4.5.1. Boolean Algebra

Let B be a non empty set having two binary operations + and ., one unary operation ' and two distinct elements 0 and 1. B is called a Boolean Algebra if the following axioms hold :

1. For every pair of elements a, b in B

   (i) $a+b=b+a$ (ii) $a\cdot b=b\cdot a$

   (Commutative laws)

2. For all elements a, b, c in B

   (i) $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$ (ii) $a+(b\cdot c)=(a+b)\cdot(a+c)$

   (Distributive laws)

3. For every element a in B

   (i) $a+0=a$ (ii) $a\cdot 1=a$

   (Identity law)

4. For every element a there is an element a' in B such that

   (i) $a+a'=1$ (ii) $a\cdot a'=0$

   (Complement laws)

Note : (1) *We designate it as "(B, +, ., ', 0, 1) is a Boolean Algebra".*

(2) The element 1 is called ***unit element*** and the element 0 is called the ***zero element.***

(3) *a' is called **complement** of a.*

(4) *We usually drop the symbol . and use juxtaposition, i.e., instead of writing $a\cdot b$ we shall write ab in practice.*

(5) *+ is called 'sum' (not arithmetic sum) and . is called 'product' (not arithmetic product) and ' is called complement.*

(6) *Conventionally the operation has precedence over the operation . and . has precedence over +. For example $x\cdot y'$ means $x\cdot(y')$ and not $(x\cdot y)$; $x+y\cdot z$ means $x+(y\cdot z)=x+yz$.*

Illustrative Examples.

(i) *Let S be a set. Then its power set $P(S)$ (set of all subsets of S) is a Boolean algebra under the operations $\cup$ (union), $\cap$ (Intersection),' (complement) with Φ (null set) as zero element and S as unit.*

Since $A\cup B = B\cup A$ and $A\cap B = B\cap A$ so the commutative property hold :

Since $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$ and $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ so the distributive law hold.

Since $A\cup\Phi = A$ and $A\cap S = A$ so the Identity law is valid.

Since $A\cup A' = S$ and $A\cap A' = \Phi$ so the complement laws are hold.

So $(P(S), \cup, \cap, ', \Phi, S)$ is a Boolean Algebra.

(ii) *Let $D_{40} = \{1, 2, 4, 5, 8, 10, 20, 40\}$ is the set of divisors of 40. Let $\vee, \wedge$ and — are defined as $a\vee b = LCM$ of a, b; $a\wedge b = HCF$ of a, b and $\bar{a} = \dfrac{40}{a}$. Then $(D_{40}, \vee, \wedge,$ —, 1, 40) is **not** a Boolean Algebra.*

Since *LCM* of $a, b = LCM$ of b,a and *HCF* of $a, b = HCF$ of b,a so $a\vee b = b\vee a$ and $a\wedge b = b\wedge a$ showing the validity of commutative property.

Since *HCF* of *a* and *LCM* of $b, c = LCM$ of (*HCF* of *a, b*) and (*HCF* of *a, c*) so $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$. Similarly, $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$ hold. This shows that the distributive law hold.

Since *LCM* of *a* and $1 = a$ and *HCF* of *a* and $40 = a$ (as *a* is a factor of 40) so $a\vee 1 = a, a\wedge 40 = a$. This shows Identity laws are valid

But $2\vee\bar{2} = 2\vee 20 = LCM$ of 2 and $20 = 20 \neq 40$. This shows complement laws are invalid.

Hence $(D_{40}, \vee, \wedge,$ —, 1, 40) is not a Boolean Algebra.

(iii) Let the set of all factors of 70, $D_{70} = \{1, 2, 5, 7, 10, 14, 35, 70\}$. Let $\vee$ and $\wedge$ are defined as $a\vee b = LCM$ of a, b ; $a\wedge b = HCF$ of a,b ; $\bar{a} = \dfrac{70}{a}$. Then $(D_{70}, \vee, \wedge,$ —, 1, 70) is a Boolean Algebra.

(iv) (An Important Boolean Algebra)

Let $B = \{0, 1\}$ be a set. The three operations +, . and ' are defined as

+	1	0
1	1	1
0	1	0

.	1	0
1	1	0
0	0	0

'	1	0
	0	1

B is a Boolean Algebra w.r.t the above operations.

From the very definition of the operation we see all the axioms of Boolean Algebra are valid. For instance $0+1 = 1 = 1+0$ etc. So (B, +, ., ', 0, 1) is a Boolean Algebra.

(v) Let $B = \{0, 1\}$ be a Boolean Algebra as defined in the previous example. Let $B^n = B\times B\times B\times\cdots\times B = \{(x_1, x_2, \cdots, x_n) :$ each $x_i \in B\}$. Then B^n is a Boolean Algebra w.r.t the operation +, . and ' defined by

$$(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n);$$

$$(x_1, x_2, \cdots, x_n)\cdot(y_1, y_2, \cdots, y_n) = (x_1\cdot y_1, x_2\cdot y_2, \cdots, x_n\cdot y_n) \text{ and}$$

$$(x_1, x_2, \cdots, x_n)' = (x_1', x_2', \cdots, x_n')$$

Since $x_i + y_i \in B, x_i\cdot y_i \in B$ and $x_i' \in B$ for all x_i, y_i in B.

So +, . are binary operations and ' is uniary operation on B^n.

Let us verify the axioms of Boolean Algebra :

1. (i) $(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$

$$= (y_1 + x_1, y_2 + x_2, \cdots, y_n + x_n) = (y_1, y_2, \cdots, y_n) + (x_1, x_2, \cdots, x_n)$$

(ii) $(x_1, x_2, \cdots, x_n)\cdot(y_1, y_2, \cdots, y_n) = (x_1\cdot y_1, x_2\cdot y_2, \cdots, x_n\cdot y_n)$

$= (y_1\cdot x_1, y_2\cdot x_2, \cdots, y_n\cdot x_n)$

$= (y_1, y_2, \cdots, y_n)\cdot(x_1, x_2, \cdots, x_n)$

So, commutative law hold.

2. (i) $(x_1, x_2, \cdots, x_n)\cdot\{(y_1, y_2, \cdots, y_n) + (z_1, z_2, \cdots, z_n)\}$

$= (x_1, x_2, \cdots, x_n)\cdot\{(y_1 + z_1, y_2 + z_2, \cdots, y_n + z_n)\}$

$= (x_1\cdot(y_1 + z_1), x_2\cdot(y_2 + z_2), \cdots, x_n\cdot(y_n + z_n))$

$= (x_1\cdot y_1 + x_1\cdot z_1, x_2\cdot y_2 + x_2\cdot z_2, \cdots, x_n\cdot y_n + x_n\cdot z_n)$

$= (x_1\cdot y_1, x_2\cdot y_2, \cdots, x_n\cdot y_n) + (x_1\cdot z_1, x_2\cdot z_2, \cdots, x_n\cdot z_n)$

$= (x_1, x_2, \cdots, x_n)\cdot(y_1, y_2, \cdots, y_n) + (x_1, x_2, \cdots, x_n)\cdot(z_1, z_2, \cdots, z_n)$

(ii) Similarly it can be shown that

$(x_1, x_2, \cdots, x_n) + \{(y_1, y_2, \cdots, y_n)\cdot(z_1, z_2, \cdots, z_n)\}$

$= \{(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n)\}\cdot\{(x_1, x_2, \cdots, x_n) + (z_1, z_2, \cdots, z_n)\}$

So distributive law hold.

