DISCRETE MATHEMATICS

NOTE:

MAKAUT course structure and syllabus of 4th semester has been changed from 2020. Previously **DISCRETE MATHEMATICS** was in 3rd Year both Semesters. This subject has been completely redesigned and shifted in 4th semester in present curriculum. Taking special care of this matter we are providing chapterwise relevant MAKAUT university solutions and some model questions & answers for newly introduced topics, so that students can get an idea about university questions patterns.

SET, RELATION & FUNCTION

Chapter at a Glance

Introduction to Relations and Functions

This topic "*Relations and Functions*" is a foundation or fundamental of algebra in mathematics. Relations and functions are two different words having different meaning mathematically. Many us you might be confused in their difference. We shall study both these concepts in detail here. Same as the relations which we have in our daily life, a kind of relations also exists in algebra. In daily life, relations are like brother and sister, friends, student and teacher and many more. In mathematics also we see some relations like a line is parallel or perpendicular to another, any one variable is greater or less than the another variable. Any Set A is subset of B, all these are examples of relations.

One thing which we see in common while studying relations, that it required two different objects to link two different objects via relations.

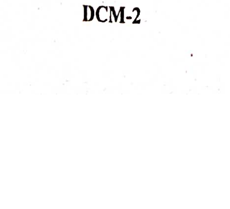
What is the meaning of Relation in math?

Understanding Relations requires basic knowledge of sets. *A Set is a collection of well defined objects of particular kind.* **For example** a set of outcomes of dice, a set of English alphabet. Relation is always studied between two sets. If we have two non void (or null/empty) sets A and B then the relation **R** from set A to set B is represented by **aRb**, where a is the set of elements belonging to set A while b belongs to set B.

Relation from a set A to a set B is the subset of the ***Cartesian product*** of A and B i.e. subset of A x B. Relation in other way can also be defined as an collection of ordered pairs (a, b) where a belongs to the elements from set A and b from set B and the relation is from A to B but not vice versa.

For Example

Consider a set A containing elements as {1, 2, 3} and set B contains elements as {2, 4, 6}. Then the relation between Set A and B from A to B will be set of any combinations from Set A to set B.



From the above diagram, we can see that Relation from A to B i.e. **R** will be set of {(1,4). (1,2), (3,4), (3,2)}. This relation is a subset of the Cartesian product of two sets A X B.

Let's take another example where, set A = {1, 2, 3} and set B = {1, 2, 3, 4, 5, 6, 7, 8, 9}.

If the Relation between A and B is as: elements of B is the squares of elements of set A, then the relation is written in the form of sets as:

R = {(a,b): where b is square of a and a ∈ A & b ∈ B}

Then R = {(1,1), (2,4), (3,9)}

Same as sets, relation may also be represented algebraically either by the ***Roster method*** or by the ***Set-builder method***.

Relation can also be defined as a linear operation which establishes relationship between the element's of two set's according to some definite rule of relationship.

R : {(a, b) | (a, b) ∈ A x B and a R b}

Eg: 1 A is {2, 3, 5}

B is {1, 4, 9, 25, 30}

If a R b → b is square of a

Discrete element of relation are {(2, 4), (3, 9). (5, 25)}

Eg: 2 A = {Jaipur, Lucknow, Kanpur, Bhopal}

B = {Rajasthan, Uttar pradesh, Madhya Pradesh}

aRb → a is capital of b,

A X B = {(Jaipur, Rajasthan), (Lucknow, Uttar Pradseh), (Bhopal, Madhya Pradesh}

Total Number of Relation from A to B

Let the number of relations from A to B be x.

Let A contain 'm' elements and B contains 'n' elements

Number of element's in A x B → m x n

Number of non-void subset $= C_1^{mn} + C_2^{mn} + \dots\dots + C_{mn}^{mn} = 2^{mn} - 1$

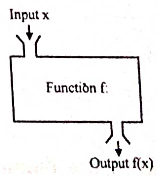
Thus, for A = {1, 2, 3) & B = {x, y, z}

Number of non-void subset's or the number of possible relations = $=2^9 - 1 = 511$

What are Functions?

Functions are the special class of relation or we can say that *special types of relations are called as **Functions***. Function is one of the most important concepts in mathematics as every situation in real life are solved and analysed first by writing its mathematical equation or function.

A function is like a machine which gives unique output for each input that is fed into it. But every machine is designed for certain defined inputs for e.g. a washing machine is designed for washing cloths and not the wood. Similarly the functions are defined for certain inputs which are called as its ***domain*** and corresponding outputs are called ***Range.***



Let A and B be two sets and let there exist a rule or manner or correspondence 'f' wihch associates to each element of A to a unique element in B, then f is called a ***Function*** or ***Mapping*** from A to B. It is denoted by symbol

$f:(A,B)$ or $f:A\rightarrow B$ or $A\xrightarrow{f}B$

Which reads 'f is a function from A to B' or' f maps A to B.

If an elements $a \in A$ is associated with an element $b \in B$ then b is called 'the f ***image*** of a' or 'image of a under f' or 'the value of the function f at a'. Also a is called the ***pre – image*** of b or argument of b under the function f. we write it as

f:(a, b) or f:a → b or b = f(a)

A relation f from a set A to a set B is called as the function if it satisfies the below conditions:

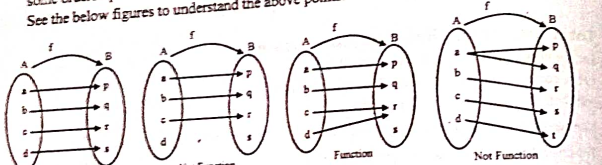
- All the elements of A should be mapped with the elements of B. That is, there should not be any element in A which is being unmapped with B.

  i.e. ∀a, (a, f(a)) ∈ f, where, a is the elements of set A
- Elements of set A should be uniquely mapped with the elements of set B.

  i.e. if (a, b) ∈ f & (a, c) ∈ f, ⇒ b = c

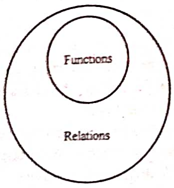
  Thus the ordered pairs of *f* must satisfy the property that each element of A appears in some ordered pair and no two ordered pairs have same first elements.

  See the below figures to understand the above points.



Function Not Function Function Not Function

Note: Every function is a relation but every relation is not necessarily a function.



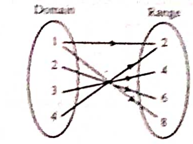
What is the domain of a function? Also explain its co-domain and Range

For a relation from set A to set B i.e. a*R*b, all the elements of set A are called as the ***domain*** of the relation R while all the elements of set B are called as the ***co-domain*** of the relation R.

Range is the set of all second elements from the ordered pairs (a, b) in the relation aRb.

Domain of f = {a | a ∈ A, (a, f(a)) ∈ f }

Range of f = {f (a) |a ∈ A, f(a) ∈ B, (a, f (a)) ∈ f}



For the relation aRb, domain is considered as the input to relation R while the co-domain is the possible outputs and range is the actual output.

Similarly for a function, f: A → B, elements of set A are the inputs and B is the set of possible outputs. But the second elements of all ordered pair of *f(A, B)* will be the actual outputs.

It should be noted that range is a subset of co-domain. If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

Which is not the graph of a function?

To identify any graph, weather it is a function or not, we must understand its definition once again but in terms of graphical meaning.

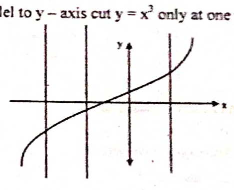
First condition of function says that – "All the elements of A should be mapped with the elements of B". That means graphically, for every input in its domain, function must give or provide the corresponding output.

Second condition of the function says that – "Elements of set A should be uniquely mapped with the elements of set B". These means that, for any input x, we must have one and only one output. The best way to check this condition for the function y = f(x), is draw a line parallel to y – axis. If it cuts the graph at two or more distinct points, this means, for one value of x, we are getting more than one outputs. And hence it will not be a function.

Example

$y = x^3$

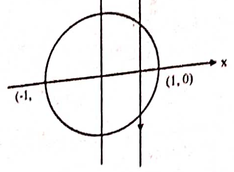
Here all the straight lines parallel to y – axis cut $y = x^3$ only at one point.



Example

$x^2 + y^2 = 1$

Here line parallel to y –axis is intersecting the circle at two points hence it is not a function.



Even Function: Let f(x) be a real valued function of a real variable. Then f is even if the following equation holds for all x and -x in the domain of f:

f(x) = f(-x)

Geometrically, the graph of an even function is symmetric with respect to the y-axis.

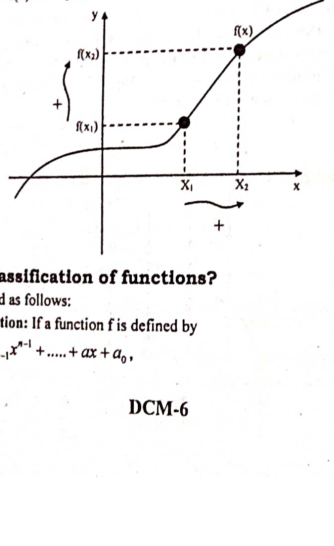
Odd Function: Again, let f(x) be a real valued function of a real variable. Then f is odd if the following equation holds for all x and -x in the domain of f:

$f(-x) = -f(x)$ or $f(x) + f(-x) = 0$

Increasing function: A function f is said to be increasing if whenever a >b, then $f(a) = f(b)$.

Further a function is said to be strictly increasing if

When a > b, then f(a) > f(b). The graph of an increasing function looks somewhat like this:



What are the Classification of functions?

Functions are classified as follows:

- **Polynomial Function:** If a function f is defined by

$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + ax + a_0$,

where n is a non negative integer and $a_n, a_{n-1}, ..., a, a_0$ are real number and $a_n \neq 0$, then f is called a **Polynomial Function of degree n.** A polynomial function is always continuous.

- **Algebraic Function:** A function f is called an algebraic function if it can be constructed using algebraic operations such as addition, subtraction, multiplication, division and taking roots, started with polynomials.

Example

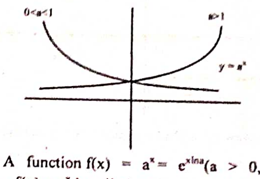
$$f(x) = \sqrt{x^2+1} \quad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x-2)\sqrt[3]{x+1}$$

Note: All the polynomials are algebraic but converse is not true. Functions which are not algebraic, are known as **Transcidental Function.**

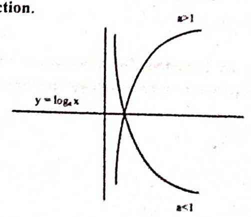
Fractional Rational Function: A rational function is a function of the form, $y = f(x) = \frac{f(x)}{h(x)}$, where $g(x)$ & $h(x)$ are polynomials & $h(x) \neq 0$. The domain of f(x) is set of real x such that $h(x) \neq 0$.

Example

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}; \quad D = \{x | x \neq \pm 2\}$$



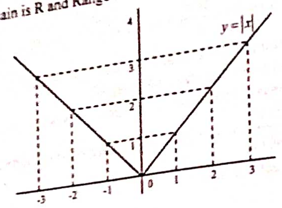
- **Exponential Function:** A function f(x) = $a^x = e^{x\ln a}$(a > 0, a ≠ 1, x ∈ R) is called an **Exponential Function.** f(x) = a^x is called an exponential function because the variable x is the exponent. It should not be confused with power function. g(x) = x^2 in which variable x is the base. For f(x)= e^x domain in R and range is R^+.
- **Logarithmic Function:** A function of the form y=$\log_a$ x, x > 0, a > 0, a ≠ 1, is is called **Logarithmic Function.**

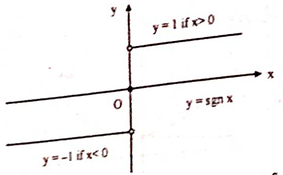


- **Absolute Value Function (or Modulus Function):** A function $y = f(x) = |x|$ is called the **Absolute Value Function** or **Modulus Function**. It is defined as:

$$y = |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

For f(x) = |x|, domain is R and Range is $R^+ \cup \{0\}$. See below for its figure.





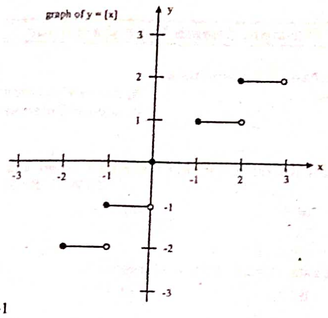
- **Signum Function:** A function $y = f(x) = \text{sgn}(x)$ is defined as follows:

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

It is also written as

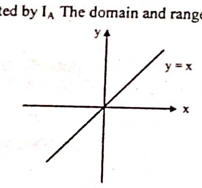
$$\text{sgn}(x) = \frac{|x|}{x} \text{ or } \frac{x}{|x|}, x \ne 0 : f(0) = 0$$

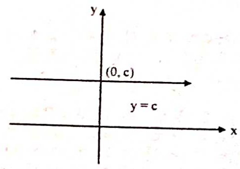
- **Greatest Integer Or Step up Function:** The function y = f(x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for:



$-1 \le x < 0$; [x] = -1
$0 \le x < 1$; [x] = 0
$1 \le x < 2$; [x] = 1
$2 \le x < 3$; [x] = 2 and so on.
For f(x) = [x], domain is R and range is I.

- **Identity Function:** The f: A → B defined by f(x) = x, ∀ x ∈A is called the identity of A and is denoted by I_A The domain and range of identity function is entire real range i.e. R **f(x) = x**





- **Constant Function:** The function f: A → B is said to be a constant function if every element of A has the same f image in B. Thus f : A → B; f(x) = c, ∀ x ∈ A, c ∈ B is a constant function. Note that the range of a constant function is a singleton and a constant function may be one – one or many – one, onto or into.

Example

$f(x) = [\{x\}]$; $g(x) = \sin^2 x + \cos^2 x$; $h(x) = \text{sgn}(x^2 - 3x + 4)$ etc., all are constant functions.

Multiple Choice Type Questions

1. Every non-empty subset of $\mathbb{N}$ contains a [WBUT 2014(ODD)]
a) maximal element
b) minimal element
c) least element
d) greatest element
Answer: (c)

2. The number of binary relations on a set having 3 elements is [WBUT 2013(EVEN), 2016(EVEN)]
a) 3^3
b) 3^9
c) $3!$
d) none of these
Answer: (d)

3. The number of subsets of a set with n elements is [WBUT 2015(EVEN)]
a) n
b) $2n$
c) $2^n - 1$
d) 2^n
Answer: (d)

4. In the set $S = \{1, 2, 3, 4, 6, 9\}$ defines a relation R by ${}_aR_b$ if and only if b is a multiple of a. Then which one of the following statements is correct? [WBUT 2016(ODD)]
a) 3 and 4 are comparable
b) 9 succeeds 3
c) 3 succeeds 9
d) 4 and 6 are comparable
Answer: (c)

5. If P = {2, 4, 6, 7, 8, 9}, Q = {1, 2, 6, 9}, then $P - Q$ is [MODEL QUESTION]
a) {4, 7, 8}
b) {4, 6, 8, 9}
c) {1}
d) {2, 4, 6, 7, 8, 9}
Answer: (a)

6. If A = {1, 2, 3}, B = {a, b}, A × B is given by [MODEL QUESTION]
a) {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}
b) {(a, 1), (a, 2), (a, 3), (b,1), (b, 2), (b, 3)}
c) {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b), ( )}
d) {1, 2, 3, a, b}
Answer: (a)

7. If $A = \{1,2,3,4,8\}$, $B = \{2,4,6,7\}$, then $A \Delta B$ is [MODEL QUESTION]
a) {2, 4}
b) {1, 2, 3, 4, 6, 7, 8}
c) ϕ
d) (1, 3, 6, 7, 8)
Answer: (d)

8. If $A = \{1, 2, 3\}$, $B = \{2, 3, 6\}$, then $A \cup B$ is [MODEL QUESTION]
a) $\{1, 2, 3\}$
b) $\{2, 3\}$
c) $\{1, 2, 3, 6\}$
d) none of these
Answer: (c)

9. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 6\}$, then $A \cup B$ is [MODEL QUESTION]
a) $\{1, 2, 3\}$
b) $\{2, 3\}$
c) $\{1, 2, 3, 6\}$
d) none of these
Answer: (c)

10. If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$, then $A \Delta B$ is [MODEL QUESTION]
a) $\{1, 2\}$
b) $\{1, 2, 3, 6\}$
c) $\{1, 3, 6\}$
d) $\{6\}$
Answer: (c)

11. If $A = \{2, 4, 6\}$ and $B = \{1, 3, 5, 7\}$, then $A \cup B$ is [MODEL QUESTION]
a) $\{0\}$
b) $\{1, 2, 3, 4, 5, 6, 7\}$
c) $\{1, 2, 4, 5, 6, 7\}$
d) $\{0, 2\}$
Answer: (b)

12. If $P = \{2, 4, 6, 7, 8, 9\}$, $Q = \{1, 2, 6, 9\}$ then $P \cap Q$ is [MODEL QUESTION]
a) $\{1, 2, 6\}$
b) $\{2, 6, 9\}$
c) $\{1, 6, 9\}$
d) $\{4, 6, 9\}$
Answer: (b)

13. Which of the following is a null set? [MODEL QUESTION]
a) $A = \{0\}$
b) $A = \{\phi\}$
c) $A = \{x : x$ is an integer & $1 < x < 2\}$
d) none of these
Answer: (c)

14. If {1, 2, 3, 4, 5, 6, 7} be universal set and $A = \{4, 3, 2, 1\}$, $B = \{2, 4, 6\}$ then $(A \cup B)'$ is [MODEL QUESTION]
a) {5, 7}
b) {1, 3, 7}
c) {1, 3, 5, 6, 7}
d) none of these
Answer: (a)

15. If $P = \{2, 4, 6, 7, 8, 9\}$, $Q = \{1, 2, 6, 9\}$, then $P - Q$ is [MODEL QUESTION]
a) {4, 7, 8}
b) {4, 6, 8, 9}
c) {1}
d) {2, 4, 6, 7, 8, 9}
Answer: (a)

16. Which of the following is a null set? [MODEL QUESTION]
a) $A = \{0\}$
b) $A = \{x : x$ is an integer and $1 < x < 2\}$
c) $A = \{\phi\}$
d) None of these
Answer: (b)

17. Let $f: z \to z$ be a mapping defined by $f(x) = 2x - 3$. Then the mapping f is **[MODEL QUESTION]**

a) one to one
b) onto
c) neither one-one nor onto
d) both

Answer: (d)

18. ρ is a relation on the set $R \times R$ of ordered pairs of real numbers as follows; **[MODEL QUESTION]**

For all (a,b), $(c,d) \in R \times R$ $(a,b)(c,d) \Leftrightarrow a = c$

Then ρ is

a) symmetric only
b) symmetric but not reflective
c) equivalence relation
d) none of these

Answer: (c)

19. Let $A = R - \{3\}$ and $B = R - \{1\}$, **[MODEL QUESTION]**

If $f: A \to B$; $f(x) = \frac{x-2}{x-3}$ then

a) f is into
b) f is surjective
c) f is bijective
d) none of these

Answer: (c)

Short Choice Type Questions

1. Find the number of integers between 1 and 720 both inclusive that are not divisible by any of the integers 2, 3 and 5. **[MODEL QUESTION]**

Answer:

No. of integers between 1 to 720 divisible by 2 is

$$\frac{720}{2} = 360 = n(A) \text{ (say)}$$

No. of integers between 1 to 720 divisible by 3 is

$$\frac{720}{3} = 240 = n(B) \text{ (say)}$$

and no. of integers between 1 to 720 divisible by 5 is

$$\frac{720}{5} = 145 = n(C) \text{ (say)}$$

Now, $n(A \cap B) = \frac{720}{6} = 120$

$$n(A \cap C) = \frac{720}{10} = 72$$

$$n(B \cap C) = \frac{720}{15} = 48$$

$$n(A \cap B \cap C) = \frac{720}{30} = 24$$

So the required no. of integers is 360+240+145-(120+72+48+24)=481

2. Find the number of natural numbers not greater than 1000 which are not divisible by 3, 5 or 7. **[MODEL QUESTION]**

Answer:

Let A donate the set of natural nos. divisible by 3,
B donate the set of natural nos. divisible by 5,
and C donate the set of natural nos. divisible by 7,

Then $n(A) = 333$, $n(B) = 200$, $n(C) = 142$

$n(A \cap B) = 66$, $n(B \cap C) = 25$, $n(C \cap A) = 47$ and $n(A \cap B \cap C) = 9$

Hence

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$
$$= 333 + 200 + 142 - 66 - 25 - 47 + 9 = 546.$$

Therefore the no of natural numbers not divisible by 3, 5 or 7 is $1000 - 546 = 454$.

3. Write down the quantifiers in predicate calculus and symbolize the following statements;

i) **Every rational number is real number.**
ii) **There exists a number which is prime.** **[MODEL QUESTION]**

Answer:

The quantifiers are

For all: $\forall$ (universal quantifier)
There exists: $\exists$ (existential quantifier)

Let $\mathbb{Q}$ denote the set of rational numbers, $\mathbb{R}$ denote the set of real numbers and P denote the set of prime numbers.

Then (i) $\forall x \in \mathbb{Q},\ x \in \mathbb{R}$

(ii) $\exists x,\ x \in P$

4. In a survey concerning the smoking habits of consumers it was found that 55% smoke cigarette-A, 50% smoke cigarette-B, 42% smoke cigarette-C, 28% smoke cigarette-A & B, 20% smoke cigarette-A & C, 12% smoke cigarette-B & C and 10% smoke all the three cigarette. What percentage do not smoke? **[MODEL QUESTION]**

Answer:

$n(A) = 55$; $n(A \cap B) = 28$; $n(A \cap B \cap C) = 10$;
$n(B) = 50$; $n(A \cap C) = 20$;
$n(C) = 42$; $n(B \cap C) = 12$;

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C)$$
$$- n(B \cap C) + n(A \cap B \cap C)$$

$= 147 - 60 + 10 = 97$

Therefore, $100 - 97 = 3\%$ students do not smoke.

5. If $A = \{a, b, c, d, e\}$, $B = \{c, a, e, g\}$ and $C = \{b, e, f, g\}$, then show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. **[MODEL QUESTION]**

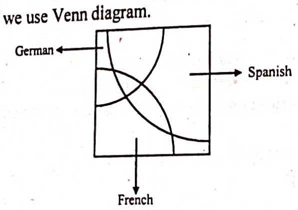
Answer:

$$A \cup B = \{a,b,c,d,e\} \cup \{c,a,e,g\} = \{a,b,c,d,e,g\}$$
$$(A \cup B) \cap C = \{a,b,c,d,e,g\} \cap \{b,e,f,g\} = \{b,e,g\}$$
$$A \cap C = \{a,b,c,d,e\} \cap \{b,e,f,g\} = \{b,e\}$$
$$B \cap C = \{c,a,e,g\} \cap \{b,e,f,g\} = \{e,g\}$$
$$(A \cap C) \cup (B \cap C) = \{b,e\} \cup \{e,g\} = \{b,e,g\}$$
$$\therefore (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

6. Out of 440 students, 112 students read German, 120 students read French and 168 read Spanish. Of these 32 read French and Spanish, 40 read German and Spanish, 20 read German and French, while 12 read all the three subjects. How many students **[MODEL QUESTION]**
a) do not read any of the three languages
b) read just one language?

