Module-5

Graph and its Properties

5.1

5.1.1. Introduction to Graph

In a locality let there be eight poles forming a network of telephone. Let $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ and P_8 be the poles. The poles are connected by cables in following way :

P_1 is connected with P_4	P_2 is connected with P_3
P_1 is connected with P_6	P_2 is connected with P_5
P_1 is connected with P_3	P_2 is connected with P_8 with double line
P_3 is connected with P_7	P_4 is connected with P_8
P_4 is connected with P_6	

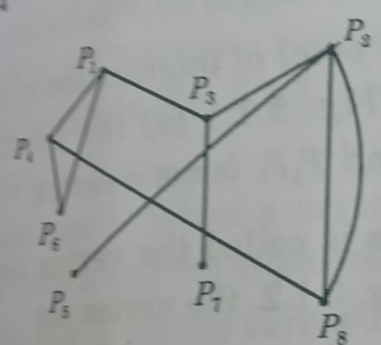

Fig. 5.1.1 (a)

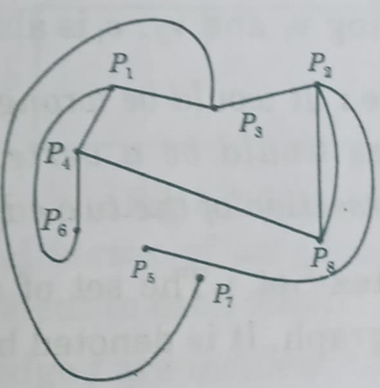

Fig. 5.1.1 (b)

We may represent this network by either of the two diagrams shown in Fig. 5.1.1 (a) or Fig. 5.1.1 (b). The dots (P_1, P_2 etc.) represent the pole and the line joining two dots represent a cable connecting two poles. These lines may not be straight line. In Fig. 5.1.1 (a) all are straight lines whereas in Fig. 5.1.1 (b) some are curve. This diagram of Fig. 5.1.1(a) or Fig. 5.1.1(b) is known as *Graph*. The dots P_1, P_2 etc. are known as vertices and the line joining any two dots are known as edge.

Thus we come to the formal definition of Graph and other related terms.

5.1.2 Graph and Related Terms

Graph : A diagram consisting of a finite number of dots or points together with lines (not necessarily straight line) joining certain pairs of these dots is called Graph. Fig. 5.1.2 shows a graph.

Vertex : The dots used in a Graph are called vertices. These are denoted by v_1, v_2, etc. or any other letters.

Note : *In fact the vertices are representation of some others. e.g. in the model stated in Art 5.1.1, the vertices are representation of pole.*

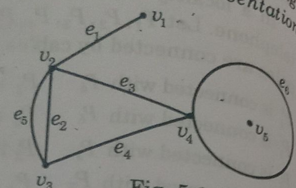

Fig. 5.1.2

Edge : The lines (not necessarily straight line) joining certain pair of the dots in a Graph are called edges. These are denoted by $e_1, e_2, \cdots$ or $a, b, c, \cdots$ etc.

In Fig. 5.1.2, e_1 is an edge joining v_1 and v_2. e_1 is also represented by (v_1, v_2) etc..

Note : *It would be wrong to say the point of intersection of two edges would be a vertex; e.g. in Fig. 5.1.1 (a) the point of intersection of the two edges P_2P_5 and P_3P_7 is not a vertex.*

Vertex-Set : The set of all vertices is called the vertex set of the graph. It is denoted by V. In Fig. 5.1.2, the vertex set

$$V = \{v_1, v_2, v_3, v_4, v_5\}.$$

Edge-Set : The set of all edges is called the Edge-set of the graph. It is denoted by E. In Fig. 5.1.2, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

Note : (1) For convenience we write a graph as $G(V, E)$

(2) In a graph it is immaterial whether the lines are drawn straight or curved, long or short.

In fact, from mathematical point of view a graph is defined in the following way :

Formal Definition of Graph :

A graph $G = (V,E)$ consists of two finite sets $V(\neq \phi)$ and E such that each element (called edge) of E is assigned an unordered pair of elements (called vertices) of V. V is called vertex set and E is called edge set.

End vertices of an Edge : Every edge would be associated with two vertices. The vertices associated with an edge are called the end vertices of the edge. In Fig. 5.1.2 v_2 and v_4 are end vertices of the edge e_3.

Self-loop or Loop : An edge whose two end vertices are same is called a self loop or Loop. In Fig. 5.1.2 the edge e_6 is a loop.

Parallel Edges : There may exist more than one edge associated with a given pair of vertices. Such edges are called parallel edges or parallels. In Fig 5.1.2 the edges e_2 and e_5 are parallels.

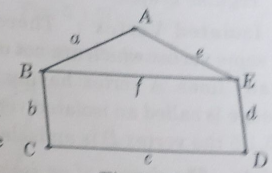

Fig 5.1.3

Simple Graph : A graph having no parallel edges and no self loops is called simple graph. The graph in Fig. 5.1.2 is not a simple graph. The graph in Fig. 5.1.3 is a simple graph.

Incidence : When a vertex is an end vertex of an edge then the vertex and the edge are called incident to each other.

In Fig. 5.1.3 the vertex B and the edge f are incident to each other. We say also f is incident to E.

Note that in Fig 5.1.2 no edge is incident to the vertex v_5.

Adjacent Vertices : Two vertices are said to be adjacent if they are end vertices of the same edge. In Fig 5.1.3 the vertices C and D are adjacent; A and C are not adjacent. In Fig. 5.1.2 the vertex v_5 is adjacent to no vertex.

Adjacent Edges : Two non parallel edges are said to be adjacent if they are incident to a common vertex. In Fig. 5.1.3 the edges c and d are adjacent since they are incident to the common vertex D. Here c and f are not adjacent.

Degree of Vertex : The number of edges incident to a vertex, with self-loops counted twice, is called degree of the vertex. It is denoted by $\deg(v_i)$. In Fig. 5.1.2 $\deg(v_3) = 3$ and $\deg(v_4) = 4$ and $\deg(v_2) = 4$; $\deg(v_5) = 0$.

Even and Odd Vertex : A vertex is called odd or even depending on whether its degree is odd or even. In Fig. 5.1.3 the vertex B is odd vertex whereas the vertex C is even.

Regular Graph : A graph in which all vertices are of equal degree is called a regular graph. The graph in Fig 5.1.4 is regular of degree 3. Some time we call it 3-regular graph.

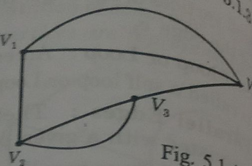

Fig. 5.1.4

Isolated Vertex : There may exist some vertex which are not connected by any lines. A vertex having no incident edge is called an isolated vertex. In Fig. 5.1.5 the vertex D is an isolated vertex.

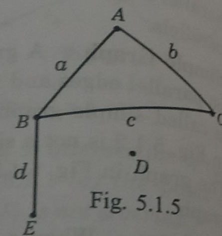

Fig. 5.1.5

Note : *The degree of an isolated vertex is 0.*

Pendant Vertex and Pendant Edge (or, End Vertex) : A vertex whose degree is 1 is called pendant vertex. An edge incident to a pendant vertex is called a pendant edge. In Fig. 5.1.5 the vertex E is a pendant vertex ; d is pendant edge.

Edges in Series : Two adjacent edges are said to be in seires if their common vertex is of degree 2. In Fig. 5.1.5 the edges a and b are in series whereas the edges a and d are not in series.

Null Graph : A graph may have no edge. A graph having no edges is called a Null Graph. Obviously every vertex in a null graph is an isolated vertex. In Fig. 5.1.6(a) we show such a null graph

Note : *(1) The Edge set of a null graph is a null set.*

(2) The vertex set of a graph cannot be empty set.

Complete Graph : A simple graph is called a complete graph if each pair of distinct vertices is joined by an edge.

A complete graph with n vertices is denoted by K_n.

The graph shown in Fig. 5.1.6 (b) is not a complete graph because the two vertices C and D are not joined by any edge. The graph shown in Fig. 5.1.4 is not a complete graph also.

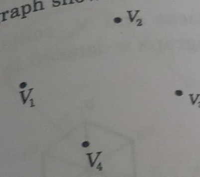

Fig. 5.1.6(a)

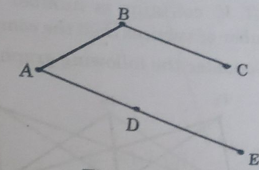

Fig. 5.1.6(b)

The graph given in Fig. 5.1.6(c) is a complete graph.

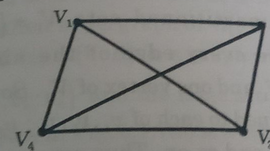

Fig. 5.1.6(c)

Note : *A complete graph is always a regular graph.*

Bipartite Graph: A simple Graph G is called a bipartite graph if the vertex set V of G can be partitioned into two non empty subsets V_1 and V_2 such that each edge of G is incident to one vertex in V_1 and one vertex in V_2.

The simple graph shown in Fig 3.1.6(b) is a bipartite graph because the vertex set $V=\{A,B,C,D,E\}$ can be partioned into two subsets $V_1=\{A,C,E\}$ and $V_2=\{B,D\}$. We see the edge BC is incident to $B\in V_2$ and to $C\in V_1$. The edge AB is incident to $B\in V_2$ and to $A\in V_1$. The edge AD is incident to $D\in V_2$ and to $A\in V_1$. The edge DE is incident to $D\in V_2$ and to $E\in V_1$.

Note that the the graph shown in Fig 3.1.6(c) is also a bipartite graph as $\{V_1,V_2,V_3,V_4\}=\{V_1,V_4\}\cup\{V_2,V_3\}$.

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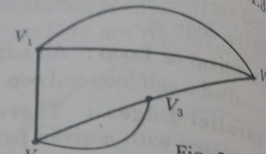

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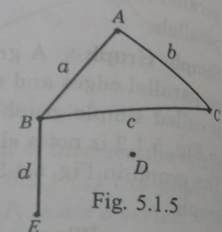

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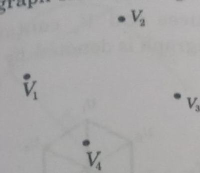


Fig. 5.1.6(a)

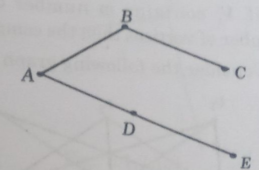
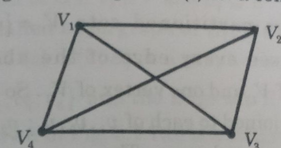


Fig. 5.1.6(b)

The graph given in Fig. 5.1.6(c) is a complete graph.



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Note that the the graph shown in Fig 3.1.6(c) is also a bipartite graph as $\{V_1,V_2,V_3,V_4\}=\{V_1,V_4\}\cup\{V_2,V_3\}$.

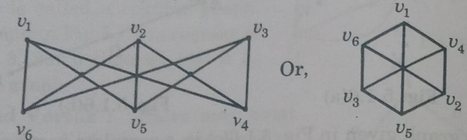
The simple graph shown in Fig 3.1.3 is not a bipartite graph.

Complete Bipartite Graph : A bipartite graph G with the two partioned vertex set V_1 and V_2 is called complete bipartite graph if every vertex of V_1 is joined to every vertex of V_2.

If V_1 contains m number of vertices and V_2 contains n number of vertices then the complete bigraph is denoted by $K_{m,n}$

Consider the following graph G

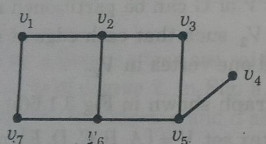
G :



Consider the two partitioned sets $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$. We see every edge of the above graph G connects one vertex of V_1 and one vertex of V_2. So G is bipartite. Moreover we see v_1 is joined to each of v_4, v_5, v_6; v_2 is also joined to each of v_4, v_5 and v_6 and so on. Thus every vertex of V_1 is joined to every vertex of V_2. So G is a complete Bigraph which can be denoted as $K_{3,3}$.

Consider the following graph G'

G′



Here consider the two vertex set $V_1 = \{v_1, v_6, v_3, v_4\}$ and $V_2 = \{v_7, v_2, v_5\}$. We see every edge of G' connects one vertex of V_1 with one vertex of V_2. So G' is bipartite graph. Observe that the vertex v_1 of V_1 is not joined with v_5 of V_2. So we cannot say this graph G' is a complete Bigraph.

Note : (1) *No two of the vertices in a partioned vertex set of a bipartite graph are adjacent.*

(2) *A $K_{m,n}$ complete Bi-graph has mn edges.*

Subgraph and Supergraph : Let $G(V, E)$ be a graph with vertex set V and edge set E.

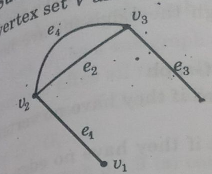


Fig. 5.1.7 (a)

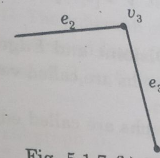


Fig. 5.1.7 (b)

A graph $G_1(V_1, E_1)$ with vertex set V_1 and edge set E_1 is called subgraph of G if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Here G is called supergraph of G_1. We denote $G_1 \subset G$.

In Fig. 5.1.7 (a) $G(V, E)$ is a graph where the vertex-set $V = \{v_1, v_2, v_3, v_4\}$ and the Edge set $E = \{e_1, e_2, e_3, e_4\}$.

The graph shown in Fig. 5.1.7(b) is a subgraph of $G(V, E)$ because its vertex set $\{v_2, v_3, v_4\} \subseteq V$ and its edge-set $\{e_1, e_3\} \subseteq E_G$.

Note : (1) *Every graph is its own subgraph*

(2) *A single vertex of a graph G may be considered as a sub graph of G*

(3) *Any single edge of a graph G together with its end vertices is a sub-graph of G.*

Spanning Subgraph : A subgraph G_1 of a graph G is called a spanning subgraph of G if G and G_1 have same vertex set.

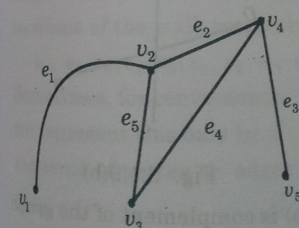


Fig. 5.1.8 (a)

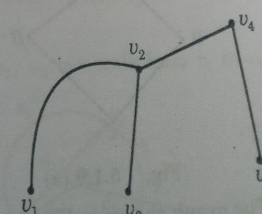


Fig. 5.1.8 (b)

The graph in Fig. 5.1.8 (b) is a spanning subgraph of the graph shown in Fig 5.1.8(a) because both the graph have same vertex-set $V = \{v_1, v_2, v_3, v_4, v_5\}$ though the Edge-set are not same.

Vertex Disjoint and Edge-disjoint Graph

Two graphs are called vertex disjoint if they have no vertex in common.

Two graphs are called edge-disjoint if they have no edge in common.

In Fig 5.1.8 (a) we show a graph $G(V, E)$ where the vertex set $v = \{v_1, v_2, v_3, v_4, v_5\}$ and the edge set $E = \{e_1, e_2, e_3, e_4, e_5\}$. Consider the two subgraphs of G, viz.