3. (i) $\{(x_1, x_2, \cdots, x_n) + (0, 0, \cdots, 0)\} = (x_1 + 0, x_2 + 0, \cdots, x_n + 0)$

$= (x_1, x_2, \cdots, x_n)$

(ii) $(x_1, x_2, \cdots, x_n)\cdot(1, 1, \cdots, 1) = (x_1\cdot 1, x_2\cdot 1, \cdots, x_n\cdot 1)$

$= (x_1, x_2, \cdots, x_n)$

So, the identity law hold. Here $(0, 0, \cdots, 0)$ is the zero element and $(1, 1, \cdots, 1)$ is unit element.

4. (i) $(x_1, x_2, \cdots, x_n) + (x_1, x_2, \cdots, x_n)'$

$= (x_1, x_2, \cdots, x_n) + (x_1', x_2', \cdots, x_n')$

$= (x_1 + x_1', x_2 + x_2', \cdots, x_n + x_n') = (1, 1, \cdots, 1) =$ the unit element.

(ii) $(x_1, x_2, \cdots, x_n)\cdot(x_1, x_2, \cdots, x_n)'$

$= (x_1, x_2, \cdots, x_n)\cdot(x_1', x_2', \cdots, x_n') = (x_1\cdot x_1', x_2\cdot x_2', \cdots, x_n\cdot x_n')$

$= (0, 0, \cdots, 0) =$ the zero element.

Thus all the axioms are valid in B''.

So B'' is a Boolean Algebra.

4.5.2. Duality in Boolean Algebra.

In a Boolean Algebra $(B, +, \cdot, ', 0, 1)$ the dual of any statement is defined to be the statement that is obtained by interchanging + and . ; 0 and 1 in the original statement. For example the dual of the statement '$(a+0)+(1\cdot a')=1$' is the statement '$(a\cdot 1)\cdot(0+a')=0$'.

Principle of Duality in a Boolean Algebra

If a Theorem hold in a Boolean Algebra then another Theorem is obtained which is nothing but the dual of the former.

This follows from the fact that the dual of each axiom of a Boolean Algebra also comes as an axiom.

Theorem 1. (Idempotent Laws)

In a Boolean algebra $(B, +, \cdot, ', 0, 1)$

(i) $a + a = a$ (ii) $a\cdot a = a$

Proof. (i) $a + a = (a + a)\cdot 1$ $\because$ 1 is unit element

$= (a + a)\cdot(a + a')$ $\because a'$ is complement of a

$= a + (a\cdot a')$ by distributive law

$= a + 0 = a$.

Though (ii) follows from (i) by principle of duality we give the independent proof.

(ii) $a\cdot a=(a\cdot a)+0$ $\because$ 0 is zero element

$=a\cdot a+(a\cdot a')$ $\because$ a' is complement.

$=a\cdot(a+a')$ by distributive property

$=a\cdot 1=a$.

Theorem 2. (Boundedness Law)

In a Boolean Algebra $(B,+,\cdot,',0,1)$

(i) $a+1=1$ (ii) $a\cdot 0=0$

Proof. (i) $a+1=(a+1)\cdot 1$ $\because$ 1 is unit element

$=(a+1)\cdot(a+a')$ $\because$ a' is complement of a

$=a+(1\cdot a')$ by distributive law

$=a+(a'\cdot 1)$ by commutative law

$=a+a'=1$ by complement law

(ii) Similar or follows from (i) by principle of duality.

Theorem 3. (Absorption Laws)

In a Boolean Algebra $(B,+,\cdot,',0,1)$

(i) $a+(a\cdot b)=a$

(ii) $a\cdot(a+b)=a$ [W.B.U.Tech 2007]

Proof. (i) Similar to (ii)

(ii) $a\cdot(a+b)=(a+0)\cdot(a+b)=a+(0\cdot b)$ by distributive property

$=a+(b\cdot 0)=a+0=a$.

Theorem 4. (Associative Laws)

In a Boolean Algebra $(B,+,\cdot,',0,1)$

(i) $(a+b)+c=a+(b+c)$

(ii) $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ [W.B.U.Tech 2007]

Proof. (i) Let $x=(a+b)+c$ and $y=a+(b+c)$ we shall first prove that $a\cdot x=a\cdot y$.

Now, $a\cdot x=a\cdot((a+b)+c)=a\cdot(a+b)+a\cdot c$

$=a+a\cdot c$ by Absorption Law

$=a$ by Absorption Law

and $a\cdot y=a\cdot(a+(b+c))=(a\cdot a)+(a\cdot(b+c))$

$=a+(a\cdot(b+c))$ by idempotent Law

$=a+(a\cdot z)$ putting $z=b+c$

$=a$ by Absorption Law

Thus $a\cdot x=a\cdot y$ ... (1)

Next we shall show $a'\cdot x=a'\cdot y$

Now, $a'\cdot x=a'\cdot((a+b)+c)$

$=a'\cdot(a+b)+(a'\cdot c)$ by distributive Law

$=((a'\cdot a)+(a'\cdot b))+(a'\cdot c)$ by distributive Law

$=(0+(a'\cdot b))+(a'\cdot c)$ $=(a'\cdot b)+(a'\cdot c)$ since 0 is zero element

$=a'\cdot(b+c)$

and $a'\cdot y=a'\cdot(a+(b+c))=(a'\cdot a)+a'\cdot(b+c)$

$=0+a'\cdot(b+c)=a'\cdot(b+c)$

Thus $a'\cdot x=a'\cdot y$ ... (2)

Now, $x=1\cdot x=(a+a')\cdot x$

$=(a\cdot x)+(a'\cdot x)=(a\cdot y)+(a'\cdot y)$ by (1) and (2)

$=(a+a')\cdot y=1\cdot y=y$

$\therefore x=y$. So, $(a+b)+c=a+(b+c)$.

(ii) Similar (Follow by principle of duality from (i))

Theorem 5. (Uniqueness of Complement)

Let a be any element in a Boolean Algebra $(B,+,\cdot,',0,1)$. If $a+x=1$ and $a\cdot x=0$ then x is complement of a.

Proof. The complement of a,

$a' = a' + 0 = a' + (a \cdot x)$ by hypothesis

$= (a' + a) \cdot (a' + x)$ by distributive law

$= 1 \cdot (a' + x) = a' + x$ ... (1)

and $x = x + 0 = x + (a \cdot a') = (x + a) \cdot (x + a')$

$= 1 \cdot (x + a') = x + a' = a' + x$ by commutative law ... (2)

From (1) and (2) we have $x = a'$ i.e., x is complement of a.

Theorem 6. (Involution Law)

In a Boolean Algebra $(B, +, \cdot, ', 0, 1)$, $(a')' = a$ for each element a in B.

Proof. Now, $a + a' = 1,\ a \cdot a' = 0$

$\therefore\ a' + a = 1,\ a' \cdot a = 0$ by commutative law

$\therefore$ by uniqueness of complement $a = (a')'$.

Theorem 7. In a Boolean Algebra $(B, +, \cdot, ', 0, 1)$

(i) $0' = 1$ (ii) $1' = 0$

Proof. By boundedness law $0 + 1 = 1$.

We have also $0 \cdot 1 = 0$ $\because$ 1 is unit element

So, by uniqueness of complement 1 is the complement of 0, i.e. $1 = 0'$.

