5.5 PLANAR & DUAL GRAPHS

5.5.1. Introduction

In this chapter we shall study when a graph can be drawn on a plane such that its edges do not cross over. This has great significance from a theoretical point of view. These are also useful in many practical situations.

5.5.2. Planar Graphs

A graph is called *plane graph* if it can be drawn on a plane in such a way that any two of its edges either meet only at their end vertices or do not meet at all.

A graph which is isomorphic to a plane graph is called planar graph.

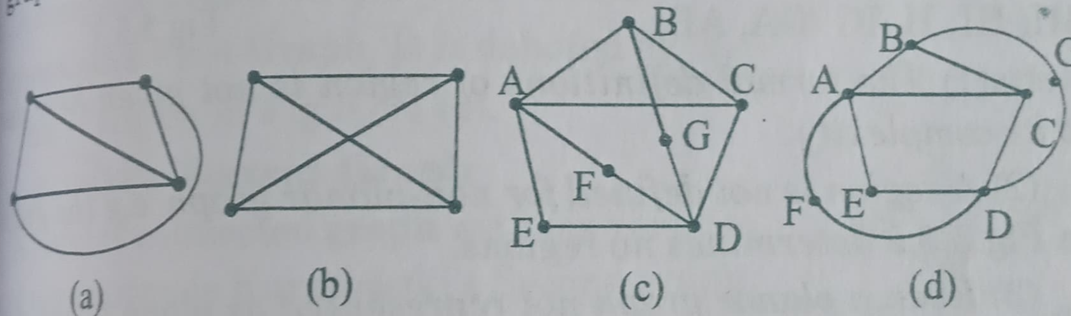


Fig 5.5.1

In Fig 5.5.1 (a) the graph is a plane graph. In Fig 5.5.1 (b) though two edges intersect it is isomorphic to the plane graph shown in Fig 5.5.1 (a). So the graph in Fig 5.5.1 (b) is a planar graph. Similarly the graph shown in Fig 5.5.1 (c) is isomorphic to the graph shown in Fig 5.5.1 (d) which is a plane graph. So the graph in Fig 5.5.1 (c) is also a planar graph. Note that the graphs shown in Fig 5.5.1 (a) and (d) are obviously planar graph.

Nonplanar Graph

A graph which is not planar is called nonplanar graph.

The graph shown in Fig 5.5.2 can not be redrawn without crossing over the edges. So this graph is a non planar graph.

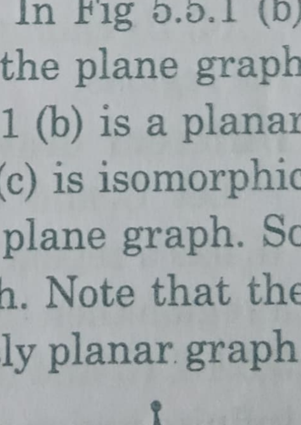


Fig 5.5.2

Note. *To declare a graph as a planar we must see whether it can be drawn on a plane without a cross over between its edges (as we see in Fig 5.5.1).*

If it is not possible to do so then it is nonplanar.

Region (or, Face) of a Planar Graph

A plane graph divides the plane into regions. In Fig 5.5.3. the regions of the graph are shown by the numbers 1,2,3,........ e.g Region 2 is determined (or, bounded) by the edges AB, BH, HF, FG and GA. We say AB, BH, HF, FG and GA are the boundaries of this region. Region 6 is called infinite region. Its boundaries are the edges BH, HI, II, IG, GA, AB,

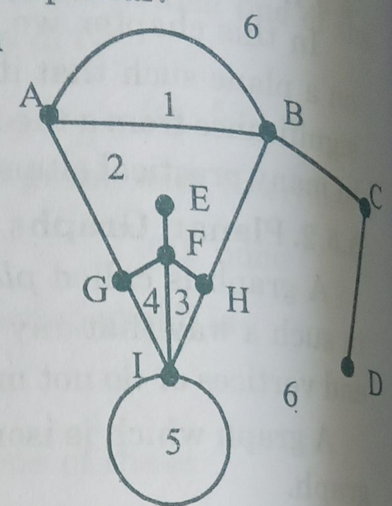


Fig 5.5.3

Note.(1) *The formal definitions of region is not given here due to its complexity.*

(2) *A region is not defined for non-planar graph. e.g the graph in Fig 5.5.2 determines no regions.*

(3) *Even a planar graph not represented as plane graph does not determine region.e.g the graph in Fig 5.5.1 (b) does not determine regions unless it is represented in Fig 5.5.1 (a).*

(4) *By changing the drawing of a planar graph we can change the regions.*

Different drawing of a Planar Graph.

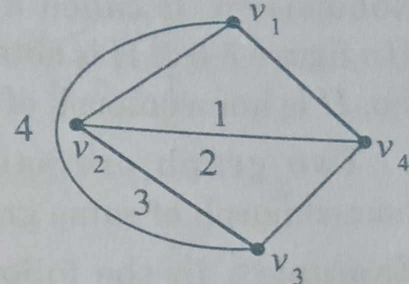
Let G_1 and G_2 be two drawings of a same planar graph. If G_1 has a region with k number of boundaries and G_2 has no such a region then the drawings G_1 and G_2 are called different. In Fig 5.5.13, (i) and (ii) are different drawings of a graph because the infinite region of (i) has 4 boundaries where as the infinite region of (ii) has 5 boundaries.

Illustrative Examples

Draw a complete graph of four vertices to show it is a planar graph. Show the regions determined by the graph.

Let v_1, v_2 , v_3 and v_4 be the four vertices.

We first form the quadrilateral joining them consecutively. Join $v_2 v_4$ by a line segment lying inside the quadrilateral. We do not join v_1 v_3 through the inside of the quadrilateral because it would cross over the edge $v_2 v_4$.



We insert the edge $v_1 v_3$ lying outside the quadrilateral. Thus the graph is planar.

The regions are shown by 1, 2, 3 and 4. Region 4 is infinite region.

5.5.3 Kuratowski's Graph

Kuratowski's First Graph

A complete graph with five vertices is Kuratowski's First Graph. It is denoted by K_5. This is shown in Fig 5.5.4 (a).

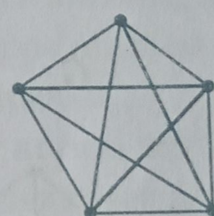


Fig 5.5.4(a)

Kuratowski's Second Graph

A regular connected graph with six vertices and nine edges is Kuratowski's Second graph. This is shown in Fig 5.5.4 (b).

Note that it is nothing but the complete bipartite-graph $K_{3,3}$

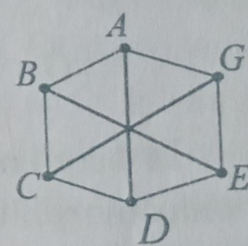


Fig 5.5.4 (b)

Homeomorphic Graph

Let e be an edge joining the two vertices u and v in a graph G. Let a new graph H be formed by deleting the edge e and introducing a new vertex w and two new edges, one joining u and w and the other joining v and w. This operation of replacement of an edge by two edges and a new vertex is called **edge subdivision.**

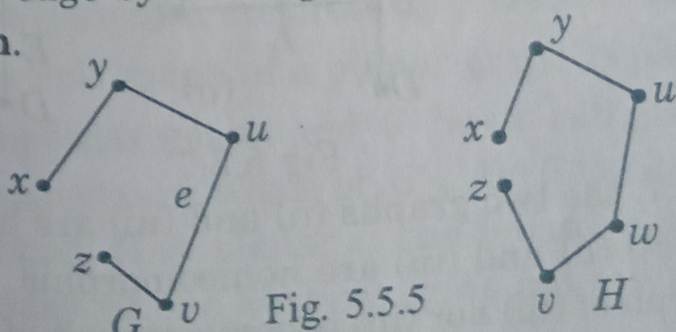


Fig. 5.5.5

A graph, obtained from a graph G by a sequance of edge subdivision, is called a **Homeomorph** (or a **subdivision**) of G. (In figure 5.5.5. H is obtained from G by only one edge subdivision. So, H is homeomorph of G)

Two graph are said to be **Homeomorphic** if each is a homeomorph of same graph.

