Asymptotics & Perturbation Methods
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Lecture notes compiled by
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Preface

This document serves as my notes of the online lectures on Asymptotics & Perturbation Methods delivered by Prof. Steven Strogatz. These are techniques for solving complicated problems approximately by exploiting the presence of a small or large parameter. These methods can be applied to integrals, ordinary differential equations (ODEs), partial differential equations (PDEs).

These notes have not been proofread by anyone other than myself and readers are advised to be wary of errors. I will periodically make improvements to this document by fixing typographical errors, adding skipped steps, new methods related to, but not covered in, this course (will go into the Appendix).

The source codes/files used to generate various plots and schematics have been made available openly through GitHub. If you wish to contribute to this document, please do so in one of the two ways:

- email the updated `.tex/.svg/.py` file(s) to arka.bokshi@gmail.com
- make changes using `Git` and open a pull request to `merge` them

Thank you and happy reading!

Lecture 1

Asymptotic Expansions

Consider an integral that depends on a large parameter x

$$F(x) = \int_0^\infty \frac{\mathrm{e}^{-xt}}{1+t}\mathrm{d}t \qquad (x > 0) \tag{1}$$

This integral – Laplace transform of $1/(1+t)$ – cannot be evaluated in closed form in terms of elementary functions. To make progress, try writing

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots$$

This series converges for $|t| < 1$. In the integral this series is being used outside its region of convergence and we are therefore likely to run into trouble. Ignoring that nonetheless, the integral yields[1]

$$\begin{aligned} F(x) &= \int_0^\infty \mathrm{e}^{-xt}\left[1 - t + t^2 - t^3 + \dots\right]\mathrm{d}t \\ &= \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \dots + \frac{(n-1)!}{x^n}\dots \end{aligned} \tag{2}$$

Using the definition of the Gamma function

$$\int_0^\infty t^n \mathrm{e}^{-xt}\mathrm{d}t = \frac{n!}{x^{n+1}}$$

Does the series $F(x)$ given by eqn. 2 converge? We perform the ratio test to ascertain if $|a_{n+1}/a_n| < 1$ as $n \to \infty$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{x^{n+1}}\frac{x^n}{(n-1)!} = \frac{n}{x} \to \infty \tag{3}$$

as $n \to \infty$ for all $x > 0$. The series clearly diverges. This is down to using the expansion for $1/(1+t)$ in a region it is not valid. But divergent series are rather

[1] No rigorous justification for interchanging the integral and summation; we are not 'formal'.

useful as we shall see (even more so than convergent series)! This is because we would not take infinitely many terms and only use the first few leading order terms. Let us take some moderately large value of x and see what the series predicts. Take $x = 10$ (as powers of 10 recognizable in decimals) and keep the first 5 terms of the series.

$$\begin{aligned} F(10) &= 0.1000 - 0.0100 + 0.0020 - 0.0006 \\ &= 0.0914 \end{aligned}$$

A simple numerical quadrature integration of the function[2] yields

$$F(10) = 0.09156$$

So why did this work so well, especially considering this was a divergent series? Consider the partial sum

$$\begin{aligned} S_n(x) &= \sum_{k=0}^{n-1} (-1)^k \frac{k!}{x^{k+1}} \\ &= \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^2} + \cdots + (-1)^{n-1} \frac{(n-1)!}{x^n} \\ R_n(x) &= F(x) - S_n(x) \end{aligned}$$

where $R_n(x)$ is the remainder/error we make by stopping at the n^{th} term. In order to do this, we deal with $F(x)$ more carefully now. Now

$$\left[1 - t + t^2 - t^3 + \cdots + (-1)^{n-1} t^{n-1}\right] + \frac{(-1)^n t^n}{1+t} = \frac{1}{1+t}$$

and therefore

$$F(x) = \int_0^\infty \mathrm{e}^{-xt} [\ldots] \mathrm{d}t + \underbrace{\int_0^\infty (-1)^n \mathrm{e}^{-xt} \frac{t^n}{1+t} \mathrm{d}t}_{R_n}$$

We are interested in evaluating

$$\begin{aligned} |R_n(x)| &= \int_0^\infty \frac{t^n \mathrm{e}^{-xt}}{1+t} \mathrm{d}t \\ &\leq \int_0^\infty t^n \mathrm{e}^{-xt} \mathrm{d}t = \frac{n!}{x^{n+1}} \end{aligned}$$

since for $t > 0$ we have $1 > 1/(1+t)$. It is interesting to note from eqn. 2 that the remainder term R_n is less than the first neglected term in the infinite series!

$$|R_n| \leq |a_{n+1}|$$

[2] Code available through repo.

In other words, $S_4(10) = 0.0914$ must lie within $a_5 = 4!/10^5$ of the true answer. So how many terms should we take for optimal accuracy (i.e. as close to the answer as possible)? Since $R_n(x)$ alters with n, exact value of $F(x)$ must lie between any two consecutive S_n even though the series diverges! To see this note

$$\begin{aligned} F(x) &= S_n + R_n \\ &= S_{n+1} + R_{n+1} \end{aligned}$$

If, say, $R_n > 0$ and $R_{n+1} < 0$, then

$$S_{n+1} > F(x) > S_n$$

Therefore, any two partial sums bound the answer. And we should stop where the partial sums are the tightest (roughly around $n \sim x$ as suggested by eqn. 3). From the numerically calculated partial sums we find (see code)

$$0.09158 > F(10) > 0.09154$$

Comments

1. $f(x) \sim g(x)$ or f is asymptotic to g as $x \to x_0$ means

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

   Here f is usually complicated and g is relatively simple. Example $\sin x \sim x$ as $x \to 0$. Equivalently

$$\sin x \sim x - \frac{x^3}{3!} \quad \text{as} \quad x \to 0$$

2. $f(x) = O(g(x))$ ("big oh") means $f(x)/g(x)$ is bounded as $x \to x_0$. Useful for orders of magnitude when we are not interested in the prefactor. Example

$$\sin x = x + O(x^3)$$

   since $(\sin x - x)/x^3$ is bounded as $x \to 0$.

3. $f(x) = o(g(x))$ "little oh" means $f(x) \ll g(x)$ as $x \to x_0$. This implies f tends to 0 faster than g does. Equivalently

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

   Example $\sin x = o(1)$ as $x \to 0$.

4. $\{\phi_j(x)\}$ is called an "asymptoic sequence" as $x \to x_0$ if
$$\phi_{j+1}(x) \ll \phi_j(x)$$
as $x \to x_0$. Example
$$\frac{1}{x} \gg \frac{1}{x^2} \gg \frac{1}{x^3} \dots \qquad \text{as} \quad x \to \infty$$

5. We finally define an "asymptotic expansion" (Poincaré 1886)
$$f(x) \sim a_1\phi_1(x) + a_2\phi_2(x) + \dots \qquad \text{as} \quad x \to x_0$$
if $\{\phi_j(x)\}$ is an asymptotic sequence and
$$f(x) - \underbrace{\sum_{j=1}^{n} a_j\phi_j(x)}_{S_n(x)} \ll \phi_n(x)$$
for each $n = 1, 2, \dots$ i.e. the remainder sum/error is much less than the last term in the sum. Equivalently
$$\begin{aligned} R_n(x) &= f(x) - S_n(x) \\ &\ll \phi_n(x) \end{aligned}$$
This also implies that
$$R_n(x) = O(\phi_{n+1}(x))$$
i.e. the error is of the order of the first neglected term.

In fact, we have shown already that
$$F(x) = \int_0^\infty \frac{\mathrm{e}^{-xt}}{1+t}\mathrm{d}t \sim \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \dots$$
as $x \to \infty$. Two things to check:

1. Here, ignoring the prefactors a_j
$$\frac{1}{x} \gg \frac{1}{x^2} \gg \frac{1}{x^3} \dots$$
is the asymptotic sequence $\{\phi_j(x)\}$.

2. Also
$$|R_n(x)| \le \frac{n!}{x^{n+1}} \ll \frac{1}{x^n} = \phi_n(x)$$
as $x \to \infty$ for any fixed n.

Lecture 2

Properties of Asymptotic Series

This lecture is about things that can go bad when working with asymptotics.

Comment 1 Important to appreciate the distinction between *convergent* and *asymptotic* series. Suppose

$$S_n(x) = \sum_{j=1}^{n} a_j \phi_j(x)$$

is an n-term approximation to a given $f(x)$. A series is convergent if

$$S_n(x) \to f(x) \qquad \text{as} \quad n \to \infty$$

for each fixed x. However, asymptotic behaviour implies

$$S_n(x) \sim f(x) \qquad \text{as} \quad x \to x_0$$

for each fixed n. The base-point is x_0 about which the expansion is taking place.

Comment 2 If $\{\phi_j(x)\}$ is an asymptotic sequence such that

$$\phi_1(x) \gg \phi_2(x) \gg \dots \qquad \text{as} \quad x \to x_0$$

then if

$$f(x) \sim a_1\phi_1(x) + a_2\phi_2(x) + \dots$$

then the coefficients $\{a_j\}$ are uniquely determined as follows:

$$f(x) \sim a_1\phi_1(x) \qquad \text{as} \quad x \to x_0$$
$$\implies a_1 = \lim_{x \to x_0} \frac{f(x)}{\phi_1(x)}$$

Likewise

$$f(x) \sim a_1\phi_1(x) + a_2\phi_2(x) \qquad \text{as} \quad x \to x_0$$
$$\implies a_2 = \lim_{x\to x_0} \frac{f(x) - a_1\phi_1(x)}{\phi_2(x)}$$

Example: Consider the basis/Gauge functions

$$\cos\epsilon \gg \sin\epsilon \gg \sin^2\epsilon \qquad \text{as} \quad \epsilon \to 0$$

and suppose we want to represent $\sqrt{9+\epsilon}$ using these basis functions. Then

$$\sqrt{9+\epsilon} = a_1\cos\epsilon + a_2\sin\epsilon + a_3\sin^2\epsilon + o(\sin^2\epsilon)$$

and

$$a_1 = \lim_{\epsilon\to 0} \frac{\sqrt{9+\epsilon}}{\cos\epsilon} = 3$$
$$a_2 = \lim_{\epsilon\to 0} \frac{\sqrt{9+\epsilon} - 3\cos\epsilon}{\sin\epsilon} = \frac{1}{6}$$
$$a_3 = \lim_{\epsilon\to 0} \frac{\sqrt{9+\epsilon} - 3\cos\epsilon - 1/6\sin\epsilon}{\sin^2\epsilon} = \frac{323}{216}$$

where we have used the L'Hôpital rule once in calculating a_2 and twice in determining a_3.

Comment 3 If we use a different asymptotic sequence $\{\psi_j(x)\}$ then we get different coefficients. So now if we consider

$$\sqrt{9+\epsilon} = a_1 + a_2\epsilon + a_3\epsilon^2 + O(\epsilon^3)$$

then

$$a_1 = \lim_{\epsilon\to 0} \frac{\sqrt{9+\epsilon}}{\epsilon^0} = 3$$
$$a_2 = \lim_{\epsilon\to 0} \frac{\sqrt{9+\epsilon} - 3}{\epsilon} = \frac{1}{6}$$
$$a_3 = \lim_{\epsilon\to 0} \frac{\sqrt{9+\epsilon} - 3 - 1/6\epsilon}{\epsilon^2} = -\frac{1}{216}$$

Comment 4 Note that every convergent Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x)$$

is asymptotic to $f(x)$ as $x \to x_0$ with basis

$$1 \gg (x-x_0) \gg (x-x_0)^2 \gg (x-x_0)^3 \ldots \qquad \text{as} \quad x \to x_0$$

Asymptoticity is a local property (in the neighbourhood of x_0).

Comment 5 Transcedentally small terms (TST) are terms which are much smaller than any non-negative power of ϵ as $\epsilon \to 0$, or of $1/x$ as $x \to \infty$. More precisely, $f(x)$ is a TST if

$$f(x) \ll \frac{1}{x^n} \qquad \text{as} \quad x \to 0$$

for any $n \geq 0$. We may also say that $f(x)$ is subdominant to $1/x^n$. TST are important to understand as they cause a very serious kind of "non-uniqueness" and other kinds of trouble. Many different functions can have the same asymptotic expansion!

Example: e^{-x} is a TST as $x \to \infty$. We need to show

$$\frac{\mathrm{e}^{-x}}{1/x^n} = \frac{x^n}{e^x} \to 0$$

as $x \to \infty$. We can apply L'Hôpital's rule n times. The numerator disappears wheras the denominator remains e^x. Alternatively, look at

$$\ln(x^n \mathrm{e}^{-x}) = n \ln x - x$$

As $x \to \infty$ keeping n fixed, this term goes to $-\infty$ (x grows much faster than $\ln x$). Which implies the term inside the ln goes to zero.

Using this fact, we can show

$$\mathrm{e}^{-x} \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \dots$$

(asymptotic power series in $1/x$, such as that derived in eqn. 2). The bizarre solution to the above becomes

$$\mathrm{e}^{-x} \sim 0 + \frac{0}{x} + \frac{0}{x^2} + \dots$$

However, this is not an invalid solution since

$$R_n(x) = \mathrm{e}^{-x} \ll \frac{1}{x^n}$$

and this qualifies as an asymptotic expansion. This is odd since this makes the statement that the e^{-x} function is asymptotically indistinguishable from the zero function. This means that if

$$f(x) \sim \sum_{j=0}^{\infty} a_j x^{-j}$$

as $x \to \infty$, then, a new function

$$g(x) = f(x) + C\mathrm{e}^{-x}$$

i.e. defined by adding any (astronomical) amount of e^{-x} to $f(x)$, would still have the same asymptotic expansion as $f(x)$. TST[1] are invisible to an asymptotic power series and if present with big pre-factors, can cause issues.

Trouble occurs because e^{-z} is not analytic at $z = \infty$ in complex plane. It is an "essential singularity".

Comment 6 Asymptotic expansions can be added, subtracted, multiplied, divided and even intergated term by term. BUT need to be wary of substitution and differentiation.

Example (sub): Suppose we have

$$f(x) = e^{x^2}$$
$$x(\epsilon) = \frac{1}{\epsilon} + \epsilon$$

and we wish to consider $\epsilon \to 0$. The exact result

$$\begin{aligned} f(x(\epsilon)) &= \exp\left[\frac{1}{\epsilon^2} + 2 + \epsilon^2\right] \\ &= e^{1/\epsilon^2} e^2 e^{\epsilon^2} \\ &= e^{1/\epsilon^2} e^2 \left(1 + \epsilon^2 + \epsilon^4/2! + \dots\right) \\ f(x(\epsilon)) &\sim e^2 e^{1/\epsilon^2} \end{aligned}$$

But what if we naively write

$$x(\epsilon) \sim \frac{1}{\epsilon} \qquad \text{as} \quad \epsilon \to 0$$

i.e. take asymptotic first and then substitute, we get

$$f(x(\epsilon)) \sim e^{1/\epsilon^2}$$

which is not correct (missing the correct pre-factor e^2).

NB In exponentials, may need to keep higher order terms – not just the leading order terms. Same is true for sin, cos, sinh, cosh (they are all exponentials in the complex plane). Refer back to the example of $\sqrt{9+\epsilon}$. If we replaced $\cos, \sin, \sin^2$ with $1, \epsilon, \epsilon^2$, we do not recover the same $\{a_j\}$ for this very reason – the trignometric functions need care.

Example (diff): Consider the example

$$f(x) = x + \sin x$$

[1] "Hyper-asymptotics" or "asymptotics beyond all orders" calculates these subdominant terms (research frontier).

as $x \to \infty$, $f(x) \sim x$ since $f(x)/x = 1 + \sin x/x \to 1$. But

$$f'(x) = 1 + \cos x$$

whereas applying asymptotics first we see $f'(x) \sim 1$ (the correct result oscillates). Trouble because $\sin z$ has an essential singularity at $z = \infty$.

Tauberian theorems allow us to differentiate asymptotic forumlae. Roughly, if $f(z)$ is analytic in some sector on the complex plane, it is alright to differentiate term by term there. Also, if $f(z)$ and $f'(z)$ have asymptotic expansions in a $\{\phi_j\}$, alright to differentiate term by term.

Lecture 3

Integration by parts

Chapter on asymptotic expansion of integrals. Many different special functions have “integral representation”. For example, the Bessel function

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin t)\,\mathrm{d}t$$

which arises when solving, say, the wave equation on a circular drumhead (circular or cylindrical symmetry). The error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^{x} \mathrm{e}^{-t^2}\mathrm{d}t$$

arises when we look at probability or flow of heat/diffusion in 1D. The Gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

generalizes the factorial (can interpolate between the integer factorials).

Often we solve linear ODEs and PDEs with transform (Laplace or Fourier) methods. These express the solution as an inverse transform, i.e. an integral. Asymptotics tell us the long-time ($t \gg 1$) or far-field ($x \gg 1$) behaviour of the solution. Useful when we cannot solve the integral in closed form.

<u>Example:</u> Find the small-x and large-x expansions for

$$I(x) = \int_x^{\infty} \mathrm{e}^{-t^4}\mathrm{d}t \qquad x > 0$$

For small-x, can we expand the integral? This may be a good idea if the dominant contribution to the integral is coming from the region near $t = 0$. This will allow us to truncate early. However the series expansion to the exponential is still convergent and more generally we can write

$$I(x) = \int_x^\infty \left[1 - t^4 + \frac{t^8}{2!} - \dots\right] \mathrm{d}t$$

However each term is divergent! Now they should all cancel term by term, to give us something finite, but that is not at all obvious using this approach. An alternative would be to split the problem into two integrals

$$\begin{aligned} I(x) &= \underbrace{\int_0^\infty}_{I_1} - \underbrace{\int_0^x}_{I_2} \\ &= (1/4)\,\Gamma\,(1/4) - \big(x - x^5/5 + x^9/18 - \dots\big) \end{aligned} \tag{4}$$

- where in order to evaluate the first integral I_1, with an exponential inside, we think in terms of the Gamma function and let $\tau = t^4$
- and the integral I_2 has been evaluated by integrating term by term

NB The above series is convergent for *all* x but *useless* for large x. Since, say for $x = 10$, the numerator grows rapidly, and we would need an enormous number of terms before the denominator builds up enough to give decent convergence.

For large-x, we will try integration by parts:

$$\begin{aligned} I(x) &= \int_x^\infty \frac{\mathrm{d}(\mathrm{e}^{-t^4})}{-4t^3} \\ &= \underbrace{\frac{1}{4x^3}\mathrm{e}^{-x^4}}_{I_0(x)} \underbrace{-\frac{3}{4}\int_x^\infty \frac{1}{t^4}\mathrm{e}^{-t^4}\,\mathrm{d}t}_{R(x)} \\ &= \frac{1}{4x^3}\mathrm{e}^{-x^4} - \frac{3}{16x^7}\mathrm{e}^{-x^4} + \dots \end{aligned} \tag{5}$$

Note[1] that the remainder term $R(x) \ll I_0(x)$ because of the extra $1/t^4$ in the integrand and the fact that $t \geq x \gg 1$. In fact, we show that

$$I(x) \sim \frac{1}{4x^3}\mathrm{e}^{-x^4}$$

by estimating the size of the remainder term.

$$\begin{aligned} |R(x)| &= \frac{3}{4}\left|\int_x^\infty \frac{1}{t^4}\mathrm{e}^{-t^4}\mathrm{d}t\right| \\ &= \frac{3}{16}\left|\int_x^\infty \frac{1}{t^7}\mathrm{d}(\mathrm{e}^{-t^4})\right| \\ &< \frac{3}{16x^7}\mathrm{e}^{-x^4} \end{aligned}$$

[1] We could have integrated by parts with the functions reversed, but that does not give us a series ordered in $1/x$.

since $t > x$ in $t \in [x, \infty)$ implying $1/t^7 < 1/x^7$. The remainder term is less than the first term ignored in the series.

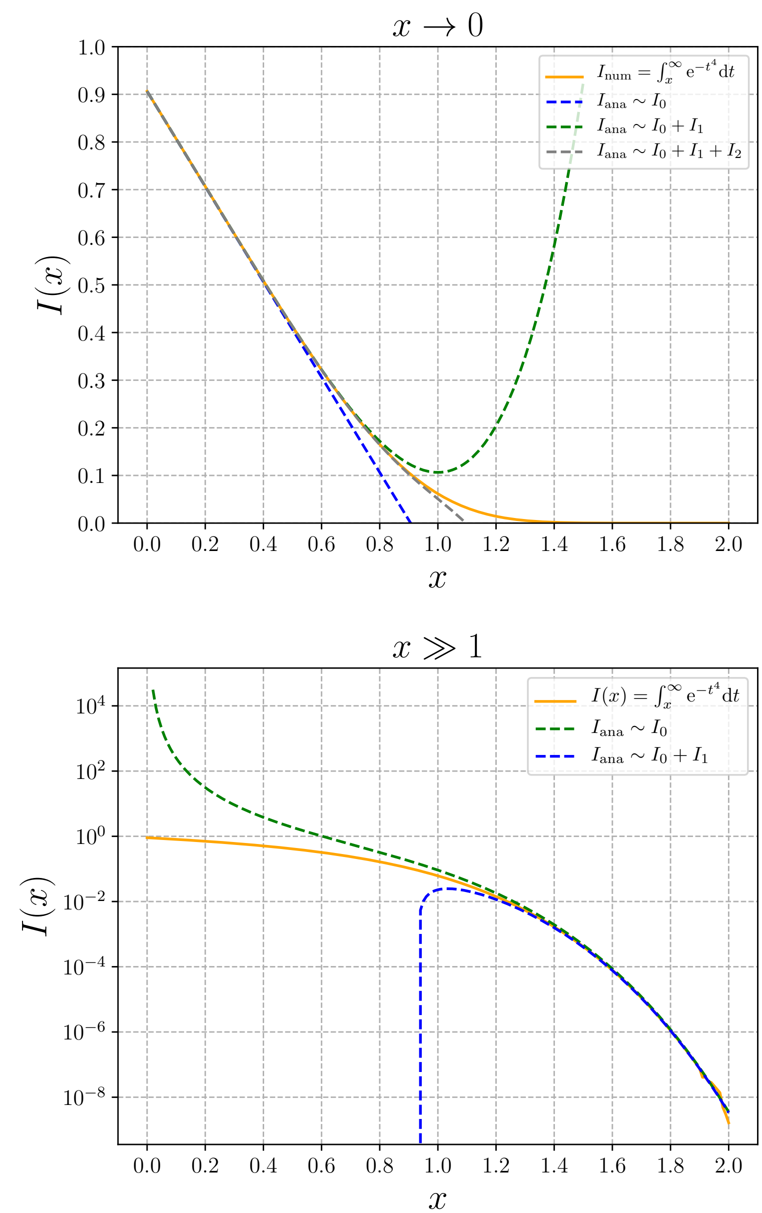

Figure 3.1: Exact and asymptotic solutions to the integral in the small x and large x limits as given by the expressions 4 and 5 respectively.

General procedure for integration by parts on a Laplace integral

$$
\begin{aligned}
I(x) &= \int_a^b f(t)\mathrm{e}^{x\phi(t)}\mathrm{d}t \qquad x \to \infty \\
&= \int_a^b \frac{f(t)}{x\phi'(t)}\mathrm{d}(\mathrm{e}^{x\phi(t)}) \\
&= \frac{1}{x}\frac{f(t)}{\phi'(t)}\mathrm{e}^{x\phi(t)}\bigg|_a^b - \frac{1}{x}\int_a^b \left[\frac{f}{\phi'}\right]' \mathrm{e}^{x\phi(t)}\mathrm{d}t
\end{aligned}
$$

We need to assume

- $\phi'(t) \neq 0$ anywhere inside the domain of integration; neither is $(f/\phi')'$
- $f(b)$ and $f(a)$ are not both zero

Further, if the second integral exists and is asymptotically smaller than the first term, then

$$
I(x) \sim \frac{1}{x}\frac{f(t)}{\phi'(t)}\mathrm{e}^{x\phi(t)}\bigg|_a^b
$$

NB This method generates *integer* powers of $1/x$. Will fail on problems where the correct asymptotics involve fractional powers of x or $\log x$.

Lecture 4

Laplace's method

We will calculate the asymptotic to the Gamma function – Sterling formula[1].

For *sharply peaked integrands* where the dominant contribution to the integral comes from the neighbourhood of a single point. As some parameter x becomes large, the "bell-shaped" integrand becomes narrower, giving a much better approximation.

Example 1: Take the integral

$$I(x) = \int_{-10}^{10} \mathrm{e}^{-xt^2} \mathrm{d}t \qquad x \to \infty$$

Note the limit which are not from $-\infty$ to ∞ which is a known integral (can be done exactly). This is a Laplace style integral discussed in the previous lecture with $f(t) = 1$ and $\phi(t) = -t^2$. Integration by parts breaks down since $\phi'(t) = 0$ in the domain of integration. Anyway, as we shall see, we get an expansion in *fractional* powers of x (not integer powers), which means the Laplace method would not have worked. Now a very crude estimate of this integral is the area under the curve which is simply the height ($= 1$) times the width (where the integrand drops to $1/e$ of its peak value)

$$I(x) \sim O\left(1/\sqrt{x}\right)$$

The scaling as we shall see is right, but we are missing a prefactor which we next determine. The region where $|t| \gg 1/\sqrt{x}$ contributes exponentially small terms (TST) which will not affect the asymptotics ("subdominant" to $I(x)$).

[1]First calculated by de Moivre, but Sterling contributed by calculating a certain pre-factor.

This allows us to add/subtract them!

$$\begin{aligned} I(x) &= \int_{-10}^{10} \mathrm{e}^{-xt^2}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-xt^2}\mathrm{d}t + \mathrm{TST} \\ &\sim \sqrt{\frac{\pi}{x}} \end{aligned}$$

To check that the remainder terms are indeed TST, observe

$$\int_{-10}^{10}() = \int_{-\infty}^{\infty}() - 2\underbrace{\int_{10}^{\infty}()}_{\text{bound ?}}$$

To find the bounds on the above integral, note that for $t > 10$, the parabola t^2 always lies above the straight line $10t$ and therefore for positive x

$$\begin{aligned} \int_{10}^{\infty} \mathrm{e}^{-xt^2}\mathrm{d}t < \int_{10}^{\infty} \mathrm{e}^{-10xt}\mathrm{d}t &= \left.\frac{\mathrm{e}^{-10xt}}{-10x}\right|_{10}^{\infty} \\ &< \frac{1}{10x}\mathrm{e}^{-100x} \ll O\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Example 2: Consider

$$I(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-x\cosh t}\mathrm{d}t$$

The function $\cosh t$ is itself like a parabola which is growing exponentially rather than quadratically. Now for small t

$$\cosh t \approx 1 + \frac{t^2}{2!} \tag{6}$$

so the integrand is like a bell-shaped curve, except now the height is order e^{-x}. So as x becomes large, the integrand becomes narrower and lower. To the leading order, it is OK to approximate $\cosh t$ using eqn. 6 in the neighbourhood of its maximum as $t \to 0$. Of course when t is large, our approximation is pretty bad, but it does not matter since that part of the integration is contributing TST anyway (for large x)!

$$\begin{aligned} I(x) &\sim \mathrm{e}^{-x}\int_{-\infty}^{\infty} \mathrm{e}^{-xt^2/2}\mathrm{d}t \\ &\sim \sqrt{\frac{2\pi}{x}}\mathrm{e}^{-x} \end{aligned}$$

But we can go higher order: eqn. 6 motivates an exact change of variable such that

$$\cosh t = 1 + \tau^2$$

Since Gaussian integrals are helpful, we make sure we get one by defining the new variable in the above manner and solve this τ integral.

$$I(x) = \sqrt{2}\mathrm{e}^{-x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+\frac{1}{2}\tau^2}} \mathrm{e}^{-x\tau^2} \mathrm{d}\tau$$

Worth noting that at the lower and upper limits, we are faced with a choice of sign. We pick $\tau = \pm\infty$ corresponding to $t = \pm\infty$ (distortion of t axis). We may also take twice the area under the curve from $t = 0$ to ∞ and that also helps us select the limits $\tau = 0$ to ∞.

Now as $x \to \infty$, we see that $\mathrm{e}^{-x\tau^2}$ gets more sharply peaked about $\tau = 0$ and we can Taylor expand the function in the denominator about this point. Note that the MacLaurin series

$$(1 + \tau^2/2)^{-1/2} = 1 - \frac{\tau^2}{4} + \frac{3}{32}\tau^4 - \dots$$

converges iff $|\tau^2/2| < 1$, i.e. $-\sqrt{2} < \tau < \sqrt{2}$. However, the integral over the domain $-\infty < \tau < \infty$ is justified since the region outside the convergence interval would only contribute TST[2] (expansion diverges but does no harm).

$$\begin{aligned} I(x) &\sim \sqrt{2}\mathrm{e}^{-x} \int_{-\infty}^{\infty} \mathrm{e}^{-x\tau^2} \left(1 - \frac{\tau^2}{4} + \frac{3}{32}\tau^4 - \dots\right) \mathrm{d}t \\ &\sim \sqrt{\frac{2\pi}{x}}\mathrm{e}^{-x} \left(1 - \frac{1}{8x} + \frac{9}{128x^2} - \dots\right) \end{aligned}$$

To evaluate the Gaussian integrals, we use the well known result

$$\int_{-\infty}^{\infty} \mathrm{e}^{-xt^2} \mathrm{d}t = \sqrt{\frac{\pi}{x}}$$

and apply repeated differentiation with respect to the parameter x to derive

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{e}^{-xt^2} t^{2n} \mathrm{d}t &= \sqrt{\pi}(-1)^n \frac{\partial}{\partial x^n} x^{-\frac{1}{2}} \\ &= \sqrt{\frac{\pi}{x}} \frac{(2n-1)!!}{(2x)^n} \end{aligned} \tag{7}$$

[2]These steps are along the lines of the "Watson Lemma".

Example 3: The Stirling formula for $\Gamma(x+1) = x!$ as $x \to \infty$

$$\Gamma(x+1) = \int_0^\infty t^x \mathrm{e}^{-t} \mathrm{d}t$$

Not obvious where this integral sharply peaks or in fact has a peak as $x \to \infty$. The *first* trick is a slight rearrangement to express the function as

$$\Gamma(x+1) = \int_0^\infty \mathrm{e}^{x \ln t} \mathrm{e}^{-t} \mathrm{d}t$$

The strategy for such Laplace-type integrals is to find where the exponential peaks and expand about the point. So we proceed to determine the point where the exponential is maximized, i.e. where $x \ln t - t$ has a maxima. Therefore the bell is centered about $t = x$, which itself is moving to infinity (movable maxima).