By law of duality $0 = 1'$ (or, pose an independent proof)

Theorem 8. (De Morgan's Law)

In a Boolean Algebra $(B, +, \cdot, ', 0, 1)$

(i) $(a + b)' = a' \cdot b'$ (ii) $(a \cdot b)' = a' + b'$

Proof. (i) We shall prove that

$(a + b) + (a' \cdot b') = 1$ and $(a + b) \cdot (a' \cdot b') = 0$

Now, $(a + b) + (a' \cdot b') = (b + a) + (a' \cdot b')$ by commutative law

$= b + (a + a' \cdot b')$ by Associative law (which is shown)

$= b + ((a + a') \cdot (a + b'))$ by distributive law

$= b + (1 \cdot (a + b')) = b + (a + b') = b + (b' + a)$

$= (b + b') + a = 1 + a = 1$

And $(a + b) \cdot (a' \cdot b') = ((a + b) \cdot a') \cdot b'$. Since Associative law hold in Boolean Algebra

$= ((a \cdot a') + (b \cdot a')) \cdot b'$ by distributive law

$= (0 + (b \cdot a')) \cdot b'$

$= (b \cdot a') \cdot b' = (a' \cdot b) \cdot b' = a' \cdot (b \cdot b') = a' \cdot 0 = 0$.

Thus $(a + b) + (a' \cdot b') = 1$ and $(a + b) \cdot (a' \cdot b') = 0$

So, by uniqueness of complement $(a + b)' = a' \cdot b'$.

(ii) Left to reader as exercise.

Theorem 9. In a Boolean Algebra $(B, +, \cdot, ', 0, 1)$ the following four results are equivalent :

(1) $a + b = b$ (2) $a \cdot b = a$

(3) $a' + b = 1$ (4) $a \cdot b' = 0$

Proof. I. Let (1) be true. $\therefore\ b = a + b$

So, $a \cdot b = a \cdot (a + b) = (a \cdot a) + (a \cdot b) = a + (a \cdot b)$ by Idempotent Law

$= a$ by Absorption law in Boolean Algebra.

Thus (2) is true.

II. Let (2) be true. $\therefore\ a \cdot b = a$

$\therefore\ (a \cdot b)' = a'$ $\therefore\ a' + b' = a'$ by D' Morgans Law

or, $a' + b' + b = a' + b$ or, $a' + 1 = a' + b$ by complement Law

or, $1 = a' + b$ by Boundedness Law

$\therefore\ a' + b = 1$. Proving (3).

III. Let (3) hold. $\therefore\ a'+b=1$

$\therefore\ (a'+b)'=1'$ or, $(a')'\cdot b'=1'$ by D' Morgans

or, $a\cdot b'=0$ by Theorem 7. This proves (4)

IV. Suppose (4) is true. Establish (1). This is left as an exercise.

Illustrative Example.

Ex. 1. *Let $(B,+,\cdot,')$ be a Boolean Algebra and $a,b,c\in B$. Prove that if $a+b=a+c$ and $a\cdot b=a\cdot c$ then $b=c$.*

$\because\ a+b=a+c \qquad \therefore\ (a+b)\cdot b=(a+c)\cdot b$

or, $a\cdot b+b\cdot b=a\cdot b+c\cdot b$

or, $a\cdot b+b=a\cdot b+c\cdot b$ by Idempotent Law

or, $b=a\cdot c+b\cdot c$ by Absorption law and by hypothesis

or, $b=a\cdot c+b\cdot c$ or, $b=(a+b)\cdot c$

or, $b=(a+c)\cdot c$ by hypothesis

$b=a\cdot c+c\cdot c=a\cdot c+c=c$. Hence $b=c$.

Ex. 2. *In a Boolean Algebra $(B,+,\cdot,',0,1)$ if x, y, z belong to B simplify the expression $(x+y)(x+z)(x'y')'$.*

$(x+y)(x+z)(x'y')'$

$=(x+y)(x+z)\left((x')'+(y')'\right)$ by D'Morgans La

$=(x+y)(x+z)(x+y)=(x+y)(x+y)(x+z)$

$=(x+y)(x+z)$ by Idempotent Law

$=xx+xz+yx+yz=x+xz+yx+yz$ by Idempotent Law

$=x+yx+yz$ by Absorption Law

$=x+yz$ by Absorption Law, which is the simplified form.

Note. *Though it looks simple we do not know whether it can be simplified more. This answer is obtained in.*

4.5.3. Boolean Ring

A Ring $(R,+,\cdot)$ with *unit* element is called Bollean Ring if $a^2=a$ for all a in R (a^2 is defined as $a\cdot a$)

Example. The set $B=\{0,1\}$ is a Boolean Ring w.r.t the compositions + and · and ' defined in the following table :

+	1	0
1	1	1
0	1	0

·	1	0
1	1	0
0	0	0

$1'$	=	0
$0'$	=	1

From the composition table we see all the axioms of Ring are satisfied by B. So B is a Ring with unit element which is 1.

Moreover, $1^2=1\cdot 1=1$ by table

and $0^2=0\cdot 0=0$ by table

So the set B is a Bollean Ring.

4.5.4. Relation with Bollean Algebra and Bollean Ring.

Theorem1. (Boolean algebra is made into Bollean Ring)

In a Boolean algebra (B,+,·, ', 0, 1) we defined $a\oplus b=(a\cdot b')+(a'\cdot b)$ and $a\odot b=a\cdot b$. Then $(B,\oplus,\odot)$ is a Boolean Ring.

Proof : We shall first show $(B,\oplus,\odot)$ is a Ring with unit element.

I. Let $a, b, c\in B$ be arbitrary.

(i) Since B is Boolean algebra so the complements a' and b' are in B. Again since the coomposition · is a binary relation so $a\cdot b'$ and $a'\cdot b$ belong to B. So $a\cdot b'+a'\cdot b\in B$ as + is a binary relation on B. Thus $a\oplus b\in B$.

(ii) $a\oplus(b\oplus c)=a\oplus((b\cdot c')+(b'\cdot c))$

$=a((b\cdot c')+(b'\cdot c))'+a'\cdot((b\cdot c')+(b'\cdot c))$

[by definition of $\oplus$]

$=a\cdot(b'+(c')')\cdot((b')'+c')+a'\cdot((b\cdot c')+(b'\cdot c))$

(using D'Morgass law)

$= a\cdot((b'+c)\cdot(b+c'))+a'\cdot(b\cdot c'+b'\cdot c)$

$= a\cdot(b'\cdot b+b'\cdot c'+c\cdot b+c\cdot c')+a'\cdot b\cdot c'+a'\cdot b'\cdot c$

(using Distributive property)

$= a\cdot(0+b'\cdot c'+c\cdot b+0)+a'\cdot b\cdot c'+a'\cdot b'\cdot c$

$= a\cdot(b'\cdot c'+c\cdot b)+a'\cdot b\cdot c'+a'\cdot b'\cdot c$

$= a\cdot b'\cdot c'+a\cdot c\cdot b+a'\cdot b\cdot c'+a'\cdot b'\cdot c$

$\therefore a\oplus(b\oplus c)=a\cdot b\cdot c+a\cdot b'\cdot c'+a'\cdot b\cdot c'+a'\cdot b'\cdot c$ ... (1)

(using commutative property)

Again $(a\oplus b)\oplus c$

$= c\oplus(a\oplus b)$

$= c\cdot a\cdot b+c\cdot a'\cdot b'+c'\cdot a\cdot b'+c'\cdot a'\cdot b$ using (1)

$= a\cdot b\cdot c+a\cdot b'\cdot c'+a'\cdot b\cdot c'+a'\cdot b'\cdot c$ ... (2)

(using commutative property)

From 1 and (2) we get

$a\oplus(b\oplus c)=(a\oplus b)\oplus c$

(iii) We see $a\oplus 0=a\cdot 0'+a'\cdot 0$ by definition of $\oplus$

$= a\cdot 1+0$

$= a$

$\therefore$ 0 is null element / zero element in B.

