5.6 GRAPH COLOURING

5.6.1. Introduction

Suppose Indian Airlines has seven flights $F_1, F_2, F_3 ..... F_7$ all starting from Delhi. Folllowing table shows the towns touched by each of the flight.

Flight	Towns Touched
F_1	Chandigarh → Mumbai → Bangalore → Kochi
F_2	Lucknow → Patna → Kolkata → Kochi
F_3	Ahmedabad → Bangalore → Kochi → Chennai
F_4	Ahmedabad → Nagpur → Patna
F_5	Ahmedabad → Panaji → Bangalore
F_6	Jaipur → Kolkata → Kochi
F_7	Jaipur → Bhopal → Patna

For the servicing of the planes the company decides the flights to fly only on Monday,Wednesday and Friday.As per Market-Research it is also decided that no more than one flight per day will touch any of the towns.

Indian Airlines wants whether this is possible and,if possible,how the flights can be alloted on these three days.To solve this problem we first construct the following graph (Fig 5.6.1) having seven vertices.Each vertex represents a 'flight'.If two have a common town to touch then they are joined by an edge.

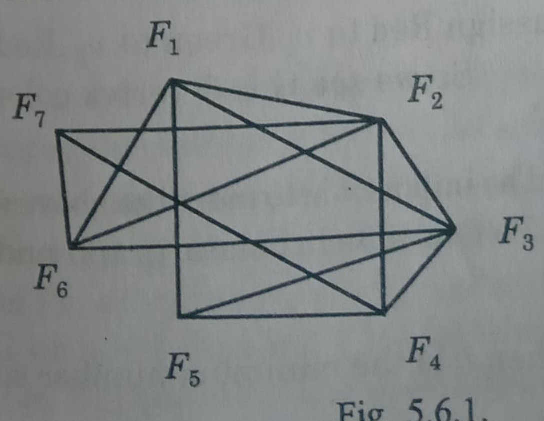


Fig. 5.6.1.

To give the above problem a mathamatical model we replace the three days by the colours colour-1(say red),colour-2(say green) and colour-3(say blue). Then the problem becomes 'colouring the seven vertices so that adjacent vertices have different colours'.

In this capter we deal with these problems and try to find the reliable solution of the above type of problems.

5.6.2. Vertex Colouring of Graphs.

A simple graph is said to be **k-Vertex Colourable** if it is possible to assign one colour from a set of k number of colours to each vertex such that no two adjacent vertices are assigned the same colour. *We say the graph is coloured.*

If a simple graph G is k-vertex colourable but not (k-1)-vertex colourable we say the graph G is a k-chromatic graph and that its **Chromatic number** χ **(G) is k.**

Illustrations. Let us consider the following graph G.

G:

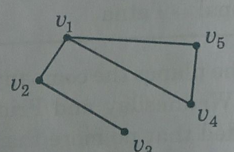


Fig.5.6.2

Consider the set of four colours.{Red,Green,Black and Blue}.If we assign Red to v_1,Green to v_2,Blue to v_3,Black to v_4 and Blue to v_5 then we see no two adjacent vertices are assigned the same colour.So G is a 4-vertex colourable graph.

Again we could assign Red to v_1,Green to v_2,Red to v_3, Black to v_4 and Green to v_5. So we see G is 3-vertex colourable graph also.

Note that it would be impossible to meet the above requirements by only two colours.So G is a 3-chromatic graph and $\chi(G)=3$.

Note.

(1) If $\chi(G)=k$ then k is the minimum number such that G is k-vertex-colourable.

(2) A k-chromatic graph is a graph that needs at least k colours for its colouring.

(3) A k-vertex-colourable graph is a graph that does not need more than k colours for its colouring. If the graph G has n number of vertices then $\chi(G)\le n$

(4) If G is single-vertex graph then G is 1-chromatic graph.

(5) If G is a Null Graph then G is 1-chromatic graph also.

(6) If a graph G has a loop at a vertex v_i then v_i is adjacent to itself and so no colouring of G is possible. To avoid this uninteresting case we remove the loops of a graph if it has so.

(7) If there exist some parallel edges joining two vertices of a graph G then remove all except one because solution for both the cases would be same.

Because of Note (6) and (7) we defined graph-colouring for a simple graph.

If H is a subgraph of G then $\chi(H)\le\chi(G)$.

A graph may be coloured in different ways with same number of colours.Two colourings of a graph with k number of colours will be considered different if at least one of the vertices is assigned different colours. For example if, in the above graph G, we assign Red to v_1,Green to v_2, Blue to v_3, Blue to v_4 and Black to v_5 then we see the graph G is coloured with the four colours in different way.

Theorem 1.

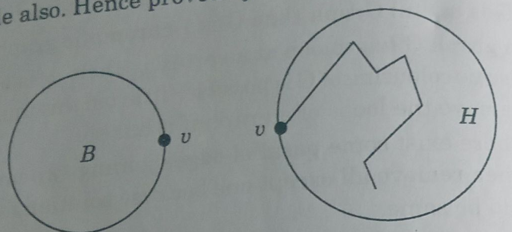
A graph is k-vertex colourable if and only if each block in it is k-vertex colourable.

Proof.

If a graph is k vertex colourable then obviously each block in it is also k vertex colourable since a block is nothing but a subgraph.

The converse part is established by induction. The result is true if there is only one block in the graph. Now assume that the result is true for any graph with r number of blocks. Consider any graph G with r+1 number of blocks where each of the blocks is k-colourable. Suppose B is the block obtained by deleting only one vertex v from G. H be the union of the remaining blocks of G. By induction hypothesis H is k-colourable.

Since every block of G is k-colourable, B is k-colourable. Now colour the block B in such a way that the common vertex v is assigned with same colour in both B and H. Then G will be k-colourable also. Hence proved by method of induction.



Theorem2.

The chromatic number of a graph is k if the chromatic number of each block of it is k.

Proof. Let G be the graph. Its block are $G_1, G_2, \ldots G_r$.

Let $\chi(G_i) = k$ for $i = 1, 2, \ldots r$.

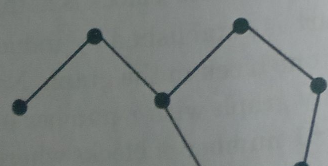
$\therefore$ Every block G_i is k colourable. So by the previous theorem G is k colourable. Let, if possible, G is m-colourable where $m < k$.

Again by the previous theory every G_i is m-colourable, which contradicts the hypothessis that $\chi(G_i) = k$ and $m < k$. So G cannot be m-colourable if $m < k$.

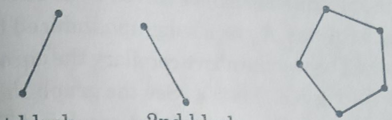
$\therefore \chi(G) = k$.

Note.

Converse of the above theorem is not true. For example the chromatic number of the following graph is 3.



But its three blocks are the followings.



1st block 2nd block 3rd block

We see the chromatic numbers of the 1st block is 2 not 3.

5.6.3. Chromatic Number of a Complete Graph (K_n).

Theorem 1. The chromatic number of a complete graph with n vertices (K_n) is n. [W.B.U.T 2013]

Proof. Let K_n be a complete graph with n number of vertices $v_1, v_2, \ldots v_n$.

Let a colour C_1 be assigned to v_1. Since v_2 is adjacent to v_1 so a different colour is needed to be assigned to v_2. Let colour C_2 be assigned to v_2. Since v_3 is adjacent to both v_1 and v_2 so another colour (other than C_1 and C_2) is needed to be assigned to v_3.

Let this colour be named C_3. In this way the different colours $C_4, C_5, \ldots C_n$ are needed to be assigned to the vertices $v_4, v_5, \ldots v_n$ respectively.

Thus n number of colours are needed for K_n. If any of the colours $C_1, C_2, \ldots C_n$ is removed then they can not colour K_n. Thus $\chi(K_n) = n$.

Theorem 2. The chromatic number of the graph $K_n - v_i$ obtained by deletion of a vertex v_i from K_n is $n-1$.

Proof. If a vertex is deleted from a complete graph then the graph remains to be complete. So, $K_n - v_i$ is a complete graph having $n-1$ number of vertices. So by the previous theorem $\chi(K_n - v_i) = n-1$.

Corollary. If a complete graph K_n is a subgraph of a graph G then $\chi(G) \geq n$.

Proof. Since $K_n \subset G$ $\therefore \chi(G) \geq \chi(K_n)$

$\therefore \chi(G) \geq n$

Note. From the above corollary the problem of Indian Airlines stated in the Introduction can be solved now. The corresponding graph in Fig 5.6.1 has K_4 as a subgraph, induced by the vertices F_1, F_2, F_3 and F_6. So by the above corollary the chromatic number of the graph (in Fig 5.6.1) is ≥ 4. See the graph in Fig 5.6.1 is 4-colourable. So chromatic number of that graph is 4. Thus the seven flights can be scheduled on 4 days but not 3 as wanted by Indian Airlines, subject to their restrictions.

5.6.4. Chromatic number of a cycle (C_n).

Theorem. The chromatic number of a cycle with n vertices (C_n) is
(i) 2 if n is even.
(ii) 3 if n is odd. [W.B.U.T. 2015]

Proof. Let the cycle C_n has the vertices $v_1, v_2, \ldots v_n$ appearing in order of the cycle. If we assign colour-1 to v_1, v_2 must be coloured with different colour say with colour-2 (because v_2 is adjacent to v_1). Now v_3 is adjacent to v_2 but not to v_1, So we can assign colour-1 to the vertex v_3. In this way we can alternately assign colour-2, colour-1, colour-2, ... to the vertices $v_4, v_5, \ldots$ respectively

(i) Obviously if n is even the last vertex v_n which is again adjacent to v_1 will be assigned colour-2. Thus if n is even C_n can be coloured by 2 colours only but not less than 2. Hence $\chi(C_n) = 2$ if n is even.

(ii) Obviously if n is odd the last vertex can not be assigned any of colour-1 and colour-2 because this vertex is adjacent to v_{n-1} (which is assigned colour-2) and v_1 (which is assigned colour-1). So the last vertex v_n should be assigned a new colour, say colour-3. Thus if n is odd then C_n can be coloured by 3 colours but not less than 3. Hence $\chi(C_n) = 3$ if n is odd.

Corollory. If C_n is a subgraph of a graph G then

(i) $\chi(G) \geq 2$ if n is even (ii) $\chi(G) \geq 3$ if n is odd

Proof. Since $C_n \subset G$ so $\chi(G) \geq \chi(C_n)$

$\therefore$ (i) $\chi(G) \geq 2$ if n is even (ii) $\chi(G) \geq 3$ if n is odd

Note. The converse of the above theorem is not true. For example consider the two graph G_1 and G_2 shown below

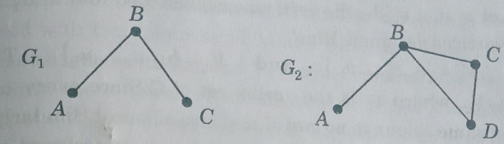


Fig.5.6.3.

Observe that none of G_1 and G_2 is cycle at all. But G_1 can be coloured in the following way:

A is assigned 'Red'
B is assigned 'Black'
C is assigned 'Red'

So $\chi(G_1) = 2$

Again G_2 can be coloured in the following way:

A is assigned 'Red'
B is assigned 'Black'
C is assigned 'Red'
D cannot be assigned 'Red' or 'Black'

So D is assigned 'Green'

Thus $\chi(G_2) = 3$

5.6.5 Chromatic Number of a Bi-Partite Graph $(K_{m,n})$.

Theorem.1. The chromatic number of a non-null graph is 2 if and only if the graph is bi-partite. [W.B.U.T. 2012]

Proof. Let G be a bipartite graph. So its vertex set V can be partitioned in two sets $V_1 = \{v_1, v_2, \ldots v_m\}$ and $V_2 = \{u_1, u_2, \ldots u_n\}$ such that every edge of G joins some v_i to some u_j. Obviously no two of the vertices in V_i are adjacent. So we can assign a same colour, say colour-1 to each v_i.

Since u_i is adjacent to a v_j and since no two of u_is are adjacent we can assign colour-2 to each u_i. Thus the graph is coloured only with colour -1 and colour-2, Hence $\chi(G) = 2$.

Conversely, let G be a graph such that $\chi(G)=2$. So G is 2-colourable and let it be coloured with the two colours Red and Blue. Let $v_1, v_2, \ldots v_m$ be the vertices assigned 'Red' and $u_1, u_2, \ldots u_n$ be the vertices assigned 'Blue'.