$G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ where $V_1 = \{v_1, v_2, v_3\}$;

$E_1 = \{e_1, e_5\}$ and $V_2 = \{v_2, v_3, v_4, v_5\}$; $E_2 = \{e_2, e_5, e_4, e_3\}$.

Then we see the two subgraphs G_1 and G_2 are not disjoint.

Note : (1) *Two edge disjoint graphs may not be vertex disjoint because they may have common vertices.*

(2) *Two vertex disjoint graphs are edge disjoint.*

Complement of a Graph : Let G be a simple graph. Another graph say $\overline{G}$ is said to be complement of G if $\overline{G}$ has same vertex set as G and two vertices in $\overline{G}$ would be adjacent if they are not adjacent in G.

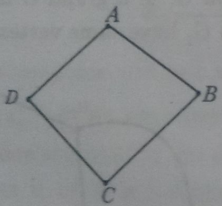


Fig. 5.1.9 (a)

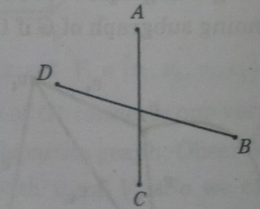


Fig. 5.1.9(b)

The graph shown in Fig. 5.1.9 (b) is complement of the graph shown in Fig. 5.1.9(a).

Note : $G \cup \overline{G}$ *is a complete graph.*

Complement of a Sub-Graph in the Graph.

Let G_1 be a sub graph of a graph G. The subgraph of G which remains after all the edges of G_1 are removed from G (keeping the vertices of G_1) is called complement of G_1 in G. It is denoted by $G - G_1$.

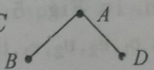
In the graph 5.1.9 (a) consider the subgraph G_1 :

Then $G - G_1$ is the graph

Deletion of a vertex from a graph.

Let G be a graph and v be vertex of it. The subgraph of G obtained by deleting v and the edges incident on v is known as the graph obtained by deletion of v. It is denoted by $G - v$.

In the graph 5.1.9 (a) the graph obtained by deletion of the vertex C is $G - C$



Walk (or Chain) : A finite alternating sequence of vertices and edges, begining and ending with vertices, such that each edge is incident to its preceeding and following vertices is called Walk. In Fig. 5.1.10 the sequence $v_5, e_5, v_1, e_2, v_2, e_3, v_3$ is a walk in the graph. This is denoted by $v_5 - v_3$ walk v_5 and v_3 are origin and *terminus* of the walk respectively.

$v_1, e_5, v_5, e_6, v_3, e_3, v_2, e_1, v_1, e_7, v_3$ is a walk from v_1 to v_3.

Sometimes, for convenience, we represent this only by a sequence of vertex or edges, e.g.,

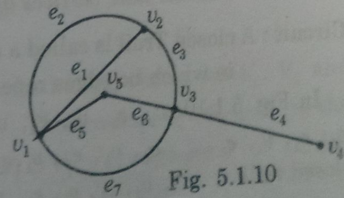


Fig. 5.1.10

Note : (1) *The origin and terminus of a walk may be same. In this case we do not treat them as appearing twice.*

(2) *A vertex may appear twice or more in a walk.*

(3) *An edge may appear twice or more in a walk.*

(4) *A self-loop can be included in a walk.*

Length of Walk : The number of edges in a walk is called the length of the walk. In the graph Fig. 5.1.10 the length of the walk $v_5, e_5, v_1, e_2, v_2, e_3, v_3$ is 3.

Closed and Open Walk : If a walk begins and ends at the same vertex then the walk is called closed walk. A walk that is not closed is called open walk. In Fig. 5.1.10 the walk v_1, e_2, v_2, e_1, v_1 is closed walk but the walk v_1, e_2, v_2, e_3, v_3 is an open walk.

Trail : A walk is called a trail if all the edges are distinct. In Fig. 3.1.10 the walk v_1, e_2, v_2, e_3, v_3 is a trail whereas the walk $v_3, e_3, v_2, e_2, v_1, e_1, v_2, e_3, v_3, e_4, v_4$ is not a trail.

Path (or simple path) : A walk in which no vertex appears twice or more is called a path. In Fig. 5.1.10 the walk v_1, e_2, v_2, e_3, v_3 is a path ; the walk $v_1, e_2, v_2, e_1, v_1, e_5, v_5$ is not a path as v_1 repeats twice

Note : (1) *Every path is a Trail but every Trail may not be a path.*

(2) *A single edge (having two adjacent vertices) is a path of length 1.*

(3) *A self loop is a closed path.*

(4) *Considering a path as a subgraph we see the terminal vertices of a open path are of degree one and the intermediate vertices are of degree 2.*

Circuit : A closed Trail is called a circuit. i.e., it is a closed walk in which no edges repeat.

In Fig. 5.1.10. $v_2, e_1, v_1, e_5, v_5, e_6, v_3, e_3, v_2$ is a circuit; v_2, e_1, v_1, e_5 ... $e_6, v_3, e_7, v_1, e_2, v_2$ is also a circuit but the closed walk ... $, v_2, e_1, v_1, e_2, v_2, e_3, v_3$ is not a circuit.

Cycle **:** A closed path is called a cycle; i.e., it is a closed walk in which no vertices except the two terminal vertex repeat.

In Fig. 5.1.10 $v_2, e_1, v_1, e_5, v_5, e_6, v_3, e_3, v_2$ is a cycle (it was a circuit also) but $v_2, e_1, v_1, e_5, v_5, e_6, v_3, e_7, v_1, e_2, v_2$ is not a cycle though it was a circuit.

Note : (1) *Every cycle is a circuit but every circuit may not be a cycle (seen from the above examples).*

(2) *A circuit is a cycle if no vertices (except the two terminal vertices repeat.*

Distance between two vertices : The distance between two vertices v_i and v_j, denoted by $d(v_i, v_j)$ is the length of the shortest path between v_i and v_j.

In Fig. 5.1.11 there exist path of lengths 2, 3, 4, 5 between v_2 and v_7.

So $d(v_2, v_7) = 2$.

Diameter of a Graph : The maximum distance between any two points in a graph G is called Diameter of G. It is denoted by $\dim(G)$.

In Fig. 5.1.11 the distance between v_2 and v_6 is 3 which is greatest of all distances between any pair of vertices in G. So $\dim(G) = 3$.

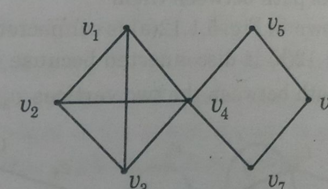


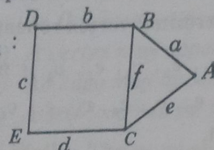
Fig. 5.1.11

Union of two Graphs

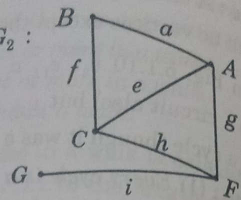
Let G_1 and G_2 be two graphs having vertex set V_1, V_2 and e-set E_1, E_2 ... ctively. Their union $G = G_1 \cup G_2$ is the ... h having ... et $= V_1 \cup V_2$ ... l edge-set ... $\cup E_2$

For example

Let G_1 :

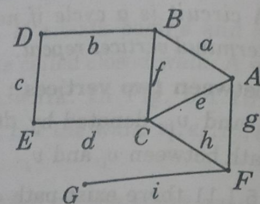


and G_2 :



be two graphs. Then their union

$G_1 \cup G_2$:



Note : (1) $G \cup G = G$ *and most of the properties of union on set are satisfied by this operation.*

(2) *Similarly Intersection of two graphs can be defined.*

Connected and Disconnected Graph : A graph is called connected if there exists at least one path between every pair of vertices in the graph.

A graph is called disconnected if there exists at least two vertices having no path between them

The graph shown in Fig. 5.1.12(a) is connected and the graph shown in Fig 5.1.12(b) is disconnected because in Fig 5.1.12(b) there exists no path between the two vertices v_3 and v_5.

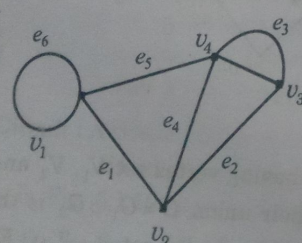


Fig. 5.1.12(a)

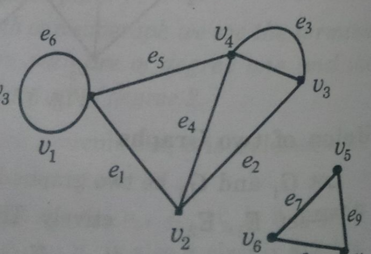


Fig. 5.1.12(b)

Note : (1) *A graph which is not connected is disconnected*

(2) *A disconnected graph is consisted of two or more connected graph.*

(3) *Obviously a complete graph is connected.*

Component : We noted that a disconnected graph is formed by two or more connected graph. Each of these connected sub graph is called a component of the graph. In Fig. 5.1.12(b) the graph consists of two components.

Fundamental Number of a Graph :

Let G be a graph. The number of vertices n, number of edges e and number of component k are called Fundamental Number of G. These three numbers are independent.

Note : (1) $n \geq k$

(2) $e \geq n-k$

(3) *The fundamental numbers alone are not enough to specify a graph.*

(4) $n-k$ *is called* ***rank of*** G, $e-n+k$ *is called* ***nullity of*** G.

Euler line, Euler circuit and Euler graph : A trail in a graph is called an Euler line if it includes every edge of the graph. A circuit that contains all the edges is called Euler circuit A graph is called Euler graph if it has an Euler circuit.

In the graph shown in Fig 5.1.13 (a) the sequence of vertices and edges $A, b, B, k, G, g, F, h, B, c, C, d, D, e, E, l, C, i, F, j, E, f, G, a, A$ is a closed Trail (a vertex may appear more than once in a Trail) containing all edges of the graph. So this is an Euler circuit of the graph and this graph is an Euler graph.

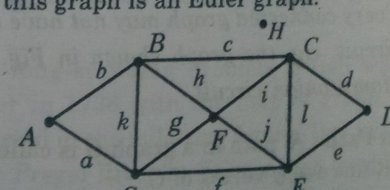


Fig. 5.1.13(a)

The graph found in Fig. 5.1.13(b) is itself an Euler circuit because $A_1, a_1, A_2, a_2, A_3, a_3, A_1, a_4, A_4, a_5, A_5, a_6, A_1$ form a circuit consisting of all vertices and edges of the graph.

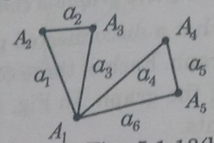


Fig. 5.1.13(b)

Note : (1) *Every Euler graph is connected if the isolated vertices (if exists) are eliminated.*

(2) *The famous problem, Konisberg bridge problem was solved by Euler in light of Euler graph.*

(3) *We often face Euler line in various puzzles.*

Hamiltonian Circuit : A circuit in a connected graph G is said to be Hamiltonian circuit if it includes every vertex of G exactly once.

In the graph shown in Fig. 5.1.14 the circuit form a Hamiltonian circuit.

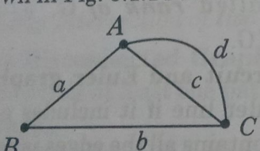


Fig. 5.1.14

Note : (1) *A disconnected graph can not have Hamiltonian circuit.*

(2) *A Hamiltonian circuit in a graph of n vertices consists of exactly n number of edges.*

(3) *Every connected graph may not have a Hamiltonian circuit, e.g. the graph shown in Fig. 5.1.15 has no Hamiltonian circuit.*

Hamiltonian Path : A path in a graph G is called Hamiltonian Path if it contains every vertex of G.

In Fig. 5.1.14 the path C, c, A, a, B is a Hamiltonian path.

Note : (1) *If any one edge is removed from a Hamiltonian circuit then a open Hamiltonian path is obtained.*

(2) *Every graph having a Hamiltonian circuit has also a Hamiltonian Path.*

(3) *A graph may have a Hamiltonian path but no Hamiltonian circuit, e.g the graph shown in Fig. 5.1.16 has a Hamiltonian Path but no Hamiltonian circuit.*

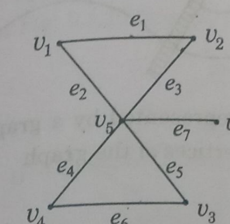


Fig. 5.1.15

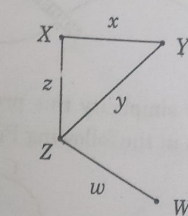


Fig. 5.1.16

(4) *The length of a open Hamiltonian path in a connected graph of n vertices is $n-1$.*

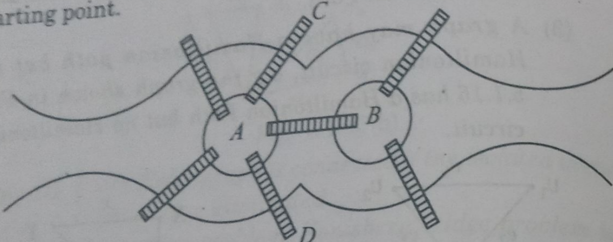
(5) *A Hamiltonian circuit or Path cannot include a self loop or parallel edge. Because of this, to look for a Hamiltonian circuit in a graph we first remove all the loops and parallel edges.*

Hamiltonian Graph : A graph is called Hamiltonian if it has a Hamiltonian Circuit. The Graph in Fig. 5.1.14 is a Hamiltonian graph.

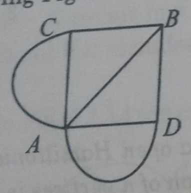
A Best-known Example of Graph (Konigsberg Bridge Problem)

This was a long-standing problem which was solved by Leonhard Euler in 1736 with the help of graph theory. The problem is depicted in the following figure. Two islands A and B , formed by the Pregel River in Kaliningrad (earlier known as Konigsberg ; in West Soviet Russia) are connected to each other and to the banks C and D with seven bridges.

For some purpose (maintanence or etc.) a problem was faced - starting from any of the four lands (A, B, C and D) walk over each of the seven bridges exactly once and come back to the starting point.



For simplicity this problem was represented by a graph as shown in the following Figure. The vertices of the graph



represents the land and the edges represent the bridges, e.g. the edge AD represents the bridge connecting the two lands A and D. Note that there are two bridges connecting A and D. Hence there are two parallels between A and D.

Here we are not giving the solution of this problem ; but we like to mention that Euler proved that a solution for this problem does not exist, i.e., it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

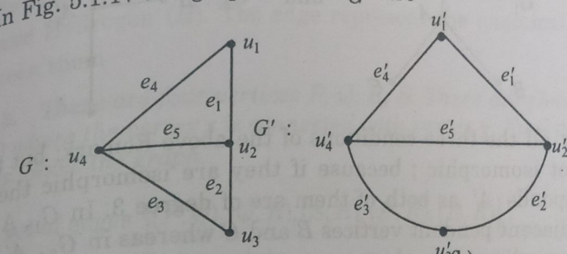
5.1.3. Isomorphic Graphs

Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be isomorphic if there exists an injective mapping $f : V \to V'$ such that there is an edge between $f(v_1)$ and $f(v_2)$ in G' if and only if there is an edge between v_1 and v_2 in G. That is two graphs G and G' are isomorphic if (i) we get one and only one vertex in G' corrsponding to each vertex of G (ii) we get one and only one edge in G' corresponding to each edge of G (iii) the incidence relationship between vertex and edge is preserved.

