Module-2

2.1 COUNTING TECHNIQUES

2.1.1. Introduction

In this chapter we are mainly concerned with determining the number of logical possibilities of some event without necessarily identifying every case.

For example, suppose we are to create password which is six to eight characters long, where each character is an English letter (upper or lower case) or a special character from the set {#, @, &}. Here we are concerned with the number of such possible passwords without identifying every password. Finding this type of number is counting. There are two basic counting principles used throughout.

2.1.2. Sum Rule Principle

Let some event A can occur in n_1 ways and the event B can occur in n_2 ways; A and B cannot occur simultaneously. Then 'A or B' can occur in n_1+n_2 ways.

More, generally let an event A_1 can occur in n_1 ways, A_2 can occur in n_2 ways, A_3 can occur in n_3 ways...and so on. Suppose no two of the events can occur at the same time. Then one of the events A_1, A_2, A_3 ...can occur in $n_1+n_2+n_3+\cdots$ ways.

Illustrations

(i) Suppose there are 8 characters each of English letters and 5 special symbols. A person can choose a character in $8+5=13$ ways.

(ii) Suppose $A=$ "Choosing a prime number between 10 and 20", $B=$ "Choosing an even number between 10 and 20". A can occur in 4 ways $\{11,13,17,19\}$ and B can occur in 4 ways $\{12,14,16,18\}$. Note that A and B can not occur simulaneously. So 'A or B' can occur in $4+4=8$ ways.

(iii) Suppose A = "Choosing a prime number less than 10" and B = "Choosing an even number less than 10". Then A can occur in four ways $\{2, 3, 5, 7\}$ and B can occur in 4 ways $\{2, 4, 6, 8\}$. Here A and B may occur simultaneously because $\{2, 3, 5, 7\} \cap \{2, 4, 6, 8\} = \{2\}$ containing 1 element. In these cases A or B can occur in $4+4-1=7$ ways.

2.1.3. Product Rule Principle

Let an event A can occur in m ways and, independent of A, the event B can occur in n ways. Then combinations of A and B can occur in mn ways.

In general let an event A_1 can occur in n_1 ways, independent of A_1 let A_2 can occur in n_2 ways, A_3 can occur in n_3 ways,...Then all the event can occur simultaneously in $n_1 \cdot n_2 \cdot n_3 \cdots$ ways.

Illustrations

(i) Suppose a code contains two letters followed by three digits with the first digit non-zero. Each letter can be choosen in 26 different ways, the first digit in 9 ways and each of the other two digits in 10 ways. Hence total number of possible codes $= 26 \times 26 \times 9 \times 10 \times 10 = 608400$.

(ii) Let an organisation contains 26 members. The organisation is to elect a president, a vice-president and a secretary. The president can be elected in 26 different ways; following this, the vice president can be elected in 25 ways (since the person choosen president is not eligible to be vice president); and following this, the secretary can be elected in 24 different ways. Then by this principle of counting, there are $26 \times 25 \times 24 = 15600$, different ways in which the organisation can elect the office bearers.

2.1.4. Expression of the above principle by set

If $n(A)$ is the cardinal number of a set that is if $n(A)$ is the number of elements in a set A then

(1) Sum Rule Principle: $n(A \cup B) = n(A) + n(B)$ if A and B are disjoint sets. i.e. if $A \cap B = \phi$

In general, let $S = A_1 \cup A_2 \cup .... \cup A_n$ be a non-empty set where each $A_i (i = 1 \text{ to } n)$ is a non-empty set such that $A_i \cap A_j = \phi$, $i, j = 1$ to n and $i \neq j$

Then $n(S) = n(A_1) + n(A_2) + ... + n(A_n) = \sum_{i=1}^{n} n(A_i)$

(2) Product Rule Principle: $n(A \times B) = n(A) \cdot n(B)$ where $A \times B$ is the cartesian product of the two sets A and B.

In general, let $A_1, A_2, ... A_n$ be n non-empty sets. Then

$$n(A_1 \times A_2 \times \cdots \times A_n) = n(A_1) . n(A_2) \cdots n(A_n) = \prod_{i=1}^{n} n(A_i)$$

where $A_1 \times A_2 \times \cdots \times A_n$ is the cartesian product of the sets $A_1, A_n ... A_n$,

Illustrations.

Let us find the number of two digit or three digit numbers which can be formed using the digits 1,2,3,4,5,6 (repetition is not allowed).

Number of ways in which two-digit numbers can be formed are $6 \times 5 = 20$, sine ten's place can be filled in 6 ways and unit;s place can be filled in 5 ways, as repitition of digits is not allowed.

Similarly, the number of ways in which 3 digit number can be formed using the given six digits are

$$6 \times 5 \times 4 = 120.$$

Hence, by sum rule, the number of two digit or three digit number can be formed

$$= 30 + 120 = 150 \text{ ways}.$$

2.1.5. Permutations

Any arrangements of a set of n objects in a given order is called a permutation of the objects.

Arrangement of any $r (\leq n)$ number of these objects in a given order is called a 'permutation of the n objects taken r at a time'.

Illustrations

(i) Consider the set of letters x, y, z and w. Then

(a) xyzw, yxzw, wzxy and wyxz are some permutations of the four letters taken four at a time.

(b) xyz, yzw, wzx, are some of the permutation of the four lettes taken three at a time.

(c) xy, yz, wy, zy, are some of the permutation of the four letters taken two at a time.

The number of permutation of n number of objects taken r at a time is denoted by ^nP_r .

(ii) Suppose permutations are being done taking three from the 6 different object A, B, C, D, E and F. To find the number of such permutations we consider the following three boxes :

$\boxed{}\qquad\boxed{}\qquad\boxed{}$

Now first letter can be choosen in 6 different ways; following this, the second letter can be choosen in 5 different ways; and, following this, the last letter can be choosen in 4 different ways. We write each number in its appropriate box as follow :

$\boxed{6}\qquad\boxed{5}\qquad\boxed{4}$

This by principle of counting (Product Rule Principle) there are $6 \times 5 \times 4 = 120$ possible permutation of six objects taking 3 at a time. Thus $^6P_3 = 120$.

Theorem. $^nP_r = \dfrac{n!}{(n-r)!}$

Proof. (This proof follows the procedure in the preceeding example).

When r objects are being chosen from n objects the first element can be chosen in n different ways; following this, the second element in the permutations can be chosen in $n-1$ ways; and, following this, the third element in the permutation can be chosen in $n-2$ ways.

Continuing this process, we have that the rth (last) element in the permutation can be chosen in $n-(r-1) = n-r+1$ ways. Thus, by principle of counting (Product Rule Principle) we have

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

Corollary : There are $n!$ number of permutation of n objects taken all at a time.

Proof : Number of this permutation $= {}^nP_n = \dfrac{n!}{0!} = n!$

Illustrations

(i) Number of different 3-digit number formed from the digits 2, 5, 6, 8, 9.

= Number of permutation of 5 digits taking 3 at a time

$$= {}^5P_3 = \frac{5!}{2!} = \frac{5 \times 3 \times 2!}{2!} = 15$$

(ii) Number of possible ways of sitting of 6 students in a row

= number of permutation of 6 students taken 6 at a time

$= {}^6P_6 = 6! = 720$

Permutation with Repetitions

In this Article we are going to find the number of permutation of a set of objects some of which are alike.

We use the notation $P(n; n_1, n_2, \cdots n_r)$ to denote the number of permutations of n objects of which n_1 are alike, n_2 are alike,... n_r are alike. In the following theorem we find the value of this permutation.

Theorem. $P(n; n_1, n_2, \cdots n_r) = \dfrac{n!}{n_1! n_2! \cdots n_r!}$

Proof. Beyond the scope of the book.

Illustration : Number of 10–letter words formed using the letters of the word STATISTICS $= \dfrac{10!}{3!3!2!}$ because here S repeats 3 times, T repeats 3 times and I repeats 2 times.

2.1.6. Combinations

Let we have n number of objects. A combination of these n objects taken r at a time is any selection of r number of objects where order does not count.

Illustrations

(i) The combination of the letters x, y, z, w taken three at a time are

xyz, xyw, xzw, yzw,

observe that the following combinations are same :

xyz, yxz, zxy, xzy and yzx

The number of combinations of n objects taken r at a time is denoted by ^nC_r

Before we go to the following theorem we consider an example :

Example : *Find the number of combination of four objects, x, y, z, w taken three at a time.*

Each combination consisting of 3 objects determines $^3P_3 = 3! = 6$ permutation of the objects in the combination as shown in the following table : (shown in the next page)

Thus the number of combinations multiplied by 3! equals the number of permutations; that is

$$^4C_3 \times 3! = {^4P_3}$$

Combination	Permutations
xyz	xyz, yxz, yzx, zyx, zxy, xzy
xyw	xyw, xwy, ywx, wyx, yxw, wxy
xzw	xzw, zxw, zwx, xwz, wxz, wzx
yzw	yzw, zyw, zwy, wyz, wzy, ywz

Theorem : $^nC_r = \dfrac{n!}{r!(n-r)!}$

Proof. Since any combination of n number of objects taken r at a time determines r! number of permutations of the objects in the combination, we can say

$$^nP_r = r!\,{^nC_r}$$

or, $$^nC_r = \frac{^nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!r!}$$

Illustrations.

(i) Let us find the number of committees of four that can be formed from nine people.

Each committee is, essentially, a combination of the nine people taken four at a time. Thus the number committee that can be formed is

$$^9C_4 = \frac{9!}{5!4!} = \frac{9\times8\times7\times6\times5!}{5!\times4\times3\times2} = 126$$

(ii) A man buys 3 cows, 2 pigs and 4 hens from another person who has 7 cows, 6 pigs and 8 hens. Let us find the number of choices that the man can do.

The farmer can choose cows in $^7C_3$ ways, the pigs in $^6C_2$ ways and the hens in $^8C_4$ ways.

