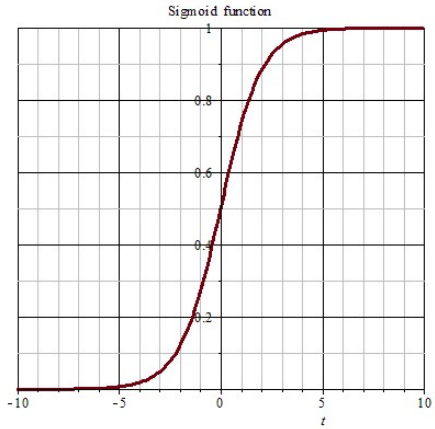
Gompertz function

The **Gompertz curve** or **Gompertz function,** is a type of mathematical model for a time series and is named after Benjamin Gompertz (1779-1865). It is a sigmoid function which describes growth as being slowest at the start and end of a given time period. The right-hand or future value asymptote of the function is approached much more gradually by the curve than the left-hand or lower valued asymptote. This is in contrast to the simple logistic function in which both asymptotes are approached by the curve symmetrically. It is a special case of the generalised logistic function. The function was originally designed to describe human mortality, but since has been modified to be applied in biology, with regards to detailing populations.

Contents



The sigmoid function serves as the basis of the Gompertz function, in which initial growth is rapid followed by a levelling-off.

History

Benjamin Gompertz originally designed the function to detail his law of human mortality for the Royal Society in 1825. The law rests upon an *a priori* assumption that a person's resistance to death decreases as his age increases. The model can be written in this way:

$$N(t) = N(0)\exp(-c(\exp(at) - 1))$$

where:

- *N(0)* is the initial number of cells/organisms when time is zero
- *a* is an asymptote
- *b* and *c* are positive numbers
- *b* denotes the displacement across the x-axis
- *c* denotes the rate of growth
- *exp* is the exponential function.

N(*t*) represents the number of individuals in the given time period, *t*. The letters *c* and *a* are constants. This model is a modification of a demographic model of Robert Malthus. It was commonly used by insurance companies to calculate the cost of life insurance. This equation is known as a Gompertz function.

Formula

$$f(t) = ae^{-be^{-ct}},$$

where

- *a* is an asymptote, since $\lim_{t\to\infty} ae^{-be^{-ct}} = ae^0 = a$
- *b, c* are positive numbers
- *b* sets the displacement along the *x*-axis (translates the graph to the left or right)
- *c* sets the growth rate (*y* scaling)
- *e* is Euler's Number (*e* = 2.71828...)

Properties

The halfway point is found by solving $f(t) = a/2$ for t.

- $t_{hwp} = -\frac{\ln(\ln(2)/b)}{c}$

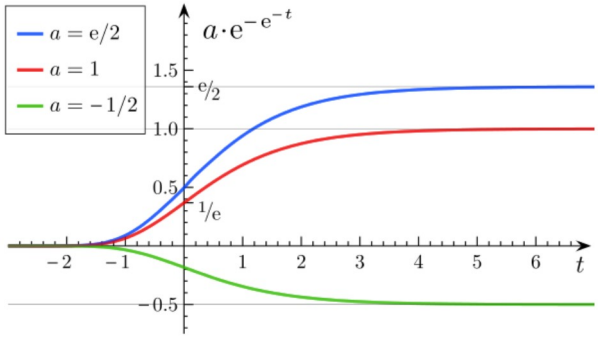
The point of maximum rate of increase is found by solving $\frac{d^2}{dt^2} f(t) = 0$ for t.

