 1.3$	2.0692	----	2.0692	---
$x_2 = 1.4$	2.1143	----	---	2.1143
$x_3 = 1.5$	2.1667	----	2.1667	---
$x_4 = 1.6$	2.2250	2.2250	----	---

i) Simpson's one-third Rule is

$$I_S = \frac{h}{3}\left[(y_0 + y_4) + 2(y_1 + y_3) + 2y_2\right]$$

$$= \frac{0.1}{3}\left[4.2583 + 4 \times 4.2359 + 2 \times 2.1143\right]$$

$$= 0.84768$$

$$= 0.85 \text{ (Approx)}$$

(ii) Trapezoidal Rule

$$I_T = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.1}{2}\left[4.2583 + 2\times(4.2359 + 2.1143)\right]$$

$$= 0.84794$$

$$I_T^{\,C} = 0.85$$