5.2 TREES & BINARY TREES

5.2.1. Introduction

Application of graph is not possible without the knowledge of tree. In this chapter we define tree and discuss several theorems on it. Later we introduce spanning tree which is an another important notion in graph theory. This appears in numerous instances.

5.2.2. Trees and Related terms :

Tree. A connected graph without any cycle is called a tree.

The graph shown is Fig. 5.2.1 is a tree.

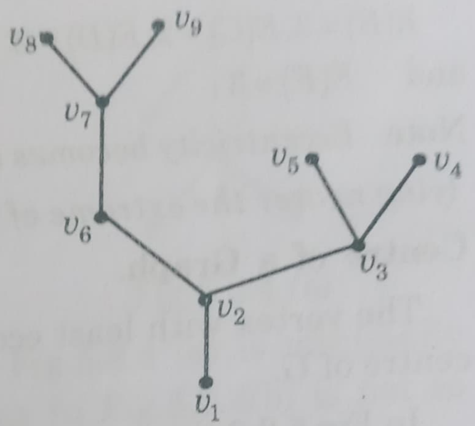


Fig.5.2.1

Remark. (1) We suppose tree contains at least one vertex

(2) Study of trees with infinite number of vertices is beyond the scope of this book.

(3) A tree is always a simple graph because a loop or a pair of parallel edges form a cycle.

(4) To pose a practical example we can say that a river with its tributaries and subtributaries can be represented by tree.

Forest. A collection of some trees is called a forest.

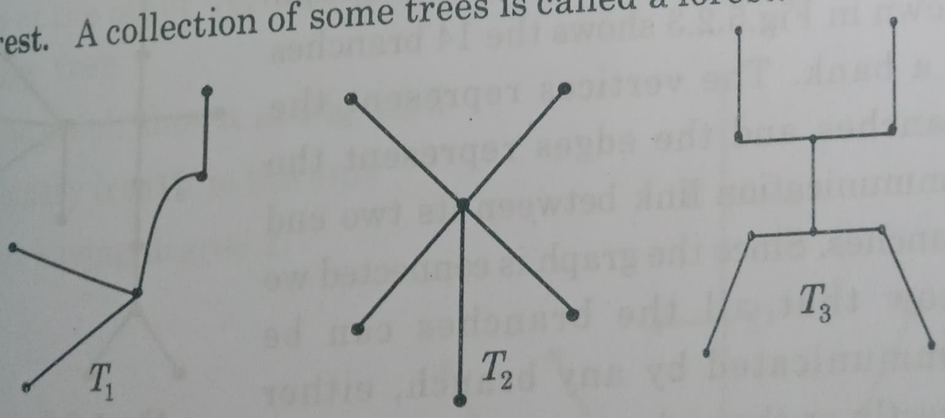


Fig. 5.2.1 (a)

In Fig 5.2.1 (a) the graph T is a collection of three trees T_1, T_2 and T_3. So the graph T is a forest.

Eccentricity of a Vertex.

Let v be a vertex of a graph G. The eccentricity of v, $E(v)$ is the distance from v to the vertex farthest from v in G.

Mathematically $E(v) = \max_{v_j \in G} d(v, v_j)$.

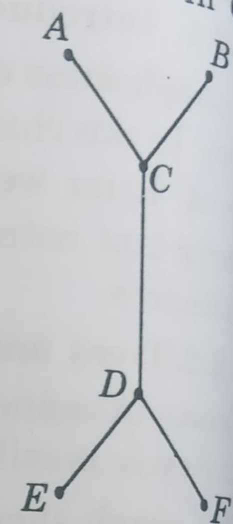


Fig.5.2.2

In Fig.5.2.2 ; $d(A,C) = 1, d(A,B) = 2$,

$d(A, D) = 2, d(A, E) = 3, \ D(A,F) = 3$.

So E or F is farthest from A. So, $E(A) = 3$.

Similarly

$E(B) = 3, E(C) = 2, E(D) = 2, \ E(E) = 3$

and $\quad E(F) = 3$.

Note. *Eccentricity becomes larger as vertex lying nearer the extreme of a graph.*

Centre of a Graph.

The vertex with least eccentricity in a graph G is called the centre of G.

In Fig.5.2.2 the vertices C and D are two centres of the graph.

Note. (1) *From the above example it is evident that a graph may have many or unique centre.*

(2) *The centre of graph having only one vertex is nothing but that vertex itself. The eccentricity of that vertex is 0.*

Illustrative Example : Let the graph shown in Fig.5.2.3 shows the 14 branches of a bank. The vertices represent the branches and the edges represent the communication link between its two end branches. Since the graph is connected we know that all the branches can be communicated by any branch, either directly or through some other branches.

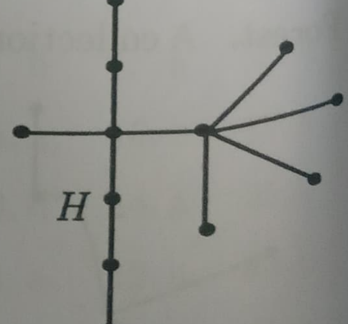


Fig.5.2.3

The eccentricity of each vertex shows how close a branch to the farthest branch of the group of fourteen.

If it is decided to promote one of the branches to the head-office where closeness of communication is the only criterion then we must think about the branch H because H is the centre of the graph (so far communication link is concerned).

Minimally Connected graph. A connected graph is said to be minimally connected if the graph becomes disconnected when one edge is removed.

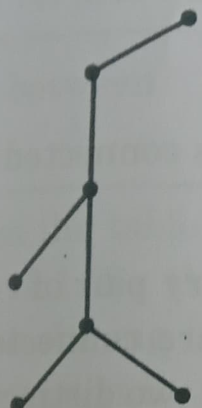


Fig.5.2.4(a)

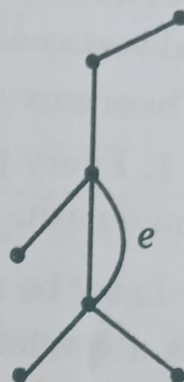


Fig.5.2.4 (b)

The connected graph shown in Fig.5.2.4 (a) is minimally connected whereas the graph shown in Fig.5.2.4(b) is not so because if the edge e is removed the graph is still connected.

Note. *A minimally connected graph can not have a cycle, i.e. a minimally connected graph is a tree.*

5.2.3. Binary Trees and Rooted Trees

A tree in which there is exactly one vertex of degree two and each of the other vertices is of degree one or three is called a **binary tree.**

The graph shown in Fig.5.2.5 is a binary tree. V is the only vertex having degree 2.

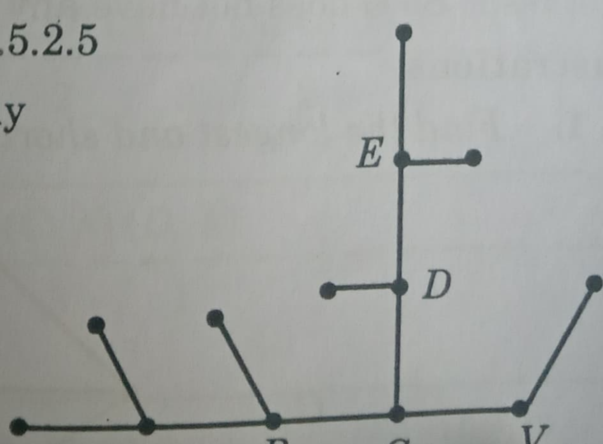


Fig.5.2.5

*The vertex of degree 2 in a binary tree is called **Root** of the tree. So a binary tree is a Rooted Tree.*

One of the most useful applications of rooted trees is in search procedures.

Internal Vertex of a Tree.