- $t_{max} = \ln(b)/c$

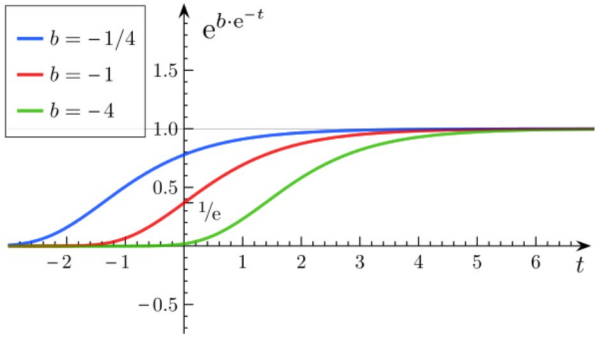
The increase at t_{max} is

- $\max\left(\frac{d}{df}\right) = \frac{ac}{e}$

Graphs of Gompertz curves, showing the effect of varying one of a,b,c while keeping the others constant



Varying a



Varying b

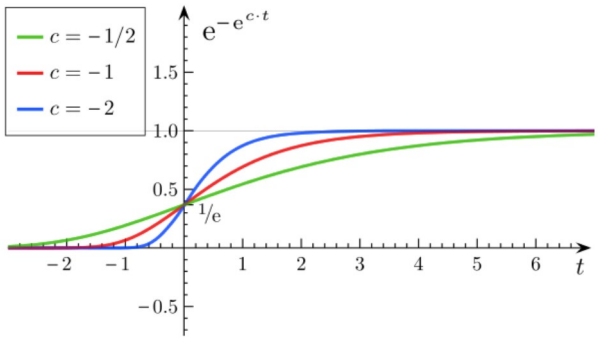
Derivation

The function curve can be derived from a Gompertz law of mortality, which states the rate of absolute mortality (decay) falls exponentially with current size. Mathematically,

$$k^r \propto \frac{1}{y(t)},$$

where

- $r = \frac{y'(t)}{y(t)}$ is the rate of growth
- k is an arbitrary constant.



Varying c

Example uses

Examples of uses for Gompertz curves include:

- Mobile phone uptake, where costs were initially high (so uptake was slow), followed by a period of rapid growth, followed by a slowing of uptake as saturation was reached.[1]
- Population in a confined space, as birth rates first increase and then slow as resource limits are reached.[2]
- Modelling of growth of tumors (http://cancerres.aacrjournals.org/content/70/1/46)
- Modelling market impact in finance.[3]
- Detailing population growth in animals of prey, with regards to predator-prey relationships.
- Modelling bacterial cells within a population
- Examining disease spread

Applications

Gompertz Curve

Population biology is especially concerned with the Gompertz Function. This function is especially useful in describing the rapid growth of a certain population of organisms while also being able to account for the eventual horizontal asymptote, once the carrying capacity is determined (plateau cell/population number).

It is modeled as follows:

$$N(t) = N_0(\exp \ln(N_I/N_0)(1 - \exp(-bt))$$

where:

- t is time
- N_0 is the initial amount of cells
- N_I is the plateau cell/population number
- b is the initial rate of tumor growth

This function consideration of the plateau cell number makes it useful in accurately mimicking real-life population dynamics. The function also adheres to the sigmoid function, which is the most widely accepted convention of generally detailing a population's growth. Moreover, the function makes use of initial growth rate, which is commonly seen in populations of bacterial and cancer cells, which undergo the log phase and grow rapidly in numbers. Despite its popularity, the function initial rate of tumor growth is difficult to predetermine given the varying microcosms present with a patient, or varying environmental factors in the case of population biology. In cancer patients, factors such as age, diet, ethnicity, genetic predispositions, metabolism, lifestyle and origin of metastasis play a role in determining the tumor growth rate. The carrying capacity is also expected to change based on these factors, and so describing such phenomena is difficult.

