Module-3

3.1 PROPOSITIONAL LOGIC

3.1.1. Introduction :

Logic is the basis of all mathematical reasoning. It provides rules and techniques for determining whether a given argument is valid. Logic has practical applications to the design of computing machines. It is directly involved in computer programming and used to verify the correctness of the programs.

3.1.2. Proposition or Statement

A declarative sentence which is either true or false, but not both is called a Proposition or Statement. Proposition is denoted by the letters p,q,r etc.

Illustrations: (1) p : It is hot. Here p is a proposition because the sentence 'It is hot' is declared. It is either true or false.

(2) p : 'who are you?'. Here p is not a proposition because this is neither true not false.

(3) $p: a+b=3\ \forall a,b$. Here p is not a proposition because it is not true if $a=1$, $b=4$ and it is not false if $a=1.5$, $b=1.5$.

(4). q: Kolkata is the capital of West Bengal. Here q is a proposition because the declared sentence 'Kolkata is the Capital of West Bengal' is true.

Truth value of a proposition.

The truth or falsity of a proposition is called the Truth value of the proposition. So a proposition may take two Truth values: *'true' which is denoted by T or 1 and 'false' which is denoted* by F or 0.

Illustrations: (1) Let $p: 4+5=10$. Here p is a proposition which is false. So p takes the Truth value F or 0

(2) q : Every action has an equal and opposite reaction. This is a proposition which is true. So the truth value of q is T or 1.

Note : The letters p, q, ... etc. are also known as *propositional variables* because these may take the two different values T or F.

3.1.3. Truth Table : The table which displays all the truth values assumed by proposition is known as Truth Table.

Illustration: If p be a proposition then the Truth Table of p is

p
T
F

Remark. Truth table are especially valuable in the determination of truth values of the propositions which are modified from many other proposition. These will be seen in the subsequent discussion.

3.1.4. 'Logical Connectives' or 'Operations on statements'.

Propositions or statements may be modified or combined in various ways to form new proposition or statement. Different type of *connectives* or *operations* are used for this purpose. These are discussed below :

Conjunction. If p and q are two propositions then the proposition '*p and q*' is called conjunction of p and q. This is denoted by $p \wedge q$ (this will be read as p and q)

Illustration : Let p : *It is raining*; q : *It is hot* be two propositions. Then $p \wedge q$: *It is raining and hot* is conjunction of p and q

Truth Table for Conjunction. Let p and q be two propositions. Then the proposition $p \wedge q$ is true whenever p and q are true, $p \wedge q$ is false when p is true and q is false, $p \wedge q$ is false whenever p is false and q is true, $p \wedge q$ is false when both of p and q are false. So the truth table of the conjuction $p \wedge q$ is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction. If p and q are two propositions then the proposition '*p or q* or both, is called disjunction of p and q. This is denoted by $p \vee q$ (this will be read as p or q)

Illustration : Let p : *Sukrit is sincere* and q : *Sukrit is intelligent* be two propositions. Then $p \vee q$: *Sukrit is sincere or he is intelligent* is disjunction of p and q.

Truth Table for disjuction.

Let p and q be two propositions. Then the proposition $p \vee q$ is true whenever p and q both are true, $p \vee q$ is true when at least one of p and q is true, $p \vee q$ is false when p and q both are false. So the truth table of the disjunction $p \vee q$ is

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation. If p is a proposition then the proposition '*not p*' is called negation of p. It is denoted by $\sim p$ (this will be read as '*not p*'). $\sim p$ is also denoted by $\overline{p}$, $\neg p$ and p'.

Illustration : Let '*p: Pappu is a student of Mathematics*' be a proposition. Then $\sim p$: *Pappu is not a student of Mathematics* is negation of p.

Truth Table for Negation. Let p be a proposition. Then the proposition $\sim p$ is false when p is true, $\sim p$ is true when p is false. So the truth table of the negation $\sim p$ is

p	$\sim p$
T	F
F	T

Note: $(p \vee q) \wedge \sim (p \wedge q) \equiv$ either p or q

3.1.5. Conditional Connectives :

If p and q be two propositions then the proposition ' If p then q' is called implication. It is denoted by $p \rightarrow q$ or $p \Rightarrow q$ (this will be read as 'p implies q') [W.B.U.T.2012]

Illustrations : (1) Let p: *Buku works hard* and q : *Buku comes first in exam* be two propositions.

Then $p \to q$: *If Buku works hard, he comes first in exam* is implication.

(2) Let p : *It is raining* and q : *I go to library* be two propositions. Then $p \to q$: *If it is raining then I go to library* is implication.

Truth Table for Implication.(conditional proposition) [W.B.U.T.2012]

Let p and q be two propositions. Then $p \to q$ is false if and only if p is true and q is false otherwise it is true. So the truth table of the proposition $p \to q$ is

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse. If $p \to q$ is an implication then its *converse* is the implication $q \to p$.

Illustrations. In the previous illustration $p \to q$ is *"If Buku works hard, he comes first in exam."* Its converse $q \to p$ is *"If Buku comes first then he works hard."*

Truth Table for Converse

Let $p \to q$ be implication. Then its converse is the implication $q \to p$. So its truth table is

p	q	$q \to p$
T	T	T
T	F	T
F	T	F
F	F	T

Contrapositive. Let $p \to q$ be an implication. Then its *contrapositive* is the implication $\sim q \to \sim p$.

Illustration. In the previous illustration $p \to q$ is *"If Buku works hard, he comes first"*. Its contrapositive is $\sim q \to \sim p$ which means *"If Buku does not come first, he does not work hard"*.

Truth Table for Contrapositive

Let $p \to q$ be an implication. Then its contrapositive is $\sim q \to \sim p$. So its truth table is

p	q	$\sim q \to \sim p$
T	T	T
T	F	F
F	T	T
F	F	T

Inverse. Let $p \to q$ be an implication. Then its *inverse* is the implication $\sim p \to \sim q$.

Illustration : If $p \to q$ means *'If Buku works hard, he comes first'*. Its inverse $\sim p \to \sim q$ is *'If Buku does not work hard, he does not come first'.*

Truth Table for Inverse

p	q	$\sim p \to \sim q$
T	T	T
T	F	T
F	T	F
F	F	T

3.1.6. Biconditional Proposition. [W.B.U.T.2012]

If p and q be two propositions then the propositions ' If p then q and if q then p' is called biconditional propositions of p and q. It is denoted by $p \leftrightarrow q$ or $p \Leftrightarrow q$ (this will be read as 'p if and only if q').

Illustration : Let *p: Sukrit is tall and q: Sukrit is intelligent* be two propositions. Then $p \leftrightarrow q$: *'If sukrit is tall then he is intelligent and if Sukrit is intelligent then he is tall'* is bioconditional proposition.

Truth Table for bioconditional proposition.

[W.B.U.T.2012]

Let p and q be two propositions. Now $p \leftrightarrow q$ is true whenever p and q are both true or both false but $p \leftrightarrow q$ is false whenever p is true, q is false or q is true, p is false. So, the truth table of $p \leftrightarrow q$ is

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note : The operations $\wedge, \vee, \sim, \rightarrow$ and $\leftrightarrow$ are know as Connectives.

3.1.7. Propositional formula or Statement formula

The proposition obtained by connecting one or more propositions p, q, r... by the connectives $\vee, \wedge, \sim, \rightarrow$ or $\leftrightarrow$ is called a propositional formula or statement formula. This is denoted by $f(p,q,r,....)$ where p, q, r... are known as *variable*.

This is called **Compound proposition** or **Compound statement**. when more than one propositions are connected.

This is called **Simple proposition** when only one proposition is involved

Illustrations : (1) If p, q and r be three propositions then the proposition $f(p,q,r) = p \vee (\sim q) \vee r \rightarrow p$ is a proposition formula of the variables p, q and r. This is a compound proposition.

(2) p or $\sim p$ is a simple proposition since only one proposition is involved here.

Truth Table for Propositional formula.

By using the truth table of the connectives (i.e. the conjunction, disjunction etc.) we get the truth table for propositional formula. This is illustrated by the following example:

Illustration: $f(p,q) = \sim(\sim p \wedge q)$ is a proposition formula of the variables p and q. Then the Truth Table of $f(p,q)$ is

p	q	$\sim p$	$\sim p \wedge q$	$\sim(\sim p \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

3rd column is formed from 1st row using truth value of $\sim$, 4th column is formed from 3rd and 2nd row using truth value of $\wedge$.

Finally the last column is formed from 4th row by using truth value of $\sim$. In fact the 1st, 2nd and the last columns constitute the truth table of $f(p,q)$. Other columns are made for computation of the truth values.

Note. The truth table of a propositional formula of n variables will have 2^n number of rows. e.g. in the above illustration $f(p,q)$ is a formula of 2 variables p and q; so the truth table of $f(p,q)$ had $2^2 = 4$ rows.

Illustrative Examples :

Example: 1. Let p be 'Amit speaks Bengali' and let q be 'Amit speaks Urdu '. Give simple verbal sentence which describes each of the following:

(i) $p \vee q$ (ii) $p \wedge q$

Solution: (i) Amit speaks Bengali or Urdu

(ii) Amit speaks Bengali and Urdu

Example: 2 Let p: Suman is good and q: Suman is honest. Write the following statement in symbolic form :

(i) Suman is good and honest
(ii) Suman is good but not honest
(iii) Suman is neither good nor honest
(iv) Suman is good or Suman is bad and honest
(v) It is not true that Suman is bad or honest

Solution: (i) $p \wedge q$ (ii) $p \wedge \sim q$ (iii) $\sim p \wedge \sim q$

(iv) $p \vee (\sim p \wedge q)$ (v) $\sim(\sim p \vee q)$

Example: 3. Let p: Mr. X is gentlemen, q: Mr. X is happy.

Give a simple verbal sentence which describes each of the following statements:

(i) $p \to q$ (ii) $\sim p \to \sim q$ (iii) $\sim p \to q$ (iv) $p \vee \sim q$
(v) $\sim p \wedge q \to p$ (vi) $q \leftrightarrow q$

Solution: (i) If Mr. X is gentlemen, he is happy
(ii) If Mr. X is not gentlemen, he is not happy
(iii) If Mr. X is not gentlemen, he is happy.
(iv) Mr. X is gentlemen or Mr. X is not happy
(v) If Mr. is not gentlemen and happy, he is gentlemen.
(vi) Mr. X is gentlemen if and only if Mr. X is happy

Example: 4. Construct the truth table for each of the following compound proposition:

(i) $(p \vee q) \wedge q$ (ii) $(\sim p \wedge q) \vee p$ (iii) $(p \vee q) \vee \sim q$

(iv) $\sim(p \vee q) \vee (\sim p \wedge \sim q)$ (v) $(p \wedge q) \vee (p \wedge r)$ (vi) $(\sim p \wedge q) \wedge \sim r$

Solution: The following table shows the required truth tables:

(i) Truth Table for $(p \vee q) \wedge q$

p	q	$p \vee q$	$(p \vee q) \wedge q$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

(ii) Truth Table for $(\sim p \wedge q) \vee p$

p	q	$\sim p$	$\sim p \wedge q$	$(\sim p \wedge q) \vee p$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

(iii) Truth Table for $(p \vee q) \vee \sim q$

p	q	$\sim q$	$p \vee q$	$(p \vee q) \vee \sim q$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	T
F	F	T	F	T

(iv) Truth Table for $\sim(p \vee q) \vee (\sim p \wedge \sim q)$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$(\sim p \wedge \sim q)$	$\sim(p \vee q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

(v) Truth Table for $(p \wedge q) \vee (p \wedge r)$

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

(vi) Truth Table for $(\sim p \wedge q) \wedge \sim r$

p	q	r	$\sim p$	$\sim p \wedge q$	$\sim r$	$(\sim p \wedge q) \wedge \sim r$
T	T	T	F	F	F	F
T	T	F	F	F	T	F
T	F	T	F	F	F	F
T	F	F	F	F	T	F
F	T	T	T	T	F	F
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	T	F	T	F

Example:5 Construct the truth table for each of the following statements:

(i) $(p \vee q) \rightarrow p$ (ii) $(p \vee q) \rightarrow (p \wedge q)$ (iii) $(p \rightarrow q) \rightarrow (q \rightarrow p)$ (iv) $\sim(p \rightarrow q) \rightarrow r$ (v) $\sim(p \wedge q) \rightarrow (\sim q \wedge p)$

Solution:

(i) Truth Table for $(p \vee q) \rightarrow p$

p	q	$p \vee q$	$(p \vee q) \rightarrow p$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

(ii) Truth Table for $(p \vee q) \rightarrow (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(iii) Truth Table for $(p \rightarrow q) \rightarrow (q \rightarrow p)$

p	q	$p \rightarrow q$	$(q \rightarrow p)$	$(p \rightarrow q) \rightarrow (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

(iv) Truth Table for $\sim(p \rightarrow q) \rightarrow r$

p	q	r	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim(p \rightarrow q) \rightarrow r$
T	T	T	T	F	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	T	F	T

(v) Truth Table for $\sim(p \wedge q) \rightarrow (\sim q \wedge p)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim q$	$\sim q \wedge p$	$\sim(p \wedge q) \rightarrow (\sim q \wedge p)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	F	T	F	F	F
F	F	F	T	T	F	F

Example: 6 Construct the table for the following statements:

(i) $(\sim p \wedge \sim q) \leftrightarrow p$

(ii) $(p \rightarrow q) \leftrightarrow (\sim q \leftrightarrow \sim p)$

(iii) $(q \rightarrow \sim p) \leftrightarrow (p \leftrightarrow q)$

(iv) $\sim(p \leftrightarrow \sim q) \leftrightarrow (p \leftrightarrow q)$

Solution:

(i) Truth Table for $(\sim p \wedge \sim q) \leftrightarrow p$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(\sim p \wedge \sim q) \leftrightarrow p$
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	T
F	F	T	T	T	F

(ii) Truth Table for $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim p \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

(iii) Truth Table for $(q \rightarrow \sim p) \leftrightarrow (p \leftrightarrow q)$

p	q	$\sim p$	$q \rightarrow \sim p$	$p \leftrightarrow q$	$(q \rightarrow \sim p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

(iv) Truth Table for $(\sim p \leftrightarrow \sim q) \leftrightarrow (p \leftrightarrow q)$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$	$p \leftrightarrow q$	$(\sim p \leftrightarrow \sim q) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

3.1.8. Tautology

A proposition formula or statement formula is called tautology if it assumes only the truth value T i.e.in the Truth Table of the formula every entry in the last column will be T.