Example . In the following figure we show two sequences of edge-subdivision on a graph G.

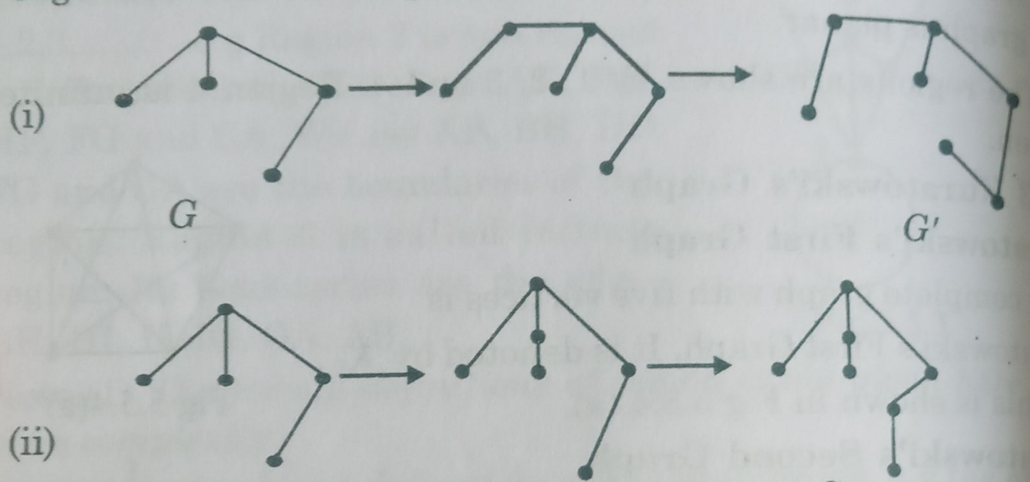


Fig 5.5.6

In sequence (i) G' is homeomorph of G and in (ii) G'' is homeomorph of same G. Therefore the two graph G' and G'' in Fig 5.5.6. are homeomorphic.

Note. If two graphs G_1 and G_2 are homeomorphic, it is not necerary that G_1 is homeomorph of G_2 or G_2 is homeomorph of G_1 This can be noticed from the following graph.

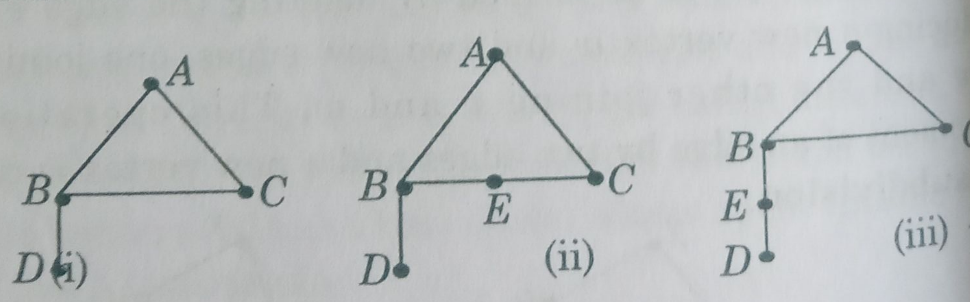


Fig 5.5.7

In Fig 5.5.7. the two graphs (ii) and (iii) are homeomorph of the graph (i). So (ii) and (iii) are homeomorphic. But neither (ii) is homeomorph of (iii) nor (iii) is homeomorph of (ii).

Kuratowski's Graph

Any homeomorph of kuratowski's First Graph or Second Graph is called kuratowski's Graph.

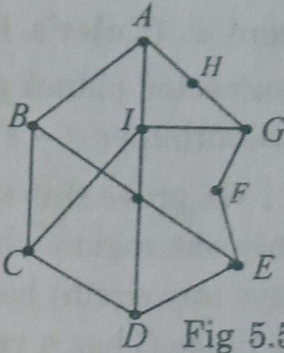


Fig 5.5.8(iv)

Exapmle. The graph in Fig 5.5.8(iv) is a kuratowski's graph :

5.5.4 Theorems on Planarity.

Theorem 1. If a graph H is obtained from the graph G by an edge subdivision then G is planar if and only if H is planar.

Proof : Let H be planar. Let a,b be two arbitrary edges of G. If neither of these two was not divided then they are also edges of H and so they either meet at end vertices or do not meet at all. If any of these two say b is divided into two edges a_1,b_1 in H then each of the pairs a,a_1 and a,b_1 either meet at end vertices or do not meet at all. Hence in G a,b will also either meet at end vertices or do not meet at all. So G is planar.

Conversely let G be planar; a,b be two arbitrary edges of H. If they were edges of G then the proof is obvious. Otherwise let any of them say a be a divided part of an edge e of G. e and b, being two edges of G, either meet at end vertices or do not meet at all. So a and b either meet at end vertices or do not meet. Hence H is planar.

Theorem 2. A graph is planar if and only if every homeomorph of it is planar.

Proof: Let G be a graph and H be a homeomorph of it. So we get H from G by a finite sequence of edge subdivisions: $G \to G_1 \to G_2 \to \cdots \to G_n = H$. G_1 is obtained from G by an edge subdivision. So by the previous theorem G_1 is planar if and only if G is planar. This complete the proof by method of induction.

Theorem 3. Every sub graph of a planar graph is planar.

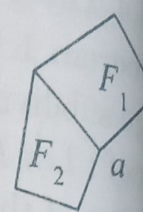
Proof: Let G be a planar graph and G' be its sub graph. Let a and b be any two arbitrary edges of G'. As $G' \subset G$ so a, b are two edges in G. Since G is planar a and b can be placed such that they either meet at end vertices or do not meet at all (i.e. they have no cross-over). So G' is planar.

Theorem 4. (Euler's Formula)

A connected planar graph G with n vertices and e number of edges determines $f = e - n + 2$ number of regions. [W.B.U.T 2012]

Proof : We prove this theorem using induction on f. If $f = 1$, G has only one region which is the only infinite region. So G can not have any circuit because a circuit bounded a region. So G is a tree. Since G has n vertices so G has $n-1$ edges,

i.e. $e = n - 1$.



Thus $e - n + 2 = (n-1) - n + 2 = 1 = f$.

So the theorem is proved for $f = 1$.

Now, let $f > 1$ and the theorem is true for all connected plane graphs having less than f regions. Since $f > 1$, G is not a tree. So G has at least one circuit. Let a be an edge of a circuit. Then $G - a$ is still connected. Now since G is planar so $G - a$ is also planar because $G - a \subset G$. Now due to removal of an edge 'a' two regions of G will be combined [shown in the figure as illustration] and so the number of regions of $G - a$ is $f - 1$ and the number of edges of $G - a$ is $e - 1$. Therefore by our hypothesis, we get for the graph $G - a$, $f - 1 = (e-1) - n + 2$

i.e. $f = e - n + 2$.

Hence by induction, the result is true for all connected planar graphs.

Theorem 5. (Generalised Euler's Formula)

A planar graph G with n vertices, e number of edges and k number of connected components determines $f = e - n + k + 1$ number of region. [W.B.U.T 2016]

Proof : Let the connected components of G be $G_1, G_2, \cdots G_k$. Let G_i has n_i number of verties, e_i number of edges and f_i number of regions (for $i = 1, 2 \cdots k$) Then by the previous theorem $f_i = e_i - n_i + 2$ for $i = 1, 2 \cdots k$. Now the exterior region is same for all components. If the exterior regions are not considered then 'Number of interior regions of each G_i ' $= f_i - 1$

Total number of interior regions of G_k

$$=(f_1-1)+(f_2-1)+\cdots+(f_k-1)=\sum_{i=1}^{k} f_i-k$$

$$=\sum_{i=1}^{k}(e_i-n_i+2)-k=\sum_{i=1}^{k} e_i-\sum_{i=1}^{k} n_i+\sum_{i=1}^{k} 2-k$$

$$=e-n+2k-k=e-n+k$$

So, the total number of regions (including the exterior region) of G is $f=e-n+k+1$

Theorem 6. Let G be a simple connected planar graph with n vertices, e edges and f regions (faces).