In such problems it is always good to pick a coordinate such that the peak is not moving. We make the substitution

$$s = \frac{t}{x}$$

which has the advantage that the maxima is now at $s = 1$ for all x. This is the *second* trick. With this change

$$\Gamma(x+1) = x^{x+1} \int_0^\infty \mathrm{e}^{x(\ln s - s)} \mathrm{d}s \tag{8}$$

The *third* trick is to expand the term $\ln s - s$ about its maximum at $s = 1$ (see Fig. 4.1). The Taylor expansion reads

$$\ln s - s = -\left[1 + \frac{(s-1)^2}{2} \dots\right] \tag{9}$$

The integral simplifies to

$$\Gamma(x+1) \sim x^{x+1} \mathrm{e}^{-x} \int_0^\infty \mathrm{e}^{-x\frac{(s-1)^2}{2}} \mathrm{d}s$$

The *fourth* trick is to add the tails that contribute TST and expand the lower limit of the integral from 0 to $-\infty$. This readily yields

$$\Gamma(x+1) \sim \sqrt{\frac{2\pi}{x}} x^{x+1} \mathrm{e}^{-x} \tag{10}$$

which for integers provides the well know identity

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \tag{11}$$

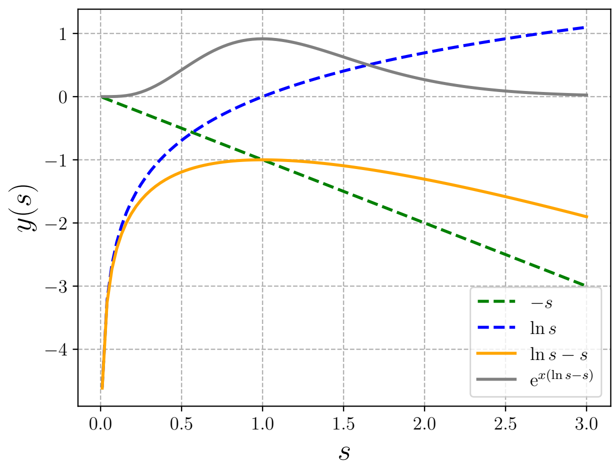

Figure 4.1: Plotting the integrand (gray) in the integral given by eqn. 8. This is scaled by a multiplicative factor for visualization. For very large values of s the linear term dominates, whereas for $s \to 0$ the logarithmic term is dominant.

The above is a one-term Stirling approximation to the Gamma integral. If we want to get higher order correction terms, we take eqn. 8 and, inspired by eqn. 9, perform the exact change of variable

$$\ln s - s = 1 + \tau^2$$

This converts the exponential term to an exact Gaussian and allows us to expand the newly introduced prefactor algebraically (cf. previous example).

[To do]

- Higher order expansion...

Lecture 5

Stationary phase

The Laplace method gave us a way of evaluating the integral when the integrand was sharply peaked (dominant contribution was coming from the region underneath the peak). The stationary phase method is used when the integrand is *rapidly oscillating*. Here the integral is dominated by the region where the rapid oscillations slow down (rapid oscillations cancel out in other regions).

Example 1: Consider

$$\begin{aligned} I(\omega) &= \int_a^b \cos \omega t \,\mathrm{d}t \\ &= \frac{1}{\omega}(\sin \omega b - \sin \omega a) \sim O\left(\frac{1}{\omega}\right) \end{aligned}$$

As $\omega \to \infty$, $I(\omega) \to 0$ because of lots of rapid cancellations. The amount that does not cancel is roughly related to the spacing – time period T – which is $O(1/\omega)$.

Example 2: Now consider a slowly varying amplitude modulation to the rapidly varying cos function

$$\begin{aligned} I(\omega) &= \int_a^b f(t) \cos \omega t \,\mathrm{d}t \\ &= \left| f(t) \frac{\sin \omega t}{\omega} \right|_a^b - \frac{1}{\omega} \int_a^b f'(t) \sin \omega t \,\mathrm{d}t \\ &= O\left(\frac{1}{\omega}\right) + o\left(\frac{1}{\omega}\right) \end{aligned}$$

Again expect $I(\omega) = O(1/\omega)$ as $\omega \to \infty$ due to oscillatory cancellations. The second term decays faster by an extra $1/\omega$ factor due to a similar reasoning.

Example 3: More generally, we can estimate

$$I(x) = \int_a^b f(t)\mathrm{e}^{\mathrm{i}x\psi(t)} \qquad x \to \infty$$

Start by trying integration by parts on the problem:

$$\begin{aligned} I(x) &= \left.\frac{f}{\mathrm{i}x\psi'}\mathrm{e}^{\mathrm{i}x\psi}\right|_a^b - \frac{1}{\mathrm{i}x}\int_a^b \mathrm{e}^{\mathrm{i}x\psi}\left(\frac{f}{\psi'}\right)' \mathrm{d}t \\ &= O\left(\frac{1}{x}\right) + o\left(\frac{1}{x}\right) \end{aligned}$$

Following caveats should be noted:

- f/ψ' should be smooth on $[a, b]$
- and $\neq 0$ at both $t = a$ and $t = b$ so that the first term may provide the dominant contribution
- and need ψ to be continuously differentiable on $[a, b]$ and not constant on any subinterval

NB But what if $\psi' = 0$ somewhere on $[a, b]$? That is the point of stationary phase and of most interest to us. $\psi(t)$ is often called the phase of $\mathrm{e}^{\mathrm{i}x\psi}$ (really, $x\psi$ is the phase, but we do not make the distinction). The major contribution to $I(x)$ comes from neighbourhood of such points. Why? Because the "oscillatory cancellations effect" weakens/disappears near points of stationary phase. To see this, expand $\psi(t)$ about some point t_0

$$\mathrm{e}^{\mathrm{i}x\psi(t)} = \mathrm{e}^{\mathrm{i}x\left[\psi(t_0)+(t-t_0)\psi'(t_0)+\frac{1}{2}(t-t_0)^2\psi''(t_0)+\ldots\right]}$$

Note that $x\psi'(t_0)$ is the local oscillation frequency at t_0. $x\psi(t_0)$ is just a constant multiplicative factor. Can show that if $\psi'(c) = 0$ but $\psi''(c) \neq 0$, then $I = O(x^{-1/2})$ as $x \to \infty$ (and not $O(1/x)$ as seen from integration by parts).

If $\psi'(c) = 0$ and $\psi''(c) = 0$, but $\psi'''(c) \neq 0$, we get $I = O(x^{-1/3})$!

Example 4: We will now look at the Bessel function $J_0(x)$ as $x \to \infty$. Bessel functions look like sin or cos waves except they decay as $x \to \infty$. The damping is algebraic and not exponential.

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin t)\,\mathrm{d}t \tag{12}$$

We will derive this integral representation later in the course (see sec. 8). Observe that

$$J_0(x) = \Re\left[\frac{1}{\pi}\int_0^\pi \mathrm{e}^{\mathrm{i}x\sin t}\mathrm{d}t\right]$$

The phase $\psi(t) = \sin t$ is stationary at $t = \pi/2$. Let us now Taylor expand ψ about $t = \pi/2$:

$$\sin t \approx 1 - \frac{1}{2}(t - \pi/2)^2$$

As $x \to \infty$

$$\begin{aligned}
\frac{1}{\pi}\int_0^{\pi} \mathrm{e}^{\mathrm{i}x\sin t}\mathrm{d}t &\sim \frac{\mathrm{e}^{\mathrm{i}x}}{\pi}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \mathrm{e}^{-\frac{\mathrm{i}x}{2}\left(t-\frac{\pi}{2}\right)^2}\mathrm{d}t \\
&\sim \sqrt{\frac{2}{x}}\frac{\mathrm{e}^{\mathrm{i}x}}{\pi}\int_{-\epsilon\sqrt{x/2}}^{\epsilon\sqrt{x/2}} \mathrm{e}^{-\mathrm{i}s^2}\mathrm{d}s \\
&\sim \sqrt{\frac{2}{x}}\frac{\mathrm{e}^{\mathrm{i}x}}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}s^2}\mathrm{d}s \\
&\sim \sqrt{\frac{2}{x}}\frac{2}{\pi}\mathrm{e}^{\mathrm{i}x}\underbrace{\int_0^{\infty} \mathrm{e}^{-\mathrm{i}s^2}\mathrm{d}s}_{\frac{\sqrt{\pi}}{2}\mathrm{e}^{-\mathrm{i}\pi/4}}
\end{aligned}$$

Here we performed the change of variable $s = \sqrt{x/2}(t - \pi/2)$ and noted that $\epsilon \ll 1$ (independent of x). Then we study the behaviour as $x \to \infty$. The integral goes by the name 'Fresnel integral'. Taking the real part we arrive at

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right) \qquad \text{as} \quad x \to \infty \tag{13}$$

Even for very low values of $x = O(1)$, the agreement is remarkable as can be seen from Fig. 5.1.

Example 5: Next consider the function

$$I(x) = \int_0^1 t\mathrm{e}^{\mathrm{i}xt^2}\mathrm{d}t \qquad x \to \infty$$

Here $\psi(t) = t^2$ has $\psi' = 0$ at $t = 0$. But $f(t) = t$ also vanishes there so we may not get the $O(x^{-1/2})$ scaling. This is a contrived example and can be done exactly. Indeed

$$\begin{aligned}
I(x) &= \frac{1}{2\mathrm{i}x}\left(\mathrm{e}^{\mathrm{i}x} - 1\right) \\
&= O\left(\frac{1}{x}\right)
\end{aligned}$$

where we have made use of the substitution $\mathrm{i}xt^2 = y$.

- What is the significance of Example 5?

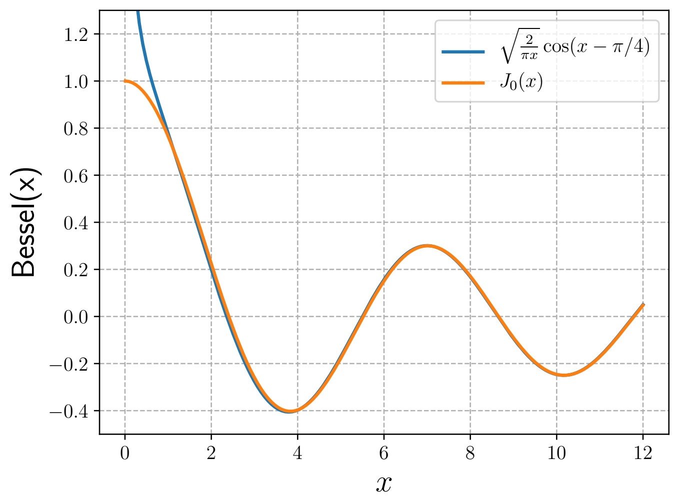

Figure 5.1: Plotting the exact Bessel integral defined by eqn. 12 and its asymptotic approximation given by eqn. 13.

Example 6: Consider

$$I(x) = \int_0^\infty \cos\left[x\left(\frac{t^3}{3} - t\right)\right] \mathrm{d}t \tag{14}$$

Begin by plotting the integrand (Fig. 5.2).
The phase is stationary when $t = \pm 1$. Since $0 \leq t \leq \infty$, we only expand about the positive root. Taylor expanding

$$\psi(t) \approx -\frac{2}{3} + (t-1)^2 \tag{15}$$

This results in

$$\begin{aligned} I(x) &\sim \int_0^\infty \cos\left[-\frac{2x}{3} + x(t-1)^2\right] \mathrm{d}t \\ &\sim \int_{-1}^\infty \cos\left[-\frac{2x}{3} + xs^2\right] \mathrm{d}s \end{aligned}$$

We can extend the limit -1 to $-\infty$ as we expect it to not affect the leading order term (affects higher order terms). Also note that we Taylor expand only about one of the stationary points in writing the relation 15. So even as we extend the domain to include $t = -1$, it won't provide a dominant contribution.

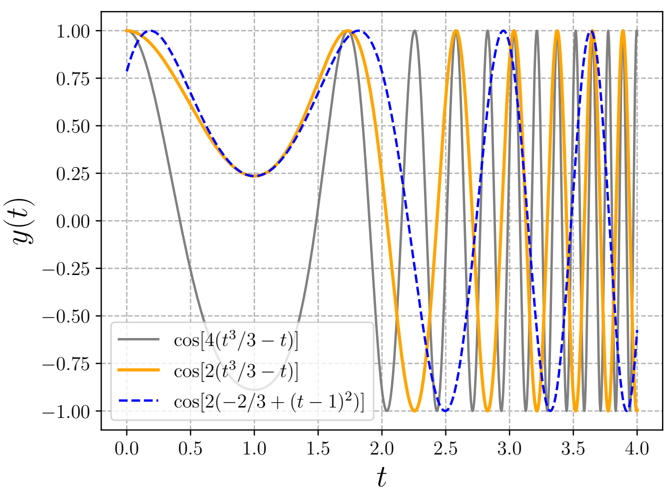

Figure 5.2: Plotting the integrand of eqn. 14 for two different values of x. As x doubles, so does the frequency. The Taylor expansion about the stationary point $t = 1$, eqn. 15, is also plotted.

Then

$$\begin{aligned} I(x) &\sim \int_{-\infty}^{\infty} \cos\left[-\frac{2x}{3} + xs^2\right] \mathrm{d}s \\ &\sim \cos\left(\frac{2x}{3}\right) \underbrace{\int_{-\infty}^{\infty} \cos(xs^2)\mathrm{d}s}_{\sqrt{\frac{\pi}{2x}}} + \sin\left(\frac{2x}{3}\right) \underbrace{\int_{-\infty}^{\infty} \sin(xs^2)\mathrm{d}s}_{\sqrt{\frac{\pi}{2x}}} \\ &\sim \sqrt{\frac{\pi}{x}} \cos\left(\frac{2x}{3} - \frac{\pi}{4}\right) \end{aligned}$$

These Fresnel integrals are easily worked out using the complex Gaussian integral (eqn. **??**) with $-\mathrm{i} \to \mathrm{i}$ and the scale transformation $\sqrt{x}s = t$.

Lecture 6

Steepest Descent

The Laplace and Stationary phase methods can be seen as the real and imaginary parts of the method of "steepest descent". Quick recap:

1. The Laplace method is used to solve integrals that look like

$$\int_a^b f(t)\mathrm{e}^{x\phi(t)}\mathrm{d}t \qquad x \to \infty$$

   and can give the whole asymptotic expansion. But needs $\phi(t)$ to be real and bell-shaped with the dominant contribution coming from the neighborhood of the peak as $x \to \infty$ (bell narrows). **NB** We can write any $\phi(t) = 1 + \tau^2$ if its maxima exists in the domain of integration. Or else the dominant contribution should come from the boundary terms.

2. The Stationary phase method deals with integrals of the form

$$\int_a^b f(t)\mathrm{e}^{\mathrm{i}x\psi(t)}\mathrm{d}t \qquad x \to \infty$$

   This only gives us the leading term plus requires pure imaginary exponent. The rapid oscillations typically cancel except near the point of stationary phase, which provides the dominant contribution. The oscillations becomes more rapid with increasing x.

With the steepest descent method we can recover higher order terms cf. stationary phase method (improves and generalizes upon both methods). Consider

$$I(x) = \int_a^b h(t)\mathrm{e}^{x\rho(t)}\mathrm{d}t \qquad x \to \infty$$

where $\rho(t) = \phi(t) + \mathrm{i}\psi(t)$ is an *analytic function* and we perform the integral in the complex t plane with $t = u + \mathrm{i}v$.

Idea We wish to deform the integral $\int_a^b$ from the real axis to a new contour C on which $\Im[\rho(t)] = \psi(t)$ is constant! This constant can be different on different pieces of the contour. Then

$$I(x) = \mathrm{e}^{\mathrm{i}x\psi} \int_a^b h(t)\mathrm{e}^{x\phi(t)}\mathrm{d}t$$

and the problem becomes amenable to Laplace's method. It is worth noting that the function $\phi(t)$ changes most rapidly along curves of constant $\psi(t)$ – hence the name! This is illustrated through an example.

Example: Consider

$$\begin{aligned} I(x) &= \int_0^1 \mathrm{e}^{\mathrm{i}xt^2}\mathrm{d}t \\ &= \int_0^1 \mathrm{e}^{\mathrm{i}x(u^2-v^2)}\mathrm{e}^{-2xuv}\mathrm{d}t \end{aligned}$$

We want to replace the current path with a new contour – connecting the two end points – such that $\psi = u^2 - v^2$ is constant on each piece of the contour. Note that

$$\psi(0,0) = 0 \qquad \psi(1,0) = 1$$

which means the contour must be broken up into at least two pieces. At the origin since $\psi = 0$, the stationary phase requires that $u = \pm v$. At $(1,0)$, we find $v = \pm\sqrt{u^2-1}$.

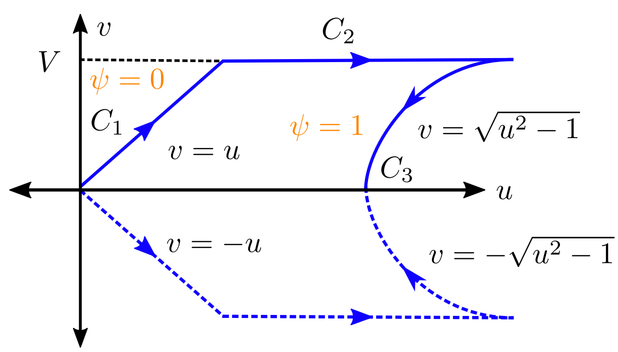

Figure 6.1

Let us see how $\phi(t)$ behaves on these pieces as we take the bridge to infinity (Fig. 6.1):

1. C_1: On $u = v$, $\psi = 0$ and $\phi = -2v^2$. As v runs from 0 to ∞, the function decreases monotonically. This is good since the maximum value occurs at (0,0). If instead we chose $u = -v$, the function would have increased as we moved towards infinity, making it unwieldy and requiring careful cancellations of the infinities.

2. C_3: On the $\psi = 1$ piece given by the hyperbola, $u = \sqrt{v^2+1}$. We see $\phi = -2v\sqrt{v^2+1}$ which peaks at $v = 0$ and decreases as we move towards infinity.

3. On the bridge given by the contour C_2, ψ is not constant. The contribution of this term is

$$\begin{aligned}\int_{C_2} \mathrm{e}^{-\mathrm{i}xt^2}\mathrm{d}t &\leq \int_{C_2} \underbrace{|\mathrm{e}^{\mathrm{i}x(u^2-v^2)}|}_{1}\, |\mathrm{e}^{-2xuv}|\mathrm{d}t \\ &= \int_{u_1}^{u_2} \mathrm{e}^{-2xuV}\mathrm{d}u \to 0\end{aligned}$$

as $V \to 0$ and $x \to \infty$.

Therefore

$$\begin{aligned}I(x) &= (1+\mathrm{i})\int_0^\infty \mathrm{e}^{-2xv^2}\mathrm{d}v + \mathrm{e}^{\mathrm{i}x}\int_\infty^0 \mathrm{e}^{-2xv\sqrt{v^2+1}}\left[\frac{v}{\sqrt{v^2+1}} + \mathrm{i}\right]\mathrm{d}v \\ &= \frac{(1+\mathrm{i})}{2}\sqrt{\frac{\pi}{2x}} + \mathrm{e}^{\mathrm{i}x}\int_\infty^0 \mathrm{e}^{-2xv(1+\dots)}(\mathrm{i}+\dots)\mathrm{d}v \\ &\sim \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\frac{\pi}{x}} - \frac{\mathrm{i}}{2x}\mathrm{e}^{\mathrm{i}x}\end{aligned}$$

where in evaluating the second integral, we note that the integrand is sharply peaked about $v = 0$ and is set up for Laplace's method. To the leading order, and most crudely, we have simply set $v = 0$.

There is a neater way to find the integral along C_3: since the contribution is coming from the infinitesimal neighborhood of the maxima at $t = 1$, we do not care about the full hyperbolic equation, instead approximate C_3 by its tangent line – the detailed global shape of C_3 is irrelevant. So we can directly write

$$t = 1 + \mathrm{i}v \qquad v \in [0, \epsilon)$$

and

$$\begin{aligned}I_3 = \int_{C_3} \mathrm{e}^{\mathrm{i}xt^2}\mathrm{d}t &= \int_\epsilon^0 \mathrm{e}^{\mathrm{i}x(1+\mathrm{i}v)^2}\mathrm{i}\mathrm{d}v \\ &= -\mathrm{i}\mathrm{e}^{\mathrm{i}x}\int_0^\infty \mathrm{e}^{-2xv}\mathrm{e}^{-\mathrm{i}xv^2}\mathrm{d}v \\ &= -\mathrm{i}\mathrm{e}^{\mathrm{i}x}\int_0^\infty \mathrm{e}^{-2xv}\left[1 - \mathrm{i}xv^2 + \dots\right]\mathrm{d}v \\ &\sim -\frac{\mathrm{i}}{2x}\mathrm{e}^{\mathrm{i}x}\left[1 - \frac{\mathrm{i}}{2x}\right]\end{aligned}$$

Lecture 7

Saddle Points

NB Notes are incomplete and need both checking and expansion. Several steps missed and some (small) sections in lecture not yet included.

Extension of the method of steepest descent. As in the previous lecture, consider integrals of the form

$$I(x) = \int_a^b h(t) \mathrm{e}^{x\rho(t)} \mathrm{d}t \qquad x \to \infty$$

where $\rho(t) = \phi(t) + \mathrm{i}\psi(t)$ is an analytic function of a complex variable $t = u + \mathrm{i}v$.

Along contours of constant ψ, points of maximum ϕ make dominant contributions to $I(x)$. This was discussed in the previous lecture with end point maximas. Here we ask the question what happens if the maximum lies in the interior of the domain.

A point t_0 where $\rho'(t_0) = 0$ is a "saddle point". Recall from complex analysis that if the function $\rho(t) = \phi(t) + \mathrm{i}\psi(t)$ is analytic, and $t = u + \mathrm{i}v$, then from Cauchy-Riemann equations

$$\frac{\partial \phi}{\partial u} = \frac{\partial \psi}{\partial v} \qquad \frac{\partial \phi}{\partial v} = -\frac{\partial \psi}{\partial u}$$

This provides two key conclusions:

1. The Laplacian $\nabla^2 \psi = \nabla^2 \phi = 0$. Therefore

$$\underbrace{\frac{\partial^2 \phi}{\partial u^2}}_{>0} + \underbrace{\frac{\partial^2 \phi}{\partial v^2}}_{<0} = 0$$

   i.e. a maxima along one axis implies a minima along the other.

2. Further

$$\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial u}+\frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial v}=\nabla\phi.\nabla\psi=0$$

Therefore $\nabla\phi$ points in the direction of ψ =const. This means that the contours of constant ψ pass through the saddle in ϕ.

Example: Consider the Bessel function with this new method.

$$J_0(x)=\frac{1}{\pi}\Re\int_0^{\pi}\mathrm{e}^{\mathrm{i}x\sin t}\mathrm{d}t$$

With the stationary phase method we got the result (eqn. 13)

$$J_0(x)\sim\sqrt{\frac{2}{\pi x}}\cos\left(x-\pi/4\right)\qquad x\to\infty$$

We can obtain higher order terms using the saddle point method. Here

$$\rho(t)=\mathrm{i}\sin t$$

We want to deform the contour such that $\Im[\rho]=\Im[\mathrm{i}\sin t]$ is constant on each piece of countour.

$$\begin{aligned}\sin t&=\sin(u+\mathrm{i}v)\\&=\sin u\cos(\mathrm{i}v)+\cos u\sin(\mathrm{i}v)\\&=\sin u\cosh v+\mathrm{i}\cos u\sinh v\end{aligned}$$

Therefore

$$\begin{aligned}\Im[\rho]&=\psi=+\sin u\cosh v\\\Re[\rho]&=\phi=-\cos u\sinh v\end{aligned}$$

and we want to move along a new contour such that ψ = const. Begin by calculating ψ at the end points:

$$\begin{aligned}\psi(0,0)&=0\\\psi(\pi,0)&=0\end{aligned}$$

Note that $\cosh v\neq 0$. Therefore we must require that $u=0$ or $u=\pi$, i.e. the constant phase lines are *vertical lines*. So starting at $t=0$, which way is descending in ϕ? Recall

$$\phi=-\cos 0\sinh v=-\sinh v$$

Therefore if we move towards positive v, we would move downhill. Likewise at the other end point $(\pi,0)$

$$\phi=-\cos\pi\sinh v=+\sinh v$$

and we needs to move towards negative v. We need a bridge that connects the two pieces. And ideally, the bridge should be a curve of constant ψ.

$$\nabla\psi = \left\langle \frac{\partial}{\partial u}(\sin u \cosh v), \frac{\partial}{\partial v}\dots \right\rangle = 0$$

$$\cos u \cosh v = 0 \qquad \& \qquad \sin u \sinh v = 0$$

$$\boxed{u = \frac{\pi}{2},\ v = 0}$$

At this point, $\psi = 1$ and this is the contour level we are interested in. We can expand about this saddle point using Taylor series:

$$u - \frac{\pi}{2} = s \ll 1 \qquad v \ll 1$$

and therefore

$$\begin{aligned}
\psi &= \sin\left(\frac{\pi}{2} + s\right)\cosh v = \cos s \cosh v \\
&= \left(1 - \frac{s^2}{2}\dots\right)\left(1 + \frac{v^2}{2}\right) \\
&= \cancel{1} + \frac{v^2}{2} - \frac{s^2}{2}\dots = \cancel{1}
\end{aligned}$$

Therefore $s = \pm v$. Refer (figure?) where we will allow $V \to \infty$. We will find that C_2 and C_4 are zero. I_1 and I_5 are pure imaginary and do not affect $J_0(x)$ whose real part is what interests us.

$$\begin{aligned}
\text{e.g.}\quad I_1 &= \frac{1}{\pi}\int_0^\infty \mathrm{e}^{\mathrm{i}x\sin(\mathrm{i}v)}\mathrm{i}\mathrm{d}v \\
&= \underbrace{\frac{\mathrm{i}}{\pi}\int_0^\infty \mathrm{e}^{-x\sinh v}\mathrm{d}v}_{\text{purely imaginary}}
\end{aligned}$$

On the final curve of interest, which is C_3, the integral is dominated by contribution from the saddle point. As before in Lecture 6, we replace C_3 by its tangent at the saddle point.

$$t = \frac{\pi}{2} + (1-\mathrm{i})r \qquad r \in [-\epsilon, \epsilon]$$

Using the above parameterization

$$I_3 = \int_{-\epsilon}^{\epsilon} \mathrm{e}^{\mathrm{i}x\sin[\pi/2 + (1-\mathrm{i})r]}(1-\mathrm{i})\mathrm{d}r$$

The power is expanded as

$$\begin{aligned}
\sin\left[\frac{\pi}{2} + (1-\mathrm{i})r\right] &= \cos\left[(1-\mathrm{i})r\right] \\
&= 1 - \frac{(1-\mathrm{i})^2 r^2}{2} + \frac{(1-\mathrm{i})^4 r^4}{4!}\dots \\
&= 1 + \mathrm{i}r^2 - \frac{1}{6}r^4\dots
\end{aligned}$$

And finally

$$\begin{aligned} I_3 &\sim 2(1-\mathrm{i})\mathrm{e}^{\mathrm{i}x} \int_0^{\epsilon} \mathrm{e}^{-xr^2} \mathrm{e}^{-\mathrm{i}xr^4/6} \mathrm{d}r \\ &\sim 2(1-\mathrm{i})\mathrm{e}^{\mathrm{i}x} \int_0^{\infty} \mathrm{e}^{-xr^2} \left[1 - \frac{\mathrm{i}xr^4}{6} \dots \right] \mathrm{d}r \end{aligned}$$

yielding

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{\pi}{4}\right) + \frac{1}{8x} \sin\left(x - \frac{\pi}{4}\right) + ...\right]$$

Lecture 8

Integral Representations and Introduction to Dominant Balance

Part I of the course was asymptotic expansions and their properties. Part II was aymptotic expansions of integrals. Part III is aymptotic analysis of linear ODEs (with non-constant coefficients).

Integral representation: Bessel function

Derivation of

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin t)\mathrm{d}t$$

from Bessel's equation

$$xy'' + y' + xy = 0$$

This equation has two linearly indepedent solutions:

- $J_0(x) \to 1$ as $x \to 0$
- $Y_0(x) \to -\infty$ as $x \to 0$

Power series can give us the approximation to $J_0(x)$ whereas we would need the more sophisticated Frobenius series to determine $Y_0(x)$ (allows non-integer powers of x). A more powerful method is to seek solutions of the form

$$y(x) = \int_c \mathrm{e}^{xz} P(z)\mathrm{d}z$$

where we are free to choose $P(z)$ and a suitable contour c in the complex plane z to make this work. Our ODE then reads

$$\int_c \left[xz^2 + z + x\right] \mathrm{e}^{xz} P(z)\, \mathrm{d}z = 0 \qquad \forall\, x$$

Suppose we could find a function $S(z)$ such that the integrand becomes a perfect differential, then the Bessel's equation would become

$$0 = \int_c \frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{e}^{xz} S(z)\right] \mathrm{d}z = \mathrm{e}^{xz} S(z)|_{\partial c}$$

and we'd only have to choose c such that this holds (for instance, when the contour is closed or S vanishes at both ends). Note that our definition of S is such that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{e}^{xz} S(z)\right] = (xS + S')\, \mathrm{e}^{xz} = (xz^2 + z + x)\mathrm{e}^{xz} P$$

Since this is true for all x, we can equate the coefficients:

$$S = (z^2+1)P \qquad \text{and} \qquad S' = zP$$

Solving for S:

$$\begin{aligned} \frac{S'}{S} &= \frac{z}{z^2+1} \\ \ln S(z) &= \frac{1}{2}\ln(z^2+1) + \text{const.} \end{aligned}$$

For now we assume $P(z) \neq 0$. Also, here we can get $J_0(x)$ by setting const $= 0$. Note that we are not looking for a general solution, rather "a" integral representation.

$$\begin{aligned} S(z) &= \sqrt{z^2+1} \\ P(z) &= \frac{1}{\sqrt{z^2+1}} \end{aligned}$$

In summary

$$y(x) = \int_c \mathrm{e}^{xz} \frac{1}{\sqrt{z^2+1}} \mathrm{d}z$$

satisifes the Bessel's equation <u>if</u> we can find a contour c such that

$$\mathrm{e}^{xz}\sqrt{z^2+1}\Big|_{\partial c} = 0 \qquad \forall\, x$$

One way is to simply pick a contour that goes from $[-\mathrm{i}, \mathrm{i}]$ – a vertical line segment for example. Then with the substitution $z = \mathrm{i}w$

$$\begin{aligned} y(x) &= \int_{-\mathrm{i}}^{\mathrm{i}} \mathrm{e}^{xz} \frac{1}{\sqrt{z^2+1}} \mathrm{d}z \\ &= \int_{-1}^{1} \mathrm{e}^{\mathrm{i}xw} \frac{1}{\sqrt{1-w^2}} \mathrm{i}\mathrm{d}w \end{aligned}$$

Now, is $y(x)$ equal to $J_0(x)$ or $Y_0(x)$ (multplied by some scalar) or a linear combination of both? Way to check is to evaluate $y(0)$:

$$y(0) = \mathrm{i}\int_{-1}^{1} \frac{\mathrm{d}w}{\sqrt{1-w^2}} = \mathrm{i}\pi$$

where we have made the substitution $w = \sin\theta$. Therefore

$$y(x) = \mathrm{i}\pi J_0(x)$$

and

$$\begin{aligned} J_0(x) = \frac{1}{\mathrm{i}\pi} y(x) &= \frac{1}{\pi}\int_{-1}^{1} \frac{\mathrm{e}^{\mathrm{i}xw}}{\sqrt{1-w^2}}\mathrm{d}w \\ &= \frac{2}{\pi}\int_{0}^{1} \frac{\cos(xw)}{\sqrt{1-w^2}}\mathrm{d}w \\ &= \frac{2}{\pi}\int_{0}^{\pi/2} \cos(x\sin t)\mathrm{d}t \end{aligned}$$

since the odd sin function goes to zero and we have made the change $w = \sin t$. Finally, noting

$$\int_{0}^{\pi/2} \cos(x\sin t)\mathrm{d}t = \int_{\pi/2}^{\pi} \cos(x\sin t)\mathrm{d}t$$

with the help of the substitution $u = \pi - t$, we can show that

$$\begin{aligned} J_0(x) &= \frac{1}{\pi}\left[\int_{0}^{\pi/2} + \int_{\pi/2}^{\pi}\right] \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos(x\sin t)\mathrm{d}t \end{aligned}$$

- Note that at the end points $P(z) \to \infty$. It may be worth thinking if this is OK since the integral is fine as we approach the ends points from the interior side where the function is not badly behaved.