Illustration: Consider the formula $f(p,q) = p \rightarrow (q \rightarrow p)$

The truth table of $f(p,q)$ is

p	q	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

Note : 3rd column is formed from 2nd and 1st column by using the truth value of $\rightarrow$ .Finally the 4th column is constructed from 1st and 3rd column by using the truth value of $\rightarrow$.

Since all the entries in the last column are T so $f(p,q)$ assumes the only truth value T. So $f(p,q)$ is a Tautology.

3.1.9. Contradiction : A proposition formula is called contradiction if it assumes only the truth value F i.e. in the truth table of the formula every entry in the last column will be F.

Illustration: Consider the formula $f(p,q) = (p \wedge q) \wedge \sim (p \vee q)$

The truth table of $f(p,q)$ is

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim (p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

We see every entry in the last column is F. So $f(p,q)$ is a contradiction.

3.1.10. Logical Equivalence

Two propositions P and Q (simple or compound proposition) are said to be logical equivalence if they have identical truth tables. We denote this by $P \equiv Q$

Illustration: Consider the two compound proposition $p \to q$ and $\sim p \vee q$. We see the truth tables of $p \to q$ and $\sim p \vee q$ are

Truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth table of $\sim p \vee q$

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

which are identicals. Therefore $p \to q \equiv \sim p \vee q$

Illustrative Examples:

Example:1. Using truth table, determine which of the following compound propositions are tautologies and which of them are contradiction:

(i) $(\sim q \wedge p) \vee \sim (p \wedge q)$ (ii) $(p \vee q) \wedge (\sim p \wedge \sim q)$

(iii) $(\sim q \wedge (p \to q)) \to \sim p$ (iv) $(\sim p \wedge q) \wedge (\sim q \wedge (p \to q))$

(v) $((p \to q) \wedge (q \to r)) \to (p \to r)$

(vi) $\sim (q \to r) \wedge r \wedge (p \to q)$

Truth Table for $(\sim q \wedge p) \vee \sim (p \wedge q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim q$	$\sim q \wedge p$	$(\sim q \wedge p) \vee \sim(p \wedge q)$
T	T	T	F	F	T	T
T	F	F	T	T	T	T
F	T	F	T	F	F	T
F	F	F	T	T	T	T

Since the truth value of the given compound proposition is T for all combination of p and q, the proposition is a tautology.

(ii) Truth Table for $(p \vee q) \wedge (\sim p \wedge \sim q)$

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

Thelast column contains only F as the truth value of the given proposition. Hence given proposition is a contradicdiction.

(iii) Truth Table for $(\sim q \wedge (p \to q)) \to \sim p$

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \wedge (p \to q)$	$(\sim q \wedge (p \to q)) \to \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

As the truth value of the given compound proposition is T for all values of p and q, so it is a tautology.

(iv) Truth Table for $(\sim p \wedge q) \wedge (\sim q \wedge (p \to q))$

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$p \to q$	$\sim q \wedge (p \to q)$	$(\sim p \wedge q) \wedge (\sim q \wedge (p \to q))$
T	T	F	F	F	T	F	F
T	F	F	T	F	F	F	F
F	T	T	F	T	T	F	F
F	F	T	T	F	T	T	T

Since all the entries in the last column are F 's, the given statement is a contradiction.

(v) Truth Table for $((p \to q) \wedge (q \to r)) \to (p \to r)$

p	q	r	$p \to q$	$p \to r$	$q \to r$	$(p \to q) \wedge (q \to r)$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Illustration: Consider the two compound proposition $p \to q$ and $\sim p \vee q$. We see the truth tables of $p \to q$ and $\sim p \vee q$ are

Truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth table of $\sim p \vee q$

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

which are identicals. Therefore $p \to q \equiv \sim p \vee q$

Illustrative Examples:

Example: 1. Using truth table, determine which of the following compound propositions are tautologies and which of them are contradiction:

(i) $(\sim q \wedge p) \vee \sim (p \wedge q)$ (ii) $(p \vee q) \wedge (\sim p \wedge \sim q)$

(iii) $(\sim q \wedge (p \to q)) \to \sim p$ (iv) $(\sim p \wedge q) \wedge (\sim q \wedge (p \to q))$

(v) $((p \to q) \wedge (q \to r)) \to (p \to r)$

(vi) $\sim (q \to r) \wedge r \wedge (p \to q)$

Truth Table for $(\sim q \wedge p) \vee \sim (p \wedge q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim q$	$\sim q \wedge p$	$(\sim q \wedge p) \vee \sim(p \wedge q)$
T	T	T	F	F	T	T
T	F	F	T	T	T	T
F	T	F	T	F	F	T
F	F	F	T	T	T	T

Since the truth value of the given compound proposition is T for all combination of p and q, the proposition is a tautology.

(ii) Truth Table for $(p \vee q) \wedge (\sim p \wedge \sim q)$

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

Thelast column contains only F as the truth value of the given proposition. Hence given proposition is a contradicdiction.

(iii) Truth Table for $(\sim q \wedge (p \to q)) \to \sim p$

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \wedge (p \to q)$	$(\sim q \wedge (p \to q)) \to \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

As the truth value of the given compound proposition is T for all values of p and q, so it is a tautology.

(iv) Truth Table for $(\sim p \wedge q) \wedge (\sim q \wedge (p \to q))$

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$p \to q$	$\sim q \wedge (p \to q)$	$(\sim p \wedge q) \wedge (\sim q \wedge (p \to q))$
T	T	F	F	F	T	F	F
T	F	F	T	F	F	F	F
F	T	T	F	T	T	F	F
F	F	T	T	F	T	T	T

Since all the entries in the last column are F's, the given statement is a contradiction.

(v) Truth Table for $((p \to q) \wedge (q \to r)) \to (p \to r)$

p	q	r	$p \to q$	$p \to r$	$q \to r$	$(p \to q) \wedge (q \to r)$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the truth value of the given statement is T for all combination of truth values of p, q and r, it is a tautology.

(vi) Truth Table for $\sim(q \to r) \wedge r \wedge (p \to q)$

p	q	r	$p \to q$	$q \to r$	$\sim(q \to r)$	$\sim(q \to r) \wedge r$	$\sim(q \to r) \wedge r \wedge (p \to q)$
T	T	T	T	T	F	F	F
T	T	F	T	F	T	F	F
T	F	T	F	T	F	F	F
T	F	F	F	T	F	F	F
F	T	T	T	T	F	F	F
F	T	F	T	F	T	F	F
F	F	T	T	T	F	F	F
F	F	F	T	T	F	F	F

The last column contains only F as the truth value of the given proposition. Hence it is a contradiction.

Example:2. Prove the following equivalences:

(i) $\sim(p \to q) \equiv p \wedge \sim q$

(ii) $(p \vee q) \wedge (p \vee \sim q) \equiv p$

(iii) $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$

Solution: Let us construct the following truth tables.

(i)

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From the truth table it is observed that the columns corresponding to $\sim(p \to q)$ and $p \wedge \sim q$ identical.

Hence

$$\sim(p \to q) \equiv p \wedge \sim q$$

(ii)

p	q	$\sim q$	$p \vee q$	$p \vee \sim q$	$(p \vee q) \wedge (p \vee \sim q)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

Since the last column and the first column are identical, so $(p \vee q) \wedge (p \vee \sim q)$ is equivalent to p.

$$\therefore (p \vee q) \wedge (p \wedge \sim q) \equiv p$$

(iii)

p	q	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	T	F	F	F	F	F	F
T	F	F	T	F	T	T	F	T
F	T	F	T	T	F	F	T	T
F	F	T	F	T	T	F	F	F

From the truth table, it should be noted that the fourth column and last column are identical.

Hence

$$\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q).$$

Example.3 Among the two restauants next to each other, one has a sign that says 'Good food is not cheap' and the other has a sign that says 'Cheap food is not good'. Investigate the signs regarding their equivalance.

Solution: Let p: Food is good.

q : Food is cheap.

The first sign says _ statement is $p \to \sim q$

The second one says _ statement is $q \to \sim p$

Let us construct the truth table for $p \to \sim q$ and $q \to \sim p$

p	q	$\sim p$	$\sim q$	$p \to \sim q$	$q \to \sim p$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	T	T
T	T	F	F	F	F

The truth values by $p \to \sim q$ and $q \to \sim p$ are identical. Hence both the signs are equivalent.

3.1.11. Algebraic laws of 'Connectives'

The 'Logical connectives' $\vee, \wedge, \sim, \to$ etc. obey the following laws: If p, q, r are any propositions then

(i) $p \vee p \equiv p$ and $p \wedge p \equiv p$ (Idempotent law)

(ii) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ and $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ (Associative law)

(iii) $p \vee q \equiv q \vee p$ and $p \wedge q \equiv q \wedge p$ (Commutative law)

(iv) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (Distributive Laws)

(v) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$ and $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$ (D' Morgan's law)

(vi) $p \vee (p \wedge q) \equiv p$ and $p \wedge (p \vee q) \equiv p$ (Absorption law)

(vii) $p \to q \equiv \sim q \to \sim p$ (Contrapositive law)

(viii) $p \vee$ (any contradiction) $\equiv p$

$p \vee$ (any Tautology) $\equiv$ The Tautology

$p \wedge$ (any Tautology) $\equiv p$

$p \wedge$ (any contradiction) $\equiv$ The contradiction (Identity laws)

(ix) $p \vee (\sim p) \equiv$ Tautology

$p \wedge (\sim p) \equiv$ Contradiction (Complement Laws)

(x) $\sim$ (a tautology) $\equiv$ a contradiction

$\sim$ (a contradiction) $\equiv$ a tautology

(xi) $\sim(\sim p) \equiv p$.

Proof. (iv) We shall proof the distributive law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ using Truth Table:

Truth table of $p \wedge (q \vee r)$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	F
F	F	T	T	F
F	T	F	T	F
F	F	F	F	F
T	T	F	T	T
T	F	F	F	F

Truth table of $(p \wedge q) \vee (p \wedge r)$

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	F	T	F	T	T
F	T	T	F	F	F
F	F	T	F	F	F
F	T	F	F	F	F
F	F	F	F	F	F
T	T	F	T	F	T
T	F	F	F	F	F

From above we see the two propositions $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ have same truth table. Hence

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$$

(v) We shall proof the D' Morgan's law

$\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$: using truth table

Truth Table of $\sim (p \vee q)$

p	q	$p \vee q$	$\sim (p \vee q)$
T	T	T	F
F	F	F	T
T	F	T	F
F	T	T	F

Truth Table of $(\sim p) \wedge (\sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	F	F	F
F	F	T	T	T
T	F	F	T	F
F	T	T	F	F

From above we see the truth tables of $\sim (p \vee q)$ and $(\sim p) \wedge (\sim q)$ are identical. Hence

$\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$. So D' Morgan's law is verified.

Illustration: *Simplification of a compound proposition.*

Here we shall show how the above laws can be used to simplify a compound proposition like $(p \wedge q) \vee (\sim p)$

$(p \wedge q) \vee (\sim p) \equiv (\sim p) \vee (p \wedge q)$ (using commutative law)

$\equiv (\sim p \vee p) \wedge (\sim p \vee q)$ (using distributive law)

$\equiv (p \vee \sim p) \wedge (\sim p \vee q)$ (using commutative property)

$\equiv$ Tautology $\wedge (\sim p \vee q)$ by complement laws.

$\equiv (\sim p \vee q) \wedge$ Tautology, using commutative law.

$\equiv \sim p \vee q$ which is very simplified proposition.

Theorem 1. $p \to q \equiv \sim p \vee q$

Proof. Truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth table for $\sim p \vee q$

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

We see the truth tables of $p \to q$ and $\sim p \vee q$ are same. Hence $p \to q \equiv \sim p \vee q$.

Theorem 2. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Proof. Truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth table for $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

We see both the two truth tables are identical. So

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p).$$

3.1.12. Functionally complete set of connectives

The set of connectives for which every propositional formula or expression can be rewritten in terms of an equivalent formula or expression containing the connectivies from that set is known as functionally complete set of connectives. This set is called a minimal functionally complete set of connectives if it does not contain a connective that can be expressed in terms of other connectives of the set.

Illustration:

We have from Theorem -2,

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \qquad (1)$$

Again, from Theorem -1

$$p \to q \equiv \sim p \vee q \qquad (2)$$

$\therefore$ In virtue of (2), we have from (1),

$$\therefore p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p). \qquad (3)$$

Thus the connectives $\to$ and $\leftrightarrow$ can be expressed in terms of the other connectives $\sim, \vee, \wedge$.

