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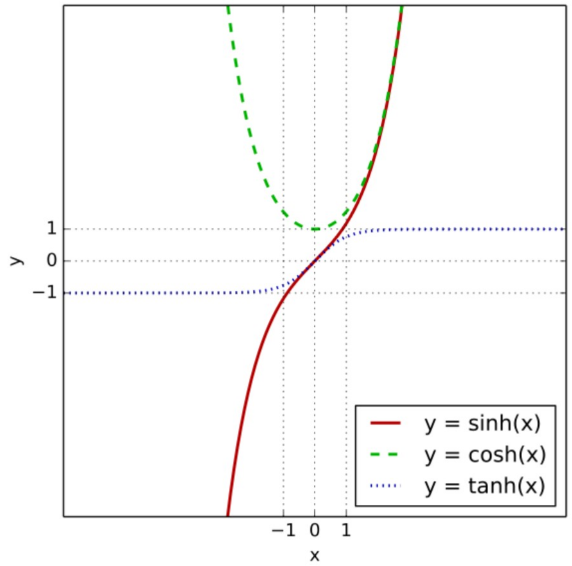
Hyperbolic function

In mathematics, **hyperbolic functions** are analogs of the ordinary trigonometric, or circular, functions.



The basic hyperbolic functions are:

- **hyperbolic sine** "sinh" (/sɪntʃ, ʃaɪn/),[1]
- **hyperbolic cosine** "cosh" (/kɒʃ, koʊʃ/),[2]

from which are derived:

- **hyperbolic tangent** "tanh" (/tæntʃ, θæn/),[3]
- **hyperbolic cosecant** "csch" or "cosech" (/ˈkoʊʃɛk/[2] or /ˈkoʊsɛtʃ/)
- **hyperbolic secant** "sech" (/ʃɛk, sɛtʃ/),[4]
- **hyperbolic cotangent** "coth" (/koʊθ, kɒθ/),[5][6]

corresponding to the derived trigonometric functions.

The inverse hyperbolic functions are:

- **area hyperbolic sine** "arsinh" (also denoted "$\sinh^{-1}$", "asinh" or sometimes "arcsinh")[7][8][9]
- and so on.

Just as the points ($\cos t$, $\sin t$) form a circle with a unit radius, the points ($\cosh t$, $\sinh t$) form the right half of the equilateral hyperbola. The hyperbolic functions take a real argument called a hyperbolic angle. The size of a hyperbolic angle is twice the area of its hyperbolic sector. The hyperbolic functions may be defined in terms of the legs of a right triangle covering this sector.

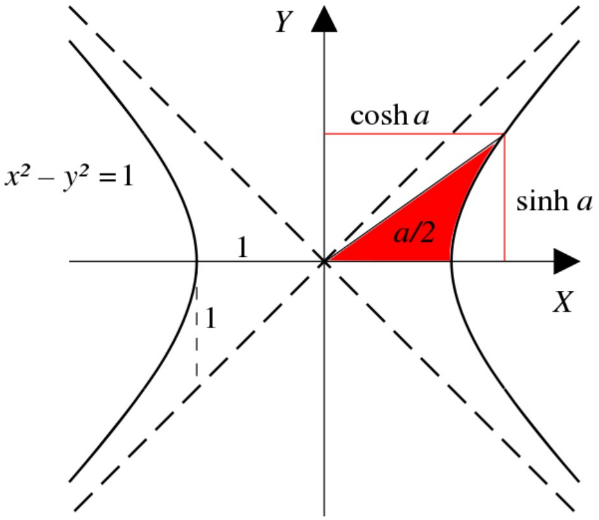
Hyperbolic functions occur in the solutions of many linear differential equations (for example, the equation defining a catenary), of some cubic equations, in calculations of angles and distances in hyperbolic geometry, and of Laplace's equation in Cartesian coordinates. Laplace's equations are important in many areas of physics, including electromagnetic theory, heat transfer, fluid dynamics, and special relativity.