Answer:

To solve this problem, we use Venn diagram.



$$\therefore \quad n(G) = 112 \quad n(F) = 120 \quad n(S) = 168$$
$$n(F \cap S) = 32 \quad n(G \cap S) = 40$$
$$n(G \cap F) = 20 \quad n(G \cap F \cap S) = 12$$

Now, $n(G \cup F \cup S) = n(G) + n(F) + n(S)$
$$- n(F \cap G) - n(F \cap S) - n(G \cap S) + n(F \cap G \cap S)$$
$$= 112 + 120 + 168 - (32 + 40 + 20) + 12 = 320.$$

$\therefore$ $(440 - 320) = 120$ students does not read any of the three languages.

Now, $n(G) - n(G \cap S) - n(G \cap F) + n(G \cap F \cap S)$

$$= 112 - 40 - 20 + 12 = 64$$
$$n(F) - n(F \cap S) - n(F \cap G) + n(G \cap F \cap S)$$
$$= 120 - 32 - 20 + 12 = 80$$
$$n(S) - n(S \cap G) - n(S \cap F) + n(G \cap F \cap S)$$
$$= 168 - 40 - 32 + 12 = 108$$

Hence 64 students read only German, 80 students read only French and 108 students read only Spanish.

7. Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. **[MODEL QUESTION]**

Answer:

Let (x, y) be an arbitrary element of the set $(A \times B) \cap (C \times D)$

Then $(x, y) \in (A \times B) \cap (C \times D)$

$\Rightarrow (x, y) \in (A \times B)$ *and* $(x, y) \in (C \times D)$

$\Rightarrow \{x \in A$ *and* $y \in B\}$ *and* $\{x \in C$ *and* $y \in D\}$

$\Rightarrow \{x \in A$ *and* $x \in C\}$ *and* $\{y \in B$ *and* $y \in D\}$

$\Rightarrow x \in (A \cap C)$ *and* $y \in (B \cap D)$

$\Rightarrow (x, y) \in (A \cap C) \times (B \cap D)$

$\therefore \quad (A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$ ... (1)

Again, (a, b) be any arbitrary element of the set $(A \cap C) \times (B \cap D)$. then

$(a,b) \in (A \cap C) \times (B \cap D) \Rightarrow a \in (A \cap C)$ *and* $b \in (B \cap D)$

$\Rightarrow \{a \in A$ *and* $a \in C\}$ *and* $\{b \in B$ *and* $b \in D\}$

$\Rightarrow \{a \in A$ *and* $b \in B\}$ *and* $\{a \in C$ *and* $b \in D\}$

$\Rightarrow (a,b) \in (A \times B)$ *and* $(a,b) \in (C \times D)$

$\Rightarrow (a,b) \in (A \times B) \cap (C \times D)$

$\therefore (A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$ ... (2)

From (1) and (2) we get,

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

8. For any three sets A, B, C show that $A - (B - C) = (A - B) \cup (A \cap C)$ **[MODEL QUESTION]**

Answer:

L.H.S $= A - (B - C)$
$$= A \cap (B - C)^c$$
$$= A \cap \left(B \cap C^c\right)^c$$
$$= A \cap \left(B^c \cup C\right) \quad \text{[By De-Morgan's rule]}$$

$=(A\cap B^c)\cup(A\cap C)$

$=(A-B)\cup(A\cap C)$

9. In a survey of 320 persons, number of person taking tea is 210, taking milk is 100 and coffee is 70. Number of persons who take tea and milk is 50, milk and coffee is 30, tea and coffee is 50. The number of persons all three together is 20. Find the number of people who take neither tea not coffee nor milk. [MODEL QUESTION]

Answer:

Let M, T and C be the sets denote respectively number of persons taking milk, tea and coffee.

Therefore, n(M) = 100, n(T) = 210, n(C) = 70. $n(M\cap T)=50$, $n(M\cap C)=30$, $n(T\cap C)=50$, $n(M\cap T\cap C)=20$.

We know,

$n(M\cup T\cup C)=n(M)+n(T)+n(C)-n(M\cap T)-n(T\cap C)$

$n(C\cap M)+n(M\cap T\cap C)$

$=100+210+70-50-50-30+20=270$

Thus, number of persons who take neither Milk not Tea nor Coffee is 320-270 = 50.

10. Let $A=\{1,2,3,4\}$ and $R=\{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(3,2)\}$. Is R is equivalence relation? Explain. [MODEL QUESTION]

Answer:

Here $R=\{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(3,2)\}$.

Clearly R is reflexive, transitive but not Symmetric as $(1,2)\in R$ but $(2,1)\notin R$.

Hence R is not an equivalence relation.

11. If B is countable and $A\subset B,(A\neq\phi)$, then A is countable. [MODEL QUESTION]

Answer:

If B is finite, A is clearly finite. If B is countably infinite, there is a bijection $f:B\to N$

Then $f(A)\subset N$, so by the proposition $f(A)$ is either finite or countably infinite. Since $A\sim f(A)$ (given that f is injective), it follows that A is countable.

As the following results shows, to establish that a set A is countable it is enough to find a function from N onto A, or a one-to-one function from A into N; this is easier than exhibiting a bijection $N\to A$.

12. Let A be a nonempty set. The following are equivalent:

(1) A is countable (that is, there is a bijection $h:A\to I_N$, or $h:A\to N$).

(2) There exists $f:N\to A$ surjective.

(3) There exists $g:A\to N$ injective.

[MODEL QUESTION]

Answer:

(1) $\Rightarrow$ (2): If A is countably infinite, we may take $f=h$. If A is finite with cardinality N, consider the surjective function:

$f_N:\mathbb{N}\to I_N$, $f(n)=n$ for $1\le n,\le N$, $f(n)=N$ for $n\ge N$

Then, $f=h^{-1}\circ f_N:\mathbb{N}\to A$ is surjective.

(2) $\Rightarrow$ (3): Let f be as in (2), and define $g:A\to\mathbb{N}$ as follows. Given $a\in A$, the preimage $f^{-1}(\{a\})$ is a non-empty subset of $\mathbb{N}$ (since f is surjective). By the Well-Ordering Principle, this set has a smallest element; we let $g(a)$ be this smallest element. g is injective, since for two elements $a_1\neq a_2$ in A the preimages $f^{-1}(\{a_1\})$ and $f^{-1}(\{a_2\})$ are disjoint, and hence their smallest elements are distinct.

(3) $\Rightarrow$ (1): Let g be as in (3). $g(A)$ is a non-empty subset of $\mathbb{N}$, hence (by Proposition 1) $g(A)$ is countable. Since $A\sim$ g(A) (given that A is injective), it follows that A is countable.

13. $\mathbb{Q}\sim\mathbb{N}$: the set of rational numbers is countably infinite. [MODEL QUESTION]

Answer:

Define $f:\mathbb{Z}\times\mathbb{N}\to\mathbb{Q}$ by $f(p,q)=p/q$. Since each rational number is of the form p/q for some $p\in\mathbb{Z}$ and some $q\in\mathbb{N}$ (not necessarily unique ones), it follows that f is surjective. Since $\mathbb{Z}\times\mathbb{N}\sim\mathbb{N}$ (using examples 1, 2 and 3), it follows from part (2) of Proportion 2 and Remark 1 that $\mathbb{Q}$ is countable (hence countably infinite)

14. Let $A_1,A_2,A_3,\ldots$ be non-empty countably infinite sets (not necessarily disjoint), then their union is countably infinite: [MODEL QUESTION]

$$\bigcup_{i\ge 1}A_i\sim\mathbb{N}$$

Answer:

We're given that, for each $i\in\mathbb{N}$, there exists a bijection $f_i:\mathbb{N}\to A_i$. Define

$f:\mathbb{N}\times\mathbb{N}\to\bigcup_{i\ge1}A_i$, $\quad f(i,j)=f_i(j)$.

To see that f surjective, let $x\in\bigcup_{i\ge1}A_i$ be arbitrary. Then $x\in A_i$ for some $i\in\mathbb{N}$, so $x=f_i(j)$ for some $j\in\mathbb{N}$; thus $x=f(i,j)$.

Before describing the proof of the fact that $\mathbb{R}$ is uncountable, we need to review some basic facts about decimal expansions of real numbers. This expansion is not unique. For example:

$0.9999999\ldots=1.$

This equality is not approximate, it is EXACT, a fact that some students find surprising; both sides are different decimal representations (in base 10) of the same rational number. One way to see this is:

$$0.99999... = 9\sum_{j=1}^{\infty} 10^{-j} = \frac{9/10}{1-1/10} = 1,$$

using the well-known formula for the sum of a convergent geometric series $\sum_{j=0}^{\infty} aq^{-j} = \frac{a}{1-q}$ if $|q|<1$. The same would happen for any rational number with a terminating decimal expansion, for example:

$$0.1234567 = 0.12345669999999...$$

Thus if we want to assign to each real number a unique decimal expansion, we need to make a choice; for the purpose of the argument that follows, we'll choose the "non-terminating expansion", that is, the one that is eventually an infinite sequence of 9's. With this choice, it is then a theorem about the real numbers (which won't be proved here) that every real number in the interval (0, 1] has a unique non-terminating expansion: an infinite sequence taking values in {0, 1,... 9}, arbitrary except for the fact that it can't be eventually all 0's. Thus, there is a bijection between the interval (0, 1] and the set:

$$\{f:\mathbb{N}\to\{0,1,...,9\} \mid (\forall n\in\mathbb{N})(\exists m>n)(f(m)\neq 0)\}.$$

Under these correspondence between nonterminating decimal expansions and sequences we have, for example:

$0.314515199999... \leftrightarrow (3,1,4,5,1,5,1,9,9,9,9,9...)$

15. The set $\{x\in\mathbb{R} \mid 0<x\leq 1\}$ is uncountable. **[MODEL QUESTION]**

Answer:

Arguing by contradiction, suppose a bijection $f:\mathbb{N}\to(0,1]$ exists. Listing the $f(n)$ by their nonterminating decimal expansions, we build a bi-infinite array:

$f(1) = 0.a_{11}a_{12}a_{13}a_{14}a_{15}\cdots$

$f(2) = 0.a_{21}a_{22}a_{23}a_{24}a_{25}\cdots$

$f(3) = 0.a_{31}a_{32}a_{33}a_{34}a_{35}\cdots$

$f(4) = 0.a_{41}a_{42}a_{43}a_{44}a_{45}\cdots$

$f(5) = 0.a_{51}a_{52}a_{53}a_{54}a_{55}\cdots$

...

Given the array we can explicitly exhibit a real number $x\in(0,1]$ that it can't possibly include. Namely, let x be the number with nonterminating decimal expansion:

$$x = 0.d_1d_2d_3d_4d_5...$$

where the d_n are defined using the diagonal entries of the array, modified as follows:

$$d_n = a_{nn}+1 \text{ if } a_{nn}\in\{0,1,...,8\}; \quad d_n = 8 \text{ if } a_{nn} = 9.$$

Note that $d_n \neq 0$ for all n, so this nonterminating decimal expansion os of the allowed kind, and defines a real number in (0, 1].

We claim that all $n\in\mathbb{N}$ $f(n)\neq x$, contradicting the fact that f is onto. To see this, observe that the n-th digits in the decimal expansion of x is d_n, and in the expansion of $f(n)$ is d_{nn}; these are different (from the construction above). This concludes the proof.

Numerical example. We have no control over the "listing" f that is assumed to exist at the start of the argument, but suppose (for example) the first five entries were (highlighting the diagonal entries of the array):

$f(1) =$ 0.12034506...

$f(2) =$ 0.13579017...

$f(3) =$ 0.24608046...

$f(4) =$ 0.31415926...

$f(5) =$ 0.21784143...

...

Then the first five digits of x would be: $x = 0.24725...$, and it is clear that x can't be any of the first five elements on the list (and indeed can't be any element on the list).

Long Choice Type Questions

1. Answers for the following questions:

a) If $A=\{1,2,3\}$ and $B=\{x,y\}$, list all members of $A\times B$. [MODEL QUESTION]

Answer:

$A=\{1,2,3\}$ & $B=\{x,y\}$

$A\times B=\{(1,x),(1,y),(2,x),(2,y),(3,x),(3,y)\}$

b) If $A=\{2,4,6\}$ and $B=\{1,3,5,7\}$, find $A\cap B$. **[MODEL QUESTION]**

Answer:

$A=\{2,4,6\}$ & $B=\{1,3,5,7\}$

$A\cap B=\phi$

c) If Z is the set of all integers and $f(x)=|x|$ as $x\in Z$. Show that f is not one to one. **[MODEL QUESTION]**

Answer:

Let, $x_1,x_2\in Z$

Now $f(x_1)=f(x_2) \Rightarrow |x_1|=|x_2|$

or $x_1=\pm x_2$

Therefore f is not one to one.

2. In a class of 50 students, 15 read Physics, 20 read Chemistry and 20 read Mathematics. 3 read Physics and Chemistry, 6 read Chemistry and Mathematics and 5 read Physics and Mathematics. 7 read none of the subjects. How many students read all the subjects? **[MODEL QUESTION]**

Answer:

Since 7 students read none of the subjects, the total number of students who read at least one subject is $50-7=43$

Now using Venn diagram, we get

$n(P\cup C\cup M)=n(P)+n(C)+n(M)-n(P\cap C)-n(P\cap M)-n(C\cap M)+n(P\cap C\cap M)$

or, $43=15+20+20-3-5-6+n(P\cap C\cap M)$

or, $n(P\cap C\cap M)=2$, which is the required no. of students who read all the subjects.

3. If A = {a, b, c, d, e}, B = {c, a, e, g} and C = {b, e, f, g}, then show that (A ∪ B) ∩ C = (A ∩ C) ∪ (B ∩ C) **[MODEL QUESTION]**

Answer:

$A=\{a,\ b,\ c,\ d,\ e\}$

$B=\{c,\ a,\ e,\ g\}$

$C=\{b,\ e,\ f,\ g\}$

$A\cup B=\{a,\ b,\ c,\ d,\ e,\ g\}$

$(A\cup B)\cap C=\{b,\ e,\ g\}$

Now $A\cap C=\{b,\ e\}$

$B\cap C=\{e,\ g\}$

$(A\cap C)\cup(B\cap C)=\{b,e,g\}$

Therefore, $(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$

4. In a survey of 320 persons, number of persons taking tea is 210, taking milk is 100 and coffee is 70. Number of persons who take tea and milk is 50, milk and coffee is 30, tea and coffee is 50. The number of persons all three together is 20. Find the number of people who take neither tea nor coffee nor milk. **[MODEL QUESTION]**

Answer:

persons taking milk, coffee, tea is 20
persons taking tea, milk is 50
persons taking only tea, milk is (50-20)=30
persons taking milk, coffee is 30
persons taking only milk, coffee is (30-20)=10
persons taking tea, coffee is 50
persons taking only tea, coffee is (50-20)=30
persons taking milk is 100
persons taking only milk is (100-10-20-30)=40
persons taking coffee is 70
persons taking only coffee is (70-30-20-10)=10

persons taking tea is 210
persons taking only tea is (210-30-20-30)=130
Total person taking tea, coffee, milk is (130+40+10+30+30+10+20)=270
Person not taking tea, coffee, milk is (320-270)=50

5. If $A=\{a,b,c,d,e\}$, $B=\{c,a,e,g\}$ and $C=\{b,e,f,g\}$, then show that $(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$. **[MODEL QUESTION]**

Answer:

$A=\{a,b,c,d,e\}$

$B=\{c,a,e,g\}$

$C=\{b,e,f,g\}$

$A\cup B=\{a,b,c,d,e,g\}$

$(A\cup B)\cap C=\{b,e,g\}$

$A\cap C=\{b,e\}$

$B\cap C=\{e,g\}$

$(A\cap C)\cup(B\cap C)=\{b,e,g\}$

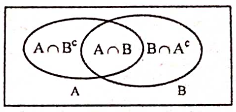
$\therefore\ (A\cup B)\cap C=(A\cap C)\cup(B\cap C)$

6. a) Prove that $|A\cup B|=|A|+|B|-|A\cap B|$ where A and B are two non-empty sets. **[MODEL QUESTION]**

Answer:

Let, A and B are two subsets of a universal set S.

Here $A\cap B^c$, $A\cap B$ and $B\cap A^c$ are disjoint.



Now, $n(A)=n(A\cap B^c)+n(A\cap B)$

$n(B)=n(A\cap B)+n(B\cap A^c)$

Again, $A\cup B=(A\cap B^c)\cup(A\cap B)\cup(B\cap A^c)$

$\therefore n(A\cup B)=n(A\cap B^c)+n(A\cap B)+n(B\cap A^c)$

$=n(A)-n(A\cap B)+n(A\cap B)+n(B)-n(A\cap B)$

$=n(A)+n(B)-n(A\cap B)$

b) If $A=\{a,b,c,d\}$ $B=\{b,c,p,q\}$, then find out $A\times B$, $B\times A$ and $A\,\Delta\,B$. **[MODEL QUESTION]**

Answer:

$A\times B=\{(a,b)\ ;\ a\in A \text{ and } b\in B\}$

$=\{(a,b),(a,c),(a,p),(a,q),(b,b),(b,c),(b,p),(b,q),(c,b),(c,c),(c,p),(c,q),(d,b),(d,c),(d,p),(d,q)\}$

$B - A = \{$[illegible]$\}$

[illegible]

$A \oplus B = $ [illegible]

6. Define power set. Find the power set of $\{a,b,c\}$. **[MODEL QUESTION]**

Answer:

A set formed of all the sub-sets of a set as its elements is called power set of S and is symbolically denoted by $P(S)$.

Let $S = \{a,b,c\}$

$$P(S) = \{\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},\phi\}$$

7. Let A, B be sets, and let $f: A \to B, g: B \to A$, then $A \equiv B$. **[MODEL QUESTION]**

Answer:

(Note we let $\overline{X}$ denote the complement of X.)

Let $S = \bigcup_{i=0}^{\infty} (fg)^i(\overline{f(A)}) \subseteq B$, where $(fg)^0$ is the identity by convention. We then define

$$h: A \to B,\ x \mapsto \begin{cases} g^{-1}(x) & \text{if } x \in g(S) \\ f(x) & \text{otherwise} \end{cases}$$

and propose that h is a bijection.

It is clear that $g^{-1}|_{g(S)}$ serves as a bijection from $g(S)$ to S. It is also clear that since f is a injection, that it is a bijection onto whatever its image is. Thus it suffices to show that the image of $\overline{g(S)}$ under f is exactly the complement to the image of $g^{-1}|_{g(S)}$, that is $\overline{S}$. In other words, I claim: $f(\overline{g(S)}) = \overline{S}$ and showing this completes the proof.

$\subseteq$. Suppose, that the left hand side is not contained in the right hand side, then $\exists x \in f(\overline{g(S)}) \cap S$. Since $x \in S, x = (fg)^m(z), z \in \overline{f(A)}, m \in N$. But is $m > 0$, then $x \in f(g(S))$, which is a contradiction to $x \in \overline{g(S)}$. Thus $x = z \in \overline{f(A)}$, but we see that $f^{-1}(x)$ is defined and $x \in \overline{f(A)}$ are precisely the elements where f^{-1} is not defined. Thus this also a contradiction and therefore no such x exists.

Note that this also says that h is injective, that the map defined as f does not overlap with the map defined as g^{-1}.

$\supseteq$. Let $x \in \overline{S}$ and notice that $\overline{S} = \overline{\bigcup_{i=0}^{\infty} (fg)^i(\overline{f(A)})} = \bigcap_{i=0}^{\infty} \overline{(fg)^i(\overline{f(A)})}$. Then since $(fg)^0 = idB$, we may say that $x \in f(A)$. Thus we may write $x = f(a), a \in A$ an we want to show that $a \in g(S)$, but this must be true, otherwise $a \in g(S)$ implies $x = f(a) \in S$, contradictory to our original choice of x.

Note that this also says that h *is surjective, that the map defined as* f *is onto the values of* B, *that is not mapped to by* g^{-1}.

8. Prove that $(0,1)$, $(0,1]$, $[0,1]$ and $\mathbb{R}$ are equivalent sets. **[MODEL QUESTION]**

Answer:

The easiest equivalence is $(0,1) \sim \mathbb{R}$, one possible bijection is given by $f:(0,1) \to \mathbb{R}$,

$$f(x) = \begin{cases} 2 - \frac{1}{x} & \text{for } 0 < x < \frac{1}{2} \\ \frac{1}{1-x} - 2 & \text{for } \frac{1}{2} \le x < 1. \end{cases}$$

with inverse function

$$f^{-1}(y) = \begin{cases} \frac{1}{2-y} & \text{for } y < 0, \\ 1 - \frac{1}{2+y} & \text{for } y \ge 0. \end{cases}$$

To show $(0,1] \sim (0,1)$, one possible bijection $g:(0,1] \to (0,1)$ is given by

$$g(x) = \begin{cases} \frac{1}{n+1} & \text{for } x = \frac{1}{n}, n \in \mathbb{N}, \\ x & \text{if } x \ne \frac{1}{n} \text{ for all } n \in \mathbb{N}, \end{cases}$$

with inverse

$$g(x) = \begin{cases} \frac{1}{n-1} & \text{for } y = \frac{1}{n}, n \in \mathbb{N}, n \ge 2, \\ y & \text{if } y \ne \frac{1}{n} \text{ for all } n \in \mathbb{N}, n \ge 2. \end{cases}$$

Then $h:[0,1] \to [0,1)$ defined by

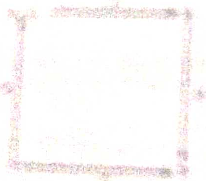
$$h(x) = \begin{cases} g(x) & \text{for } x \ne 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is again a bijection, so $[0,1] \sim [0,1)$. But $F:[0,1) \to (0,1]$, $F(x) = 1 - x$ is a bijection too (with $F^{-1} = F$, so $[0,1) \sim (0,1]$. By transitivity the claimed equivalences follow.

9. Assume f:A→B is one-to-one & g:B→A is also one-to-one. Then show that A and B have the same cardinality and there is a one-to-one function h from A onto B. **[MODEL QUESTION]**

Answer:

Consider the function f:(0,1)→[0,1] defined by f(x) = x. f would be one-to-one. The function g:[0,1]→(0,1), defined by g(x)=(1/2)x+1/8 is also one-to-one.



[illegible]

10. Let A be any set and 2^A be the power set of A. Then card $A <$ card 2^A **[MODEL QUESTION]**

Answer:
The map $a \mapsto \{a\}$ is a one-one function from A into 2^A. Hence, card $A \le$ card 2^A. We show that card $A \neq$ card 2^A. On the contrary, suppose that card $A =$ card 2^A. Then there is a onto function $\alpha : A \to 2^A$. Define a function $f : A \times A \to 2^A$ by

$f(a,b)=1$ if $a \in \alpha(b)$ and $f(a,b)=0$ if $a \notin \alpha(b)$.