Hence altogether he can choose the animals in

$= {^7C_3} \times {^6C_2} \times {^8C_4}$ ways

(Here product Rule Principle is applicable)

$$= \frac{7!}{4!3!} \times \frac{6!}{4!2!} \times \frac{8!}{4!4!} = 36750 \text{ ways}$$

Combination with Repetitions

Consider the following problem : Let we have the letters a, b, c and d. We are going to find the number of combinations of these letters, with repetitions, taking 8 at a time. Four such combinations are :

$$r_1 = aabbccdd,\ r_2 = aaacddd,\ r_3 = bbbbcddd, \text{ and } r_4 = aaaaaddd.$$

Counting the number of such combinations is not easy. Suppose we want to code the above combinations using only two symbols, say 0 and 1. This can be done by letting 0 deno : a letter (among a, b, c, d) and letting 1 denote a change from one kind of letters to its succuding letter (e.g b to c or d to a etc). Then each combination will require $r=8$ zeros and $4-1=3$ ones $(1s')$, where the first 1 denotes the change from a to b, the second 1 from b to c, and the third 1 from c to d. Thus the above four combinations will be coded as follow :

$$r_1 = 00100100100; \quad r_2 = 0001101000;$$

$$r_3 = 10000101000; \quad r_4 = 00000111000.$$

Each of these type of code words contains $8+4-1=11$ digits. Accordingly required number of these combination = Number of selection of 3 rooms from 11 rooms $= {}^{11}C_3 = 165$.

This problem leads us to get the following theorem.

Theorem. Suppose there are n kinds of objects. Then the number of combinations of r such objects (where any object may repeat any time) $= {}^{n+r-1}C_r$.

Proof. Beyond the scope of the book.

Illustration: *Let us find the number of non-negative integral solution of the equation $x+y+z=18$.*

We see $x=3$, $y=7$ and $z=8$ is a solution of the equation. We can view this solution as a combination of 18 objects consisting of three 'a', seven 'b' and eight 'c'.

Similarly we can view the solution $x=5$, $y=2$, $z=11$ as a combination of 18 objects consisting of five 'a', two 'b' and eleven 'c'.

Thus required number of integral solution

= number of combination of 18 objects (where any object may repeat any time) from three kinds of objects (say a, b and c) $= {}^{3+18-1}C_{18} = {}^{20}C_{18} = 190$

2.1.7. Binomial Co-efficients

The symbols ${}^nC_r$ is defined as

$${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1.2.3\cdots(r-1)r}$$

We have some important relation :

(1) ${}^nC_r = {}^nC_{n-r}$

(2) ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$

Illustrations.

(i) ${}^7C_2 = \frac{7!}{2!5!} = \frac{7\times 6}{2} = 2!$

(ii) ${}^{12}C_5 = \frac{12!}{7!5!} = \frac{12.11.10.9.8}{5.4.3.2} = 792$

Note that ${}^nC_r$ has exactly r number of factors in both the numerator and the denominator.

Binomial co-efficients

The number ${}^nC_r$ are called binomial co-efficients since they appear as the co-efficients in the expansion of $(A+B)^n$. Specifically, it can be proved that

$$(A+B)^n = A^n + {}^nC_1 A^{n-1}B + {}^nC_2 A^{n-2}B + \cdots + {}^nC_{n-1} AB^{n-1} + B^n$$

$$= \sum_{r=0}^{n} {}^nC_r A^{n-r} B^r$$

Theorem. ${}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n = 2^n - 1$

Proof. From the expansion of $(1+1)^n$ the result follows.

Selection of one on more object.

The number of ways of selecting one on more items, for a group of n distinct item is

$$2^n - 1.$$

Selection of one or more objects.

The number of ways of selecting one or more items, for a group of n distinct item is

$$= 2^n - 1$$

Illustration.

The number of ways one student can solve one or more question out of 6 question is

$${}^6C_1 + {}^6C_2 + \ldots + {}^6C_6$$

$$= 2^6 - 1 \qquad = 63$$

Circular permutations.

The permutations discussed so far are known as linear permutations as the objects were assumed to be arranged in a line. When the objects are arranged in a circle or in a closed curve, we get circular permutation.

In a circular permutation, we consider one object is fixed and the remaining objects are arranged in the same way as arrange are made in linear permutation. We have two types of circular permutations:

(A) Those in which anticlockwise and clockwise are distinguishable. In this case the number of different circular arrangements of n objects is $(n-1)!$

(B) On the otherhand when anticlockwise and clockwise arrangment are not distinguishable, then the number of different circular arrangments of n objects is $\frac{1}{2}(n-1)!$

Illustrations.

(i) 8 people can be arranged about a circular table in $(8-1)!$ i.e $7!$ ways.

(ii) 8 differfent beads can be arranged to form a necklace in $\frac{1}{2}(8-1)!$

Illustrative Examples:

Ex.1. How many seven letter words can be formed using the letters of the word BENZENE?

Solution. Required number of words = number of Permutation of the letters in BENZENE

$$= \frac{7!}{3!2!} \text{ (Since E repeats 3 times, N repeats 2 times)}$$

$$= 420$$

Ex.2. Find the number of ways that a party of seven persons can arrange themselves

(i) in a row of seven chairs

(ii) around a circular table

Solution. (i) The seven persons can be arranged in seven chair in ${}^7P_7 = 7! = 5040$ ways.

(ii) One person can sit at any place in the circular table. The other six persons can then sit in ${}^6P_6 = 6! = 46656$ ways.

Ex.3. Find the number of distinct permuations that can be formed from all the letters of the word (i) RADAR (ii) UNUSUAL.

Solution. (i) Among the 5 letters, R repeats 2 times, A repeats 2 times.

Therefore total number ofpermutation $= \frac{5!}{2!2!} = 30$

(ii) Among the 7 letters U repeats 3 times.

$\therefore$ total number of permutation $= \frac{7!}{3!} = 840$.

Ex.4. In how many ways can 4 physics books, 3 mathematics book, 3 chemistry books and 2 biology books be arranged on a shelf so that all books of the same subject are together?

Solution. 'Physics', 'Mathematics', 'Chemistry' and 'Biology' books can be arranged among themselves in ${}^4P_4 = 4! = 24$ ways.

Now 'Physics' contains 4 books which can be arranged among themselves in 4! = 24 ways. Similarly the 3 mathematics books can be arranged among themselves in 3! = 6 ways. 3 Chemistry and 2 biology books can be arranged among themselves in 3! = 6 and 2! =2 ways respectively. So by Product Rule Principle the required number of arrangement

$$= 24 \times 24 \times 6 \times 6 \times 2 = 41472$$

Ex.5. 6 boys and 6 girls are to be seated in a row. How many ways can they be seated if

(i) all boys are to be seated together and all girls are to be seated together.

(ii) no two girls should be seated together.

(iii) the boys occupy extreme positions.

Solution. (i) All possible ways girls can be seated $= {}^6P_6 = 6!$.

All possible arrangement among boys = 6!.

Now 'boys' and 'girls' can be arranged among themselves in 2! = 2 ways.

$\therefore$ the required number of arrangements

$$= 6! \times 6! \times 2 = 1036800$$

(ii) The boys can be seated among themselves in

${}^6P_6 = 6!$ ways.

There are 7 in between positions among boys (including the positions at both end).

Now if the 6 girls be seated among these positions then no two girls will be together.

This can be done in ${}^7P_6$ ways.

$\therefore$ required number of arrangements

$$= 6! \times {}^7P_6 = 6! \times \frac{7!}{1!} = 720 \times 5040$$

(iii) Two boys may be seated in two extreme positions in ${}^6P_2$ ways. The remaining ten people (4 boys and 6 girls) will be seated in the remaining 10 positions in ${}^{10}P_{10} = 10!$ ways.

$\therefore$ by Product Rule Principle required number of sitting $= {}^6P_2 \times 10! = 30 \times 10!$

Ex.6. If all the permutations of the letters of the word DIRECTOR be written down as in a dictionary what is the rank of the word?

Solution. When the permutations of the letters of DIRECTOR are written down as in a dictionary we have the following sequence of words :

C * * * * * * * (1)
D C * * * * * * (2)
D E * * * * * * (3)
D I C * * * * * (4)
D I E * * * * * (5)
D I O * * * * * (6)
D I R C * * * * (7)
D I R E C O * * (8)
D I R E C R * * (9)
D I R E C T O R (10)

Number of words in (1) is $\frac{7!}{2!} = 2520$

Number of words in (2) is $\frac{6!}{2!} = 360$

Number of words in (3) is $\frac{6!}{2!}=360$

Number of words in (4) is $\frac{5!}{2!}=60$

Number of words in (5) is $\frac{5!}{2!}=60$

Number of words in (6) is $\frac{5!}{2!}=60$

Number of words in (7) is $4! = 24$

Number of words in (8) is $2! = 2$

Number of words in (9) is $2! = 2$

So number of permuted words written down before DIRECTOR is written

$=2520+360+360+60+60+60+24+2+2$

$=3448$

$\therefore$ The word DIRECTOR will be written at 3449 th position.

Ex. 7. How many ways can the letters in the word MISSISSIPPI can be arranged? If P's are to be sepeated then find the number of arrangements.

Solution. Total ways of arrangement of all the letter

$$=\frac{11!}{4!4!2!}=34650$$

Keeping P together two number of arrangements

$$=\frac{10!}{4!4!}=6300$$

$\therefore$ Number of arrangement where the Ps' are separated

$=34650-6300=28350$

Ex. 8. How many different signals, each consisting of 8 flags hung in a vertical line, can be formed from a set of 4 identical red flags, 3 identical white flags and a blue flag?

Solution. Required number of different signals

= No of permutation of the 8 flags

$=\frac{8!}{4!3!}$ ($\because$ red flags repeats 4 times and white flags repeats 3 times.)

$=280$

Ex.9. If repetitions are not permitted, how many four digit numbers can be formed from the digits 1, 2, 3, 7, 8 and 5 that contain both the digits 3 and 5.

Solution. After inclusion on 3 and 5 other two digits can be selected from 1, 2, 7, 8 in ${}^4C_2$ ways. After selecting the two digits all the four digits can be arranged among themselves in ${}^4P_4=!$ ways.

$\therefore$ total number of required numbers $={}^4C_4\times 4!=6\times 24=144$

Ex.10. Find the number of different words of six letters taken from 10 different letters such that in each of which at least one letter is repeated.

Solution. Number of words of six different letters taken from 10 different letters $={}^{10}P_6$.

