PROBABILITY THEORY

Multiple Choice Type Questions

1. If $P(A)=\frac{1}{3}, P(B)=\frac{1}{4}$ and $P(A\cup B)=\frac{1}{2}$, then $P(B|A)$ is [WBUT 2012]
a) $\frac{3}{4}$ b) $\frac{4}{3}$ c) $\frac{1}{4}$ d) $\frac{1}{3}$
Answer: (c)

2. The inverse of the permutation $\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{pmatrix}$ is [WBUT 2012]
a) $\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}$ b) $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}$ c) $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix}$ d) none of these
Answer: (c)

3. The probability of an event A is $\frac{1}{3}$, that of $A+B$ is $\frac{1}{2}$ and that of AB is $\frac{1}{4}$. Then the probability of B is [WBUT 2013]
a) $\frac{1}{12}$ b) $\frac{5}{12}$ c) $\frac{1}{6}$ d) none of these
Answer: (b)

4. A problem in Mathematics is given to three students A, B and C. The chances of solving the problem by A, B and C are $\frac{1}{3},\frac{1}{4}$ and $\frac{1}{5}$ respectively. The probability that the problem will be solved is [WBUT 2014]
a) $\frac{2}{5}$ b) $\frac{3}{5}$ c) $\frac{1}{60}$ d) $\frac{47}{60}$
Answer: (b)

5. Let A and B be two events with $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}, P(A\cap B)=\frac{1}{4}$. Then $P(A/B)$ is [WBUT 2014]
a) 3/4 b) 5/4 c) 7/4 d) 2
Answer: (a)

6. Let s be a finite set containing n elements. Then the probability that a mapping $f:s\to s$ will be a birective mapping is [WBUT 2014]
a) $\frac{n^n}{n!}$ b) $\frac{n!}{n^n}$ c) $\frac{n-1}{n!}$ d) $\frac{n+1}{n!}$
Answer: (b)

7. If A and B are two events with $P(A)=0.4$, $P(B)=0.3$ and $P(A\cap B)=0.2$, then $P(A^C\cap B)$ is [WBUT 2015]
a) 0.1 b) 0.2 c) 0.3 d) 0.4
Answer: (a)

8. If two events A and B are independent, then [WBUT 2015]
a) $P(A\cap B)=P(A)P(B)$ b) $P(A+B)=P(A)+P(B)$
c) $P(A-B)=P(A)P(B)$ d) $P(A\cap B)=P(A)P(B/A)$
Answer: (a)

9. A fair die is thrown. The probability that either an odd number or a number greater than 4 will turn up is [WBUT 2015, 2017, 2018]
a) $\frac{2}{5}$ b) $\frac{3}{7}$ c) $\frac{2}{7}$ d) $\frac{2}{3}$
Answer: (d)

10. A fair die is thrown 180 times. The expected number of sixes is [WBUT 2016]
a) 10 b) 20 c) 30 d) 40
Answer: (c)

11. The probability that A passes the exam is 0.9 and B passes the exam. Is 0.8. The probability that at least one of them passes is [WBUT 2016]
a) 0.98 b) 0.97 c) 0.9 d) 0.72
Answer: (a)

12. If A and B are two events with $P(A)=0.4$, $P(B)=0.3$ and $P(A\cap B)=0.2$, then $P(A\cap B^c)$ is [WBUT 2017]
a) 0.1 b) 0.2 c) 0.3 d) 0.4
Answer: (b)

13. Inverse of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ is [WBUT 2017]

a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ d) none of these

Answer: (c)

14. Let A and B be two events such that $P(A \cup B) = \frac{7}{8}$; $P(A \cap B) = \frac{1}{4}$ and $P(\bar{A}) = \frac{5}{8}$, [WBUT 2018]

then $P(A/\bar{B})$ is

a) $\frac{1}{9}$ b) $\frac{1}{8}$ c) $\frac{1}{2}$ d) None of these

Answer: (c)

Short Answer Type Questions

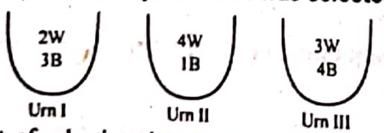
1. If $P(A \cap B) = P(A)P(B)$, then prove that $P(A^c \cap B^c) = P(A^c)P(B^c)$. [WBUT 2012]

Answer:

$$P(A^c \cap B^c) = P([A \cup B]^c)$$
$$= 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$$
$$= 1 - P(A) - P(B) + P(A)P(B) \quad \text{[By hypothesis]}$$
$$= 1 - P(A) - P(B)\{1 - P(A)\}$$
$$= \{1 - P(A)\}\{1 - P(B)\} = P(A^c)P(B^c).$$

2. Urn I has 2 white and 3 black balls. Urn II has 4 white and 1 black and Urn III has 3 white and 4 black balls. An urn is selected at random and ball drawn at random is found to be white. Find the probability that Urn I was selected. [WBUT 2015]

Answer:



Let A_i denote the event of selecting the ith urn and B denote the event of getting a white ball.

Then the required probability

$$P(A_1 | B) = \frac{P(A_1)P(B|A_1)}{\sum_{i=1}^{3} P(A_i)P(B|A_i)} = \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{4}{5} + \frac{1}{3} \times \frac{3}{7}} = \frac{14}{57}$$

3. A speaks truth 4 out of 5 times. A die is tossed. He reports that there is a six. What is the probability that it was actually a six? [WBUT 2016]

Answer:

Let A_1 denote the event of a six and A_2 denote the event no six.

Let B denote the event of A's reporting truth.

Then the required prob. $= P(A_1 | B) = \frac{P(A_1 \cap B)}{P(B)}$

$$= \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$= \frac{\frac{1}{6} \times \frac{4}{5}}{\frac{1}{6} \times \frac{4}{5} + \frac{5}{6} \times \frac{1}{5}} = \frac{2}{15} \times \frac{30}{9} = \frac{4}{9}$$

4. Show that the permutation $\begin{pmatrix} 1 & 23 & 4 & 5 \\ 3 & 21 & 5 & 4 \end{pmatrix}$ is even. [WBUT 2017]

Answer:

The given permutation

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$
$$= (1 \ 3)(4 \ 5).$$

Clearly, the given permutation has 2 transpositive.

Hence it is even.

5. An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the latter. What is the probability that it is a white ball? [WBUT 2017]

Answer:

Let B denote the event of drawing a white ball from the second urn.

Let A_1 denote the event of drawing 2 white balls from the first urn, A_2 denote that of drawing 2 blue balls and A_3 denote that of drawing 1 white and 1 blue balls from the first urn.

Then the required prob.

$$= P((A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B))$$
$$= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$

Since the events $A_1 \cap B$, $A_2 \cap B$ and $A_3 \cap B$ are mutually exclusive.

$$= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$

$$=\frac{{}^{10}C_2}{{}^{13}C_2}\times\frac{5}{10}+\frac{{}^{3}C_2}{{}^{13}C_2}\times\frac{3}{10}+\frac{{}^{10}C_1\times{}^{3}C_1}{{}^{13}C_2}\times\frac{4}{10}$$

$$=\frac{10\times9}{13\times12}\times\frac{5}{10}+\frac{3\times2}{13\times12}\times\frac{3}{10}+\frac{10\times3\times2}{13\times12}\times\frac{4}{10}$$

$$=\frac{15}{52}+\frac{3}{260}+\frac{2}{13}=\frac{59}{130}=0.4538$$

6. If A and B are two events such that $P(A^c\cup B^c)=\frac{5}{6}$, $P(A)=\frac{1}{2}$ and $P(B^c)=\frac{2}{3}$, show that A and B are independent. **[WBUT 2018]**

Answer:

We see $P(A\cap B)=1-P\left(\left[A^c\cup B^c\right]\right)=1-\frac{5}{6}=\frac{1}{6}$

$$P(A)=\frac{1}{2},\ P(B)=1-P(B^c)=1-\frac{2}{3}=\frac{1}{3}$$

Clearly, $P(A\cap B)=P(A)P(B)$

Hence A and B are independent.

7. If 10 unbiased dice are thrown at random, find the probability that the sum of the points shown by all of them is between 30 and 40.
Given $\phi(1.018)=0.3461$ **[WBUT 2018]**

Answer:

Let $X_1, X_2, \cdots\cdots, X_{10}$ be 10 variables representing the points turned up by a dice.

Now $P(30\le X_1+X_2+X_3+X_4+X_5+X_6+X_7+X_8+X_9+X_{10}\le 40)$

$$=P\left(20\le 10\overline{X}\le 30\overline{X}\right)=P\left(2\le\overline{X}\le3\right)$$

8. State Baye's theorem. Three identical boxes I, II and III contain respectively 4 white and 3 red balls, 3 white and 7 red balls, 2 white and 3 red balls. A box is chosen at random and a ball is drawn out of it. If the ball is found to be white, what is the probability that box II is selected? **[WBUT 2018]**

Answer:

1st Part: ***Refer to Question No. 1(b) (1st Part) of Long Answer Type Questions.***

2nd Part:

$\boxed{\begin{matrix}4W\\3R\end{matrix}}$ $\boxed{\begin{matrix}3W\\7R\end{matrix}}$ $\boxed{\begin{matrix}2W\\3R\end{matrix}}$

Let A_i denote the event of choosing the i^{th} box, where $i=1, 2, 3$ and B denote the event of drawing a white ball.

The required probability

$$=P(A_2\mid B)=\frac{P(B\mid A_2)P(A_2)}{\sum_{i=1}^{3}P(B\mid A_i)P(A_i)}=\frac{\frac{1}{3}\times\frac{3}{10}}{\frac{1}{3}\times\frac{4}{7}+\frac{1}{3}\times\frac{3}{10}+\frac{1}{3}\times\frac{2}{5}}=\frac{105}{132}$$

9. State the axiomatic definition of probability. **[MODEL QUESTION]**

Answer:

Let E be a random experiment with sample space S, then probability of any event point A is denoted by $P(A)$, satisfying the following conditions;

1. $P(A)\ge 0$
2. $P(S)=1$
3. For any finite or infinite sequence of pairwise mutually exclusive and exhaustive events $A_1, A_2, A_3, \ldots\ldots$; $P(A_1\cup A_2\cup A_3\cup\ldots\ldots)=P(A_1)+P(A_2)+P(A_3)+\ldots\ldots$

10. If two events A and B are independent, then prove that the events A^C and B^C are also independent. **[MODEL QUESTION]**

Answer:

We see

$$P(A^C\cap B^C)=P(A\cup B)^C=1-P(A\cup B)$$
$$=1-P(A)-P(B)+P(A\cap B)$$
$$=1-P(A)-P(B)+P(A)P(B)\ [\because A \text{ and } B \text{ are independent}]$$
$$=\{1-P(A)\}\{1-P(B)\}$$
$$=P(A^C)P(B^C)$$

Hence A^C and B^C are independent.

11. If A box contains 10 pairs of shoes and 8 shoes are randomly selected, what is the probability that there is (i) no complete pair, (ii) exactly three complete pairs? **[MODEL QUESTION]**

Answer:

The eight shoes are chosen from 20 shoes. This can be done in ${}^{20}C_8$ which are equally likely, mutually exclusive and exhaustive.

i) In order that no pair comes up, we have to choose 8 shoes from 8 different pairs of shoes and that can be done in ${}^{10}C_8$ different ways.

Again, one shoe can be drawn from a pair of shoes in 2 ways.

Hence the required probability is $\frac{{}^{10}C_8.2^8}{{}^{20}C_8}=.09145$.

ii) For exactly three complete pairs; we have to draw 3 pairs of shoes from 10 pairs and 2 shoes from remaining 7 pairs.

Hence the required probability is $\frac{{}^{10}C_3 \cdot {}^{7}C_2 \cdot 2^2}{{}^{20}C_8}$.

12. Show that the probability of occurrence of only one of the events A and B is $P(A)+P(B)-2P(AB)$. **[MODEL QUESTION]**

Answer:

We know $A\cup B=(A\cap B^C)\cup(A\cap B)\cup(A^C\cap B)$ and $A\cap B^C, A\cap B, A^C\cap B$ are disjoint sets.

Hence, $P(A\cup B)=P(A\cap B^C)+P(A\cap B)+P(A^C\cap B)$

Now, $P(\text{Only one event happens})=P\left((A^C\cap B)\cup(A\cap B^C)\right)$

$=P(A^C\cap B)+P(A\cap B^C)$

$=P(A\cup B)-P(A\cap B)$

$=P(A)+P(B)-2P(A\cap B)$

13. A can hit a target 4 times in 5 shots, B 3 times in 4 shots and C twice in 3 shots. If they fire at a target, find the probability that at least two shots hit the target. **[MODEL QUESTION]**

Answer:

Let A denote the event of A's hitting the target and B denote the event of B's hitting the target and C denote the event of C's hitting the target.

Then $P(A)=\frac{4}{5}, P(B)=\frac{3}{4}, P(C)=\frac{2}{3}$

$\therefore P(\text{at least two hits the target})$

$=P\left(A\cap B\cap C^C\right)\cup\left(A\cap B^C\cap C\right)\cup\left(A^C\cap B\cap C\right)\cup\left(A\cap B\cap C\right)$

$=P\left(A\cap B\cap C^C\right)+\left(A\cap B^C\cap C\right)+\left(A^C\cap B\cap C\right)+\left(A\cap B\cap C\right)$

$=P(A)P(B)P(C^C)+P(A)P(B^C)P(C)+P(A^C)P(B)P(C)+P(A)P(B)P(C)$

14. There were three candidates A, B and C for the position of a manager whose chances of getting the appointment are in the proportion 4:2:3. the probability that A, if selected would launch a new product in the market is 0.3. The probabilities of B and C doing the same as 0.5 and 0.8 respectively. What is the probability that the new product was launched in the market by C? **[MODEL QUESTION]**

Answer:

Let A, B and C denote respectively the events of A, B and C's getting appointment. The $P(A)=\frac{4}{9}, P(B)=\frac{2}{9}, P(C)=\frac{3}{9}$

P (a new product will be launched) = P (it is launched by any one of A, B, C)

= P (A or B or C is selected)

$=P(A\cup B\cup C)$

$=P\left(\left(A^C\cap B^C\cap C^C\right)^C\right)$

$=1-P\left(A^C\cap B^C\cap C^C\right)$

$=1-P(A^C)P(B^C)P(C^C)$

$=1-\frac{5}{9}\times\frac{7}{9}\times\frac{6}{9}=1-\frac{70}{243}=\frac{173}{243}$

15. A and B throw alternatively a pair of dice. A wins if he throws 8 before B throws 5 and B wins if he throws 5 before A throws 8. Find the probability of A wins. **[MODEL QUESTION]**

Answer:

P (A wins) = P (A throws 8 but B throws 2 or 3 or 4 or 5)

= P (A throws 8 and B throws 2) + P (A throws 8 and B throws 3) + P (A throws 8 and B throws 4) + P (A throws 8 and B throws 5)

= P (A throws 8) P (B throws 2) + P (A throws 8) P (B throws 3) + P (A throws 8) P (B throws 4) + P (A throws 8) P (B throws 5)

$=\frac{5}{36}\times\frac{1}{36}+\frac{5}{36}\times\frac{2}{36}+\frac{5}{36}\times\frac{3}{36}+\frac{5}{36}\times\frac{4}{36}=\frac{25}{648}$

16. If A and B are not mutually exclusive events, then show that $P(A\cup B)=P(A)+P(B)-P(A\cap B)$. **[MODEL QUESTION]**

Answer:

As $n(A\cup B)=n(A)+n(B)-n(A\cap B)$, if Ω denotes the sample space, then dividing both sides by $n(\Omega)$ we get

$$\frac{n(A\cup B)}{n(\Omega)}=\frac{n(A)}{n(\Omega)}+\frac{n(B)}{n(\Omega)}-\frac{n(A\cap B)}{n(\Omega)}$$

or, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

17. A bag contains 5 white and 4 black balls. If 3 balls are drawn at random, what are the probabilities of the following? **[MODEL QUESTION]**
i) 2 of them are white
ii) atmost one of them is white
iii) at least two are white.

Answer:

i) Now, P (2 of them are white) $= \frac{{}^5C_2\,{}^4C_1}{{}^9C_3}$.

P (at most one of them is white) = P (none is white exactly one is white)

$= \text{P (none is white)} + \text{P (exactly one is white)} = \frac{{}^4C_3}{{}^9C_3} + \frac{{}^5C_1\,{}^4C_2}{{}^9C_3} = \frac{1}{21} + \frac{5}{14} = \frac{17}{42}$

ii) The total no. of equally likely, mutually exclusive and exhaustive cases is

$${}^9C_3 = \frac{9\times 8\times 7}{6} = 84$$

iii) P (at heat two are white)
= P (Two are white or all three are white only)
= P (Two are white) + P (all three are white)

$$= \frac{{}^5C_2\,{}^4C_1}{{}^9C_3} + \frac{{}^5C_3}{{}^9C_3} = \frac{40}{84} + \frac{10}{84} = \frac{50}{84} = \frac{25}{42}.$$

18. One ticket is selected at random from 100 tickets numbered 00, 01, 02, ... 98, 99. If X and Y denote the sum and product of the digit on the tickets respectively. Find P (X = 9 / Y = 0). **[MODEL QUESTION]**

Answer:

$$P(X=9/Y=0) = \frac{P(X=9)\cap(Y=0)}{P(Y=0)}$$

Now, $P(X=9) = P\{09,18,27,36,45,54,63,72,81,90\} = \frac{10}{100} = \frac{1}{10}$

$P(Y=9) = P\{00,01,02,03,04,05,06,07,08,09,10,20,30,40,50,60,70,80,90\} = \frac{19}{100}$

$P((X=9)\cap(Y=0)) = P(\{09,90\}) = \frac{2}{100} = \frac{1}{50}$

$$\therefore P(X=9/Y=0) = \frac{\frac{2}{100}}{\frac{19}{100}} = \frac{2}{19}$$

19. Three groups of children consists of 3 girls and 1 boy, 2 girls and 2 boys and 1 girl and 3 boys respectively. One child is selected from each group. Show that the probability that the 3 selected children consisting of 1 girl and 2 boys is 13/32. **[MODEL QUESTION]**

Answer:

Here Group I: 3 girls, 1 boy
Group II: 2 girls, 2 boys
Group III: 1 girl, 3 boys

$\therefore$ P (1 girl and 2 boys) $= P((G_1\cap B_2\cap B_3)\cup(B_1\cap G_2\cap B_3)\cup(B_1\cap B_2\cap G_3))$

Where G_i denotes a girl from the i[th] group and B_j denotes a boy from the j[th] group.

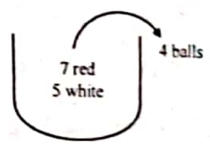
$$= P(G_1)P(B_2)P(B_3) + P(B_1)P(G_2)P(B_3) + P(B_1)P(B_2)P(G_3)$$

$$= \frac{3}{4}\times\frac{2}{4}\times\frac{3}{4} + \frac{1}{4}\times\frac{2}{4}\times\frac{3}{4} + \frac{1}{4}\times\frac{2}{4}\times\frac{1}{4} = \frac{9+3+1}{32} = \frac{13}{32}$$

20. A bag contains 7 red and 5 white balls. 4 balls are drawn at random. What is the probability that (i) all of them are red (ii) two of them be white and two red? **[MODEL QUESTION]**

Answer:

Clearly 4 balls can be drawn from the bag in ${}^{12}C_4$ ways which are equally likely, mutually exclusive and exhaustive.



Now, P (all are red) $= \frac{{}^7C_4}{{}^{12}C_4}$

P (Two white & 2 red) $= \frac{{}^7C_2\times{}^5C_2}{{}^{12}C_4}$.

Long Answer Type Questions

1. a) A box contains 5 red balls and 10 white balls. Two balls are drawn at random from the box without replacement.
What is the probability that
i) the second ball is white,
ii) the first ball drawn is red, given the second ball drawn is white? **[WBUT 2012]**

Answer:

Let R_1 denote the event of a red ball in the 1[st] drawing and W_1 and W_2 denote the event of a white ball in the 1[st] and 2[nd] drawings respectively.

Then the required probability $= P(W_2) = P(R_1 \,\&\, W_2) + P(W_1 \,\&\, W_2)$

$$= P(R_1)P(W_2|R_1) + P(W_1)P(W_2|W_1)$$

$$= \frac{5}{15}\times\frac{10}{14} + \frac{10}{15}\times\frac{9}{14} = \frac{2}{3}.$$

Similarly, $P(R_1|W_2)=\frac{P(R_1 \& W_2)}{P(W_2)}=\frac{50/210}{2/3}=\frac{5}{14}$.

b) State and prove Bayes' theorem. **[WBUT 2012]**

Answer:

Bayes' theorem

If an event B can occur in conjunction with and subsequent to one of the n mutually exclusive and exhaustive events $A_1, A_2, \ldots, A_n$ and if B actually happens, i.e., $P(B)>0$, then the probability that B is predicted by a particular event A_i is given by

$$P(A_i|B)=\frac{P(B\cap A_i)}{\sum_{j=1}^{n}P(A_j)P(B|A_j)}=\frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n}P(A_j)P(B|A_j)}.$$

Proof: By the theorem of conditional probability

$$P(A_i\,|\,B)=\frac{P(A_i\cap B)}{P(B)}\quad, P(B)\neq 0$$

Since $A_1, A_2, \cdots, A_n$ are mutually exclusive and exhaustive

$$\Omega=\bigcup_{i=1}^{n}A_i \text{ where } \Omega \text{ is the sample space}$$

$\therefore$ $B=B\cap\Omega=B\cap\left(\bigcup_{i=1}^{n}A_i\right)=\bigcup_{i=1}^{n}(B\cap A_i)$ and

$(B\cap A_i)\cap(B\cap A_j)=\phi$ for $i\neq j$

Hence $P(B)=P\left(\bigcup_{i=1}^{n}(A_i\cap B)\right)=\sum_{i=1}^{n}P(A_i\cap B)$

$\therefore$ $P(A_i\,|\,B)=\frac{P(A_i\cap B)}{P(B)}=\frac{P(A_i\cap B)}{\sum_{i=1}^{n}P(A_i\cap B)}=\frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n}P(A_i)P(B|A_i)}$

This completes the proof.

2. a) In a bolt factory, machines A,B,C manufacture respectively 25%, 35%, 40%. Of the total of their output 5%, 4%, 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probability that it was manufactured by machines A,B and C? **[WBUT 2012]**

Answer:

Let D denote the event of a bolt being defective.

By Bayes' theorem,

Then $P(A|D)=\frac{P(D|A)P(A)}{P(D|A)P(A)+P(D|B)P(B)+P(D|C)P(C)}$

$$=\frac{0.05\times0.25}{0.05\times0.25-0.04\times0.35+0.02\times0.40}=\frac{0.0125}{0.0125+0..0140+0.0080}$$

$$=\frac{0.0125}{0.0345}=0.3623$$

Similarly, $P(B|D)$ & $P(C|D)$ can be calculated.

b) The lifetime of a certain brand of an electric bulb may be considered as a random variable with mean 1200h and s.d. 250h. Find the probability, using Central Limit theorem, that the average lifetime of 60 bulbs exceeds 1250h. **[WBUT 2012]**

Answer:

Let X denote the lifetime of the electric bulb.

Then $\mu=1200$ hrs., $\sigma=250$ hrs. Further $n=60$.

The required probabilities

$$=P(\overline{X}>1250\text{ hrs.})$$

$$=P\left(\frac{\overline{X}-1200}{250/\sqrt{60}}>\frac{1250-1200}{250/\sqrt{60}}\right)$$

$$=P\left(z>\frac{\sqrt{60}}{50}\right)\qquad \text{where } z=\frac{\overline{X}-1200}{250/\sqrt{60}}\sim N(0,1) \text{ by CLT}$$

$$=P\left(z>\frac{\sqrt{15}}{25}\right)$$

$$=P(z>0.15)=0.47 \text{ approx.}$$

3. a) A box contains 5 defective and 10 non-defective lamps, 8 lamps are drawn at random in succession without replacement. What is the probability that the 8^{th} lamp drawn is the 5^{th} defective one? **[WBUT 2015]**

Answer:

Since 8^{th} lamp is the 5^{th} defective, four defectives happened in the earlier seven lamp drawn.

The required prob = P (4 defs in 7 drawings & 5^{th} defective in the 8^{th} drawing)

$$=\frac{{}^5C_4\,{}^{10}C_3}{{}^{15}C_7}\times\frac{1}{8}=\frac{5\times\frac{\lfloor 10}{\lfloor 7\lfloor 3}}{\frac{\lfloor 15}{\lfloor 8\lfloor 7}}\times\frac{1}{8}=\frac{5}{429}$$

b) It is seen that a cricketer becomes out within 10 runs in 3 out of 10 innings. If he plays 4 innings, what is the probability that he will becomes (i) out twice (ii) out at least once within 10 runs. **[WBUT 2015]**

Answer:

Let p denote the prob. That a cricketer becomes out within 10 runs

Then $p = 3/10$

The cricketer plays 4 innings.

Now P (He will be out 2 times) $= {}^4C_2 (3/10)^2 (7/10)^2$

P (He will be out at least once) $= 1 - P$ (He will not be out in 4 innings)

$$= 1 - {}^4C_0 (3/10)^0 (7/10)^4 = 1 - (7/10)^4$$

4. Experience shows that 20% of the people reserving tables at a certain restaurant never show up. If the restaurant has 50 tables and it takes 52 reservations, then find the probability that it will be able to accommodate everyone. **[WBUT 2016]**

Answer:

Let p denote the probability that a person who booked a table will not turn up.

Then $= \frac{20}{100} = \frac{1}{5}$.

$\therefore P$ (Everyone of 52 reservers will be accommodated)

$= P$ (2 will not turn up)

$$= {}^{52}C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{50}$$

RANDOM VARIABLE & DISTRIBUTION FUNCTION

Multiple Choice Type Questions

1. The variance of a random variable x is **[WBUT 2012, 2015]**

a) $\{E(x)\}^2$ b) $E(x^2)$ c) $E(x^2) - \{E(x)\}^2$ d) $E(x^2) - E(x)$

Answer: (c)

2. A random variable x has the following p.d.f: **[WBUT 2012, 2015]**

$$f(x) = \begin{cases} k, & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

then the value of k is

a) $\frac{1}{12}$ b) $\frac{1}{2}$ c) $\frac{1}{4}$ d) $\frac{1}{8}$

Answer: (c)

3. If the exponential distribution is given by the probability density function $f(x) = e^{-x}$, $0 < x < \infty$, then the mean of the distribution is **[WBUT 2013]**

a) 1 b) 3 c) $\frac{1}{3}$ d) none of these

Answer: (a)

4. The distribution for which the mean and variance are equal is **[WBUT 2013]**

a) poisson b) normal c) binomial d) exponential

Answer: (a)

5. In a Binomial (n, p) distribution, if its mean and variance are 2 and 4/3 respectively, then the values of n and p are **[WBUT 2013]**

a) $8, \frac{1}{4}$ b) $6, \frac{1}{3}$ c) $4, \frac{1}{2}$ d) none of these

Answer: (b)

6. The mean of Binomial variate is **[WBUT 2013]**

a) np b) $np(1-p)$ c) $\sqrt{np}$ d) none of these

Answer: (a)

7. The probability that a leap year selected at random will contain 53 Sundays is [WBUT 2014]

a) 2/53 b) 52/53 c) 1/7 d) 2/7

Answer: (d)

8. The mean and variance of a distribution is given to be 10 and 6 respectively. Then the distribution is [WBUT 2014]

a) Standard Normal Distribution b) Binomial Distribution
c) Poisson Distribution d) None of these

Answer: (b)

9. A random variable X has the following probability density function: [WBUT 2014]

$$f(x)=\begin{cases} kx & 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}$$

The value of k is

a) 1 b) 2 c) 4 d) none of these

Answer: (b)

10. In a Poisson distribution if $2P(x=1)=P(x=2)$, then the variance is [WBUT 2014]

a) 0 b) –1 c) 4 d) 2

Answer: (c)

11. The mean of a uniform distribution with parameters a and b is [WBUT 2015]

a) $b-a$ b) $b+a$ c) $\frac{(a+b)}{2}$ d) $\frac{b-a}{2}$

Answer: (c)

12. Let X be a Poisson Random Variate and $E(X)=\lambda$. Then $E\left[(X+1)^2\right]$ will be [WBUT 2015]

a) λ b) $\lambda^2+2\lambda$ c) $\lambda^2+2\lambda+1$ d) $\lambda^2+3\lambda+1$

Answer: (d)

13. Let X be a random variable and λ be a real number. Then $\frac{\operatorname{var}(\lambda X)}{\operatorname{var}(X)}$ is [WBUT 2016]

a) λ b) λ^2 c) $\frac{1}{\lambda}$ d) $\frac{1}{\lambda^2}$

Answer: (b)

14. For a random variable X with mean 0, the value of $P(-4\sigma<X<4\sigma)$ will be at least [WBUT 2016]

a) $\frac{1}{16}$ b) $\frac{1}{4}$ c) $\frac{15}{16}$ d) $\frac{13}{16}$

Answer: (c)

15. The variance of a rectangular uniform distribution with parameters a and b is [WBUT 2017]

a) $\frac{b-a}{2}$ b) $\frac{(b-a)^2}{12}$ c) $\frac{(a+b)^2}{12}$ d) $\frac{(b-a)^2}{2}$

Answer: (b)

16. A random variable X has the following probability density function:

$$f(x)=\begin{cases} kx, & 1\le x\le 2 \\ 0, & \text{otherwise} \end{cases}$$

The value of k is [WBUT 2017]

a) 1 b) 2 c) 2/3 d) none of these

Answer: (c)

17. The mean and standard deviation of standard normal variable are [WBUT 2017]

a) 0, –1 b) 1, 1 c) 0, 1 d) 1, 0

Answer: (c)

18. If $Var(X)=\frac{2}{3}$, then $Var(3X+5)$ is [WBUT 2018]

a) 8 b) 2 c) 6 d) 11

Answer: (c)

19. If X be binomially distributed with mean 2 and variance $\frac{4}{3}$, then value of n and p are respectively [WBUT 2018]

a) $5, \frac{1}{2}$ b) $7, \frac{1}{5}$ c) $6, \frac{1}{3}$ d) None of these

Answer: (c)

20. A random variable X has uniform distribution over $(-4, 4)$, then $P(1\times 1\le 2)$ is [WBUT 2018]

a) $\frac{1}{2}$ b) $\frac{1}{8}$ c) $\frac{7}{8}$ d) None of these

Answer: (b)

21. If $F(x)$ is the distribution function of a random variable, then [MODEL QUESTION]
   a) $F(x)$ is continuous at all points
   b) $F(x)$ is monotonic decreasing function
   c) $F(-\infty)=1$ d) $F(+\infty)=1$

Answer: (d)

22. If X is a discrete random variable, then [MODEL QUESTION]
   a) $E(|X|)\le|E(X)|$ b) $E(|X|)\ge|E(X)|$
   c) $E(|X|)=|E(X)|$ d) none of these

Answer: (b)

23. The value of k for which $f(x)=kx(1-x),\ 0<x<1=0$, otherwise will be the p.d.f. of a random variable X is [MODEL QUESTION]
   a) 6   b) 2   c) 1   d) 3

Answer: (a)

24. If the exponential distribution is given by the probability density function
$f(x)=e^{-x},\ 0<x<\infty$ [MODEL QUESTION]
then the mean of the distribution is
   a) 1   b) 3   c) $\frac{1}{3}$ d) 4

Answer: (a)

25. The probability $P(a<x\le b)$ (where $F(x)$ is the distribution function of the random variable x) is given by [MODEL QUESTION]
   a) $F(b)-F(a)$ b) $F(b)+F(a)$ c) $F(a)-F(b)$ d) $F(a)F(b)$

Answer: (a)

26. The mean of Binomial distribution Bin (n,p) (where n and p are the Number of trails and probability of success) is [MODEL QUESTION]
   a) $\frac{n}{p}$ b) 0   c) np d) 1

Answer: (c)

27. A random variable X has Poisson distribution such that P (1) = P (2). Then the standard deviation of X is [MODEL QUESTION]
   a) 0   b) 2   c) $\sqrt{2}$ d) –2

Answer: (c)

28. The mean of the Poisson distribution is μ, then its standard deviation is [MODEL QUESTION]
   a) $\frac{1}{\sqrt{\mu}}$ b) $\sqrt{\mu}$ c) μ d) $\frac{1}{\mu}$

Answer: (b)

29. The variance of the binomial distribution $B(n,p)$ is [MODEL QUESTION]
   a) np b) npq c) $(n-1)p$ d) $n(p-1)$

Answer: (b)

30. The mean of a Poisson distribution with parameter μ is [MODEL QUESTION]
   a) μ b) μ^2 c) $-\mu$ d) $-\mu^2$

Answer: (a)

31. The mean of the binomial distribution Bin $\left(10,\frac{2}{5}\right)$ is [MODEL QUESTION]
   a) 4   b) 6   c) 5   d) 0

Answer: (a)

Short Answer Type Questions

1. Find the mean and variance of Poisson distribution with parameter λ. [WBUT 2012]

Answer:
Let $\phi(t)$ be the m.g.f. of the Poisson distribution $P(\lambda)$.

Then $\phi(t)=E(e^{tx})=\sum_{x=0}^{\infty}e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}=e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}=e^{-\lambda}\cdot e^{\lambda e^t}=e^{\lambda(e^t-1)}$

Now, Mean $=E(X)=\phi'(0)$ where $\phi'(t)=e^{\lambda(e^t-1)}\cdot\lambda e^t$
$=\lambda$

Again, $E(X^2)=\phi''(0)$ where $\phi''(t)=\lambda^2e^te^{\lambda(e^t-1)}+\lambda e^te^{\lambda(e^t-1)}$
$=\lambda^2+\lambda$

Hence $\mathrm{Var}(X)=E(X^2)-\{E(X)\}^2=\lambda^2+\lambda-\lambda^2=\lambda$

Therefore, $\sigma_X=\sqrt{\lambda}$.

2. A normal population has a mean 0.1 and standard deviation 2.1. Find the probability that the mean of a sample of size 900 will be negative. Given that **[WBUT 2012]**

$P(|z|=1.43)=0.847.$

Answer:

Let $\overline{X}$ denote the mean of a sample of size 900.

Then the required probability $=P(\overline{X}>0)=P\left(\frac{\overline{X}-0.1}{2.1/\sqrt{900}}\right)<\frac{-0.1}{2.1/\sqrt{900}}$

$$=P(z<-1.43)=0.5+0.4235=0.9235$$

3. Examine whether the function $|x|$ in (-1, 1) and zero elsewhere is a density function. **[WBUT 2013]**

Answer:

The given function is

$$f(x)=\begin{cases}|x| & \text{for } x\in(-1,1)\\ 0 & \text{elsewhere}\end{cases}$$

Clearly $f(x)\ge 0$ on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f(x)dx=\int_{-1}^{0}(-x)dx+\int_{0}^{1}x\,dx=1$

Hence $f(x)$ is a density function as $f(x)\ge 0$ and $\int_{-\infty}^{\infty} f(x)dx=1$.

4. In a certain city, the daily consumption of electric power (in millions of kilowatt hours) is a random variable having the probability density **[WBUT 2014]**

$$f(x)=\frac{1}{9}xe^{-x/3},x>0$$
$$=0,\quad x\le 0$$

If the city's power plant has a daily capacity of 12 million kilowatt-hours, what is the probability that this power supply will be inadequate on any given day?

