Module-1

1.1 SET THEORY

1.1.1. Basic Concept of Sets

The concept of set is fundamental in all branches of mathematics. *A well defined collection of distinct objects is defined to be a set when the objects of collection are independent of the order of their arrangement.* By a well-defined collection we mean that there exists a definite rule with the help of which it is possible to tell whether a particular object belongs tc the collection or not. The objects constituting the set are called *elements* or *members* of the set.

We shall denote the sets by capital letter A, B, C, X, Y, Z, P, etc. and their elements by small letters a, b, c, x, y, z, p, q etc.

If an object a is a member of a set A, then we write $a \in A$ which may be read as 'a belongs to A' or 'a is an element of A'. On the other hand, if a is not a member of a set A, then we write $a \notin A$.

Illustration. **(i)** "The collection of BCA- 1st year students of your college" is well defined, because given a student of your college, we see if he (or she) is a student of BCA-1st year or not. If the answer is 'yes', he or she belongs to the collection, otherwise not.

(ii) Let $A = \{1, 8, 3, 2\}$, $B = \{4, 4, 5, 1\}$.

Here B is not a set but A is

(iii) The set of vowels of English alphabet can be represented as $X = \{a, e, i, o, u\}$ or $\{e, o, u, i, a\}$.

(iv) $P = \{x : x$ is a positive even number less than $10\}$ which can also be written as $P = \{2, 4, 6, 8\}$

(v) $X = \{1, 3, 5, 7, \cdots\}$ is a set of all positive odd integers.

A set can be expressed in two forms, one is *tabular form* where the elements are within brackets { } separated by commas as Ex (ii) and (iii).

Another form is *set-builder form* as first form of Ex (iv) where P is the set consisting of the elements x such that x satisfies the property stated within the brackets. The notation " : " stands for "such that ".

A set having no elements in it is called a ***null set*** or ***void set*** or ***empty set***. It is usually denoted by ϕ.

Illustration : Let $A = \{x : x^2 + 2 = 0 \text{ and } x \text{ is real}\}$

Since there is no real number which satisfies the equation $x^2 + 2 = 0$, so the set A is empty, i.e., $A = \phi$.

A set consisting of a single element is called a ***singleton***. Thus the set $\{2\}$ is a singleton.

A set having finite number of elements in it is called a ***finite* set**. Otherwise a set is ***infinite***. Ex (iii), (iv) is a finite set whereas Ex (v) is infinite set.

Two sets A and B are said to be **equal** if every element of A is an element of B and also every element of B is an element of A. Then we write $A = B$. Symbolically, $A = B$ if $x \in A \Leftrightarrow x \in B$.

We shall now give a list of important sets of numbers which will be often used later :

N = the set of all natural numbers

Z or I = the set of all integers

Q = the set of all rational numbers

R = the set of all real numbers

C = the set of all complex numbers.

1.1.2. Subset

If A, B are two sets such that every element of A is also an element of B, then A is said to be a subset of B. Symbolically this relation is denoted by writting $A \subseteq B$ and is read as 'A is a subset of B' or 'A is contained in B'

Note : (1) *From the definition of subset it is obvious that every set is a subset of itself i.e., $A \subseteq A$. Here A is called the improper subset of A. Any other subset of A is said to be a proper subset of A.*

(2) *When A is a proper subset of B, then we denote this by $A \subset B$. Obviously the null set ϕ is a proper subset of every set except ϕ itself.*

Illustration : If $B = \{a, b, c\}$, then all the subsets of B are $\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}$ and ϕ.

Theorem 1. If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

Proof. If $A \subseteq B$, then every element of A is an element of B. Also if $B \subseteq A$, then every element of B is an element of A. Combining these two, we can say that $A = B$.

Theorem 2. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof. Let x be an arbitrary element of the set A. Since $A \subseteq B$, so $x \in A \Rightarrow x \in B$. But $B \subseteq C$ (given). Therefore $x \in B \Rightarrow x \in C$. Thus $x \in A \Rightarrow x \in C$. Hence by definition of subset $A \subseteq C$.

1.1.3. Power Set

If S is any set, then the set of all the subsets of S is called the power set of S and is denoted by $P(S)$. Obviously ϕ and S are both the elements of $P(S)$.

Illustration . If $S = \{1, 2, 3\}$ then

$$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$$

Theorem. If a finite set S has n elements, then its power set has 2^n elements.

Proof. Since the finite set S has n elements, so there are ${}^nC_r$ subsets consisting of r elements where $r = 0, 1, 2, \cdots, n$. Hence the total number of subset of S i.e., the total number of elements in $P(S)$ is ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = (1+1)^n = 2^n$.

1.1.4. Universal Set :

A non-empty set of which all the sets under consideration are subsets is called the Universal set and is denoted by U.

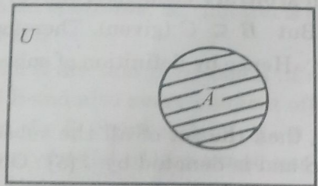
Illustration (i) All the people in the world constitute the universal set in any study of human population.

(ii) If $U = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$; $B = \{1, 3, 5, \cdots\}$, $C = \{2, 4, 6, \cdots\}$ then U is the Universal set for B and C.

1.1.5. Venn-Euler Diagram

John-Venn, an English logician (1834-1923) invented the diagram to present pictorial representation of sets and the operations on sets also. These pictures consist of rectangle and closed curves. These combinations of rectangles and closed regions are called Venn-Euler *diagram or simply Venn-diagram.*

In Venn-diagram we shall denote the universal set U by a large rectangle and we write the letter 'U' in one corner of that rectangle. Also we denote any subset A of the Universal set by by circle or closed curve and write the letter A within the circle.



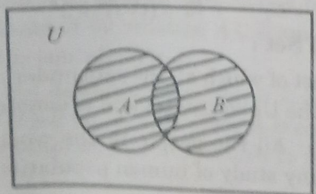
1.1.6. Set Operations

Union of Sets

The union of two sets A and B is the set of all elements which are in the set A or in B or in both. This set is denoted by $A \cup B$. Thus

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The union of two sets A and B is shown with shaded area in the adjoining Venn diagram.



In general the union of n sets $A_1, A_2, \cdots, A_n$ is denoted by $A_1 \cup A_2 \cup \cdots \cup A_n$ i.e., $\bigcup_{i=1}^{n} A_i$ and

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some value of } i, 1 \le i \le n\}$$

Using the Venn diagram the following properties can be easily verified :

(i) $A \cup A = A$ (idempotent property)

(ii) $A \cup \phi = A$

(iii) $A \cup B = B \cup A$ (commutative property)

(iv) If $A \subseteq B$, then $A \cup B = B$.

(v) $A \subset A \cup B$, $B \subset A \cup B$

(vi) $(A \cup B) \cup C = A \cup (B \cup C)$, (associative property)

Illustration (i) If $A = \{1, 3, 4, 6\}$, $B = \{2, 3, 4\}$, then

$$A \cup B = \{1, 2, 3, 4, 6\}.$$

(ii) If $P = \{x : x \text{ is a positive integer less then } 10\}$ and

$$Q = \{x : x \text{ is an integer and } 5 \le x \le 15\}$$

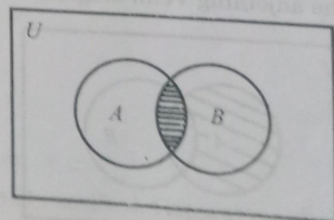
then $P \cup Q = \{x : x \text{ is an integer and } 1 \le x \le 15\}$

Intersection of Sets.

The intersection of two sets A and B is the set of all those elements which are both in A and B and is denoted by $A \cap B$.

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

The intersection of two sets A and B is shown with shaded area in the adjoining Venn diagram.



In general the intersection of n sets $A_1, A_2, \cdots, A_n$ is denoted by $A_1 \cap A_2 \cap \cdots \cap A_n$ or, $\bigcap_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i = \{x : x \in A_i, \forall i, 1 \le i \le n\}$.

Using the Venn diagram the following properties can be easily verified:

(i) $A \cap A = A$ (idempotent property)

(ii) $A \cap \phi = \phi$

(iii) $A \cap B = B \cap A$ (commutative property)

(iv) If $A \subseteq B$, then $A \cap B = A$

(v) $A \cap B \subseteq A$, $A \cap B \subseteq B$

(vi) $(A \cap B) \cap C = A \cap (B \cap C)$ (associative property)

Illustration : (i) If $A = \{2, 4, 8, 10\}$, $B = \{2, 3, 4, 5, 7\}$, $C = \{1, 3, 5, 7\}$ and $A \cap B = \{2, 4\}$, $A \cap C = \phi$, $B \cap C = \{3, 5, 7\}$

(ii) If $A = \{x : 0 < x < 4\}$ and $B = \{x : 1 \le x \le 6\}$ then $A \cap B = \{x : 1 \le x < 4\}$.

Disjoint sets

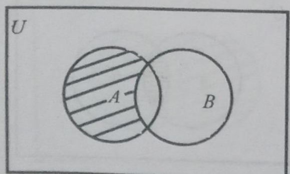
Two sets A and B *are* said to be disjoint if they have no common element i.e. if $A \cap B = \phi$.

Difference of two Sets

The difference of two sets A and B in that order is the set of all those elements of A which do not belong to B and is denoted by $A - B$. Thus $A - B = \{x : x \in A \text{ and } x \notin B\}$.

$A - B$ is also called the *complement* of B with respect to A.

The difference $A - B$ of two sets A and B is shown with shaded area in the adjoining Venn diagram



It can be easily verified that $A - B = \phi$ if $A \subset B$ and $A - B = A$ if $A \cap B = \phi$.

Illustration

If $A = \{a, b, c, d\}$, $B = \{b, f, d, g\}$

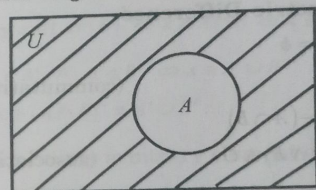
then $A - B = \{a, c\}$, $B - A = \{f, g\}$

Obviously $A - B \neq B - A$.

Complement of a Set relative to the Universal Set

The complement of a set A with respect to the Universal set U is the difference of U and A and is denoted by A' or A^c. Thus $A' = U - A = \{x : x \in U \text{ and } x \notin A\}$ or simply $A' = \{x : x \notin A\}$

The complement of a set A is shown with shaded area in the adjoining Venn diagram.



Example : If $U = \{1, 2, 3, 4, \cdots\}$ and $A = \{2, 4, 6, 8, \cdots\}$, then $A' = \{1, 3, 5, \cdots\}$

Properties :

(i) $U' = \phi$

(ii) $\phi' = U$

(iii) $A \cap A' = \phi$

(iv) $A \cup A' = U$

(v) $(A')' = A$

(vi) If $A \subset B$ then $B' \subset A'$.

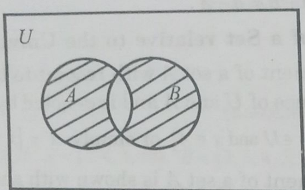
These can be easily verified. General proof are not given.

Symmetric Difference of two Sets.

The symmetric difference of two set A and B is denoted by $A \Delta B$ and is defined by $A \Delta B = (A-B) \cup (B-A)$.

Thus $A \Delta B$ is the set of all those elements which belongs either to A or to B but not to both.

The shaded area in the adjoining Venn diagram is the symmetric difference of two sets A and B.



Properties of Symmetric Difference

(i) $A \Delta \phi = A$, $A \Delta A = \phi$

(ii) $A \Delta B = B \Delta A$ (commutative property)

(iii) $A \Delta B = (A \cup B) - (A \cap B)$

(iv) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$ (associative property)

Proof : Beyond the scope of the book.

Illustration.

If $A = \{1,2,3,4,5\}$, $B = \{3,4,6,7\}$ then $A - B = \{1,2,5\}$, $B - A = \{6,7\}$

$\therefore$ $A \Delta B = \{1,2,5,6,7\}$.

1.1.7. Laws of the Algebra of Sets.

1. $A \cap (A \cap B) = A$, $A \cup (A \cap B) = A$ (absorptive property)

2. (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (distributive property)

3. (a) $(A \cup B)' = A' \cap B'$

   (b) $(A \cap B)' = A' \cup B'$

   (De-Morgan's laws)

4. (a) $A - (B \cup C) = (A - B) \cap (A - C)$

   (b) $A - (B \cap C) = (A - B) \cup (A - C)$

   (generalisation of De-Morgan's law)

Proof :

1. Left as an exercise.

2. (a) Let x be any arbitrary element of $A \cap (B \cup C)$.

Then $x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $x \in (B \cup C)$

$\Leftrightarrow x \in A$ and $(x \in B$ or $x \in C$ or both$)$

$\Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$ or both

$\Leftrightarrow x \in (A \cap B)$ or $x \in (A \cap C)$ or both

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

3. (b). Let x be any arbitrary element of $(A \cap B)'$

Then $x \in (A \cap B)' \Leftrightarrow x \notin A \cap B \Leftrightarrow x \notin A$ or, $x \notin B$

$\Leftrightarrow x \in A'$ or, $x \in B' \Leftrightarrow x \in A' \cup B'$

Hence $(A \cap B)' = A' \cup B'$.

4. (a) Let x be any arbitrary element of $A - (B \cap C)$. Then

$x \in A - (B \cup C) \Leftrightarrow x \in A$ and $x \notin B \cup C$

$\Leftrightarrow x \in A$ and $(x \notin B$ and $x \notin C)$

$\Leftrightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$

$\Leftrightarrow x \in A - B$ and $x \in A - C$

$\Leftrightarrow x \in (A - B) \cap (A - C)$

Hence $A - (B \cup C) = (A - B) \cap (A - C)$.

1.1.8. Ordered Pair

Let A, B be two non-empty sets and $a \in A$, $b \in B$. Then (a, b) is called an ordered pair and a, b are called the first and second co-ordinate of the ordered pair (a, b). If (a, b) and (c, d) are two ordered pairs, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Cartesian Product of two Sets

Let A and B be two non-empty sets. Their cartesian product denoted by $A \times B$ is the set of all distinct ordered pairs whose first co-ordinate is an element of A and the second co-ordinate is an element of B.

Thus $A \times B = \{(a, b) ; a \in A \text{ and } b \in B\}$.