Number of words of six letters where the letters may repeat any number of times $=10\times10\times10\times10\times10\times10=10^6$.

$\therefore$ Required number of words $=10^6-{}^{10}P_6=848800$.

Ex.11. 15 different balls are to be placed in 15 different holes one in each hole. If 6 holes are too small for 8 of the balls, how many ways the placement of all the balls in the hole is possible?

Solution. 8 balls can not be placed in the 6 small holes. From the remaining 7 balls these 6 holes are to be filled in which can be done in ${}^7P_6$ ways. Alter filling in the 6 holes we have 1 + 8 = 9 balls which are to be placed in 9 holes (big) which can be done in ${}^9P_9=9!$ ways.

$\therefore$ the required number of ways of placement

$={}^7P_6\times 9!=1828915200$

Ex.12. There are 4 buses plying between two places A and B; and 3 buses plying between two places B and C. In how many ways can a man travel (i) by bus from A to C via B (ii) by bus from A to C to A via B.

(iii) by bus from A to C to A via B if he does not want to use a bus more than once ?

Solution. (i) The man may go from A to B in 4 ways and from B to C in 3 ways.

So required number of way of travel = 4 × 3 = 12

(ii) Number of ways of journey from A to C (via B) = 12.

Similarly Number of ways of journey from C to A (via B) = 3 × 4 = 12.

∴ by product rule principle, required number of journey = 12 ×12 = 144.

(iii) Number of ways of journey from A to B = 4

Number of ways of journey from B to C = 3

Number of ways of journey from C to B = 2

(∵ the bus used from B to C is not used from C to B)

Number of ways of journey from B to A = 3

(∵ the bus used from A to B is not used from B to A)

∴ the required number of journey $= 4 \times 3 \times 2 \times 3 = 72$.

Ex. 13. How many three digit number can be formed from the six digits 2, 3, 5, 6, 7 and 9 if repetitions are not permitted? How many of these are less than 400? How many of these are even?

Solution. Number of 3-digit number (without repetitions) = Number of Permutation of 3 digits taken from six

$$= {}^6P_3 = \frac{6!}{3!} = 120$$

The numbers of these which are less than 400 are started by 2 or 3.

So, first position can be occuppied in 2 ways. Keeping 2 or 3 at 1st position the remaining two positions can be filled in ${}^5P_2$ ways.

Therefore, by Product Rule Principle, Number of Numbers which are less than 400 is $2 \times {}^5P_2 = 2 \times 20 = 40$.

Even number ends with 2 or 6.

So, last position can be occupied in 2 ways. Keeping 2 or 6 at last position the 1st two positions can be filled in ${}^5P_2$ ways. Therefore, by Product Rule Principle, number of even numbers (of three digits) $= {}^5P_2 \times 2 = 40$.

Ex.14. How many number can be formed by using all of the digits 5,4, 3 , 2, 1, 4, 3, 2, 1 such that odd digits always occupy odd places.

Solution. The odd places are 1st, 3rd, 5th, 7th and 9th places.

These places are occupied by 5, 3, 1, 3, 1 in $\frac{5!}{2!2!}$ ways since 3 repeats twice and 1 repeatε twice.

After placing the odd digits the even digits 4, 2, 4, 2 may occupy 2nd, 4th, 6th and 8th places in $\frac{4!}{2!2!}$ way.

∴ required number of numbers

$$= \frac{5!}{2!2!} \times \frac{4!}{2!2!} = 30 \times 6 = 180$$

Ex.15. Out of 12 employes, a group of four trainees is to be sent for a software training. In how many ways these can be selected if there are two employees who refuse to go together?

Solution. From 12 employees 4 can be selected in ${}^{12}C_4 = 495$ ways.

Now the number of selection in which the two employees are sent together $= {}^{10}C_2$ (because here 2 employees are to be selected from 10 employees) = 45.

∴ required Number of ways of selection $= 495 - 45 = 450$

Ex.16. Out of 12 employees, a group of four trainees is to be sent for a training. In how many ways these can be selected if

(i) there are two employees either they both go or both do not go.

(ii) there are two employees who want to go together and there are two employees who refuse to go together.

Solution. (i) Let A = set of all selections in which the two employees both go and B = set of all selections in which the two employees do not go.

$\therefore \quad n(A) = {}^{10}C_2$ and $n(B) = {}^{10}C_4$

Obviously $A \cap B =$ null set

$\therefore$ by principle of inclusion, the required number of ways of selection $= n(A) + n(B)$

$$= {}^{10}C_2 + {}^{10}C_4 = 45 + 210 = 255$$

(ii) Let x_1, x_2 be two employees who refuse to go together and x_3, x_4 are two who want to go together. The required selection can be done in the following way :

x_1, x_2 both do not go and x_3, x_4 both go; this can be done in ${}^8C_2$ ways.

x_1, x_2 and also x_3, x_4 do not go; this can be done in ${}^8C_4$ ways.

x_3, x_4 both and either of x_1 or x_2 go for the training; this can be done in $2 \times {}^8C_1$ ways. x_3 and x_4 both do not go for training and either x_1 or x_2 go for training; this can be done in $2 \times {}^8C_3$ ways.

$\therefore$ the required number of selection

$$= {}^8C_4 + {}^8C_2 + 2 \times {}^8C_1 + 2 \times {}^8C_3 = 226$$

Ex.17. How many committee of three can be formed from 8 gentlemen?

Solution. Number of required committee

$$= \text{Number of combination of 3 out of 8} = {}^8C_3 = \frac{8!}{3!5!} = 56$$

Ex.18. 10 points on a straight line g_1 are joined to 12 points on another straight line g_2. Find the number of points of intersection of the line segments thus formed.

Solution. Let A, B be two points on the line g_1 and C, D be two points on g_2. AC, AD and BC, BD are the line segments. The points of intersection of them are A, B on g_1, C, D on g_2 and P not on g_1 or g_2. Thus every point on g_1 and on g_2 is a point of intersection of the segments.

The point of intersection P occurs for every pair of choices in g_1 and in g_2.

So, number of such points is ${}^{10}C_2 \times {}^{12}C_2 = 45 \times 66 = 2970$.

$\therefore$ the total number of point of intersection

$$= 2970 + 10 + 12 = 2992$$

Ex.19. In how many ways can nine student be partitioned into three teams containing 4, 3 and 2 students respectively?

Solution. 4 students of 1st team can be selected from 9 students in ${}^9C_4$ ways; 3 students of 2nd team can be selected from the remaining $9 - 4 = 5$ students in ${}^5C_3$ ways; 2 students of 3rd team can be selected from the residual 2 students in ${}^2C_2$ ways.

Therefore by, Product Rule Principle, the required number of ways of partition $= {}^9C_4 \times {}^5C_3 \times {}^2C_2 = 126 \times 10 \times 1 = 1260$

Ex.20. There are 12 students in a class. In how many ways can the 12 students take 4 different tests if 3 students are to take each test?

Solution. There are ${}^{12}C_3$ ways to choose 3 students to take 1st test; following this ${}^9C_3$ ways to choose three students to take the 2nd test, and ${}^6C_3$ ways to choice 3 students to take the 3rd test.

The remaining 3 students may take 4th test in ${}^3C_3$ ways. Thus altogether there are ${}^{12}C_3 \times {}^9C_3 \times {}^6C_3 \times {}^3C_3 = 369600$ ways for the students to take the test.

Ex.21. There are 50 students in each of the first year and second year classes. Each class has 25 boys and 25 girls students. In how many ways can an 8 students committee be formed so that there are 4 girls and three second year students in the committee.

Solution. The committee can be formed in the following way.

	First Year		Second Year	
	Boys	Girls	Boys	Girls
Way 1 :	4	1	0	3
Way 2 :	3	2	1	2
Way 3 :	2	3	2	1
Way 4 :	1	4	3	0

Way 1 can be formed in ${}^{25}C_4 \times {}^{25}C_1 \times {}^{25}C_0 \times {}^{25}C_3$

$$= {}^{25}C_4 \times {}^{25}C_1 \times {}^{25}C_3$$

Way 2 can be formed in ${}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1 \times {}^{25}C_2$

Way 3 can be formed in ${}^{25}C_2 \times {}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1$

Way 4 can be formed in ${}^{25}C_1 \times {}^{25}C_4 \times {}^{25}C_3 \times {}^{25}C_0$

$$= {}^{25}C_1 \times {}^{25}C_4 \times {}^{25}C_3$$

$\therefore$ Required number of committees

$$= 2 \times {}^{25}C_4 \times {}^{25}C_1 \times {}^{25}C_3 + 2 \times {}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1 \times {}^{25}C_2$$

Ex.22. There are 12 bulbs in a room each of which is operated independently by 12 different switches. In how many ways the room can be illuminated?

Solution. If we choose any 1 switch from 12 switches the room can be illuminated. 1 switch can be choosen from 12 switches in ${}^{12}C_1$ way.

Similarly any 2 switches can illuminate the room and 2 switches can be choosen in ${}^{12}C_2$ ways. Thus total number of ways of illumination

$$= {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 + \cdots + {}^{12}C_{12} = 2^{12} - 1 = 4095$$

Ex.23. In how many ways 6 men and 6 women be seated along a circular table, so that are alternate?

Solution: 6 men can be seated along a circle

$$= (6-1)! \text{ ways } = 5! \text{ ways.}$$

Now, there are 6 places for 6 women. So they can be seated in 6! ways.

$\therefore$ The required number is $5! \times 6! = 86400$

Ex.24. If 10 people are seated about a round table, how many different circular arrangements are possible if arrangements are considered the same when one can be obtained from the other by rotation? If 5 of them are female and others are males, in how many arrangements do the sexes alternate?

Solution: The required number of circular arrangements is $(10-1)! = 9!$

5 male can be seated about a round table is $(5-1)! = 4!$

Then there are 5 places for 5 female.

So they can be seated in 5! ways.

$\therefore$ Total no. of required circular arrangements is

$$4! \times 5! = 2880$$

Ex.25. Find the number of ways in which 7 different flowers can be arranged in a garland.

Solution: Since the arrangment of flowers in a garland is one direction, so the required number of ways is

$$\frac{1}{2}(7-1)! = \frac{1}{2} \times 6 = 360$$

Ex.26. In how many ways 10 different beads can be arranged to form a necklace?