Let $g : A \to \{0,1\}$ be any function. This defines a subset of A, namely, $B=\{a \in A : g(a)=1\}$. Since $\alpha : A \to 2^A$ is onto, there is $b \in A$ such that $\alpha(b)=B$. Now $f(a,b)=g(a)$ for every $a \in A$. Let us verify it. If

$a \in B=\alpha(b)$, then $f(a,b)=1$ and in this case, $g(a)=1$, by the definition of B. On the other hand, if $a \notin B$ (with $a \in A$), then $a \notin \alpha(b)$, i.e., $f(a,b)=0$ and in this case, $g(a)=0$, again by the definition of B. Thus for every $a \in A$, $g(a)=f(a,b)$. Hence every function $g : A \to \{0,1\}$ is representable by f. BY CLT, every function $\phi : \{0,1\} \to \{0,1\}$ has a fixed point. However, the negation function $\neg : \{0,1\} \to \{0,1\}$ defined by $\neg(0)=1$, $\neg(1)=0$ has no fixed point. Therefore, card $A \neq$ card 2^A

11. Let A, B be sets and $f : A \times A \to B$ be any function such that all functions $g : A \to B$ are representable by f. Then every function $\phi : B \to B$ has a fixed point. **[MODEL QUESTION]**

Answer:
Suppose that $f : A \times A \to B$ is such a function that all functions $g : A \to B$ are representable by f. Let $\phi : B \to B$ be any function. Define a function $\psi : A \to A \times A$, called Cantor's diagonalization, by $\psi(a)=(a,a)$. Let $h=\phi \circ f \circ \psi$; look at the diagram below.

$$\begin{tikzcd} A \times A \arrow[r, "f"] & B \arrow[d, "\phi"] \\ A \arrow[u, "\psi"] \arrow[r, "h"] & B \end{tikzcd}$$

Since the function $h : A \to B$ is representable by f, we have an $a \in A$ such that for all $x \in A$, $h(x)=f(x,a)$. In particular, $h(a)=f(a,a)$. But $h(a)=\phi(f(\psi(a)))=\phi(f(a,a))$. Writing $f(a,a)=b$, we have $\phi(b)=b$. Thus ϕ has a fixed point, namely, b.

Then, there is a bijective function h from $[0,1)$ onto $[0,1]$. Let $h(x)=(1/2)x+1/8$. Observe that $[0,1]\sim[1/8,5/8]$. Since $[1/8,5/8]\subset[0,1)$ it follows that $[0,1]\sim[1/8,5/8]\leq[0,1)$.
Thus, $[0,1)\leq[0,1]$ and $[0,1]\leq[0,1)$, so $[0,1)\sim[0,1]$.

10. Let A be any set and 2^A be the power set of A. Then card $A <$ card 2^A.
[MODEL QUESTION]

Answer:
The map $x \mapsto \{x\}$ is a one-one function from A into 2^A. Hence, card $A \leq$ card 2^A. We show that
card $A \neq$ card 2^A. ON the contrary, suppose that card $A =$ card 2^A. Then there is a one-one function $\alpha : A \to 2^A$. Define a function $f : A \times A \to 2^A$ by

$f(x,y)=1$ if $x \in \alpha(y)$ and $f(x,y)=0$ if $x \notin \alpha(y)$.

Let $g : A \to \{0,1\}$ be any function. This defined a subset of A, namely, $B=\{x \in A : g(x)=1\}$. Since $\alpha : A \to 2^A$ is onto, there is $z \in A$ such that $\alpha(z)=B$. Now $f(x,z)=g(x)$ for every $x \in A$. Let us verify it. If

$x \in B=\alpha(z)$, then $f(x,z)=1$ and in this case, $g(x)=1$, by the definition of B. On the other hand, if $x \notin B$ (with $x \in A$), then $x \notin \alpha(z)$, i.e., $f(x,z)=0$ and in this case, $g(x)=0$, again by the definition of B. Thus, for every $x \in A$, $g(x)=f(x,z)$. Hence every function $g : A \to \{0,1\}$ is representable by f. BY CLT, every function $\phi : \{0,1\} \to \{0,1\}$ has a fixed point. However, the negation function $\neg : \{0,1\} \to \{0,1\}$ defined by $\neg(0)=1, \neg(1)=0$ has no fixed point. Therefore, card $A \neq$ card 2^A.

11. Let A, B be sets and $f : A \times A \to B$ by any function such that all functions $g : A \to B$ are representable by f. Then every function $\phi : B \to B$ has a fixed point.
[MODEL QUESTION]

Answer:
Suppose that $f : A \times A \to B$ is such a function that all functions $g : A \to B$ are representable by f. Let $\phi : B \to B$ be any function. Define a function $\psi : A \to A \times A$, called Cantor's diagonalization, by $\psi(x)=(x,x)$. Let $h=\phi \circ f \circ \psi$; look at the diagram below.

$$\begin{tikzcd}
A \times A \arrow[r, "f"] & B \arrow[d, "\phi"] \\
A \arrow[u, "\psi"] \arrow[r, "h"] & B
\end{tikzcd}$$

Since the function $h : A \to B$ is representable by f, we have an $a \in A$ such that for all $x \in A$, $h(x)=f(x,a)$. In particular, $h(a)=f(a,a)$. But $h(a)=\phi(f(\psi(a)))=\phi(f(a,a))$. Writing $f(a,a)=b$, we have $\phi(b)=b$. Thus ϕ has a fixed point, namely, b.

PRINCIPLE OF MATHEMATICAL INDUCTION

Chapter at a Glance

One key basis for mathematical thinking is deductive reasoning. An informal, and example of deductive reasoning, borrowed from the study of logic, is an argument expressed in three statements: (a) Socrates is a man.
(b) All men are mortal, therefore,
(c) Socrates is mortal.
If statements (a) and (b) are true, then the truth of (c) is established.
To make this simple mathematical example, we could write:
(i) Eight is divisible by two.
(ii) Any number divisible by two is an even number, therefore,
(iii) Eight is an even number.
Thus, deduction in a nutshell is given a statement to be proven, often called a conjecture or a theorem in mathematics, valid deductive steps are derived and a proof may or may not be established, i.e., deduction is the application of a general case to a particular case. In contrast to deduction, inductive reasoning depends on working with each case, and developing a conjecture by observing incidences till we have observed each and every case. It is frequently used in mathematics and is a key aspect of scientific reasoning, where collecting and analysing data is the norm. Thus, in simple language, we can say the word induction means the generalisation from particular cases or facts.
In algebra or in other discipline of mathematics, there are certain results or statements that are formulated in terms of n, where n is a positive integer. To prove such statements the well-suited principle that is used-based on the specific technique, is known as the principle of mathematical induction. Motivation In mathematics, we use a form of complete induction called mathematical induction. To understand the basic principles of mathematical induction, suppose a set of thin rectangular tiles are placed vertically one by one. When the first tile is pushed in the indicated direction, all the tiles will fall. To be absolutely sure that all the tiles will fall, it is sufficient to know that
(a) The first tile falls, and
(b) In the event that any tile falls its successor necessarily falls.
This is the underlying principle of mathematical induction. We know, the set of natural numbers N is a special ordered subset of the real numbers. In fact, N is the smallest subset of R with the following property: A set S is said to be an inductive set if $1 \in S$ and $x + 1 \in S$ whenever $x \in S$. Since N is the smallest subset of R which is an inductive set, it follows that any subset of R that is an inductive set must contain N.
Suppose we wish to find the formula for the sum of positive integers 1, 2, 3,...,n, that is, a formula which will give the value of $1 + 2 + 3$ when $n = 3$, the value $1 + 2 + 3 + 4$, when $n = 4$ and so on and suppose that in some manner we are led to believe that the formula

$1 + 2 + 3 + ... + n = n(n+1)/2$ is the correct one.
How can this formula actually be proved? We can, of course, verify the statement for as many positive integral values of n as we like, but this process will not prove the formula for all values of n. What is needed is some kind of chain reaction which will have the effect that once the formula is proved for a particular positive integer the formula will automatically follow for the next positive integer and the next indefinitely. Such a reaction may be considered as produced by the method of mathematical induction.
The Principle of Mathematical Induction Suppose there is a given statement P(n) involving the natural number n such that
(i) The statement is true for $n = 1$, i.e., P(1) is true, and
(ii) If the statement is true for $n = k$ (where k is some positive integer), then the statement is also true for $n = k + 1$, i.e., truth of P(k) implies the truth of P (k + 1).
Then, P(n) is true for all natural numbers n. Property (i) is simply a statement of fact. There may be situations when a statement is true for all $n \geq 4$. In this case, step 1 will start from $n = 4$ and we shall verify the result for $n = 4$, i.e., P(4). Property (ii) is a conditional property. It does not assert that the given statement is true for $n = k$, but only that if it is true for $n = k$, then it is also true for $n = k + 1$. So, to prove that the property holds, only prove that conditional proposition: If the statement is true for $n = k$, then it is also true for $n = k + 1$. This is sometimes referred to as the inductive step. The assumption that the given statement is true for $n = k$ in this inductive step is called the inductive hypothesis.
For example, frequently in mathematics, a formula will be discovered that appears to fit a pattern like
$1 = 1^2 = 1$
$4 = 2^2 = 1 + 3$
$9 = 3^2 = 1 + 3 + 5$
$16 = 4^2 = 1 + 3 + 5 + 7$, etc
It is worth be noted that the sum of the first two odd natural numbers is the square of second natural number, sum of the first three odd natural numbers is the square of third natural number and so on. Thus, from this pattern it appears that $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$, i.e, the sum of the first n odd natural numbers is the square of n.
Let us write P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$.
We wish to prove that P(n) is true for all n. The first step in a proof that uses mathematical induction is to prove that P (1) is true. This step is called the basic step. Obviously $1 = 1^2$, i.e., P(1) is true. The next step is called the inductive step. Here, we suppose that P (k) is true for some positive integer k and we need to prove that P (k + 1) is true. Since P (k) is true, we have
$1 + 3 + 5 + 7 + ... + (2k - 1) = k^2$ ... (1)
Consider,
$1 + 3 + 5 + 7 + ... + (2k - 1) + \{2(k + 1) - 1\}$ ... (2) $= k^2 + (2k + 1) = (k + 1)^2$ [Using (1)]
Therefore, P (k + 1) is true and the inductive proof is now completed. Hence P(n) is true for all natural numbers n.

Multiple Choice Type Questions

1. If 6^{16} is divided by 17, the remainder will be [WBUT 2013(EVEN)]
a) 15 b) 10 c) 1 d) none of these
Answer: (c)

2. If k is a positive integer, $\gcd(ka, kb) =$ [WBUT 2014(ODD)]
a) $k \cdot \gcd(ka, b)$ b) $k \cdot \gcd(a, b)$ c) $k \cdot \gcd(a, kb)$ d) none of these
Answer: (b)

3. If $\gcd(a, b) = d$, then $\frac{a}{d}, \frac{b}{d}$ are [WBUT 2014(ODD)]
a) relatively prime b) prime c) composite d) both (a) and (b)
Answer: (a)

4. For every integer x, $\gcd(x, x+2) =$ [WBUT 2015(ODD)]
a) 0 b) 2 c) 1 d) none of these
Answer: (d)

5. The number 9420544 is divisible by [WBUT 2015(ODD), 2019(EVEN)]
a) 36 b) 28 c) 24 d) none of these
Answer: (b)

6. If $7x \equiv 3(\bmod 5)$, then x can take the value [WBUT 2015(ODD)]
a) 17 b) 19 c) 21 d) 22
Answer: (b)

7. If $ap = bq$ and a is prime to b then [WBUT 2015(ODD)]
a) $a|p$ and $b|q$ b) $a|b$ and $p|q$ c) $a|q$ and $b|q$ d) none of these
Answer: (d)

8. The number of positive divisors of 252 is [WBUT 2015(ODD)]
a) 9 b) 5 c) 18 d) 10
Answer: (c)

9. If 6^8 is divided by 17, the remainder will be [WBUT 2016(EVEN)]
a) 15 b) 10 c) 1 d) none of these
Answer: (d)

10. If we divide –10 with 6 then the remainder will be [WBUT 2016(ODD)]
a) –4 b) 4 c) 2 d) –2
Answer: (c)

11. The total number of positive divisors of 9216 is [WBUT 2016(ODD)]
a) 33 b) 20 c) 12 d) 14
Answer: (a)

12. If $68 \equiv 4(\bmod n)$, then n can be [WBUT 2016(ODD)]
a) 12 b) 17 c) 13 d) 16
Answer: (d)

13. The remainder when the sum 4! + 5! + 6! + ...+ 50! is divided by 4 is [WBUT 2017(EVEN)]
a) 1 b) 2 c) 3 d) 0
Answer: (d)

14. The number 8955795758 is divisible by [WBUT 2017(ODD)]
a) 7 only b) 13 only c) 7 or 13 or 37 d) none of these
Answer: (c)

15. If n is an odd integer, $\gcd(3n, 3n+2)$ is [WBUT 2018(EVEN)]
a) 1 b) 2
c) 3 d) cannot be determined uniquely
Answer: (a)

16. Division algorithm states that for any positive integer d, there exist unique integer q and r such that $n = d.q + r$ and [WBUT 2018(ODD)]
a) $0 \le r < d$ b) $0 < r < d$ c) $0 \le d < r$ d) None of these
Answer: (a)

17. Suppose we're proving that $2^n + 4$ is divisible by 2 for all natural numbers, n using mathematical induction. We've already proven our base step (that it is true for n = 1). What is the next step? [MODEL QUESTION]
a) We would set the expression $2^k + 4$ equal to 2 and solve for k
b) We would assume that $2^k + 4$ is divisible by 2, so $2^k + 4 = 2m$, where m is a whole number
c) The base step is the only step in mathematical induction, so we've already proven that $2^n + 4$ is divisible by 2 for all natural numbers, n and there is no next step
d) We would subtract 4 from the expression $2^n + 4$
Answer: (b)

18. What is mathematical induction? [MODEL QUESTION]
   a) A proof technique used to prove a property is true for a well-ordered set by showing that if it is true for an element n and n + 1 in a set, then it is true for all elements
   b) A way of combining addition and subtraction in a mathematical expression
   c) A sum of two functions
   d) An operation performed on functions when we want to divide a polynomial function by another polynomial function

Answer: (a)

19. Which of the following statements is NOT true? [MODEL QUESTION]
   a) 100 is divisible by 10 b) 7 divides 40
   c) 12 is divisible by 4 d) 7 divides 28

Answer: (b)

20. If we were proving that $6^n + 4$ is divisible by 5 for all natural numbers, n, using mathematical induction, what would be the first step? [MODEL QUESTION]
   a) We would set $6^n + 4$ equal to 1 and solve for n
   b) We would assume that it is true for all integers
   c) We would subtract 4 from the expression
   d) We would show that it is true for the first element in the set of natural numbers (n = 1)

Answer: (d)

21. If we are using mathematical induction to prove that a mathematical expression A is divisible by a number b for all natural numbers n, then step 1 is to show it's true for n = 1. Step 2 is to assume that it's true for a natural number k. What is the third step in this process? [MODEL QUESTION]
   a) We show that it is true for k + 1
   b) We assume that it is true for all integers
   c) We set A equal to b and solve for the variable
   d) There is no third step in mathematical induction

Answer: (a)

22. Let P(n) be the statement that $1^2+2^2+...+n^2 = n(n+1)(2n+1)/6$ for $n > 0$. If you can't prove the base case, for which values of n can you prove that P(n) is true using mathematical induction? [MODEL QUESTION]
   a) none b) all $n > 0$
   c) just n = 1 d) all $n > k$

Answer: (b)

23. Let P(n) be the statement that $1^2+2^2+...+n^2 = n(n+1)(2n+1)/6$ for $n > 0$. What is the statement P(1)? [MODEL QUESTION]
   a) $1^2+2^2+...+n^2 = n(n+1)(2n+1)/6$ b) n=1

   c) $1^2 = 1(1+1)(2+1)/6$ d) It doesn't exist

Answer: (c)

Short Answer Type Questions

1. If m is a positive integer and $a \equiv b$ (mod m) and $c \equiv d$ (mod m). Show that
   a) $a+c \equiv b+d$ (mod m) [WBUT 2013(EVEN)]
   b) $ac \equiv bd$ (mod m)

Answer:

a) $a \equiv b (\text{mod } m)$ implies $a - b = mk$ for some $k \in \mathbb{Z}$. and $c \equiv d (\text{mod } m)$ implies $c - d = mt$ for some $t \in \mathbb{Z}$

Hence adding we get $(a+c)-(b+d) = m(k+t)$ where $k+t \in \mathbb{Z}$.

Hence $a+c \equiv b+d \,(\text{mod } m)$

b) Again if $-b = mk$, then $ac - bc = mkc$. and if $c - d = mt$, then $bc - bd = mtb$.

Adding we get $ac - bd = m(kc + tb)$ when $kc + tb \in \mathbb{Z}$

Hence $ac \equiv bd \,(\text{mod } m)$

2. Find the remainder when the sum $1^5 + 2^5 + 3^5 + ... + 100^5$ is divided by 5. [WBUT 2013(EVEN), 2019(EVEN)]

Answer:

The given expression is

$$1^5 + 2^5 + \ldots + 100^5 = \left(1^5 - 1\right) + \left(2^5 - 2\right) + \ldots + \left(100^5 - 100\right) + \left(1 + 2 + \ldots + 100\right)$$

$$= 5k_1 + 5k_2 + \ldots\ldots + 5k_{100} + \frac{100 \times 101}{2} \text{ [By Fenots Theo.]}$$

$$= 5\left\{k_1 + k_2 + \ldots\ldots + k_{100} + 1010\right\}.$$

Hence $1^5 + 2^5 + \ldots\ldots + 100^5$ leaves remainder 0 when divided by 5.

3. Find the remainder when the sum $1! + 2! + 3! + ... + 100!$ is divided by 5. [WBUT 2013(ODD)]

Answer:

Here the expression is $1! + 2! + 3! + 4! .... + 100!$.

Clearly from 5! onwards each term is divisible by 5.

So we consider only $1! + 2! + 3! + 4!$.

By Wilson's theory $4! \equiv -1 (\text{mod } 5)$, 5 being a prime no.

i.e. $4! + 1$ is divisible by 5.

So we are to consider only $2!+3!$ i.e. 8 which clearly leaves 3 as remainder when divided by 5.
So the remainder asked for is 3.

4. Find integers u and v satisfying $\gcd(272, 119) = 272u + 119v$.
[WBUT 2014(ODD), 2017(ODD)]

Answer:
We have

$$272 = 2\times 119 + 34$$
$$119 = 3\times 34 + 17$$
$$34 = 2\times 17$$
$$\therefore \quad \gcd(272, 119) = 17$$

Now $\quad 17 = 119 - 3\times 34 = 19 - 3\{272 - 2\times 119\} = 7(119) - 3(272)$.

Thus $\gcd(272, 119) = 272u + 119v$ where $u = -3,\ v = 7$.

5. Prove that the product of any m consecutive integers is divisible by m.
[WBUT 2014(ODD)]

Answer:
Let all natural numbers be grouped as
$\{1, 2, \ldots, m-1, m\}, \{m+1, m+2, \ldots, 2m\}, \{2m+1, 2m+2, \ldots, 3m\}, \{3m+1, \ldots, 3m\}$ If the sequence of m consecutive integers begin with 1, evidently the product contains m as a factor and hence is divisible by m.
Every other string of m consecutive integers starting with 2 or 3 etc. upto m contains m as a factor and hence is divisible by m.
If the sequence of m consecutive integers starting with $2m+1$ or $2m+2$ upto $2m$ contains $2m$ as a factor and hence is divisible by m.
The argument is similar for every other strings of m consecutive integers.
Hence the result.

. Find the number of integer's n, $1 \le n \le 1000$ that are not divisible by 5, 6 and 8.
[WBUT 2015(ODD)]

swer:
.M. of 5,6 and 8 is 120
dently 120, 240, 360, 480, 600,720, 840, 960 are the numbers between 1 and 1000
sible by 5, 6 and 8.
e the no. of integers between 1 and 1000 both inclusive which are not divisible by
f the integers 5, 6 and 8 are
$-8)$ or 992.

7. Find the $\gcd(595, 252)$ and express it in the form 252m + 595n.
[WBUT 2016(ODD)]

Answer:
By division algorithm

$$595 = 2\times 252 + 91$$
$$252 = 2\times 91 + 70$$
$$91 = 1\times 70 + 21$$
$$70 = 3\times 21 + 7$$
$$21 = 3\times 7 + 0$$

Since the last non-zero remainder is 7, $\gcd(595, 252) = 7$.

Now, $\quad 7 = 70 - 3\times 21 = 70 - 3\times(91 - 1\times 70) = 4\times 70 - 3\times 91 = 4\times(252 - 2\times 91) - 3\times 91$
$= 4\times 252 - 11\times 91 = 4\times 252 - 11\times(595 - 2\times 252) = 26\times 252 - 11\times 595$

8. Find the remainder when the sum $1! + 2! + 3! + \ldots\ldots + 100!$ is divided by 18.
[WBUT 2016(ODD)]

Answer:
We know $6! = 720 \equiv 0 \pmod{18}$
When $n \ge 6$, $n! = 18k$; k is a non-zero integer.

$$\therefore \quad n! \equiv 0 \pmod{18};\ n \ge 6$$
$$\therefore \quad (1! + 2! + 3! + 4! + 5! + \ldots\ldots + 100!) \pmod{18}$$
$$\equiv (1 + 2 + 6 + 24 + 120) \pmod{18}$$
$$\equiv 153 \pmod{18}$$
$$\equiv 17$$

9. Find integers m and n such that $512m + 320n = 64$. **[WBUT 2016(ODD)]**

Answer:
We know

$$512 = 1\times 320 + 192 \quad \ldots (1)$$
$$320 = 1\times 192 + 128 \quad \ldots (2)$$
$$192 = 1\times 128 + 64 \quad \ldots (3)$$
$$128 = 2\times 64 + 0 \quad \ldots (4)$$

Thus 64 is the gcd of 512 and 320. Hence there exist integers, m and n such that

$$m\ 512 + n\ 320 = 64$$

Now, from equation (3) we have

$$64 = 192 - 128$$
$$= 192 - (320 - 192) \quad \text{[from equation (2)]}$$

$= 2\times 192 - 320$
$= 2\times(512-320) - 320$ [from equation (1)]
$= 2\times 512 - 2\times 320 - 320 = 2\times 512 - 3\times 320$
$\therefore m = 2,\ n = -3$

10. a) If $\gcd(a,b)=1$, prove that $\gcd(a^2,b^2)=1$. [WBUT 2017(EVEN)]

Answer:
Let $\gcd(a^2,b^2)=d$ then $a^2 = dk_1,\ b^2 = dk_2,\ k_1, k_2 \in N$
$\therefore a = \sqrt{dk_1},\ b = \sqrt{dk_2}$
As $\gcd(a,b)=1$, we get $\gcd\left(\sqrt{d}\sqrt{k_1}, \sqrt{d}\sqrt{k_2}\right)=1$
So $\sqrt{d}=1$ or $d=1$

b) Find two integers u and v satisfying $63u+55v=1$. [WBUT 2017(EVEN)]

Answer:
Since $\gcd(63,55)=1$, the equation $63u+55v=1$ has integral solutions.
Now, $63 = 1.55 + 8$
$55 = 6.8 + 7$
$8 = 7.1 + 1$
$\therefore 1 = 8 - 7.1 = (63-1.55)-(55-6.8).1 = 63 - 2.55 + 6(63-1.55) = 7.63 - 8.55$
So $u = 7, v = -8$

11. State division algorithm. Show that every square integer is of the from $5k, 5k\pm 1$ for some integer K. [WBUT 2017(EVEN)]

Answer:
1st Part:
Division of one integer by another plays an important role in the study of integers.
If $b \neq 0$ and a are integers, then we can divide a by b. If q is the quotient and r is the remainder, then we say a is completely divisible by b if $r=0$. indeed we can state the following:
Theorem:
If $a, b \in Z, b \neq 0$, then there exist $q, r \in Z$ such that $a = bq + r,\ 0 \le r < |b|$
For example, if $a = 37, b = 5$, then $q = 7, r = 2$ as $37 = 5\times 7 + 2$
Again, if $a = -37, b = 5$, then $q = -8, r = 3$ as $-37 = 5\times(8) + 3$.