Let $V_1=\{v_1, v_2, \ldots, v_m\}$ and $V_2=\{u_1, u_2, \ldots, u_n\}$. Then $V=V_1 \cup V_2$ where V is the vertex set of G.Since every u_i is assigned same colour so no two of $u_i s'$ are adjacent.Similarly no two of $v_j s'$ are adjacent.Since a u_i and a v_j is assigned with distinct colours, each of the edges of G joined some u_i with some v_j. Therefore G becomes a bipartite Graph.

Theorem.2. A Tree with two or more vertices has chromatic number 2. [W.B.U.T. 2016]

Proof. Let $V=\{v_1, v_2, \ldots v_n\}$, $n \geq 2$

be the vertex set of a tree T. Two subsets V_1 and V_2 are being formed from V in the following way:

The vertex $v_1 \in V_1$. If v_2 is adjacent to v_1 then $v_2 \in V_2$. Otherwise it $\in V_1$. If v_3 is adjacent to at least one vertex in V_1 then $v_3 \in V_2$ otherwise it $\in V_1$. In this way each vertex of V is classified into the two non-empty classes V_1 and V_2 where $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = V$.

Since no two of the vertices in V_1 are adjacent we can assign a same colour, say colour-1 to each member of V_1. Since no two vertices in V_2 are adjacent and every vertex in V_2 is adjacent to some vertex in V_1 we can assign another colour, say colour-2 to each vertex in V_2. Thus T is coloured only with the two colours colour- 1 and colour-2. Hence $\chi(T)=2$

Corollary . The chromatic number of a complete bigraph is 2, that is $\chi(K_{m,n})=2$.

Theorem 3. (Konig's Theorem).

The chromatic number of a graph with at least one edge is 2 if and only if the graph contains no cycles of odd length.

Proof. Without loss of generality we suppose G be a connected graph with at least on edge and G contains no odd cycle. Let T be a spanning tree of G. Since chromatic number of a tree in 2 so T is coloured with two colour say Red and Blue. Let V_1 = set of all vertices assigned with Red and V_2 = set of vertices assigned with Blue. Since the vertex set of G is identical with that of its spanning tree T, $V=V_1 \cup V_2$ (V is vertex set of G). Let e be an edge of G.Its two end vertices be v_i and v_j. Let, if possible, both of v_i and v_j belong to a same class say V_1 .So v_i and v_j are assigned Red. Let P be a path in T from v_i to v_j. Then P and e togetherly form a cycle whose length is odd.This centradicts our hypothesis. Thus v_i and v_j will belong to two opposite classes; one in V_1 and another one in V_2. So G becomes a bi-partite graph. Therefore, $\chi(G)=2$

Conversely let $\chi(G)=2$. If G contains a cycle of odd length then its chromatic number should be ≥ 3 because the chromatic number of an odd cycle is 3. So G cannot contain a cycle of odd length.

Compiling the above two theorems we reach the following theorem.

Theorem.4. For a non-null graph G, the following statements are equivalent :

(i) $\chi(G)=2$ (ii) G is bipartite

(iii) G contains no cycle of odd length

5.6.6. Upper Bound of Chromatic Number

Though we become able to find the exact chromatic number of K_n, C_n and $K_{m,n}$, no efficient and convenient procedure is known for finding the chromatic number of any arbitary Graph.

However there are various results which give *Upper Bounds* for the chromatic number of an arbitary Graph, provided we know the degrees of their vertices.

Before going to those results we need the following notations: for a graph $G, \Delta(G)$ = maximum of the degrees of all the vertices of G and $\delta(G)$ = minimum of the degrees of all the vertices of G.

Theorem 1. For any Graph G, $\chi(G) \leq \Delta(G)+1$

Proof. Beyond the scope of this Book.

Theorem 2. The inequality in the above theorem becomes an equality when G is a

(i) complete graph (K_n)

(ii) cycle with odd number of vertices (C_n, n is odd)

Proof.

(i) For a complete graph K_n the degree of each vertex is $n-1$.

$$\therefore \Delta(K_n) = n-1$$

By the Theorem 1 introduced in Art 5.6.3

$$\chi(K_n) = n = (n-1)+1 = \Delta(K_n)+1$$

(ii) For a cycle C_n the degree of each vertex is 2. So $\Delta(C_n) = 2$.

By the Theorem introduced in Art 5.6.4 for an odd integer n, $\chi(C_n) = 3 = 2+1 = \Delta(C_n)+1$.

In 1941, R.L.Brooks brought the following theorem which reduces the upper bound of $\chi(G)$ by 1 for a particular type of graph G.

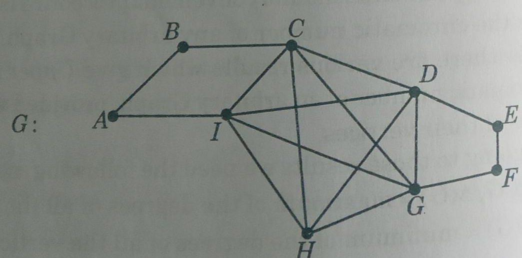
Theorem 3. (Brooks Theorem).

Let G be a connected graph with $\Delta(G) \geq 3$. If G is not complete then $\chi(G) \leq \Delta(G)$.

Proof. Beyond the scope of the book.

Illustrative Examples.

Consider the following graph G



Here we see the graph induced by the Five vertices C, I, H, G and D is a complete sub graph of G. $\therefore 5 \leq \chi(G)$

On the other hand in G,

deg (A)=2, deg (B)=2, deg (C)=5,
deg (D)=5, deg (E)=2, deg (F)=2,
deg (G)=5, deg (H)=4 and deg (I)=5.

So $\Delta(G) = 5$

Since G is not a connected graph and since $\Delta(G) \geq 3$ so by Brooke's theorem $\chi(G) \leq 5$

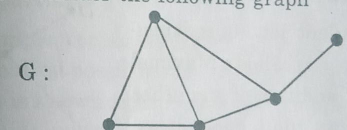
Thus $5 \leq \chi(G) \leq 5$. Hence $\chi(G) = 5$.

Theorem 4 (Szekeres-Wif Theorem):

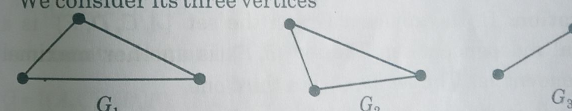
For any graph G, $\chi(G) \leq 1 + \max \delta(G')$ where the maximum is taken over all subgraphs G' of G.

Proof. Beyond the scope of the book.

Illustration. Consider the following graph



We consider its three vertices



Here $\delta(G_1)$ = minimum of the degrees of all the vertices of G_1

$= \min\{2,2,2\}=2$

Similarly, $\delta(G_2) = 2$ and $\delta(G_3) = 1$

Then according to above Theorem,

$$\chi(G) \leq 1 + \max\{2,2,1\} = 1+2 = 3$$

Note that this obeys the fact $\chi(G) = 3$.

5.6.7. Lower bounds of Chromatic Number.

Independent set of vertices : Let G be a graph, A be a subset of its vertex set. If no two vertices in A are adjacent (in G) then A is called an independent set of vertices.

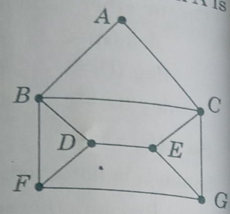


Illustration : In the adjacent graph G the set of vertices S={A,E,F} is independent as the pairs A, E and E, F and F, A are not adjacent. Note that the set {A,E,G} is not independent as the two vertics E,G are adjacent.

Independence Number : The cardinal number of the largest independent set in a graph G is called the Independence Number or vertex-independence number of G. It is denoted by α (G).

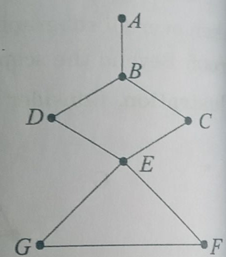


Illustration: In the adjacent graph, the set $\{A, C, D, F\}$ is the largest indepent set. So its independence number $\alpha(G) = 4$.

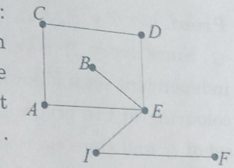
Maximal Independent set. An independent set of vertiecs A is called a Maximal Independent set if addition of any other vertex to A destroys the independence property of A.

Illustration: In the adjacent Graph the set $\{A, C, D, F\}$ is a maximal independent set. The set $\{B, F\}$ is another maximal indenpendent set. The set $\{B, G\}$ is a third one.

Thus it is clear that a graph may have many maximal independent sets; and they may have different cardinal number.

Finding a Maximal Independent set.

A reasonable method of finding a maximal independent set in a graph G is to start with any vertex v of G in the set. Add more vertices, one-by-one, to the set, selecting at each slagt a vertex that is not adjacent to any of those already selected. This procedure will ultimately produce a maximal independent set.



This is shown by the following example: Consider the adjacent graph. Start with the vertex A. The vertices C and E are adjacent to A. But the vertex B is not adjacent to A. So we form the set $\{A, B\}$.

Next the vertex D is not adjacent to any of A and B. So we form the set $\{A, B, D\}$.

Similarly we form the set $\{A, B, D, I\}$. We do not take F.

Therefore, the vertex set $\{A, B, D, I\}$ is a maximal independent set of the graph.

Similarly if we would start with the vertex C we get another maximal independent set $\{C, E, F\}$.

Note. A mathematical method for obtaining all maximal independent sets in any graph can be developed using Boolean arithmetic on the vertices. This method is not discussed as it is beyond this text.

Chromatic partioning of a Graph.

Given a simple, connected graph G, partion the vertex set of G into the smallest possible number of disjoint, independent sets. This problem is known as chromatic partioning of G.

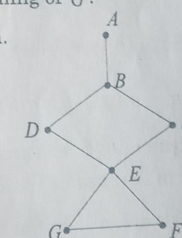


Illustration: Consider the adjacent graph.

$$P = \{(A, C, D, F), (B, G), (E)\}$$

is a chromatic partion but,

$$Q = \{(A, C), (D, F), (B, G), (E)\},$$

$$R = \{(A, C, F), (D, C, G), (B, F)\}$$

are not chromatic partition of the graph.

Following Theorems 2 and 3 give the lower bound of chromatic number of G.

Theorem 1. Corresponding to a chromatic partion of a graph G, the chromatic number $\chi(G) =$ Number of independent sets in the partion.

Proof. Let $P=\{P_1, P_2, ...P_k\}$ be a chromatic partition of the graph G. Since every P_i is independent set, from the definition of the independent set if follows that one colour is needed for each P_i for colouring of the graph G. Since $P_1 \cup P_2 \cup P_3 \cup ... \cup P_k =$ the vertex set of G and since every pair of $P_1, P_2, ... P_k$ is disjoint so minimum number of colours needed for colouring the graph is k.

$$\therefore \ \chi(G)=k.$$

Theorem 2. For agraph G with n vertices and $\alpha(G)$ independence number, $\chi(G) \geq \dfrac{n}{\alpha(G)}$ where $\chi(G)$ is the chromatic number of G.

Proof: Let P be a chromatic partition of G where

$P=\{P_1, P_2, ...P_k\}$. Then $k=\chi(G)$ (by the previous theorem).

Now, $P_1 \cup P_2 \cup P_3 \cup ... \cup P_k = V$, the vertex set

or, $n(p_1)+n(p_2)+n(p_3)+...+n(p_k)=n(V)$

($\because$ every pair p_i, p_j is disjoint)

Since $\alpha(G)=\max\{n(p_1), n(p_2), ... n(p_k)\}$

$\therefore$ $n(p_i) \leq \alpha(G)$ for $i=1, 2, 3 ... k$.

So, $n(p_1)+n(p_2)+n(p_3)+...+n(p_k) \leq k.\alpha(G)$

or, $n(V) \leq \chi(G).\alpha(G)$ or, $n \leq \chi(G).\alpha(G)$ or, $\chi(G) \geq \dfrac{n}{\alpha(G)}$

Clique : Let G be a graph, Z be a subset of its vertex set. If every pair of vertices in Z are adjacent then Z is called a clique of G.

Illustration : In the above graph G the set {A,B,C} is a clique but the set {B,C,G,F} is not a clique.

Clique Number : The cardinal number of the largest clique in a graph G is called the clique number of G. It is denoted by ω (G).