We have, from De Morgan's laws,

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

and $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Hence, $p \wedge q \equiv \sim (\sim p \vee \sim q)$

and $p \vee q \equiv \sim (\sim p \wedge \sim q)$ $[\because \sim (\sim p) \equiv p]$

Thus the equivalent expression (3) can also be witten as

$$p \leftrightarrow q \equiv \sim (\sim (\sim p \vee q) \vee \sim (\sim q \vee p)) \qquad (4)$$

or, $p \leftrightarrow q \equiv \sim (\sim (\sim p) \wedge \sim q) \wedge \sim (\sim (\sim q) \wedge \sim p)$

i.e. $p \leftrightarrow q \equiv \sim (p \wedge \sim q) \wedge \sim (q \wedge \sim p)$ $[\because \sim (\sim p) \equiv p]$ (5)

The equivalent expression (4) and (5) contain the minimal complete set of connectives $\{\sim, \vee\}$ and $\{\sim, \wedge\}$ respectively,

Remark.

In any propositional formula, we can eliminate the conditional $(\to)$, biconditional $(\leftrightarrow)$ connectives and then all conjuction $(\wedge)$ or all disjunction $(\vee)$ to get an equivalent formula containing only the set of connectives $\{\sim, \vee\}$ or $\{\sim, \wedge\}$. Thus the sets $\{\sim, \vee\}, \{\sim, \wedge\}$ are the minimal functionaliy complete set of connectives.

Illustrative Examples.

Example.1 Find an equivalent expression for $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p)$ that contains minimal complete set of connectives. [W.B.U.T.2013]

Solution. We know

$$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p) \qquad (1)$$

$$\therefore p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p)$$

$$\equiv p \wedge ((\sim q \vee r) \wedge (\sim r \vee q)) \vee ((\sim r \vee p) \wedge (\sim p \vee r)) \quad \text{using} \quad (1)$$

$$\equiv \sim (\sim p \vee (\sim (\sim q \vee r) \wedge (\sim r \vee q))) \vee \sim (\sim (\sim r \vee p) \vee \sim (\sim p \vee r)),$$

using De Morgan's law

$$\equiv \sim (\sim p \vee (\sim (\sim q \vee r) \vee \sim (-r \vee q))) \vee \sim (\sim (\sim r \vee p) \vee \sim (\sim p \vee r))$$

which is the required equivalent expression.

Example.2 Write an equivalent formula for $(p \to q \wedge r) \vee (r \leftrightarrow q)$ which involves only the connectives $\{\sim, \vee\}$

Solution. We have

$$p \to q \equiv \sim p \vee q \qquad (1)$$

and $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \wedge p)$ (2)

Using (1) and (2), the given expression can be rewritten as

$$(p \to q \wedge r) \vee (r \leftrightarrow q)$$

$$\equiv ((\sim p \vee q) \wedge r) \vee ((\sim r \vee q) \wedge (\sim q \vee r))$$

$$= \sim (\sim (\sim p \vee q) \vee \sim r) \vee \sim (\sim (\sim r \vee q) \vee \sim (\sim q \vee r))$$

which is the required equivalent formula

3.1.13. Normal Forms

Conjunctive Normal Form (CNF).

A compound proposition P is said to be conjunctive normal form (CNF) of r simple propositions $p_1, p_2, \ldots p_r$ if the expression p can be expresed as $P = P_1 \wedge P_2 \wedge \ldots \wedge P_n$ where each $P_i =$ Disjunction of a finite number of the simple propositions of each of $p_1, p_2, \ldots p_n$ or their negations and $P_i \neq P_j$ for all distinct i, j

Illustrations:

(i) $p=(\sim p_1 \vee p_3 \vee p_2)\wedge(\sim p_1 \vee \sim p_2 \vee \sim p_3)\wedge(p_1 \vee p_2 \vee p_3)$ is a CNF of the simple propositions p_1, p_2, p_3

But $p=(\sim p_1 \vee p_2)\wedge(\sim p_1 \vee \sim p_2 \vee \sim p_3)$ is not a CNF since the expression $(\sim p_1 \vee p_2)$ does not contain p_3 or its negation.

(ii) $p=p\vee(\sim p\wedge q)$ is not a CNF

(iii) $(\sim p\vee q)\wedge(\sim q\vee p)$ is a CNF of the two simple propositions p and q.

Disjunctive Normal Form (DNF). A compound proposition P is said to be Disjunctive Normal Form (DNF) of r simple propositions $p_1, p_2, \ldots, p_r$ if the expression P can be expressed as $P=P_1\vee P_2 \vee \ldots \vee P_n$.

where each p_i is conjunction of some of the simple propositions or their negations without any repetition and no p_i is contained in any $p_j (j\neq i)$.

Illustrations: (i) $P=(p\wedge \sim r)\vee(\sim q\wedge p\wedge r)\vee(\sim p\wedge \sim q)$ is a DNF.

Here $P_1=p\wedge \sim r$ is conjunction of p and $\sim r$, $P_2=\sim q\wedge p\wedge r$ is conjunction of $\sim q, p$ and r and $P_3=\sim p\wedge \sim q$ etc.

(ii) $(p\wedge \sim r)\vee(p\wedge \sim q\wedge r)\vee(\sim r\wedge p\wedge q)$ is not DNF since $P_1=p\wedge \sim r$ is contained in $P_3=\sim r\wedge p\wedge q=p\wedge \sim r\wedge q=(p\wedge \sim r)\wedge q$.

3.1.14. Arguments : Deduction of a proposition Q from a set of propositions $P_1, P_2, \ldots P_n$ (compound or simple) is called argument. Symbollically we write $p_1, p_2, \ldots p_n \vdash q$; $p_1, p_2, \ldots p_n$ are called *premises* and Q is called *conclusion*.

Valid Argument : An argument $p_1, p_2, \ldots p_n \vdash q$ is called valid argument if $(p_1 \wedge p_2 \wedge \ldots \wedge p_n)\to q$ is a tautology.

An argument which is not valid is called *invalid argument. or fallacy*

Illustration. Let $P_1=p$ and $P_2=p\to q$ be two propositions (We see P_1 is simple and P_2 is compound proposition). We doubt whether q can be deduced from P_1 and P_2. Therefore $P_1, P_2 \vdash q$ is an argument.

Now we shall test whether it is a valid argument. For this purpose we want to see whether $(p_1\wedge p_2)\to q$ is a tautology.

Truth Table of $(p_1\wedge p_2)\to q$

$p_1=p$	q	$p_2=p\to q$	$p_1\wedge p_2$	$(p_1\wedge p_2)\to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

In the this truth table we see all the entries in the last column are T. So we conclude $(p_1\wedge p_2)\to q$ is a tautology. Thus $p_1, p_2 \vdash q$ is a valid argument.

Illustrative Examples:

Example.1 Show that $\sim p$ is a valid conclusion from the premises $p\to q$, $r\to\sim q$ and r.

Solution: We shall prove that the argument $(p\to q)\wedge(r\to\sim q)\wedge r\to\sim p$ is a tautology.

Let us construct the following truth table:

p	q	r	$\sim p$	$\sim q$	$p\to q$ (p_1)	$r\to\sim q$ (p_2)	$p_1\wedge p_2$	$p_1\wedge p_2\wedge r$	$p_1\wedge p_2\wedge r\to\sim p$
T	T	T	F	F	T	F	F	F	T
T	T	F	F	F	T	T	T	F	T
T	F	T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	F	F	T
F	T	T	T	F	T	F	F	F	T
F	T	F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T	F	T
F	F	F	T	T	T	T	T		T

From the last column we see that $(p \to q) \wedge (r \to \sim q) \wedge r \to \sim p$ is a tautology. Hence $\sim p$ is a valid conclusion from the premises $p \to q, r \to \sim q$ and r.

Example.2 Show that the following argument is not valid.

$p \to q, \sim p \dashv \sim q$

Solution: We know construct the truth table of $(p \to q) \vee \sim p \to \sim q$ as given below.

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge \sim p$	$(p \to q) \wedge \sim p \to \sim q$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

We see that all the entries in the last column are not *T*. So $(p \to q) \wedge \sim p \to \sim q$ is not a tautology. Thus the argument $p \to q, \sim p \dashv \sim q$ is not valid

Example.3 Using truth table, show that the following argument is valid:

$p \to \sim q,\ r \to q,\ r \vdash \sim p$

Solution: To test the validity of the given argument, we construct the following truth table.

p	q	r	$\sim p$	$\sim q$	$p \to q$ (P_1)	$r \to q$ (P_2)	$P_1 \wedge P_2$	$P_1 \wedge P_2 \wedge r$	$P_1 \wedge P_2 \wedge r \to \sim p$
T	T	T	F	F	F	T	F	F	T
T	T	F	F	F	F	T	F	F	T
T	F	T	F	T	T	F	F	F	T
T	F	F	F	T	T	T	T	F	T
F	T	T	T	F	T	T	T	T	T
F	T	F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	F	F	T
F	F	F	T	T	T	T	T	F	T

From the last column of the truth table, it is observed that $p_1 \wedge p_2 \wedge r \to \sim p$ is a tautology.

Thus, $p_1, p_2, r \vdash \sim p$

i.e. $p \to \sim q,\ r \to q,\ r \to \sim p$ is a valid argument.

3.1.15. Theory of Inference.

Inference is the process of deriving logical conclusions from certain basic assumptions, called ***premises***, by employing certain principles of reasoning 'called ***rules of inference***. For example, the rule of infrence, called ***modus ponens*** taken two premises, one in the from 'if p then q' and another in the form 'p' and returns the conclusion 'q' The rule is valid with respect to the semantics of ***classical logic*** , in the sense that if the premises are true, then so is the conclusion. Any conclusion which is obtained by the rules of inference is called a ***valid conclusion*** and the argument is called a ***valid argument.***

Rules of inference.

To check the validity of an argument we shall frequently use three basic rules of inference called rules P,T and CP as given below:

Rule P: A premise may be introduced at any step in the derivation.

Rule T: A formula s may be introduced in the derivation if s is tautologically implied by any one or more preceding formulas in the derivation.

Rule CP: If we can derive a formula s from another formula r and a set of premises, then the statement $r \to s$ can be devived from the set of premises alone.

The rule CP follows from the equivalence proposition $(p \wedge r) \to s \equiv p \to (r \to s)$

Note. When the conclusion is of the form $r \to s$, we will take r as an additional premise and derives using the given premises and r.

A list of rules of inference which are frequently used to check the validity of an argument is given in the following table.*(Shown in the next page)*

Table: Rules of inference.

Rules of inference	Tautological form	Name
1. $\dfrac{p}{\therefore p\vee q}$ or $\dfrac{q}{\therefore p\vee q}$	$p\to p\vee q$ or, $q\to p\vee q$	Addition
2. $\dfrac{p\wedge q}{\therefore p}$ or $\dfrac{p\wedge q}{q}$	$(p\wedge q)\to p$ or, $(p\wedge q)\to q$	Simplification
3. $\dfrac{\begin{matrix}p\\ q\end{matrix}}{\therefore p\wedge q}$	$((p)\wedge(q))\to(p\wedge q)$	Conjunction
4. $\dfrac{\begin{matrix}p\to q\\ r\end{matrix}}{\therefore q}$	$(p\to q)\wedge p\to q$	Modus Ponens
5. $\dfrac{\begin{matrix}p\to q\\ \sim q\end{matrix}}{\therefore \sim p}$	$((p\to q)\wedge\sim q)\to\sim p$	Modus Tollens
6. $\dfrac{\begin{matrix}p\to q\\ q\to r\end{matrix}}{p\to r}$	$((p\to q)\wedge(q\to r))\to(p\to r)$	Hypothetical syllogism
7. $\dfrac{\begin{matrix}p\vee q\\ \sim p\end{matrix}}{\therefore q}$	$((p\wedge q)\wedge\sim p)\to q$	Disjunction syllogism
8. $\dfrac{\begin{matrix}p\vee q\\ p\to r\\ q\to r\end{matrix}}{\therefore r}$	$((p\vee q)\wedge(p\to r)\wedge(q\to r))\to r$	Dilemma
9. $\begin{matrix}p\vee r\\ p\to q\\ r\to s\end{matrix}$	$((p\vee r)\wedge(p\to q)\wedge(r\to s))\to(q\vee s)$	Constructive Dilemma

10. $\dfrac{\begin{matrix}\sim q\vee\sim s\\ p\to q\\ r\to s\end{matrix}}{\therefore\sim p\vee\sim r}$	$((\sim q\vee\sim s)\wedge(p\to q)\wedge(r\to s))\to(\sim p\vee\sim r)$	Destructive Dilemma
11. $\dfrac{\begin{matrix}p\vee q\\ \sim p\vee r\end{matrix}}{\therefore q\vee r}$	$((p\vee q)\wedge(\sim p\vee r))\to(q\vee r)$	Resolution

Illustrations

1. We now discus the validity of the following argument:

$$\dfrac{\begin{matrix}p\to q\\ r\to\sim q\end{matrix}}{\therefore p\to\sim r}$$

Given that $r\to\sim q$. This leads $q\to\sim r$, by contrapositive law. Thus $p\to a$ and $q\to\sim r$ give $p\to\sim r$ by the law of hypothetical syllogism. Hence the given argument is valid.