(iv) We see $a\oplus a=a\cdot a'+a'\cdot a$ by definition of $\oplus$

$0+0=0$

$\therefore$ Additive inverse of a is a itself.

(v) Now by definition of $\oplus$,

$a\oplus b=a\cdot b'+a'\cdot b$

and $b\oplus a=b\cdot a'+b'\cdot a$

$= a'\cdot b+a\cdot b'$ using commutative property

These imply $a\oplus b=b\oplus a$

Thus B is ablian group under the composition $\oplus$

II. Let $a,b,c\in B$ be arbitrary. By definition of $\odot$

(i) $a\odot b=a\cdot b\in B$ $\because$ the composition $\cdot$ is a binary relation

(ii) $a\odot(b\odot c)=a\cdot(b\cdot c)$

$=(a\cdot b)\cdot c$ by Associative law of $\cdot$

$=(a\odot b)\odot c$ according to definition of $\odot$

III. Let $a,b,c\in B$ be arbitrary.

(i) $a\odot(b\oplus c)=a\cdot(b\cdot c'+b'\cdot c)$ by definition of $\odot$ and $\oplus$

$= a\cdot(b\cdot c')+a\cdot(b'\cdot c)$ by distributive property

$= a\cdot b\cdot c'+a\cdot b'\cdot c$ (3)

Again $(a\odot b)\oplus(a\odot c)$

$=(a\cdot b)\oplus(a\cdot c)$ by definition of $\odot$

$=(a\cdot b)\cdot(a\cdot c)'+(a\cdot b)'\cdot(a\cdot c)$

$=(a\cdot b)\cdot(a'+c')+(a'+b')\cdot(a\cdot c)$ by D'Morgan's law

$=(a\cdot b)\cdot a'+(a\cdot b)+c'+a'\cdot(a\cdot c)+b'\cdot(a\cdot c)$ by distributive law

$= b\cdot(a\cdot a')+a\cdot b\cdot c'+(a'\cdot a)\cdot c+a\cdot b'\cdot c$

$= b\cdot 0+a\cdot b\cdot c'+0\cdot c+a\cdot b'\cdot c$

$= a\cdot b\cdot c'+a\cdot b'\cdot c$ (4)

From (3) and (4) we get

$a\odot(b\oplus c)=(a\odot b)\oplus(a\odot c)$ (5)

(ii) $(b\oplus c)\odot a=(b\oplus c)\cdot a$

$= a\cdot(b\oplus c)$ by commutilative property of $\cdot$

$= a\odot(b\oplus c)$

$=(a\odot b)\oplus(a\odot c)$ using (5) (6)

Thus all the axioms of Ring are satisfied the compositive $\oplus$ and $\odot$ in B. So $(B,\oplus,\odot)$ is a Ring.

Next we see for arbitrary element a in B,

$a \odot 1$ (1 is the unit element in the Boolean Algebra)

$= a \cdot 1$ by definition of $\odot$

$= a$

Thus the ring $(B, \oplus, \odot)$ has unit element.

Again, for arbitrary a in B,

$a^2 = a \odot a = a \cdot a$

$= a$ by Idempotent law in Boolean Algebra.

So $(B, \oplus, \odot)$ is a Boolean Ring

Theorem 2. *If $(R, +, \cdot)$ is a Boolean ring prove that*

(i) $a + a = 0 \ \forall \ a \in R$

i.e. each element of R is its own additive inverse

(ii) $a + b = 0 \Rightarrow a = b$

(iii) *R is commutative ring.*

(i) $a \in R \Rightarrow a + a \in R$

$\Rightarrow (a+a)^2 = a + a$, by the given condition

$\Rightarrow (a+a)\cdot(a+a) = a + a$

$\Rightarrow (a+a)\cdot a + (a+a)\cdot a = a + a$, using distributive law

$\Rightarrow \left(a^2 + a^2\right) + \left(a^2 + a^2\right) = a + a \ \left[\because a \cdot a = a^2\right]$

$\Rightarrow (a+a) + (a+a) = (a+a) + 0 \ \left[\because a + 0 = a\right]$

$\Rightarrow a + a = 0$, (by left cancellation law for addition in R)

(ii) Now $a + b = 0 \quad \Rightarrow a + b = a + a$ [by (i)]

$\Rightarrow b = a$ (by left cancellation law)

(iii) We have, $(a+b)^2 = a + b$

$\Rightarrow (a+b)\cdot(a+b) = a + b$

$\Rightarrow (a+b)\cdot a + (a+b)\cdot b = a + b$ (by distributive law)

$\Rightarrow a^2 + b \cdot a + a \cdot b + b^2 = a + b$ (by distributive law)

$\Rightarrow (a + b \cdot a) + (a \cdot b + b) = a + b$ (by given condition)

$\Rightarrow (a+b) + (b \cdot a + a \cdot b) = (a+b) + 0$, using commutative and associative property

$\Rightarrow b \cdot a + a \cdot b = 0$, by left cancellation law

$\Rightarrow a \cdot b = b \cdot a$ by (ii)

$\therefore$ R is a commutative ring.

Theorem 3. (Boolean Ring is made into Boolean Algebra).

A Boolean Ring $(R, +, \cdot)$ is a Boolean Algebra with respect to the compositions $\vee$, and $\wedge$ defined by

$a \vee b = a + b + a \cdot b$; $a \wedge b = a \cdot b$; $a' = 1 + a$ where 1 is the unit element in R.

Proof. By axioms of Ring $a + b$, $a \cdot b$ and $1 + a \in R$ and so $a \vee b$ and $a \wedge b$ and a' all belong to R for all element a in R. So the compositions $\vee$ and $\wedge$ are binary operations and the composition ' is uniary composition on R.

Let us verify the axioms of Boolean Algebra :

1. For all a, b in R

(i) $a \vee b = a + b + a \cdot b$

$= b + a + b \cdot a$ [$\because$ + and $\cdot$ are commutative]

$= b \vee a$

(ii) $a \wedge b = a \cdot b$

$= b \cdot a$ [$\because$ $\cdot$ is commutative]

2. For all elements a, b, c in R

(i) $a \wedge (b \vee c) = a \wedge (b + c + b \cdot c)$

$= a \cdot (b + c + b \cdot c)$

$= (a \cdot b) + (a \cdot c) + a \cdot (b \cdot c)$ [by distributive law of $\cdot$ over +] (1)

Again $(a\wedge b)\vee(a\wedge c)$

$=(a\cdot b)\vee(a\cdot c)$ by definition of $\wedge$

$=(a\cdot b)+(a\cdot c)+(a\cdot b)\cdot(a\cdot c)$

$=(a\cdot b)+(a\cdot c)+((a\cdot b)\cdot a)\cdot c$ [$\because\ \cdot$ is Associative]

$=(a\cdot b)+(a\cdot c)+(a\cdot(a\cdot b))\cdot c$ [$\because\ \cdot$ is commutative]

$=(a\cdot b)+(a\cdot c)+(a^2\cdot b)\cdot c$

$=(a\cdot b)+(a\cdot c)+(a\cdot b)\cdot c$ [$\because$ R Boolean Ring]

$=(a\cdot b)+(a\cdot c)+a\cdot(b\cdot c)$ (2)

From (1) and (2) we have

$a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$

(ii) $a\vee(b\wedge c)=a\wedge(b\cdot c)$

$=a+(b\cdot c)+a\cdot(b\cdot c)$ (3)

and $(a\vee b)\wedge(a\vee c)$

$=(a\vee b)\cdot(a\vee c)$ by definition of $\wedge$

$=(a+b+a\cdot b)\cdot(a+c+a\cdot c)$ by definition of $\vee$

$=a\cdot a+a\cdot c+a\cdot(a\cdot c)+b\cdot a+b\cdot c+b\cdot(a\cdot c)+(a\cdot b)\cdot a+(a\cdot b)\cdot c$

$+(a\cdot b)\cdot(a\cdot c)$

[using distributive property]

$=a^2+a\cdot c+a^2\cdot c+b\cdot a+b\cdot c+b\cdot(a\cdot c)+b\cdot a^2+(a\cdot b)\cdot c+b\cdot(a^2)\cdot c$

$=a+a\cdot c+a\cdot c+b\cdot a+b\cdot c+b\cdot(a\cdot c)+b\cdot a+(a\cdot b\cdot c)+b\cdot(a\cdot c)$

[$\because$ R is Boolean Ring]

$=a+0+(b\cdot c)+(b\cdot a+b\cdot a)+(b\cdot(a\cdot c)+b\cdot(a\cdot c))+(a\cdot b\cdot c)$

[$\because a+a=0$ in a Boolean Ring.]