Then (i) $e \geq \frac{3}{2} f$ (ii) $e \leq 3n-6$. [W.B.U.T 2013]

Proof : If $n=3$ then G may have 2 or 3 edges. If G has 3 edges then G has 2 faces otherwise G has 1 regions. Thus if $e=3$, $f=2$ and if $e=2$ then $f=1$. In any case result (i) is true.

So we assume $n = 4$ or $n > 4$. If G is tree then $e=n-1$; and $f = 1$ as there is no circuit to enclose a finite region. In that case $e=n-1 \geq 4-1=\frac{3}{2}.2>\frac{3}{2}.1=\frac{3}{2} f$ proving the result (i).

If G is not a tree then it must contain a circuit and at least one circuit all of whose edges are boundary of the infinite region of G. Now since G has no loop or parallels so "number of boundary-edges of each region of G" ≥ 3". So "sum of the number of boundary-edges of each region" $\geq 3f$ ... (1)

In LHS of (1) each edge is counted either once or twice.

$\therefore$ LHS of (1) $\leq 2e$. So, from (1) we have $2e \geq 3f$ i.e. $e \geq \frac{3f}{2}$

(ii) Now, from a previous theorem, we have $f=e-n+2$. Then from result (i) we have $e \geq \dfrac{3(e-n+2)}{2}$

or, $2e \geq 3e-3n+6$ or, $e \leq 3n-6$.

Theorem 7. All the drawings of a connected planar graph having vertex connectivity 3 are same.

Proof : *Beyond the scope of the book.*

Theorem 8. A complete graph with 5 vertices is non planar. [OR, Kuratowski's First Graph K_5 is non-planar]

Proof : Let n = number of vertices and e = number of edges of the graph. Since this is a complete graph so applying the formula

"Sum of degrees of all vertices" $= 2e$ we get $4 \times n = 2e$

or, $4 \times 5 = 2e$ or, $e = 10$. Here $3n - 6 = 3 \times 5 - 6 = 9$.

This shows that $e > 3n - 6$. So this graph is non-planar (by Theorem 6-above)

Illustration

Re-Drawing Kuratowski's First Graph show that it is non-planar.

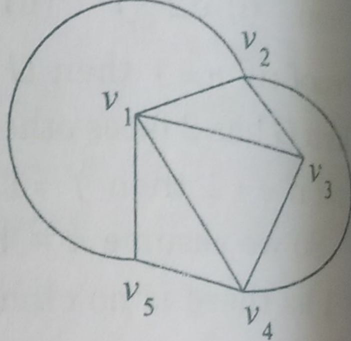


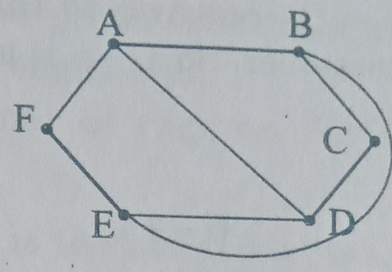
Fig 5.5.9

Proof : Let v_1, v_2, v_3, v_4 and v_5 be five vertices. Since the graph is complete each vertex to be joined with every vertex. So there must exists a pentagon joining v_1, v_2, v_3, v_4 and v_5 as its 5 vertices. This pentagon divides the plane (on which it is drawn) into two regions, 'inside' and 'outside'. Since there would be an edge connecting v_1 and v_3 (without intersecting any of the five sides of the pentagon) this edge may be 'inside' or 'outside' the polygon. Suppose we draw this edge $v_1 v_3$ inside the pentagon. Similarly draw the edge $v_1 v_4$ lying inside the pentagon without crossing over any of the edges previously drawn. This v_1 is joined to each of the remaining four vertices.

Next to connect v_2 with v_5 and v_4 we draw the edges $v_2 v_5$ and $v_2 v_4$ lying outside the pentagon such that they do not cross over the edges previously drawn. Thus v_2 is joined with each of the other four vertices.

Next to connect v_3 with v_5 we cannot place the edge $v_3 v_5$ inside or outside the pentagon without cross over. Thus the graph is not a planar graph.

Theorem 9. Kuratowski's Second Graph is non planar.



Proof : The six vertices of Kuratowski's Second Graph form the hexagon with the 6 out of 9 edges. The edge AD is kept inside the hexagon. The edge BE is kept outside such that it does not cross-over any edge. Now the edge FC can be drawn neither 'inside' nor 'outside' the hexagon without a cross-over. So this graph is not planar.

Corollary : Any Kuratowski's Graph is nonplanar.

Proof : Follows from Theorem 2 and Theorem 8 above.

Note. *In Kuratowski's Second Graph* $n=6$, $e=9$. *So* $3n-6=18-6=12$, *i.e.,* $e \le 3n-6$. *Yet this graph is non-planar. This shows that the condition* $e \le 3n-6$ *is not a sufficient condition for planarity.*

Theorem 10. Both the Kuratowski's graph are regular graph.

Proof : Obvious.

Theorem 11. Removal of one edge or vertex makes each of the two Kuratowski's graphs a planar graph.

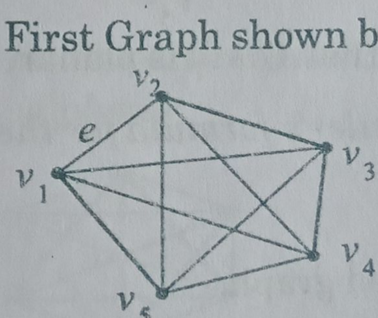
Proof of this theorem is omitted but the result is shown by the following examples.

Illustrative Examples

Ex 1. *Remove an edge of Kuratowski's First Graph and show it becomes planar.*

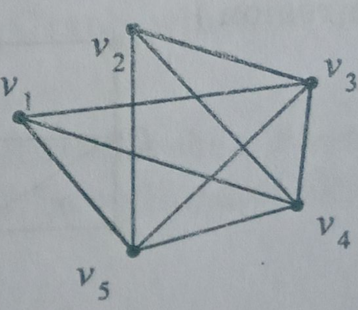
Consider the Kuratowski's First Graph shown below.

G :



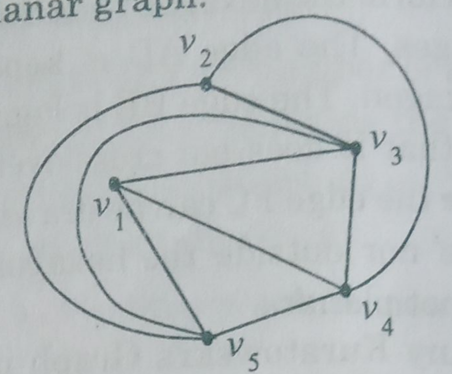
Remove the edge $v_1v_2 \equiv e$. Then the graph is drawn below :

$G-e$:



$G-e$ is redrawn so that edges are drawn without having any cross-over. So $G-e$ is a planar graph.

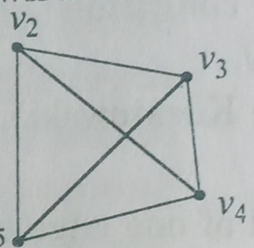
$G-e$:



Ex 2. *Remove a vertex of Kuratowski's First Graph and show it becomes a planar graph.*

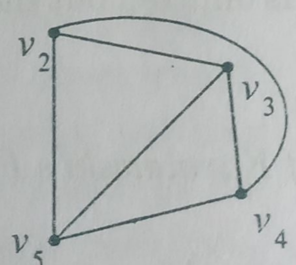
Remove the vertex v_1 from G (shown in the previous Ex.1). Then the graph $G-v_1$ is drawn below :

$G-v_1$:



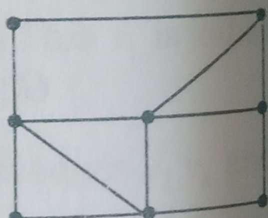
$G-v_1$ is redrawn below so that edges drawn without any cross-over.