- For calculation of $Y_0(x)$, which involves a different contour, see book by Olver on Intro to Asymptotics and Special Functions (p.241)

Local analysis of second order ODE

The goal is to obtain asymptotic behavior of $y(x)$ directly from ODE (second order, linear, homogeneous) rather than from integral representation.

$$y'' + p_1(x)y' + p_0(x)y = 0$$

The downside of this method is that we won't know the constant (unlike the integral representation). But often this is good enough if we wish to understand scaling behavior etc. Classification of "singular points":

- Ordinary point: x_0, where $p_1(x)$ and $p_0(x)$ are analytic in a neighborhood of x_0 (i.e. have convergent power series expansions there). Then the linearly independent solutions $y_1(x)$ and $y_2(x)$ are also expressible as convergent Taylor series.
- Regular singular point: Not an ordinary point, but $(x - x_0)p_1(x)$ and $(x - x_0)^2 p_0(x)$ are analytic around x_0. The solutions can be expressed as Frobenius series (p.63 B&O).
- Irregular singular point: Neither of the above. There is no general theory. This is the case of interest.

Example: Consider

$$x^n y'' + xy' + y = 0 \qquad n \geq 0, \text{integer}$$

in the vicinity of $x_0 = 0$. Rewrite

$$y'' + x^{1-n}y' + x^{-n}y = 0$$

I'll get ordinary point if $1 - n \geq 0$ and $-n \geq 0$. This implies $n \leq 0$. From the original constraint $n \geq 0$, the only possibility is when $n = 0$.

Next $x_0 = 0$ is a regular singular point if $n = 1$ or $n = 2$.

Finally, $x_0 = 0$ is an irregular singular point if $n = 3, 4, 5, \ldots$.

Example: Classify the singular point at $x_0 = \infty$ for Airy's equation

$$y'' = xy$$

The standard move is to bring the infinity into the origin by letting $t = 1/x$. Then classify $t = 0$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = -t^2\dot{y} \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-t^2\dot{y}\right)\frac{\mathrm{d}t}{\mathrm{d}x} = t^2(2t\dot{y} + t^2\ddot{y})
\end{aligned}$$

The Airy's equation now reads

$$t^4\ddot{y} + 2t^3\dot{y} = \frac{1}{t}y$$

In standard form

$$\ddot{y} + \frac{2}{t}\dot{y} - \frac{1}{t^5}y = 0$$

This is an irregular singular point at $t = 0$ and hence at $x_0 = \infty$.

Asymptotic expansion near an irregular singular point

The strategy is to proceed with the ansatz

$$y = \mathrm{e}^{S(x)}$$

Then

$$y' = S'\mathrm{e}^{S}$$
$$y'' = \left[S'' + (S')^2\right]\mathrm{e}^{S}$$

Next examine ODE for S and usually $S'' \ll (S')^2$ and we can start neglecting terms.

Example:

$$x^4 y'' + 2x^3 y' - y = 0$$

This equation has exact solutions that are $\mathrm{e}^{\pm 1/x}$. But we proceed without this knowledge. This equation has an irregular singular point at $x_0 = 0$. Proceeding with our solution $y = \mathrm{e}^{S}$ we arrive at

$$x^4\left[S'' + (S')^2\right] + 2x^3 S' - 1 = 0$$

which gives the standard form

$$S'' + (S')^2 + \frac{2}{x}S' - \frac{1}{x^4} = 0$$

The idea of dominant balance is that two terms balance each other as $x_0 \to 0$ whereas the others are asymptotically small. Now we can see that $1/x^4$ is a big term. We can compare this with the other terms and check for consistency:

1. Balance the last two terms: This means that $S' = O(1/x^3)$ and
$$(S')^2 = O\left(\frac{1}{x^6}\right) \gg \frac{1}{x^4}$$
Hence this is not consistent.
2. Balance the first and last: also inconsistent.
3. Balance the second and last:
$$S' = \pm\frac{1}{x^2} \quad \Longrightarrow \; S = \mp\frac{1}{x}$$
$$S'' = O\left(\frac{1}{x^3}\right) \ll \frac{1}{x^4}$$
$$\frac{2}{x}S' = O\left(\frac{1}{x^3}\right) \ll \frac{1}{x^4}$$
The ordering is consistent.

Normally this would be our first approximation to $S(x)$, but our solution here, $y = e^{\pm 1/x}$, is exact.

Lecture 9

Dominant Balance (contd.)

Example: Solve

$$y'' = \frac{1}{x^3} y$$

asymptotically as $x \to 0^+$. We have an irregular singular point at $x_0 = 0$ (since more singular than $1/x^2$). We lead with the solution

$$y(x) = \mathrm{e}^{S(x)}$$

Note that if $1/x^3$ was instead some constant, our solution would have looked like e^λ. This form is what motivates our ansatz. Then

$$(S')^2 + S'' = \frac{1}{x^3} \tag{16}$$

Now usually two of the terms are bigger than the third, but worthwhile noting that in cases all the terms may be in balance. Leading with the ansatz $(S')^2 \gg S''$ we arrive at

$$\begin{aligned} S' &\sim \pm \frac{1}{x^{3/2}} \\ \implies S &= \mp 2x^{-1/2} + c(x) \end{aligned} \tag{17}$$

Note that

$$\underbrace{O\left(\frac{1}{x^3}\right)}_{(S')^2} \gg \underbrace{O\left(\frac{1}{x^{5/2}}\right)}_{S''}$$

which means that our ordering is consistent.

- Since S is singular as $x \to 0^+$, we do not need to put the constant (correction term) which is asymptotically smaller, i.e. $c(x) \ll x^{-1/2}$.

- As we derive higher order terms, we would find that the singularities become weaker, providing us with a natural place to stop.

We will focus on eqn. 17 with the "+" sign for now: this is the exponentially growing term and of more interest than the other decaying term. Now

$$\begin{aligned} S &= 2x^{-1/2} + c(x) & \qquad c &\ll x^{-1/2} \\ S' &= -x^{-3/2} + c'(x) & c' &\ll x^{-3/2} \\ S'' &= \frac{3}{2}x^{-5/2} + c''(x) & c'' &\ll x^{-5/2} \end{aligned}$$

as $x \to 0^+$. Now usually differentiation can cause trouble, but in the case of x to some power, this is alright. We insert these into eqn. 16 to derive

$$\underbrace{\frac{3}{2}x^{-5/2} + c''}_{\gg} + \cancel{x^{-3}} + \underbrace{(c')^2 - 2c'x^{-3/2}}_{\ll} = \cancel{x^{-3}}$$

We are left with

$$\begin{aligned} \frac{3}{2}x^{-5/2} &\sim 2c'x^{-3/2} \\ \implies c(x) &\sim \ln x \end{aligned}$$

At this stage we should do a consistency check: using L'Hopital's rule

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = -2x^{1/2} \to 0$$

etc. More precisely

$$c(x) = \frac{3}{4}\ln x + d(x) \qquad d(x) \ll \ln x$$

as $x \to 0^+$. Altogether

$$S(x) = 2x^{-1/2} + \frac{3}{4}\ln x + d(x) \tag{18}$$

with each term milder (less singular) than the previous term. Continuing in this way by inserting eqn. 18 into eqn. 16 we derive

$$\begin{gathered} \underbrace{-\frac{3}{16}x^{-2} + d''}_{\gg} + \underbrace{(d')^2 + \frac{3}{2}x^{-1}d'}_{\ll} - 2x^{-3/2}d' = 0 \\ -\frac{3}{16}x^{-2} + \underbrace{\frac{3}{2}x^{-1}d' - 2x^{-3/2}d'}_{\ll} = 0 \end{gathered}$$

The balancing equation yields

$$d(x) \sim -\frac{3}{16}x^{1/2} + e$$

Collecting terms

$$S(x) = 2x^{-1/2} + \frac{3}{4}\ln x + e + \delta(x)$$
$$\lim_{x\to 0^+} \delta(x) = -\frac{3}{16}x^{1/2} \to 0$$

This is a good place to stop since the correction term is no longer singular and our solution of interest

$$\begin{aligned} y(x) &= \mathrm{e}^{S(x)} \\ &= c_1 x^{3/4}\mathrm{e}^{2x^{-1/2}}\mathrm{e}^{\delta(x)} \\ &\sim c_1 x^{3/4}\mathrm{e}^{2x^{-1/2}}\left(1 - \frac{3}{16}x^{1/2}\right) \end{aligned}$$

for $\delta \ll 1$ and as $x \to 0^+$. This is only the growing solution. To obtain the full asymptotic expansion, we isolate the "leading order" (the wildly varying singular term) and denote everything else by $w(x)$, i.e.[1]

$$y(x) = c_1 x^{3/4}\mathrm{e}^{2x^{-1/2}}w(x)$$

We can proceed to solve for $w(x)$ using the series (p.84)

$$w(x) = \sum_{n=0}^{\infty} a_n (x^{1/2})^n$$

NB This method does not give us the prefactor c_1, just the x dependence. To determine the pre-factor, techniques like the "integral representation" must be employed.

[1] B&O call this "peeling off the leading behavior".

Lecture 10

Perturbation methods for Algebraic equations

Part IV of the course.

Example 1: Solve the "regular" problem

$$x^2 + \epsilon x - 1 = 0 \qquad \epsilon \ll 1$$

The exact solution is of course

$$\begin{aligned} x &= -\frac{\epsilon}{2} \pm \sqrt{1 + \frac{\epsilon^2}{4}} \\ &= -\frac{\epsilon}{2} \pm \left(1 + \frac{\epsilon^2}{8} \dots\right) \end{aligned}$$

where we have used the Binomial theorem

$$(1+\delta)^\alpha = 1 + \alpha\delta + \frac{\alpha(\alpha-1)}{2!}\delta^2 + \dots$$

which is valid for $|\alpha| < 1$. Therefore the series converge for $\epsilon < 2$.

Method 1: Expansion in a power series in ϵ, i.e. try

$$x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \dots$$

disregarding $O(\epsilon^3)$. We know $x_0 = \pm 1$ which is easily seen by setting $\epsilon = 0$. Expanding

$$(1 + \epsilon x_1 + \epsilon^2 x_2 + \dots)^2 + \epsilon(1 + \epsilon x_1 + \epsilon^2 x_2 + \dots) - 1 = 0 + 0\epsilon + 0\epsilon^2 + \dots$$
$$(1 + \epsilon^2 x_1^2 + 2\epsilon x_1 + 2\epsilon^2 x_2) + (\epsilon + \epsilon^2 x_1) - 1 = 0$$

Equating powers:

$$\begin{aligned} O(\epsilon^0): &\quad 1 - 1 = 0 \\ O(\epsilon^1): &\quad 2x_1 + 1 = 0 \\ O(\epsilon^2): &\quad x_1^2 + 2x_2 + x_1 = 0 \end{aligned}$$

Therefore

$$x_1 = -\frac{1}{2}, \qquad x_2 = \frac{1}{8}$$

yielding

$$x = 1 - \epsilon\frac{1}{2} + \epsilon^2\frac{1}{8} + \dots$$

NB Here we guessed the asymptotic sequence which was ordered as

$$1, \epsilon, \epsilon^2, \epsilon^3, \dots$$

such that as $\epsilon \to 0$, $1 \gg \epsilon \gg \epsilon^2 \gg \epsilon^3, \dots$ and this would typically work for most "regular" problems.

Method 2: This is an iteration method which is useful when it is difficult to guess the sequence. We proceed to rewrite our problem as

$$x^2 = 1 - \epsilon x$$

and start with with a naïve guess say $x_0 = 1$ (and take the plus root):

$$x_{n+1} = \sqrt{1 - \epsilon x_n}$$

and

$$\begin{aligned} x_1 &= \sqrt{1-\epsilon} \\ &\sim 1 - \frac{\epsilon}{2} \underbrace{- \frac{\epsilon^2}{8}}_{!!} + \dots \end{aligned}$$

NB This is wrong! We should truncate at $O(\epsilon)$. Next

$$\begin{aligned} x_2 &= \sqrt{1 - \epsilon\left[1 - \frac{\epsilon}{2} + O(\epsilon^2)\right]} \\ &= \left[1 - \left(\epsilon - \frac{\epsilon^2}{2}\right) + O(\epsilon^3)\right]^{1/2} \\ &= 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \dots \end{aligned}$$

So this method is somewhat tricky and needs to be carefully wielded (plus requires increasing amount of work at each stage).

Example 2: Consider now a "singular" (as opposed to "regular") problem:

$$\epsilon x^2 + x - 1 = 0$$

Clearly if $\epsilon = 0$, $x = 1$. But we should obtain two roots for $\epsilon \neq 0$ (even if $\epsilon \ll 1$). Such big qualitative change are typical of singularly perturbed problems. Consider dominant balance: may be the ϵx^2 term is big for the missing root?

1. If $\epsilon x^2 \sim 1$, then $x \gg 1$ which is not consistent with the ordering.

2. $\epsilon x^2 \sim x$ would instead imply $x \gg 1$ which is consistent.

[To do] If we solve this, we would derive

$$x \sim -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 + \dots$$

Example 3: Next consider non-integer powers

$$(1-\epsilon)x^2 - 2x + 1 = 0$$

As $\epsilon = 0$, we get $x^2 - 2x + 1 = 0$, i.e. $(x-1)^2 = 0$. Double roots typically spell danger! To illustrate, we perform the naïve expansion about $x = 1$. Try

$$x = 1 + \epsilon x_1 + \dots \tag{19}$$

to derive

$$(1-\epsilon)(1+\epsilon x_1 + \dots)^2 - 2(1 + \epsilon x_1 + \dots) + 1 = 0$$

Collecting orders:

$$\begin{aligned} O(\epsilon^0): &\quad 1 - 2 + 1 = 0 \\ O(\epsilon^1): &\quad 2x_1 - 1 + 2x_1 = 0 \end{aligned}$$

which is a contradiction! This is since our anstaz – eqn. 19 – was wrong. In this case we want to try

$$x \sim 1 + \epsilon^{1/2} + \epsilon + \epsilon^{3/2} + \dots$$

It is wiser to assume

$$x \sim 1 + \epsilon^{\alpha} x_1 + \epsilon^{\beta} x_2 + \dots$$

or more generally

$$x \sim 1 + \phi_1(\epsilon) x_1 + \phi_2(\epsilon) x_2 + \dots$$

[To do] Try either approach, or iteration with $\epsilon^{1/2}$.

Example 4: Logarithms! Solve

$$x\mathrm{e}^{-x} = \epsilon \qquad \text{as} \quad \epsilon \to 0^+$$

[To do] We can begin by sketching the curve $y(x) = x\mathrm{e}^{-x}$. The intersection of $y(x) = \epsilon$ at two places give us the two solutions. And as $\epsilon \to 0$, one roots goes to 0 and the other goes to ∞. For the root near $x = 0$, we can use iteration:

$$\begin{aligned} x_{n+1} &= \epsilon \mathrm{e}^{x_n} \\ x_1 &= \epsilon \\ x_2 &= \epsilon \mathrm{e}^{\epsilon} = \epsilon(1 + \epsilon + \dots) \end{aligned}$$

and we end up with

$$x \sim \epsilon + \epsilon^2 + \frac{3}{2}\epsilon^3 + \frac{8}{3}\epsilon^4 + \dots$$

The other root: try dominant balance.

$$\underbrace{\ln x - x}_{\ll} = \ln \epsilon$$

Therefore

$$x \sim \ln \epsilon^{-1}$$

Dominant balance then suggests the iteration

$$\begin{aligned} x_{n+1} &= \ln x_n + \ln \frac{1}{\epsilon} \\ x_1 &= \ln \ln \frac{1}{\epsilon} + \ln \frac{1}{\epsilon} \\ x_2 &= \ln x_1 + \ln \frac{1}{\epsilon} \\ &= \ln \ln \frac{1}{\epsilon} + \ln \left[1 + \frac{\ln \ln(1/\epsilon)}{\ln(1/\epsilon)} \right] + \ln \frac{1}{\epsilon} \\ &\sim \ln \frac{1}{\epsilon} + \ln \ln \frac{1}{\epsilon} + \frac{\ln \ln(1/\epsilon)}{\ln(1/\epsilon)} \dots \end{aligned}$$

Lecture 11

Regular Perturbation methods for ODEs

This is the heart of the course. The "regular" theory is the "vanilla" variant. To understand, consider the projectile problem

Example 1: Launch a toy rocket straight up. How long does it take to hit the ground if g is varying with height x above the Earth's surface?

$$
\begin{aligned}
\cancel{m}\ddot{x} &= -\frac{GM\cancel{m}}{(R+x)^2} = -\frac{GM}{R^2}\frac{R^2}{(R+x)^2} \\
&= -g\frac{R^2}{(R+x)^2} \qquad (20)
\end{aligned}
$$

The initial conditions are

$$
\begin{aligned}
x(0) &= 0 \\
\dot{x}(0) &= v_0
\end{aligned}
$$

Assume initial velocity is small such that $x \ll R$ for all t. Naïvely

$$
\begin{aligned}
\ddot{x} &\approx -g \\
x(t) &\approx -\frac{1}{2}gt^2 + v_0 t \qquad (21)
\end{aligned}
$$

What is the leading correction to this solution? It is better to scale the problem and work with dimensionless units: the characteristic time t_c is, say, the time it takes to reach the peak and the characteristic scale x_c is the peak height. Ignoring any $O(1)$ factor, we see

$$
t_c = \frac{v_0}{g} \qquad x_c = \frac{v_0^2}{g}
$$

and introduce the new dimensionless variables

$$\tau = \frac{t}{t_c} \qquad y = \frac{x}{x_c}$$

In terms of these our governing eqn. 20 reads:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = -\frac{1}{\left(1 + \frac{x_c y}{R}\right)^2}$$

Defining $x_c/R = \epsilon$ we arrive at the form

$$y'' = -\frac{1}{(1+\epsilon y)^2}$$

The parameter

$$\epsilon = \frac{v_0^2}{gR} \ll 1$$

also suggests that the initial velocity is much smaller than the escape velocity. The initial conditions now read

$$\begin{aligned} y(0) &= \frac{x(0)}{x_c} = 0 \\ y'(0) &= \frac{t_c}{x_c}\dot{x}(0) = 1 \end{aligned}$$

In summary, we are left to solve the nonlinear ODE subject to the BCs

$$y'' = -\frac{1}{(1+\epsilon y)^2}, \qquad y(0) = 0, \qquad y'(0) = 1 \tag{22}$$

We try the solution which is a power series in ϵ

$$y(\tau, \epsilon) = y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \dots \tag{23}$$

with our IC reading

$$\begin{aligned} \underbrace{y(0,\epsilon)}_{0} &= y_0(0) + \epsilon y_1(0) + \dots = 0 + 0\epsilon + \dots \\ \underbrace{y'(0,\epsilon)}_{1} &= y_0'(0) + \epsilon y_1'(0) + \dots = 1 + 0\epsilon + \dots \end{aligned}$$

Inserting our ansatz – eqn. 23 – into the ODE:

$$\begin{aligned} y_0'' + \epsilon y_1'' + \dots &= -(1+\epsilon y)^{-2} \\ &= -1 + 2\epsilon y + O(\epsilon^2) \\ &= -1 + 2\epsilon y_0 + \dots \end{aligned}$$

Our hierarchy of equations read:

$$\begin{aligned} O(\epsilon^0): \quad & y_0'' = -1 \\ O(\epsilon^1): \quad & y_1'' = 2y_0 \end{aligned}$$

Solving the $O(1)$ equation:

$$y_0(\tau) = -\frac{1}{2}\tau^2 + \tau$$

which is simply the dimensionless version of eqn. 21. The $O(\epsilon^1)$ equation gives us the leading order correction to this:

$$y_1(\tau) = -\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3$$

Pulling everything together

$$y(\tau, \epsilon) = \tau - \frac{\tau^2}{2} + \epsilon\left(\frac{\tau^3}{3} - \frac{\tau^4}{12}\right) + \ldots \tag{24}$$

Now in our dimensionless variables, the time it takes for the rocket to return is simply when $y(\tau) = 0$, which yields

$$\tau = 0 \qquad \tau = 2$$

What is the time it takes for the rocket to hit the ground when the leading order correction given by eqn. 24 is considered? We must now solve the algebraic equation

$$0 = \tau - \frac{\tau^2}{2} + \epsilon\left(\frac{\tau^3}{3} - \frac{\tau^4}{12}\right)$$

which we attempt to solve with the solution form, following sec. 10 ex. 1

$$\tau^* = 2 + \epsilon\tau_1^* + O(\epsilon^2)$$

Then, neglecting $O(\epsilon^2)$ terms

$$0 = (2 + \epsilon\tau_1^*) - \frac{1}{2}(4 + 4\epsilon\tau_1^*) + \epsilon\left[\frac{1}{3}(8 + 12\epsilon\tau_1^*) - \frac{1}{12}(16 + 32\epsilon\tau_1^*)\right]$$

which yields $\tau_1^* = 4/3$ and

$$\tau = 2 + \frac{4}{3}\epsilon + \ldots$$

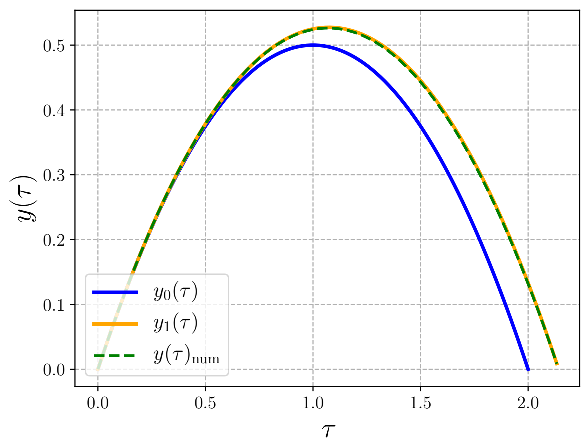

Figure 11.1: Plotting the numerical solution to eqn. 22 along with the analytical approximations given by eqn. 24 for $\epsilon = 0.1$. The numerical and higher order solutions have been plotted to $\tau = 2+4\epsilon/3$, which is the corrected time it takes for the projectile to hit the ground.

Example 2: Solve

$$y' + y + \epsilon y^2 = 0$$

with the IC $y(0) = 1$ on the interval $[0, \infty)$ ignoring $O(\epsilon^2)$. We try a regular perturbation series

$$y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

The ICs translate to

$$y(0, \epsilon) = 1 + 0\epsilon + \cdots = y_0 + \epsilon y_1 + \ldots$$

and our ODE becomes

$$(y_0' + \epsilon y_1' + \ldots) + (y_0 + \epsilon y_1 + \ldots) + \epsilon(y_0 + \epsilon y_1 + \ldots)^2 = 0$$

which yields the hierarchy of equations

$$\begin{aligned} O(\epsilon^0): &\qquad y_0' + y_0 = 0 \\ O(\epsilon^1): &\qquad y_1' + y_1 + y_0^2 = 0 \end{aligned}$$

Solving these sequentially:

$$y_0(x) = \mathrm{e}^{-x}$$

and

$$y_1' + y_1 = -\mathrm{e}^{-2x}$$

Such first order ODEs can be straightforwardly solved using the integrating factor method. This yields

$$y_1(x) = -\mathrm{e}^{-x} + \mathrm{e}^{-2x}$$

and our solution

$$y(x,\epsilon) = \mathrm{e}^{-x} + \epsilon(-\mathrm{e}^{-x} + \mathrm{e}^{-2x}) \tag{25}$$

Now our contrived example can be solved exactly by writing as

$$\int \left[\frac{1}{y} - \frac{\epsilon}{1+\epsilon y}\right] \mathrm{d}y = -x + c$$

which with the proper boundary condition yields

$$\ln\left[\frac{y}{1+\epsilon y}(1+\epsilon)\right] = -x$$

Some rearrangement yields

$$y(x,\epsilon) = \frac{1}{-\epsilon + (1+\epsilon)\mathrm{e}^{x}} \tag{26}$$

In the complex ϵ plane, the function $y(x,\epsilon)$ converges for all x with the radius of convergence $R(x)$[1] when

$$-\epsilon + (1+\epsilon)\mathrm{e}^{x} = 0$$
$$\epsilon_{\text{singular}} = \frac{\mathrm{e}^{x}}{1-\mathrm{e}^{x}}$$

Observe that for any $x \geq 0$

$$|\epsilon_{\text{singular}}| = \left|\frac{\mathrm{e}^{x}}{\mathrm{e}^{x}-1}\right| > 1$$

i.e. $|\epsilon| < 1$ ensures convergence for all x. Refer to the solution plots in Fig. 11.2. The perturbative solution matches well over the full domain in x – this is referred to as "uniform approximation/validity". This feature typifies "regular" perturbation problems.

Example 3: However, there are bad x. So our solution is asymptotic, but not uniformly valid. Consider

$$f(x,\epsilon) = x + \mathrm{e}^{-x/\epsilon}$$

where $x \in (0,1)$. As $\epsilon \to 0^{+}$

$$f(x) \sim x$$

and further the dropped term is a TST. Despite this, as can be seen from Fig. 11.3, the approximate solution is not uniformly valid.
When this kind of phenomenon happens, we have a "boundary layer" and are very important in fluid dynamics.

[1] $R(x)$ is the distance from the origin to the nearest singularity of $y(x,\epsilon)$.

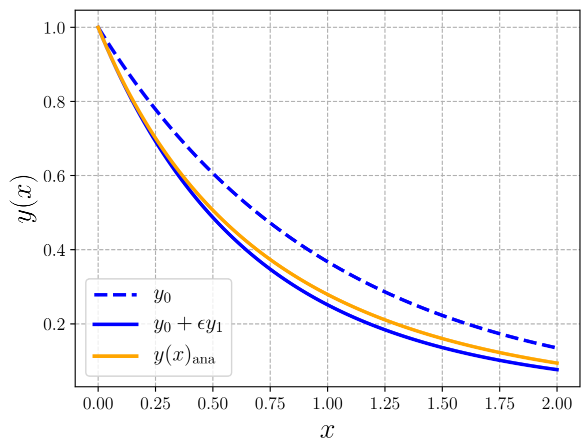

Figure 11.2: The exact soln. 26 and perturbative soln. 25 plotted for $\epsilon = 0.5$.

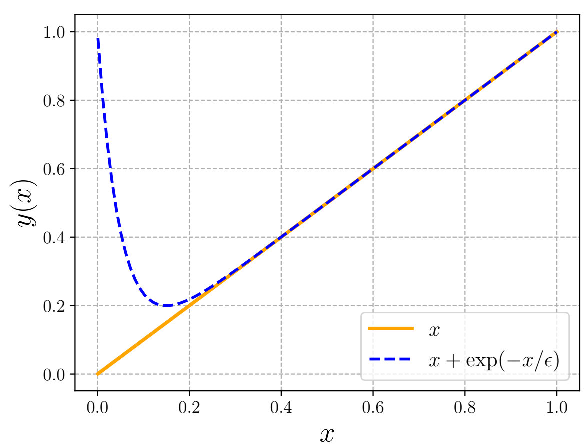

Figure 11.3: $f(x, \epsilon)$ and $f(x) \sim x$ plotted for $\epsilon = 0.05$.

Lecture 12

Introduction to Boundary Layer Theory (BLT)

Part VI of the course. Even appears in neurobiology or cardiology. Example, when two different time-scales such as the time for a neuron to charge up (gradual) and then discharge (spike). Boundary layer is when the spike occurs – matching the slow to fast solutions (could be in time or space).

The Boundary Layer theory is a singlular perturbation method that is useful when $\epsilon \ll 1$ multiplies the highest derivative in a differential equation.