In complex analysis, the hyperbolic functions arise as the imaginary parts of sine and cosine. The hyperbolic sine and the hyperbolic cosine are entire functions. As a result, the other hyperbolic functions are meromorphic in the whole complex plane.

By Lindemann–Weierstrass theorem, the hyperbolic functions have a transcendental value for every non-zero algebraic value of the argument.[10]

Hyperbolic functions were introduced in the 1760s independently by Vincenzo Riccati and Johann Heinrich Lambert.[11] Riccati used *Sc.* and *Cc.* (*sinus/cosinus circulare*) to refer to circular functions and *Sh.* and *Ch.* (*sinus/cosinus hyperbolico*) to refer to hyperbolic functions. Lambert adopted the names but altered the abbreviations to what they are today.[12] The abbreviations sh, ch, th, cth are also at disposition, their use depending more on personal preference of mathematics of influence than on the local language.

Contents



A ray through the unit hyperbola $x^2 - y^2 = 1$ in the point $(\cosh a, \sinh a)$, where a is twice the area between the ray, the hyperbola, and the x-axis. For points on the hyperbola below the x-axis, the area is considered negative (see animated version with comparison with the trigonometric (circular) functions).

Definitions

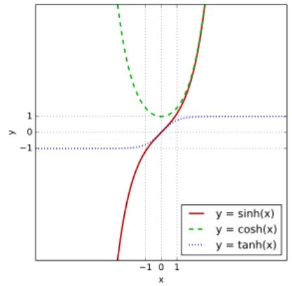
There are various equivalent ways for defining the hyperbolic functions. They may be defined in terms of the exponential function:

- Hyperbolic sine: the odd part of the exponential function, that is

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}.$$

- Hyperbolic cosine: the even part of the exponential function, that is

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}.$$



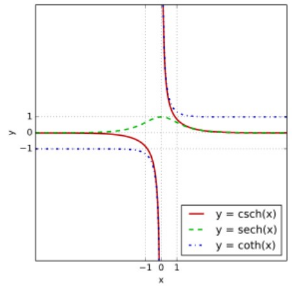
sinh, cosh and tanh

- Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

- Hyperbolic cotangent: for $x \neq 0$,

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}.$$



csch, sech and coth

- Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1}.$$

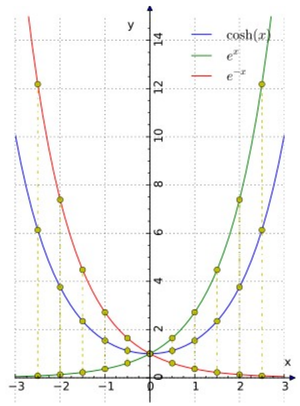
- Hyperbolic cosecant: for $x \neq 0$,

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x} - 1}.$$

The hyperbolic functions may be defined as solutions of differential equations: The hyperbolic sine and cosine are the unique solution (s, c) of the system

$$\begin{aligned} c'(x) &= s(x) \\ s'(x) &= c(x) \end{aligned}$$

such that $s(0) = 0$ and $c(0) = 1$.



(a) $\cosh x$ is the average of e^x and e^{-x}

They are also the unique solution of the equation $f''(x) = f(x)$, such that $f(0) = 1, f'(0) = 0$ for the hyperbolic cosine, and $f(0) = 0, f'(0) = 1$ for the hyperbolic sine.

Hyperbolic functions may also be deduced from trigonometric functions with complex arguments:

- Hyperbolic sine:

$$\sinh x = -i \sin(ix)$$

- Hyperbolic cosine:

$$\cosh x = \cos(ix)$$

- Hyperbolic tangent:

$$\tanh x = -i \tan(ix)$$

- Hyperbolic cotangent:

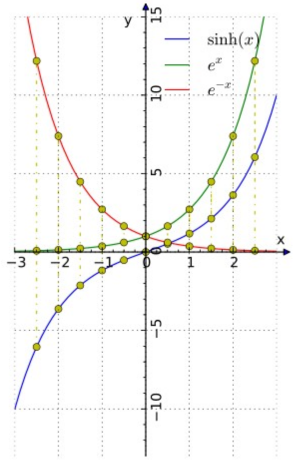
$$\coth x = i \cot(ix)$$

- Hyperbolic secant:

$$\operatorname{sech} x = \sec(ix)$$

- Hyperbolic cosecant:

$$\operatorname{csch} x = i \csc(ix)$$



(b) $\sinh x$ is half the difference of e^x and e^{-x}

where i is the imaginary unit with the property that $i^2 = -1$.