The vertex of a tree is called internal vertex which is not pendant vertex. In Fig.5.2.5 the vertices A, B, C, D, E and V are all internal vertices.

5.2.4. Theorems on Trees.

Theorem 1. Every pair of vertices in a tree is connected by one and only one path.

Proof. Let T be a tree ; A, B be an arbitrary pair of vertices. Since T is a connected graph so A and B are connected by a path. Let, if possible, A and B be connected by two distinct paths. These two paths togetherly form a cycle and then T can not be a tree. So there is only one path connecting A and B.

Theorem 2 (Converse of Theorem 1)

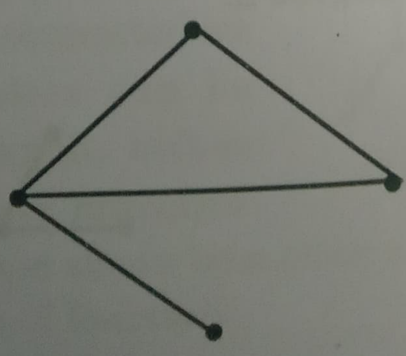
If there is one and only one path between every pair of vertices in a graph G then G is a tree.

Proof. Since there is a path between every pair of vertices so the graph G is connected. Let, if possible, G posses a cycle. So there exists at least one pair of vertices A, B such that there are two distinct paths between A and B. This contradicts the hypothesis. So G does not have any cycle. So, G is a tree.

Illustrations.

Ex. 1. *Find the longest and shortest path found in the graph*

G :



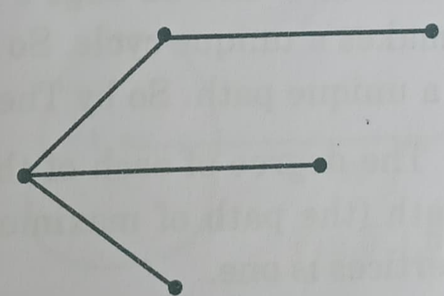
We get several lengths between for a pair of vertices. However we compute them in the following table :

Pair of vertices :	(A, B)	(A, D)	(A, D)	(A, C)	(A, C)	(B, C)	(B,C)	(B, D)
Length between : the pair	1	2	3	2	3	1	2	1
Pair of vertices :	(B, D)	(C, D)	(C, D)					
Length between : the pair	2	1	2					

From the table we see the greatest length is 3 and smallest length is 1. So there exist several longest paths and several shortest. Among these $A-B-D$ is a longest path and $C-D$ is a shortest path.

Ex. 2. *Find the longest and shortest path in the tree T.*

T :



Here we see the length is unique for a pair of vertices. We compute them in the following table :

Pair of vertices :	(A, B)	(A, D)	(A, E)	(A, C)	(B, C)	(B, D)	(B, E)
Length between the pair :	1	2	3	2	1	1	2
Pair of vertices :	(C, D)	(C, E)	(D, E)				
Length between the pair :	2	3	1				

We see length of (C,E) is one of the maximum . So, $C-B-D-E$ is one of two longest paths. $A-B$ is a shortest path in the given tree.

Note. *In a graph which is not a tree there are generally many paths between two vertices. But in a tree there is only one path between any two vertices. So the determination of length between two vertices and consequently finding of shortest and longest path becomes more easier in a tree.*

Theorem 3. A connected graph is a tree if and only if addition of an edge between any two vertices in the graph creates exactly one cycle.

Proof. G be the connected graph. Let G be a tree and A, B be two arbitrary vertices in G.

Let an edge e be added between A and B. Since G is a tree, by Theorem 1, we have an unique path P in G joining A, B. Since $e \notin G \therefore e \notin P$. So, P together with e creates a unique cycle.

Conversly, let addition of an edge between any two vertices in the graph creates a unique cycle. Let A, B be any two arbitrary vertices in G and an edge e be added between A and B. Then this makes a unique cycle. So A and B must already be connected by a unique path. So by Theorem 2, G is a tree.

Theorem 4. The degree of each of the origin and terminus of the longest path (the path of maximum length) in a tree with at least two vertices is one.

Proof. Let T be a tree and $P: v_0, e_0, v_1 e_1; \cdots v_{m-1}, e_{m-1}, v_m$ be a longest path in T where v_i are vertices and e_i are edges. Since the tree has at least two vertices so $v_0 \neq v_m$. We shall show $\deg(v_0) = \deg(v_m) = 1$.

Let, if possible, $\deg(v_0) \neq 1$. Since degree of vertices of a tree with at least two vertices cannot be 0, so $\deg(v_0) > 1$.

So there must be another edge $e \neq e_0$ joining v_0 to a vertex v of T. If this $v = v_i$ for some i then the path $v_0, e_0; v_1, e_1; \cdots v_i e, v_0$ forms a cycle. This is impossible since T cannot have any cycle. If v is not equal to any v_i of the path P then $v, e; v_0, e_0; v., e_1; \cdots v_{m-1}, e_{m-1}, v_m$ becomes a path of length $m+1$.

This is again a contradiction since the longest path in T has length m. Thus $\deg(v_0)=1$.

Similarly, we can show $\deg(v_m)=1$ also.

Theorem 5. Any tree with two or more vertices contains at least two pendant vertices.

Proof. Any two vertices in a tree is connected by one and only one path. Since the tree is supposed to be a finite graph (having finite number of vertices) so there exists a longest path

$$P : v_0, e_0 \,;\, v_1, e_1 \,;\cdots v_{m-1}, e_{m-1} \,;\, v_m$$

in the tree. Then from the previous theorem we see v_0 and v_m are pendant and they are distinct. This completes the proof.

Theoem 6. A tree with n number of vertices has $n-1$ number of edges. [W.B.U.T. 2013]

Proof. Let T be the tree. The result will be proved by method of induction on n. Clearly the result is true for $n=1,2$.

We assume the result is true for k number of vertices whenever $k<n$. In T let a be an edge with end vertices A and B.

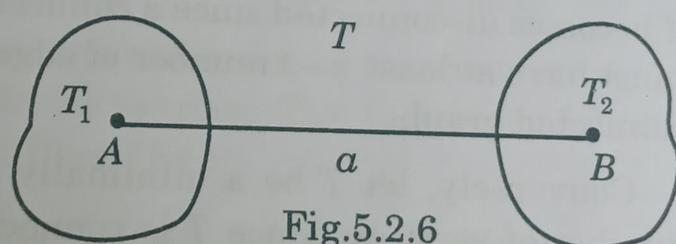


Fig.5.2.6

Since two vertices in a tree are connected by only one path so there is no other path between A and B ; a is the only path joining A and B.

So, $T-a$ i.e. the graph obtained from T by deleting the edge a, becomes a disconnected graph. Now the graph $T-a$ has exactly two components, say T_1 and T_2. Let T_1 and T_2 contains n_1 and n_2 number of vertices. So, $n=n_1+n_2$. If the component T_1 contains a cycle then T would have a cycle, which is not possible. So T_1 is a tree. Similarly T_2 is a tree also. So by our hypothesis T_1 has n_1-1 and T_2 has n_2-1 number of edges. Thus $T-a$ consists of $(n_1-1)+(n_2-1)=n_1+n_2-2=n-2$ number of edges. Hence T has $n-2+1=n-1$ edges.

Theorem 7. (Converse of the above theorem)

A connected graph with n vertices and $n-1$ edges is a tree.

Proof. Let G be a connected graph with n vertices and $n-1$ edges. Let, if possible, G be not a tree. Then G contains a cycle. Let, e be an edge of this cycle. Then the subgraph $G-e$ (the subgraph obtained from G by deleting the edge e) is still connected.