Solution: In a necklace, the beads are arranged either in clockwise or in anticlockwise direction. So the required number of arrangent is $\frac{1}{2}(10-1)! = 181440$

Ex.27. There are four coplanar lines. Five distinct points are there on each of these lines. Find the maximum number of triangles with vertices at these points.

Solution. There are two mutually exclusive cases to consider:

Case 1 : A triangle is formed with the three vertices where each vertex lies on a single line. Number of these types of triangle = Number of selection from points on the three separate lines $= {}^5C_1 \times {}^5C_1 \times {}^5C_1 \times {}^4C_3 = 500$

Case 2 : A triangle is formed with two vertices on one line and the third vertex on any of the remaining lines. Number of these types of triangles = Number of selection of two points from a selected line and a point from the 5 points on any of the remaining lines

$$= {}^5C_2 \times {}^5C_1 \times {}^4C_1 \times {}^3C_1 = 600$$

Note that if some of the three vertices selected above (in any case) become collinear then the number of triangles will be diminished. So the maximum number of triangles

$$= 500 + 600 = 1100$$

Ex.28. A car company manufactures 5 models of cars. Find the number of ways a customer can buy 8 cars.

Solution. The models will repeat several times. From an earlier theorem, number of ways of purchase of 8 cars from the cars of 5 models = Number of combination (with repeatitions) of 8 from $5 = {}^{8+5-1}C_8 = {}^{12}C_8 = 495$.

Ex.29. Find the number of nonnegetive solutions to

$x + y + z = 18$ with the condition $x \geq 3, y \geq 2$ and $z \geq 1$.

Solution. Let $x' = x - 3,\ y' = y - 2,\ z' = z - 1$.

Then $x + y + z = 18 \Rightarrow x' + y' + z' = 12$

Then required number of solution

= number of nonnegative solution to $x' + y' + z' = 12$

= number of combination of 12 objects (where any object may repeat any time) from three kinds of objects (say a, b and c)

$$= {}^{3+12-1}C_{12} = {}^{14}C_{12} = 91.$$

Ex.30. Find the number of non-negative integral solutions of the inequality

$$x_1 + x_2 + x_3 + x_4 < 8? \quad xi \geq 0, \quad i = 1 \text{ to } 4$$

Solution: We convert the inequality into an equality by introducing an auxiliary variable $x_5 > 0$ i.e $x_5 \geq 1$.

Thus we get

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8 \qquad (1)$$

where $xi \geq 0, \ = 1, 2, 3, 4, \ x_5 \geq 1$.

Putting $x_5 - 1 = x_5'$ in (1), we get

$$x_1 + x_2 + x_3 + x_4 + x_5' = 7$$

where $x_i \geq 0,\ i = 1 \text{to} 4,\ x_5' \geq 0$

$\therefore$ The required number of non-negative integral solution is

$${}^{7+5-1}C_7 = {}^{11}C_7 = {}^{11}C_4 = 330$$

Ex.31. Find the number of ways in which 4 science students and 17 commerce students can sit together in a queue such that between any two science students at least two commerce students can sit.

Solution. Let S_1, S_2, S_3 and S_4 be the four science students; they are kept in a queue. Let n_1 number of commerce students sit before S_1, n_2 commerce students sit between S_1 and S_2, n_3 commerce students sit between S_2 and S_3, n_4 commerce students sit between S_3 and S_4, n_5 commerce students sit after S_4.

We have to find n_1, n_2, n_3, n_4 and n_5 such that

$$n_1 + n_2 + n_3 + n_4 + n_5 = 17 \qquad \ldots (1)$$

and $n_1 \geq 0, n_5 \geq 0;\ n_2 \geq 2, n_3 \geq 2, n_4 \geq 2.$

Let $n_1' = n_1,\ n_2' = n_2 - 2,\ n_3' = n_3 - 2,\ n_4' = n_4 - 2,\ n_5' = n_5.$

$\therefore$ We get from above, $n_1' + n_2' + n_3' + n_4' + n_5' = 17 - 6 = 11$

where $n_1', n_2', n_3', n_4', n_5' \geq 0.$

Then required number of solution of (1)

= Number of nonnegative solution to

$$n_1' + n_2' + n_3' + n_4' + n_5' = 11$$

= Number of combination of 11 objects (where any object may repeat any time) from 5 kinds of objects

$$= {}^{5+11-1}C_{11} = {}^{15}C_{11} = 1365$$

Now the science students may permute themselves in 4! ways. The commerce students may permute themselves in 17! ways.

$\therefore$ required number of arrangements $= 17! \times 4! \times 1365$

Ex.32. Four dice are thrown at a time. In how many way a total of 16 can be obtained?

Solution. Let n_1 = points of the face shown by 1st die

n_2 = points of the face shown by 2nd die

n_3 = points of the face shown by 3rd die

n_4 = points of the face shown by 4th die

Therefore, we have to find the number of solutions of

$n_1 + n_2 + n_3 + n_4 = 16$. where $1 \leq n_1 \leq 6,\ 1 \leq n_2 \leq 6,\ 1 \leq n_3 \leq 6$ and $1 \leq n_4 \leq 6$.

This number = co-efficient of x^{16} in the expansion

$$(n + n^2 + n^3 + n^4 + n^5 + n^6)(n + n^2 + n^3 + n^4 + n^5 + n^6)$$
$$(n + n^2 + n^3 + n^4 + n^5 + n^6)(n + n^2 + n^3 + n^4 + n^5 + n^6)$$

= co-efficient of n^{16} in the expansion

$$(n + n^2 + n^3 + n^4 + n^5 + n^6)^4$$

= co-efficient of n^{16} in $n^4\left(1 + n + n^2 + n^3 + n^4 + n^5\right)^4$

= co-efficient of n^{12} in $\left(1 + n + n^2 + n^3 + n^4 + n^5\right)^4$

= co-efficient of n^{12} in $\left\{\dfrac{1\left(n^6 - 1\right)}{n - 1}\right\}^4$

= co-efficient of n^{12} in $\left(n^6 - 1\right)^4 (n - 1)^{-4}$

= co-efficient of n^{12} in $\left(1 - n^6\right)^4 (1 - n)^{-4}$ ... (1)

Now $\left(1 - n^6\right)^4 = \left\{\left(1 - n^6\right)^2\right\}^2$

$$= \left(1 - 2n^6 + n^{12}\right)^2$$

$$= 1 + 4n^{12} + n^{24} - 4n^6 - 4n^{18} + 2n^{12}$$

$$= 1 - 4n^6 + 6n^{12} - 4n^{18} + n^{24}$$

$\therefore$ from (1) we get the required number of solution

= co-efficient of n^{12} in $\left(1 - 4n^6 + 6n^{12} - 4n^{18} + n^{24}\right)(1 - n)^{-4}$

= co-efficient of n^{12} in

$$\left(1 - 4n^6 + 6n^{12} - 4n^{18} + n^{24}\right)\left\{1 + \sum_{r=1}^{\infty} \frac{4.5.6\cdots(3+r)}{r!} n^r\right\}$$

$$= \frac{4.5.6.7\cdots14.15}{12!} - 4 \cdot \frac{4.5\cdots9}{6!} + 6$$

$$= \frac{4.5.6.7.8.9.10.11.12.13.14.15}{12!} - 4 \cdot \frac{4.5.6.7.8.9}{6!} + 6$$

$$= \frac{13.14.15}{2.3} - 4 \cdot \frac{7.8.9}{2.3} + 6$$

$= 455 - 336 + 6 = 125$

So, in 125 way the total may be 16.

Ex.33. Find the number of divisors of the number 2160

(i) if 1 is not supposed to be a factor; find the sum of the factors.

(ii) if 1 and the number itself are not supposed to be factor; find the sum of these factors.

Solution. We see $2160 = 2^4 \times 3^3 \times 5$

If we take any number of 2 or no 2 from the four 2, any number of 3 or no 3 from the three 3 and the 5 or no 5 and multiply them we get a factor.

From the four $2s'$ we can take one 2 or two 2 or three 2 or four 2 or no 2. So there are 5 choices of selection from $2s'$. Similarly there are 4 choices from 3 and two choices on 5. Therefore number of choices $= 5 \times 4 \times 2$. Here one case may occur where none of 2, 3 or 5 are taken.

(i) $\therefore$ total number of factors, excluding $1 = 5 \times 4 \times 2 - 1 = 39$.

Sum of these factors

$$= \left(1+2+2^2+2^3+2^4\right)\left(1+3+3^2+3^3\right)(1+5)-1$$

$$= \frac{1\left(2^5-1\right)}{2-1} \times \frac{1\left(3^4-1\right)}{3-1} \times 6 - 1 \quad = 31 \times \frac{80}{2} \times 6 - 1 = 7439$$

(ii) $\therefore$ total number of factors, excluding 1 and itself

$$= 5 \times 4 \times 2 - 1 - 1 = 38$$

Sum of these factors

$$= \left(1+2+2^2+2^3+2^4\right)\left(1+3+3^2+3^3\right)(1+5)-1-2160$$

$$= 7439 - 2160 = 5279$$

2.1.8. Pigeonhole Principle

Many results of Counting Technique come from the following statement.

Pigeonhole Principle : If n pigeonholes are occupied by $n+1$ or more pigeons, then at least one pigeonhole is occupied by more than one pigeon. [W.B.U.T. 2015]

Remark: The pigeonhole principle can also be stated in the set theoretic terms as follows: (shown in the next page)

Let X and Y be any two finite sets. If $n(X) > n(Y)$, then any function $f : X \to Y$ cannot be one-to-one, i.e. there exits at least two distinct elements x_1 and x_2 is X such that $f(x_1) = f(x_2)$.

Illustrations

(i) Suppose an institute contains 13 teachers. Then at least two of the teachers (pigeons) were born in the same month (pigeonholes).

(ii) Suppose a busket contains many red, white and blue balls. Then one need only grap four balls (pigeons) to be sure of getting at least two of same 'colour balls' (pigeonholes).

(iii) Find the minimum number of elements that you need to take from the set $A = \{1, 2, 3, \cdots 9\}$ to be sure that two of the numbers (not necessarily distinct) add upto 10. Here the pigeonhole are the five sets $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$, $\{5, 5\}$. Thus any choice of six elements (pigeons) of A will gurantee that two of the numbers add upto 10.