Illustration. (i) Let $A = \{1, 2\}$, $B = \{x, y\}$

Then $A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$

Also $B \times A = \{(x, 1), (x, 2), (y, 1), (y, 2)\}$

Hence in general $A \times B \neq B \times A$

(ii) $N \times N$ is the set of all ordered pairs $\{(x, y) : x \in N, \ y \in N\}$.

Illustrative Examples

Ex. 1. *If* $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 5, 6, 9\}$, $B = \{1, 2, 3, 4, 10\}$, $C = \{4, 6, 7, 8, 9\}$ *then*

(i) *Find* A'

(ii) *Verify* $(A')' = A$

(iii) *Verify De' Morgan's law* $(A \cap B)' = A' \cup B'$

(iv) *Verify Distributive law* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(i) $A' = U - A = \{1, 3, 4, 7, 8, 10\}$

(ii) $(A')' = U - A' = \{2, 5, 6, 9\} = A$

(iii) $A \cap B = \{2\}$ $\therefore$ $(A \cap B)' = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$ ... (i)

Again $A' = \{1, 3, 4, 7, 8, 10\}$ and $B' = \{5, 6, 7, 8, 9\}$

$\therefore$ $A' \cup B' = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$ ... (ii)

From (i) and (ii) we have $(A \cap B)' = A' \cup B'$

(iv) Now $B \cap C = \{4\}$ $\therefore$ $A \cup (B \cap C) = \{2, 4, 5, 6, 9\}$ ... (i)

Again, $A \cup B = \{1, 2, 3, 4, 5, 6, 9, 10\}$

and $A \cup C = \{2, 4, 5, 6, 7, 8, 9\}$ ... (ii)

$\therefore$ $(A \cup B) \cap (A \cup C) = \{2, 4, 5, 6, 9\}$

$\therefore$ From (i) and (ii), we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Ex. 2. *If* $P = \{x : x \text{ is an integer and } x \geq 6\}$, $Q = \{x : x \text{ is an integer and } -3 \leq x \leq 10\}$ *then find* $P \cup Q$, $P \cap Q$. *and verify that* $(P \cup Q) \cap (P \cup Q') = P$.

We have $P = \{6, 7, 8, 9, 10, 11, \cdots\}$

$Q = \{-3, -2, -1, 0, 1, 2, 3, \cdots, 9, 10\}$

$\therefore$ $P \cup Q = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \cdots\}$

$= \{x : x \text{ is an integer and } x \geq -3\}$

$P \cap Q = \{6, 7, 8, 9, 10 \cdots\}$

Also, $Q' = \{\cdots, -6, -5, -4, 11, 12, 13 \cdots\}$

$= \{x : x \text{ is an integer and } x \leq -4 \text{ and } x \geq 11\}$

$\therefore$ $P \cup Q' = \{\cdots, -6, -5, -4, 6, 7, 8, 10, \cdots\}$

$\therefore$ $(P \cup Q) \cap (P \cup Q') = \{6, 7, 8, 9, \cdots\} = P$.

Ex. 3. *If* $A = \{1, 3, 5\}$, $B = \{1, 3, 8\}$, *then find*

(i) $A \times B$ (ii) $B \times A$

(iii) $(A \times B) - (B \times A)$ (iv) $A \,\Delta\, B$

(v) *also verify that* $A \,\Delta\, B = (A \cup B) - (A \cap B)$

(i) $A \times B = \{(1, 1), (1, 3), (1, 8), (3, 1), (3, 3), (3, 8), (5, 1), (5, 3), (5, 8)\}$

(ii) $B \times A = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (8, 1), (8, 3), (8, 5)\}$

(iii) $(A \times B) - (B \times A) = \{(1, 8), (3, 8), (5, 1), (5, 3), (5, 8)\}$

(iv) $A - B = \{5\}$, $B - A = \{8\}$

$\therefore$ $A \,\Delta\, B = (A - B) \cup (B - A) = \{5, 8\}$

(v) Now $A \cup B = \{1, 3, 5, 8\}$, $A \cap B = \{1, 3\}$

$\therefore\ (A \cup B) - (A \cap B) = \{5, 8\}$

Hence $A \Delta B = (A \cup B) - (A \cap B)$.

Ex. 4. *In an examination, out of 90 students 65 passed in Physics and 50 passed in Mathematics and 35 passed in both Physics and Mathematics.*

(i) How many students passed in exactly one of the two subjects ?

(ii) How many students failed in both the subjects ?

Let $U =$ No. of all students, $P =$ No. of students of Physics, $M =$ No. of students of Mathematics.

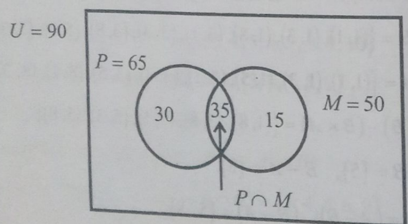
Here $U = 90$, $P = 65$, $M = 50$ and $P \cap M = 35$.

So the number of students who passed in Physics only is $P - (P \cap M) = 65 - 35 = 30$ and the no. of students who passed in Mathematics only is $M - (P \cap M) = 50 - 35 = 15$.

(i) Thus the total number of students who passed in exactly one of the two subjects is 30 + 15 = 45.

(ii) Hence the number of students who failed in both the subjects is $90 -$ (No. of students who passed in only one subject + No. of students who passed in both the subjects) $= 90 - (45 + 35) = 10$.

Note : *Using Venn-Diagram, the above problem can be solved easily as shown in the following figure*



Ex. 5 *If A and B are sets, then prove that $A - B$, $A \cap B$ and $B - A$ are pairwise disjoint*

First we shall prove that $A - B$ and $A \cap B$ are disjoint

i.e., $(A - B) \cap (A \cap B) = \phi$.

If possible, let x be any arbitrary element of $(A - B) \cap (A \cap B)$

Then $x \in (A - B) \cap (A \cap B)$

$\Rightarrow x \in (A - B)$ and $x \in (A \cap B)$

$\Rightarrow (x \in A \text{ and } x \notin B)$ and $(x \in A \text{ and } x \in B)$.

But there is no element x which satisfies both $x \in B$ and $x \notin B$.

Hence there is no element in the set $(A - B) \cap (A \cap B)$

i.e., $(A - B) \cap (A \cap B) = \phi$.

Similarly we can prove that $B - A$ and $A \cap B$ are disjoint i.e., $(B - A) \cap (A - B) = \phi$.

Finally we shall prove that $(A - B) \cap (B - A) = \phi$. If possible, let y be any arbitrary element of $(A - B) \cap (B - A)$. Then $y \in (A - B) \cap (B - A)$

$\Rightarrow y \in A - B$ and $y \in B - A$

$\Rightarrow (y \in A \text{ and } y \notin B)$ and $(y \in B \text{ and } y \notin A)$

But there is no element y which satisfies both $y \in A$ and $y \notin A$ or which satisfies both $y \in B$ and $y \notin B$.

Hence there is no element in the set $(A - B) \cap (B - A)$

i.e., $(A - B) \cap (B - A) = \phi$.

Thus the sets $A - B$, $A \cap B$ and $B - A$ are pairwise disjoint.

Ex. 6. *A, B, C are subsets of a Universal set U. Prove that $A - (B \cup C) = (A - B) \cap (A - C)$.*

Let x be any arbitrary element of the set $A - (B \cup C)$

Then $x \in A - (B \cup C)$

$\Leftrightarrow x \in A$ and $x \notin (B \cup C)$

$\Leftrightarrow x \in A$ and $\{x \notin B \text{ and } x \notin C\}$

$\Leftrightarrow \{x \in A \text{ and } x \notin B\}$ and $\{x \in A \text{ and } x \notin C\}$

$\Leftrightarrow x \in (A-B)$ and $x \in (A-C) \Leftrightarrow x \in (A-B) \cap (A-C)$

Hence $A-(B \cup C) = (A-B) \cap (A-C)$.

Ex. 7. *If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, prove that $B = C$.*

$B = B \cap (B \cup A)$, by absorptive property

$= B \cap (A \cup B)$, by commutative property

$= B \cap (A \cup C)$, given $A \cup B = A \cup C$

$= (B \cap A) \cup (B \cap C)$, by distributive property

$= (A \cap B) \cup (B \cap C)$, given $A \cap B = A \cap C$

$= (C \cap A) \cup (C \cap B)$

$= C \cap (A \cup B)$

$= C \cap (A \cup C)$

$= C \cap (C \cup A)$

$= C$, by absorptive property.

Ex. 8. *If $A \cup B = A \cup C$ and $A' \cup B = A' \cup C$, prove that $B = C$*

We have $A \cup B = A \cup C$ (1)

and $A' \cup B = A' \cup C$ (2)

$\therefore$ From (1) and (2),

$(A \cup B) \cap (A' \cup B) = (A \cup C) \cap (A' \cup C)$

or, $(B \cup A) \cap (B \cup A') = (C \cup A) \cap (C \cup A')$, by commutative property

or, $B \cup (A \cap A') = C \cup (A \cap A')$, by distributive property

or, $B \cup \phi = C \cup \phi$

$\therefore B = C$.

Ex. 9. *Let P, Q, R be subsets of a Universal set U.*

Prove that (i) $P \times (Q \cap R) = (P \times Q) \cap (P \times R)$

(ii) $(P-Q) \times R = (P \times R) - (Q \times R)$

(i) Let (x, y) be any arbitrary element of the set $P \times (Q \cap R)$.

Then, $(x, y) \in P \times (Q \cap R) \Leftrightarrow x \in P$ and $y \in Q \cap R$

$\Leftrightarrow x \in P$ and $(y \in Q \text{ and } y \in R)$

$\Leftrightarrow (x \in P \text{ and } y \in Q) \text{ and} (x \in P \text{ and } y \in R)$

$\Leftrightarrow (x, y) \in P \times Q$ and $(x, y) \in P \times R$

$\Leftrightarrow (x, y) \in (P \times Q) \cap (P \times R)$

$\therefore\ P \times (Q \cap R) = (P \times Q) \cap (P \times R)$

(ii) Let (x, y) be any arbitrary element of the set $(P-Q) \times R$.

Then $(x, y) \in (P-Q) \times R \Leftrightarrow x \in (P-Q)$ and $y \in R$

$\Leftrightarrow (x \in P \text{ and } x \notin Q)$ and $y \in R$

$\Leftrightarrow (x \in P \text{ and } y \in R)$ and $(x \notin Q \text{ and } y \in R)$

$\Leftrightarrow (x, y) \in P \times R$ and $(x, y) \notin Q \times R$

$\Leftrightarrow (x, y) \in (P \times R) - (Q \times R)$

$\therefore\ (P-Q) \times R = (P \times R) - (Q \times R)$

Ex. 10. *Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$*

Let (x, y) be any arbitrary element of $(A \times B) \cap (C \times D)$

Then $(x, y) \in (A \times B) \cap (C \times D)$

$\Leftrightarrow (x, y) \in A \times B$ and $(x, y) \in C \times D$

$\Leftrightarrow (x \in A \text{ and } y \in B)$ and $(x \in C \text{ and } y \in D)$

$\Leftrightarrow (x \in A \text{ and } y \in C)$ and $(x \in B \text{ and } y \in D)$

$\Leftrightarrow x \in A \cap C$ and $y \in B \cap D$

$\Leftrightarrow (x, y) \in (A \cap C) \times (B \cap D)$

$\therefore (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Ex. 11. *Prove that $(A \cap C) \cup (B \cap C') = \phi \Rightarrow A \cap B = \phi$*

$(A \cap C) \cup (B \cap C') = \phi \Rightarrow A \cap C = \phi$ *and* $B \cap C' = \phi$

Now $B \cap C' = \phi \Rightarrow B \subseteq C$ $[\because \text{ If } x \in B \Rightarrow x \notin C' \Rightarrow x \in C]$

$\Rightarrow A \cap B \subseteq A \cap C \Rightarrow A \cap B \subseteq \phi$ $[\because A \cap C = \phi]$

$\Rightarrow A \cap B = \phi$

Ex. 12. *For any three sets A, B, C, show that*

$$A-(B-C)=(A-B)\cup(A\cap C).$$

Let x be any arbitrary element of the set $A-(B-C)$.

Then $x\in A-(B-C)\Leftrightarrow x\in A$ and $x\notin B-C$

$\Leftrightarrow x\in A$ and $(x\notin B$ or $x\in C)$

$\Leftrightarrow (x\in A$ and $x\notin B)$ or $(x\in A$ and $x\in C)$

$\Leftrightarrow x\in A-B$ or $x\in A\cap C \quad \Leftrightarrow x\in(A-B)\cup(A\cap C)$

$\therefore A-(B-C)=(A-B)\cup(A\cap C)$

Ex. 13. *Using Venn diagram, prove that*

$$(A-C)\cup(B-C)=(A\cup B)-C.$$

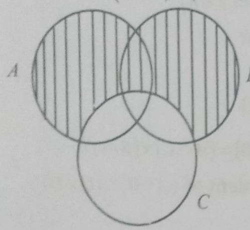


Fig. 1

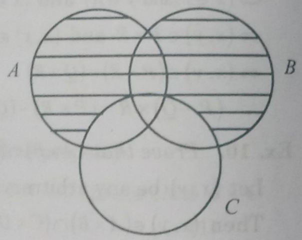


Fig. 2

In Fig.1, the shaded region with the vertical lines is the set $(A-C)\cup(B-C)$.

In Fig. 2, the shaded region with horizontal lines is the set $(A\cup B)-C$.

Hence from Fig.1 and Fig. 2, the result is proved.

1.1.9. Relations

Definition

Let A and B be two non-empty sets. A relation between two sets A and B is a subset of $A\times B$ and is denoted by R (or ρ). Thus $R\subseteq A\times B$. When $R\subseteq A\times A$, then R is said to be a relation on the set A.

If $(a,b)\in R$, then obviously $(a,b)\in A\times B$ and we write aRb and read as 'a is related to b'.