If two graphs are non-isomorphic then we say they are **distinct.**

More precisely, suppose u_1 and u_2 are two vertices joined by an edge e_1 in the graph G. Then we would get two vertices u_1' and u_2' in G' corresponding to u_1 and u_2 and edge e_1' in G' such that e_1' will be incident to u_1' and u_2'.

In Fig. 5.1.17 two graphs G and G' are



(a) Fig. 5.1.17 (b)

shown. Vertex u_i of G corresponds vertex u_i' of G' ; edge e_i of G corresponds edge e_i' of G'. Then we see, as an example, e_5' is incident to u_4' and u_2' where e_5', u_4' and u_2' are obtained in G' corresponding to e_5, u_4 and u_2 in G where e_5 is incident to u_4 and u_2. Thus we get one and only one vertex in G' corresponding to each vertex in G and the incidence relationship is preserved. So in Fig. 5.1.17 the two graph G and G' are isomorphic.

Note : (1) *Two isomorphic graphs are supposed to be same graph, drawn differently only.*

(2) *In Art 5.1.1 we see a problem can be graphed in different ways [As shown by Fig. 5.1.1(a) and (b)] but they are isomorphic.*

(3) *Just by looking at two graphs we cannot always tell that they are isomophic or not.*

(4) *If the two subgraphs obtained from the two graph G_1 and G_2 by deleting one or more vertices of same degree are distinct then G_1 and G_2 are non-isomorphic.*

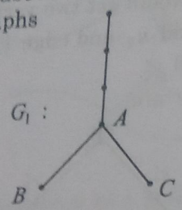
Remark : It is immediately seen (from the definition of isomorphism) that two isomorphic graphs must have

(i) the same number of edges.

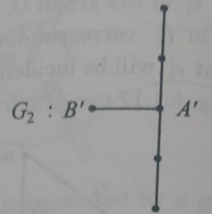
(ii) the same number of vertices.

(iii) an equal number of vertices with a given degree.

The above three conditions are not sufficient. For example the two graphs



and

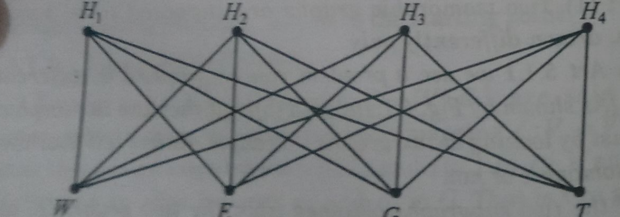


satisfy all the three conditions of the above Remark; but they are not isomorphic ; because if they are isomorphic then A corresponds A' as both of them are of degree 3. In G_1, A has two adjacent pendant vertices B and C whereas in G_2, A' has only one adjacent pendant vertex B'.

Illustrative Examples

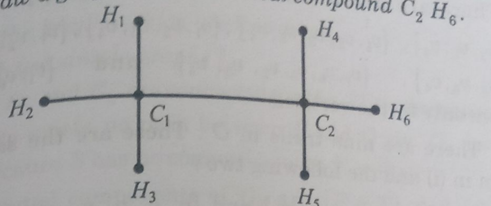
Ex. 1. *Draw a graph representing four houses and four utilities : water, electricity, gas and telephone. Keep a problem, if possible.*

Let H_1, H_2, H_3 and H_4 represents four houses ; W, E, G and T represents supply-point of water, electricity, gas and telephone service. The above figure represents a graph showing an way of connecting the houses to the four utilities. Here we see every house is connected to every utility.



A proposed problem : Is it possible to make connections without crossing over the lines ?

Ex. 2. *Draw a graph of the chemical compound C_2H_6.*

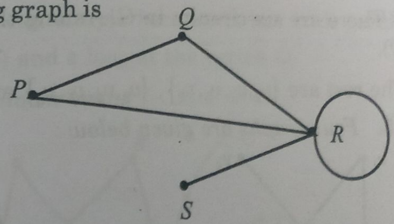


We draw the graph where the vertices are atoms of Carbon (C) and Hydrogen (H). The edge represent the chemical bond between them

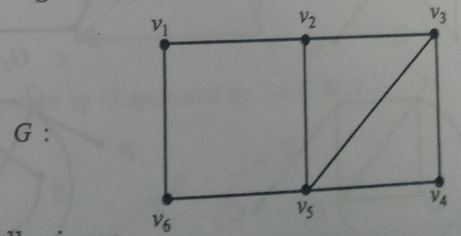
Ex. 3. *There are four vertices P, Q, R, S. There are five pairs (x, y) where the vertex x is connected with vertex y. Find a set of edges. Show the graph.*

A set of edges $\{(P,Q),(Q,R),(S,R),(P,R),(R,R)\}$.

The corresponding graph is



Ex. 4. *You are given a graph, G below :*



Find the followings :

(i) all paths from v_1 to v_4 (ii) all trails from v_1 to v_4 (iii) the distance between v_1 and v_4 (iv) dim (G) (v) all cycles which include vertex v_1 (vi) all cycles in G. [W.B.U.Tech 2006]

(i) There are seven paths between v_1 and v_4. These are $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_5, v_4\}$, $\{v_1, v_2, v_5, v_4\}$, $\{v_1, v_2, v_5, v_3, v_4\}$, $\{v_1, v_6, v_5, v_4\}$, $\{v_1, v_6, v_5, v_2, v_3, v_4\}$ and $\{v_1, v_6, v_5, v_3, v_4\}$ (shown only by the vertices)

(ii) There are nine trails in G. These are the seven paths shown in (i) and the following two :

$$\{v_1, v_6, v_5, v_2, v_3, v_5, v_4\} \text{ and } \{v_1, v_6, v_5, v_3, v_2, v_5, v_4\}.$$

(iii) We see the path $\{v_1, v_2, v_3, v_4\}$ from v_1 to v_4 is of length 3. There are no other shorter path from v_1 to v_4.

So $d(v_1, v_4) = 3$.

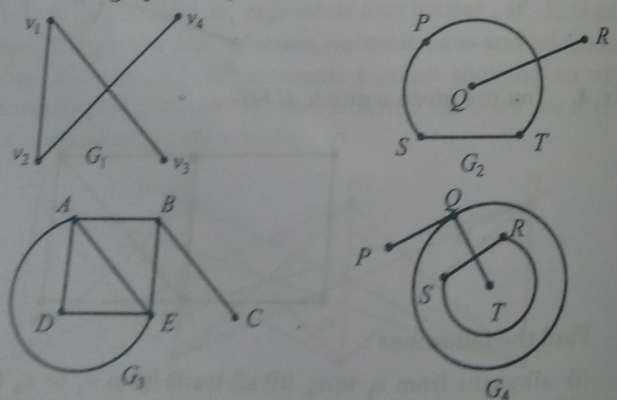
(iv) The distance between any two vertices of G is ≤ 3 where $d(v_1, v_4) = 3$. Hence $\dim(G) = 3$.

(v) There are three cycles which include v_1. These are $\{v_1, v_2, v_5, v_6, v_1\}$, $\{v_1, v_2, v_3, v_5, v_6, v_1\}$ and $\{v_1, v_2, v_3, v_4, v_5, v_6, v_1\}$.

(vi) There are six circuits in G. Among these three are shown in (v).

The rest are $\{v_2, v_3, v_5, v_2\}$, $\{v_3, v_4, v_5, v_3\}$ and $\{v_2, v_3, v_4, v_5, v_2\}$.

Ex. 5. *Four graphs are given below :*



(i) Among those find which one of these are connected ? Find the connected components of those which are not connected.

(ii) Find the graphs having no circuit.

(iii) Find the graphs which are loop-free.

(iv) Which are simple graphs ?

(i) Only G_1 and G_3 are connected graphs. G_2 is disconnected because there exists no path between P and Q etc. G_4 is not connected because S has no connection with T or Q etc.

The connected components of G_2 are $\{P, S, T\}$ and $\{Q, R\}$ [shown only by vertex set]. The two connected components of G_4 are $\{P, Q, T\}$ and $\{S, R\}$.

(ii) G_2 has the circuit $\{P, S, T, P\}$ and G_3 has the circuit $\{A, B, E, A\}$. The loop $\{S, R\}$ is circuit in G_4. The graph G_1 has no circuit.

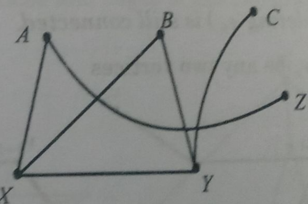
(iii) Only G_4 has a loop.

(iv) G_1 and G_2 are simple graph.

G_3 in not simple because there are two parallels between A and E vertices. G_4 is not simple because it has both parallel edges (joining R and S) and a loop at the vertex Q.

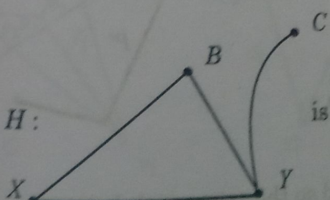
Ex. 6. *A graph G is shown below :*

G:



Find a subgraph of G spanned by $\{B, C, X, Y\}$.

The graph H :



is a required sub graph.

Ex.7. *Let G be a finite graph with at least one edge and without having any circuit. Prove that G has at least two vertices of degree 1.*

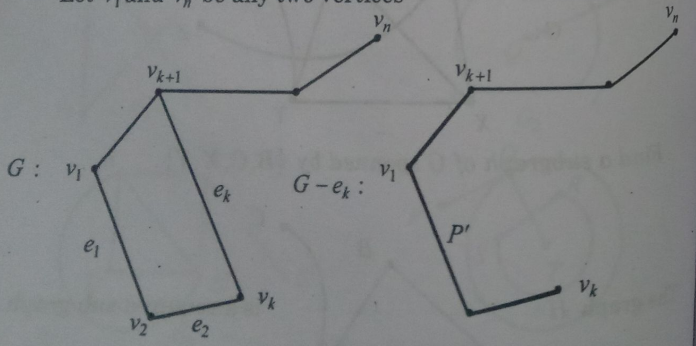
Since G has no circuit so we must get a vertex, say v_1 at which only one edge is incident, i.e $\deg(v_1)=1$. Let e_1 be this edge which is incident at v_1. Since G has no circuit so other end of e_1 is not v_1. Let it be v_2. If there exist no other edge which is incident at v_2 then $\deg(v_1)=1$; otherwise let e_2 be the edge which is incident at v_2. Arguing similar way and proceeding in this way we get a vertex v_k having degree 1 and $v_k \neq v_1$.

Ex. 8. *If* $v_1, e_1, v_2, e_2, \cdots, v_k, e_k, v_{k+1}, e_{k+1}, \cdots, v_n, e_n, v_{n+1} = v_1$ *constitute a cycle (where v_i are vertices and e_i are corresponding incident edges) then prove that if e_k is deleted, the two end vertices (v_k and v_{k+1}) of e_k are connected by a path.*

If e_k is deleted then obviously the sequence $v_{k+1}, e_{k+1}, \cdots, v_n, e_n, v_{n+1}, = v_1, e_1, v_2, e_2, \ldots v_k$ constitute the path. Hence proved.

Ex. 9. *If G be a connected graph, prove that if G contains a circuit which contains an edge e_k then $G-e_k$ (i.e. the subgraph obtained by deleting e_k) is still connected.*

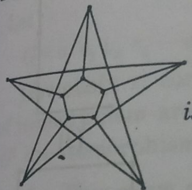
Let v_1 and v_n be any two vertices



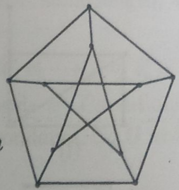
Since G is connected so there is a path between v_1 and v_n.

Say P be this path. If e_k is not included in P then deletion of e_k from G cannot disconnect v_1 and v_n. They would be also connected by the path P in the graph $G-e_k$. If e_k is included in P then after deletion of e_k its two end vertices v_k and v_{k+1} would be connected by a path, say P' (shown in bold in the example figure of $G-e_k$). If v_1 is included in P' then obviously v_1 and v_n are connected by a path (comprising of a part of P and a part of P'). If v_1 is not included in P' then suppose P_1 is the part of P connecting v_{k+1} and v_n ; P_2 is the part of P connecting v_k and v_1 (or v_1, v_n and v_{k+1}, v_k). Then the path joined by P_1, P and P_2 connects v_1 and v_n. Thus two arbitrary vertices of $G-e_k$ are connected by a path. So $G-e_k$ is connected.

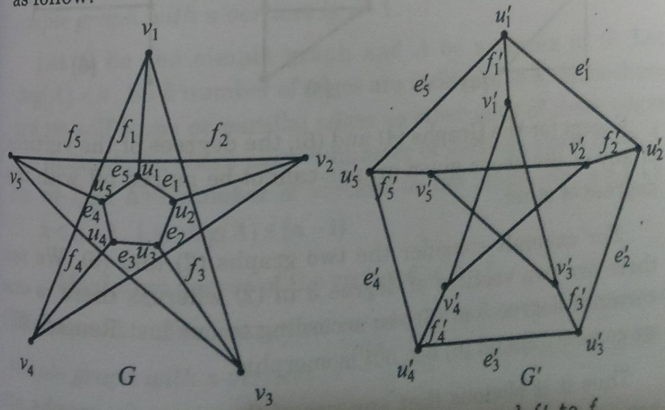
Ex. 10. *Prove that the graph*



is isomorphic to the graph



We name the vertices and edges of the two graphs G and G' as follow:



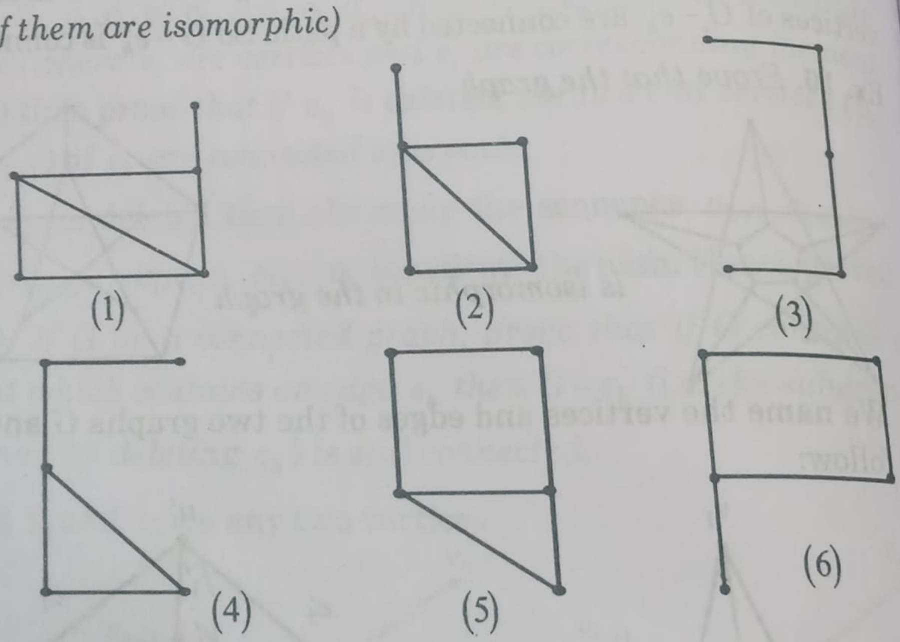
We correspond u_i' to u_i, v_i' to v_i ; e_i' to e_i and f_i' to f_i.