$G-e$ has $n-2$ edges and n vertices. This is not possible since we know a connected graph with n vertices has at least $n-1$ edges.

Theorem 8. A graph is a tree if and only if it is minimally connected. [*W.B.U.T. 2013*]

Proof. Let T be a tree having n vertices. So, by Theorem 7. T has $n-1$ edges.

If one edge is removed from T then it has $n-2$ edges. Then T becomes disconnected since a connected graph with n vertices must have at least $n-1$ number of edges. Thus T is a minimally connected graph.

Conversely, let T be a minimally connceted graph with n number of vertices. Since T is connected graph so, Number of Edges of $T \geq n-1$. Let, if possible, T be not a tree. Then T contains a cycle. T becomes still connected if one edge of this cycle is removed from T. This contradicts our hypothesis that T is a minimally connected graph. Hence T is a tree.

Illustrations.

Ex. 1. *Find the minimum number of line segments to interconnect 100 distinct points.*

From the previous theorem the answer is $100-1=99$.

Ex. 2. *We need only $n-1$ pieces of wire to short electrically n pins together. The resulting structure is a tree.*

Theorem 9. A graph with n number of vertices, $n-1$ number of edges and without any cycle is connected.

Proof. Let G be such a graph. Let, if possible, G be disconnected. Then G has two or more connected components. Without loss of generality suppose G_1 and G_2 be two such components. Since G_1 and G_2 are subgraphs of G so they also do not contain any cycle.

Let v_j and v_k be two vertices in the components G_1 and G_2 respectively. Add an edge e between v_j and v_k (shown in Fig 5.2.7).

G :

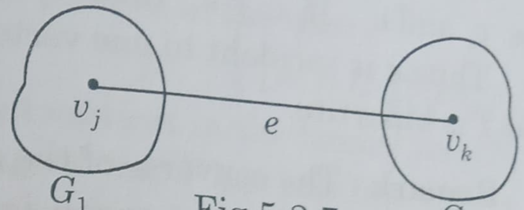


Fig.5.2.7

Since there is no path between v_j and v_k in G so adding e would not create a cycle. Thus the graph together with G and e(mathematically speaking $G \cup e$) becomes a cycleless connected graph i.e. a tree. We see this tree is having n number of vertices and $(n-1)+1=n$ number of edges. This contradicts the fact that a tree having n number of vertices must have $n-1$ number of edges (See Theorem 7). Hence T is a connected graph.

Theorem 10. Every tree has either one or two centres.

Proof. Left to the reader as exercise (See the steps of finding the centre in subsequent Illust. Ex.)

Note. *If a tree has two centres then the two centres must be adjacent.*

Theorem 11. A tree with two or more vertices is a bipartite graph

Proof. Let $V=\{v_1, v_2, \ldots. v_n\}, n \geq 2$, be the vertex set of a tree T. Two subsets V_1 and V_2 are being formed from V in the following way :

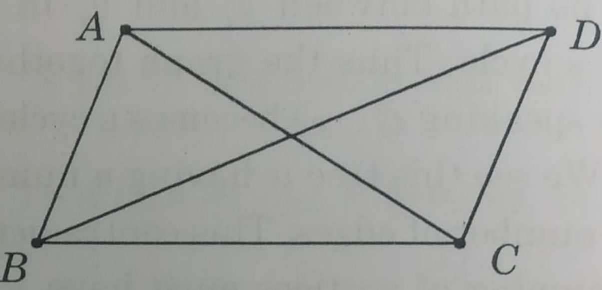
The vertex $v_1 \in V_1$. If v_2 is adjacent to v_1 then $v_2 \in V_2$ otherwise it $\in V_1$. If v_3 is adjacent to at least one vertex in V_1 then $v_3 \in V_2$ otherwise it $\in V_1$. In this way, in general, if v_r is adjacent to at least one vertex in V_1 then $v_r \in V_2$ otherwise $v_r \in V_1$. Thus each of the vertices of $v_1, v_2, \ldots. v_n$ is classified into the two classes V_1 and V_2. Obviously $V_1 \cap V_2 = \phi$.

Since v_1 is not isolated and T has at least two vertices so there exists at least one vertex which is adjacent to v_1. $\therefore V_2 \neq \phi$. Thus V is partitioned into two non-null disjoint subsets V_1 and V_2.

Now let e be an arbitrary edge of the graph T. Its two ends be v_i and v_j. If $v_i \in V_1$ then $v_j \notin V_1$ because v_j is adjacent to v_i. Thus e is incident to one vertex in V_1 and one vertex in V_2. So T is bipartite.

Remark : The converse of the above theorem is not true. For example the following graph is a Bipartite graph, with the partitions $\{A, B\}$ and $\{D, C\}$ of the vertex set, which is not a tree.

G :



Theorem 12. If a Tree is a complete bipartite Graph $K_{m,n}$, then either m or n is 1.

Proof : Let T be a tree which is a complete bipartite graph $K_{m,n}$. Then number of vertices of T is $m+n$. Hence it has $m+n-1$ number of edges. Again $K_{m,n}$ has mn number of edges. Since

$$T \equiv K_{m,n} \text{ so } mn = m + n - 1.$$

$$a,(m-1)(n-1) = 0 \Rightarrow \text{either } m = 1 \text{ or } n = 1$$

Corollary. $K_{1,n}$ or $K_{n,1}$ are the only complete bipartite graph which are Tree also.

5.2.5. Theorems on Binary Trees or Rooted Trees

Theorem 1 The number of vertices in a binary tree is always odd.

[W.B.U.T. 2014, 2010, 2007]

Proof. Let a binary tree has n number of vertices. Since a binary tree has one vertex of degree 2 and the other vertices are of degree 1 or 3 so there are $n-1$ number of odd vertices in the tree.

Since we know the number of odd vertices in a graph is even, $n-1$ is an even integer. So n is an odd integer, that is the number of vertices is odd.

Theorem 2. The number of pendant vertices in a binary tree is $\frac{n+1}{2}$ where n is the number of vertices in the tree.

[W.B.U.T. 2015]

Proof. Let $x=$ number of pendant vertices in the binary tree T, i.e., $x=$ number of vertices of degree one. T has only one vertex of degree 2 and each of the other vertices is of degree three. So, sum of the degrees of all vertices of

$$T = x\times 1+1\times 2+(n-x-1)\times 3 = 3n-2x-1.$$

We know in a graph, sum of all degrees $=2\times$ number of edges.

Since number of edges in a tree is $n-1$, $3n-2x-1=2(n-1)$

or, $x=\dfrac{n+1}{2}$.

Theorem 3. The number of internal vertices in a binary tree is one less than the number of pendant vertices. [WBUT 2012]

Proof. A non-pendant vertex in a binary tree is an internal vertex. Let the binary tree contains $x+y$ vertices, where $x=$ Number of pendant vertices, $y=$ Number of non-pendant vertices, i.e. internal vertices in the binary tree. $\therefore$ the total number of vertices, $n=x+y$.