Illustration : In the above graph G, the set {A,B,C} in one of the largest cliques in G. So its clique number ω (G) = 3.

Note : (1) A single vertex set in an independent set.

(2) A complete subgraph of a graph G is a clique of G.

(3) A null graph has no clique.

(4) $\omega(G) \geq 2$ for any graph G.

(5) A clique in a graph G is independent set of vertices in the complement of G.

Theorem 3. For a graph G with n vertices and $\alpha(G)$ independence number, $\chi(G) \geq \omega(G)$.

Proof. Obvious.

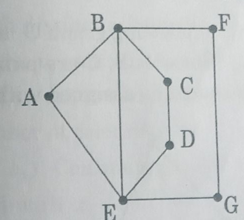
Note : Above two theorems give two *lower bounds* of $\chi(G)$ where G is a graph with n vertices.

5.6.8. Perfect Graphs.

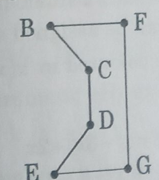
A graph G is called perfect if for every subgraph H of G, $\omega(H)=\chi(H)$.

Illustration : (i) Consider the following graph

G :

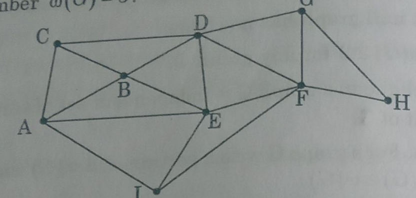


If we take any subgraph H of G e.g. the sub-graph



we shall see $\chi(H)=\omega(H)$ hold. So this graph is a perfect graph.

(ii) The following graph G has chromatic number $\chi(G)=4$ and clique number $\omega(G)=3$.



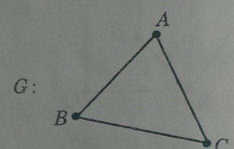
So this graph is not perfect (as G itself is a subgraph of G).

5.6.9. Chromatic Polynomials (Chromials).

In Note of Art 5.6.2 we discussed that a graph may be coloured in different ways with the same number of colours.

Let $f(G,x)$ be the number of different colourings of a graph G with x or fewer colours.Then it can be shown that $f(G,x)$ will be a polynomial of x .This $f(G,x)$ is called chromatic polynomial of G.

Illustration. Consider the following graph G and the x number of colours $C_1, C_2, C_3 \dots C_x$. Since A, B, C are pairwise adjacent to each other,each of them should be assigned with distinct colour.



Now A can be assigned with x number of colour; B can be assigned with $x-1$ and C can be assigned with $x-2$ number of colours. Therefore G may be coloured in $x(x-1)(x-2)$ way with x or fewer colours.

Thus $f(G,x)=x(x-1)(x-2)=x^3-3x^2+2x$ is the chromatic polynomial of G .

Note. The smallest x for which $f(G,x)\neq 0$ is nothing but $\chi(G)$ because $\chi(G)$ is the least number of colours with which G

can be coloured.In otherwords $f(G,x)=0$ if $x<\chi(G)$. This can be seen in the previous Illustration.The graph shown in the above Illustration is nothing but K_3. So $\chi(G)=3$.

See that $f(G,2)=0, f(G,1)=0$ etc.

Notice that the chromatic polynomial $f(G,x)=x^n$ where G is a null graph with n number of vertices.

5.6.10. Evaluation of Chromatic Polynomial.

First we evaluate the chromatic polynomial for a complete graph K_n. This is a generalized result of the previous Illustration. Next in the subsequent theorems we find the chromatic polynomial for an arbitary Graph.

Theorem 1. Chromatic polynomial for a complete graph with n vertices (K_n) is $f(K_n,x)=x(x-1)(x-2)\dots\dots(x-n+1)$

Proof. Let the vertices of K_n be $v_1, v_2, v_3 \dots\dots v_n$ and we have x number of colours $c_1, c_2, c_3 \dots\dots c_x$. Obviously $x \geq n$. Since the vertices are pairwise adjacent to each other each of them should be assigned with distinct colours.Now v_1 can be assigned with x number of colours; v_2 can be assigned with $x-1$ number of colours (as the colour of v_1 can not be assigned to v_2). v_3 can be assigned with $x-2$ number of colours (as the colour of v_1 and v_2 can not be assigned to v_3). Continuing this way we arrive at v_n which can be assigned with $x-(n-1)=x-n+1$ number of colours.

So, by principle of counting, K_n can be coloured in $x(x-1)(x-2)\dots\dots(x-n+1)$ ways with x or fewer colours.

Thus $f(K_n,x)=x(x-1)(x-2)\dots\dots(x-n+1)$

Theorem 2. The chromatic polynomial of a tree with n vertices is $x(x-1)^{n-1}$.

Proof. Let G be a tree with n vertices. We shall prove its chromatic polymial is $x(x-1)^{n-1}$. We prove this by method of induction. For $n=1$ the result is obvious.

For $n = 2$, G has two vertices v_1 and v_2. v_1 may be assigned with x number of colours from a collection of x colours. Since v_2 is adjacent to v_1, v_2 can be assigned with $x-1$ colours from the collection of x colours. $\therefore f(G,x) = x(x-1)$ proving the result for $n = 2$. Let the result be true for $n = m$. We shall prove the result for $n = m+1$. Since G is a tree it must have at least one pendant vertex say v. Then the graph $G-v$ (obtained by deleting v from G) is also a tree with m vertex. Then by hypothesis its $f(G-V,x) = x(x-1)^{m}$. Since v is pendant of G so it is adjacent to only one vertex say v'. Since v can not be assigned with the colour of v' so v can be assigned with $x-1$ number colours from a collection of x colours.

So by principle of counting $f(G,x) = x(x-1)^{m}(x-1) = x(x-1)^{m+1}$.

Theorem 3. The Chromatic Polynomial of any Graphis a polynomial.

Proof. Let G be a graph with n vertices. Let G be coloured with i number of colours in c_i different way. Since i colours can be chosen from x number of colours in ${}^{x}C_i$ ways, there are $c_i\,{}^{x}C_i$ ways of colouring with i number of colours taken (in different way) from x number of colours. Since G has n vertices so it is not possible to use more than n colours for the colouring of G. Thererfore $i = 1, 2, \ldots, n$. Therefore, the chromatic polynomial,

$$f(G,x) = c_1\,{}^{x}C_1 + c_2\,{}^{x}C_2 + c_3\,{}^{x}C_3 + \ldots + c_n\,{}^{x}C_n$$

Each c_i has to be evaluated individually for the given graph but it can be mentioned that each c_i is positive integer.

Again $\quad {}^{x}C_i = \dfrac{x(x-1)(x-2)\ldots\ldots(x-i+1)}{i!}$

is a polynomial of x (of degree i).

So $f(G,x)$ is a polynomial of degree at most n.

Ex. *Find the chromatic polynomial of the following graph given in the Fig 5.6.6. Hence find its chromatic number.*

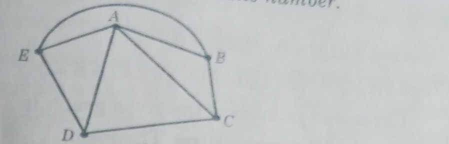


Fig.5.6.6

Since the given graph G has 5 vertices so the chromatic polynomial

$$f(G,x) = c_1\,{}^{x}C_1 + c_2\,{}^{x}C_2 + c_3\,{}^{x}C_3 + c_4\,{}^{x}C_4 + c_5\,{}^{x}C_5.$$

G can not be coloured with one colour $\therefore c_1 = 0$.

G can not be coloured with two colours $\therefore c_2 = 0$.

To find c_3 suppose that we have three colours λ, μ, ∂. These three colours can be assigned to A, B, C in $3! = 6$ different ways.

Now E must have same colour as C as D must have same colour as B. So after assigning colour to A, B, C we have no more choices left. Therefore $c_3 = 6$.

To find c_4 suppose that we have four colours $\lambda, \mu, \partial, \tau$.

The distinct colours are required for A, B and C. This can be chosen from four in $4 \times 3 \times 2 = 24$ ways. After assigning three colours, say λ, μ, ∂ to A, B and C we have one colour τ which can be assigned to D or E, thus providing 2 choice. The fifth vertex provides no addition choice. So, $c_4 = 24 \times 2 \times 1 = 48$.

To find c_5 we see five colours can be assigned to five vertices in $5! = 120$ ways. $\therefore c_5 = 120$.

So the chromatic polynomial

$$f(G,x) = 6\,{}^{x}C_3 + 48\,{}^{x}C_4 + 120\,{}^{x}C_5$$

$$= 6.\frac{x(x-1)(x-2)}{3!} + 48.\frac{x(x-1)(x-2)(x-3)}{4!}$$

$$+120.\frac{x(x-1)(x-2)(x-3)(x-4)}{5!}$$

$$= x(x-1)(x-2) + 2x(x-1)(x-2)(x-3)$$
$$+ x(x-1)(x-2)(x-3)(x-4)$$

we see $f(G,1) = 0,\ f(G,2) = 0$

but $f(G,3) = 3(3-1)(3-2) = 3 \times 2 \times 1 = 6 \neq 0$

$\therefore$ Chromatic number of the given graph is 3.

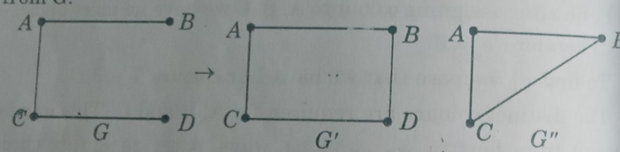
Theorem 4 (Decomposition Theorem).

Let v_1 and v_2 be two non-adjacent vertices of a simple graph G. Let G' be a graph obtained from G by adding an edge between v_1 and v_2. Let G'' be a graph obtained from G by merging the vertices v_1 and v_2 together. Then $f(G,x) = f(G',x) + f(G'',x)$

Proof. Beyond the scope of the book. [W.B.U.T.2014]

Illustration.

Consider the following graph G and the graph G', G'' obtained from G:



(obtained by adding(obtained by merging the edge e)Dwith B)

G' is obtained from G by adding the edge e between the two vertices B and D. G'' is obtained from G by merging D with B, that is in G'', $D \equiv B$ (and so the edge CD takes the position CB).

Since G is a tree with 4 vertices, $f(G,x) = x(x-1)^3$.

Since G'' is K_3 $\therefore f(G'',x) = x(x-1)(x-2)$

By the above theorem we have

$$f(G,x) = f(G',x) + f(G'',x)$$

or $f(G',x) = f(G,x) - f(G'',x) = x(x-1)^3 - x(x-1)(x-2)$

$$= x^4 - 4x^3 + 6x^2 - 3x$$

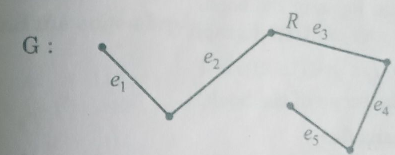
5.6.11. Edge Colouring of Graphs

Recall that two edges of a graph of a graph are called *adjacent* if they have an end vertex in common.

A simple graph is said to be k-Edge Colourable if it is possible to assign a colour from a set of k number of colours to each edge such that no two adjacent edges are assigned the same colour. We say the graph is *edge-coloured.*

If a simple graph G is k-edge colourable but not (k-1)-edge colourable we say the graph G is a **k-Edge Chromatic Graph** and that its Edge Charomatic Number $\chi_1(G)$ is k.

Illusration. Consider the following graph G.



Consder the set of four colours {Red, Green, Black and Blue}. If we assign Red to e_1, Green to e_2, Blue to e_3, Black to e_4 and Red to e_5, we see no two adjacent edges are assigned the same colour. So G is a 4-edge colourable graph.

Again we could assign Red to e_1, Green to e_2, Red to e_3, Green to e_4 and again Red to e_5. Then G is 2-edge colourable also. Note that it would be impossible to meet the above requirements by only one colour. So G is a 2-edge chromatic graph and $\chi_1(G) = 2$.

5.6.12. Theorems on Edge Colouring

Following interesting and useful results are obtained for the Edge Chromatic Number of a non trivial graph.

Theorem 1. If H is a subgraph of G then $\chi_1(H) \leq \chi_1(G)$

Proof. Obvious.

Theorem 2. For any simple graph G, $\chi_1(G) \geq \Delta(G)$ where $\Delta(G)$ is the maximum of the degrees of all the vertices of G.