Again if $(a,b)\notin R$ then we write $a(\sim R)b$ or $a\overline{R}b$ and read as " a is not related to b. "

We may described a relation R in the set builder notation as

$$R=\{(a,b):\ a\in A,\ b\in B\}.$$

Illustration. (i) If $A=\{2,3\}$, $B=\{1,2,3,4\}$, then

$$A\times B=\{(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4)\}.$$

Then $R=\{(2,2),(2,3),(2,4),(3,3),(3,4)\}$, a subset of $A\times B$ be a relation between two set sets A and B which can also be written as $R=\{(a,b):a\in A,b\in B \text{ and } a\le b\}$.

(ii) If $A=\{a,b,c\}$, then

$$A\times A=\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$$

Let $R=\{(x,y):x,y\in A \text{ and } x=y\}$ be a relation.

Then $R=\{(a,b),(b,b),(c,c)\}$.

Note : With the help of the Theorem of Art 1.3 we can say there exist 2^{n^2} number of relations on a set A having n number of elements. [*W.B.U.Tech 2008*]

Domain and Range of a Relation

The *domain* of a relation is the set of all first co-ordinate of the ordered pairs in the relation and the range is the set of all second co-ordinate of the ordered pairs in the relation.

In the above Illustration (i), the domain is $\{2,3\}$ and the range is $\{1,2,3,4,\}$.

Universal Relation.

A relation R in a set A is said to be *universal relation* if $R=A\times A$ i.e., if aRb for all $a,b\in A$.

Illustration. (iii) Let $A=\{a,c\}$. Then

$R=\{(a,a),(a,c),(c,a),(c,c)\}$ is a universal relation.

(iv) Consider the set N of natural numbers.

Then $R = \{(a,b): a, b \in N \text{ and } a+b \in N\}$ is a universal relation.

Void Relation.

If $R = \phi$ (void set), then $a\,R\,b$ does not hold for any pair of elements a, b of a set A. This relation R is said to be *void or null* relation in A.

Illustration.(v) If A be the set of all women and if

$R = \{(a,b): a, b \in A \text{ and } a \text{ is a brother of } b\}$, then $R = \phi$ and hence R is a void relation.

Identity Relation

A relation in a set A is said to be identity relation if $a\,R\,b$ holds for all $a, b \in A$ only when $a = b$ and is denoted by I_A.

Thus $I_A = \{(a,b): a, b \in A \text{ and } a = b\}$

In the illustration (ii), the relation R is the identity relation.

Inverse Relation

Let R be a relation from A to B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A is such that bRa hold if $a\,R\,b$ hold.

That is, $R^{-1} = \{(b,a): b \in B, a \in A \text{ and } (a,b) \in R\}$.

Illustration (vi) Let $A = \{a, b, c\}, B = \{1, 2, 3\}$ and

$R = \{(a,2)\,(a,3)\,(b,1)\,(c,3)\}$

Then $R^{-1} = \{(2,a), (3,a), (1,b), (3,c)\}$.

Reflexive Relation.

Let R be a relation on a set A. Then R is said to be a *reflexive* relation if $(a,a) \in R\ \forall\, a \in A$ i.e., if $a\,R\,a\ \ \forall\ a \in A$.

Illustration (vii) Let $A = \{a, b, c\}$.

Then $R = \{(a,a), (b,a), (b,b), (a,c), (c,c)\}$ is reflexive as $(a,a), (b,b)$ and $(c,c) \in R$ but $R = \{(a,a), (b,a), (a,c), (c,c), (b,c)\}$ is not reflexive, as $(b,b) \notin R$.

Symmetric Relation.

Let R be a relation on a set A. Then R is said to be a symmetric relation if $(a,b) \in R \Rightarrow (b,a) \in R\ \forall\ (a,b) \in R$, i.e., $a\,R\,b \Rightarrow b\,R\,a$.

Illustration (viii) Let $A = \{1, 2, 3, 4\}$.

Then $R = \{(1,2), (1,3), (3,3), (2,1), (3,1)\}$ is symmetric but

$R = \{(1,2), (1,3), (3,3), (3,1)\}$

is not symmetric, as $(2,1) \notin R$ though $(1,2) \in R$.

Transitive Relation

Let R be a relation on a set A. Then R is said to be a transitive relation. if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$;

i.e., if $a\,R\,b$ and $b\,R\,c \Rightarrow a\,R\,c$.

Illustration (ix) Let N be the set of all natural numbers and let R be the relation in N defined by "x is less or equal to y"

i.e., $R = \{(x,y): x, y \in N \text{ and } x \le y\}$

Let $(a,b), (b,c) \in R$.

Then $a \le b,\ b \le c \Rightarrow a \le c \Rightarrow (a,c) \in R$

Thus, $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$.

Hence R is transitive relation.

Anti Symmetric Relation

Let R be a relation on a set A. Then R is said to be a anti-symmetric relation if $(a,b) \in R, (b,a) \in R \Rightarrow a = b$,

i.e., $a\,R\,b$ and $b\,R\,a \Rightarrow a = b$.

In illustration (ix), $(a,b) \in R,\ (b,a) \in R\ \Rightarrow a \le b,\ b \le a \Rightarrow a = b$

So the relation R in (ix) is anti-symmetric.

1.1.10. Equivalence Relations

Let R be a relation on a set A. Then R is said to be an equivalence relation if R is reflexive, symmetric and transitive.

Illustration (x) Let A be the set of all triangles in a plane and R be the relation in A defined as $x\,R\,y$ if x is similar (or congruent) to y, $\forall\, x, y \in A$. Then since every triangle is similar to itself, so $x\,R\,x\ \forall\, x \in A$. So R is reflexive. Now if a triangle x is similar to a triangle y, then y is similar to x.

So $x\,R\,y \Rightarrow y\,R\,x\ \forall\ x, y \in A$. Hence R is symmetric. Again if a triangle x is similar to a triangle y and y is similar to a triangle z, then x must be similar to z

i.e., $x\,R\,y$ and $y\,R\,z \Rightarrow x\,R\,z\ \forall\, x, y, z \in A$. So, R is transitive.

Thus R is an equivalence relation in A.

Equivalence class or Equivalence set

Let R be an equivalence relation on a non-empty set A and a be an arbitrary element of A. Then the elements $x \in A$ satisfying $x\,R\,a$ constitute a subset of A. This subset is called an equivalence class or equivalence set and is denoted by $[a]$ or $\bar{a}$ or $cl(a)$ or C_a or A_a.

Thus $[a] = \{x : x \in A \text{ and } x\,R\,a\}$.

Illustration In illustration (x), the equivalence class $[a]$ is the set of all triangles of A which are similar to the triangle 'a'. Thus if $b \in [a]$, then the triangle b is similar to the triangle a.

Theorem. Let A be a non-empty set and R be an equivalence relation in A. Then for all $a, b \in A$.

(i) $a \in [a]$

(ii) If $b \in [a]$, then $[a] = [b]$

(iii) $[a] = [b]$ iff $a\,R\,b$

(iv) Either $[a] = [b]$ or $[a] \cap [b] = \phi$

i.e., two equivalence classes are either disjoint or identical.

(v) $A = \bigcup [a]$, the union all equivalence classes is the set A.

Proof: Beyond the scope of the book.

Partitions.

Let A be a non-empty set. Then a set $P = \{A_1, A_2, A_3, \cdots\}$ of non-empty subsets of A will be called a partition of A if

(i) $\bigcup_i A_i = A$ and (ii) $A_i \cap A_j = \phi$ for $i \neq j$.

Illustration. Let $A = \{a, b, c, d, e, f\}$

Then $P_1 = \{\{a, c\}, \{b\}, \{d, e, f\}\}$, $P_2 = \{\{a\}, \{b, f\}, \{c, d, e\}\}$ etc are partitions of A.

Fundamental Theorem on Equivalence Relations.

An equivalence relation R on a non-empty set A determines a partition of A and conversely, a partition of A defines an equivalence relation on A.

Proof. Beyond the scope of the book.

1.1.11. Illustrative Examples.

Ex. 1. *Give an example of relation which is reflexive but is neither symmetric nor transitive.*

Let $A = \{1, 2, 3\}$ and R be the relation defined as $R = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$. Then R is reflexive as $(x, x) \in R\ \forall\ x \in A$ but is not symmetric as $(1,2) \in R$ and $(2,1) \notin R$. Again $(1,2) \in R$ and $(2,3) \in R$ but $(1, 3) \notin R$. Hence R is not transitive.

So the relation R is reflexive but is neither symmetric nor transitive.

Ex. 2. *Give an example of a relation which is reflexive and transitive but not symmetric.*

Let N be the set of all natural numbers and R be the relation defined as

$R = \{(x, y) : x, y \in N \text{ and } x \text{ is a divisor of } y\}$

As x is a divisor of x, so $(x, x) \in R$.

Therefore R is reflexive.

Again, if x is a divisor of y, then y cannot be a divisor of x.

Thus $(x, y) \in R \not\Rightarrow (y, x) \in R\ \forall\ x, y \in N$.

Hence R is not symmetric.

Finally x is a divisor of y and y is a divisor of z implies x is divisor of z. Thus $(x,y)\in R,(y,z)\in R \Rightarrow (x,z)\in R$.

Therefore R is transitive.

Ex. 3. *Let R be a relation in the set of integers Z defined by xRy if $x-y$ is divisible by 5 for all $x,y\in Z$*

i.e. $R=\{(x,y): x,y\in Z$ and $x-y$ is divisible by $5\}$

Prove that R is an equivalence relation. Find all the distinct equivalence classes of R.

Let $x\in Z$. Then $x-x=0$ and 0 is divisible by 5

So, $(x,x)\in R\ \forall x\in Z$. Thus R is reflexive.

Again let $x,y\in Z$ and $(x,y)\in R$

Then $(x,y)\in R \Rightarrow (x-y)$ is divisible by 5

$\Rightarrow -(y-x)$ is divisible by 5 $\Rightarrow (y-x)$ is divisible by 5

$\Rightarrow (y,x)\in R$.

Therefore R is symmetric.

Finally let $x,y,z\in Z$ and $(x,y)\in R$, $(y,z)\in R$

$\therefore$ $(x,y)\in R$ and $(y,z)\in R \Rightarrow (x-y)$ and $(y-z)$ both are divisible by 5

$\Rightarrow \{(x-y)+(y-z)\}$ is divisible by $5 \Rightarrow (x-z)$ is divisible by 5

$\Rightarrow (x,z)\in R$.

Hence R is transitive. Thus R is an equivalence relation.

For this relation, the equivalence classes are $Z_r=\{r+5n : n\in Z\}$ and $r=0,1,2,3,4$. Thus all the distinct equivalence classes are

$$Z_0=\{0,\pm 5,\pm 10,\pm 15,\cdots\}=\{\cdots -15,-10,-5,0,5,10,15\cdots\}$$

$$Z_1=\{1,1\pm 5,1\pm 10,\cdots\}\ =\{\cdots -9,-4,1,6,11,\cdots\}$$

$$Z_2=\{\cdots 2,2\pm 5,2\pm 10,\cdots\}\ =\{\cdots,-8,-3,2,7,12,\cdots\}$$

$$Z_3=\{\cdots 4,4\pm 5,4\pm 10,\cdots\}\ =\{\cdots,-7,-2,3,8,13,\cdots\}$$

$$Z_4=\{\cdots 4,4\pm 5,4\pm 10,\cdots\}\ =\{\cdots,-6,-1,4,9,14,\cdots\}$$

Note. (1) *The above relation R is said to be the relation of congruence (mod 5). Thus, if $(a,b)\in R$, then a is said to be congruent to $b \pmod 5$ and is expressed as $a\equiv b \pmod 5$. e.g* $29\equiv 4 \pmod 5$, $16\equiv 1 \pmod 5$.

(2) *The above distinct classes Z_0, Z_1, Z_2, Z_3, Z_4 are also called the classes of Residues of Z (mod 5).*

Ex. 4. *On the set Z of integers, define a binary relation ρ by $a\rho b$ if and only if a+b be even. Show that ρ is an equivalence relation.*

Here the relation ρ in the set Z is defined as

$$\rho=\{(a,b): a+b \text{ is even } \forall\ a,b\in Z\}.$$

As $a+a=2a$ be even $\forall\ a\in Z$, so $a\rho a\ \forall\ a\in Z$. Therefore ρ is reflexive

Let $a,b\in Z$ and $a\rho b$. Then $a+b$ is even and so $b+a$ is even. Hence $b\rho a$.

$\therefore$ $a\rho b \Rightarrow b\rho a\ \forall\ a,b\in Z$

$\therefore$ ρ is symmetric.

Next let $a,b,c\in Z$ and $a\rho b$, $b\rho c$

$\therefore$ $a+b$ and $b+c$ are even.

$\Rightarrow a+2b+c$ is even $\Rightarrow a+c$ is even. $\Rightarrow a\rho c$

$\therefore$ $a\rho c$ and $b\rho c \Rightarrow a\rho c\ \forall\ a,b,c\in Z$

$\therefore$ ρ is transitive.

Hence ρ is an equivalence relation.

Ex. 5. *A relation ρ on the set of integers Z is defined by $\rho=\{(a,b): a,b\in Z$ and $|a-b|\le 5\}$. Is the relation reflexive, symmetric and transitive?* [W.B.U.T. 2007]

Let $a \in Z$. Then $|a-a| = 0 < 5$. Therefore $a \rho a$ holds $\forall\ a \in Z$.

So ρ is reflexive.

Next let $a, b \in Z$ and $(a, b) \in \rho$.

Then $|a-b| \le 5 \Rightarrow |b-a| \le 5 \Rightarrow (b, a) \in \rho$

$\therefore$ ρ is symmetric.

Again let $a, b, c \in Z$ and $(a, b) \in \rho$, $(b, c) \in \rho$

Then $|a-b| \le 5$ and $|b-c| \le 5$

Now, $|a-c| = |(a-b)+(b-c)| \le |a-b| + |b-c| \le 5+5 = 10$

$\therefore$ $|a-c| \le 5$ does not hold always.

$\therefore$ $(a, b) \in \rho, (b, c) \in \rho$ do not always imply $(a, c) \in \rho$

Hence ρ is non transitive.

Ex. 6. *Determine the nature of the relation R on the set Z defined by aRb iff $a, b \in Z$ and $ab \ge 0$.*

Let $a \in Z$. Then $a \cdot a \ge 0$. So aRa holds $\forall\ a \in Z$

Therefore R is reflexive.

Next let $a, b \in Z$ and aRb. Then $ab \ge 0 \Rightarrow ba \ge 0 \Rightarrow bRa$.