Answer:

Here Prob. (Power supply is inadequate on a day) $=P(X>12)=\frac{1}{9}\int_{12}^{\infty}xe^{-\frac{x}{3}}dx$

5. If the weekly wage of 10,000 workers in a factory follows normal distribution with mean and standard deviation Rs. 70 and Rs. 5 respectively, then find the expected number of workers whose weekly wages are

i) between Rs. 66 and Rs. 72

ii) less than Rs. 66.

[Given that the area under the standard normal curve between z = 0 and z = 0.4 is 0.1554 and z = 0 and z = 0.8 is 0.2881]. **[WBUT 2014]**

Answer:

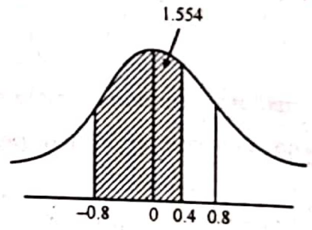
Let X denote weekly wage of a worker.

Then $X\sim N(70,25)$

Now $P(66<X<72)=P\left(\frac{66-70}{5}<\frac{X-70}{5}<\frac{72-70}{5}\right)$

$$=P(-0.8<Z<0.4)=0.6554-(1-0.7881)$$
$$=0.6554-0.2119=0.4435$$
$$=0.1554+0.2881$$

$$P(X<66)=P\left(\frac{X-70}{5}<\frac{66-70}{5}\right)=P(Z<-0.8)=0.2119$$



Hence the expected no. of workers with weekly wage

(a) between Rs.66 and Rs.72 $=0.4435\times 10{,}000=4435$

(b) less than Rs.60 $=0.2119\times 10{,}000=2119$.

6. The probability density of a random variable *z* is given by **[WBUT 2014]**

$$f(z)=\begin{cases}kze^{-z^2}; & \text{for } z>0\\ 0 & \text{for } z\le 0\end{cases}$$

Find the value of *k* and find out the corresponding distribution function of *z*.

Answer:

As $f(z)$ is a p.d.f. of a.r.v, we have $f(z)\ge 0$ and $\int_{-\infty}^{\infty} f(z)dz=1$

Thus $\int_{0}^{\infty} kze^{-z^2}dz=1$

Putting $t=z^2$, $2z\,dz=dt$, when $z=0,t=0$ and when $z\to\infty, t\to\infty$.

$\therefore\quad \int_{0}^{\infty} e^{-t}dt=\frac{2}{k}$

or, $\left[-e^{t}\right]_{0}^{\infty}=\frac{2}{k}$

or, $l=\frac{2}{k}$

or, $k=2$

The distribution function $F(z)$ is given by

$$F(z)=\int_{-\infty}^{z} f(x)dx$$

$$=\begin{cases}\int_0^z 2xe^{-x^2} & \text{if } z>0\\ 0 & \text{if } z\le 0\end{cases}$$

$$=\left[-e^{-x^2}\right]_0^z \text{ if } z>0,=0 \text{ if } z\le 0 \qquad =\begin{cases}1-e^{-z^2} & \text{if } z>0\\ 0 & \text{if } z\le 0\end{cases}$$

7. If $X_1, X_2, X_3, \ldots\ldots, X_n$ constitute a random sample of size n from an infinite population with mean μ and variance σ^2, then prove that $E(\overline{X})=\mu$ and $\text{var}(\overline{X})=\frac{\sigma^2}{n}$. **[WBUT 2016]**

Answer:

We see $\overline{X}=\frac{1}{n}(X_1+X_2+\ldots\ldots+X_n)$

$$\therefore E(\overline{X})=\frac{1}{n}\{E(X_1)+E(X_2)+\ldots+E(x_n\}=\frac{1}{n}\{\mu+\mu+\ldots+\mu\}=\frac{1}{n}\cdot n\mu=\mu$$

$$Var(\overline{X})=\frac{1}{n^2}\{Var(X_1)+Var(X_2)+\ldots+Var(X_n)\}$$

$$=\frac{1}{n^2}\{\sigma^2+\sigma^2+\ldots..+\sigma^2\}=\frac{n\sigma^2}{n^2}=\frac{\sigma^2}{n}$$

Recall $Var(X)=E(x-\overline{X})^2$

8. Determine the mean and variance of exponential distribution. **[WBUT 2017]**

Answer:

The p.d.f. of the exponential distribution is

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & \text{for} \quad 0<x<\infty,\ \lambda>0\\ 0 & \text{for} \quad x\le 0\end{cases}$$

Now Mean $=E(x)=\int_0^\infty xf(x)dx$

$$=\lambda\int_0^\infty xe^{-\lambda x}dx=\int_0^\infty xd\left(-e^{-\lambda x}\right)=\left[-xe^{-\lambda x}\right]_0^\infty+\int_0^\infty e^{-\lambda x}dx$$

$$=0+\left[e^{-\lambda x}\left(\frac{-1}{\lambda}\right)\right]_0^\infty=\frac{1}{\lambda}$$

Now, $E(X^2)=\int_0^\infty x^2f(x)dx=\lambda\int_0^\infty x^2e^{-\lambda x}dx=\int_0^\infty x^2d\left(-e^{-\lambda x}\right)$

$$=\left[-x^2e^{-\lambda x}\right]_0^\infty+\int_0^\infty 2xe^{-\lambda x}dx=0+2\int_0^\infty xe^{-\lambda x}dx=\frac{2}{\lambda}\int_0^\infty xe\left(-e^{-\lambda x}\right)$$

$$=\frac{2}{\lambda}\left[xe^{-\lambda x}\right]_0^\infty+\frac{2}{\lambda}\int_0^\infty e^{-\lambda x}dx=0+\frac{2}{\lambda}\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^\infty=\frac{2}{\lambda^2}$$

Hence $\text{var}(X)=E(X^2)-\{E(X)\}^2=\frac{2}{\lambda^2}-\left(\frac{1}{\lambda}\right)^2=\frac{1}{\lambda^2}$

9. Define probability density function of a random variable. **[MODEL QUESTION]**

Answer:

The probability density function $f(x)$ of a random variable X is defined to be a function which satisfies the conditions.

1) $f(x)\ge 0$

2) $\int_{\infty}^{\infty} f(x)dx=1$

3) $P(a<x\le b)=\int_a^b f(x)dx$

10. The life of a tyre manufactured by a company follows a continuous distribution given by the density function **[MODEL QUESTION]**

$$f(x)=\frac{k}{x^3}, \qquad 1000\le x\le 1500$$
$$=0, \qquad \text{elsewhere}$$

Find the value of k and find the probability that a randomly selected tyre would function for at least 1200 hours.

Answer:

Since $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_{1000}^{1500} \frac{k}{x^2}dx = 1$$

or, $$\int_{1000}^{1500} x^{-2}dx = \frac{1}{k}$$

or, $$\left[\frac{-1}{x}\right]_{1000}^{1500} = \frac{1}{k}$$

or, $$\frac{1}{1000} - \frac{1}{1500} = \frac{1}{k}$$

$\therefore k = 3000$

$$\therefore P(x \geq 1200\,\text{hrs}) = \int_{1200}^{1500} \frac{3000}{x^2}dx = 3000\left[-\frac{1}{x}\right]_{1200}^{1500} = 3000\left(\frac{1}{1200} - \frac{1}{1500}\right) = \frac{1}{2}$$

11. If the r.v X takes the values 1, 2, 3 and 4 such that 2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4), find the probability distribution. **[MODEL QUESTION]**

Answer:

Since $P(X=1)+P(X=2)+P(X=3)+P(X=4)=1$

$$P(X=1)+\frac{2}{3}P(X=1)+2P(X=3)+\frac{2}{5}P(X=1)=1$$

$$P(X=1)=\frac{15}{61}$$

$$\therefore P(X=2)=\frac{2}{3}\times\frac{15}{61}=\frac{10}{61}$$

$$P(X=3)=2\times\frac{15}{61}=\frac{30}{61}$$

$$P(X=4)=\frac{2}{5}\times\frac{15}{61}=\frac{6}{61}.$$

12. A r.v X has the following probability distribution **[MODEL QUESTION]**

x:	-2	-1	0	1	2	3
P(x):	0.1	k	0.2	2k	0.3	3k

Find P(-2<X<2) and the mean of X.

Answer:

Since $\sum_{x=-2}^{3} P(x) = 1$,

$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$

$6k = 0.4$

$$k = \frac{0.4}{6} = \frac{1}{15}$$

Now, $P(-2<X<2)=P(X=-1)+P(X=0)+P(X=1)=\frac{1}{15}+\frac{2}{10}+\frac{2}{15}=0.4$

Mean of $X = E(X) = (-2)(0.1)+(-1)\times\frac{1}{15}+0\times(0.2)+1\times\frac{2}{15}+3\times\frac{3}{15}$

$$= -\frac{2}{10}-\frac{1}{15}+\frac{2}{15}+\frac{9}{15}=0.46$$

13. Show that a function $f(x)=\begin{cases}|x| \text{ when } -1<x<1\\ 0 \quad \text{elsewhere}\end{cases}$ **is a p.d.f find its distribution function.** **[MODEL QUESTION]**

Answer:

We see $f(x)\geq 0$

And $$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{-1} 0dx + \int_{-1}^{1}|x|dx + \int_{1}^{\infty} 0dx = 2\int_{0}^{1} xdx = 2\left[\frac{x^2}{2}\right]_0^1 = 1$$

Hence $f(x)$ is a p.d.f

We know the distribution function $F(x)$ is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

$$= \begin{cases} 0 & \text{if } x \leq -1 \\ \int_{-1}^{x}|t|dt & \text{if } -1<x<1 \\ 1 & \text{if } x \geq 1 \end{cases} = \begin{cases} 0 & \text{if } x \leq -1 \\ \frac{x^2-1}{2} & \text{if } -1<x<0 \\ \frac{x^2+1}{2} & \text{if } 0<x<1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

14. If the weekly wages of 10,000 workers in a family follow normal distribution with mean and st-dev. Rs. 70 and Rs. 5 respectively. Find the expected no. of workers whose weekly wages are (i) between Rs. 66 and Rs. 72 (ii) less than Rs. 66 (iii) mere than Rs. 72. **[MODEL QUESTION]**

Answer:

Let X denote wage of a worker

Then $X \sim N(70,25)$

Now, $$P(66 \leq X \leq 72) = P\left(\frac{66-70}{5} \leq \frac{X-70}{5} \leq \frac{72-70}{5}\right)$$

$$= P(-0.8 \le Z \le 0.4), \text{writing } Z = \frac{X-70}{5}$$

$$P(X < 66) = P\left(\frac{X-70}{5} < \frac{66-70}{5}\right) = P(Z < -08)$$

$$P(X > 72) = P\left(\frac{X-70}{5} > \frac{72-70}{5}\right) = P(Z > 0.4)$$

Hence the expected no. of workers whose monthly ways lies between Rs.66 and Rs.72
$= 10,000 \times P(66 \le X \le 72)$

The expected no. of workers with monthly wage less than Rs.66
$= 10,000 \times P(X < 66)$

The expected no. of workers with monthly wage more than Rs.72
$= 10,000 \times P(X > 72)$

15. If the probability density function of a random variable X is given by $f(x) = ce^{-(x-2)^2/2}$ $-\infty < x < \infty$, find the value of c, the expectation and variance of the distribution. **[MODEL QUESTION]**

Answer:

As $\int_{-\infty}^{\infty} f(x)dx = 1$, we have

$$C\int_{-\infty}^{\infty} e^{-\frac{(x-2)^2}{2}} dx = 1$$

Now putting $\frac{x-2}{\sqrt{2}} = z$

$dx = \sqrt{2}dz$ and when $x \to -\infty, z \to \infty$ and when $x \to \infty, z \to \infty$.

Thus $C\int_{-\infty}^{\infty} e^{-z^2}\sqrt{2}dz = 1$ or, $C\int_{-\infty}^{\infty} e^{-z^2}dz = \frac{1}{\sqrt{2}C}$

16. If the mean of a binomial distribution is 3 and the variance is $\frac{3}{2}$, find the probability of obtaining atmost 3 success. **[MODEL QUESTION]**

Answer:

By the given condition $np = 3$, $npq = \frac{3}{2}$ $\therefore q = \frac{1}{2}$

Hence $p = \frac{1}{2}, n = 6$

Let x denote the number of success

$\therefore P(x \le 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$$= {}^6C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^6 + {}^6C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5 + {}^6C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 + {}^6C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3$$

$$= \left(\frac{1}{2}\right)^6\{1+6+15+20\} = \frac{42}{64} = \frac{21}{32}$$

17. The p.d.f. of a continuous distribution of a random variable is given by **[MODEL QUESTION]**

$f(x) = x(2-x), 0 < x < 2$

= 0, otherwise

compute mean and variance of x.

Answer:

Let X denote the r.v.

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 x^2(2-x)dx = \frac{4}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 x^3(2-x)dx = \frac{8}{3}$$

$$\therefore \text{var}(X) = E(X^2) - \{E(X)\}^2 = \frac{8}{3} - \frac{16}{9} = \frac{24-16}{9} = \frac{8}{9}$$

18. A random variable X has the following probability distribution:

X = x:	0	1	2	3	4	5	6	7
p (x) :	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2$+k

Obtain the value of k and find P (X < 6) and P (0 < X < 5). **[MODEL QUESTION]**

Answer:

By definition, $\sum_{x=0}^{z} f(x) = 1$

or, $k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

or, $10k^2 + 9k - 1 = 0$

or, $(10k-1)(k+1) = 0$

$\therefore k = \frac{1}{10}$ as k can not be negative

$\therefore P(X < 6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = 0.81$$

$$P(0<X<5)=P(X=1)+P(X=2)+P(X=3)+P(X=4)$$
$$=\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}=0.8$$

19. A fair coin is tossed 400 times. Using normal approximation to binomial distribution find the probability of obtaining (i) exactly 200 heads (ii) between 190 and 210 heads, both inclusive. Given that the area under standard normal curve between z = 0 and z = 0.05 is 0.0199 and between z = 0 and z = 1.05 is 0.3531. **[MODEL QUESTION]**

Answer:

Let X denote the no. of heads in 400 doses

Than $n=400, p=q=½, E(X)=np=200, \text{var}(X)=npq=100$ $\therefore \sigma_x=10.$

$$P(X=200)=P(199.5<\hat{X}<200.5)$$
$$=P\left(\frac{199.5-200}{10}<z<\frac{200.5-200}{10}\right)$$
$$=P(-0.05<z<0.05)$$
$$=2\times0.0199=0.0398$$
$$P(190<X<210)=P\left(\frac{189.5-200}{10}<z<\frac{210.5-200}{10}\right)$$
$$=P(-1.05<z<1.05)=2\times0.3531=0.7062$$

20. A random variable *X* has the following probability function:

$X=$	-2	-1	0	1	2	3
$P(X)=$	0.1	k	0.2	$2k$	0.3	$3k$

i) Calculate *k*
ii) Find $P(X<2)$, $P(X\ge2)$, $P(-2<X\ge2)$. **[MODEL QUESTION]**

Answer:

As $\sum_{x=-2}^{3}P(x)=1$, we get $0.1+k+0.2+2k+0.3+3k=1$

or, $6k=0.4=\frac{4}{10}=\frac{2}{5}$

$\therefore k=\frac{1}{15}$

$$P(X<2)=1-P(X=2)-P(X=3)=1-0.3-3\times\frac{1}{15}=1-0.3-0.2=0.5$$
$$P(X\ge2)=P(X=2)+P(X=3)=0.3+0.2=0.5$$
$$P(-2<X\le2)=P(X=-2)-P(X=3)=1-0.1-0.2=0.7$$

21. Find the mathematical expectation of the number of the points obtained in a single throw of an unbiased die. **[MODEL QUESTION]**

Answer:

Let X denote the no of points obtained by a single throw of an unbiased die.
Then probability distribution is as follow:

X	1	2	3	4	5	6
Prob.	1/6	1/6	1/6	1/6	1/6	1/6

$$\therefore E(X)=1.\frac{1}{6}+2.\frac{1}{6}+3.\frac{1}{6}+4.\frac{1}{6}+5.\frac{1}{6}+6.\frac{1}{6}=\frac{6\times7}{2\times6}=\frac{7}{2}=3.5$$

22. The mean weight of 500 male students at a certain college is 150 lbs and the standard deviation is 15 lbs. Assuming that the weight is normally distributed find how many students weigh. **[MODEL QUESTION]**
i) between 120 and 155 lbs
ii) more than 155 lbs.
[Given $\phi(2)=0.9772$; $\phi(0.33)=0.6293$]

Answer:

Let X denote weight of a student.

Then $X\sim N(150, 225)$, So, $Z=\frac{X-150}{15}\sim N(0,1)$

Now, $P(120\le X\le155)=P\left(\frac{120-150}{15}\le\frac{X-150}{15}\le\frac{155-150}{15}\right)$
$$=P(-2\le Z\le0.33)=(0.9772-0.5)+(0.6293-0.5)$$
$$=0.4772+0.1293=0.6065$$
$$P(X>155)=P\left(\frac{X-150}{15}>\frac{155-150}{15}\right)=P(Z>0.33)=1-0.6293=0.3707$$

Hence the no. of students having weight between 120 lbs and 155 lbs
$=500\times0.6065=303.25$
Also the no. of students having weight more than 155 lbs
$=500\times0.3707=185.35$

23. A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with average number of demand per day 1.5. Calculate the proportion of days on which neither car is used and the proportion of days on which some demand is refused. **[MODEL QUESTION]**

Answer:

Let X denotes the no of demands.
By hypothesis $X\sim P(1.5)$

$$P(0<X<5)=P(X=1)+P(X=2)+P(X=3)+P(X=4)$$
$$=\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}=0.8$$

19. A fair coin is tossed 400 times. Using normal approximation to binomial distribution find the probability of obtaining (i) exactly 200 heads (ii) between 190 and 210 heads, both inclusive. Given that the area under standard normal curve between z = 0 and z = 0.05 is 0.0199 and between z = 0 and z = 1.05 is 0.3531. **[MODEL QUESTION]**

Answer:

Let X denote the no. of heads in 400 doses

Than $n=400, p=q=\frac{1}{2}, E(X)=np=200, \text{var}(X)=npq=100$

$$P(X=200)=P(199.5<\hat{X}<200.5) \qquad \therefore \sigma_x=10.$$
$$=P\left(\frac{199.5-200}{10}<z<\frac{200.5-200}{10}\right)$$
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ii) Find $P(X<2)$, $P(X\ge2)$, $P(-2<X\ge2)$. **[MODEL QUESTION]**

Answer:

As $\sum_{x=-2}^{3}P(x)=1$, we get $0.1+k+0.2+2k+0.3+3k=1$

$$\text{or, } 6k=0\,4=\frac{4}{10}=\frac{2}{5}$$
$$\therefore k=\frac{1}{15}$$

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X	1	2	3	4	5	6
Prob.	1/6	1/6	1/6	1/6	1/6	1/6

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i) between 120 and 155 lbs

ii) more than 155 lbs.

[Given $\phi(2)=0.9772$; $\phi(0.33)=0.6293$]

Answer:

Let X denote weight of a student.

Then $X\sim N(150, 225)$, So, $Z=\frac{X-150}{15}\sim N(0,1)$

$$\text{Now, } P(120\le X\le155)=P\left(\frac{120-150}{15}\le\frac{X-150}{15}\le\frac{155-150}{15}\right)$$
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Answer:

Let X denotes the no of demands.

By hypothesis $X\sim P(1.5)$

$\therefore P(X=0)=\frac{e^{-1.5}(1.5)^0}{0!}=e^{-1.5}$

$P(X>2)=1-P(X\le 2)=1-P(X=0)-P(X=1)-P(X=2)$

$=1-\frac{e^{-1.5}.(1.5)^0}{0!}-\frac{e^{-1.5}.(1.5)^1}{1!}-\frac{e^{-1.5}.(1.5)^2}{2!}$

24. If a random variable X has Binomial distribution, find the mean and variance of the distribution with parameters (n, p). **[MODEL QUESTION]**

Answer:

The m. g. f of the Binomial distribution $B(n, p)$ is given by $\varphi(t)=(q+pe^t)^n$

$\therefore$ Mean $=E(X)=\varphi'(0)=np$

$E(X^2)=\varphi''(0)=n(n-1)p^2+np$

$\therefore \operatorname{var}(X)=E(X^2)-\{E(X)\}^2=n(n-1)p^2+np-n^2p^2=npq$

25. Let X be a Poisson distributed random variable with the particular μ; then show that E (X) = μ and Var (X) = μ. **[MODEL QUESTION]**

Answer:

The m. g. f of the Poisson distribution $P_0(\mu)$ is

given by $\varphi(t)=e^{\mu(e^t-1)}$

Now Mean $=E(X)=\varphi'(0)=\mu e^0 e^{\mu(e^0-1)}$

$E(X^2)=\varphi''(0)=(\mu e^0)^2 e^{\mu(e^0-1)}+\mu e^0 e^{\mu(e^0-1)}$

$\therefore \operatorname{var}(X)=E(X^2)-\{E(X)\}^2=\mu^2+\mu-\mu^2=\mu$

26. If the chance of being killed by flood during a year is 1/3000, use Poisson distribution to calculate probability that out of 3000 persons living in a village, at least one will die in flood in a year. **[MODEL QUESTION]**

Answer:

Here $p=\frac{1}{3000}$, $n=3000$.

$\therefore \lambda=np=1$

Hence $P(X\ge 1)=1-P(X=0)=1-\frac{e^{-1}1^0}{0!}=1-e^{-1}$

27. If x follows a Normal Distribution with mean 12 and variance 16, find $P(x\ge 20)$.

[Given: $\int_{-\infty}^{2} 1/\sqrt{2\pi}\, e^{-1/2t^2}\, dt=0.977725$] **[MODEL QUESTION]**

Answer:

Here $X\sim N(12,16)$

$\therefore P(X\ge 20)=P\left(\frac{X-12}{4}\ge\frac{20-12}{4}\right)=P(Z\ge 2)=1-0.977725=0.022275$

28. A random variable follows binomial distribution with mean 4 and standard deviation $\sqrt{2}$. Find the probability of assuming non-zero value of the variable. **[MODEL QUESTION]**

Answer:

Let X be the said r.v.

Then $X\sim B(n, p)$ where $np=4$ and $npq=2$

Solving we get $q=\frac{1}{2}$, so $p=\frac{1}{2}$, $n=8$

Hence $P(X>0)=1-P(X=0)=1-{}^8C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^8=1-\left(\frac{1}{2}\right)^8$

29. If a random variable follows Poisson Distribution such that $P(1)=P(2)$.
Find (i) mean of the distribution **[MODEL QUESTION]**
(ii) $P(4)$.

Answer:

Let X be the r.v. following Poisson distribution with $P(1)=P(2)$. Let λ be its parameter

Then $P(1)=P(2)$ gives

$\frac{e^{-\lambda}.\lambda}{1!}=\frac{e^{-\lambda}.\lambda^2}{2!}$

or, $\lambda^2-2\lambda=0$

or, $\lambda(\lambda-2)=0$

$\therefore \lambda=2$ since $\lambda=0$ is not admissible

Now, $P(4)=\frac{e^{-\lambda}.\lambda^4}{4!}=\frac{e^{-2}.2^4}{4!}=\frac{e^{-2}.\cancel{16}}{\cancel{24}}=\frac{2}{3}e^{-2}$

Mean of the distribution $=\lambda=2$

30. The probability density function of a continuous distribution is given by $f(x)=\frac{3}{4}x(2-x)$. Compute mean and variance of the distribution. **[MODEL QUESTION]**

Answer:

Let X denote the r.v. Its p.d.f. is $f(x)=\frac{3}{4}x(2-x)$

Now, Mean $= E(X)=\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} x.\frac{3}{4}x(2-x)dx = \frac{3}{4}\int_{-\infty}^{\infty} x^2(2-x)dx$

$$E(x^2)=\int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{\infty} x^2.\frac{3}{4}x(2-x)dx = \frac{3}{4}\int_{-\infty}^{\infty} x^3(2-x)dx$$

Hence $\text{Var}(X)=E(X^2)-\{E(X)\}^2$

Long Answer Type Questions

1. a) Find the mathematical expectation of the number of points obtained in a single throw of an unbiased die. **[WBUT 2013]**

Answer:

Let X denote the number of points obtained in a single throw of an unbiased die. Then the probability table of X will be

X	1	2	3	4	5	6
Prob.	1/6	1/6	1/6	1/6	1/6	1/6

Hence $E(X)=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=3\frac{1}{2}$

b) Define Poisson distribution and find its mean and variance. **[WBUT 2013]**

Answer:

The Poisson distribution with parameter λ is defined by the probability mass function $f(x)$ given by $f(x)=\frac{-e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,\cdots\cdots,n,n+1,\cdots\cdots$ to ∞.

The constant λ is called the parameter of the distribution.

Let X be a Poisson variate, i.e., $X\sim P(\lambda)$

Then Mean $=E(X)=\sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\left\{\frac{1\cdot\lambda}{1!}+\frac{2\cdot\lambda^2}{2!}+\frac{3\cdot\lambda^3}{3!}+\cdots\cdots\right\}$

$$=\lambda e^{-\lambda}\left\{1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\cdots\cdots\right\}=\lambda e^{-\lambda}\cdot e^{\lambda}=\lambda.$$

Again, $E(X^2)=\sum_{x=0}^{\infty} x^2\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\{x(x-1)+x\}\frac{\lambda^x}{x!}$

$$=e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^x}{(x-2)!}+e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!}=e^{-\lambda}\cdot\lambda^2\cdot e^{\lambda}+e^{-\lambda}\cdot\lambda\cdot e^{\lambda}$$

$$=\lambda^2+\lambda$$

Hence, $\text{var}(X)=E(X^2)-\{E(X)\}^2=\lambda^2+\lambda-\lambda^2=\lambda.$

2. a) Determine the mean and variance of exponential distribution. **[WBUT 2013]**

Answer:

The exponential distribution $E(\mu)$ is given by the p.d.f.

$$f(x)=\begin{cases}\mu e^{-\mu x} & \text{if } x\ge 0\\ 0 & \text{if } x<0\end{cases},\quad \mu>0$$

Now Mean $=E(X)=\int_0^{\infty} x\mu e^{-\mu x}dx$

Putting $\mu x=y$, $\mu dx=dy$

$\therefore$ $E(X)=\int_0^{\infty} ye^{-y}\frac{dy}{\mu}=\frac{1}{\mu}\int_0^{\infty} yd\left(-e^{-y}\right)=\frac{1}{\mu}\left[-ye^{-y}\right]_0^{\infty}=\frac{1}{\mu}$

Similarly, $E(X^2)=\int_0^{\infty} x^2\mu e^{-\mu x}dx=\frac{2}{\mu^2}$

Hence, $\text{var}(X)=E(X^2)-\{E(X)\}^2=\frac{2}{\mu^2}-\frac{1}{\mu^2}=\frac{1}{\mu^2}.$

b) The probability density function of a random variable X is assumed to be of the form $f(x)=cx^{\alpha}, 0\le x\le 1$ for some number and constant c. If $\{X_1,X_2,\cdots X_n\}$ is a random sample of size n, find the maximum likelihood estimate of α. **[WBUT 2013, 2018]**

Answer:

Here $f(x)=cx^a$, $0\le x\le 1$, $c>0$

As $f(x)$ is a p.d.f. we get

$$\int_{-\infty}^{\infty} f(x)dx=1 \quad \text{i.e.,} \int_0^1 cx^a dx=\frac{c}{a+1}=1 \quad \text{or, } c=a+1$$

If L denotes the likelihood function, we have

$$L(x_1,x_2,\ldots\ldots,x_n;a)=cx_1^a\cdot cx_2^a\ldots\ldots cx_n^a=(a+1)^n(x_1x_2\ldots\ldots x_n)^a$$

$\therefore$ $\log L=n\log(a+1)+\sum_{i=1}^{n} a\log x_i$

Now, $\frac{\partial}{\partial a}(\log L)=0$ gives $\frac{n}{a+1}+\sum_{i=1}^{n}\log x_i=0$

or, $a = -1 - \dfrac{n}{\sum\limits_{i=1}^{n} \log x_i}$

Clearly, $\dfrac{\partial^2}{\partial a^2}(\log L) = -\dfrac{n}{(a+1)^2} < 0$ for $a = -\left(1 + \dfrac{n}{\sum \log x_i}\right)$

So, $-\left(1 + \dfrac{n}{\sum\limits_{i=1}^{n} \log x_i}\right)$ is the maximum likelihood estimate of a.

3. a) Suppose that an airplane engine will fail, when in flight, with probability $(1-p)$ independently from engine to engine; suppose that the airplane will make a successful flight if at least 50% of its engines remain operative. For what, values of p is a four-engine plane preferable to a two-engine plane? **[WBUT 2014]**

Answer:

A four-engine plane is preferable to a two-engine plane if 4-engine plane flies more successfully than a 2-engine plane i.e., if two or more engines of a 4-engine plane remain operative

P(1 or more engines of a 2-engine plane remain operative)

But, P(2 or more engines remains operative)

$$= {}^4C_2 p^2 (1-p)^2 + {}^4C_3 p^3 (1-p) + {}^4C_4 p^4$$

P(1 or more engines remains operative) $= {}^2C_1 p(1-p) + {}^2C_2 p^2$

Hence the required condition is

$${}^4C_2 p^2 (1-p)^2 + {}^4C_3 p^3 (1-p) + {}^4C_4 p^4 > {}^2C_1 p(1-p) + {}^2C_2 p^2$$

or, $6p^2(1-p)^2 + 4p^3(1-p) + p^4 - 2p(1-p) - p^2 > 0$

or, $6p^2 - 12p^3 + 6p^4 + 4p^3 - 4p^4 + p^4 - 2p + 2p^2 - p^2 > 0$

or, $3p^4 - 8p^3 + 7p^2 - 2p > 0$

or, $3p^3 - 8p^2 + 7p - 2 > 0$ [As $p > 0$ by definition]

$3p^2 - 5p + 2 > 0$ as $p \neq 1$

$p^2 - \frac{5}{3}p + \frac{2}{3} > 0$

$\left(p - \frac{5}{6}\right)^2 > \frac{25}{36} - \frac{2}{3} = \frac{1}{36}$

or, $p > \frac{5}{6} + \frac{1}{6}$

or, $p < \frac{5}{6} - \frac{1}{6}$ or, $< \frac{2}{3}$

b) Find the mean of an uniform distribution. **[WBUT 2014]**

Answer:

The uniform distribution $U(a,b)$ of a.r.v. X is given by the p.d.f.

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean} = \int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{a+b}{2}$$

4. a) Define Normal distribution and find its mean, variance and standard deviation. **[WBUT 2015]**

Answer:

A random variable X is said to have normal distribution if its probability density function is given by $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \quad -\infty < x < +\infty$.

Where m, $\sigma(>0)$ are two parameters of this distribution. If any random variable have normal distribution with parameters m and σ then we write $X \sim N(m,\sigma)$.

Mean: $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \dfrac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-m) e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx + \frac{m}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$$

Now we substitute $\dfrac{x-m}{\sigma\sqrt{2}} = z$ for the first integral;

$$= \sigma\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} z e^{-z^2}\,dx + m \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$$

$= 0 + m$

$= m$

Variance: $E(X-m)^2 = \int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx = \dfrac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-m)^2 e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$

Substitute: $\dfrac{x-m}{\sigma\sqrt{2}} = z$;

$$=\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} z^2 e^{-z^2}dx=\frac{4\sigma^2}{\sqrt{\pi}}\int_0^{\infty} z^2 e^{-z^2}dx$$

Subs: $z^2=t \Rightarrow 2zdz=dt$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\int_0^{\infty} t^{\frac{1}{2}}e^{-t}dx=\frac{2\sigma^2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)=\frac{2\sigma^2}{\sqrt{\pi}}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\sigma^2.$$

Standard deviation $=\sqrt{\text{Variance}}=\sigma$.

b) A normal population has mean 0.1 and standard deviation 2.1. Find the probability that the mean of a sample of size 900 will be negative. [Given that $P(|z|<1.43=0.847)$] [WBUT 2015]

Answer:

Here $X\sim N(0,1,2.1^2)$, $n=900$. That is, $\mu=0.1$, $\sigma=2.1$

The required probability $=P(\bar{X}<0)$

$$=P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}<\frac{-\mu}{\sigma/\sqrt{n}}\right)=P\left(Z<\frac{-0.1}{2.1/30}\right)$$

$$=P(Z<-1.43)=\frac{1}{2}(1-0.847)=-0.0765$$

5. The mean yield for one acre plot is 662 kilos with a standard deviation 32 kilos. Assuming normal distribution how many one acre plots in a batch of 1000 plots would you expect to have yield over 700 kilos? (Given that $\phi(1.19)=0.3830$) [WBUT 2016]

Answer:

Here, $\mu=662$ kilos, $\sigma=32$ kilos.

Let X denote yield of an one acre plot.

Then $P(X>700 \text{ kilos})=P\left(\frac{X-\mu}{\sigma}>\frac{700-662}{32}\right)$

$$=P\left(Z>\frac{38}{32}\right),\ \text{denoting } Z=\frac{X-\mu}{\sigma}$$

$$=P(Z>1.19)=0.5-0.3850=0.115$$

Hence no of plots yielding more than 700 kilos $=1000\times 0.115=115$

6. a) Let X be uniformly distributed over the interval $([1,2]$ and $\bar{X}=E(X))$. Find out the value of a so that $P(X>a+\bar{X})=\frac{1}{6}$. [WBUT 2016]

Answer:

Given $X\sim U(1,2)$. Then its $p-d-f$ $f(x)$ is given by

$$f(x)=\begin{cases}1 \text{ if } 1<x<2\\ 0 \text{ if otherwise}\end{cases}$$

$$E(X)=\int_{-\infty}^{\infty} x f(x)dx=\int_1^2 x\,dx=\left[\frac{x^2}{2}\right]_1^2=\frac{3}{2}$$

Now, $P\left(X>a+\frac{3}{2}\right)=\frac{1}{6}$

or, $\int_{a+3/2}^{\infty} f(x)dx=\frac{1}{6}$

or, $\int_{a+3/2}^{2} 1.dx=\frac{1}{6}$

or, $2-a-3/2=\frac{1}{6}$

or, $a=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$

b) A random sample of size $n=100$ is taken from an infinite population with the mean $\mu=75$ and the variance $\sigma^2=256$. Based on Chebyshev's theorem with what probability can we assert that the value we obtain for $\bar{X}$ will fall between 67 and 83? [WBUT 2016]

Answer:

Here, $\mu=75$, $\sigma^2=256$, $n=100$

By Chebyshev's theorem, we get

$$P\left(\left|\bar{X}-E(\bar{X})\right|\ge\varepsilon\right)\le\frac{Var(\bar{X})}{\varepsilon^2}$$

or, $P\left(\left|\bar{X}-75\right|\ge\varepsilon\right)\le\frac{256/100}{\varepsilon^2}$

or, $P\left(75-\varepsilon<\bar{X}<75+\varepsilon\right)\ge 1-\frac{2.56}{\varepsilon^2}$

Taking $\varepsilon=8$, we get $P(67<\bar{X}<83)\ge 1-0.04=0.96$

So the probability that $\bar{X}$ lies between 67 and 83 is at least 0.96.

c) If X has the standard normal variate, then find the probability density function of $Z = X^2$. **[WBUT 2016]**

Answer:

In real variable $z = x^2$

So $\frac{dz}{dx} = 2x, \; -\infty < x < \infty$

Now $P(z < Z \le z + dz) = P\left(x^2 < X^2 \le (x+dx)^2\right)$

If $x > 0$,

then the event $(x < X \le x + dx) \cup (-(x+dx) \le X, < -x) = (x^2 < X^2 \le (x+dx)^2)$

and $(x < X \le x + dx) \& (-(x+dx) < X < -x)$ are mutually exclusive.