Proof. Let v be the vertex of G for which degree of v, deg (v) is maximum, i.e., $\deg(v) = \Delta(G)$. Then $\Delta(G)$ number of edges are adjacent with end vertex v in common. Each of these edges must have a different colour in any edge colouring of G.

$\therefore$ minimum number of colour should be $\geq \Delta(G)$

$\therefore \chi_1(G) \geq \Delta(G)$

Theorem 3. For any non empty biparite graph G, $\chi_1(G) = \Delta(G)$.

Proof. Beyond the scope of the book.

Theorem 4. For a complete graph K_n with n vertices, $n \geq 2$,

$$\chi_1(G) = \Delta(G) = n-1 \text{ if } n \text{ is even}$$

$$= \Delta(G)+1 = n \text{ if } n \text{ is odd}$$

Proof. Beyond the scope of the book.

Theorem 5. For any nontrivial graph G

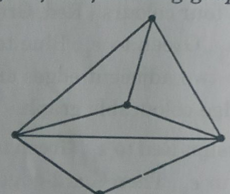
$$\Delta(G) \leq \chi_1(G) \leq \Delta(G)+1$$

Proof. Beyond the scope of the book.

Illustrative Examples :

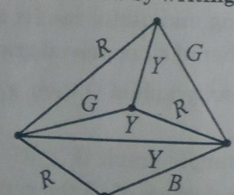
Ex.1. *Colour the edges of the following graph with the colour set {G, R, Y, B}.*

G :



Try to colour the edges with any three colours in the set. Hence write down the value of $\chi_1(G)$

The colouring is shown below by writing the colour name beside each edge



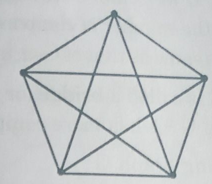
The maximum value of degree of the vertices is 4, i.e., $\Delta(G) = 4$.

So $\chi_1(G) \geq 4$

$\therefore$ G can not be coloured with any three of the given colours.

So, $\chi_1(G) = 4$.

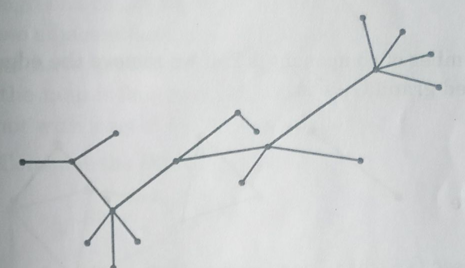
Ex.2. *What is the edge chromatic number of the following graph*



The given graph is a complete graph. Since 5 is odd so chormatic number of the graph is 5.

Ex.3. *Find the edge-chromatic number of the following graph :*

G :



G is a bipartite graph as G does not contian any cycle of odd length (in fact this is a tree). Here $\Delta(G) = 5$

So, by a previous theorem $\chi_1(G) = 5$

Ex.4. *A complete graph consists of 45 number of edges. Find the edge chromatic number of the graph.*

Let G be the complete graph having n number of vertices. Then

number of edges of G $= \dfrac{n(n-1)}{2}$.

By problem, $\dfrac{n(n-1)}{2} = 45$ or, $n(n-1) = 90$ which gives $n = 10$.

$\therefore G$ is a complete graph with 10 vertices.

$\therefore \chi_1(G) = 10$ (by a previous theorem)

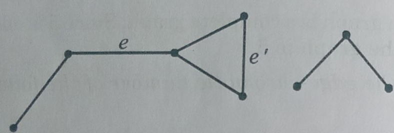
5.6.13. Map Colouring

In this section we have to first introduce the term **Bridge** in a graph. Let G be a graph. If we remove an edge e (keeping its two end vertices intact) then the number of components in the obtained graph $G - e$ may either be same as that of G or increase by exactly 1.

A edge e of a graph G is called a **bridge** (or a **cut edge** or or an **isthmus**) if the subgraph $G - e$ has more component than G has.

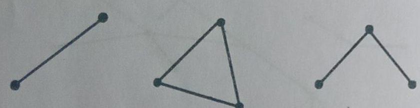
Example. In the following graph G,

G :



the number of component is 2. If we remove the edge e then the obtained graph $G - e$ is

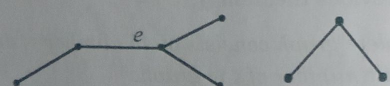
G – e :



We see $G - e$ has 3 components. So here the edge e is a bridge of G.

But if we remove the edge e' from G then the obtained graph $G - e'$ will be

G – e′ :

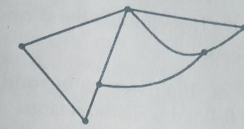


We see $G - e'$ has 2 components. So here the edge e' is not a bridge of G.

Obviously if e is a bridge of a graph G then number of component of $G - e$ = number of components of $G + 1$.

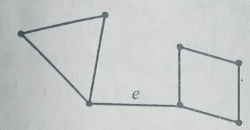
Map. A plane connected graph with no bridges is called a **map.**

Example (1)



The above graph is a map because it is plane connected having no bridge.

Example (2)



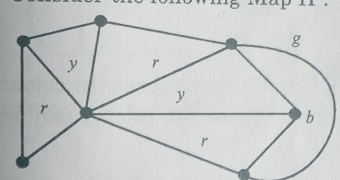
This graph is a plane graph but not a map. Here e is bridge.

Colouring a map

A map is said to be **face colourable** with a set of colours if it is possible to assign one colour from the set to each face or region such that no two adjacent faces (i.e., two faces sharing a common boundary edge) have same colour. A map is said to be **k-face colourable** if the map is face colourable with a set of k number of colour but not with a set of $k - 1$ numbers of colours.

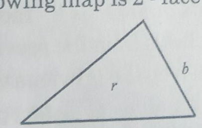
Example (1) Consider the following Map H :

H:



This graph is 4 - face colourable as it is face colourable with the set of four colours $\{r, y, b, g\}$.

Example (2) The following map is 2 - face colourable with the set of colours $\{r, b\}$



In the next Article we present some useful results on map colouring.

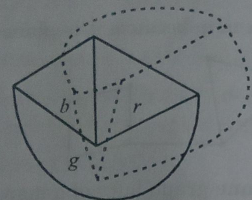
5.6.14. Theorems on Map Colouring

Theorem 1. A map G is k - face colourable if and only if its dual G^* has chromatic number k.

Proof. We keep the proof of this theorem beyond the scope of the book but go through the following Illustration.

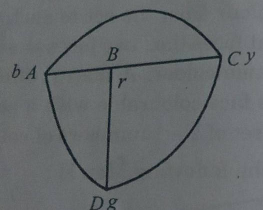
Illustration.

G :



This map G is 4 - face colourable with the colours b, r, g and y. Now its dual is

G^* :



The vertices A, B, C and D are assigned the colours b, r, y and g respectively. No three of these four colours can not colour the vertices. $\therefore \chi(G^*) = 4$.

Note. In fact colouring the regions or faces of a map is equivalent to colouring the vertices of its dual.

Theorem 2. A plane connected simplegraph G has chromatic number k if and only if its dual G^* is k - face colourable.

In fact this theorem follows from the previous one. Since G has no loops, G^* is a map. Then by the previous theorem G^* is k - face colourable if and only if its dual $(G^*)^*$ has chromatic number k. Since G is connected it is isomorphic to $(G^*)^*$. So G has also chromatic number k.

Theorem 3. A map G is 2 - face colourable if and only if it is an Euler graph.

Proof. Let G be 2 - face colourable map. Then by a previous theorem $\chi(G^*) = 2$ where G^* is dual of G.

Therefore by a previous theorem G^* is bipartite.

Therefore, its dual $(G^*)^*$ is Euler graph. Since G and $(G^*)^*$ are isomorphic G is also Euler graph.

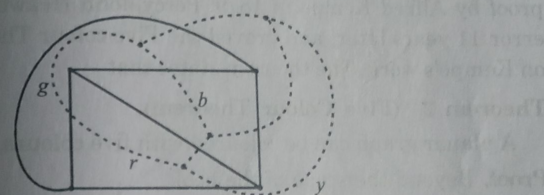
Conversely, let G be Euler graph. So its double dual $(G^*)^*$ is Euler also and so G^* is bi-partite. Here $\chi(G^*) = 2$.

Therefore by a previous theorem G is 2 - face colourable.

Illustrative Example

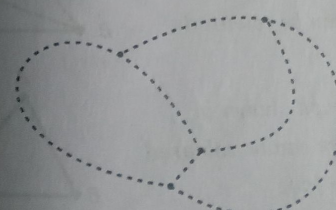
Colour the faces of the complete graph with four vertices. Can it be face colourable with only three colours.

The complete graph K_4 is shown below :



The four colours g, b, r, y are assigned to the above graph

K_4^* :



We see the dual K_4^* is also K_4. So its chromatic number is 4. Therefore K_4 is 4 - face colourable. It can not be face colourable with only three colours.

5.6.15. Four and Five Colour Theorem.

The conjecture of Four Colour Theorem was first proposed in 1852 when **Francis Guthrie** was trying to colour the map of countries of England. He noticed that only four different colours are needed for this colouring of map. There were many mathematician failed to prove this conjecture. Finally in 1976 this was proved by **Kenneth Appel** and **Wolfan Haken** as a Theorem, known as *Four colour Theorem*. The stament of this theorem, in term of Graph Theory, is given below:

Theorem 1 (Four Colour Theorem).

Every planar graph has a chromatic number Four or less.

Proof. Beyond the scope of this book

In fact the Five colour Theorem,which will be stated here, is implied by the Four colour Theorem. But it is considerably easier to prove.It was based in a failed attempt at the Four colour Theorem proof by Alfred Kempe in 1879. Percy John Heawood found an error 11 years later, and proved the Five colour Theorem based on Kempe's work. The theorem states that

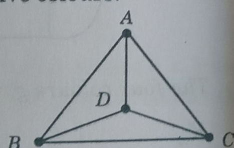
Theorem 2 (Five Colour Theorem).

A planar graph can be coloured with five colours.

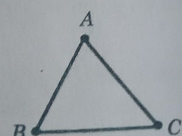
Proof. Beyond the scope of the book.

Illustrations.

(i) *The planar graph shown in the adjacent figure can be coloured with Four colours.*



(ii) *The planar graph shown in the following figure can be coloured with three colours (<4).*



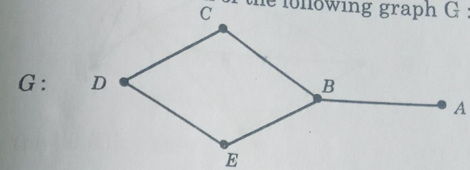
Four Colour Theorem on Map Colouring

From the Four colour theorem on vertex colouring the following result immediately follows :

Every map is almost 4 - colourable.

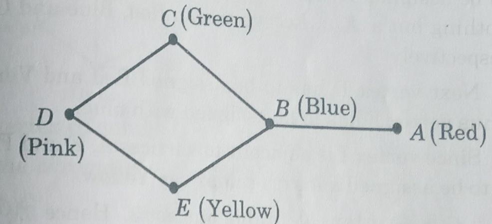
5.6.16. Illustrative Examples

Ex. 1. Show three different colouring with the colour Red, Blue, Yellow, Green and Pink of the following graph G :

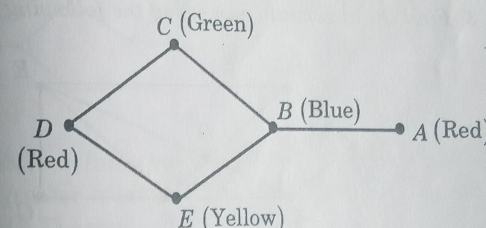


The three different colouring of G are shown below :

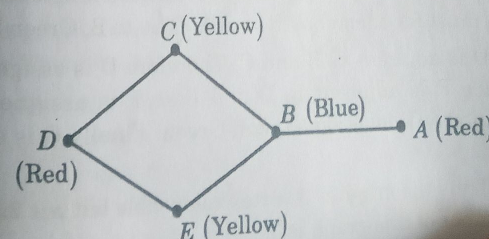
Type I:



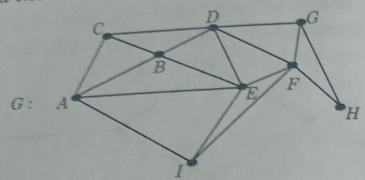
Type II:



Type III:



Ex. 2. *Find the choromatic number of the following graph G:*



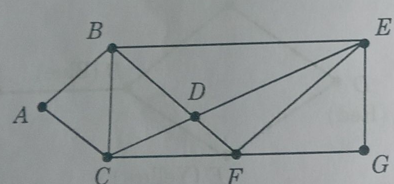
We see the vertices A,B,C form a triangle. So three colour are to be assigned to the vertices A, B and C (because a triangle is nothing but a K_3). Let we assign Red, Blue and Green to them respectively.