2. We shall show that r is a valid conclusion from the premises $p\vee q,\ q\to r,\ p\to s,\ \sim s$

The from of argument given as follows shows that the premises lead to the conclusion

Step No.	Statement	Reason
1.	$p\to s$	Rule P
2.	$\sim s$	Rule P
3.	$\sim p$	Rule T,steps 1,2 and Modus Tollers
4.	$p\vee q$	Rule P
5.	q	Rule T, steps 3,4 and Disjunctive syllogism
6.	$q\to r$	Rule P
7.	r	Rule T, steps 5,6 and Modus Ponens

Thus r is a valid conclusion from the given premises. Hence the given argument is valid. It should be noted that the rules of inference are used in steps 3,5 and 7.

Illustrative Examples.

Example.1 Show that $(a\vee b)$ follows logically from the premises $p\vee q$, $(p\wedge q)\to\sim r$, $\sim r\to(s\wedge\sim t)$ and $(s\wedge\sim t)\to(a\vee b)$.

Solution. The following steps are used to derive the result

Step No.	Statement	Reason
1.	$(p\vee q)\to\sim r$	Rule P
2.	$\sim r\to(s\wedge\sim t)$	Rule P
3.	$(p\vee q)\to(s\wedge\sim t)$	Rule T, steps 1,2 and by hypothetical syllogism
4.	$p\vee q$	Rule P
5.	$s\wedge\sim t$	Rule T,steps 3,4 and by Modus Ponens
6.	$(s\wedge\sim t)\to(a\vee b)$	Rule P
7.	$a\vee b$	Rule T, steps 5,6 and by Modus Ponens

Thus $(a\vee b)$ follows logically from the given premises.

Example.2 Show that $r\vee s$ is a valid conclusion from the premises $p\vee q$, $p\vee q\to\sim w$, $\sim w\to(u\wedge\sim v)$ and $(u\wedge\sim v)\to r\vee s$.

Solution. The following steps are used for deducting the valid conclusion from the given premises:

Step No.	Statement	Reason
1.	$p\vee q\to\sim w$	Rule P
2.	$\sim w\to(u\wedge\sim v)$	Rule P
3.	$(p\vee q)\to(u\wedge\sim v)$	Rule T, steps 1,2 and by hypothetical syllogism
4.	$(u\wedge\sim v)\to(r\vee s)$	Rule P
5.	$(p\vee q)\to(r\vee s)$	Rule T, steps 3,4 and by hypothetical syllogism
6.	$p\vee q$	Rule P
7.	$r\vee s$	Rule T, steps5,6 and Modus ponens

Hence $r\vee s$ is a valid conclusion from the given premises.

Example.3 Show that s is a valid conclusion from the premises $p\to\sim q$, $q\vee r$, $\sim s\to p$, $\sim r$ [W.B.U.T 2012, 2015, 2016]

Solution. The steps for deducting whether s is a valid conclusion from the given premises are as given below.

Step No.	Statement	Reason
1.	$q\vee r$	Rule P
2.	$\sim r$	Rule P
3.	q	Rul T, steps 1,2 and by disjunction syllogism
4.	$p\to\sim q$	Rule P
5.	$\sim(\sim q)$	Rule T and using $\sim(\sim q)\equiv q$
6.	$\sim p$	Rule T, steps 4,5 and by Modus Tollens
7.	$\sim s\to p$	Rule P
8.	$\sim(\sim s)$	Rule T, steps 6,7 and by Modus Tollens
9.	s	Rule T and using $\sim(\sim s)\equiv s$

Thus s is a valid conclusion from the given premises.

Example.4 Show that $(t\wedge s)$ can be derived from the premises $p\to q$, $q\to\sim r$, r, $p\sim(t\wedge s)$

Solution. To derive $(t\wedge s)$ from the premises $p\to q$, $q\to\sim r$, r $p\vee(t\wedge r)$, let us consider the following steps.

Step No.	Statement	Reason
1.	$p\to q$	Rule P
2.	$q\to\sim r$	Rule P
3.	$p\to\sim r$	Rule T, steps 1,2 and by Hypothetical syllogism
4.	$r\to\sim p$	Rule 7, step 3 and by contrapositive law
5.	r	Rule P

6.	$\sim p$	RuleT, steps 4,5 and by Modus ponens
7.	$p \vee (t \wedge s)$	Rule P
8.	$t \wedge s$	Rule T,steps 6,7 and by Disjunctive syllogism.

Hence $t \wedge s$ can be derived from the given premises.

Example.5 Show that t is a valid conclusion from the premises $p \to q,\ q \to r,\ r \to s,\ \sim s$ and $p \vee t$

[W.B.U.T. 2012,2016]

Solution. The following steps are used for deducting whether t is a valid conclusion from the given premises.

Step No.	Statement	Reason
1.	$p \to q$	Rule P
2.	$q \to r$	Rule P
3.	$p \to r$	Rule T,steps 1,2 and by hypothetical syllogism
4.	$r \to s$	Rule P
5.	$p \to s$	Rue T,steps 3,4 and by hypothetical syllogism
6.	$\sim s$	Rule P
7.	$\sim p$	Rule T, steps 5,6 and by Modus Tollens
8.	$p \vee t$	Rule P
9.	t	RuleT,steps 7,8 and bydisjunctive syllogism

Hence t is a valid conclusion from the given premises.

Example.6 Using CP-rule, show that $p \to (q \to s)$ from the premises $p \to (q \to r)$ and $q \to (r \to s)$

Solution. Here we arsume p as an additional premise. Using p and the two given premises we will derive $(q \to s)$.Then by CP- rule , we shall prove that $p \to (q \to s)$.

Step No.	Statement	Reason
1.	p	Rule P (additional)
2.	$p \to (q \to r)$	Rule P
3.	$q \to r$	RuleT,steps 1,2 and by modus ponens
4.	$\sim q \vee r$	Rule T, step 3
5.	$q \to (r \to s)$	Rule P
6.	$\sim q \vee (r \to s)$	Rule T, step 5
7.	$\sim q \vee (r \wedge (r \to s))$	Rule T, steps 4,6 and using distributive law
8.	$\sim q \vee s$	Rule T,step 7 and by modus ponens
9.	$q \to s$	Rule T, step 8
10.	$p \to (q \to s)$	Rule T, step 9 and using CP-rule.

3.1.16. Predicate and Propositional Function

Recall the concept of function $f: A \to B$ where $f(x) \in B$ for all x in A. x was variable for the function f; $f(x)$ was the image of x. Now, let A be a non-null set and S be the set of all *proposition*. A mapping $P: A \to S$ is called a Propositional Function. So for any $x \in A$, $P(x) \in S$ i.e., $P(x)$ is a proposition.

The set A is called **Domain** or **Universe of Discourse** of the propositional function P. $P(x)$ is called **Predicate** and x is called **Predicate Variable** of the propositional functon P.

Example : (1) Let R be the set of all real numbers. S is Set all propositions. Let $P(x)$:'$x > 10$'. Then $P(2)$:'$2 > 10$' is a proposition (which is False). $P(18\cdot1)$:'$18\cdot1 > 10$' is a proposition (which is True).

$\therefore P(2), P(18\cdot1) \in S$ Thus for every $x \in R$ we get a $P(x)$ in S.

So the mapping $P:R \to S$ defined by $P(x)$:'$x>10$' is a *propositional function*. '$x>10$' is the predicate and x (any number) is predicate variable of the function. The set of reals R is the Domain or the Universe of Discourse of the function.

(2) Let A be the set of all persons in India. S is set of all propositions. Consider $P(x)$:'x is a graduate'. Then P(Ram) : Ram is a graduate is a proposition (which may be True or False). $P(Ram) \in S$. Thus $P(x) \in S$ i.e., $P(x)$ is a proposition where x is a person of India. Therefore the mapping $P: A \to S$ defined by $P(x): x$ is a *graduate* is a propositional function, 'x is a graduate' is the predicate and x (persons of India) is predicate variable of the function. The set of all persons of India is Domain.

(3) Propositional function can be defined for two or more predicate variable also. For example consider $P((x,y)): x+2y=1$ where x, y are real number. Then here the mapping $P: R \times R \to S$ defined by $P((x,y)): x+2y=1$ is propositional function, '$x+2y=1$' is predicate, (x, y) is predicate variable.

3.1.17. Quantification and Quantifier

We have seen a proposition can be created from a propositional function $P(x)$ by assigning particular values to the variable x.

Quantification is another way of creating a proposition from a propositional function. There are two types of quantifications, namely, ***Universal Quantifications*** and ***Existential Quantifications***.

Universal Quantification.

The universal quantification of the propositional function $P(x)$ is the proposition "$P(x)$ is true for all values of x in the domain". This quantification is denoted by $\forall x, P(x)$. The symbol $\forall$ is called ***Universal Quantifier***.

Illustrations.

(1) Consider the propositional function $P(x)$ stating "x is a good batsman" where the domain is set of all students in a class. Then its universal quantification $\forall x, P(x)$ is the statement "Every student in the class is a good batsman".

(2) Let $P(x) \equiv$ "x is mortal" be a propositional function. Then its universal quantification $\forall x, P(x)$ is the proposition "All human beings are mortal", where the domain is the set of all human beings. Obviously this proposition $\forall x, P(x)$ is true.

(3) Let $P(x)$ be the propositional function "$x+4<15$". Then its universal quantification is

$$"x+4<15" \ \forall x \in \{1,3,5,7,9,11,13,15\}.$$

Here the domain is the set $S=\{1,3,5,7,9,11,13,15\}$. Obviously this proposition $\forall x, P(x)$ is false because the value 13 in the domain does not satisfy the relation $x+4<15$. $\forall x, P(x)$ would be true if the domain is taken as $\{1,2,3,4,5,6\}$.

Existential Quantification.

The existential quantification of the propositional function $P(x)$ is the proposition "There exists a value of x in the domain for which $P(x)$ is true". This quantificaion is denoted by $\exists x, P(x)$. The symbol $\exists$ is called ***Existential Quantifier***.

Illustrations.

(1) Consider the propositional function $P(x)$ stating "x is a good batsman" where the domain is the set of all students in a class. Then its existentional quantification $\exists x, P(x)$ is the statement "There exists a student in the class who is a good batsman".

(2) Let $P(x)$ be the propositional function "$x+4<15$". Then its existential quantification $\exists x, P(x)$ is "There exists a value of x for which $x+4<15$ is valid" where the domain is the set $\{1,2,3,4,5,6,7,8,9,10,11,12\}$. Obviously this proposition $\exists x, P(x)$ is true. But it would be false if the domain is taken as the set $\{12,13,14,15\}$.

(3) For the propositional function $P(x)$ stating "$x+2=x$" the existential quantification is "there exists a value of x for which $x+2=x$ is valid"; the domain is the set of all real numbers. Obviously this proposition $\exists x, P(x)$ is false because no real value x satisfies the relation $x+2=x$.

Negation of Quantification.

If $P(x)$ be a propositional function on a domain A then it can be proved that

(i) $\sim(\forall x\, P(x)) \equiv \exists x(\sim P(x))$ for $x \in A$

(ii) $\sim(\exists x\, P(x)) \equiv \forall x(\sim P(x))$ for $x \in A$

Illustrations.

(i) Let $A = \{1,2,3,4\}$ be the domain of $P(x): x+3=10$.

Then $\sim(\exists x, P(x)) \equiv \sim(\exists x,\ x+3=10)$

$\equiv (\forall x,\ \sim(x+3=10))$

$\equiv \forall x, x+3 \neq 10$ for $x \in A$

$\because\ 1+3\neq10,\ 2+3\neq10,\ 3+3\neq10,\ 4+3\neq10$ so this negation $\sim(\exists x,\ P(x))$ is True.

(ii) Let $A = \{1,2,3,4\}$ be the universe of discourse of the function $P(x)$ stating '$x+3<10$'.

Then $\sim(\forall x,\ P(x)) \equiv \sim(\forall x, x+3<10)$

$\equiv \exists x,\ \sim(x+3<10)$

$\equiv \exists x,\ x+3 \geq 10$ for $x \in A$.

Since $1+3 \ngeq 10,\ 2+3 \ngeq 10,\ 3+3 \ngeq 10,\ 4+3 \ngeq 10$.

$\therefore$ this negation $\sim(\forall x,\ P(x))$ is false.

3.1.18. Miscellaneous Examples.

Ex.1. Let p:*It is cold* and q : *It is raining* be two propositions. Write the sentences which describe

(i) $q \wedge \sim p$ (ii) $\sim(p \vee q)$ (iii) $p \vee q$

Solution. (i) It is raining or it is not cold

(ii) It is not true that it is cold or it is not raining.

(iii) It is cold or it is raining.

Ex.2. Let p: *He is intelligent* and q: *He is tall* be two propositions. Write each of the following statements in symbolic form using p and q:

(i) He is tall but not intelligent

(ii) He is neither tall nor intelligent.

(iii) He is intelligent or he is tall

(iv) It is not true that he is intelligent or tall.

(v) It is not true that he is not tall or not intelligent.

Solution.

(i) $q \wedge \sim p$ (ii) $\sim q \wedge \sim p$ (iii) $p \vee q$

(iv) $\sim(p \vee q)$ (v) $\sim(\sim q \vee \sim p)$

Ex.3. If p: *Madhu goes to cinema* and q: *Ram goes to cinema* be two propositions then construct symbolic form of the statements :

(i) Both of Madhu and Ram goes to cinema.

(ii) At least one of Madhu and Ram goes to cinema.

(iii) Madhu does not goes to cinema but Ram goes to cinema.

(iv) Either Madhu or Ram goes to cinema.