$\therefore P\left(x^2 < X^2 \le (x+dx)^2\right) = P(x < X \le x + dx) + P(-(x+dx) < X < -x)$

$= 2P(x < X \le x + dx)$ due to symmetry

$\therefore f_Z(z)dz = 2f_X(x)$, where $f_X^{(x)}$ and $f_Z^{(z)}$ are the density functions of X and Z.

Now, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

So, $f_z(z) = 2\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{dx}{dz} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{2}{2x} = \frac{e^{-z} z^{-\frac{1}{2}}}{\sqrt{2\pi}}$ as $x = \sqrt{z}$

Thus if $x > 0$, $f_z(z) = \frac{e^{-z} z^{-1/2}}{\sqrt{2\pi}}, 0 < z < \infty$

It can be similarly shown that if $x < 0$, $f_z(z) = \frac{e^{-z} z^{-1/2}}{\sqrt{2\pi}}, \; 0 < z < \infty$

Thus we get in either case $f_z(y) = \frac{e^{-z} z^{\frac{1}{2}-1}}{\sqrt{2\pi}}, 0 < z < \infty$

7. a) The mean of a normal distribution is 50 and 5% of the values are greater than 60. Find the standard deviation of the distribution (Area under standard normal curve between $z = 0$ and $z = 1.64$ is 0.45). **[WBUT 2017]**

Answer:

Let X denote the normal variate whose mean is 50.

Then $Z = \frac{X - 50}{\sigma}$ is the standard normal variate.

By the given condition $P(X > 60) = 0.05$

or, $P\left(\frac{X-50}{\sigma} > \frac{60-50}{\sigma}\right) = 0.05$

or, $P\left(Z > \frac{10}{\sigma}\right) = 0.05$

But it is given $P(Z > 1.64) = 0.05$

So, $\frac{10}{\sigma} = 1.64$

or, $\sigma = \frac{10}{1.64} = 6.098$

b) Show that for the exponential distribution **[WBUT 2017]**

$$f(x) = \frac{1}{\sigma} e^{\left(-\frac{x}{\sigma}\right)}, \; 0 < x < \infty$$

mean and standard deviation both equal to σ.

Answer: ***Refer to Question No. 8 of Short Answer Type Questions***

8. a) The p.d.f. of a random variable X is $f(x) = k(x-1)(2-x)$, $1 \le x \le 2$. Find (i) the value of the constant k (ii) the distribution function $F(x)$ (iii) $P\left(\frac{5}{4} \le X \le \frac{3}{2}\right)$. **[WBUT 2018]**

Answer:

Here $f(x) = \begin{cases} k(x-1)(2-x) & \text{when } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

$\therefore \int_{-\infty}^{\infty} f(x)dx = 1$ gives $k\int_1^2 \left(-2 + 3x - x^2\right)dx = 1$

or, $\left[-2x + \frac{3}{2}x^2 - \frac{x^3}{3}\right]_1^2 = \frac{1}{k}$

or, $-4 + 6 - \frac{8}{3} + 2 - \frac{3}{2} + \frac{1}{3} = \frac{1}{k}$ $\quad \therefore k = 6$

Again, $F(x) = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} (x-1)(2-x)dx$

$P\left(\frac{5}{4} \le X \le \frac{3}{2}\right) = \int_{3/2}^{5/4} f(x)dx = \int_{3/2}^{5/4} (x-1)(2-x)dx$

b) If X is the number scored in a throw of a fair die, show that the Tchebycheff's inequality gives $P(|X - \mu| > 2.5) < 0.47$, where μ is the mean of X, while the actual probability is zero. **[WBUT 2018]**

Answer:
Here we see

X:	1	2	3	4	5	6
p:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$\mu = E(X) = \frac{3\times 7}{6} = 3.5,\ E(X^2) = \frac{6\times 7\times 13}{6} = 91,\ \sigma^2 = 91 - 12.25 = 78.75$

Also, $k\sigma = 2.5$

or, $k = \frac{2.5}{\sigma}$

$\therefore\ \frac{1}{k^2} = \frac{\sigma^2}{6.25} = \frac{78.75}{6.25} = \frac{63}{5}$

By Tchebycheff's inequality

$$P(|X-\mu| > 2.5) < \frac{1}{k^2} = \frac{63}{5}$$

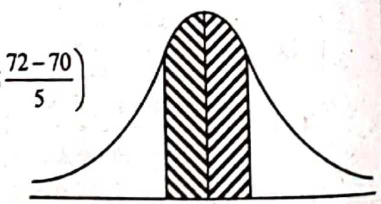
9. a) If the weekly wages of 10,000 workers in a factory follow normal distribution with mean and s.d. ₹70 and ₹5 respectively, find the expected number of workers whose weekly wages are (i) between ₹66 and ₹72 (ii) less that ₹66 and (iii) more than ₹72. **[WBUT 2018]**

Given that $\frac{1}{\sqrt{2\pi}}\int_0^z e^{-t^2/2}dt = 0.1554$ **and** 0.2881 **according as** $z = 0.4$ **and** $z = 0.8$.

Answer:
$n = 10{,}000,\ \mu = 70,\ \sigma = 5$

Let X denote the weekly wage of the workers.

$$P(60 \le X \le 72) = P\left(\frac{66-70}{5} \le \frac{X-70}{5} \le \frac{72-70}{5}\right)$$
$$= P(-0.8 \le Z \le 0.4)$$

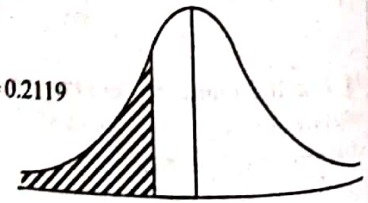


where, $Z = \frac{X-70}{5}$

$0.1554 + 0.2881 = 0.4435$

So the expected number $= 10{,}000 \times 0.4435 = 4435$

$$P(X \le 66) = P\left(\frac{X-70}{5} \le \frac{66-70}{5}\right)$$
$$= P(Z \le -0.8) = 0.5 - 0.2881 = 0.2119$$



So the expected number $= 10{,}000 \times 0.2119 = 2119$

$$P(X > 72) = P\left(\frac{X-70}{5} > \frac{72-70}{5}\right)$$

$$= P(Z > 0.4) = 0.5 - 0.1554 = 0.3446$$

Hence, the expected number $= 10{,}000 \times 0.3446 = 3446$

b) The following table gives the number of aircraft accidents that occurred during various days of the week. Test whether the accidents are uniformly distributed over the week.

Given: $\chi^2 0.05(\gamma = 6) = 12.59$ **[WBUT 2018]**

Day	Sun	Mon	Tue	Wed	Thurs	Fri	Sat
No. of accidents	13	14	19	12	11	15	14

Answer:
We make the following table

Day	No. of accidents (f_o)	Expected no. of accidents (f_e)	$f_o - f_e$	$\frac{(f_o-f_e)^2}{f_e}$
1	13	14	−1	$\frac{1}{14}$
2	14	14	0	0
3	19	14	5	$\frac{25}{14}$
4	12	14	−2	$\frac{4}{14}$
5	11	14	−3	$\frac{9}{14}$
6	15	14	1	$\frac{1}{14}$
7	14	14	0	0
Total	98			

So, $\chi^2 = \sum\frac{(f_o-f_e)^2}{f_e} = \frac{40}{14} = 2.86$ which is less than the tabulated value 12.59

Hence the accidents are uniformly distributed overall the days.

TRANSFORMATION OF RANDOM VARIABLES

Short Answer Type Questions

1. If a random variable X follows uniform distribution with parameters 0 and 1, find the p.d.f of the random variable $U=\sqrt{x}$: **[WBUT 2015]**

Answer:

We have the p.d.f. of X

$$f(x)=\begin{cases}1 \text{ if } & 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

Further, $U=\sqrt{X}$ or, $X=U^2$ or, $\frac{dx}{du}=2u$

Therefore the p.d.f. of u is given by

$$f_u(u)=f_X(x)\left|\frac{dx}{du}\right|$$

$$=\begin{cases}1.2u & \text{when } 0<u<1\\ 0.\ 2u & \text{when } 1\le u<\infty\end{cases}$$

i.e. $$f_u(u)=\begin{cases}2u & \text{if } 0<u<1\\ 0 & \text{if } 1\le u<\infty\end{cases}$$

2. If the random variable X has the *p.d.f.* **[MODEL QUESTION]**

$f(x)=3x,\ 0<x<1$

$\quad =0$**, elsewhere**

then find the *p.d.f.* of $y=4x+3$.

Answer:

Given $$f(x)=\begin{cases}3x & \text{if } 0<x<1\\ 0 & \text{elsewhere}\end{cases}$$

This cannot be a probability density function as $\int_{-\infty}^{\infty} f(x)\,dx\neq 1$

So let us take, $$f(x)=\begin{cases}2x & \text{if } 0<x<1\\ 0 & \text{elsewhere}\end{cases}$$

$\therefore$ Here $y=4x+3$ $\quad\therefore \frac{dy}{dx}=4>0$

$\therefore$ The p.d.f. of Y is given by $f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right|$

$$=\begin{cases}\frac{1}{4}\cdot\frac{3}{4}\cdot(y-3) & \text{if } 3<y<7\\ 0 & \text{elsewhere}\end{cases}$$

$$=\begin{cases}\frac{3}{16}(y-3) & \text{if } 3<y<7\\ 0 & \text{elsewhere}\end{cases}$$

3. If the random variable X has the *p.d.f.* **[MODEL QUESTION]**

$f(x)=3x,\ 0<x<1$

$\quad =0$, **elsewhere**

then find the *p.d.f.* of $y=4x+3$.

Answer:

Given, $$f(x)=\begin{cases}3x & \text{if } 0<x<1\\ 0 & \text{elsewhere}\end{cases}$$

This cannot be a probability density function as $\int_{-\infty}^{\infty} f(x)\,dx\neq 1$

So let us take

$$f(x)=\begin{cases}2x & \text{if } 0<x<1\\ 0 & \text{elsewhere}\end{cases}$$

$\therefore$ Here $y=4x+3$ $\quad\therefore \frac{dy}{dx}=4>0$

$\therefore$ The p.d.f. of Y is given by

$$f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right|$$

$$=\begin{cases}\frac{1}{4}\cdot\frac{3}{4}\cdot(y-3) & \text{if } 3<y<7\\ 0 & \text{elsewhere}\end{cases}$$

$$=\begin{cases}\frac{3}{16}(y-3) & \text{if } 3<y<7\\ 0 & \text{elsewhere}\end{cases}$$

4. Let X be a standard normal variate. Find the probability density function of y where $y=\frac{1}{2}X^2$. **[MODEL QUESTION]**

Answer:

We have $X \sim N(0,1)$ and $Y = \frac{1}{2}X^2$

$\therefore \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

If $F(y)$ denote the distribution function of y, then

$$F(y) = P(Y \le y) = P\left(\frac{1}{2}X^2 \le y\right) = P(X^2 \le 2y)$$

$$= P(-\sqrt{2y} \le X \le \sqrt{2y}) = P(X \le \sqrt{2y}) - P(X \le -\sqrt{2y})$$

$\therefore \quad F'(y) = \left\{F'(\sqrt{2y}) - F'(-\sqrt{2y})\right\} / \sqrt{2y}$

$\therefore$ The p.d.f. of $Y = f(y) = \left\{f(\sqrt{2y}) + f(-\sqrt{2y})\right\} / \sqrt{2y}$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}}(e^{-y} + e^{-y}) & \text{if } y > 0 \\ 0 & \text{if otherwise} \end{cases}$$

$$= \begin{cases} \sqrt{\frac{2}{\pi}} e^{-y} & \text{if } y > 0 \\ 0 & \text{if } y \le 0. \end{cases}$$

Long Answer Type Questions

1. If a random variable X has a uniform distribution within the interval [2, –2] then determine the distribution of Min (X, 1). **[MODEL QUESTION]**

Answer:

Let $Z = \text{Min}(X, 1)$, when $x < 1$, $z = x$ and $\frac{dz}{dx} = 1$.

Therefore, $P(z < Z \le z + dz) = P(x < X \le x + dx) = f_x(x)dx$

or, $f_z(z)dz = f_x(x)dx$ or, $f_z(z) = f_x(x) = \int_{-2}^{1} \frac{1}{4} dx = \frac{1}{4}$ when $x > 1$, $z = 1$

Therefore, $P(x < X \le x + dx) = P(z = 1)$

or, $P(z = 1) = \int_{1}^{2} \frac{1}{4} dx = \frac{1}{4}$

Hence Z has a continuous distribution in $-2 < z < 1$ with density function $f_z(z) = \frac{1}{4}$ and a probability mass $\frac{1}{4}$ at $z = 1$

LIMIT THEOREMS

Short Answer Type Questions

1. A random sample of size $n = 100$ is taken from an infinite population with the mean $\mu = 75$ and the variance $\sigma^2 = 256$. Based on Chebyshev's theorem with what probability can we assert that the value we obtain for $\overline{X}$ will fall between 67 and 83? **[WBUT 2014]**

Answer:

Here $\mu = 75, \sigma^2 = 256$ i.e., $\sigma = 16$

By Chebyshev's inequality, we have

$$P(|\overline{X} - 75| < \varepsilon) \ge 1 - \frac{256/100}{\varepsilon^2}$$

or, $P(75 - \varepsilon < \overline{X} < 75 + \varepsilon) = 1 - \frac{256}{\varepsilon^2}$, putting $\varepsilon = 8$ we get

$$P(67 < \overline{X} < 75) \ge 1 - 0.04 = 0.06$$

2. Show by Tchebycheff's inequality that in 2000 throws with a coin, the probability that the number of heads lies between 900 and 1100 is at least $\frac{19}{20}$. **[WBUT 2017]**

Answer:

From Tchebycheff's inequality we have

$$P(|X - E(X)| \le \varepsilon) \ge 1 - \frac{\text{var}(X)}{\varepsilon^2}$$

Let X denote the no. of heads in 2000 throws.

$$E(X) = np = 2000 \times \frac{1}{2} = 1000$$

$$\text{var}(X) = npq = 2000 \times \frac{1}{2} \times \frac{1}{2} = 500$$

Now from the inequality, we get

$$P(|X - 1000| \le 100) \ge 1 - \frac{500}{10000}, \text{ taking } \varepsilon \text{ as } 100$$

i.e., $P(900 \le X \le 1100) \ge 1 - \frac{1}{20} = \frac{19}{20}$.

Hence the proof.

3. Consider a random variable X to be uniformly distributed over $\left(1-\frac{1}{\sqrt{3}},1+\frac{1}{\sqrt{3}}\right)$.

Find $P\left(|x-\mu|\geq\frac{3}{2}\sigma\right)$. Compare it with the upper bound obtained by Tchebycheff's inequality. **[MODEL QUESTION]**

Answer:

We first see that $X\sim U\left(1-\frac{1}{\sqrt{3}},1+\frac{1}{\sqrt{3}}\right)$ means its p.d.f is

$$f(x)=\begin{cases}0 & \text{if } x\leq 1-\frac{1}{\sqrt{3}}\\ \frac{\sqrt{3}}{2} & \text{if } 1-\frac{1}{\sqrt{3}}<x<1+\frac{1}{\sqrt{3}}\\ 0 & \text{if } x\geq 1+\frac{1}{\sqrt{3}}\end{cases}$$

$$E(X)=\int_{-\infty}^{\infty}xf(x)dx=\int_{1-\frac{1}{\sqrt{3}}}^{1+\frac{1}{\sqrt{3}}}x.\frac{\sqrt{3}}{2}dx=1=\mu$$

$$E(X^2)=\int_{-\infty}^{\infty}x^2f(x)dx=\int_{1-\frac{1}{\sqrt{3}}}^{1+\frac{1}{\sqrt{3}}}x^2\frac{\sqrt{3}}{2}dx=\frac{10}{9}$$

$$\therefore\sigma^2=E(X^2)-\{E(X)\}^2=\frac{10}{9}-1=\frac{1}{9}\ \therefore\sigma=\frac{1}{3}$$

$$\therefore P\left(|X-\mu|\geq\frac{3}{2}\sigma\right)=1-P\left(|X-1|<\frac{3}{2}\cdot\frac{1}{3}\right)=1-P\left(\frac{1}{2}<X<\frac{3}{2}\right)$$

Now, $P\left(\frac{1}{2}<X<\frac{3}{2}\right)=\int_{\frac{1}{2}}^{\frac{3}{2}}f(x)dx=\frac{\sqrt{3}}{2}\int_{1-\frac{1}{\sqrt{3}}}^{1+\frac{1}{\sqrt{3}}}dx$

So, $P\left(|X-\mu|\geq\frac{3\sigma}{2}\right)=1-1=0=\frac{\sqrt{3}}{2}\times\frac{2}{\sqrt{3}}=1$

By Tchebycheff's inequality, $P(|X-\mu|\geq\varepsilon)\leq\frac{\sigma^2}{\varepsilon^2}$. So, $P\left(|X-\mu|\geq\frac{3\sigma}{2}\right)\leq\frac{\sigma^2}{\frac{9}{4}\sigma^2}=\frac{4}{9}$

4. Write Tchebycheff's inequality. **[MODEL QUESTION]**

Answer:

If X is a random variable and $\varepsilon>0$, then $P(|X-E(X)|\geq\varepsilon)\leq\frac{\text{var}(X)}{\varepsilon^2}$

5. State Weak Law of Large Numbers (WLLN). **[MODEL QUESTION]**

Answer:

If $X_1,X_2,\cdots,X_n$ are n independent identically distributed random variables with respective finite means $\mu_1,\mu_2,\cdots,\mu_n$ and $B_n=\text{var}(X_1+X_2+\cdots+X_n)$ then for $\varepsilon>0$, $\eta>0$ arbitrarily small.

$$P\left(\left|\frac{X_1+X_2+\cdots+X_n}{n}-\frac{X_1+X_2+\cdots+X_n}{n}\right|<\varepsilon\right)>1-\eta \text{ for all } n\geq n_0 \text{ if } \lim_{n\to\infty}\frac{B_n}{n^2}=0$$

6. If X_k, k=1, 2, ..., n are mutually independent and identically distributed random variables with mean μ and finite variance σ, prove that WLLN holds for the sequence $\{S_n\}$ where $S_n=X_1+X_2+\cdots+X_n$. **[MODEL QUESTION]**

Answer:

Here $E(S_n)=\sum_{k=1}^{\infty}E(X_k)=n\mu$

$\text{var}(S_n)=\sum_{k=1}^{n}\text{var}(X_k)=n\sigma^2$

$\therefore\frac{\text{var}(S_n)}{n^2}=\frac{\sigma^2}{n}$

$\lim_{n\to\infty}\frac{Var(S_n)}{n^2}=\lim_{n\to\infty}\frac{\sigma^2}{n}=0$

Hence WLLN holds for $\{S_n\}$ by Tchebycheff's inequality.

7. Find the probability using Tchebycheff's theorem that the number of driving licenses X issued by RTA in a specific month is between 64 and 184, if the X is a r.v. with E(X) = 124 and std. Div. = 7.5. **[MODEL QUESTION]**

Answer:

Let X denote the no. of license issued in a month.

$$P(64<x<184)=P(-60<x-124<60)$$
$$=P(|X-124|<60)$$
$$=1-P(|X-124|\geq 60)$$

But by Chebyschev's inequality

$$P(|X-124|\geq 60)\leq\frac{56.25}{60\times 60}=0.015625$$
$$P(64\leq X\leq 184)\geq 1-0.015625\geq 0.984$$

8. Show by Tchebycheff's inequality that in 2000 throws with a coin, the probability that the number of heads lies between 900 and 1100 is at least $\frac{19}{20}$. **[MODEL QUESTION]**

Answer:
Let X denote the no. of heads in 2000 throws

Then $X \sim B\left(2000, \frac{1}{2}\right) n = 2000, p = \frac{1}{2}, q = \frac{1}{2}$

$$\overline{X} = E(X) = np = 2000 \times \frac{1}{2} = 1000$$

$$\text{var}(X) = npq = 2000 \times \frac{1}{2} \times \frac{1}{2} = 500$$

$$P(900 < X < 1100) = P(900 - 1000 < X - 1000 < 1100 - 1000)$$
$$= P(-100 < X - 1000 < 100)$$
$$= P(|X - 1000| < 100)$$
$$\geq 1 - \frac{500}{100 \times 100} = 1 - \frac{1}{20} = \frac{19}{20}$$

9. Use Tchebycheff's inequality to show that for $n \geq 36$, the probability that in n throws of a fair die the number of sixes lies between $\frac{1}{6}n - \sqrt{n}$ and $\frac{1}{6}n + \sqrt{n}$ is at least $\frac{31}{36}$. **[MODEL QUESTION]**

Answer:
Let X denote the no. of sixes in n throws of a fair die.

Here $p = \frac{1}{6}$ $\therefore E(X) = \frac{n}{6}$, $\text{var}(X) = n \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5n}{36}$

By Chebyshev's inequality,

$$P\left(\frac{1}{6}n - \sqrt{n} < X < \frac{1}{6}n + \sqrt{n}\right) = P\left(\left|X - \frac{n}{6}\right| < \sqrt{n}\right)$$
$$= 1 - P\left(\left|X - \frac{n}{6}\right| \geq \sqrt{n}\right) \geq 1 - \frac{5}{36} = \frac{31}{36},$$

Since the no. of sixes cannot be negative, $\frac{1}{6}n - \sqrt{n} \geq 0$

or, $n^2 \geq 36n$ or, $n \geq 36$.

[Recall: $P(|X - E(X)| \geq \varepsilon) \leq \frac{\text{var}(X)}{\varepsilon^2}$ or, $P(|X - E(X)| < \varepsilon) \geq 1 - \frac{\text{var}(X)}{\varepsilon^2}$]

10. 100 unbiased coins are tossed. Using normal approximation to binomial distribution calculate the probability of getting (i) exactly 40 heads, (ii) 55 heads or more.
[Given $\Phi(2.1) = 0.9821, \Phi(1.9) = 0.9713, \Phi(0.9) = 0.8159$] **[MODEL QUESTION]**

Answer:
Let X denotes the number of heads in 100 tosses.

Then $n = 100, p = q = \frac{1}{2}, E(X) = 50, \text{var}(X) = 25$

Now $P(X = 40) = P(39.5 < \hat{X} < 40.5) = P\left(\frac{39.5 - 50}{5} < \frac{\hat{X} - 50}{5} < \frac{40.5 - 50}{5}\right)$

$$= P(-2.1 < z < -1.9) = 0.9821 - 0.9713 = 0.0108$$

$$P(X \geq 55) = P(\hat{X} \geq 54.5) = P\left(\frac{\hat{X} - 50}{5} \geq \frac{54.5 - 50}{5}\right)$$
$$= P(z \geq 0.9) = 1 - 0.8159 = 0.1844$$

Long Answer Type Questions

1. Show by Tchebyshev's inequality that if a die is thrown 3,600 times, the probability of number of sixes that lies between 550 and 650 is at least. **[WBUT 2017]**

Answer:
The Tchebyshey's inequality is

$$P(|X - E(X)| \leq \varepsilon) \geq 1 - \frac{\text{var}(X)}{\varepsilon^2}$$

Let X denote the no. of sizes in 3600 throws of a die

$\therefore$ $E(X) = np = 3600 \times \frac{1}{6} = 600$

$$\text{var}(X) = npq = 3600 \times \frac{1}{6} \times \frac{5}{6} = 500$$

$\therefore$ The inequality gives

$$P(|X - 600| \leq 50) \geq 1 - \frac{500}{50 \times 50}, \text{ taking } \varepsilon = 50$$

or, $P(550 \leq X \leq 650) \geq 1 - \frac{1}{5} = \frac{4}{5}$.

2. A random variable x has the function e^{-x}, $x \geq 0$. **[MODEL QUESTION]**
Show that Tchebycheff's inequality gives $P[|X - 1| > 2] < \frac{1}{4}$ and show that actual probability is e^{-3}.

Answer:

Let X be the r.v. with probability function $f(x)=\begin{cases} e^{-x} & \text{when } x\ge 0 \\ 0 & \text{otherwise} \end{cases}$

Then $E(X)=\int_{-\infty}^{\infty} xf(x)\,dx=\int_0^{\infty} xe^{-x}dx=\left[-xe^{-x}\right]_0^{\infty}+\int_0^{\infty} e^{-x}dx=1$

$E(X^2)=\int_0^{\infty} x^2e^{-x}dx=\left[-x^2e^{-x}\right]_0^{\infty}+2\int_0^{\infty} xe^{-x}dx=2$

$\therefore$ Var $(X)=E(X^2)-\{E(X)\}^2=2-1=1.$

Now the Tchebycheff's inequality is

$$P(|X-E(X)|>\varepsilon)<\frac{\text{Var}(X)}{\varepsilon^2} \quad \text{or} \quad P(|X-1|>2)<\frac{1}{4}$$

Again, $P(|X-1|>2)=1-P(|X-1|\le 2)=1-P(-1\le X\le 3)=1-\int_{-1}^{3} f(x)\,dx$

$$=1-\int_0^3 e^{-x}dx=1-\left[-e^{-x}\right]_0^3=1-\left(1-e^{-3}\right)=e^{-3}$$

3. Write note on Central Limit Theorem. **[MODEL QUESTION]**

Answer:

The central limit theorem is one of the most remarkable results in Probabilities theory. It states that the sum of a large number of independent random variables has a distribution that is approximately normal. So it not only provides a simple method for computing approximate probabilities for sums of independent random variables but also it helps explain the fact that the empirical frequencies of so many natural populations exhibit bell-shaped curves. The simplest form of the central limit theorem is

If X_1, X_2 be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2 then the distribution of $\dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}$ tends to the standard normal as n tends to infinity.

SAMPLING THEORY

Multiple Choice Type Questions

1. The standard deviation of a sample mean for SRSWR is **[WBUT 2013]**

a) σ^2/n b) $\sigma/\sqrt{n}$ c) σ/n d) n

Answer: (b)

2. If t is a statistic such as $E(t^2)=5$ and $E(t)=2$, then the standard error of t is **[WBUT 2013]**

a) 0 b) 1 c) 2 d) none of these

Answer: (b)

3. In 'Goodness of fit' which of the following is used as test statistic **[WBUT 2014]**

a) normal variate b) t variate
c) Poisson variate d) X^2 variate

Answer: (d)

4. A random sample of two individuals is to be drawn from a population of size 43. The possible numbers of distinct samples under SRSWR is **[MODEL QUESTION]**

a) 43×43 b) ${}^{43}C_2$ c) ${}^{43}P_2$ d) none of these

Answer: (b)

5. In random sampling with replacement from a population with s.d. σ, if the sample size is equal to the population size (= N), then the standard error of sample mean will be **[MODEL QUESTION]**

a) 0 b) σ c) $\sigma/\sqrt{n}$ d) none of these

Answer: (a)

6. The expected value of the sample variance of size n drawn from a population with mean m and standard deviation σ is **[MODEL QUESTION]**

a) σ^2 b) $n\sigma^2$ c) $\dfrac{n-1}{n}\sigma^2$ d) $\dfrac{\sigma^2}{n}$

Answer: (c)

Short Answer Type Questions

1. Prove that the chromatic number of a circuit with n vertices is **[WBUT 2018]**
(i) 2 if n is even (ii) 3 if n is odd.

Answer:

G be a circuit made of even number of vertices $v_1, v_2, v_3, \ldots\ldots\ldots, v_n$.

It is observed that every odd numbered vertex can be assigned one and the same colour and every even numbered vertex can be assigned a different colour to colour the circuit. Hence $\chi(G)=2$.

As it is a circuit. v_1 and v_n get different colours. Hence $\chi(G)=2$. One the other hand if G consists of odd number of vertices, the last vertex is an odd numbered vertex. So it gets the colour of v_1, but G being a circuit for proper colouring v_n has to have a different colour. So v_n must be assigned a third colour. Thus $\chi(G)=3$.

2. Write the mean and variance of the sampling distribution of sample means for SRSWR and SRSWOR? **[MODEL QUESTION]**

Answer:

$E(\bar{X})=\mu$ for SRSWR and also for SRSWOR.

$$Var(\bar{X})=\frac{\sigma^2}{n} \text{ for SRWR } =\frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right) \text{ for SRSWOR.}$$

Long Answer Type Questions

1. Let $\bar{X}$ be the sample mean of samples of size n drawn at random from a population which is normally distributed with mean μ and variance σ^2. Find the standard error of the statistics $\bar{X}$. **[WBUT 2013]**

Answer:

Let the r.v. assume the values $X_1, X_2, \cdots\cdots, X_N$ and $x_1, x_2, \cdots\cdots, x_n$ constitute a sample (Note then $x_1, \cdots\cdots, x_n$ are n r.v.'s assuming value $X_1, X_2, \cdots\cdots, X_N$).

Then $$E(\bar{x})=E\left(\frac{x_1+x_2+\cdots\cdots+x_n}{n}\right)=\frac{1}{n}\{E(x_1)+E(x_2)+\cdots\cdots+E(x_n)\}$$

But $$E(x_i)=\frac{X_1}{N}+\frac{X_2}{N}+\cdots\cdots+\frac{X_N}{N},\ i=1,2,\ldots\ldots,n=\frac{1}{N}\sum_{j=1}^{N}X_j=\mu$$

So, $$E(\bar{x})=\frac{1}{n}\{\mu+\mu+\cdots\cdots+\mu\}=\frac{1}{n}\cdot n\mu=\mu$$

$$\mathrm{var}(\bar{x})=\mathrm{var}\left\{\frac{x_1+x_2+\cdots\cdots+x_n}{n}\right\}$$

$$=\frac{1}{n^2}\mathrm{var}(x_1)+\frac{1}{n^2}\mathrm{var}(x_2)+\cdots\cdots+\frac{1}{n^2}\mathrm{var}(x_n)+\frac{2}{n^2}\mathrm{cov}(x_1,x_2)+\frac{2}{n^2}\mathrm{cov}(x_1,x_3)+\cdots\cdots+\frac{2}{n^2}\mathrm{cov}(x_{n-1},x_n)$$

$$=\frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}(x_i) \text{ since } x_1,x_2,\ldots\ldots,x_n \text{ are i.i.d.}$$

$$\mathrm{var}(x_i)=E(x_i^2)-\{E(x_i)\}^2 \text{ for } i=1,2,\ldots\ldots,n=E(x_i^2)-\mu^2$$

Now, $$E(x_i^2)=\frac{X_1^2}{N}+\frac{X_2^2}{N}+\cdots\cdots+\frac{X_N^2}{N}=\frac{1}{N}\sum_{j=1}^{N}X_j^2.$$

Also, $$\sigma^2=\frac{1}{N}\sum_{j=1}^{N}X_j^2-\left\{\frac{1}{N}\sum_{j=1}^{N}X_j\right\}^2=\frac{1}{N}\sum_{j=1}^{N}X_j^2-\mu^2.$$

$\therefore$ $$E(x_i^2)=\sigma^2+\mu^2$$

So, $$\mathrm{var}(x_i)=\sigma^2+\mu^2-\mu^2=\sigma^2$$

and $$\mathrm{var}(\bar{x})=\frac{1}{n^2}\{\sigma^2+\sigma^2+\cdots\cdots+\sigma^2\}=\frac{1}{n^2}\cdot n\sigma^2=\frac{\sigma^2}{n}$$

$\therefore$ $$SE(\bar{x})=\frac{\sigma}{\sqrt{n}}.$$

ESTIMATION OF PARAMETERS

Multiple Choice Type Questions

1. A statistic t is said to be an unbiased estimator of a population parameter θ when **[WBUT 2012, 2014, 2017]**

a) $E(t)=\theta$ b) $E(t^2)=\theta$

c) $E(t^2)=\{E(\theta)\}^2$ d) $\{E(t)\}^2=E(\theta^2)$

Answer: (a)

2. The maximum likelihood estimate is a solution of the equation **[WBUT 2012]**

a) $\frac{d\angle(\theta)}{d\theta}=0$ b) $\frac{d\angle(\theta)}{d\theta}=$ constant

c) $\frac{d\angle(\theta)}{d\theta}=\theta$ d) none of these

Answer: (a)

3. If $E(T)=5\theta+6$ then the unbiased estimator of θ is **[WBUT 2016]**

a) $\frac{1}{5}T-\frac{6}{5}$ b) $T-\frac{6}{5}$ c) $\frac{1}{5}T$ d) $T-2$

Answer: (a)

4. If T_1 and T_2 be statistic with $E(T_1)=2\theta_1+3\theta_2$ and $E(T_2)=\theta_1+\theta_2$ then the unbiased estimator of θ_1 is **[WBUT 2018]**

a) $3T_2-T_1$ b) $3T_2+2T_1$ c) $3T_2-3T_1$ d) $3T_2+T_1$

Answer: (b)

4. The area under the standard normal curve beyond the lines $z=\pm1.96$ is **[MODEL QUESTION]**

a) 95 per cent b) 90 per cent c) 5 per cent d) 10 per cent

Answer: (a)

Short Answer Type Questions

1. Let $x_1, x_2, \ldots\ldots, x_n$ be the values of a random sample from an exponential population that the $f(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}}$ for $x_i>0$. Then find the maximum likelihood estimator of its parameter θ. **[WBUT 2014]**

Answer:

The m. l. f. $L(x_1, x_2, \ldots\ldots, x_m\theta)=\frac{1}{\theta}e^{-\frac{x_1}{\theta}}\cdot\frac{1}{\theta}e^{-\frac{x_2}{\theta}}\ldots\ldots\frac{1}{\theta}e^{-\frac{x_n}{\theta}}=\frac{1}{\theta^n}e^{-\frac{1}{\theta}(x_1+x_2+\ldots\ldots+x_n)}$

Taking log both sides we get

$$\log L=-\frac{1}{\theta}\left(\sum x_i\right)-n\log\theta$$

Now, $\frac{\partial \log L}{\partial\theta}=0$ gives $\frac{\sum x_i}{\theta^2}=\frac{n}{\theta}$ or, $\theta=\frac{1}{n}\sum x_i=\bar{x}$.

2. If T is an unbiased estimator of θ, then show that T^2 is a biased estimator of θ^2. **[WBUT 2018]**

Answer:

If T is an unbiased estimator of θ, then $E(T)=\theta$

But $E(T^2)\neq\theta^2$ and hence T^2 is an unbiased estimator of θ^2.

3. From the random sample of size 49 drawn from a normal population of standard deviation2, find the 99% confidence interval of the population mean. Find the interval if the mean of such a sample is 3.