$=a+0+0+(b\cdot c)+(a\cdot b\cdot c)$

$=a+(b\cdot c)+a\cdot(b\cdot c)$ (4)

By (3) and (4) we have

$a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$

3. (i) We see for the null element / zero element 0 in R,

$a\vee 0=a+0+a\cdot 0$

$=a+0$ [$\because\ a\cdot 0=0$ in Ring]

$=a$ $\because$ 0 is null element

$\therefore a\vee 0=a$

(ii) For the unit element 1 in R,

$a\wedge 1=a\cdot 1=1$

4. (i) We see $a\vee a'=a+a'+a\cdot a'$

$=a+(1+a)+a\cdot(1+a)$ by definition of a'

$=1+(a+a)+a\cdot 1+a\cdot a$

$=1+0+a+a$ $\because$ R is Boolean Ring

$=1+0$ [$\because a+a=0$ in Boolean Ring]

$=1$

(ii) $a\wedge a'=a\wedge(1+a)=a\cdot(1+a)$

$=(a\cdot 1)+(a\cdot a)=a+a^2$

$=a+a$ [$\because$ R is Boolean Ring]

$=0$ [by the previous theorem]

Thus all the axious of Boolean Algebra are satisfied. So $(R,\vee,\wedge,1,0)$ is a Boolean Algebra.

Ex. In a Boolean Ring $(R,+,\cdot)$ prove that $(a+b)^2=a^2+2ab+b^2$ $(ab\equiv a\cdot b)$.

Solution. $a^2+2ab+b^2=a^2+ab+ab+b^2$

$=a(a+b)+(a+b)b$ by distributive proprety

$=a(a+b)+b(a+b)$ $\because R$ is commulative Ring

$=(a+b)(a+b)=(a+b)^2$

4.5.4. Boolean Function.

Boolean Expression.

Let $(B,+,\cdot,',0,1)$ be a Boolean Algebra. Let $x_1, x_2, \cdots, x_n$ be n variables on B, i.e. they take values from B. An expression built up from these x_i using the operations $+$, $\cdot$ and $'$ is called a Boolean Expression. For example $E = (xz'y + xy')' + (x' + zy')'$ is a Boolean expression of x, y, z which are variables on B. Here the operation $(\cdot)$ is kept understood.

Boolean Function.

Let $(B,+,\cdot,',0,1)$ be a Boolean Algebra. A mapping $f : B^n \to B$ is called a Boolean function of n number of variables.

Illustration.

Consider the Boolean Algebra $B = \{0,1\}$ as cited in an earlier example.

Let $f(x,y) = x'y + y$ where x,y are variables on B. Then this f is a Boolean function of two variables.

If $x = 0, y = 1$ then $f(0,1) = 0'1 + 1 = 11 + 1 = 1 + 1 = 1$ etc.

Note. *The value of a Boolean function of some variable is a Boolean expression.*

4.5.5. Simplification of a Boolean Expression Literal.

A single variable or its complement is called a literal. For example x, x', y, y' are literal of x, y etc.

Conjunctive Normal Form (Product of Sum)

A Boolean expression E is said to be in conjunctive normal form (*CNF*) of r variables $x_1, x_2, \cdots, x_r$ if the expression, E can be expressed as $E = E_1 E_2 \cdots E_n$ where each E_i = sum of a finite number of literals of each of $x_1, x_2, \cdots, x_n$ and $E_i \neq E_j$ for all distinct i, j.

Illustration.

(i) $E = (x' + z + y)(x' + y' + z')(x + y + z)$ is a *CNF* of the variables x, y and z.

Here $E_1 = x' + z + y$, $E_2 = x' + y' + z'$, $E_3 = x + y + z$.

But $E = (x' + z)(x' + y' + z')(x + y + z)$ is not so as the expression $x' + z$ does not contain the literal of y.

(ii) The expression $E = x + x'y$ is not a *CNF*.

(iii) Express the Boolean expression $(x + y + z)(xy + x'z)'$ in *CNF*

$$(x + y + z)(xy + x'z)' = (x + y + z)\ \{(xy)'(x'z)'\}$$

$$= (x + y + z)\{(x' + y')((x')' + z')\}$$

$$= (x + y + z)(x' + y')(x + z')$$

$$= (x + y + z)\{(x' + y') + 0\}\{(x + z') + 0\}$$

$$= (x + y + z)\{(x' + y' + zz')(x + z' + yy')\}$$

$$= (x + y + z)\{(x' + y' + z)(x' + y' + z')(x + z' + y)(x + z' + y')\}$$

by distributive law

(iv) Express the Boolean function $(x' + y')(x + y)$ in *CNF* in the variables x, y and z

$$(x' + y')(x + y) = \{(x' + y') + zz'\}\{(x + y) + zz'\}$$

$= (x' + y' + z)(x' + y' + z')(x + y + z)(x + y + z')$ by distributive law

(v) Express $ab + a'b'$ in *CNF* of the variables a, b

$ab + a'b' = (ab + a')(ab + b')$ [By distributive law

$$= (a' + ab)(b' + ab)$$

$$= (a' + a)(a' + b)(b' + a)(b' + b)$$

$= 1(a' + b)(b' + a)1 = (a' + b)(b' + a)$ which is the required *CNF*.

Fundamental Product.

A literal or product of two or more literals in which no two literals involve the same variable is called Fundamental Product.

For example $z'x, x, x'yz, y'$ are all fundamental product. But $xyx'z$ and $yxyz$ are not fundamental product.

Minterm.

A Fundamental Product which involves all variables is called a Minterm. For example, for these variables x, y, z the product $x'yz$ is a minterm but xy' is not a minterm.

Inclusion

Let P and Q be two fundamental products. P is said to be contained in or included in Q if every literal of P is also a literal of Q.

For example $x'z$ is contained in $x'yz$ but $x'z$ is not contained in $xy'z$.

Theorem. If a fundamental product P is contained in another fundamental product Q then $P+Q=P$.

Proof. Since P is contained in Q we can express $Q=P\cdot P_1$ where P_1 is another product.

Then $P+Q=P+P\cdot P_1=P$ by Absorption law.

Illustration.

(i) $xz'+z'yx=xz'+xz'y$ by commutative property
$=(xz')+(xz')y=xz'$.

(ii) $xyz'yx$ is not a fundamental product. But it can be expressed into a fundamental product as follow :

$xyz'yx=xyyz'x=xyz'$ (by idempotent law)

which is a fundamental product. Note that it is a min-term also.