From Theorem 2 $x=\dfrac{n+1}{2}$

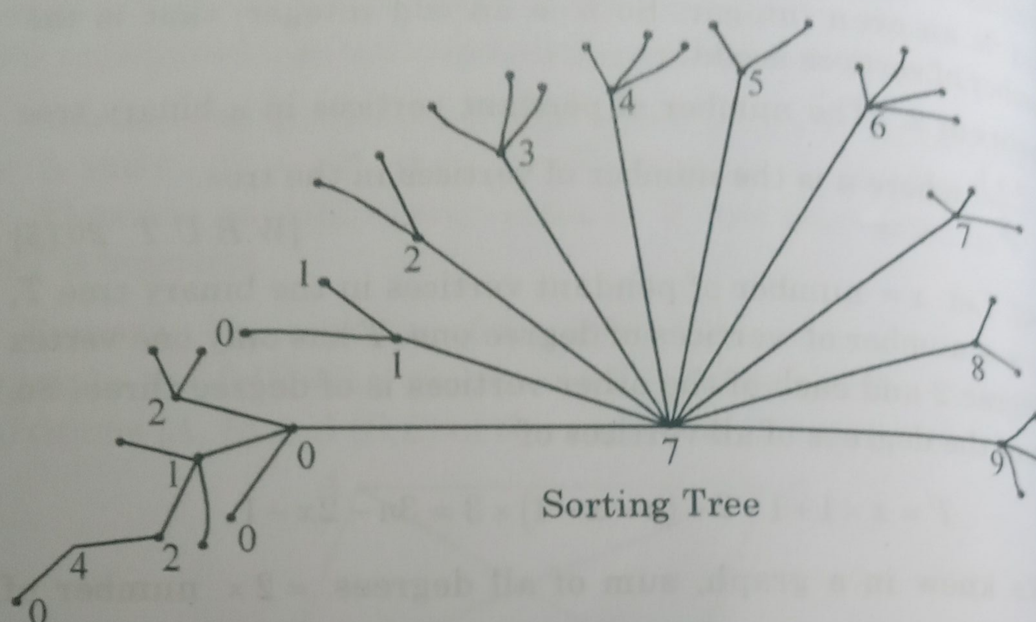
or, $x=\dfrac{x+y+1}{2}$ or, $2x=x+y+1$ or, $y=x-1$

$\therefore$ No. of internal vertices $=$ No. of pendant vertices -1.

5.2.6. Sorting Tree or Decision Tree.

The sorting of letters which are to be delivered by Postal Service is done according to Pin-Code / Zip-Code. This flow of letters is according to a trees called Sorting Tree or Decision Tree.

The following figure is an example of such a flow

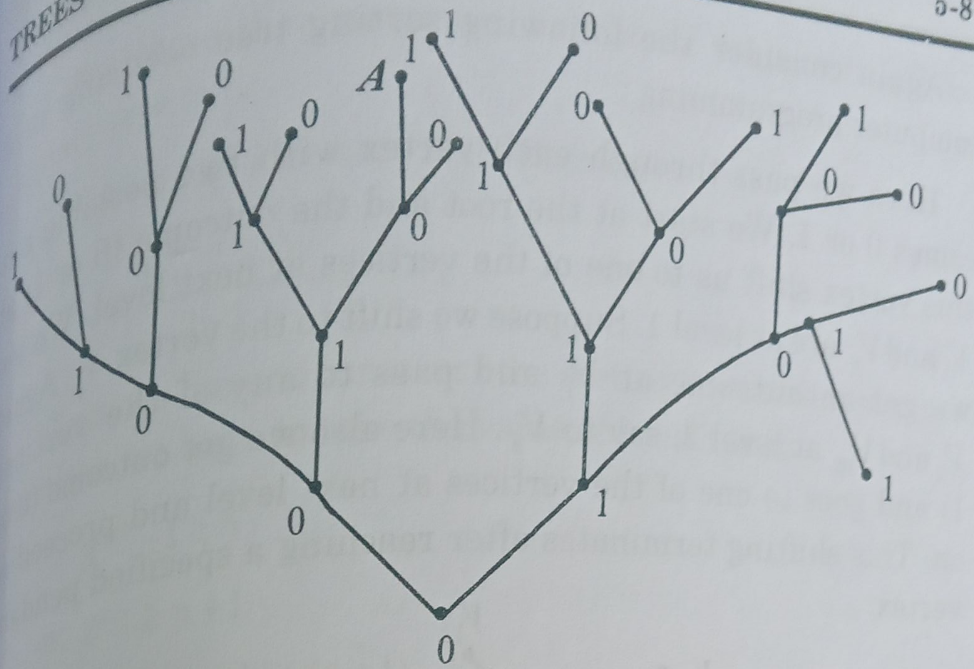


Sorting Tree

Suppose a letter 'L' bears the Pin Code / Zip Code 701240. All the letters bearing the first digit as 7 in Pin Code arrive at the Head office, vertex '7'. If the digit 7 stands for West Bengal then it will be the head office of West Bengal. At this office all the letters are divided into 10 piles '0', '1', '2', '3', ...'9'. Now this letter 'L' will be sent to the office located at the office for which the digit '0' stands. This office is under control of the office '7' i.e., this will be adjacent to the vertex '7'. Letters arrived the office at this vertex '0' are divided into 10 piles again which will be sent to its adjacent vertices '0', '1', '2',......'9'(all are not shown in the above figure). The letter 'L' will be sent to such an adjacent office at the vertex '1'. Next it will be dispatched to the office at '2', next to '4' and to the final destination '0' which is adjacent to the preeceding vertex '4'.

Since the PIN Code is a 6-digit number so at the final stage the number of subdivision into piles $= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$ (note that at the final destination no pileing is not done)

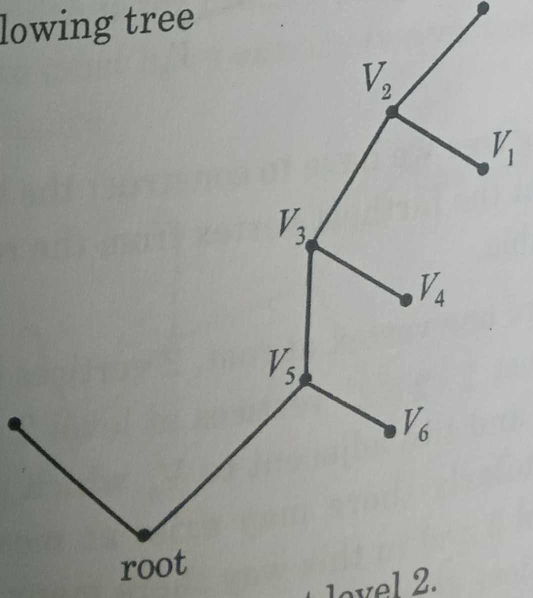
Ex. Construct a sorting tree where the PIN Code is a 5 digit number formed by the digit 0 and 1 only. Mention the PIN Code of a destination. Find the total number of possible destination starting with the digit 0.



The PIN Code of the destination A is 00101. Total number of possible destination $= 2 \times 2 \times 2 \times 2 = 16$.

