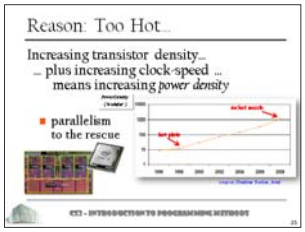
CS 2

Introduction to Programming Methods

Last Time

Concurrency and Parallelism
- synchronization of multiple threads



Today's Lecture

Numerics: when good math goes wrong

Computer programmers are fallible
- losing satellite due to public vs. private var.
  - self.setValue(.) bypassed; worked at the time...
  - but code changed, so value overwrote other data
- losing a Mars orbiter due to conversion
  - we have switched to the metric system, right?

Computers are perfect with numbers...
- right?

Integer Paradise

Dealing with bounded ints is great
- exact computations easy
  - as long as integers below a prescribed value (often, 32 bits)
    - 2 bits: 0 to 3
    - 8 bits: 0 to 255
    - n bits: 0 to 2^n-1
  - signed int
    - most commonly: **two's complement**
    - why?
      - » add/sub painless (bit overflow ignored)

$-x \equiv \bar{x}+1$

$v = -d_{31}2^{31} + \sum_{n=0}^{30} d_n 2^n$

Most significant bit

0	1	1	1	1	1	1	1	=	127
0	1	1	1	1	1	1	0	=	126
0	0	0	0	0	0	1	0	=	2
0	0	0	0	0	0	0	1	=	1
0	0	0	0	0	0	0	0	=	0
1	1	1	1	1	1	1	1	=	-1
1	1	1	1	1	1	1	0	=	-2
1	0	0	0	0	0	0	1	=	-127
1	0	0	0	0	0	0	0	=	-128

What About Reals?

Not so simple...

```
Python 2.5.1
Type "help", "copyright", "credits" or "license" for more information.
>>> 0.1
0.10000000000000001
>>>
```

- what's wrong?? [by the way, newer versions say: 0.1]

Tenths not very easy to represent in binary
- 0.1 = 0b0.0001100110011001100110011001100110011001100110011001100110011010....

So, computations often not perfect
- and we can send people to the moon??

```
>>> sum = 0.0
>>> for i in range(10):
...     sum += 0.1
...
>>> sum
0.9999999999999999
```


Fixed-Point Representation

Approximate positive reals w/ powers of 2
- first, decide how many bits to use
  - 32 bits or 64 bits quite common
- then, decide where to put the "binary point"
  - for instance, point before last bit → quantum of 0.5

Signed reals?

$b_i\ b_{i-1}\ \cdots\ b_2\ b_1\ b_0\ .\ b_{-1}\ b_{-2}\ b_{-3}\ \cdots\ b_{-j}$

$\sum_{k=-j}^{i} b_k \cdot 2^k$

- same two's complement idea
  - e.g., 32 bits for numbers between -1 and $1-2^{-31}$

$v = -d_0 2^0 + \sum_{n=1}^{32} d_n 2^{-n}$

Adaptive Representation?

Fixed-point is fast and simple
- but limited!
  - fixed accuracy, quite limited range
  - in fact, trade precision for range

Large range and high precision?
- would require lots of bits
- potentially wasteful

Floating-Point Representation

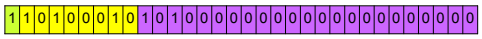
IEEE 754 definition

Variants:
- single precision: 8 `exp` bits, 23 `mant.` bits
  - 32 bits total
- double precision: 11 `exp` bits, 52 `mant.` bits
  - 64 bits total
- extended precision: 15 `exp` bits, 63 `mant.` bits
  - mostly in Intel-compatible machines; stored in 80 bits
    - 1 bit wasted

Single Precision FP

For 32 bits

- represents a value of $(-1)^s\ (1.m)_2\ 2^{e-127}$
  - $1 \le (1.m)_2 < 2$ $0 < e < 255$
- example: 0xD1500000