We get one u_i' in G' corresponding to one u_i in G, one v_i' in G' corresponding to one v_i in G; one e_i' in G' corresponding to one e_i in G and one f_i' corresponding to one f_i in G.

We also see when two end vertices of e_i are u_i and u_{i+1}, two end vertices of e_i' are u_i' and u_{i+1}' ; when two end vertices of f_i are u_i and v_i, two end vertices of f_i' are u_i' and v_i'.

Thus the incidence relationship between vertex and edge is preserved under the given correspondence from G to G'. So, G and G' are isomorphic.

Ex. 11. *Prove that the following six graphs are distinct (i.e. no two of them are isomorphic)*

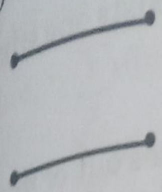
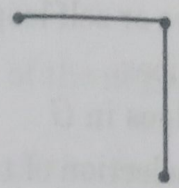


Except for the Graphs (4) and (6), the degrees of the vertices of any of the above given graphs cannot be paired off with the degrees of other.

For example consider the two graphs (2) and (6). We see there are two vertices of degree 3 in (2) whereas there is one vertex of degree 3 in (6) ; so according to our last Remark (iii) we can say (2) and (6) are not isomorphic.

Thus it is obvious that any two of the given six graphs are non-isomorphic except passibly (4) and (6).

Now, if we delete the 3-degree vertex in the graphs (4) and (6) we get the following two subgraphs

 and  which are distinct. Thus the graphs (4) and (6) are also non-isomorphic (See last Note (4)). Hence all the six given graphs are mutually non isomorphic, i.e. all are distinct.

Ex. 12. *Prove that two simple connected graphs with same number of vertices, all of degree two, are isomorphic.*

Let G and G' be the two graphs. Let each has n number of vertices. Since they are connected and every vertex has degree 2 so each of them is nothing but a closed path of same length, n. Let

$G = v_1\, e_1\, v_2\, e_2 \cdots e_{n-1}\, v_n\, e_n\, v_1$ and $G' = v_1'\, e_1'\, v_2'\, e_2' \cdots e_{n-1}'\, v_n'\, e_n'\, v_1'$

Obviously v_i corresponds v_i' ; e_i corresponds e_i'. Conditions of isomorphism are satisfied by this correspondence. So G and G' are isomorphism.

5.1.5. Theorems of Graph.

Theorem. 1 *Prove that the maximum degree of any vertex in a simple graph with n vertices is* $n-1$.

Let G be the simple graph and A be a vertex of G. Let $\deg(A) = k$. So k number of edges are incidence to A. Since there are no self loops or parallel edges so other ends of these edges are disticnt. So there exist k number of vertices apart from A. So we find $k+1$ number of vertices in the graph. So, $n \geq k+1$

$\therefore\ k \leq n-1$, i.e. $\deg(A) \leq (n-1)$

$\therefore$ Maximum degree of the vertex A is $n-1$.

Theorem. 2 *The maximum number of edges in a connected simple graph with n vertices is* $\dfrac{n(n-1)}{2}$. [W.B.U.T 2006]

Let G be the connected simple graph.

Now maximum number of edges in G

= Number of edges joining any two vertices of G

If there are no parallel edges or self loop, there exists one edge corresponding to any two vertices.

So maximum number of edges in G

= Number of all possible selection of two vertices out of n

$$= {}^nC_2 = \frac{n(n-1)}{2}.$$

Theorem. 3. (First Theorem of Graph Theory) : The sum of the degrees of all vertices in a graph is twice the number of edges in the graph.

Proof : Let G be a graph. Let e be an edge of G. If e is a self-loop then it contributes 2 to 'sum of degrees of all vertices'. If e is not a self-loop, it would be incident to two distinct vertices say v_1 and v_2. Then e contributes 1 to the degrees of v_1 and 1 to the degree of v_2, that is e contributes 2 to the 'sum of degrees of all vertices'. Therefore 2 × (Number of all edges) = Sum of the degrees of all vertices. Thus

Sum of the degrees of all vertices = Twice the number of edges in the graph.

Illustration : For the graph shown in Fig. 5.1.19 sum of the degrees of all vertices

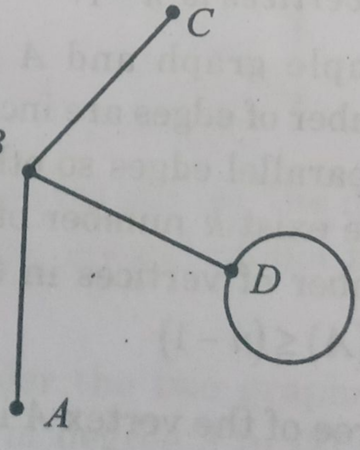


Fig. 5.1.18

$$= \deg(A) + \deg(B) + \deg(C) + \deg(D)$$

$= 1 + 3 + 1 + 3 = 8 = 2 \times 4 = 2 \times$ Number of edges.

Theorem 4 The number of odd vertices in any graph is even.

Proof : Let in a graph, $O=$ Set of all odd vertices and $E=$ Set of all even vertices. [*WBUT 2011*]

Let, $\sum_{O} d(v_i)=$ Sum of the degrees of all odd vertices

and $\sum_{E} d(v_j)=$ Sum of the degrees of all even vertices

$=$ An even number

[$\because d(v_j)$ is even for every v_j in E]

$\therefore \sum_{O} d(v_i)+\sum_{E} d(v_j)=$ Sum of the degrees of all vertices

$= 2\times$ Number of edges [From the previous theorem]

$=$ An even number.

$\therefore \sum_{O} d(v_i)=$ difference of two even number

$=$ An even number (1)

Now, $d(v_i)$ is odd for every vertex v_i in O.

Since only a sum of even number of some odd numbers is even, so, we can say there are even number of terms in the summation of *LHS* of (1).

That is the number of odd vertices in the graph is even.

Note : *The number of even vertices may be odd or even. (See the following two examples)*

Illustration :

(i) In the graph shown in Fig. 5.1.19 (a) the number of odd vertex $=2$ (which are v_1 and v_5)

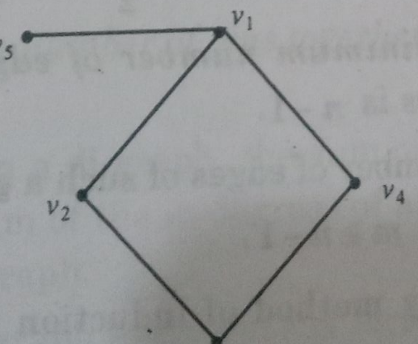


Fig. 5.1.19 (a)

Here the number of even vertices $=3=$ odd.

(ii) In the graph shown in Fig. 5.1.19 (b) the number of odd vertex = 2 (A and B).

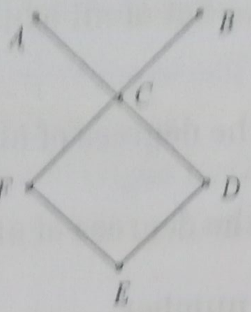


Fig. 5.1.19 (b)

Here the number of even vertices = 4 = even (C, D, E, F)

(iii) Draw a graph with seven vertices having degrees 3, 5, 2, 7, 4, 6, 8.

We see there are three odd vertices in the graph. But we know any graph contains an even number of odd vertices. So it is not possible to draw such a graph.

Theorem. 5 *Prove that a complete graph with n vertices consists of* $\frac{n(n-1)}{2}$ *number of edges.* [W.B.U.T. 2010, 2008]

Proof : Since the graph is complete and since there is no loop/parallel edge so every vertex of the graph has $n-1$ adjacent vertices, i.e. degree of each vertex is $n-1$.

$\therefore$ Sum of all degrees of the graph $= n(n-1)$.

or, 2 × No. of edges of the graph $= n(n-1)$

$\therefore$ No. of edges of the graph $= \dfrac{n(n-1)}{2}$.

Theorem.6 *The minimum number of edges in a connected graph with n vertices is* $n-1$. [W.B.U.T. 2005]

Proof : Let $m =$ number of edges of such a graph.

We have to show $m \geq n-1$.

We prove this by method of induction on m. If $m=0$ then obviously $n=1$ (otherwise G will be disconnected).

Clearly then $m \geq n-1$ (as $0 \geq 1-1$).

Let the result be true for $m = 0, 1, 2, 3, \cdots, k$. We shall show that the result is true for $m = k+1$. Let G be a graph with $k+1$ edges. Let e be an edge of G. Then the subgraph $G-e$ (i.e. the graph obtained by deleting only e from G) has k edges and n number of vertices. If $G-e$ is also connected then by our hypothesis $k \geq n-1$, i.e. $k+1 \geq n > n-1$.

If $G-e$ becomes disconnected then it would have two connected components. Let the two components have k_1 and k_2 number of edges and n_1, n_2 number of vertices respectively.

So, by our hypothesis $k_1 \geq n_1 - 1$ and $k_2 \geq n_2 - 1$.

These two imply $k_1 + k_2 \geq n_1 + n_2 - 2$ that is,

$k \geq n-2$ $(\because k_1 + k_2 = k,\ n_1 + n_2 = n)$ i.e., $k+1 \geq n-1$.

Thus the result is true for $m = k+1$.

Theorem. 7 The minimum number of edges in a simple graph (not necessarily connected) with n vertices is $n-k$, where k is the number of connected components of the graph.

Proof: Left to the reader.

Illustration

(i) Let a *connected simple graph has 7 vertices.*

Then $7-1 \leq$ No. of edges $\leq \dfrac{7(7-1)}{2}$;

i.e. $6 \leq$ No. of edges ≤ 21.

So, minimum and maximum number of edges of this graph are 6 and 21 respectively.

(ii) *To short n number of pins together, we need at least* $n-1$ *pieces of wire.*

Theorem. 8 In a di-graph, the sum of the out-degrees of all vertices = the sum of the in-degrees of all vertices = number of edges in the di-graph.

Proof: In a di-graph every edge is incident into at least some vertex. So every edge contributes in-degree 1 in the di-graph. So sum of all in-degrees = Number of edges.

Similarly every edge is incident out of at least some vertex. So sum of all out degrees = Number of edges.

This completes the proof.

Theorem. 9 A graph G with vertex set V is disconnected if and only if there exist two non-empty disjoint sets V_1 and V_2 such that $V = V_1 \cup V_2$ and there exists no edge in G whose one end vertex belongs to V_1 and other end belongs to V_2.

Proof : First, let where $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$ and $V_1 \neq \phi$, $V_2 \neq \phi$ and there exists no edge in G whose one end vertex belongs to V_1 and other end belongs to V_2. Let a and b be two arbitrary vertices of G such that $a \in V_1$ and $b \in V_2$.

Let, if possible, there exists a path

$P = \{a, e_1, u_1 e_2, \cdots, u_k e_k, u_{k+1} e_{k+1}, \cdots, b\}$ joining a and b [u_i's are vertices, e_i's are edges] where $u_k \in V_1$ and $u_{k+1} \in V_2$. Then e_k is the edge joining u_k and u_{k+1}, contradicting our hypothesis. So there exists no path between a and b. So G is disconnected.

Conversely, let G be disconnected. So there exist at least two vertices say a and b which are not connected by any path. Let V_1 be the set of all vertices of G which are connected to a by a path. Let $V_2 = G - V_1$ (Set of all vertices of G which are not in V_1). Since $b \notin V_1$ so $b \in V_2$. $\therefore$ $V_2 \neq \phi$

By definition of V_1 we can say no vertex in V_1 is joined to any vertex in V_2. So, no vertex of V_1 is connected to any vertex of V_2 by an edge. Hence the converse part is proved.

Theorem. 10 If a graph has exactly two vertices of odd degree there must be a path joining these two vertices. [i.e. the two vertices are connected]

Proof : Let G be the graph with all even vertices except the two say a and b which are the only two odd-vertices. If G is a connected graph then there exists a path joining a and b (by definition of connected graph) and the theorem is proved. If G is not connected then suppose G_1 and G_2 are two of the components of G such that $a \in G_1$ and $b \in G_2$. Then G_1 itself becomes a graph having only one odd vertex (a). But we know number of odd vertex in a graph is even. So, this is not possible. So both of a and b must belong to same component, say G_1. Since component G_1 is itself a connected graph so there is a path joining a and b.

Theorem. 11 A simple graph with n number of vertices and k number of components can have maximum $\frac{(n-k)(n-k+1)}{2}$ number of edges. [*W.B.U.T. 2008*]

Proof: Let each of the k number of components of the graph contains $n_1, n_2, \cdots, n_k$ number of vertices respectively.

So, $n_1 + n_2 + \cdots + n_k = n$ and $n_i \geq 1$.

Now we shall prove the following inequality

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \qquad (1)$$

Now, $\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k \quad \therefore \left\{\sum_{i=1}^{k}(n_i - 1)\right\}^2 = n^2 + k^2 - 2nk$

or, $\sum_{i=1}^{k}(n_i - 1)^2 + \sum_{i \neq j}\sum (n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$

or, $\sum_{i=1}^{k}\left(n_i^2 - 2n_i + 1\right) + \sum_{i \neq j}\sum (n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$

or, $\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 + \sum_{i \neq j}\sum (n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$

or, $\sum_{i=1}^{k} n_i^2 - 2n + k + \sum_{i \neq j}\sum (n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$

or, $\sum_{i=1}^{k} n_i^2 - 2n + k \le n^2 + k^2 - 2nk$ $\left[\because n_i - 1 \ge 0,\ n_j - 1 \ge 0\right]$

or, $\sum_{i=1}^{k} n_i^2 \le n^2 + k^2 - 2nk + 2n - k$

or, $\sum_{i=1}^{k} n_i^2 \le n^2 - (k-1)(2n-k)$, thus (1) is proved.

Let G_i be the component of G having n_i number of vertices. Now maximum number of edges in G_i = Number of possible edges joining any two vertices.

Since there are no parallel edges or self loop ($\because$ the graph is simple) so there exists exactly one possible edge corresponding to any two vertices of G_i.

$\therefore$ Max number of edges in G_i

$= $ Number of selection of two vertices from n_i

$$= {}^{n_i}C_2 = \frac{n_i(n_i-1)}{2}$$

$\therefore$ the maximum number of edges in G is

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k}\left(n_i^2 - n_i\right) = \frac{1}{2}\left\{\sum n_i^2 - \sum n_i\right\}$$

$$\le \frac{1}{2}\left\{n^2 - (k-1)(2n-k) - n\right\}, \text{ by (1)}$$

$$= \frac{1}{2}(n-k)(n-k+1).$$

Theorem. 12. A bipartite graph cannot contain a cycle of odd length.

Proof: Let G be a bipartite graph. V be its vertex set.

We can express $V = V_1 \cup V_2$ where V_1 and V_2 are disjoint. Each edge connects one vertex of V_1 with one vertex of V_2.