Next vertex D has to be assigned Red and Vertex E Green. Thus vertex F has to be assigned with blue.

Since vertex I is adjacent to vertices A, E and F, this vertex I is to be assigned a fourth colour say Yellow.

$\therefore G$ is 4-vertex colourable but not 3. Hence $\chi(G) = 4$.

Ex. 3. *Find the chromatic number of the following graph.*

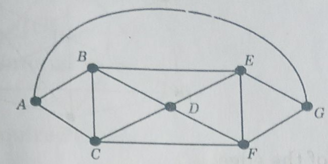


Since the vertices A, B and C form a triangle so three colours are must for them. Say Red to A, Blue to B, Green to C

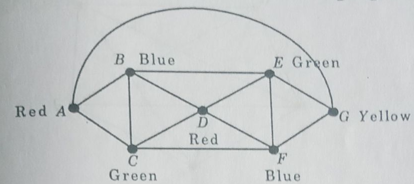
D is adjacent to B and C. Therefore D is assigned with Red. Since F is adjacent to D and C so F is assigned with Blue. Similarly E is assigned with Green. Finally G is assigned with Red.

Thus the graph is 3 vertex colourable but not 2. So chromatic number of this graph is 3.

Ex. 4. *Label the vertices with colour for the following graph. Hence state how-much colourable in this.*

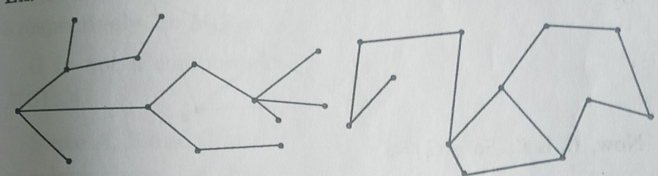


We show the labelling only in the following figure:



So it is 4-vertex colourable.

Ex. 5. *Find the chromatic of the following graph*



The graph is nothing but a tree. The number of vertices in the tree is greater than 2. So its chromatic number is 2 (See corollary 2 of a Theorem

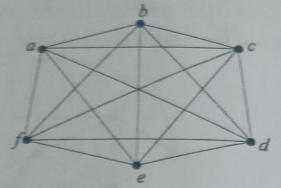
Ex. 6. *Find the chromatic number of the following graph*

The given graph does not contain any circuit of odd length. So by Konigs' Theorem its chromatic number is 2.

Ex. 7. *Consider K_6, the complete graph on the six vertices $a, b, c, d, e, f,$. The graph G_1 is obtained from K_6, by deleting the edge ab. The graph G_2 is obtained from G_1 by deleting the edge ed. What are the chromatic numbers of G_1 and G_2?*

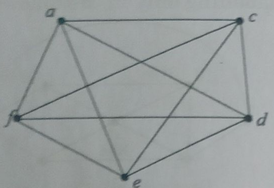
[W.B.U.T.2012, 2016]

K_6 :



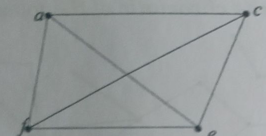
After deletions of the edge ab,

G_1 :



After deletion of the edge cd,

G_2 :



Now, G_1 is K_5. So $\chi(G_1)=5$

and G_2 is K_4 $\therefore \chi(G_2)=5$

Ex. 8. *Let the graph G have connected components $G_1, G_2,...G_k$. Prove that $\omega(G) = max\{\omega(G_1), \omega(G_2)...\omega(G_k)\}$.*

Let $r = \max\{\omega(G_1), \omega(G_2)...\omega(G_k)\}$.

$\therefore$ $r = \omega(G_p)$ for some p, $1 \le p \le k$.

Let Z be the largest clique of G_p whose cardinal number is r.

Let Z' be the largest clique of some G_i. Obviously Z and Z' will be cliques of G. But $n(Z) \ge n(Z')$. So Z is the largest chique of G. Therefore $\omega(G)=r$. Hence proved.

Ex. 9. *Find the clique of a non-null bipartite graph.*

We know if G is a bipartite graph then $\chi(G)=2$.

Now $\omega(G) \le \chi(G)$. $\therefore$ $\omega(G) \le 2$

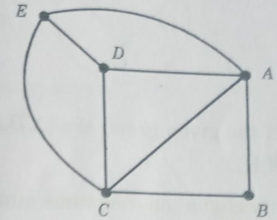
Since G is non-null so $\omega(G) \ne 1$

$\therefore$ $\omega(G)=2$

Thus the required clique number is 2.

Ex. 10. *Find the chromatic number and the clique number of the following graph G*

G :



The Degree of the vertices A, B, C, D and E are 4, 2, 4, 3 and 3 respectively. So $\Delta(G)=4(\ge 3)$.

G is not a complete graph. Therefore by Brooks theorem $X(G) \le 4$.

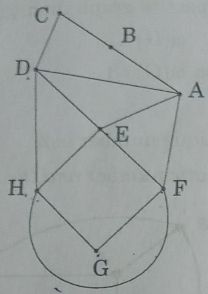
Since A, B and C form a triangle so three colours are needed to colour them. Let they are C_1, C_2 and C_3 which colour A, B and C respectively. Since D is not adjacent to B (but to A and C) so C_2 can be assigned to D. But E is adjacent to A, C and D assigned with the colours C_1, C_3 and C_2. Therefore E is to be assigned with a fourth colour say C_4. Thus at least Four colours are needed to colour G. Hence $X(G) \ge 4$.

$\therefore 4 \le \chi(G) \le 4$. So $\chi(G)=4$

In the given graph the set {A,C,D,E} is a clique. If we add B to this then the set {B,A,C,D,E} is not a clique. So {A,C,D,E} is the largest clique of the given graph.

$\therefore$ The clique number $\omega(G)=4$.

Ex. 11. *Find all the cliques and the clique numbers of the following graph.*



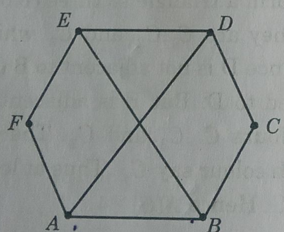
The clique of the given graph are {A,D,E}, {A,E,F}, {D,E,H}, {FG,H} and {E,F,H}.

Note that every clique contains same number of elements.

So the largest clique contains three elements.

$\therefore$ the clique number of G, $\omega(G) = 3$.

Ex. 12. *Find the chromatic number of the following graph. Find whether this graph is perfect.*

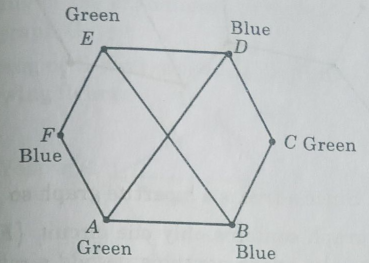


[*WBUT 2013*]

Let the vertex A be assigned a colour say Green. Since B, D and F are adjacent to A so they should be assigned with an another colour say Blue.

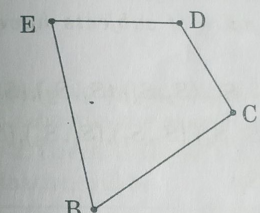
Since E and C are not adjacent to A they are assigned Green again.

The final labelling is shown below.



Hence the chromatic number is 2.

Let H :



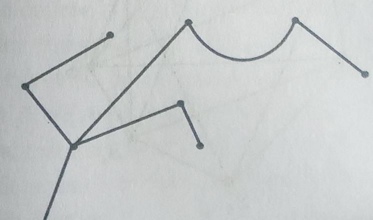
be a subgraph of the given graph G. Obviously $\omega(H) = 2$ and $\chi(H) = 2$ which shows $\omega(H) = \chi(H)$.

Similarly if we take any subgraph of the given we shall see the chromatic number and the clique number of each will be equal. So the given graph is perfect graph.

Ex. 13. *Find the chromatic number of the following graph*

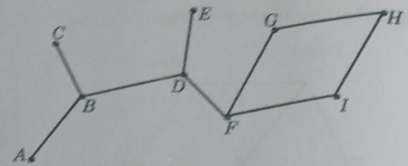
(i)

G_1:



(ii)

G_2:



(i) G_1 is a tree. Since a tree is a bipartite graph so $\chi(G_1)=2$

(ii) We see the graph contains only one circuit $\{F,G,H,I\}$ which is even. Thus the graph contains no odd circuit. So its chromatic number is 2.

Ex. 14. *In an examination seven subjects are to be scheduled; $S_1, S_2, S_3, \ldots S_7$. Following pairs of subjects have common students:*

$(S_1,S_2),(S_1,S_3),(S_1,S_4),(S_1,S_7),(S_2,S_3),(S_2,S_3),(S_2,S_4),$

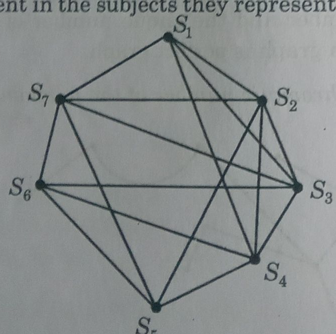
$(S_2,S_5),(S_2,S_7),(S_3,S_4),(S_3,S_6),(S_3,S_7),(S_4,S_5),(S_4,S_6),$

$(S_5,S_6),(S_5,S_7)$ *and* (S_6,S_7).

How can the examination be scheduled so that no student has two examination at the same day?

The above subjects are represented as vertices of the following graph G. The edge between two vertices is drawn if there is a common student in the subjects they represent.

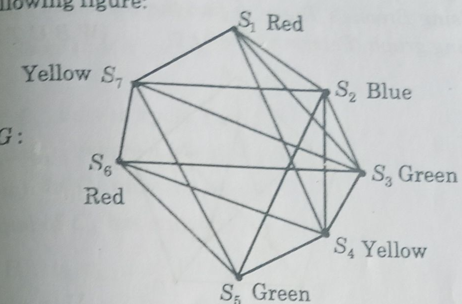
G:



Suppose a 'day of examination' is represented by a 'colour'. So scheduling of the examination corresponds to a colouring of the above graph G.

A colouring of G with minimum number of colours is shown in the following figure:

G:



We see G is 4-colourable. Therefore we understand 4 days are sufficient to schedule the examination. Suppose Red $\equiv$ Day-1; Blue $\equiv$ Day-2; Green $\equiv$ Day-3, Yellow $\equiv$ Day-4. The time-lable of the examination will be

Exam-Day	Day-1	Day-2	Day-3	Day-4
Subjects	S_1,S_6	S_2	S_3,S_5	S_4,S_7

Ex. 15. *Show that every planar graph is 6-colourable.*

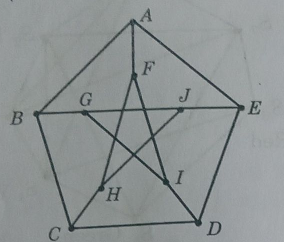
Let G be a planar graph with n number of vertices.

The result is true for $n \le 6$. We shall prove the result by method of induction of n. Let every planar graph of $n-1$ number of vertices is 6-colourable. Now since G is planar, it must have a vertex v whose degree is ≤ 5 (as established an earliar problem). Now consider the graph $G' = G - v$. Therefore G' has $n-1$ number of vertices. So, by hypothesis, G' is 6-colourable. Since v has at most 5 adjacent vertices in G' so colouring of G is possible by assigning a colour from the set of 6 colours required for G'. Thus G is also 6-colourable. Thus the result follows from the method of induction.

Ex. 16. *Find the chromatic number of a circuit (cycle) with 107 edges.*

A circuit with 107 edges will have 107 vertices. That is this circuit is C_{107}. Since 107 is odd. $\therefore \chi(C_{107}) = 3$

Ex. 17. *Using Brooke's Theorem find the chromatic number of the following graph (Petersen Graph)* Γ: [W B U T 2014]



The graph Γ is neither complete nor a circuit with odd number of vertices. It is not bipartite. So $\chi(\Gamma) \neq 2$. Obviously $\chi(\Gamma) \neq 1$. Here $\Delta(\Gamma) = 3$, the maximum degree of all the vertices. So by Brooke's theorem $\chi(\Gamma) \leq 3$. Hence $\chi(\Gamma) = 3$.