Example 1: Consider

$$\begin{aligned} \epsilon y'' + (1+\epsilon)y' + y &= 0 \\ y(0) = 0 \qquad y(1) &= 1 \end{aligned} \tag{27}$$

Let us try the regular perturbation series, meaning we *naïvely* guess a power series of the form

$$y(x,\epsilon) = y_0 + \epsilon y_1(x) + \dots$$

This yields

$$\begin{aligned} \epsilon(y_0'' + \epsilon y_1'' + \dots) + (1+\epsilon)(y_0' + \epsilon y_1' + \dots) + (y_0 + \epsilon y_1 + \dots) &= 0 \\ y_0(0) + \epsilon y_1(0) + \dots &= 0 + 0\epsilon + \dots \\ y_0(1) + \epsilon y_1(1) + \dots &= 1 + 0\epsilon + \dots \end{aligned}$$

resulting in the hierarchy of equations

$$\begin{aligned} O(\epsilon^0): &\qquad y_0' + y_0 = 0 \\ O(\epsilon^1): &\qquad y_0'' + y_0' + y_1' + y_1 = 0 \end{aligned}$$

And this is trouble already! Since we have a first order equation at $O(\epsilon^0)$ yet two BCs (over-determined)! To check explicitly,

$$y_0 = c\mathrm{e}^{-x}$$

and since $y_0(0) = 0$, it must be that $c = 0$. The solution $y_0(x) = 0$ does not satisfy the $y_0(1) = 1$. Thus the "regular" perturbation method fails here and we refer to such problems as "singular".

To understand what is going on, we look at the exact solution obtained by trying the soln. form $y = \mathrm{e}^{\lambda x}$:

$$\epsilon\lambda^2 + (1+\epsilon)\lambda + 1 = 0$$

This readily yields the roots

$$\lambda = -1 \qquad \lambda = -\frac{1}{\epsilon}$$

On application of the two BCs, we arrive at the exact solution

$$y(x) = \frac{\mathrm{e}^{-x} - \mathrm{e}^{-x/\epsilon}}{\mathrm{e}^{-1} - \mathrm{e}^{-1/\epsilon}} \tag{28}$$

Notice the essential singularity $\mathrm{e}^{-x/\epsilon}$. While it appears harmless as it would approach $1/\infty$ as $\epsilon \to 0^+$, it does not have a convergent Taylor series about $\epsilon = 0$. Therefore the regular perturbative approach, where we can get away with a power series, does not work.

Explanation: If we fix $x > 0$ and let $\epsilon \to 0^+$, and say $x \geq 10\epsilon$ then

$$y(x) \to \mathrm{e}^{1-x} \tag{29}$$

If $x \sim \epsilon$ then the $\mathrm{e}^{-x/\epsilon}$ term cannot be ignored. From Fig. 12.1 it is evident that as x gets smaller, ϵ must decrease to get a closer match with the asymptotic expression. This is point-wise convergence and not uniform convergence. For any ϵ, the match is quite good in the region $x \geq 10\epsilon$.
The region of rapid change near $x = 0$ (namely $0 < x < O(\epsilon)$) is called a "boundary layer". The boundary layer is the "inner" region and the region away from it is called the "outer" region.

Solution: Using a technique called "matched asymptotic expansions". First attempt the <u>outer region</u> solution using regular perturbation theory: we think of x as fixed and let $\epsilon \to 0^+$ and solve the ODE with the right BC at $x = 1$. So we solve

$$y_0' + y_0 = 0 \qquad y_0(1) = 1$$

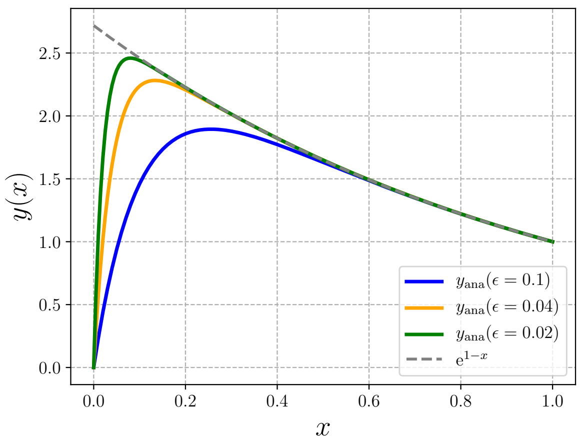

Figure 12.1: Plotting the exact soln. 28 and the asymptotic soln. 29.

yielding

$$y_0(x)_{\text{out}} = \mathrm{e}^{1-x} \tag{30}$$

Going to the $O(\epsilon)$ problem

$$\begin{aligned} y_1' + y_1 &= -y_0' - y_0'' \\ &= -(\underbrace{y_0 + y_0'}_{0})' \end{aligned}$$

which with the appropriate BC at $x = 1$ yields

$$y_1(x) = 0$$

For this problem $y_2 = y_3 = \cdots = 0$.

Now we look at the inner region. We introduce a new 'stretched' variable $X = x/\epsilon$, since this is the natural length-scale defined in the question. In our X scale, the BL is at $X \to \infty$. Our transformed ODE reads

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} + (1+\epsilon)\frac{\mathrm{d}Y}{\mathrm{d}X} + \epsilon Y = 0 \tag{31}$$

Observe that when $\epsilon = 0$, we do not loose the highest derivative! In some sense, the rapid variation through Y_{XX} multiples with the ϵ term to be finite and not zero. We now use regular perturbation theory on this re-scaled inner ODE. The BC is now

$$y(0) = 0 \qquad Y(X = 0) = 0 + 0\epsilon + \ldots$$

Let

$$Y_{\text{inn}} = Y_0 + \epsilon Y_1 + \dots$$

which yields

$$(Y_0'' + \epsilon Y_1'' + \dots) + (1+\epsilon)(Y_0' + \epsilon Y_1' + \dots) + \epsilon(Y_0 + \epsilon Y_1 + \dots) = 0$$

and the ordered set of equations

$$\begin{aligned} O(\epsilon^0): &\qquad Y_0'' + Y_0' = 0 \\ O(\epsilon^1): &\qquad Y_1'' + Y_1' + Y_0' + Y_0 = 0 \end{aligned}$$

The $O(\epsilon^0)$ equation is solved by one trivial integration and then solving the resultant first order ODE with the integrating factor method. This yields

$$Y_0 = A + B\mathrm{e}^{-X}$$

With the BC imposed we derive

$$Y_0(X) = A(1 - \mathrm{e}^{-X})$$

where the parameter A is what would help match the inner and outer solutions.

Overlap region & composite expansion: The condition on the matching region is

$$\begin{aligned} \lim_{X\to\infty} Y_0(X) &= \lim_{x\to 0} y(x)_{\text{out}} \\ A &= \mathrm{e} \end{aligned}$$

NB: We have effectively done a zeroth order matching! In some problems we may need to match solutions at higher orders too. That said, zeroth order matching is usually sufficient in practice. The uniformly valid *composite* approximation is expressed as

$$\begin{aligned} y_c &= y_{\text{out}} + Y_{\text{inn}} - y_{\text{match}} \\ &= \mathrm{e}^{1-x} + \mathrm{e}(1 - \mathrm{e}^{x/\epsilon}) - \mathrm{e} \\ &= \mathrm{e}(\mathrm{e}^{-x} - \mathrm{e}^{-x/\epsilon}) \end{aligned}$$

which is related to our exact solution eqn. 28 as

$$y(x) = \frac{y_c}{1 - \mathrm{e}^{1-1/\epsilon}}$$

For a moderately low $\epsilon = 0.1$, the denominator is $1 - \mathrm{e}^{-9} \approx 1 - 10^{-4} \approx 1$.

- Note that the overlap region ceases to exist as ϵ is increased. **Why?**

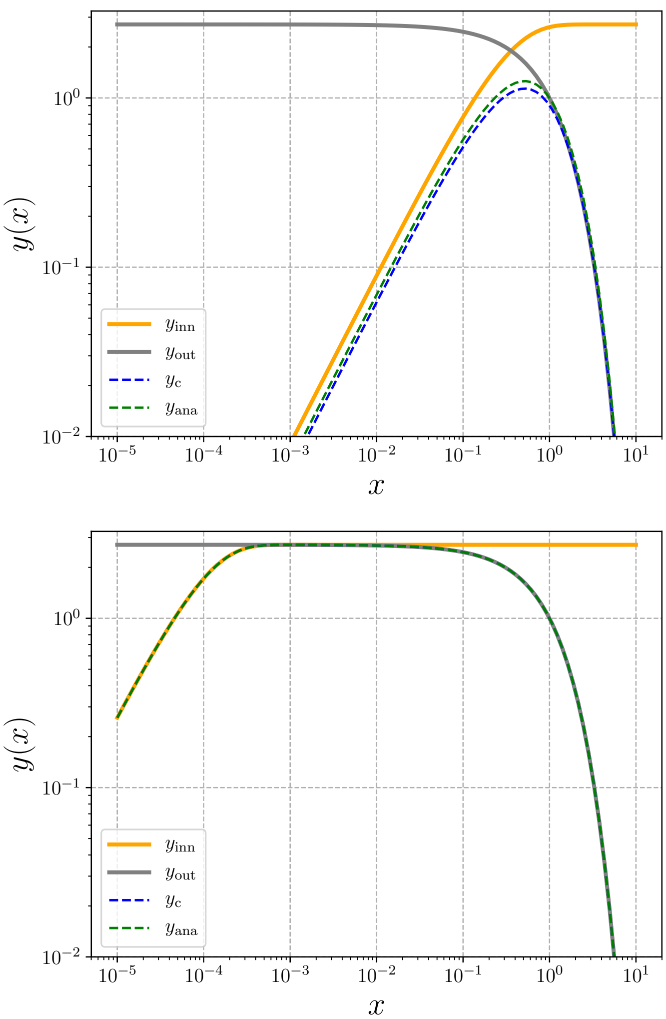

Figure 12.2: The various solution forms plotted for (top) $\epsilon = 0.3$ and (bot) $\epsilon = 10^{-4}$.

Lecture 13

Higher-order matching in Boundary Layer Theory

Several ways to match. No set way!

Example 1: Consider

$$\epsilon y'' + (1+x)y' + y = 0 \tag{32}$$
$$y(0) = 0 \qquad y(1) = 1$$

Such problems where ϵ multiplies the highest derivative typifies "singular" problems. We will learn techniques in the next lecture which will help us identify the thickness and other characteristics of the BL. For this problem the BL is at $x = 0$ and has thickness $O(\epsilon)$.

Outer soln. Try the regular perturbation series

$$y = y_0 + \epsilon y_1 + O(\epsilon^2)$$

We can first rewrite the ODE as

$$\epsilon y'' + [(1+x)y]' = 0$$

The $O(\epsilon^0)$ solution can be written directly

$$(1+x)y_0 = a$$

At $O(\epsilon^1)$

$$[(1+x)y_1]' = -y_0''$$
$$(1+x)y_1 = -y_0' + b$$

which yields

$$y_1 = \frac{a}{(1+x)^3} + \frac{b}{(1+x)}$$

The outer solution will use the BC at $x = 1$

$$y(1) = y_0(1) + \epsilon y_1(1) + \cdots = 1 + 0\epsilon + \ldots$$

which determines both constants:

$$y(x,\epsilon) = \frac{2}{1+x} + \epsilon\left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right] + O(\epsilon^2)$$

Inner soln. Let us scale the problem by introducing $X = x/\epsilon$.

$$Y'' + (1+\epsilon X)Y' + \epsilon Y = 0$$

And use regular perturbation method since we will no longer lose the highest derivative

$$Y = Y_0 + \epsilon Y_1 + O(\epsilon^2)$$
$$Y(0) = 0 + 0\epsilon + \cdots = Y_0 + \epsilon Y_1 + \ldots$$

Substituting

$$(Y_0'' + \epsilon Y_1'' + \ldots) + (1+\epsilon X)(Y_0' + \epsilon Y_1' + \ldots) + \epsilon(Y_0 + \epsilon Y_1 + \ldots) = 0$$

This leads to

$$\begin{aligned} O(\epsilon^0): &\qquad Y_0'' + Y_0' = 0 \\ O(\epsilon^1): &\qquad Y_1'' + Y_1' = -XY_0' - Y_0 \end{aligned}$$

At $O(\epsilon^0)$, we integrate once and then use the integrating factor method to solve the first order ODE with the BCs

$$Y_0(X) = A(1 - \mathrm{e}^{-X})$$

Our $O(\epsilon^1)$ equation reads

$$Y_1'' + Y_1' = -AX\mathrm{e}^{-X} - A(1 - \mathrm{e}^{-X})$$

Proceed by integrating both side

$$Y_1' + Y_1 = A(X\mathrm{e}^{-X} + \cancel{\mathrm{e}^{-X}}) - A(X + \cancel{\mathrm{e}^{-X}}) + B$$

The integrating factor method yields

$$\begin{aligned} Y_1 \mathrm{e}^X &= \int \mathrm{e}^X [AX\mathrm{e}^{-X} - AX + B] \mathrm{d}X \\ &= \int [AX - AX\mathrm{e}^X + B\mathrm{e}^X] \mathrm{d}X \\ &= \frac{1}{2} AX^2 - A[X\mathrm{e}^X - \mathrm{e}^X + C] + B\mathrm{e}^X \\ Y_1 &= \frac{1}{2} AX^2 \mathrm{e}^{-X} - A(X-1) - AC\mathrm{e}^{-X} + B \\ &= AX\left(\frac{1}{2} X\mathrm{e}^{-X} - 1\right) + \underbrace{A + B - AC\mathrm{e}^{-X}}_{E(1-\mathrm{e}^{-X})?} \end{aligned}$$

Intermediate region matching: The simplest, or "primitive" matching (given by Prandtl) is the lowest order match.

$$\begin{gathered} \lim_{X\to\infty} Y_0(X) = \lim_{x\to 0} y_0(x) \\ A = 2 \end{gathered}$$

Then our composite solution would be

$$\begin{aligned} y_c(x) &= Y_{\text{inn}} + y_{\text{out}} - y_{\text{match}} \\ &= 2(1 - \mathrm{e}^{-x/\epsilon}) + \frac{2}{1+x} - 2 \\ &= 2\left[\frac{1}{1+x} - \mathrm{e}^{-x/\epsilon}\right] \end{aligned} \tag{33}$$

NB. In the higher order match, y_{match} may be a function and not a number. [To do] Use Wronskian method too?
In the higher order match, noting that $\epsilon \to 0^+$ and $x \to 0$, we keep terms of $O(x, \epsilon)$ and discard terms of $O(x^2, \epsilon x, \epsilon^2)$. Going to a still higher order, we would discard $O(x^3, \epsilon^3)$ terms. Note that the implicit assumption is

$$x^2 \ll \epsilon$$

B&O mention that at each higher order of matching, the matching region gets smaller and smaller. The ordering

$$\epsilon \ll x \ll \epsilon^{1/2}$$

emerges since the overlap region is multiple ϵ to the right of the left boundary. Contrast this with the $O(\epsilon^0)$ overlap region

$$\epsilon \ll x \ll 1$$

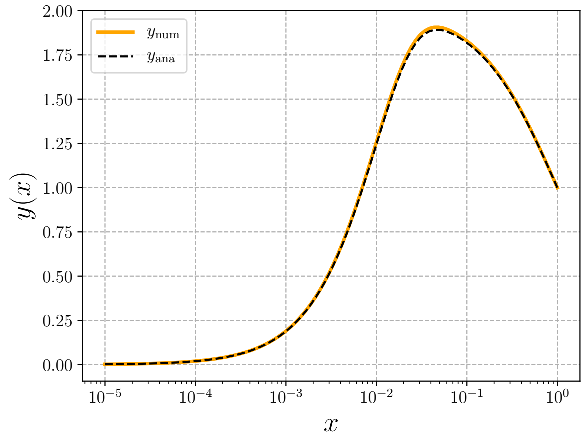

Figure 13.1: Solving eqn. 32 in Python using the `scipy.integrate.solve_bvp` module along with the $O(\epsilon^0)$ composite solution given by eqn. 33.

Expanding the outer solution for small x, ϵ:

$$\begin{aligned} y_{\mathrm{out}} &= \frac{2}{1+x} + \epsilon\left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right] + O(\epsilon^2) \\ &= 2[1 - x + O(x^2)] + \epsilon[2(1-3x) - (1-x)/2] + O(\epsilon^2) \\ &= 2(1-x) + \frac{3}{2}\epsilon + O(x^2, \epsilon x, \epsilon^2) \end{aligned}$$

Similarly for the inner region, $X \gg 1$, but again, $O(\epsilon^2 X^2, \epsilon^2, \epsilon^2 X)$ terms are neglected

$$\begin{aligned} Y_{\mathrm{inn}} &= 2(1 - \mathrm{e}^{-X}) + \epsilon\left[2X\left(\frac{1}{2}X\mathrm{e}^{-X} - 1\right) + E(1 - \mathrm{e}^{-X})\right] + O(\epsilon^2) \\ &= 2 + \epsilon\left[-2X + E\right] + O(\epsilon^2) + \mathrm{TST} \\ &= 2 - 2x + \epsilon E \end{aligned}$$

Now the inner and outer solutions match for

$$E = \frac{3}{2}$$

and

$$y_{\mathrm{match}} = 2 - 2x + \frac{3}{2}\epsilon + \dots$$

Lecture 14

Location and Thickness of Boundary Layers

Example 1: Consider

$$\epsilon y'' - y' + y = 0$$
$$y(0) = 0 \qquad y(1) = 1$$

First we find the outer solution, i.e. $\epsilon \to 0^+$:

$$-y_0' + y_0 = 0 \qquad \implies y_0 = A\mathrm{e}^x$$

But which BC to apply? In BL we expect the solution to decay exponentially as we move away from the BL. Since the solution is growing, we expect the BL to be at the right boundary such that as we move into the domain, the solution decays. So we expect the BL to be at $x = 1$.

But let's suppose the BL is at $x = 0$. Then we apply the BC at $x = 1$, i.e.

$$y_0(x)_{\text{out}} = \mathrm{e}^{x-1}$$

The inner solution: Still assuming the BL at $x = 0$, we assume thickness $O(\delta)$, where $\delta = \delta(\epsilon)$ is to be determined. Let

$$X = \frac{x}{\delta}$$

and we are interested in the ODE

$$y(x)_{\text{inn}} = Y(X)$$

which is achieved through

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}x} = \frac{1}{\delta}Y_X$$
$$y'' = \frac{\mathrm{d}y'}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}x} = \frac{1}{\delta^2}Y_{XX}$$

Note that since the ODE is linear, we can always write $Y = cy$ without affecting the equation. But in nonlinear (and some linear) problems, this c is to be determined. Our ODE becomes

$$\epsilon \frac{1}{\delta^2} Y_{XX} - \frac{1}{\delta} Y_X + Y = 0$$

We now make a dominant balance argument: if we have chosen δ correctly, it would mean that Y, Y_X, Y_{XX} are $O(1)$. Is it then

1. As $\epsilon \to 0^+$

$$\frac{\epsilon}{\delta^2} \approx \frac{1}{\delta} \gg 1?$$

This would mean $\delta \approx \epsilon$ and $\ll 1$ which is consistent.

2. It is also possible that

$$\frac{\epsilon}{\delta^2} \approx 1 \gg \frac{1}{\delta}$$

Therefore $\delta \approx \epsilon^{1/2} \ll 1$. But the ordering suggests $\delta \gg 1$, i.e. inconsistent.

3. Finally,

$$\frac{1}{\delta} \approx 1 \gg \frac{\epsilon}{\delta^2}$$

This means $\delta \approx 1$ and $\epsilon \ll 1$. The latter is OK, but $\delta \approx 1$ means the outer solution and not the inner, i.e. layer, solution!

So we conclude that the BL has a thickness

$$\delta = O(\epsilon) \qquad \Longrightarrow \delta = \epsilon$$

and if only *one case* gives a consistent and non-trivial balance (i.e. not the outer solution), we refer to it as a "distinguised limit".

- Whilst this analysis is predicated on the wrong assumption about the location of the BL, nevertheless, the argument to determine the thickness is correct.
- We know the position of the BL is incorrect as we will fail to match the inner and outer solutions.

Our ODE for the inner problem is finally

$$Y'' - Y' + \epsilon Y = 0$$
$$Y(0) = 0$$

Proceeding with the usual ansatz we derive

$$(Y_0'' + \epsilon Y_1'' + \dots) - (Y_0' + \epsilon Y_1' + \dots) + \epsilon(Y_0 + \epsilon Y_1 + \dots) = 0$$

This yields at $O(\epsilon^0)$

$$Y_0'' - Y_0' = 0$$

Solving this

$$Y_0(X) = B(1 - \mathrm{e}^X)$$

Our solution is growing exponentially away from $x = 0$ – this is a sign of trouble! Solutions usually decay away from the end-point of the BL. Let's try to match:

$$\lim_{X\to\infty} Y_0(X)_{\mathrm{inn}} = \lim_{x\to 0} y_0(x)_{\mathrm{out}}$$
$$-B(\infty) = \mathrm{e}^{-1}$$

The matching is impossible and this is because we incorrectly guessed the location of the BL. If instead we took the BL at $x = 1$

$$y_0(x)_{\mathrm{out}} = A\mathrm{e}^x$$

and the BC of interest is that at $x = 0$, i.e. $A = 0$ and

$$y(x)_{\mathrm{out}} = 0$$

to all orders as can be easily checked. Now we are used to $X \to \infty$ in our BL when the BL is near $x = 0$.

If the BL is at $x = 1$, we take

$$X = \frac{1-x}{\delta}$$
$$0 \le x \le 1 \qquad 0 \le X \le \frac{1}{\delta}$$

Proceeding like previously

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}x} = -\frac{1}{\delta}Y_X$$
$$y'' = \frac{\mathrm{d}y'}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}x} = \frac{1}{\delta^2}Y_{XX}$$

and our ODE becomes

$$Y_{XX} + Y_X + \epsilon Y = 0$$

Notice the positive sign on Y_X with was negative previously. Solving this at $O(\epsilon^0)$

$$Y_0(x) = C + D\mathrm{e}^{-X}$$
$$Y_0(0) = 1 = C + D$$

since we are looking at $x = 1$ or $X = 0$. Therefore

$$Y_0(X) = C + (1 - C)\mathrm{e}^{-X}$$

and matching this

$$\begin{gathered}\lim_{X\to\infty} Y_0(X) = \lim_{x\to 1} y_0(x)\\ C = 0\end{gathered}$$

The composite solution at $O(\epsilon^0)$ is therefore

$$\begin{aligned} y_c &= y_{\text{out}} + Y_{\text{inn}} - y_{\text{match}}\\ &= 0 + \mathrm{e}^{-X} - 0\\ &= \mathrm{e}^{(x-1)/\epsilon} + O(\epsilon)\end{aligned}$$

Note that our matched asymptotic does not satisfy the value at the left boundary. It just so happens to be transcedentally small and gets the BC $\sim$negligibly wrong.

Useful shortcut: For general second order BVP of the form

$$\begin{gathered}\epsilon y'' + a(x)y' + b(x)y = 0\\ y(0) = A \qquad y(1) = B\end{gathered}$$

Note this equation is (1) linear in y and (2) homogeneous. Then

- if $a(x) > 0$ for all $x \in [0, 1]$ then BL is at $x = 0$
- if $a(x) < 0$ for all $x \in [0, 1]$ then BL is at $x = 1$
- if $a(x) = 0$ somewhere in $x \in [0, 1]$ then we can have an <u>interior layer</u>.

Example 2:

$$\begin{gathered}\epsilon y'' + x^2 y' - y = 0\\ y(0) = 1 \qquad y(1) = 1\end{gathered}$$

Note that $a(x) = 0$ at the left end point and not strictly greater than zero. This only means there is a layer at zero. Proceeding as usual, the <u>outer</u> solution is found by setting $\epsilon = 0$, which yields

$$y_0 = A\mathrm{e}^{-1/x}$$

Applying the BC at $x = 1$

$$y_0(x) = \mathrm{e}^{1-1/x}$$

To solve the <u>inner</u> problem, set

$$X = \frac{x}{\delta}$$
$$X = O(1) \qquad \delta \ll 1$$

which leads to

$$\epsilon \frac{1}{\delta^2} Y_{XX} + X^2 \delta^2 \frac{1}{\delta} Y_X - Y = 0$$

Next consider the three balances again. The consistent dominant balance is

$$\frac{\epsilon}{\delta^2} \approx 1 \gg \delta \qquad \Longrightarrow \delta \approx \epsilon^{1/2}$$

Our governing innder ODE is then

$$Y'' + \epsilon^{1/2} X^2 Y' - Y = 0$$

with our trial solution reading

$$Y = Y_0 + \epsilon^{1/2} Y_1 + \epsilon Y_2 + \ldots$$

At $O(\epsilon^0)$

$$Y_0'' - Y_0 = 0 \qquad Y_0(0) = 1$$

Solving this we arrive at

$$Y_0(X) = A\mathrm{e}^{X} + B\mathrm{e}^{-X}$$

Now observe that our matching condition requires taking $X \to \infty$. Therefore, with this forethought, we set $A = 0$. The BC now means that $B = 1$. Together

$$Y_0(X)_{\mathrm{inn}} = \mathrm{e}^{-X}$$

Observe that both inner and outer solutions match, with $y_{\mathrm{match}} = 0$, in the overlap region. The composite solution is then

$$y_c = \mathrm{e}^{1-1/x} + \mathrm{e}^{-x/\sqrt{\epsilon}} + O(\epsilon^{1/2})$$

NB. Suppose we wanted to go to the next order, we would need $O(\epsilon)$ terms in y_{out}, which means we would need one more term in the outer, yet two more terms in the inner: $O(\sqrt{\epsilon})$ and $O(\epsilon)$. The inner solution takes twice the effort!

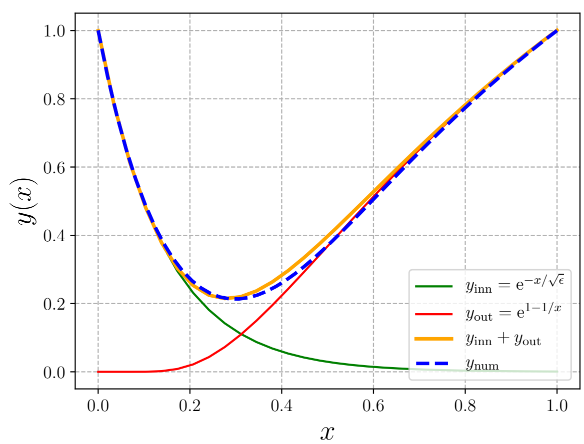

Figure 14.1: Various solution terms with $\epsilon = 0.02$. Notice the thickness of the boundary layer.

Lecture 15

Corner Layers

What if the BL does not appear at $x = 0$ or $x = 1$ (i.e. boundary of domain) and the rapid variation appears somewhere in the interior of the domain?

Example:

$$\epsilon y'' + xy' - y = 0 \qquad -1 \leq x \leq 1$$
$$y(-1) = 1 \qquad y(1) = 2$$

In the sub-interval $x \in [-1, 0)$, $a(x) = x < 0$, so expect the BL to occur at the right end point. Likewise, in the sub-interval $x \in (0, 1]$, $a(x) > 0$ and the BL is at the left end point. Hence some kind of "interior layer" occurs at $x = 0$. The point $x = 0$ is referred to as a "turning point" since a change in the sign of $a(x)$ occurs.

In the outer region, the governing equation at $O(\epsilon^0)$ is

$$xy_0' - y_0 = 0$$

Separation of variable yields

$$y_0 = cx$$

These are straight line solutions going through the origin. Since the BL is at $x = 0$, we can safely apply the BC at both ends.

$$y_0^R = 2x \qquad y_0^L = -x$$

Note that the outer solution (y_0^L and y_0^R) are continuous at $x = 0$, but has a discontinuity in the derivative. It is called a "corner layer" for this reason – the two slopes are meeting at a corner (as $\epsilon \to 0^+$).

The next order of business is to find the scale-length $\delta = \delta(\epsilon)$ in the inner layer. We re-write the ODE in terms of the re-scaled variable $X = x/\delta$. This yields

$$\epsilon \frac{1}{\delta^2} Y'' + X\delta \frac{1}{\delta} Y' - Y = 0$$

It is apparent that the second and third terms are of the same order. Therefore it may be that the dominant balance argument involves all three terms.

$$\frac{\epsilon}{\delta^2} \approx 1 \approx 1$$

The inner layer ODE is then

$$Y_{XX} + XY_X - Y = 0$$

with $\delta = \sqrt{\epsilon}$. The general solution of this can be written in terms of *parabolic cylinder functions*[1]. Our trial solution

$$Y = \mathrm{e}^{-X^2/4} W$$

yields

$$\begin{aligned} Y' &= \mathrm{e}^{-X^2/4}\left(W' - \frac{XW}{2}\right) \\ Y'' &= \mathrm{e}^{-X^2/4}\left(W'' - XW' - \frac{W}{2} + \frac{X^2 W}{4}\right) \end{aligned}$$

leading to the ODE

$$W'' + \left(-\frac{3}{2} - \frac{X^2}{4}\right) W = 0 \tag{34}$$

which can be solved for $W(X)$ to determine $Y(X)$. The above equation has a known solution in terms of the parabolic cylinder function:

$$W(X) = AD_\nu(X) + BD_\nu(-X)$$

where $\nu + 1/2 = -3/2$. We next find A and B by matching the inner solution to y^L and y^R. As we go the overlap region where the matching takes place, $X \to \pm\infty$. So we need to understand the large X behaviour of D_{-2}. Quoting B&O

$$\begin{aligned} D_\nu(t) &\sim t^\nu \mathrm{e}^{-t^2/4} & t \to \infty \\ D_\nu(-t) &\sim \frac{1}{t^{\nu+1}} \mathrm{e}^{t^2/4} \frac{\sqrt{2\pi}}{\Gamma(-\nu)} & t \to \infty \end{aligned}$$

The above results can be derived by taking the asymptotic behavior of eqn. 34 as done in the earlier part of this course (integral representation or $y = \mathrm{e}^S$ etc).

[1] Related to quantum mechanical harmonic oscillators and Hermite polynomials.

$$\begin{aligned} Y(X) &\sim \mathrm{e}^{-X^2/4}\left[A\frac{1}{X^2}\mathrm{e}^{-X^2/4} + BX\mathrm{e}^{X^2/4}\sqrt{2\pi}\right] \\ &\sim \underbrace{A\frac{1}{X^2}\mathrm{e}^{-X^2/2}}_{\text{TST}} + B\sqrt{2\pi}X \\ &\sim B\sqrt{2\pi}X \end{aligned}$$

Likewise

$$Y(X) \sim -A\sqrt{2\pi}X$$

as $X \to -\infty$. Matching the left boundary

$$\begin{aligned} -X\sqrt{\epsilon} &= -A\sqrt{2\pi}X \\ A &= \sqrt{\frac{\epsilon}{2\pi}} \end{aligned}$$

Similarly, at the right boundary

$$B = 2\sqrt{\frac{\epsilon}{2\pi}}$$

Pulling everything together

$$\begin{aligned} &Y_{\text{inn}} \sim \sqrt{\frac{\epsilon}{2\pi}}\mathrm{e}^{-x^2/(4\epsilon)}\left[D_{-2}\left(\frac{x}{\sqrt{\epsilon}}\right) + 2D_{-2}\left(-\frac{x}{\sqrt{\epsilon}}\right)\right] \\ &y_{\text{out}}^L \sim -x \\ &y_{\text{out}}^R \sim 2x \end{aligned}$$

On the right boundary (as well as the left boundary)

$$\begin{aligned} y_c(x,\epsilon) &= Y_{\text{inn}} + y_{\text{out}}^R - y_{\text{match}} \\ &= Y_{\text{inn}} + 2x - 2x \\ &= Y_{\text{inn}} \end{aligned}$$

i.e. the inner solution gives the correct asymptotic in the outer region as well! This is the uniformly valid solution.