Let $C = \{v_1, e_1, v_2, e_2, v_3 \ldots v_n, e_n, v_1\}$ be a cycle contained in G.

We shall show that C contains even number of edges i.e. n is even.

As e_1 connects v_1 and v_2 so v_1 and v_2 cannot belong to same set of V_1 and V_2.

Suppose $v_1 \in V_1$ and $v_2 \in V_2$.

Similarly, as e_2 connects v_2 and v_3, $v_2 \in V_2$ and $v_3 \in V_1$

Similarly, as connects v_3 and v_4, $v_3 \in V_1$ and $v_4 \in V_2$

Similarly, $v_4 \in V_2$ and $v_5 \in V_1$ and so on

Thus we see if $v_r \in V_2$ then r is even.

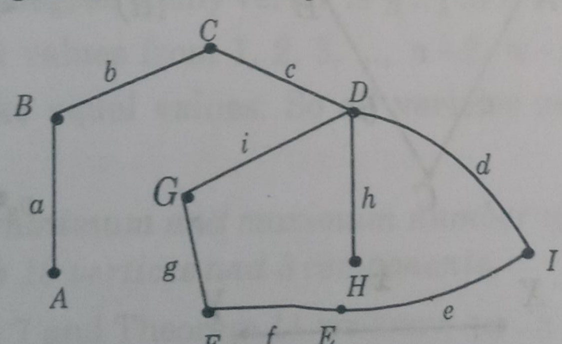
Nor since e_n connects v_n and v_1 and since $v_1 \in V_1$ therefore $v_n \in V_2$. So n is even. Hence proved.

Theorem. 13 (Converse of the above theorem)

If a simple graph does not contain any cycle of odd length then the graph is a bipartite graph.

Proof: Beyond the scope of this book.

Illustration. Using the above two theorems we can say that the following simple graph cannot be a bipartite graph as it contains the cycle $\{D, i, G, g, F, f, E, e, I, d, D\}$ of length 5 (odd)



Theorem. 14 Every vertex, except the origin and terminus vertices of a walk whose all edges are distinct (i.e. Trail) is even vertex.

Proof: Let $u - v$ be a walk. Let v_k be a vertex of this walk such that $v_k \neq u$ and $v_k \neq v$. If v_k occurs only once in the walk then there must exist one edge preceeding v_k and one edge succeeding v_k.

Clearly then v_k becomes a 2-degree vertex. Next let v_k occurs more than once in the walk, say p times. At each time of occurence v_k gets 2-degree (since no edge repeats) and so the degree of v_k is $2p$. Thus in any case any vertex has degree even. (In the adjacent example figure v_k has degree $2 \times 3 = 6$.)

Theorem. 15 Every $u-v$ Trail contains a $u-v$ path.

Proof: In the $u-v$ Trail, u is the origin and v is terminus. Both u and v are of 1-degree. We know every vertex, except u and v is of even-degree. So the $u-v$ Trail contains only two odd vertices. By Theorem 10, the $u-v$ trail has a path joining u and v because we can think the trail $u-v$ itself as a graph ($\because$ the vertices and edges other than those in $u-v$ trail have no influence in this proof)

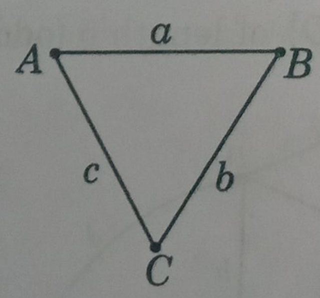
Theorem. 16 A connected graph is an Euler graph if and only if all vertices of G are of even degree. [W.B.U.T. 2013]

Proof: Beyond the scope.

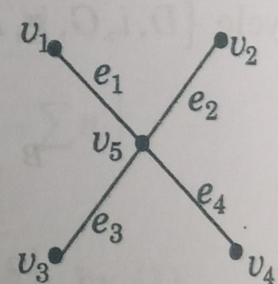
5.1.6. Illustrative Examples

Ex. 1. *State which of the following graphs are bipartite graph:*

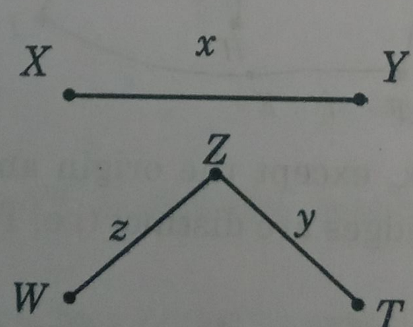
(i)



(ii)



(iii)



(i) This graph cannot be a bipartite graph. Let this be a bipartite graph. Then the vertex set $V = \{A, B, C\}$ can be partitioned into two sets V_1 and V_2.

Since the edge a connects A and B so A and B cannot belong to same set. Suppose $A \in V_1$ and $B \in V_2$. Again since the edge b connects B and C so they cannot belong to same set. Since $B \in V_2$ therefore C belongs to V_1. Again since the edge c connects A and C therefore they cannot belong to same set. Since $C \in V_1$ so $A \in V_2$. This contradicts the fact that $A \in V_1$.

(ii) This is a bipartite graph because the vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ can be partitioned into two sets $V_1 = \{v_1, v_2, v_3, v_4\}$ and $V_2 = \{v_5\}$ and we see each edge of the graph is incident to one vertex of V_1 and one vertex of V_2.

(iii) This is a bipartite graph since its vertex set $V = \{X, Y, Z, W, T\}$ can be partitioned into two sets $V_1 = \{Y, Z\}$ and $V_2 = \{X, W, T\}$ and each edge joins a vertex of V_1 to a vertex in V_2.

Ex. 2. *Does there exist a simple graph with two or more vertices such that all the vertices are of different degree?*

Let there be n number of vertices. Then from Theorem 1 we see the maximum degree of any vertex is $n-1$. If n vertices have to take n integral values from 1, 2, 3, ..., $n-2$, $n-1$ then two vertices must take equal values. So all vertices cannot have different degree.

Ex. 3. *Find the minimum and maximum number of edges of a simple graph with 10 vertices and 3 components.*

From Theorem 7 and Theorem 11 we know $10-3 \leq$ Number of edges $\leq \dfrac{(10-3)(10-3+1)}{2}$

or, $7 \leq$ Number of edges $\leq \dfrac{7 \times 8}{2}$

or, $7 \leq$ Number of edges ≤ 28

So, minimum number of edges $= 7$ and

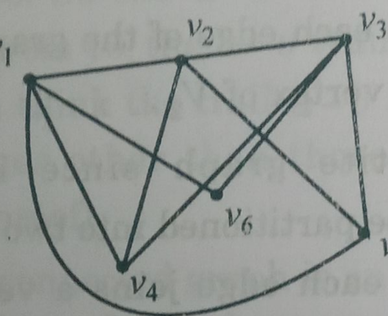
Maximum number of edges $= 28$.

Ex. 4. *Draw a connected graph with 10 vertices (i) having 7 edges (ii) having 50 edges.*

(i) Impossible ; since, by Th 6., the minimum number of edges is $10-1=9$.

(ii) Impossible ; Since, by Th 2., the maximum number of edges is $\dfrac{10(10-1)}{2}=45$.

Ex. 5. *Draw a simple graph having a vertex of degree 2, two vertices of degree 3 each and three vertices of degree 4 each.*



The graph is drawn above.

Here $\deg(v_1)=\deg(v_2)=\deg(v_3)=4$; $\deg(v_4)=\deg(v_5)=3$ and $\deg(v_6)=2$.

Ex. 6. *Prove that there exist no simple graph with five vertices having degrees 4, 4, 4, 2, 2.*

Let, if possible, there exists a graph having vertices v_1, v_2, v_3, v_4 and v_5 such that $\deg(v_1)=\deg(v_2)=\deg(v_3)=4$ and $\deg(v_4)=\deg(v_5)=2$. Since the graph has no parallel edge or loop so each of v_1, v_2 and v_3 has four adjacent vertices.

So both of v_4 and v_3 must be adjacent vertex of v_1, v_2 and v_3 .So, $\deg(v_4)$ is at least 3. ($\deg(v_5)$ is also so).

This contradicts our hypothesis. Hence there exists no such graph.

Ex. 7. *Prove that there exists no graphs with four edges having vertices of degree 4, 3, 2, 1.*

Let, if possible, there exists a graph, G with the given properties.

Then the sum of degrees of all vertices $= 4+3+2+1 = 10$.

But we know the sum of degrees of all vertices $= 2 \times$ Number of edges $= 2 \times 4 = 8$. This is a contradiction.

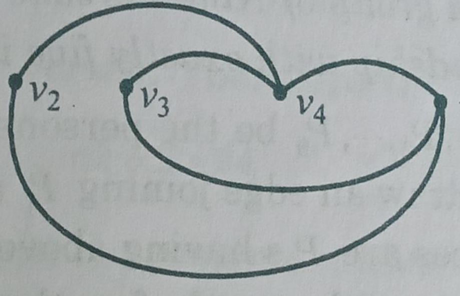
Ex. 8. Draw, if possible, a simple graph with five vertices having degrees 2, 3, 3, 3, 3.

Let, v_1, v_2, v_3, v_4, v_5 be the vertices where $\deg(v_1) = 2$. Since there is no loop so v_1 must be connected with two of the rest, say v_2 and v_3.

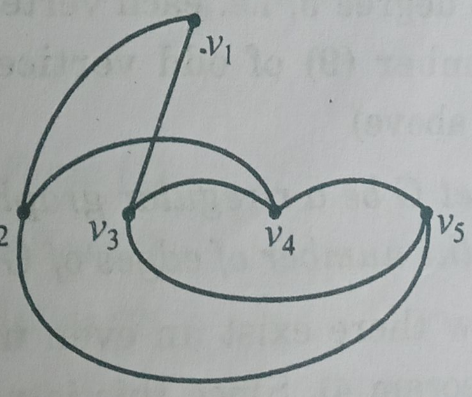
So removing v_1 with all its incidence edges we are left with the graph with vertices v_2, v_3, v_4, v_5 where

$$\deg(v_2) = \deg(v_3) = 2 \,;\, \deg(v_4) = \deg(v_5) = 3\,.$$

This can be drawn as follow



So, the required graph is



where $\deg(v_1) = 2, \deg(v_2) = 3, \deg(v_3) = 3, \deg(v_4) = 3, \deg(v_5) = 3$.

Ex. 9. Find the number of vertices in a graph with 15 edges if each vertex has degree 2.

Let $n =$ number of vertices. So, sum of the degree of all vertices $= 2n$, By Theorem 3, $2n = 2 \times$ Number of edges.

$n = 15$

$\therefore$ Number of vertices is 15.

Ex. 10 . *Let G be a graph with n vertices and e edges. Prove that G has a vertex of degree m such that* $m \geq \frac{2e}{n}$.

By Theorem 3 we have,

Sum of the degrees of all vertices $= 2 \times$ Number of all edges $= 2e$ (1)

Let v_k be that vertex of G whose degree is maximum, say m

$\therefore$ Sum of the degrees of all vertices $\leq m + m + \cdots + m$ (n times)

$$= mn$$

$\therefore$ from (1) we get $mn \geq 2e$, i.e., $m \geq \frac{2e}{n}$

where $m = \deg(v_k)$, greatest of all degrees.

Ex. 11. *In a group of Nine Persons it is impossible that every one has friendship with exactly five in the group - Prove it.*

Let $P_1, P_2, P_3, \cdots, P_9$ be the persons. If P_i has friendship with P_j then we draw an edge joining P_i and P_j. Consider the graph whose vertices are P_is having above type of edges. If every one has friendship with exactly five then in the graph each vertex would have degree 5, i.e. each vertex is odd. So this graph has an odd number (9) of odd vertices. This is impossible (by Theorem 4 above)

Ex. 12. *Let G be a r-regular graph where r is an odd integer. Show that the number of edges of G is a multiple of r.*

We know there exist an even number of odd vertices in a graph (Theorem 4). Since this is r-regular so each vertex has degree r which is odd. So number of vertices is even say $2k$, k is an integer. Therefore $2kr =$ Sum of degrees of all vertices of $G = 2 \times$ Number of edges. Thus, Number of edges $= k \times r =$ A multiple of r.

Ex. 13. *Let G be graph having n number of vertices and n-1 number of edges. Prove that G has either a pendant vertex or an isolated vertex.*

If $v_1, v_2, \cdots, v_n$ be the vertices then by Th 3, $\sum d(v_i) = 2(n-1)$

Let, if possible, G has no vertex whose degree is 1 or 0.

So ; $\deg(v_i) \geq 2$ for all i. $\therefore \sum_{i=1}^{n} d(v_i) \geq \sum_{i=1}^{n} 2 = 2n$

So, from above, $2(n-1) \geq 2n$,

i.e., $-2 \geq 0$, which is a contradiction.

Thus G would have at least one vertex whose degree is either 1 or 0 ; i.e., G would have at least one vertex which is pendant or isolated.

Ex. 14. *If a simple regular graph has n vertices and 24 edges, find all possible values of n. Draw a graph against each of such values.* [W.B.U.T. 2015, 2005]

Let $k =$ degree of every vertex. Therefore the sum of the degrees of all vertices $= nk$. So $nk = 2 \times 24$ $\therefore n = \dfrac{48}{k}$ ... (1)

On the other hand we know the maximum number of edges in a simple graph is $\dfrac{n(n-1)}{2}$ $\therefore \dfrac{n(n-1)}{2} \geq 24$

$\therefore n(n-1) \geq 48$. ... (2)

Now k is positive integer. From (1)

$k = 1 \Rightarrow n = 48$ and this satisfies (2)

$k = 2 \Rightarrow n = 24$ and this satisfies (2)

$k = 3 \Rightarrow n = 16$ and this satisfies (2)

$k = 4 \Rightarrow n = 12$ but this satisfies (2)

$k = 5 \Rightarrow n$ is not integer.

$k = 6 \Rightarrow n = 8$ and this satisfies (2) and so on.

$k = 7 \Rightarrow$ n is not integer

$k = 8 \Rightarrow n = 6$ but this does not satisfy (2) and so on.

$\therefore$ the possible values of n are 48, 24, 16, 12, 8.

Ex. 15. *Let G be a simple graph with n vertices and G' be its complement. For an arbitrary vertex v of G, prove that deg (v) in G+deg (v) in G' = n−1.*

Let deg (v) in $G = k$. So there are k adjacent vertices of v in G ($\because$ G is simple). So in the graph G there are $(n-1)-k$ number of vertices which are not adjacent to v. So, in the graph G' these $n-k-1$ number of vertices would be adjacent to v. So, in the graph G', $\deg(v) = n-k-1$. Hence proved.

Ex. 16. *Let G be a simple graph with n vertices and G' be its complement. If G has exactly one even vertex, how many odd vertices does G' have?*

Now G has exactly $n-1$ number of odd vertices. So, $n-1$ must be even. So n must be odd.

We get, deg (v) in $G' = n-1-\deg(v)$ in G. (from the previous example)

So, we get $\deg(v)$ in $G' =$ an even No. $-\deg(v)$ in G.

So a vertex which is odd in G is odd in G' ; a vertex which is even in G is even in G' and $(n-1)$ No. of odd vertices.