Ex. 18. *Show that a cycle with 3 vertices is perfect.*

C_3 is cycle with three vertices. We know its chromatic number $\chi(C_3) = 3$.

If the vertices v_1, v_2, v_3 form the cycle then every pair of these are adjacent. So $\{v_1, v_2, v_3\}$ form a clique of C_3. Obviously it is largest. $\therefore$ its clique number $\omega(C_3) = 3$. This shows that $\chi(H) = \omega(H)$ for every subgraph H of C_3. Thus C_3 is perfect.

Ex. 19. *Show that a cycle with n vertices is not perfect if n is odd and greater then 3.*

Let C_n be the cycle formed by the succesive vertices $v_1, v_2, v_3 \ldots v_n, v_1$.

We know $\chi(C_n) = 3$ (See an earlier Theorem).

In C_n only two consecutive vertices are adjacent. So any clique of C_n has only two vertices.

Therefore $\omega(C_n) = 2$.

Hence C_n is not a perfect graph.

Ex. 20. *Show that a cycle with n vertices is a perfect graph if n is even.*

Let C_n be the cycle formed by the successive vertices $v_1, v_2, v_3 \ldots v_n, v_1$ where n is odd. We know $\chi(C_n) = 2$ (see an earlier Theorem). In C_n only two consecutive vertices are adjacent. So any clique of C_n has only two vertices. $\omega(C_n) = 2$.

Let H : $v_k, v_{k+1}, \ldots v_r$ be an arbitrary, subgraph of C_n. Then obviously $\chi(H) = 2$ and $\omega(H) = 2$.

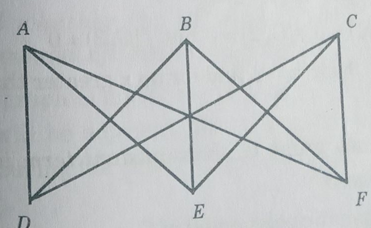
Thus C_n is perfect graph.

Ex. 21. *Show that a complete graph is a perfect graph.*

Let K_n be a complete graph with n vertices. Then $\chi(K_n) = n$. Since every pair of vertices of a complete graph are adjacent so K_n itself is the largest clique of K_n.

$\therefore \omega(K_n) = n$. $\therefore \chi(K_n) = \omega(K_n)$ $\therefore$ K_n is perfect graph.

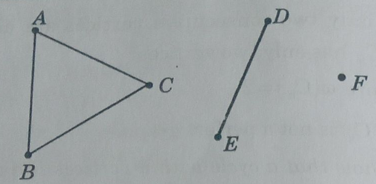
Ex. 22. *Find the chromatic number for the following graph* G:



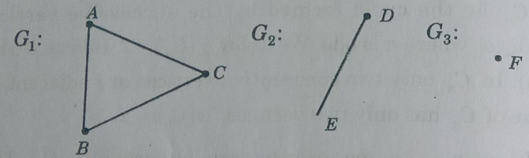
We see G is a bipartite graph. So its chromatic number is 2.

Ex. 23. *Find the chromatic polynomial and hence the chromatic number for the following graph :*

G:



G is made up of the three compoments G_1, G_2 and G_3 where



The first components G_1 is complete and can be coloured in $x(x-1)(x-2)$.

The second component G_2 can be coloured in

${}^xP_2 = \dfrac{x!}{(x-2)!} = x(x-1)$ ways.

G_3 is null graph. $\therefore G_3$ can be coloured in x ways.

$\therefore$ by principle of counting G can be coloured in

$$x(x-1)(x-2).x(x-1).x = x^3(x-1)^2(x-2)$$

$\therefore$ the chromatic polynomial is

$f(G,x) = x^3(x-1)^2(x-2)$

Now, $f(G,1)=0, f(G,2)=0$ but $f(G,3)\neq 0$

$\therefore f(G,x)=0$ for $x=2$ and for no x greater than 2.

$\therefore \chi(G)=2$

Ex. 24. *For each graph G, then constant term in its chromatic polynomial is zero*

Let the chromatic polynomial of G is

$$f(G,x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

Now $\chi(G)\geq 1$ for any graph G. (Note that $\chi(G)=1$ when G is null graph).

$\therefore f(x,G)=0$ for $x=0$

or, $a_0 0^n + a_1.0^{n-1} + \ldots + a_{n-1}.0 + a_n = 0$

or, $a_n = 0$. Proved.

Ex. 25. *If a graph G has at least one edge then the sum of the coefficients in its chromatic polynomial is 0.*

Let the chromatic polynomial of G is

$$f(G,x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

If G has at least one edge then at least two vertices of G are adjacent. So at least two colours are required for the colouring of G.

$\therefore \chi(G)\geq 2 \quad \therefore f(G,1)=0$

or, $a_0.1^n + a_1 1^{n-1} + \ldots + a_{n-1}.1 + a_n = 0$

or, $a_0 + a_1 + \ldots + a_{n-1} + a_n = 0$

Ex. 26. *Find whether the polynomial x^3+5x^2-3x+5 is a chromatic polynomial of some graph.*

Now $\chi(G)\geq 1$ for any graph G (Note that $\chi(G)=1$ when G is null graph.)

$\therefore f(G,x)=0$ for $x=0$.

But here we see $f(G,0)=5$. So the given polynomial can not be a chromatic polynomial of any graph.

Ex. 27. *Find whether the polynomial $x^4+2x^3-4x^2$ is a chromatic polynomial of some non-null graph.*

A non-null graph G must have at least one edge i.e. at least two vertices will be adjacent. So at least two colours will be needed for colouring the graph. $\therefore \chi(G)\geq 2$

$\therefore f(G,1)=0$

But here $f(G,1) = 1^4 + 2.1^3 - 4.1^2 = -1$.

Hence the given polynomial can not be a chromatic polynomial of any non-null graph.

Ex. 28. *Find the chromatic polynomial of a connected graph on three vertices.* [W.B.U.T. 2013]

Since the given graph G has 3 vertices so the chromatic polynomial is

$$f(G,x)=c_1\,{}^xC_1++c_2\,{}^xC_2+c_3\,{}^xC_3$$

Since G is connected so it must have an edge. So only one colour cannot colour G. Hence $c_1=0$.

Case 1. Now if the three vertices form a triangle then $c_2=0$ and $c_3=3!=6$. Then the chromatic polynomial will be

$$f(G,x)=6.\,{}^xC_3=6\frac{x(x-1)(x-2)}{3!}=x(x-1)(x-2)$$

Case 2. Now if the three vertices does not form a triangle then it makes a path. In that case $c_2=2$ and $c_3=3!=6$. Then the chromatic polynomial is $f(G,x)=2\,{}^xC_2+6\,{}^xC_3$

$$=2\frac{x(x-1)}{2!}+6\frac{x(x-1)(x-2)}{3!}=x(x-1)+x(x-1)(x-2)=x(x-1)^2$$

Ex. 29. *How many ways a tree on 5 vertices can be coloured with at most 4 colours.*

Let G be the tree. The required number is nothing but the value of the chromatic polynomial $f(G,x)$ for $x=4$.

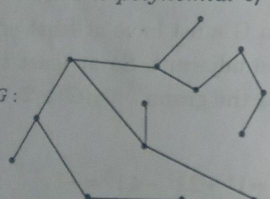
We know the chromatic polynomial of G on 5 vertices is

$$f(G,x)=x(x-1)^4$$

$$\therefore f(G,4)=4(4-1)^4=324 \text{ ways.}$$

Ex. 30. *Find the chromatic polynomial of the following graph G :*

G :

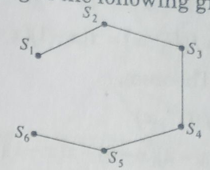


G is a tree with 14 vertices. Therefore its chromatic polynomial

$$f(G,x)=x(x-1)^{13}$$

Ex. 31. *A new flag is to be designed with 6 vertical stripes using 4 colours.In how many ways can this be done so that no two adjacent stripes have the some colour?* [W B U T 2012]

Let S_1, S_2, S_3, S_4, S_5, and S_6 be the stripes. The problem is nothing but the coloring of the following graphs with four colors:



Note that S_1 and S_6 are not adjacent stripes. This is a tree with 6 vertices.

The chromatic polynomial of this tree is

$$f(x)=x(x-1)^{6-1}=x(x-1)^5$$

No. of ways of colouring with 4 or fewer colors is

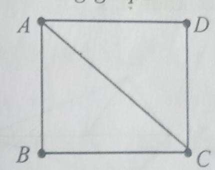
$$f(4)=4(4-1)^5=972$$

No. of ways of coloring with 3 or fewer colors

$$=f(3)=3(3-1)^5=96$$

No. of ways of coloring with four colors $=972-96=876$.

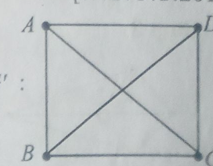
Ex. 32. *Find, using Decomposition Theorem, the chromatic polynomial of the following graph*



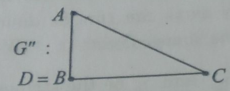
Hence find chromatic number of G. [W.B.U.T.2014]

Let G be the given graph. We obtain the following graph G' from G by adding the edge BD between the two vertices B and D.

G' :



The graph G'' is obtained from G by merging B and D together.



Since G' is K_4, So $f(G',x) = x(x-1)(x-2)(x-3)$

Since G'' is K_3, So $f(G'',x) = x(x-1)(x-2)$

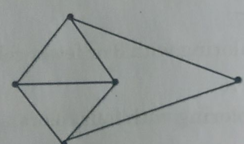
By Decomposition Theorem,

$$f(G,x) = f(G',x) + f(G'',x)$$
$$= x(x-1)(x-2)(x-3) + x(x-1)(x-2)$$
$$= x(x-1)(x-2)(x-2) = x(x-1)(x-2)^2$$

We see $f(G,1) = 0, f(G,2) = 0, f(G,3) \neq 0$

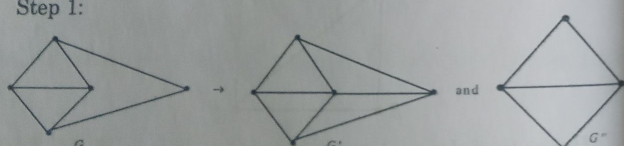
$\therefore \chi(G) = 3$

Ex. 33. *Find the chromatic polynomial of the following graph*

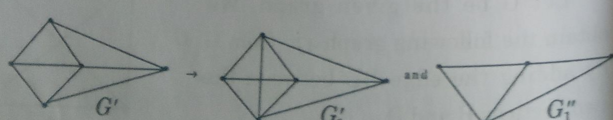


We show the following iteration to reach the graphs whose chromatic polynomial are known:

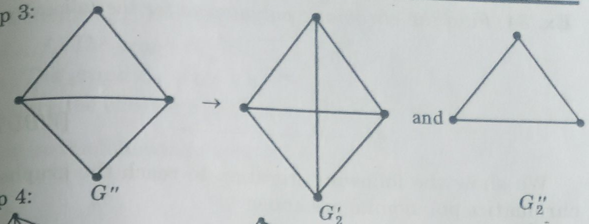
Step 1:



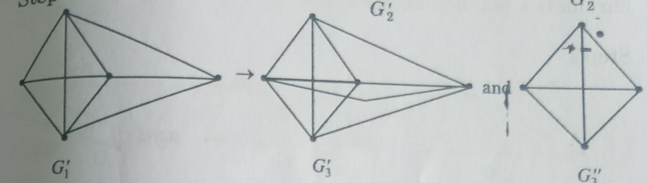
Step 2:



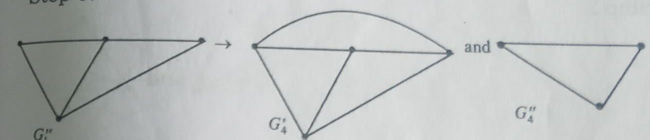
Step 3:



Step 4:



Step 5:



By Decomposition Theorem,

$f(G,x) = f(G',x) + f(G'',x)$ from Step 1

$= \{f(G_1',x) + f(G_1'',x)\} + \{f(G_2',x) + f(G_2'',x)\}$ from Step 2 and Step 3