(v) If Ram goes to cinema then Madhu also goes to cinema.

(vi) If Madhu does not go to cinema then Ram will go to cinema.

Solution.

(i) $p \wedge q$ (ii) $p \vee q$

(iii) $\sim p \wedge q$ (iv) $(p \wedge q') \vee (p' \wedge q)$

(v) $q \to p$ (vi) $\sim p \to q$

Ex. 4. If p: Today is Friday

q: It is raining

r: It is hot,

write the statement against the following symbol

(i) $\sim q \to (r \wedge p)$ (ii) $(p \vee q) \leftrightarrow r$ (iii) $(p \wedge \sim q) \to \sim r$

Solution. (i) $\sim q$: It is not raining

$r \wedge p$: It is hot and today is Friday

$\therefore \sim q:(r \wedge p)$: If it is not raining then it is hot and today is Friday.

(ii) $p \vee q$: Today is Friday or it is raining today.

$\therefore (p \vee q) \leftrightarrow r$: Today is Friday or it is raining if and only if it is hot today.

(iii) $p \wedge \sim q$: Today is Friday and it is not raining today

$\sim r$: It is not hot

$\therefore (p \wedge \sim q) \to \sim r$: If today is Friday and not raining then it is not hot.

Ex. 5. What is the negation of the proposition 'Some people have no scooter' ? [W.B.U.T. 2006]

Solution. The negation is " All people have scooter'.

Ex. 6. Write the negation of the statement 'If it is hot then we will not go to cinema'.

Solution: Let p: *It is hot* and q: *We will go to cinema* be the two propositions.

Then the given statement is $p \to \sim q$

Its negation is $\sim(p \to \sim q) \equiv \sim(\sim p \vee \sim q)$ $[\because p \to q \equiv \sim p \vee q]$

$\equiv \sim\sim(p \wedge q)$ by D' Morgan's Rule

$\equiv p \wedge q$

$\therefore$ the negation is " It is hot and we will go to cinema ".

Ex. 7. If p is true and q is false then find the truth value of $(p \wedge q) \to (p \vee q)$ [W.B.U.T 2006]

Solution. Here p takes truth value T; q takes the truth values F.

Then from the truth table of conjunction we see $p \wedge q$ assumes the truth value F and from the truth table of Disjunction we see $p \vee q$ assumes T.

From the truth table of conditional proposition we see $(p \wedge q) \to (p \vee q)$ assumes the truth value T.

Ex. 8. Express the following problem in logical notation :

All men are mortal.

Socrates is a man.

Therefore socrates is mortal. [W.B.U.T 2007]

Solution. If p: all men are mortal; q: Socrates is a man and r: Socrates is mortal.

Then this above argument is $p, q \vdash r$ is a valid argument.

Ex. 9. If the two propositions p and q are false then find whether the proposition $(p \vee q) \wedge (\sim p) \vee (\sim q)$ is true or false.

Solution. We construct a truth table of the given expression by

p	q	$\sim p$	$\sim q$	$p \vee q$	$(\sim p) \vee (\sim q)$	$(p \vee q) \wedge (\sim p) \vee (\sim q)$
F	F	T	T	F	T	F

taking only the value F against p and q.

Looking at the last column of the above table we conclude that the proposition $(p \vee q) \wedge (\sim p) \vee (\sim q)$ is false.

Ex. 10. Find the truth tables of the followings :

(i) $\sim(p \wedge \sim q)$ (ii) $(p \to q) \leftrightarrow (\sim q \to \sim p)$ (iii) $[(p \wedge q) \vee (\sim r)] \leftrightarrow p$

Solution.

(i) In $\sim(p \wedge \sim q)$ two propositions p and q are involved.

So there are $2^2 = 4$ rows in its truth Table

Truth Table for $\sim(p \wedge \sim q)$

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(ii) Here also two rows will be there

Truth table for $(p \to q) \leftrightarrow (\sim q \to \sim p)$

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Note : To construct the 5th, 6th and 7th column see the Truth table for $\to$ and $\leftrightarrow$ discussed in Art 3.1.4.

(iii) Since $[(p \wedge q) \vee (\sim r)] \leftrightarrow p$ involves 3 simple propositions p, q, r so there will be $2^3 = 8$ rows in the Truth Table :

Truth Table for $[(p \wedge q) \vee (\sim r)] \leftrightarrow p$

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee (\sim r)$	$[(p \wedge q) \vee (\sim r)] \leftrightarrow p$
T	T	T	T	F	T	T
T	T	F	T	T	T	T
T	F	T	F	F	F	F
F	T	T	F	F	F	T
T	F	F	F	T	T	T
F	T	F	F	T	T	F
F	F	T	F	F	F	T
F	F	F	F	T	T	F

Ex.11. Show that $\{(p \wedge \sim q) \to r\} \to \{p \to (q \vee r)\}$ is a tautology.

[W.B.U.T. 2006]

Solution: If all the truth values of

$$\{(p \wedge \sim q) \to r\} \to \{p \to (q \vee r)\}$$

are T then it will be a tautology. We are constructing the truth table of this statement. Since it involves 3 propositions so the truth table will have $2^3 = 8$ rows.

Truth Table of $\{(p \wedge \sim q) \to r\} \to \{p \to (q \vee r)\}$

p	q	r	$\sim q$	$p \wedge \sim q$	$q \vee r$	$(p \wedge \sim q) \to r$	$p \to (q \vee r)$	$\{(p \wedge \sim q) \to r\} \to \{p \to (q \vee r)\}$
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T	T
T	F	T	T	T	T	T	T	T
F	T	T	F	F	T	T	T	T
F	T	F	F	F	T	T	T	T
F	F	T	T	F	T	T	T	T
T	F	F	T	T	F	F	F	T
F	F	F	T	F	F	T	T	T

From the truth table we see the given statement has the only truth value. T. So this is a tautology.

Ex.12. Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

[W.B.U.T. 2007, 2014]

Solution. We are proving this without using Truth table :

$(p \wedge q) \to (p \vee q) \equiv \sim (p \wedge q) \vee (p \vee q)$ $\quad [\because p \to q \equiv \sim p \vee q]$

$\equiv (\sim p \wedge \sim q) \vee (p \vee q)$ by D'Morgan's law

$\equiv \{(\sim p \wedge \sim q) \vee p\} \vee q$ by associative law

$\equiv \{(\sim p \vee p) \wedge (\sim q \vee p)\} \vee q$ using distributive laws

$\equiv \{$ (tatulogy) $\wedge (\sim q \vee p)\} \vee q$ by complement laws

$\equiv (\sim q \vee p) \vee q$ by Identity law

$\equiv (p \vee \sim q) \vee q$ by commutative law

$\equiv p \vee (\sim q \vee q)$ by Assocciative law

$\equiv p \vee$ (a tautology) $=$ Tautology by Identity law.

Ex.13. Prove that $\neg(P \vee Q) \vee (\neg P \wedge Q) \vee P$ is a tautology.

[W.B.U.T. 2008]

Solution. Note "$\neg p$ stands for $\sim p$" $\neg(P \vee Q) \vee (\neg P \wedge Q)$

$\equiv (\neg P \wedge \neg Q) \vee (\neg P \wedge Q)$ Using D' Morgan's law

$\equiv (\neg P) \wedge (\neg Q \vee Q)$ Using distributive law

$\equiv \neg P \wedge$ Tautology by complement law

$\equiv \neg P$ by Identity Law

$\therefore \neg(P \vee Q) \vee (\neg P \wedge Q) \vee P \equiv \neg P \vee P$, from above

$\equiv$ Tautology by complement law.

Ex.14. Show that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is contradiction.

Solution. If all the truth value assumed by the given expression are False (F) then it will be a contradiction.

Truth Table for $(p \vee q) \wedge (\sim p \wedge \sim q)$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

From the above truth table we see all the values assumed by $(p \vee q) \wedge (\sim p \wedge \sim q)$ are False. So, this is a contradiction.

Ex. 15. Using truth table show that

$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$$

Solution. Truth Table for $p \to (q \vee r)$

p	q	r	$q \vee r$	$p \to (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
F	T	T	T	T
T	F	F	F	F
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

Truth Table for $(p \to q) \vee (p \to r)$

p	q	r	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
F	T	T	T	T	T
T	F	F	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

The truth values taken by $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$ are identical.

$\therefore p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

Ex.16. Prove the following equivalence $p \equiv (p \wedge q) \vee (p \wedge \sim q)$

[W.B.U.T. 2006]

Solution. RHS $= (p \wedge q) \vee (p \wedge \sim q)$

$\equiv p \wedge \{q \vee \sim q\}$ using distributive property.

$\equiv p \wedge$ (a tautology) using complement law

$= p$ using Identity law.

$\therefore (p \wedge q) \vee (p \wedge \sim q) \equiv p$

i.e. $p \equiv (p \wedge q) \vee (p \wedge \sim q)$

Ex.17. Prove the following equivalence

$$P \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$$ [W.B.U.T 2008]

Solution. RHS $= (P \to Q) \vee (P \to R) \equiv (\sim P \vee Q) \vee (\sim P \vee R)$

$\equiv (Q \vee \sim p) \vee (\sim P \vee R)$ by commutative law.

$\equiv Q \vee \{\sim P \vee (\sim P \vee R)\}$ by Associative law.

$\equiv Q \vee \{(\sim P \vee \sim P) \vee R\}$ " " "

$\equiv Q \vee \{\sim P \vee R\}$ by Idempotent law

$\equiv Q \vee \{R \vee \sim P\}$ by commutative law

$\equiv (Q \vee R) \vee \sim P$ by Associative law

$\equiv \sim P \vee (Q \vee R)$ by commutative law

$\equiv P \rightarrow (Q \vee R)$ [$\because$ by a previous theorem,

$P \rightarrow Q \equiv \sim P \vee Q$]

Ex. 18. Construct truth table and determine whether the following proposition is tautology or contradiction.

$$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$$ [W.B.U.T 2007]

Solution. Since there are three simple propositions p, q and r so the truth table will have $2^3 = 8$ rows.

Truth table for $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$p_1 \wedge p_2$	$p_1 \wedge p_2 \wedge p_3$	$p_4 \rightarrow r$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	F	T
F	F	T	F	T	T	F	F	T
F	F	F	F	T	T	F	F	T

From the last column of the above truth table we see all the truth values of the given proposition are 'T'. So it is a tautology.

Ex.19. Prove the following equivalences without using truth tables:

(i) $p \vee (p \wedge q) \equiv p$

(ii) $\sim(\sim(p \vee q) \vee (\sim p \wedge q)) \equiv p$

(iii) $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$ [W.B.U.T 2013,2013]

(iv) $(\sim p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$

(v) $(p \wedge r) \vee (q \wedge r) \vee (\sim p \wedge (\sim q \wedge r)) \equiv r$

(vi) $(q \wedge \sim (p \wedge q)) \vee (p \wedge (\sim p \vee q)) \equiv q$

Solution: (i) $p \vee (p \wedge q)$

$\equiv (p \wedge \text{tautology}) \vee (p \wedge q)$ [$\because p \wedge \text{tautology} \equiv p$]

$\equiv p \wedge (\text{tautology} \vee q)$ [by Distributive property]

$\equiv p \wedge \text{tautology}$ [$\because$ tautology $\vee q \equiv$ tautology]

$\equiv p$

(ii) We have

$\sim(p \vee q) \vee (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$, by De Morgan's law

$\equiv \sim p \wedge (\sim q \vee q)$, by Distributive property

$\equiv \sim p \wedge$ tautology [$\because \sim q \vee q \equiv$ tautology]

[$\because p \wedge \text{tautology} \equiv p$]

$\therefore \sim(\sim(p \vee q) \vee (\sim p \wedge q))$

$\equiv \sim(\sim p)$

$\equiv p$

(iii) $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q)$

$\equiv \sim((\sim p \wedge \sim q) \vee (\sim p \wedge q)) \vee (p \wedge q)$, by commutative law

$\equiv \sim(\sim p) \vee (p \wedge q)$, by (ii)

$\equiv p \vee (p \wedge q)$

$\equiv p$, by (i)

(iv) $(\sim p \vee q) \wedge (p \wedge (p \wedge q))$

$\equiv (\sim p \vee q) \wedge ((p \wedge p) \wedge q)$, by associative law

$\equiv (\sim p \vee q) \wedge (p \wedge q)$, by idempotent law

$\equiv (p \wedge q) \wedge (\sim p \vee q)$, by commutative law

$\equiv((p\wedge q)\wedge\sim p)\vee((p\wedge q)\wedge q)$, by distributive law

$\equiv(\sim p\wedge(p\wedge q))\vee(p\wedge(q\wedge q))$, by commutative and associative law

$\equiv((\sim p\wedge p)\wedge q)\wedge(p\wedge q)$, by associative and idempotent law

$\equiv(\text{contradiction}\wedge q)\vee(p\wedge q)$

$\equiv(\text{contradiction})\vee(p\wedge q)$

$\equiv p\wedge q$

(v) $(p\wedge r)\vee(q\wedge r)\vee(\sim p\wedge(\sim q\wedge r))$

$\equiv((p\vee q)\wedge r)\vee((\sim p\wedge\sim q)\wedge r)$, by distributive and associative law