2nd Part:
Let x be any positive integer
Then $x = 5q$ or $x = 5q+1$ or $x = 5q+4$ for integer x.
If $x = 5q, x^2 = (5q)^2 = 25q^2 = 5(5q^2) = 5n$ (where $n = 5q^2$)
If $x = 5q+1, x^2 = (5q+1)^2 = 25q^2 + 10q + 1 = 5(5q^2+2q)+1$
$= 5n+1$ (where $n = 5q^2 + 2q$)
If $x = 5q+4, x^2 = (5q+4)^2 = 25q^2 + 40q + 16 = 5(5q^2+8q+3)+1$
$= 5n+1$ (where $n = 5q^2 + 8q + 3$)
$\therefore$ In each of three cases x^2 is either of the form $5q$ or $5q+1$ or $5q+4$ and for integer q.

12. When n is a positive integer, show that $3^{2n+1} \equiv 3\cdot 2^n \pmod 7$. Obtain a similar result for 2^{n+2} and deduce that $(3^{2n+2}+2^{n+2})$ is a multiple of 7, for all n. [WBUT 2018(ODD)]

Answer:
Let $P_n : 3^{2n+1} = 3.2^n$ (Mod z), i.e., $3^{n+1} - 3.2^n$ is divisible by 7
Clearly, P_n is true when $n=0$ as $3-3$ is divisible by 7.
Suppose P_n is true for m i.e., $3^{2m+1} - 3.2^m$ is divisible by 7.
Then $3^{2(m+1)+1} - 3.2^{m+1} = 9.3^{2m+1} - 3^2.2^m = 2\{3^{2m+1} - 3.2^m\} + 7.3^{2m+1}$
$= 2.7k + 7.3^{2m+1} = 7\{2k + 3^{2m+1}\}$
Hence, $P(m+1)$ is true.
By induction the result is true for every $n \in N$
Now, $3^{2n+1} + 2^{n+2} = (3^{2n+1} - 3.2^n) + 3.2^n + 2^{n+2} = 7k + 7.2^n$
Hence $3^{2n+1} + 2^{n+2}$ is a multiple of 7 for all n.

13. Show that for all primes p and all integers a and b, if $p|ab$ then $p|a$ or $p|b$ of both. [WBUT 2019(EVEN)]

Answer:
Suppose p is prime and $p|ab$. If $p|a$ we are done. If not then $\gcd(p,a)=1$ and by Euclid's Lemma, we know that —

If $\gcd(a, b)=1$ and $a|bc$ then $a|c$ and if p is prime then $\gcd(a, p)=1$ if and only if p does not divide a.
So, we can conclude that $p|b$.

14. Using well ordering principle prove the Archimedean property of natural numbers, viz. for any two $a,b \in N, \exists n \in N$ such that $na \geq b$. **[WBUT 2018(ODD)]**

Answer:
Given, $a,b \in N$
Consider the set $S=\{x; xa \geq b\}$
Clearly, $S \neq \phi$ since $ba \geq b$.
By the well ordering principle, S is bounded below and hence has a first element say n.
Thus there exist $n \in N$ such that $na \geq b$.

Long Choice Type Questions

1. a) By mathematical induction, prove that $6^{n+2}+7^{2n+1}$ is divisible by 43, for each natural number n. **[WBUT 2013(ODD)]**

Answer:
Let $P(n)=6^{n+2}+7^{2n+1}$
Clearly $P(0)$ is true as $P(0)=36+7=43$

Also $P(1)$ is true as $P(1)=559$ which is divisible by 43.
So assume $P(m)$ is divisible by 43 i.e. $6^{m+2}+7^{2m+1}=43k, k \in Z$
Now $P(m+1)=6^{m+3}+7^{2m+3}=6.6^{m+2}+49.7^{2m+1}$
$=6\{6^{m+2}+7^{2m+1}\}+43.7^{2m+1}=6\times 43k+43.7^{2m+1}$
$=43(6k+7^{2m+1})=43k'$ where $k' \in Z$.
Thus $P(m+1)$ is divisible by 43.
Hence by induction principle, $6^{n+2}+7^{2n+1}$ is divisible by 43.

b) If $\gcd(a,b)=1$, show that $\gcd(a+b, a^2-ab+b^2)=1$ or 3. **[WBUT 2014(ODD)]**

Answer:
Given $\gcd(a,b)=1$.
Now let $d=\gcd(a+b, a^2-ab+b^2)$
Then $d \mid a+b$. So $d \mid (a+b)^2$

Again, $d \mid a^2-ab+b^2$.
So $d \mid \{(a+b)^2-(a^2-ab+b^2)\}$ i.e., $d \mid 3ab$.
Since 3 is a prime, i.e., $d=3$ or $d \mid ab$.
But as $\gcd(a,b)=1, d=1$.
Thus d is 1 or 3.

2. a) Define GCD of two integers a and b. Use Eucledian algorithm to find integers u and v such that $\gcd(72, 120)=72u+120v$.
b) (i) State and prove Fundamental Theorem of Arithmetic.
(ii) If $a|b$ and $a|c$ then prove that $a|(bx+cy)$ for arbitrary integers x and y.
(iii) If $a|bc$ and $\gcd(a,b)=1$ then prove that $a|c$. **[WBUT 2015(EVEN)]**

Answer:
a) When a and b are (non-zero) integer, then an integer $d(70)$ is said to be the greatest common divisor of a and b.if $d \mid a$ and $d \mid b$.

We know, $120 = 1\times72+48$
$72 = 1\times48+24$
$48 = 2\times24+0$

Now, $24 = 72 - 1\times 48$
$= 72 - 1\times (120 - 1 \times 72)$
$= 3 \times 72 - 1 \times 120$

$\therefore u=3, v=-1$

b) i) Fundamental Theorem of arithmetic:
Every integer $n > 1$ can be written uniquely as a product of prime numbers.
Proof:
We shall prove the theorem by the principle of induction.
Let $n = 2$. Since 2 is prime, $n(=2)$ is a product of primes ($\because$ a product may consist of a single factor)
Let $n > 2$.
If n is prime, it is a product of primes i.e., a sing factor product.
If n is not a prime, i.e., composite, let us assume than the theorem holds good for positive integers less than n and n = ab. since a, $b < n$, each of a and b can be expressed as the product of primes (by the assumption).
$\therefore n = ab$ is also a product of primes.

ii) Since $a|b$, we have $b = ak$, where k is an integer.
Again, $a|c$ then $c = al$, where l is an integer.
$\therefore a|(bx + cy)$ for some integers x and y.

iii) Let $\gcd(a,b) = 1$. Therefore $\exists$ two integers m and n such that

$$ma + nb = 1 = \gcd(a,b) \quad \ldots (i)$$

Multiplying equation (i) by c we get

$$mac + nbc = c \quad \ldots (ii)$$

Now $a|bc \Rightarrow a|nbc$

$\therefore a|mac \Rightarrow a|c$ (Proved)

3. a) Find the number of integers between 1 and 1000 both inclusive that are not divisible by any of the integers 2, 3 and 7.
b) Find all possible value of x, for $345x \equiv 138 \pmod{912}$. **[WBUT 2016(ODD)]**

Answer:

a) We first see which are the numbers divisible by 2, 3 and 7 i.e., by 42.
Evidently, 42, 84, 126, 168, 210, 252, 294, 336, 378, 420, 462, 504, 546, 588, 630, 672, 714, 756, 798, 840, 882, 924, 966 are the numbers between 1 and 1000 divisible by 2, 3 and 7.
Hence the no. of integers between 1 and 1000 both inclusive which are not divisible by any of the integers 2, 3 and 7 are $1000 - 23 = 977$.

b) Since $\gcd(345, 912) = 3$, the given congruence equation has 3 solutions. We denote

$345x \equiv 138 \pmod{912}$ ...(i)

$115x \equiv 46 \pmod{304}$ ...(ii)

We first solve (ii) and for this note that $\gcd(115, 304) = 1$.

Now,

[illegible]

Hence [illegible]

This implies [illegible] $\equiv 1 \pmod{304}$

i.e., $x_0 = 88$ is a solution of (ii)

Hence $x_0 = 88$ is also a solution of (i)

The three solutions of (i) can now be given as

$x = 88 + \frac{912}{3}t \pmod{912}, t = 0, 1, 2$

or, $x = 88 + 304t \pmod{912}, t = 0, 1, 2$

4. Let m, n be integers not both zero. Prove that $\gcd(km, kn) = k\gcd(m, n)$ for any positive integer k. **[WBUT 2016(ODD)]**

Answer:

Let $d = \gcd(m, n)$

Then $mx + ny = d$ where x and y are integers.

$\therefore x(km) + y(kn) = kd$

i.e., $\gcd(km, kn) = kd = k\gcd(m, n)$

If k is any integer, then result becomes

$\gcd(km, kn) = |k|\gcd(m, n)$

5. a) State and prove the recursion theorem of $\gcd$. **[WBUT 2018(EVEN)]**
b) i) Prove that $\gcd(a, b) = \gcd(a, a - b)$
ii) Show that two consecutive integers are prime to each other.
c) Calculate $\gcd(567, 315)$ and have hence express $\gcd(567, 315)$ as $567x + 315y$ where x and y are integers.

Answer:

a) GCD Algorithm 2: Euclid's Algorithm

The basis of the algorithm is the following fact:

$$\text{For } m \geq n > 0,\ \gcd(m, n) = \begin{cases} n & \text{if } n \text{ divides } m \text{ with no remainder} \\ \gcd\left(n, \text{remainder of } \frac{m}{n}\right) & \text{otherwise} \end{cases}$$

ii) Since $a|b$, we have $a|xb$, where x is an integer.
Again, $a|c$ then $a|yc$, where y is an integer.
$\therefore a|(xb+yc)$ for two integers x and y.

iii) Let $\gcd(a,b)=1$. Therefore $\exists$ two integers m and n such that

$$ma+nb=1=\gcd(a,b) \quad \text{.... (i)}$$

Multiplying equation (i) by c we get

$$mac+nbc=c \quad \text{.... (ii)}$$

Now $a|bc \Rightarrow a|nbc$
$\therefore a|mac \Rightarrow a|c$ (Proved)

3. a) Find the number of integers between 1 and 1000 both inclusive that are not divisible by any of the integers 2, 3 and 7.
b) Find all possible value of x, for $345x \equiv 18(\mathrm{mod}\,912)$. [WBUT 2015(ODD)]

Answer:
a) We first see which are the numbers divisible by 2, 3, and 7, i.e., by 42.
Evidently, 42, 84, 126, 168, 210, 252, 294, 336, 378, 420, 462, 504, 546, 588, 630, 672, 714, 756, 798, 840, 882, 924, 966 are the numbers between 1 and 1000 divisible by 2, 3 and 7.
Hence the no. of integers between 1 and 1000 both inclusive which are not divisible by any of the integers 2, 3 and 7 are 1000 – 23 or 977.

b) Since gcd(345, 912) = 3, the given congruence equation has 3 solutions. We denote

$345x \equiv 18(\mathrm{mod}\,912) \quad \cdots(i)$

$115x \equiv 6(\mathrm{mod}\,104) \quad \cdots(ii)$

We first solve (ii) and for this note that gcd(115, 104)=1

Now,

```
104)115(1
    104
    ---
     11)104(9
         99
         --
          5)11(2
            10
            --
             1
```


Hence

$$\begin{aligned} 1 &= 11-5\times 2 \\ &= 11-(104-9\times 11) \\ &= 10\times 11-104 \\ &= 10\,(115-1\times 104)-104 \\ &= 10\times 115-11\times 104 \end{aligned}$$

This implies $115\times 10 \equiv 1 \pmod{104}$
i.e., $x_0=10$ is a solution of (ii)
Hence $x_0=10$ is also a solution of (i)
The three solutions of (i) can now be given as

$$x=10+\frac{912}{3}i(\mathrm{mod}\,912), i=0,1,2$$

or, $x=10+304i(\mathrm{mod}\,912), i=0,1,2$

4. Let m,n be integers not both zero. Prove that $\gcd(km,kn)=k.\gcd(m,n)$ for any positive integer k. [WBUT 2016(ODD)]

Answer:
Let $d=\gcd(m,n)$
Then $ma+nb=d$ when a and b are integers.
$\therefore$ $a(km)+b(kn)=kd$
i.e., $\gcd(km,kn)=kd=k\gcd(m,n)$
If k is any integer, then result becomes
$\gcd(km,kn)=k\gcd(m,n)$

5. a) State and prove the recursion theorem of gcd. [WBUT 2018(EVEN)]
b) i) Prove that $\gcd(a,b)=\gcd(a,a-b)$
ii) Show that two consecutive integers are prime to each other.
c) Calculate $\gcd(567,315)$ and have hence express $\gcd(567,315)$ as $567x+315y$ where x and y are integers.

Answer:
a) GCD Algorithm 2: Euclid's Algorithm
The basis of the algorithm is the following fact:

$$\text{For } m\ge n>0,\ \gcd(m,n)=\begin{cases} n & \text{if } n \text{ divides } m \text{ with no remainder} \\ \gcd\left(n, \text{remainder of } \dfrac{m}{n}\right) & \text{otherwise} \end{cases}$$

We can rewrite m as follows:

$$m = n\left[\frac{m}{n}\right] + \text{remainder of } \frac{m}{n}$$

Now any divisor d common to m and n must divide the first term with no remainder, since it is the product of n and an integer. Therefore, d must also divide the second term since d divides m and m is the sum of the two terms.

Since any divisor common to m and n must divide the remainder of m/n, we know that, in particular, the gcd does, since it is a common divisor. It just happens to be the greatest such divisor.

So by taking the GCD (n, remainder of m/n), we can "close in quickly" on the GCD of m and n.

Now we can write:

```
int gcd(int m, int n) {
  if ((m % n) == 0)
    return n;
  else
    return gcd(n, m % n);
}
```

gcd(468, 24)
gcd(24, 12)
=> 12
gcd(135, 19)
gcd(19, 2)
gcd(2, 1)
=> 1

Euclid's GCD algorithm is very fast, but division (taking remainders) is a more time-consuming operation than simple addition and subtraction.

b) (i) Let $p = \gcd(a,b)$. Then $p|a, p|b$. So $p|a-b$.

If $h|a, h|a-b$, then $h|a-(a-b)$ i.e., $h|b$

But $p = \gcd(a,b)$. So $h|p$

Hence $p = \gcd(a, a-b)$

(ii) Let n and $n+1$ be two consecutive integers. If possible let n and $n+1$ be not prime to each other. Then there exists $k \in \mathbb{Z}$, $k \neq 1$, such that $k = \gcd(n, n+1)$. This implies $k|n$ and $k|n+1$. So $k|(n+1)-n$ or $k|1$. But the only integer which divides 1 is 1. So $k = 1$. This contradiction therefore proves the result.

c) We have

$315\,\underline{|\,567}$ $\quad 252\,\underline{|\,315}$ $\quad 63\,\underline{|\,252}$

$\quad 252 \qquad\qquad 63 \qquad\qquad 0$

i.e., $567 = 315\times1+252$ i.e., $315 = 252\times1+63$

So, $\gcd(567, 315) = 63$

Now, $63 = 315 - 252\times1 = 315-(567-315) = 315(2)+(-1)567$

Thus, $x=-1$, $y=2$.

6. a) State the principle of well-ordering.
b) State and prove the division algorithm by the well-ordering principle.
c) In a round –robin tournament where every player plays every other player exactly once and each match has a winner and a loser, a cycle is said to exist if P_1 beats P_2, P_2 and P_3 and so on upto P_{m-1} beats P_m and P_m and P_1. The cycle is of length $m(m \geq 3)$. Prove that if a cycle exists among m players then there must be a cycle of 3 of these players. **[WBUT 2018(EVEN)]**

Answer:

a) Every non-empty subset of $\mathbb{N}$ has a first element.

b) If a and b are integers, $b>0$, then there exist unique integers q and r such that

$$a = bq + r,\ 0 \leq r < b$$

Let $S = \{a - tb;\ t\in\mathbb{Z},\ a-tb \geq 0\}$

We now prove that S is a non-empty subset of $\mathbb{N}$.

Case 0: $a \geq 0$

Clearly $a \in S$ as $a = a - 0b$

Case 1: $a < 0$

Clearly $a - ab = a(1-b) \geq 0$ as $a<0$ and $1<b$

Hence $a - ab \in S$.

Thus S is a non-empty subset of $\mathbb{N}$ and hence by the well ordering principle, S has a first element r say.

Then $r = a - qb$ for some integer qv, $a = bq + r$ and $r \geq 0$.

We now show that $r < b$.

If $r \geq b$, then $a-(q+1)b = (a-qb)-b = r-b \geq 0$

This means $a-(q+1)b \in S$, i.e., $r - b \in S$.

But, $r - b < r$ (as $b > 0$)

This contradicts the fact that r is the first element of S.

Hence $r < b$.

To prove uniqueness, let there exist $q_1, q_2, r_1, r_2 \in \mathbb{Z}$ such that $a = q_1 b + r_1$ and $a = q_2 b + r_2$, $0 \le r_1 < b$, $0 \le r_2 < b$.

Then $q_1 b + r_1 = q_2 b + r_2$

or, $(q_1 - q_2) b = r_2 - r_1$

This implies b divides $r_2 - r_1$.

But $-b < r_2 - r_1 < b$.

Hence $r_2 - r_1 = 0$ or $r_1 = r_2$

Then $q_1 b + r_1 = a_2 b + r_2$ or $(q_1 - q_2) b = 0$

This implies b divides $r_2 - r_1$:

Then $q_1 b + r_1 = a_2 b + r_2$ or $(q_1 - q_2) b = 0$

Since $b > 0$, we have $q_1 - q_2 = 0$ or $q_1 = q_2$

This proves the uniqueness.

c) Let us define order by win/loss. If A wins B and B wins C then it is ordered as ...>A>B>C>...

Consider $n \ge 3$ teams, $n(n-1)/2$ games are played and $2^{(n(n-1)/2)}$ unique results might be possible. Out of those $2^{(n(n-1)/2)}$ possible results, how many unique outcomes are there such that we can order **every** player (only once) such that it forms a cycle. i.e.

$P_1 > P_2 > P_3 > .... > P_n > P_{m-1}$

There cannot be cycle if there is Total Winner or Total Looser. I found (by using Principle of Inclusion Exclusion) out there exists

$2^{(n(n-1)/2)} - (2n)\, 2^{((n-1)(n-2)/2)} + n(n-1)\, 2^{((n-2)(n-3)/2)}$

Possible results of the tournament where there is no Total Winner or no Total looser.

If three players played game, player (1), (2), and (3) the outcomes are

[(1, 2), (1, 3), (2, 3)] ← (1) is total winner so there can't be cycle.

[(1, 2), (1, 3), (3, 2)]

[(1, 2), (3, 1), (2, 3)]

[(1, 2), (3, 1), (3, 2)]

[(2, 1), (1, 3), (2, 3)]

[(2, 1), (1, 3), (3, 2)]

[(2, 1), (3, 1), (2, 3)]

[(2, 1), (3, 1), (3, 2)]

Here, (1, 2) means match was played by (1) and (2) and (1) won the game.

The following forms a cycle (also it is the only outcome where there are no total winner and total looser).

BASIC COUNTING TECHNIQUES

☞ Chapter at a Glance

Here we conceptualize some counting strategies that culminate in extensive use and application of permutations and combinations. The questions raised all require that we count something, yet each involves a different approach.

The Addition Principle

If I order one vegetable from the menu at Big Basket, how many vegetable choices does Basket offer?

Here we select one item from a collection of items. Because there are no common items among the two sets Basket has called Greens and Potatoes, we can pool the items into one large set. We use addition, here 4+5, to determine the total number of items to choose from.

This illustrates an important counting principle.

The Addition Principle
If a choice from Group I can be made in **n** ways and a choice from Group II can be made in **m** ways, then the number of choices possible from Group I or Group II is **n+m**.
Necessary Condition: No elements in Group I are the same as elements in Group II.

This can be generalized to a single selection from more than two groups, again with the condition that all groups, or sets, are *disjoint*, that is, have nothing in common.

Examples to illustrate The Addition Principle:

Here are three sets of letters, call them sets I, II, and III:

- Set I: {a,m,r}
- Set II: {b,d,i,l,u}
- Set III: {c,e,n,t}

How many ways are there to choose one letter from among the sets I, II, or III? Note that the three sets are disjoint, or *mutually exclusive*: there are no common elements among the three sets.

Here are two sets of positive integers:

- A={2,3,5,7,11,13}
- B={2,4,6,8,10,12}.

How many ways are there to choose one integer from among the sets A or B? Note that the two sets are not disjoint. What modification can we make to the Addition Principle to accommodate this case? Try to write that modification.

The Multiplication Principle

A "meal" at the Bistro consists of one soup item, one meat item, one green vegetable, and one dessert item from the a-la-karte menu. If Basket's friend Pierre always orders such a meal, how many different meals can be created?

We can enumerate the meals that are possible, preferably in some organized way to assure that we have considered all possibilities. Here is a sketch of one such enumeration, where {V,O}, {K,R}, {S,P,B,I}, and {L,A,C,F} represent the items to be chosen from the soup, meat, green vegetable, and dessert menus, respectively.

VKSL	VKPL	VKBL	VKIL	...and so on to...	ORIL
VKSA	VKPA	VKBA	VKIA		ORIA
VKSC	VKPC	VKBC	VKIC		ORIC
VKSF	VKPF	VKBF	VKIF		ORIF

Take note of the enumeration process used in the table. How could you describe it in words?
How else could we complete the count without identifying all possible options? A map or tree to illustrate the enumeration process provides a bridge to such a method.
We have two ways to select a soup item, two ways to select a meat item, four green vegetables to choose from, and four desserts to choose from. The matching of one soup with each meat, then each of those pairs with each of four possible green vegetables, and each of those triples with each of four possible desserts leads to the use of multiplication as a quick way to count all the possible meals we could assemble at Basket's.
This suggests we use another counting principle to describe this technique.

> **The Multiplication Principle**
> If a task involves two steps and the first step can be completed in **n** ways and the second step in **m** ways, then there are **n*m** ways to complete the task.
> Necessary Condition: The ways each step can be completed are *independent* of each other.

This can be generalized to completing a task in more than two steps, as long as the condition holds.
Example to illustrate The Multiplication Principle:
Recall our three sets I, II, and III: {a,m,r}, {b,d,i,l,u}, and {c,e,n,t}. Determine the number of three-letter sets that can be created such that one letter is from set I, one letter in from set II, and one letter is from set III. Note that our choice in each set is independent of our choice in the other sets. If necessary, we could enumerate the possible three-letter, or three-element, sets.

Permutations

In how many ways can the letters within just one set, from among I, II, and III, be ordered? In set I, we have these possibilities:

amr arm mar mra ram rma

We use the Multiplication Principle to describe our selection. We have three letters to choose from in filling the first position, two letters remain to fill the second position, and just one letter left for the last position: $3\times2\times1=6$ different orders are possible. Likewise, for set II there are 120 different ways to order the five letters and there are 24 different ways to order the letters in set III.
This above discussion exemplifies the concept of another basic counting strategy.

> **Permutation**
> A linear arrangement of elements for which the order of the elements must be taken into account.

We also point out the availability of **factorial notation** to compactly represent the specific multiplication we just carried out: $3\times2\times1 = 3!$, $5\times4\times3\times2\times1 = 5!$, and so on. So $n(n-1)(n-2)...(2)(1) = n!$.

> **Factorial Notation**
> A compact representation for the multiplication of consecutive integers. We use **n!** torepresent the product **n(n-1)(n-2)...(2)(1)**, where **n** is some positive integer.