The complex forms in the definitions above derive from Euler's formula.

Characterizing properties

Hyperbolic cosine

It can be shown that the area under the curve of the hyperbolic cosine over a finite interval is always equal to the arc length corresponding to that interval:[13]

$$\text{area} = \int_a^b \cosh x \, dx = \int_a^b \sqrt{1 + \left(\frac{d}{dx} \cosh x\right)^2} \, dx = \text{arc length.}$$

Hyperbolic tangent

The hyperbolic tangent is the solution to the differential equation $f' = 1 - f^2$ with $f(0) = 0$ and the nonlinear boundary value problem:[14][15]

$$\tfrac{1}{2} f'' = f^3 - f; \quad f(0) = f'(\infty) = 0.$$

Useful relations

Odd and even functions:

$$\begin{aligned}\sinh(-x) &= -\sinh x \\ \cosh(-x) &= \cosh x\end{aligned}$$

Hence:

$$\begin{aligned}\tanh(-x) &= -\tanh x \\ \coth(-x) &= -\coth x \\ \operatorname{sech}(-x) &= \operatorname{sech} x \\ \operatorname{csch}(-x) &= -\operatorname{csch} x\end{aligned}$$

It can be seen that $\cosh x$ and $\operatorname{sech} x$ are even functions; the others are odd functions.

$$\begin{aligned}\operatorname{arsech} x &= \operatorname{arcosh}\left(\frac{1}{x}\right) \\ \operatorname{arcsch} x &= \operatorname{arsinh}\left(\frac{1}{x}\right) \\ \operatorname{arcoth} x &= \operatorname{artanh}\left(\frac{1}{x}\right)\end{aligned}$$

Hyperbolic sine and cosine satisfy:

$$\begin{aligned}\cosh x + \sinh x &= e^x \\ \cosh x - \sinh x &= e^{-x} \\ \cosh^2 x - \sinh^2 x &= 1\end{aligned}$$

the last of which is similar to the Pythagorean trigonometric identity.

One also has

$$\begin{aligned}\operatorname{sech}^2 x &= 1 - \tanh^2 x \\ \operatorname{csch}^2 x &= \coth^2 x - 1\end{aligned}$$

for the other functions.

Sums of arguments

$$\begin{aligned}\sinh(x+y) &= \sinh x\cosh y + \cosh x\sinh y\\ \cosh(x+y) &= \cosh x\cosh y + \sinh x\sinh y\\[6px] \tanh(x+y) &= \frac{\tanh x + \tanh y}{1+\tanh x\tanh y}\end{aligned}$$

particularly

$$\begin{aligned}\cosh(2x) &= \sinh^2 x + \cosh^2 x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1\\ \sinh(2x) &= 2\sinh x\cosh x\end{aligned}$$

Also:

$$\begin{aligned}\sinh x + \sinh y &= 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)\\ \cosh x + \cosh y &= 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)\end{aligned}$$

Subtraction formulas

$$\begin{aligned}\sinh(x-y) &= \sinh x\cosh y - \cosh x\sinh y\\ \cosh(x-y) &= \cosh x\cosh y - \sinh x\sinh y\end{aligned}$$

Also:[16]

$$\begin{aligned}\sinh x - \sinh y &= 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)\\ \cosh x - \cosh y &= 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)\end{aligned}$$

Half argument formulas

$$\sinh\left(\frac{x}{2}\right) = \frac{\sinh(x)}{\sqrt{2(\cosh x + 1)}} \qquad = \operatorname{sgn} x \sqrt{\frac{\cosh x - 1}{2}}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh\left(\frac{x}{2}\right) = \frac{\sinh x}{\cosh x + 1} \qquad = \operatorname{sgn} x \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} = \frac{e^x - 1}{e^x + 1}$$

where sgn is the sign function.