$\equiv((p\vee q)\wedge r)\vee(\sim(p\vee q)\wedge r)$, by D' Morgan's law

$\equiv((p\vee q)\vee\sim(p\vee q))\wedge r$, by distributive law

$\equiv(\text{Tautology}\wedge r)$

$\equiv r$

(vi) $(q\wedge\sim(p\wedge q))\vee(p\wedge(\sim p\vee q))$

$\equiv(q\wedge(\sim p\vee\sim q))\vee((p\wedge\sim p)\vee(p\wedge q))$, by D' Morgan's law and distributive law

$\equiv((q\wedge\sim p)\vee(q\wedge\sim q))\vee((\text{contradiction})\vee(p\wedge q))$

$\equiv((q\wedge\sim p)\vee(\text{contradiction}))\vee(p\wedge q)$

$\equiv(q\wedge\sim p)\vee(q\wedge p)$, by commutative law

$\equiv q\wedge(\sim p\vee p)$, by distributive law

$\equiv q\wedge$ (tautology)

$\equiv q$

Ex.20 Without using truth table, prove that

(i) $p\to q\equiv p\to(p\wedge q)$

(ii) $(p\vee r)\to q\equiv(p\to q)\wedge(r\to q)$

(iii) $p\to(q\to r)\equiv p\wedge q\to r$ [W.B.U.T.2013,2015]

(iv) $p\leftrightarrow q\equiv(p\vee q)\to(p\wedge q)$

(v) $(p\to q)\to q\equiv p\vee q$

(vi) $(q\to(p\wedge\sim p))\to(r\to(p\wedge\sim p))\equiv r\to q$

Solution: (i) $p\to q$

$\equiv\sim p\vee q$

$\equiv(\text{tautology})\wedge(\sim p\vee q)$

$\equiv(\sim p\vee p)\wedge(\sim p\vee q)$ [$\because$ tautology $\equiv\sim p\vee p$]

$\equiv\sim p\vee(p\wedge q)$, by distributive law

$\equiv p\to(p\wedge q)$

(ii) $(p\vee r)\to q$

$\equiv\sim(p\vee r)\vee q$

$\equiv(\sim p\wedge\sim r)\vee q$ by D' Morgan's law

$\equiv(\sim p\vee q)\wedge(\sim r\vee q)$, by distributive law

$\equiv(p\to q)\wedge(r\to q)$

(iii) $p\to(q\to r)$

$\equiv p\to(\sim q\vee r)$

$\equiv\sim p\vee(\sim q\vee r)$

$\equiv(\sim p\vee\sim q)\vee r$, by associative law

$\equiv\sim(p\wedge q)\vee r$, by D' Morgan's law

$\equiv(p\wedge q)\to r$

(iv) We have

$p\leftrightarrow q$

$\equiv(p\to q)\wedge(q\to p)$

$\equiv(\sim p\vee q)\wedge(\sim q\wedge p)$

$\equiv((\sim p\vee q)\wedge\sim q)\vee((\sim p\vee q)\wedge p)$, by distributive property

$\equiv(\sim q\wedge(\sim p\vee q))\vee(p\wedge(\sim p\vee q))$, by commutative property

$\equiv((\sim q\wedge\sim p)\vee(\sim q\wedge q))\vee((p\wedge\sim p)\vee(p\wedge q))$, by distributive

$\equiv\sim(q\vee p)\vee\text{contradiction}\vee(\text{contradiction}\vee(p\wedge q))$

$\equiv\sim(p\vee q)\vee(p\wedge q)$ [$\because p\vee\text{contradiction}\equiv p$]

$\equiv(p\vee q)\to(p\wedge q)$

(v) $(p \to q) \to q$

$\equiv (\sim p \vee q) \to q$

$\equiv \sim(\sim p \vee q) \vee q$

$\equiv (p \wedge \sim q) \vee q$, by D Morgan's law

$\equiv (p \vee q) \wedge (\sim q \vee q)$, by Distributive law

$\equiv (p \vee q) \wedge (\text{tautology})$

$\equiv p \vee q$

(vi) $(q \to (p \wedge \sim p)) \to (r \to (p \wedge \sim p))$

$\equiv (\sim q \vee (p \wedge \sim p)) \to (\sim r \vee (p \wedge \sim p))$

$\equiv (\sim q \vee \text{contradiction}) \to (\sim r \vee \text{contradiction})$

$\equiv \sim q \to \sim r$

$\equiv \sim(\sim q) \vee \sim r$

$\equiv q \vee \sim r$

$\equiv \sim r \vee q$

$\equiv r \to q$

Ex.21. Write DNF and CNF of the following statement

$$p \to (p \wedge (q \to p)) \qquad \text{[W.B.U.T. 2008,2006]}$$

Solution. $p \to (p \wedge (q \to p)) \equiv \sim p \vee \{p \wedge (\sim q \vee p)\}$ $\because p \to q \equiv \sim p \vee q$

$\equiv (\sim p \vee p) \wedge \{\sim p \vee (\sim q \vee p)\}$

$\equiv \text{Tautology} \wedge \{\sim p \vee (\sim q \vee p)\}$

by complement law

$\equiv \sim p \vee (\sim q \vee p)$ by Identity Law.

$\equiv \sim p \vee (p \vee \sim q)$ by commutative Law.

$\equiv (\sim p \vee p) \vee \sim q$ by Associative Law.

$\equiv \text{Tautology} \vee \sim q \equiv \text{Tautology}$

$\equiv p \vee \sim p$ which is a DNF

Ex. 22. Write a CNF (Conjunctive normal form) of the following statements :

(i) $p \wedge (p \to q)$

(ii) $\{q \vee (p \wedge r)\} \wedge \sim \{(p \vee r) \wedge q\}$

Solution. (i) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$ which is the required CNF

(ii) $\{q \vee (p \wedge r)\} \wedge \sim \{(p \vee r) \wedge q\}$

$\equiv \{q \vee (p \wedge r)\} \wedge \{\sim (p \vee r) \vee \sim q\}$ by D' Morgan's Law

$\equiv \{q \vee (p \wedge r)\} \wedge \{\sim (p \wedge \sim r) \vee \sim q\}$ by D' Morgan's law.

$\equiv (q \vee p) \wedge (q \vee r) \wedge \{(\sim p \vee \sim q) \wedge (\sim r \vee \sim q)\}$

$\equiv (q \vee p) \wedge (q \vee r) \wedge (\sim p \vee \sim q) \wedge (\sim r \vee \sim q)$ which is required CNF.

Ex.23. Find the CNF of the following statement :

$$\neg(p \vee q) \leftrightarrow (p \wedge q)$$

Solution.

$\neg(p \vee q) \leftrightarrow (p \wedge q) \equiv \sim (p \vee q) \leftrightarrow (p \wedge q)$

$\equiv \{\sim (p \vee q) \to (p \wedge q)\} \wedge \{(p \wedge q) \to \sim (p \vee q)\}$

$\equiv \{\sim\sim (p \vee q) \vee (p \wedge q)\} \wedge \{\sim (p \wedge q) \vee \sim (p \vee q)\}$

$[\because p \to q \equiv \sim p \vee q]$

$\equiv \{(p \vee q) \vee (p \wedge q)\} \wedge \sim \{(p \wedge q) \wedge (p \vee q)\}$ . . (1)

by D'Morgan's law

Now, $(p \vee q) \vee (p \wedge q) \equiv \{(p \vee q) \vee p\} \wedge \{(p \vee q) \vee q\}$

$\equiv \{(q \vee p) \vee p\} \wedge \{p \vee (q \vee q)\}$

$\equiv \{q \vee (p \vee p)\} \wedge \{p \vee q\}$

$\equiv (q \vee p) \wedge (p \vee q)$ using Idempotent law

$\equiv (p \vee q) \wedge (p \vee q)$ using Commutative law

$\equiv p \vee q$ again using idempotent law.

Again $(p\wedge q)\wedge(p\vee q) \equiv p\wedge\{q\wedge(p\vee q)\}$

$\equiv p\wedge\{(q\wedge p)\vee(q\wedge q)\} \equiv p\wedge\{(q\wedge p)\vee q\}$

$\equiv\{p\wedge(q\wedge p)\}\vee(p\wedge q) \equiv \{p\wedge(p\wedge q)\}\vee(p\wedge q)$

$\equiv\{(p\wedge p)\wedge q\}\vee(p\wedge q) \equiv (p\wedge q)\vee(p\wedge q) = p\wedge q$

$\therefore \sim\{(p\wedge q)\wedge(p\vee q)\} \equiv\sim(p\wedge q)\equiv(\sim p\vee\sim q)$

So, from (1) we have $\neg\ (p\vee q)\leftrightarrow(p\wedge q)$

$\equiv(p\vee q)\wedge(\sim p\vee\sim q)$ which is a CNF

Ex. 24. Obtain the DNF (Disjunctive Normal form) of the following statement :

(i) $p\wedge(p\to q)$ (ii) $p\vee\{\sim p\to(q\vee(q\to\sim r))\}$

(iii) $p\to(p\to q)\wedge\{\sim(\sim q\vee\sim p)\}$

Solution. (i) $p\wedge(p\to q)\equiv p\wedge(\sim p\vee q)\equiv(p\wedge\sim p)\vee(p\wedge q)$

which is required DNF.

(ii) $p\vee\{\sim p\to(q\vee(q\to\sim r))\}$

$\equiv p\vee\{\sim p\to(q\vee(\sim q\vee\sim r))\}$ $\because$ we know $p\to q\equiv\sim p\vee q$

$\equiv p\vee\{\sim\sim p\vee(q\vee(\sim q\to\sim r))\} \equiv p\vee\{p\vee((q\vee\sim q)\vee\sim r)\}$

$\equiv p\vee p\vee q\vee\sim q\vee\sim r \equiv p\vee q\vee\sim q\vee\sim r$ which is the required DNF.

(iii) $p\to\{(p\to q)\wedge\sim(\sim q\vee\sim p)\}$

$\equiv\sim p\vee\{(p\to q)\wedge\sim(\sim q\vee\sim p)\}$ $\because$ we know $p\to q\equiv\sim p\vee q$

$\equiv\sim p\vee\{(p\to q)\wedge(\sim\sim q\wedge\sim\sim p)\}$ by D'Morgan's law.

$\equiv\sim p\vee\{(p\to q)\wedge(q\wedge p)\} \equiv\sim p\vee\{(\sim p\vee q)\wedge(q\wedge p)\}$

$\equiv\sim p\vee\{[\sim p\wedge(q\wedge p)]\vee[q\wedge(q\wedge p)]\}$ using distributive property.

$\equiv\sim p\vee\{[\sim p\wedge(p\wedge q)]\vee[(q\wedge q)\wedge p]\}$ using Commutative law.

$\equiv\sim p\vee\{[(\sim p\wedge p)\wedge q]\vee[(q\wedge p)]\}$

$\equiv\sim p\vee\{\ [\text{False}\wedge q]\vee(q\wedge p)\}$

$\equiv\sim p\vee\{\ \text{False}\vee(q\wedge p)\}$

$\equiv\sim p\vee(q\wedge p)$ which is the reuired DNF.

Ex.25. Obtain the DNF of the following statements

(i) $\sim\{\sim(p\leftrightarrow q)\wedge r\}$ (ii) $\{p\wedge\sim(q\wedge r)\}\vee(p\to q)$

Solution.(i) $\sim\{\sim(p\leftrightarrow q)\wedge r\}\equiv\sim\sim(p\leftrightarrow q)\vee\sim r$ by D'Morgan's law

$\equiv(p\leftrightarrow q)\vee\sim r\equiv\{(p\to q)\wedge(q\to p)\}\vee\sim r$

$\equiv\{(\sim p\vee q)\wedge(\sim q\vee p)\}\vee\sim r=\{(\sim p\vee q)\wedge\sim q\vee(\sim p\vee q)\wedge p\}\vee\sim r$

$=(\sim p\wedge\sim q)\vee(q\wedge\sim q)\vee(\sim p\wedge p)\vee(q\wedge p)\vee\sim r$ which is required DNF.

(ii) $\{p\wedge\sim(q\wedge r)\}\vee(p\to q)$

$\equiv\{p\wedge(\sim q\vee\sim r)\}\vee(\sim p\vee q)$ $[\because p\to q\equiv\sim p\vee q]$

$\equiv\{(p\wedge\sim q)\vee(p\wedge\sim r)\}\vee(\sim p\vee q)$

$\equiv(p\wedge\sim q)\vee\{(p\wedge\sim r)\vee(\sim p\vee q)\}$

$\equiv(p\wedge\sim q)\vee\{(p\wedge\sim r)\vee(\sim p)\vee q\}$ by Associative law

$\equiv(p\wedge\sim q)\vee(p\wedge\sim r)\vee(\sim p)\vee q$ which is the required DNF.

Ex. 26. Find whether the argument $p\to\sim q, r\to q, r\perp\sim p$ is valid or not. *[W.B.U.T.2014]*

Solution. This argument will be valid if the proposition

$[(p\to\sim q)\wedge(r\to q)\wedge r]\to\sim p$ is tautology

Now, $[(p\to\sim q)\wedge(r\to q)\wedge r]\to\sim p$

$\equiv[(\sim p\vee\sim q)\wedge(\sim r\vee q)\wedge r]\to\sim p$

$=[(\sim p\vee\sim q)\wedge\{(\sim r\wedge r)\vee(q\wedge r)\}]\to\sim p$

$=[(\sim p\vee\sim q)\wedge\{\ \text{False}\vee(q\wedge r)\}]\to\sim p$

$=[(\sim p\vee\sim q)\wedge(q\wedge r)\}]\to\sim p=\sim[(\sim p\vee\sim q)\wedge(q\wedge r)\}]\vee\sim p$

$=[(\sim p \vee \sim q) \wedge q \wedge r\}] \to \sim p$

$=[(\sim p \wedge q) \vee (\sim q \wedge q) \wedge r\}] \to \sim p = [(\sim p \wedge q) \vee \text{ False } \wedge r] \to \sim p$

$=[(\sim p \wedge q) \wedge r] \to \sim p = \sim[(\sim p \wedge q) \wedge r] \vee \sim p$

$=[(p \vee \sim q) \vee \sim r] \vee \sim p = \sim p \vee [(p \vee \sim q) \vee \sim r]$

$=[\sim p \vee (p \vee \sim q)\ ] \vee \sim r = [(\sim p \vee p) \vee \sim q] \vee \sim r$

$=[\text{ Tautology } \vee \sim q] \vee \sim r = \text{ Tautology } \vee \sim r = \text{ Tautology.}$

$\therefore$ The argument is valid.