Example to illustrate use of Permutations:
Almost every morning or evening on the news I hear about the State of Illinois DCFS, the Department of Children and Family Services. I get confused, because our mathematics department has a committee called the Department Faculty Status Committee, or DFSC. Can you see why I'm confused? How many different 4-letter ordered arrangements, or permutations, exist for the set of letters {D, F, S, C}?
Thinking of four positions to fill, __ __ __ __ , we have 4 letters to choose from for the first position, 3 for the next, 2 letters for the next position, and 1 choice for the last position. Using the multiplication principle, there are 4x3x2x1=24 different 4-letter ordered arrangements for the set of letters {D, F, S, C}.
We can extend this application to consider ordered arrangements of only some of the elements in a set. For example, returning to the beverages menu of Big Basket's. If Basket will post only four possible soda chocies, how many different ordered arrangements of the four sodas are there?
Thinking of four positions to fill, __ __ __ __, we have 6 sodas to choose from for the first position, 5 for the next, 4 sodas for the next, and 3 sodas for the last position. Using the multiplication principle, there are 6x5x4x3=360 different ways to select and order four of the six sodas on the menu.
In general, we use the notation **P(n,r)** to represent the number of ways to arrange r objects from a set of n objects. In the first problem above, we determined that P(4,4)=24, and in the second we calculated P(6,4)=360. The general value of P(n,r) is n(n-1)(n-2)...([n-(r-1)] or P(n, r) = n(n-1)(n-2)...(n-r+1). Note that n can be any nonnegative integer. Are there any restrictions on the value of r?
There is a step of arithmetic we can apply to the general pattern for P(n,r) to help streamline permutation calculations. In the second line below, we have multiplied by

$\frac{(n-r)(n-r-1)...(2)(1)}{(n-r)(n-r-1)...(2)(1)}$, which is just the value 1 because the numerator and denominator are equal. In the fourth line below we see how the expression can be simplified using factorial notation.

$$P(n,r)=n(n-1)(n-2)...(n-r+1)$$

$$=n(n-1)(n-2)...(n-r+1).\frac{(n-r)(n-r-1)...(2)(1)}{(n-r)(n-r-1)...(2)(1)}$$

$$=\frac{n(n-1)(n-2)...(n-r+1)(n-r)(n-r-1)...(2)(1)}{(n-r)(n-r-1)...(2)(1)}$$

$$=\frac{n!}{(n-r)!}$$

Thus, we have P(6,2)=6!/4! and P(40,8)=40!/32!. What about P(4,4)? The result above suggests P(4,4)=4!/0!. We already know that P(4,4)=4x3x2x1=4!, So we have 4!=4!/0!. For this to be true, it must be the case that 0!=1. As strange as that may appear, we need 0!=1 in order to maintain consistency within the calculations we wish to carry out.

Combinations

What is the distinction between asking these two questions?

(i) In how many ways can a 5-card poker hand be dealt?

(ii) How many different 5-card poker hands exist?

The first question considers the order or arrangement of the cards as they are dealt. In the second question, the end result when dealt 2H,4D,JC,3S,10D in that order is the same as being dealt 4D,3S,JC,10D,2H in that order. In each case, the same 5-card poker hand exists. The questions help illustrate the difference between a permutation and a combination.

Combination
A collection of elements whose order does not matter.

We found P(52,5) as the solution to the first problem. That is, we arranged 5 objects selected from among 52 cards. For the second question, there are many arrangements that yield the same 5-card hand. We need to account for this. Let's consider a simpler problem.

How many ordered arrangements exist for the letters of the set {A,B,C,D,E}?

Using permutations, we have P(5,5) = 5! = 120 ways to arrange the five letters.

How many ordered arrangements are there of 3 items from the 5-element set?

We have P(5,3) = 543 = 5!/2! = 60 arrangements. For example, for the three letters {A,B,C} we have these arrangements: ABC, ACB, BAC, BCA, CAB, CBA. This represents 6 of the 60 arrangements, yet each involves the same selection of three letters. Likewise for the three letters {A,C,E}: We have ACE, AEC, CAE, CEA, EAC, ECA.

It seems that for each 3-letter subset of {A,B,C,D,E} there are 6 arrangements of the same three letters.

This is a helpful observation in exploring the following question:

How many ways can we select three items from the 5-element set {A,B,C,D,E} when the order of the three items is disregarded?

One way is to list the unique 3-element subsets of {A,B,C,D,E}: ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE. There are 10 such 3-element subsets.

Another way to consider the count is to use the fact that:

(i) there are P(5,3) = 60 ordered arrangements of the 5-element set into 3-element subsets, and

(ii) within the 60 ordered arrangements, there are 10 groups of 6 arrangements that use the same 3-letter subset. That is, 60 ÷ 6 = 10 unique 3-element subsets. Using combinatorics notation, we have

$$C(5,3)=\frac{P(5,3)}{P(3,3)}=\frac{\frac{5!}{2!}}{\frac{3!}{0!}}=\frac{5!0!}{2!3!}=\frac{5!}{2!3!}=C(5,3)$$

In general, we have a way to determine the number of combinations of n items selected r at a time, where the order of selection or the arrangement of the r items is not considered:

$$C(n,r)=\frac{n!}{(n-r)!r!}$$

and we note that

$$C(n,r)=\frac{P(n,r)}{r!}=\frac{P(n,r)}{P(r,r)} \text{ because } P(n,r)=\frac{n!}{(n-r)!}$$

The Relationship Between Permutations and Combinations
If **r** elements are to be collected or arranged from a set of **n** elements, then the number of combinations of **n** elements taken **r** at a time, **C(n,r)**, related to the number of permutations of **n** elements taken **r** at a time, **P(n,r)**, according to the equation $C(n,r)=\frac{P(n,r)}{r!}=\frac{P(n,r)}{P(r,r)}$.

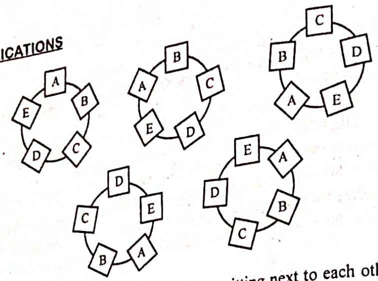
Circular Permutations

How many ways are there to arrange 5 children at a round table?

If we consider the case in a linear fashion,

___, ___, ___, ___, ___,

we have P(5,5) = 5! arrangements. Now extend this to a circle:



Notice that in each of these cases, the same people are sitting next to each other. Although there has been a change--a rotation--about the table, the five children are still in the same positions relative to each other. How many ways are there to rotate the unique linear relationship ABCDE ? There are five such ways, all pictured in the drawing.

Thus we have 5! unique linear arrangements of the children, but we can group those so each group has 5 arrangements that show the children in the same position relative to each other. Therefore, we have $5!/5 = 4!$ **circular permutations** of the five children.

What if we arrange in a circle an r-element subset from an n-element set? Suppose we arrange 3 of the 5 children. In the linear case, there are P(5,3) = 60 arrangements, but we can group those so each group has 3 arrangements that show the children in the same position relative to each other. Therefore, we have P(5,3)/3 = 5!/(2!*3) circular permutations of the five children into 3-children subsets.

In general,

> **Circular Permutation**
> A circular permutation is a circular arrangement of elements for which the order of the elements must be taken into account.
> In general:
> - For **n** elements, there are **(n-1)!** circular permutations.
>
> The number of circular permutations of r-elements taken from an n-element set is P(n,r)/r.

Multiple Choice Type Questions

1. Which of the following is congruent to 28 modulo 3? **[WBUT 2018(EVEN)]**
a) 5 b) 8 c) 13 d) 15
Answer: (c)

2. Which of the following is congruent to 56 modulo 5? **[WBUT 2019(EVEN)]**
a) 20 b) 21 c) 22 d) 23
Answer: (b)

3. If 12 distinct points are placed on the circumference of a circle and all the chords connecting these points are drawn, at how many points do the chords intersect? Assume that no three chords intersect at the same point. **[WBUT 2012(ODD)]**
a) $C(12, 2)$ b) $C(12, 4)$ c) 2^{12} d) $12!/2$
Answer: (b)

4. How many ways are there to travel in *xyz* space from the origin (0, 0, 0) to the point (4, 3, 5) by taking unit steps in positive x, y, z directions only? **[WBUT 2012(ODD), 2016(EVEN)]**
a) 4!.3!.5! b) 60 c) 12!/(5!4!3!) d) 3^{12}
Answer: (a)

5. In how many ways can 7 women and 3 men be arranged in a row if 3 men must always stand next to each other? **[WBUT 2013(EVEN), 2015(EVEN)]**
a) $7!\times 3!$ b) $7!+3!$ c) $3!\times 8!$ d) $7!\times 8!$
Answer: (a)

6. The number of non-negative integral solution of the equation $x+y+z=17,\ x,y,z\geq 0$ is **[WBUT 2013(EVEN), 2015(EVEN)]**
a) 170 b) 171 c) 172 d) none of these
Answer: (b)

7. The number of permutations of a set with k elements is **[WBUT 2013(ODD)]**
a) $k!$ b) $(k-1)!$ c) $(k+1)!$ d) none of these
Answer: imprecise

8. In how many ways 7 different beads can be arranged to form a necklace? **[WBUT 2013(ODD)]**
a) 250 b) 300 c) 360 d) 350
Answer: (c)

9. The number of ways an even sum is obtained when 2 indistinguishable dice are thrown is **[WBUT 2015(EVEN)]**
a) 18 b) 12 c) 16 d) 14
Answer: (a)

10. Total number of functions from a set of 10 elements of another set of 15 elements is **[WBUT 2016(ODD)]**
a) 10^{15} b) 15^{10} c) 2^{15} d) 2^{10}
Answer: (b)

11. In how many ways can 5 letters be posted in 3 letter boxes? **[WBUT 2018(EVEN)]**

a) 25 b) 15 c) 243 d) 720

Answer: (c)

12. How many different ways can three of the letters of the word BYTES be chosen if the first letter must be B? **[WBUT 2018(ODD)]**

a) P(4, 2) b) P(2, 4) c) C(4, 2) d) 4!

Answer: (c)

13. Among 200 people, 150 either swim or jog or both. If 85 swim and 60 swim and jog, how many job? **[WBUT 2018(ODD)]**

a) 125 b) 225 c) 85 d) 25

Answer: (a)

Short Choice Type Questions

1. Use theory of congruence to prove that for $n>1,\ 17|\left(2^{3n+1}+3\cdot 5^{2n+1}\right)$. **[WBUT 2016(EVEN)]**

Answer:

$$2^{3n+1}=2\cdot 2^{3n}=2\cdot 8^n$$

$$3\cdot 5^{2n+1}=3\cdot 5\cdot 5^{2n}=15\cdot 5^{2n}=15\cdot(25)^n=(17-2)25^n$$

$$\therefore\quad 3\cdot 5^{2n+1}+2^{3n+1}=17\cdot(25)^{2n}-2\cdot(25)^n-2\cdot 8^n=17\cdot(25)^{2n}-2\left((25)^n-8^n\right)$$

Now $17\cdot(25)^n\equiv 0(\text{mod}17)$

and $2\left((25)^n-8^n\right)\equiv 0\,\text{mod}(17)$

Hence $17|\left(2^{3n+1}+3\cdot 5^{2n+1}\right)$

2. Solve the linear congruence $6x=3(\text{mod}9)$. **[WBUT 2017(EVEN)]**

Answer:

As $\gcd(6,9)=3$ and 3 divides 3 of the RHS, the congruence has 3 incongruent solutions,

Note $6x\equiv 3(\text{mod}9)$ is equivalent to $2x\equiv 1(\text{mod}3)$.

As $\gcd(2,3)=1$, the congruence $2x\equiv 1(\text{mod}3)$ has only one solution.

We know that there exist $u,v\in\mathbb{Z}$ such that $2u+3v=1$ which is satisfied by $u=-1, v=1$ so $2(-1)\equiv 1(\text{mod}3)$.

Thus $x=-1$ is a solution of the given congruence.

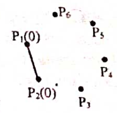
The three incongruent solutions are

$x=-1,-1+3,-1+6(\text{mod}9)$

i.e., $x=-1,2,5(\text{mod}9)$

3. Assume that in a group of six people, each pair of individuals are either friends or enemies. Show that there are either three mutual friends or three mutual enemies. **[WBUT 2013(ODD)]**

Answer:



Let us take six people as $P_1, P_2, P_3, P_4, P_5, P_6$ represented by six points. Let us start with P_1 and P_2 and assume that they are friends & label friends by 0 and enemies by 1.

Then P_2 and P_3 are also friends and proceeding this way we get all of them friends. Exactly a same arguments will lead to a situation when all are enemies and there cannot be a situation where one is a friend and the other is an enemy for a particular pair. So the problem is wrong.

4. State principle of inclusion and exclusion and use it to find the total number of integers between 1 and 1000 which are neither perfect squares nor perfect cubes. **[WBUT 2014(ODD)]**

Answer:

We use the Inclusion-Exclusion Principle. There are 1000 integers from 1 to 1000; among these numbers, 31 are perfect squares (indeed, 312 = 961 ≤ 1000, but 322 > 1000), 10 are perfect cubes (this is because 103 = 1000), and 3 are both squares and cubes (these three numbers are 16 = 1, 26 = 64, and 36 = 729). Thus, by the Inclusion-exclusion principle, there are 1000 − (31 + 10) + 3 = 962 numbers that are neither perfect squares nor perfect cubes.

5. If there are 200 faculty members that speak French, 50 that speak Russian, 100 that speak Spanish, 20 that speak French and Russian, 60 that speak French and Spanish, 35 that speak Russian and Spanish, while only 10 speak French, Russian and Spanish, how many speak either French or Russian or Spanish? **[WBUT 2015(EVEN)]**

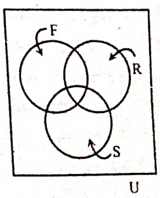
Answer:

Let F be the set containing the number of faculty who speak in French and R, S that of for Russian and Spanish respectively.

$\therefore n(F)=200,\ n(R)=50,\ n(S)=100$

$n(F\cap R)=20,\ n(R\cap S)=35,\ n(F\cap S)=60,$

$n(F\cap S\cap R)=10$



$\therefore$ No. of faculty who speak only in French is
$n(F)-n(F\cap R)-n(F\cap S)+n(F\cap R\cap S)=200-20-60+10=130$

Similarly, number of faculty who speak only in Russian is
$n(R)-n(R\cap F)-n(R\cap S)+n(R\cap F\cap S)=50-20-35+10=5$

and number of faculty who speak only in Spanish is
$n(S)-n(S\cap F)-n(S\cap R)+n(S\cap R\cap F)=100-60-35+10=15$

6. State Pigeonhole Principle and solve: **[WBUT 2015(ODD)]**
A box contains 10 blue balls, 20 red balls, 8 green balls, 15 yellow balls and 25 white balls. How many balls must we choose to ensure that we have 12 balls of the same colour?

Answer:

Pigeonhole Principle: If n pigeonholes are occupied by $kn+1$ or more pigeons, where $k\in N$, then at least one pigeonhole is occupied by $k+1$ or more pigeons.

Here $n=5, k+1=12$, So, $k=11$. Hence $nk+1=56$

So 56 balls are to be drawn.

7. State pigeonhole principle. **[WBUT 2017(EVEN)]**
Suppose that a patient is given a prescription of 45 capsules with the instructions to take at least one capsule per day for 30 days. Then prove that there must be a period of consecutive days during which the patient takes exactly 14 capsules.

Answer:

1st Part: *Refer to Question No. 4(1st Part) of Short Answer Type Questions.*

2nd Part:

Let a_1 be the number of capsules takes until the day 1, and so on, a_i be the number of capsules takes until day i.

Consider a sequence like $a_1, a_2, \ldots, a_{30}$ where $1\le a_i\le 45, \forall a_i$.

Add 14 to each elements of the sequence $a_1+14, a_2+14, \ldots, a_{30}+14$ where $15\le a_i+14\le 59, \forall a_i$.

Now we have two sequences

1. $a_1, a_2, \ldots, a_{30}$ and
2. $a_1+14, a_2+14, \ldots, a_{30}+14$

having 60 elements in total with each elements taking a value ≤ 59.

So according to pigeon hole principle there must be at least two elements taking a same value ≤ 59 i.e., $a_i=a_j+14$ for some i and j.

Therefore, there exist at one period a_j to a_i, in which the patient takes 14 capsules.

8. Prove that any sequence of six integers must contain a subsequence whose sum is divisible by six. **[WBUT 2017(EVEN)]**

Answer:

Let $\{a\_n\}$ be a sequence of integers.

Now consider the sums

$b\_0=0$
$b\_1=a\_1$
$b\_2=a\_1+a\_2$
$b\_3=a\_1+a\_2+a\_3$

etc.... up to

$b\_n=a\_1+a\_2+a\_3+\ldots+a\_n$

There are $n+1$ of the b_i s and only n possible remainders modulo n, so two of the b_i are congruent modulo n. So suppose $b\_i=b\_k(\bmod n)(k>j)$

Then $b\_k-b\_j=a\_1+a\_2+\ldots+a\_k.(a\_1+a\_2+\ldots+a\_j)$
$=a\_[j+1]+a\_[j+2]+a\_[j+3]+\ldots+a\_k$

is divisible by n

So the sequence contains a subsequence

$c\_i=a\_[j+i]$ for $i=1$ to $k-j$

whose sum is divisible by n.

9. Find the total number of integers lying between 1 and 1000 that are divisible by at least one of 2, 3, 7. **[WBUT 2017(ODD)]**

Answer:

Let A, B, C denote the numbers between 1 and 1000, divisible by 2, 3 and 7 respectively.

Then $A=\{2,4,6,\ldots\ldots,1000\}$ and $n(A)=500$
$B=\{3,6,9,\ldots\ldots,999\}$ and $n(B)=333$
$C=\{7,14,21,\ldots\ldots,994\}$ and $n(C)=142$
$A\cap B=\{6,12,18,\ldots\ldots,996\}$ and $n(A\cap B)=166$
$B\cap C=\{21,42,\ldots\ldots,987\}$ and $n(B\cap C)=47$
$C\cap A=\{14,28,\ldots\ldots,994\}$ and $n(C\cap A)=71$
$A\cap B\cap C=\{42,84,\ldots\ldots,966\}$ and $n(A\cap B\cap C)=23$.

$\therefore$ The total number of integers lying between 1 and 1000 and divisible by 2, 3 or 7
$=n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)$

$$-n(C\cap A)+n(A\cap B\cap C)$$
$$=500+333+142-166-47-71+23=714$$

10. a) State the principle of Inclusion and Exclusion. **[WBUT 2018(EVEN)]**
b) Show that if A and B are subsets of some universal set.
S then $|A\cup B|=|A|+|B|-|A\cap B|$

Answer:
a) *Refer to Question No. 2(1st Part) of Short Answer Type Questions.*

b) If A and B are two sets, then
$|A\cup B|=|A|+|B|-|A\cap B|$
Let A contains x elements common with B and there are y elements in A, not common with B.
Then $|A|=x+y, |A\cap B|=x$
If z be the no. of elements in B, not common with A, then
$|B|=x+z, |A\cup B|=x+y+z$
Hence $|A\cup B|=|A|+|B|-|A\cap B|$

11. a) State the "Pigeon Hole Principle" and the "Generalized Pigeon Hole principle". **[WBUT 2018(EVEN)]**
b) Prove the "Generalized Pigeon Hole Principle." **[WBUT 2018(EVEN), 2019(EVEN)]**

Answer:
a) If n pigeons are assigned to m pigeonholes and $n>m$, then at least one pigeonhole contains two or more pigeons.
If there are n pigeonholes occupied by $nk+1$ pigeons, then there must be at least one pigeonhole occupied by $k+1$ or more pigeons.

b) The generalized pigeonhole principle: If N objects are placed into k boxes, then there is at least one box containing at least? N/k objects.
Proof: Suppose none of the boxes contains N/k or more objects.
Then every box contains at most $N/k-1$ objects.
So, the total number of objects is at most $k(N/k-1)$.
But $N/k-1<N/k$.
Thus, the total number of objects is less than $k(N/k)$,
i.e. less than N.
This is a contradiction. End of proof.

12. Obtain the CNF of $\neg(p\to(q\wedge r))$. **[WBUT 2018(ODD)]**

Answer:
We make the truth table for $\neg\{p\to(q\wedge r)\}$

p	q	r	$q\wedge r$	$p\to(q\wedge r)$	$\neg\{p\to(q\wedge r)\}$
1	1	1	1	1	0
1	1	0	0	0	1
1	0	1	0	0	1
1	0	0	0	0	1
0	1	1	1	1	0
0	1	0	0	1	0
0	0	1	0	1	0
0	0	0	0	1	0

The required CNF is
$$(\neg p\vee\neg q\vee\neg r)\wedge(p\vee\neg q\vee\neg r)\wedge(\neg p\vee q\vee\neg r)\wedge(p\vee q\vee r)$$

Long Answer Type Questions

1. a) Prove the Pascal's identity: **[WBUT 2013(EVEN), 2019(EVEN)]**
$c(n,r)=c(n-1,r)+c(n-1,r-1)$, where the notation carries usual meaning.

Answer:
C(n–1, r) + C(n–1, r–1) = (n–1)!/ [r! (n–1–r)!] + (n–1)!/[(r–1)! (n–r)!]
= (n–r) * (n–1)!/ [r! * (n–r)(n–1–r)!] + r * (n–1)!/[r(r–1)! (n–r)!]
= (n–r) * (n–1)! + r * (n–1)!] / [r! (n–r)!]
= (n–1)! * [(n–r) + r] / [r! (n–r)!] = n!/[r! (n–r)!] = C(n,r)

b) State Pigeonhole principle. **[WBUT 2013(EVEN), 2016(EVEN), 2019(EVEN)]**
Using that prove that if any five numbers from 1 to 8 are selected, then two of them will add to 9. **[WBUT 2013(EVEN), 2016(EVEN), 2019(EVEN)]**
OR,
Show that if 5 integers from 1 to 8 are chosen, then at least 2 of them will add to 9. **[WBUT 2019(EVEN)]**

Answer:
Pigeonhole Principle: If n number of pigeonhole be accommodated by m number of pigeon $(n<m)$ then at least one of the pigeonhole must be filled up by more than one pigeon. Here, 8 numbers (1 to 8) will be filled up in 5 places. According to pigeonhole principle, at least one of places will be filled up by more than one numbers. So there exist at least one place where sum of the two or more digits (in that place where more than digits are placed) is 9.

c) Using Pigeonhole principle, find the total number of natural numbers that must be chosen to be sure of getting at least two of them whose difference is divisible by 11. **[WBUT 2013(EVEN)]**

Answer:

We consider the congruence classes $[0],[1],[2],\ldots\ldots,[10]$ Pigeonhole as 11. If any 12 natural numbers are chosen, at least two of them will occupy the same class. Hence their difference will be divisible by 11.

Thus the required number is 12.

2. a) Using principle of inclusion and exclusion show that for any three sets A, B and C $n(A\cup B\cup C)=n(A)+n(B)+n(C)$ if they are pair wise mutually disjoint. **[WBUT 2013(ODD)]**

Answer:

By the principle of inclusion and exclusion,

$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C)$

But A, B, C are pairwise mutually disjoint means

$$P(A\cap B)=P(B\cap C)=P(C\cap A)=0 \text{ and } P(A\cap B\cap C)=0$$

Hence $n(A\cup B\cup C)=n(A)+n(B)+n(C)$

b) Let D be a square drawn in the plane with sides of length $\sqrt{2}$. Prove that in every set of 5 distinct points in D, there exist two points whose distance from one another is at most 1. **[WBUT 2013(ODD)]**

Answer:

Let D be a square with all the four sides of length $\sqrt{2}$. To see whether it is possible to get two points with distance more than 1, we first place four points as farthest as possible and the best option is putting the points on the four vertices. In that case if we try to choose the firth point as apart as possible from the other four vertices, the only choice is the centre of the square. But even then there are two points viz. the centre and any one vertex will have distance 1. In all offer cases availability of two points with mutual distance ≤ 1 is obvious. Hence the proof.

c) Find the minimum number of students in a class to be sure that six of them are born in the same month. **[WBUT 2013(ODD)]**

Answer:

By the Piegeon hole principle, it is clear that if there are at least 61 students.