[Given $\int_0^{2.58}\phi(z)\,dz=0.495$] **[MODEL QUESTION]**

Answer:

Here $\bar{x}=3, \sigma=2$ and the sample is a large one $(\nu\geq30)$

$\therefore$ The confidence interval is

$$\left(\bar{x}-z_\alpha\frac{\sigma}{\sqrt{n}}, \bar{x}+z_\alpha\frac{\sigma}{\sqrt{n}}\right)$$

i.e. $\left(3-2.58\times\frac{2}{\sqrt{49}}, 3+2.58\times\frac{2}{\sqrt{49}}\right)$

i.e. $\left(3-\frac{5.16}{7}, 3+\frac{5.16}{7}\right)$

i.e. $\left(\frac{21-5.16}{7}, \frac{21+5.16}{7}\right)$

i.e. $\left(\frac{15.84}{7}, \frac{26.6}{7}\right)$

i.e. $(2.2628, 3.8)$

4. What will be the likelihood function for estimating population proportion P of a binomially distributed population with parameters P and N when N is known? **[MODEL QUESTION]**

Answer:

$$L(x_1,x_2,\cdots,x_n;p)={}^{N}C_{x_1}\,{}^{N}C_{x_2}\cdots{}^{N}C_{x_n}\,p^{x_1+x_2+\cdots+x_n}(1-p)^{nN-\sum_{i=1}^{n}x_i}$$

5. Define point estimation and interval estimation. **[MODEL QUESTION]**

Answer:
When a parameter is estimated by a single value it is called a point estimation. But when an interval is taken in which the parameter is expected to lie, it is called an interval estimation.

6. A population has a normal distribution with parameters μ and σ^2. Find the MLE (maximum likelihood estimator) of σ^2 when μ is known. Also find the mean and variance of the MLE and show that the MLE is unbiased and consistent. **[MODEL QUESTION]**

Answer:
The likelihood function is

$$L(x_1,x_2,\ldots,x_n;\sigma^2)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x_1-\mu)^2}{2\sigma^2}}\cdot\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x_2-\mu)^2}{2\sigma^2}}\cdots\cdots\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x_n-\mu)^2}{2\sigma^2}}$$

$$=\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$\therefore \log L=-\frac{n}{2}\log\sigma^2-\frac{n}{2}\log(2\pi)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\therefore \frac{\partial}{\partial\sigma^2}(\log L)=-\frac{n}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2$$

Equating to zero we therefore get

$$\sigma^2=\frac{1}{n}\sum(x_i-\mu)^2=S^2,\text{ say,}$$

Further $\frac{\partial}{\partial\sigma^2}\left(\frac{\partial}{\partial\sigma^2}(\log L)\right)=-\frac{n}{2\sigma^4}=-\frac{n}{2S^4}$ at $\sigma^2=S^2$

which the negative.

Hence S^2 is the maximum likelihood estimator of σ^2.

Observe that $E(S^2)=\sigma^2$ which proves (i) the mean of the likelihood estimator is σ^2 and (ii) this estimator is unbiased.

7. A sample of 600 screws is taken from a large consignment and 75 are found to be defective. Set up a 99% confidence interval for the proportion of defectives for the population. **[MODEL QUESTION]**

Answer:

Here the sample is large and $P=\frac{75}{600}=\frac{1}{8}, n=600$

The 99% confidence interval is, $\left(p-2.58\sqrt{\frac{pq}{n}},\, p+2.58\sqrt{\frac{pq}{n}}\right)$

i.e. $\left(\frac{1}{3}-2.58\sqrt{\frac{\frac{1}{3}\times\frac{2}{3}}{600}},\frac{1}{3}+2.58\sqrt{\frac{\frac{1}{3}\times\frac{2}{3}}{600}}\right)$

i.e. $\left(\frac{1}{3}-2.58\sqrt{\frac{0.22}{600}},\frac{1}{3}+2.58\sqrt{\frac{0.22}{600}}\right)$

i.e. $(0.284, 0.379)$

8. Find out the Maximum Likelihood Estimate for a population having Poisson distribution. **[MODEL QUESTION]**

Answer:

Let λ be the parameter of the Poisson distribution

$$L(x_1,x_2,x_n;\lambda)=\frac{e^{-\lambda}\lambda^{x_1}}{x_1!}\cdot\frac{e^{-\lambda}\lambda^{x_2}}{x_2!}\cdots\cdots\frac{e^{-\lambda}\lambda^{x_n}}{x_n!}=\frac{e^{-n\lambda}\lambda^{x_1+x_2+\cdots x_n}}{x_1!x_2!\ldots x_n!}$$

$\frac{\partial L}{\partial\lambda}=0$ now gives $-n^{-n\lambda}\lambda^{\sum x_i}+e^{-n\lambda}\cdot\left(\sum x_i\right)\lambda^{\sum x_i-1}=0$

or, $-n+\frac{\sum x_i}{\lambda}=0$ or, $\lambda=\frac{\sum x_i}{n}=\bar{x}$

Clearly $\left(\frac{\partial^2 L}{\partial\lambda^2}\right)_{\lambda=\bar{x}}<0$. Hence the m. l. e. of λ is $\bar{x}$.

9. A sample of 600 screws in taken from a large consignment and 75 are found to be defective. Set up a 99% confidence interval for the proportion of the defectives in the population. **[MODEL QUESTION]**

Answer:

Here, $n=600$, $p=\frac{75}{600}=\frac{1}{8}$ $\therefore q=\frac{7}{8}$

So the 99% confidence interval for the proportion of defectives

$$=p\pm 2.58\sqrt{\frac{pq}{n}}=0.125\pm 2.58\times 0.0135$$

i.e. $(0.0902,\ 0.15983)$.

10. Find the maximum likelihood estimate for the parameter λ of a Poisson distribution on the basis of a sample of size n. **[MODEL QUESTION]**

Answer:

The m.l.f. of a Poisson distribution

$$L(x_1,x_2,x_n,\lambda)=\frac{e^{-\lambda}\lambda^{x_1}}{x_1!}\cdot\frac{e^{-\lambda}\lambda^{x_2}}{x_2!}\cdots\cdots\frac{e^{-\lambda}\lambda^{x_n}}{x_n!}=\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}}{x_1!x_2!\ldots x_n!}$$

$$\therefore\frac{\partial L}{\partial\lambda}=\frac{1}{x_1!x_2!\ldots x_n!}\left\{e^{-n\lambda}(-n)\lambda^{\sum_{i=1}^{n}x_i}+\left(\sum_{i=1}^{n}x_i\right)\lambda^{\sum_{i=1}^{n}x_i-1}.e^{-n\lambda}\right\}$$

Now $\frac{\partial L}{\partial\lambda}=0$ gives $e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}\left\{-n+\frac{\sum_{i=1}^{n}x_i}{\lambda}\right\}=0$; or, $\lambda=\frac{1}{n}\sum_{i=1}^{n}x_i=\bar{x}$

11. The marks obtained by 17 students is an examination have a mean 57 and variance 64. Find 99% confidence interval for the mean of the population of marks assuming it to be normal. [Given that $P(t>3.250)=0.005$ for 16 degree of freedom]. **[MODEL QUESTION]**

Answer:

Here $\bar{x}=57,\ s^2=64,\ n=17$

$\therefore$ The 99% confidence interval for the population mean is

$$\left(\bar{x}-t_{0.005}\frac{s}{\sqrt{n-1}},\ \bar{x}+t_{0.005}\frac{s}{\sqrt{n-1}}\right)$$

i.e., $\left(57-3.250\times\frac{8}{4},\ 57+3.250\times\frac{8}{4}\right)$ i.e., $(50.75,\ 63.25)$

Long Answer Type Questions

1. a) If T is an unbiased estimator of θ, show that $\sqrt{T}$ is biased estimate of $\sqrt{\theta}$. **[WBUT 2012]**

OR,

If T is an unbiased estimator of θ, then show that T^2 is a biased estimator of θ^2. **[WBUT 2014]**

Answer:

Since $E(X^2)\geq\{E(X)\}^2$, we get $\left\{E(\sqrt{T})\right\}^2\leq E(T)$

or, $E(\sqrt{T})\leq\sqrt{E(T)}=\sqrt{\theta}$.

Hence $\sqrt{T}$ is not an unbiased estimator of $\sqrt{\theta}$.

b) If a population has normal distribution with parameter μ and σ, then prove that the statistic $\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$ is maximum likelihood estimate of σ^2 where μ is known. **[WBUT 2012, 2015]**

Answer:

Let $X_1, X_2, \ldots\ldots, X_n$ be a sample of size n and $X_i\sim N(\mu,\sigma^2)$.

Then the likelihood function is

$$L(\sigma^2)=\frac{1}{\left(\sigma\sqrt{2\pi}\right)^n}e^{-\frac{\sum(x_i-\mu)^2}{2\sigma^2}}\quad\text{for }\mu\text{ known.}$$

$$\therefore\ln L=-\frac{n}{2}\ln(\sigma^2 2\pi)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\therefore\frac{\partial}{\partial\sigma^2}(\ln L)=-\frac{n}{2\sigma^2}+\frac{\sum(x_i-\mu)^2}{2\sigma^4}$$

Equating this to zero we get

$$-\frac{n}{2\sigma^2}+\frac{\sum(x_i-\mu)^2}{2\sigma^4}=0$$

$$\text{or,}\ \sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

Hence the maximum likelihood estimator of σ^2 is $\frac{1}{n}\sum_{i}^{n}(x_i-\mu)^2$

Further,

$$E\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right\}=\frac{1}{n}\sum_{i=1}^{n}\left(x_i^2-2X_i\mu+\mu^2\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{E(X_i^2)-2\mu E(X_1)+E(\mu^2)\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left(\mu_2'-2\mu\mu+\mu^2\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left(\mu_2'-\mu_2\right)$$

$$=\frac{1}{n}.n\mu_2'-\frac{1}{n}.n\mu_2=\mu_2'-\mu_1'=\mu_2=\sigma^2$$

Hence $\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$ is an unbiased estimator of σ^2.

c) Prove that the sample mean $\bar{x}$ is an unbiased estimator of the population mean. **[WBUT 2012]**

Answer:

Since $X_1, X_2, \ldots X_n$ are r-v-s having the same mean μ, say, 'i' is $E(X_i)=\mu$ for every i,

we have $E(\overline{X}) = E\left(\frac{X_1+X_2+\ldots\ldots+X_n}{n}\right) = \frac{1}{n}\sum E(X_i) = \frac{1}{n}n\mu = \mu.$

Hence $\overline{X}$ is an unbiased estimated of the population mean μ.

2. Let $x_1, x_2, \ldots\ldots, x_n$ be the values of a random sample from an exponential population that is $f(x_i)=\frac{1}{\theta}e^{-\frac{x}{\theta}}$ for $x_i>0$. Then find the maximum likelihood estimator of its parameter θ. **[WBUT 2016]**

Answer:

The likelihood function is

$$L(x_1, x_2, \ldots, x_n; \theta) = \frac{1}{\theta}\cdot e^{-\frac{x_1}{\theta}} \cdot \frac{1}{\theta}e^{-\frac{x_2}{\theta}} \ldots\ldots \frac{1}{\theta}e^{-\frac{x_n}{\theta}} = \left(\frac{1}{\theta}\right)^n e^{-\frac{1}{\theta}(x_1+x_2+\ldots\ldots+x_n)}$$

$$\log L = -n\log\theta - \frac{1}{\theta}\left(\sum x_i\right)$$

$$\therefore \frac{\partial}{\partial\theta}(\log L) = -\frac{n}{\theta}+\frac{1}{\theta^2}\left(\sum x_i\right)$$

Equating to zero we get

$$-\frac{n}{\theta}+\frac{1}{\theta^2}\left(\sum x_i\right)=0$$

$$\text{or,} \quad \theta = \frac{1}{n}\sum x_i = \bar{x}$$

Clearly, $\frac{\partial^2}{\partial\theta^2}(\log L)<0$ at $\theta=\bar{x}$

Hence the maximum likelihood estimator of θ is $\bar{x}$

3. a) If a population has Poisson distribution with parameter λ, then show that the sample mean is the maximum likelihood estimate of λ. **[WBUT 2017]**

Answer:

The p.m.f. of the Poisson distribution

$$= f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \; x=0,1,2,\ldots\ldots,\infty$$

$\therefore$ The likelihood function is

$$L(x_1, x_2, \ldots\ldots, x_n, \lambda) = f(x_1, \lambda)\ldots\ldots f(x_n, \lambda)$$

$$= \frac{e^{-\lambda}\lambda^{x_1}}{x_1!}\cdot\frac{e^{-\lambda}\lambda^{x_2}}{x_2!}\cdot\ldots\ldots\cdot\frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$

$$= \frac{e^{-n\lambda}\lambda^{\sum x_i}}{x_1!x_2!\ldots\ldots x_n!}$$

$$\therefore \log L = -n\lambda + n\bar{x}\log\lambda - \log(x_1!x_2!\ldots\ldots x_n!)$$

$$\therefore \frac{\partial}{\partial\lambda}(\log L) = -n - \frac{n\bar{x}}{\lambda} - 0$$

Equating to zero, we get

$$-n+\frac{n\bar{x}}{\lambda}$$

$$\text{or,} \quad \lambda=\bar{x}$$

Again, $\frac{\partial^2}{\partial\lambda^2}(\log L) = \frac{-n\bar{x}}{\lambda^2} = -\frac{n}{\bar{x}}<0$ at $\lambda=\bar{x}$.

Hence $\log L$ and therefore L is maximum at $\lambda=\bar{x}$.

Thus the maximum likelihood estimator of λ is $\bar{x}$.

b) If $x_1, x_2, x_3, x_4, x_5, x_6$ be an independent simple random sample from a normal population with unknown variance σ^2, find K so that $K\left[(X_1-X_2)^2+(X_3-X_4)^2+(X_5-X_6)^2\right]$ is an unbiased estimator of σ^2. **[WBUT 2017]**

Answer:

Let μ denote the mean of the normal variate X

$$\text{var}(X)=\sigma^2$$

Since $x_1, x_2, \ldots\ldots x_n$ are independent, we have

$$E(X_iX_j) = E(X_i)E(X_j) = \mu^2 \text{ for } i\neq j$$

and $\quad E(X_i^2) - \{E(X_i)\}^2 = \sigma^2$ for each i

or, $\quad E(X_i^2) = \sigma^2+\mu^2$

Since $K\left\{(X_1-X_2)^2+(X_3-X_4)^2+(X_5-X_6)^2\right\}$ is an unbiased estimator of σ^2, we have

$$E\left\{K\left[(X_1-X_2)^2+(X_3-X_4)^2+(X_5-X_6)^2\right]\right\}=\sigma^2$$

or, $\quad KE\left(X_1^2+X_2^2+X_3^2+X_4^2+X_5^2+X_6^2-2X_1X_2-2X_3X_4-2X_5X_6\right)=\sigma^2$

or, $K\{E(X_1^2)+E(X_2^2)+E(X_3^2)+E(X_4^2)+E(X_5^2)+E(X_6^2)$
$-2E(X_1X_2)-2E(X_3X_4)-2E(X_5X_6)\}=\sigma^2$

or, $K(6\sigma^2+6\mu^2-2\mu^2-2\mu^2-2\mu^2)=\sigma^2$

or, $6K=1$

or, $K=\frac{1}{6}$

c) The weights in pound of a sample of 12 packets of butter are 13, 7, 22, 15, 12, 18, 14, 21, 8, 17, 10, 23 taken at random from its population having standard deviation 5. Find 95% confidence interval for the mean of the population. [Given $Z_{0.05}=1.96$] [WBUT 2017]

Answer:

Given that, $\sigma=5$, $Z_{0.05}=1.96$, $n=12$

$$\bar{x}=\frac{(13+7+22+15+12+18+14+21+8+17+10+23)}{12}=\frac{180}{12}=15$$

$\therefore$ Confidence interval for the population mean μ is,

$$\left(\bar{x}-\frac{\sigma}{\sqrt{n}}\times1.96, \bar{x}+\frac{\sigma}{\sqrt{n}}\times1.96\right)$$
$$=\left(15-\frac{5}{\sqrt{12}}\times1.96, 15+\frac{5}{\sqrt{12}}\times1.96\right)$$
$$=(12.17, 17.82)$$

TESTING OF HYPOTHESIS

Multiple Choice Type Questions

1. If $H_1(\mu>60)$ is an alternative hypothesis, then the null hypothesis is [WBUT 2012]
   a) $H_0(\mu<60)$ b) $H_0(\mu\geq60)$ c) $H_0(\mu\leq60)$ d) none of these

Answer: (d)

2. A null hypothesis is a statistical hypothesis which is setup and whose validity is tested for possible [WBUT 2015]
   a) acceptance b) rejection c) testing d) none of these

Answer: (b)

3. If μ is a parameter and $H_0(\mu=5)$ is null hypothesis then which one of the following is a left-sided alternative hypothesis? [WBUT 2018]
   a) $H_1(\mu\neq5)$ b) $H_1(\mu<5)$ c) $H_1(\mu>5)$ d) None of these

Answer: (b)

3. The power of the test in case of testing of hypothesis, is [MODEL QUESTION]
   a) $1-P$ (Type I Error) b) P (Type I Error) P (Type II Error)
   c) $1-P$ (Type II Error) d) $1-P$ (Type I Error) P (Type II Error)

Answer: (c)

4. Which of the following is type II error? [MODEL QUESTION]
   a) The error of accepting H_0 when H_0 is true
   b) The error of rejecting H_0 when H_0 is false
   c) The error of accepting H_0 when H_0 is false
   d) The error of rejecting H_0 when H_0 is true

Answer: (c)

Short Answer Type Questions

1. A random sample with observations 65, 71, 64, 71, 70, 69, 64, 63, 67, 68 is drawn from a normal population with variance 7.056. Test the hypothese's that the population mean is 69 at 1% level of significance. [WBUT 2012]
[Given that $P(0<z<2.58)=0.495$]

Answer:

Here the null hypothesis is $H_0(\mu=69)$

We have $\bar{X}=67.2$ $\sigma^2=7.056$

The test statistic, $z=\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}=\frac{67.2-69}{2.07/\sqrt{10}}=\frac{-1.8}{2.07/3.162}=-\frac{1.8}{0.65}=-2.769$

2. Suppose that 100 tires made by a certain manufacturer lasted on the average 21819 miles with a standard deviation of 1295 miles. Test the null hypothesis $\mu=22000$ miles against the alternate hypothesis $\mu<22000$ miles at the 0.05 level of significance. **[WBUT 2014]**

Answer:

Set the null hypothesis H_0 as $H_0(\mu=22,000)$ against $H_1(\mu<22,000)$

The test statistics

$$Z=\frac{21819-22000}{1295/\sqrt{100}}\sim N(0,1)\text{ approx}$$

$$=\frac{-181}{129.5}=-1.398$$

As this value is greater than 1.645, we accept the null hypothesis.

3. A random sample of size 20 from a normal population gives the sample mean of 42 and the sample standard deviation of 6. Test the hypothesis that the population mean is 44. Value of t distribution with 19 degrees of freedom at 5% level is 2.09. **[WBUT 2015]**

Answer:

Null hypothesis is that the population mean is 44. $H_0(\mu=44)$. Alternative hypothesis is that the population mean is different from 44. $H_2(\mu\neq 44)$, i.e., both-sided alternatives.

Since the population s.d. (σ) is not known, we use t test. Here, $n=20, \overline{x}=42$ and $S=6$. Therefore, $t=\frac{42-44}{\frac{6}{\sqrt{19}}}=-1.45$

Degrees of freedom $=(20-1)=19$

From the tables of t distribution, we find for 19 d.f. the percentage points of t are respectively $t_{.025}=2.09$ and $t_{.005}=2.86$. We have to use a two-tailed test, because the alternatives are both-sided. Since $|t|=1.45$ is less than the 5% tabulated value corresponding to the two tails (viz. 2.09) there is no reason to reject the null hypothesis at 5% level of significance and we conclude that the population mean may be 44.

[99% confidence limits for μ are $42\pm 2.86\times\left(\frac{6}{\sqrt{19}}\right)=42\pm 3.94=38.06$ and 45.94]

4. What are Type I error and Type II error. **[MODEL QUESTION]**

Answer:

The error made in rejecting the null hypothesis H_0 when it is true is called Type I error.
The error made in accepting the null hypothesis H_0 when it is false is called Type II error.

5. If $x\geq 1$ is the critical region for testing the null hypothesis $H_0:\theta=2$ against the alternative hypothesis $H_1:\theta=1$, on the basis of a single observation from the population $f(x,\theta)=\theta\exp[-\theta_x]$, $0\leq x<\infty=0$, elsewhere, then find the type I error and type II error. **[MODEL QUESTION]**

Answer:

P (Type I error) = P (Rejecting H_0 when H_0 is true)

$$=P(x\geq 1 \text{ where } \theta=2)$$

$$=\int_1^{\infty}2e^{-2x}dx=\left[2\frac{e^{-2x}}{-2}\right]_1^{\infty}=e^{-2}$$

P (Type II error) = P (Accepting H_0 when H_0 is false)

$$=P(x<1 \text{ when } \theta=1)=\int_0^1 e^{-x}dx=\left[-e^{-x}\right]_0^1=1-\frac{1}{e}$$

6. To test the unbiasedness of a die it is thrown six times and is accepted as unbiased if not more than one six is obtained. Find the probability of type I error. **[MODEL QUESTION]**

Answer:

P (Type I error) = P (Rejecting H_0 when H_0 is true)

= P (More than one six is obtained in 6 throws with P=1/6)

= 1-P (Less than one six is obtained when P = 1/6)

$$=1-{}^6C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6-{}^6C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^5$$

$$=1-\left(\frac{5}{6}\right)^6-\left(\frac{5}{6}\right)^5$$

7. The following table gives the number of road accidents that occurred during various days of the week in Kolkata. Test whether the accidents are uniformly distributed over the week. **[MODEL QUESTION]**

Day	Sun	Mon	Tues	Wed	Thurs	Fri	Sat
No. of accidents	13	14	19	12	11	15	14

[Given $\chi^2_{0.05}(v=6)=12.59$.]

Answer:

Here the null hypothesis H_0 is that the accidents are uniformly distributed over the week.
Thus we have

Day	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
f_0	13	14	19	12	11	15	14	98
f_e	14	14	14	14	14	14	14	98

$$X_6^2=\sum\frac{(f_0-f_e)^2}{f_e}=\frac{1}{4}(1+0+25+4+9+1+0)=\frac{40}{14}=2.8$$

Since the observed value of χ^2 is less than the tabulated value 12.59, we conclude that the accidents can be taken to be uniformly distributed.

8. The length of life X of certain computers is approximately normally distributed with mean 800 hours and st. dev. 40 hrs. If a random sample of 30 computers has an average life of 788 hrs, test the null hypothesis that $\mu = 800$ against the alternative hypothesis $\mu \neq 800$ hrs at 5% level of significance (Given $Z_{0.025} = 1.96$) **[MODEL QUESTION]**

Answer:
Here the null hypothesis $H_0 : \mu = 800$ against $H_1 : \mu \neq 800$
So it is a case of two tailed test.
The test statistic Z is given by

$$Z = \frac{\bar{X} - E(\bar{X})}{S \cdot E \cdot (\bar{X})} = \frac{788 - 800}{40/\sqrt{30}}$$

Since $E(\bar{X}) = \mu$ and $S \cdot E(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{3\sqrt{30}}{10} = 1.643$

Since this observed value is numerically less than 1.96, the null hypothesis is accepted.

9. In order to test whether a coin is perfect the coin is tossed 5 times. The null hypothesis of perfectness is rejected if more than 4 heads are obtained. What is the probability of Type 1 error? Find the probability of Type II error when corresponding probability of head is 0.2. **[MODEL QUESTION]**

Answer:
P (Type I error) = P (Rejecting H_0 when H_0 is true)
= P (getting 5 heads in 5 tosses of the coin)

$$= {}^5C_5\left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

P (Type II error) = P (Accepting H_0 when H_1 is true)
= P (getting at most 4 heads when p = 0.2)
= 1 - P (getting 5 heads when p = 0.2)

$$= 1 - {}^5C_5(0.2)^5$$

$$= 1 - (0.2)^5$$

10. The following table gives the number of aircraft accidents that occurred during various days of by the week. Test whether the accidents are uniformly distributed over the week.

Days	Sun	Mon	Tues	Wed	Thurs	Fri	Sat
No. of accidents	13	14	19	12	11	15	14

[Given $\chi^2_{0.5,6} = 12.59$.] **[MODEL QUESTION]**

Answer:
Assume that the accidents are uniformly distributed. Thus the null hypothesis H_0 : (The accidents are uniformly distributed) against the alternative hypothesis H_1 (they are not).
The test statistics is therefore

$$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e}$$

where $f_e = \frac{13+14+19+12+11+15+14}{7} = \frac{98}{7} = 14$

$$= \frac{(13-14)^2}{14} + \frac{(14-14)^2}{14} + \frac{(19-14)^2}{14} + \frac{(12-14)^2}{14} + \frac{(11-14)^2}{14} + \frac{(15-14)^2}{14} + \frac{(14-14)^2}{14}$$

$$= \frac{1}{14}(1+0+25+4+9+1+0) = \frac{40}{14} = 2.857 \text{ with 6 df.}$$

Since the observed values 2.857 is much less than the tabulated value 12.59 of χ^2 at 5% level of significance, we accept the null hypothesis and conclude that the accidents are uniformly distributed.

11. In a certain city 100 men in a sample of 400 were found to be smokes. In another city the number of smokes was 300 in a random sample of 800. Does this indicate that there is a greater proportion of smokes in the second city than in the first city? **[MODEL QUESTION]**

Answer:
Here $n_1 = 400,\ p_1 = 100/400 = 1/4;\ n_2 = 800,\ p_2 = 300/800 = 3/8$
We set up

$$\therefore \quad p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{400 \times \frac{1}{4} + 800 \times \frac{3}{8}}{1200} = \frac{1}{3} \quad \therefore q = \frac{2}{3}$$

$H_0 : P_1 = P_2$ against $H_1 : P_1 < P_2$ where P_1 and P_2 are the population proportions.
The test statistic

$$z = \frac{p_1 - p_2}{pq\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{\frac{1}{4} - \frac{3}{8}}{\frac{1}{3} \times \frac{2}{3}\sqrt{\frac{1}{400} + \frac{1}{800}}} = -9.186$$

Since this observed value is much less than the tabulated value of z at 1% level, we reject the null hypothesis and conclude that there is a greater proportion of smokers in the second city than in the first city.

12. The proportion of defective items in a large lot of items is p. To test the hypothesis $p=0.2$, we take a random sample of 3 items and accept the hypothesis if the number of defectives in the sample is 6 or less. Find the probability of Type-I – error of the test. Find the probability of Type-II – error if $p=0.3$? **[MODEL QUESTION]**

Answer:
Here, $p=0.2,\ n=3$

$$P(\text{Type I error}) = P(\text{rejecting } H_0 \text{ when it is free})$$
$$= P(\text{rejecting when } p=0.2)$$
$$= P(\text{getting 7 or 8 defectives when } p=0.2)$$
$$= {}^8C_7(0.2)^7\cdot(0.8)^1 + {}^8C_8(0.2)^8\cdot(0.8)^0$$
$$= 8(0.2)^7(0.8)+(0.2)^8.$$

$$P(\text{Type II error}) = P(\text{accepting } H_0 \text{ when it is false})$$
$$= P(\text{getting less than or equal to 6 heads when } H_1 \text{ is true})$$
$$= 1-P(\text{getting 7 or 8 heads when } p=0.3)$$
$$= 1-{}^8C_7(0.3)^7(0.7)^1-{}^8C_8(0.3)^8(0.7)^0.$$

13. A random sample with observations 65, 71, 64, 71, 70, 69, 64, 63, 67, 68 is drawn from a normal population with standard deviation $\sqrt{7.056}$. Test the hypothesis that the population mean is 69 at 1% level of significance.
[Given: $P(0<Z<2.58)=0.495$]. **[MODEL QUESTION]**

Answer:
Here

$$\overline{X}=\frac{(65+71+64+71+70+69+64+63+67+68)}{10}=67.2$$

We set the null hypothesis is $H_0(\mu=69)$ against the alternative hypothesis $H_1(\mu\neq 69)$.
The test statistic

$$Z=\frac{\overline{X}-E(\overline{X})}{\frac{\sigma}{\sqrt{n}}}=\frac{67.2-69}{\frac{\sqrt{7.056}}{\sqrt{10}}}=\frac{-1.8}{0.84}=-2.143.$$

Since the observed value of $|Z|<2.58$, we accept the null hypothesis and conclude therefore the population mean may be taken as 69 at 1% level.

14. Intelligence tests on two groups of boys and girls gave the following results:

	Mean	SD	N
Boys	70	20	250
Girls	75	15	150

Is there any significant difference in the mean scores obtained by boys and girls? **[MODEL QUESTION]**

Answer:
We set the null hypothesis H_0 as H_0 (There is no difference in the mean scores by boys and girls) against the alternative hypothesis H_1 as Hi (There is difference in the said mean scores)
We now have the test statistic Z as

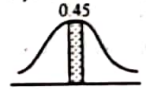
$$Z=\frac{\overline{x}_1-\overline{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}=\frac{70-75}{\sqrt{\frac{400}{250}+\frac{225}{150}}}=\frac{-5}{\sqrt{1.6+1.5}}=\frac{-5}{\sqrt{3.1}}=\frac{-5}{1.76}=-2.84$$

As the value of Z is numerically greater than 2.58, we reject the null hypothesis and conclude that there is significant difference in the mean scores.

Long Answer Type Questions

1. a) A machine part was designed to withstand an average pressure of 120 units. A random sample of size 100 from a large batch was tested and it was found that the average pressure which these parts can withstand is 105 units with a standard deviation of 20 units. Test at 5% level of significance whether the batch meet the specification. Suppose the population has normal distribution and given that $\phi(1.645)=0.45$. **[WBUT 2015]**

Answer:
Here the null hypothesis $H_0:(\mu=120)$ against $H_1(\mu<120), n=100$ where μ denotes the average pressure that the machine part can withstand.



The test statistic $Z=\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}=\frac{105-120}{20/\sqrt{100}}=\frac{-15}{2}=-7.5$

Since the deserved value -7.5 is less than the tabulated value of Z at 5% level, we reject the null hypothesis and conclude that the average pressure that the machine part can withstand is less than 120.

b) Survey of 320 families with 5 children each revealed the following distribution:

No. of boys:	5	4	3	2	1	0
No. of girls:	0	1	2	3	4	5
No. of family:	14	56	110	88	40	12

Is the result consistent with the hypothesis that male and female births are equally probable? The 5% value of X^2 with 5 d.o.f. is 11.07. **[WBUT 2015]**

Answer:
We set the null hypothesis as H_0 (male & female births are equally probable) of 5 children,

$$P(\text{all are boys}) = \left(\frac{1}{2}\right)^5$$

$$P(4\text{ boys } \& \text{ 1 girl}) = {}^5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 = 5\left(\frac{1}{2}\right)^5$$

$$P(3\text{ boys } \& \text{ 2girls}) = {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 10\left(\frac{1}{2}\right)^5$$

$$P(2\text{boys } \& \text{ 3 girls}) = {}^5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = 10\left(\frac{1}{2}\right)^5$$

$$P(1\text{ boy } \& \text{ 4 girls}) = {}^5C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 = 5\left(\frac{1}{2}\right)^5$$

$$P(\text{ no boy } \& \text{ 5 girls}) = {}^5C_0\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^5$$

Therefore of 320 families,

No. of families having all five boys $= \left(\frac{1}{2}\right)^5 \times 320 = 10$

No. of families having 4 boys & 1 girl $= 5\left(\frac{1}{2}\right)^5 \times 320 = 50$

No. of families having 3 boys & 2 girls $= 10\left(\frac{1}{2}\right)^5 \times 320 = 100$

No. of families having 2 boys & 3 girls $= 10\left(\frac{1}{2}\right)^5 \times 320 = 100$

No. of families having 1 boy & 4 girls $= 5\left(\frac{1}{2}\right)^5 \times 320 = 50$

No. of families having only girls $= \left(\frac{1}{2}\right)^5 \times 320 = 10$

So we have

$$\chi^2 = \sum\frac{(f_0 - f_e)^2}{f_e} = \frac{(14-10)^2}{10} + \frac{(56-50)^2}{50} + \frac{(110-100)^2}{100} + \frac{(88-100)^2}{100} + \frac{(40-50)^2}{50} + \frac{(12-10)^2}{10} = 7.16$$

As the observed value 7.16 is less than the tabulated value 11.07 of χ^2 with 5 d.f.

We accept the null hypothesis and conclude that male and female births are equally probable.

2. A random sample with observations 65, 71, 64, 71, 70, 69, 64, 63, 67, 68 is drawn from a normal population with a standard deviation $\sqrt{7.056}$. Test the hypothesis that the population mean is 69 at 1% level of significance.
[Given that $P(0 < z < 2.58) = 0.495$]. **[WBUT 2016]**

Answer:
Here $\overline{X} = (65+71+64+71+70+69+64+63+67+68)/10 = 67.2$
We set the null hypothesis H_o as $H_o(\mu = 69)$ against $H_1(\mu \neq 69)$
The test statistics is

$$Z = \frac{\overline{X} - E(\overline{X})}{SE(\overline{X})} = \frac{67.2-69}{\sqrt{7.056/10}} = 6.776$$

Since this value is greater than 2.58, it is significant at 1% level. So we reject the null hypothesis.

3. The following figures show the distribution of digits in numbers chosen at random from a telephone directory:

Digits	0	1	2	3	4	5	6	7	8	9	Total
Frequency	1026	1107	997	966	1075	933	1107	972	964	853	10,000

Test whether the digits may be taken to occur equally frequently in the directory.
[The tabulated $\chi^2_{0.05}$ for 9 d.f. $= 16.919$] **[WBUT 2017]**

Answer:
We set the null hypothesis H_0 as
H_0 (The digits occur equally frequently) against H_1 (They are not)

Here
$$\chi^2 = \sum\frac{(f_0 - f_e)^2}{f_e} = \frac{(1026-1000)^2}{1000} + \frac{(1107-1000)^2}{1000} + \frac{(997-1000)^2}{1000} + \frac{(966-1000)^2}{1000} + \frac{(1075-1000)^2}{1000} + \frac{(933-1000)^2}{1000} + \frac{(1107-1000)^2}{1000} + \frac{(972-1000)^2}{1000} + \frac{(964-1000)^2}{1000} + \frac{(853-1000)^2}{1000}$$

$$= \frac{1}{1000}\{26\times26 + 107\times107 + 3\times3 + 34\times34 + 25\times25 + 67\times67 + 107\times107 + 28\times28 + 56\times56 + 147\times147\}$$

$$= \frac{1}{1000}\{676+11,449+9+1,156+625+4,489+11,449 +784+3,136+21,609\}$$

$$= 55.382$$

Since $\chi^2_{0.05}$ for 9 d.f. = 16.919 and observed $\chi^2 = 55.382 > 16.919$, we reject the null hypothesis and conclude that the digits do not occur equally frequently in the directory.