So R is symmetric.

Again let $a, b, c \ge 0$ and aRb, bRc

Then $ab \ge 0, bc \ge 0 \Rightarrow (ab)\cdot(bc) \ge 0 \quad \Rightarrow ac \cdot b^2 \ge 0$

$\Rightarrow ac \ge 0$, provided $b^2 \ne 0$ i.e., $b \ne 0$

e.g. if $a = 1, b = 0, c = -3$, then $ab = 1 \cdot 0 = 0 \ge 0$, $bc = 0 \cdot (-3) = 0 \ge 0$

but $ac = 1 \cdot (-3) = -3 < 0$

Hence, $ab \ge 0$, $bc \ge 0$ do not always imply $ac \ge 0$. So R is not transitive.

Ex. 7. *Show that the following relation R on Z is an equivalence relation:*

$$R = \{(a, b) : a, b \in Z \text{ and } a^2 + b^2 \text{ is a multiple of } 2\}$$

[W.B.U.T. 2008]

Let $a \in Z$. Then $a^2 + a^2 = 2a^2$ which is a multiple of 2.

Therefore $(a, a) \in R, \forall a \in Z$. $\therefore$ R is reflexive.

Again let $a, b \in Z$ and $(a, b) \in R$.

Then $a^2 + b^2$ is a multiple of $2 \Rightarrow b^2 + a^2$ is a multiple of $2 \Rightarrow (b, a) \in R$

$\therefore$ R is symmetric.

Lastly let $a, b, c \in Z$ and $(a, b) \in R$, $(b, c) \in R$

Then $a^2 + b^2$ and $b^2 + c^2$ both are multiple of 2

$\Rightarrow (a^2 + b^2) + (b^2 + c^2)$ is a multiple of 2

$\Rightarrow (a^2 + c^2) + 2b^2$ is a multiple of $2 \Rightarrow a^2 + c^2$ is a multiple of 2

$\Rightarrow (a, c) \in R$

$\therefore$ $(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R$

$\therefore$ R is transitive. Hence R is an equivalence relation.

Ex. 8. *Let R be a relation defined on $Z \times Z$ by "$(a, b)\, R\, (c, d)$" if and only if $a + d = b + c$ for $(a, b), (c, d) \in Z \times Z$. Show that R is an eqivalence relation.*

Let $(a, b) \in Z \times Z$

Then $a + b = b + a \Rightarrow (a, b)\, R\, (a, b) \quad \forall\ (a, b) \in Z \times Z$

$\therefore$ R is reflexive.

Again let $(a, b), (c, d) \in Z \times Z$ and $(a, b)\, R\, (c, d)$

Then $a + d = b + c \Rightarrow c + b = d + a \Rightarrow (c, d)\, R\, (a, b)$

$\therefore$ R is symmetric.

Lastly let $(a, b), (c, d), (m, n) \in Z \times Z$

and $(a, b)\, R\, (c, d)$, $(c, d)\, R\, (m, n)$

Then $a + d = b + c$ and $c + n = d + m$

$\Rightarrow (a + d) + (c + n) = (b + c) + (d + m)$

$\Rightarrow (a+n)+(c+d)=(b+m)+(c+d)$

$\Rightarrow a+n=b+m \Rightarrow (a,b)R(m,n)$

$\therefore (a,b)R(c,d)$ and $(c,d)R(m,n) \Rightarrow (a,b)R(m,n)$

$\therefore R$ is transitive. Hence R is an equivalence relation.

Ex. 9. *Show that the intersection of two equivalence relations is also an equivalence relation.*

Let R_1 and R_2 be two equivalence relations on the set A. Then $(a,a) \in R_1$, $(a,a) \in R_2\ \forall\ a \in A \Rightarrow (a,a) \in R_1 \cap R_2\ \forall\ a \in A$

$\therefore R_1 \cap R_2$ is reflexive

Again let $a, b \in A$ and $(a,b) \in R_1 \cap R_2$

Then $(a,b) \in R_1$ and $(a,b) \in R_2$

$\Rightarrow (b,a) \in R_1$ and $(b,a) \in R_2$ [$\because$ R_1, R_2 are symmetric]

$\Rightarrow (b,a) \in R_1 \cap R_2$

$\therefore (a,b) \in R_1 \cap R_2 \Rightarrow (b,a) \in R_1 \cap R_2$

$\therefore R_1 \cap R_2$ is symmetric

Lastly let $a,b,c \in A$ and $(a,b) \in R_1 \cap R_2$, $(b,c) \in R_1 \cap R_2$.

Then $\{(a,b) \in R_1 \text{ and } (a,b) \in R_2\}$ and $\{(b,c) \in R_1 \text{ and } (b,c) \in R_2\}$

$\Rightarrow \{(a,b) \in R_1 \text{ and } (b,c) \in R_1\}$ and $\{(a,b) \in R_2 \text{ and } (b,c) \in R_2\}$

$\Rightarrow (a,c) \in R_1$ and $(a,c) \in R_2$ [$\because R_1, R_2$ are transitive]

$\Rightarrow (a,c) \in R_1 \cap R_2$

$\therefore R_1 \cap R_2$ is transitive.

Hence $R_1 \cap R_2$ is an equivalence relation.

1.1.12. Partial Ordering Relation.

Let S be a non empty set and $\preccurlyeq$ be a relation in S. $\preccurlyeq$ is called a 'partially order' if the following three axioms are satisfied:

(i) For any a in S we have $a \preccurlyeq a$ (Reflexive)

(ii) If $a \preccurlyeq b$ and $b \preccurlyeq a$ then $a=b$ (Antisymmetric)

(iii) If $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$ (Transitive)

The set S with the partially order $\preccurlyeq$ is called a **Partially Ordered Set** or ***PO* Set.**

We write $(S, \preccurlyeq)$ is a PO set in short form.

Note. (1) *In the above definition we say* $\preccurlyeq$ *defines a **partially ordering** of S.*

(2) As discussed above every two elements of S may not be related by $\preccurlyeq$. *Because of this, the word 'partial' is used.*

(3) ***The symbol for partial order relation '$\preccurlyeq$' is often replaced by the symbol*** $\leq$

'Precedes & Succeeds'

In a PO set $(S, \preccurlyeq)$ if for two elements a and b, $a \preccurlyeq b$ we say 'a precedes b' and 'b succeeds a.'

If $a \preccurlyeq b$ and $a \neq b$ we say 'a strictly precedes b' and 'b strictly succeeds a'. These are written as $a \prec b$.

Illustrations.

(i) Let S be a set and $P(S)$ be its power set, i.e. $P(S)$ is collection of all subsets of S. Then the relation $\subseteq$ (subset) is a partially order in $P(S)$ because :

(a) $A \subseteq A$ for all A in $P(S)$

(b) $A \subseteq B$ and $B \subseteq A$ imply $A = B$

(c) $A \subseteq B$, $B \subseteq C$ imply $A \subseteq C$.

(ii) Let Z be set of all integers. Define ρ in such a way that $a \rho b$ hold if b can be expressed as $b = a^r$ for some positive integer r. (e.g. $2 \rho 8$).

Now

(a) since $a = a^1$ so $a \rho a$.

(b) $a \rho b$ and $b \rho a$ imply $b = a^{r_1}$ and $a = b^{r_2}$ (where r_1, r_2 are positive integers). This imply $b^{\frac{1}{r_1}} = b^{r_2}$.

Since r_1, r_2 are positive integers so $r_1 = 1$, $r_2 = 1$

$\therefore a \rho b$ and $b \rho a \Rightarrow b = a$.

(c) $a \rho b, b \rho c \Rightarrow b = a^r, c = b^s$ (r, s are positive integers)

$\Rightarrow c = a^{rs} \Rightarrow a \rho c$ ($\because rs$ is a positive integer)

Thus ρ is a partially order and (Z, ρ) is a PO set.

(iii) Let Z be set of all integers and '/' be a relation defined in such a way that a/b means b is divisible by a. (e.g. 5 / 60). Then we see

(a) a/a for all a in Z.

(b) $3 / -3$ and $-3/3$ but $3 \neq -3$, i.e. the relation '/ 'is not antisymmetric.

[W.B.U.T. 2006]

So $(Z, /)$ is not a PO set.

Dual of a PO set.

Let $(S, \leq)$ be a poset. Let us define an another relation '$\geq$' on S by "$a \geq b$ if and only if $b \leq a$" for all $a, b \in S$.

Then $(S, \geq)$ is a poset and is known as the dual of the poset $(S, \leq)$.

We know prove that $(S, \geq)$ is a poset.

(i) Since $(S, \leq)$ is a poset, so

$a \leq a \; \forall a \in S$

$\therefore a \geq a \; \forall a \in S$

$\therefore \geq$ is reflexive

(ii) Since $a \leq b$ and $b \leq a \Rightarrow a = b \; \forall a, b \in S$,

[$\because \leq$ is antisymmetric]

so $b \geq a$ and $a \geq b \Rightarrow a = b \;\; \forall a, b \in S$

$\Rightarrow \geq$ is antisymmetric.

(iii) $a \leq b$ and $b \leq c \Rightarrow a \leq c \, \forall a, b, c \in S$ [$\because \leq$ is transitive]

$\therefore$ $c \geq b$ and $b \geq a \Rightarrow c \geq a$

$\Rightarrow \geq$ is transitive.

Thus the $(S, \geq)$ is a poset

Comparable and Non-comparable elements in a PO set.

Two elements a and b in a PO set $(S, \preccurlyeq)$ are said to be comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

Two elements are non- comparable if they are not comparable.

We write this as $a \| b$.

Illustration :

Let $S = \{1, 2, 3, 4\}$ be the set. Then the two elements $A = \{2, 4\}$ and $B = \{1, 4\}$ are non-comparable in the PO set $(P(S), \subseteq)$.

Totally Ordered or Linearly Ordered Sets.

A PO set $(S, \preccurlyeq)$ is called totally ordered set if every pair of elements in S is comparable, i.e. for any two elements a, b in S either $a \preccurlyeq b$ or $b \preccurlyeq a$.

Illustration.

(i) The PO set $(Z, \leq)$ is a totally ordered set because for any two integers a and b either $a \leq b$ or $b \leq a$.

(ii) The PO set $(P(S), \subseteq)$ is not totally ordered.

1.1.13. Hasse Diagram of PO set

Immediate predecessor & Immediate successor

Let $(S, \preccurlyeq)$ be a PO set and suppose $a, b \in S$. We say a is an immediate predecessor of b or b is an immediate successor of a if $a \prec b$ and there is no element c in S such that $a \prec c \prec b$, i.e. if a precedes b and S contains no element which lies between a and b.

This is written as $a \prec\prec b$. Sometimes we say b covers a.

Illustration.

(i) Let $S = \{1, 2, 3, 4, 5\}$ be a set. Consider the PO set $(S, \subseteq)$. We see the two elements $A = \{2, 4\}$ and $B = \{2, 4, 5\}$ are such that A is a immediate predecessor of B, i.e. $A \subset\subset B$. On the other hand if $C = \{2, 3, 4, 5\}$ then A is a predecessor of C but not immediate because $A \subset B \subset C$.

(ii) In the totally ordered set $(Z, \leq)$ every element has immediate predecessor and immediate successor.

Hasse diagram of Partial Ordering on a finite set

Let $(S, \preccurlyeq)$ be a finite PO set. We place the points on a plane which represent the elements of S. If an element y is an immediate successor of the element x then we place y higher than x and draw a line joining them. Thus a diagram is created whose vertices represents the elements of S and edge represents the immediate predecessor/ successor relationship. This diagram is known as Hasse diagram of the PO set.

Illustration :

(i) Let $S = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be a set and '/' be a relation defined in S such that a/b mean b is divisible by a. *e.g.* 2/12 but 6/9 is not true etc.

Clearly $(S, /)$ is a PO set. We show the Hasse diagram in the adjacent figure.

Note that '6' is placed higher than '3' because 6 is an immediate successor of 3 and so on. The vertices '2' and '12' are not joined since 2/12 but 2/12.

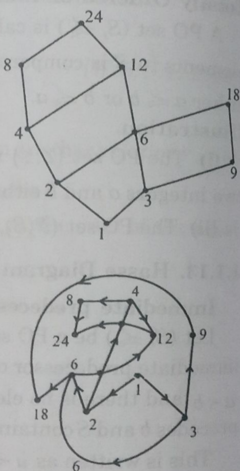


Fig. 3

Note.(1) *Instead of placing the immediate successor at a higher position we could draw arrow from the immediate predecessor to the immediate successor.*

Thus we show an alternate Hasse diagram in the adjacent figure.

This is nothing but a Di-graph

(2) *There can be no circuits (directed) in this Di-graph.*

(3) *The Di-graph of a PO set may not be connected.*

(4) *If $a \prec b$ then there must be a path (directed path) from the vertex a to b. e.g. in Fig.3 we see there is a path $1 \to 2 \to 4 \to 8 \to 24$ because $1/24$.*

(5) *Any PO set gives a Hasse diagram and every Hasse diagram gives a PO set.*

(ii) $A = \{1, 3, 9, 29, 81\}$. Then the *Hasse diagram* of the poset $(A, /)$ is shown in the adjacent figure

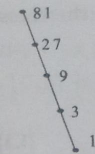


Fig. 4

1.1.14. Illustrative Examples.

Ex1. Find which of the followings relations are partial order:

(i) The relation '<' on Z^+

(ii) The relation ρ on Z defined by $a \rho b$ means $a \le b$

Solution.

(i) Since $a < a$ does not hold good for all $a \in Z^+$, so the relation '<' is not reflexive. Thus the relation '<' is not partial order on Z^+

(ii) Since $a \rho a$, as $a \le a \forall a \in Z$

$\therefore$ ρ is reflxive.

Again

$\therefore$ $a \rho b$ and $b \rho a \Rightarrow a \le b$ and $b \le a$ for all $a, b \in Z$

$\Rightarrow a = b$ $\quad$ $\therefore \rho$ is antisymmetric.

Lastly $a \rho b$ and $b \rho c$

$\Rightarrow a \le b$ and $b \le c$.

$\Rightarrow a \le c$.