Generalized Pigeonhole Principle

If n pigeonholes are occupied by $kn+1$ or more pigeons (k is a positive integer), then at least one pigeonhole is occupied by $k+1$ or more pigeons.

Illustrations

(i) Find the minimum number of teachers in an institute to be sure that four of them are born in the same month. Here $n = 12$ months are the pigeonholes and $k+1=4$ or $k=3$.

So among any $kn+1 = 3 \times 12 + 1 = 37$ teachers (pigeons) four of them are born in the same months

(ii) Let a box contains many red, white and blue balls. Find the minimum number of balls that one needs to choose in order to get four balls are of the same colour.

Here $n = 3$ colours (pigeonholes) and $k+1=4$ or $k=3$. Thus among any $kn+1=10$ balls (pigeons), four of them have the same colour.

(iii) Consider six people. Any two of them are either friends or strangers. Find the number of selected persons where the persons are either mutual friends or mutual stranger.

[W.B.U.T.2013]

Let x_1 be one of the people.

Also let $X = \{x : x \text{ is friends of } x_1\}$ $Y = \{x : x \text{ is stranger of } x_1\}$

There are 5 people who are friend or stranger of x_1. Here X and Y supposed to be pigeonhole i.e. here $n = 2$ pigeonholes.

If $5 = 2 \times 2 + 1$ peoples (pigeons) occupies these pigeonholes then at least one of X and Y contains $2 + 1 = 3$ or more people (by Generalized Pigeonhole Principle). Suppose X contain 3 people. If two of them are friends, then these two and x_1 become 3 mutual friends. If not then X has three mutual strangers. Again let Y has three people. If two of them are strangers then these two and x_1 become 3 mutual strangers. If not, then X has three mutual friends. So there are three of the six people which are either mutual friend or mutual strangers.

2.1.9. Illustrative Examples:

Ex. 1. Assume there are 50 distinct pair of shoes in a rack. How many single shoes can be chosen from the rack, that we are certain to have a pair.

Solution. The 50 distinct pairs constitute 50 pigeonholes. We think the single shoes as pigeons. Therefore if we choose 51 single shoes, by pigeonhole principle, at least two shoes will go to one pigeonhole i.e. we certainly get a pair of shoes.

Ex.2. Show that at least 3 people out of 25 must have their birthday in the same month when they are assembled in a room.

Solution: Here we may consider birth months of people as pigeom and the calendar months ($n = 12$) as the pigeonholes.

According $k + 1 = 3 \therefore k = 2$

$$\therefore kn + 1 = 2.12 + 1 = 25$$

Thus the minium number of people who may have their birthday in the same month is 25

Ex.3. Find the minimum number of students needed to guarantee that 5 of them belong to the same class (1st year, 2nd year, 3rd year an 4th year).

Solution. Here the number of pigeonholes (classes) is $n = 4$. In generalised pigeonhole principle minimum number of pigeons (here students) to go to a pigeon is $k + 1 = 5$ or, $k = 4$.

Thus among $kn + 1 = 17$ students (pigeons) 5 of them belong to the same class.

Ex.4. If three men and five women are lined up in a row, show that at least two women will be next to each other.

Solution. First, place the three men in a row. Then the three men generate four locations (pigeonholes) in which the women (pigeon) are going to be placed. Since there are five women (pigeons), at least one slot will contain two women who must, therefore, be next to each other.

Ex.5. Find the minumum number n of integers to be selected from $s = \{1, 2, \cdots 9\}$ so that (a) the sum of two of the n integers is even (b) the difference of two of the n integers is 5.

Solution. (a) Consider the sets $S_1 = \{1, 3, 5, 7, 9\}$ and $S_2 = \{2, 4, 6, 8\}$. These two are pigeonholes.

So if at least 3 integers (pigeons) are chosen from S two will belong to S_1 or S_2 giving the sum even.

Thus the minimum number $n = 3$.

(b) Consider the sets

$S_1 = \{1, 6\}$, $S_2 = \{2, 7\}$, $S_3 = \{3, 8\}$,

$S_4 = \{4, 9\}$, $S_5 = S - (S_1 \cup S_2 \cup S_3 \cup S_4) = \{5\}$.

Note that it is not necessary to select 5

So we consider are 4 pigeonholes S_1, S_2, S_3, S_4. So if at least 5 integers (pigeon) are chosen from S two will belong to one of the above sets and their difference will be 5 (Note that no two will belong to S_5).

Hence the minimum number $n = 5$

Ex.6. How many students, each of whome comes from one of 29 states, must be enrolled in a college to guarantee that there are at least 35 who come from the same state.

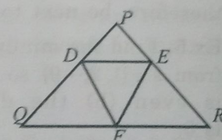
Solution. Here $n = 29$ states are the pigeonholes and $k+1 = 35$, i.e. $k = 34$

$$\therefore \ kn+1 = 34 \times 29 + 1 = 987$$

So by the generalised pigeonhole principle among 987 students (pigeons) at least 35 of them come from the same state.

Ex.7. PQR is an equilateral triangle with $PQ = 1$. Show that if five points are selected from the interior of the triangle, there are at least two points whose distance apart is less then $\frac{1}{2}$

Solution. Let D, E, F are the mid points of the sides PQ, PR, QR respectively. Then the four smaller triangle, say $\Delta PDE, \Delta DFQ, \Delta EFR$ and ΔDEF are congruent euqilateral triangles with each side $= \frac{1}{2}$. Here 4 equilateral triangles created be considered as pigeon holes and five interior points as pegions.



Therefore, by pigeon hole principle, for any five interior points of the triangle ΔPQR, at least two points must be in or on the same small triangle.

Thus the distance between them is less than $\frac{1}{2}$.

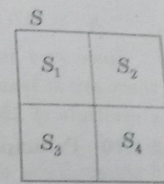
Ex.8. Let S be a square where each side has length 2 inches. Find the minimum number of points to be chosen from the interior of S such that distance between two of the points will be less than $\sqrt{2}$ inches. [W.B.U.T 2013]

Solution. Draw two lines beween the opposite sides of S which partitions S into four subsquares S_1, S_2, S_3 and S_4.

Each side of every S_r is 1.

So the diagonal of every S_r is $\sqrt{2}$.

So if any two points belong to any of S_1, S_2, S_3 and S_4 then their distance will be less than $\sqrt{2}$.



Consider S_1, S_2, S_3, S_4 as four pigeonholes.

So by pigeonholes principle if at least five points (pigeons) of the interior of S are chosen two will belong to any of S_1, S_2, S_3 and S_4 i.e., the distance between two of the five points will be less than $\sqrt{2}$.

Hence the minimum number of points is 5.

Ex.9. 36 candidates appears in a competitive examination. Show that there exists at least two among them whose roll number differ by a multiple of 17.

Solution. Let X be the set of roll numbers of 36 candidates and Y be the set of remainders when any positive integer is divided by 17.

$$\therefore Y = \{1, 2, \ldots 16\}.$$

Now we define a function f as follows: $f: X \to Y$ such that

$$f(x) = \text{ remainder when } x \text{ is divided by } 17$$

$\therefore$ As $n(X) = 36$ and $n(Y) = 16$,

so $n(X) > n(Y)$.

$\therefore$ Therefore by Pigeonhole principle, function f is not one-to-one. Thus there exist two distinct roll numbers x_1 and x_2 such that $f(x_1) = f(x_2)$.

By Euclidean algorithm, We have

$$x_1 = 17p + f(x_1)$$

$$x_2 = 17q + f(x_2)$$

where $p, q \in Z^+$, the set of positive integer.

$\therefore x_1 - x_2 = 17(p-q)$ $[\because f(x_1) = f(x_2)]$

Since $p-q$ is an integer, $x_1 - x_2$ is the multiple of 25.Hence there are at least two candidates whose roll numbers differ by a multiple of 25.

2.1.10. Principle of Inclusion and Exclusion

[W.B.U.T.2014]

Let A_1 and A_2 be two sets. $n(A_1)$, $n(A_2)$ are cardinal number of A_1 and A_2 respectively (that is the number of elements in the set).

Then $n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$

For three sets A_1, A_2, A_3,

$$n(A_1 \cup A_2 \cup A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_2 \cap A_3) - n(A_3 \cap A_1) + n(A_1 \cap A_2 \cap A_3)$$

In general for m number of sets $A_1, A_2, A_3 \cdots A_m$

$$n(A_1 \cup A_2 \cup .... \cup A_m) = \sum_{i=1}^{m} n(Ai) - \sum_{\substack{i,i=1 \\ i \neq j}}^{m} n(A_i \cap A_j) + \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{m} n(A_i \cap A_j \cap A_k) .... + (-1)^{m-1} n(A_1 \cap A_2 \cap .... A_m)$$

Remark. If $A_1, A_2, ... A_m$ are mutually disjoint finite set then

$$n(A_1 \cup A_2 \cup .... \cup A_m) = \sum_{i=1}^{m} n(A_i)$$

Illustrations

(i) Let $U = \{1, 2, 3, \cdots 1000\}$. Then find $n(S)$ where S = set of such integers of U which are not divisible by 3,5 or 7

Let A = set of integers of U which are divisible by 3;

B = set of integers of U which are divisible by 5 and

C = set of integers U which are divisible by 7.

Then $S = A^c \cap B^c \cap C^c$.

Since $\frac{1000}{3} = 333\frac{1}{3}$, so $n(A) = 333$

Again since $\frac{1000}{5} = 200$ and $\frac{1000}{7} = 142\frac{6}{7}$,

so $n(B) = 200$ and $n(C) = 142$

Now, $A \cap B$ = set of all integers of U which are divisible by 3 and 5 both.

= set of all integers of U which are divisible by 15 (l.c.m of 3 and 5)

Since $\frac{1000}{15} = 66\frac{2}{3}$, so $n(A \cap B) = 66$

Similarly, since $\frac{1000}{35} = 28\frac{4}{7}$, so $n(B \cap C) = 28$

and also since $\frac{1000}{21} = 47\frac{13}{21}$, so $n(C \cap A) = 47$

Now, $A \cap B \cap C$ = Set of integers of U which are multiple of 105 (l.c.m of 3, 5 and 7).