If $x \neq 0$, then[17]

$$\tanh\left(\frac{x}{2}\right) = \frac{\cosh x - 1}{\sinh x} = \coth x - \operatorname{csch} x$$

Inverse functions as logarithms

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$$

$$\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad |x| < 1$$

$$\operatorname{arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad |x| > 1$$

$$\operatorname{arsech}(x) = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) \qquad 0 < x \leqslant 1$$

$$\operatorname{arcsch}(x) = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) \qquad x \neq 0$$

Derivatives

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x}$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\operatorname{csch}^2 x = -\frac{1}{\sinh^2 x} \qquad x \neq 0$$

$$\frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x$$

$$\frac{d}{dx}\operatorname{csch} x = -\coth x \operatorname{csch} x \qquad x \neq 0$$

$$\frac{d}{dx}\operatorname{arsinh} x = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}} \qquad 1 < x$$

$$\frac{d}{dx}\operatorname{artanh} x = \frac{1}{1-x^2} \qquad |x| < 1$$

$$\frac{d}{dx}\operatorname{arcoth} x = \frac{1}{1-x^2} \qquad 1 < |x|$$

$$\frac{d}{dx}\operatorname{arsech} x = -\frac{1}{x\sqrt{1-x^2}} \qquad 0 < x < 1$$

$$\frac{d}{dx}\operatorname{arcsch} x = -\frac{1}{|x|\sqrt{1+x^2}} \qquad x \neq 0$$

Second derivatives

Sinh and cosh are both equal to their second derivative, that is:

$$\frac{d^2}{dx^2}\sinh x = \sinh x$$

$$\frac{d^2}{dx^2}\cosh x = \cosh x\,.$$

All functions with this property are linear combinations of sinh and cosh, in particular the exponential functions e^x and e^{-x}, and the zero function $f(x) = 0$.

Standard integrals

$$\begin{aligned}
\int \sinh(ax)\,dx &= a^{-1}\cosh(ax) + C \\
\int \cosh(ax)\,dx &= a^{-1}\sinh(ax) + C \\
\int \tanh(ax)\,dx &= a^{-1}\ln(\cosh(ax)) + C \\
\int \coth(ax)\,dx &= a^{-1}\ln(\sinh(ax)) + C \\
\int \operatorname{sech}(ax)\,dx &= a^{-1}\arctan(\sinh(ax)) + C \\
\int \operatorname{csch}(ax)\,dx &= a^{-1}\ln\left(\tanh\left(\frac{ax}{2}\right)\right) + C = a^{-1}\ln\left|\operatorname{csch}(ax) - \coth(ax)\right| + C
\end{aligned}$$

The following integrals can be proved using hyperbolic substitution:

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2+u^2}}\,du &= \operatorname{arsinh}\left(\frac{u}{a}\right) + C \\
\int \frac{1}{\sqrt{u^2-a^2}}\,du &= \operatorname{arcosh}\left(\frac{u}{a}\right) + C \\
\int \frac{1}{a^2-u^2}\,du &= a^{-1}\operatorname{artanh}\left(\frac{u}{a}\right) + C \qquad u^2 < a^2 \\
\int \frac{1}{a^2-u^2}\,du &= a^{-1}\operatorname{arcoth}\left(\frac{u}{a}\right) + C \qquad u^2 > a^2 \\
\int \frac{1}{u\sqrt{a^2-u^2}}\,du &= -a^{-1}\operatorname{arsech}\left(\frac{u}{a}\right) + C \\
\int \frac{1}{u\sqrt{a^2+u^2}}\,du &= -a^{-1}\operatorname{arcsch}\left|\frac{u}{a}\right| + C
\end{aligned}$$

where C is the constant of integration.

Taylor series expressions

It is possible to express the above functions as Taylor series:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

The function sinh x has a Taylor series expression with only odd exponents for x. Thus it is an odd function, that is, $-\sinh x = \sinh(-x)$, and $\sinh 0 = 0$.