4. a) Let a be a group. If $a, b \in G$ such that $a^4 = e$, the identity element of G and $ab = ba^2$, then prove that $a = e$. **[WBUT 2018]**

Answer:

We see $ab = ba^2$

or, $(ab)a^2 = (ba^2)a^2$

or, $a(ba^2) = ba^4$

or, $a(ab) = b \quad [\because a^4 = e]$

or, $a^2b = b$

or, $a^2bb^{-1} = bb^{-1} = e$

or, $a^2 = e$

b) If $(R, +, \cdot)$ be a ring such that $a^2 = a \forall a \in R$, then show that
(i) $b = -b \forall b \in R$ (ii) $a - b = 0 \Rightarrow a = b$ **[WBUT 2018]**

Answer:

We see $(b+b)^2 = b+b \quad [\because a^2 = a \forall a \in R]$

or, $b^2+b^2+b^2+b^2 = b+b$

or, $b+b+b+b = b+b$

or, $b+b = 0$

or, $b = -b$

The other quality follows trivially

ADVANCED GRAPH THEORY

Multiple Choice Type Questions

1. A complete graph is called Kuratowski's first graph if it has **[WBUT 2012, 2015, 2017]**
a) 5 vertices b) 4 vertices c) 6 vertices d) 7 vertices
Answer: (c)

2. If G is a non-planar graph then the possible number of vertices of G is
a) 2 b) 3 c) 4 d) 6 **[WBUT 2012, 2014]**
Answer: (d)

3. The chromatic number of a graph containing an odd circuit is **[WBUT 2012]**
a) 3 b) 2
c) greater than or equal to 3 d) grater than or equal to 2.
Answer: (a)

4. If G is a connected planar graph with n vertices, e edges and f faces, then $n - e + f = 2$. This statement is **[WBUT 2013]**
a) true b) false
Answer: (a)

5. Kuratowski's graph is a **[WBUT 2013]**
a) planar graph b) regular graph c) tree d) none of these
Answer: (b)

6. If a simple graph has 15 edges then sum of the degrees of all the vertices is **[WBUT 2014]**
a) 25 b) 24 c) 50 d) 30
Answer: (d)

7. Chromatic number of a complete graph with 15 vertices is **[WBUT 2014]**
a) 12 b) 13 c) 14 d) 15
Answer: (d)

8. Let C_{97} be a circuit with 97 vertices. Then $\chi(C_{97})$ is equal to **[WBUT 2015, 2017]**
a) 97 b) 98 c) 2 d) 3
Answer: (d)

9. The maximum number of edges in a simple connected graph with n vertices is
a) $2 \cdot {}^nC_2$ b) ${}^nC_2$ c) $(n-1)$ d) n^2 **[WBUT 2015]**
Answer: (b)

10. If G is a non-planar graph then the possible number of vertices of G is [WBUT 2016]
a) 2 b) 3 c) 4 d) 6
Answer: (d)

11. The chromatic number of a graph containing an odd circuit is [WBUT 2016]
a) 3 b) 2
c) greater than or equal to 3 d) greater than or equal to 2
Answer: (a)

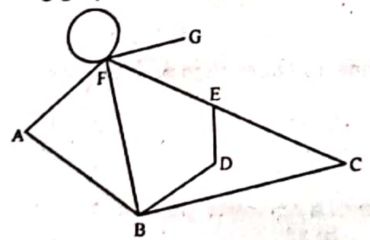
12. The dual of a disconnected graph is [WBUT 2018]
a) regular b) connected c) complete d) disconnected
Answer: (b)

13. A connected planar graph with 8 vertices determines 4 regions. The number of edges of this graph is [WBUT 2018]
a) 4 b) 8 c) 10 d) 12
Answer: (d)

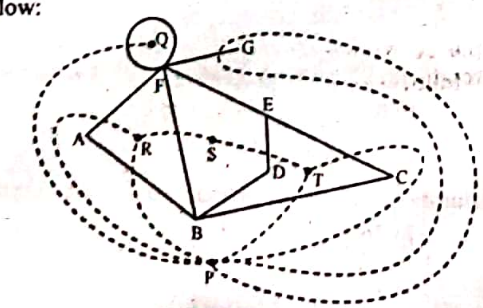
14. The chromatic polynomial of a tree with n vertices is [WBUT 2018]
a) λ^n b) λ^{n-1} c) $\lambda(\lambda-1)^{n-1}$ d) $\lambda(\lambda-1)^n$
Answer: (c)

Short Answer Type Questions

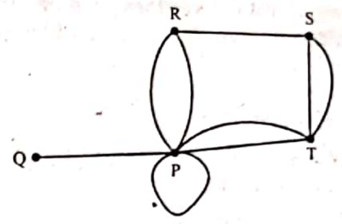
1. Draw the dual of the following graph: [WBUT 2012, 2018]



Answer:
The dual is shown below:



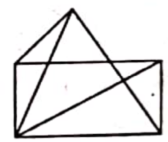
So its dual is



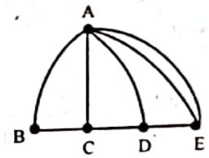
2. Show that a connected graph is Eulerian if and only if each of its vertices is of even degree. [WBUT 2013]

Answer:
Let G be a connected graph and let G be Eulerian. So an Eulerian tour exists which passes through every edge and hence through every vertex. So when a path comes to a vertex, it has to go out of it to complete the tour.
Thus every vertex has degree 2 or 4 or 6 etc. i.e., the degree of each vertex is even.
Conversely, let G be a connected graph having each of its vertex of even degree. Since the graph is connected, starting with a vertex a path must exist which goes out of each vertex and again comes back to it again. Thus there exists an Euler tour crossing each vertex and passing through each edge only once. Hence G is Eulerian.

3. Draw the dual of the graph. [WBUT 2015]



Answer:
The dual of the given graph is



4. Let G be a simple connected planar graph with n vertices, e edges and f faces. Prove that the following inequalities must hold [WBUT 2015]

(a) $n \geq \frac{3}{2} f$;

(b) $e \leq (3n-6)$

Answer:
Refer to Question No. 4(b) of Long Answer Type Questions.

5. Prove that a planer graph G with n vertices, e number of edges and k number of connected components determines $f = e - n + k + 1$ number of regions. **[WBUT 2016]**

Answer:
We prove the result by induction on k.
By Euler's theorem, $n - e + f = 2$ when the graph is connected
i.e. $k = 1$. Clearly $n - e + f = k + 1$ is true.
Now let, $k = 2$, i.e., the graph has 2 components.
In this case, $n_1 - e_1 + f_1 + n_2 - e_2 + f_2 = 2$ where n_i e_i f_i are the no. of vertices, edges and faces of the i^{th} components.
or, $n - e + f - 1 = 2$ since the outside region has been taken twice.
Here again, $n - e + f = k + 1$ as $k = 2$
Now suppose the result is true for k components. The $n - e + f = k + 1$.
For additional one components i.e. k + 1 components, we get as above

$$\sum n_i - \sum e_i + \sum f_i - 1 = k + 1$$

or, $n - e + f = (k+1) + 1$

Thus the result is true for $k + 1$ components also. Hence the result.

6. Let G be a simple connected planar graph with e edges and f regions. Then prove that $e \geq \frac{3}{2} f$. **[WBUT 2017]**

Answer:
Let φ denote a face of the simple graph G and let $d(\varphi)$ denotes the degree of the face φ, i.e., the no. of edges in the boundary of φ.
Let $s = \sum d(\varphi)$, the summation being taken over all faces of G.
Since each face has at least three edges, we have

$$s \geq 3f$$

where f is the number of faces of G.
However when we sum up to get s, each edge of G was counted either once or twice (i.e., when it occurred as a boundary edge for two faces) and so

$$s \leq 2e$$

Combining the above two inequalities, we get

$$2e \geq s \geq 3f$$

or, $e \geq \frac{3}{2} f$

7. Let G be a simple connected planar graph with n vertices, e edges and f regions. Then prove that $e \leq 3n - 6$. **[WBUT 2018]**

Answer:
By Euler's formula, $n - e + f = 2$

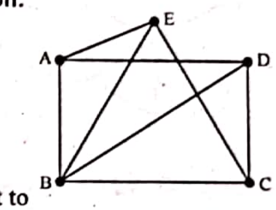
We know $e \geq \frac{3}{2} f$

or, $e \geq \frac{3}{2}(2 - n + e)$

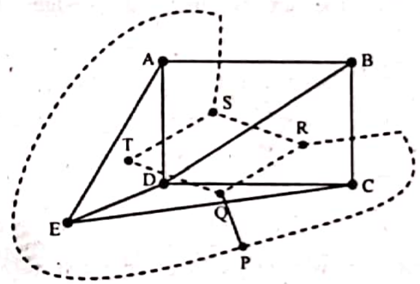
or, $2e \geq 6 - 3n + 3e$

or, $e \geq 3n - 6$

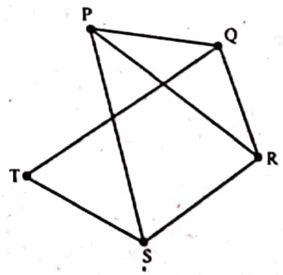
8. Draw the dual of the graph: **[WBUT 2018]**



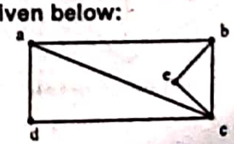
Answer:
The given graph is equivalent to



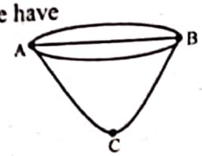
Its dual is



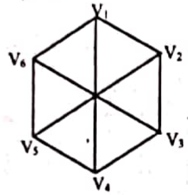
9. Find the dual of the graph given below: **[MODEL QUESTION]**



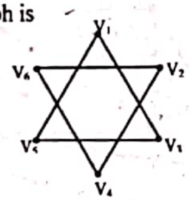
Answer:
For the dual of the given graph we have
Thus the dual is



10. Define complement of a graph. Find the complement of the graph. **[MODEL QUESTION]**

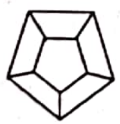


Answer:
The complement of a graph G is a graph G' which has the same vertex set as of G but the edges of G' are the joins of the vertices which are not edges of G so that $G \cup G'$ is a complete graph.
The complement of the given graph is



Note $G \cup G'$ is a complete graph where G is the given graph and G' is its complement.

11. Draw the dual of the graph. **[MODEL QUESTION]**



Answer:
The dual of the given graph is

Long Answer Type Questions

1. a) A regular graph G determines 8 regions, degree of each vertex being 3. Find the number of vertices of G. **[WBUT 2012]**

Answer:
Let the no. of vertices be n

We know $f = e - n + 2 \Rightarrow 8 = \frac{3n}{2} - n + 2 \Rightarrow \frac{n}{2} = 6$ or, $n = 12$.

b) Prove that the chromatic polynomial of a tree with n vertices $x(x-1)^{n-1}$, whose x is the no. of colours. **[WBUT 2012, 2018]**

OR,

Prove that a graph with n vertices is a tree if and only if its chromatic polynomial $\rho_n(\lambda) = \lambda(\lambda-1)^{n-1}$. **[WBUT 2014]**

Answer:
The chromatic polynomial of a tree with n vertices $x(x-1)^{n-1}$.
Proof: If $n = 1$ i.e., there is only one vertex, then evidently

$$P_1(x) = \sum_{i=1}^{1} W_i\,{}^{x}C_i = W_1\,{}^{x}C_1$$

The number of ways a vertex can be coloured with one is 1 i.e., $W_1 = 1$. Hence

$$P_1(x) = 1 \cdot x = x(x-1)^{1-1}.$$

If $n = 2$, i.e., there are two vertices, then

$$P_2(x) = W_1\,{}^{x}C_1 + W_2\,{}^{x}C_2 = 0 \cdot {}^{x}C_1 + 2 \cdot \frac{x(x-1)}{2!} \text{ since } W_1 = 0, W_2 = 2$$
$$= x(x-1)^{2-1}$$

Now let us assume, $P_r(x) = x(x-1)^{r-1}$
Now adding one vertex, we observe

$$P_{r+1}(x) = W_1\,{}^{x}C_1 + W_2\,{}^{x}C_2 + \cdots + W_r\,{}^{x}C_r + W_{r+1}\,{}^{x}C_{r+1}$$
$$= x(x-1)^{r-1} + x(x-1)^{r-1}(x-2) = x(x-1)^{r-1}(x-2) = x(x-1)^{r}.$$

Hence the result is true for $r+1$.
So by induction the result holds for any $n \in \mathbb{N}$.
Therefore, $P_n(x) = x(x-1)^{n-1}$.

2. a) Show that every planar graph is 6 colourable. **[WBUT 2012, 2018]**

Answer:

From the definition of colourability of a graph, it is clear that if a graph is k colourable, then it is $(k+1)$ colourable.

As by the Four Colour theorem, every planar connected graph is 4-colourable, it is 5-colourable and hence it is 6-colourable.

b) If G be a connected graph with n vertices, e edges and r faces, prove that $n-e+r=2$. **[WBUT 2012, 2017]**

OR,

If G be a connected planar graph with n vertices, e number of edges and f number of faces, prove that $n-e+f=2$. **[WBUT 2014]**

OR,

State and prove Euler's formula for a connected planar graph. **[WBUT 2015]**

Answer:

Euler's Formula: For every connected planar graph with n vertices, e edges and r regions, the following is true $n-e+r=2$.

Proof: It suffices to prove the formula for a simple graph since addition of a loop or a parallel edge simply adds one region but simultaneously it adds one edge also and hence this extra 1 with r and also with e balances each other. All edges that do not form boundary of any region are also disregarded as in this case too n is in increased by 1 and also e is increased by 1, ultimately balancing each other.

Since every simple planar graph can have a plane representation such that each edge is a straight line any planar graph can be drawn such that each region is a polygon i.e., polygonal set.

Let the polygonal set representing the given graph consist of r regions and let k_p be the number of p sided regions.

Clearly $3k_3+4k_4+\cdots+pk_p=2e$

Also, $k_3+k_4+\cdots+k_p=r$

We now note that the sum of all angles subtended at all vertices of the polygonal sets is $2\pi n$.

Recall that the sum of the interior angles of a polygon of sides is $\pi(p-2)$ and the sum of the exterior angles of a polygon of p sides is $\pi(p+2)$.

$\therefore$ The grand sum of all interior angles of $p-1$ regions plus the sum of the exterior angles of the polygon defining the infinite region

$$=\pi(3-2)k_3+\pi(4-2)k_4+\cdots+\pi(p-2)k_p+4\pi=\pi(2e-2r)+4\pi$$

But $\quad 2\pi(e-r)+4\pi=2\pi n \quad$ or, $\quad n-e+r=2$

3. a) Prove that a planar graph with n vertices, e number of edges and k number of components determines f number of regions, where $f=e-n+k+1$. **[WBUT 2013]**

Answer:

If G is a planar graph with n vertices e edges, f faces and k connected components, then

$$n-e+f=k+1$$

Proof: Since G has k components, g can be treated as composed of k simple connected graphs. Interestingly if the no. of faces for components of the graph are $f_1, f_2, \cdots, f_k$ respectively, then $f_1+f_2+\cdots\cdots+f_k-(k-1)$ is the total no. of faces of the graph.

Hence, $\quad n-e+\{f_1+f_2+\cdots\cdots f_k-(k-1)\}=2$

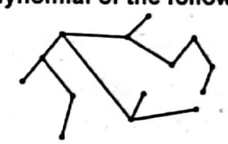
or, $\quad n-e+f=k-1+2=k+1 \qquad$ where, $f=\sum f_i$

b) Show that any simple connected planar graph with n vertices $(n\geq 3)$ has at most $(3n-6)$ edges. **[WBUT 2013]**

Answer:

Refer to Question No. 4(b) (ii) of Long Answer Type Questions.

4. a) Find out the chromatic polynomial of the following graph G: **[WBUT 2016]**



Answer:

Since the given graph is a tree, its chromatic polynomial is $\lambda(\lambda-1)^{14-1}$ i.e., $\lambda(\lambda-1)^{13}$

b) Prove that if G is a simple planar graph then G has at least one vertex v such that $\deg(v)\leq 5$. **[WBUT 2016]**

Answer:

If V(G) is a singleton set, then degree of the vertex is zero. If G has only two vertices, then both must have degree at most one. So we can suppose that G has at least three vertices. Now if the degree of each vertex of G is at least six, we get

$$\sum_{v\in V(G)} d(v)\geq 6n$$

But, we know $\sum_{v\in V(G)} d(v)=2e$ Thus $2e\geq 6n$ or, $6\geq 3n$

But this is impossible since we know $e\leq 3n-6$

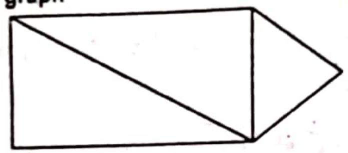
This contradiction thus leads to the fact that G must have at least one vertex of degree less than 6.

c) A regular graph G determines 8 regions, degree of each vertex being 3. Find the number of vertices of G. **[WBUT 2016]**

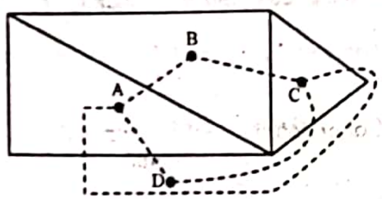
Answer:
We know by Euler's theorem that $n-e+f=2$. Assume, n to be the no. of vertices.

Then $f=8, e=\frac{3n}{2}$. Hence we get, $n-\frac{3n}{2}+8=2$ or, $n=12$.

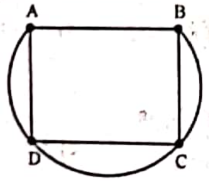
5. a) Draw the dual of the graph **[WBUT 2017]**



Answer:
The given graph is
We take one point in each region and join the points so that the join cross only one edge as shown below:



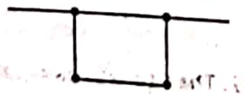
When represented the dotted lines given the dual as



b) Define planar graph. Construct a planar graph with 6 vertices. **[WBUT 2017]**

Answer:
1st Part: In graph theory, a planar graph is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.

2nd Part:
A planar graph with 6 vertices.



GROUP THEORY

Multiple Choice Type Questions

1. A group G is commutative iff **[WBUT 2012]**
a) $ab=ba$ b) $(ab)^{-1}=b^{-1}a^{-1}$ c) $(ab)^{-1}=a^{-1}b^{-1}$ d) $(ab)^2=ab$
Answer: (a)

2. The generators of the cyclic group $(Z,+)$ are **[WBUT 2012]**
a) 1, –1 b) 0, 1 c) 0, –1 d) 2, –2
Answer: (a)

3. Which of the following sets is closed under multiplication? **[WBUT 2012]**
a) $\{1,-1,0,2\}$ b) $\{1,i\}$ c) $\{1,\omega,\omega^2\}$ d) $\{\omega,1\}$
Answer: (c)

4. The number of generations of an infinite cyclic group is [WBUT 2013, 2015, 2017]
a) 1 b) 2 c) infinite d) none of these
Answer: (b)

5. Which one of the following sets forms a group under usual multiplication of complex numbers? **[WBUT 2013]**
a) $\{1, i\}$ b) $\{1, \omega, \omega^2\}$ c) $\{1, \omega^2\}$ d) $\{1, \omega\}$
Answer: (b)

6. The order of the dihedral group D_4 is **[WBUT 2013, 2015]**
a) 4 b) 6 c) 8 d) 64
Answer: (c)

7. If A and B are two subgroups of a group G, then which of the following is always a subgroup of G? **[WBUT 2013]**
a) $A\cup B$ b) $G-A$ c) $G-B$ d) $A\cap B$
Answer: (d)

8. The symmetric group S_3 has **[WBUT 2013]**
a) 6 elements b) 8 elements c) 9 elements d) none of these
Answer: (a)

9. Let G be a Group and $a,b\in G$. Then $(a^{-1}b)^{-1}$ is equal to **[WBUT 2014]**
a) ab^{-1} b) $b^{-1}a$ c) $a^{-1}b^{-1}$ d) $b^{-1}a^{-1}$
Answer: (b)

10. Which one of the following is not a cyclic group. [WBUT 2014]
a) $(Z,+)$ b) $(Z_4,+)$ c) $(Q,+)$ d) $(Z_{15},+)$
Answer: (c)

11. Let G be a Group and $x \in G$ be such that $o(x)=5$. Then [WBUT 2016]
a) $o(x^{15})=4$ b) $o(x^{10})=6$ c) $o(x^{23})=5$ d) $o(x^{20})=3$
Answer: (c)

12. The set $\{[2], [4], [6], [8]\}$ is a group under multiplication modulo 10. The identity element of the group is [WBUT 2016]
a) $[2]$ b) $[4]$ c) $[6]$ d) $[8]$
Answer: (c)

13. Let G be a finite group of even order. Then the number of elements of order 2 in G is [WBUT 2016]
a) 2 b) 4 c) even d) odd
Answer: (d)

14. The generators of the cyclic group $(Z, +)$ are [WBUT 2016]
a) $1, -1$ b) $0, 1$ c) $0, -1$ d) $2, -2$
Answer: (a)

15. In the group S_3 of all permutations on $\{1, 2, 3\}$; the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ is [WBUT 2016]
a) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$
Answer: (c)

16. If H is a subgroup of a group G and a, b are two distinct elements of G, then indicate which of the following statements is true [WBUT 2015]
a) $aH = Ha$
b) $Ha \cap Hb = \phi$
c) $Ha \cap Hb \neq \phi$ and $Ha \neq Hb$
d) $aH = bH$
Answer: none of these

17. If a is an element of a group G and order of a is 35, then the order of a^{10} is [WBUT 2017]
a) 5 b) 7 c) 35 d) none of these
Answer: (b)

18. The number of generators of a cyclic group of order 7 is [WBUT 2017]
a) 2 b) 6 c) 7 d) 5
Answer: (b)

19. A group contains 12 elements. Then the possible number of elements in a subgroup is [WBUT 2018]
a) 5 b) 3 c) 11 d) 7
Answer: (b)

20. Which of the following statements is true [WBUT 2018]
a) A semi-group with identity element is called monoid
b) A groupoid is said to be a semi-group if the binary operation is commutative
c) $(Q,+)$ is a cyclic group
d) Identity element in a group is not unique
Answer: (a)

21. If the cyclic group G contains 11 distinct elements, then it has [WBUT 2018]
a) 2 generators b) 7 generators c) 9 generators d) 10 generators
Answer: (d)

22. The number of unit elements of the ring $(Z,+,\cdot)$ [MODEL QUESTION]
a) 2 b) 3 c) 1 d) infinite
Answer: (a)

23. If S and T are two subgroups of a group G, then which of the following is a subgroup? [MODEL QUESTION]
a) $S \cup T$ b) $S \cap T$ c) $S-T$ d) $G-S$
Answer: (b)

24. In the additive group Z_6 the order of the element [4] is [MODEL QUESTION]
a) 0 b) 2 c) 3 d) 6
Answer: (c)

25. Let G be a group and $a \in G$. If $o(a)=17$, then $o(a^8)$ is [MODEL QUESTION]
a) 17 b) 16 c) 8 d) 5
Answer: (a)

27. In a group $(G,0)$ if $(a\,o\,b)^{-1} = a^{-1}\,o\,b^{-1}$, then [MODEL QUESTION]
a) G is finite b) G is infinite c) G is abelian d) none of these
Answer: (c)

Short Answer Type Questions

1. If G be a group such that $(ab)^2 = a^2b^2$ for $a,b \in G$; show that the group G is Abelian. **[WBUT 2012]**

OR,

Prove that a group $(G,*)$ is commutative if and only if $(a*b)^2 = a^2*b^2$, for all $a,b \in G$. **[WBUT 2014]**

Answer:

Let $a,b \in G$ be arbitrary. Then by hypothesis

$$abab = aabb.$$
$$\text{or } a^{-1}(abab)b^{-1} = a^{-1}(aabb)b^{-1}$$
$$\text{or } \quad ba = ab.$$

Hence G is abelian.

2. Let $(Q,+)$ be the additive group of rational numbers and $(Q^+,\cdot)$ be the multiplicative group of positive rational numbers. Are these two groups isomorphic? Justify your answer. **[WBUT 2013]**

Answer:

No. We prove the result by contradiction.

If possible let $Q:(Q,+)\to(Q^+,\bullet)$ be an isomorphism and $\varphi(a)=2$.

Now $2=\varphi(a)=\varphi\left(\frac{a}{2}+\frac{a}{2}\right)=\varphi\left(\frac{a}{2}\right)\cdot\varphi\left(\frac{a}{2}\right)=\left\{\varphi\left(\frac{a}{2}\right)\right\}^2$ i.e., $\varphi\left(\frac{a}{2}\right)=\sqrt{2}$

Clearly $\sqrt{2}\notin Q^+$. This is a contradiction. Hence the result.

3. Prove that every nontrivial subgroup of the additive group Z of integers is cyclic. **[WBUT 2013]**

Answer:

Let H be a subgroup of $\mathbb{Z}$.

If H consists of integral multiples of an integer only it is a cyclic group generated by a single element. If H consists of at least two elements one of which is not an integral multiple of the other, then by taking integral multiples of them and adding we can get any integer.

This implies that $H=\mathbb{Z}$.

Hence the subgroups of $\mathbb{Z}$ are cyclic.

4. Show that the 7th roots of unity form a cyclic group. Find all the generators of this group. **[WBUT 2014]**

Answer:

The 7th roots of unity are the roots of the equation $x^7=1=\cos 2\pi + i\sin 2k\pi,\ k\in\mathbb{Z}$.

Thus the roots are $\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7},\ k=0,1,2,\ldots,6$.

Denoting $\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$ by α, the roots are $1,\alpha,\alpha^2,\ldots,\alpha^6$.

So G denotes the set of all 7th roots of units, then

$$G=\{1,\alpha,\alpha^2,\ldots,\alpha^6\} \text{ where } \alpha^7=1$$

Clearly G I closed w.r.t. multiplication and is associative, 1 is the identity and the inverse of $\alpha^r=\alpha^{7-r},\ 1\le r\le 6$.

Thus G is a cyclic group. The generators are $\alpha^2,\alpha^3,\alpha^4,\alpha^5,\alpha^6$.

5. Show that for any two subgroups H and K of a group G, $H\cap K$ is also a subgroup of G. **[WBUT 2015]**

Answer:

Let H and K be any two subgroups of G. The $H\cap K\neq\phi$, since at least the identity element e is common to both H and K.

In order to prove that $H\cap K$ is a subgroup it is sufficient to prove that $a\in H\cap K, b\in H\cap K\Rightarrow ab^{-1}\in H\cap K$.

Now $a\in H\cap K\Rightarrow a\in H$ and $a\in K$.

$b\in H\cap K\Rightarrow b\in H$ and $b\in K$.

But H, K are subgroups. Therefore

$a\in H, b\in H\Rightarrow ab^{-1}\in H$,

$a\in K, b\in K\Rightarrow ab^{-1}\in K$.

Finally, $ab^{-1}\in H, ab^{-1}\in K\Rightarrow ab^{-1}\in H\cap K$.

Thus we have shown that $a\in H\cap K,\ b\in H\cap K\Rightarrow ab^{-1}\in H\cap K$.

Hence $H\cap K$ is a subgroup of G.

6. Let $(G,*)$ be a group and $a,b\in G$. Suppose that $a^2=e$ and $a*b^4*a=b^7$. Prove that $b^{33}=e$. **[WBUT 2016]**

Answer:

$$b^7*b^7=(a*b^4*a)*(a*b^4*a)$$
$$=a*b^4*(a*a)*b^4*a \quad =a*b^4*b^4*a$$
$$=a*b^8*a \ =a*b*(a*b^4*a)*a \ =a*b*a*b^4$$

Therefore,

$$b^{28}=a*b^{16}*a$$

Now,

$$b^{33}=b^5*b^{28}=b^5*(a*b^{16}*a)=a*a=e$$

7. Find out order of the element $\begin{pmatrix}1 & 2 & 3\end{pmatrix} \in S_3$. Also find out inverse of $\begin{pmatrix}1 & 2 & 3\end{pmatrix} \in S_3$. **[WBUT 2016]**

Answer:

Clearly (123) $\quad (123) = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$

and $\{(123)(123)\}(123) = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}(123) = \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$

Hence $0((123)) = 3$

Next, $(1\ 2\ 3)^{-1} = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$

8. Prove that a group G is Abelian, iff $(ab)^2 = a^2b^2$ for all $a, b \in G$. **[WBUT 2017]**

Answer:

Let G be an Abelian group.

Then, $a, b \in G$, $ab = ba$

$\therefore \quad (ab)^2 = a(ba)b = a(ab)b = a^2b^2$

Conversely, let $(ab)^2 = a^2b^2$ for all $a, b \in G$.

This gives,

$abab = a^2b^2$

or, $ba = ab$ for all $a, b \in G$

Hence G is Abelian.

9. Show that in a group (G, o), $(a \, o \, B)^{-1} = b^{-1} o \, a^{-1}$ for all $a, b \in G$. **[WBUT 2018]**

Answer:

We observe that

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$$

$$(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b = b^{-1}eb = b^{-1}b = e$$

Hence $b^{-1}a^{-1}$ is the inverse of ab i.e., $(ab)^{-1} = b^{-1}a^{-1}$

10. Define subgroup of a group. Let $GL(2, \mathbb{R})$ be the multiplicative group of all real non-singular matrices of order 2. Show that the Set $H = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} | ad - bc = 1\right\}$ is a subgroup of $GL(2, \mathbb{R})$. **[WBUT 2018]**

Answer:

Here $\quad H = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}, |ad - bc| = 1\right\}$

Let $\quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}, \begin{pmatrix}p & q\\ r & s\end{pmatrix} \in H$

Then $\quad |ad - bc| = 1, |ps - rq| = 1$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix}^{-1} = \pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}s & -q\\ -r & p\end{pmatrix} = \pm\begin{pmatrix}as - br & -aq + bp\\ cs - dr & -cq + dp\end{pmatrix}$$

Also, $\quad |(as - br)(-cq + dp) - (cs - dr)(-aq + bp)|$

$$= |(ad - bc)(ps - rq)| = |ad - bc||ps - rq| = 1$$

Hence H is a subgroup of $GL(2, \mathbb{R})$

11. Let G be a group, if $a, b \in G$ such that $a^4 = e$, the identity element of G and $ab = ba^2$, prove that $a = e$. **[MODEL QUESTION]**

Answer:

From $ab = ba^2$ we get

$b^{-1}ab = a^2$

$\therefore \quad e = (a^2)^2 = (b^{-1}ab)(b^{-1}ab) = a^2$

$\therefore \quad ab = ba^2$ gives $ab = b \quad [\because a^2 = e]$

$\therefore \quad a = e$.

12. Show that the mapping $f : (\mathbb{Z}, \bullet) \to (\mathbb{R}, \bullet)$ defined by $f(x) = x^2 \forall x \in \mathbb{Z}$ is a monomorphism but not isomorphism. **[MODEL QUESTION]**

Answer:

We first prove that f is a homomorphism.

To this end we see

$$f(xy) = (xy)^2 = x^2y^2 = f(x)f(y) \forall x, y \in \mathbb{Z}$$

Hence f is a homomorphism.

Next, we observe if $x \neq y$, then $x^2 = y^2$ for $x, y \in \mathbb{Z}$.

Hence $\quad f(x) \neq f(y)$

Thus f is injective.

But f is not surjective as $-1 \in \mathbb{R}$ has no preimage in $\mathbb{Z}$.

Thus f is a monomorphism and not an isomorphism.

13. Show that the set of all possible permutations of the elements of the set $S = \{a, b, c\}$ form a group with the respect to the binary operation of composition of permutations. **[MODEL QUESTION]**

Answer:

The permutations are

$$p_1 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix},\ p_2 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix},\ p_3 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix},\ p_4 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix},$$

$$p_5 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix},\ p_6 = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}.$$

Let $G = \{p_1, p_2, p_3, p_4, p_5, p_6\}$.

Define composition in the usual way we see G is closed with respect to this composition, has the associative property, p_1 is the identity and every element has an inverse (e.g. $p_6^{-1} = p_5, p_2^{-1} = p_1$). Hence G is a group.

14. Show that the mapping $f:(\mathbb{Z},\cdot)\to(\mathbb{R},\cdot)$ defined by $f(x)=x^2, \forall x \in \mathbb{Z}$ is a monomorphism but not an isomorphism. **[MODEL QUESTION]**

Answer:

The given mapping is neither a monomorphism nor an isomorphism as it is not a homomorphism

Note: $f(m+n) \neq f(m)\cdot f(n)$ i.e., $(m+n)^2 \neq m^2.n^2$

15. If $f:G\to G'$ be a group homomorphism from a group G to the Group G', then show that $\ker f$ is a normal subgroup of G. **[MODEL QUESTION]**

Answer:

Let e and e' be the identity elements in $(G,*)$ and $(G',\bullet)$ respectively.

$\therefore f(e)=e'$. Therefore $e \in$ kerf. i.e. kerf is non empty.

Now let $a, b \in$ kerf. $\therefore f(a)=e', \quad f(b)=e'$,

Now, $f(a*b^{-1}) = f(a).f(b^{-1}) = f(a).(f(b))^{-1} = e'.(e'^{-1}) = e'.e' = e'$

$\therefore a*b^{-1} \in$ kerf i.e. kerf is a subgroup of $(G,*)$.

Now we have to show that kerf forms a Normal subgroup $(G,*)$.

Let $a \in G$ and $k \in$ kerf.

Therefore, $f(a*k*a^{-1}) = f(a).f(k).f(a^{-1}) = f(a).e'.[f(a)]^{-1} = f(a).[f(a)]^{-1} = e'$.

Hence $a*k*a^{-1} \in$ kerf i.e. kerf is a Normal Subgroup of $(G,*)$.

Long Answer Type Questions

1. a) Prove that a subgroup H of a group G is said to be normal if $aH = Ha$ for all $a \in G$. **[WBUT 2012]**

Answer:

A subgroup $(N,*)$ of the group $(G,*)$ is called a normal subgroup. If for any $g \in G$, $gN = Ng$.

Since the index of H in G is 2, therefore there are only two distinct left cosets of H in G. These two left cosets are H and G-H.

Also, there exist two distinct right cosets, they are H, G-H.

Let $a \in H$, then $aH = H$ and $Ha = H$.

Therefore $aH = Ha$.

Now, if $a \notin H$, then $a \in G - H$.

$\therefore aH = G - H$, Since G–H is only left coset.

And $Ha = G - H$, Since G–H is only the right coset.

Thus, $aH = Ha \quad \forall a \in G$.

Hence, H is a normal subgroup of G.

b) Define a cyclic group. Prove that every cyclic group is abelian. **[WBUT 2012]**

Answer:

1st Part:

A group (G , $\cdot$) is called *cyclic* if every element of G can be obtained from a particular element of G by repeatedly applying the operation , i.e., there is a special element g in G so that for every $a \in G$, there exists $m \in Z$ such that $a = g^m$. The element g is then called a generator.

Clearly the group (G, $\cdot$) where G = { 1, -1, i, -i } is cyclic as every element of G can be obtained from i e.g., $1 = i^4, -1 = i^2, -i = i^3$. Note –i is another generator.

2nd Part:

Let G be a cyclic group and let $a,b \in G$.

If g is a generator of G, then $a=g^m$, $b=g^n$ for some $m,n \in Z$.

$\therefore a.b = g^m.g^n = g^{m+n} = g^{n+m}$ since Z is commutative w.r.t. +

$= g^n.g^m = b.a$

c) Show that the group $(z_9,+)$ is a homomorphic image of the group $(z,+)$. **[WBUT 2012]**

Answer:

We define a mapping

$$\varphi:(\mathbb{Z}_1 +)\to(\mathbb{Z}_9 ;+)$$

as $\quad \varphi(m) = [m](\text{mod}\, 9)$

where $m \in \mathbb{Z}_9 [m]$ denotes the class $[m]$ in $\mathbb{Z}_9$.

Clearly φ is a homomorphism as

$$\begin{aligned}\varphi(m+n) &= [m+n](\bmod 9)\\ &= [m]+[n]\\ &= \varphi(m)+\varphi(n)\end{aligned}$$

Next, φ is surjective as for any class $[m] \in \mathbb{Z}_9$, we see $m \in \mathbb{Z}$.

Hence $(\mathbb{Z}_9, +)$ is a homomorphic image of $(\mathbb{Z}, +)$.

2. a) Show that every cyclic group is commutative. **[WBUT 2013]**

Answer:

Let G be a cyclic group and g be a generator

Set $a, b \in G$ be two arbitrary elements.