Since $\frac{1000}{105} = 9\frac{11}{21}$, so $n(A \cap B \cap C) = 9$

By Principle of Inclusion and Exclusion

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$$= 333 + 200 + 142 - 66 - 28 - 47 + 9 = 543$$

By D' Morgan's law, $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$

$$\therefore \quad n(S) = n(A \cup B \cup C)^c = n(U - (A \cup B \cup C))$$

$$= n(U) - n(A \cup B \cup C) \quad [\because \ A \cup B \cup C \subset U]$$

$$= 1000 - 543 = 457$$

2.1.11. Illustrative Examples.

Ex.1. Of 32 students who play football or cricket (or both), 30 play football and 14 play cricket. Find the number of students who (i) play both the game (ii) play only football and (iii) play only cricket.

Solution. Let F = Set of students playing football

C = Set of students playing cricket

$\therefore\ n(F \cup C) = 32, n(F) = 30, n(C) = 14.$

(i) $F \cap C$ = Set of students playing both the game. By principle of Inclusion and Exclusion,

$$n(F \cup C) = n(F) + n(C) - n(F \cap C)$$

or, $32 = 30 + 14 - n(F \cap C) = 44 - n(F \cap C)$

$\therefore\ n(F \cap C) = 44 - 32 = 12$

$\therefore$ No. of students playing both the game = 12.

(ii) Now, $F - F \cap C$ = Set of students playing only football.

$\therefore$ Required number $= n(F - F \cap C)$

$= n(F) - n(F \cap C)\ [\because\ F \cap C \subset F]$

$= 30 - 12 = 18$

(iii) Similarly, No. of students playing only cricket

$$= n(C - F \cap C) = n(C) - n(F \cap C) = 14 - 12 = 2$$

Ex.2. Let A, B, C, D denote, respectively English, Japanese, French and Russian language courses. Let 12 take A, 20 take B, 5 take A and B, 7 take A and C, 3 take A, B and C, 2 take A, B and D; 20 take C, 8 take D, 4 take A and D, 16 take B and C, 4 take B and D, 3 take C and D, 2 take B, C, D, 3 take A, C, D, 2 take all the four and 71 take none.

Find the number of candidates who take at least one courses. Hence find the total number of candidates on which the survey is done.

Solution.

Let A = Set of candidates taking language-course A

B = Set of candidates taking language-course B

C = Set of candidates taking language-course C

D = Set of candidates taking language-course D

$\therefore\ A \cup B \cup C \cup D$ = set of all candidates taking at least one course.

By Principle of Inclusion and Exclusion,

$$n(A \cup B \cup C \cup D)$$
$$= n(A) + n(B) + n(C) + n(D) - n(A \cap B)$$
$$- n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D) - n(C \cap D)$$
$$+ n(A \cap B \cap C) + n(A \cap B \cap D) + n(B \cap C \cap D)$$
$$+ n(C \cap D \cap A) - n(A \cap B \cap C \cap D)$$
$$= (12 + 20 + 20 + 8) - (5 + 7 + 4 + 16 + 4 + 3)$$
$$+ (3 + 2 + 2 + 3) - 2$$
$$= 60 - 39 + 10 - 2 = 29$$

Total number of candidates = 29+71=100

Ex.3. Find the total number of integers lying between 1 and 1000 that are divisible by at least one of 2, 3 and 7.

Solution. Let A = set of integers, lying between 1 and 1000, which are divisible by 2.

B = set of such integers divisible by 3

C = set of such integers divisible by 7

Now $\dfrac{1000}{2} = 500\ \therefore\ n(A) = 500$

$\dfrac{1000}{3} = 333\frac{1}{3}\ \therefore\ n(B) = 333$

$\dfrac{1000}{7} = 142\frac{6}{7}\ \therefore\ n(C) = 142$

Now l.c.m of 2 and 3 = 6

l.c.m of 3 and 7 = 21

l.c.m of 7 and 2 = 14

l.c.m of 2, 3 and 7 = 42

Now, $\frac{1000}{6}=166\frac{2}{3}$, $\frac{1000}{21}=47\frac{13}{21}$, $\frac{1000}{14}=71\frac{3}{7}$

and $\frac{1000}{42}=23\frac{17}{21}$

$\therefore\ n(A\cap B)=166,\ n(B\cap C)=47,\ n(C\cap A)=71,$

$n(A\cap B\cap C)=23.$

$\therefore$ Required number of integers $= n(A\cup B\cup C)$

$$= n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C)$$

$$= 500+333+142-166-47-71+23=714$$

Ex.4. Using Principle of Inclusion and Exclusion show that for any three sets A, B and C,

$n(A\cup B\cup C)=n(A)+n(B)+n(C)$ if they are pairwise mutually disjoint. [W.B.U.T.2013]

Solution. If A, B, C are pairwise mutually disjoint then $A\cap B=B\cap C=C\cap A=A\cap B\cap C=\phi$, the null set,

Then $n(A\cap B)=n(B\cap C)=n(C\cap A)=n(A\cap B\cap C)=0$

Then by Principle of Inclusion and Exclusion,

$$n(A\cup B\cup C)$$

$$= n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C)$$

$$= n(A)+n(B)+n(C)$$

Ex.5. (i) Let A and B be sets such that $n(A)=6$ and $n(B)=4$. Find the number of surjective mappings from A to B.

Solution. Let $B=\{x_1,x_2,x_3,x_4\}$. Let U=set of all mapping from A to B.

Let F_1 = Set of mappings which do not send any element of A to x_1.

F_2 = Set of mappings which do not send any element of A to x_2.

Similarly F_3, F_4 are formed.

Then F_1^c = Set of mappings which send at least one element of A to x_1.

F_2^c = Set of mappings which send at least one element of A to x_2. and so on.

$\therefore\ F_1^c\cap F_2^c\cap F_3^c\cap F_4^c$ = Set of mapping form A to B where x_1, x_2, x_3 and x_4 are images of some element of A.

= Set of all surjective mapping from A to B.

For each mapping in U there are 4 choices for each of the 6 elements in A. Hence $n(U)=4^6=4096$.

For each mapping in F_1 there are 3 choices for each of the 6 elements in A. Hence $n(F_1)=3^6=729$.

Similarly $n(F_2)=729$, $n(F_3)=729$ and $n(F_3)=729$. For each mapping in $F_1\cap F_2$ there are 2 choices for each of the 6 elements in A. Hence $n(F_1\cap F_2)=2^6=64$.

Similarly $n(F_2\cap F_3)=n(F_3\cap F_4)=\cdots=64$.

Similarly $n(F_1\cap F_2\cap F_3)=n(F_2\cap F_3\cap F_4)=\cdots=1^6=1$

Oviously $F_1\cap F_2\cap F_3\cap F_4$ is null set.

Hence $n(F_1\cap F_2\cap F_3\cap F_4)=0$.

By Inclusion Exclusion Principle,

$$n(F_1\cup F_2\cup F_3\cup F_4)$$

$$=\sum n(F_1)-\sum n(F_1\cap F_2)+\sum n(F_1\cap F_2\cap F_3)-n(F_1\cap F_2\cap F_3\cap F_4)$$

$$=4\times 729-{}^4C_2\times 64+{}^4C_3\times 1-0$$

$$=2916-384+4=2536$$

$\therefore$ Number of surjective mapping from A to B.

$$= n\left(F_1^c \cap F_2^c \cap F_3^c \cap F_4^c\right) = n\left(F_1 \cup F_2 \cup F_3 \cup F_4\right)^c$$

$$= n\left(U - \left(F_1 \cup F_2 \cup F_3 \cup F_4\right)\right) = n(U) - n\left(F_1 \cup F_2 \cup F_3 \cup F_4\right)$$

$$= 4096 - 2536 = 1560$$

(ii) Find the number of ways a company can assign 7 projects to 4 people so that each person gets at least one project.

Solution. Let A = set of seven projects and B = set of four people. If f is a surjective mapping from A to B then we understand every people of B is assigned at least one project. Therefore required number of ways the company can assign 7 projects

= Number of surjective mapping from A to B

= 8400 (as the problem (i)).

Ex.6 Find the total number of integars between 1 and 1000 which are neither perfect squere nor perfect cubes.

[W.B.U.T.2014]

Solution. Let $U = \{1, 2, \ldots 1000\}$. Also set $A =$ set of integers, lying between 1 and 1000, which are perfect squares and $B =$ set of integers, lying between 1 and 1000, which are perfect cubes.

$\therefore$ The set of integers which are either perfect squares or perfect cubes is $A \cup B$ and the set of integers which are both perfect square and perfect cube is $A \cap B$.

Now, the set A is obtained by squaring each element of the set $\{1, 2, 3, \ldots, 31\}$ since $31^2 = 961$ but $32^2 = 1024$

$$\therefore n(A) = 31$$

Similarly the let B is obtained by cubing each element of the set $\{1, 2, 3, \ldots 10\}$, since $10^3 = 1000$

$$\therefore n(B) = 10$$

An integer which is both a perfect square and a perfect cube must be a perfect power of $\text{lcm}(2,3) = 6$

$\therefore$ The $A \cap B$ is obtained by raising to the sixth power of each element of the set $\{1, 2, 3\}$ since $3^6 = 729$ but $4^6 = 4096$

$$\therefore n(A \cap B) = 3.$$

$\therefore$ By Principle of Inclusion and Exclusion,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 31 + 10 - 3 = 38$$

By D' Morgan's law,

$$A^c \cap B^c = (A \cup B)^c$$

$$\therefore n\left(A^c \cap B^c\right) = n(A \cup B)^c = n(U - (A \cup B)) = n(U) - n(A \cup B)$$

$$= 1000 - 38 = 962$$

Ex.7. How many solution the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, x_3 are non-negative such that $x_1 \le 3$, $x_2 \le 4$ and $x_3 \le 6$? Use the principle of inclusion-exclusion.