So, G' has exactly one even vertex .

Ex. 17. *Prove that if the number of edges in a connected graph having at least two vertices, is less than the number of vertices, the graph has pendant vertex.*

Let, if possible, the graph G has no pendant vertex, i.e. no vertex having degree 1. Since, G is connected so it has no vertex having degree 0. So degree of every vertex ≥ 2.

Let G has n number of vertices and d_i is degree of i-th vertex.

$\therefore$ Sum of all degrees $= \sum_{i=1}^{n} d_i$,

or, $2\times$ No. of edges $\geq \sum_{r=1}^{n} 2 \quad (\because d_i \geq 2)$

$\therefore$ $2\times$ No. of edges $\geq 2n$

$\therefore$ No. of edges $\geq n$

This contradicts the hypothesis that the No. of edges is less than n.

Ex. 18. *Show that a connected graph is a cycle if the degree of each vertex is 2.*

Let G be the connected graph each of whose vertices is of degree 2. Let A and B be two arbitrary vertices of G. Since G is connected there is a path from A to B.

Let G_1 be the sub graph consisting of all the vertices and edges of this path. Similarly if we take any other distinct pair of vertices C, D we get a path from D to C.

Let G_2 be the subgraph formed by this path. Since G is connected so, C, A and B, D are connected by path, say G_3 and G_4 respectively.

Now, G_1, G_4, G_2 and G_3 togetherly form a closed walk $A-A$. So there exists a cycle from this walk. Now we shall prove that every vertex of G belongs to this cycle say X.

Let v be any vertex of G that does not belong to X. Since G is connected so there is an edge whose one end is v and other end is a vertex of X, say u. So degree of u becomes 2+1=3, which contradicts our hypothesis. So every vertex, consequently every edge of G is included in the cycle, i.e. the graph G becomes itself a cycle.

Ex. 19. *Suppose G is a non-directed graph with 12 edges. If G has 6 vertices each of degree 3 and the rest have degree less than 3, find the minimum number of vertices G can have.*

[W.B.U.T. 2003,2006]

Let the number of vertices = $6+k$.

Let $x_1, x_2, \cdots, x_k$ be the degrees of rest of the vertices respectively.

$\therefore$ Sum of degrees of all vertices $= 6 \times 3 + x_1 + x_2 + \cdots + x_k$

$\therefore 2 \times 12 = 18 + x_1 + x_2 + \cdots + x_k$

or, $x_1+x_2+\cdots+x_k=6$ or, $6=x_1+x_2+\cdots+x_k$

or, $6<3+3+\cdots+3$ (k times) $[\because\ x_i<3\ \ \forall i]$

or, $6<3k\ \therefore\ k>2$.

Since k can get only integral valus so the minimum value of k is 3.

$\therefore$ the minimum number of vertices $=6+3=9$.

Ex. 20. *Show that a simple graph having n number of vertices must be connected if it has more than* $\dfrac{(n-1)(n-2)}{2}$ *edges.*

Let G be the graph. By próblem

the number of edges of G $>\dfrac{(n-1)(n-2)}{2}$.

Let G has k number of components.

We know G can have at most $\dfrac{(n-k)(n-k+1)}{2}$ number of edges (see Th 11)

So, $\dfrac{(n-1)(n-2)}{2}<\dfrac{(n-k)(n-k+1)}{2}$

i.e., $(n-1)(n-2)<(n-k)(n-k+1)$, where k must be a positive integer. We see this inequality is satisfied only for $k=1$. This is not satisfied for $k=2,3,\cdots$ etc. Thus number of component of G is one. So G is a connected graph.

Ex. 21. *Let G be a graph with 15 vertices and 4 components. Show that G has at least one component having at least 4 vertices. Find the largest number of vertices that a component of G can have.*

If the four components contain n_1, n_2, n_3 and n_4 number of vertices then $n_1+n_2+n_3+n_4=15$ ... (1)

If each of n_1,n_2,n_3 and n_4 is ≤ 3 then $n_1+n_2+n_3+n_4\leq 12$ which contradicts (1). So at least one of n_1,n_2,n_3 and n_4 must be 4.

If one component contains exactly 4 vertices then the other components togetherly contain $15-4=11$ vertices. So, component among the remaing three must contain at least 4 vertices (can be discussed as above).

If one component of those three contains exactly 4 vertices then the remaining two components togetherly contains $15-(4+4)=7$ vertices.

Now these two components would share between themseleves $(1,6),(2,5),(3,4),\ (4,3),(5,2)$ and $(6,1)$ number of vertices.

So the maximum number of vertices in a component is 6.

Ex. 22. *Let G be a simple graph having atmost 2n vertices. If the degree of every vertex of $G \geq n$ then prove that G is connected.*

Let, if possible, G be not connected. Then G would have a finite number of components, say $k_1, k_2, \cdots, k_m$; m must be ≥ 2 .

Since degree of each vertex is at least n and since the graph has no loop or parallel edges so each vertex has at least n number of adjacent vertices. So each component has at least $n+1$ number of vertices. So G has at least $m(n+1)$ number of vertices.

Now since $m \geq 2$, so $m(n+1) \geq 2(n+1) > 2n$ .Thus G has more than $2n$ number of vertices. This contradicts a given condition. So the graph is connected.

Ex. 23. *Prove that if A be an odd vertex of a graph then there exists an another odd vertex, B in G which is connected with A by a path.*

Let G' be a component of G containing A as one of its vertices. G' is a connected graph itself.

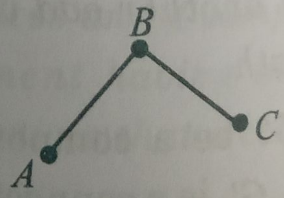
Since A is an odd vertex in G', so the graph G' contains at least one more odd vertex, say B because the number of odd vertices in a graph is even.

Since G' in connected so there exists a path connecting the vertices A and B.

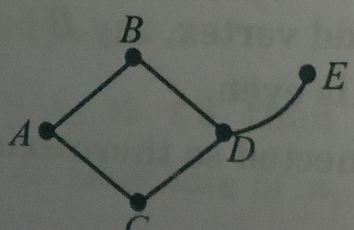
Exercise

[I] Short Answer Questions

1. Define vertex-set of a graph. Give an example.
2. Define edge-set of a graph. Give an example.
3. Define parallel edges. Draw a graph showing three parallel edges.
4. Define simple graph. Give an example.
5. Define adjacent edges in a graph. Draw a graph showing four adjacent edges.
6. Draw a graph showing the existence of no adjacent edges
7. Define even and odd vertices of a graph.
8. Draw a regular graph with five vertices.
9. Draw a graph, which is not simple, having two isolated vertex
10. Draw a regular graph having at least one isolated vertex.
11. Draw a non-simple regular graph with three vertices
12. Define pendant vertex. Draw a graph with five vertices having four pendant vertices. Indicate the pendant edges.
13. When two edges are said to be in series. Explain with an example.
14. Define complete graph. Give an example with five vertices
15. Draw a graph which is regular but not complete.
16. Draw a supergraph of the graph

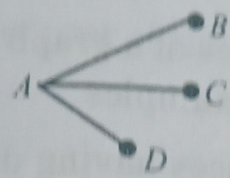


17. Draw a graph which is a spanning subgraph of the graph



18. Draw, with explanation, two edge disjoint graphs which are not vertex disjoint

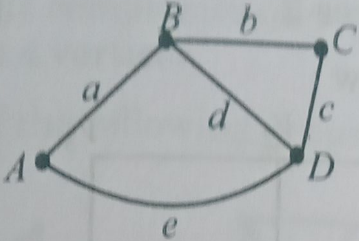
19. If G be the graph



then find

20. Define a walk in a graph. Show an example.

21. In the graph



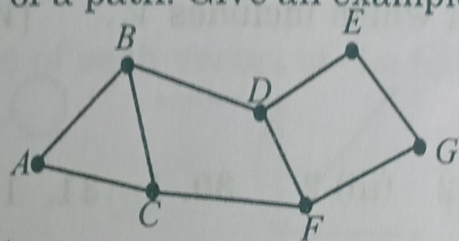
find a walkwhich is not a path whose origin and terminus are A and C respectively.

22. Give an example of a walk which is not a Trail.

23. Define path. Give an example of a walk which is not a path.

24. Define length of a path. Give an example of a path of length 4.

25. In the graph



(i) construct a path with origin A and terminus C of length 4. Give an another path with same length.

(ii) find the distance between A and F

(iii) find the distance between D and C

26. Draw a graph with four vertices and three components.

27. Define connected graph. Draw a connected graph which is not complete.

28. Construct two graphs which are not isomorphic. Give reasons.

29. Prove that the maximum degree of any vertex in a simple graph with 3 vertices is 2. [Hint : Theorem 1]

30. G be a simple connected graph having 5 vertices. Find the maximum number of edges of G. [Hint : Follow Theorem 2]

31. Find the number of edges in a complete graph with 6 vertices [Hint : Follow Th 3]

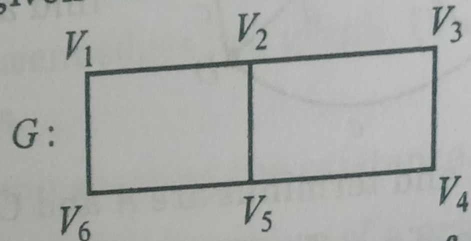
32. Prove that the sum of the degrees of all vertices in a graph with 10 edges is 20. [See the proof of Theorem 6.]

33. The number of even vertices of a graph may be even or odd – Show this by exhibiting two examples.

34. Draw a graph with 9 vertices having degree

35. Determine the number of edges with 6 nodes (vertices), two of degree four and 4 of degree 2. [W.B.U.T. 2005]

36. A graph G is given below :

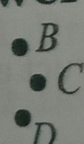


(a) Find the distance between V_1 and V_4 of G.

(b) Find the dim (G)

(c) Find one circuit which includes V_1. [W.B.U.T. 2006]

Answers

19. 

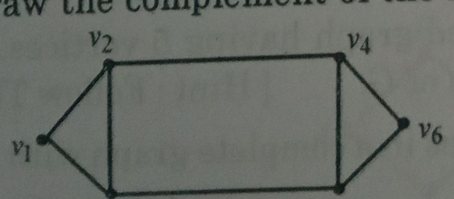
25. (ii) 2 (iii) 2 30. 1 31. 15 34. Impossible

35. 8 36. (a) 3 (b) 3 (c) one is $\{V_1, V_2, V_5, V_6, V_1\}$,

another one is $\{V_1, V_2, V_3, V_5, V_6, V_1\}$ etc.

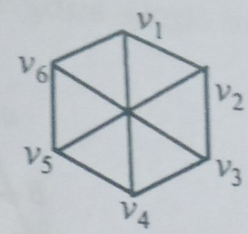
[II] Long Answer Questions

1. Draw a graph representing two houses and four utilities : gas, telephone, water and electricity.
2. Draw a simple graph with (i) three (ii) one (iii) four vertices.
3. Draw graphs of the following chemical compounds
   (i) C_6H_6 (ii) CH_4.
4. (a) Draw the complement of the following graph :



[W.B. U. T, 2003]

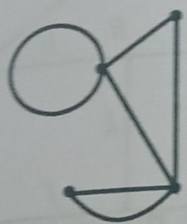
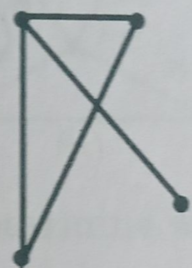
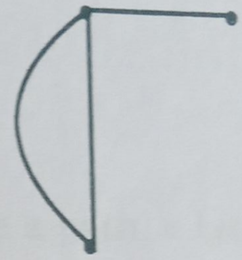
(b) Find the complement of the graph



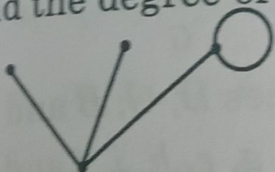
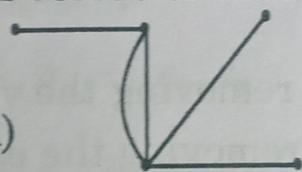
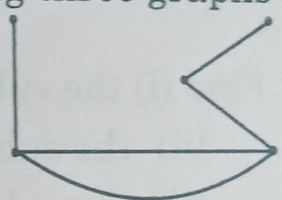
[W.B. U. T. 2007, 2012]

(c) Draw a self complementary graph(i.e,isomorphic to its complement) with 4 vertices.

5. State which of the following graphs are simple

(a)  (b)  (c) 

6. Find the degree of each vertex of the following three graphs

(i)  (ii)  (iii) 

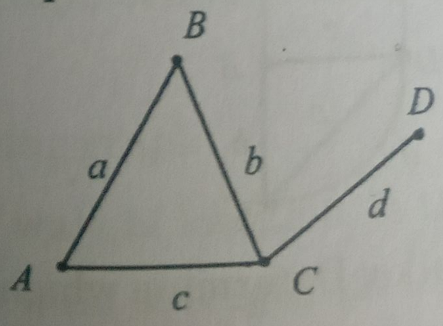
7. (i) Draw a simple graph having four vertices each of degree 2

(ii) Draw a connected graph with 3 vertices and 4 edges.

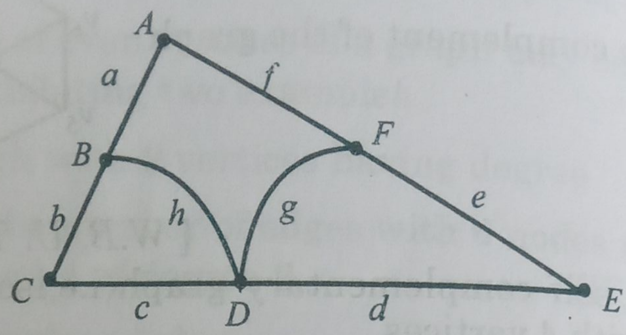
(iii) Find the minimum No. of connected component of a simple graph with 16 vertices and 10 edges.

8. (i) Draw a simple graph with five vertices having degree 4, 3, 3, 3, 3

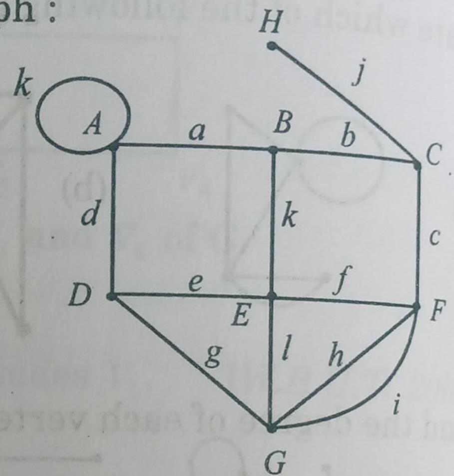
9. Find three subgraphs of the following graph



10. Find three subgraphs of the following graph



11. Consider the following graph :



Find (i) the subgraph removing the vertices D, G, B and F

(ii) the subgraph removing the edges g, e, k, i, f and j.