$G-v_1$:



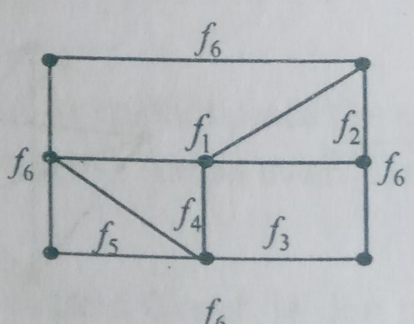
This shows that $G-v_1$ is planar.

Ex 3. *Verify Euler's formula for the graph*



It is a connected graph.

We draw the graph showing all the regions f_1, f_2, f_3, f_4, f_5 and f_6 (the exterior region.)



Therefore, Number of regions of the graph is 6. Now, the number of vertices, $n=8$ and number of edges, $e=12$.

So, $e-n+2=12-8+2=6 = f$, the number of regions. Hence Euler's formula is verified.

Theorem 12. Kuratowski's First Graph is the nonplanar graph with the smallest number of vertices.

Proof : Obviously any graph with four or lesser number of vertices is planar. Hence proved.

Theorem 13. Kuratowski's Second Graph is the nonplanar graph with the smallest number of edges.

Proof : Beyond the scope of the book.

But note that any graph with 8 or lesser number of edges are planar.

Theorem 14. (Kuratowski's Theorem)

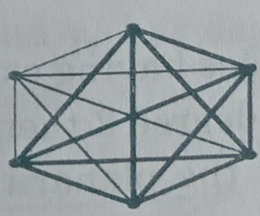
A graph G is planar if and only if G does not contain either of the Kuratowski's two graphs.

Proof : Let G be planar. Then by Theorem 1 every subgraph of G is planar. Since either of the Kuratowski's two graph is non-planar so they can not be contained by G.

The converse part is left as an exercise.

Illustrative Examples.

If K_n is a complete graph with n vertices then find all integral values of $n > 2$ for which K_n is planar.

K_6 : 

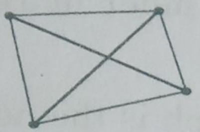
K_5 is Kuratowski's First Graph which is non-planar. We consider K_6 which is shown :

In this graph we see the Kuratowski's First Graph K_5 is contained (shown by bold lines). So by Kuratowski's Theorem K_6 is non-planar.

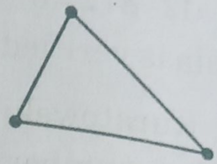
Similarly K_7, K_8 ..... are all non-planar.

Now K_3 is obviously planar K_4 is

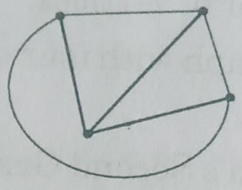
K_4 :



K_3 :



This can be redrawn as



So K_4 is planar. So the required values of n are 3 and 4.

5.5.5 Detection for Planarity

Given any graph G we reduce it to a simple form through the following simplifying steps. Then it becomes easy to detect whether the graph is planar or not.

Step 1. If G has several components, consider only one component at a time, [since G is planar if and only if each of its components is planar.]

Step 2. If any component (or, G itself) is separable it would have several blocks. Consider only one block at a time [$\because$ a component is planar if and only if each of its blocks is planar]

Step 3. Remove all self-loops from G since addition or deletion of self-loops does not affect planarity of G.

Step 4. Keep only one edge between any two vertices by removing all parallel edges between them, since this does not affect planarity of G.

Step 5. If G has two edges having exactly one vertex in common and if this vertex is of degree 2 then elimination of such vertex from G does not affect the planarity of G. So remove all such vertices from G.

Step 6. Repeat the above steps so long we can.

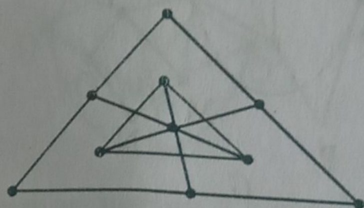
After going through the above steps a block (or component) of G would be look like (1) A single edge OR (2) A complete graph of four vertices OR (3) A graph with number of vertices ≥ 5 and number of edges ≥ 7.

If it is looked like (1) or (2) then obviously it would be planar. If it is looked like (3) then further detection for planarity is needed. In that case we take help of Theorem 1 to Theorem 11

Illustrative Examples

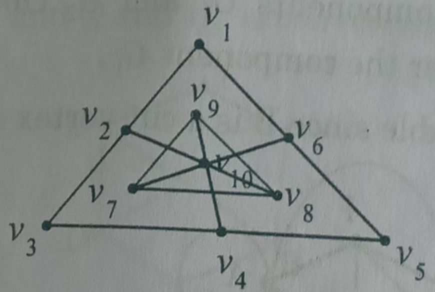
Ex 1. *Find whether the graph, is planar?*

$G:$

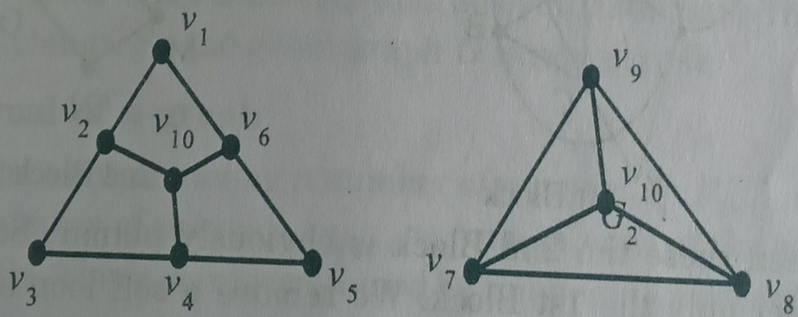


The given graph is

$G:$

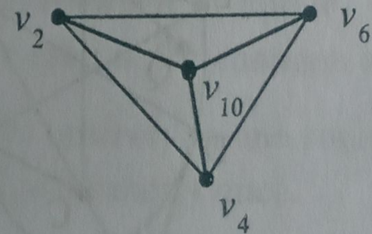


We see the vertex v_{10} in G is a cut vertex. So G is separable. Two blocks of G are



G_1

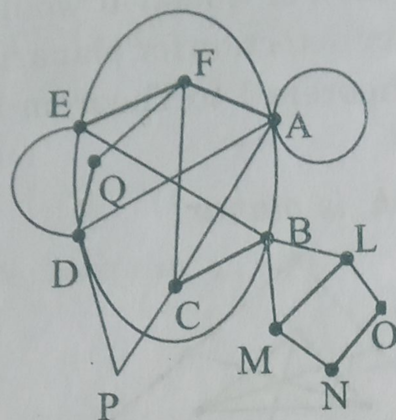
In G_1 the vertices v_1, v_3 and v_5 has degree 2 each. These are also common vertices of the edges $v_1 v_2$, $v_1 v_6$ etc. So in G_1 we remove the vertices v_1, v_3 and v_5 and G_1 is converted to



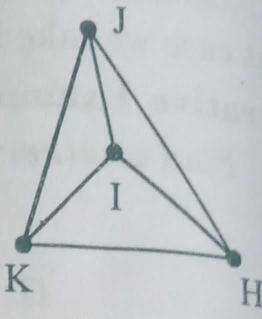
This graph is planar since every edge of this subgraph is drawn without having any cross-over. Similarly other block G_2 is also planar. Thus the given graph is a planar graph.