Solution. Let $S =$ the set of all non-negative solution of

$$x_1 + x_2 + x_3 = 11$$

$\therefore n(S) =$ no. of combination of 11 objects from three kinds of objects

$$= {}^{11+3-1}C_{11} = {}^{13}C_{11} = {}^{13}C_2 = 78$$

Let S_1, S_2, S_3 be the sets of non-negative integral solution of $x_1 + x_2 + x_3 = 11$ when $x_1 \ge 4$, $x_2 \ge 5$, $x_3 \ge 7$ respectivly.

$\therefore n(S_1) =$ no. of negative solution of $x_1 + x_2 + x_3 = 11$ when $x_1 \ge 4$,

i.e, $x_1' + x_2 + x_3 = 7$ [putting $x_1' = x_1 - 4$]

$$= {}^{7+3-1}C_7 = {}^9C_7 = {}^9C_2$$

$$= 36$$

$n(S_2) =$ no. of non-negative integral solution of $x_1 + x_2 + x_3 = 11$ when $x_2 \ge 5$

i.e. $x_1 + x_2' + x_3 = 6$ [putting $x_2' = x_2 - 5$]

$$= {}^{6+3-1}C_6 = {}^8C_6 = {}^8C_2 = 28$$

Similarly $n(S_3) = {}^{4+3-1}C_4 = {}^6C_4 = {}^6C_2 = 15$

Now $n(S_1 \cap S_2) =$ no. of non-negative solution of

$x_1 + x_2 + x_3 = 11$ when $x_1 \geq 4,\ x_2 \geq 5$

i.e. $x_1' + x_2' + x_3 = 2$

$= {}^{2+3-1}C_2 = {}^4C_2 = 6$

$n(S_2 \cap S_3) =$ no. of non-negative solution of $x_1 + x_2 + x_3 = 11$

when $x_2 \geq 5,\ x_3 \geq 7$

i.e. $x_1 + x_2' + x_3' = -1 = 0$

Similarly $n(S_1 \cap S_2 \cap S_3) =$ no. of non-negative solution of $x_1 + x_2 + x_3 = 11$ when $x_1 \geq 4,\ x_2 \geq 5,\ x_3 \geq 7$

$= 0$

$\therefore$ By the Principle of inclusion – exclusion we have

$$n(S_1 \cup S_2 \cup S_3) = \sum n(S_i) - \sum n(S_i \cap S_2) + n(S_1 \cap S_2 \cap S_3)$$

$$= 36 + 28 + 15 - (6 + 0 + 1) + 0 = 72$$

$\therefore$ The required number of solution

$$= n(S) - n(S_1 \cup S_2 \cup S_3) = 78 - 72 = 6$$

Exercise

I Short and Long Answer Questions

1. Find n if (a) ${}^nP_2 = 72$ (b) $2^nP_2 + 50 = {}^{2n}P_2$

2. (i) Find the number of ways in which five persons can sit in a row.

(ii) Find the number of ways in which five persons can sit on a row if two of the persons want to sit together.

(iii) Find the number of ways in which five persons sit around a circular table if two of the persons do not sit together.

3. In how many ways can 5 examination be scheduled in a week so that no two examinations are scheduled on the same day considering Sunday as holiday?

4. Find the number of possible codes made up of two different letters followed by a digit.

5. Find the number of license plates that can be made if each plate contains two different letters followed by three different digits.

6. Find the number of license plates that can be made if each plate contains two different letters followed by three different digits if the first digit can not be 0.

7. There are 6 roads from Kolkata to Burdwan and 4 roads between Burdwan and Durgapur. Find the number of ways that a person can drive :

(i) from Kolkata to Durgapur via Burdwan

(ii) from to Kolkata Durgapur via Burdwan and returns from Durgapur to Kolkata via Burdwan

(iii) from Kolkata to Durgapur and returns back via Burdwan without using the same road more than once.

8. In how many ways can letters a, b, c, d, m and n be arranged if (i) m and n always together (ii) m is always before n.

9. In how many ways 6 people can occupy the seats of a car if one of three particular people of them will drive.

10. If $A = \{1, 2, 3, \cdots 2n\}$ then find the number of dearrangements of the elements of A such that the first n elements of each of the dearrangement are the last n elements of A.

11. Find the number of ways in which five books of Mathematics, four books of Physics and three books of Computer science can be placed on a shelf so that the books of same subject are together.

12. (i) Find the number of words that can be formed from the letters of the word ELEVEN

(ii) How many of them begin and end with E?

(iii) How many of them have the three E's together?

(iv) How many begin with E and end with N?

13. Find the number of permutations made with the letters of the word MISSISSIPPI taken all together. In how many of these will the vowels occupying the even places?

14. How many different words containing all the letters of the word TRIANGLE can be formed so that

(i) consonants are never separated

(ii) consonants never come together

(iii) vowels occupy odd places.

15. How many different words can be formed with the letters of the word DOGMATIC? How many of these

(i) letters G will occupy odd places?

(ii) letters D, O, G occupy only first three places?

16. If all the permutations of the letters of the word

(i) CHALK be written down as in a dictionary, what is the rank of this word?

(ii) MODESTY be written down as in a disctionary, what is the rank of this word?

17. (i) In how many ways can three 1st year students and two second year students sit in a row?

(ii) In how many ways can they sit in a row if the 1st year students and 2nd year students are each to sit together?

(iii) In how many ways can the students sit in a row if just the 2nd year students sit together?

18. Find the number of ways that 4 digit number can be formed with 0, 4, 5, 6, 7, 8, 9; no digit being repeated. How many of them are not divisible by 5?

19. How many number not more than 5 digits taken from the digits 1, 2, 3, 4, 5, 6, 7 in each of which digits may repeat any number of times.

20. How many seven digit telephone numbers are possible, if–

(i) only odd digits may be used.

(ii) The number is multiple of 100.

(iii) the first three digits are 481.

21. In how many ways can 4 students win the 6 prizes assuming that

(i) one student gets only one prize?

(ii) one may get all the six prizes.

22. How many eight digit even telephone numbers are possible if the number has at least one non zero digit and at least one of their digit repeated.

23. Find the number of divisors of 42336 excluding 1 and the number itself. Find the sum of these divisors.

24. Find the number of words with five letters formed from 10 different letters in each of which at least one letter is repeated.

25. Find the number of 9 different digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each of which 2 is placed after 1.

26. In how many ways can 5 programmers and 3 software engineers sit around a table so that no two software engineers are together?

27. In how many ways can 15 members of a council sit around a circular table, when the president is to sit one side of the secretary and the vice President on the other side?

28. Find the number of different committees of 5 members from 10 members in each of which 2 particular members will always be included.

29. In how many ways can a committee consisting of three men and two women be chosen from seven men and five women?

30. A busket countains six blue balls and five yellow balls. Find the number of ways four marbles can be drawn from the busket if (i) they can be any color (ii) two must be blue and two yellow (iii) they must all be of the same color.

31. How many committees of 5 with a given chairperson can be selected from 12 persons?

32. How many triangles can be made by joining 12 points in a plane, given that 7 are in one line.

33. 8 points on a straight line g_1 are joined to 10 points on another straight line g_2. Find the number of points of inter section of the line segments.

(i) not lying on g_1 or g_2

(ii) lying on g_1 and g_2

34. Out of 5 males and 6 females, a committee of 5 is to be formed. Find the number of ways in which it can be formed so that among the persons chosen in the committee there are

(i) exactly 3 males and 2 females

(ii) at least 2 males and one female

35. A staff of a bank consisits of a manager, a deputy manager and 10 other officers. A committee of r members is to be formed. Find the number of ways in which this can be formed so as to always include

(i) Manager (ii) Manager but not the deputy manager (iii) neither the manager nor the deputy manager.

36. Sukrit has 11 friends. Find the number of ways :

(i) Sukrit invites 5 of them to a party.

(ii) Sukrit invites 5 of them if two of his friends are married and will not attend separately.

(iii) Sukrit invites 5 of them if two of them are not on speaking terms and will not attend together.

37. Sayani has 11 close friends of whom six are also girls. Find the number of ways that she

(i) can invite three or more to her birth day party.

(ii) can invite three or more if she wants the same number of boys as girls (including herself).

38. Pappu has 8 friends of whom 5 are girls and 3 are boys. Buku has 7 friends of whom 4 are girls and 3 are boys. In how many ways Pappu and Buku invite their friends in Pappu's birthday party of 4 girls and 4 boys so that there are 4 of Pappu's friend and 4 of Buku's friends?

39. A student is to answer 10 out of 13 questions in an examination.

(i) How may way he can answer.

(ii) How many ways he can answer if he must answer Q.No.1 and Q.No.2.

(iii) How many ways he can answer if he answer Q.No.1 or Q.No.2 but not both.

(iv) How many ways he can answer if he answer exactly three out of the first five questions?

(v) How many ways he can answer if he answer at least three of the first five questions?

40. Twelve biscuits of different shapes are distributed among four children so that the youngest child receives 4 biscuits and the eldest child receives 2 biscuits while each of the remaining children receives 3 biscuits. Find the number of ways of distributions.

41. In how many ways 10 students can be divided into three teams, one containing four students and the other two contains three each.

42. In how many ways can 12 students be partitioned into four teams A, B, C and D so that each team contains 3 students?

43. There are 9 students in a class. Find the number of ways : the 9 students can take 3 different tests if 3 students are to take each test.

44. Find the number of ways of dividing 14 persons into 6 committees where two of the committees contain three members and the others two.

45. In how many ways can a set with three elements be divided into three cells (a cell can be empty).

46. A candidate is required to answer 6 out of 10 questions which are divided into two groups containing 5 questions each and he is permitted to attempt not more than four questions from any group. In how many ways can he select the questions?

47. There are three coplanar lines. Four distinct points are on each of these lines. Find the maximum number of triangles with vertices at these points.

48. In how many ways can a cricket team of eleven be chosen out of 14 players? How many of them will

(i) include a particular player.

(ii) exclude a particular player.

49. In a triangle ABC four, five and six points (other than A, B, C) are taken on the sides AB, BC and CA respectively. How many triangles can be constructed using these points as vertices? How many triangles considering the points A,B,C along with the above 15 points as vertices can be constructed?

50. Assume there are n distinct pairs of shoes in a box. How many single shoes can be chosen from the box so that we are certain to have a pair.

51. What is the least number of students that can be admitted to a college so that there are at least 15 students from one of the 50 districts.