3. Obtain the CNF of $\neg(p\rightarrow(p\rightarrow r))$ **[WBUT 2019(EVEN)]**

Answer:

We make the truth table for $\neg(p\rightarrow(p\rightarrow r))$

p	r	$p\rightarrow r$	$p\rightarrow(p\rightarrow r)$	$\neg(p\rightarrow(p\rightarrow r))$
1	1	1	1	0
1	0	0	0	1
0	1	1	1	0
0	0	1	1	0

Hence the required CNF is

$$(\sim p\vee\sim r)\wedge(\sim p\vee r)\wedge(p\vee r)$$

INTRODUCTION TO PROPOSITIONAL CALCULUS

Chapter at a Glance

In propositional logic we consider declarative sentences Examples
"The sun is red"
"All sparrows are birds"
"My name is Torben"
"It is raining"
Note: There are other sorts of sentences
Syntax of propositional logic:
The propositional symbols p, q, r, ... stands for propositions Propositional symbols are atomic formulas with which compound formulas are built using the connectives ¬, ∧, ∨, ⇒ together with parentheses The connectives stand for respectively "not", "and", "or", "implies".
Examples of formulas:
¬p, (¬p∨q), (p⇒q), (((p⇒q)∧p)⇒q) The formula ¬p is built using the connective ¬ and the atomic formula p, The formula(¬p∨q) is built using the connective ∨ and the formulas ¬p and q, etc.
Parentheses are often omitted like in the formula ¬p∨q
Declarative sentences are symbolized using formulas.
Examples of symbolizations:
If p and q symbolizes the sentences "The sun is red" and "My name is Torben" Then the formula ¬p symbolizes the sentence "The sun is not red" and ¬p ∨ q symbolizes "The sun is not red or my name is Torben"
Semantics of propositional logic:
We call T or F truth-values to assign a propositional symbol the truth-value T is to assume that it stands for a true proposition Analogously, to assign a propositional symbol the truth value F is to assume that it stands for a false proposition A propositional symbol is assigned either T or F

Truth-tables: To each connective there is a truth-table

□ ¬□	□ ψ □∧ψ	□ ψ □∨ψ	□ ψ □⇒ψ
--------	----------	----------	-------------
T F	T TTTTTTTT		
F T	T F F	T F T	T F F
	F T F	F T TFT T	
	F FFFFFFF T		

Thereby any compound formula can be given a truth-table (Greek letters □, ψ, θ, ... stand for arbitrary formulas)
Examples of truth tables

p	¬p	¬¬p	p	¬p	p∨¬p
T	F	T	T	F	T
F	T	F	F	T	T

p	q	¬p	¬p∨q	p	q	¬q	p∧¬q	¬(p∧¬q)
T	T	F	T	T	T	F	F	T
T	F	F	F	T	F	T	T	F
F	T	T	T	F	T	F	F	T
F	F	T	T	F	F	T	F	T

Alternative to truth-tables:
¬□ is true if and only if □ is not true
□∧ψ is true if and only if □ is true and ψ is true
□∨ψ is true if and only if □ is true or ψ is true
□⇒ψ is true if and only if □ is true implies that ψ is true
Such truth-conditions contain the same information as the truth-tables.
A couple of definitions:
A formula is a tautology if and only if it is true whatever truth values the involved propositional symbols are assigned
Two formulas are logically equivalent if and only if they have the same truth-table (That is, the formulas □ and ψ are logically equivalent if and only if the formula (□⇒ ψ)∧(ψ⇒ □) is a tautology).

Multiple Choice Type Questions

1. $A \wedge B$ is equivalent to which of the following? [WBUT 2012(ODD), 2016(EVEN)]
a) $\neg A \rightarrow \neg B$ b) $\neg A \rightarrow B$ c) $\neg B \rightarrow A$ d) $\neg(A \rightarrow \neg B)$.
Answer: (d)

2. A disjunctive normal form of $P \rightarrow Q$ is
[WBUT 2013(EVEN), 2016(EVEN), 2016(ODD)]
a) $\sim P \vee Q$ b) $P \vee \sim Q$
c) $(\sim P \wedge Q) \vee (P \wedge \sim Q)$ d) $(P \wedge Q) \vee (P \wedge \sim Q)$
Answer: (a)

3. $\sim(p\vee q)\vee(\sim p\wedge q)\equiv$ [WBUT 2013(EVEN), 2016(EVEN)]
   a) $\sim p$ b) p c) $\sim q$ d) q

Answer: (d)

4. $\sim(p\vee q)\vee(p\wedge\sim q)\equiv$ [WBUT 2013(ODD), 2015(EVEN)]
   a) $\sim p$ b) p c) $\sim q$ d) none of these

Answer: (c)

5. The proposition $p\wedge(q\wedge\sim q)$ is a [WBUT 2014(ODD), 2017(EVEN)]
   a) contradiction b) tautology c) both (a) and (b) d) none of these

Answer: (a)

6. $A\wedge B$ is equivalent to [WBUT 2014(ODD)]
   a) $\sim A\to\sim B$ b) $\sim A\to B$ c) $\sim B\to A$ d) $\sim(A\to\sim B)$

Answer: (d)

7. The truth value of the statement '$x^2+4=0$ hold for some real values of x' is [WBUT 2015(ODD)]
   a) true b) false
   c) both (a) and (b) d) none of these

Answer: (d)

8. If p: 'Anil is rich' and q: 'Kanchan is poor' then the symbolic form of the statement 'Either Anil or Kanchan is rich' is [WBUT 2015(ODD), 2017(EVEN)]
   a) $p\vee q$ b) $p\vee\neg q$ c) $\neg p\vee q$ d) $\neg(p\wedge q)$

Answer: (a)

9. $P\to(P\vee Q)$ is a [WBUT 2016(ODD)]
   a) tautology b) contradiction c) contingency d) none of these

Answer: (d)

10. Contrapositive of '$\sim p\to q$' is [WBUT 2017(EVEN)]
   a) $p\to q$ b) $\sim q\to\sim p$ c) $\sim q\to p$ d) $q\to\sim p$

Answer: (c)

11. The truth of the statement '$x^2=x$ hold for all real values of x' is [WBUT 2017(EVEN)]
   a) T b) F c) both (a) and (b) d) none of these

Answer: (a)

12. $\neg p\vee(q\wedge\neg q)$ is equivalent ($\equiv$) to [WBUT 2017(ODD)]
   a) q b) $\neg q$ c) p d) $\neg p$

Answer: (d)

13. Let $L(x, y)$ be the statement x likes y and domain for x, y consists of all people in the world. Then we can express Nobody likes everybody as [WBUT 2017(ODD)]
   a) $\forall x\exists yL(x, y)$ b) $\forall x\exists y\neg L(x, y)$
   c) $\forall x\forall y\neg L(x, y)$ d) none of these

Answer: (c)

14. The statement "The sun rises in the north" is [WBUT 2018(EVEN)]
   a) not a proposition b) true proposition
   c) false proposition d) None of these

Answer: (d)

15. Contrapositive of '$p\to\sim q$' is [WBUT 2018(EVEN)]
   a) $\sim q\to p$ b) $q\to\sim p$ c) $\sim p\to q$ d) $\sim q\to\sim p$

Answer: (d)

16. Which of the following statement is correct? [WBUT 2018(EVEN)]
   a) $\sim\forall xP(x)\equiv\exists x\sim P(x)$ b) $\sim\forall_x P(x)\equiv\sim\exists_x P(x)$
   c) $\sim\exists xP(x)\equiv\forall xP(x)$ d) $\exists xP(x)\equiv\sim\forall xP(x)$

Answer: (a)

17. If $p=$ It is raining, q = She will go to college, then "It is raining and she will not go to college" will be denoted by [WBUT 2018(ODD)]
   a) $p\wedge\neg q$ b) $p\wedge q$ c) $\neg(p\wedge q)$ d) $\neg p\wedge q$

Answer: (a)

18. The statement $p\leftrightarrow q\equiv(p\to q)\wedge(q\to p)$ describes [WBUT 2018(ODD)]
   a) Commutative Law b) Implication Laws
   c) Exportation Law d) Equivalence

Answer: (d)

19. The statement 'Please close the door' is [WBUT 2019(EVEN)]
   a) not a proposition b) true proposition
   c) false proposition d) none of these

Answer: (a)

20. Not p unless q is represented as **[WBUT 2019(EVEN)]**

a) $q \to p$ b) $p \to q$ c) $p \leftrightarrow q$ d) none of these

Answer: (d)

21. Inverse of $\sim p \to q$ is **[WBUT 2019(EVEN)]**

a) $q \to \sim p$ b) $\sim q \to \sim p$ c) $\sim q \to p$ d) $p \to \sim q$

Answer: (d)

22. If 5 is less than 2 then sun rises in the west. The statement is **[WBUT 2019(EVEN)]**

a) true b) false
c) this is not a statement d) cannot be determined

Answer: (a)

23. If p is a statement then which of the following is a tautology? **[WBUT 2019(EVEN)]**

a) $p \wedge \sim p$ b) $p \wedge T$ c) $p \vee F$ d) $p \vee \sim p$

Answer: (d)

Short Answer Type Questions

1. Show that the following pair of propositions are logically equivalent:

a) $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q)$ and p

b) $p \to (q \to r)$ and $(p \wedge q) \to r$

[WBUT 2013(EVEN), 2013(ODD), 2015(EVEN), 2015(ODD)]

Answer:

a) We construct the truth table as follows:

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$\sim p \wedge \sim q$	$\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q))$	$\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q)$
1	1	0	0	0	0	1	1
1	0	0	1	0	0	1	1
0	1	1	0	1	0	0	0
0	0	1	1	0	1	0	0

Since the columns for $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q)$ and p are identical, they are logically equivalent.

b) The truth tables for the given propositional functions:

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$
1	1	1	1	1	1	1
1	1	0	0	0	1	0
1	0	1	1	1	0	1
1	0	0	1	1	0	1
0	1	1	1	1	0	1
0	1	0	0	1	0	1
0	0	1	1	1	0	1
0	0	0	1	1	0	1

As the columns for $p \to (q \to r)$ and $(p \wedge q) \to r$ are identical,

$$p \to (q \to r) \Leftrightarrow (p \wedge q) \to r.$$

2. Write an equivalent formula for $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p)$ which involves only the connectives $(\sim, v)$. **[WBUT 2013(ODD)]**

Answer: Use $p \to q = \rceil p \vee q$ and $p \wedge q = \sim(\sim p \vee \sim q)$

3. Show that $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology. **[WBUT 2014(ODD)]**

Answer:

We make the following truth table.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	Formula
1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	0	1	0	1	1	0	1
1	0	0	0	1	0	0	1
0	1	1	1	1	1	1	1
0	1	0	1	0	1	0	1
0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1

Hence the given formula is a tautology.

4. Find the PDNF and PCNF of the following statement: **[WBUT 2015(EVEN)]**

$$(p \wedge q) \vee (7p \wedge q) \vee (q \wedge r).$$

Answer:

Truth table:

p	q	r	p∧q	~p	~p∧q	q∧r	(p∧q)∨(~p∧q)∨(q∧r)
1	1	1	1	0	0	1	1
1	1	0	1	0	0	0	1
1	0	1	0	0	1	0	1

p	q	r	p∧q	~p	~p∧q	q∧r	(p∧q)∨(~p∧q)∨(q∧r)
1	0	0	0	0	1	1	1
0	1	1	0	1	1	1	1
0	1	0	0	1	1	0	1
0	0	1	1	1	0	0	1
0	0	0	1	1	0	1	1

So the PDNF is $pqr + pqr' + pq'r + pq'r' + p'qr + p'qr' + p'q'r + p'q'r'$.
This is a tautology, so PCNF is not possible.

5. Show that the premises "one student in this class knows how to write programs in JAVA" and "Everyone who knows how to write programs in JAVA can get a high paying job" imply the conclusion "Someone in this class can get a high paying job". **[WBUT 2015(EVEN)]**

Answer:
Let C(x) represent 'x in this class' J(x) represent 'x knows JAVA Programming' and H(x) represent 'x get high salaried job'.
Then the given premises are $\exists x(C(x)\wedge J(x))$ and $\forall x\ (J(x)\rightarrow H(x))$. The conclusion is $\exists x(C(x)\wedge H(x))$.

6. Show that the inverse of an element n in Z_m will exist if and only if $\gcd(n, m)=1$. **[WBUT 2015(ODD)]**

Answer:
We note $\mathbb{Z}_m=\{0,1,2,\cdots,m-1\}\pmod m$
Suppose we wish to find modular multiplicative inverse x of 3 modulo 11.
$x\equiv 3^{-1}\pmod{11}$
This is the same as finding x such that
$3x\equiv 1\pmod{11}$
Working in $\mathbb{Z}_{11}$ we find one value of x that satisfies this congruence is 4 because
$3(4)=12\equiv 1\pmod{11}$
And there are no other values of x in $\mathbb{Z}_{11}$ that satisfy this congruence. Therefore, the modular multiplicative inverse of 3 modulo 11 is 4.

7. Obtain the conjunctive normal form of the following statement:
$(p\rightarrow(q\wedge r))\wedge(\sim p\rightarrow(\sim q\wedge\sim r))$ **[WBUT 2015(ODD)]**

Answer:
We make the truth table of the given formula:

p	q	r	~p	~q	~r	q∧r	~q∧~r	p→(q∧r)	~p→(~q∧r)	f
1	1	1	0	0	0	1	0	1	1	1
1	1	0	0	0	1	0	0	0	1	0
1	0	1	0	1	0	0	0	0	1	0
1	0	0	0	1	1	0	1	0	1	0
0	1	1	1	0	0	1	0	1	0	0
0	1	0	1	0	1	0	0	1	0	0
0	0	1	1	1	0	0	0	1	0	0
0	0	0	1	1	1	0	1	1	1	1

Hence the CNF is
$$(x'+y'+z)(x'+y+z')(x'+y+z)(x+y'+z')(x+y'+z)(x+y+z')$$

8. Prove that $\left[(p\vee q)\rightarrow(p\rightarrow r)\rightarrow(q\rightarrow r)\rightarrow r\right]$ is a tautology.[WBUT 2016(EVEN)]

Answer:
Let us construct the truth table:

p	q	r	$p\rightarrow r$	$p\vee q$	$(p\vee q)\rightarrow(p\rightarrow r)$	$q\rightarrow r$	$(p\vee q)\rightarrow(p\rightarrow r)\rightarrow(q\rightarrow r)$	$(p\vee q)\rightarrow(p\rightarrow r)\rightarrow(q\rightarrow r)\rightarrow r$
1	1	1	1	1	1	1	1	1
1	1	0	0	1	0	0	1	1
1	0	1	1	1	1	1	1	1
1	0	0	0	1	0	1	1	1
0	1	1	1	1	1	1	1	1
0	1	0	1	1	1	0	0	1
0	0	1	1	0	1	1	1	1
0	0	0	1	0	1	1	1	i

So, $(p\vee q)\rightarrow(p\rightarrow r)\rightarrow(q\rightarrow r)\rightarrow r$ is a tautology.

9. Show that $(\neg q\wedge(p\rightarrow q))\rightarrow\neg p$ is a tautology **[WBUT 2016(ODD)]**

Answer:

p	q	$\neg p$	$\neg q$	$(p\rightarrow q)$	$\neg q\wedge(p\rightarrow q)$	$(\neg q\wedge(p\rightarrow q))\rightarrow\neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since the truth values of the given compound proposition is T for all combinations of p and q, it is a tautology.

10. Find CNF of $\neg(p\vee q)\leftrightarrow(p\wedge q)$ using laws of proposition. [WBUT 2016(ODD)]

Answer:

$$\neg(p\vee q)\leftrightarrow(p\wedge q)$$
$$\equiv(\neg p\wedge\neg q)\leftrightarrow(p\wedge q)$$
$$\equiv((\neg p\wedge\neg q)\wedge(p\wedge q))\vee(\neg(\neg p\wedge\neg q)\wedge\neg(p\wedge q))$$
$$\equiv((\neg p\wedge\neg q)\wedge(p\wedge q)\vee(p\vee q)\wedge(\neg p\vee\neg q)$$
$$\equiv(\neg p\wedge\neg q)\wedge(p\wedge q)\vee(p\vee q)\wedge(\neg p\vee\neg q)$$
$$\equiv(p\wedge\neg p)\wedge(q\wedge\neg q)\vee(p\vee q)\wedge(\neg p\vee\neg q)$$
$$\equiv F\wedge F\vee(p\vee q)\wedge(\neg p\vee\neg q)$$
$$\equiv(p\vee q)\wedge(\neg p\vee\neg q)$$

which is the required CNF.

11. Verify the validity of the following statements: [WBUT 2017(EVEN)]
Every living thing is a plant or an animal
My cat is an animal and it is not a plant
All animals have lung
My cat has lung

Answer:
Let Li denote the set of lining things, P denote the set of plants and A denote the set of animals. Let L denote set of animal having lungs.

Then we have $\forall(x\in Li\Rightarrow x\in P\vee A)$

$c\in A,\ c\notin P$

$\forall(x\in A\Rightarrow x\in\perp)$

$\therefore c\in\perp$

So the argument is valid.

12. Using the laws of prepositional logic show that $\models(p\wedge q)\rightarrow(p\vee q)$. [WBUT 2017(ODD)]

Answer:
We have

$$(p\wedge q)\rightarrow(p\vee q)=\neg(p\vee q)\vee(p\vee q)$$
$$=(\neg p\wedge q)\vee(p\vee q)=[\neg p\vee(p\vee q)]\wedge[\neg q\vee(p\vee q)]$$
$$=[(\neg p\vee p)\vee q]\wedge[p\vee(q\vee\neg q)]=[1\vee q]\wedge[p\vee 1]$$
$$=1\wedge 1=1.$$

13. a) Justify the statement "converse of inverse of an implication is equivalent to its inverse of converse."
b) Draw the truth table of "not p unless q". [WBUT 2018(EVEN)]

Answer:
a) If $\to a$ p, then its inverse is $\sim p\to\sim q$ and the converse of the inverse is $\sim q\to\sim p$.
Again if $p\to q$, then its converse is $q\to p$. the inverse of the converse is $\sim q\to\sim p$.
Hence the proof.

b) Not p unless q is nothing but not $p\leftarrow q$ is $\sim p\leftarrow q$. Its truth table is

p	q	$\sim p$	$q\to\sim p$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	T

14. Two restaurant have respectively the following advertisements:
a) "Good food is not cheap".
b) "Cheap food is not good".
Examine whether they say equivalent statements. [WBUT 2018(EVEN)]

Answer:
Let g denote good food and c denote cheap food.
The statements are $g\Rightarrow\sim c$ and $c\Rightarrow\sim g$

But $c\Rightarrow\sim g$ gives $g\Rightarrow\sim c$ [contra positive]
So the statements are equivalent.

15. Prove that $(p\wedge(p\Leftrightarrow q))\Rightarrow q$ [WBUT 2018(ODD)]

Answer:
We have $p\wedge(p\leftrightarrow q)$
As $p\leftrightarrow q$ implies $p\to q$ and $q\to p$
So we get $p, p\to q, q\to p$. The first two gives by Modus pollens q.

16. Verify the validity of the argument 'in drive to work then I will arrive in time. I do not drive to work. Therefore, I will not arrive in time'. [WBUT 2019(EVEN)]

Answer:
Let us symbolise statement as follows:
p: drive to work
q: arrive in time

Step No.	Statement	Reason
1	$p \to q$	p
2	$\neg p$	p
3	$\neg p \to q$	$\$$, 1

[WBUT 2019(EVEN)]

17. Explain the following with example:
i) Fallacy of denying the hypothesis
ii) Fallacy of affirming the conclusion

Answer:

i) Fallacy of denying the hypothesis:
This hypothesis is an incorrect reasoning in proving $p \to q$ by starting with assuming $\neg p$ and $\neg q$ with proving. For example:

Show that if x is irrational, then $\frac{x}{2}$ is irrational. A fallacy of denying the hypothesis argument would start with:
"Assume that x is rational. Then ……..".

ii) Fallacy of affirming the conclusion:
This is an incorrect reasoning in proving $p \to q$ by starting with assuming q and proving p. For example: Show that if $x+y$ is odd, then either x or y is odd, but not both. A fallacy of affirming the conclusion argument would start with: "Assume that either x or y is odd, but not both. Then …….".

Long Answer Type Questions

1. a) Show that s is a valid conclusion from the premises $p \to \sim q, q \vee r, \sim s \to p$.
[WBUT 2012(ODD), 2015(ODD), 2016(EVEN)]

Answer:
We have $p \to \sim q,\ \ q \vee r, \sim s \to p$
We start with $q \vee r$, Say q
$p \to \sim q \ \ \therefore q \to \sim p$
$\sim s \to p \ \ \ \therefore \sim p \to s$
Again, start with $q \vee r$, say, r; r has not implication.
Hence the conclusion s is valid.

b) Show that t is a valid conclusion from the premises $p \Rightarrow q, q \Rightarrow r, r \Rightarrow s$ and $p \vee t$.
[WBUT 2012(ODD), 2016(EVEN), 2016(ODD)]

Answer:
We have $p \to q,\ q \to r,\ r \to s,\ p \vee t$
Let us start with $p \vee t$
Assume p.
Then $p \to q$
$q \to r$
$r \to s$ gives $\underline{s}$
Next assume t
As t does not imply anything.
Hence the conclusion s is valid.

2. a) Check the validity of the following arguments: [WBUT 2012(ODD)]
"If my program runs successfully then I will submit my project. I can appear the examination only if I submit my project. Either my program runs successfully or the computer crashes then I cannot appear in examination."

Answer:
Let us denote the statements symbolically as
p: My program runs successfully
q: I will submit my project
r: I can appear the examination.
s: The computer crashes.
The premises are
$p \to q,\ \ r \uparrow q,\ \ p \vee s.$
The conclusion is $\sim r$
Assume p
Then $p \to q$
Also, $r \to q$ gives $\sim q \to \sim r$. Also s has no implication
So it is not a valid conclusion.

b) Write down the truth table for conditional and bi-conditional proposition.
[WBUT 2012(ODD)]

Answer:
The truth table for conditional $(\to)$ with p and q as the propositions:

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

The truth table for the bi-conditional $(\leftrightarrow)$ with p and q as the propositions is

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

3. a) Obtain the CNF of $\sim(p \to (q \wedge r))$. **[WBUT 2013(EVEN)]**

Answer:

We make the truth table for $\sim(p \to (q \wedge r))$ as follows

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$\sim(p \to (q \wedge r))$
1	1	1	1	1	0
1	1	0	0	0	1
1	0	1	0	0	1
1	0	0	0	0	1
0	1	1	1	1	0
0	1	0	0	1	0
0	0	1	0	1	0
0	0	0	0	1	0

Hence the required CNF is

$$(\sim p \vee \sim q \vee \sim r) \wedge (p \vee q \vee r) \wedge (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$$

b) Prove that $(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$ is a tautology. **[WBUT 2013(EVEN), 2018(ODD)]**

Answer:

Let us construct the truth table

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \wedge (p \to r) \wedge (q \to r) = f$	$(f \to r)$
1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	0	1	1	1	1	1	1
1	0	0	1	0	1	0	1
0	1	1	1	1	1	1	1
0	1	0	1	1	0	0	1
0	0	1	0	1	1	0	1
0	0	0	0	1	1	0	1

Since the column corresponding to the given expression has only 1's, the given expression is a tautology.

c) Check the validity of the following argument:
"If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been cancelled and Alice would have been angry. If the party were cancelled, then refunds would have to be made. No refunds were made." **[WBUT 2013(EVEN), 2018(ODD), 2019(EVEN)]**

Answer:

Let we denote

p : The band can play rock music

q : The refreshments were delivered on time.

r : The New Year's party will be held.

t : Alice will be angry.

s : Refunds will be made.

