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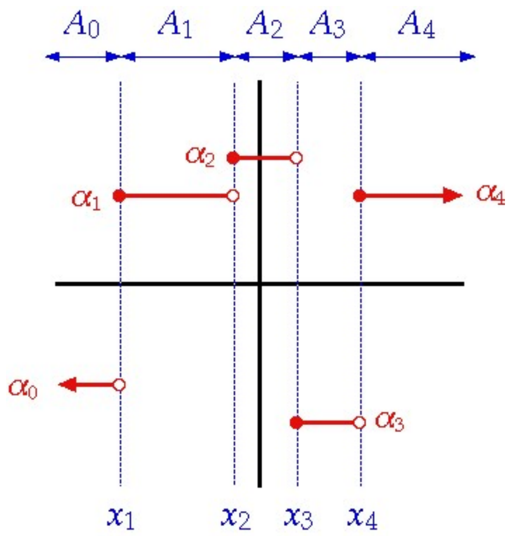
Step function

In mathematics, a function on the real numbers is called a **step function** (or **staircase function**) if it can be written as a finite linear combination of indicator functions of intervals. Informally speaking, a step function is a piecewise constant function having only finitely many pieces.



Example of a step function (the red graph). This particular step function is right-continuous.

Contents

Definition and first consequences

A function $f\colon \mathbb{R} \rightarrow \mathbb{R}$ is called a **step function** if it can be written as

$$f(x) = \sum_{i=0}^{n} \alpha_i \chi_{A_i}(x) \quad \text{for all real numbers } x$$

where $n \geq 0$ and α_i are real numbers, A_i are intervals, and χ_A is the indicator function of A:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

In this definition, the intervals A_i can be assumed to have the following two properties:

1. The intervals are pairwise disjoint: $A_i \cap A_j = \emptyset$ for $i \neq j$
2. The union of the intervals is the entire real line: $\bigcup_{i=0}^{n} A_i = \mathbb{R}$.

Indeed, if that is not the case to start with, a different set of intervals can be picked for which these assumptions hold. For example, the step function

$$f = 4\chi_{[-5,1)} + 3\chi_{(0,6)}$$

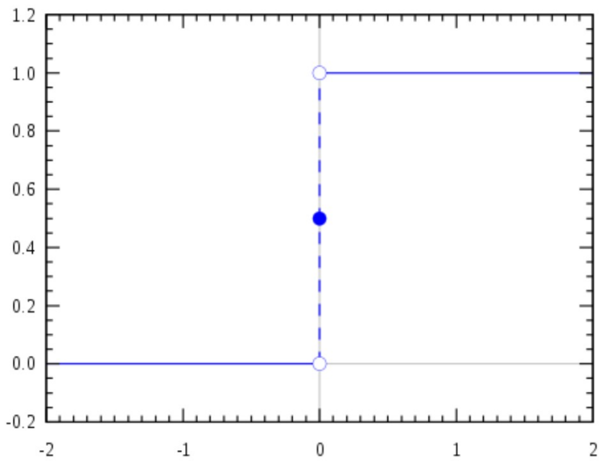
can be written as

$$f = 0\chi_{(-\infty,-5)} + 4\chi_{[-5,0]} + 7\chi_{(0,1)} + 3\chi_{[1,6)} + 0\chi_{[6,\infty)}.$$

Examples

- A constant function is a trivial example of a step function. Then there is only one interval, $A_0 = \mathbb{R}$.
- The sign function $\operatorname{sgn}(x)$, which is −1 for negative numbers and +1 for positive numbers, and is the simplest non-constant step function.
- The Heaviside function $H(x)$, which is 0 for negative numbers and 1 for positive numbers, is equivalent to the sign function, up to a shift and scale of range ($H = (\operatorname{sgn} + 1)/2$). It is the mathematical concept behind some test signals, such as those used to determine the step response of a dynamical system.

- The rectangular function, the normalized boxcar function, is used to model a unit pulse.



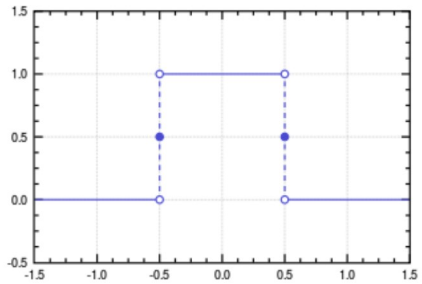
The Heaviside step function is an often-used step function.

Non-examples

- The integer part function is not a step function according to the definition of this article, since it has an infinite number of intervals. However, some authors[1] also define step functions with an infinite number of intervals.[1]

Properties

- The sum and product of two step functions is again a step function. The product of a step function with a number is also a step function. As such, the step functions form an algebra over the real numbers.
- A step function takes only a finite number of values. If the intervals A_i, for $i = 0, 1, \dots, n$ in the above definition of the step function are disjoint and their union is the real line, then $f(x) = \alpha_i$ for all $x \in A_i$.
- The definite integral of a step function is a piecewise linear function.
- The Lebesgue integral of a step function $f = \sum_{i=0}^{n} \alpha_i \chi_{A_i}$ is $\int f\,dx = \sum_{i=0}^{n} \alpha_i \ell(A_i)$, where $\ell(A)$ is the length of the interval A, and it is assumed here that all intervals A_i have finite length. In fact, this equality (viewed as a definition) can be the first step in constructing the Lebesgue integral.[2]
- A discrete random variable is defined as a random variable whose cumulative distribution function is piecewise constant.[3]



The rectangular function, the next simplest step function.

See also

- Unit step function
- Crenel function
- Simple function
- Piecewise defined function
- Sigmoid function
- Step detection

References

1. Bachman, Narici, Beckenstein. "Example 7.2.2". *Fourier and Wavelet Analysis*. Springer, New York, 2000. ISBN 0-387-98899-8.
2. Weir, Alan J. "3". *Lebesgue integration and measure*. Cambridge University Press, 1973. ISBN 0-521-09751-7.
3. Bertsekas, Dimitri P. (2002). *Introduction to Probability* (https://www.worldcat.org/oclc/51441829). Tsitsiklis, John N., Τσιτσικλής, Γιάννης N. Belmont, Mass.: Athena Scientific. ISBN 188652940X. OCLC 51441829 (https://www.worldcat.org/oclc/51441829).