$\Rightarrow a \rho c$

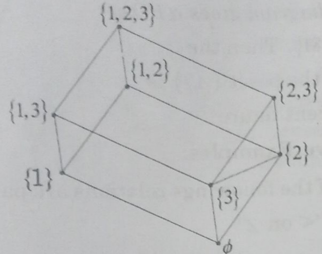
$\therefore \rho$ is transitive.

Hence the relation ρ is partial order

Ex.2. If $S = \{1, 2, 3\}$, then draw the Hasse diagram of the poset $(P(S), \subseteq)$ where $P(S)$ is the power set of S [W.B.U.T.2014]

Solution: Here $P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\} \{1,3\}, \{1,2,3\}\}$.
since the null set ϕ is the subject of all elements of $P(S)$, so ϕ is the lowest point of the Hasse diagram. Again $\{1, 2, 3\}$ is the super set of all sets is $P(S)$, so it is the highest point of the Hasse diagram.

Thus the Hasse diagram of $P(S)$ is drawn below :



Ex. 3. Show that the set of all rational numbers with usual order '$\leq$' is a PO set. Is it totally ordered ? Prove that no element in this set has an immediate successor or an immediate predecessor.

Solution. Let $Q =$ set of all rational numbers.

Obviously '$\leq$' is reflexive, antisymmtric and transitive. So first part is obvious.

For any two rational numbers a and b either $a \leq b$ or $b \leq a$. So, $(Q, \leq)$ is totally ordered.

Let a be an arbitrary element of Q. If possible, suppose b is an immediate successor of a. So $a \leq b$ and $a \neq b$ i.e., $a < b$.

Now $a < \frac{a+b}{2} < b$ and $\frac{a+b}{2} \in Q$.

This contradicts the definition of 'immediate successor'. Thus there exists no immediate successor of a. Similarly a has no immediate predecessor.

Ex. 4. Let (N, ρ) be a PO set where N is set of all natural numbers and 'ρ' stands for divisibility. (i) State which one of the following two subsets of N are linearly (totally) ordered.

$$A = \{2, 8, 32, 4\}, \; B = \{3, 15, 20\}$$

Solution.

The set $A = \{2, 8, 32, 4\}$ is totally ordered because for any two elements a, b we have $a \rho b$ or $b \rho a$.

$B = \{3, 15, 20\}$ is not linearly ordered because $3 \not\rho\, 20$ or $20 \not\rho\, 3$.

Ex. 5. Let ρ be a relation on the set C of all complex numbers and is defined by $(a_1 + ib_1)\rho(a_2 + ib_2)$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$ for all $(a_1 + ib_1)$, $(a_2 + ib_2) \in C$. Prove that (C, ρ) is a poset .

Solution: Since $a_1 \leq a_1$ and $b_1 \leq b_1$, so

$(a_1 + ib_1)\rho(a_1 + ib_1)$.

$\therefore \rho$ is reflexive in C .

Next. let $(a_1 + ib_1)\rho(a_2 + ib_2)$

and $(a_2 + ib_2)\rho(a_1 + ib_1) \forall (a_1 + ib_1) \in C$ and $(a_2 + ib_2) \in C$.

$\therefore \quad (a_1 \leq a_2, b_1 \leq b_2)$ and $(a_2 \leq a_1, b_2 \leq b_1)$

$\Rightarrow a_1 = a_2$ and $b_1 = b_2$

$\Rightarrow (a_1 + ib_1) = (a_2 + ib_2)$

$\therefore \rho$ is antisymmetric in C .

Lastly let $(a_1 + ib_1), (a_2 + ib_2), (a_3 + ib_3) \in C$.

and $\quad (a_1 + ib_1)\rho(a_2 + ib_2)$ and $(a_2 + b_2)\rho(a_3 + ib_3)$.

$\Rightarrow (a_1 \leq a_2, \; b_1 \leq b_2)$ and $(a_2 \leq a_3, \; b_2 \leq b_3)$

$\Rightarrow a_1 \leq a_3$ and $b_1 \leq b_3$

$\Rightarrow (a_1 + ib_1)\rho(a_3 + ib_3)$.

$\therefore \rho$ is transitive

Hence (C, ρ) is a poset.

Ex. 6. Prove that $(N \times N, \preccurlyeq)$, is a PO set where $(a, b) \preccurlyeq (a', b')$ if $a \leq a'$ and $b \leq b'$; N is the set of all natural numbers.

Insert the correct symbol $\prec$, $\succ$ or $\|$ between each of the following two elements of $N \times N$:

(a) (5, 5) ______ (4, 5) (b) (7, 9) ______ (8, 2)(c) (1, 3) ______ (7, 2)

(d) (4, 6) ______ (4, 2) (e) (5, 7) ______ (7, 10) (f) (7, 0) ______ (4, 1)

Solution. Since $a \leq a$ and $b \leq b$ so $(a, b) \preccurlyeq (a, b)$

Thus $\preccurlyeq$ is reflexive.

Let $(a,b) \preccurlyeq (a',b')$ and $(a',b') \preccurlyeq (a,b)$.

Then $a \le a'$ and $a' \le a$; $b \le b'$ and $b' \le b$.

This give i.e. $(a,b) = (a',b')$. Thus $\preccurlyeq$ is anti symmetric.

Let $(a,b) \le (c,d)$ and $(c,d) \le (e,f)$.

Then $a \le c \le e$ and $b \le d \le f$.

This give $a \le e$, $b \le f$ i.e. $(a,b) \le (e,f)$.

Thus $\preccurlyeq$ is transitive. Hence ($N \times N$, $\preccurlyeq$) is a PO set.

The correct symbols are

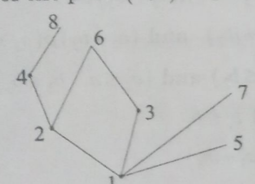
(a) $\succ$ (b) $\|$ (c) $\|$ (d) $\succ$ (e) $\succ$ (f) $\|$.

Ex.7. Draw the Hasse diagram of the poset $(A, /)$ where

$$A = \{1,2,3,4,5,6,7,8\}.$$

Solution.

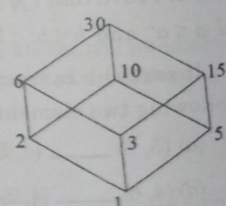
The Hasse diagram of the poset $(A, /)$ is given below



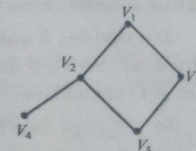
Ex.8. Draw the Hasse diagram of the poset (S_{30}, R) where S_{30} is the set of all positive divisors of 30 and R is the relation division.

Solution. Here $S_{30} = \{1,2,3,5,6,10,15,30\}$.

$\therefore$ The Hasse diagram of the poset (S_{30}, R) is



Ex. 9. Let $A = \{a,b,c,d,e\}$ Determine the relation represented by the given Hasse diagram.



Solution. If R be the relation then R is a subset of $A \times A$ and

$R = \{(a,b), (b,c), (b,d), (c,e), (d,e)\}$

i.e. bRd hold etc.

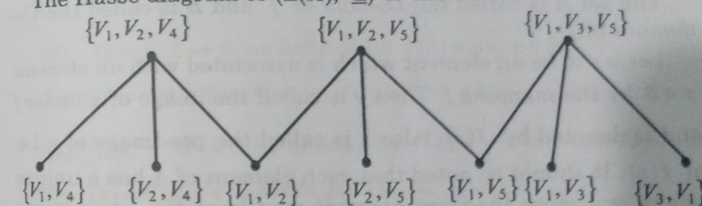
Ex. 10. Let $A = \{V_1, V_2, V_3, V_4, V_5\}$ This set is ordered by the following Hasse diagram

If $L(A)$ is the collection of totally ordered subset of A with 2 or more elements. Find $L(A)$. Draw the Hasse diagram of the PO set $(L(A), \subseteq)$.

Solution. $L(A) = \{\{V_1, V_2, V_4\}, \{V_1, V_2, V_5\}, \{V_1, V_3, V_5\} \{V_1, V_2\}, \{V_1, V_4\}, \{V_1, V_3\}, \{V_1, V_5\}, \{V_2, V_4\} \{V_2, V_5\}, \{V_3, V_5\}\}$

(Note that $\{V_3, V_4\}$ is not totally ordered)

The Hasse diagram of $(L(A), \subseteq)$ is shown below :



Hasse diagram of $L(A)$

The set $\{V_1, V_4\}$ $\{V_2, V_4\}$ $\{V_1, V_2\}$, ... are all minimal element ; the sets $\{V_1, V_2, V_4\}$, $\{V_1, V_2, V_5\}$, $\{V_1, V_3, V_5\}$ are maximal elements of $L(A)$.

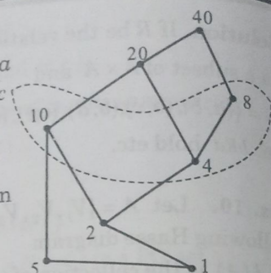
Ex. 11. Let D_{40} be the set of all positive divisors of 40.

Find whether D_{40} is a PO set w.r.t the relation ρ where $a \rho b$ means a divides b.

Draw the Hasse diagram of the PO set (D_{40}, ρ).

Solution. $D_{40} = \{1, 2, 4, 5, 8, 10, 20, 40\}$.

Now a divides a, so $a \rho a$, so ρ is reflexive. Since D_{40} contains only positive integers so $a \rho b$ and $b \rho a$ imply $a = b$.



Hasse diagram of D_{40}

Thus ρ antisymmetric.

If a divides b and b divides c then a divides c. So $a \rho b$ and $b \rho c$ imply $a \rho c$ i.e. ρ is transitive.

So (D_{40}, ρ) is a PO set.

The Hasse diagram of D_{40} is drawn in the given figure.

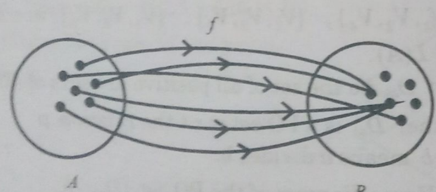
1.1.15. Function or Mapping

Let A and B be two non-empty sets. If there exists a correspondence i.e. a rule, denoted by f, which associates to each element of A a unique element of B, then f is called a function from A to B and is denoted by

$f: A \to B$ or by $A \xrightarrow{f} B$.

The set A is called the *Domain* of f and B is called the *Co-domain* of f.

Let $x \in A$ be an element which is associated with an element $y \in B$ by the mapping f. Then y is called the image of x under f and is denoted by $f(x)$. Also x is called the pre-image of y i.e. of $f(x)$. It should be noted that each element of A has a unique image and more than one element may have the same image but each element of B may not be the image of an element of A (as shown in the following fig.)



The set of all images under f is called the ***Range set*** or ***Image set*** and is denoted by $f(A)$. Thus $f(A) = \{f(x) : x \in A\}$ and $f(A) \subseteq B$.

Function or Mapping can also be defined as sets of ordered pairs as follows :

Let A and B be two non-empty sets. Then a sub-set f of $A \times B$ is called a mapping from A to B if, to each element $a \in A$ there exists an element $b \in B$ such that the ordered pair $(a, b) \in f$.

Illustration (1) Let $A = \{a, b, c\}, B = \{x, y, z, w\}$.

Then (i) $f = \{(a, x), (b, w), (a, z), (c, z)\}$ is not a function, as the same element a of A has different image x and z.

(ii) $f = \{(a, w), (c, z), (a, y)\}$ is not a function on A as the element b has no image.

(iii) $f = \{(a, x), (b, z), (c, y)\}$ is a function as each element of A has a unique image. Here $f(a) = x, f(b) = z, f(c) = y$.

The domain of f is the set A and the range set of f.

i.e., $f(A) = \{x, z, y\}$.

(2) Let $f: Z \to R$ be defined by $f(x) = \sqrt{x}, x \in Z$.

Then f is not a function as $f(-1) = \sqrt{-1} \notin R$.

(3) Let $f: N \to R$ be defined by $f(x) = \frac{1}{x}, x \in N$ is a function since $f(x) \in R \ \forall x \in N$. Here $f(N) \subseteq R$.

Into function : A function $f: A \to B$ is said to be an into function if the range set $f(A)$ is a proper subset of the co-domain B i.e. if $f(A) \subset B$ and we say that f maps A into B.

Illustration. Let $f: N \to N$ be defined by $f(x) = 3x, x \in N$. Then f is an into function as the range set $f(N)$ (the set of all integers multiple of 3) is a proper subset of the co- domain N.

Onto or Surjective Function. A function $f: A \to B$ is said to be an onto function if the range set $f(A)$ is equal to the domain B i.e. every element of B is an image of some element of A and we say that f maps A onto B.

Illustration. Let $f: Z \to Z$ be defined by

$f(x) = x+3, x \in Z$. Then f is an onto function as $f(Z) = Z$.

One-to-One or Injective Function. A function $f: A \to B$ is said to be one-to-one if different elements of A have different images in B i.e. if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or, in otherwords $f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \forall\ x_1, x_2 \in A$.

Illustration. The function $f: Z \to Z$ defined by $f(x) = x+3$ is one-to-one, as

$$f(x_1) = f(x_2) \Rightarrow \quad x_1 + 3 = x_2 + 3 \Rightarrow x_1 = x_2 \ \forall\ x_1, x_2 \in Z$$

Bijective Function. A function $f: A \to B$ is said to be bijective if f is one-to-one and onto.

Illustration. The function $f: Z \to Z$ defined by $f(x) = x+3$ is bijective, as f is one-to-one and onto.

Identity Function : A function $f: A \to A$ is said to be an identity function if $f(x) = x \ \forall\ x \in A$.

We denote this identity function by I_A.

Illustration. Let $A = \{1, 2, 3, 4\}$.

Then $f = \{(1,1), (2,2), (3,3), (4,4)\}$ is an identity function of A.

Constant Function : A function $f: A \to B$ is said to be a constant function if the image of every element of A is same

i.e. if

$$f(x) = c \text{ (constant)} \ \forall\ x \in A.$$

Illustration. Let $f: R \to R$ be defined by $f(x) = 6 \ \forall\ x \in R$. Then f is a constant function.

Composite of Function. Let f and g be two functions such that $f: A \to B$ and $g: B \to C$. Then the composite of the function f and g denoted by $(g \circ f)$ is a function given by $(g \circ f): A \to C$

such that $(g \circ f)(x) = g(f(x)) \ \forall\ x \in A$.

