Singular Value Decomposition (SVD)

Ariel Cintrón, Ph.D.

Reference

Chapter 10 Eigenvalues and Singular Values of C. Moler's *Numerical Computing with MATLAB*.

`http://www.mathworks.com/moler/chapters.html`

Zombie Math

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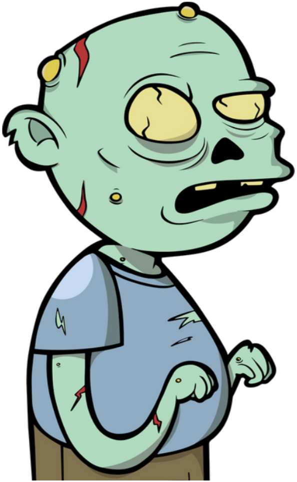

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$$A = S\Lambda S^{-1}.$$

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where A^* denotes the *Hermitian transpose* or the complex conjugate transpose of A (we have been calling this operator the adjoint). In other words, $A^* = A^H = \overline{A^T} = \overline{A}^T$.

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Therefore,

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Example

Let $A = \begin{bmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{bmatrix}$.

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- Clearly, the rank of A is one, i.e., $r = 1$.

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$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

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- Each column of E_k is a multiple of u_k, the kth column of U, and each row is a multiple of v_k^T, the transpose of the kth column of V.

Principal Component Analysis (PCA)

- The component matrices are orthogonal in the sense that for $j \neq k$
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- the result is A_r, a rank r approximation to the original matrix A.

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- PCA is used in a wide range of fields, including statistics, earth sciences, archeology, and image processing.

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$$\text{Let } A = \begin{bmatrix} 47 & 15 \\ 93 & 35 \\ 53 & 15 \\ 45 & 10 \\ 67 & 27 \\ 42 & 10 \end{bmatrix}, \text{ where } \quad U = \begin{bmatrix} -0.3153 & 0.1056 \\ -0.6349 & -0.3656 \\ -0.3516 & 0.3259 \\ -0.2929 & 0.5722 \\ -0.4611 & -0.4562 \\ -0.2748 & 0.4620 \end{bmatrix},$$

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$S = \begin{bmatrix} 156.4358 & 0 \\ 0 & 8.7658 \end{bmatrix}$, and $V = \begin{bmatrix} -0.9468 & 0.3219 \\ -0.3219 & -0.9468 \end{bmatrix}$.

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A rank one approximation to A is

$$E_1 = \sigma_1 \mathrm{u}_1 \mathrm{v}_1^T = \begin{bmatrix} 46.7021 & 15.8762 \\ 94.0315 & 31.9657 \\ 52.0806 & 17.7046 \\ 43.3857 & 14.7488 \\ 68.2871 & 23.2139 \\ 40.6964 & 13.8346 \end{bmatrix}$$

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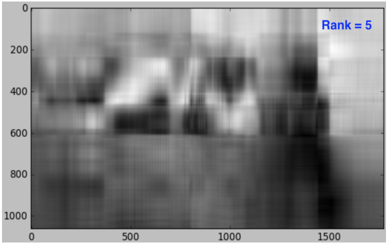

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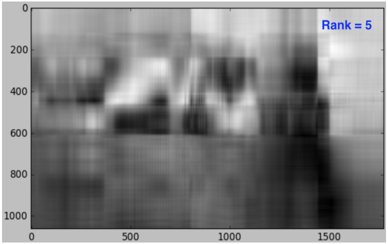

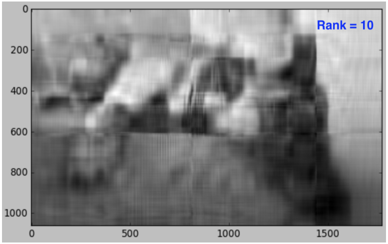

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