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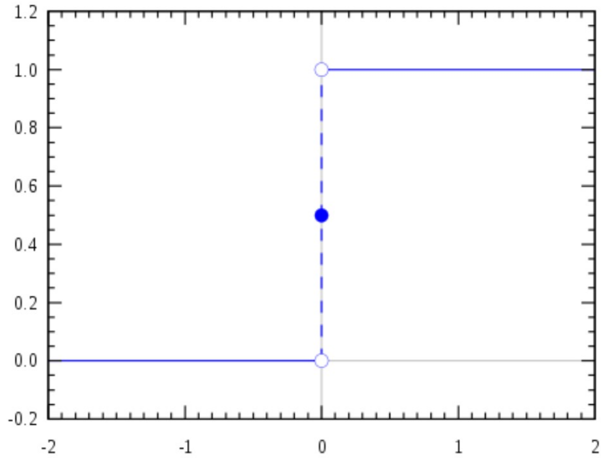
Heaviside step function

The **Heaviside step function**, or the **unit step function**, usually denoted by H or θ (but sometimes u, $\mathbf{1}$ or $\mathbb{1}$), is a discontinuous function, named after Oliver Heaviside (1850–1925), whose value is zero for negative arguments and one for positive arguments. It is an example of the general class of step functions, all of which can be represented as linear combinations of translations of this one.



The Heaviside step function, using the half-maximum convention

The function was originally developed in operational calculus for the solution of differential equations, where it represents a signal that switches on at a specified time and stays switched on indefinitely. Oliver Heaviside, who developed the operational calculus as a tool in the analysis of telegraphic communications, represented the function as $\mathbf{1}$.

The simplest definition of the Heaviside function is as the derivative of the ramp function:

$$H(x) := \frac{d}{dx} \max\{x, 0\} \quad \text{for } x \neq 0$$

The Heaviside function can also be defined as the integral of the Dirac delta function: $H' = \delta$. This is sometimes written as

$$H(x) := \int_{-\infty}^{x} \delta(s)\, ds$$

although this expansion may not hold (or even make sense) for $x = 0$, depending on which formalism one uses to give meaning to integrals involving δ. In this context, the Heaviside function is the cumulative distribution function of a random variable which is almost surely 0. (See constant random variable.)

In operational calculus, useful answers seldom depend on which value is used for $H(0)$, since H is mostly used as a distribution. However, the choice may have some important consequences in functional analysis and game theory, where more general forms of continuity are considered. Some common choices can be seen below.

Contents

Discrete form

An alternative form of the unit step, as a function of a discrete variable n:

$$H[n] = \begin{cases} 0, & n < 0, \\ 1, & n \ge 0, \end{cases}$$

or using the half-maximum convention: [1]

$$H[n] = \begin{cases} 0, & n < 0, \\ \tfrac{1}{2}, & n = 0, \\ 1, & n > 0, \end{cases}$$

where n is an integer. Unlike the usual (not discrete) case, the definition of $H[0]$ is significant.

The discrete-time unit impulse is the first difference of the discrete-time step

$$\delta[n] = H[n] - H[n-1].$$

This function is the cumulative summation of the Kronecker delta:

$$H[n] = \sum_{k=-\infty}^{n} \delta[k]$$

where

$$\delta[k] = \delta_{k,0}$$

is the discrete unit impulse function.

Analytic approximations

For a smooth approximation to the step function, one can use the logistic function

$$H(x) \approx \tfrac{1}{2} + \tfrac{1}{2}\tanh kx = \frac{1}{1+e^{-2kx}},$$

where a larger k corresponds to a sharper transition at $x = 0$. If we take $H(0) = \tfrac{1}{2}$, equality holds in the limit:

$$H(x) = \lim_{k\to\infty} \tfrac{1}{2}(1 + \tanh kx) = \lim_{k\to\infty} \frac{1}{1+e^{-2kx}}.$$

There are many other smooth, analytic approximations to the step function.[2] Among the possibilities are:

$$\begin{aligned} H(x) &= \lim_{k\to\infty} \left(\tfrac{1}{2} + \tfrac{1}{\pi}\arctan kx\right) \\ H(x) &= \lim_{k\to\infty} \left(\tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf} kx\right) \end{aligned}$$

These limits hold pointwise and in the sense of distributions. In general, however, pointwise convergence need not imply distributional convergence, and vice versa distributional convergence need not imply pointwise convergence. (However, if all members of a pointwise convergent sequence of functions are uniformly bounded by some "nice" function, then convergence holds in the sense of distributions too.)