$= [\{f(G_3',x) + f(G_3'',x)\} + \{f(G_4',x) + f(G_4'',x)\}] + \{f(G_2',x) + f(G_2'',x)\}$

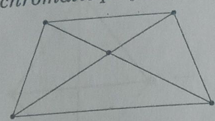
from Step 4 and Step 5

$$= [\{x(x-1)(x-2)(x-3)(x-4) + x(x-1)(x-2)(x-3)\}$$
$$+ \{x(x-1)(x-2)(x-3) + x(x-1)(x-2)\}]$$
$$+ \{x(x-1)(x-2)(x-3) + x(x-1)(x-2)\}$$

[$\because G_3', G_3'', G_4', G_4'', G_2'$ and G_2'' all are complete Graph]

$$= x(x-1)(x-2)(x-3)(x-4) + 3x(x-1)(x-2)(x-3) + 2x(x-1)(x-2)$$
$$= x^5 - 7x^4 + 19x^3 - 23x^2 + 10x$$

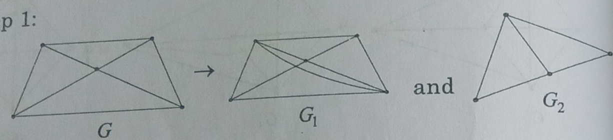
Ex. 34. *Find the chromatic polynomoal for the following graph*



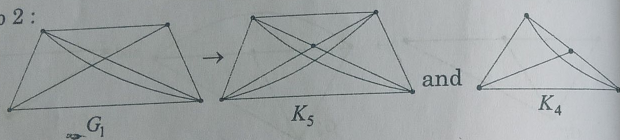
[WBUT 2015]

We show the following iteration to reach the graphs whose chromatice polynomial are know:

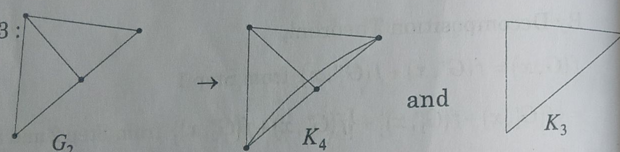
Step 1:



Step 2 :



Step 3 :



We stop at step 3 as G_5 and G_6 are complete graph.

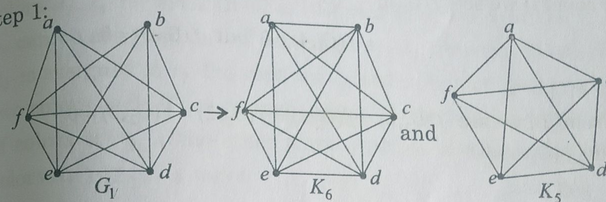
By decomposition theorem,

$$f(G,x)=f(G_1,x)+f(G_2,x)$$
$$=\{f(K_5,x)+f(K_4,x)\}+\{f(K_4,x)+f(K_3,x)\}$$
$$=f(K_5,x)+2f(K_4,x)+f(K_3,x)$$
$$=x(x-1)(x-2)(x-3)(x-4)+2x(x-1)(x-2)(x-3)\ +x(x-1)(x-2)$$
$$=x(x-1)(x-2)\{(x-3)(x-4)+2(x-3)+1\}$$
$$=x(x-1)(x-2)\{(x-3)(x-2)+1\}$$

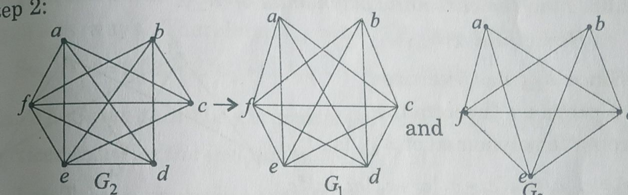
Ex. 35. *Consider K_6, the complete graph on the six vertices a, b, c, d, e, f. The graph G_1 is optained from K_6 by deleting the edge ab. The graph G_2 is optained from G_1 by deleting the edge cd. What are the chromatic numbers of G_1 and G_2* [WBUT 2016]

We show the following decompositions:

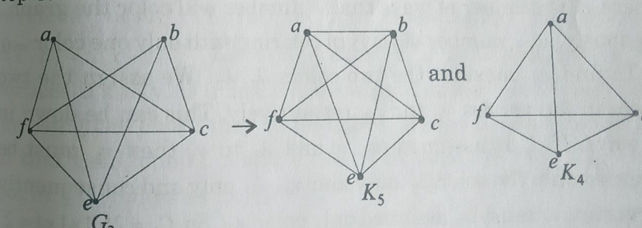
Step 1:



Step 2:



Step 3:



From step we have, by decomposition theorem,

$$f(G_1,x)=f(K_6,x)+f(K_5,x)=x(x-1)(x-2)(x-3)(x-4)(x-5)$$
$$+x(x-1)(x-2)(x-3)(x-4)=x(x-1)(x-2)(x-3)(x-4)^2 \quad (1)$$

From step 2,we have $f(G_2,x)=f(G_1,x)+f(G_3,x)$

From step 3, we have $f(G_3,x)=f(K_5,x)+f(K_4,x)$

$\therefore f(G_2,x) = f(G_1,x) + f(K_5,x) + f(K_4,x) = x(x-1)(x-2)(x-3)$
$(x-4)^2 + x(x-1)(x-2)(x-3)(x-4) + x(x-1)(x-2)(x-3) = x(x-1)$
$(x-2)(x-3)(x-4)^2 + x(x-1)(x-2)(x-3)^2$

$$= x(x-1)(x-2)(x-3)\left\{(x-4)^2 + x - 3\right\} \qquad (2)$$

From (1) we see $f(G_1,0) = f(G_1,1) = f(G_1,2) = f(G_1,3)$

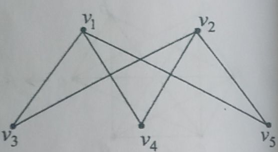
$$= f(G_1,4) = 0 \text{ but } f(G_1,5) \neq 0.$$

$\therefore \chi(G_1) = 5$

From(2), we see $f(G_2,0) = f(G_2,1) = f(G_2,2) = f(G_2,3) = 0$
but $f(G_2,4) \neq 0$

$\therefore \chi(G_1) = 4$

Ex.36. *Find the chromatic polynomial of $K_{2,3}$. Hence find $\chi(K_{2,3})$.*



Since $K_{2,3}$ has 5 vertices by a previous Theorem the chromatic polynomial of $K_{2,3}$ is

$f(x) = c_1\,{}^xc_1 + c_2\,{}^xc_2 + c_3\,{}^xc_3 + c_4\,{}^xc_4 + c_5\,{}^xc_5$

where c_i is number of ways that i number will color the graph.
Obviously, c_1 = number of ways of coloring with only one color $= 0$

To find c_2 consider the two colors λ_1, λ_2. We assign the two colors to the verties v_1 and v_3 respectively. This can be down in 2! ways. If λ_1 is assigned to v_1 and λ_2 to v_3 then v_2 must be assigned only the color λ_2 only and v_4, λ_2 only and consequently the vertex v_5 must be assigned only color λ_2. So $C_2 = 2! \times 1 \times 1 \times 1 = 2$.

To find c_3 conssider the three colors $\lambda_1, \lambda_2, \lambda_3$. Among these two would be assigned to the two vertices v_1, v_3. This can be done in ${}^3p_2$ ways. Corresponding to the assignment λ_1 to v_1, λ_2 to v_3, the vertex can be assigned λ_1 or λ_3, 2 ways. If λ_3 is assigned to v_2 then v_4, v_5 should be assigned only the color λ_2.

Thus $\qquad c_3 = {}^3p_2 \times 2 \times 1 = \dfrac{3!}{1!} \times 2 = 12.$

To find c_4 consider the four colors $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Vertices v_3 and v_1 can be assigned any two distirct colors from these ___ this can be done in ${}^4p_2 = 12$ ways. Corresponding to λ_1, λ_2 at v_1, v_3 vertex v_2 can be assigned any of the colors λ_1, λ_3, and λ_4. If λ_1 is assigned to v_2 the vertices v_4 and v_5 can be assigned any of the colors λ_3, λ_4 for each. If λ_3 is assigned to v_2 the vertices v_4 and v_5 can be assigned any of the colors λ_2, λ_4 for each. Similarly if λ_4 is assigned to v_2 the vertices v_4 and v_5 can be assigned with two colors each. Thus corresponding to a particular assignment of colors to v_3, v_1 the other three vertices can be assigned the colors in $2 \times 2 \times 2 = 8$ ways.

Therefore, the graph can be colored with the four colors in $12 \times 8 = 96$ ways. Therefore, $C_4 = 96$ Obviously $C_5 = 5! = 120$

So, $f(x) = 0.{}^xC_1 + 2.{}^xC_2 + 12.{}^xC_3 + 96\,{}^xC_4 + 120\,{}^xC_5$

Ex. 37. *Show that the chromatic polynomial of a graph G of n vertices satisfies the inequality $f(G,x) \le x(x-1)^{n-1}$.*

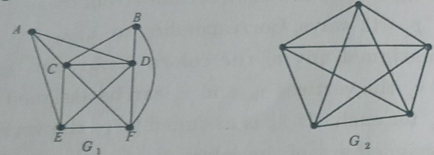
Let T be a spanning tree of G. So the tree T will have n vertices also. Therefore its chromoatic polynomial $f(T,n) = x(x-1)^{n-1}$. Let e be a chord of T in G joining the two vertices u and v. If u and v are assigned same colour then after imposing the chord they could not be assigned same colour. So, in that case, one of u and v might be coloured in $x-2$ ways; not in $x-1$ ways as for T.

Now as $x-2 < x-1$. Therefore, $f(G,x) \le x(x-1)^{n-1}$

Ex. 38. *Prove that the non-planar graph with smallest number of vertices is a complete graph with five vertex.*

K_5 denotes the complete graph with five vertices. We know $\chi(K_5) = 5$. Four colour theorem states that the chromatic number of a planar graph is ≤ 4. So K_5 can not be planar. Obviously a graph with four vertices is always planar. Hence proved.

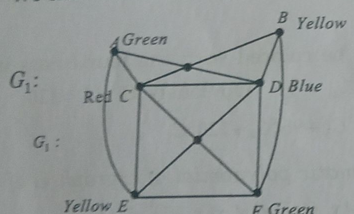
Ex. 39. *Find the chromatic number of the following two graphs G_1 and G_2. State these results do not violate Four colour Theorem.*



CDFE is a complete graph K_4.

$\therefore \chi(K_4) = 4$.

We show the labelling in the following figure



G_2 is a complete graph K_5. Therefore, $\chi(G_2) = 5$.

Four colour Theorem states that $\chi(G) \le 4$ if G is planar. But here G_2 is non-planar. So this result does not violate Four colour Theorem.

Ex. 40. *Show that every bi-partite graph is perfect.*

Let G be a bi-partite graph and H be a subgraph of G. So H is either null graph or a Bi-partite graph. Since clique number of a null graph does not exist so we assume H is a bipartite graph.

Let the vertex set of it is partioned into two sets V_1, V_2. Every clique of H consists of two vertices one of V_1 and another of V_2. So every clique of H has 2 elements.

$\therefore\ \omega(H) = 2$. Again $\chi(H) = 2$. $\chi(H) = 2$ (as we got earlier). Hence G is perfect graph.