Ex. 27. Represent the argument

If I work hard then I will get success

I work hard

I got success

Solution. Let p: I work hard

q: I will get success

Then the 1st proposition P is $p \to q$

the 2nd proposition is p

the 3rd proposition is q

$\therefore$ the argument is $P, p \vdash q$

This will be valid if $(P \wedge p) \to q$ is a tautology.

This is being shown by going through 'Algebraic Laws of connective':

$(P \wedge p) \to q \equiv \sim (P \wedge p) \vee q \quad [\because \text{ we know } p \to q \equiv \sim p \vee q]$

$\equiv \sim \{(p \to q) \wedge p\} \vee q \qquad \because P \text{ is } p \to q$

$\equiv \sim \{(\sim p \vee q) \wedge p\} \vee q$

$\equiv \{\sim (\sim p \vee q) \vee \sim p\} \vee q$ by D'Morgan's law

$\equiv \{(p \wedge \sim q) \vee \sim p\} \vee q$ by D'Morgan's law

$\equiv \{(p \vee \sim p) \wedge (\sim q \vee \sim p)\} \vee q$

$\equiv \{\text{ (Tautology) } \wedge (\sim q \vee \sim p)\} \vee q$

$\equiv (\sim q \vee \sim p) \vee q \equiv (\sim p \vee \sim q) \vee q \equiv \sim p \vee (\sim q \vee q)$

$\equiv \sim p \vee \text{ Tautology} = \text{Tautology}$

$\therefore$ the argument is valid.

Ex. 28. Express the following argument symbolically and test whether this is a valid argument :

If Sukrit solved this problem, then he scores 90

Sukrit scores 90

Sukrit solved this problem

Solution. Let p:Sukrit solved this problem.

q: He (Sukrit) scores 90

The first statement is $P \equiv p \to q$

Then the argument can be written symbolically as $P, q \vdash p$

This will be valid $(P \wedge q) \to p$ is Tautology.

Truth Table of $(P \wedge q) \to p$

p	q	$P(p \to q)$	$P \wedge q$	$(P \wedge q) \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

We see all the truth values of $(P \wedge q) \to p$ are not T. So this is not a valid argument.

Ex. 29. Represent the following argument symbolically :

If this number is divisible by 6, then it is divisible by 3.

This number is not divisible by 3.

This number is not divisible by 6

Test whether the argument is valid.

Solution. Let p: the number is divisible by 6

q: It is divisible by 3

The first premises - statement is $p \to q$

The second premise –statement is $\sim q$

The conclusion–statement is $\sim p$

$\therefore$ the argument is $p \to q, \sim q \vdash \sim p$

This will be valid if $(p \to q) \wedge (\sim q) \to \sim p$ is a tautology.

Now, $(p \to q) \wedge (\sim q) \to \sim p$

$\equiv (\sim p \vee q) \wedge (\sim q) \to \sim p \equiv (\sim p \wedge \sim q) \vee (q \wedge \sim q) \to \sim p$

$\equiv (\sim p \wedge \sim q) \vee$ False $\to \sim p \equiv \sim p \wedge \sim q \to \sim p$

$\equiv \sim (\sim p \wedge \sim q) \vee (\sim p) \equiv (p \vee q) \vee (\sim p) \equiv (q \vee p) \vee (\sim p)$

$\equiv q \vee (p \vee \sim p) \equiv q \vee$ Tautology $=$ Tautology

$\therefore$ The argument is valid.

Ex. 30. Verify the validity of the argument " If I drive to work then I will arrive in time. I do not drive to work. Therefore, I will not arrive in time".

Solution. Let the simple propositions be

p: I drive to work

q: I will arrive in time

The first Premises is $p \to q$

The second premise is $\sim p$

$\therefore$ the argument is $(p \to q), \sim p \vdash \sim q$

It will be valid if $(p \to q) \wedge (\sim p) \to \sim q$ is tautology.

Now, $(p \to q) \wedge (\sim p) \to \sim q \equiv (\sim p \vee q) \wedge (\sim p) \to \sim q$

$\equiv (\sim p \wedge \sim p) \vee (q \wedge \sim p) \to \sim q \equiv (\sim p) \vee (q \vee \sim p) \to \sim q$

$\equiv (\sim p) \vee (\sim p \vee q) \to \sim q \equiv (\sim p \vee \sim p) \vee q \to \sim q$

$\equiv (\sim p \vee q) \to \sim q \equiv \sim (\sim p \vee q) \vee (\sim q) \equiv (p \wedge \sim q) \vee (\sim q)$

Truth Table of $(p \to q) \wedge (\sim p) \to \sim q$

p	q	$p \to q$	$\sim p$	$\sim q$	$(p \to q) \wedge (\sim p)$	$(p \to q) \wedge (\sim p) \to \sim q$
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	T	F	T	F
F	F	T	T	T	T	T

$\therefore$ This is not valid argument.

Ex. 31. Find the truth value of $\forall x, P(x)$ where $P(x)$ is the statement "$x^2 < 20$" and the domain is the set $\{1,2,3,4\}$.

Solution. We see $1^2 = 1 < 20$, $2^2 = 4 < 20$, $3^2 = 9 < 20$ and $4^2 = 16 < 20$.

So $x^2 < 20$ for all values of x in the set $\{1,2,3,4\}$.

$\therefore$ The universal quantifier $\forall x\, P(x)$ is True.

Ex. 32. Find the truth value of $\forall x\, P(x)$ where $P(x)$ is the statement "$3x + 1 < 10$" and the domain is the set $\{0,1,2,3\}$.

Solution. We see for $x = 3$, $3 \times 3 + 1 = 10 \not< 10$

$\therefore$ we cannot say $3x + 1 < 10$ for all values of x in $\{0,1,2,3\}$.

$\therefore$ The proposition $\forall x\, P(x)$ is false.

Ex. 33. Find the truth value of the existential quantification of $P(x)$ where $P(x)$ is the statement '$x^2 > 20$' and the universe of discourse is $\{1,2,3,4,5\}$.

Solution. We see $5^2 = 25 > 20$.

$\therefore$ $P(5)$ is true.

$\therefore$ there exist a value of x in the domain for which $x^2 > 20$ hold.

$\therefore$ The quantification $\exists x\, P(x)$ is true.

Ex. 34. Find the truth value of the universal quantifier of the propositional function $P(x,y)$ stating "$x^2 + y^2 < 12$" and the domain is $\{1,2,3\}$.

Solution. The universal quantifier of $P(x,y)$ is "$x^2 + y^2 < 12$ for all x, y in $\{1,2,3\}$". [There are two variables in $P(x,y)$]

We see $1^2 + 1^2 = 1 < 12$; $1^2 + 2^2 = 5 < 12$; $1^2 + 3^2 = 10 < 12$; $2^2 + 2^2 = 8 < 12$ but $2^2 + 3^2 = 13 \not< 12$

$\therefore$ $x^2 + y^2 < 12$ is not valid for $x = 2$, $y = 3$.

So the universal quantifier is False.

Ex. 35. Let $P(x,y)$ states that "$x^2 > y+1$" where the domain is the set $S = \{1, 2, 3\}$. Find the truth value of the quantifier $\exists x, \forall y\ P(x,y)$.

Solution. For y = 1, 2, 3, $y + 1$ = 2, 3, 4 respectively.

Now, for $x = 3$, $x^2 = 9 > 2, 3$ and 4.

$\therefore$ for $x = 3$, $x^2 > y + 1 \forall y$ $\therefore \exists x$, such that $\forall y\ P(x,y)$ is True.

Ex. 36. Find the truth value of $\forall x, \exists y\ P(x,y)$

where $P(x,y)$ states that "$x^2 + y^2 < 12$" and the domain is $\{1,2,3\}$.

Solution. For $x = 1$, $1^2 + y^2 < 12$ or, $y^2 < 11$

For $x = 2$, $2^2 + y^2 < 12$ or, $y^2 < 8$

For $x = 3$, $3^2 + y^2 < 12$ or, $y^2 < 3$

We see for $y = 1$, $1^2 < 11$, $1^2 < 8$ and $1^2 < 3$.

$\therefore$ there exists a value of y for which $x^2 + y^2 < 12$ hold for all values of x.

$\therefore$ $\forall x, \exists P(x,y)$ is True.

Ex. 37. Determine the truth value of the quantifier $\exists x,\ x^2 - 2x + 5 = 0$; set of all real numbers being the domain.

Solution. $x^2 - 2x + 5 = 0$

or, $$x = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$

$= 1 \pm 2i$ which are not real.

So no x in the set of reals satisfies the equation $x^2 - 2x + 5 = 0$.

$\therefore$ $\exists x,\ x^2 - 2x + 5 = 0$ is False.

Exercises

I. Short and Long Answer Questions

1. Consider the following :

   p: Pranab is rich

   q: Nanda is poor

   Write each of the following statements in symbolic forms :

   (i) Pranab and Nanda are both rich

   (ii) Pranab is poor and Nanda is rich

   (iii) Pranab is not rich and Nanda is poor

   (iv) Neither Pranab nor Nanda is poor

   (v) It is not true that Pranab and Nanda are both rich

   (vi) Either Pranab is poor or Nanda is poor.

   (vii) Either Pranab or Nanda is poor.

2. Express the following symbolically:

   Lions are dangerous animals. This forest has lions. Therefore this forest has dangerous animals.

3. Let p be a proposition 'He is intelligent' and q be a proposition 'He is tall'. Write the following proposition in symbolic form :

   (i) He is intelligent and tall

   (ii) He is tall or he is not tall and intelligent

4. Consider the following

   p: This car is good

   q: This car is cheap

   Express the following statements symbolically :

   (i) This car is good and cheap

   (ii) This car is not good but cheap

   (iii) This car is costly but good

   (iv) This car is neither good nor cheap

   (v) This car is good or cheap

5. Given the following propositions

   p: It is hot day and q: The temperature is 40° c

   Write the statement against the following symbolic expression:

   (i) $p \vee q$ (ii) $\sim(p \vee q)$ (iii) $\sim p \wedge \sim q$ (iv) $\sim(\sim p \vee \sim q)$

6. p: Today is Monday

q: It is raining

r: It is cold

write the statement for the following symbolic expression

(i) $p \to q$ (ii) $\sim p \to (q \vee r)$

7. If p: I take a diploma in MBA

q: I get a job

r: I get in USA

write the statement against the following symbolic propositions:

(i) $\sim p \to \sim q$ (ii) $(p \wedge \sim q) \to \sim r$

8. If p: x is even

q: x is divisible by 2

write in sentence form for the following symbols

(i) $\sim p \wedge \sim q$ (ii) $p \leftrightarrow q$

9. Write down the negation of the statement :

(i) If she studies, she will pass

(ii) All Africans are bad

(iii) There is no dog, that can talk

(iv) Some one has visited every part of India except Kolkata

10. Find the negation of the following statement :

(i) Today is Friday

(ii) No one wants to buy my car

(iii) Every even integer greater than 2 is not prime.

11. If p and q are true and r and s are false, find the truth value of

(i) $p \vee (q \vee r)$

(ii) $\{\sim (p \wedge q) \vee \sim r\} \vee \{([\sim p \vee q] \vee \sim r) \wedge s\}$

12. If p is true and q is false, find the truth value of

(i) $(p \wedge q) \to (p \vee q)$ (ii) $\sim (p \wedge q) \vee \sim (q \leftrightarrow p)$

13. If p and q are true and r is false then find the truth value of

(i) $(p \vee q) \leftrightarrow (p \to \sim r)$ (ii) $(p \leftrightarrow r) \to r$

14. Find the Truth table of the following :

(i) $p \vee \sim q$ (ii) $(p \vee \sim q) \wedge p$

(iii) $\sim (p \vee q) \vee (\sim p \wedge \sim q)$ (iv) $(q \vee r) \wedge p$

(v) $\sim p \vee q \to \sim q$ (vi) $(\sim q \to \sim p) \to (p \to q)$

(vii) $q \vee r \leftrightarrow p \wedge \sim r$

15. Write down the converse, inverse and contrapositive of the following statements :

(i) If today is independence-day, then tomorrow is Monday.

(ii) If ABC is a right triangle then $|AB|^2 + |BC|^2 = |AC|^2$.

(iii) If P is a rectangle then it is a parallelogram.

(iv) If n is prime, then n is 2 or n is odd.

(v) If a triangle is not isoceles, then it is not equilateral.

16. Write the converse, contrapositive and inverse of the followings :

(i) If it is rainning, the grass is wet

(ii) It is rainning if it is cloudy

(iii) Rain is sufficient for it to be cloudy

(iv) Rain is necessary for it to be cloudy.