  - value = $-1.625\ 2^{35} = -5.5834e+10$

note: hexadecimal notation

Special Numbers

s	e	m	value
0	all zeros	all zeros	0
1	all zeros	all zeros	−0
0	all ones	all zeros	∞
1	all ones	all zeros	−∞
0 or 1	all ones	non-zero	NaN
0 or 1	all zeros	non-zero	$(-1)^s\ (0.m)_2\ 2^{-126}$

gradual underflow

Check: max = `3.402823466e+38`

min = `1.401298464e-45`

Examples: 0×∞, `sqrt(-1)`,…

Double-Precision FP

Same exact principles
- represents value of $(-1)^s(1.m)_2\ 2^{e-1023}$

max = `1.7976931348623157e+308`

min = `5e-324`

Floating-Point Operations

Conceptually
- First, compute exact result
- Make it fit into desired precision
  - Possibly overflow if exponent too large
  - Possibly round to fit into `mantissa`

Example: $(-1)^{s1}\ m_1\ 2^{e1} \times (-1)^{s2}\ m_2\ 2^{e2}$
- exact result: $(-1)^s\ m\ 2^e$
  - *s = s1+s2, m = m1.m2, e = e1+e2*
  - *m*≥2? shift *m* right, increment *e*; *e* out of range? overflow
  - round *m* to fit `mantissa` precision

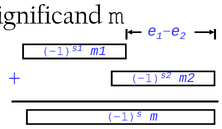
FP Addition

Operands: $(-1)^{s_1} m_1 2^{e_1}$ and $(-1)^{s_2} m_2 2^{e_2}$

- Assume $e_1 > e_2$

exact result: $(-1)^{s}\ m\ 2^{e}$

- exponent $e=e_1$; sign s; significand m
  - result of signed align & add



Normalization:

- If $m \geq 2$, shift m right, increment e
- if $m < 1$, shift m left k positions, decrement e by k
- overflow if e out of range, round m to fit precision

Math vs. Numerics

Properties of Addition

- commutative? YES
- associative? NO
  - overflow and rounding
- 0 is additive identity? YES
- always additive inverse ALMOST
  - except for ±infinity & NaNs
- $a \geq b \Rightarrow a+c \geq b+c$? ALMOST
  - except for ±infinity & NaNs

[Numerics in C]

C provides two levels

- `float` single precision
- `double` double precision

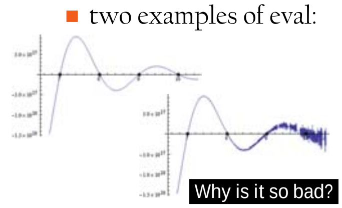
Casting between `int`, `float`, & `double`

- `double` or `float` to (64 bit) `int`
  - truncates fractional part (rounding toward zero)
- `int` to `double`
  - exact conversion
- `int` to `float`
  - depends on rounding mode…

More Numerical Catastrophes…

Understanding FP helps with accuracy…

- p(.) polynomial having x=1, 2,…, 25 as roots
  - unique one of degree 25 $p(x)=\prod_{i=1}^{25}(x-i)$
- two examples of eval:

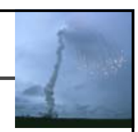


p(x) = -15511210043330985984000000 + 59190128811701203599360000 x - 10048017154835116154880000 x^2 + 1023395306017446756725760000 x^3 - 70874145319837672677196800 x^4 + 3577035564590760682636262 4 x^5 - 1374646821796792697868000 0 x^6 + 4144457803247115877036800 x^7 - 1001369304512841374110000 x^8 + 196928100451110820242880 x^9 - 31882014375298512782500 x^{10} + 4284218746244111474800 x^{11} - 480544558742733545125 x^{12} + 45145946926994481865 x^{13} - 3557372853474553750 x^{14} + 234961569422786050 x^{15} - 12972753318542875 x^{16} + 595667304367135 x^{17} - 22563937825000 x^{18} + 696829576300 x^{19} - 17247104875 x^{20} + 333685495 x^{21} - 4858750 x^{22} + 50050 x^{23} - 325 x^{24} + x^{25}

Numerics Can Ruin a Game Too

Depth a…

How Numerics Can Kill



1996: Ariane 5 explodes

- problem quickly identified as numeric
  - but this part worked just fine on Ariane 4!
- Error was due to "*a data conversion from **64-bit floating point** to **16-bit signed integer** value. The floating point number which was converted had a value greater than what could be represented by a 16-bit signed integer.*"
  - Ariane 5 was… faster
- $7.5B in development and launch lost