Sum-of-Product (Disjunctive Normal Form)

A fundamental product or the sum of two or more fundamental products none of which is contained in another is called a Sum-of-Product.

Illustration.

(i) $E=xz'+y'xz+x'y'$ is a Sum-of-Product i.e. DNF

(ii) $E_1=yz'+xy'z+z'xy$ is not a Sum-of-Product since the fundamental product yz' is contained in $z'xy=yz'x$. Of course we can express $E_1=yz'+yz'x+xy'z=yz'+xy'z$.

(iii) We can express any Boolean Expression e.g. $E=\{(xy)'z\}'\ \{(x'+z)(y'+z')\}'$ into a Sum-of-Product form. This is shown below :

$$E=\{(xy)'z\}'\{(x'+z)(y'+z')\}'$$

$$=\left\{((xy)')'+z'\right\}\{(x'+z)'+(y'+z')'\}$$ removing complement on paranthesis first.

$$=\left\{(xy)+z'\right\}\left\{(x')'z'+(y')'(z')'\right\}=\left\{(xy)+z'\right\}\left\{xz'+yz\right\}$$

$$=\left\{(x+z')(y+z')\right\}\left\{xz'+yz\right\}$$ using distributive property where ever possible

$$=(xy+xz'+z'y+z'z')(xz'+yz)=xyxz'+xyyz+z'xz'+z'yz$$

$$=xxyz'+xzyy+xz'z'+yzz'=xyz'+xzy+xz'+y0$$

$$=xyz'+xyz+xz'+0=xy(z'+z)+xz'$$

$$=xy\,1+xz'=xy+xz'$$, which is a Sum-of-Product.

[W.B.U.T 2007]

Complete Sum-of-Product (Full Disjunctive Normal Form) and Min-term

Let E be a Boolean Expression of n variables $x_1, x_2, \cdots, x_n$. If E is a Sum-of-Product where each product P involves all the n-variables then E is called a Complete- Sum-of-Product.

Thus every product in a complete Sum-of-Product is Min-term.

For example $E(x,y,z)=xy'z+yzx+x'y'z'$ is a Complete Sum-of-Product of the three variables x, y and z. Here $xy'z$, yzx are minterms of E.

Theorem. Every non-zero Boolean expression can be expressed as a complete Sum-of-Product uniquely.

Proof. Beyond the scope of the text.

Illustrative Examples :

Ex. 1. *Express the Boolean expression $z(x'y)'$ in a complete Sum-of-Product form.*

Note that the given expression is not a Sum-of-Product.

Now, $z(x'y)' = z((x')' + y') = z(x + y')$

$= zx + zy'$, this is not a comple sum of product but it is a Sum-of-Product.

$= zx1 + zy'1$

$= zx(y + y') + zy'(x + x')$

$= zxy + zxy' + zy'x + zy'x'$

$= zxy + zy'x + zy'x + zy'x'$

$= zxy + zy'x + zy'x'$ which is a complete-Sum-of Product.

Ex. 2. Express $E = y' + z(x' + y)$ as a Full disjunctive normal form.

$E = y' + zx' + zy = y'11 + zx'1 + zy1 = y'(x + x')(z + z') + zx'(y + y') + zy(x + x')$

$= xyz + xy'z + xy'z' + x'yz + x'y'z + x'y'z'$.

Ex. 3. If A, B, C are subsets of the set S then express $(A^c \cup C)^c \cap (B \cap C)^c$ as union of intersection.

Since the set of all subsets of S is Boolean Algebra under the operation $+ \equiv \cup$. and $\cdot \equiv \cap$ so we use all the properties of Boolean Algebra in this expression.

For convenience we replace $\cup$ by $+$ and $\cap$ by . and A^c by A'.

$(A^c \cup C)^c \cap (B \cap C)^c = (A' + C)' \cdot (B \cdot C)' = \{(A')' \cdot C'\} \cdot \{B' + C'\}$ by D'Morgans law

$= (A \cdot C') \cdot (B' + C') = A \cdot B' + A \cdot C' + C' \cdot B' + C' \cdot C'$

$= A \cdot C' \cdot B' + A \cdot C' \cdot C' = A \cdot C' \cdot B' + A \cdot C'$

$= (A \cap C^c \cap B^c) \cup (A \cap C^c)$

Ex. 4. *Reduce the expression* $E = xy + x'z' + x'y' + z'y$ *to a simplified form.*

We express each product into a complete Sum-of-Product:

$xy = xy(z + z') = xyz + xyz'$

$x'z' = x'z'(y + y') = x'z'y + x'z'y'$

$x'y' = x'y'(z + z') = x'y'z + x'y'z'$

$z'y = z'y(x + x') = z'yx + z'yx'$

We see the summands of $x'z'$ are $x'z'y$ and $x'z'y'$ which appear among the summands of $z'y$ and $x'y'$ respectively.

So we delete $x'z'$ and get $E = xy + x'y' + z'y$ ... (1)

We see there are no such Prime Implicants whose all summands appear among others.

So (1) is the required minimal form of the Boolean expression.

Ex. 5. *Simplify the Boolean expression* $x'y(y + z) + yz(y' + x')$

$x'y(y + z) + yz(y' + x') = x'yy + x'yz + yzy' + yzx'$

$= x'y + x'yz + z0 + x'yz = x'y + x'yz + x'yz + 0$

$= x'y + x'yz = x'y$ (by absorption law)

Ex. 6. *Express the expression* $x'yz + x'y'z' + x'y + xy'z'$ *into the simplified form.*

$x'yz + x'y'z' + x'y + xy'z'$

$= x'yz + x'y + x'y'z' + xy'z'$

$= x'y + (x' + x)y'z'$ (by absorption and distributive law)

$= x'y + 1y'z' = x'y + y'z'$.

Note. *In the last article of this chapter we shall show another process called KARNAUGH MAP process to find the minimal expression of a Boolean function.*

4.5.6. Truth Table of a Boolean Function

Let $B = \{0, 1\}$ be a Boolean Algebra w.r.t the operations +, . and ' defined as

+	1	0
1	1	1
0	1	0

.	1	0
1	1	0
0	0	0

$1' = 0$

$0' = 1$

(Earlier we have shown B is a Boolean Algebra).

Let $f(x_1, x_2, \cdots, x_n)$ be a Boolean function of the n variables $x_1, x_2, \cdots, x_n \in B = \{0, 1\}$.

The table which shows all the possible values attained by f corresponding to all possible values taken by the variables is known as Truth Table of the Boolean function f.

Illustration.

(i) Let $f(x, y) = x' + y'$ be a Boolean function on $B = \{0, 1\}$. Then the truth table of f is

x	y	x'	y'	$f(=x'+y')$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

We write $T(f) = 1110$

OR

$T(0011,\ 0101) = 1110$.

(ii) Construct the truth table of the Boolean function

$$f(x, y, z) = (yz + xz')(xy' + z)'.$$

First we find the minimal expression of $f(x, y, z)$.

$$f(x, y, z) = (yz + xz')\{(xy')'z'\}$$

$$= (yz + xz')((x' + (y')')z') = (yz + xz')((x' + y)z')$$

$$= (yz + xz')(x'z' + yz')$$

$$= yzx'z' + yzyz' + xz'x'z' + xz'yz' \quad \text{(This is not Sum-of-Product)}$$

$$= yx'0 + y0 + 0z' + xyz' = 0 + 0 + 0 + xyz' = xyz'.$$

This is complete sum-of-product and can not be reduced further.

So, $f(x, y, z) = xyz'$ is the minimal form.