Then there exist $m, n \in \mathbb{Z}$ s.t. $a = g^m$ and $b = g^n$.

$\therefore\ a \cdot b = g^m \cdot g^n = g^{m+n} = g^{n+m} = g^n \cdot g^m = ba$ since integers are commutative.

Hence, G is commutative.

b) Let H be a normal subgroup of a group G and G/H be the set of all cosets of H in G. Show that G/H forms a group under the composition $(aH).(bH) = (ab)H$ for all $a, b \in G$. **[WBUT 2013]**

Answer:

Here $G/H = \{gH;\ g \in G\}$

Let $g_1H, g_2H \in G/H$

Then by definition, $(g_1H)(g_2H) = (g_1g_2)H \in G/H$

Associativity follows trivially as so is G.

Identity property is obvious as $e \in G$ and $eH = H$ is the identity of G/H.

The inverse of gH is $g^{-1}H$ since $(gH)(g^{-1}H) = (gg^{-1})H = H$.

Therefore G/H is a group.

Note that the product is well defined, as N is a normal subgroup of G.

3. a) Let S' be the set defined by $S' = \{z \in C : |z| = 1\}$, where C is the set of all complex numbers. Show that S' forms a commutative group under usual multiplication of complex numbers. **[WBUT 2013]**

Answer:

Let $S' = \{z \in \mathbb{C};\ |z| = 1\}$.

Let $z_1, z_2 \in S'$.

Then $z_1 z_2 \in S'$ as

$z_1 z_2 \in \mathbb{C}$ and $|z_1 z_2| = |z_1||z_2| = 1 \cdot 1 = 1$

The associativity is trivial.

For $1 \in C$, we see $1 \cdot z = z \cdot 1 = z \forall z \in S'$

So 1 is the identity of S'.

Finally, $z \in S'$ implies $z \neq 0$ as $|z| = 1$ and $z^{-1} \in \mathbb{C}$.

So, $z^{-1} \in S'$ as $|z^{-1}| = \dfrac{1}{|z|} = \dfrac{1}{1} = 1$.

Hence, S' is a group.

The commutativity of S' follows from the commutativity of $\mathbb{C}$.

b) Let R be the additive group of real numbers and C^* be the multiplicative group of nonzero complex numbers. If $f = R \to C^*$ is a group homomorphism defined by $f(x) = e^{2\pi x}$ for all $x \in R$, find the kernel of f. **[WBUT 2013]**

Answer:

Here $f : \mathbb{R} \to C^*$, where $C^* = \{z \in \mathbb{C};\ z \neq 0\}$ is defined as

$$f(x) = e^{2\pi i x}$$

$\therefore\quad \ker f = \{x \in \mathbb{R};\ f(x) = 1\}$

Now $f(x) = 1$ implies $e^{2\pi i x} = 1$

or, $\cos 2\pi x + i \sin 2\pi x = 1$

or, $\cos 2\pi x = 1 = \cos 2n\pi,\ n \in \mathbb{Z}$

$\therefore\quad 2\pi x = 2n\pi$

$\therefore\quad x = n$.

Hence, $\ker f = \{n;\ n \in \mathbb{Z}\} = \mathbb{Z}$.

4. a) Prove that the order of each subgroup of a finite group is a divisor of the order of the group. **[WBUT 2014, 2017]**

Answer:

Let G be a finite group of order n and H be a subgroup.

Consider the set $\{gH,\ g \in G\}$

Clearly (i) $\bigcup_{g \in G} gH = G$ (ii) $\text{card}(gH) = \text{card}(g'H)$ where $g, g' \in G$.

Thus G is partitioned into disjoint subsets $gH, g \in G$ and each of these cosets have the same cardinality.

$\therefore n = \text{card}(G) = \text{card}(H) \times$ no. of cosets

Hence card(H) is order of H is a divisor of the card(G) i.e., order of G.

This completes the proof.

b) Prove that every group of prime order is cyclic. **[WBUT 2014, 2016]**

Answer:
A group $(G, \cdot)$ is called *cyclic* if every element of G can be obtained from a particular element of G by repeatedly applying the operation, i.e., there is a special element g in G so that for every $a \in G$, there exists $m \in Z$ such that $a = g^m$. The element g is then called a generator.
Clearly the group $(G, \cdot)$ where $G = \{1, -1, i, -i\}$ is cyclic as every element of G can be obtained from i e.g., $1 = i^4, -1 = i^2, -i = i^3$. Note -i is another generator.

c) Let $GL(2,\mathbb{R})$ denote the set of all non singular 2×2 matrices with real entries. Show that

$$SL(2,R)-\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2,R): ad-bc=1\right\}$$

is a normal subgroup of $GL(2,\mathbb{R})$. **[WBUT 2014]**

Answer:
Let $A,B \in GL(2,R)$. Then

$$\varphi(AB) = \det(AB) = \det A \det B$$
$$= \varphi(A)\varphi(B)$$

Hence φ is a homomorphism.
Since 1 is the identity of $(R-\{0\},\bullet)$,

$$\ker\varphi = \{A \in GL(2,R);\ \det A = 1\} = SL(2,R).$$

5. a) If G is a group and H is a subgroup of index 2 in G, then prove that H is a normal subgroup in G. **[WBUT 2015]**

Answer:
Let H be a subgroup of index 2 of a group G. So the left cosets of H are H and G-H and so also are the right cosets.
Now if $a \in H$, then $aH = H$ and $Ha = H$. So $aH = Ha$
If $a \in G-H$, then $aH = G-H = Ha$. So again $aH = Ha$
Thus $aH = Ha$ for $a \in G$. Hence H is a normal subgroup.

b) Show that the set G of all ordered pairs (a, b) with $a \neq 0$, of real numbers a, b forms a group with respect to '$\circ$' defined by $(a, b)\circ(c, d) = (ac, bc+d)$. **[WBUT 2015]**

Answer:
We have $G\{(a,b); a,b \in R, a \neq 0\}$
Let (a,b) & $(c,d) \in G$, then

$$(a,b), (c,d) = (ac, bc+d) \in G \text{ as } ac, bc+d \in R \text{ and } ac \neq 0$$

So closure property holds.

Let $(a,b),(c,d),(e,f) \in G$. Then

$$\{(a,b)\circ(c,d)\}\circ(e,f) = (ac, bc+d)\ (e,f) = (ace, bce+de+f)$$

And $(a,b)\circ\{(c,d)\circ(e,f)\} = (a,b)\circ(ce, de+f)\} = (ace, bce+de+f)$

Hence associative property holds
The element $(1,0) \in G$ is the identity element as

$$(a,b)\circ(1,0) = (a,b)$$
$$(1,0)\circ(a,b) = (a,b)$$

Thus the identity property is satisfied
Finally, the inverse of (a,b) is $\left(\frac{1}{a}, -\frac{b}{a}\right)$ as $(a,b)\circ\left(\frac{1}{a}, -\frac{b}{a}\right) = (1,0)$

And $\left(\frac{1}{a}, -\frac{b}{a}\right)\circ(a,b) = (1,0)$

Hence G is a group w.r.t. the operation $\circ$ defined above.

6. a) For any two integers m, n define $m \oplus n = m+n-1$ and $m \odot n = m+n-mn$. Prove that the set of integers $\mathbb{Z}$ forms a commutative ring with identity with these two binary operations. **[WBUT 2015]**

Answer:
Clearly, $(\mathbb{Z},\oplus)$ is an abelian group as
$\mathbb{Z}$ is closed w.r.t. $\oplus$ (note $m+n-1 \in \mathbb{Z}$), $\mathbb{Z}$ is associative w.r.t. $\oplus$ as $(m\oplus n)\oplus p = (m+n-1)\oplus p = m+n-1+p-1 = m+n+p-2$

and $m\oplus(n\oplus p) = m\oplus(n+p-1) = m+n+p-1-1 = m+n+p-2$

$\mathbb{Z}$ has the identity 1 as $m\oplus 1 = m+1-1 = m$ and $1\oplus m = 1+m-1 = m$
The inverse of m is $2-m$ as $m\oplus(2-m) = m+2-m-1 = 1$
and $(2-m)\oplus m = 2-m+m-1 = 1$

Next, $(\mathbb{Z}, \odot)$ is a semigroup
For this we note that $\mathbb{Z}$ has the closure property as $m\odot n = m+n-mn \in \mathbb{Z}$ and associative property of $\odot$ follows trivially.

Next, we see

$$m\odot(n\oplus p) = m\odot(n+p-1) = m+n+p-1-mn-mp+m$$
$$m\odot n = m+n-mn,\ m\odot p = m+p-mp$$

Clearly $m\odot(n\oplus p) = m\odot n \oplus m\odot p$
The other distributive property follows similarly.
Hence $(Z,\oplus,\odot)$ is a ring.

b) Let G be a group and let $a \in G$. Prove that the mapping $\phi_a : G \to G$ defined by $\phi_a(x) = a \cdot x \cdot a^{-1}$ is an isomorphism on G. **[WBUT 2015]**

Answer:

Clearly, $\varphi_a(x)$ is injective as $\varphi_a(x) = \varphi_a(y) \Rightarrow axa^{-1} = aya^{-1} \Rightarrow x = y$

$\varphi_a(x)$ is subjective as $\varphi_a(x) = y \Rightarrow axa^{-1} = y$ or $x = a^{-1}y(a^{-1})^{-1} \in G$

Further $\varphi_a(x)$ is a homomorphism as $\varphi_a(xy) = axya^{-1} = axa^{-1}aya^{-1} = \varphi_a(x)\varphi_a(y)$

Hence $\varphi_a(x)$ is an isomorphism.

7. a) Let H be a subgroup of a finite group G. Prove that the order of H divides the order of G. **[WBUT 2016]**

Answer:

Proof:

Let $|G| = t$ and $\{a_1H,\ a_2H,\ \ldots\ldots,\ a_tH\}$ be the family of all cosets of H in G. Then $G = a_1H \cup a_2H \cup \ldots\ldots \cup a_tH$, because $G = \{a_1,\ a_2,\ \ldots\ldots,\ a_t\}$ and $1 \in H$. By (ii) of the Lemma above for any two cosets a_iH and a_jH we have only two possibilities:

$$a_iH \cap a_jH = \phi$$

or, $\quad a_iH = a_jH$.

Moreover, the Lemma above it follows that all cosets have exactly $|H|$ number of elements. Therefore

$$|G| = |H| + |H| + \ldots\ldots + |H| \quad \Rightarrow \quad |G| = d|H|,$$

and the result follows.

b) Show that the group $(\mathbb{Z}_6,\ +)$ is a homomorphic image of the group $(\mathbb{Z},\ +)$. **[WBUT 2016]**

Answer:

Recall $\mathbb{Z}_6\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}$

Define $\varphi : (\mathbb{Z},+) \to (\mathbb{Z}_6,\oplus)$ as

$$\varphi(n) = \bar{n} \pmod 6$$

Clearly φ is a homomorphism.

Hence the result.

8. a)Show that center of a group G, given by **[WBUT 2016]**

$$Z(G) = \{a \in G : a * g = g * a \text{ for all } g \in G\}$$

is a normal subgroup of G.

Answer:

To demonstrate that Z is a subgroup we must demonstrate that it is closed under the group operation and that each element in the center has an inverse in the center (since associativity will be inherited and the identity is clearly in the center). To that end, let $a, b \in Z$ and let $h \in G$. Then

$$h(ab)h^{-1} = hah^{-1}hbh^{-1} = ab,$$

so Z is closed under the group operation. Now, let $a \in Z$. Then a has an inverse a^{-1} in G; we want to show that $a^{-1} \in Z$. Let $h \in G$. Then

$$\left(ha^{-1}h^{-1}\right)^{-1} = hah^{-1} = a,$$

which implies that $ha^{-1}h^{-1} = a^{-1}$, so $a^{-1} \in Z$. Therefore, we conclude that Z is, in fact, a subgroup of G.Furthermore, the calculation we did above to demonstrate that Z is closed under the group operation is precisely the calculation necessary to demonstrate that Z to normal in G.

b) Let $S_1 = \{z \in \mathbb{C} : |z| = 1\}$ is a subgroup of $(\mathbb{C}^*,.)$ where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Also consider the group $(\mathbb{R},\ +)$. Define a map $f : \mathbb{R} \to \mathbb{C}^*$ by $f(x) = e^{ix}$. Prove that f is a homomorphism and find out $\ker(f)$. **[WBUT 2016]**

Answer:

The map $f : \mathbb{R} \to C^*$ is defined as $f(x) = e^{ix}$

Clearly f is injective i.e., one-one

Further $f(x+y) = e^{i(x+y)} = e^{ix}.e^{iy} = f(x) \cdot f(y)$

Hence f is a homomorphism.

Now zerf $f = \{x \in \mathbb{R}; f(x) = 1\}$ as 1 is the identity of $(C^*,.)$

$$= \{0\}$$

9. a) Show that the group $(Z_5,\ +)$, i.e., the additive group of all integers modulo 5 is cyclic. Find all generators of Z_5. **[WBUT 2017]**

Answer:

The modular group $Z_5 = \{\bar{0},\ \bar{1},\ \bar{2},\ \bar{3},\ \bar{4}\}$

where $\bar{0}$ is the set of all integers divisible by 5, $\bar{1}$ is the set of all integers when divided by 5 leave 1 as remainder and so on.

The group $(Z_5,\ +)$ has a generator $\bar{1}$ as $\bar{1}+\bar{1}=\bar{2}$, $\bar{1}+\bar{1}+\bar{1}=\bar{3}$, $\bar{1}+\bar{1}+\bar{1}+\bar{1}=\bar{4}$ and $\bar{1}+\bar{1}+\bar{1}+\bar{1}+\bar{1}=\bar{5}=\bar{0}$.

Hence the group is cyclic.

The set of all generators $= \{\bar{1},\ \bar{2},\ \bar{3},\ \bar{4}\}$.

b) If H and K are two subgroups of a group G, then prove that HK is also subgroup of G if and only if $HK = KH$. **[WBUT 2017]**

Answer:

Suppose HK is a subgroup of G.
Let $h \in H$ and $k \in K$. Then $h = h1 \in HK$ and $k = 1k \in HK$ and since HK is closed under products, we deduce $kh \in HK$.
Thus we have $KH \subseteq HK$.
Also for the same elements $h \in H$ and $k \in K$, we have $(hk)^{-1} \in HK$, so $(hk)^{-1} = xy$ where $x \in H$ and $y \in K$. Then $hk = (xy)^{-1} = y^{-1}x^{-1} \in KH$ (as $x^{-1} \in H$ and $y^{-1} \in K$). This shows that $HK \subseteq KH$.
Hence if HK is a subgroup of G, then $HK = KH$.
Conversely suppose $HK = KH$. Let $a, b \in HK$; say $a = h_1k_1$ and $b = h_2k_2$ where $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then $k_1h_2 \in KH = HK$; say $k_1h_2 = hk$ where $h \in H$ and $k \in K$.
Now

$$ab = h_1k_1h_2k_2 = h_1hkk_2 \in HK$$

since $h_1h \in H$ and $kk_2 \in K$. Also,

$$a^{-1} = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH = HK.$$

Hence HK is closed under products and inverses, so it is a subgroup of G.

c) Let $f: G \to G'$ be a homomorphism. Show that f is one-to-one if and only if $\ker f = \{e\}$. **[WBUT 2017]**

Answer:

Let $f: G \to G'$ be a one-to-one homomorphism and e, e' be the respective identities of G and G'. Clearly $f(e) = e'$.
Since f is one-to-one, $\ker f = \{e\}$.
Conversely, let $\ker f = \{e\}$ and let $f(a) = f(b)$ for $a, b \in G$.
Then $f(a^{-1}b) = f(a^{-1})f(b) = f(a)^{-1}f(b) = f(a)^{-1}f(a) = e'$.
Thus $a^{-1}b \in \ker f$ or, $a^{-1}b = e$ or, $a = b$
Hence f is one-to-one.

10. a) Prove that $H = \left\{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \middle| x \in \mathbb{R}, x \neq 0\right\}$ forms a normal subgroup of $GL(2, \mathbb{R})$, the group of all real non-singular 2×2 matrices. **[WBUT 2017]**

Answer:

Let $Q \in H$ and $P \in GL(2, \mathbb{R})$

Then $Q = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ for some $x \in \mathbb{R}$, $x \neq 0$

and $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $a, b, c, d \in \mathbb{R}$, $ad \neq bc$

Now
$$PQP^{-1} = \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
$$= \frac{1}{ad-bc}\begin{pmatrix} ax & bx \\ cx & dx \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{x}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
$$= \frac{x}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in H$$

b) Prove that every group of prime order is commutative. **[WBUT 2017]**

Answer:

Let G be a group whose order is a prime p. Since $p > 1$, there is an element $a \in G$ such that $a \neq e$. The group $\langle a \rangle$ generated by a is a subgroup of G.
By Lagrange's theorem, the order of $\langle a \rangle$ divides $|G|$. But the only divisors of $|G| = p$ are 1 and p. Since $a \neq e$, we have $|\langle a \rangle| > 1$, so $|\langle a \rangle| = p$. Hence $\langle a \rangle = G$ and G is cyclic.
Hence it is abelian or commutative.

11. a) Define an abelian group. The non-zero rational numbers form an abelian group under multiplication. What is the identity element and what are its inverses?
b) Show that (G,*), defined as below, is an abelian group {G: the set of all non-zero real numbers} and $a.b = \frac{ab}{2}$; $a, b \in G$ .w.r.t. **[MODEL QUESTION]**

Answer:

a) A group (G, •) which satisfies the commutative property is called a *commutative group* or *abelian group*.
Let $\mathbb{Q}^*$ denote the set of non-zero rational numbers. Then $(\mathbb{Q}^*, \cdot)$ is a group. Its identity is 1 and the inverse of $\frac{p}{q} \in \mathbb{Q}^*$ is $\frac{q}{p}$.

b) Clearly $a * b = \frac{ab}{2} \in G$ where G^* is the set of non-zero real numbers. This proves the closure property and $a, b \in G^*$.
Now $(a*b)*c = \frac{ab}{2} * c = \frac{abc}{4} \in G^*$ and $a*(b*c) = a * \frac{bc}{2} = \frac{abc}{4} \in G^*$
Hence the associative property holds. Let e denote the identity, then $a * e = a$ gives $\frac{ae}{2} = a$ or $e = 2$. Clearly, $2 * a = \frac{2a}{2} = a$.

Hence, $2 \in G^*$ is the identity.

Let a' denote the inverse of a. Then $a * a' = e$ or $\frac{aa'}{2} = 2$ or $a' = \frac{4}{a}$.

Hence G^* is a group. Further $a * b = \frac{ab}{2} = b * a \forall a, b \in G^*$

Hence G^* is an abelian group.

12. a) Given Z, the group of integers with addition and H, a subset of Z consisting of all multiples of a positive integer m, i.e., km (k = ..., –1, 0, +1, 2, ...). Show that H is a sub-group of G.
b) Let G = {1, 2, 3, 4, 5, 6} under multiplication modulo 7. Is G cyclic, examine?
[MODEL QUESTION]

Answer:
a) Clearly $H \subset \mathbb{Z}$
Let $a, b \in H$. The there exists $k_1, k_2 \in \mathbb{Z}\ s \cdot t$

$$a = k_1 m,\ b = k_2 m$$

Now $a - b = (k_1 - k_2)m = k'm$ where $k' \in \mathbb{Z}$

Hence H is a subgroup of $\mathbb{Z}$.

b) Here $G = \{1, 2, 3, 4, 5, 6\} (\text{mod } 7)$
We claim G is cyclic with generator 3
Note, $3^0 = 1, 3^1 = 3, 3^2 = 2, 3^3 = 4, 3^{-1} = 5, 3^{-3} = 6$

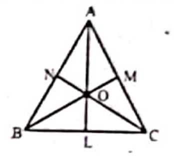
13. a) What are permutation groups, explain with examples? Can you call a permutation group a symmetric group? Form a dihedral group from the symmetries of an equilateral triangle.
b) Find a sub-group H of order 4 for the symmetric group S3 (consisting of 3! Elements). Give the statement of the theorem you use. [MODEL QUESTION]
Answer:
a) Groups whose elements are permutations are called permutation groups.
For example:

$$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ p_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Then $\Pi = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ is a permutation group w.r.t. usual multiplication.
Not all permutation groups are not symmetric groups the set of all permutations of n distinct elements is a group with respect to multiplication.

Consider an equilateral triangle ABC and let O be its centre.



Let T_0 denote the rotation by $0°$ about the centre O, T_1 denote the rotation by $60°, T_2$ denote the rotation by $120°, T_3$ denote reflection of ABC about AL, T_4 denote reflection of ABC about BM and T_5 the reflection about CN.

Then $G\{T_0, T_1, T_2, T_3, T_4, T_5\}$ is a group w.r.t. composition. This group is called the Dihedral group D_3.

b) Use Lagrange's Theorem
There cannot be a subgroup of order 4 of the group S_3. Since the order of S_3 is 6 and 4 does not divide 6, the above is not possible by Lagrange' theorem which states that the order of every subgroup of a finite group is a divisor of the order of the group.

14. a) Let (N, +) be the semi-group of natural numbers and (S, *) be the semi-group on S : {0, 1, e} with the operation · given in the table:

*	e	0	1
e	e	0	1
0	0	0	0
1	1	0	1

A mapping g: N→S given by g(0) = 1 and g(j) = 0 for j ≠ 0 is a semi-group homomorphism but not a monoid homomorphism, examine it?
b) If N is the set of positive integers and * is the operation of least common multiple (l.c.m.) on N, examine whether (N, *) is a semi-group, is it also commutative?
[MODEL QUESTION]

Answer:
a) We have

$$S = \{0, 1, e\},\ g : N \to S$$

Let $m, n \in N,\ m \neq 0,\ n \neq 0$

Then $g(mn) = 0 = g(m) * g(n)$ $\quad [\because g(m) = 0,\ g(n) = 0]$

Let now $m \in N,\ m \neq 0$

Then $g(mo) = g(0) = 1 \neq g(m) * g(o) = 0 * 1 = 0$

Hence g is not a homomorphism.

b) Let $a, b \in N$.

Then $a \cdot b = \text{lcm}\{a, b\} \in N$. Hence the closure property follows.

Next, let $a, b, c \in N$.

Then $(a \cdot b) \cdot c = [\text{lcm}\{a,b\}] \cdot c = \text{lcm}\{a,b,c\}$

$a \cdot (b \cdot c) = a \cdot [\text{lcm}\{b,c\}] = \text{lcm}\{a,b,c\}$

Hence the associative property follows.

Thus $(N, \cdot)$ is a semi group.

Yes, $(N, \cdot)$ is commutative as $a \cdot b = \text{lcm}\{a,b\} = \text{lcm}\{b,a\} = b \cdot a$

15. Define a Normal sub-group. If Z is the group of integers under addition and H be the sub-group of Z consisting of the multiples of 5. Show that H is a normal sub-group of Z. Find also the quotient group Z/H. **[MODEL QUESTION]**

Answer:

A subgroup N of a group G is called a normal subgroup of G if every left coset of N is also a right coset i.e. $gng^{-1} \in N \ \forall g \in G,\ n \in N$, when G is multiplicative.

We have H is a subgroup of $\mathbb{Z}$ where $H = 5\mathbb{Z}$.

Let $n \in H$ and $g \in \mathbb{Z}$. Then $n = 5m$ for some $m \in \mathbb{Z}$.

$$g + n + (-g) = g + 5m - g \qquad \text{where, } g, m_0 \in \mathbb{Z}$$
$$= 5m \in H$$

Hence H is a normal subgroup of $\mathbb{Z}$.

The quotient group $\mathbb{Z}/H$ now consists of all left (or right) cosets of H.

Thus $\mathbb{Z}/H = \{g + H;\ g \in \mathbb{Z}\}$.

16. Let G be a finite group and H be a sub-group of G. Prove that O(H) is a divisor of O(G), where O(G) denotes the order of G. **[MODEL QUESTION]**

Answer:

Since G is finite, H is also finite, thus

$$O(G) < \infty,\ O(H) < \infty.$$

We claim

(i) $G = \bigcup_{g \in G} gH$

(ii) $gH \cap g'H = \phi$ if gH and $g'H$ are distinct left cosets.

(iii) gH is isomorphic to H for every $g \in G$.

Then (i) and (ii) gives

$$\text{Card}(G) = \text{Card}(gH) + \text{Card}(g'H) + \cdots$$

where gH and $g'H$ are distinct left cosets

$$O(G) = nO(H)$$

where n is the number of distinct left cosets.

This proves that $O(H)$ is a divisor of $O(G)$.

17. a) Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 23 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$ **be two permutations. Show that AB ≠ BA.**

b) Let H be a sub-group of a group G. Prove that H is normal in G if and only if g∈G implies $ghg^{-1} \in H$. **[MODEL QUESTION]**

Answer:

a) Given that

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$$

$$\therefore AB = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$$

Clearly, $AB \neq BA$

b) Let H be a normal subgroup of G.

Then every left coset of H is also a right coset.

If $g \in G$, then gH is a left coset of H. As it is also a right coset $gH = Hg'$ for some $g' \in G$.

But as $Hg' \cap Hg \neq \phi$ (note $g \in Hg' \cap Hg$), $Hg' = Hg$

Thus $gH = Hg$.

Therefore if $h \in H$, then $gh = h'g$ for some $h' \in H$

or, $ghg^{-1} = h' \in H$.

Hence the proof.

18. Let (Z, +) be the additive group of all integers and $(Q - \{0\}, \cdot)$ **be the multiplicative group of non-zero rational numbers. Define f : Z → Q − {0} by f(x) = 3^x for all x∈Z. Show that f is a homorphism but not an isomorphism.** **[MODEL QUESTION]**

Answer:

Here $f : \mathbb{Z} \to Q - \{0\}$ is defined as $f(x) = 3^x$

Now let $m, n \in \mathbb{Z}$, then

$$f(m+n) = 3^{m+n} = 3^m \cdot 3^n = f(m) \cdot f(n)$$

Hence f is a homomorphism

As f is not surjective, f is not an isomorphism.

(Note: $\frac{1}{2} \in Q - \{0\}$ has no pre image in $\mathbb{Z}$).

19. a) Prove that in a group (G, o) the equation a o x = b has unique solution.
b) Let G = (Z, +), G′ = (Z, +) be two Groups; f:G→G be a mapping defined by f(x) = |x|. Is the mapping a homo-morphism? Give reasons.
c) Find the idempotent elements in the ring (Z6, +,.) where Z6 is set of residue class module 6.
d) Let (N, P) be a coset where N is set of all natural numbers and P stands for divisibility. Find the maximal and minimal element of the set {2,8,32,4} CN.
[MODEL QUESTION]

Answer:
a) We have, $a \cdot x = b$

Pre-multiplying by a^{-1} we get

$$a^{-1} \cdot (a \cdot x) = a^{-1} \cdot b$$

or, $(a^{-1} \cdot a)x = a^{-1} \cdot b$

or, $e \cdot x = a^{-1} \cdot b$

or, $x = a^{-1} \cdot b$.

Next if possible let x^* and $\bar{x}$ be two solutions of the given equation.
So, $a \cdot x^* = b = a \cdot \bar{x}$

By pre-multiplication of a^{-1} both sides we get, $x^* = \bar{x}$.
Hence the solution is unique.

b) This is not a homomorphism as $f(5+(-3)) \neq f(5)+f(-3)$
since $f(2) = 2$ but $f(5) = 5$, $f(-3) = 3$.

c) $\bar{0}, \bar{1}, \bar{3}, \bar{4}$ since $\bar{0}.\bar{0} = \bar{0}$, $\bar{1}.\bar{1} = \bar{1}$, $\bar{3}.\bar{3} = \bar{9} = \bar{3}$, $\bar{4}.\bar{4} = \overline{16} = \bar{4}$.

d) Minimal element = 2 and maximal element = 32.

20. a) Let G be an abelian group. Prove that the subset S={p∈G:p=p⁻¹} forms a subgroup of G.
b) Define cyclic group. Show that a cyclic group of prime order has no proper or nontrivial subgroup.
[MODEL QUESTION]

Answer:
a) Here S = {p∈G; p=p^{-1}}. Let $a, b \in S$. Then $(ab)^{-1} = (ba)^{-1} = a^{-1}b^{-1} = ab$. Hence $ab \in S$. Again $a^{-1} \in S$ as $a^{-1} \in G$ and $a^{-1} = (a^{-1})^{-1} = a$. Therefore S is a subgroup of G.

b) By Lagrange's theorem, the order of every subgroup of a finite group divides the order of the group. Since the order of the cyclic group is a prime number, which is divisible only by itself and 1, the group cannot have any non-trivial subgroup.

21. a) Define normal subgroup of a group. If G is a group and H is a subgroup of index 2 in G, prove that H is a normal subgroup of G.
b) Show that the set of all roots of the equation $x^4 = 1$ forms a group under multiplication.
[MODEL QUESTION]

Answer:
a) A subset H of a group (G, *) is called a *subgroup* if H is a group with respect to the same operation *.
A property of a group is called *hereditary* if it holds in every subset of the group. Clearly associative property is hereditary.
In view of the above observation, proving a subset H of a group (G, *) to be a subgroup requires verification of only three properties, viz., closure, identity and inverse properties. That these three conditions can be merged into two conditions or even one condition is clear from the following result.

Theorem
A subset H of a group (G, *) is a subgroup if and only if

(i) $a * b \in H$ ***and*** $a^{-1} \in H$.

or ***(ii)*** $a * b^{-1} \in H$.

Proof:
Let H be a subgroup of G and let $a, b \in H$.
Then clearly $ab \in H$ and $a^{-1} \in H$ as H is a group in its own right.
It also true that $ab^{-1} \in H$.
Thus the condition is necessary.
Conversely, let us assume that $ab^{-1} \in H$ for $a, b \in H$.
Then clearly, $aa^{-1} \in H$, i.e., $e \in H$.
This proves the identity property.
Again, as $b \in H$, $eb^{-1} \in H$. So, $b^{-1} \in H$, i.e., H has the inverse property.
Now, $ab = a(b^{-1})^{-1} \in H$ as $a, b^{-1} \in H$
This proves the closure property.
Thus H having closure property, identity property and inverse property and associative property by heredity, H is a subgroup.

b) The roots are obtained as

$$x^4 = 1 = \cos 2k\pi + i\sin 2k\pi, \ k \in \mathbb{Z}$$

$$\therefore \quad x = (\cos 2k\pi + i\sin 2k\pi)^{1/4} = \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}, \ k = 0, 1, 2, 3.$$

Let $\alpha = \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}$.

Then the other roots are $1, \alpha^2, \alpha^3$.

Let $G = \{1, \alpha, \alpha^2, \alpha^3\}$ where $\alpha^4 = 1$.

Clearly the closure and associative properties hold 1 is the identity. The $a^{-1}=a^3$, $(a^2)^{-1}=a^2$, $(a^3)^{-1}=a$.

Thus G is a group.

22. a) In a group G, prove that $(ab)^2=a^2b^2$, iff $(ab)^{-1}=a^{-1}b^{-1}$, where $a,b\in G$.
b) If f is a group homomorphism from $G\to G'$, then show that $f(e)=e'$ and $f(a^{-1})=[f(a)]^{-1}$, where e and e' are the identity elements of G and G' respectively and $a\in G$. **[MODEL QUESTION]**

Answer:

a) In a group G, let us assume $(ab)^2=a^2b^2$ for all $a,b\in G$.

Then $abab=aabb$

or, $ba=ab$

$\therefore$ $(ba)^{-1}=(ab)^{-1}$

Conversely, let $a^{-1}b^{-1}=(ab)^{-1}$ for all a, b in G.

Then $(a^{-1}b^{-1})^{-1}=\{(ab)^{-1}\}^{-1}$

or, $(b^{-1})^{-1}(a^{-1})^{-1}=ab$

or, $ba=ab$.

or, $a(ba)b=a(ab)b$.

or, $(ab)(ab)=(aa)(bb)$

or, $(ab)^2=a^2b^2$

b) Let $a\in G$. Then $ae=ea=a$

$\therefore$ $f(ae)=f(ea)=f(a)$

or, $f(a)f(e)=f(e)f(a)=f(a)$

So $f(e)$ is the identity of $f(G)$ but $f(G)$ being a subgroup of G, $f(e)=e'$.

Next, we know $aa^{-1}=a^{-1}a=e$

$\therefore$ $f(aa^{-1})=f(a^{-1}a)=f(e)$

or, $f(a)f(a^{-1})=f(a^{-1})f(a)=f(e)=e'$

So $f(a^{-1})$ is the inverse of $f(a)$.

That is $(f(a))^{-1}=f(a^{-1})$.

23. Let $G=\{(a,b):a\neq 0,\ b\in R\}$ and * be a binary composition defined on G by $(a,b)*(c,d)=(ac,bc+d)$. Show that $(G,*)$ is a non Abelian group. **[MODEL QUESTION]**

Answer:

We see, if $(a,b),(c,d)\in G$, then $a\neq 0, c\neq 0$.

Also $(a,b)*(c,d)=(ac,bc+d)\in G$ since $ac\neq 0, ac, bc+d\in\mathbb{R}$

This implies the closure property.

Further, $\{(a,b)\cdot(c,d)\}\cdot(e,f)=(ac,bc+d)\cdot(e,f)=(ace,bce+de+f)$

$(a,b)\cdot\{(c,d)\cdot(e,f)\}=(a,b)\cdot(ce,de+f)=(ace,bce+de+f)$

Hence he associative property is proved.

To find the identity, assume (p,q) to be the identity.

Then $(a,b)\cdot(p,q)=(a,b)$

or, $(ap,bp+q)=(a,b)$ $\quad\therefore ap=a\quad\therefore p=1$

and $bp+q=b\quad q=0$

Thus $(1,0)$ is the identity of G.

Let (u,v) be the inverse of (a,b).

Then $(a,b)\cdot(u,v)=(au,bu+v)=(1,0)$ $\quad$ or, $\quad au=1$

$\therefore b=\frac{1}{a}$ and $bu+b=0$ $\quad\therefore v=-\frac{b}{a}$.

Thus $\left(\frac{1}{a},-\frac{b}{a}\right)$ is the inverse of (a,b). So G is a group.

As $(a,b)\cdot(c,d)=(ac,bc+d)\neq(ca,da+b)=(c,d)\cdot(a,b)$, G is non-abelian.

24. a) Define group homomorphism. If G is a group of real, non-singular, n-square matrices under multiplication, show that the determinant function is a homomorphism of $GL(2,\mathbb{R})$ into G' where G' is the group of non-zero real numbers under multiplication.
b) Show that every cyclic group of order n is isomorphic to the group $(Z_n, +_n)$ where Z_n is the set of equivalence classes for the congruence modulo n over the set of integers. **[MODEL QUESTION]**

Answer:

a) Let $(G,*)$ and $(H,\circ)$ be two groups and $\phi:G\to H$ be a mapping. ϕ is called a group homomorphism from G into H if $\phi(a*b)=\phi(a)\circ\phi(b)$ for all $a,b\in G$.

We have here $\phi:GL(2,\mathbb{R})\to G'(=\mathbb{R}=\{0\})$

defined by $\phi\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

We see if $\begin{pmatrix}a & b\\ c & d\end{pmatrix},\begin{pmatrix}p & q\\ r & s\end{pmatrix}\in GL(2,\mathbb{R})$, then $\phi\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right] = \det\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\det\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

$$= \phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}\phi\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

Hence ϕ is a homomorphism.

b) Let G be a group of order n with α as a generator. Then define $\phi: G \to \mathbb{Z}_n$ as $\phi(g) = \phi(a^k) = [k]$. Clearly ϕ is a homomorphism and bijective. Hence the result.