Ex. 2. *Find whether the following graph, G is planar:*

G :



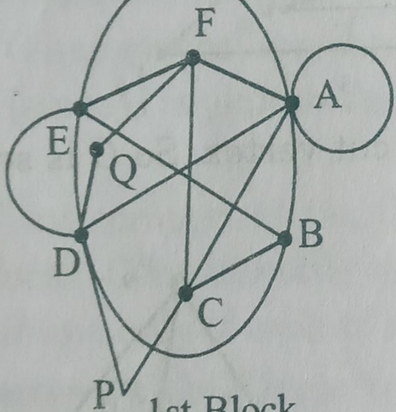
G_1



G_2

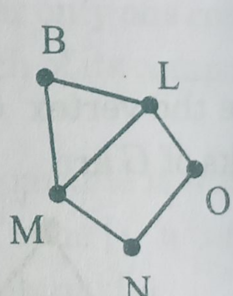
G has two components G_1 and G_2. Obviously G_2 is planar. Now we consider the component G_1.

G_1 is separable since B is a cut vertex of it. The two blocks of G_1 are



1st Block

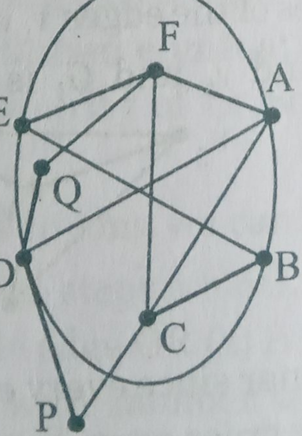
and



2nd Block

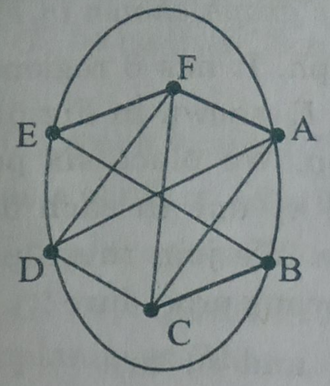
Between these the 2nd Block is obviously planar. So we need consider only the 1st Block. We remove a self loop at A and a parallel edge ED to convert the block into a simple connected subgraph,

G_3 :

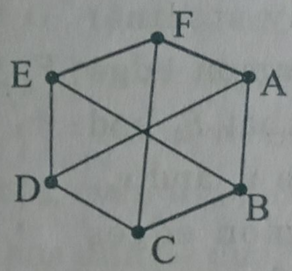


In G_3 each of the two vertices P and Q has degree 2; the edges

G_4 :



FQ and QD have one common vertex, Q; similarly for P. So we remove the vertices P and Q and G_3 is converted to G_4 contains the subgraph



which is Kuratowski's Second Graph. So by Kuratowski's Theorem G_4 is nonplanar. So the given graph G is non planar.

5.5.6. Dual of a graph

Let G be a graph having n number of regions, $F_1, F_2, \ldots F_n$. We place n number of points $v_1, v_2, \ldots v_n$ one in each of the regions respectively. These points are joined by line segment according to the following procedure:

If two region F_i and F_j are adjacent (i.e. F_i and F_j have common edge), the points v_i and v_j are joined by a line segment intersecting the common edge between F_i and F_j exactly once. If there is more than one edge common between F_i and F_j we draw one segment between v_i and v_j for each of the common edges.

If an edge e of G lies entirely in one region, F_k draw a loop at v_k intersecting the edge e exactly once.

Thus we get a new graph G^* whose vertices are $v_1, v_2, \ldots v_n$ and the line segment joining these are edges. This graph G^* is called Dual (or Geometric Dual) of G.

Illustration

Draw the Dual of the graph shown in Fig 5.5.10.

Let G be the graph. It has 6 regions F_1, F_2, F_3, F_4, F_5 and F_6 shown in Fig 5.5.11. F_6 is infinite region. We place six points $v_1, v_2, v_3, v_4,$ v_5 and v_6 one in each of the regions respectively. We join these points according to the following procedure :

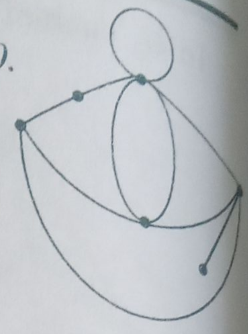


Fig. 5.5.10

If two regions F_i and F_j are adjacent (i.e. F_i and F_j have common edge), join v_i and v_j by a line (not necessarily st. line) intersecting the common edge exactly once. e.g note that F_4 and F_6 are adjacent. So join v_4 and v_6. There are two common edges between F_6 and F_3. So we draw one line between v_3 and v_6 for each of the common edges. The edge e of G lies completely within the region F_5. So we draw a loop at v_5 intersecting e exactly once.

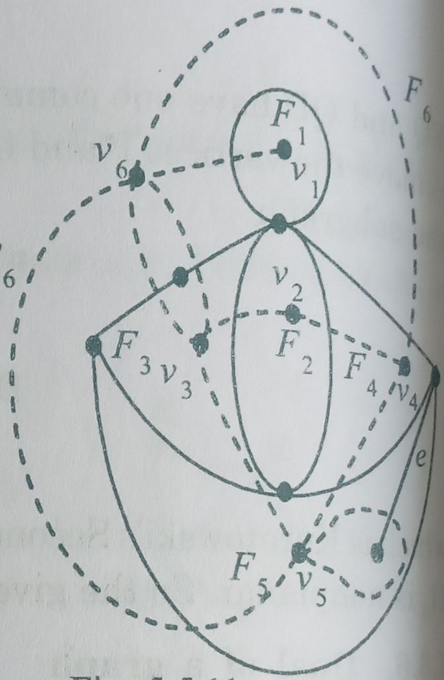


Fig. 5.5.11

Thus we get a new graph G^* consisting of six vertices $v_1, v_2, v_3,$ v_4, v_5 and v_6 and of edges joining these (the dotted lines in Fig 5.5.11). This dual, G^* is shown in Fig 5.5.12.

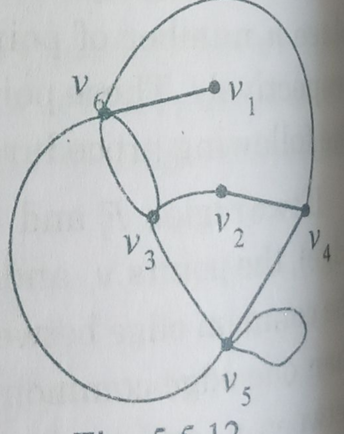


Fig. 5.5.12

Note.(1) *Since one edge of G^* intersects one and only one edge of G so there is a one-to-one correspondence between the edges of G and those of its dual G^*.*

(2) *Every planar graph has dual and vice versa. This is a well known Theorem.*

Relation between a Graph and its Dual Graph

Let G be a graph and G^* be its dual. Some relations between G and G^* are noted as:

(1) A pendant edge of G gives a loop in G^*.

(2) A self loop in G gives a pendant edge in G^*.

(3) Edges in series in G produce parallel edges in G^*.

(4) If G is planar then G^* is planar. Of course here we define the dual graph of a planar graph.

(5) If G^* is dual of G then G is dual of G^*. Because of this we call G and G^* are dual graph.

(6) (i) No. of vertices of G^* = No. of regions of G

(ii) No. of edges of G^* = No. of edges of G

(iii) No. of regions of G^* = No. of vertices of G

(7) G^* is always connected, even when G is disconnected.

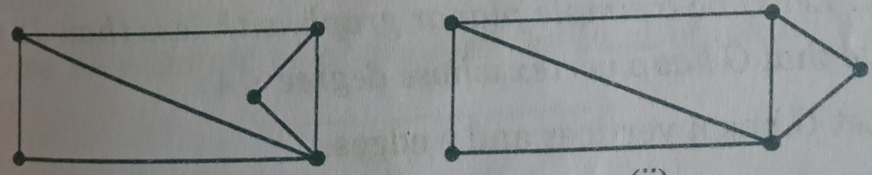
Theorem. A planar graph has unique dual if and only if it has a unique drawing on a plane.

Proof. Follows from the method of construction of duals.

Illustrations.