In general, any cumulative distribution function of a continuous probability distribution that is peaked around zero and has a parameter that controls for variance can serve as an approximation, in the limit as the variance approaches zero. For example, all three of the above approximations are cumulative distribution functions of common probability distributions: The logistic, Cauchy and normal distributions, respectively.

Integral representations

Often an integral representation of the Heaviside step function is useful:

$$\begin{aligned} H(x) &= \lim_{\varepsilon \to 0^+} -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\tau + i\varepsilon} e^{-ix\tau} \, d\tau \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\tau - i\varepsilon} e^{ix\tau} \, d\tau. \end{aligned}$$

where the second representation is easy to deduce from the first, given that the step function is real and thus is its own complex conjugate.

Zero argument

Since H is usually used in integration, and the value of a function at a single point does not affect its integral, it rarely matters what particular value is chosen of $H(0)$. Indeed when H is considered as a distribution or an element of L^∞ (see L^p space) it does not even make sense to talk of a value at zero, since such objects are only defined almost everywhere. If using some analytic approximation (as in the examples above) then often whatever happens to be the relevant limit at zero is used.

There exist various reasons for choosing a particular value.

- $H(0) = \frac{1}{2}$ is often used since the graph then has rotational symmetry; put another way, $H - \frac{1}{2}$ is then an odd function. In this case the following relation with the sign function holds for all x:

  $$H(x) = \tfrac{1}{2} + \tfrac{1}{2} \operatorname{sgn}(x).$$

- $H(0) = 1$ is used when H needs to be right-continuous. For instance cumulative distribution functions are usually taken to be right continuous, as are functions integrated against in Lebesgue–Stieltjes integration. In this case H is the indicator function of a closed semi-infinite interval:

  $$H(x) = \mathbf{1}_{[0,\infty)}(x).$$

  The corresponding probability distribution is the degenerate distribution.

- $H(0) = 0$ is used when H needs to be left-continuous. In this case H is an indicator function of an open semi-infinite interval:

  $$H(x) = \mathbf{1}_{(0,\infty)}(x).$$

- In functional-analysis contexts from optimization and game theory, it is often useful to define the Heaviside function as a set-valued function to preserve the continuity of the limiting functions and ensure the existence of certain solutions. In these cases, the Heaviside function returns a whole interval of possible solutions, $H(0) = [0,1]$.

Antiderivative and derivative

The ramp function is the antiderivative of the Heaviside step function:

$$\int_{-\infty}^{x} H(\xi) \, d\xi = x H(x) = \max\{0, x\} \,.$$

The distributional derivative of the Heaviside step function is the Dirac delta function:

$$\frac{d H(x)}{dx} = \delta(x) \,.$$

Fourier transform

The Fourier transform of the Heaviside step function is a distribution. Using one choice of constants for the definition of the Fourier transform we have

$$\hat{H}(s) = \lim_{N\to\infty} \int_{-N}^{N} e^{-2\pi i x s} H(x)\,dx = \frac{1}{2}\left(\delta(s) - \frac{i}{\pi}\mathrm{p.\,v.}\frac{1}{s}\right).$$

Here $\mathrm{p.v.}\frac{1}{s}$ is the distribution that takes a test function φ to the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{\varphi(s)}{s}\,ds$. The limit appearing in the integral is also taken in the sense of (tempered) distributions.

Unilateral Laplace transform

The Laplace transform of the Heaviside step function is a meromorphic function. Using the unilateral Laplace transform we have:

$$\begin{aligned}\hat{H}(s) &= \lim_{N\to\infty} \int_{0}^{N} e^{-sx} H(x)\,dx \\ &= \lim_{N\to\infty} \int_{0}^{N} e^{-sx}\,dx \\ &= \frac{1}{s}\end{aligned}$$

When bilateral transform is used, the integral can be split in two parts and the result will be the same.

Hyperfunction representation

This can be represented as a hyperfunction as

$$H(x) = \left(\frac{1}{2\pi i}\log(z),\ \frac{1}{2\pi i}\log(z) - 1\right).$$

See also

- Rectangular function
- Step response
- Sign function
- Negative number
- Laplace transform
- Iverson bracket
- Laplacian of the indicator
- Macaulay brackets
- Sine integral
- Dirac delta function

References

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2. Weisstein, Eric W. "Heaviside Step Function" (http://mathworld.wolfram.com/HeavisideStepFunction.html). *MathWorld*.

External links

- Digital Library of Mathematical Functions, NIST, [1] (http://dlmf.nist.gov/1.16#iv).

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