5.2.7. Sorting Tree Occuring in Computer Programming.

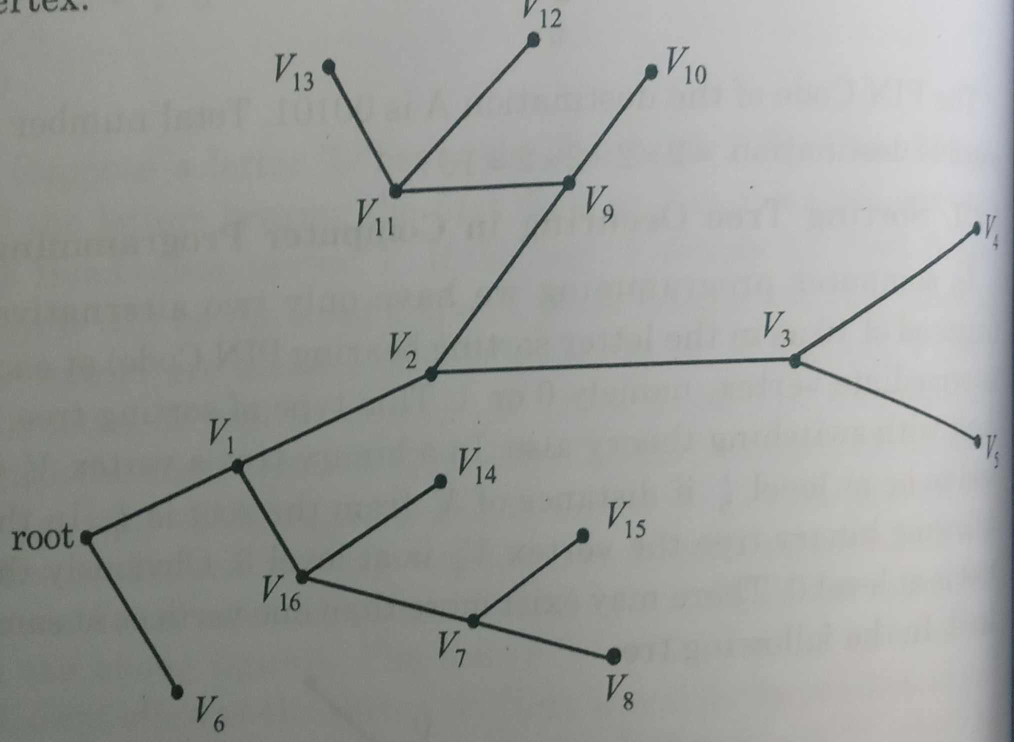
In computer programming we have only two alternatives (intstead of 10 as in the letter sorting bearing PIN Code) at each intermediate vertex, namely 0 or 1. This type of sorting tree is dealt with switching theory also. In a binary tree a vertex V_i is said to be at level l_i if distance of V_i from the root is l_i. In the following binary tree the vertex V_2 is at level 3. Obviously the root is at level 0. There may exist more than one vertices at same level. In the following tree



the two vertices V_3 and V_6 are at level 2.

Again consider the following sorting tree occuring in a computer programming :

Here we pass-through each vertex with two possible outcomes 0 or 1. We start at the root and the outcome (0 or 1) at this vertex shift us to one of the vertices at next level. We see V_1 and V_6 are at level 1. Suppose we shift to the vertex V_1. Again we get an outcome at V_1 and pass to any of the vertices V_2 and V_{16} at level 2, say to V_2. Here also we get outcome (0 or 1) and goes to one of the vertices at next level and proceed so on. This shifting terminates after reaching a specified pendant vertex.



In this *sorting tree* we have to construct the binary tree with n vertices so that the farthest vertex from the root is as close to the root as possible.

Now there are one vertex at root, 2 vertices at level 1. There may have at most $2 \times 2 = 2^2$ vertices at level 2. Two adjacent to V_1 (V_2 and V_{16}) and two adjacent to V_6 which are not shown in the figure. Similarly there may exist at most 2^3 number of vertices at level 3 and in this way there many exist at most 2^k number of vertices at level k.

Therefore ther maximum number of vertices in a k-level binary tree is

$$1+2+2^2+2^3+....+2^k \text{ (G.P series)}$$

$$=\frac{1(2^{k+1}-1)}{2-1}=2^{k+1}-1$$

Since we have to construct the sorting tree with n number of vertices, we have to take the value of k such that

$$2^{k+1}-1\geq n$$

or, $2^{k+1}\geq n+1$

or, $\log_2 2^{k+1}\geq \log_2(n+1)$

or, $k+1\geq \log_2(n+1)$ $[\because \log_2 2=1]$

or, $k\geq \log_2(n+1)-1$ (1)

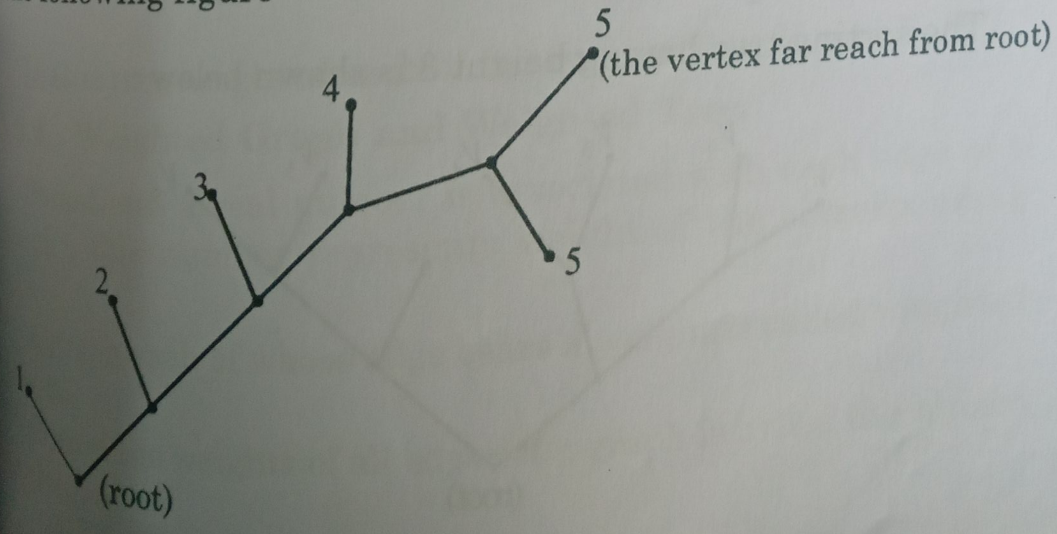
Now the maximum lavel of any vertex in the tree is called the **height** of this sorting tree.

So from (1) we get

Minimum possible height of an n-vertex sorting tree

$=[\log_2(n+1)-1]+1$ where $[x]$ represent intigral part of x.

On the other hand to get maximum value of k we have to get a vertex which is far reach from the root, as far as possible. To do this we must have exactly two vertices at each level like the following figure



So, the total number of vertices,

$$n = \underbrace{2+2+...+2}_{k\ times}+1 = 2k+1 \quad \therefore k = \frac{n-1}{2}$$

So the maximum possible height of an n-vertex sorting tree $= \frac{n-1}{2}$

Note the $\frac{n-1}{2}$ must be integer as the number of vertices in a binary tree is always odd. (See a previous Theorem)

Sum of the levels of all pendant vertices of a binary tree is called **Path length** (or **External Path length**) of the tree. In the above sorting tree Path length $= 1+2+3+4+5+5 = 20$. Path length of a sorting tree is aften directly related to the execution time of the algorithm for which the programming is done.

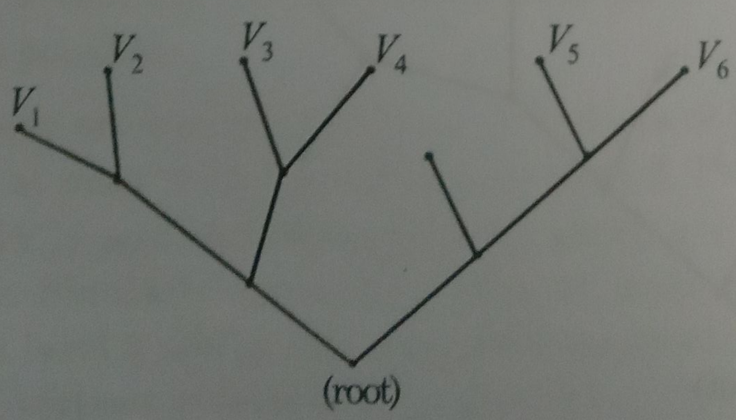
Example 1. Find the minimum possible height of a sorting tree with 13 vertices. At each intermediate vertex only two alternatives 0 or 1 may come out. Draw the diagram of the tree attaining this maximum height. Find the Path length of the tree.