Metabolic Curve

The metabolic function is particularly concerned with accounting for the rate of metabolism within an organism. This function can be applied to monitor tumor cells; metabolic rate is dynamic and is greatly flexible, making it more precise in detailing cancer growth.The metabolic curve takes in to consideration the energy the body provides in maintaining and creating tissue. This energy can be considered as metabolism and follows a specific pattern in cellular division. Energy conservation can be used to model such growth, irrespective of differing masses and development times. All taxons (a group of one or more populations of an organism) share a similar growth pattern and this model, as a result, considers cellular division, the foundation of the development of a tumor.

$$B = \sum_{C} (N_C B_C)(E_C \frac{\mathrm{d}N_C}{\mathrm{d}t})$$

- B = energy organism uses at rest
- N_C = number of cells in the given organism
- B_C= metabolic rate of an individual cell
- $N_C B_C$= energy required to maintain the existing tissue
- E_C= energy required to create new tissue from an individual cell

The differentiation between energy used at rest and metabolic rate work allows for the model to more precisely determine the rate of growth. The energy at rest is lower than the energy used to maintain a tissue, and together represent the energy required to maintain the existing tissue. The use of these two factors, alongside the energy required to create new tissue, comprehensively map the rate of growth, and moreover, lead in to an accurate representation of the lag phase.

Growth of tumors

In the 1960s A.K. Laird[4] for the first time successfully used the Gompertz curve to fit data of growth of tumors. In fact, tumors are cellular populations growing in a confined space where the availability of nutrients is limited. Denoting the tumor size as X(t) it is useful to write the Gompertz Curve as follows:

$$X(t) = K \exp\left(\log\left(\frac{X(0)}{K}\right)\exp(-\alpha t)\right)$$

where:

- X(0) is the tumor size at the starting observation time;
- K is the carrying capacity, i.e. the maximum size that can be reached with the available nutrients. In fact it is:

$$\lim_{t\to+\infty} X(t) = K$$

independently on X(0)>0. Note that, in absence of therapies etc.. usually it is X(0)<K, whereas, in presence of therapies, it may be X(0)>K;

- α is a constant related to the proliferative ability of the cells.
- log() refers to the natural log.

It is easy to verify that the dynamics of X(t) is governed by the Gompertz differential equation:

$$X'(t) = \alpha \log\left(\frac{K}{X(t)}\right) X(t)$$

i.e. is of the form when broken down:

$$X'(t) = F\left(X(t)\right) X(t), \quad \text{with} \quad F'(X) \leq 0,$$

F(X) is the instantaneous proliferation rate of the cellular population, whose decreasing nature is due to the competition for the nutrients due to the increase of the cellular population, similarly to the logistic growth rate. However, there is a fundamental difference: in the logistic case the proliferation rate for small cellular population is finite:

$$F(X) = \alpha\left(1 - \left(\frac{X}{K}\right)^{\nu}\right) \Rightarrow F(0) = \alpha < +\infty$$

whereas in the Gompertz case the proliferation rate is unbounded:

$$\lim_{X\to 0^+} F(X) = \lim_{X\to 0^+} \alpha \log\left(\frac{K}{X}\right) = +\infty$$

As noticed by Steel[5] and by Wheldon,[6] the proliferation rate of the cellular population is ultimately bounded by the cell division time. Thus, this might be an evidence that the Gompertz equation is not good to model the growth of small tumors. Moreover, more recently it has been noticed[7] that, including the interaction with immune system, Gompertz and other laws characterized by unbounded F(0) would preclude the possibility of immune surveillance.

Gompertz growth and logistic growth

The Gompertz differential equation

$$X'(t) = \alpha \log\left(\frac{K}{X(t)}\right) X(t)$$

is the limiting case of the generalized logistic differential equation

$$X'(t) = \alpha\nu \left(1 - \left(\frac{X(t)}{K}\right)^{\frac{1}{\nu}}\right) X(t)$$

(where $\nu > 0$ is a positive real number) since

$$\lim_{\nu \to +\infty} \nu \left(1 - x^{1/\nu}\right) = -\log(x).$$

In addition, there is an inflection point in the graph of the generalized logistic function when

$$X(t) = \left(\frac{\nu}{\nu+1}\right)^{\nu} K$$

and one in the graph of the Gompertz function when

$$X(t) = \frac{K}{e} = K \cdot \lim_{\nu \to +\infty} \left(\frac{\nu}{\nu+1}\right)^{\nu}.$$