52. Let 50 students with distinct heights stand in a row. Show there are 8 students which are either increasing or decreasing.

53. Let a team plays 19 games in a period of 14 days and the team plays at least one game per day. Show there is a period of consecutive days that the team played exactly 8 games.

54. Let S be an equilateral triangle where each side has length 3 inches. Find the minimum number of points to be chosen from the interior of S such that distance between two of the interior points will be less than one inch.

[Hint : Partition S into 9 equilateral triangle where each side has length one inch.]

55. Show that if four numbers from 1 to 7 are chosen, then two of them will add upto 8.

56. Show that at least 2 people out of 13 must have their birthday in the same month when they are assembled in a room.

57. Prove that a set of 37 positive integers contains 2 elements that have the same remainder upon division by 36.

58. Let $X = \{x_1, x_2, x_3 \cdots x_{25}\}$ where every x_i is a positive integer. Prove that there exist some elements of X whose sum is divisible by 25.

[Hint : Let $S_1 = x_1$, $S_2 = x_1 + x_2$, $S_3 = x_1 + x_2 + x_3 \cdots$ The result is true if some S_i is divisible by 25. Otherwise, r_i be remainder when S_i is divided by 25. Two of such remainder say r_p and r_q are equal. Let $p < q$. Then $S_q - S_p = x_{p+1} + x_{p+2} \cdots + x_q$ is divisible by 25.]

59. In a party there are 10 persons where each pair of persons are either friends or strangers. Prove that there is either a group of 4 mutual friends in the party or a group of 3 mutual strangers in the party.

60. If A and B are two disjoint sets then by principle of Inclusion and Exclusion prove that $n(A \cup B) = n(A) + n(B)$

61. Find the number of integers between 1 and 500 (inclusive) which are multiples of 3, 5 and 7.

62. Find the number of integers between 1 and 500 (inclusive) which are not divisible by 3, 5 and 7.

63. Find the number of nonnegative integral solution of $x_1 + x_2 + x_3 = 18$ where $x_1 < 7$, $x_2 < 8$, $x_3 < 9$.

64. Find the number of non-negative integral solution of $x + y + z = 20$. where $x < 8$, $y < 9$, $z < 10$.

65. Find the number of combinations of 10 items taking from five x's, four y's five z's, and seven w's.

[Hint : required number = number of non-negative integral solution of $x_1 + x_2 + x_3 + x_4 = 10$ where $x_1 \le 5$, $x_2 \le 4$, $x_3 \le 5$ and $x_4 \le 4$]

66. A store sells 4 kinds of biscuits. How many way a customer can buy (i) 15 biscuits (ii) 10 biscuits.

67. Find the number of non-negative solution to $x+y+z=20$ with the conditions $x \geq 5$, $y \geq 3$ and $z \geq 1$.

68. How many way 14 apples can be distributed among A,B,C and D such that everybody gets at least one apple.

69. Find the number of positive integers which are less or equal to 1000 and not divisible by 11, 7 or 3.

70. Find the number of ways of forming 6 committees from 14 persons such that 2 committees contain 3 persons and 4 other committee contain 2 persons.

71. A set A contains 8 distinct elements and a set B contains 3 elements. Find the number of surjective mappings from A to B.

72. Find the number of ways a company can assign 6 projects to 4 employees so that each employee gets at least one project.

73. Find the number of ways a school can give 5 prizes to 5 students so that each student gets at least one prize.

74. A set A contains 5 elements and a set B contains 7 elements. How many surjective mapping from A to B can be constructed.

75. Four dice are thrown at a time. In how many ways a total of 14 can be obtained.

76. Three dice are thrown simultaneously. In how many ways a total of 11 can be obtained so that the sum of the points shown by two dice should exceed the point shown by the third die.

[Hint : Required number of way = number of solution of $x+y+z=11$ so that $x+y>z$, $y+z>x$, $z+x>y$; that is each of x, y, z should be ≤ 5 and ≥ 1.

So required number of ways = number of integral solution of $x+y+z=11$, $1 \leq x, y, z \leq 5$.]

77. Find the number of ways in which 3 doctors and 15 engineers can sit together in a queue such that between any two doctors at least two engineers can sit.

Answers

1.(a) 9 (b) 5

2. (i) 5!=120 (ii) 2!×4! (iii) 12

3. $^6P_5 = 6$

4. $26 \times 26 \times 10 = 6500$

5. 26×25×10×9×8=468000

6. 26×25×9×9×8=421200

7. (i) 24 (ii) 576 (iii) 360

8. (i) 240 (ii) 5!

9. 3×5!=360

10. $(n!)^2$

11. 3!×5!×4!×3!=103680

12. (i) 120 (ii) 24 (iii) 24 (iv) 12

13. $\dfrac{11!}{4! \times 4! \times 2!} = 34,650; \dfrac{5!}{4!} \times \dfrac{7!}{4!2!} = 525$

14. (i) $4! \times 5!$ (ii) $8! - (5! \times 4!)$ (iii) $^4P_3 \times {}^5P_5 = 480$

15. (i) 4×7! (ii) 3!×5!

16. (i) 4!+3!+1!+1=32nd rank (ii) 6!+6!+5!+5!+1=1681

17. (i) 5!=120 (ii) 3!×2!×2=24 (iii) 4!×2!=48

18. $^7P_4 - {}^6P_3 = 720; {}^7P_4 - ({}^6P_3 - {}^5P_2) - {}^6P_3 = 30$

19. 19607

20. (i) 5^7 (ii) 10^5 (iii) 10^4

21. (i) $^6P_4 = 360$ (ii) $6^4 = 1296$

22. $(5 \times 10^7 - 1) - {}^{10}P_8 = 48185599$

23. 70;101303

24. 69760

25. $\frac{1}{2}({}^8C_7 \times 9! - {}^7C_6 \times 8!)$

26. $4! \times 3! = 144$

27. 12!×2

28. $^8C_3 = 56$

29. 350

30. (i) $^{11}C_4 = 330$ (ii) $^6C_2 \times {}^5C_2 = 150$ (iii) $^6C_4 + {}^5C_4 = 20$

31. $12 \times {}^{11}C_4 = 3960$

32. $^{12}C_3 - {}^7C_3 = 185$

33. (i) ${}^{8}C_2 \times {}^{10}C_2 = 1260$ (ii) $8 + 10 = 18$

34. (i) ${}^{5}C_3 \times {}^{6}C_2 = 150$

(ii) ${}^{5}C_4 \times {}^{6}C_1 + {}^{5}C_3 \times {}^{6}C_2 + {}^{5}C_2 \times {}^{6}C_3 = 380$

36. (i) 462 (ii) 210 (iii) $303 = {}^{9}C_5 + {}^{2}C_1 \times {}^{9}C_3 + {}^{2}C_2 \times {}^{9}C_1$

37. [(i) ${}^{11}C_3 + {}^{11}C_4 + \cdots + {}^{11}C_{11} = 2^{11} - \left({}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2\right) = 1981$

(ii) ${}^{5}C_5 \times {}^{6}C_4 + {}^{5}C_4 \times {}^{6}C_3 + {}^{5}C_3 \times {}^{6}C_2 + {}^{5}C_2 \times {}^{6}C_1 = 325$

38. 925

39. (i) 286 (ii) 165 (iii) 110 (iv) 80 (v) 276

40. ${}^{12}C_4 \times {}^{8}C_3 \times {}^{5}C_3 \times {}^{2}C_2 = 277200$

41. ${}^{10}C_4 \times {}^{6}C_3 \times {}^{3}C_3 = 4200$

42. 369600

43. 5040

44. 151351200

45. 3×3×3=27

46. $\left({}^{5}C_2 \times {}^{5}C_4\right) + \left({}^{5}C_3 \times {}^{5}C_3\right) + \left({}^{5}C_4 \times {}^{5}C_2\right) = 200$

47. $(4 \times 4 \times 4) + 4 \times {}^{4}C_2 \times {}^{3}P_2 = 208$

48. ${}^{14}C_{11} = 364$ (i) ${}^{13}C_{10} = 286$ (iii) ${}^{13}C_{11} = 78$

49.
1st part : ${}^{4}C_1 \times {}^{5}C_1 \times {}^{6}C_1 = 120$
2nd part : ${}^{18}C_3 - \left({}^{6}C_3 + {}^{7}C_3 + {}^{8}C_3\right) = 705$

50. $n+1$

51. 701

54. 10

61. 4

62. 315

63. ${}^{20}C_{18} - \left({}^{13}C_{11} + {}^{12}C_{10} + {}^{11}C_9\right) + \left({}^{5}C_3 + {}^{4}C_2 + {}^{3}C_1\right) - 0 = 10$

64. 15

65. 150

66. (i) 646 (ii) 286, 67.78

68. ${}^{13}C_{10} = 286$

69. 520

70. $\dfrac{14!}{(3!)^2 \times (2!)^5 \times 4!} = 3153150$

71. 5796

72. 1560

73. 120

74. 0

75. 146

76. 15

77. ${}^{14}C_{11} \times 15! \times 3!$.

II. Multiple Choice Question.

1. If ${}^{2n}C_3 : {}^{n}C_2 = 44:33$, the value of n is

(a) 6 (b) 5 (c) 2 (d) 7

2. The number of permutation of a set with R elements is

(a) $R!$ (b) $(R-1)!$ (c) $(R+1)!$ (d) none of these

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3. The number of diagonals that can be drawn in a polygon of n sides is

(a) $\dfrac{n(n+1)}{2}$ (b) $\dfrac{n(n+3)}{2}$ (c) $\dfrac{n(n-3)}{2}$ (d) $\dfrac{(n+3)}{2}$

4. If ${}^{n}p_r = 120 \times {}^{n}C_{n-r}$, the value of r is

(a) 5 (b) 4 (c) 6 (d) 3

5. If ${}^{n}C_1, {}^{n}C_2$ and ${}^{n}C_3$ are in A.P, the value of n is

(a) 6 (b) 7 (c) 8 (d) 4

6. The number of triangles that can be formed by joining the angular points of a decagon is

(a) 120 (b) 56 (c) 220 (d) none of these

7. A man has 6 friends . The number of ways he can invite one or more of them to a party is

(a) 64 (b) 63 (c) 32 (d) 31