Solution. For $n = 13$, the minimum possible height of the sorting tree $= [\log_2(13+1)-1]+1$

$$= [\log_2 14 - 1]+1 = \left[\frac{\log_{10} 14}{\log_{10} 2} - 1\right]+1$$

$$= \left[\frac{1 \cdot 1461}{0 \cdot 3010} - 1\right]+1 = [3 \cdot 8076 - 1]+1 = [2 \cdot 8076]+1 = 2+1 = 3$$

The sorting tree having height 3 is shown below :



See this is a sorting tree where each of the pendant vertices V_1, V_2, V_3, V_4, V_5 and V_6 is at level 3.

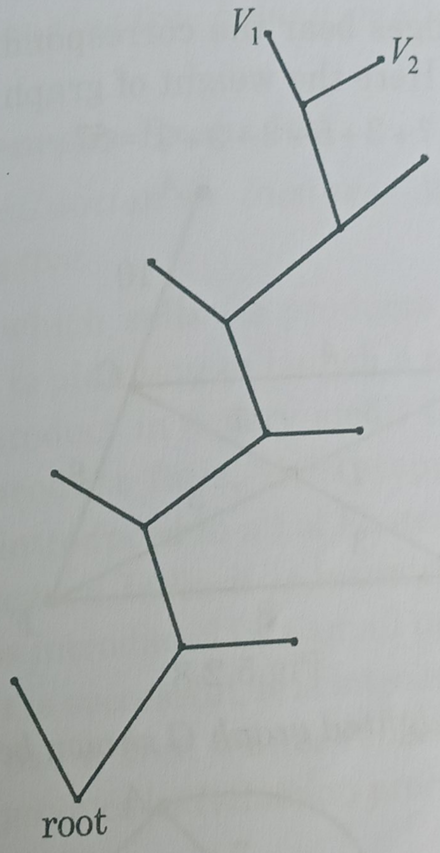
Path length of this tree $= 3+3+3+3+2+3+3 = 20$.

Example 2. Draw a sorting tree with 15 verticesused in computer programming attaining its maximum possible height.

Solution. Here $n = 15$.

The maximum possible height $= \frac{n-1}{2} = \frac{15-1}{2} = 7$.

We draw the following sorting tree attaining its maximum possible height :



In this tree height of each of the two vertices V_1 and V_2 is 7.

5.2.8. Weighted Graph and Weighted Tree.

Sometimes a real number is associated with each edge of a graph ; this number represents weight of the corresponding edge.

A graph each of whose edge bears a weight is called weighted graph.

Sum of the weights of all edges of a graph is called the weight of the graph.

Weight of a Tree.

The sum of the weigths of all edges of a weighted tree is called the weight of the tree.

Illustration.

Ex. 1. *We show a graph in Fig.5.2.8.*

Here the vertices $A, B, C, \cdots$ represent the cities of a country. The edge say $(A\,C)$ represents the road from the city A to city C and so on. Let the cost of transporation from A to C be Rs. 10,000. Then we suppose the edge $(A\,C)$ bears a weight 10. This is inserted beside the side AC.

Similarly other edges bear the corresponding transpotation cost as their weight. Here the weight of graph

$$=10+8+4+7+3+5+8+9+3=57\,.$$

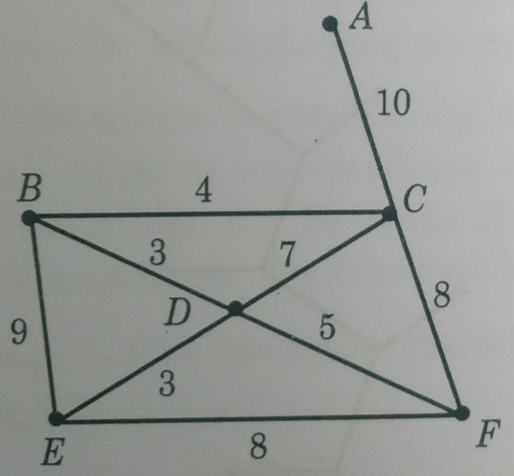
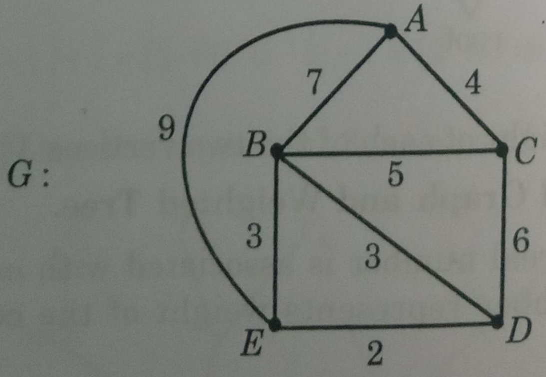


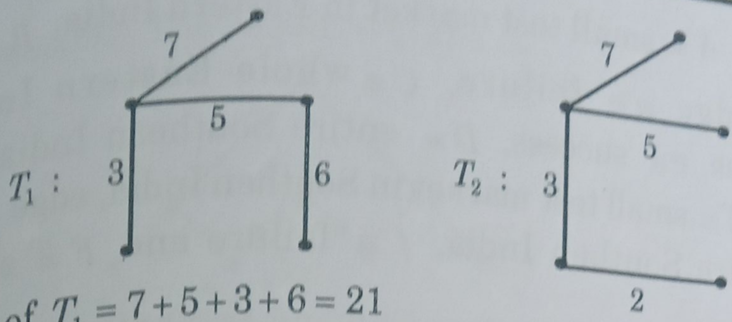
Fig.5.2.8

Ex. 2. *From the weighted graph G shown below*



pick up two spanning trees and find their weights.

We show two of several spanning trees below (actually we know how to find all the spanning tree. See a previous Illustration)



Weight of $T_1 = 7+5+3+6 = 21$

Weight of $T_2 = 7+5+3+2 = 17$.

Note. (1) *Different spanning tree of a graph has different weights.*

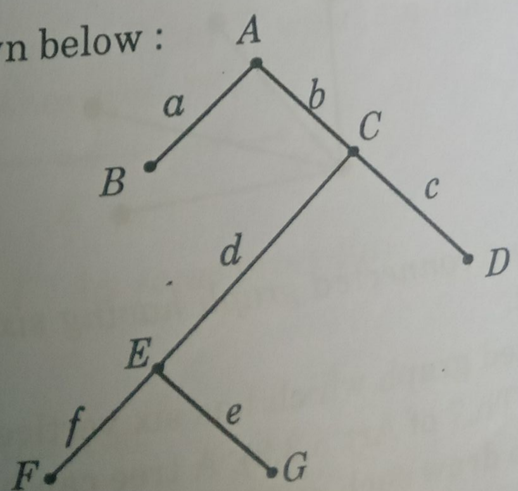
(2) *The spanning tree having smallest weight has practical significance.*

5.2.9. Miscelleneous Examples.

Ex. 1. *Cite a situation in Business-Marketing that can be represented by a tree.*

A company which sells its products in Eastern India and Southern India is planning to launch a new product. First they introduce the product in a very small test market in Eastern India. If the product fails, it is stopped from production. Otherwise it is introduced in all of Eastern India. If the product is success in Eastern India it is introduced in all of Southern India; if not, it is introduced in a small test market in Southern India. Again, if it is successful, it is introduced to entire Southern India. We can use a tree which represents all the possible outcomes of the product introduction process.

The tree is shown below :

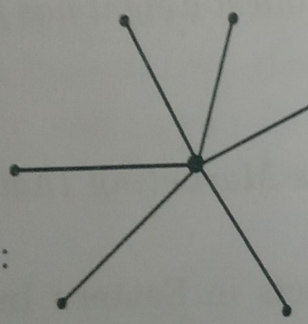


Where $A \equiv$ small test market in Eastern India, $B \equiv$ stopped. The edge $a \equiv$ failure, $C \equiv$ whole Eastern India, edge $b \equiv$ success, $c \equiv$ success, $D \equiv$ entire Southern India, edge $d \equiv$ failure, $E \equiv$ small test market in Southen India, edge $e \equiv$ success, $G \equiv$ Entire Southen India, $f \equiv$ failure and $F \equiv$ stopped the product.