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

The function cosh x has a Taylor series expression with only even exponents for x. Thus it is an even function, that is, symmetric with respect to the y-axis. The sum of the sinh and cosh series is the infinite series expression of the exponential function.

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!}, |x| < \frac{\pi}{2}$$

$$\coth x = x^{-1} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \cdots = x^{-1} + \sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}x^{2n-1}}{(2n)!}, 0 < |x| < \pi$$

$$\operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots = \sum_{n=0}^{\infty} \frac{E_{2n}x^{2n}}{(2n)!}, |x| < \frac{\pi}{2}$$

$$\operatorname{csch} x = x^{-1} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots = x^{-1} + \sum_{n=1}^{\infty} \frac{2(1-2^{2n-1})B_{2n}x^{2n-1}}{(2n)!}, 0 < |x| < \pi$$

where:

B_n is the nth Bernoulli number
E_n is the nth Euler number

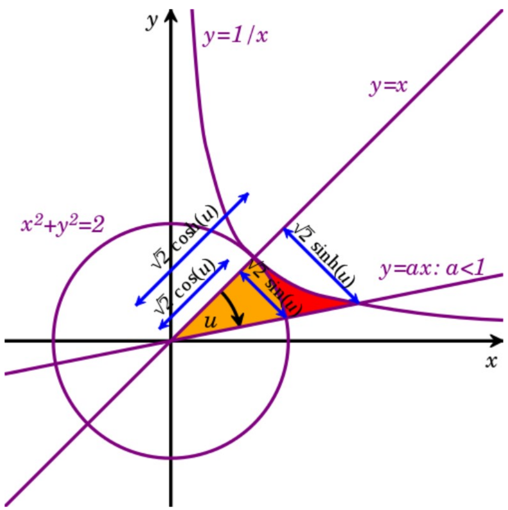
Comparison with circular functions

The hyperbolic functions represent an expansion of trigonometry beyond the circular functions. Both types depend on an argument, either circular angle or hyperbolic angle.

Since the area of a circular sector with radius r and angle u is $r^2u/2$, it will be equal to u when $r = \sqrt{2}$. In the diagram such a circle is tangent to the hyperbola $xy = 1$ at (1,1). The yellow sector depicts an area and angle magnitude. Similarly, the yellow and red sectors together depict an area and hyperbolic angle magnitude.

The legs of the two right triangles with hypotenuse on the ray defining the angles are of length $\sqrt{2}$ times the circular and hyperbolic functions.

The hyperbolic angle is an invariant measure with respect to the squeeze mapping, just as the circular angle is invariant under rotation.[18]



Circle and hyperbola tangent at (1,1) display geometry of circular functions in terms of circular sector area *u* and hyperbolic functions depending on hyperbolic sector area *u*.

Identities

The hyperbolic functions satisfy many identities, all of them similar in form to the trigonometric identities. In fact, **Osborn's rule**[19] states that one can convert any trigonometric identity into a hyperbolic identity by expanding it completely in terms of integral powers of sines and cosines, changing sine to sinh and cosine to cosh, and switching the sign of every term which contains a product of 2, 6, 10, 14, ... sinhs. This yields for example the addition theorems

$$\begin{aligned}\sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y)\\ \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y)\\ \tanh(x+y) &= \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}\end{aligned}$$

the "double argument formulas"

$$\begin{aligned}\sinh(2x) &= 2\sinh x\cosh x\\ \cosh(2x) &= \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1\\ \tanh(2x) &= \frac{2\tanh x}{1+\tanh^2 x}\\ \sinh(2x) &= \frac{2\tanh x}{1-\tanh^2 x}\\ \cosh(2x) &= \frac{1+\tanh^2 x}{1-\tanh^2 x}\end{aligned}$$

and the "half-argument formulas"[20]

$$\sinh\frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x - 1)}$$ Note: This is equivalent to its circular counterpart multiplied by −1.

$$\cosh\frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x + 1)}$$ Note: This corresponds to its circular counterpart.

$$\tanh\frac{x}{2} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} = \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x} = \coth x - \operatorname{csch} x.$$

$$\coth\frac{x}{2} = \coth x + \operatorname{csch} x.$$

The derivative of sinh *x* is cosh *x* and the derivative of cosh *x* is sinh *x*; this is similar to trigonometric functions, albeit the sign is different (i.e., the derivative of cos *x* is −sin *x*).