(iii) the subgraph *induced* by the vertices A, G, E, and F (i.e. the subgraph having vertices A, G, E, F and those edges of the given graph that have both ends as the given vertices)

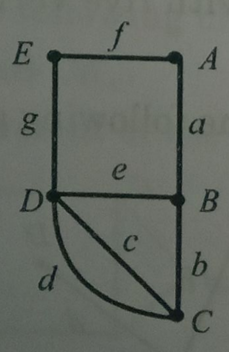
(iv) the subgraph *induced* by the edges d, g, i and c

(v) the subgraph which is complete with three vertices

(vi) the isomorphic graph (if exists) to the subgraph of G which is complete with four vertices.

(vii) a simple subgraph ; all possible simple subgraphs.

12. From the graph :



You are given some walks. Determine which are (a) path (b) trail

(c) a closed walk (d) a circuit

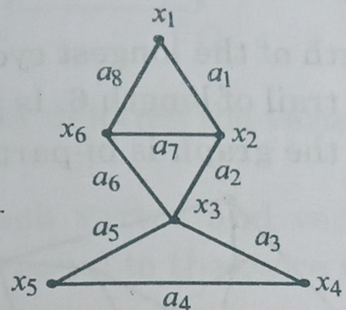
(i) A, a, B

(ii) B, b, C, c, D, d, C

(iii) $B, b, C, c, D, d, C, b, B$

(iv) $D, d, C, c, D, e, B, a, A, f, E, g, D$

13.(a) From the graph :



You are given some walks. Determine which are a path, a trail, a closed walk, a circuit ;

(i) $x_1, a_1, x_2, a_2, x_3, a_5, x_5 a_4, x_4$

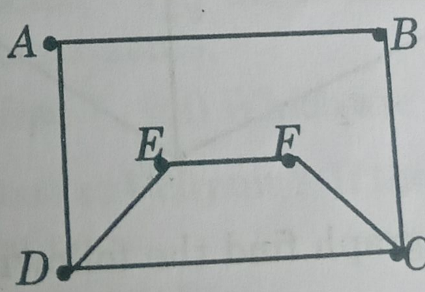
(ii) $x_2, a_7, x_6, a_6, x_3, a_3, x_4 a_4, x_5, a_5, x_3$

(iii) $x_6, a_7, x_2, a_2, x_3, a_3, x_4 a_4, x_5, a_5, x_3, a_6, x_6$

(iv) $x_1, a_1, x_2, a_2, x_3, a_6, x_6$

(v) $x_2, a_2, x_3, a_6, x_6, a_8, x_1 a_1, x_2$

(b) Find all the cyles in the following graph. Hence find whether it is a bipartite graph :



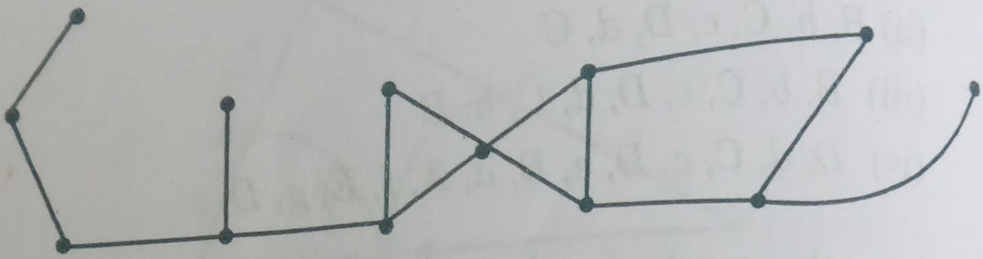
[Hint : Since the graph contains no cycle of odd length so it is a bipartite graph]

14. From the following graph

(a) find a closed walk of length 6. Is this walk a trail ?

(b) find an open walk of length 12. Is it a path ?

(c) Find the length of the longest path in the graph. Find if possible, another path in the graph having the same length.

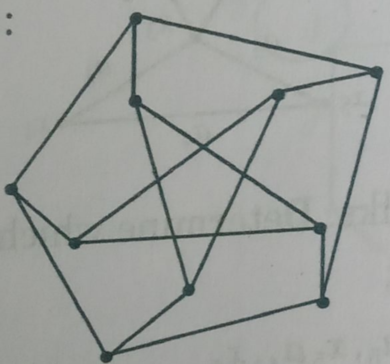


(d) Find the length of the longest cycle (circuit) in the graph.

(e) Find a closed trail of length 6. Is your trail a circuit ?

(f) Find whether the graph is bi-partite.

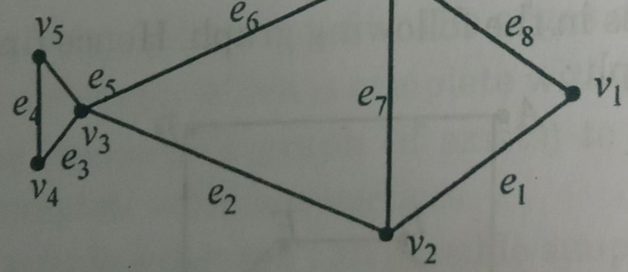
15. From the graph :



find (i) a path of length 9 (ii) a trail of length 5

(iii) a circuit of length 8 (iv) a cycle of length 6.

16. Find three cycles from v_3 to v_3 in the following graph. Hence find whether it is a bipartite graph



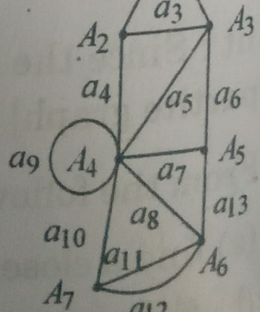
17. In the following graph find the following :

(a) a walk of length 5 from A_3 to A_4.

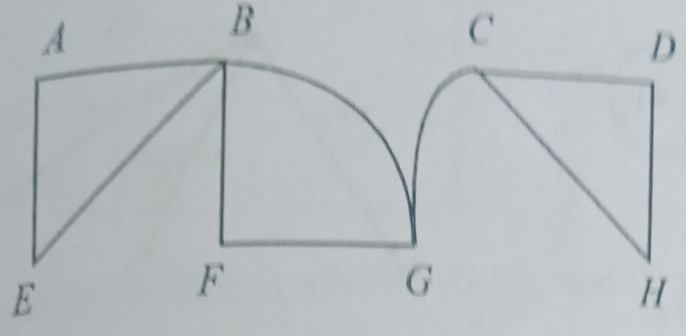
(b) a trail of length 9 from A_1 to A_3.

(c) a cycle of length 7 from A_5 to A_5.

(d) Whether the graph is bi-partite graph.



18. From the following graph, find



(a) the subgraph after deleting the vertex F, G and the edge (A, B)

(b) the degree of each vertex and verify that the sum of degrees of all vertices is equal to the twice the number of edges.

(c) all simple paths from A to G

(d) all trails from B to C

(e) the distance between A and C

(f) the diameter of the graph

(g) all cycles

(h) Find the subgraph spanned by

(i) $\{B, C, D, E, F\}$ (ii) $\{A, C, E, G, H\}$

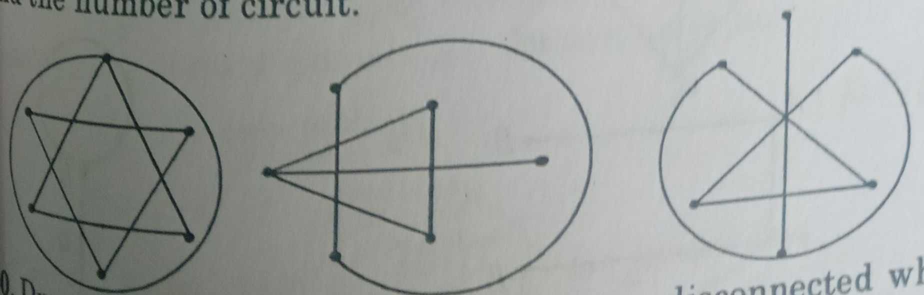
(iii) $\{B, D, E, H\}$ (iv) $\{C, F, G, H\}$.

19. Among the following three graphs find

(i) which are connected ? For dis-connected graph find the number of connected components.

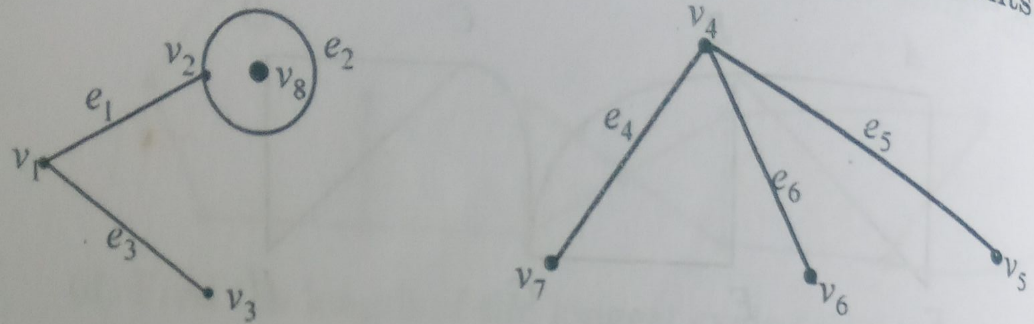
(ii) Which are loop-free ? (iii) Which are simple graphs ?

(iv) Which are without any circuit ? If they have circuit, then find the number of circuit.

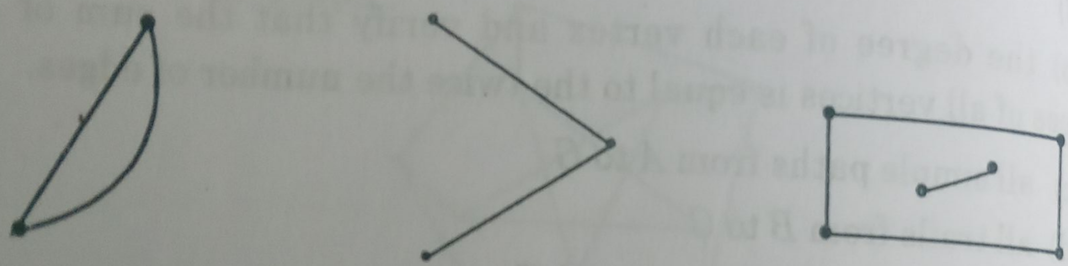


20. Draw a connected graph that becomes disconnected when any edge is removed from it.

21. **(a)** Below you are given a graph. Find its components :

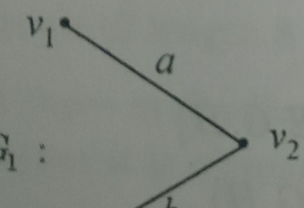


(b) Below you are given a graph. Find its components :

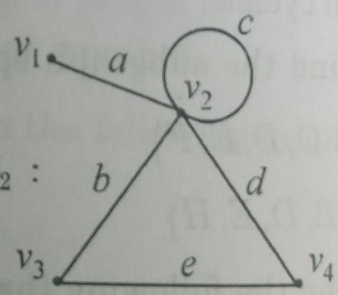


22. Find the union $G_1 \cup G_2$ when

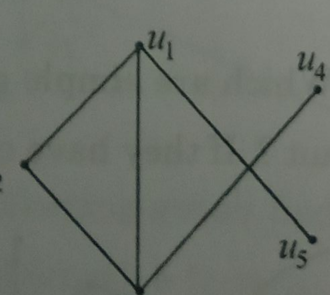
(i) G_1 :



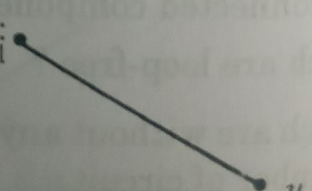
G_2 :



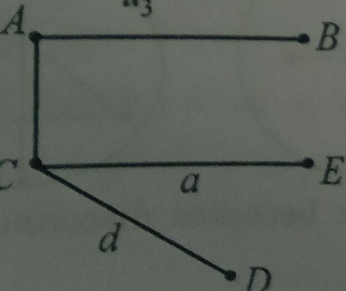
(ii) G_1 :



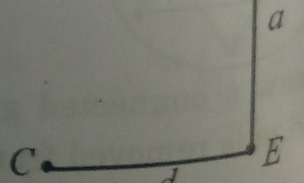
G_2



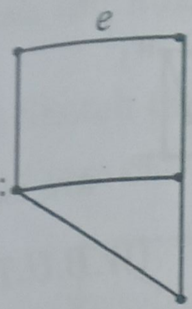
(iii) G_1 :

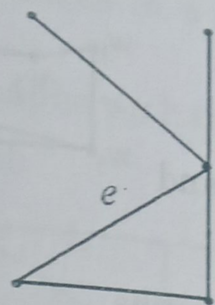


G_2 :

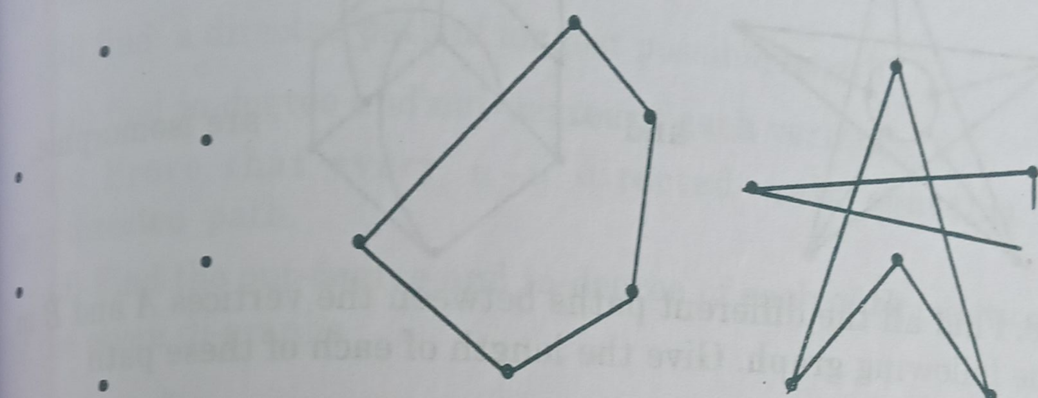


(iv) G_1 :





23. You are given the following graphs : G_1, G_2 and G_3 as



(i) Find the complement of G_1, G_2, G_3.

(ii) Find which of the graphs are *self complementary* (i.e. isomorphic to its own complement)

24. (a) Find the maximum number of vertices in a connected graph having 17 edges $[n-1 \leq 17 \Rightarrow n \leq 18]$.

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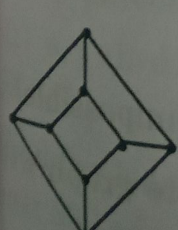
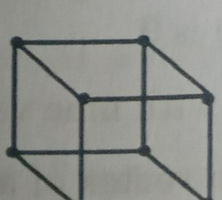
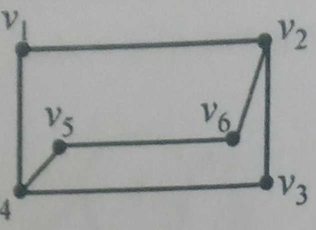
(b) Find the minimum number of edges in a connected graph having 21 vertices [see a previous theorem]

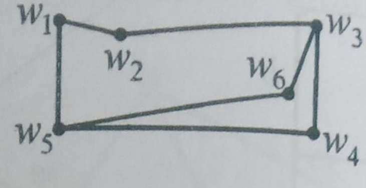
(c) Let G be a graph with 4 connected components and 24 edges. Find the maximum number of vertices G may have.