Its Truth Table is

x	y	z	z'	xy	xyz'
0	1	1	0	0	0
0	0	1	0	0	0
0	1	0	1	0	0
0	0	0	1	0	0
1	0	1	0	0	0
1	0	0	1	0	0
1	1	1	0	1	0
1	1	0	1	1	1

We write $T(f) = 00000001$

or,, $T = (00001111, 10100011, 11001010) = 00000001$

(iii) Using Truth table show that $xy' + xy + x'y = x + y$

Let the variable x, y take values on the Boolean Algebra $\{0, 1\}$ w.r.t the binary operation as mentioned earlier. If the Truth table of the two Boolean functions on *LHS* and *RHS* are same then we can claim the two sides are equal :

Truth Table of $(xy' + xy + x'y)$

x	y	x'	y'	xy'	xy	$x'y$	$xy'+xy$	$xy'+xy+x'y$
1	0	0	1	1	0	0	1	1
1	1	0	0	0	1	0	1	1
0	0	1	1	0	0	0	0	0
0	1	1	0	0	0	1	0	1

Truth Tabel of $x + y$

x	y	$x+y$
1	0	1
1	1	1
0	0	0
0	1	1

we see the Truth table of the two functions are same.

So $xy' + xy + x'y = x + y$

Construction of Boolean function from Truth Table.

This is shown by the following example :

Construct the Boolean function and simplify it given the following table:

x	y	z	$f(x,y,z)$
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

or, $T(11110000, 11001100, 10101010) = 01100110$.

f gets values 1 at 2nd, 3rd, and 6th, 7th row. The terms corresponding to these rows are $xyz', xy'z, x'yz'$ and $x'y'z$ respectively. The terms corresponding to other rows are 0.

So the Boolean expression is

$xyz' + xy'z + x'yz' + x'y'z$

$= (x + x')yz' + (x + x')y'z = 1yz' + 1\,y'z \quad = yz' + y'z$

Which is a simplified form.

∴ the required Boolean function is $f(x,y,z) = yz' + y'z$.

Exersice

I. Short Answer Questions

1. Prove that the set of divisors of 70 is a Boolean Algebra w.r.t the binary operation + and . and unary operation ', where $a+b = lcm$ of a,b, and $a \cdot b = HCF$ a, b and $a' = 70/a$.

2. Prove that the set of divisors of (i) 55 (ii) 20 Boolean algebra (operations to be introduced by you)

3. D_{110} = set of divisors of 110. Show its diagram as a Boolean algebra.

4. Prove that the power set of a set S is a Boolean Algebra w.r.t the operation $\cap$ and $\cup$.

5. Construct a Boolean Algebra with only two elements

6. Prove that in a Boolean Algebra $(B, \vee, \wedge)$ $a \vee a = a$ and $a \wedge a = a$

7. In a Boolean Algebra $(B, +, ' 01)$ prove that $a + 1 = 1$. Write the dual of this relation

8. Prove that in a Boolean Algebra the two unit elements viz 0 and 1 are unique.

9. State and prove the 'Absorption law' in a Boolean Algebra

10. State D' Morgans law in a Boolean Algebra. Prove one of the two.

11. In a Boolean algebra under + and., prove that

(i) $a' + b = 1 \Rightarrow a + b = b$.

(ii) $a.b' = 0 \Rightarrow a + b = b$

12. Define fundamental product and min-term of three variables in a Boolean algebra. Give an example of a fundamental product which is not a min-term

13. In a Boolean Algebra prove that

(i) $B + C = C + BC'$ (ii) $AB + CA = (C + A')(B + A)$

(iii) $(x' + y)\cdot(x + y) = y$ (iv) $xy + xyz + x'y + xy'z = y + zx$

(v) $b(a + c)(c' + a') + (b + c)d = (c' + abd + a'b + b'd)(c + ab + bd)$

(vi) $(f + g)'h + ghf' = hf'$

(vii) $(a + b + c)\cdot(abc)' = a \cdot b' + b \cdot c' + c \cdot a'$

14. Write the dual of the following relations in Boolean Algebra

(i) $a + a' \cdot b = a + b$ (ii) $(a \cdot 1) \cdot (0 + a') = 0$

(iii) $(a+1)(a+0) = a$ (iv) $(a+b)(b+c) = ac + b$

(v) $x'y'z + x'yz'$ (vi) $x'z + xy(y + y'z)$

(vii) $(yz + y'z')x$ (viii) $(a+b) \cdot (a+1) = a + a \cdot b + b$

(ix) $A \cdot B' = 0$ iff $A \cdot B = A$

[Hint : Replace + by . , . by +, 0 by 1 and 1 by 0 and get the result]

15. By applying D'Morgans's law find the complement of

(i) $(x + x'y)$ (ii) $C(A + B')$ (iii) $(y + u') \cdot (x' + z)$

(iv) $y'z + x'$ (v) $(A + B) \cdot (A' + B)$

16. Find the complement of the expression $xy' + xz + y'z$.

17. Simplify the following Boolean expression (Just by using properties of Boolean Algebra)

(i) $(x+y)(x+z) + xyz$ (ii) $a'bc + a'bc' + abc' + abc$

(iii) $(x + y')(y + z')(y' + x')(x' + z)$

(iv) $y'x' + y'x + yx$ [W.B.U.T. 2007] (v) $a' + (ab' + ac)'$

(vi) $xy' + xy + x'y$ (vii) $C(B + C)(A + B + C)$

(viii) $A + B(B + A) + (B + A')A$ (ix) $x + x'yz' + (y+z)'$

(x) $x[y + z(xy + xz)']$ (xi) $(AB' + C) \cdot (A + B')C$

(xii) $x'yz' + xyz' + xy'z' + x'y'z'$.

18. Express the set expression $(A \cup B)^C \cap (C^C \cup B)$ as union of intersection with the help of the Boolean Algebra.

19. For any Boolean Algebra B, Prove that

(i) $(a+b)(b+c)(c+a) = ab + bc + ca$ for all a, b, c, in B

[Hint : $LHS = (ab + ac + bb + bc)(c + a) = (ab + ac + b + bc)(c + a)$

$= abc + ab + ac + ac + bc + ba + bc + abc = abc + ab + bc + ca$

$= ab + bc + ca$ ($\because$ ab is contained in abc)]

[W.B.U.T.2001]

(ii) $(xy' + xz') + x' = (x' + y + z)(x' + y + z')(x' + y' + z')$

[W.B.U. T. 2002, 2005]

(iii) $(x+y)(x'+z)(y+z) = xz + x'y + yz = (x+y)(x'+z)$

20. Express the following expression in CNF in the variables present in the expression :

(i) $yz + x'$ (ii) $(a' + b')c + (a + c)(a' + c')$

(iii) $\{(y + z')' + x\}(z' + y + x')$ (iv) $\{(a' + b) + (a' + b)'\}'$

(v) $x' + (xz + xy')'$ (vi) $(a' + b')(a + b)(a + c)$

21. Reduce the following expression to either 0 or a fundamental product. Are the reduced form Min term ?

(i) $x\,y\,z'\,y'\,s\,t\,s$ (vi) $t\,y'\,t\,z'\,y\,x$

22. Reduce the following expression into Disjunctive Normal form (Sum-of-Product) and then to complete sum of product.

(i) $(xy' + y'z + x'y)x$ (ii) $x'y + (x' + y)'$

(iii) $(xy + x'y + y')x$ (iv) $(z' + y)(y'z + x)$

(v) $(yx' + y'x + y'z)x$ (vi) $(yx')'(x' + xyz')$

(vii) $x_1 x_2' + x_3$

(viii) $(x + y')(y + z')(z + x')(x' + y')$ [W.B.U. Tech 2005]