Then the given argument can be put as:

$\neg p \vee \neg q \Rightarrow \neg r \wedge t,\ \neg r \Rightarrow s$. Conclusion $\neg s$.

The conclusion is not valid as the only way to arrive at $\neg s$ is through r but we have the other way complication $\neg s \to r$.

An alternative method is to prove that $\{(\neg p \vee \neg q) \to (\neg r \wedge t)\} \wedge \{\neg r \to s\} \Rightarrow \neg s$ is not a tautology.

4. a) Show that $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ is a tautology.

b) Let us consider the discrete mathematics class. If a student S_1 is late, then another student S_2 is late, and if both S_1 and S_2 are late, then the class becomes boring. Suppose that the class is not boring! What conclusion can be drawn about the student S_1 using truth table? **[WBUT 2015(ODD)]**

Answer:

a)

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p\vee q$	$\sim q\vee\sim r$	$\sim p\vee(\sim q\vee\sim r)$	$\sim p\vee q$	$\sim p\vee\sim r$	$(p\vee q)\sim\vee(\sim p\vee(\sim q\vee\sim r))$	f
1	1	1	0	0	0	1	0	0	0	0	0	0
1	1	0	0	0	1	1	1	1	0	0	1	1
1	0	1	0	1	0	1	1	1	0	0	1	1
1	0	0	0	1	1	1	1	1	0	0	1	1
0	1	1	1	0	0	1	0	1	0	0	1	1
0	1	0	1	0	1	1	1	1	0	1	1	1
0	0	1	1	1	0	0	1	1	1	0	0	1
0	0	0	1	1	1	0	1	1	1	1	0	1

As we find value of the function (f) is always '1', so it is tuatology.

b) Let s_1 denote the statement that s_1 is late

s_2 denote the statement that s_2 is late

b denote the statement that the class is boring.

Then $s_1 \wedge s_2 \Rightarrow b$. So $\sim b \Rightarrow \sim s_1 \vee \sim s_2$

The truth table of the above formula asserts that either s_1 is not late or if s_1 is late, s_2 is not late.

s_1	s_2	b	$\sim s_1$	$\sim s_2$	$\sim b$	$\sim s_1 \vee \sim s_2$
1	1	1	0	0	0	0
1	1	0	0	0	1	0
1	0	1	0	1	0	1
1	0	0	0	1	1	1
0	1	1	1	0	0	1
0	1	0	1	0	1	1
0	0	1	1	1	0	1
0	0	0	1	1	1	1

5. a) Without using truth table show that $(p\wedge q)\rightarrow(p\vee q)$ is a tautology. **[WBUT 2017(EVEN)]**

Answer:

We note that $p\rightarrow q$ is equivalent to $\sim p\vee q$

Therefore we see

$(p\wedge q)\rightarrow\{p\vee q\}$

$\Leftrightarrow \sim(p\wedge q)\vee(p\vee q)$

$\Leftrightarrow (\sim p\vee\sim q)\vee(p\vee q)$

$\Leftrightarrow \{(\sim p\vee\sim q)\vee p\}\vee q$ by associativity

$\Leftrightarrow \{(\sim p\vee p)\vee\sim q\}\vee q$ by commutativity

$\Leftrightarrow (1\vee\sim q)\vee q$

$\Leftrightarrow 1\vee q\Leftrightarrow 1$

Hence the given formula is a tautology.

b) Using truth table prove that $p\rightarrow(p\vee r)=(p\rightarrow q)\vee(p\rightarrow r)$. **[WBUT 2017(EVEN)]**

Answer:

We make the following truth table

p	q	r	$q\vee r$	$p\rightarrow(q\vee r)$	$p\rightarrow q$	$p\rightarrow r$	$(p\rightarrow q)\vee(p\rightarrow r)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	0	0	0	1	1	1	1

Since the 5th and 8th columns are identical, the equality is proved.

6. a) Symbolize the following: **[WBUT 2017(ODD)]**
If either George enrols or Harry enrols then Ira does not enrol. Either Ira enrols or Harry enrols. If either Harry enrols or George does not enrol then Jim enrols. George enrols. Therefore either Jim enrols or Harry does not enrol.

Answer:

Let g denote George, h denote Harry, i denote Ira and j denote Jim.

Let E denote the set of enrolled persons. Then the statements can be symbolized as:

$g\in E\vee h\in E \;\Rightarrow\; i\notin E$

$i\in E\vee h\in E$

$h\in E\vee g\notin E \;\Rightarrow\; j\in E$

$j\in E\vee h\notin E$

b) What is a formula in proportional calculus? What is a tautology? State and prove Duality theorem. **[WBUT 2017(ODD)]**

Answer:

Throughout our treatment of formal logic it is important to distinguish between syntax and semantics. Syntax is concerned with the structure of strings of symbols (e.g. formulas and formal proofs), and rules for manipulating them, without regard to their meaning. Semantics is concerned with their meaning. Formulas are certain strings of symbols as specified below.

Definition of Propositional Formula:

1) Any atom P is a formula.
2) If A is a formula so is ¬A.
3) If A, B are formulas, so is (A ∧B).
4) If A, B are formulas, so is (A ∨B).

All (propositional) formulas are constructed from atoms using rules 2) - 4)

A formula of propositional logic is a tautology if the formula itself is always true regardless of which valuation is used for the propositional variables.

7. a) Determine whether the following argument is valid stating the rules of inference you are using:

Premises:

i) ***If it does not rain today or it is not foggy then the tournament will start and the first match will be played.***
ii) ***If the match is played then the referee will come.***
iii) ***The referee did not come.***

Conclusion: If rained today

b) Prove that there are infinitely many primes.

c) Show that the congruence relation is an equivalence relation. **[WBUT 2018(EVEN)]**

Answer:

a) Let r denote a rainy day, f denote a foggy day, t denote the starting of the tournament, p denote playing of the first match.

The given premises can be written as

$$\sim r \vee \sim f \Rightarrow t \wedge p,\ p \Rightarrow c,\ \sim c$$

where c denoted the coming of the referee.

By contra positive,

$$\sim c \Rightarrow \sim p$$

and $\quad \sim t \vee \sim p \Rightarrow r \wedge f$

From the last implication it now follows

$$\sim p \Rightarrow r \wedge f$$

Thus $\quad \sim c \Rightarrow r$

Hence the conclusion is valid.

b) If possible let there by finitely many primes and these primes be $p_1, p_2, \ldots\ldots, p_n$.

Then clearly, $\quad p_1 \cdot p_2 \cdot \ldots\ldots \cdot p_{n-1} > p_i;\ i = 1, 2, \ldots\ldots, n$

and $\quad p_1 \cdot p_2 \cdot \ldots\ldots \cdot p_{n-1}$ is not divisible by p_1 or p_2 or .... or p_n.

This implies $p_1 \cdot p_2 \cdot \ldots\ldots \cdot p_{n-1}$ is a prime greater than $p_1, p_2, \ldots\ldots, p_n$.

Hence our assumption is not correct and there are infinitely many primes.

c) Let $\equiv$ be a congruence relation defined on $\mathbb{Z}$

Then clearly $\equiv$ is reflexive as $x \equiv x(m) \forall x \in \mathbb{Z}$ as $x - x$ is divisible by m.

Again, $\equiv$ is symmetric as $x \equiv y(\text{mod}\, m)$ as $x - y$ is divisible by m implies.

Finally, $\equiv$ is transitive as $x \equiv y(\text{mod}\, m)$ and $y \equiv z(\text{mod}\, m)$ imply $x \equiv z(\text{mod}\, m)$ as $y - x$ is divisible by m.

Hence $\equiv$ is an equivalence relation.

8. Let $G = (\mathbb{Z}, +)$ and $H = (\mathbb{Z}_n, +)$ for some $n > 1$. Define $\varphi: G \to H$ by $\varphi(x) = [x]$. Then φ is a homomorphism. **[MODEL QUESTION]**

Answer:

Since operation in both groups is addition, the equation that we need to check in this case is $\varphi(x+y) = \varphi(x) + \varphi(y)$. Verification is given below:

$$\varphi(x) + \varphi(y) = [x] + [y] = [x + y] = \varphi(x + y)$$

(where quality $[x] + [y] = [x + y]$ holds by definition of addition in $\mathbb{Z}_n$).

9. Let F be a field, $n > 1$ and integer, $G = GL_n(F)$ and $H = (F \setminus \{0\}, .)$. Define the map $\varphi(A) = \det(A)$. **[MODEL QUESTION]**

Answer:

In this example φ is a homomorphism thanks to the formula $\det(AB) = \det(A)\det(B)$. Note that while this formula holds for all matrices (not necessarily invertible ones), in the example have to restrict ourselves to invertible matrices since the set $Mat_n(F)$ of all $n \times n$ matrices over F does not form a group with respect to multiplication.

10. Let $f: G \to G'$ be a homomorphism. Show that f is one-to-one if and only if $\ker f = \{e\}$. **[MODEL QUESTION]**

Answer:

Let $f:G\to G'$ be a one-to-one homomorphism and e,e' be the respective identities of G and G'. Clearly $f(e)=e'$.

Since f is one-to-one, $\ker f=\{e\}$.

Conversely, let $\ker f=\{e\}$ and let $f(a)=f(b)$ for $a,\ b\in G$.

Then $\quad f(a^{-1}b)=f(a^{-1})f(b)=f(a)^{-1}f(b)=f(a)^{-1}f(a)=e'$.

Thus $\quad a^{-1}b\in\ker f \quad$ or, $a^{-1}b=e \quad$ or, $a=b$

Hence f is one-to-one.

11. Let (Z, +) be the additive group of all integers and $(Q-\{0\},\cdot)$ be the multiplicative group of non-zero rational numbers. Define f : Z → Q − {0} by f(x) = 3^x for all x∈Z. Show that f is a homorphism but not an isomorphism. **[MODEL QUESTION]**

Answer:

Here $\quad f:Z\to Q-\{0\}$ is defined as $f(x)=3^x$

Now let $m,n\in Z$, then

$$f(m+n)=3^{m+n}=3^m\cdot 3^n=f(m)\cdot f(n)$$

Hence f is a homomorphism

As f is not surjective, f is not an isomorphism.

(Note: $\frac{1}{2}\in Q-\{0\}$ has no pre image in Z).

12. a) Prove that in a group (G, o) the equation a o x = b has unique solution.
b) Let G = (Z, +), G' = (Z, +) be two Groups; f:G→G be a mapping defined by f(x) = |x|. Is the mapping a homo-morphism? Give reasons.
c) Find the idempotent elements in the ring (Z6, +,.) where Z6 is set of residue class module 6.
d) Let (N, P) be a coset where N is set of all natural numbers and P stands for divisibility. Find the maximal and minimal element of the set {2,8,32,4} CN. **[MODEL QUESTION]**

Answer:

a) We have, $a\cdot x=b$

Pre-multiplying by a^{-1} we get

$$a^{-1}(a\cdot x)=a^{-1}\cdot b$$

or, $\quad (a^{-1}\cdot a)x=a^{-1}\cdot b$

or, $\quad e\cdot x=a^{-1}\cdot b$

or, $\quad x=a^{-1}\cdot b$.

Next if possible let x^* and $\bar{x}$ be two solutions of the given equation.

So, $a\cdot x^*=b=a\cdot\bar{x}$

By pre-multiplication of a^{-1} both sides we get, $\quad x^*=\bar{x}$.

Hence the solution is unique.

b) This is not a homomorphism as $f(5+(-3))\neq f(5)+f(-3)$

since $f(2)=2$ but $f(5)=5$, $f(-3)=3$.

c) $\bar{0},\bar{1},\bar{3},\bar{4}$ since $\bar{0}.\bar{0}=\bar{0}$, $\bar{1}.\bar{1}=\bar{1}$, $\bar{3}.\bar{3}=\bar{9}=\bar{3}$, $\bar{4}.\bar{4}=\bar{16}=\bar{4}$.

d) Minimal element = 2 and maximal element = 32.

13. a) In a group G, prove that $(ab)^2=a^2b^2$, iff $(ab)^{-1}=a^{-1}b^{-1}$, where $a,b\in G$.
b) If f is a group homomorphism from $G\to G'$, then show that $f(e)=e'$ and $f(a^{-1})=[f(a)]^{-1}$, where e and e' are the identity elements of G and G' respectively and $a\in G$. **[MODEL QUESTION]**

Answer:

a) In a group G, let us assume $(ab)^2=a^2b^2$ for all $a,b\in$G.

Then $abab=aabb$

or, $\quad ba=ab$

$\therefore\quad (ba)^{-1}=(ab)^{-1}$

Conversely, let $a^{-1}b^{-1}=(ab)^{-1}$ for all a, b in G.

Then $(a^{-1}b^{-1})^{-1}=\{(ab)^{-1}\}^{-1}$

or, $(b^{-1})^{-1}(a^{-1})^{-1}=ab$

or, $\quad ba=ab$.

or, $\quad a(ba)b=a(ab)b$.

or, $\quad (ab)(ab)=(aa)(bb)$

or, $\quad (ab)^2=a^2b^2$

b) Let $\quad a\in G$. Then $ae=ea=a$

$\therefore\quad f(ae)=f(ea)=f(a)$

or, $\quad f(a)f(e)=f(e)f(a)=f(a)$

So $f(e)$ is the identity of $f(G)$ but $f(G)$ being a subgroup of G, $f(e)=e'$.

Next, we know $aa^{-1} = a^{-1}a = e$

$\therefore \quad f\left(aa^{-1}\right) = f\left(a^{-1}a\right) = f(e)$

or, $\quad f(a)f\left(a^{-1}\right) = f\left(a^{-1}\right)f(a) = f(e) = e'$

So $f\left(a^{-1}\right)$ is the inverse of $f(a)$.

That is $\left(f(a)\right)^{-1} = f\left(a^{-1}\right)$.

14. a) Define group homomorphism. If G is a group of real, non-singular, n-square matrices under multiplication, show that the determinant function is a homomorphism of $GL(2,\mathbb{R})$ into G' where G' is the group of non-zero real numbers under multiplication.

b) Show that every cyclic group of order n is isomorphic to the group (Z_n, $+_n$) where Z_n is the set of equivalence classes for the congruence modulo *n* over the set of integers.

[MODEL QUESTION]

Answer:

a) Let $(G,*)$ and $(H,\circ)$ be two groups and $\phi: G \to H$ be a mapping. ϕ is called a group homomorphism from G into H if $\phi(a*b) = \phi(a)\circ\phi(b)$ for all $a, b \in G$.

We have here $\phi: GL(2,\mathbb{R}) \to G'\left(=\mathbb{R}=\{0\}\right)$

defined by $\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

We see if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL(2,\mathbb{R})$,

then $\phi\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right] = \det\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\det\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

$= \phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}\phi\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

Hence ϕ is a homomorphism.

b) Let G be a group of order n with α as a generator. Then define $\phi: G \to \mathbb{Z}_n$ as $\phi(g) = \phi\left(\alpha^k\right) = [k]$. Clearly ϕ is a homomorphism and bijective. Hence the result.

15. Let (N, +) be the semi-group of natural numbers and (S, *) be the semi-group on S : {0, 1, e} with the operation · given in the table:

*	e	0	1
e	e	0	1
0	0	0	0
1	1	0	1

A mapping g: N→S given by g(0) = 1 and g(j) = 0 for $j \neq 0$ is a semi-group homomorphism but not a monoid homomorphism, examine it?

[MODEL QUESTION]

Answer:

We have

$S = \{0, 1, e\}, \; g: N \to S$

Let $\quad m, n \in N, \; m \neq 0, \; n \neq 0$

Then $\quad g(m\,n) = 0 = g(m) * g(n) \qquad \left[\because g(m) = 0, \; g(n) = 0\right]$

Let now $m \in N, \; m \neq 0$

Then $\quad g(m\,o) = g(0) = 1 \neq g(m) * g(o) = 0 * 1 = 0$

Hence g is not a homomorphism.

16. Show that the group $(\mathbb{Z}_6, +)$ is a homomorphic image of the group $(\mathbb{Z}, +)$.

[MODEL QUESTION]

Answer:

Recall $\mathbb{Z}_6\left\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\right\}$

Define $\varphi: (\mathbb{Z},+) \to (\mathbb{Z}_6,\oplus)$ as

$\varphi(n) = \bar{n}$ (mod 6)

Clearly φ is a homomorphism.

Hence the result.

ALGEBRIC STRUCTURES AND MORPHISM

Chapter at a Glance

Algebraic Structure

A non empty set S is called an algebraic structure w.r.t binary operation (*) if it follows following axioms:

Closure: (a*b) belongs to S for all a,b∈ S.

Ex: S = {1,-1} is algebraic structure under *

As 1*1 = 1, 1*-1 = -1, -1*-1 = 1 all results belongs to S.

But above is not algebraic structure under + as 1+(-1) = 0 not belongs to S.

Semi Group

A non-empty set S, (S,*) is called a semigroup if it follows the following axiom:

- **Closure:** (a*b) belongs to S for all a,b∈ S.
- **Associativity:** a*(b*c) = (a*b)*c ∀a,b,c belongs to S.

Note: A semi group is always an algebraic structure.

Ex: (Set of integers, +), and (Matrix ,*) are examples of semigroup.

Monoid

A non-empty set S, (S,*) is called a monoid if it follows the following axiom:

- **Closure:** (a*b) belongs to S for all a,b∈ S.
- **Associativity:** a*(b*c) = (a*b)*c ∀a,b,c belongs to S.
- **Identity Element:** There exists e ∈ S such that a*e = e*a = a ∀ a ∈ S

Note: A monoid is always a semi-group and algebraic structure.

Ex: (Set of integers,*) is Monoid as 1 is an integer which is also identity element.

(Set of natural numbers, +) is not Monoid as there doesn't exist any identity element. But this is Semigroup.

But (Set of whole numbers, +) is Monoid with 0 as identity element.

Group

A non-empty set G, (G,*) is called a group if it follows the following axiom:

- **Closure:** (a*b) belongs to G for all a,b∈ G.
- **Associativity:** a*(b*c) = (a*b)*c ∀a,b,c belongs to G.
- **Identity Element:** There exists e ∈ G such that a*e = e*a = a ∀ a ∈ G
- **Inverses:** ∀ a ∈ G there exists a^{-1} ∈ G such that $a*a^{-1} = a^{-1}*a = e$

Note:

1. A group is always a monoid, semigroup, and algebraic structure.
2. (Z,+) and Matrix multiplication is example of group.

AbelianGroup or Commutative group

A non-empty set S, (S,*) is called aAbelian group if it follows the following axiom:

- **Closure:** (a*b) belongs to S for all a,b∈ S.
- **Associativity:** a*(b*c) = (a*b)*c ∀a,b,c belongs to S.
- **Identity Element:** There exists e ∈ S such that a*e = e*a = a ∀ a ∈ S
- **Inverses:** ∀ a ∈ S there exists a^{-1} ∈ S such that $a*a^{-1} = a^{-1}*a = e$
- **Commutative:** a*b = b*a for all a,b∈ S

Note : (Z,+) is a example of Abelian Group but Matrix multiplication is not abelian group as it is not commutative.

For finding a set lies in which category one must always check axioms one by one starting from closure property and so on.

Group Homomorphism

By homomorphism we mean a mapping from one algebraic system with a like algebraic system which preserves structures.

Definition

Let G and G′ be any two groups with binary operation ∘ and ∘′ respectively. Then a mapping f:G→G′ is said to be a homomorphism if for all a,b∈G

f(a∘b)=f(a)∘′f(b)

A homomorphism f which at the same time is also onto is said to be an epimorphism.

A homomorphism f which at the same time is also one-one is said to be anmonomorphism.

A group G′ is called a homomorphism image of a group G, if there exists a homomorphism f of G onto G′. A homomorphism of a group G into itself is called an edomorphism.

Examples:

(i) Let G be any group under binary operation ∘. If f(x)=x for every x∈G then f:G→G is a homomorphism because

f(xy)=f(x)f(y)

(ii) Let G be the group of integers under addition, let G′ be the group of integers under addition modulo n. If f:G→G′ be defined by f(x)=remainder of xx on division by n, then this is a homomorphism.

(iii) Let G be any group under addition. If f(x)=e, ∀x∈G then the mapping f:G→G is a homomorphism because for all x,y∈G, f(x,y)=e and f(x)+f(y)=e+e=e, so that

f(x+y)=f(x)+f(y)

(iv) Let G be the group of integers under addition and let G′=G. If for all x∈G, f(x)=2x, then f is a homomorphism because

f(x+y)=2(x+y)=2x+2y=f(x)+f(y)

Kernel of Homomorphism

Definition

If f is a homomorphism of a group G into a G′, then the set K of all those elements of G which is mapped by f onto the identity e′ of G′ is called the kernel of the homomorphism f.

Theorem:

Let G and G′ be any two groups and let e and e′ be their respective identities. If f is a homomorphism of G into G′, then

(i) $f(e)=e'$

(ii) $f(x-1)=[f(x)]-1$ for all $x \in G$

(iii) K is a normal subgroup of G.

Proof:

(i) We know that for $x \in G$, $f(x) \in G'$.

$f(x)\cdot e'=f(x)=f(xe)=f(x)\cdot f(e)$, and therefore by using left cancellation law we have $e'=f(e)$ or $f(e)=e'$

(ii) Since for any $x \in G$, $xx^{-1}=e$, we get

$f(x).f(x^{-1})=f(xx^{-1})=f(e)=e'$

Similarly $x^{-1}x=ex^{-1}x=e$, gives $f(x^{-1})\cdot f(x)=e'$

Hence by the definition of $[f(x)]^{-1}$ in G′ we obtain the result

$f(x^{-1})=[f(x)]^{-1}$

(iii) Since $f(e)=e'$, $e \in K$, this shows that $K \neq \square$, now let $a,b \in K$, $x \in G$, $a \in K, b \in K$,

$\Rightarrow f(a)=e', f(b)=e' \Rightarrow f(a)=e', f(b^{-1})=[f(b)]^{-1}=e' \Rightarrow f(ab^{-1})=f(a)[f(b)]^{-1}=e'\cdot e'=e' \Rightarrow ab^{-}$

$\Rightarrow f(a)=e', f(b)=e' \Rightarrow f(a)=e', f(b^{-1})=[f(b)]^{-1}=e' \Rightarrow f(ab^{-1})=f(a)[f(b)]^{-1}=e'\cdot e'=e' \Rightarrow ab^{-1} \in K$

${}^{1}\in K \Rightarrow f(a)=e', f(b)=e' \Rightarrow f(a)=e', f(b^{-1})=[f(b)]^{-1}=e' \Rightarrow f(ab^{-1})=f(a)[f(b)]^{-1}=e'\cdot e'=e' \Rightarrow ab^{-1} \in K$

This establishes that K is a subgroup of G.

Now, to show that it is also normal we prove the following:

$f(x^{-1}ax)=f(x^{-1})f(a)f(x)=[f(x)]^{-1}f(a)f(x)=[f(x)]^{-1}e'f(x)=[f(x)]^{-1}f(x)=e'$

Therefore, $x^{-1}ax \in K$, hence the result.