Gomp-ex law of growth

Based on the above considerations, Wheldon[6] proposed a mathematical model of tumor growth, called the Gomp-Ex model, that slightly modifies the Gompertz law. In the Gomp-Ex model it is assumed that initially there is no competition for resources, so that the cellular population expands following the exponential law. However, there is a critical size threshold X_C such that for $X > X_C$. The assumption that there is no competition for resources holds true in most scenarios. It can however be affected by limiting factors, that requires the creation of sub-factors variables.

the growth follows the Gompertz Law:

$$F(X) = \max\left(a, \alpha \log\left(\frac{K}{X}\right)\right)$$

so that:

$$X_C = K \exp\left(-\frac{a}{\alpha}\right).$$

Here there are some numerical estimates[6] for X_C:

- $X_C \approx 10^9$ for human tumors
- $X_C \approx 10^6$ for murine (mouse) tumors

See also

- Gompertz distribution
- Growth curve
- Von Bertalanffy function
- Sigmoid function

References

1. Islam, Towhidul; Fiebig, Denzil G.; Meade, Nigel (2002), "Modelling multinational telecommunications demand with limited data", *International Journal of Forecasting*, **18** (4): 605–624, doi:10.1016/S0169-2070(02)00073-0 (https://doi.org/10.1016%2FS0169-2070%2802%2900073-0).
2. Zwietering, M. H.; Jongenburger, I.; Rombout, F. M.; van 't Riet, K. (1990), "Modeling of the Bacterial Growth Curve" (http://aem.asm.org/content/56/6/1875), *Applied and Environmental Microbiology*, **56** (6): 1875–1881.
3. Caravelli, F.; Sindoni, L.; Caccioli, F.; Ududec, C. (2015), *Optimal leverage trajectories in presence of market impact*, arXiv:1510.05123 (https://arxiv.org/abs/1510.05123), Bibcode:2016PhRvE..94b2315C (http://adsabs.harvard.edu/abs/2016PhRvE..94b2315C), doi:10.1103/PhysRevE.94.022315 (https://doi.org/10.1103%2FPhysRevE.94.022315).
4. Laird A. K. (1964). "Dynamics of tumor growth". *Br J Cancer*. **18** (3): 490–502. doi:10.1038/bjc.1964.55 (https://doi.org/10.1038%2Fbjc.1964.55).
5. Steel, G.G. (1977). *Growth Kinetics of Tumors*. Oxford: Clarendon Press. ISBN 0-19-857388-X.
6. Wheldon, T.E. (1988). *Mathematical Models in Cancer Research*. Bristol: Adam Hilger. ISBN 0-85274-291-6.
7. d'Onofrio A. (2005). "A general framework for modeling tumor-immune system competition and immunotherapy: Mathematical analysis and biomedical inferences". *Physica D*. **208** (3–4): 220–235. arXiv:1309.3337 (https://arxiv.org/abs/1309.3337). Bibcode:2005PhyD..208..220D (http://adsabs.harvard.edu/abs/2005PhyD..208..220D). doi:10.1016/j.physd.2005.06.032 (https://doi.org/10.1016%2Fj.physd.2005.06.032).

External links

- Weisstein, Eric W. "Gompertz Curve" (http://mathworld.wolfram.com/GompertzCurve.html). *MathWorld*.
- https://archive.org/details/philtrans04942340
- http://chemoth.com/tumorgrowth

Retrieved from "https://en.wikipedia.org/w/index.php?title=Gompertz_function&oldid=880559021"

This page was last edited on 28 January 2019, at 02:09 (UTC).

Text is available under the Creative Commons Attribution-ShareAlike License; additional terms may apply. By using this site, you agree to the Terms of Use and Privacy Policy.
Wikipedia® is a registered trademark of the Wikimedia Foundation, Inc., a non-profit organization.