Illustration. Let $f: R \to R$ be defined by $f(x) = \sin x, \ x \in R$

and $g: R \to R$ be defined by $g(x) = x^2, \ x \in R$.

Then the composite function $(g \circ f): R \to R$ is defined as

$(g \circ f)(x) = g(f(x)) = g(\sin x) = \sin^2 x, \ x \in R$.

Theorem. If $f: A \to B$ and $g: B \to C$ be both bijective function, then the composite function $g \circ f: A \to C$ is bijective.

Proof. Beyond the scope of the book.

Inverse Function.

Let $f: A \to B$ be a bijective function. Then the function from B to A which associates to each element $b \in B$ the element $a \in A$ such that $f(a) = b$ is called the inverse function of the function f and is denoted by f^{-1}. Thus $f^{-1}(B) = \{a \in A : f(a) = b\}$.

Note. (1) *Every function may not have inverse. e.g. the function $f: Z \to Z$ defined by $f(z) = |z|$ is neither one-to-one nor onto and hence is not bijective. So this function does not have an inverse.*

(2) *A function which possesses an inverse is said to be invertible.*

Illustration. As the function $f: Z \to Z$ definded by $f(x) = x+3$ is one-to-one and onto (shown in a previous Example) so it is invertible

Let $y \in Z$, the co-domain.

Then $x = y - 3 \ \in Z$ and $f(x) = (y-3)+3 = y$ i.e., $f^{-1}(y) = y - 3$

i.e. $f^{-1}(x) = x - 3, \ x \in Z$, (the co-domain of f^{-1})

Theorem. If $f: A \to B$ and $g: B \to C$ be two one-to-one and onto function, then the composite function $g \circ f: A \to C$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. Beyond the scope of the book.

Illustrative Examples.

Ex. 1. *Show that the function $f: Z_+ \to Z_+$ (the set of all +ve integers) defined by $f(x) = 2x, x \in Z_+$ is injective but not surjective.*

Let x_1 and x_2 be any two elements in Z_+.

Then $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

Therefore f is injective. (i.e., one-to-one).

Let y be any arbitrary element in the co-domain set Z_+.

Then we have $x = \frac{y}{2} \notin Z_+ \ \forall\, y \in Z_+$ such that $f(x) = 2 \cdot \frac{y}{2} = y$.

So each element in the co-domain Z_+ has not pre-image under f. Therefore f is not surjective (i.e. onto)

Ex. 2. *Show that the following function g is neither surjective nor injective :*

$g: R \to R$ defined by $g(x) = x^2, x \in R$.

Let x_1 and x_2 be any two elements in R.

Then $g(x_1) = g(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

Therefore, $g(x_1) = g(x_2)$ does not imply $x_1 = x_2$

$\therefore$ g is not injective.

Let y be any arbitrary element in the co-domain R.

Then we have $x = \pm\sqrt{y} \notin R$, for negative real number y in R (co-domain), such that $f(x) = \left(\pm\sqrt{y}\right)^2 = y$.

So each element in the co-domain set R has no pre-image under f.

$\therefore$ g is not surjective.

Ex. 3. *Show that the function $f: R - \{\sqrt{2}\} \to R$ defined by $f(x) = \frac{x}{x^2 - 2}, x \neq \sqrt{2}$ is surjective but not injective.*

Let x_1 and x_2 be any two elements in $R - \{\sqrt{2}\}$.

Then

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1^2 - 2} = \frac{x_2}{x_2^2 - 2} \Rightarrow (x_1 - x_2)(x_1 x_2 + 2) = 0 \Rightarrow x_1 = x_2$$

or, $x_1 = -\frac{2}{x_2}$

Therefore we cannot say $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

$\therefore$ f is not injective.

Let y be any arbitrary element in the co-domain set R.

Let y be any arbitrary element in the co-domain set R.

If $y = 0$ then we get $x = 0$ in R for which $f(x) = y$.

If $y \neq 0$, then we have $x = \frac{1 \pm \sqrt{1 + 8y^2}}{2y} \in R - \{\sqrt{2}\}$ as $y \in R$ such that $f(x) = y$.

So each element in the co-domain set R has a pre-image under f. $\therefore$ f is surjective.

Ex. 4. *Show that the function $f: R \to R$ defined by $f(x) = 3x + 5, x \in R$ is bijective. Determine f^{-1}.*

Let x_1 and x_2 be any two elements in R. Then

$f(x_1) = f(x_2) \Rightarrow 3x_1 + 5 = 3x_2 + 5 \Rightarrow x_1 = x_2$

$\therefore$ f is one-to one.

Let y be any arbitrary element in R.

Now $y = 3x + 5 \Rightarrow x = \frac{y-5}{3} \in R$ as $y \in R$

such that $f(x) = 3\frac{y-5}{3} + 5 = y$.

So each element in the co-domain set R has a pre-image under f.

$\therefore$ f is onto.

Hence f is a bijective function.

Therefore f has an inverse function $f^{-1}: R \to R$. Since each y in the co-domain set R has a unique pre-image $\frac{y-5}{3}$ in the domain R, So, $f^{-1}: R \to R$ is defined by $f^{-1}(y) = \frac{y-5}{3}$, $y \in R$

i.e., $f^{-1}(x) = \frac{x-5}{3}$, $x \in R$.

Ex. 5. *Let $X = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $Y = [-1,1]$. Show that the function $f: X \to Y$ defined by $f(x) = \sin x \ (x \in X)$ is one-to-one and onto. Find the inverse function $f^{-1}: Y \to X$.*

Let x_1 and x_2 be any two different real numbers in X.

$\therefore \sin x_1 \neq \sin x_2$, as any two different real numbers in $X = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ have not the same sine.

$\therefore x_1 \neq x_2 \Rightarrow \sin x_1 \neq \sin x_2 \Rightarrow f(x_1) \neq f(x_2)$

$\therefore f$ is one-to-one.

Let y be any arbitrary real number in $Y = [-1,1]$.

Then there exist a real number x in $X = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin x = y$. So each element of the co-domain set Y has a pre-image in X under f. Therefore f is onto.

Hence f is a bijective function. So f has an inverse function $f^{-1}: Y \to X$.

Let y be the image of x under f.

Then $y = f(x) = \sin x \ \therefore \ x = \sin^{-1} y$.

Thus $f^{-1}: Y \to X$ is defined by $f^{-1}(y) = \sin^{-1} y$

i.e., $f^{-1}(x) = \sin^{-1} x$, $x \in Y$ is the definition of f^{-1}.

Ex. 6. *Let $A = \{a, b, c\}$ and let $f: A \to A$, $g: A \to A$ be given by $f: a \to b, b \to c, c \to a$; $g: a \to a, b \to c, c \to b$. Evaluating $f \circ g$ and $g \circ f$, show that $f \circ g \neq g \circ f$.*

Now $(f \circ g)(a) = f(g(a)) = f(a) = b$

$(f \circ g)(b) = f(g(b)) = f(c) = a$

$(f \circ g)(c) = f(g(c)) = f(b) = c$

Also $(g \circ f)(a) = g(f(a)) = g(b) = c$

$(g \circ f)(b) = g(f(b)) = g(c) = b$

$(g \circ f)(c) = g(f(c)) = g(a) = a$

Hence, $f \circ g \neq g \circ f$.

Ex. 7. *Let $f: R \to R$ and $g: R \to R$ be defined by $f(x) = x^2 + 3$, $g(x) = x + 6$, Then find the composite $f \circ g$ and $g \circ f$.*

Now, $(f \circ g)(x) = f(g(x)) = f(x+6) = (x+6)^2 + 3 = x^2 + 12x + 39$

and $(g \circ f)(x) = g(f(x)) = g(x^2+3) = x^2 + 3 + 6 = x^2 + 9$.

Ex. 8. *Find the domain of definition of f where*

$f(x) = \sqrt{(3x-1)(7-x)}$

Let $f(x)$ has real value if $(3x-1)(7-x) \geq 0$

i.e . if $3x - 1 \geq 0$ and $7 - x \geq 0$ i.e. if $x \geq \frac{1}{3}$ and $x \leq 7$

i. e. if $\frac{1}{3} \leq x \leq 7$

or, if $3x - 1 \leq 0$ and $7 - x \leq 0$

i.e. if $x \leq \frac{1}{3}$ and $x \geq 7$ which is impossible.

So, the domain of $f(x)$ is $\left[\frac{1}{3}, 7\right]$.

Ex. 9. *Find the domain of $f(x) = \sqrt{4+x} + \sqrt{9-x}$*

Let $f(x)$ has real value if $4 + x \geq 0$ and $9 - x \geq 0$

i.e. if $x \geq -4$ and $x \leq 9$ i.e. if $x \in [-4, \infty)$ and $x \in (-\infty, 9]$

Now $[-4, \infty) \cap (-\infty, 9] = [-4, 9]$

$\therefore$ The domain of $f(x)$ is $[-4, 9]$.

Ex. 10. *Find the domain of the function* $f(x)=\dfrac{x^2+1-x}{x^2-5x+6}$.

Let $f(x)$ has real value when $x^2-5x+6\neq 0$

i.e. when $(x-3)(x-2)\neq 0$ i. e. when $x\neq 3$ or, $x\neq 2$

$\therefore$ The domain is $(-\infty,\infty)-\{3,2\}$.

Ex. 11. *Find the domain of the function* $\dfrac{x^2+1-x}{\sqrt{x^2-5x+6}}$

The function has real value if $x^2-5x+6>0$ as the denominator should not be 0

or, if $x^2-3x-2x+6>0$ or, if $x(x-3)-2(x-3)>0$

or, if $(x-3)(x-2)>0$ i. e. if $x-3>0$ and $x-2>0$

i. e. if $x>3$ and $x>2$ i. e. if $x>3$

or, if $x-3<0$ and $x-2<0$ i.e. if $x<3$ and $x<2$

i. e. if $x<2$

Thus the domain is $(-\infty,2)\cup(3,\infty)$.

Ex. 12. *Find the domain of the function* $\sqrt{\dfrac{1-|x|}{2-|x|}}$

The function is real if $\dfrac{1-|x|}{2-|x|}\geq 0$ ... (1)

and if $2-|x|\neq 0$ ... (2)

First condition holds if case (i) : $1-|x|\geq 0$ and $2-|x|>0$

i. e. if $1\geq|x|$ and $2>|x|$ i. e. $-1\leq x\leq 1$ and $-2<x<2$

i. e. $x\in[-1,1]$ and $x\in(-2,2)$ i. e. $x\in[-1,1]\cap(-2,2)$

i. e. $x\in[-1,1]$ ... (3

or, : Case (ii) : $1-|x|\leq 0$ and $2-|x|<0$

i. e. if $1\leq|x|$ and $2<|x|$ i. e. if $|x|>2$ only

i. e. if $x>2$ or, $x<-2$

i. e. if $x\in(-\infty,-2)\cup(2,\infty)$ ... (4)

Now (2) holds if $|x|\neq 2$ ... (5)

i. e. $x\neq\pm 2$

From (3), (4) and (5) we get the domain as

$[-1,1]\cup(-\infty,-2)\cup(2,\infty)$.

Ex. 13. *Find the domain of the function* $f(x-5)$ *if* $f(x)$*'s domain is* [0, 1].

$\because$ the domain of $f(x)$ is [0, 1]

$\therefore 0\leq x-5\leq 1$ or, $5\leq x\leq 6$

$\therefore$ the required domain is [5, 6].

Ex. 14. *Find the domain of the function* $f(x)=\dfrac{1-\tan x}{\cos x-\sin x}$

Let $f(x)$ has real value if $\cos x-\sin x\neq 0$

or, $\dfrac{1}{\sqrt{2}}\cos x-\dfrac{1}{\sqrt{2}}\sin x\neq 0$

or, $\cos\left(x+\pi/4\right)\neq 0$ i. e. if $x+\dfrac{\pi}{4}\neq(2n+1)\pi/2$

or, $x\neq(2n+1)\dfrac{\pi}{2}-\dfrac{\pi}{4}=\left(\dfrac{2n+1}{2}-\dfrac{1}{4}\right)\pi=\dfrac{4n+2-1}{4}\pi=(4n+1)\dfrac{\pi}{4}$

$\therefore$ The domain is $(-\infty,\infty)-\left\{(4n+1)\dfrac{\pi}{4}\right\}$, where n is any integer.

Ex.15. *Find the domain of the function* $f(x)=\cos^{-1}\dfrac{2x-3}{3}$

Let $f(x)$ has real value if $-1\leq\dfrac{2x-3}{3}\leq 1$

i. e. if $-3\leq 2x-3\leq 3$ i. e. if $0\leq 2x\leq 6$

i. e. if $0\leq x\leq 3$ i. e. if $x\in[0,3]$

So the required domain is [0, 3].

Ex. 16. *Find the domain of definition of $f(x)$ where $f(x)=\sqrt{\log\frac{4x-x^2}{3}}$, x and $f(x)$ are real*

Let $f(x)$ is real if $\log\frac{4x-x^2}{3}\ge 0$

i. e. if $\frac{4x-x^2}{3}\ge 1$ ... (1)

i. e. if $4x-x^2\ge 3$ i. e. if $x^2-4x+3\le 0$

i. e. if $x-1\le 0$ and $x-3\ge 0$ i.e. if $(x-1)(x-3)\le 0$

i. e. if $x\le 1$ and $x\ge 3$ which is impossible.

or, if $x-1\ge 0$ and $x-3\le 0$ i. e. if $x\ge 1$ and $x\le 3$

i.e. if $x\in[1,3]$.

Again $\log\frac{4x-x^2}{3}$ is defined if $\frac{4x-x^2}{3}>0$

which is automatically satisfied if (1) holds.