The Gudermannian function gives a direct relationship between the circular functions and the hyperbolic ones that does not involve complex numbers.

The graph of the function *a* cosh(*x*/*a*) is the catenary, the curve formed by a uniform flexible chain hanging freely between two fixed points under uniform gravity.

Relationship to the exponential function

The decomposition of the exponential function in its even and odd parts gives the identities

$$e^x = \cosh x + \sinh x,$$

and

$$e^{-x} = \cosh x - \sinh x.$$

The first one is analogous to Euler's formula

$$e^{ix} = \cos x + i\sin x.$$

Additionally,

$$e^x = \sqrt{\frac{1+\tanh x}{1-\tanh x}} = \frac{1+\tanh\frac{x}{2}}{1-\tanh\frac{x}{2}}$$

Hyperbolic functions for complex numbers

Since the exponential function can be defined for any complex argument, we can extend the definitions of the hyperbolic functions also to complex arguments. The functions sinh *z* and cosh *z* are then holomorphic.

Relationships to ordinary trigonometric functions are given by Euler's formula for complex numbers:

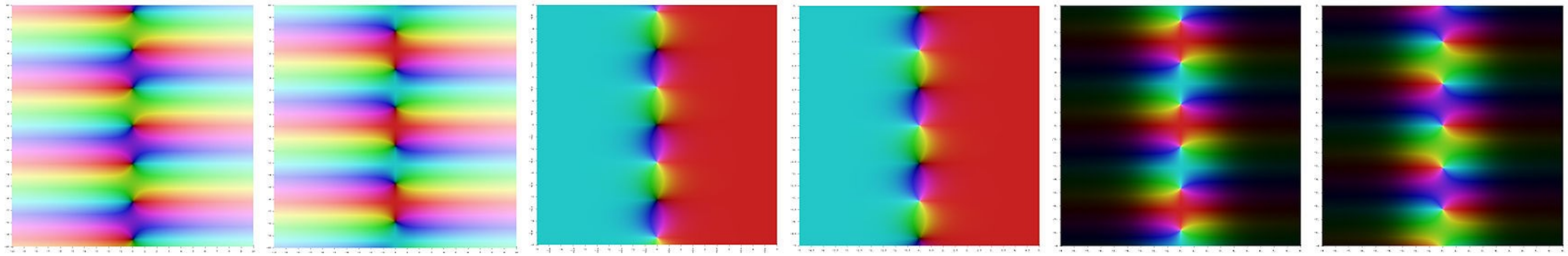
$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned}$$

so:

$$\begin{aligned} \cosh(ix) &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cos x \\ \sinh(ix) &= \frac{1}{2}\left(e^{ix} - e^{-ix}\right) = i \sin x \\ \cosh(x+iy) &= \cosh(x)\cos(y) + i \sinh(x)\sin(y) \\ \sinh(x+iy) &= \sinh(x)\cos(y) + i \cosh(x)\sin(y) \\ \tanh(ix) &= i \tan x \\ \cosh x &= \cos(ix) \\ \sinh x &= -i \sin(ix) \\ \tanh x &= -i \tan(ix) \end{aligned}$$

Thus, hyperbolic functions are periodic with respect to the imaginary component, with period $2\pi i$ (πi for hyperbolic tangent and cotangent).

Hyperbolic functions in the complex plane



$\sinh(z)$ $\cosh(z)$ $\tanh(z)$ $\coth(z)$ $\operatorname{sech}(z)$ $\operatorname{csch}(z)$

See also

- e (mathematical constant)
- Equal incircles theorem, based on sinh
- Inverse hyperbolic functions
- List of integrals of hyperbolic functions
- Poinsot's spirals
- Sigmoid function
- Trigonometric functions

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