[To do] There is an alternative way of solving this: first note that

$$y = c_1 x$$

is a solution to eqn. 34. We can then use reduction of order (see D'Alembert's reduction) to determine the other solution. The general solution is then

$$Y(X) = c_1 X - c_2\mathrm{e}^{-X^2/2} - c_2 X\sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{X}{\sqrt{2}}\right)$$

In the original unscaled variables, the derivative $\mathrm{d}y/\mathrm{d}x$ tends to -1 on the left and $+2$ on the right. So in terms of the inner variable, the gradients $\mathrm{d}Y/\mathrm{d}X$ tend to $-\sqrt{\epsilon}$ and $+2\sqrt{\epsilon}$ (rather than impose the exact BCs $y(-1) = 1$ etc.). In other words, we are performing a lowest order match and applying it to the derivative of y rather than y itself. This yields

$$c_1 - \sqrt{\frac{\pi}{2}} c_2 = 2\sqrt{\epsilon}$$
$$c_1 + \sqrt{\frac{\pi}{2}} c_2 = -\sqrt{\epsilon}$$

These yield

$$c_1 = \frac{\sqrt{\epsilon}}{2} \qquad c_2 = -\frac{3\sqrt{\epsilon}}{\sqrt{2\pi}}$$

Lecture 16

A Tricky Nonlinear Boundary Value Problem

The problem discussed today has been solved in Mark Holmes (1995) pp.69–73; the analysis here diverges from that in the book (so care needed)!

Example: This nonlinear BVP shows another kind of interior layer – a *shock layer* – which is different from a *corner layer* discussed in the previous lecture.

$$\epsilon y'' = \underline{yy'} - y$$
$$y(0) = 1 \qquad y(1) = -1$$
$$0 < x < 1 \qquad \epsilon \to 0^+$$

- In nonlinear problems, linear superpositions of solutions are not necessarily solutions anymore.
- Unlike in the linear case, there is no theory that tells us where layers are. So one approach is to try layers at $x = 0$, $x = 1$ or $x = x_0$ and try to match the solutions like usual. However, in nonlinear problems there may be multiple layers! Plus this could be a lot of work...

The approach here is to perform a *phase plane analysis*. Regard x as time t and introduce a new variable $z = y'$ which yields the coupled equation

$$y' = z$$
$$z' = \frac{1}{\epsilon} y(z-1)$$

on the (y, z) phase-plane. So at every point (y, z), there is a corresponding "velocity" vector-field (y', z') which tells us which direction to move in at that instantaneous time/point. Note the following:

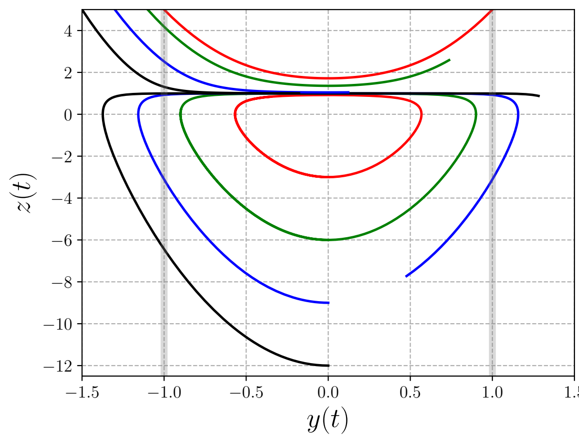

Figure 16.1: Here $\epsilon = 0.1$. The vertical gray lines at $y(t) = \pm 1$ are the boundary values. The "particles" above $z = 1$ were launched at $z = 5$ while those below this line were launched at $y = 0$. The "open" curves ($z > 1$) are run for $t = 1$; the "closed" curves ($z < 1$) are run for $t = 3$.

- The second order ODE is determined either by the boundary values ($y = \pm 1$) or initial values. The latter corresponds to specifying $(y, y') = (y, z)$ at $t = 0$. This is plotted in Fig. 16.1 for a variety of ICs.
- For a chosen value (duration) of the parameter $t = t(x) = x$, the "correct" y' is the value of $z(0)$ which satisfies the BC stipulated by our BVP.
- Alternatively, for each initial $z(0)$, we can always find a value of t which will satisfy the left BC. But since $t = t(x)$ and $x \in (0, 1)$ is prescribed, there is a "correct" $t = 1$.

We can prove that all the trajectories that lie below the *invariant* line $z = 1$ are closed curves. Use symmetry in this particular system:

$$
\begin{aligned}
y' &= z \\
z' &= \frac{1}{\epsilon} y(z - 1)
\end{aligned}
$$

My argument does not appear correct for $z < 1$ and $z > 1$. Observe that if $z > 1$, both z and $z - 1$ remain positive for all values of z and under the transformation

$$
\begin{aligned}
z &\to z \\
x &\to -x \\
y &\to -y
\end{aligned}
$$

the coupled ODE remains the same. In other words, if $(y(x), z(x))$ is a trajectory, so is $(-y(-x), z(-x))$. So as x (equivalently time t) runs backwards,

$z(-t) = z(t)$ remains unchanged whereas $y(-t) = -y(t)$, i.e. there is a reflection about $y = 0$. The trajectories are therefore *closed curves*. Such ODEs are called "reversible". However if $z < 1$, z and $z - 1$ can be $+/-$, breaking the symmetry and the curves are no longer closed.

So far we have not used the fact that $0 < \epsilon \ll 1$. Consider two case:

- If $y \neq 0$ and $z - 1 = O(1)$, i.e. we are not too close to that invariant line $z = 1$, then $1/\epsilon \to \infty$ meaning that $z' = O(1/\epsilon) \gg y'$. So the vertical velocity is huge and the variation in z is very rapid cf. y. This is the dynamical equivalent of a "layer" or <u>inner</u> solution!

- Otherwise, if $z - 1 = O(\epsilon)$, then $y' \approx z' = O(1)$. This corresponds to the <u>outer</u> solution.

To know where the layer may occur, we can try launching a particle with different $y'(0)$ as shown in Fig. 16.2. From Fig. 16.1 the slope $y'(t) = z(t)$ peaks at $y = 0$ and is therefore suggestive of an "interior layer" near $t = 0.5$ if we pick the *symmetric solution*. Note that other solutions are possible and this has been discussed in depth by Clark et al.

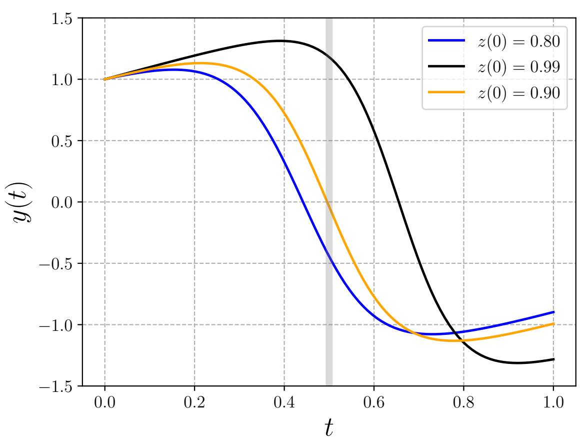

Figure 16.2: We start with different launch angles close to $z = 1$ to understand the presence of the layer.

We now revert to standard BL thinking and construct a solution of this type. The <u>outer</u> solution is obtained by formally setting $\epsilon = 0$. To the lowest order

$$y_0 y_0' - y_0 = 0$$

So either

$$y_0 = 0 \qquad \text{or} \qquad y_0 = x + a$$

Using the BCs

$$y_0^L = x + 1$$
$$y_0^R = x - 2$$

To deal with the inner region, we define

$$X = \frac{x - 1/2}{\delta}$$

yielding

$$y' = \frac{\mathrm{d}y}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}x} = \frac{1}{\delta}Y_X$$
$$y'' = \frac{1}{\delta^2}Y_{XX}$$

where $y(x) = Y(X\delta + 1/2)$ together with the governing ODE

$$\epsilon\frac{1}{\delta^2}Y'' = Y\frac{1}{\delta}Y' - Y$$

From dominant balance

$$\frac{\epsilon}{\delta^2} \approx \frac{1}{\delta} \gg 1$$

yielding $\delta = \epsilon$ and the inner equation

$$Y'' = YY' - \epsilon Y$$

To the lowest order ignoring $O(\epsilon)$ terms

$$Y_0'' = Y_0 Y_0'$$

which is integrated to derive

$$Y_0' = \frac{1}{2}Y_0^2 + A$$

i.e. the arcs in the phase-plane are parabolas to the lowest order. Observe that at $Y = 0$ our intercept has to be negative, so for convenience we can set

$$A = \frac{1}{2}b^2$$

This yields

$$\frac{\mathrm{d}Y_0}{Y_0^2 - b^2} = \frac{1}{2}\mathrm{d}X$$

Note that

$$|x| < 1 \qquad\qquad |x| > 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}x}(\coth^{-1} x) = \frac{1}{1-x^2}$$

This allows us to integrate to write

$$Y_0(X) = b \tanh\left(c - \frac{b}{2}X\right)$$

Noting that $Y_0(X = 0) = 0$ (from symmetry of this solution), we see $c = 0$. Matching at the left boundary

$$\begin{aligned}\lim_{x\to 1/2} y_0^L(x) &= \lim_{X\to -\infty} Y_0(X) \\ \frac{3}{2} &= b\end{aligned}$$

The right boundary matching condition also yields the same value for b. Collecting our solutions:

$$\begin{aligned} y_0^L(x) &\sim x + 1 \\ y_0^R(x) &\sim x - 2 \\ Y(X) &\sim -\frac{3}{2}\tanh\left(\frac{3}{4}X\right)\end{aligned}$$

On the left half

$$\begin{aligned} y_c &\sim y_0^L(x) + Y(X)_{\text{inn}} - y_{\text{match}} \\ &\sim (x+1) - \frac{3}{2}\tanh\left[\frac{3}{4}\left(\frac{x-1/2}{\epsilon}\right)\right] - \frac{3}{2}\end{aligned}$$

On the right half

$$\begin{aligned} y_c &\sim y_0^R(x) + Y(X)_{\text{inn}} - y_{\text{match}} \\ &\sim (x-2) - \frac{3}{2}\tanh\left[\frac{3}{4}\left(\frac{x-1/2}{\epsilon}\right)\right] + \frac{3}{2}\end{aligned}$$

The two composite solutions are identical and the equation is odd about $x = 1/2$.

Lecture 17

An application to systems biology: Michaelis-Menten model

This is a model for enzyme kinetics: relates the rate of formation of a biochemical product P to the concentrations of a substrate S and enzyme E, via an intermediate complex C.

$$S + E \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C \xrightarrow{k_3} E + P$$

Here k_i are the different reaction rates. The dynamical equations for this system are:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -k_1 SE + k_2 C \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= -k_1 SE + k_2 C + k_3 C \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= k_1 SE - k_2 C - k_3 C \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= k_3 C
\end{aligned}$$

The system is quadratic nonlinear due to the presence of SE. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}(E + C) = 0 \qquad \Longrightarrow E + C = E_0$$

since the total amount of initial enzyme remains the same. Further, there is no complex or product initially. Our initial conditions are therefore

$$\begin{aligned}E(0) &= E_0\\ S(0) &= S_0\\ C(0) &= 0\\ P(0) &= 0\end{aligned}$$

Since $E(t) = E_0 - C(t)$, we are left with three coupled equations. Now the natural small parameter in this system is

$$\epsilon = \frac{E_0}{S_0} \ll 1$$

as is often the case (though not always). This allows us to simplify the problem significantly. We are typically interested in the effective rate law for the production of P, i.e. a simple formula for $\mathrm{d}P/\mathrm{d}t$ in terms of the known rate constants k_i and S_0. Let us first non-dimensionalize the sytem using

$$\begin{aligned}s &= \frac{S}{S_0} \qquad & p &= \frac{P}{S_0}\\ e &= \frac{E}{E_0} \qquad & c &= \frac{C}{E_0}\end{aligned}$$

with $0 \leq s, p, e, c \leq 1$. [Check] Why can't $p > 1$? Our coupled system now reads:

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= -k_1 E_0 s(1-c) + k_2 \frac{E_0}{S_0} c\\ \frac{\mathrm{d}c}{\mathrm{d}t} &= k_1 S_0 s(1-c) - (k_2 + k_3)c\end{aligned}$$

using $e + c = 1$. Note that we do not need the evolution equation for P to solve the coupled sub-system S and C. Now $k_1 E_0$ must have the units of $1/t$. So we introduce the dimensionless time

$$\tau = k_1 E_0 t$$

which leads to the governing equation set

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\tau} &= -s(1-c) + \underbrace{\frac{k_2}{k_1 S_0}}_{\lambda} c\\ \frac{\mathrm{d}c}{\mathrm{d}\tau} &= \frac{1}{\epsilon} s(1-c) - \frac{1}{\epsilon} \underbrace{\frac{k_2 + k_3}{k_1 S_0}}_{\mu} c\\ s(0) &= 1 \qquad c(0) = 0\end{aligned}$$

For posterity also note

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \underbrace{\frac{k_3}{k_1 S_0}}_{\mu-\lambda} c$$

Now there are two ways to analyze this problem.

Phase-plane analysis Recall our equations

$$\begin{aligned} s' &= -\left[s(1-c) - \lambda c\right] \\ c' &= \frac{1}{\epsilon}\left[s(1-c) - \mu c\right] \end{aligned}$$

Since the concentration of the intermediate complex is not expected to change in the steady-state, we examine the $c' = 0$ state in the first quadrant (concentrations are positive). This yields the quasi–steady state approximation

$$s = \frac{\mu c}{1-c}$$

In perturbation theory terms this is the <u>outer</u> solution. The initial concentration is $(s, c) = (1, 0)$. Here

$$s' = -1 \qquad c' = \frac{1}{\epsilon}$$

which means the horizontal component varies rapidly except in an $O(\epsilon)$ neighborhood of $c' = 0$ where $c' = O(1)$ and comparable to s'. On the curve $c' = 0$, we can only move downwards. It can be further shown that anywhere other than the $O(\epsilon)$ region of $c' = 0$, the vector field would push the phase-space points back onto the curve [To do].

Perturbative approach Let us look at the *fast* variation. Define an inner "rapid" time scale

$$T = \frac{\tau}{\epsilon}$$

This yields

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}T} &= -\epsilon s(1-c) + \lambda \epsilon c \\ \frac{\mathrm{d}c}{\mathrm{d}T} &= s(1-c) - \mu c \end{aligned}$$

Proceeding with our usual anstaz, the lowest order equations are:

$$\begin{aligned} \frac{\mathrm{d}s_0}{\mathrm{d}T} &= 0 \\ \frac{\mathrm{d}c_0}{\mathrm{d}T} &= s_0(1-c_0) - \mu c_0 \end{aligned}$$

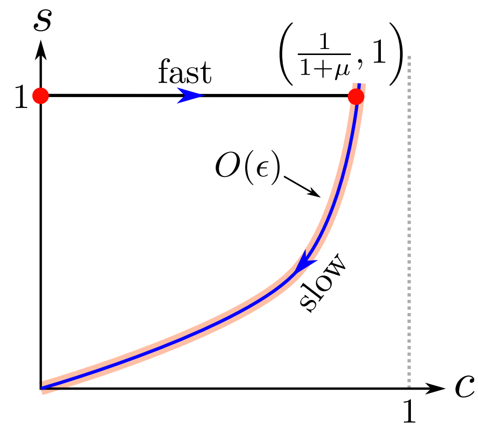

Figure 17.1: The $O(\epsilon)$ neighborhood is shown in orange about the quasi steady-state solution $c' = 0$ shown in blue.

With our initial conditions

$$
\begin{aligned}
s(0) &= 1 = s_0 + \epsilon s_1 + \dots \\
c(0) &= 0 = c_0 + \epsilon c_1 + \dots
\end{aligned}
$$

we find

$$
\begin{aligned}
s_0(T) &= 1 \\
c_0(T) &= \frac{1 - \mathrm{e}^{-(\mu+1)T}}{1+\mu} \\
\implies c(T \to \infty) &= \frac{1}{1+\mu}
\end{aligned}
$$

Now the intermediate complex in steady state does not change its concentration, i.e. $\mathrm{d}c/\mathrm{d}T = 0$. This also yields

$$
\begin{aligned}
c(T) &\approx \frac{s(T)}{\mu + S(T)} \\
c_0(T) &\sim \frac{s_0}{\mu + s_0} = \frac{1}{\mu + 1}
\end{aligned}
$$

Finally, the rate of product formation (except for the initial transient) is

$$
\frac{\mathrm{d}p_0}{\mathrm{d}\tau} = (\mu - \lambda)\frac{s_0}{\mu + s_0} = -\frac{\mathrm{d}s_0}{\mathrm{d}\tau}
$$

The above is referred to as the “Michaelis-Menten kinetics”.

Lecture 18

Introduction to WKB Theory

Part VII of the course.

- WKB theory is good for linear ODEs and PDEs in which the highest derivative is multiplied by $\epsilon \ll 1$
- In the linear context, WKB subsumes the Boundary Layer (BL) theory
- *But* the BL theory can also handle nonlinear problems
- WKB theory is ideal for problems in which there's a global (as opposed to a local) breakdown in the solution as $\epsilon \to 0^+$.
- Arises often in quantum theory, acoustics, optics or other fields that deal with slowly-modulated waves or oscillations

Example 1:

$$\epsilon y'' + y = 0$$
$$y(0) = 0 \qquad y(1) = 1$$

This is easily solved to yield

$$y(x, \epsilon) = \frac{\sin(x/\sqrt{\epsilon})}{\sin(1/\sqrt{\epsilon})} \qquad \epsilon \neq \frac{1}{(n\pi)^2}$$

Note that as $\epsilon \to 0^+$, the solution $y(x, \epsilon)$ becomes more oscillatory over the whole domain – there is no boundary layer. WKB handles problems like this.

WKB approximation We guess a solution of the form

$$y \sim \exp\left[\frac{1}{\delta}S(x)\right]$$
$$S(x) = S_0 + \delta S_1 + \delta^2 S_2 + \dots$$

with δ determined by dominant balance[1]. For posterity we record the terms

$$\begin{aligned} y' &\sim \mathrm{e}^{S(x)/\delta}\left[\frac{S'(x)}{\delta}\right] \\ &\sim \mathrm{e}^{S(x)/\delta}\left[\frac{S_0'}{\delta} + S_1' + \delta S_2' + \dots\right] \\ y'' &\sim \mathrm{e}^{S(x)/\delta}\left[\frac{S''(x)}{\delta} + \frac{1}{\delta^2}\left(S'(x)\right)^2\right] \\ &\sim \mathrm{e}^{S(x)/\delta}\left[\frac{1}{\delta^2}(S_0')^2 + \frac{1}{\delta}(S_0'' + 2S_0'S_1') + (S_1'' + (S_1')^2 + 2S_0'S_2') + \dots\right] \end{aligned}$$

Example 2: The "Schrödinger" equation

$$\epsilon^2 y'' = Q(x)y, \qquad 0 < \epsilon \ll 1$$

is a broad class of second order ODE with the $y'(x)$ term absent. For now assume $Q(x) \neq 0$ for any x. Next proceed to substitute the WKB ansatz to write

$$\frac{\epsilon^2}{\delta^2}(S_0')^2 + \frac{\epsilon^2}{\delta}(S_0'' + 2S_0'S_1') + \epsilon^2(S_1'' + (S_1')^2 + 2S_0'S_2') + \cdots = Q(x)$$

From dominant balance, setting $\delta = \epsilon$ gives us a nice hierarchy of equations:

$$\begin{aligned} O(\epsilon^0): &\qquad (S_0')^2 = Q(x) \\ O(\epsilon^1): &\qquad 2S_1'S_0' + S_0'' = 0 \\ O(\epsilon^2): &\qquad 2S_2'S_0' + (S_1')^2 + S_1'' = 0 \end{aligned}$$

At the lowest order

$$\begin{aligned} S_0' &= \pm\sqrt{Q(x)} \\ \implies S_0 &= \pm\int_a^x \sqrt{Q(t)}\mathrm{d}t \end{aligned}$$

At the next order

$$\begin{aligned} S_1' &= -\frac{1}{4}\frac{Q'(x)}{Q(x)} \\ \implies S_1 &= -\frac{1}{4}\ln Q(x) + c \end{aligned}$$

[1]Louiville and Greene introduced the $\mathrm{e}^{S(x)}$ move.

Note that $Q(x) \neq 0$. Pulling these terms together

$$\begin{aligned} y(x,\epsilon) &\sim \exp\left[\frac{1}{\epsilon}S_0 + S_1 + O(\epsilon)\right] \\ &\sim \frac{1}{Q(x)^{1/4}}\left\{c_1 \exp\left[\frac{1}{\epsilon}\int_a^x \sqrt{Q(t)}\mathrm{d}t\right] + c_2 \exp\left[-\frac{1}{\epsilon}\int_a^x \sqrt{Q(t)}\mathrm{d}t\right]\right\} \end{aligned} \tag{35}$$

Since $\mathrm{e}^{\epsilon f} \approx 1 + \epsilon f + \ldots$, the $O(\epsilon)$ terms may be neglected.

Example 3:

$$\begin{gathered} \epsilon y'' + y' + y = 0 \\ y(0) = 0 \qquad y(1) = 1 \end{gathered}$$

The BL theory provides the uniformly valid approximation

$$y_c \sim \mathrm{e}^{1-x} - \mathrm{e}^{1-x/\epsilon} + O(\epsilon)$$

which has a BL near $x = 0$. Here approach with the WKB method:

$$\begin{aligned} \epsilon\left[\frac{1}{\delta^2}(S_0')^2 + \frac{1}{\delta}(S_0'' + 2S_0'S_1') + (S_1'' + (S_1')^2 + 2S_0'S_2') + \ldots\right] \\ + \left[\frac{1}{\delta}S_0' + S_1' + \delta S_2' + \ldots\right] + 1 = 0 \end{aligned}$$

A consistent hierarchy of equations result when

$$\frac{\epsilon}{\delta^2} \approx \frac{1}{\delta}$$

yielding

$$\begin{aligned} O\left(\epsilon^{-1}\right): &\qquad (S_0')^2 + S_0' = 0 \\ O(\epsilon^0): &\qquad S_0'' + 2S_0'S_1' + S_1' + 1 = 0 \end{aligned}$$

At the lowest order

$$S_0'(S_0' + 1) = 0 \qquad \Longrightarrow \begin{cases} S_0 = \text{const} \\ S_0 = -x + \text{const} \end{cases}$$

The two cases would give us the two linearly independent solutions. At the next order

$$\begin{cases} S_1 = -x + \text{const} \\ S_1 = x + \text{const} \end{cases}$$

Linearly combining the two solutions

$$\begin{aligned} y(x,\epsilon) &\sim \exp\left[\frac{1}{\epsilon}S_0 + S_1 + \dots\right] \\ &\sim c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x/\epsilon + x} \end{aligned}$$

With the application of the BCs

$$\begin{aligned} 0 &= c_1 + c_2 \\ 1 &= c_1 \mathrm{e}^{-1} + c_2 \underbrace{\mathrm{e}^{-1/\epsilon+1}}_{\text{TST}} \end{aligned}$$

In the limit of $\epsilon \to 0^+$, $\mathrm{e}^{-1/\epsilon+1} \to 0$. This yields $c_1 = \mathrm{e}$, $c_2 = -\mathrm{e}$ and

$$\begin{aligned} y(x,\epsilon) &\sim \mathrm{e}^{1-x} - \mathrm{e}^{1+x-x/\epsilon} \\ &\sim \mathrm{e}^{1-x} - \mathrm{e}^{1-x/\epsilon} \end{aligned}$$

Since outside the BL, $x = O(1)$ and the x/ϵ term dominates (the $\mathrm{e}^{1+x-x/\epsilon}$ term is a TST). In the BL, $x \le O(\epsilon) \ll 1$. From both arguments, the term x can be ordered out and we end up with the identical BL solution. And yet, we did not have to do any matching in the overlap region. Therefore this is a *global* method as opposed to a *local* method in which we have to think about the BL.

Example 4: The "slowly aging spring"

$$\ddot{x} + \underbrace{\mathrm{e}^{-\epsilon t}}_{k(t)} x = 0 \tag{36}$$

Here, compared with Hooke's law, the mass is unity and the spring stiffness $k = k(t)$ decays very slowly. Observe that at very long times, specifically $t \gg 1/\epsilon$, $k(t) \approx 0$ and

$$x(t) = vt + \text{const}$$

That is, the oscillations are expected to stop. Let us introduce the slow time

$$T = \epsilon t$$

and rewrite our equation:

$$\epsilon^2 X'' + \mathrm{e}^{-T} X = 0$$

This is a Schrödinger equation with $Q(T) = -\mathrm{e}^{-T}$. Using our earlier results

$$\epsilon^2 \left[\frac{1}{\delta^2}(S_0')^2 + \frac{1}{\delta}(S_0'' + 2S_0'S_1') + (S_1'' + (S_1')^2 + 2S_0'S_2') + \dots\right] + \mathrm{e}^{-T} = 0$$

The choice of $\delta = \epsilon$ gives us the consistent hierarchy

$$\begin{aligned} O(\epsilon^0): &\qquad (S_0')^2 + \mathrm{e}^{-T} = 0 \\ O(\epsilon^1): &\qquad S_0'' + 2S_0'S_1' = 0 \end{aligned}$$

The first order solution is

$$\begin{aligned} S_0' &= \pm \mathrm{i}\mathrm{e}^{-T/2} \\ \implies S_0 &= \mp 2\mathrm{i}\mathrm{e}^{-T/2} \end{aligned}$$

At the next order

$$S_1 = \frac{T}{4}$$

Putting it all together

$$\begin{aligned} X(T) &\sim \exp\left[\frac{1}{\epsilon}S_0 + S_1 + \dots\right] \\ &\sim c_1 \exp\left[-\frac{2\mathrm{i}}{\epsilon}\mathrm{e}^{-T/2} + \frac{T}{4} + \dots\right] + c_2 \exp\left[+\frac{2\mathrm{i}}{\epsilon}\mathrm{e}^{-T/2} + \frac{T}{4} + \dots\right] \\ &\sim \mathrm{e}^{T/4}\left[a \sin\left(\frac{2}{\epsilon}\mathrm{e}^{-T/2}\right) + b\cos\left(\frac{2}{\epsilon}\mathrm{e}^{-T/2}\right)\right] \end{aligned} \tag{37}$$

Note the following:

- There is a slowly growing amplitude
- It may be a good idea to define a nonlinear time $\tau = 2\mathrm{e}^{-\epsilon t/2}/\epsilon$. In fact, under this transformation, the aging spring problem is exactly solvable in terms of the Bessel function!

The analytical and numerical results are compared subject to the conditions

$$x(0) = 1 \qquad x'(0) = 0 \qquad \epsilon = 0.05 \tag{38}$$

We solve for a and b using the equations

$$\begin{aligned} a\sin(2/\epsilon) + b\cos(2/\epsilon) &= 1 \\ -a\cos(2/\epsilon) + b\sin(2/\epsilon) &= -\frac{\epsilon}{4} \end{aligned}$$

The analytical and numerical results are in reasonably good agreement as can be seen from Fig. 18.1.

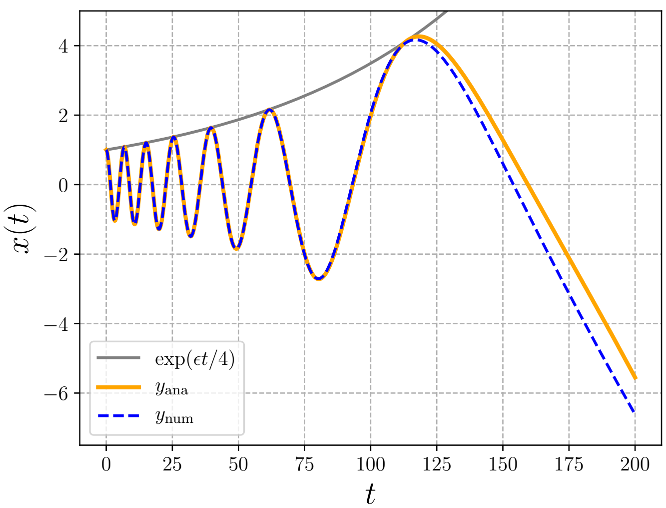

Figure 18.1: Solving eqn. 36 numerically and plotting eqn. 37 subject to the constraints given by eqns. 38.

Lecture 19

Turning Points and Airy Function

Consider

$$\epsilon^2 y'' = Q(x) y$$

There are two distinct behaviors:

- $Q(x) < 0$ gives oscillatory solutions
- $Q(x) > 0$ gives exponential growth/decay

The turning point is a point x_0 that separates the two types of behaviors, i.e.

$$Q(x_0) = 0$$

NB. The simple WKB formalism breaks down near such an x_0: we remedy this by performing a match near the "inner" layer using the Airy function.