Hence the domain of definition is [1, 3]

Ex. 17. *Find the domain of the function*

$$f(x)=\log_{2x-5}(x^2-3x-10)$$

Let $f(x)$ is defined if

$x^2-3x-10>0$ ... (1)

and $2x-5>0$ and $2x-5\ne 1$ ... (2)

(1) is valid if $x^2-5x+2x-10>0$

or, $x(x-5)+2(x-5)>0$ or, $(x-5)(x+2)>0$

if $x-5>0$ and $x+2>0$ i. e. if $x>5$ and $x>-2$

i. e. if $x>5$

or, if $x-5<0$ and $x+2<0$ i. e. if $x<5$ and $x<-2$

i. e. if $x<-2$ only

$\therefore$ (1) is valid if $x\in(5,\infty)\cup(-\infty,-2)$ ... (3)

(2) is valid if $x>\frac{5}{2}$ and $x\ne 3$

i. e. if $x\in\left(\frac{5}{2},\infty\right)-\{3\}$ ... (4)

$\therefore$ the required domain is the common part of (3) and (4) which is $(5,\infty)$

Ex. 18. *Find the range or image set of the function*

$$f(x)=\frac{1}{2-\cos 4x}.$$

Since $f(x)=\frac{1}{2-\cos 4x}$ or, $\cos 4x=\frac{2f(x)-1}{f(x)}$.

Since, $-1\le\cos 4x\le 1$ we have $-1\le\frac{2f(x)-1}{f(x)}\le 1$ ... (1)

Taking $f(x)>0$ we get $-f(x)\le 2f(x)-1\le f(x)$

From the 1st part, $3f(x)\ge 1$ or, $f(x)\ge\frac{1}{3}$

From the 2nd part $f(x)\le 1$ $\therefore\frac{1}{3}\le f(x)\le 1$.

$\therefore$ Range of $f(x)$ is $\left[\frac{1}{3},1\right]$

Taking $f(x)<0$, we get from (1) $-f(x)\ge 2f(x)-1\ge f(x)$

From the 1st part, $3f(x)\le 1$ or, $f(x)\le 1/3$

From the 2nd part, $f(x)\ge 1$ which is impossible..

Ex. 19. *Find the range of the function* $f(x)=\frac{x}{x^2+1}$

Let $y=\frac{x}{x^2+1}$ $\therefore yx^2+y=x$ or, $yx^2-x+y=0$.

x has real value if its discriminant ≥ 0 i.e. $(-1)^2-4\cdot y\cdot y\ge 0$

or, $1-4y^2\ge 0$ or, $y^2\le\frac{1}{4}$ or, $-\frac{1}{2}\le y\le\frac{1}{2}$

$\therefore$ the range is $\left[-\frac{1}{2},\frac{1}{2}\right]$.

Ex. 20. *Find the range of the function*

$$f(x)=\log_2\frac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}$$

Let $y=\log_2\dfrac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}$

or, $2^y=\dfrac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}=\left(\dfrac{1}{\sqrt{2}}\sin x-\dfrac{1}{\sqrt{2}}\cos x\right)+3$

or, $2^y=\sin\left(x-\dfrac{\pi}{4}\right)+3$ or, $2^y-3=\sin\left(x-\dfrac{\pi}{4}\right)$

Now, $-1\le\sin\left(x-\dfrac{\pi}{4}\right)\le 1$ $\quad\therefore -1\le 2^y-3\le 1$

or, $2\le 2^y\le 4$ $\quad$ or, $2^1\le 2^y\le 2^2$

$\therefore 1\le y\le 2$ $\quad$ [$\because$ the base $2>1$]

$\therefore$ the range is [1, 2]

Ex. 21. *Find the inverse function of* $f(x)=x^2, x\ge 0$

Let $y=x^2$ $\quad$ or, $x=\pm\sqrt{y}$.

To make it single valued we consider $x=\sqrt{y}$ (we may take $x=-\sqrt{y}$ also). $\quad\therefore$ the inverse function $f^{-1}(x)=\sqrt{x}, x\ge 0$.

Ex. 22. *Find the inverse of the function*

$$f(x)=\log\left(x+\sqrt{x^2+1}\right)$$

Let $y=\log\left(x+\sqrt{x^2+1}\right)$

or, $x+\sqrt{x^2+1}=e^y$ $\quad$ or, $\sqrt{x^2+1}=e^y-x$

or, $x^2+1=e^{2y}+x^2-2xe^y$ $\quad$ or, $1=e^{2y}-2xe^y$ or, $x=\dfrac{e^{2y}-1}{2e^y}$

Interchanging x and y we get the inverse function

$f^{-1}(x)=\dfrac{e^{2x}-1}{2e^x}$. $\quad$ or, $f^{-1}(x)=\dfrac{1}{2}\left(e^x-e^{-x}\right)$.

Ex. 23. *If* $f(x)=x-1$ *and* $\phi(x)=\dfrac{1}{x+1}$ *find the formula of the function* $\phi\circ f$, $f\circ f$ *and find* $\phi\circ f\left(\dfrac{1}{2}\right)$.

Let $(\phi\circ f)(x)=\phi\{f(x)\}=\dfrac{1}{f(x)+1}=\dfrac{1}{x-1+1}=\dfrac{1}{x}$.

$\therefore(\phi\circ f)(x)=\dfrac{1}{x}$.

$(f\circ f)(x)=f(f(x))=f(x-1)=(x-1)-1=x-2$

$\therefore(f\circ f)(x)=x-2$

$(\phi\circ f)\left(\dfrac{1}{2}\right)=\phi\left(f\left(\dfrac{1}{2}\right)\right)=\phi\left(\dfrac{1}{2}-1\right)=\phi\left(-\dfrac{1}{2}\right)=\dfrac{1}{-\frac{1}{2}+1}=\dfrac{1}{1/2}=2$

Exercise 1

I. Short Answer Questions

1. (a) If $A=\{1,2,3,4,5,6,7,8,9\}, B=\{2,3,4,5\}, C=\{2,4,6,8\}$ and $D=\{4,5,6,7\}$, find (i) $B\cup C$

(ii) $B\cap D$ (iii) verify that $(B\cup C)\cup(A\cup D)=A$

(iv) verify the distributive law $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

(v) verify the De-Morgan's law

$A-(B\cup C)=(A-B)\cap(A-C)$

$A-(B\cap C)=(A-B)\cup(A-C)$

(b) If $U=\{-1,-2,0,3,5,10,12,13,16\}$, $P=\{-2,3,5,12\}$, $Q=\{-1,-2,0,5,12,13\}$ then verify $(P\cap Q)'=P'\cup Q'$.

(c) If $U=\{x: x\in Z \text{ and } 1\le x\le 10\}$

$A=\{x: x\in U \text{ and } x \text{ is a prime number}\}$,

$B=\{x: x\in U \text{ and } x \text{ is even}\}$,

then find $A-B, B-A, A\,\Delta\, B$.

(d) If $A=\{1,2,3\}, B=\{2,3,4\}$, write down the following sets

(i) $\{(x,y):(x,y)\in A\times B \text{ and } x<y\}$

(ii) $\{(x,y):(x,y)\in A\times B \text{ and } x\geq y\}$

2. If A, B and C are any sets, prove that

(i) $A\cap B=A$ if and only if $A\subseteq B$

(ii) $A\cap B=\phi$ if and only if $A-B=A$

(iii) $A\cap(A\cup B)=A\cup(A\cap B)=A$

(iv) $(A-B)\cup A=A$ (v) $A\cap(B-C)=(A\cap B)-C$

3. Given that $A=\{1,2,3,4,5\}$, $B=\{1,2,4,7,8\}$ and $C=\{1,2,3,5,7,9\}$, find $(A\cup B)-C$.

4. Find $A\cup(B\Delta C)$ when $A=\{1,2,3,4,5\}$, $B=\{1,2,4,8\}$, $C=\{1,2,3,5,7\}$

5. If $X\cup Y=X$ then prove that $Y\subseteq X$.

6. Prove that if $A-B=A$ then A and B are disjoint.

7. If $A=\{1,2\}$, $B=\{2,3\}, C=\{3,4\}$, then find

(i) $(A\times B)\cup(A\times C)$ (ii) $(A\times B)\cap(A\times C)$

8. Let A, B, C be subsets of a Universal set U. Prove that

(i) $A\times(B\cup C)=(A\times B)\cup(A\times C)$

(ii) $A\times(B-C)=(A\times B)-(A\times C)$

9. If $A\cap C=B\cap C$ and $A\cap C'=B\cap C'$, prove that $A=B$.

10. Show that $A\times(B-C)=(A\times B)\cap(A\times C')$ where C' is the complement of C in U.

11. Prove that $A\cup B=A\cap B\Rightarrow A=B$.

12. Prove $(A\Delta B)'=(A'\cup B)\cap(A\cup B')$

13. Prove that $(A\cup B)\cap B'=A$ if and only if $A\cap B=\phi$.

14. Prove that $(A\cup B)\Delta(A\Delta B)=A\cap B$

15. Let P, Q, R be subsets of a universal set S and $P\Delta Q=R$. Prove that (i) $P=Q\Delta R$, (ii) $Q=R\Delta P$

16. Prove that $(A-B)'=A'\cup B$.

17. Verify $(A-B)\cup(A\cap B)=A$

18. Verify $A-(B\cup C)=(A-B)\cap(A-C)$

19. Simplify the following set : $A'\cup B'\cup(A\cap B\cap C')$

20. Define the cartesian product of two sets and give an example.

21. If the relation R is defined on the set of real numbers by the rule $a\,R\,b$ hold if $|a-b|\leq\frac{1}{2}$ then prove that R is not equivalence relation.

22. If A, B, C are sets, prove algebraically that

$$A\times(B\cap C)=(A\times B)\cap(A\times C)$$

23. Define a relation on a set and give an example.

24. If R is the relation from $A=\{1,2,3,4\}$ to $B=\{2,3,4,5\}$, find the elements in R defined by $a\,R\,b$, if a and b are both odd. Write also the domain and range of R.

25. Give an example of a relation that is both symmetric and anti-symmetric.

26. In the set $S=\{0,1,2,3,4,5,\ldots\ldots\ldots,15\}$ the relation R is defined by $2x+3y=30$ where $x,y\in S$. Find whether R is transitive, reflexive, symmetric and anti-symmetric.

27. Give an example of a relation that is symmetric and transitive but not reflexive.

28. (a) If $A=\{1,2,3,4\}\times\{1,2,3,4\}$ and the relation R is defined on A by $(a,b)\ R\ (c,d)$ if $a+b=c+d$, verify that R is an equivalence relation on A.

(b) Determine the nature of the following relations on the set $A=\{1,2,3\}$.

(i) $R=\{(3,3)\}$, (ii) $R=\{(1,2),(2,2)\}$

(iii) $R=\{(1,1),(1,2),(2,3),(2,2),(3,3),(2,1),(3,2)\}$

29. Show that the relation R on the set of integers such that a R b if and only if $a=b$ or $a=-b$ is equivalence relation.

30. If the relation R is defined on set of integers Z such that a R b hold if $a-b=$ even integer, then show that R is equivalence relation.

31. Prove that uniion and intersection of two symmetric relation is symmetric.

32. Find whether the relation R defiined on the set of real numbers as a R b hold iff $|a|>|b|$ is equivalence relation.

33. Find whether the relation "a is a multiple of b" define on the set of natural number is an equivalence relation.

34. Give an example with reasons of a relation which is

(i) reflexive, symmetric but not transitive,

(ii) symmetric, transitive but not reflexive. [W.B.U.T.2007]

35. Show that the following relation R defined on Z is symmetric. transitive but not reflexive : $R=\{(a,b):a,b\in Z \text{ and } ab>0\}$

36. Show that the relation "is perpendicular" to on the set of all straight lines in the plann is symmetric but neither reflexive nor transitive.

37. Is the relation " is brother of " an equivalence relation on a set of human beings ? Why ?

[Hints : Let x be a female human being. So x is not a brother of x and hence $x\bar{R}x$. Thus the relation is not reflexive. Also the relation is not symmetric]

38. Determine the nature of the following relations R on the set Z:

(i) $R=\{(a,b):a,b\in Z \text{ and } a\le b\}$

(ii) $R=\{(a,b):a,b\in Z \text{ and } 2a+3b \text{ is divisible by } 5\}$

39. Let R be the relation in $N\times N$ which is defined by $(a,b)R(c,d)$ if and only if $ad=bc$. Prove that R is an equivalence relation.

40. Examine the nature of the following relation R on the set Q

$R=\{(a,b):a,b\in Q \text{ and } a-b \text{ is an integer}\}$ [W.B.U.Tech.2006]

41. A relation ρ is defined on the set Z by $x\rho y$ if $x^y=y^x\ \forall\, x,y\in Z$. Show that ρ is not an equivalence relation.

42. Examine whether the following define a function :

(i) $A=\{3,1,4,5\}$, $B=\{2,4,8,9\}$ and $f(3)=2,\ f(1)=4,\ f(4)=9,$ $f(5)=8,\ f(5)=2$.

(ii) $A=\{1,2,3,4\}$, $B=\{1,4,9,16\}$ and $f(x)=x^2$ where $x\in A$.

(iii) $X=\{5,6,8\}$, $Y=\{2,0,1,4\}$ and $f=\{(5,0),(6,1),(8,1)\}$.

43. If $f:x\rightarrow$ highest prime factor of x and the domain of f is $\{18, 13, 21, 15, 16, 17\}$, find the range set of f.

44. Show that the function f is injective but not surjective

(i) $f:N\rightarrow N$ defined by $f(x)=x+1, x\in N$

(ii) $f:Z\rightarrow Q$ defined by $f(x)=2^x,\ x\in Z$.

45. Discuss the nature of the following function :

(i) $f:R\rightarrow R$ defined by $f(x)=x^3-x, x\in R$

(ii) $f:R\rightarrow R$ defined by $f(x)=|x|, x\in R$

(iii) $f:R\rightarrow R$ defined by $f(x)=\sin x,\ x\in R$.

46. Determine whether the following function is one-to-one and/ or onto :

$f:R\rightarrow R$ given by $f(x)=3x^3-x$.

47. Given that $A=\{1,2,3,4\}$ and $B=\{x,y,z\}$, how many functions $f:A\rightarrow B$ satisfy $f(1)=x$?

48. If $f:R\rightarrow R$ is given by $f(x)=x+\frac{\pi}{2}$, $g:R\rightarrow R$, is given by $g(x)=\sin x$ then find $g\circ f$ and $f\circ g$, and hence show that $f\circ g\neq g\circ f$.

49. If $f, g : R \to R$ where $f(x) = ax + b$, $g(x) = 1 - x + x^2$ and $(g \circ f)(x) = 9x^2 - 9x + 3$, find the values of a and b.