17. Show by a truth table the inverse of $p \to q$ is equivalent to converse of $p \to q$.

18. Prove that the following propositions are Tautology :

(i) $(p \wedge q) \to p$ (ii) $\sim p \to (p \to q)$

(iii) $[p \wedge (p \to q)] \to q$ (iv) $(p \wedge q) \to (p \to q)$

(v) $[(p \to q) \wedge (q \to r)] \to (p \to r)$ [W.B.U.T.2014]

(vi) $(p \to q) \leftrightarrow (\sim q \to \sim p)$ (vii) $p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$

(viii) $\{p \wedge (p \leftrightarrow q)\} \to q$ (ix) $(p \wedge q) \to (p \vee q)$

19. Show that

(i) $p \wedge (\sim q \vee q) \equiv p$ (ii) $p \vee (p \wedge q) \equiv p$

(iii) $(p \wedge q) \vee (p \wedge \sim q) \equiv p$ (iv) $\sim \{p \vee (\sim p \wedge q)\} \equiv \sim p \wedge q$

(v) $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p) \equiv p \wedge \{(q \to r) \wedge (r \to q)\} \vee \{(r \to p) \wedge (p \to r)\}$

(vi) $p \wedge \{(\sim q \vee r) \wedge (\sim r \vee q)\} \equiv p \wedge (q \leftrightarrow r)$

(vii) $\sim (p \wedge q) \wedge (p \vee q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$

(viii) $(p \to q) \vee \sim (p \vee q) \equiv (p \to q) \wedge (p \leftrightarrow q)$

(ix) $\sim \{p \vee (\sim p \wedge q)\} \equiv \sim p \wedge \sim q$

20. Show that the following propositions are contradiction

(i) $p \wedge \sim q$ (ii) $(p \vee q) \wedge (\sim q) \wedge (\sim p)$

(iii) $(p \wedge q) \wedge \{\sim (p \vee q)\}$

21. Find the DNF of the following propositions

(i) $\sim \{p \to (q \wedge r)\}$ (ii) $(\sim p \to r) \wedge (p \leftrightarrow q)$

22. Obatin the CNF of the following propositions :

(i) $\sim \{(p \vee \sim q) \wedge \sim r\}$ (ii) $\sim (p \vee q) \leftrightarrow (p \wedge q)$

23. Show that the following arguments are valid :

(i) $p \to (q \vee r), q \to s, r \to t \vdash p \to (s \vee t)$

(ii) $q \vee r, p \to \sim q, \sim s \to p, \sim r \vdash s$

(iii) $p \to q, r \to \sim q \vdash p \to \sim r$

(iv) $\sim q \to r, q, p \vee \sim q \vdash -r$

24. Show that the argument

(i) $p \to q, \sim p \vee r, \sim r \vee \sim s, s \vdash q$ is not valid

(ii) $p \to q, \sim p \vdash \sim q$ is not valid.

25. Verify the validity of the following argument :

(i) If today is Tuesday, then yesterday was Monday. Yesterday was Monday. Today is Tuesday.

(ii) If I study then I will pass in Exam
If I do not go to cinema then I will study
I failed in examination
———
I went to cinema

(iii) If a man is married then he is unhappy.
If a man is unhappy then he dies young.
———
Married man dies young

(iv) Every living thing is a plant or an animal
My cat is an animal and it is not a plant
All animals have lung
———
My cat has lung

(v) If I like Mathematics, then I will study
Either I study or I get less marks
———
If I get less marks, I do not like Mathematics.

[Hint: 2nd premises is $(q \vee r) \wedge \sim (q \wedge r)$]

(vi) If wages increase then there will be inflation
———
The cost of living will not increase if there is no inflation
If wages will increase then the cost of living will increase

Ex. 26. Translate the following sentences into symbolic forms

(i) There is a student in a school who can speak English and who knows Hindi.

(ii) No student in the school can speak English or knows Hindi.

(iii) There is a student in the school who can speak English but who does not know Hindi.

Ex. 27. Let $P(x)$ be the statement 'x can speak Englsih', $Q(x)$ be the statement 'x knows Hindi'. Translate the following symbolic statements into English language.

(i) $\forall x\, (P(x) \vee Q(x))$ (ii) $\exists x \sim (P(x) \wedge Q(x))$

Ex. 28. Translate the following language into symbolic form using quantifier, variables and predicate

(i) All healhty people drink milk a day

(ii) Nanda does not drink milk a day

(iii) Nanda is not a healthy person.

Ex. 29. The domain is set of all 4-wheelers. Let P(x) be 'x is a 4-wheeler', Q(x) be 'x is a car' and R(x) be 'x is manufactured by Tata'. Express the following statements using quantifiers.

(a) Every car is a 4-wheeler manufactured by Tata.

(b) There are cars which are not manufactured by Tata.

(c) Every 4-wheeler is a car.

Ex. 30. The universe of discourse is set of all animals. Let P(x) be 'x is a whole'; Q(x) be 'x is a fish' and R(x) be 'x lives in water'.

Translate the following in your own words :

(i) $\exists x(\sim R(x))$ (ii) $\exists x(Q(x) \wedge \sim P(x))$

(iii) $\forall x(P(x) \wedge R(x)) \rightarrow Q(x)$

Ex. 31. If the domain is set of all integers, let A(x) be '$x + y$'.

Ex. 32. Let A = {1, 2, 3, 4, 5} be the domain. Determine the truth value of each of the following statements :

(i) $\exists x(x+3=10)$ (ii) $\forall x(x+3<10)$ (iii) $\exists x(x+3<5)$

(iv) $\forall x(x+3 \leq 7)$ (v) $\exists x(x^2-4=0)$

Ex. 33. Determine the truth value of each of the following where the set {1, 2, 3} is the universe of discourse :

(i) $\exists x, \forall y, x^2 < y+1$ (ii) $\forall x \exists y, x^2+y^2<12$

Ex. 34. Determine the truth value of each of the following statements where A = {1, 2, 3,....9, 10} is the universe of discourse:

(i) $\forall x \exists y (x+y<14)$ (ii) $\forall x \forall y (x+y<14)$

Ex. 35. Find the truth value of the following quantifier; set of all numbers being the domain

(i) $\forall x P(x)$ where $P(x)$ is "$x^2 = x$".

(ii) $\exists x P(x)$ where $P(x)$ is "$2x = x$".

(iii) $\forall x P(x)$ where $P(x)$ is "$x - 4 < x$"

(iv) $\exists x, P(x)$ where $P(x)$ is "$x^2 - 4 = 0$".

Ex. 36. Negate each of the following

(i) $\sim \forall x \forall y P(x, y)$ (ii) $\sim(\exists x \exists y \sim R(x,y) \wedge \forall x \forall y P(x,y))$.

Ex. 37. Negate each of the following statements :

(i) If the teacher is absent, then some student do not complete their home work.

(ii) All the students complete their homework and the teacher is absent.

(iii) Some of the students did not complete their homework or the teacher is absent.

Ex. 38. Write negatians of each of the following statements :

(i) x is real number, if $x > 4$ then $x^2 > 16$

(ii) $x \in R$ if $x(x+1) > 0$ then $x > 0$ or $x < -1$

Answers

1. (i) $p \wedge \sim q$ (ii) $\sim p \wedge \sim q$
   (iii) $\sim p \wedge q$ (iv) $p \wedge \sim q$
   (v) $\sim(p \wedge \sim q)$ (vi) $(\sim p) \vee q$
   (vii) $(\sim p \wedge \sim q) \vee (p \wedge q)$
3. (i) $p \wedge q$ (ii) $q \vee (\sim q \wedge p)$
4. (i) $p \wedge q$ (ii) $\sim p \wedge q$
   (iii) $\sim q \wedge p$ (iv) $\sim p \wedge \sim q$
   (v) $p \vee q$
5. (i) It is a hot day or the temperature is 40°c
   (ii) This is not true that it is a hot day or temperature is 40°c
   (iii) This is neither hot day nor the temperature is 40°c
   (iv) It is false that it is not a hot day or temperature is not 40°c
6. (i) If today is Monday then it is raining
   (ii) If today is not Monay, then it is raining or it is cold
7. (i) If I do not take diploma in MBA then I do not get in USA.
   (ii) If I take diploma in MBA and do not get a job then I do not get in USA
8. (i) x is not even and x is not divisible by 2
   (ii) x is even if and only if x is divisible by 2

9. (i) She studies and she will not pass
(ii) so African is good
(iii) there is a dog that talks
(iv) Every person has either visited Kolkata or has not visited any part of India other than Kolkata.

10. (i) Today is not Friday
(ii) someone wants to buy my car
(iii) some even integer greater then 2 is not prime

11. (i) True (ii) True
12. (i) True (ii) True
13. (i) True (ii) True

14. (i)

p	q	$p \vee \sim q$
T	T	T
T	F	T
F	T	F
F	F	T

(ii)

p	q	$(p \vee \sim q) \wedge p$
T	T	T
T	F	T
F	T	F
F	F	F

(iii)

p	q	$\sim(p \vee q) \vee (\sim p \wedge \sim q)$
T	T	F
T	F	F
F	T	F
F	F	T

(iv)

p	q	r	$(q \vee r) \wedge p$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

(v)

p	q	$\sim p \vee q \to \sim q$
T	T	T
T	F	T
F	T	F
F	F	T

(vi)

p	q	$(\sim q \to \sim p) \to (p \to q)$
T	T	T
T	F	T
F	T	T
F	F	T

(vii)

p	q	r	$q \vee r \leftrightarrow p \wedge \sim r$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

15. (i) converse : If tomorrow is Monday then today is independence day.

inverse : If today is not independence day, then tomorrow is not Monday.

Contrapositive : If tomorrow is not Monday, then today is not independence day.

(ii) Converse : If $|AB|^2+|BC|^2=|AC|^2$ then the triangle ABC is a right angle triangle.

Inverse : If triangle ABC is not a right triangle then $|AB|^2+|BC|^2 \neq |AC|^2$.

Contrapositive : If $|AB|^2+|BC|^2 \neq |AC|^2$ then the triangle ABC is not a right angle.

(iii) Converse : If P is a parallelogram, then P is a rectangle.
Inverse : If P is not a rectangle, then P is not a parallelogram.
Contrapositive : If P is not a parallelogram, then P is not a rectangle.

(iv) Converse : If n is odd or n is 2, then n is prime.

Inverse : If n is not prime, then n is not odd and n is not 2.

Contrapositive : If n is not odd and n is not 2, then n is not prime.

(v) Converse : If a triangle is not equilateral, then it is not isosceles.

Inverse : If a triangle is isoceles, then it is equilateral.

Contrapositive : If a triangle is equitaleral, then it is isosceles

21. (i) $(p\wedge \sim q)\vee(p\wedge \sim r)$ (ii) $(p\wedge q)\vee(p'\wedge q'\wedge r')$

22. (i) $(\sim p\vee r)\wedge(q\wedge r)$ (ii) $(\sim p\vee \sim q)\wedge(p\vee q)$

25. (i) not valid (ii)valid (iii) valid
(iv) valid (v) not valid

26. (i) $\exists x(P(x)\wedge Q(x))$ (ii) $\forall x\sim(P(x)\vee Q(x))$
(iii) $\exists x(P(x)\wedge \sim Q(x))$

27. (i) Every student of the school either can speak English or know Hindi.
(ii) There is a student who cannot speak English and also do not know Hindi.

28. (i) $\forall x, P(x)\rightarrow Q(x)$
(ii) ~Q(Nanda)
(iii) ~P(Nanda) where P(x) is 'x is a healthy person' and Q(x) is 'x drinks milk a day'

29. (a) $\forall x(P(x)\wedge \sim Q(x))\rightarrow R(x)$
(b) $\exists x(P(x)\wedge R(x))$ (c) $\forall x(P(x)\rightarrow Q(x))$

30. (i) there exists an animal which does not live in water (ii) there exists a fish that is not a whale (iii) Every whale that lives in water is a fish.

32. (i) False (ii) True (iii) True (iv) False (v) True.

33. (i) True (ii) True

34. (i) True (ii) False

35. (i) False (ii) True (iii) True (iv) True

36. (i) $\exists x\, \exists y, \sim P(x,y)$ (ii) $(\forall x \forall y\, R(x,y)\vee(\exists x\, \exists y, \sim P(x,y))$

37. (i) The teacher is absent and all the students completed their homework.
(ii) Some of the students did not complete their homework or the teacher is absent.
(iii) All the students complete their homework and the teacher is absent.

38. (i) $\exists$ a real number x such that $x > 3$ and $x^2 \leq 9$.
(ii) $\exists$ a real number x such that $x(x+1)>0$ then $x\leq 0$ and $x\geq -1$

II. Multiple Choice Questions

1. If p: 'Anil is rich' and q: 'Kanchan is poor' then the symbolic form the statement 'Either Anil or Kanchan is rich' is

[W.B.U.T 2006,2015]

(a) $p\vee q$ (b) $p\vee \sim q$
(c) $\sim p\vee q$ (d) $\sim(p\wedge q)$

2. Let p: *It is cold* and q: *It is raining*, then the symbolic form of the statement '*It is cold or it is not raining*' is

(a) $p\vee q$ (b) $\sim p\wedge q$
(c) $\sim p\vee q$ (d) $p\vee \sim q$

3. If p: *It is cold* and q: *It is raining* then the statement. '*It is not raining and it is not cold*' has the symbolic form

(a) $\sim q\wedge p$) $\sim q\wedge \sim p$
(c) $\sim(q\wedge p)$ (d) these