(ix) $\{(xy')'z\}'$ [W.B.U. Tech 2006]

23. Find the disjunctive normal form (Sum-of-Product) for the Boolean expression

(i) $g(w,x,y,z) = wx\bar{y} + wy\bar{z} + xy$

(ii) $(x+y+z)(xy+x'z)'$ [W.B.U. Tech B 2003]

24. Transform the following CNF into DNF :

(i) $(x+y')(y+x')$

(ii) $(x'+y'+z')(y'+x'+z)(x'+y+z)\ (x+y'+z)(x+y'+z')(x+y+z')$

25. Find the complements of the following Boolean function

(i) $x'y'z' + xy'z + zyx' + xyz$ (ii) $y'x + yx'$

(iii) $x(y'z' + yz)$ (iv) $ab' + ac + b'c$

26. Find $T(f)$, the truth table of the following Boolean function :

(i) $f(x,y,z) = x'yz + xyz'$ (ii) $f = yx' + zx$ (iii) $f = x(y+x')$

27. Find the Boolean function from the following Truth Table :

(i) $T(00001111, 00110011, 01010101) = 00101111$

(ii) $T(00001111, 00110011, 01010101) = 10100110$

(iii) $T(11110000, 11001100, 10101010) = 11110010$

28. Using Truth table show that $(x+y)\cdot(x+z) = x + (y\cdot z)$.

[Hint : Show that Truth Value of LHS and RHS are same]

29. Using truth table verify

(i) idempotent law (ii) D'Morgan's law (iii) $x(x+y)(x+xy) = x$

(iv) $xy' + z + (x'+y)z' = 1$ (v) $yz + x'yz' + xy = y$

30. Find the Boolean expression in CNF from the following truth table:

x	y	$f(x,y)$
0	0	1
0	1	0
1	0	0
1	1	1

31. Show that the set $S = \{[0],[3]\}$ is a Boolean Ring with respect $+$ and $\cdot$ defined by $[a]+[b] = [a+b]$ and $[a]\cdot[b] = [ab]$ where $[x] = \{a \in z : x \equiv a (\text{mod}\, 6)\}$.

32. Test whether the Power set of a non null set S is a Boolean Ring w.r.t the compositions $\cup, \cap$ and $'$ (complement).

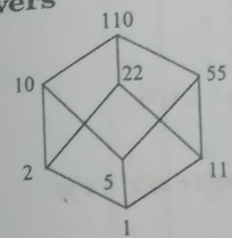
[Hint : In fact this is not a Ring at all]

33. In a Boolean Ring prove that every element is its own inverse under addition $+\cdot$.

[Hint : Follows from a theorem]

Answers

3.



7. $a \cdot 0 = 0$

14. (i) $a\cdot(a'+b) = a\cdot b$ (ii) $(a+0)+(1\cdot a') = 1$

(iii) $a\cdot 0 + a\cdot 1 = a$ (iv) $ab + bc = (a+c)b$

(v) $(x'+y'+z)(x'+y+z')$ (vi) $(x'+z)\{x+y+y(y'+z)\}$

(vii) $(y+z)(y'+z') + x$ (viii) $a\cdot b + a\cdot 0 = a\cdot(a+b)\cdot b$

(ix) $A + B' = 1$ iff $A + B = A$

15. (i) $x'y'$ (ii) $C' + A'B$ (iii) $xz' + y'u$

(iv) $x(y+z')$ (v) $A\cdot B + A\cdot B' + A'\cdot B'$

16. $(x'+y).(x+z).(y+z')$

17. (i) $x+yz$ (ii) b (iii) $x'y'z'$ (iv) $xy+y'$ (v) $a'+bc'$ (vi) $x+y$ (vii) C (viii) $A+B$ (ix) $x+z'$ (x) xy (xi) $AC+B'C$ (xii) z'

18. $A^c \cap B^c \cap C^c$.

20. (i) $(x'+y+z)(x'+y+z')(x'+y'+z)$

(ii) $(a+b+c)(a+b'+c)(a'+b'+c')$

(iii) $(x+y'+z)(x+y'+z')(x+y+z)(x'+y+z')$

(iv) $(a+b)(a+b')(a'+b)(a'+b')$

(v) $(x'+y+z)(x'+y+z')(x'+y'+z')$

(vi) $(a'+b'+c)(a+b+c)\ (a+b+c')(a+b'+c)$

21. (i) 0 (ii) 0

22. (i) $xy'z'+xy'z$ (ii) $x'yz+x'yz'+xy'z+xy'z'$

(iii) $xyz+xyz'+xy'z+xy'z'$ (iv) $xy+xz'=xyz+xyz'+xy'z'$

(v) $xy'+xy'z=xy'z'+xy'z$

(vi) $xyz'+x'y'=xyz'+x'y'z+x'y'z'$

(vii) $x_1x_2'x_3+x_1x_2'x_3'+x_1x_2x_3+x_1'x_2x_3+x_1'x_2x_3'$

(viii) $x'y'z'$ (ix) $xy+z'$

23. (i) $xy'z+xy'z'+x'y'z+x'yz$ (ii) $xy'z+xy'z'+x'yz'$

24. (i) $x'y'+xy$ (ii) $x'y'z'+xy'z$

25. (i) $x'y'z+x'yz'+xy'z'+xyz'$ (ii) $x'y'+xy$

(iii) $x'+(y+z)(y'+z')$ (iv) $(a'+b)(a+c)(b+c')$

26. (i) 00001010 (ii) 00110101 (iii) 1000

27. (i) $xz+y$, (ii) $yz'+x'z'+xy'z$, (iii) $xy+y'(x+z)$

30. $(x'+y)(x+y')$ **32.** no.

II. Long Question Answers

1. Prove that the set of divisors of m is a Boolean Algebra with respect to the composition $+$, . and ', where $a+b=LCM$ of (a, b), $a\cdot b=HCF$ of (a, b) and $a'=\frac{m}{a}$, where m is a product of distinct prime integers.

[Hint : Show the set is bounded, distributive complemented lattice]

2. Prove that the set of divisors of (i) 99 are not Boolean algebra (ii) 130 are Boolean algebra (operations to be introduced by you)

3. Express the expression $(a+b)(a+b')(a'+c)$ in the smallest possible number of variables.

4. Find $T(f)$, the truth table of the following Boolean function :

(i) $f(x,y,z)=zy'x+zx'y'+zyx$ (ii) $f(x,y,z)=(yz+z')x'+(x'y)'yz'$

[W.B.U.T. 2005, 2008]

(iii) $f=z'yx+xy'+y$

(iv) $f=y'xz'+y'z'x'+yz'x$ (v) $f=z'x'y+y'x'z$

(vi) $f=y(x'+z)+y'x$

5. Find the Disjunctive Normal form of the Boolean function having truth table :

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

6. Find the Boolean expression in CNF from the following truth table:

(i)

x	y	z	f
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

(ii) $T(11110000, 11001100, 10101010)=11101001$

Answers

3. ac

4. (i) 01000101 (ii) 10101010

   (iii) 00111111 (iv) 10001010 (v) 01000000 (vi) 10111100

5. $xyz + xyz' + xy'z + x'yz + x'yz'$. 6. (i) $(x' + y' + z')(x + y + z)$

III. Multiple Choice Questions

1. Number of operations required in a Boolean Algebra is
   (a) 1 (b) 2
   (c) 3 (d) 4

2. Number of Unary operation required in a Boolean Algebra is
   (a) 1 (b) 2
   (c) 3 (d) none