50. Determine whether the function $f : R \to R$ defined by $f = \{(x, y) : y = x^3\}$ is invertible and if so, determine f^{-1}

51. Define partially ordered set. Illustrate with an example.

52. Prove that the set of integers is a PO set w.r.t the relation $\geq$.

53. Which of the following are PO sets ?

(i) $(Z, >)$, (ii) $(Z, \neq)$, (iii) $(Z, =)$ where Z is set of all integers.

54. Write down all possible partial order relation of the set $\{a, b\}$.

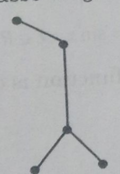
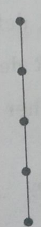
55. Prove that the set $Y = \{1, 2, 3, 4, 6, 9\}$ forms a PO set w.r.t the 'divide' relation.

Draw the Hasse diagram for each.

56. Draw the Hasse diagram for the PO set $(A, /)$ where / stand for divisibility and (i) $A = \{2, 3, 6, 12, 24, 36\}$ [W.B.U.T. 2006]

(ii) $A = \{3, 6, 12, 24, 48\}$

57. Let $X = \{a, b, c, d, e\}$. Determine the relation represented by the following Hasse diagram

(i)  (ii) 

58. Find whether the following PO sets are totally ordered :

(i) $(N, /)$ where N is set of all positive integers and a/b means b is divisible by a.

(ii) The power set $P(A)$ of a set A with two or more elements, respect to $\subseteq$.

59. Draw the Hasse diagram of the PO set $\{P(S), \subseteq\}$ where $S = \{x, y, z\}$

60. Draw the Hasse diagram of the PO set $\{A, /\}$ where $A = \{81, 27, 9, 3, 1\}$.

61. Draw the Hasse Diagram for the divisibility relation on the set $A = \{2, 3, 6, 12, 24, 36\}$

62. In the following PO set $(P(A), \subseteq)$ where $A = \{0, 1, 2\}$ find two non-comparable elements.

Answers

1. (a) (i) $B \cup C = \{2, 3, 4, 5, 6, 8\}$ (ii) $B \cap D = \{4, 5\}$

(c) $\{3, 5, 7\}, \{4, 6, 8, 10\}, \{3, 4, 5, 6, 7, 8, 10\}$

(d) (i) $\{(1, 2), (2, 3), (3, 4), (1, 3), (1, 4), (2, 4)\}$ (ii) $\{(2, 2), (3, 3), (3, 2)\}$

3. $\{4, 8\}$ 4. $\{1, 2, 3, 4, 5, 7, 8\}$

7. (i) $\{(1, 2), (1, 3), (2, 2), (2, 3), (1, 4), (2, 4)\}$ (ii) $\{(1, 2), (2, 3)\}$

19. $A' \cup B' \cup C'$

24. $\{(1,3), (1,5), (3,3), (3,5)\}$; dom $(R) = \{1, 3\}$, Range $(R) = \{3, 5\}$

25. $\{(1,1), (2,2)\}$ on $\{1,2\}$ 26. anti-symmetric

27. $\{(1,1), (2,2), (1,3), (3,1)\}$ on $\{1,2,3\}$

28. (b) (i) symmetric, transitive but not reflexive

(ii) trannsitive but neither reflexive nor symmetric

(iii) reflexive, symmetric but not transitive

32. not eequivalence 33. not symmetric, i.e., not-equivalence

34. (i) Let $A = \{1, 2, 3\}$ and the relation is

$$R = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2), (1, 2), (2, 1)\}.$$

(ii) Let $A = \{a, b, c\}$ and the relation is

$$R = \{(a, a), (b, b), (a, b), (b, a)\}.$$

37. No.

38. (i) reflexive, transitive but not symmetric
(ii) Equivalence relation.

40. Equivalence relation

42. (i) No, (ii) bijective, (iii) one-to-one, into

43. {3, 13, 7, 5, 2, 17}

45. (i) onto but not one-to-one
(ii) Neither one-to-one nor onto
(iii) Neither one-to-one nor onto

46. onto but not one-to-one

47. 27

48. $(f\circ g)(x)=\sin x+\frac{\pi}{2}, (g\circ f)(x)=\cos x, \forall x\in R$

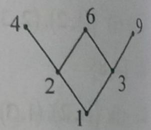
49. $a=3, b=-1$ or, $a=-3, b=2$

50. yes; $f^{-1}(x)=x^{\frac{1}{3}}$

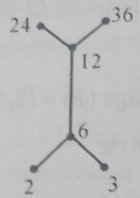
53. (i) no (ii) no, (iii) yes

54. $aRa, bRb, aRa, bRb, aRb; aRa, bRb, bRa$

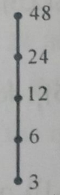
55. Maximal elements are 4, 6, 9, miniml elements are 1; $l.b$ is 1, no u,b, no Sup, Inf is 1



56. (i)



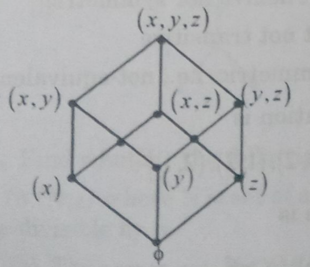
(ii)



57. (i) $(a,c),(b,c),(c,d),(d,e)$ (ii) $(a,b),(b,c),(c,d),(d,e)$

58. (i) Not totally ordered (ii) Not totally ordered.

59.



60.

62. {0} and {1}

II. Long Answer Questions

1. If A, B, C are subsets of a Universal Set U, then show that
(i) $(A-C)\cup(B-C)=(A\cup B)-C$
(ii) $(A-C)\cap(B-C)=(A\cap B)-C$
(iii) $A\cap(B-C)=(A\cap B)-(A\cap C)$

2. Prove that $(A-B)\cup(B-A)=(A\cup B)-(A\cap B)$

3. Prove that $A\cup B=(A\cap B)\,\Delta\,(A\,\Delta\,B)$

4. Using Venn-diagram prove that
(i) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
(ii) $(A-B)\cup(B-A)=(A\cup B)-(A\cap B)$

5. Using the set theory, show that H.C.F and L.C.M. of the numbers 18, 27 are 9 and 54 repectively.
[Hints : H.C.F.= Largest element of $A\cap B$ where A, B are set of factors of the given two numbers. L.C.M. = Least element of $S\cap T$ where S,T are set of multiples of the given two numbers]

6. In a class there are 100 students. Of them 41 can speak in English, 53 can speak in Bengali and 45 can speak in Hindi. Also 16 can speak in both English and Bengali, 17 can speak in both Bengali and Hindi, 12 can speak in both English and Hindi and 5 can speak in all the three languages.
(i) How many persons cannot speak in any of the three languages ?
(ii) How many persons can speak in exactly one of the three languages ?
(iii) How many persons can speak in exactly two of the three languages ?

7. In a survey of 320 persons, number of persons taking tea is 210, taking milk 100 and coffee is 70. Number of persons who take tea and milk is 50, milk and coffee is 30, tea and coffee is 50. The number of persons taking all the three together is 20. Find the number of persons who take neither tea nor milk nor coffee.

8. In a survey concerning the smoking habits of consumers it was found that 55% smoke cigarette A, 50% smoke cigarette B, 42% smoke C, 28% smoke A and B, 20% smoke A and C, 12% smoke B and C and 10% smoke all the three cigarattes.

(i) What percentage do not smoke ?

(ii) What percentage smoke exactly 2 brands of cigarettes ?

9. If R be an equivalence relation in a set A, then show that R^{-1} is also an equivalence relation.

10. Prove that a relation ρ defined on a set A is an equivalence relation if and only if ρ be reflexive and such that $a\rho b$ and $b\rho c$ imply $c\rho a$ for all $a,b,c \in A$.

11. Let $h : R \to R$ be defined by $h(x) = |x| + 20$ and $k : R \to R$ be defined by $k(x) = |x| - x, \forall x \in R$. Find $h \circ k$ and $k \circ h$.

12. Show that $f : R - \{3\} \to R - \{1\}$ given by $f(x) = \dfrac{x-2}{x-3}$ is a bijective and find its inverse.

13. Show that the mapping f is bijective . Determine f^{-1}.

(i) $f : R \to R$ defined by $f(x) = x^3,\ x \in R$

(ii) $f : Q \to Q$ defined by $f(x) = 5x + 2, x \in R$.

14. If $f(x) = x + 5$ and $g(x) = x^2 - 3$ find

(i) $f \circ g$ (ii) $g \circ f$ (iii) $g \circ g$

(iv) $(f \circ f)(-5)$ (v) $(g \circ g)(2)$

15. If $f(x) = 4x - 5$, $g(x) = x^2$ and $\phi(x) = \dfrac{1}{x}$ find the formula for the composite functions :

(i) $f \circ g \circ \phi$ (ii) $\phi \circ f \circ g$ (iii) $g \circ \phi \circ f$

16. Find the inverse of the following functions

(i) $f(x) = 5^{\log x}$ (ii) $f(x) = \sin^{-1}\dfrac{x}{3}, -3 \le x \le 3$

(iii) $f(x)\dfrac{1-x}{1+x}$ (iv) $f(x) = \sin(3x-1)$

(v) $f(x) = x^2 + 1, x \ge 0$ (vi) $f(x) = x^3 - 1$

(vii) $f(x) = (x+1)^2, x \ge -1$ (viii) $f(x) = 2^{x(x-1)}, 1 \le x < \infty$

17. Find which of the following are partial order set

(i) (Z, ρ) when $a\rho b$ means $|a| \le |b|$

(ii) (Z, ρ) when $a\rho b$ means $|a-b| \le 1$

(iii) (Z, ρ) when $a\rho b$ means $a - b \le 0$

(iv) (Z, ρ) when $a\rho b$ means $a + b$ is an even integer.

18. Let $A = \{a,b,c,d\}$ and consider the relation

$R = \{(a,a),(a,b),\ (a,c),(b,b),(c,b),(c,c),(d,b),(d,c),(d,d)\}$.

Show that R is a PO relation. Draw its Hasse diagram.

19. Prove that the sets

$X = \{2, 3, 5, 30, 60, 120, 180, 360\}$

forms a PO set w.r.t the 'divide' relation.

Draw the Hasse diagram for each. Find the maximal and minimal element; greatest and least element; $u.b.$, $l.b.$ Supremum and Infimum of each of the set X.

20. Let $X = \{24, 18, 12, 9, 8, 6, 4, 3, 2, 1\}$ be ordered by the relation 'x divides y'. Find the Hasse diagram.

21. (a) Draw the Hasse diagram for the P.O.Set $(A, /)$ where $/$ stand divisibility

(i) $A =$ set of all factors of 30 (including 1 and 30)

(ii) $A =$ set of all factors of 17

(b) Let $A = \{2,3,5,30,60,120,180,360\}$.

Prove that $(A, /)$ is a PO set. Is it well ordered ? Find the

(i) Successors of 30

(ii) Immediate successor of 120

(iii) Predecessors of 180

(iv) Immediate predecessor of 5

22. In the B. Tech course we say $A < B$ if the paper A is must for studying paper B. In the B. Tech course there are eight papers on Mathematics. The paper codes and their prerequisities are given below:

Paper	: M101	M201	M250	M251	M340	M341	M450	M500
Prerequisities	: None	M101	M101	M250	M201	M340	M201, M250	M450, M251

Construct a PO set regarding this problem and draw a Hasse diagram

23. Find whether the following set is a PO set w.r.t the relation mentioned : $(N, /)$ where $N =$ set of all positive integers and a/b means a divides b.

24. Let D_{36} be the set of all positive divisors of 36. Show that D_{36} is a PO set w.r.t the relation 'divisor'. Draw the Hasse diagram of this PO set. Find (i) the Supremum and Infimum of the set $\{4, 9\}$

Answers

6. (i) 1, (ii) 64, (iii) 30 7. 50, 8. 3, 30

11. $(k \circ h)(x) = 0 \;\forall\, x \in R$

$(h \circ k)(x) = 20$ if $x \geq 0$

$= -2x + 20$ if $x < 0$.

12. $f^{-1}(x) = \dfrac{2-3x}{1-x}$, $\forall x \in R - \{1\}$

13. (i) $f^{-1}(x) = \sqrt[3]{x}, x \in R$ (ii) $f^{-1}(x) = \dfrac{x-2}{5}$, $x \in Q$

14. (i) $x^2 + 2$ (ii) $(g \circ f)(x) = x^2 + 10x + 22$

(iii) $x^4 - 6x^2 + 6$ (iv) 5 (v) -2

15. (i) $\dfrac{4}{x^2} - 5$ (ii) $\dfrac{1}{4x^2-5}$ (iii) $\left(\dfrac{1}{4x-5}\right)^2$

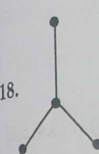
16. (i) $f^{-1}(x) = x^{1/\log 5}\,(x > 0)$ (ii) $f^{-1}(x) = 3\sin x$

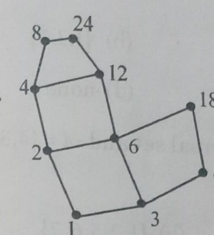
(iii) $f^{-1}(x) = \dfrac{1-x}{1+x}$ (iv) $f^{-1}(x) = \dfrac{1+\sin^{-1}x}{3}$

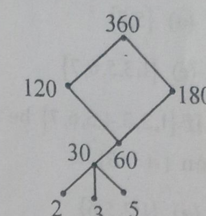
(v) $f^{-1}(x) = \sqrt{x-1}, x \geq 1$ (vi) $f^{-1}(x) = \sqrt[3]{x+1}$

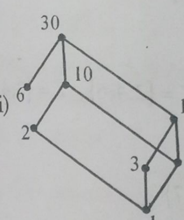
(vii) $f^{-1}(x) = \sqrt{x} - 1, x \geq 0$ (viii) $f^{-1}(x) = \dfrac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$

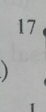
17. (i) no, (ii) no, (iii) yes (iv) no (ρ is not antisymmetric)

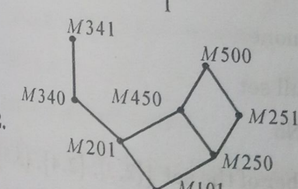
18. 

20. 



21. (i) 

(ii) 

22. 

23. not a PO set.

24.