25. a) Let $G = \{(a,b): a \neq 0,\ b \in R\}$ and $*$ be a binary composition defined on G by $(a,b)*(c,d) = (ac, bc+d)$.

Show that $(G,*)$ is a non-Abelian group.

b) Let G be a group. If $a, b \in G$ such that $a^4 = e$, the identity element of G and $ab = ba^2$, prove that $a = e$. **[MODEL QUESTION]**

Answer:

a) If $(a,b) \in G$ and $(c,d) \in G$, then $a \neq 0, c \neq 0$ and so $ac \neq 0$. Thus G is closed w.r.t. $*$ as $(a,b)*(c,d) = (ac, bc+d), ac \neq 0$.

To see the associative property let $(a,b),(c,d),(e,f) \in G$.

Then $\{(a,b)*(c,d)\}*(e,f) = (ac, bc+d)*(e,f) = (ace, bce+de+f)$

and $(a,b)*\{(c,d)*(e,f)\} = (a,b)*(ce, de+f) = (ace, bce+de+f)$

Hence G is associative w.r.t. $*$.

Clearly $(1,0) \in G$ is the identity element of G, as

$$(a,b)*(1,0) = (a,b) \text{ and } (1,0)*(a,b) = (a,b).$$

Let (a', b') be the inverse of (a,b).

Then $\quad (a,b)*(a',b') = (aa', a'b+b') = (1,0)$

$aa' = 1 \quad$ or $\quad a' = 1/a$

$a'b + b' = 0 \quad$ or $\quad b' = -a'b = -b/a$.

Thus the inverse of (a,b) is $(1/a, -b/a)$.

Next, we observe that

$(a,b)*(c,d) = (ac, bc+d)$

$(c,d)*(a,b) = (ac, ad+b)$

Clearly $(a,b)*(c,d) \neq (c,d)*(a,b)$

Hence $(G,*)$ is a group which is non-abelian.

b) We see $\quad ab = ba^2$

or, $\quad aba^2 = ba^4$ (post multiplying by b^2)

or, $\quad aba^2 = b \qquad (\because a^4 = e)$

Now $\quad ab = ba^2$ gives

$\quad aaba^2 = ba^2$

or, $\quad a^2 = e$ (cancelling ba^2 from both sides).

Thus we get $\quad ab = ba^2 = be = b$.

Now the right cancellation property in G gives $a = e$.

26. a) Show that the set D of all odd integers forms a commutative group with respect to the composition θ defined $a\theta b = a+b-1 \forall a, b \in D$.

b) Prove that the identity element and the inverse of an element in a group is unique.

c) Prove that in a group (G, o) (aob)-1=b-1oa-1. **[MODEL QUESTION]**

Answer:

a) Here $a \circ b = a+b-1 \quad \forall a, b \in D$

Clearly D has the closure property w.r.t. $\circ$ as $a+b-1 \in D$ for $a, b \in D$

Further $(a \circ b) \circ c = (a+b-1) \circ c = a+b-1+c-1 = a+b+c-2$.

Again, $a \circ (b \circ c) = a \circ (b+c-1) = a+b+c-1-1 = a+b+c-2$

Hence $(a \circ b) \circ c = a \circ (b \circ c) \cdot \forall a, b, c \in D$

This implies the associative property

Next, clearly 1 is the identity of D since

$a \circ 1 = a + \cancel{1} - \cancel{1} = a$

$1 \circ a = \cancel{1} + a - \cancel{1} = a$

That is, $a \circ 1 = 1 \circ a = a \quad \forall a \in D$.

Finally, every element a of D has an inverse $2 - a \in D$.

This is clear, as $a \circ (2-a) = a + 2 - a - 1 = 1$

and $\quad (2-a) \circ a = 2 - a + a - 1 = 1$.

Hence D is a group w.r.t. $\circ$.

b) If possible let e and e' be two identities of G.

Then $\quad ae = ea = a$ and $ae' = e'a = a \ \forall a \in G$

Now $\quad e = ee' = e' \qquad \begin{bmatrix} \because ea = a \Rightarrow ee' = e' \\ ae' = a \Rightarrow ee' = e \end{bmatrix}$

Hence the identity is unique.

Let there exist two inverses viz. a' and $\bar{a}$ of a

Then $\quad aa' = e = a'a$ and $a\bar{a} = e = \bar{a}a$

Now $a' = a'e = a'(a\bar{a}) = (a'a)\bar{a} = e\bar{a} = \bar{a}$
Hence the inverse of a is unique.

c) We observe

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$$
$$(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b = b^{-1}eb = b^{-1}b = e$$

Hence $(ab)^{-1} = b^{-1}a^{-1}$.

27. a) A function f is defined as follows: **[MODEL QUESTION]**
$f:(C-\{0\},\times)\to(C-\{0\},\times)$
$f(z)=z^4$
C : Set of all complex numbers, ×: usual multiplication.
Is f a homomorphism? If so, find Ker f.
b) Show that the set $G = \{1, 2, 3, 4, 5, 6\}$ form an abelian group with respect to multiplication modulo 7. Is it a cyclic group?

Answer:

a) Here $f:\mathbb{C}-\{0\}\to\mathbb{C}-\{0\}$ is defined by $f(z)=z^4$.

Let $z_1, z_2 \in \mathbb{C}-\{0\}$.

Then $f(z_1 z_2) = (z_1 z_2)^4 = z_1^4 z_2^4$
Hence f is a homomorphism.

Now, $\ker f = \{\xi \in \mathbb{C}-\{0\};\ f(\xi)=0\}$
$= \{\xi \in \mathbb{C}-\{0\};\ \xi^4 = 0\}$.

b) Here $G = \{1,2,3,4,5,6\}$
Clearly, G is closed w.r.t. multiplication
For example, $3\cdot 5 = 15(\text{mod}\,7) = 1 \in G$.
The associativity follows trivially, 1 is the multiplicative identity.
The inverse property is also clear.
For example,

$2^{-1} = 4$ since $2\cdot 4(\text{mod}\,7) = 1$

$3^{-1} = 5$ since $3\cdot 5(\text{mod}\,7) = 1$.

The commutating is also obvious. Hence G is an abelian group. Yes, it is cyclic.
Note: 3 is a generator.

28. a) Prove that the identity elements and the inverse of an element in a group is unique.
b) Prove that in a group $(G,*)$, $(a*b)^{-1} = b^{-1}*a^{-1}$ **[MODEL QUESTION]**

Answer:
a) *Proof:*
(i) Let G be a group and let e and e' be two identities of G. Then

$e = ee'$ $\because e'$ is an identity

$= e'$ $\because e$ is an identity.

Hence the proof.
(ii) Let G be a group and let a' and a'' be two inverses of a in G. Then $aa' = a'a = e$ and $aa'' = a''a = e$
Hence $a' = a'e = a'(aa'') = (a'a)a'' = ea'' = a''$
This completes the proof.

b) Let, $(ab)^2 = a^2 b^2$
or, $abab = aabb$
or, $a^{-1}(abab)b^{-1} = a^{-1}(aabb)b^{-1}$
or, $ba = ab$
or, $(ba)^{-1} = (ab)^{-1}$
or, $a^{-1}b^{-1} = (ab)^{-1}$
or, $(ab)^{-1} = a^{-1}b^{-1}$

29. a) Define normal subgroup of a group. If G is a group and H is a subgroup of index 2 in G, prove that H is a normal subgroup of G. **[MODEL QUESTION]**

Answer:
A subgroup $(N,*)$ of the group $(G,*)$ is called a normal subgroup. If for any $g\in G$, $gN = Ng$.
Since the index of H in G is 2, therefore there are only two distinct left cosets of H in G.
These two left cosets are H and G-H.
Also, there exist two distinct right cosets, they are H, G-H.
Let $a\in H$, then $aH = H$ and $Ha = H$.
Therefore $aH = Ha$.
Now, if $a \notin H$, then $a \in G - H$.
$\therefore$ $aH = G - H$, Since G–H is only left coset.
And $Ha = G - H$, Since G–H is only the right coset.
Thus, $aH = Ha$ $\quad \forall a\in G$.
Hence, H is a normal subgroup of G.

b) Let G be a group. If $a,b\in G$ such that $a^4 = e$, the identity element of G and $ab = ba^2$, prove that $a = e$. **[MODEL QUESTION]**

Answer:
We have, $ab = ba^2$

or, $(ab)a^2=(ba^2)a^2$

or, $a(ba^2)=b(a^2.a^2)$

or, $a(ab)=ba^4\left[\because ab=ba^2\right]$

or, $(a.a)b=b.e\left[\because a^4=e\right]$

or, $a^2b=b$

or, $(a^2b)b^{-1}=bb^{-1}\left[\because b^{-1}\text{exist}\right]$.

or, $a^2(bb^{-1})=e$

or, $a^2.e=e$

or, $a^2=e$

Again, $ab=ba^2$

or, $ab=b.e$ $\left[\because a^2=e, \text{ using(1)}\right]$

or, $ab=b$ or, $a=e$ **(proved)**

30. If two operations * and 0 on the set Z of integers are defined as follows: $a*b=a+b-1$, $aob=a+b-ab$, prove that $(Z,*,o)$ is commutative ring with unit element. **[MODEL QUESTION]**

Answer:

Clearly Z is closed w.r.t * since a, b ∈ Z=> a+b-1 ∈ Z, i.e, a*b ∈ Z

Next, (a*b)* c= (a+b-1) * c= a+b-1+c-1 = a+b+c-2

a*(b*c)=a*(b+c-1) = a+b+c-1-1 = a+b+c-2

Hence Z is associative w.r.t *

(Z,*) has the identity 1 as a*1= a+1-1=a, and 1*a=1+a-1=a.

(Z,*) has the inverse property also as 2-a is the inverse of a as a*(2-a)= a+2-a-1=1

And (2-a)*a= 2-a+a-1= 1.

Further a*b= a+b-1= b+a-1=b*a

Here (Z,*) is an abelian group.

Next, (Z,0) has the closure property as a, b∈ Z => a+b –a b ∈ Z, i.e. a_0b ∈ Z.

Further, $(a_0b)_0c$= $(a+b-ab)_0c$ =a+b – ab +c –ac+ bc+ abc

$a_0(b_0c)$= $a_0(b+c-bc)$= a+b+c-bc-ab-ac+abc

Hence, (Z,0) is associative.

Again, $a_0(b*c)$ = $a_0(b+c-1)$= a+b+c-1-ab-ac+a

and $(a_0b)*(a_0c)$= a+b-ab+a+c-ac-1

Hence $a_0(b*c)$= (a0b)*(a_0c), i.e. , the left distributive property holds.

Similarly the right distributive property holds.

So (Z, *, 0) is a ring.

Since a_0b= a+b-ab=b+a-ba=b_0c, Z is a commutative ring.

Further a_0o= a= o_0a, o is the unit element of (Z, *, a).

RING & FIELD

Multiple Choice Type Questions

1. If R is a ring without zero divisors, then $x.y=0$ implies **[WBUT 2012, 2015]**

a) $x=0$ or $y=0$ **b)** $x=0$ and $y=0$

c) $x=0,\ y\neq 0$ **d)** $x\neq 0,\ y=0$

Answer: (a)

2. Every finite integral domain is a field. This statement is **[WBUT 2013]**

a) true **b) false**

Answer: (a)

3. The number of unit elements of the ring $(Z,+,.)$ **[WBUT 2014, 2016]**

a) 2 **b) 3** **c) 1** **d) infinite**

Answer: (a)

4. An element a in a ring R is zero divisor if **[WBUT 2017]**

a) $a\cdot b=0$

b) $a\cdot b\neq 0$ **for all element b in R**

c) $a\cdot b=0$ **for some non-zero element b in R**

d) none of these

Answer: (d)

Short Answer Type Questions

1. Prove that every finite integral domain is a field. **[WBUT 2012, 2014]**

Answer:

Let D be a finite integral domain. Since every integral domain is a commutative ring with unity, it is enough to prove only that every non-zero element of D has an (multiplicative) inverse in D.

So let a be a non-zero element of D.

Consider the set $S=\{ab; b\in D\}$

Since D is closed with respect to multiplication, $S\subset D$.

Now, if $b\neq c$, then

$ab\neq ac$ because otherwise $ab=ac$, then

$a(b-c)=0 \quad \therefore b-c=0$

as D has no divisor of zero and hence $b=c$. Thus the elements of S are distinct but as D is finite, S will have as many elements as D has viz., S = D. So these exists an element a' such that $aa'=e$ where e is the unity of D. Clearly a' is the inverse of a, as

$aa' = a'a = e$ by the commutativity of D. Since 'a' is arbitrary, every non-zero element has an inverse. Hence D is a field.

2. Show that a field does not contain any zero divisor. **[WBUT 2013, 2015]**

Answer:

Let F be a field and let $a, b \in F$, $ab = 0$, $a \neq 0$.

Then a^{-1} exists in F. Multiplying both sides by a^{-1} we get

$$a^{-1}(ab) = a^{-1} \cdot 0 \text{ or, } (a^{-1}a)b = 0 \qquad \text{or, } b = 0$$

Thus there cannot exist $a, b \in F$, $a \neq 0$, $b \neq 0$ but $ab = 0$.

This implies F has no divisor of zero.

3. If in a ring R with unity, $(xy)^2 = x^2y^2$ for all $x, y \in R$, then show that R is commutative. **[WBUT 2015]**

Answer:

Let $x, y \in R$ be any elements

then $y+1 \in R$ as $1 \in R$

By given condition

$$(x(y+1))^2 = x^2(y+1)^2$$
$$\Rightarrow \quad (xy+x)^2 = x^2(y+1)^2$$
$$\Rightarrow \quad (xy)^2 + x^2 + xyx + xxy = x^2(y^2+1+2y)$$
$$\Rightarrow \quad x^2y^2 + x^2 + xyx + xxy = x^2y^2 + x^2 + 2x^2y$$
$$\Rightarrow \quad xyx = x^2y \qquad \text{...(1)}$$

Since (1) holds for all x, y in R, it holds for $x+1, y$ also. Thus replacing x by $x+1$, we get

$$(x+1)y(x+1) = (x+1)^2 y$$
$$\Rightarrow \quad (xy+y)(x+1) = (x^2+1+2x)y$$
$$\Rightarrow \quad xyx + xy + yx = y = x^2y + y + 2xy$$
$$\Rightarrow \quad yx = xy \qquad \text{using (1)}$$

Hence R is commutative.

4. In a ring (R, +, .) show that (–a).(–b) = a.b for all a, b ∈ 1R.[MODEL QUESTION]

Answer:

We see

$$0 = a \cdot 0 = a \cdot (b + (-b)) = a \cdot b + a \cdot (-b)$$

$\therefore \quad a \cdot (-b) = -a \cdot b.$

Hence $(-a) \cdot (-b) = -(-a) \cdot b = -\{-a \cdot b\} = a \cdot b.$

5. Examine whether the set of even integers form an integral domain with respect to ordinary addition and multiplication. **[MODEL QUESTION]**

Answer:

Let E denote the set of all even integers. Clearly E is an additive abelian group, a multiplicative semigroup and has the two distributive properties. Further, E has no divisor of zero and E is commutative with respect to multiplication. But E has not multiplicative identity, i.e., unity. Hence E is not an integral domain.

6. If $a^2 = a$, for every element a in a ring R, then show that $b = -b$, for every $b \in R$. **[MODEL QUESTION]**

Answer:

Since b + b ∈ R for b ∈R, we get by hypothesis

$$(b+b)^2 = b+b.$$
or $(b+b).(b+b) = b + b.$

or $b^2 + b^2 + b^2 + b^2 = b + b.$

or $b + b + b + b = b + b.$

$\therefore\ b = -b$

7. Show that the set of matrices $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ is a subring of the ring of matrices.

[MODEL QUESTION]

Answer:

Let M_2 denote the ring of matrices of order 2.

Let S denote the set of matrices $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$

Let $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \in S$

Then $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} - \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} a-c & 0 \\ b-d & 0 \end{bmatrix} \in S$

and $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} ac & 0 \\ bc & 0 \end{bmatrix} \in S$

Hence S is a sub-ring of M_2.

8. Show that the set of matrices $S = \left\{\begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} : \alpha, \beta \in R\right\}$ is a left ideal but not a right ideal of 2 × 2 real matrices. **[MODEL QUESTION]**

Answer:

A subring S of a ring R is said to be a left ideal of R if $a \in S, r \in R \Rightarrow r \cdot a \in S$ and a right ideal of R if $a \in S, r \in R \Rightarrow a \cdot r \in S$.

Let $S_1 = \begin{pmatrix} \alpha_1 & 0 \\ \beta_1 & 0 \end{pmatrix}, \frac{1}{2} = \begin{pmatrix} \alpha_2 & 0 \\ \beta_2 & 0 \end{pmatrix}$ be 2elements of S.

Then $S_1 - S_2 = \begin{pmatrix} \alpha_1 - \alpha_2 & 0 \\ \beta_1 - \beta_2 & 0 \end{pmatrix} \in S$

and $S_1 \cdot S_2 = \begin{pmatrix} \alpha_1 \cdot \alpha_2 & 0 \\ \beta_1 \cdot \beta_2 & 0 \end{pmatrix} \in S$

$\therefore \quad S_1 \in S,\ S_2 \in S \Rightarrow S_1 - S_2 \in S \ \&\ S_1 \cdot S_2 \in S$

$\therefore$ S is a subring of 2×2 real matrices.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 real matrix.

Then $S_1 \cdot A = \begin{pmatrix} \alpha_1 & 0 \\ \beta_1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_1 a & \alpha_1 b \\ \beta_1 a & \beta_2 b \end{pmatrix} \notin S.$

$$AS_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 & 0 \\ \beta_1 & 0 \end{pmatrix} = \begin{pmatrix} a\alpha_1 + b\beta_1 & 0 \\ c\alpha_1 + d\beta_1 & 0 \end{pmatrix} \in S$$

$\therefore$ S is a left ideal but not a right ideal of 2×2 real matrices.

Long Answer Type Questions

1. a) Prove that the set of all even integers form a commutative ring. **[WBUT 2012]**

Answer:

We know $2\mathbb{Z}$ is an additive commutative group.

Next, let $m, n \in 2\mathbb{Z}$. Then $m = 2p,\ n = 2q$ for some $p, q \in \mathbb{Z}$.

$\therefore \quad mn = (2p)(2q) = 2(2pq) \in 2\mathbb{Z}$

Associatively follows hereditary from $\mathbb{Z}$.

The two distributive properties also follow hereditary.

Hence $2\mathbb{Z}$ is a ring.

Farther it is commutative since $m \cdot n = nm \forall m, n \in 2\mathbb{Z}$.

b) Prove that the intersection of two subrings is a subring. **[WBUT 2012, 2017]**

Answer:

Let $(G, *, \cdot)$ be a ring and $(H, *, \cdot)$, $(K, *, \cdot)$ be two subgrings of $(G, *, \cdot)$.

Since $(H, *, \cdot)$ and $(K, *, \cdot)$ are two rings, therefore $(H, *)$ and $(K, *)$ are two commutative groups and $(H, \cdot)$ and $(K, \cdot)$ are semigroups.

Again, for any $a, b, c \in H$

i) $a.(b * c) = (a.b) * (a.c)$

ii) $(b * c).a = (b.a) * (c.a)$

and same properties holds for k also.

Now, let $h.k \in H \cap K$.

Since $(H, *)$ is a group, therefore $h * K^{-1} \in H$.

Again $(G, *)$ is a group, thus $h * K^{-1} \in G$.

Hence, $h * K^{-1} \in H \cap K$.

Therefore $(H \cap K, *)$ is a group.

Also $h * K = K * h$ for H and K.

Therefore $h * K = K * h$ for $H \cap K$.

Thus $(H \cap K, *)$ is a commutative group.

With the similar arguments we can say that $(H \cap K, \cdot)$ forms a semigroup.

Again, condition (i) and (ii) holds for H and K both.

Therefore for any three elements $a, b, c \in H \cap K$, these condition must be satisfied.

Hence $(H \cap K, *, \cdot)$ is a semigroup of $(G, *, \cdot)$.

2. a) Let f be a ring homomorphism from the ring Z of integers into itself such that $f(1) = 1$. Determine the homomorphism f. **[WBUT 2013]**

Answer:

Here we observe that $f(0) = f(0+0) = f(0) + f(0) \quad \therefore f(0) = 0$

Also, $f(2) = f(1+1) = f(1) + f(1) = 1 + 1 = 2$

$$0 = f(0) = f(1-1) = f(1) + f(-1) \qquad \therefore f(-1) = f(1)$$

Similarly, $f(n) = n$ for $n \in \mathbb{Z}$.

Hence f is the identity homomorphism.

b) Let R and S be two rings and $f: R \to S$ be a ring homomorphism. Show that kernel of f is a subring of R. **[WBUT 2013]**

Answer:

Here $\ker f = \{x \in R;\ f(x) = 0\}$ where $f: R \to S$ is a ring homomorphism.

Let $a, b \in \ker f$. Then $f(a) = 0,\ f(b) = 0$.

Now $f(0) = f(0+0) = f(0) + f(0) \quad \therefore f(0) = 0$

Also, $0 = f(b - b) = f(b) + f(-b) \quad \therefore f(-b) = -f(b)$

So, $f(a - b) = f(a) + f(-b) = f(a) - f(b) = 0 - 0 = 0$.

Hence, $a - b \in \ker f$

Further $f(a \cdot b) = f(a) \cdot f(b) = 0 \cdot 0 = 0$

So, $ab \in \ker f$

Therefore $\ker f$ is a subring of R.

3. a) Let $A=\begin{pmatrix}1&2&3&4&5\\2&3&1&5&4\end{pmatrix}$; $B=\begin{pmatrix}1&2&3&4&5\\1&3&4&5&2\end{pmatrix}$ be two permutations. Show that $AB \neq BA$. **[WBUT 2018]**

Answer:

Here $A=\begin{pmatrix}1&2&3&4&5\\2&3&1&5&4\end{pmatrix}$, $B=\begin{pmatrix}1&2&3&4&5\\1&3&4&5&2\end{pmatrix}$

$AB=\begin{pmatrix}1&2&3&4&5\\3&4&1&2&5\end{pmatrix}$, $BA=\begin{pmatrix}1&2&3&4&5\\2&1&5&4&3\end{pmatrix}$

Clearly $AB \neq BA$

b) Let $f:(c-\{0\},\cdot)\to(c-\{0\},\cdot)$ be a function defined by $f(z)=z^4$.

(i) Show that f is a homomorphism.

(ii) Find the Kernel of f. **[WBUT 2018]**

Answer:

Let $z_1, z_2 \in \mathbb{C}-\{0\}$

Then $f(z_1 z_2)=(z_1 z_2)^4 = z_1^4 z_2^4 = f(z_1)f(z_2)$

Hence f is a homomorphism.

For kernelf, we have

$f(z)=1$ as 1 is the identity of $(\mathbb{C}-\{0\};\ )$

i.e., $z^4=1=\cos 2k\pi + i\sin 2k\pi,\ k\in N$

or, $z=\cos\frac{2k\pi}{4}+i\sin\frac{2k\pi}{4},\ k=0,1,2,3$

Hence $\ker f=\left\{1, \cos\frac{4\pi}{2}+i\sin\frac{\pi}{2}, -1, \cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right\}=\{1,i,-1,-i\}$

4. a) Prove that cancellation laws hold in a ring R if and only if R has no divisor of zero.

b) Let S, T be two sub-rings of a ring R. Prove that $S \cap T$ is also a sub-ring of R. **[MODEL QUESTION]**

Answer:

a) Let the cancellation laws hold in R.

Let $a\cdot b=0$ where $a\neq 0,\ a,b\in R$

Then $a\cdot b = a\cdot 0$

$b=0$ (canceling a form both sides)

Hence R has no divisor of zero conversely, let R has no divisor of zero.

Conversely, let R has no divisor of zero.

Let $ab=ac$ where, $a,b,c\in\mathbb{R},\ a\neq 0$

or, $ab-ac=0$

or, $a(b-c)=0$

Since R has no divisor of zero, $b-c=0$ $[\because a\neq 0]$

Hence the left cancellation law holds.

The light cancellation law can be proved similarly.

b) Let S and T be two subrings of R.

Since S and T are additively abelian groups $S\cap T$ is also an additively abelian group.

Since S and T are multiplicatively semigroups, $S\cap T$ is also a multiplicative semigroup.

Since two distributive properties hold in S and T and these properties are hereditary, $S\cap T$ has the two distributive properties.

Hence $S\cap T$ is a subring of R.

5. Answer the following questions.

a) Show that the set of matrices $\begin{bmatrix}a&0\\b&0\end{bmatrix}$ is a subring of the ring of 2×2 matrices.

b) Prove that a finite integral domain is a field. **[MODEL QUESTION]**

Answer:

a) Let $\begin{bmatrix}a&0\\b&0\end{bmatrix},\begin{bmatrix}c&0\\d&0\end{bmatrix}\in M_2(\mathbb{R})$ the set of all 2×2 matrices with real entries of the form $\begin{bmatrix}a&0\\b&0\end{bmatrix}$

Clearly, $\begin{bmatrix}a&0\\b&0\end{bmatrix}-\begin{bmatrix}c&0\\d&0\end{bmatrix}=\begin{bmatrix}a-c&0\\b-d&0\end{bmatrix}\in M_2(\mathbb{R})$.

and $\begin{bmatrix}a&0\\b&0\end{bmatrix}\begin{bmatrix}c&0\\d&0\end{bmatrix}=\begin{bmatrix}ac&0\\bc&0\end{bmatrix}\in M_2(R)$

Hence the set of matrices $\begin{bmatrix}a&0\\b&0\end{bmatrix}$, $a,b\in\mathbb{R}$, is a subring of all 2×2 matrices.

b) Let D be a finite integral domain with n elements $a_0, a_1, a_2, \ldots, a_{n-1}$ of which a_0 is the zero element and a_1 is the identity. We first prove that $xa_{i_0}=a_{j_0}$ for some $a_{i_0}, a_{j_0}\in\{a_1,a_2,\ldots,a_n\}$ has a unique solution. To prove this we claim $a_1a_{i_0}, a_2a_{i_0}, \ldots, a_na_{i_0}$ are distant. If not, then say, $a_1a_{i_0}=a_2a_{i_0}$ or $(a_1-a_2)a_{i_0}=0$.

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But D does not have zero-divisor and $a_{i_0} \neq 0$, hence $a_1 - a_2 = 0$ or $a_1 = a_2$, which is a contradiction.
Hence the products $a_1a_{i_0}, a_2a_{i_0}, \ldots, a_na_{i_0}$ are distinct. So one of these must be by closure property equal to a_{j_0}. Thus $xa_{i_0} = a_{j_0}$ has a solution. The uniqueness follows by a similar argument. Now consider the equation $xa_{i_0} = a_1$. This has a solution, say, a'_{i_0}. We claim this is the inverse of a_{i_0}. This is so because of commutativity of D, $a'_{i_0}a_{i_0} = a_{i_0}a'_{i_0} = a_1$.
Hence the existence inverse is established. This means D is a field.

6. a) If a ring R consists of all integral multiples of 2 and R' consists of all integral multiples of 3, show that R is not isomorphic to R'.
b) When does a ring become a field? Does multiplication in a field obeys cancellation law, examine? What is the field of quotients of the integral domain of integers? **[MODEL QUESTION]**

Answer:
a) Here $R = 2\mathbb{Z}$, $R' = 3\mathbb{Z}$
Let f be a mapping from the ring R of all multiples of 2 of the ring R' of all multiples of 3 defined by $f: R \to R'$ by $f(2a) = 3a \forall a \in R$.
Then for any 2 elements $2a_1$ & $2a_2 \in R$ we have,

$$f(2a_1) = 3a_1,\ a_1 \in R \ \&\ f(2a_2) = 3a_2,\ a_2 \in R.$$

Now if $f[2a_1 + 2a_2] = f(2(a_1 + a_2)) = 3(a_1 + a_2) = 3a_1 + 3a_2 = f(2a_1) + f(2a_2)$

And $f[2a_1 \cdot 2a_2] = 2 \cdot 3a_1a_2 = 6a_1a_2 f(2a_1) \cdot f(2a_2) = 3a_1 \cdot 3a_2 = 9a_1a_2$

Clearly $f(2a_1 \cdot 2a_2) \neq f(2a_1) f(2a_2)$

Here f is not a homomorphism and so R is not isomorphic to R'.

b) A ring becomes a field if it has identity, it is commutative and every non-zero element of it has an inverse.
Yes. It does obey since it has no divisor of zero. The set of rational numbers i.e. $\mathbb{Q}$.

7. a) Define ideal of a ring. Let S be the set of all (2x2) real matrices defined by

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$ **[MODEL QUESTION]**

Show that S is a left ideal but not a right ideal of $M_2(\mathbb{R})$
b) Prove that every finite integral domain is a field.

Answer:
a) S is clearly a subring of $M_2(\mathbb{R})$

Since, if $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \in S$, then $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} - \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} a-c & 0 \\ b-d & 0 \end{pmatrix} \in S$

and $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} ac & 0 \\ bc & 0 \end{pmatrix} \in S$

Further, let $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in S$, $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in M_2(\mathbb{R})$,

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} pa+qb & 0 \\ ar+sb & 0 \end{pmatrix} \in S$$

Hence S is a left ideal.

Since $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap & aq \\ bq & bq \end{pmatrix} \notin S$,

S is not a right ideal.

b) Let D be a finite integral domain. Since every integral domain is a commutative ring with unity, it is enough to prove only that every non-zero element of D has an (multiplicative) inverse in D.
So let a be a non-zero element of D.
Consider the set $S = \{ab; b \in D\}$
Since D is closed with respect to multiplication, $S \subset D$.
Now, if $b \neq c$, then $ab \neq ac$ because otherwise $ab = ac$,
then $a(b-c) = 0$ $\therefore b - c = 0$ as D has no divisor of zero and hence $b = c$. Thus the elements of S are distinct but as D is finite, S will have as many elements as D has viz., S = D. So these exists an element a' such that $aa' = e$ where e is the unity of D. Clearly a' is the inverse of a, as $aa' = a'a = e$ by the commutativity of D. Since a is arbitrary, every non-zero element has an inverse. Hence D is a field.

8. a) Show that $S = \{6x;\ x \in \mathbb{Z}\}$ is an ideal $\mathbb{Z}$. **[MODEL QUESTION]**
b) Prove that the ring of intergers is not a field.
c) Prove that in a field *F* the equations *a. x = b, y.a = b* have unique solution.
Answer:
a) Here $S = \{6x;\ x \in \mathbb{Z}\}$
Clearly, S is a subring of $\mathbb{Z}$ since $6m - 6n = 6(m-n) \in 6\mathbb{Z}$
and $6m \cdot 6n = 6(6mn) \in 6\mathbb{Z}$.
Next, if $p \in \mathbb{Z}$, $s \in S$, then $s = 6m$ for some $m \in \mathbb{Z}$.
Now, $ps = p6m = 6pm \in S$ and $rp = 6mp = 6mp \in S$.

Hence S is an ideal of $\mathbb{Z}$.

b) $(\mathbb{Z},+,\circ)$ is not a field
Since every non-zero integer does not have an inverse in $\mathbb{Z}$.

c) We have the equation $ax=b,\ a\neq 0$.
Since $a\neq 0$, a has an inverse, say a^{-1}.

So $a^{-1}(ax)=a^{-1}b$

or, $(a^{-1}a)x=a^{-1}b$

or, $ex=a^{-1}b$

or, $x=a^{-1}b$.

Thus existence of a solution is proved.
For uniqueness, assume that x^* and $\bar{x}$ are two solutions of the above equation.
Then $ax^*=b$ and $a\bar{x}=b$

So $ax^*=a\bar{x}$

or, $a(x^*-\bar{x})=0$

$\therefore x^*-\bar{x}=0$ as $a\neq 0$ and F has no divisor of zero.

or, $x^*=\bar{x}$.

Thus the solution is unique.
Argument for the other equation $xa=b$ is very much similar.

QUESTION 2014

GROUP – A
(Multiple Choice Type Questions)

1. Choose the correct alternatives for any ten of the following:

i) A problem in Mathematics is given to three students A, B and C. The chances of solving the problem by A, B and C are $\frac{1}{3},\frac{1}{4}$ and $\frac{1}{5}$ respectively. The probability that the problem will be solved is

a) $\frac{2}{5}$ ✓b) $\frac{3}{5}$ c) $\frac{1}{60}$ d) $\frac{47}{60}$

ii) Let G be a Group and $a,b\in G$. Then $(a^{-1}b)^{-1}$ is equal to

a) ab^{-1} ✓b) $b^{-1}a$ c) $a^{-1}b^{-1}$ d) $b^{-1}a^{-1}$

iii) If a simple graph has 15 edges then sum of the degrees of all the vertices is

a) 25 b) 24 c) 50 ✓d) 30

iv) The probability that a leap year selected at random will contain 53 Sundays is

a) 2/53 b) 52/53 c) 1/7 ✓d) 2/7

v) The mean and variance of a distribution is given to be 10 and 6 respectively. Then the distribution is

a) Standard Normal Distribution ✓b) Binomial Distribution
c) Poisson Distribution d) None of these

vi) A random variable X has the following probability density function:

$$f(x)=\begin{cases} kx & 0\le x\le 1 \\ 0 & \text{otherwise}\end{cases}$$

The value of k is

a) 1 ✓b) 2 c) 4 d) none of these

vii) The statistic t is said to be unbiased estimator of a population parameter θ when

✓a) $E(t)=\theta$ b) $E(t^2)=\theta$
c) $E(t^2)=[E(\theta)]^2$ d) $[E(t)]^2=[E(\theta)]^2$

viii) The number of unit elements of the ring $(Z,+,.)$

✓a) 2 b) 3 c) 1 d) infinite

ix) Chromatic number of a complete graph with 15 vertices is

a) 12 b) 13 c) 14 ✓d) 15

x) In a Poisson distribution if $2P(x=1)=P(x=2)$, then the variance is

a) 0 b) −1 ✓c) 4 d) 2

xi) Let A and B be two events with $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}, P(A\cap B)=\frac{1}{4}$. Then $P(A/B)$ is

✓a) 3/4 b) 5/4 c) 7/4 d) 2

xii) In 'Goodness of fit' which of the following is used as test statistic

a) normal variate b) t variate

c) Poisson variate ✓d) X^2 variate

xiii) Let s be a finite set containing n elements. Then the probability that a mapping $j \rightarrow s$ will be a birective mapping is

a) $\frac{n^n}{n!}$ ✓b) $\frac{n!}{n^n}$ c) $\frac{n-1}{n!}$ d) $\frac{n+1}{n!}$

xiv) If G is a non-planar graph, then the number of vertices of G is

a) 2 b) 3 c) 4 ✓d) 6

xv) Which one of the following is not a cyclic group.

a) $(Z,+)$ b) $(Z_4,+)$ ✓c) $(Q,+)$ d) $(Z_{15},+)$

GROUP – B
(Short Answer Type Questions)

2. Prove that a group $(G,*)$ is commutative if and only if $(a*b)^2=a^2*b^2$, for all $a,b\in G$.
See Topic: GROUP THEORY, Short Answer Type Question No. 1.