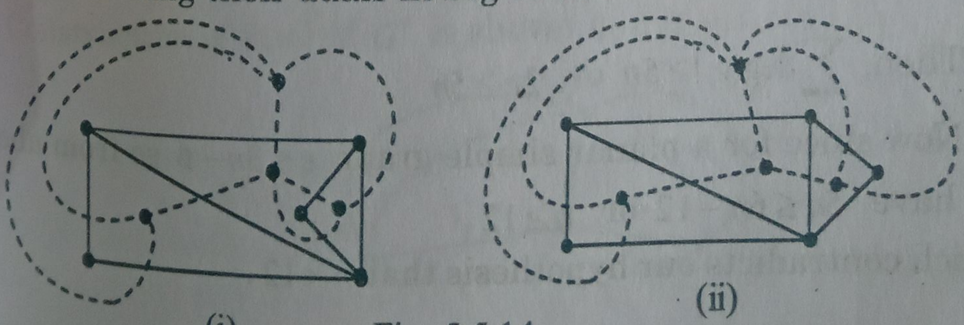
Give an example to show a graph is drawn in two different ways as planar graph. Its dual are different (non-isormphic).

In Fig 5.5.13 a graph is drawn in two different ways. They are planar.



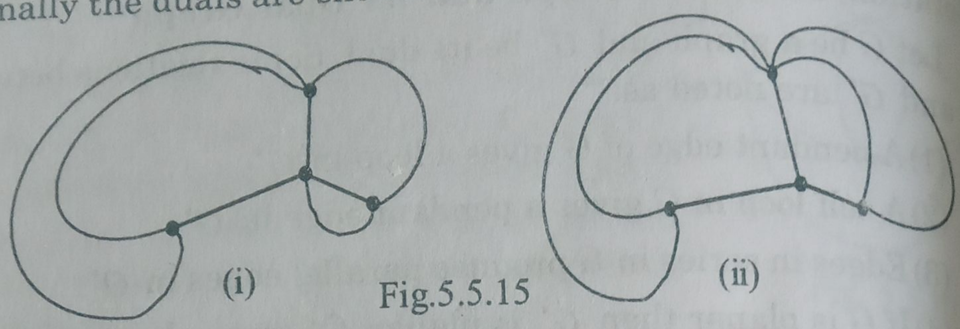
(i) Fig. 5.5.13 (ii)

We are finding their duals in Fig 5.5.14



(i) Fig. 5.5.14 (ii)

Finally the duals are shown below :



(i) Fig.5.5.15 (ii)

The graph of Fig 5.5.15(i) and 5.5.15(ii) are duals of the graph in Fig 5.5.13(i) and 5.5.13(ii) respectively. We see the two graphs of Fig 5.5.14 are not same even not isomorphic simply because one vertex of Fig 5.5.14(ii) has degree 5 whereas there is no vertex in Fig 5.5.14(i) having such degree.

4.1.7 Miscelleneous Examples.

Ex 1. *Let G be a regular graph, the degree of each of its vertices being 4. Determine the number of vertices of G if G determines 10 regions.*

Let $n =$ No. of vertices, $e =$ No. of edges, $f =$ No. of regions $= 10$

Sum of degrees of all vertices $= 2e$ or, $n \times 4 = 2e \quad \therefore e = 2n$.

The Euler's formula is $f = e - n + 2$

or, $10 = 2n - n + 2$ or, $n = 8$ $\quad \therefore$ No. of vertices $= 8$.

Ex 2. *Let G be a simple planar graph with less than 12 vertices. Prove that G has a vertex whose degree ≤ 4.*

Let G has n vertices and e edges.

$\therefore n < 12$. $v_1, v_2, \dots v_n$ be the vertices.

Let, if possible, deg $(v_i) \geq 5$ for all i.

Then, $\sum \deg(v_i) \geq 5n$ or, $2e \geq 5n$

Now since for a planar simple graph $e \leq 3n - 6$ so from above we have $5n \leq 6n - 12$ or, $n \geq 12$.

which contradicts our hypothesis that $n < 12$.

So there is at least one vertex v of G such that $\deg(v) \le 4$ (remembering degree of a vertex cannot be fraction)

Ex 3. *If every region of a simple planar graph having n vertices and e edges drawn in a plane bounded by k edges, prove that* $k(n-2) = e(k-2)$.

Let G be the graph. Its dual be G^*

No. of vertices of G^* = No. of regions of $G = f$ (say)

No. of edges of G^* = No. of edges of $G = e$

Since every region is bounded by k edges so degree of each vertex of $G^* = k$.

Applying the formula,

Sum of degrees of all vertices $= 2 \times$ No. of edges on the graph G^* we get $f \times k = 2e$. We know, $f = e - n + 2$.

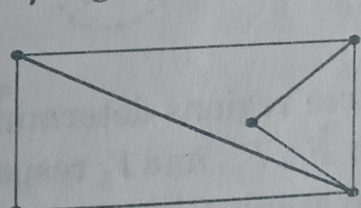
$\therefore k(e-n+2) = 2e$ or, $ek - k(n-2) = 2e$ or, $k(n-2) = e(k-2)$.

Ex 4. *If G_1 and G_2 be two homeomorphic graph then show that G_1 is planar if and only if G_2 is planar.*

Since G_1 and G_2 are homeomorphic they are homeomorph of the same graph, say G. Let G_1 be planar. Therefore by a Theorem G is planar and so by the same theorem, G_2 is planar. Converse case is same.

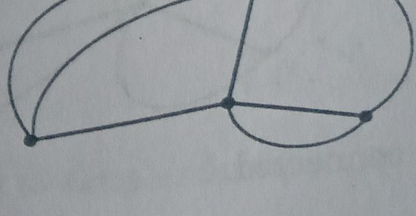
Ex 5. *Give an example of a graph G the dual of whose dual is again G.*

Consider G :

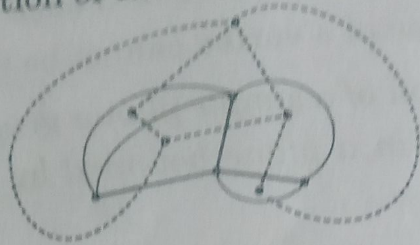


Construction of dual of G^* is shown in Fig 4.1.13(i) :

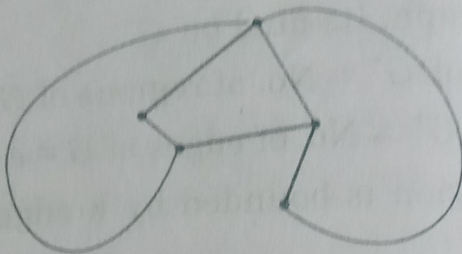
G^* :



Now the construction of the dual of G^* is

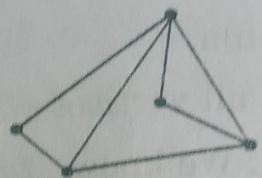


That is G^{**} :



which is same as

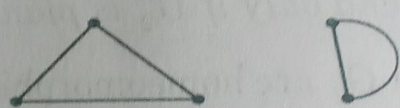
G^{**} :



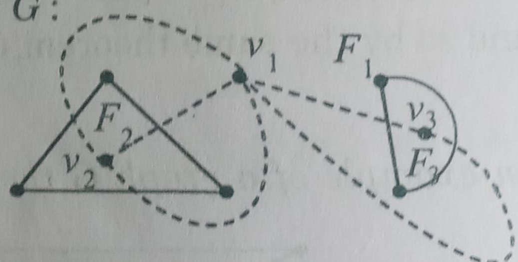
which is nothing but G.

Ex.6 *A disconnected graph G is given below. Construct its dual and show that the dual is connected.*

G :



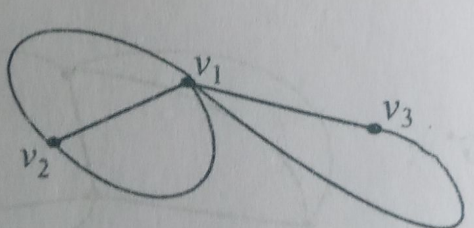
Construction of dual of G :



F_1, F_2, F_3 are the three regions determined by G. v_1, v_2, v_3 are three points taken in F_1, F_2, and F_3 respectively. They are joined accordingly and dual graph G^* is formed.