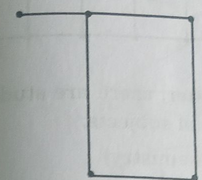
Exercise

1. Find $\chi(K_{14})$

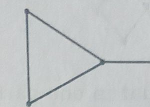
2. Find $\chi(K_5, K_{14})$

3. Find the chromatic number of the following graphs:

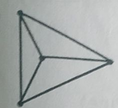
(i)



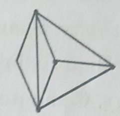
(ii)



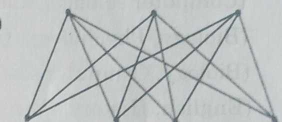
(iii)



(iv)

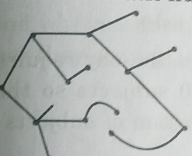


(v)

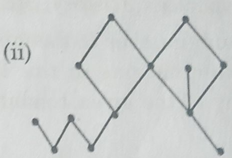


4. Find the chromatic number of the graph

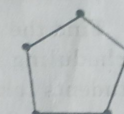
(i)



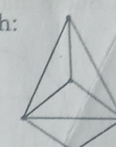
(ii)



(iii)



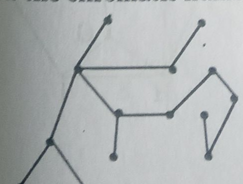
5. Find the chromatic number of the following graph:



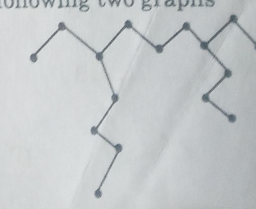
6. Find the chromatic number of a circuit (cycle) with (i) 38 edges (ii) 45 vertices (iii) 1000 vertices

7. Find the chromatic number of the following two graphs

(i)

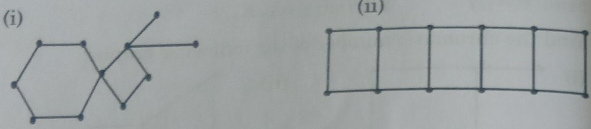


(ii)



8. Find the chromatic number of the following two graphs:

(i) (ii)



9. Suppose that in one particular semester, there are students taking each of the following combination of subjects:

(English, Mathematics, Biology and Chemistry)

(Computer science, Mathematics, English, Geography)

(Biology, Phychology, Geography, Spanish)

(Biology, Computer science, History, French)

(English, History, Psychology, Computer science)

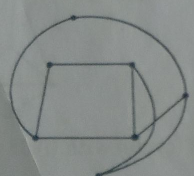
(Psychology, Computer science, Chemistry, French)

(Geography, Psychology, History, Spanish)

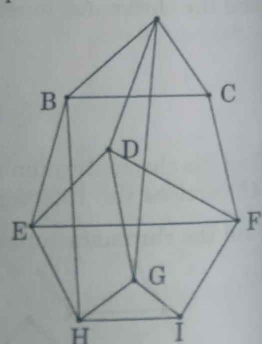
Find the minimum number of Examination-day required for scheduling the examinations in the 10 subjects so that no students taking any of the given combination of subjects have no conflicts?

10. Show that each of the following is perfect

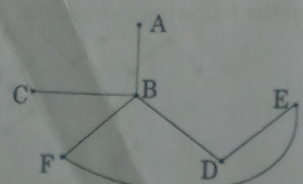
(i)



(ii)

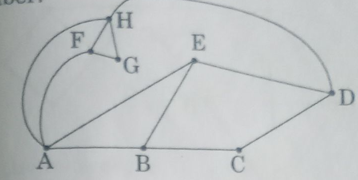


(iii)

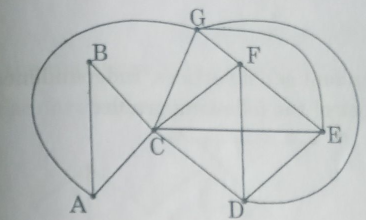


11. Find all the cliques of the following graphs. Hence find their clique number.

(i)

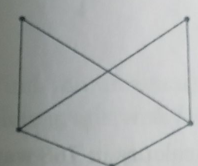


(ii)

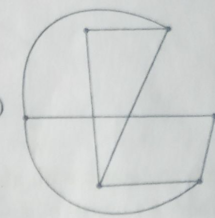


12. Find whether the following graphs are bipartite. Hence find whether they are perfect graphs.

(i)

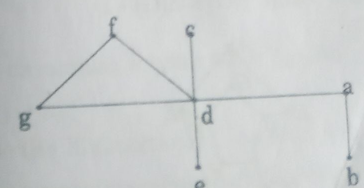


(ii)

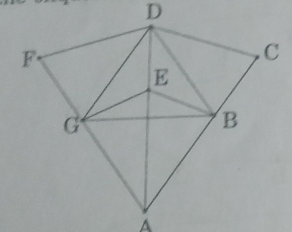


[Hint: (ii) This contains a cycle of length 3]

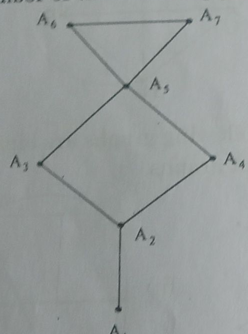
13. Find four independent sets of vertices and independence number of the following graph G;



14. Find all the cliques and the clique number of the following graph :



15. Find four independent set of vertices, Independence number and chromatic number of the following graph :

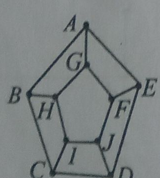


16. A tree can be coloured with at most 3 colours in 768 ways. Find the number of vertices of the tree.

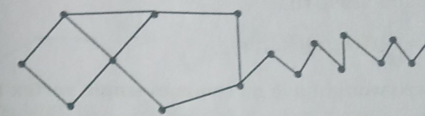
[Hint: $f(G,3)=3.2^{n-1}$ or, $768=3.2^{n-1}$]

17. Find the chromatic number of the following graph G and show that how it obeys the result $\chi(G)\le\Delta(G)+1$

α:



18. Show that the following graph G

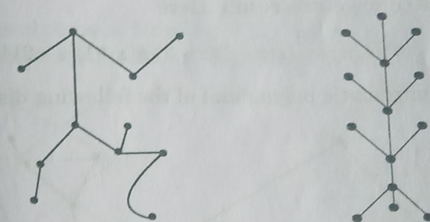


Obeys the two results (i) $\chi(G)\le\Delta(G)+1$

(ii) $\chi(G)\le\max\delta(G')+1$

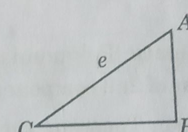
where the maximum is taken over all subgraphs G' of G

19. Find the chromatic polynomial of the following two graphs



20. Find the chromatic polynomial of K_4-e where K_4 is a complete graph with four vertices and e be an edge of K_4.

21. If G be the following graph then find the chromatic polynomial of $G-e$



22. Find the chromatic polynomial of G where G is a circuit with 4 vertices.

[Hint: See the Illustration of Theorem 4 in Art 4.2.8]

23. Find the chromatic polynomial of a circuit with three vertices

24. (i) If a graph G is union two graphs G_1 and G_2 that share a single vertex then prove that

$$f(G,x)=\frac{1}{x}f(G_1,x)f(G_2,x)$$

[Hint: If there would have no common single vertex then the chromatic polynomial be $f(G_1,x)f(G_2,x)$. Now the common vertex may be assigned x number of ways. So $xf(G,x)=f(G_1,x)f(G_2,x)$]

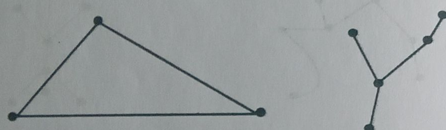
(ii) If G is the connected graph obtained by getting union of two triangles so that they share one vertex in common, find the chromatic polynomial of G.

[Hint: Use (i) to get the result. Here

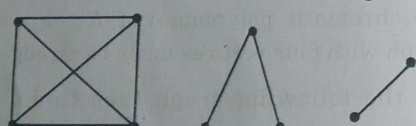
$$f(G,x)=\frac{1}{x}x(x-1)(x-2).x(x-1)(x-2)$$

25. Find the chromatic polynomial of the following disconnected graphs:

(i)



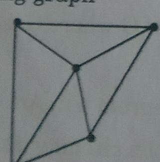
(ii)



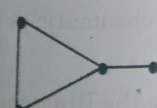
[Hint for (ii): Chromatic Polynomial of 1st component is $x(x-1)(x-2)(x-3)$, that of 2nd component is $x(x-1)(x-2)$ etc.]

26. Find by Decomposition Theorem, the chromatic polynomial of the following graph

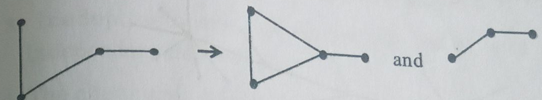
(i)



(ii)

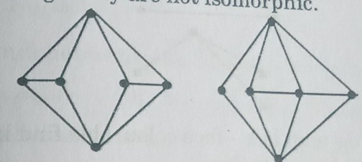


[Hint: Use the decomposition

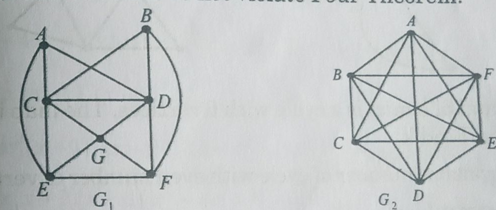


]

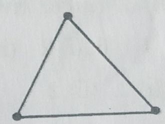
27. Prove that the following two graphs have same chromatic polynomial though they are not isomorphic.



28. Find the chromatic Number of the following two graphs and show how these results do not violate Four Theorem.

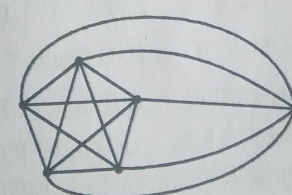


29. What is the edge chromatic number of the graph



30. Find the edge chromatic number of the following graph

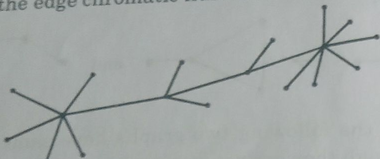
G :



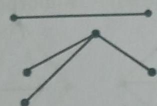
Hence colour the edges of G with minimum number of colours.

31. Find the edge chromatic number of the following graphs

(i)

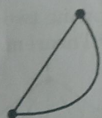


(ii)

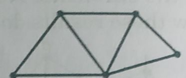


32. If the following map is k - face colourable find k. Hence find the chromatic number of its dual.

(i)



(ii)



33. The dual of a map is a cycle with 6 vertices. The map is k - face colourable. Find k.

[Hint. Chromatic number of cycle with even number of vertices is 2]

Answers

1. 14 2. 2 3. (i) 2, (ii) 3, (iii) 4, (iv) 4, (v) 2

4. (i) 2, (ii) 2, (iii) 3 5. 4 6. (i) 2 (ii) 3 (iii) 2

7. (i) 2, (ii) 2 8. (i) 2, (ii) 2

9. Minimum 6 Exam-day

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6
M, P	E, S, F	B	C, G Comp,	H	

11. (i) The cliques are {A,B,E}, {F,G,H} and {A,F,H}. clique no. = 3

(ii) The cliques are {A,B,C}, {A,C,G}, {C,F,G},{C,D,E}, {E,E,F}, {C,D,F}, {D,E,F}, {E,F,G}, {D,E,G}, {C,D,G}, {C,E,G}, {D,F,G}, {C,D,E,F}, {C,D,F,G}, {D,E,F,G}.{C,E,F,G}, {C,D,E,G}, {C,D,E,F,G}, The clique no. = 5.

12. (i) bipartite and perfect.(ii) bipartite and perfect.

13. Inedep. sets are {a,c,e}, {a,c,f}, {c,f,e} and {a,c,e,g}. Independent no. is 4.

14. The clique are {A,B,G}, {A,B,E}. {A,E,G}, {B,E,G}, {B,C,D}, {D,F,G}, {B,D,G}, {B,E,D}, {G,E,D}, {B,D,E,G}.

Clique No. is 4.

15. The independent set of vertices are $\{A_1, A_3, A_4, A_6\}$, $\{A_1, A_3, A_4, A_7\}$, $\{A_1, A_7\}$, $\{A_1, A_5\}$. Clique No. is 4, chromatic number is 3.

16. 9 17. $\chi(\alpha) = 3$ 19. (i) $x(x-1)^{10}$ (ii) $x(x-1)^{13}$

20. $x(x-1)(x-2)^2$ 21. $x(x-1)^2$

22. $x^4 - 4x^3 + 6x^2 - 3x$ 23. $x(x-1)(x-2)$

24. (ii) $x(x-1)^2(x-2)^2$

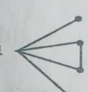
25. (i) $x^2(x-1)^5(x-2)$, (ii) $x^3(x-1)^3(x-2)^2(x-3)$

26. (i) $x(x-1)(x-2)(x^2-5x+7)$, (ii) $x(x-1)^2(x-2)$

28. $\chi(G_1) = 3, \chi(G_2) = 6$ G_2 is non-planar.

29. 3 30. 5 31. (i) 7 (ii) 3 32. (i) 2 (ii) 3 33. 2

Multiple Choice Questions

1. The graph  is

(a) 2-vertex colourable (b) 1-vertex colourable

(c) 4-vertex colourable (d) 0-vertex colourable

2. If a graph is k vertex-colourable then it is k+1 vertex colourable

(a) True (b) False