3. If T is an unbiased estimator of θ, then show that T^2 is a biased estimator of θ^2.
See Topic: ESTIMATION OF PARAMETERS, Long Answer Type Question No. 1(a).

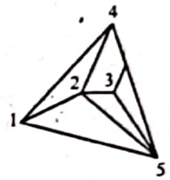
4. In a certain city, the daily consumption of electric power (in millions of kilowatt hours) is a random variable having the probability density

$$f(x)=\frac{1}{9}xe^{-x/3}, x>0$$
$$=0, \quad x\le 0$$

If the city's power plant has a daily capacity of 12 million kilowatt-hours, what is the probability that this power supply will be inadequate on any given day?
See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Short Answer Type Question No. 4.

5. Show that the 7^{th} roots of unity form a cyclic group. Find all the generators of this group.
See Topic: GROUP THEORY, Short Answer Type Question No. 4.

6. Find the dual of the following graph:



The given graph is incomplete.

GROUP – C
(Long Answer Type Questions)

7. a) If G be a connected planar graph with n vertices, e number of edges and f number of faces, prove that $n-e+f=2$.

b) Suppose that an airplane engine will fail, when in flight, with probability $(1-p)$ independently from engine to engine; suppose that the airplane will make a successful flight if at least 50% of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane?

c) Find the mean of an uniform distribution.

a) See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 2(b).
b) See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 3(a) & (b).

8. a) Prove that a graph with n vertices is a tree if and only if its chromatic polynomial $P_n(\lambda)=\lambda(\lambda-1)^{n-1}$.

b) If the weekly wage of 10,000 workers in a factory follows normal distribution with mean and standard deviation Rs. 70 and Rs. 5 respectively, then find the expected number of workers whose weekly wages are,

i) between Rs. 66 and Rs. 72
ii) less than Rs. 66.

[Given that the area under the standard normal curve between $z = 0$ and $z = 0.4$ is 0.1554 and $z = 0$ and $z = 0.8$ is 0.2881].

c) Prove that the order of each subgroup of a finite group is a divisor of the order of the group.

a) See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 1(b).
b) See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Short Answer Type Question No. 5.
c) See Topic: GROUP THEORY, Long Answer Type Question No. 4(a).

9. a) Give an example to show that a graph is drawn in two different ways as planar graph, but its dual are non isomorphic.
b) Prove that every group of prime order is cyclic.
c) Let $GL(2,\mathbb{R})$ denote the set of all non singular 2×2 matrices with real entries. Show that

$$SL(2,R)-\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2,R): ad-bc=1\right\}$$

is a normal subgroup of $GL(2,\mathbb{R})$.

a) The question is not clear.
b) & c) See Topic: GROUP THEORY, Long Answer Type Question No. 4(b) & (c).

10. a) The probability density of a random variable z is given by

$$f(z)=\begin{cases} kze^{-z^2}, & \text{for } z>0 \\ 0 & \text{for } z\le 0 \end{cases}$$

Find the value of k and find out the corresponding distribution function of z.
b) A random sample of size $n=100$ is taken from an infinite population with the mean $\mu=75$ and the variance $\sigma^2=256$. Based on Chebyshev's theorem with what probability can we assert that the value we obtain for $\overline{X}$ will fall between 67 and 83?
c) Prove that every finite integral domain is a field.

a) See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Short Answer Type Question No. 6.
b) See Topic: LIMIT THEOREMS, Short Answer Type Question No. 1.
c) See Topic: RING & FIELD, Short Answer Type Question No. 1.

11. a) Show that there does not exist any isomorphism from the group $(\mathbb{R},+)$ to group $(\mathbb{R}^{*},-)$.
($\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^{*}$ is the set of all non zero real numbers).
b) Suppose that 100 tires made by a certain manufacturer lasted on the average 21819 miles with a standard deviation of 1295 miles. Test the null hypothesis $\mu=22000$ miles against the alternate hypothesis $\mu<22000$ miles at the 0.05 level of significance.

c) Let $x_1, x_2, \ldots\ldots, x_n$ be the values of a random sample from an exponential population that the $f(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}}$ for $x_i>0$. Then find the maximum likelihood estimator of its parameter θ.

a) The question is wrong as $(\mathbb{R}^{*},-)$ is not a group.
b) See Topic: TESTING OF HYPOTHESIS, Short Answer Type Question No. 2.
c) See Topic: ESTIMATION OF PARAMETERS, Short Answer Type Question No. 1.

QUESTION 2015

GROUP – A
(Multiple Choice Type Questions)

1. Answer any *ten* questions.

i) If A and B are two events with $P(A)=0.4$, $P(B)=0.3$ and $P(A\cap B)=0.2$, then $P(A^C\cap B)$ is
✓a) 0.1 b) 0.2 c) 0.3 d) 0.4

ii) If two events A and B are independent, then
✓a) $P(A\cap B)=P(A)P(B)$
b) $P(A+B)=P(A)+P(B)$
c) $P(A-B)=P(A)P(B)$
d) $P(A\cap B)=P(A)P(B/A)$

iii) A fair die is thrown. The probability that either an odd number or a number greater than 4 will turn up is
a) $\frac{2}{5}$ b) $\frac{3}{7}$ c) $\frac{2}{7}$ ✓d) $\frac{2}{3}$

iv) The mean of a uniform distribution with parameters a and b is
a) $b-a$ b) $b+a$ ✓c) $\frac{(a+b)}{2}$ d) $\frac{b-a}{2}$

v) The variance of a random variable X is
a) $[E(X)]^2$
b) $E(X^2)$
✓c) $E(X^2)-[E(X)]^2$
d) $E(X^2)-[E(X)]$

vi) A random variable x has the following pdf

$$f(x) = k \qquad -2 < x < 2$$
$$= 0 \qquad \text{otherwise}$$

Then the value of the constant k is

a) $\frac{1}{8}$ b) $\frac{1}{2}$ ✓c) $\frac{1}{4}$ d) $\frac{1}{12}$

vii) The number of generators of an infinite cyclic group is

a) 1 ✓b) 2 c) infinite d) none of these

viii) If R is a ring without zero divisors, then $x.y = 0$ implies

✓a) $x = 0$ or $y = 0$ b) $x = 0$ and $y = 0$
c) $x = 0,\ y \neq 0$ d) $x \neq 0,\ y = 0$

ix) The order of the dihedral group D_4 is

a) 4 b) 6 ✓c) 8 d) 64

x) Let X be a Poisson Random Variate and $E(X) = \lambda$. Then $E\left[(X+1)^2\right]$ will be

a) λ b) $\lambda^2 + 2\lambda$ c) $\lambda^2 + 2\lambda + 1$ ✓d) $\lambda^2 + 3\lambda + 1$

xi) A complete graph is called Kuratowski's first graph if it has

✓a) 5 vertices b) 4 vertices c) 6 vertices d) 7 vertices

xii) Let C_{97} be a circuit with 97 vertices. Then $\chi(C_{97})$ is equal to

a) 97 b) 98 c) 2 ✓d) 3

xiii) The maximum number of edges in a simple connected graph with n vertices is

a) $2 \cdot {}^nC_2$ ✓b) ${}^nC_2$ c) $(n-1)$ d) n^2

xiv) If H is a subgroup of a group G and a, b are two distinct elements of G, then indicate which of the following statements is true

a) $aH = Ha$ b) $Ha \cap Hb = \phi$
c) $Ha \cap Hb \neq \phi$ and $Ha \neq Hb$ d) $aH = bH$

Answer: none of these

xv) A null hypothesis is a statistical hypothesis which is setup and whose validity is tested for possible

a) acceptance ✓b) rejection c) testing d) none of these

GROUP – B
(Short Answer Type Questions)

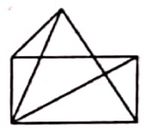
2. Show that for any two subgroups H and K of a group G, $H \cap K$ is also a subgroup of G.
See Topic: GROUP THEORY, Short Answer Type Question No. 5.

3. If in a ring R with unity, $(xy)^2 = x^2y^2$ for all $x, y \in R$, then show that R is commutative.
See Topic: RING & FIELD, Short Answer Type Question No. 3.

4. A random sample of size 20 from a normal population gives the sample mean of 42 and the sample standard deviation of 6. Test the hypothesis that the population mean is 44. Value of t distribution with 19 degrees of freedom at 5% level is 2.09.
See Topic: TESTING OF HYPOTHESIS, Short Answer Type Question No. 3.

5. Urn I has 2 white and 3 black balls. Urn II has 4 white and 1 black and Urn III has 3 white and 4 black balls. An urn is selected at random and ball drawn at random is found to be white. Find the probability that Urn I was selected.
See Topic: PROBABILITY THEORY, Short Answer Type Question No. 2.

6. Draw the dual of the graph.



See Topic: ADVANCED GRAPH THEORY, Short Answer Type Question No. 3.

GROUP – C
(Long Answer Type Questions)

7. a) State and prove Euler's formula for a connected planar graph.
b) If G is a group and H is a subgroup of index 2 in G, then prove that H is a normal subgroup in G.
c) A box contains 5 defective and 10 non-defective lamps, 8 lamps are drawn at random in succession without replacement. What is the probability that the 8th lamp drawn is the 5th defective one?

a) See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 2(b).
b) See Topic: GROUP THEORY, Long Answer Type Question No. 5(a).
c) See Topic: PROBABILITY THEORY, Long Answer Type Question No. 3(a).

8. a) Show that the set G of all ordered pairs (a, b) with $a \neq 0$, of real numbers a, b forms a group with respect to '$\circ$' defined by $(a, b)\circ(c, d)=(ac, bc+d)$.

b) Define Normal distribution and find its mean, variance and standard deviation.

c) It is seen that a cricketer becomes out within 10 runs in 3 out of 10 innings. If he plays 4 innings, what is the probability that he will becomes (i) out twice (ii) out at least once within 10 runs.

a) See Topic: GROUP THEORY, Long Answer Type Question No. 5(b).
b) See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 4(a).
c) See Topic: PROBABILITY THEORY, Long Answer Type Question No. 3(b).

9. a) Let G be a simple connected planar graph with n vertices, e edges and f faces. Prove that the following inequalities must hold

(a) $n \geq \frac{3}{2}f$;

(b) $e \leq (3n-6)$

b) A normal population has mean 0.1 and standard deviation 2.1. Find the probability that the mean of a sample of size 900 will be negative. (Given that $P(|z|<1.43=0.847))$

c) For any two integers m, n define $m \oplus n = m+n-1$ and $m \odot n = m+n-mn$. Prove that the set of integers $\mathbb{Z}$ forms a commutative ring with identity with these two binary operations.

a) See Topic: ADVANCED GRAPH THEORY, Short Answer Type Question No. 4.
b) See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 4(b).
c) See Topic: GROUP THEORY, Long Answer Type Question No. 6(a).

10. a) If a population has normal distribution with parameter μ and σ, then show that the statistic $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$ is a maximum likelihood estimator of σ^2 when μ is known.

b) A machine part was designed to withstand an average pressure of 120 units. A random sample of size 100 from a large batch was tested and it was found that the average pressure which these parts can withstand is 105 units with a standard deviation of 20 units. Test at 5% level of significance whether the batch meet the specification. Suppose the population has normal distribution and given that $\phi(1.645)=0.45$.

c) Let G be a group and let $a \in G$. Prove that the mapping $\phi_a : G \to G$ defined by $\phi_a(x) = a \cdot x \cdot a^{-1}$ is an isomorphism on G.

a) See Topic: ESTIMATION OF PARAMETERS, Long Answer Type Question No. 1(b).
b) See Topic: TESTING OF HYPOTHESIS, Long Answer Type Question No. 1(a).
c) See Topic: GROUP THEORY, Long Answer Type Question No. 6(b).

11. a) Show that a field does not contain any zero divisor.

b) Survey of 320 families with 5 children each revealed the following distribution:

No. of boys:	5	4	3	2	1	0
No. of girls:	0	1	2	3	4	5
No. of family:	14	56	110	88	40	12

Is the result consistent with the hypothesis that male and female births are equally probable? The 5% value of X^2 with 5 d.o.f. is 11.07.

c) A random variable X follows uniform distribution with parameters 0 and 1. Find the pdf of the random variable $U=\sqrt{x}$.

a) See Topic: RING & FIELD, Short Answer Type Question No. 2.
b) See Topic: TESTING OF HYPOTHESIS, Long Answer Type Question No. 1(b).
c) See Topic: TRANSFORMATION OF RANDOM VARIABLES, Short Answer Type Question No. 1.

QUESTION 2016

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternatives for any *ten* of the following:

i) A fair die is thrown 180 times. The expected number of sixes is
a) 10 b) 20 ✓c) 30 d) 40

ii) Let G be a Group and $x \in G$ be such that $o(x)=5$. Then
a) $o(x^{15})=4$ b) $o(x^{10})=6$ ✓c) $o(x^{23})=5$ d) $o(x^{20})=3$

iii) The set $\{[2], [4], [6], [8]\}$ is a group under multiplication modulo 10. The identity element of the group is
a) $[2]$ b) $[4]$ ✓c) $[6]$ d) $[8]$

iv) Let G be a finite group of even order. Then the number of elements of order 2 in G is
a) 2 b) 4 c) even ✓d) odd

v) The probability that A passes the exam is 0.9 and B passes the exam. Is 0.8. The probability that at least one of them passes is
✓a) 0.98 b) 0.97 c) 0.9 d) 0.72

vi) Let X be a random variable and λ be a real number. Then $\frac{\text{var}(\lambda X)}{\text{var}(X)}$ is

a) λ ✓b) λ^2 c) $\frac{1}{\lambda}$ d) $\frac{1}{\lambda^2}$

vii) If $E(T)=5\theta+6$ then the unbiased estimator of θ is

✓a) $\frac{1}{5}T-\frac{6}{5}$ b) $T-\frac{6}{5}$ c) $\frac{1}{5}T$ d) $T-2$

viii) If G is a non-planar graph then the possible number of vertices of G is

a) 2 b) 3 c) 4 ✓d) 6

ix) The chromatic number of a graph containing an odd circuit is

✓a) 3 b) 2
c) greater than or equal to 3 d) greater than or equal to 2

x) The generators of the cyclic group $(Z, +)$ are

✓a) $1, -1$ b) $0, 1$ c) $0, -1$ d) $2, -2$

xi) In the group S_3 of all permutations on $\{1, 2, 3\}$; the inverse of $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$ is

a) $\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$ b) $\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}$ ✓c) $\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$ d) $\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$

xii) The number of unit elements of the ring $(Z, +, .)$ is

a) 2 b) 3 c) 1 ✓d) infinite

xiii) For a random variable X with mean 0, the value of $P(-4\sigma < X < 4\sigma)$ will be at least

a) $\frac{1}{16}$ b) $\frac{1}{4}$ ✓c) $\frac{15}{16}$ d) $\frac{13}{16}$

xiv) Corresponding to a pendant edge in G we get

a) two parallel edges in its dual ✓b) a loop in its dual
c) an isolated vertex in its dual d) another pendant edge in its dual

Group – B
(Short Answer Type Questions)

2. Let $(G, *)$ be a group and $a, b \in G$. Suppose that $a^2 = e$ and $a * b^4 * a = b^7$. Prove that $b^{33} = e$.
See Topic: GROUP THEORY, Short Answer Type Question No. 6.

3. If $X_1, X_2, X_3, \ldots\ldots, X_n$ constitute a random sample of size n from an infinite population with mean μ and variance σ^2, then prove that $E(\bar{X}) = \mu$ and $\text{var}(\bar{X}) = \frac{\sigma^2}{n}$.
See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Short Answer Type Question No. 7.

4. Prove that every finite integral domain is a field.
See Topic: RING & FIELD, Short Answer Type Question No.1.

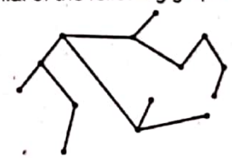
5. Prove that a planer graph G with n vertices, e number of edges and k number of connected components determines $f = e - n + k + 1$ number of regions.
See Topic: ADVANCED GRAPH THEORY, Short Answer Type Question No. 5.

6. Find out order of the element $(1 \quad 2 \quad 3) \in S_3$. Also find out inverse of $(1 \quad 2 \quad 3) \in S_3$.
See Topic: GROUP THEORY, Short Answer Type Question No. 7.

7. A speaks truth 4 out of 5 times. A die is tossed. He reports that there is a six. What is the probability that it was actually a six?
See Topic: PROBABILITY THEORY, Short Answer Type Question No. 3.

Group – C
(Long Answer Type Questions)

8. a) Find out the chromatic polynomial of the following graph G:



b) Let H be a subgroup of a finite group G. Prove that the order of H divides the order of G.
c) Show that the group $(\mathbb{Z}_6, +)$ is a homomorphic image of the group $(\mathbb{Z}, +)$.
a) See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 4(a).
b) See Topic: GROUP THEORY, Long Answer Type Question No. 7(a).
c) See Topic: GROUP THEORY, Long Answer Type Question No. 7(b).

9. a) Prove that if G is a simple planar graph then G has at least one vertex v such that $\deg(v) \le 5$.

b) Prove that every group of prime order is cyclic.

c) Show that center of a group G, given by $Z(G) = \{a \in G : a * g = g * a \text{ for all } g \in G\}$ is a normal subgroup of G.

a) See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 4(b).
b) See Topic: GROUP THEORY, Long Answer Type Question No. 4(b).
c) See Topic: GROUP THEORY, Long Answer Type Question No. 8(a).

10. a) Experience shows that 20% of the people reserving tables at a certain restaurant never show up. If the restaurant has 50 tables and it takes 52 reservations, then find the probability that it will be able to accommodate everyone.

b) The mean yield for one acre plot is 662 kilos with a standard deviation 32 kilos. Assuming normal distribution how many one acre plots in a batch of 1000 plots would you expect to have yield over 700 kilos? (Given that $\phi(1.19) = 0.3830$)

c) A regular graph G determines 8 regions, degree of each vertex being 3. Find the number of vertices of G.

a) See Topic: PROBABILITY THEORY, Long Answer Type Question No. 4.
b) See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 5.
c) See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 4(c).

11. a) Let X be uniformly distributed over the interval $([1, 2]$ and $\bar{X} = E(X))$. Find out the value of a so that $P(X > a + \bar{X}) = \frac{1}{6}$.

b) A random sample of size $n = 100$ is taken from an infinite population with the mean $\mu = 75$ and the variance $\sigma^2 = 256$. Based on Chebyshev's theorem with what probability can we assert that the value we obtain for $\bar{X}$ will fall between 67 and 83?

c) If X has the standard normal variate, then find the probability density function of $Z = X^2$.

See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 6(a), (b) & (c).

12. a) Let $S_1 = \{z \in \mathbb{C} : |z| = 1\}$ is a subgroup of $(\mathbb{C}^*, .)$ where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Also consider the group $(\mathbb{R}, +)$. Define a map $f : \mathbb{R} \to \mathbb{C}^*$ by $f(x) = e^{ix}$. Prove that f is a homomorphism and find out $\ker(f)$.

b) A random sample with observations 65, 71, 64, 71, 70, 69, 64, 63, 67, 68 is drawn from a normal population with a standard deviation $\sqrt{7.056}$. Test the hypothesis that the population mean is 69 at 1% level of significance.
[Given that $P(0 < z < 2.58) = 0.495$]

c) Let $x_1, x_2, \ldots\ldots, x_n$ be the values of a random sample from an exponential population that is $f(x_i) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$ for $x_i > 0$. Then find the maximum likelihood estimator of its parameter θ.

a) See Topic: GROUP THEORY, Long Answer Type Question No. 8(b).
b) See Topic: TESTING OF HYPOTHESIS, Long Answer Type Question No. 2
c) See Topic: ESTIMATION OF PARAMETERS, Long Answer Type Question No. 2.

QUESTION 2017

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternatives for any *ten* of the following:

i) If A and B are two events with $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$, then $P(A \cap B^C)$ is

a) 0.1 ✓b) 0.2 c) 0.3 d) 0.4

ii) A fair die is thrown. The probability that either an odd number or a number greater than 4 will turn up is

a) $\frac{2}{5}$ b) $\frac{3}{7}$ c) $\frac{2}{7}$ ✓d) $\frac{2}{3}$

iii) The variance of a rectangular uniform distribution with parameters a and b is

a) $\frac{b-a}{2}$ ✓b) $\frac{(b-a)^2}{12}$ c) $\frac{(a+b)^2}{12}$ d) $\frac{(b-a)^2}{2}$

iv) A random variable X has the following probability density function:

$$f(x) = \begin{cases} kx, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

The value of k is

a) 1 b) 2 ✓c) 2/3 d) none of these

v) If a is an element of a group G and order of a is 35, then the order of a^{10} is
a) 5 ✓b) 7 c) 35 d) none of these

vi) Inverse of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ is
a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ ✓c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ d) none of these

vii) The number of generators of an infinite cyclic group is
a) 1 ✓b) 2 c) 0 d) infinite

viii) The mean and standard deviation of standard normal variable are
a) 0, –1 b) 1, 1 ✓c) 0, 1 d) 1, 0

ix) The statistic t is said to be an unbiased estimator of a population parameter θ when
✓a) $E(t)=\theta$ b) $E(t^2)=\theta$
c) $E(t^2)=[E(\theta)]^2$ d) $[E(t)]^2=[E(\theta)]^2$

x) A complete graph is called Kuratowski's first graph if it has
✓a) 5 vertices b) 6 vertices c) 4 vertices d) 8 vertices

xi) The number of generators of a cyclic group of order 7 is
a) 2 ✓b) 6 c) 7 d) 5

xii) An element a in a ring R is zero divisor if
a) $a \cdot b = 0$ b) $a \cdot b \neq 0$ for all element b in R
c) $a \cdot b = 0$ for some non-zero element b in R ✓d) none of these

xiii) If C_{97} be a circuit with 97 number of vertices, then chromatic number of C_{97} is
a) 97 b) 2 ✓c) 3 d) none of these

Group – B
(Short Answer Type Questions)

2. Determine the mean and variance of exponential distribution.
See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Short Answer Type Question No. 8.

3. Let G be a simple connected planar graph with e edges and f regions. Then prove that $e \geq \frac{3}{2} f$.
See Topic: ADVANCED GRAPH THEORY, Short Answer Type Question No. 6.

4. Show by Tchebycheff's inequality that in 2000 throws with a coin, the probability that the number of heads lies between 900 and 1100 is at least $\frac{19}{20}$.
See Topic: LIMIT THEOREMS, Short Answer Type Question No. 2.

5. Prove that a group G is Abelian, iff $(ab)^2 = a^2b^2$ for all $a, b \in G$.
See Topic: GROUP THEORY, Short Answer Type Question No. 8.

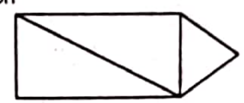
6. Show that the permutation $\begin{pmatrix} 1 & 23 & 4 & 5 \\ 3 & 21 & 5 & 4 \end{pmatrix}$ is even.
See Topic: PROBABILITY THEORY, Short Answer Type Question No. 4.

7. An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the latter. What is the probability that it is a white ball?
See Topic: PROBABILITY THEORY, Short Answer Type Question No. 5.

Group – C
(Long Answer Type Questions)

8. a) Show that the group $(Z_5, +)$, i.e., the additive group of all integers modulo 5 is cyclic. Find all generators of Z_5.
See Topic: GROUP THEORY, Long Answer Type Question No. 9(a).

b) Draw the dual of the graph



See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 5(a).

c) Prove that order of each subgroup of a finite group is divisor of the order of the group.
See Topic: GROUP THEORY, Long Answer Type Question No. 4(a).

9. a) Show by Tchebyshev's inequality that if a die is thrown 3,600 times, the probability of number of sixes that lies between 550 and 650 is at least.
See Topic: LIMIT THEOREMS, Long Answer Type Question No. 1.

b) The mean of a normal distribution is 50 and 5% of the values are greater than 60. Find the standard deviation of the distribution (Area under standard normal curve between $z=0$ and $z=1.64$ is 0.45)
See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 7(a).

c) If a population has Poisson distribution with parameter λ, then show that the sample mean is the maximum likelihood estimate of λ.
See Topic: ESTIMATION OF PARAMETERS, Long Answer Type Question No. 3(a).

10. a) Define planar graph. Construct a planar graph with 6 vertices.
See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 5(b).

b) If G be a connected graph with n vertices, e edges and r faces, prove that $n-e+r=2$.
See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 2(b).

c) If $x_1, x_2, x_3, x_4, x_5, x_6$ be an independent simple random sample from a normal population with unknown variance σ^2, find K so that $K\left[(X_1-X_2)^2+(X_3-X_4)^2+(X_5-X_6)^2\right]$ is an unbiased estimator of σ^2.
See Topic: ESTIMATION OF PARAMETERS, Long Answer Type Question No. 3(b).

11. a) The weights in pound of a sample of 12 packets of butter are 13, 7, 22, 15, 12, 18, 14, 21, 8, 17, 10, 23 taken at random from its population having standard deviation 5. Find 95% confidence interval for the mean of the population. [Given $Z_{0.05}=1.96$]
See Topic: ESTIMATION OF PARAMETERS, Long Answer Type Question No. 3(c).

b) If H and K are two subgroups of a group G, then prove that HK is also subgroup of G if and only if $HK=KH$.
See Topic: GROUP THEORY, Long Answer Type Question No. 9(b).

c) Let $f:G\to G'$ be a homomorphism. Show that f is one-to-one if and only if $\ker f=\{e\}$.
See Topic: GROUP THEORY, Long Answer Type Question No. 9(c).

12. a) Prove that $H=\left\{\begin{pmatrix} x & 0 \\ 0 & x\end{pmatrix} \middle| x\in\mathbb{R},\ x\neq 0\right\}$ forms a normal subgroup of $GL(2,\mathbb{R})$, the group of all real non-singular 2×2 matrices.
See Topic: GROUP THEORY, Long Answer Type Question No. 10(a).

b) Prove that the intersection of two subrings is a subring.
See Topic: RING & FIELD, Long Answer Type Question No. 1(b).

c) A random sample with observations 65, 71, 64, 71, 70, 69, 64, 63, 67, 68 is drawn from a normal population with standard deviation. Test the hypothesis that population mean is 69 at 1% level of significance.
[Given that $P(0<z<2.58)=0.495$]
Incomplete / Insufficient Data

13. a) Prove that every group of prime order is commutative.
See Topic: GROUP THEORY, Long Answer Type Question No. 10(b).

b) Show that for the exponential distribution

$$f(x)=\frac{1}{\sigma}e^{\left(-\frac{x}{\sigma}\right)},\ 0<x<\infty$$

mean and standard deviation both equal to σ.
See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 7(b).

c) The following figures show the distribution of digits in numbers chosen at random from a telephone directory.

Digits	0	1	2	3	4	5	6	7	8	9	Total
Frequency	1026	1107	997	966	1075	933	1107	972	964	853	10,000

Test whether the digits may be taken to occur equally frequently in the directory.
[The tabulated $\chi^2_{0.05}$ for 9 d.f. $=16.919$]
See Topic: TESTING OF HYPOTHESIS, Long Answer Type Question No. 3.

QUESTION 2018

Group – A
(Multiple Choice Type Questions)

1. Answer *any ten* questions:

i) Let A and B be two events such that $P(A\cup B)=\frac{7}{8}$; $P(A\cap B)=\frac{1}{4}$ and $P(\bar{A})=\frac{5}{8}$, then $P(A/\bar{B})$ is

a) $\frac{1}{9}$ b) $\frac{1}{8}$ ✓c) $\frac{1}{2}$ d) None of these

ii) A group contains 12 elements. Then the possible number of elements in a subgroup is

a) 5 ✓b) 3 c) 11 d) 7

iii) Which of the following statements is true

✓a) A semi-group with identity element is called monoid
b) A groupoid is said to be a semi-group if the binary operation is commutative
c) $(Q,+)$ is a cyclic group
d) Identity element in a group is not unique

iv) If $Var(X)=\frac{2}{3}$, then $Var(3X+5)$ is

a) 8 b) 2 ✓c) 6 d) 11

v) The dual of a disconnected graph is

a) regular ✓b) connected c) complete d) disconnected

vi) If the cyclic group G contains 11 distinct elements then it has

a) 2 generators b) 7 generators c) 9 generators ✓d) 10 generators

vii) If X be binomially distributed with mean 2 and variance $\frac{4}{3}$, then value of n and p are respectively

a) $5,\frac{1}{2}$ b) $7,\frac{1}{5}$ ✓c) $6,\frac{1}{3}$ d) None of these

viii) A connected planar graph with 8 vertices determines 4 regions. The number of edges of this graph is

a) 4 b) 8 c) 10 ✓d) 12

ix) If T_1 and T_2 be statistic with $E(T_1)=2\theta_1+3\theta_2$ and $E(T_2)=\theta_1+\theta_2$ then the unbiased estimator of θ_1 is

a) $3T_2-T_1$ ✓b) $3T_2+2T_1$ c) $3T_2-3T_1$ d) $3T_2+T_1$

x) A random variable X has uniform distribution over $(-4,4)$, then $P(1\times 1\le 2)$ is

a) $\frac{1}{2}$ ✓b) $\frac{1}{8}$ c) $\frac{7}{8}$ d) None of these

xi) The chromatic polynomial of a tree with n vertices is

a) λ^n b) λ^{n-1} ✓c) $\lambda(\lambda-1)^{n-1}$ d) $\lambda(\lambda-1)^n$

xii) A fair die is thrown. The probability that either an odd number or a number greater than 4 will turn up is

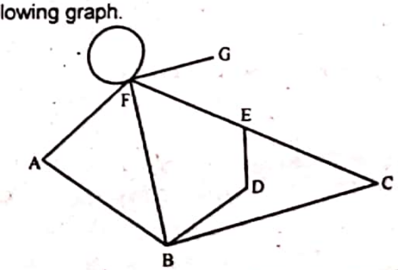
a) $\frac{2}{5}$ ✓b) $\frac{2}{3}$ c) $\frac{2}{7}$ d) None of these

xiii) If μ is a parameter and $H_0(\mu=5)$ is null hypothesis then which one of the following is a left-sided alternative hypothesis?

a) $H_1(\mu\neq 5)$ ✓b) $H_1(\mu<5)$ c) $H_1(\mu>5)$ d) None of these

Group – B
(Short Answer Type Questions)

2. Draw the dual of the following graph.



See Topic: ADVANCE GRAPH THEORY, Short Answer Type Question No. 1.

3. If A and B are two events such that $P(A^c \cup B^c)=\frac{5}{6}$, $P(A)=\frac{1}{2}$ and $P(B^c)=\frac{2}{3}$, show that A and B are independent.
See Topic: PROBABILITY THEORY, Short Answer Type Question No. 6.

4. If T is an unbiased estimator of θ, then show that T^2 is a biased estimator of θ^2.
See Topic: ESTIMATION OF PARAMETERS, Short Answer Type Question No. 2.

5. Show that in a group (G,o), $(a\,o\,B)^{-1}=b^{-1}o\,a^{-1}$ for all $a,b\in G$.
See Topic: GROUP THEORY, Short Answer Type Question No. 9.

6. Let G be a simple connected planar graph with n vertices, e edges and f regions. Then prove that $e\le 3n-6$.
See Topic: ADVANCE GRAPH THEORY, Short Answer Type Question No. 7.

7. If 10 unbiased dice are thrown at random, find the probability that the sum of the points shown by all of them is between 30 and 40.
Given $\phi(1.018)=0.3461$
See Topic: PROBABILITY THEORY, Short Answer Type Question No. 7.

Group – C
(Long Answer Type Questions)

8. a) State Baye's theorem. Three identical boxes I, II and III contain respectively 4 white and 3 red balls, 3 white and 7 red balls, 2 white and 3 red balls. A box is chosen at random and a ball is drawn out of it. If the ball is found to be white, what is the probability that box II is selected?
See Topic: PROBABILITY THEORY, Short Answer Type Question No. 8.

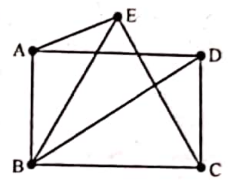
b) The p.d.f. of a random variable X is $f(x)=k(x-1)(2-x)$, $1\le x\le 2$. Find (i) the value of the constant k (ii) the distribution function $F(x)$ (iii) $P\left(\frac{5}{4}\le X\le \frac{3}{2}\right)$.
See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 8(a).

c) Show that every planar graph is 6-colourable.
See Topic: ADVANCED GRAPH THEORY, Long Answer Type Question No. 2(a).

9. a) Let a be a group. If $a,b\in G$ such that $a^4=e$, the identity element of G and $ab=ba^2$, then prove that $a=e$.

b) If $(R,+,\cdot)$ be a ring such that $a^2=a\forall a\in R$, then show that
(i) $b=-b\forall b\in R$ (ii) $a-b=0\Rightarrow a=b$
See Topic: TESTING OF HYPOTHESIS, Long Answer Type Question No. 4(a) & (b).

c) Draw the dual of the graph:



See Topic: ADVANCE GRAPH THEORY, Short Answer Type Question No. 8.

10. a) The p.d.f. of a random variable X is assumed to be of the form $f(x)=cx^{\alpha}$, $0\le x\le 1$ for some α and constant c. If $X_1, X_2, X_3, \cdots\cdots\cdots, X_n$ is a random sample of size n, find the Maximum Likelihood Estimate of α.
See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 2(b).

b) Define subgroup of a group. Let $GL(2,\mathbb{R})$ be the multiplicative group of all real non-singular matrices of order 2. Show that the Set $H=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \mid ad-bc=1\right\}$ is a subgroup of $GL(2,\mathbb{R})$.
See Topic: GROUP THEORY, Short Answer Type Question No. 10.

c) If X is the number scored in a throw of a fair die, show that the Tchebycheff's inequality gives $P(|X-\mu|>2.5)<0.47$, where μ is the mean of X, while the actual probability is zero.
See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 8(b).

11. a) Prove that the chromatic polynomial of a tree with n vertices is $x(x-1)^{n-1}$.
See Topic: ADVANCE GRAPH THEORY, Long Answer Type Question No. 1(b).

b) If the weekly wages of 10,000 workers in a factory follow normal distribution with mean and s.d. ₹ 70 and ₹ 5 respectively, find the expected number of workers whose weekly wages are (i) between ₹ 66 and ₹ 72 (ii) less that ₹ 66 and (iii) more than ₹ 72.

Given that $\frac{1}{\sqrt{2\pi}}\int_0^z e^{-t^2/2}dt=0.1554$ and 0.2881 according as $z=0.4$ and $z=0.8$.

See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 9(a).

c) Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$ be two permutations. Show that $AB \neq BA$.

See Topic: RING & FIELD, Long Answer Type Question No. 3(a).

12. a) The following table gives the number of aircraft accidents that occurred during various days of the week. Test whether the accidents are uniformly distributed over the week.

Given: $\chi^2 0.05(\gamma = 6) = 12.59$

Day	Sun	Mon	Tue	Wed	Thurs	Fri	Sat
No. of accidents	13	14	19	12	11	15	14

See Topic: RANDOM VARIABLE & DISTRIBUTION FUNCTION, Long Answer Type Question No. 9(b).

b) Prove that the chromatic number of a circuit with n vertices is
(i) 2 if n is even (ii) 3 if n is odd.

See Topic: SAMPLING THEORY, Short Answer Type Question No. 1.

c) Let $f:(c-\{0\},\cdot) \to (c-\{0\},\cdot)$ be a function defined by $f(z) = z^4$.

(i) Show that f is a homomorphism.

(ii) Find the Kernel of f.

See Topic: RING & FIELD, Long Answer Type Question No. 3(b).