The dual graph is

G^* :



and we see G^* is connected.

Ex. 7 Give an example of a graph G such that $G^{**} = \left(G^{*}\right)^{*}$ is not G, not even isomorphic to G.

The above example serves this.

Exercise

I. Short Answer Questions

1. A regular graph G determines 8 regions, degree of each vertex being 3. Find the number of vertices of G.

[Hints: As Misc Ex. 1] [W.B.U.T. 2012, 2005]

2. (a) Draw a complete graph of 3 vertices to show it is planar.

(b) Draw a regular connected graph with six vertices and nine edges which is planar.

3. Draw a graph with 4 vertices and 10 edges to show it is planar.

4. Draw a graph with 3 vertices or 6 edges to show it is planar.

5. Draw a graph of 8 edges to show it is planar.

6. (a) Remove a vertex from Kuratwoski's First Graph and show it becomes a planar graph.

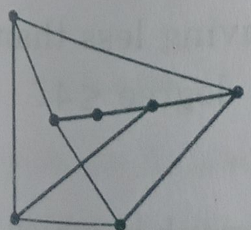
(b) Remove an edge from Kuratwoski's First graph and show it becomes a planar graph. Show the regions determined by this.

(c) Remove a vertex from Kuratwoski's second graph and show it becomes a planar graph.

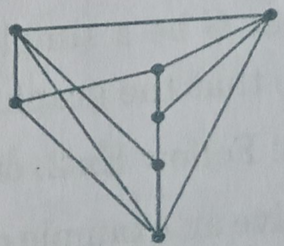
(d) Remove an edge from Kuratwoski's second graph and show it becomes a planar graph. Show the regions determined by this.

7. Find whether the following graphs are planar.

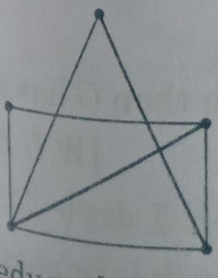
(i)



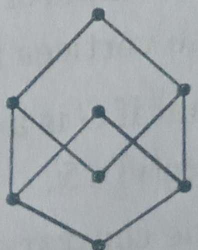
(ii)



(iii)



(iv)



[Hint: Reduce the graph to simpler form]

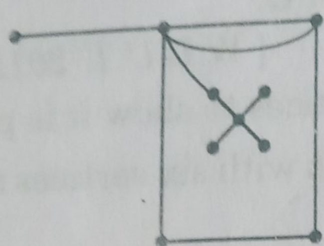
8. Give two drawings of a graph which are identical.

9. Give two drawings of a planar graph which are different. [Follow the drawings given in Fig 4.12 where (i) and (ii) are different]

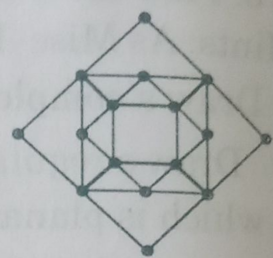
10. Give three drawings of a graph which are all different.

11. (a) Verify Euler's formula for the following graph :

(i)



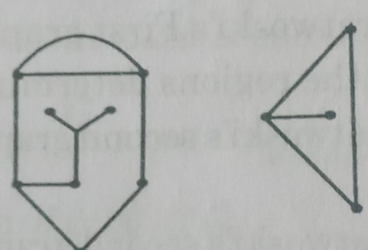
(ii)



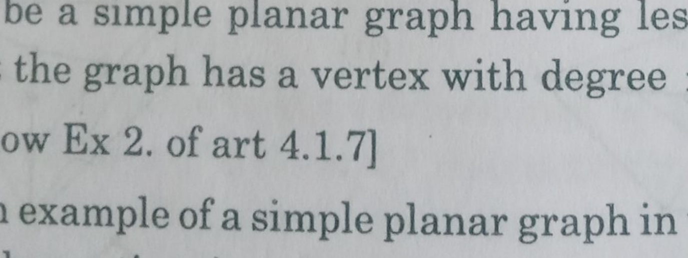
[Hint : Euler's formula is $f = e - n + 2$. Count the No. of region, No. of vertices and edges of the given graph. Then verify the result]

(b) Verify Euler's formula for the following two disconnected graph :

(i)



(ii)



12. Let G be a simple planar graph having less than 30 edges. Prove that the graph has a vertex with degree ≤ 4.

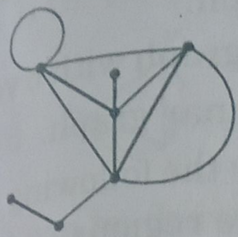
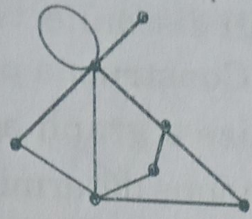
[Hint: Follow Ex 2. of art 4.1.7]

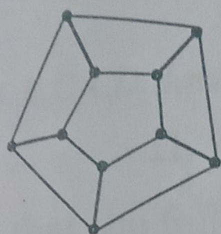
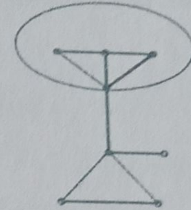
13. Give an example of a simple planar graph in which the degree of each of the vertices is at least 5.

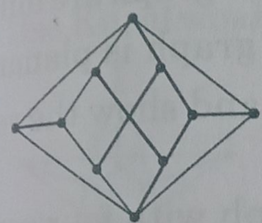
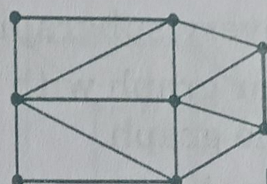
14. Prove that if G is a simple planar graph then G has a vertex v such that $\deg(v) \leq 5$. [W.U.B.T 2016]

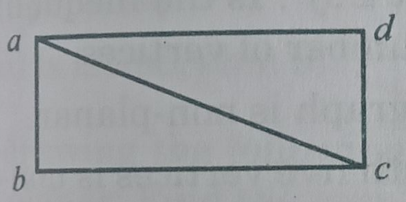
[Hint: On the contrary, let $\deg(v_i) \geq 6$ then $\Sigma \deg(v_i) \geq 6n$ or, $2e \geq 6n$ or, $e \geq 3n$ or, $3n - 6 \geq e \geq 3n$ which is impossible]

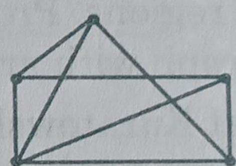
15. Draw the dual of the following graphs :

(i)  (ii)  [W.B.U. T. 2012, 2003]

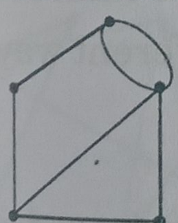
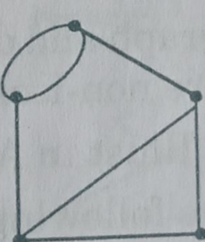
(iii)  [W.B.U. Tech 2007] (iv) 

(v)  (vi)  [W.B.U.T. 2005]

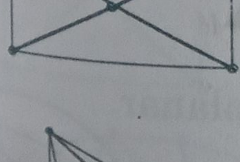
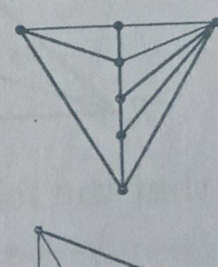
(vii) 

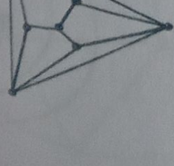
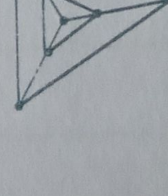
(viii)  [W.B.U.T. 2008]

16. Construct the dual of the following two graphs G_1 and G_2. Are the two duals isomorphic? Are G_1 and G_2 isomorphic?