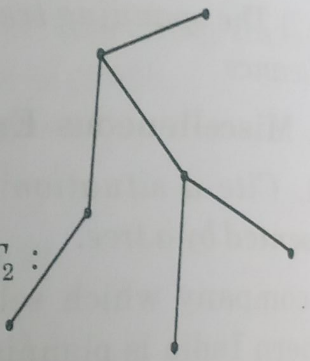
Ex. 2. *Draw two distinct (i.e. non-isomorphic) trees with 7 vertices.*

The two trees T_1 and T_2 shown below are the required trees :

T_1 :



and T_2 :



T_1 , T_2 are not isomorphic because T_1 has a vertex of degree 6 whereas T_2 has no vertex of degree 6.

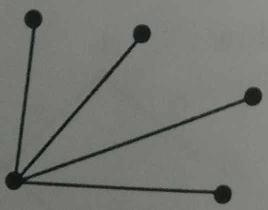
Ex. 3. *Draw a tree with five vertices and total degree 8.*

We know total degree $= 2 \times$ No. of edges

$\therefore\ 8 = 2 \times$ No. of edges

$\therefore$ No. of edges $= 4 = 5 - 1$. So this type of trees do exist.

One is

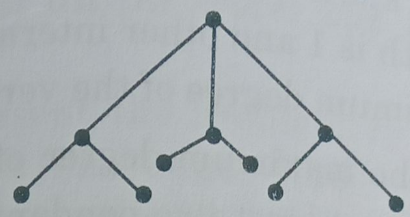


Ex. 4. *Draw a connected graph having six vertices, five edges and a circuit.*

A connected graph which has six vertices and $6 - 1 = 5$ edges is a tree (by Th.7 of Art 5.3.3). A tree can have no cycle. So it is impossible to draw such graph.

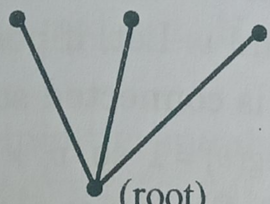
Ex. 5. Draw a tree with four internal vertices and six terminal vertices

This is possible. The required graph is

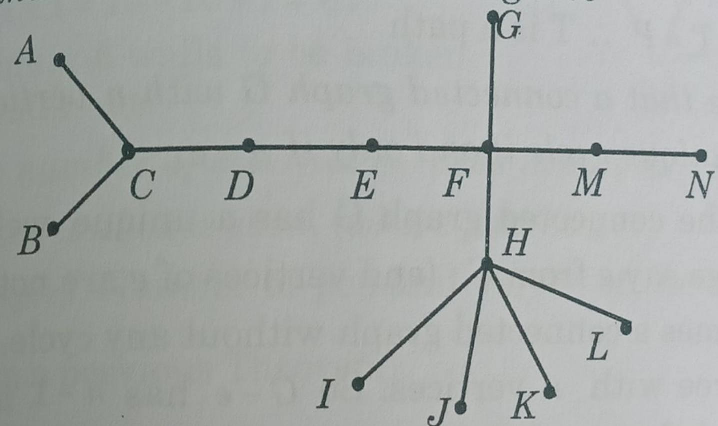


Ex. 6. Draw a rooted tree having 4 vertices

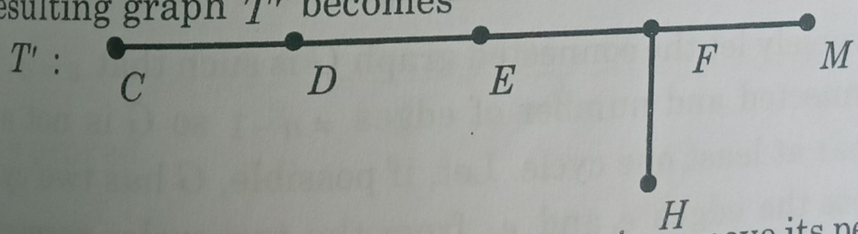
The graph is



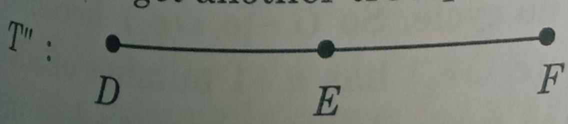
Ex. 7. Find the centre of the following tree



Let T be the given tree. Remove all pendant vertices from T. The resulting graph T' becomes



T' becomes a tree also. From T' we again remove its pendant vertices and get another tree T''.



From T'' we get T''' by same procedure

T''' : $\dot{E}$

E is the centre of the tree.

Ex. 8. *Let T be a tree. Prove that T is a path if and only if the maximum degree of the vertices of T is 2.*

Let T be a path. We know the degree of the origin and terminus vertices of a path is 1 and other intermediate vertices has degree 2. So the maximum degree of the vertices is 2.

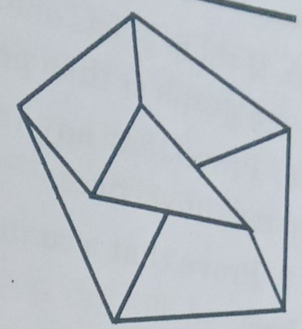
Conversely, suppose the maximum degree of the vertices of T is 2. We know a tree has at least two pendant vertices. Let u and v be the two such vertices of T. Since T is connected so there exists a path P connecting u and v. Let, if possible, there exists an edge say $e \in T-P$. Since T is connected so one end vertex of e say $v_i \in P$. Since $\deg(u)=\deg(v)=1$ so $v_i \neq u, v$. Now number of incident edges to v_i in P is 2. Since e is also an edge incident to v_i so $\deg(v_i)=2+1=3$. This contradicts our hypothesis. So $T-P=\phi$. $\therefore$ $T=P$ $\therefore$ T is a path.

Ex. 9. *Prove that a connected graph G with n vertices and e edges has a unique cycle if and only if $n=e$.*

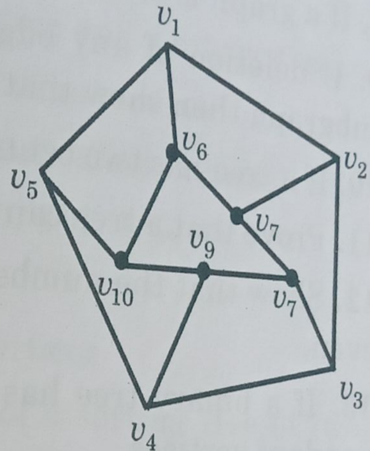
First let the connected graph G has a unique cycle, say C. Delete an edge say e from C; (end vertices of e are not deleted). Then G becomes a connected graph without any cycle, i.e. $G-e$ becomes a tree with n vertices. So $G-e$ has $n-1$ number of edges. Since only one edge was deleted from G to obtain $G-e$ so G has $(n-1)+1=n$ number of edges. $\therefore$ $e=n$.

Conversely let the connected graph G is such that $e=n$. Since G is connected and number of edges $\neq n-1$ so G is not a tree, i.e., G has at least one cycle. Let, if possible, G has two cycle. If we delete the edges e_1 and e_2 from the two cycles respectively (keeping their end vertices intact) we get a subgraph $G-(e_1 \cup e_2)$ which is connected having no cycle. So $G-(e_1 \cup e_2)$ becomes a trees with n vertices. So $G-(e_1 \cup e_2)$ has $n-1$ number of edges. Since two edges were deleted from G so number of edges in G would be $(n-1)+2=n+1=e+1$. This is a contradiction. So G cannot have two cycles. Similarly we can show G cannot have more than two cycles. So G contains unique cycle.