Example:

$$\epsilon^2 y'' = (\sinh x \cosh^2 x) y$$

$$y(0) = 1 \qquad y(x \to +\infty) \to 0$$

In quantum theory we assume

$$\int_{-\infty}^{\infty} y^2 \mathrm{d}x$$

is finite, which provides a global (boundary) condition on y. Recall from eqn. 35 that in regions where $Q(x) \neq 0$

$$y(x, \epsilon) \sim \frac{1}{Q(x)^{1/4}} \left\{ c_1 \exp\left[\frac{1}{\epsilon} \int_a^x \sqrt{Q(t)} \mathrm{d}t\right] + c_2 \exp\left[-\frac{1}{\epsilon} \int_a^x \sqrt{Q(t)} \mathrm{d}t\right] \right\}$$

Now consider the two regions:

$\underline{x > 0}$

$$\int_a^x \sqrt{Q(t)}\mathrm{d}t = \int_0^x \sinh^{1/2} t \cosh t \mathrm{d}t$$
$$= \frac{2}{3}(\sinh x)^{3/2}$$

Therefore

$$y(x) \sim \frac{1}{(\sinh x \cosh^2 x)^{1/4}} \left\{ c_1 \exp\left[\frac{2}{3\epsilon}(\sinh x)^{3/2}\right] + c_2 \exp\left[-\frac{2}{3\epsilon}(\sinh x)^{3/2}\right] \right\}$$
$$\sim \frac{1}{(\sinh x \cosh^2 x)^{1/4}} \left\{ c_2 \exp\left[-\frac{2}{3\epsilon}(\sinh x)^{3/2}\right] \right\}$$

Note that the BC $y \to 0$ as $x \to +\infty$ requires that $c_1 = 0$. Next consider its behavior as $x \to 0^+$ (noting $\sinh x \to x$ and $\cosh x \to 1$):

$$y(x)_R \sim \frac{A}{x^{1/4}} \exp\left[-\frac{2}{3\epsilon} x^{3/2}\right]$$

$\underline{x < 0}$

$$\int_0^x \sqrt{Q(t)}\mathrm{d}t = \int_0^{x<0} \pm \mathrm{i}\sqrt{\sinh |t|} \cosh t \mathrm{d}t$$
$$= \pm\frac{2\mathrm{i}}{3}[\sinh(-x)]^{3/2}$$

Note that this is purely imaginary.

$$y(x)_L \sim \frac{1}{(\sinh |x| \cosh^2 x)^{1/4}} \left\{ B \cos\left[\frac{2}{3\epsilon}\sinh^{3/2}|x|\right] + C \sin\left[\frac{2}{3\epsilon}\sinh^{3/2}|x|\right] \right\}$$

As $x \to 0^-$

$$y(x)_L \sim \frac{1}{|x|^{1/4}} \left\{ B \cos\left[\frac{2}{3\epsilon}|x|^{3/2}\right] + C \sin\left[\frac{2}{3\epsilon}|x|^{3/2}\right] \right\}$$

Inner region

Perform the usual change of variable $X = x/\delta$, where $\delta = \delta(\epsilon)$ is determined by dominant balance. Our ODE in the layer reads

$$\epsilon^2 \frac{1}{\delta^2} Y'' = [\sinh(\delta X)\cosh^2(\delta X)]Y$$
$$= [(\delta X + O(\delta^3))(1 + O(\delta^2))]Y$$
$$= [\delta X + O(\delta^3)]Y$$

The only possible balance is

$$\frac{\epsilon^2}{\delta^2} \approx \delta \qquad \Longrightarrow \delta \approx \epsilon^{2/3}$$

and

$$X = \frac{x}{\epsilon^{2/3}} = \left[\frac{x^{3/2}}{\epsilon}\right]^{2/3}$$

where the term inside [.] has shown up in the calculation of y_L and y_R. With this choice of δ, Y satisfies

$$Y'' = XY$$

This is the famous Airy's equation and is typical in turning point problems. Its general solution is

$$Y(X) = \alpha \mathrm{Ai}(X) + \beta \mathrm{Bi}(X)$$

For matching, we need their behavior as $X \to \pm\infty$. This is discussed in B&O. As $X \to \infty$

$$\begin{aligned} \mathrm{Ai}(X) &\sim \frac{1}{2\sqrt{\pi}} \frac{1}{X^{1/4}} \exp\left[-\frac{2}{3}X^{3/2}\right] \\ \mathrm{Bi}(X) &\sim \frac{1}{\sqrt{\pi}} \frac{1}{X^{1/4}} \exp\left[+\frac{2}{3}X^{3/2}\right] \end{aligned}$$

As $X \to -\infty$

$$\begin{aligned} \mathrm{Ai}(X) &\sim \frac{1}{\sqrt{\pi}} \frac{1}{(-X)^{1/4}} \sin\left[\frac{2}{3}(-X)^{3/2} + \frac{\pi}{4}\right] \\ \mathrm{Bi}(X) &\sim \frac{1}{\sqrt{\pi}} \frac{1}{(-X)^{1/4}} \cos\left[\frac{2}{3}(-X)^{3/2} + \frac{\pi}{4}\right] \end{aligned}$$

Match $y_R(x \to 0^+)$ and $Y(X \to \infty)$. Clearly there cannot be a growing Bi term and hence $\beta = 0$.

$$\begin{aligned} \frac{\alpha}{2\sqrt{\pi}} \frac{1}{X^{1/4}} \exp\left[-\frac{2}{3\epsilon}x^{3/2}\right] &\sim \frac{A}{x^{1/4}} \exp\left[-\frac{2}{3\epsilon}x^{3/2}\right] \\ \Longrightarrow \alpha &= \frac{2\sqrt{\pi}}{\epsilon^{1/6}} A \end{aligned}$$

Match $y_L(x \to 0^-)$ and $Y(X \to -\infty)$

$$\begin{aligned} \frac{1}{(-x)^{1/4}} \left\{ B \cos\left[\frac{2}{3\epsilon}(-x)^{3/2}\right] + C \sin[.] \right\} &\sim \frac{\alpha}{\sqrt{\pi}} \left(-\frac{\epsilon^{2/3}}{x}\right)^{1/4} \sin\left[\frac{2}{3\epsilon}(-x)^{3/2} + \frac{\pi}{4}\right] \\ &\sim \frac{\alpha \epsilon^{1/6}}{\sqrt{\pi}} \frac{1}{(-x)^{1/4}} \left\{ \sin[.] \frac{1}{\sqrt{2}} + \cos[.] \frac{1}{\sqrt{2}} \right\} \end{aligned}$$

Clearly

$$B = C = \frac{\alpha \epsilon^{1/6}}{\sqrt{2\pi}} = \sqrt{2} A$$

Altogether

$$\begin{aligned}
y(x)_R &\sim \frac{A}{(\sinh x \cosh^2 x)^{1/4}} \exp\left[-\frac{2}{3\epsilon}(\sinh x)^{3/2}\right] \\
y(x)_L &\sim \frac{2A}{(\sinh |x| \cosh^2 x)^{1/4}} \sin\left[\frac{2}{3\epsilon}\sinh^{3/2}|x| + \frac{\pi}{4}\right] \\
y(x)_{\text{layer}} &\sim \frac{2\sqrt{\pi}A}{\epsilon^{1/6}} \mathrm{Ai}\left[\frac{x}{\epsilon^{2/3}}\right]
\end{aligned}$$

Lecture 20

WKB for Eigenvalue Problems

Example 1: A Sturm-Liouville problem

$$\begin{aligned} y'' + EQ(x)y &= 0 \qquad 0 \leq x \leq \pi \\ y(0) = y(\pi) &= 0 \qquad Q(x) > 0 \end{aligned}$$

The ODE has solutions for a discrete set of eigenvalues E_n and corresponding eigenfunctions y_n ($E_n \to \infty$ as $n \to \infty$).

Interpret y'' as curvature (measure of concavity): if $y'' > 0$ it is a parabola opening upwards, etc. Now

$$y'' = [-EQ(x)]y$$

- if $E > 0$, when $y > 0$, $y'' < 0$, i.e. we curve downwards
- when $y < 0$, $y'' > 0$ and we curve upwards

This implies that we end up with something *sinusoidal*. If the choice of E is not correct, we won't satisfy the right-hand BC. Therefore for a large n, we expect E_n to be large which causes the large number of wiggles. This is suggestive of a natural small parameter

$$\epsilon^2 = \frac{1}{E} \ll 1$$

This translates our problem to a Schrödinger-type equation

$$\epsilon^2 y'' = -Q(x)y$$

Using WKB theory (eqn. 35)

$$y(x) \sim \frac{1}{[Q(x)]^{1/4}} \left\{ c_1 \sin\left[\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\mathrm{d}t\right] + c_2 \cos\left[\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\mathrm{d}t\right] \right\}$$

To satisfy $y(0) = 0$, $c_2 = 0$. To satisfy $y(\pi) = 0$, we require

$$0 = c_1 \sin\left[\frac{1}{\epsilon}\int_0^\pi \sqrt{Q(t)}\mathrm{d}t\right]$$

The non-trivial ($c_1 \neq 0$) solution requires that

$$\frac{1}{\epsilon}\int_0^\pi \sqrt{Q(t)\mathrm{d}t} = n\pi, \qquad n = 1, 2, 3 \ldots.$$

We cannot have negative n since the LHS is positive. Our eigenvalues are therefore

$$E_n = \frac{n^2\pi^2}{\left[\int_0^\pi \sqrt{Q(t)\mathrm{d}t}\right]^2} \qquad \text{as } n \to \infty$$

Example 2: Two turning points: an example from quantum mechanics

$$\epsilon^2 y'' = [V(x) - E]y$$
$$y \to 0 \quad \text{as} \quad |x| \to \infty$$

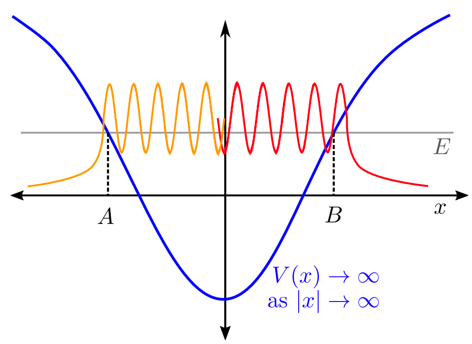

Figure 20.1

From Fig. 20.1, and following the example in sec. 19, we note that to the left of the turning point B, the wave solution must match with the wave solution to the right of the turning point at A. This is what leads to the quantization.

Schrödinger's idea: Solutions $y(x)$ to this BVP exist only if $E = E_n$, where E_n is an eigenvalue. These are the only allowed energies, i.e. the energy is "quantized" (whereas classically, any E is allowed as long as $E \geq V_{\min}$).

Strategy: Using WKB, solve near A and B separately, and then match in the classically permissible region $A < x < B$.

<u>Near $x = B$</u>

Recall from eqn. 35

$$y(x) \sim \frac{c_1}{[V(x)-E]^{1/4}} \exp\left[-\frac{1}{\epsilon}\int_B^x \sqrt{V(t)-E}\,\mathrm{d}t\right] \qquad x > B$$

Then using the "connection formula" (eqn. 10.4.13 in B+O for summary), the matching solution for $A < x < B$ is

$$y(x) \sim \frac{2c_1}{[E-V(x)]^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^B \sqrt{E-V(t)}\mathrm{d}t + \frac{\pi}{4}\right] \tag{39}$$

<u>Near $x = A$</u>

$$y(x) \sim \frac{c_2}{[V(x)-E]^{1/4}} \exp\left[-\frac{1}{\epsilon}\int_x^A \sqrt{V(t)-E}\,\mathrm{d}t\right] \qquad x < A$$

Similarly, the connection formula through $x = A$ yields

$$y(x) \sim \frac{2c_2}{[E-V(x)]^{1/4}} \sin\left[\frac{1}{\epsilon}\int_A^x \sqrt{E-V(t)}\,\mathrm{d}t + \frac{\pi}{4}\right] \tag{40}$$

Equations 39 and 40 are essentially describing the same behavior and must be identical. This gives us the eigenvalue condition on E. The constants c_1 and c_2 need to be adjusted appropriately, but it is the x dependence through the sin[.] terms which need careful consideration.

$$\begin{aligned}
\sin\left[\frac{1}{\epsilon}\int_A^x + \frac{\pi}{4}\right] &= \sin\left[\frac{1}{\epsilon}\int_A^B - \frac{1}{\epsilon}\int_x^B + \frac{\pi}{4}\right] \\
&= -\sin\left[\frac{1}{\epsilon}\int_x^B + \frac{\pi}{4} - \left\{\frac{1}{\epsilon}\int_A^B + \frac{\pi}{2}\right\}\right] \\
&= -\sin\left[\frac{1}{\epsilon}\int_x^B + \frac{\pi}{4}\right]\cos K + \cos\left[\frac{1}{\epsilon}\int_x^B + \frac{\pi}{4}\right]\sin K
\end{aligned}$$

where

$$K = \frac{1}{\epsilon}\int_A^B + \frac{\pi}{2}$$

is a constant (no x dependence). Now <u>iff</u>

$$\begin{aligned}
\sin K &= 0 \\
\implies K &= n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots
\end{aligned}$$

would the x dependence match. Therefore

$$\begin{aligned}\frac{1}{\epsilon}\int_A^B \sqrt{E_n - V(t)}\,\mathrm{d}t &= \left(n - \frac{1}{2}\right)\pi, \qquad n = 1, 2, 3\ldots \\ &= \left(n + \frac{1}{2}\right)\pi, \qquad n = 0, 1, 2, \ldots\end{aligned}$$

since the integrand > 0 on $A < x < B$. The formula works (asymptotically) if either $\epsilon \ll 1$, with E fixed, or, $n \gg 1$ and $E_n \gg 1$ with ϵ fixed (B+O, p.521). This can be understood from the rescaling of the equation

$$\tilde{\epsilon}^2 = \frac{\epsilon^2}{E} \qquad Q(x) = \frac{V(x) - E}{E}$$

Example 3: The quantum-harmonic oscillator

$$V(x) = x^2$$

Here $E = V(x)$ at $x = \pm\sqrt{E}$. Here take $\epsilon = 1$ and $n \gg 1$. The eigencondition is

$$\int_{-\sqrt{E}}^{+\sqrt{E}} \sqrt{E - x^2} = \left(n + \frac{1}{2}\right)\pi$$

Let $x = u\sqrt{E}$

$$E\underbrace{\int_{-1}^{1} \sqrt{1 - u^2}\,\mathrm{d}u}_{\pi/2} = \left(n + \frac{1}{2}\right)\pi$$

$$\implies E_n = 2n + 1, \qquad n = 0, 1, 2, \ldots$$

Here the integral is simply the area of a unit semi-circle in the $u - v$ plane, such that $u^2 + v^2 = 1$. Now WKB tells us that this is valid for $n \gg 1$, but this is in fact exact for all n (can be shown). In physics, the *time-independent 1D Schrödinger* equation reads

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi$$

where $\hbar$ is the Planck's constant divided by 2π. The energy levels come out to be

$$E_n = (2n + 1)\frac{\hbar\omega}{2}$$

Lecture 21

Delayed Bifurcation

Consider a bistable system with operating characteristics as shown in Fig. 21.1.

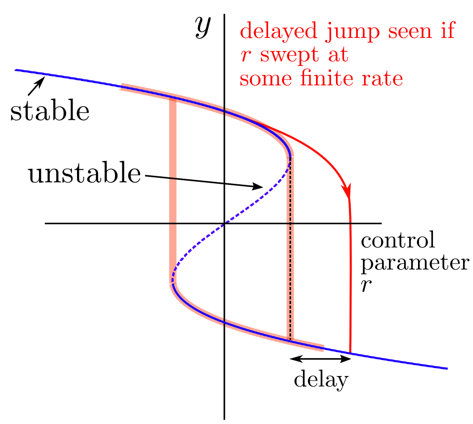

Figure 21.1: The solid/dashed blue curves show the stable/unstable points. As the control parameter r is varied infinitesimally slowly, the translucent "hysteresis" curve is traced. For a non-zero sweep-rate shown in red, a *delayed bifurcation* is seen.

How does the "delay" depend on the sweep-rate of the control parameter? Here the sweep-rate is taken to be small which lets us perform asymptotic analysis. We will show that the delay $\propto \epsilon^{2/3}$ where ϵ is the rate of sweeping ($\epsilon \to 0^+$). Methods from the Boundary Layer theory and Airy's functions will be used.

The curve in Fig. 21.1 is sketched using a functional form $f(y) = y^3/3 - y$. To introduce a time-dependence to our model, we may write

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = y - \frac{y^3}{3} - \epsilon\tau$$

where our control parameter $r = \epsilon\tau$ varies slowly through the introduction of ϵ. Let us first understand the phase-portrait when $\epsilon = 0$ (Fig. 21.2). It is worth noting that the *fixed points* occur when $\mathrm{d}y/\mathrm{d}\tau = 0$, i.e.

$$y = 0 \qquad y = \pm\sqrt{3}$$

Note that when $f(y) > 0$, $y' > 0$, so y increases and the particle is pushed towards the right (etc.), ergo, the *stable* fixed points are $y = \pm\sqrt{3}$. The max/min occur at $f'(y) = 0$, i.e.

$$y = \pm 1 \qquad \implies f(y) = \pm\frac{2}{3}$$

Now if the parameter $r(\tau)$ is slowly varying, this is simply the intercept (constant). The stable and unstable points are seen to come together and eventually coalesce. Eventually, we are left with only one stable point. Therefore, a system which would have started at a large value ($> O(1)$), moves towards the stable point $y = \sqrt{3}$, but as the stable point moves downwards due to $r(\tau)$, it jumps all the way to $y \sim -\sqrt{3}$.

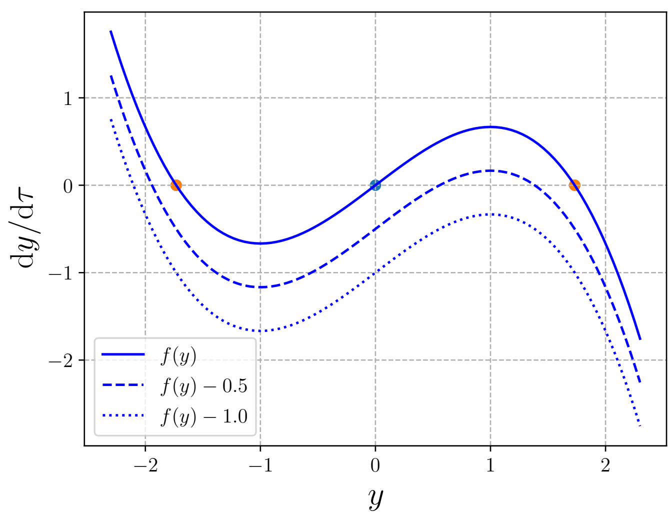

Figure 21.2: Here $r(\tau) = 0.5$. The stable fixed points are shown in pink and the unstable fixed point is shown in blue.

Observe that since $f(y) = 2/3$, we expect the "jump" when $\epsilon\tau \approx 2/3$ (the curve needs to lower by this amount for the stable/unstable points to coalesce). We proceed to analyze this more closely using asymptotics.

Let $t = \epsilon\tau$, which transforms our ODE to

$$\epsilon\frac{\mathrm{d}y}{\mathrm{d}t} = y - \frac{y^3}{3} - t \tag{41}$$

We can see immediately that the <u>outer</u> solution is when $\epsilon = 0$. With the ansatz

$$y = y_0 + \epsilon y_1 + \dots$$

at $O(\epsilon^0)$ we have a simple algebraic equation

$$y_0 - \frac{y_0^3}{3} = t \tag{42}$$

From Fig. 21.1, we see that the sytem has a corner layer and an interior layer. We start at the upper branch where $y_0 > 1$. Recall that the jump occurs at $y_0 = 1$ which corresponds to $t = 2/3$.

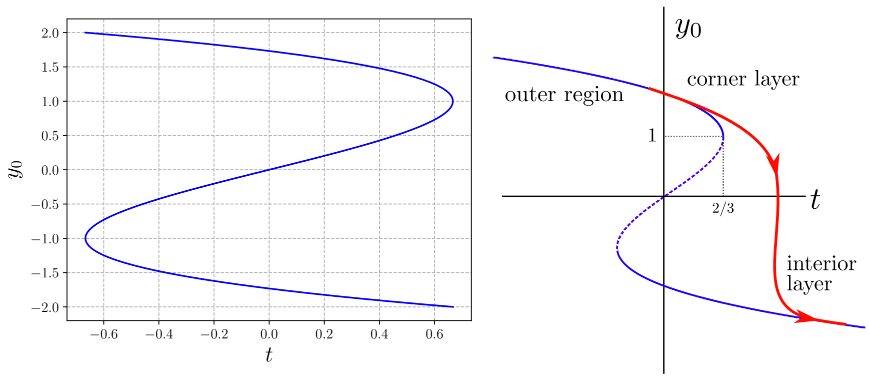

Figure 21.3: (Left) Plot of eqn. 42. (Right) The various regions in the language of asymptotics.

Corner layer: Use "tilde" to denote corner variables. We look near $t \approx 2/3$ and $y \approx 1$. Define

$$\begin{aligned} \tilde{t} &= \frac{t - 2/3}{\epsilon^\alpha} \\ y(t) = \tilde{Y}(\tilde{t}) &= 1 + \epsilon^\gamma \tilde{Y}_1 + \epsilon^{2\gamma}\tilde{Y}_2 + O(\epsilon^{3\gamma}) \end{aligned} \tag{43}$$

In terms of our new variables, eqn. 41 reads[1]

$$\begin{aligned} \frac{\epsilon}{\epsilon^\alpha}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\left(\epsilon^\gamma\tilde{Y}_1 + \epsilon^{2\gamma}\tilde{Y}_2 + \dots\right) &= \left(1 + \epsilon^\gamma\tilde{Y}_1 + \epsilon^{2\gamma}\tilde{Y}_2 + \dots\right) \\ &\quad - \frac{1}{3}\left(1 + \epsilon^\gamma\tilde{Y}_1 + \epsilon^{2\gamma}\tilde{Y}_2 + \dots\right)^3 \\ &\quad - \left[\frac{2}{3} + \epsilon^\alpha\tilde{t}\right] \\ \epsilon^{1-\alpha+\gamma}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\left(\tilde{Y}_1 + \epsilon^\gamma\tilde{Y}_2 + \dots\right) &= \cancel{\left(1 - \frac{1}{3} - \frac{2}{3}\right)} - \epsilon^\alpha\tilde{t} \\ &\quad + \left(\cancel{\epsilon^\gamma\tilde{Y}_1} + \cancel{\epsilon^{2\gamma}\tilde{Y}_2}\right) \\ &\quad - \frac{1}{3}\left(3\cancel{\epsilon^\gamma\tilde{Y}_1} + 3\epsilon^{2\gamma}\tilde{Y}_1^2 + 3\cancel{\epsilon^{2\gamma}\tilde{Y}_2}\right) + O(\epsilon^{3\gamma}) \\ &= -\epsilon^\alpha\tilde{t} - \epsilon^{2\gamma}\tilde{Y}_1^2 + O(\epsilon^{3\gamma}) \end{aligned}$$

[1] Using $(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3ac^2 + 3a^2c + 3bc^2 + 3b^2c$

A three-term dominant balance arises when (no evolution equation for $\tilde{Y}_2$)

$$1-\alpha+\gamma=2\gamma=\alpha$$

This yields

$$\gamma=\frac{1}{3}\qquad \alpha=\frac{2}{3} \tag{44}$$

which already indicates that the corner layer width

$$\Delta t\propto\epsilon^{2/3}$$

Also note that the neglected terms of $O(\epsilon^{3\gamma})=O(\epsilon)\ll\epsilon^{2/3}$. With this choice of α and γ, our ODE in the corner layer reads

$$\frac{\mathrm{d}\tilde{Y}_1}{\mathrm{d}\tilde{t}}=-\tilde{Y}_1^2-\tilde{t} \tag{45}$$

This is the famous "Riccati equation" and is solved with the neat substitution

$$\tilde{Y}_1=\frac{W'}{W} \tag{46}$$

Our corner layer eqn. now reads

$$W''=-\tilde{t}W \tag{47}$$

This is the Airy's equation (except with $-\tilde{t}$ instead of $\tilde{t}$). Also note that our nonlinear ODE (eqn. 45) has been transformed into a linear ODE (eqn. 47) through the clever transform (eqn. 46). The solution to eqn. 47 is simply

$$W(\tilde{t})=a_0\mathrm{Ai}(-\tilde{t})+a_1\mathrm{Bi}(-\tilde{t})$$

The corner solution at the lowest order is then

$$\tilde{Y}_1=\frac{-[a_0\mathrm{Ai}'(-\tilde{t})+a_1\mathrm{Bi}'(-\tilde{t})]}{a_0\mathrm{Ai}(-\tilde{t})+a_1\mathrm{Bi}(-\tilde{t})}$$

Next, we determine a_0 and a_1 to match this to the outer solution. From Fig. 21.3 note that for the outer solution $t<2/3$ everywhere; therefore we want to approach $2/3$ from the negative side, i.e. $t\to 2/3^-$, so with respect to the new variable $\tilde{t}\to-\infty$. Also note that $y_0>1$ and y comes down from above.

From intuition, since $\mathrm{Bi}(\infty)$ tends to blow up, we expect $a_1=0$. But this can we worked out more systematically.

It is worth noting the following asymptotic behavior [in Mathematica notebook by Prof. Strogatz.]

$$\lim_{\tilde{t}\to-\infty}\left[-\frac{\mathrm{Ai}'(-\tilde{t})}{\mathrm{Ai}(-\tilde{t})}\right]=+\sqrt{-\tilde{t}}$$

$$\lim_{\tilde{t}\to-\infty}\left[-\frac{\mathrm{Bi}'(-\tilde{t})}{\mathrm{Bi}(-\tilde{t})}\right]=-\sqrt{-\tilde{t}}$$

To match in the corner layer, we find the asymptotics of the outer solution for $t \to 2/3^-$ and $y_0 \to 1^+$. From eqns. 42, 43 and 44

$$\left(1 + \epsilon^{1/3}\tilde{Y}_1 + \dots\right) - \frac{1}{3}\left(1 + \epsilon^{1/3}\tilde{Y}_1 + \dots\right)^3 = \frac{2}{3} + \epsilon^{2/3}\tilde{t}$$

Comparing powers

$$-\frac{\tilde{Y}_1^3}{3} = 0 \qquad \tilde{Y}_1 = \pm\sqrt{-\tilde{t}}$$

Since we know $y_0 > 1$ on the upper branch, we pick the positive non-zero root. This also implies that we are interested in the $[-\mathrm{Ai}'(-\tilde{t})/\mathrm{Ai}(-\tilde{t})]$ solution which asymptotes to $+\sqrt{-\tilde{t}}$ and discard the Bi terms by setting $a_1 = 0$.

$$\begin{aligned} y_0 &\sim 1 - \epsilon^{1/3}\frac{\mathrm{Ai}'(-\tilde{t})}{\mathrm{Ai}(-\tilde{t})} \\ &\sim 1 + \epsilon^{1/3}\sqrt{-\tilde{t}} \end{aligned}$$

Now

$$\begin{aligned} t &= \frac{2}{3} + \epsilon^{2/3}\tilde{t} \\ &= \frac{2}{3} + \underbrace{\epsilon^{2/3}(2.338\dots)}_{\text{delay}} \end{aligned}$$

since $\tilde{t} = 2.338\dots$ is the point when the function is ‘falling off a cliff’.

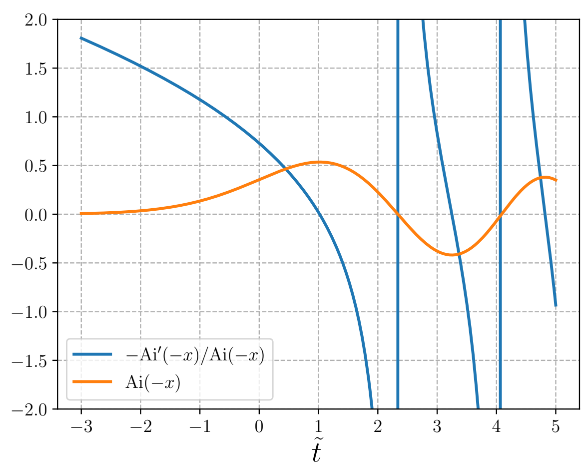

Figure 21.4: The quantity $[-\mathrm{Ai}'(-\tilde{t})/\mathrm{Ai}(-\tilde{t})]$ blows up when Ai crosses its zeros.

Interior layer: From Fig. 21.4 it is clear that the corner layer solution $y_0 \to -\infty$

and a third *interior layer* is therefore needed to match onto the lower branch in Fig. 21.3. In this region we look in the neighborhood of $t > t_0$:

$$t^* = \frac{t - t_0}{\epsilon^\kappa} \qquad t_0 = \frac{2}{3} + \epsilon^{2/3}\tilde{t}_0$$
$$y(t) = Y^*(t^*) \sim Y_0^*(t^*) + \dots$$

The jump is $O(1)$. Our governing ODE becomes

$$\epsilon^{1-\kappa}\frac{\mathrm{d}Y_0*}{\mathrm{d}t^*} = Y_0^* - \frac{Y_0^*}{3} - \left[\epsilon^\kappa t^* + \frac{2}{3} + \epsilon^{2/3}\tilde{t}_0\right] \tag{48}$$

The Y_0^* terms can only balance on both sides when $\kappa = 1$ (usual scaling in layer: time-scale $O(\epsilon)$).

$$\frac{\mathrm{d}Y_0*}{\mathrm{d}t^*} = Y_0^* - \frac{Y_0^*}{3} - \frac{2}{3} \tag{49}$$

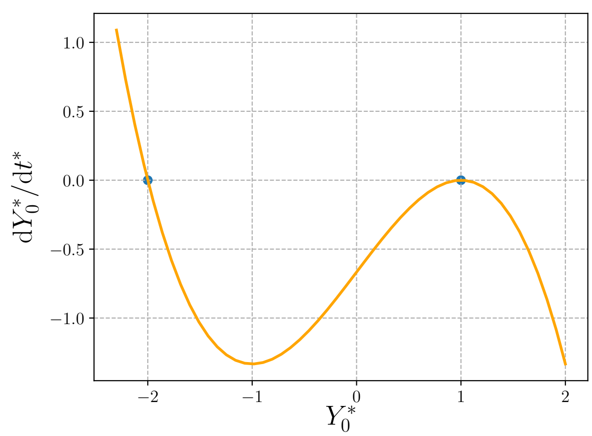

Figure 21.5: Plot of eqn. 49. The left fixed point is stable whereas the right fixed point is half-stable.

Now after we have gone around the corner, $Y_0^* < 1$; this is our initial condition, which then implies that we end up at the left stable fixed point.

Lecture 22

Introduction to the Method of Multiple Scales

Part VIII of the course. Many of these multiple-scale techniques (and related ideas like the Poincaré-Lindstedt method) are conveniently illustrated on problems involving nonlinear oscillators. This is a global perturbation scheme useful in systems characterized by disparate time scales, such as weak dissipation in an oscillator. Classical perturbation methods generally break down because of resonances that lead to what are called *secular*[1] terms.