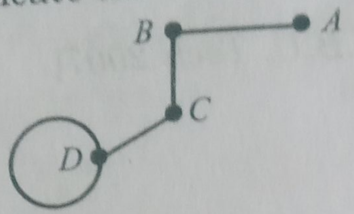
G_1 :  G_2 : 

17. Prove that the following graphs are self-dual :

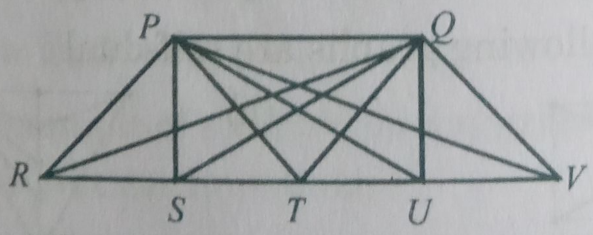
(i)  (ii) 

(iii)  (iv) 

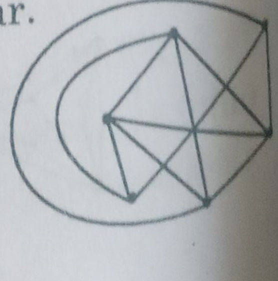
18. Give an example of a disconnected graph whose dual is connected. Is it true for all disconnected graph.
19. Define planar graph. Construct a planar graph with 6 vertices
20. How will you recognize a graph as a planar graph.
21. Find the number of regions determined by the following graph. Indicate them. Indicate the boundaries of the regions.



22. When we call the drawings of a same planar graph are different.
23. Prove that every subgraph of a planar graph is planar.
24. Draw a planar graph with five vertices and show the regions determined by the graph.
25. Let G be a simple connected planar graph with four vertices, e edges and f regions. Prove that $2e \geq 3f$. Is the inequality valid for all such graph with arbitrary number of vertices.
26. Prove that Kuratowski's First graph is non-planar.
27. Prove that a complete graph with five vertices is non-planar.
28. What is Kuratwoski's second graph. Discuss why it is non-planar.
29. Construct a non planar graph with more than four vertices.
30. Construct a graph and draw it in two different ways as planar graph– its dual are non-isomorphic.
[Hint : follow an Illust in Art 4.1.6]
31. Show that the following graph is planar, by redrawing it :

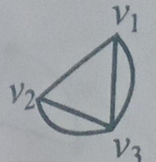


32. Show that the following graph is not planar.



Answers

1\. 12. 7. (i), (ii) are non planar, (iii) is planar (iv) is planar

15\. (vii) 

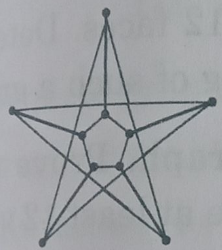
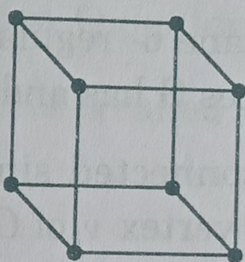
(viii) 
18\. Yes

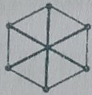
20\. if it can be drawn on plane without a cross over between its edges.

21\. 2, AB, B, C, CD and DD are boundaries of a region.

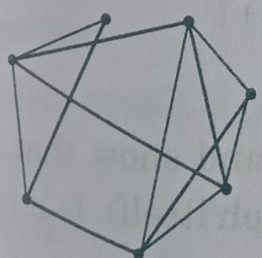
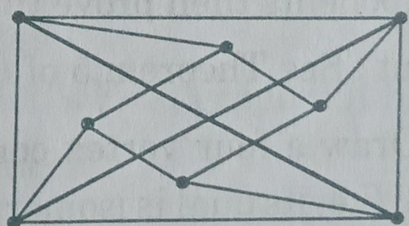
II. Long Answer Questions

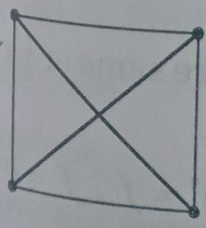
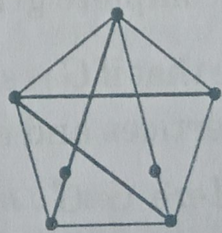
1\. Show that the following graphs are non-planar
[Hint : Use Kuratwoski's Theorem]

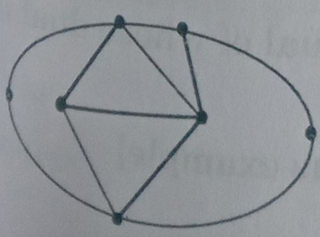
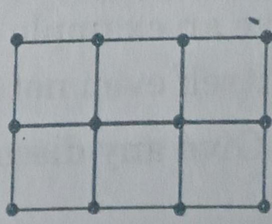
(i) 
(ii) 

[Hint : (ii) is isomorphic to ]

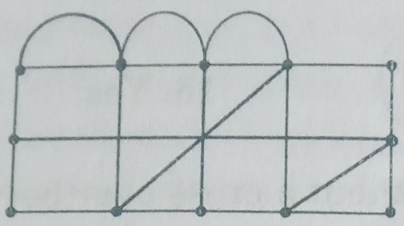
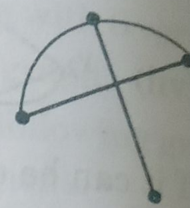
2\. By redrawing the following graphs as plane graph show that they are planar: Find the number of regions determined by each. Show the regions determined by these graph.
[Hint : Use the formula $f = e - n + 2$]

(i) 
(ii) 

(iii) 
(iv) 

(v) 
(vi) 

3. Find, by reducing to more simpler form, whether the following graphs are planar :

(i)  (ii) 

4. Give an example to show a planar graph has two different drawings and exhibit dual of each drawing showing that they are different (even non isomorphic). [Hint: Follow an Illustration Art 4.1.6]

5. Let G be a plane 6- regular graph with 12 faces. Determine how many vertices G has and give a drawing of such a graph.

6. Let G be a connected simple planar graph. Prove that if $\deg(v_i) \geq 5$ for all vertex v_i of G then there are at least 12 vertices of degree 5 in G.

7. Let G be a simple connected planar graph. Prove that if $n \geq 4$ and $\deg(v_i) \geq 3$ for all vertices v_i of G then G has at least 4 vertices of degree less than 6.

8. If G is a planar graph with n vertices, f regions, e edges and k components then prove that $n - e + f = k + 1$.

[Hint : See Theorem 5 of Art 4.1.4.]

9. Draw a four vertex complete graph and show that it is self-dual (i.e. its dual is isomorphic to the graph itself). Is it true for all 4 vertex complete graph?

10. Prove that if G is self dual (i.e. G and G^* are same or isomorphic) with n vertices and e edges than $e = 2n - 2$.

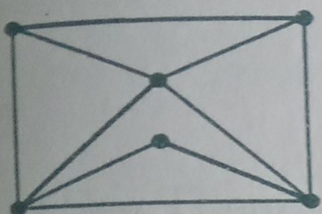
[Hint : Here $G \equiv G^* \therefore n^* = n$, apply the result $n^* = f$, $f = e - n + 2$]

11. Give an example of a graph the dual of whose dual is not the graph itself even not isomorphic.

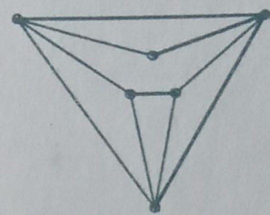
[Hint: Give any disconnected graph as example]

12. Prove that the following two graphs G_1 and G_2 are isomorphic but G_1^* and G_2^* are non-isomorphic :

G_1 :



and G_2 :



13. G is non-planar graph and $G-v$ is planar for every vertex v in G. Prove that G is connected and has no cut vertex.

Answers

3. (ii) Planar 5. 5 9. Yes

III. Multiple Choice Questions

1. A complete graph with five vertices is called
   (a) regular graph
   (b) Kuratowski's First graph
   (c) Kuratowski's second graph
   (d) none

graph with 9 edges is called Kuratowski's