Ex. 10. *Suppose we have a firm consisting of plots of land which are full of water (as shown in the figure). How many walls to be broken so that all the water can be drained out ?*



We first represent the problem by a graph as follow : The edges like $(v_1 v_2)$ or $(v_6 v_7)$ are walls etc.. If it is converted to a spanning tree then all water will be drained out. Here $n = 10$, $e = 15$



By Theorem 3, minimum number of edges to be removed to get a spanning tree is

$$e - n + 1 = 15 - 10 + 1 = 6 .$$

So minimum 6 walls to be broken to get the water out.

Ex. 11. *The number of internal vertices in a binary tree having $4p+1$ vertices is always even. (where p is integer)*

Let $n = 4p+1$. Number of pendant vertex of the binary tree is $\frac{n+1}{2}$ (from a previous Theorem).

The vertex of a tree which is not pendant is called internal vertex. So, number of internal vertices

$$= n - \frac{n+1}{2} = \frac{n-1}{2} = \frac{4p+1-1}{2} = 2p = \text{even.}$$

Hence proved.

Exercise

[I] Short Answer Questions

1. Define tree. Draw a tree with four vertices.
2. Define a minimally connected graph. Draw a non-tree which is minimally connected
3. Define Binary Tree. Draw a binary tree with four edges
4. Prove that there exists a unique path joining two vertices of a tree.

5. If there exists one and only one path between any two vertices in a graph G then prove that G is a tree.

6. Prove that any tree with four vertices contains at least two pendant vertices.

7. Prove that a connected graph with 6 vertices and 5 edges is a tree.

8. If a graph is minimally connected prove that it is a tree.

9. If deletion of any edge of a graph G gives a disconnected subgraph then show that G is a tree.

10. If a tree has two centres then show that they are adjacent.

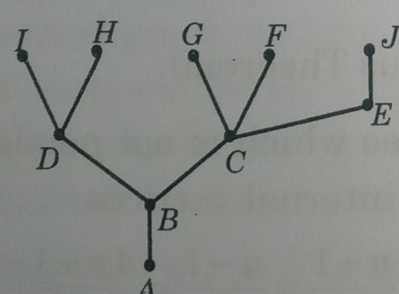
11. Prove that a tree can not have more than two centres.

12. Show that the number of vertices of a binary tree cannot be even.

13. If a binary tree has 23 vertices then prove that it has 12 pendant vertices

14. Show that the number of internal vertices of a binary tree with 27 vertices is 13.

15. You are given the following tree.



(i) Draw the subtree whose root is at C

(ii) Find the internal vertices

16. Draw all trees with exactly six vertices.

17. Draw all trees with five or fewer vertices.

18. Find the number of trees with seven vertices.

19. Draw three distinct trees (i.e. non-isomorphic trees) with 8 vertices.

20. Let T be a tree with more than k edges. How many connected components are there in the subgraph of T obtained by deleting k edges of T?

21. (a) Sketch all binary trees with six pendant vertices. Find the path length of each.

(b) How many binary trees are possible with three vertices ? Draw them.

(c) Draw two different binary trees with five vertices having maximum number of pendant vertices.

(d) Differentiate between a general tree and a binary tree.

22. Draw a tree (i) with nine edges and nine vertices

(ii) with six vertices and sum of degrees of all vertices 14.

(iii) with six vertices having degree 1, 1, 1, 1, 3, 3

(iv) with all vertices of degree 2

23. Draw two rooted trees having four vertices

24. Find the minimum possible height of a sorting tree of two alternatives with

(i) 9 vertices (ii) 17 vertices

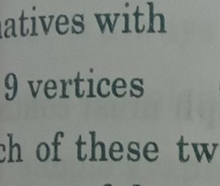
In each of these two cases draw the sorting tree showing the attainment of the minimum possible height. Determine the path length of the drawn tree in each case.

25. Find the maximum possible height of a sorting tree occurred in a computer programming with (i) 11 vertices (ii) 7 vertices

26. Draw two 11-vertex binary tree attaining the maximum and minimum height respectively

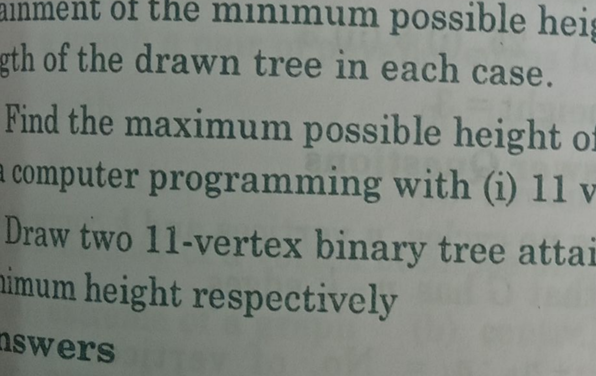
Answers

15. (i)



(ii) *A, B, C, D, E*

16.

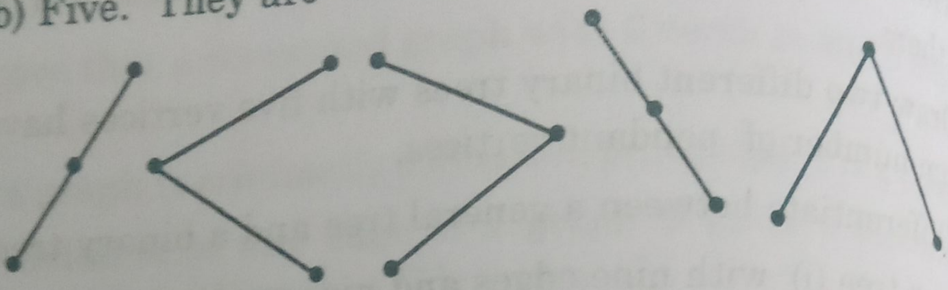


17. There are eight such trees including the null graph.

18. 10

21. (a) Six such binary trees

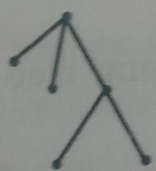
(b) Five. They are



22. (i) No such tree since a tree of 9 vertices would have $9-1=8$ edges.

(ii) No such tree since a tree with 6 vertices has 5 edges and hence total degree of 10, not 14.

(iii)



(iv) No such tree exists because such a graph must contain a circuit (by a previous Th)]

24. (i) 3 (ii) 4

25. (i) 5 (ii) 3

26. max height = 5, min height = 3

[II] Long Answer Questions

1. Let G be a graph having no cycles, n vertices and k connected components. Then prove that G has $n-k$ edges.

[Hints : $n = n_1 + n_2 + \cdots + n_k$; n_i = No. of vertices in i-th component. Applying Theorem 7 of Art 5.3.3 the No. of edges in the i-th component is $n_i - 1$

$$\therefore \text{total No. of edges} = \sum_{i=1}^{k}(n_i - 1) = n - k \,]$$

2. Let T be a tree and let v be a vertex of maximum degree in T. If $\deg(v) = k$ prove that T has at least k vertices of degree 1.

3. Prove that a graph G is a tree iff G is connected but $G-e$ is disconnected for any edge e of G.

[Hint : Nothing but the Theorem 8]

4. Let u and v be two non-adjacent vertices of a tree. Prove that if u and v are joined by an edge then the newly formed graph contains a cycle.

[III] Multiple Choice Questions

1. Tree is a connected grap without any [*WBUT 2012*]

(a) pendant vertex (b) circuit

(c) odd vertex (d) even vertex

[Hint : See definition of Tree. Remember every cycle is a circuit.]