Example: Duffing oscillator

$$\begin{gathered}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y + \epsilon y^3 = 0, \qquad 0 < \epsilon \ll 1 \\ y(0) = 1 \qquad y'(0) = 0\end{gathered} \tag{50}$$

This can be thought of as a spring problem with a weakly nonlinear restoring force

$$F(y) = -y - \epsilon y^3$$

The cubic term makes it a "hardening spring", i.e. as the displacement y increases, the restoring force becomes much stronger than that in a linear spring ($\epsilon < 0$ makes it a "softening spring"). The Duffing oscillator can be solved exactly in terms of elliptical functions, which in a way serves as a paradigmatic example for demonstrating multiple-scale analysis. Multiplying eqn. 50 with $\mathrm{d}y/\mathrm{d}t$ we see

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\dot{y}^2 + \frac{1}{2}y^2 + \frac{1}{4}\epsilon y^4\right] = 0 \tag{51}$$

[1] *def.* (Latin) of or denoting slow changes in the motion.

i.e. the quantity inside the square bracket – total energy in fact – is conserved (no external damping/forcing present)! Therefore the system is bounded and any admissible solution cannot grow.

Standard analysis: secular growth Now consider the standard (regular) perturbative power-series expansion

$$y(t,\epsilon) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots = \sum_{n=0}^{\infty} \epsilon^n y_n(t) \tag{52}$$

which we assume exists with $y_0(0) = 1$, $\dot{y}_0(0) = 0$ and $y_n(0) = \dot{y}_n(0) = 0$ for $n > 0$. Substituting eqn. 52 into eqn. 50 and equating coefficients of like powers of ϵ up to the first two orders:

$$\ddot{y}_0 + y_0 = 0 \tag{53}$$

$$\ddot{y}_1 + y_1 = -y_0^3 \tag{54}$$

Solution of eqn. 53 which satisfies the boundary conditions is

$$y_0(t) = \cos(t) \tag{55}$$

which implies that we must now solve[2]

$$\ddot{y}_1 + y_1 = -\frac{1}{4}\left[\cos 3t + 3\cos t\right] \tag{56}$$

First, we understand this heuristically: the homogenous solution y_h to the above eqn. is

$$y_h = A\cos t + B\sin t$$

To determine the particular solution y_p, first add a driving term which is not in resonance with the natural frequency in y_h, so

$$y_c = y_h + C\cos 3t + \ldots$$

Now what happens in response to the natural frequency being driven at its resonant frequency by the $\cos t$ term on the right hand of eqn. 56? Since $\cos t$ is already part of the homogeneous term, we plug in a $t\sin t$ form. It is this term which breaks down the regular perturbative treatment as it grows unbounded.

More precisely, we first write the solution to its homogeneous part

$$y_1 = \mathrm{e}^{\mathrm{i}t} \qquad y_2 = \mathrm{e}^{-\mathrm{i}t}$$
$$W = -2\mathrm{i}$$

[2]For such problems, we want the cos and sin in harmonics and not powers. A slick way of converting them is to write $\cos t = (\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t})/2$ and cubing the exponential terms.

The particular solution Y_p is found in Python using the symbolic math module SymPy (Github repo). The constants of integration are absorbed into the homogeneous solution, which upon the application of initial conditions yield

$$\begin{aligned} y &\simeq y_0 + \epsilon y_1 + \cancel{\epsilon^2 y_2 + \dots} \\ &\simeq \cos t + \epsilon \left[\frac{1}{32} \cos 3t - \frac{1}{32} \cos t - \frac{3}{8} t \sin t \right] \end{aligned} \tag{57}$$

The t dependence in y_1 is known as secular growth and arises whenever there is a resonance between y_0 and y_1. There are two clear issues with this treatment:

- The secular term $\epsilon t \sin t$ which grows unbounded as $t \to \infty$
- When $t = O(1/\epsilon)$, y_0 and $y_1 \approx \epsilon t \sin t$ become comparable, but we want $y_0 \gg \epsilon y_1$ for good asymptotics.

So what causes the problem? Our model

$$\ddot{y} + (1 + \epsilon y^2) y = 0$$

has a spring stiffness $k = 1 + \epsilon y^2$. Since the spring gets stiffer with displacement, we expect its frequency to increase too (softer springs will oscillate slowly). We therefore expect the ϵy^2 term to alter the frequency cf. the $\epsilon = 0$ case. But regular perturbation theory forced us to assume the frequency is given by the unperturbed frequency ($\omega = 1$). This is how the errors start accumulating.

NB. If we keep infinitely many terms ($O(\epsilon^2)$ etc) we expect to get the right solution through cancellations. But this is laborious and motivates several alternative approaches.

Poincaré-Lindstedt method Early (no longer state-of-art) method but is instructive and good for approximating periodic solutions, but not for analyzing transients or stability. Begin by letting

$$\tau = \omega t$$

where $\omega = \omega(\epsilon)$ is a power series in ϵ :

$$\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots$$

We will determine ω by insisting that it's the *true frequency* and hence the solution is 2π periodic in τ. This converts the problem to one we can solve with regular perturbation theory:

$$\begin{gathered} \omega^2 \frac{\mathrm{d}^2 Y}{\mathrm{d}\tau^2} + Y + \epsilon Y^3 = 0 \\ Y(0) = 1 \qquad \omega Y'(0) = 0 \end{gathered}$$

where $Y = y(\tau(t))$. Now using a regular perturbative expansion for Y

$$\begin{aligned}(\omega_0 + \epsilon\omega_1 + \dots)^2(Y_0'' + \epsilon Y_1'' + \dots)&\\ +(Y_0 + \epsilon Y_1 + \dots)&\\ +\epsilon(Y_0 + \epsilon Y_1 + \dots)^3 &= 0\end{aligned}$$

with the ICs

$$\begin{aligned} Y_0(0) + \epsilon Y_1(0) + \dots &= 1 \qquad \forall\epsilon \\ (\omega_0 + \epsilon\omega_1 + \dots)(Y_0'(0) + \epsilon Y_1'(0) + \dots) &= 0 \qquad \forall\epsilon \end{aligned}$$

This results in the hierarchy of equations:

$$\begin{aligned} O(\epsilon^0):& \qquad \omega_0^2 Y_0'' + Y_0 = 0 \\ O(\epsilon^1):& \qquad \omega_0^2 Y_1'' + Y_1 = -Y_0^3 - 2\omega_0\omega_1 Y_0'' \end{aligned}$$

The $O(1)$ equation must satisfy the ICs $Y_0(0) = 1$ and $Y_0'(0) = 0$, yielding

$$Y_0(\tau) = \cos\left(\frac{\tau}{\omega_0}\right)$$

Note that in the second IC, we required that $\omega_0 Y_0'(0) = 0$. Since $\omega_0 = 0$ would imply that $Y_0 = 0$ for all time ($O(1)$ equation), violating the first IC $Y_0(0) = 1$, it must be that $\omega_0 \neq 0$.

Now we impose the requirement – taken true at every order – that $Y_0(\tau)$ is 2π periodic in τ. Since we may express periodic functions as

$$\cos\left(\frac{2\pi}{T}t\right) = \cos\left(\frac{2\pi}{T}\frac{\tau}{\omega}\right)$$

where the time-period $T = 2\pi/\omega$, this forces

$$\omega_0 = 1$$

All this is saying that we have chosen ω to be the correct frequency such that $2\pi/\omega$ is the fundamental period of the oscillation. Next, the $O(\epsilon)$ equation reads

$$\begin{aligned} Y_1'' + Y_1 &= -Y_0^3 - 2\omega_1 Y_0'' \\ &= -\frac{1}{4}\cos 3\tau - \underbrace{\left[\frac{3}{4} - 2\omega_1\right]}_{\text{secular}}\cos\tau \end{aligned}$$

Recall that it was this resonant term which was causing the secular growth. Here in fact we have the option of choosing $\omega_1 = 3/8$ to get rid of this term! This also predicts a frequency that varies as

$$\omega = 1 + \frac{3}{8}\epsilon + O(\epsilon^2)$$

We solve the ODE for Y_1'':

$$\begin{aligned} y_h &= A\sin\tau + B\cos\tau \\ y_p &= C\cos 3\tau \end{aligned}$$

First determine C through

$$\begin{aligned} -9C\cos 3\tau + C\cos 3\tau &= -\frac{1}{4}\cos 3\tau \\ \implies C &= \frac{1}{32} \end{aligned}$$

Now the equation

$$Y_1(\tau) = A\sin\tau + B\cos\tau + \frac{1}{32}\cos 3\tau$$

must satisfy the ICs

$$Y_1(0) = 0 \qquad Y_1'(0)\cancelto{1}{\omega_0} + \omega_1 \cancelto{0}{Y_0'(0)} = 0$$

yielding

$$A = 0 \qquad B = -\frac{1}{32}$$

Pulling it all together

$$\begin{aligned} Y(\tau) &= Y_0 + \epsilon Y_1(\tau) + O(\epsilon^2) \\ &= \cos\tau + \epsilon\left[\frac{1}{32}\left(\cos 3\tau - \cos\tau\right)\right] + O(\epsilon^2) \end{aligned}$$

where

$$\tau = \left(1 + \frac{3}{8}\epsilon + \dots\right)t$$

The first term, neglecting $O(\epsilon)$ and higher terms, are also given by eqn. 65 and plotted in Fig. 22.1. The agreement at even this order is quite excellent.

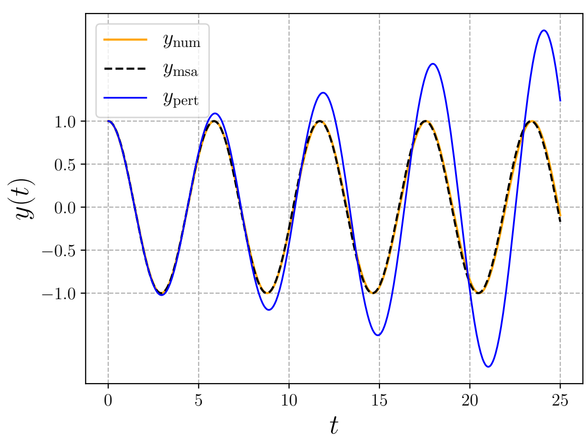

Figure 22.1: Plots of direct numerical solution to eqn. 50, simple perturbative analysis yielding eqn. 57 and solution using multi-scale analysis eqn. 65 for $\epsilon = 0.2$.

Lecture 23

Two Timing

This is the simplest multiple-scale analysis. We can have three or even four timing etc. The two time-scales are

$$\begin{aligned} t &= \text{fast time} \\ \tau = \epsilon t &= \text{slow time} \end{aligned}$$

Regard t and τ as <u>independent</u> variables[1] and write

$$y(t,\epsilon) = Y(t,\tau) = Y_0(t,\tau) + \epsilon Y_1(t,\tau) + \ldots$$

Observing that

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = \frac{\partial}{\partial t} y(t) = \frac{\partial}{\partial t} Y(t,\tau) = \frac{\partial Y}{\partial t}\cancelto{1}{\frac{\partial t}{\partial t}} + \frac{\partial Y}{\partial \tau}\cancelto{\epsilon}{\frac{\partial \tau}{\partial t}}$$

we note for posterity

$$\begin{aligned} \dot{y} &= Y_t + \epsilon Y_\tau \\ \ddot{y} &= (Y_t + \epsilon Y_\tau)_t + \epsilon (Y_t + \epsilon Y_\tau)_\tau \\ &= Y_{tt} + 2\epsilon Y_{t\tau} + \epsilon^2 Y_{\tau\tau} \end{aligned} \tag{58}$$

Example 1: Weakly damped linear oscillator

$$\begin{gathered} \ddot{y} + y + 2\epsilon \dot{y} = 0 \\ y(0) = a \qquad \dot{y}(0) = 0 \end{gathered} \tag{59}$$

Our ODE is transformed to (ignoring $O(\epsilon^2)$ terms)

$$[Y_{tt} + 2\epsilon Y_{t\tau} + \cancel{\epsilon^2 Y_{\tau\tau}}] + Y + 2\epsilon[Y_t + \cancel{\epsilon Y_\tau}] = 0$$

[1] One intuition behind this is that at the faster time scale, the slowly varying dynamics are practically invariant.

Further letting

$$Y = Y_0 + \epsilon Y_1 + O(\epsilon^2)$$

we see that

$$(Y_0 + \epsilon Y_1)_{tt} + 2\epsilon (Y_0)_{t\tau} + (Y_0 + \epsilon Y_1) + 2\epsilon (Y_0)_t = 0$$

Solving the $O(1)$ problem

$$(Y_0)_{tt} + Y_0 = 0$$

yields[2]

$$Y_0 = A\mathrm{e}^{\mathrm{i}t} + \underbrace{A^*\mathrm{e}^{-\mathrm{i}t}}_{\text{c.c}}$$

where $A = A_r + \mathrm{i}A_i$ is a complex number and A^* its complex conjugate. Since $Y_0 = Y_0(t, \tau)$, it must be that the "constant" $A = A(\tau)$. This further provides the intuition that the amplitude is changing on the slow time scale τ. The $O(\epsilon)$ ODE is

$$\begin{aligned}(Y_1)_{tt} + Y_1 &= -2(Y_0)_t - 2(Y_0)_{t\tau} \\ &= -2\mathrm{i}(A + A_\tau)\mathrm{e}^{\mathrm{i}t} + \text{c.c.}\end{aligned}$$

Note that we have a resonant forcing term on the right hand side, which would lead to a secular growth unless this is forced to zero. This provides us with an "amplitude equation"

$$A_\tau + A = 0 \qquad \Longrightarrow A(\tau) = A(0)\mathrm{e}^{-\tau}$$

Satisfying this removes the secularity at this order in ϵ. The next task is to work out the ICs:

$$\begin{aligned}y(0) = Y(0,0) = Y_0(0,0) + \epsilon Y_1(0,0) + \ldots &= a \qquad \forall\epsilon \\ Y_0(0,0) &= a \\ Y_1(0,0) &= 0 \\ &\vdots \\ \dot{y}(0) = Y_t + \epsilon Y_\tau = (Y_0)_t + \epsilon[(Y_1)_t + (Y_0)_\tau] + \ldots &= 0 \qquad \forall\epsilon \\ (Y_0)_t(0,0) &= 0 \\ (Y_1)_t(0,0) + (Y_0)_\tau(0,0) &= 0 \\ &\vdots\end{aligned}$$

[2] Expanding the expression below it is easy to see that this is the most general *real* solution: $Y_0 = 2A_r \cos t - 2A_i \sin t$. This form is advantageous as we would only need to do half as much writing.

We next use these ICs on our $O(1)$ solution

$$
\begin{aligned}
Y_0 &= A(0)\mathrm{e}^{-\tau}\mathrm{e}^{\mathrm{i}t} + \text{c.c.}\\
Y_0(0,0) &= A(0) + A^*(0) = 2\mathrm{Re}[A(0)] = a\\
(Y_0)_t(0,0) &= -2\mathrm{Im}[A(0)] = 0
\end{aligned}
$$

Altogether

$$
\begin{aligned}
Y_0 &= \frac{a}{2}\mathrm{e}^{-\epsilon t}\mathrm{e}^{\mathrm{i}t} + \text{c.c.} + O(\epsilon)\\
&= a\mathrm{e}^{-\epsilon t}\cos t + O(\epsilon) \qquad (60)
\end{aligned}
$$

This result is asymptotically valid for $\epsilon \to 0^+$, uptil $\tau = \epsilon t = O(1)$. Comparison of the analytical and numerical solutions are shown in Fig. 23.1.

It is worth noting that in calculating Y_1, the secular term was eliminated. The solution will remain bounded at $O(\epsilon)$. The secularity is in fact pushed to the $O(\epsilon^2)$ term and appears at a $\epsilon^2 t$ time scale. To push the secular term even further out in time, we would need to perform a three time scale analysis!

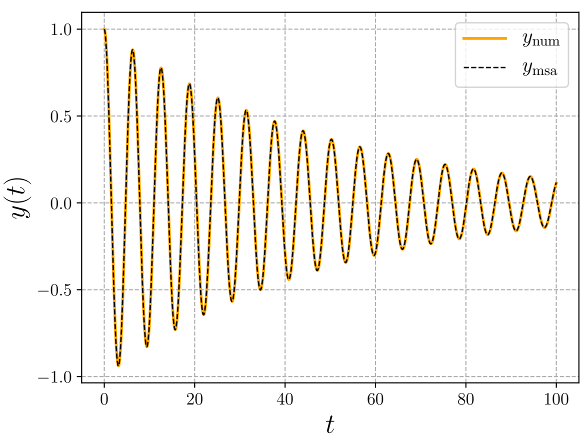

Figure 23.1: Plots of direct numerical solution to eqn. 59 and its analytical solution eqn. 60 for $\epsilon = 0.02$ and $a = 1$.

Example 2: The van der Pol oscillator

$$
\ddot{y} + \epsilon\dot{y}(y^2 - 1) + y = 0, \qquad 0 < \epsilon \ll 1 \qquad (61)
$$

Observe that if $|y| > 1$, the nonlinear ϵ term acts like an ordinary damping. However as the amplitude falls to $|y| < 1$, energy is pumped into the oscillator.

We therefore expect the system to settle into some "limit cycle", which is a self-sustained oscillation whose amplitude is independent of the ICs. We will answer a couple of questions:

1. How do solutions approach the limit cycle?
2. For $\epsilon = 0$, any amplitude solution is admissible. However for $0 < \epsilon \ll 1$, an approximate amplitude of the limit cycle can be determined.

Proceeding as previously, recall

$$\begin{aligned}\dot{y} &= Y_t + \epsilon Y_\tau \\ \ddot{y} &= Y_{tt} + 2\epsilon Y_{t\tau} + \epsilon^2 Y_{\tau\tau}\end{aligned}$$

which transforms the vdP equation to

$$\begin{gathered}(Y_{tt} + 2\epsilon Y_{t\tau} + \dots) + \epsilon(Y_t + \dots)(Y^2 - 1) + Y = 0 \\ (Y_0 + \epsilon Y_1)_{tt} + 2\epsilon (Y_0)_{t\tau} + \epsilon (Y_0)_t (Y_0^2 - 1) + (Y_0 + \epsilon Y_1) + O(\epsilon^2) = 0\end{gathered}$$

Collecting terms

$$\begin{aligned}O(\epsilon^0):& \qquad \ddot{Y}_0 + Y_0 = 0 \\ O(\epsilon^1):& \qquad \ddot{Y}_1 + Y_1 = -2(Y_0)_{t\tau} - (Y_0^2 - 1)(Y_0)_t\end{aligned}$$

The $O(1)$ solution is

$$Y_0(t, \tau) = A(\tau)\mathrm{e}^{\mathrm{i}t} + A^*(\tau)\mathrm{e}^{-\mathrm{i}t}$$

The $O(\epsilon)$ equation is

$$\begin{aligned}\ddot{Y}_1 + Y_1 =& -2(\mathrm{i}A_\tau \mathrm{e}^{\mathrm{i}t} + \text{c.c.}) \\ &- (A^2\mathrm{e}^{2\mathrm{i}t} + (A^*)^2\mathrm{e}^{-2\mathrm{i}t} + 2|A|^2 - 1)(\mathrm{i}A\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A^*\mathrm{e}^{-\mathrm{i}t}) \\ =& \mathrm{e}^{\mathrm{i}t}\left[-2\mathrm{i}A_\tau + \mathrm{i}A^2A^* - \mathrm{i}A(2|A|^2 - 1)\right] + \mathrm{e}^{3\mathrm{i}t}[-\mathrm{i}A^3] + \text{c.c.}\end{aligned}$$

The resonant forcing is the term which multiplies the $\mathrm{e}^{\mathrm{i}t}$ term. This will produce secular terms in Y_1 unless we force this to zero.

$$2A_\tau + A|A|^2 - A = 0 \tag{62}$$

The solution of the *complex* amplitude equation is found by proceeding with the ansatz

$$A(\tau) = R(\tau)\mathrm{e}^{\mathrm{i}\theta(\tau)} \tag{63}$$

We think the problem will have a circular limit cycle (weakly deviating from the simple harmonic motion), so the polar coordinate ansatz is a natural move. With this eqn. 62 reads

$$2(R' + \mathrm{i}R\theta')\mathrm{e}^{\mathrm{i}\theta} = (R - R^3)\mathrm{e}^{\mathrm{i}\theta}$$

Equating the real and imaginary parts:

$$
\begin{aligned}
R' &= \frac{1}{2}R(1-R^2) \\
\theta' &= 0
\end{aligned}
$$

At this order of perturbation theory, there is no phase drift (unlike the Duffing oscillator), i.e. $\theta = \theta_0$ (constant). The system in R is a Bernoulli differential equation and can be solved with the substitution

$$u = \frac{1}{R^2}$$

This yields

$$u' = 1 - u$$

Solving with the integrating factor

$$u(\tau) = 1 + C\mathrm{e}^{-\tau}$$

With the initial condition $R(0) = R_0$, we derive $C = R_0^{-2} - 1$ to write

$$R(\tau) = [1 + (R_0^{-2} - 1)\mathrm{e}^{-\tau}]^{-1/2}$$

Pulling everything together

$$
\begin{aligned}
Y_0 &= A(\tau)\mathrm{e}^{\mathrm{i}t} + \text{c.c} \\
&= R(\tau)\mathrm{e}^{\mathrm{i}(\theta_0 + t)} + \text{c.c} \\
&= 2R(\tau)\cos(t + \theta_0) \\
y(t,\epsilon) &\sim \frac{2\cos(t+\theta_0)}{\sqrt{1 + (R_0^{-2} - 1)\mathrm{e}^{-\epsilon t}}} + O(\epsilon) \qquad (64)
\end{aligned}
$$

Therefore as $t \to \infty$, $y \to 2\cos(t + \theta_0)$, i.e. the amplitude goes to 2.

Example 3: Duffing (eqn. 50) revisited...

$$
\begin{gathered}
\ddot{y} + y + \epsilon y^3 = 0, \qquad 0 < \epsilon \ll 1 \\
y(0) = 1 \qquad \dot{y}(0) = 0
\end{gathered}
$$

Using eqns. 58 our ODE becomes

$$
\begin{gathered}
(Y_{tt} + 2\epsilon Y_{t\tau} + \dots) + y + \epsilon y^3 = 0 \\
(Y_0 + \epsilon Y_1)_{tt} + 2\epsilon (Y_0)_{t\tau} + (Y_0 + \epsilon Y_1) + \epsilon Y_0^3 + O(\epsilon^2) = 0
\end{gathered}
$$

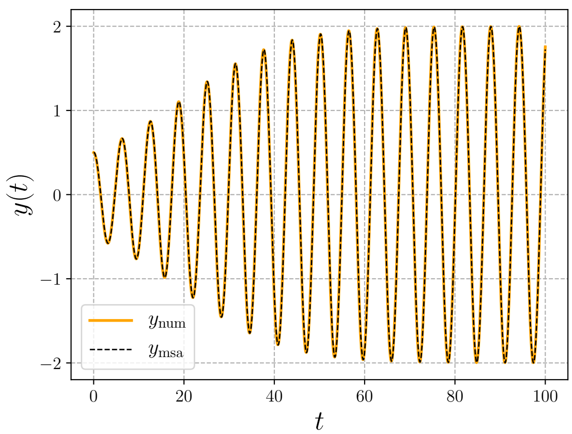

Figure 23.2: Plots of direct numerical solution to eqn. 61 and its analytical solution eqn. 64 for $\epsilon = 0.1$ and $y(0) = 1/2$ and $\dot{y}(0) = 0$.

Equating powers of ϵ:

$$
\begin{aligned}
(Y_0)_{tt} + Y_0 &= 0 \\
(Y_1)_{tt} + Y_1 &= -Y_0^3 - 2(Y_0)_{t\tau}
\end{aligned}
$$

The general solution to the $O(1)$ eqn. is

$$Y_0 = A(\tau)\mathrm{e}^{\mathrm{i}t} + A^*(\tau)\mathrm{e}^{-\mathrm{i}t}$$

which upon substitution into the $O(\epsilon)$ yields

$$
\begin{aligned}
(Y_1)_{tt} + Y_1 = &- A^3\mathrm{e}^{3\mathrm{i}t} - A^{*3}\mathrm{e}^{-3\mathrm{i}t} \\
&+ \left(-3A^2A^* - 2\mathrm{i}A_\tau\right)\mathrm{e}^{\mathrm{i}t} + \left(-3AA^{*2} + 2\mathrm{i}A^*_\tau\right)\mathrm{e}^{-\mathrm{i}t}
\end{aligned}
$$

To ensure that there are no secular terms in Y_1, the resonant terms on the right hand side are forced to zero, i.e.

$$2\mathrm{i}A_\tau + 3|A|^2A = 0$$

Substitute eqn. 63 into the above equation:

$$-2R\theta_\tau + 2\mathrm{i}R_\tau + 3R^3 = 0$$

Upon equating the real and imaginary parts to zero and integrating

$$
\begin{aligned}
R(\tau) &= R_0 \\
\theta(\tau) &= \theta_0 + \frac{3}{2}R_0^2\tau
\end{aligned}
$$

Collecting everything

$$Y_0 = 2R_0 \cos\left(t + \theta_0 + \frac{3}{2}R_0^2 \epsilon t\right)$$

Applying the initial conditions we derive

$$1 = 2R_0 \cos\theta_0$$
$$0 = -2R_0\left(1 + 3R_0^2\epsilon/2\right)\sin\theta_0$$

Therefore $R_0 = 1/2$ and $\theta_0 = 0$, yielding

$$y(t) = \cos\left(t + 3\epsilon t/8\right) + \mathcal{O}(\epsilon) \tag{65}$$

Higher order terms in the expansion are generated by systematically eliminating secular terms at each order. The *method of averaging* can also be used to derive this result (sec. A).

Lecture 24

Aging Spring and Adiabatic Invariants

We will spend this lecture revisiting the aging spring problem, first introduced in lecture 18 (eqn. 36) and solved with the help of WKB. The two-timing method brings out new ideas such as "adiabatic invariance".

$$\ddot{y} + \mathrm{e}^{-\epsilon t} y = 0, \qquad 0 < \epsilon \ll 1$$
$$y(0) = 1 \qquad \dot{y}(0) = 0$$

The stiffness $\mathrm{e}^{\epsilon t}$ is slowly decaying to zero on a very long time-scale. The instantaneous frequency ω satisfies

$$\omega^2 = \mathrm{e}^{-\epsilon t}$$

The slow scale is $\tau = \epsilon t$. But what is the correct fast time-scale? Should it be t? ωt? Since ω itself is changing. Something else?

Let the fast time s itself vary on the long time scale, i.e.

$$\frac{\mathrm{d}s}{\mathrm{d}t} = g(\tau, \epsilon) \tag{66}$$

where the function g is to be determined (it plays the role of instantaneous ω). Let

$$y(t, \epsilon) = Y(s, \tau)$$

and calculate the first and second derivatives.

$$\begin{aligned}
\dot{y} &= Y_s \frac{\mathrm{d}s}{\mathrm{d}t} + Y_\tau \frac{\mathrm{d}\tau}{\mathrm{d}t} \\
&= gY_s + \epsilon Y_\tau \\
\ddot{y} &= Y_s \frac{\mathrm{d}g}{\mathrm{d}t} + g\frac{\mathrm{d}Y_s}{\mathrm{d}t} + \epsilon \frac{\mathrm{d}Y_\tau}{\mathrm{d}t} \\
&= Y_s \epsilon g_\tau + g(gY_{ss} + \epsilon Y_{s\tau}) + \epsilon(gY_{\tau s} + \epsilon Y_{\tau\tau}) \\
&= g^2 Y_{ss} + \epsilon[2gY_{s\tau} + Y_s g_\tau] + \epsilon^2 Y_{\tau\tau}
\end{aligned}$$

We further want to expand everything as a regular perturbative series

$$\begin{aligned}
Y &= Y_0 + \epsilon Y_1 + \dots \\
g &= g_0 + \epsilon g_1 + \dots
\end{aligned}$$

With this, our ODE without $O(\epsilon^2)$ terms, becomes

$$(g_0 + \epsilon g_1)^2 (Y_0 + \epsilon Y_1)_{ss} + \epsilon[2g_0(Y_0)_{s\tau} + (Y_0)_s(g_0)_\tau] + \mathrm{e}^{-\tau}(Y_0 + \epsilon Y_1) = 0$$

The hierarchy becomes

$$\begin{aligned}
O(\epsilon^0): &\qquad g_0^2 (Y_0)_{ss} + \mathrm{e}^{-\tau} Y_0 = 0 \\
O(\epsilon^1): &\qquad g_0^2 (Y_1)_{ss} + \mathrm{e}^{-\tau} Y_1 = -[2g_0(Y_0)_{s\tau} + (Y_0)_s(g_0)_\tau + 2g_0 g_1 (Y_0)_{ss}]
\end{aligned}$$

The $O(1)$ equation suggests that it would help to choose

$$g_0^2 = \mathrm{e}^{-\tau} \qquad \implies g_0(\tau) = \mathrm{e}^{-\tau/2}$$

This was what we called ω from "common-sense" thinking. Integrating the $O(1)$ equation with the ICs [check]

$$Y_0(0,0) = 1 \qquad (Y_0)_s(0,0) = 0$$

we arrive at

$$Y_0(s,\tau) = A(\tau)\cos(s) \qquad A(0) = 1$$

The next order equation is

$$\mathrm{e}^{-\tau}[(Y_1)_{ss} + Y_1] = 2g_0 A_\tau \sin(s) + (g_0)_\tau A \sin(s) + 2\mathrm{e}^{-\tau/2} g_1 A \cos(s)$$

To avoid the secularity that would otherwise occur, the coefficients of both $\sin(s)$ and $\cos(s)$ resonant terms need to be forced to zero, i.e.

$$g_1 = 0, \qquad 2g_0 A_\tau + A(g_0)_\tau = 0$$

Observe that upon multiplication of the "amplitude" equation with A, the equation becomes a perfect partial derivative with respect to τ

$$\frac{\partial}{\partial \tau}(g_0 A^2) = 0 \qquad \implies g_0 A^2 = \text{const. in } \tau$$

This invariance is on time-scales up to and including $O(1/\epsilon)$ and accurate to within $O(\epsilon)$ and is often referred to as the "action". The *adiabatic invariance* means that we are adjusting some parameter so slowly that it gives the system sufficient time to equilibriate. To understand what this invariant term means, recall

$$\begin{aligned}\dot{y} &= gY_s + O(\epsilon)\\ &= -g_0 A\sin(s) + O(\epsilon)\\ y &= A\cos(s) + O(\epsilon)\end{aligned}$$

Now note that the energy of the spring

$$\begin{aligned}E &\propto \dot{y}^2 + ky^2\\ &\propto g_0^2 A^2 \sin^2 s + g_0^2 A^2 \cos^2 s\\ &\propto g_0^2 A^2\end{aligned}$$

which is <u>not</u> constant[1]. Now at $\tau = 0$,

$$g_0 = \mathrm{e}^{-\tau/2} = 1 \qquad A(0) = 1$$

Therefore $A(\tau)^2 = 1/g_0$ and

$$Y_0(s,\tau) = \mathrm{e}^{\tau/4}\cos(s) + O(\epsilon)$$

where

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= g_0 + \cancel{\epsilon g_1} + O(\epsilon^2) = \mathrm{e}^{-\epsilon t/2}\\ \implies s &= -\frac{2}{\epsilon}\mathrm{e}^{-\epsilon t/2} + c\end{aligned}$$

Putting it together

$$Y_0 = \mathrm{e}^{\epsilon t/4}\left[A\cos\left(\frac{2}{\epsilon}\mathrm{e}^{-\epsilon t/2}\right) + B\sin\left(\frac{2}{\epsilon}\mathrm{e}^{-\epsilon t/2}\right)\right]$$

which is what the WKB method had recovered (eqn. 37).