25. Draw two 3-regular graphs with eight vertices. Find whether these two are same or distinct (i.e. isomorphic or not)

26. Prove that the two graphs

(i)  and  are isomorphic.

(ii) 

and 

are isomorhic.

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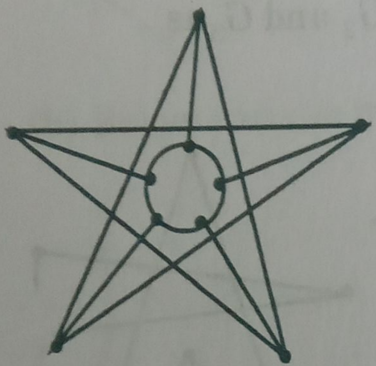
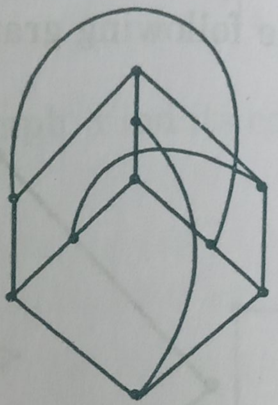
27. Prove that the two graphs

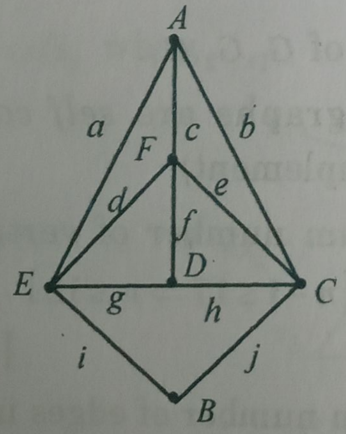


and



are isomorphic.

28. Find all the different paths between the vertices A and B in the following graph. Give the length of each of these path :



29. Let v_1, v_2 and v_3 be three distinct vertices of a graph G. Prove that there exists a path from v_1 to v_3 if there exists a path from v_1 to v_2 and from v_2 to v_3.

30. What about the degree of a simple graph in which there is no pair of adjacent edges.

31. Draw a graph in which the length of the shortest cycle is 4 and that of the longest cycle is 9.

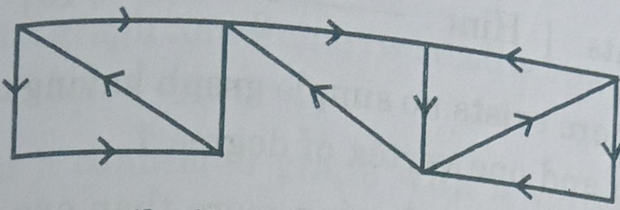
32. Draw a 3-regular graph with nine vertices. [impossible]

33. Find the degree of each vertex of a complete graph with n vertices.

34. (A) Find the number of edges in a complete graph with n vertices.

(B) (a) Sketch three digraphs with three vertices

(b) From the di-graph



(i) Find a directed trail of length 10

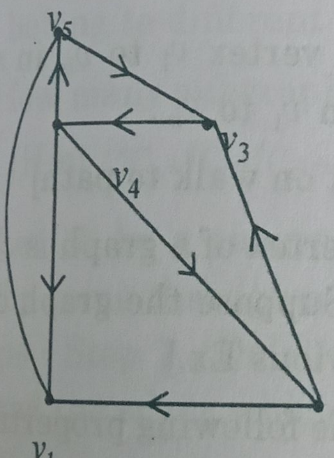
(ii) find a directed path of longest possible length

(iii) find in-degree and out-degree of each vertex.

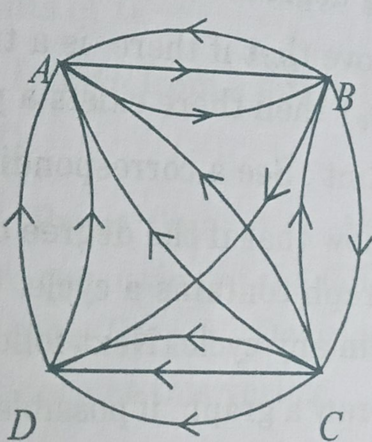
(c) Prove that every $u-v$ directed walk contains a $u-v$ directed path.

(d) Find the out-degree and in-degree of each of the vertices of the given di-graphs :

(i)



(ii)



(e) The vertex set and edge-set of a digraph, G are

$V=\{A,B,C,D,E,F,G\}$ and

$E=\{(A,A),(B,E),(A,E),(E,B),(G,C),(A,E),(D,F),(D,B),(G,G)\}$.

Find

(i) any paralles or loops

(ii) a sub-digraph of G determined by the vertex set $\{A,B,C,D\}$

35. Find the least number of vertices of a complete graph having at least 500 edges.

36. (a) Find the number of edges in a complete graph with 5 vertices.

(b) Find the max number of edges of a graph with 7 vertices and 3 components. [Hint : $\frac{(7-3)\times(7-3+1)}{2}=10$]

37. Show that there exists no simple graph having three vertices of degree 3 each and one vertex of degree 1.

38. If G be a simple graph having more than one vertices then prove that G has two vertices whose degree are equal. [OR, Prove that a simple graph with at least two vertices has at least two vertices of same degree]

39. Prove that the complement of a disconnected graph is connected.

40. Find the number of edges of a graph with seven vertices having degrees 4, 1, 2, 2, 3, 5, 5.

41. Prove that if there is a trail form vertex v_1 to v_k in a graph $(v_1 \neq v_k)$ then there exists a path from v_1 to v_k.

[Hint : See a corresponding result on walk to path]

42. Show that if the degree of each vertex of a graph is ≥ 2 then the graph contains a cycle. [Hint : Suppose the graph does not contain any cycle. Next follow a previous Ex.]

43. Draw a graph, if possible, with the following properties [Give explanation where the graph is not possible]

(i) Five vertices each of degree 4.

(ii) Five vertices each of degree 2

(iii) Simple graph, has 7 edges, 9 vertices each of degree at least 1.

(iv) Simple graph, six vertices having degrees 5, 5, 4, 2, 2, 2

(v) Simple graph, six vertices having degree 5, 5, 4, 2, 1, 1

(vi) Six vertices, four edges.

(vii) Five vertices having degree 1, 2, 2, 4, 5

(viii) Four edges, four vertices with degree- sequence 1, 2, 3, 4

44. Find the number of vertices in a graph with 20 edges if each vertex is of degree 4.

45. Show that there do not exist any graph with 10 vertices and 22 edges.

46. Prove that a graph containing two distinct paths from a vertex A to a vertex B has a cycle.

47. Prove that a connected graph with n vertices has at least $n-1$ edges.

48. Two islands, C and D, formed by a river are connected to each other and to the banks A and B with seven bridges. Represent this by means of a graph (Konigsberg Bridge problem)

49. Let G be a connected graph. If e is an edge whose two vertices are v_1 and v_2 and $G-e$ (i.e. the subgraph obtained from G by deleting the edge e from G) is disconnected. Then prove that v_1 and v_2 belong to different components of $G-e$

50. (a) How many subgraphs can be formed from a graph having e sides. [Hint : ${}^eC_0+{}^eC_1+{}^eC_2+\cdots+{}^eC_e=2^e$]

(b) Let G' be a subgraph of G. Prove that G' is obtained from G by going through a sequencce consisting of the following two steps : Step 1 : Deletion of an edge. Step 2 : Deletion of a vertex and the edges which are incidence at this vertex.

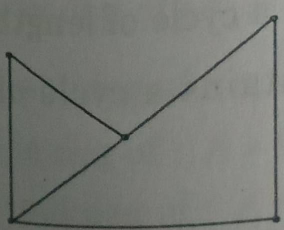
51. Find the number of vertices in a graph with 15 degrees if each vertex has degree 3.

52. Let v be a vertex in a graph and T be a closed trail from v to v. Prove that T contains a cycle from v to v.

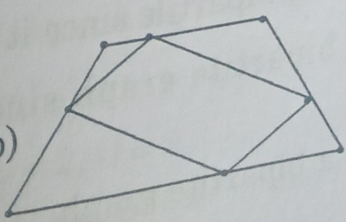
[Hint : This is modification of a theorem]

53. Find out which one of the following two graphs is Euler graph.

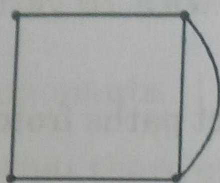
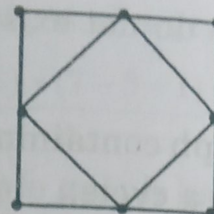
(a)



(b)



54. Determine whether the following figures are Euler cycle or not

(i)  (ii) 

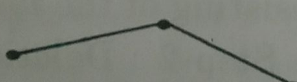
55. P.T a connected graph with n vertices and e edges contain a unique cycle if $n = e$.

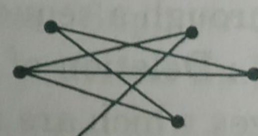
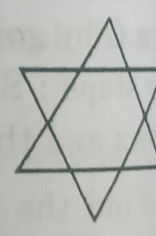
56. Show that if a connected graph is decompossed into two subgraphs then there exists at least one vertex common between the two subgraphs.

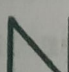
[Hint : $G = G_1 \cup G_2$. Let $u \in G_1$, $u \in G_2$. P is a path connecting u and v.

Let $p = \{u, e_1, u_1, e_2, u_2, \cdots, u_{k-1}, e_k, u_k \cdots, u_{n-1}, e_n, v\}$ be the path where $u, u_1, u_2, \cdots, u_{k-1} \in G_1$ and $u_k, u_{k+1}, \cdots, u_{n-1}, v \in G_2$. Proved $u_{k+1} \equiv u_k$.]

Answers

2. (i) 

4. (a)  (b) 

(c) 

5. only (b) 6. (i) 1, 1, 3, 3 (ii) 1, 3, 4, 1, 1 (iii) 1, 3, 3, 2, 1

7. (iii) $16 - k \leq 10$ i.e., $k \geq 6$ $\therefore$ min No. of comp $= 6$.

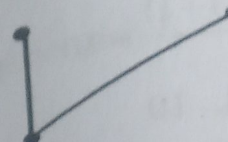
12. (i) path, (ii) trail, not path,

(iii) clsoed walk not circuit (iv) circuit

14. (f) It is not bipartile since it contains a cycle of length 3 (odd)

16. Not a bipartite graph since it contains a cycle of length 3 (odd)

17. (d) not a bipartite graph.

18. (a) 

(b) 2, 4, 3, 2, 2, 2,

(b) 2, 4, 3, 2, 2, 2,

(d) $BGC, BFGC, B$

(f) 4 (g) AB

(ii) $\{AE, FG, GC\}$

19. (i) 3rd is conn components

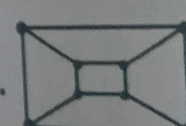
(ii) 2nd and 3rd

20. 

21. (a) There are 3

(b) There are 4 c

24. (a) 18

25.  and

These are not is 5 and the forme

30. at most 1. 32.

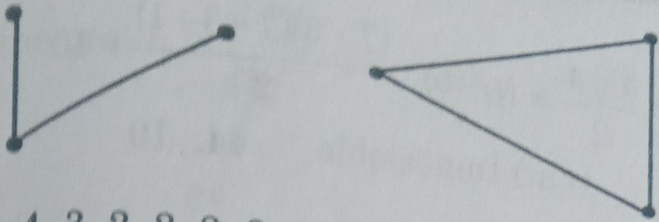
34. A. ${}^nC_2 = \frac{n(n-}{2}$ simple

(d) (i) in deg

out de

(ii) in deg

out de

18. (a) 

(b) 2, 4, 3, 2, 2, 2, 3, 2 (c) $ABG, ABFG, AEBG, AEBFG$

(b) 2, 4, 3, 2, 2, 2, 3, 2 (c) $ABG, ABFG, AEBG, AEBFG$

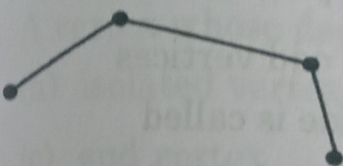
(d) $BGC, BFGC, BAEBGC, BAEBFGC$ (e) 3

(f) 4 (g) ABEA, BFGB, CDHC (h) (i) $\{BE, BF, CD\}$

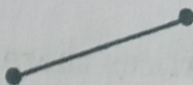
(ii) $\{AE, FG, GC\}$ (iii) $\{BE, DH\}$ (iv) $\{FG, GC, CH\}$

19. (i) 3rd is connected, 1st and 2nd have two connected components

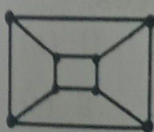
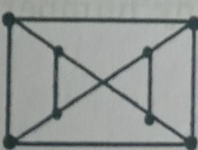
(ii) 2nd and 3rd (iii) 3rd (iv) none.

20. 

21. (a) There are 3 components . Note that $\{V_8\}$ is a compnent.

(b) There are 4 comp. Note that  is also a comp.

24. (a) 18 (b) $21-1=20$.

25.  and .

These are not isomorphic since the latter has a cycle of length 5 and the former has no such.

30. at most 1. 32. impossible 33. $n-1$

34. A. ${}^{n}C_2 = \dfrac{n(n-1)}{2}$ remembering that a complete graph is simple

(d) (i) in deg of : $v_1 = 2,\ v_2 = 2,\ v_3 = 2,\ v_4 = 2,\ v_5 = 0$

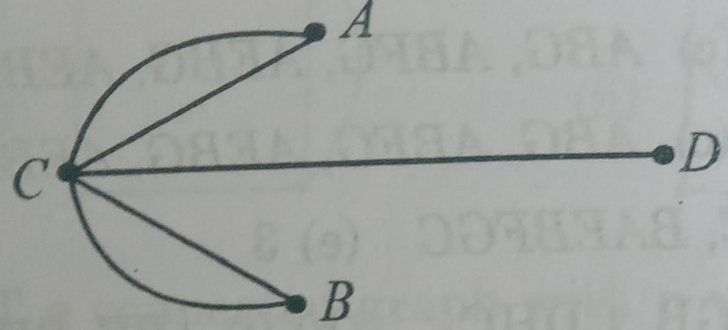
out deg of : $v_1 = 1,\ v_2 = 2,\ v_3 = 1,\ v_4 = 1,\ v_5 = 3$

(ii) in deg of : $A = 6, B = 1, C = 2, D = 2$

out deg of : $A = 1, B = 5, C = 5, D = 2$

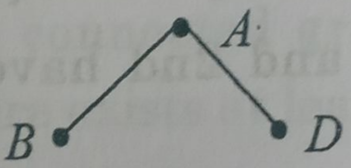
35. 33 36. (a) $\frac{5 \times 4}{2} = 10$ (b) $\frac{(7-3)(7-3+1)}{2} = 10$ 40.

43. (vi) possible (viii) Impossible 44. 10

48. 

51. 10

[III] Multiple Choice Questions

1. In the graph 

the vertices A, B are

(a) end vertices of an edge (b) pendant vertices