[1]Note that in qunatum mechanics $E = \hbar\omega$, and the ratio E/ω is conserved. This can be seen to correspond to E/g_0 ($g_0 = \omega$).

Lecture 25

Difference Equations and Multiple Scales

Arises in fields where, for instance, time is measured in integers (e.g. stock prices measured daily, or recurrence relations).

Example:

$$y_{n+1} - 2y_n + y_{n-1} = -2\omega^2(y_n + \epsilon y_n^3) \tag{67}$$
$$0 < \epsilon \ll 1 \qquad 0 < \omega < 1$$

This could be viewed as a discretization of Duffing's equation $\ddot{u} + u + \epsilon u^3 = 0$. To see this, let

$$\begin{aligned} t &= n\Delta t \\ u(t) &= u(n\Delta t) = y_n \end{aligned}$$

where Δt is the step-size. The derivatives are

$$\begin{aligned} \dot{u} &\approx \frac{u(t+\Delta t) - u(t)}{\Delta t} = \frac{u([n+1]\Delta t) - u(n\Delta t)}{\Delta t} \\ &\approx \frac{y_{n+1} - y_n}{\Delta t} \\ \ddot{u} &\approx \frac{1}{\Delta t}\left[\left(\frac{y_{n+1} - y_n}{\Delta t}\right) - \left(\frac{y_n - y_{n-1}}{\Delta t}\right)\right] \\ &\approx \frac{y_{n+1} - 2y_n + y_{n-1}}{(\Delta t)^2} \end{aligned}$$

Therefore the Duffing equation has the discrete analog[1]

$$y_{n+1} - 2y_n + y_{n-1} + (\Delta t)^2(y_n + \epsilon y_n^3) = 0$$

[1] Worth noting that the 'logistic map' and its continuous version the 'logistic differential equation' have very different behavior.

This is identical to eqn. 67 with $(\Delta t)^2 = 2\omega^2$. By analogy with the Duffing equation, we expect the solutions for y_n to be nearly sinusoidal, with a frequency that depends on amplitude. Let

$$s = \epsilon n$$

be the slow time (like $\tau = \epsilon t$ in the continuous case). The fast time time is simply 1. First assume

$$y_n \sim Y_0(n, s) + \epsilon Y_1(n, s) + \dots$$

then

$$\begin{aligned} y_{n\pm 1} &\sim Y_0(n \pm 1, s \pm \epsilon) + \epsilon Y_1(n \pm 1, s \pm \epsilon) + \dots \\ &\sim Y_0(n \pm 1, s) \pm \epsilon \partial_s Y_0(n \pm 1, s) + \epsilon Y_1(n \pm 1, s) + O(\epsilon^2) \end{aligned}$$

Note that the ϵ correction from the Y_1 term is multipled with the ϵ and lumped into the $O(\epsilon^2)$ correction. With this the $O(1)$ ODE reads

$$Y_0(n + 1, s) - 2Y_0(n, s) + Y_0(n - 1, s) = -2\omega^2 Y_0(n, s)$$

To ease notation, let $a_n = Y_0(n, s)$. Then

$$a_{n+1} - 2(1 - \omega^2)a_n + a_{n-1} = 0$$

This is a linear second order difference equation with constant coefficients. In a way analogous to

$$\mathrm{e}^{rt} = (\mathrm{e}^r)^t \to (\mathrm{e}^r)^n$$

we guess

$$a_n = \lambda^n$$

to derive

$$\lambda^{n+1} - 2b\lambda^n + \lambda^{n-1} = 0, \qquad 0 < b < 1$$

where $b = 1 - \omega^2$. The above quadratic is easily solved to yield

$$\lambda = b \pm \mathrm{i}\sqrt{1 - b^2}$$

In the complex plane we may represent as

$$|\lambda| = 1 \qquad \cos\alpha = \frac{b}{1} = 1 - \omega^2$$
$$\lambda = \mathrm{e}^{\pm \mathrm{i}\alpha}$$

So my general solution to the $O(1)$ difference equation is

$$\begin{aligned} a_n &= c_1 \mathrm{e}^{\mathrm{i}\alpha n} + c_2 \mathrm{e}^{-\mathrm{i}\alpha n} \\ &= A\cos(\alpha n + \theta) \end{aligned}$$

Also note that 'constants' are constant on the fast time-scale; strictly

$$A = A(s) \qquad \theta = \theta(s)$$

In summary

$$\begin{aligned} Y_0(n,s) &\sim A(s)\cos(\alpha n + \theta(s)) \\ \cos\alpha &= 1 - \omega^2 \end{aligned}$$

At $O(\epsilon)$, the governing ODE is

$$\begin{aligned} Y_1(n+1) - 2(1-\omega^2)Y_1 + Y_1(n-1) = &- 2\omega^2 Y_0^3 \\ &+ \frac{\partial Y_0(n-1)}{\partial s} \\ &- \frac{\partial Y_0(n+1)}{\partial s} \end{aligned}$$

wherein Y_0 and Y_1 are indexed at (n,s) unless otherwise specified. Now expand the right hand side and look for resonant terms proportional to $\cos(\alpha n + \theta)$ and $\sin(\alpha n + \theta)$. Defining $\phi = \alpha n + \theta$

$$\begin{aligned} -2\omega^2 Y_0^2 + \partial_s Y_0(n-1) - \partial_s Y_0(n+1) = &- \frac{1}{2}\omega^2 A^3[\cos(3\phi) + 3\cos\phi] \\ &+ A'\cos(\phi - \alpha) - A\theta'\sin(\phi - \alpha) \\ &- A'\cos(\phi + \alpha) + A\theta'\sin(\phi + \alpha) \\ = &- \frac{3}{2}\omega^2 A^3 \cos\phi + \ldots \\ &+ 2A'\sin\phi\sin\alpha \\ &+ 2A\theta'\cos\phi\sin\alpha \end{aligned}$$

NB. In taking the <u>s derivative</u>, *no discretization* was performed: this is to say that Y_0 changes sufficiently slowly in in s to be able to regard the variation as continuous. The slow time equations, setting the secular terms to zero, read

$$\theta'(s) = \frac{3}{4}\frac{\omega^2 A^2}{\sin\alpha}, \qquad A'(s) = 0$$

This implies that on the slow time-scale, $A(s) = A_0$ (const.) and further

$$\theta(s) = \frac{3}{4}\frac{\omega^2 A_0^2}{\sin\alpha}s + \theta_0$$

where

$$\begin{aligned} \sin\alpha &= \sqrt{1 - \cos^2\alpha} = \sqrt{1 - (1 - \omega^2)} \\ &= \omega\sqrt{2 - \omega^2} \end{aligned}$$

Simplifying

$$\begin{aligned}\theta(s) &= \underbrace{\frac{3}{4}\frac{\omega A_0^2}{\sqrt{2-\omega^2}}}_{\gamma} s + \theta_0 \\ &= \gamma \epsilon n + \theta_0\end{aligned}$$

NB. For a linearly stable system, a weak nonlinearity (e.g. small forces) does not change anything. So we want the system to be *neutral* to begin with: the *small* nonlinear terms can accumulate over a long time scale. For example, perturbations to a simple harmonic motion. These statements are mostly true for multiple-timescales and Poincaré-Lindstedt, *not* WKB or Boundary Layer.

Lecture 26

PDE and Boundary Layers

Example: Flame front propagation[1].

- The independent variables are τ (time) and ξ (1-D space)
- The temperature $\theta(\xi, \tau)$ and gas (fuel) concentration $y(\xi, \tau)$
- The Zeldovich number $\epsilon^{-1} \gg 1$ ($\epsilon \approx 0.1$) and Lewis number $L = D_T/D_Y$ which is the ratio of the heat diffusivity to the gas diffusivity

The scaled equations for this model read

$$\begin{aligned}
\frac{\partial \theta}{\partial \tau} &= \underbrace{\frac{\partial^2 \theta}{\partial \xi^2}}_{\text{heat diffusion}} + \underbrace{\frac{1}{\epsilon^2} y f(\theta)}_{\text{heat generated}} \\
\frac{\partial y}{\partial \tau} &= \underbrace{L^{-1} \frac{\partial^2 y}{\partial \xi^2}}_{\text{gas diffusion}} - \underbrace{\frac{1}{\epsilon^2} y f(\theta)}_{\text{reaction consumes } y}
\end{aligned}$$

where the reaction rate (from Arrhenius law) is

$$f(\theta) = \exp\left[\frac{\theta - 1}{\epsilon} \frac{1}{1 + \gamma(\theta - 1)}\right]$$

Here θ has been normalized by the temperature at which the gas ignites, i.e. $\theta \in [0, 1]$. So the curve of $f(\theta)$ vs. θ is essentially 0 upto a thin $O(\epsilon)$ neighborhood near the ignition temperature of 1, at which point it shoots to its maximum reaction rate. Intuitively, we expect the flame to propagate as a travelling wave of burning. We are especially interested in the wave speed (along with the profiles of gas concentration and temperature). Now was $\epsilon \to 0^+$, we expect something sharp at the flame front (see Fig. 26.1). However, it is not clear if the gas concentration y should, in the buring zone, have an *inner layer*

[1] J. Keener "Principles of Applied Mathematics" (1988), pp. 530-533

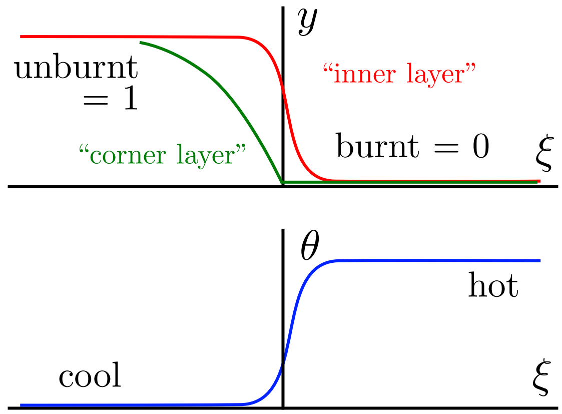

Figure 26.1: (Top) Gas concentration y as a function of space ξ at a fixed time. (Bot) The corresponding temperature θ profile.

(rapid change in y) or a *corner layer* (rapid change in its derivative). These are naturally resolved in the course of our treatment.

Since we are interested in a travelling wave (to the left), we move into a co-moving frame with the transformation

$$s = v\tau + \xi, \qquad v > 0$$

The advantage of this method is that it converts the PDE to an ODE; the value of v is to be determined. Assume

$$\theta = \theta(s) \qquad y = y(s)$$

i.e. the waves maintain their shapes and propagate leftwards. Using chain-rule to note

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial\theta}{\partial s}\frac{\partial s}{\partial\tau} = v\theta_s$$
$$\frac{\partial\theta}{\partial\xi} = \frac{\partial\theta}{\partial s}\frac{\partial s}{\partial\xi} = \theta_s$$

the PDEs become

$$v\theta_s = \theta_{ss} + \frac{1}{\epsilon^2}yf(\theta)$$
$$vy_s = L^{-1}y_{ss} - \frac{1}{\epsilon^2}yf(\theta)$$

The above are a set of nonlinear ODEs with the (intuitively derived) BCs

$$y(\infty) = 0, \qquad y(-\infty) = 1$$
$$\theta(\infty) = 1, \qquad \theta(-\infty) = 0$$

The hope is that the correct choice of the unknown velocity v helps determine the BCs correctly. Define $s = 0$ as the position of the wavefront and there should be some kind of internal layer near $s = 0$.

Outer solns: On the left, assume $\theta < 1$ (it seems implausible that θ gets hotter ahead of the flame). For this $f(\theta)$ is a TST and no reaction is happening (flame has not arrived yet). With this

$$\begin{aligned} v\theta_s &= \theta_{ss} \\ vy_s &= L^{-1}y_{ss} \end{aligned} \tag{68}$$

which is simple uncoupled diffusion. Let us introduce

$$\begin{aligned} \theta &= \theta_0 + \epsilon\theta_1 + \dots \\ y &= y_0 + \epsilon y_1 + \dots \\ v &= v_0 + \epsilon v_1 + \dots \end{aligned}$$

With this the $O(1)$ equations are

$$\begin{aligned} v_0\theta_0' &= \theta_0'' \\ v_0 y_0' &= L^{-1}y_0'' \end{aligned}$$

where the notation $f' = \mathrm{d}f/\mathrm{d}s$ has been introduced. We can straightforwardly integrate to write

$$\begin{aligned} \theta_0(s) = a\mathrm{e}^{v_0 s} + c_1, \qquad \theta_0(-\infty) = 0 \\ \implies \theta_0(s) = a\mathrm{e}^{v_0 s} \end{aligned}$$

This is exponential decay from $s = 0$ as s goes left. Similarly solving for the gas concentration with $y_0(-\infty) = 1$ we see

$$y_0(s) = 1 - b\mathrm{e}^{Lv_0 s}$$

On the right, $s > 0$, $y = 0$ and $\theta = 1$ – therefore $f(\theta) = 0$. This also yields the equation set 68. Like before, an exponential term arises and these must be set to zero to prevent blow-up as $s \to \infty$. This yields the solution

$$\theta_0(s) = 1, \qquad y_0(s) = 0$$

If the solutions to the left and right are to match at $s = 0$, it must be that $a = 1$ and $b = 1$. Altogether, for $s \leq 0$

$$\begin{aligned} \theta_0(s) &= \mathrm{e}^{v_0 s} \\ y_0(s) &= 1 - \mathrm{e}^{Lv_0 s} \end{aligned}$$

- The pictures are suggesting a corner layer at $s = 0$
- $\theta_0 < 1$ for $s < 0$ as we hoped (self-consistent)
- v is not determined yet: the $f(\theta)$ was missing.

So we need to get into the **inner region** where the burning is happening. To find v_0 we need to match more carefully near $s = 0$. The term $yf(\theta)/\epsilon^2$ becomes significant near $s = 0$. Now θ behaves linearly[2] with s in the immediate left neighborhood of $s = 0$. Since we are interested in [check argument]

$$\theta \approx v_0 s \sim 1 - \epsilon$$

(such that $f(\theta) \sim O(1)$ and not TST) we are interested in $s \leq O(\epsilon)$.

The form of $f(\theta)$ thereby motivate the strected variables

$$\eta = \frac{s}{\epsilon} \qquad \theta - 1 = \epsilon\phi \qquad y = \epsilon u \tag{69}$$

Intuitively, the burning zone would look like a sharply peaked Gaussian: to the

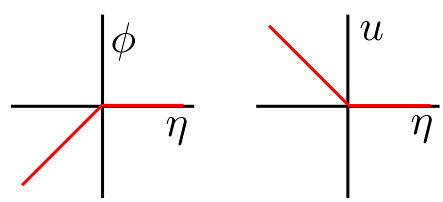

Figure 26.2: The corner layer variables using the scalings in eqns. 69 and Fig. 26.1.

left and right the reaction rate is almost zero. With these transformations the derivatives are

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}(\epsilon\eta)}(\epsilon u) = \frac{\mathrm{d}u}{\mathrm{d}\eta}\\ \frac{\mathrm{d}^2y}{\mathrm{d}s^2} &= \frac{\mathrm{d}}{\mathrm{d}(\eta\epsilon)}\frac{\mathrm{d}u}{\mathrm{d}\eta} = \frac{1}{\epsilon}\frac{\mathrm{d}^2u}{\mathrm{d}\eta^2}\end{aligned}$$

implying

$$\begin{aligned}\phi'' - \epsilon v\phi' + uf(1+\epsilon\phi) &= 0\\ L^{-1}u'' - \epsilon vu' - uf(1+\epsilon\phi) &= 0\end{aligned}$$

and

$$\begin{aligned}f(1+\epsilon\phi) &= \mathrm{e}^{\phi/(1+\gamma\epsilon\phi)} = \mathrm{e}^{(\phi_0+\dots)(1-\gamma\epsilon\phi_0+)}\\ &\sim \mathrm{e}^{\phi_0+O(\epsilon)} \approx \mathrm{e}^{\phi_0}[1+O(\epsilon)]\end{aligned}$$

[2] $\mathrm{e}^{v_0 s} \approx v_0 s + \dots$

So at $O(1)$ we get

$$\phi_0'' + u_0 e^{\phi_0} = 0 \tag{70}$$
$$L^{-1}u_0'' - u_0 e^{\phi_0} = 0 \tag{71}$$

Adding eqns. 70 and 71 we can get rid of the nonlinear term to write

$$\begin{aligned}
&\phi_0'' + L^{-1}u_0'' = 0 \\
\implies\ &\phi_0 + L^{-1}u_0 = c_1\eta + c_2
\end{aligned}$$

With the help of Fig. 26.1, and noting that u, ϕ are just scaled versions of y, θ (latter offset by one), we conclude that as $\eta \to \infty$, both $u, \phi \to 0$. Therefore, since

$$\begin{aligned}
\phi_0 + L^{-1}u_0 &\to 0 \\
\phi_0' + L^{-1}u_0' &\to 0
\end{aligned}$$

it must be that $c_1 = c_2 = 0$ yielding

$$\phi_0 = -L^{-1}u_0 \tag{72}$$

With this insight we can write eqn. 70 as

$$\phi_0'' - L\phi_0 e^{\phi_0} = 0$$

NB. This is of the form $\ddot{x} = -kx$ except that the force depends on the displacement in some non-linear way. The trick to find the constant of motion for such sytems is to multiply through with ϕ_0' and integrate[3]:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\frac{1}{2}\phi_0'^2 - Lg(\phi_0)\right] = 0$$

such that

$$\begin{aligned}
g(\phi_0) &= \int \phi_0\phi_0' e^{\phi_0}\mathrm{d}\eta = \int \phi_0 \frac{\mathrm{d}e^{\phi_0}}{\mathrm{d}\eta}\mathrm{d}\eta \\
&= \phi_0 e^{\phi_0} - \int \phi_0' e^{\phi_0}\mathrm{d}\eta \\
&= (\phi_0 - 1)e^{\phi_0} + c
\end{aligned}$$

Therefore

$$\frac{1}{2}\phi_0'^2 - L(\phi_0 - 1)e^{\phi_0} = c$$

Using the fact that as $\eta \to \infty$, $\phi_0 \to 0$

$$-\frac{1}{2}(0)^2 - L(-1)e^0 = c \qquad \implies c = L$$

[3]Such as in deriving the energy for the Duffing oscillator (eqn. 51).

We arrive at the implicit equation for $\phi_0(\eta)$. What's remaining is to match the slope ϕ_0' at $\eta \to -\infty$ to the incoming slope from the outer solution on the left. The figure helps us understand that $\phi_0(\eta) \to -\infty$ as $\eta \to -\infty$. Therefore

$$\frac{1}{2}\phi_0'^2 - \underbrace{L(\phi_0 - 1)\mathrm{e}^{\phi_0}}_{\text{TST}} = L$$
$$\implies \phi_0'(-\infty) = \pm\sqrt{2L}$$

We keep the positive root as is once again seen from the figure. Matching slopes

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{v_0 s})\right|_{s=0^-} = \left.\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right|_{\eta=-\infty}$$
$$\implies v_0 = \sqrt{2L}$$

Lecture 27

Renormalization and Envelopes

The idea is that renormalization may possibly subsume/unify everything that has been done in this course: WKB, BL, multiple scales etc.! The Renormalization Group approach to singularly perturbed PDEs and ODEs was pioneered by Chen, Goldenfeld and Oono. But in this lecture we will focus on a paper by T. Kunihiro[1]:

> **Abstract.** On the basis of classical theory of envelopes, we formulate the Renormalization Group (RG) method of global analysis, recently proposed by Goldenfeld et al. It is clarified in a generic way why the RG equation improves the global nature of function obtained in the perturbation theory.

Example 1: Find the envelope of the family of straight lines which are of unit length between their x and y intercepts (like a ladder sliding down a wall). This looks like there is a bounding curve "envelope" such that each tangent line (ladder equation at some t) is tangent to this envelope.

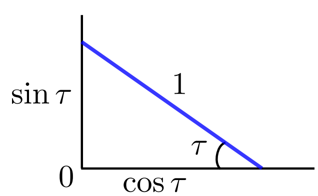

Figure 27.1: Ladder of unit length.

[1] "A Geometrical Formulation of the Renormalization Group Method for Global Analysis", Prog. Theo. Phys. **94** (1995) 503.

Noting that the slope m and intercept c are

$$m = -\frac{\sin\tau}{\cos\tau} \qquad c = \sin\tau$$

The equation of the line in terms of the parameter τ reads

$$\frac{y}{\sin\tau} + \frac{x}{\cos\tau} = 1$$

We rewrite the family as $F(x, y, \tau) = 0$ where

$$F(x, y, \tau) = y\cos\tau + x\sin\tau - \sin\tau\cos\tau$$

The envelope condition is

$$F = 0, \qquad \frac{\partial F}{\partial \tau} = 0$$

This yields

$$\begin{aligned} -y\sin\tau + x\cos\tau + (\sin^2\tau - \cos^2\tau) &= 0 \\ y\cos\tau + x\sin\tau - \sin\tau\cos\tau &= 0 \end{aligned}$$

Multiplying the first equation with $\cos\tau$ and the second with $\sin\tau$ and adding, we derive

$$x = \cos^3\tau \qquad y = \sin^3\tau$$

The envelope equation is finally

$$x^{2/3} + y^{2/3} = 1$$

To do:

- Derive envelope equation and add to appendix
- plot sliding ladder and envelope solution

Example 2: To understand how renormalization is different, let us revisit eqn. 27

$$\begin{aligned} \epsilon y'' + (1+\epsilon)y' + y &= 0 \\ y(0) = 0 \qquad y(1) &= 1 \end{aligned}$$

which we saw admits the exact solution eqn. 28

$$y(x, \epsilon) = \frac{\mathrm{e}^{-x} - \mathrm{e}^{-x/\epsilon}}{\mathrm{e}^{-1} - \mathrm{e}^{-1/\epsilon}}$$

Unlike the singular perturbation theory, as we shall shortly see, in the RG method there is no need for matching! This is somewhat like WKB except more general since this method is also applicable for nonlinear problems.

Strategy: Start by writing the inner equation (such that $\epsilon \to 0$ does not get rid of the highest derivative) and solve it with regular perturbation theory ("naïve" expansion), generating secular terms to our heart's content! We will "renormalize" them away later...

Define the stretched variable $X = x/\epsilon$ as previously. With this scaling, the governing ODE is given by eqn. 31 and the ordered equations

$$\begin{aligned} O(\epsilon^0): &\qquad Y_0'' + Y_0' = 0 \\ O(\epsilon^1): &\qquad Y_1'' + Y_1' = -Y_0' - Y_0 \end{aligned}$$

are generated. So far this looks like BL theory. It is at this point that we start to reason differently and make a rather intriguing move:

> Take an arbitrary point X_0 and impose the "boundary conditions" $Y_0(X_0)$, which would in principle determine two constants at this order. Since the solution is uniquely determined, it cannot depend on this parameter X_0. In other words, taking a partial derivative wrt X_0 should not change anything (like the envelope condition).

With the boundary condition $Y_0(X_0) = A_0$, we write the solution

$$\begin{aligned} Y_0(X) &= A_0 + B_0\left(\mathrm{e}^{-X} - \mathrm{e}^{-X_0}\right) \\ Y_1(X) &= -(A_0 - B_0\mathrm{e}^{-X_0})(X-1) + C_0 + D_0\mathrm{e}^{-X} \end{aligned}$$

It is not clear what BCs motivate the form for Y_0 as given in the reference. For pedagogical reasons we plough on with this:

[To do]

Appendix A

Method of averaging

Following Rozman, we consider the second order non-linear differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \epsilon\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3 + x = 0, \qquad \epsilon > 0 \tag{73}$$

with the initial conditions

$$x(0) = 1 \qquad \dot{x}(0) = 0 \tag{74}$$

The equation describes a nonlinear oscillator with the 'friction force' proportional to the third power of the velocity. Multiply eqn. 73 with $\dot{x}$ to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2\right) = -\epsilon\dot{x}^4$$

where the term inside the parenthesis is the mechanical energy of the oscillator $E(t) \geq 0$ with $E(0) = 1/2$. Clearly this energy decays, the rate of which is controlled by the parameter ϵ.

Numerical solution

It is always instructive to first study the numerical solution to the problem. This is achieved by rewriting the equation as a system of two coupled first order differential equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\epsilon y^3 - x\end{aligned}$$

with appropriate intial conditions. The solution is plotted in Fig. A.1 for $\epsilon = 0.2$. Again a standard perturbative approach yields a secular term. To obtain an approximate analytic solution, we use the *method of averaging*.

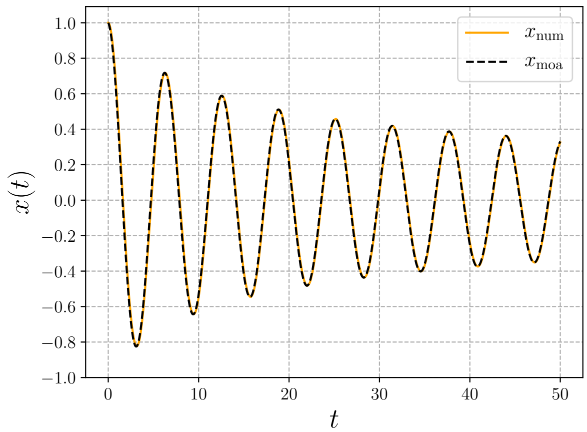

Figure A.1: Numerical solution to eqn. 73 and analytical approximation eqn. 79. Code available through Bitbucket.

Analytical solution

This approach is applicable to equations of the general form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x = \epsilon F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right), \tag{75}$$

where in our case

$$F = -\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3$$

We seek a solution to eqn. 73 of the form

$$x(t) = a(t)\cos\left(t + \psi(t)\right) \tag{76}$$

The motivation for this ansatz is that when $\epsilon = 0$, eqn. 75 has its solution of the form eqn. 76 with a and ψ constants. For small values of ϵ, we expect the same form of the solution to be approximately valid, but now $a(t)$ and $\psi(t)$ are expected to slowly vary with time t. Differentiating eqn. 76

$$\dot{a}\cos(t+\psi) - a(1+\dot{\psi})\sin(t+\psi) = -a\sin(t+\psi)$$

where in the rhs, we have assumed slow variations in a and ψ. Therefore

$$\dot{a}\cos(t+\psi) - a\dot{\psi}\sin(t+\psi) = 0 \tag{77}$$

With the hierarchy of equations

$$
\begin{aligned}
x &= a\cos(t+\psi) \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= -a\sin(t+\psi) \\
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\dot{a}\sin(t+\psi) - a(1+\dot{\psi})\cos(t+\psi)
\end{aligned}
$$

eqn. 73 yields

$$
-\dot{a}\sin(t+\psi) - a\dot{\psi}\cos(t+\psi) = \epsilon a^3 \sin^3(t+\psi) \tag{78}
$$

Using eqns. 77 and 78 we obtain two differential equations for $\dot{a}$ and $\dot{\psi}$

$$
\begin{aligned}
\frac{\mathrm{d}\psi}{\mathrm{d}t} &= -\epsilon a^2 \sin^3(\phi)\cos(\phi) \\
\frac{\mathrm{d}a}{\mathrm{d}t} &= -\epsilon a^3 \sin^4(\phi)
\end{aligned}
$$

where $\phi = t+\psi$. Now the key approximation is this: since both $\dot{a}$ and $\dot{\psi}$ are slowly varying, we can replace them with their averages:

$$
\langle \dots \rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} \dots \mathrm{d}\phi
$$

Using the definition of β function (sec. **??**)

$$
\begin{aligned}
\frac{\mathrm{d}\psi}{\mathrm{d}t} &= -\epsilon a^2 \frac{1}{2\pi}\int_0^{2\pi} \sin^3(\phi)\cos(\phi)\,\mathrm{d}\phi = 0 \\
\frac{\mathrm{d}a}{\mathrm{d}t} &= -\epsilon a^3 \frac{1}{2\pi}\int_0^{2\pi} \sin^4(\phi)\,\mathrm{d}\phi = -\epsilon a^3 \frac{1}{2\pi} 2\beta\left(\frac{5}{2},\frac{1}{2}\right) \\
&= -\epsilon a^3 \frac{3}{8}
\end{aligned}
$$

We thereby derive

$$
\begin{aligned}
\psi(t) &= \text{const.} \\
a(t) &= \frac{1}{\sqrt{\frac{3}{4}\epsilon t + \frac{1}{a^2(0)}}}
\end{aligned}
$$

We choose $a(0) = 1$ and $\psi(t) = 0$ to match the numerical solution (Fig. A.1), yielding

$$
x(t) = \frac{\cos t}{\sqrt{\frac{3}{4}\epsilon t + 1}} \quad . \tag{79}
$$

Duffing oscillator revisited

Equation 77 is unchanged and only the rhs of eqn. 78 is replaced with $-\epsilon a^3 \cos^3(t+\psi)$. The equations for $\dot{a}$ and $\dot{\psi}$ are

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}t} &= \frac{1}{2\pi}\int_0^{2\pi} \epsilon a^2 \cos^4(t+\psi)\,\mathrm{d}t \\ &= \frac{3}{8}\epsilon a^2 \\ \frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{1}{2\pi}\int_0^{2\pi} \epsilon a^3 \sin(t+\psi)\cos^3(t+\psi)\,\mathrm{d}t = 0\end{aligned}$$

where we can employ a straightforward change of variable $t+\psi=\phi$ with $\mathrm{d}\phi=\mathrm{d}t$. Applying the initial condition

$$y(t) = \cos\left(t + 3\epsilon t/8\right) \quad .$$