Examples of Group Homomorphism

Here's some examples of the concept of group homomorphism.

Example 1:

Let $G=\{1,-1,i,-i\}$, which forms a group under multiplication and I= the group of all integers under addition, prove that the mapping f from I onto G such that $f(x)=i^n \forall n \in I$ is a homomorphism.

Solution:

Since $f(x)=i^n, f(m)=i^m$, for all $m,n \in I$

$f(m+n)=i^{m+n}=i^m \cdot i^n=f(m)\cdot f(n)$

Hence f is a homomorphism.

Group Isomorphism

Definition

Let G and G′ be any two groups with binary operation ∘ and ∘′, respectively. If there exists a one-one onto mapping $f:G \to G'$ such that

$f(a \circ b)=f(a) \circ' f(b), \forall a, b \in G$

In this case, the group G is said to be isomorphic to the group G′, and the mapping ff is said to be an isomorphism. If G is isomorphic to G′, we write $G \simeq G'$ or $G \cong G'$.

Multiple Choice Type Questions

1. An element x in a ring R is a zero divisor if **[MODEL QUESTION]**

a) $x \cdot b=0$

b) $x \cdot b=0$, for some non zero element in R

c) $x \cdot b \neq 0$, for all elements b in R

d) none of these

Answer: (b)

2. If R is a ring without zero divisors, then $x.y=0$ implies **[MODEL QUESTION]**

a) $x=0$ or $y=0$

b) $x=0$ and $y=0$

c) $x=0,\ y \neq 0$

d) $x \neq 0,\ y=0$

Answer: (a)

3. Every finite integral domain is a field. This statement is **[MODEL QUESTION]**

a) true

b) false

Answer: (a)

4. The number of unit elements of the ring $(Z,+,..)$ **[MODEL QUESTION]**

a) 2

b) 3

c) 1

d) infinite

Answer: (a)

Short Answer Type Questions

1. Prove that every finite integral domain is a field. **[MODEL QUESTION]**

Answer:

Let D be a finite integral domain. Since every integral domain is a commutative ring with unity, it is enough to prove only that every non-zero element of D has an (multiplicative) inverse in D.

So let a be a non-zero element of D:

Consider the set $S=\{ab; b \in D\}$

Since D is closed with respect to multiplication, $S \subset D$.

Now, if $b \neq c$, then

$ab \neq ac$ because otherwise $ab=ac$, then

$a(b-c)=0 \quad \therefore b-c=0$

as D has no divisor of zero and hence $b=c$. Thus the elements of S are distinct but as D is finite, S will have as many elements as D has viz., S = D. So these exists an element a' such that $aa'=e$ where e is the unity of D. Clearly a' is the inverse of a, as

$aa' = a'a = e$ by the commutativity of D. Since 'a' is arbitrary, every non-zero element has an inverse. Hence D is a field.

2. Show that a field does not contain any zero divisor. **[MODEL QUESTION]**

Answer:

Let F be a field and let $a, b \in F$, $ab = 0$, $a \neq 0$.

Then a^{-1} exists in F. Multiplying both sides by a^{-1} we get

$$a^{-1}(ab) = a^{-1} \cdot 0 \text{ or, } (a^{-1}a)b = 0 \qquad \text{or, } b = 0$$

Thus there cannot exist $a, b \in F$, $a \neq 0$, $b \neq 0$ but $ab = 0$.

This implies F has no divisor of zero.

3. If in a ring R with unity, $(xy)^2 = x^2y^2$ for all $x, y \in R$, then show that R is commutative. **[MODEL QUESTION]**

Answer:

Let $x, y \in R$ be any elements

then $y + 1 \in R$ as $1 \in R$

By given condition

$$(x(y+1))^2 = x^2(y+1)^2$$
$$\Rightarrow \quad (xy + x)^2 = x^2(y+1)^2$$
$$\Rightarrow \quad (xy)^2 + x^2 + xyx + xxy = x^2(y^2 + 1 + 2y)$$
$$\Rightarrow \quad x^2y^2 + x^2 + xyx + xxy = x^2y^2 + x^2 + 2x^2y$$
$$\Rightarrow \quad xyx = x^2y \qquad \text{...(1)}$$

Since (1) holds for all x, y in R, it holds for $x+1, y$ also. Thus replacing x by $x+1$, we get

$$(x+1)y(x+1) = (x+1)^2 y$$
$$\Rightarrow \quad (xy + y)(x+1) = (x^2 + 1 + 2x)y$$
$$\Rightarrow \quad xyx + xy + yx = y = x^2y + y + 2xy$$
$$\Rightarrow \quad yx = xy \qquad \text{using (1)}$$

Hence R is commutative.

4. In a ring (R, +, .) show that (-a).(-b) = a.b for all a, b ∈ 1R. **[MODEL QUESTION]**

Answer:

We see

$$0 = a \cdot 0 = a \cdot (b + (-b)) = a \cdot b + a \cdot (-b)$$

$\therefore \quad a \cdot (-b) = -a \cdot b.$

Hence $(-a) \cdot (-b) = -(-a) \cdot b = -\{-a \cdot b\} = a \cdot b.$

5. Examine whether the set of even integers form an integral domain with respect to ordinary addition and multiplication. **[MODEL QUESTION]**

Answer:

Let E denote the set of all even integers. Clearly E is an additive abelian group, a multiplicative semigroup and has the two distributive properties. Further, E has no divisor of zero and E is commutative with respect to multiplication. But E has not multiplicative identity, i.e., unity. Hence E is not an integral domain.

6. If $a^2 = a$, for every element a in a ring R, then show that $b = -b$, for every $b \in R$. **[MODEL QUESTION]**

Answer:

Since b + b ∈ R for b ∈R, we get by hypothesis

$$(b+b)^2 = b+b.$$

or $(b+b).(b+b) = b + b.$

or $b^2 + b^2 + b^2 + b^2 = b + b.$

or $b + b + b + b = b + b.$

$\therefore\ b = -b$

7. Show that the set of matrices $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ is a subring of the ring of matrices. **[MODEL QUESTION]**

Answer:

Let M_2 denote the ring of matrices of order 2.

Let S denote the set of matrices $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$

Let $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \in S$

Then $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} - \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} a-c & 0 \\ b-d & 0 \end{bmatrix} \in S$

and $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} ac & 0 \\ bc & 0 \end{bmatrix} \in S$

Hence S is a sub-ring of M_2.

8. Show that the set of matrices $S = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} : \alpha, \beta \in R \right\}$ is a left ideal but not a right ideal of 2 × 2 real matrices. **[MODEL QUESTION]**

Answer:

A subring S of a ring R is said to be a left ideal of R if $a \in S, r \in R \Rightarrow r \cdot a \in S$ and a right ideal of R if $a \in S, r \in R \Rightarrow a \cdot r \in S$.

Let $S_1=\begin{pmatrix}\alpha_1 & 0\\ \beta_1 & 0\end{pmatrix}, \frac{1}{2}=\begin{pmatrix}\alpha_2 & 0\\ \beta_2 & 0\end{pmatrix}$ be 2elements of S.

Then $S_1-S_2=\begin{pmatrix}\alpha_1-\alpha_2 & 0\\ \beta_1-\beta_2 & 0\end{pmatrix}\in S$

and $S_1\cdot S_2=\begin{pmatrix}\alpha_1\cdot\alpha_2 & 0\\ \beta_1\cdot\beta_2 & 0\end{pmatrix}\in S$

$\therefore\quad S_1\in S,\ S_2\in S\Rightarrow S_1-S_2\in S\ \&\ S_1\cdot S_2\in S$

$\therefore S$ is a subring of 2×2 real matrices.

Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ be a 2×2 real matrix.

Then $S_1\cdot A=\begin{pmatrix}\alpha_1 & 0\\ \beta_1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}\alpha_1 a & \alpha_1 b\\ \beta_1 a & \beta_2 b\end{pmatrix}\notin S.$

$AS_1=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}\alpha_1 & 0\\ \beta_1 & 0\end{pmatrix}=\begin{pmatrix}a\alpha_1+b\beta_1 & 0\\ c\alpha_1+d\beta_1 & 0\end{pmatrix}\in S$

$\therefore S$ is a left ideal but not a right ideal of 2×2 real matrices.

9. Given a division ring D and a D (left)vector space V, then given any bases for V, B_1, B_2, then the coordinates $|B_1|=|B_2|$. Thus dimension is an invariant.
[MODEL QUESTION]

Answer:

Suppose $|B_2|>|B_1|$. Notice that given any element $v\in V$, we may write $v=\sum_{b_i\in E_j} d_i b_i$, where $d_i\in D$ and $E_j\subseteq B_1, |E_j|<\infty$. Note that the collection of all E_j (all finite subsets of B_1) has the same cardinality as B_1. Thus there is a $b\in B_2$ which is not the linear combination of any of the elements of B_1. Thus $B_1\cup\{b\}$ is a linearly independent set, contradicting B_1 a basis.

10. Let D be a division ring, and let V_1, V_2 be (left) vector-spaces over D, such that there are linear monomorphisms $\varphi_1: V_1\to V_2$ and $\varphi_2: V_2\to V_1$. Then $V_1\cong V_2$.
[MODEL QUESTION]

Answer:

Let B_i be a basis for V_i. Then consider that $\varphi_1(B_1)$ is a linearly independent set in V_2 and so extends to a basis C_2 in V_2. Similarly, $\varphi_2(B_2)$ extends to a basis C_1 of V_1.

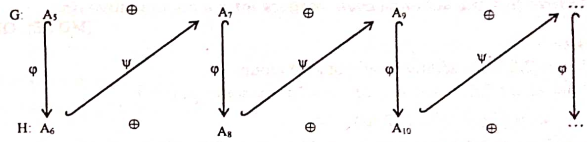
So by the dimension theorem, we have bijections: $\psi_i: B_i\to C_i$. Then $\psi_2^{-1}\circ\psi_1: B_1\to B_2$ is an injection. Similarly, $\psi_1^{-1}\circ\psi_2: B_2\to B_1'$ is also an injection. So by the Cantor-Schroeder-Bernstein theorem, there is a bijection $\tau: B_1\to B_2$.

This bijection between bases τ, extends uniquely to a linear isomorphism T and thus $V_1\cong V_2$

11. Let $G:=\bigoplus_{n=1} A_{2n+3}$, $H:=\bigoplus_{n=1} A_{2n+4}$, where A_i is the alternating group on i elements. Then there are injections from G into H and vice versa, but these groups are non-isomorphic. **[MODEL QUESTION]**

Answer:

To see that these groups embed into one another, we observe that there is a natural embeding $\iota: A_i\to A_{i+1}$ for any choice i, since the even permutations on the an $i+1$ element set include the even permutations that do not act on the $i+1st$ element. Thus we have embeddings $\varphi: G\to H$ and $\psi: H\to G$.



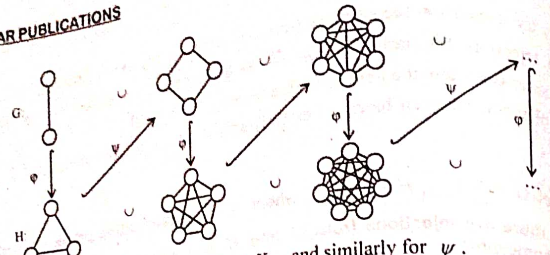
where $\varphi|_{A_k}: A_k\to A_{k+1}$ is the natural embedding and similarly for ψ.

However, to show that they are not isomorphic, let $\chi: H\to G$ be a group homomorphism and let $\rho_5: G\to A_5$ be the projection homomorphism. Recall that each A_i is simple for $i\ge 5$ and each $A_{2n+4}, n\ge 1$ contains strictly greater than $|A_5|$ elements. Because A_{2n+4} is simple, $\rho_5\circ\chi|_{A_{2n+4}}$ must be isomorphic to either A_{2n+4} or $\{e\}$ and so $\rho_5\circ\chi|_{A_{2n+4}}$ must be the trivial map. Thus χ is not onto, and since this is true for any χ, G and H are not isomorphic groups.

12. Let $G:=\bigcup_{i=1} K_{2i}$, $H:=\bigcup_{j=1} K_{2j+1}$, the disjoint union of non-trivial even and odd degree complete graphs, respectively. Then G, H embed into one another but are non-isomorphic graphs. **[MODEL QUESTION]**

Answer:

We first show the existence of embeddings $G\to H$ and $H\to G$. Notice that given any positive integer n, there is a natural embedding $\iota: K_n\to K_{n+1}$, by mapping the n vertices of K_n to any n vertices of K_{n+1}. Thus we may define φ, ψ:



where $\varphi|K_n$ is the natural embedding $K_n \to K_{n+1}$ and similarly for ψ.
However, every vertex of G has odd degree and every vertex of H has even degree.
Thus G and H cannot be isomorphic.

Long Answer Type Questions

1. a) Prove that the set of all even integers form a commutative ring. **[MODEL QUESTION]**

Answer:
We know $2\mathbb{Z}$ is an additive commutative group.
Next, let $m, n \in 2\mathbb{Z}$. Then $m = 2p$, $n = 2q$ for some $p, q \in \mathbb{Z}$.

$\therefore \quad mn = (2p)(2q) = 2(2pq) \in 2\mathbb{Z}$

Associatively follows hereditary from $\mathbb{Z}$.
The two distributive properties also follow hereditary.
Hence $2\mathbb{Z}$ is a ring.
Farther it is commutative since $m \cdot n = nm \forall m, n \in 2\mathbb{Z}$.

b) Prove that the intersection of two subrings is a subring. **[MODEL QUESTION]**

Answer:
Let $(G, *, \cdot)$ be a ring and $(H, *, \cdot)$, $(K, *, \cdot)$ be two subgrings of $(G, *, \cdot)$.
Since $(H, *, \cdot)$ and $(K, *, \cdot)$ are two rings, therefore $(H, *)$ and $(K, *)$ are two commutative groups and $(H, \cdot)$ and $(K, \cdot)$ are semigroups.
Again, for any $a, b, c \in H$

i) $a.(b * c) = (a.b) * (a.c)$

ii) $(b * c).a = (b.a) * (c.a)$

and same properties holds for k also.
Now, let $h, k \in H \cap K$.
Since $(H, *)$ is a group, therefore $h * K^{-1} \in H$.
Again $(G, *)$ is a group, thus $h * K^{-1} \in G$.

Hence, $h * K^{-1} \in H \cap K$.
Therefore $(H \cap K, *)$ is a group.
Also $h * K = K * h$ for H and K.
Therefore $h * K = K * h$ for $H \cap K$.
Thus $(H \cap K, *)$ is a commutative group.
With the similar arguments we can say that $(H \cap K, \cdot)$ forms a semigroup.
Again, condition (i) and (ii) holds for H and K both.
Therefore for any three elements $a, b, c \in H \cap K$, these condition must be satisfied.
Hence $(H \cap K, *, \cdot)$ is a semigroup of $(G, *, \cdot)$.

2. a) Let f be a ring homomorphism from the ring Z of integers into itself such that $f(1) = 1$. Determine the homomorphism f. **[MODEL QUESTION]**

Answer:
Here we observe that $f(0) = f(0+0) = f(0) + f(0) \quad \therefore f(0) = 0$

Also, $\quad f(2) = f(1+1) = f(1) + f(1) = 1 + 1 = 2$

$\qquad 0 = f(0) = f(1-1) = f(1) + f(-1) \quad \therefore f(-1) = f(1)$

Similarly, $f(n) = n$ for $n \in \mathbb{Z}$.
Hence f is the identity homomorphism.

b) Let R and S be two rings and $f: R \to S$ be a ring homomorphism. Show that kernel of f is a subring of R. **[MODEL QUESTION]**

Answer:
Here $\ker f = \{x \in R;\ f(x) = 0\}$ where $f: R \to S$ is a ring homomorphism.
Let $a, b \in \ker f$. Then $f(a) = 0$, $f(b) = 0$.

Now $\quad f(0) = f(0+0) = f(0) + f(0) \quad \therefore f(0) = 0$

Also, $\quad 0 = f(b-b) = f(b) + f(-b) \quad \therefore f(-b) = -f(b)$

So, $\quad f(a-b) = f(a) + f(-b) = f(a) - f(b) = 0 - 0 = 0$.

Hence, $a - b \in \ker f$

Further $f(a \cdot b) = f(a) \cdot f(b) = 0 \cdot 0 = 0$

So, $\quad ab \in \ker f$

Therefore $\ker f$ is a subring of R.

3. a) Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$ be two permutations. Show that $AB \neq BA$. **[MODEL QUESTION]**

Answer:

Here $A=\begin{pmatrix}1&2&3&4&5\\2&3&1&5&4\end{pmatrix}$, $B=\begin{pmatrix}1&2&3&4&5\\1&3&4&5&2\end{pmatrix}$

$AB=\begin{pmatrix}1&2&3&4&5\\3&4&1&2&5\end{pmatrix}$, $BA=\begin{pmatrix}1&2&3&4&5\\2&1&5&4&3\end{pmatrix}$

Clearly $AB \neq BA$

b) Let $f:(c-\{0\},\cdot)\to(c-\{0\},\cdot)$ be a function defined by $f(z)=z^4$.

(i) Show that f is a homomorphism.

(ii) Find the Kernel of f. **[MODEL QUESTION]**

Answer:

Let $z_1, z_2 \in \mathbb{C}-\{0\}$

Then $f(z_1z_2)=(z_1z_2)^4=z_1^4z_2^4=f(z_1)f(z_2)$

Hence f is a homomorphism.

For kernelf, we have

$f(z)=1$ as 1 is the identity of $(\mathbb{C}-\{0\};\ )$

i.e., $z^4=1=\cos 2k\pi+i\sin 2k\pi,\ k\in\mathbb{N}$

or, $z=\cos\frac{2k\pi}{4}+i\sin\frac{2k\pi}{4},\ k=0,1,2,3$

Hence $\ker f=\left\{1,\ \cos\frac{4\pi}{2}+i\sin\frac{\pi}{2},\ -1,\ \cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right\}=\{1,i,-1,-i\}$

4. a) Prove that cancellation laws hold in a ring R if and only if R has no divisor of zero.

b) Let S, T be two sub-rings of a ring R. Prove that $S\cap T$ is also a sub-ring of R. **[MODEL QUESTION]**

Answer:

a) Let the cancellation laws hold in R.

Let $a\cdot b=0$ where $a\neq 0,\ a,b\in R$

Then $a\cdot b=a\cdot 0$

$b=0$ (canceling a form both sides)

Hence R has no divisor of zero conversely, let R has no divisor of zero.

Conversely, let R has no divisor of zero.

Let $ab=ac$ where, $a,b,c\in\mathbb{R},\ a\neq 0$

or, $ab-ac=0$

or, $a(b-c)=0$

Since R has no divisor of zero, $b-c=0$ $[\because a\neq 0]$

Hence the left cancellation law holds.

The light cancellation law can be proved similarly.

b) Let S and T be two subrings of R.

Since S and T are additively abelian groups $S\cap T$ is also an additively abelian group.

Since S and T are multiplicatively semigroups, $S\cap T$ is also a multiplicative semigroup.

Since two distributive properties hold in S and T and these properties are hereditary, $S\cap T$ has the two distributive properties.

Hence $S\cap T$ is a subring of R.

5. Answer the following questions.

a) Show that the set of matrices $\begin{bmatrix}a&0\\b&0\end{bmatrix}$ is a subring of the ring of 2×2 matrices.

b) Prove that a finite integral domain is a field. **[MODEL QUESTION]**

Answer:

a) Let $\begin{bmatrix}a&0\\b&0\end{bmatrix},\begin{bmatrix}c&0\\d&0\end{bmatrix}\in M_2(\mathbb{R})$ the set of all 2×2 matrices with real entries of the form $\begin{bmatrix}a&0\\b&0\end{bmatrix}$

Clearly, $\begin{bmatrix}a&0\\b&0\end{bmatrix}-\begin{bmatrix}c&0\\d&0\end{bmatrix}=\begin{bmatrix}a-c&0\\b-d&0\end{bmatrix}\in M_2(\mathbb{R})$.

and $\begin{bmatrix}a&0\\b&0\end{bmatrix}\begin{bmatrix}c&0\\d&0\end{bmatrix}=\begin{bmatrix}ac&0\\bc&0\end{bmatrix}\in M_2(R)$

Hence the set of matrices $\begin{bmatrix}a&0\\b&0\end{bmatrix}$, $a,b\in\mathbb{R}$, is a subring of all 2×2 matrices.

b) Let D be a finite integral domain with n elements $a_0, a_1, a_2, \ldots, a_{n-1}$ of which a_0 is the zero element and a_1 is the identity. We first prove that $xa_{k_0}=a_{k}$ for some $a_{k_0}, a_{k_0}\in\{a_1,a_2,\ldots,a_n\}$ has a unique solution. To prove this we claim $a_1a_{k_0}, a_2a_{k_0},\ldots,a_na_{k_0}$ are distant. If not, then say, $a_1a_{k_0}=a_2a_{k_0}$ or $(a_1-a_2)a_{k_0}=0$.

But D does not have zero-divisor and $a_{k_0}\neq 0$, hence $a_1-a_2=0$ or $a_1=a_2$, which is a contradiction.

Hence the products $a_1a_{k_0}, a_2a_{k_0},\ldots,a_na_{k_0}$ are distinct. So one of these must be by closure property equal to a_{k_0}. Thus $xa_{k_0}=a_{k}$ has a solution. The uniqueness follows by a similar argument. Now consider the equation $xa_{k_0}=a_1$. This has a solution, say, a'_{k_0}. We claim this is the inverse of a_k. This is so because of commutativity of D, $a'_{k}a_{k} = a_{k}a'_{k} = a$

Hence the existence inverse is established. This means D is a field.

6. a) If a ring R consists of all integral multiples of 2 and R′ consists of all integral multiples of 3, show that R is not isomorphic to R′.
b) When does a ring become a field? Does multiplication in a field obey cancellation law, examine? What is the field of quotients of the integral domain of integers? **[MODEL QUESTION]**

Answer:

a) Here $R = 2\mathbb{Z}$, $R' = 3\mathbb{Z}$

Let f be a mapping from the ring R of all multiples of 2 of the ring R' of all multiples of 3 defined by $f: R \to R'$ by $f(2a) = 3a \forall a \in R$.

Then for any 2 elements $2a_1$ & $2a_2 \in R$ we have,

$$f(2a_1) = 3a_1,\ a_1 \in R \ \&\ f(2a_2) = 3a_2,\ a_2 \in R.$$

Now if $f[2a_1 + 2a_2] = f(2(a_1 + a_2)) = 3(a_1 + a_2) = 3a_1 + 3a_2 = f(2a_1) + f(2a_2)$

And $f[2a_1 \cdot 2a_2] = 2 \cdot 3a_1a_2 = 6a_1a_2 f(2a_1) \cdot f(2a_2) = 3a_1 \cdot 3a_2 = 9a_1a_2$

Clearly $f(2a_1 \cdot 2a_2) \neq f(2a_1) f(2a_2)$

Here f is not a homomorphism and so R is not isomorphic to R'.

b) A ring becomes a field if it has identity, it is commutative and every non-zero element of it has an inverse.

Yes. It does obey since it has no divisor of zero. The set of rational numbers i.e. $\mathbb{Q}$.

7. a) Define ideal of a ring. Let S be the set of all (2×2) real matrices defined by

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

[MODEL QUESTION]

Show that S is a left ideal but not a right ideal of $M_2(\mathbb{R})$
b) Prove that every finite integral domain is a field.

Answer:

a) S is clearly a subring of $M_2(\mathbb{R})$

Since, if $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \in S$, then $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} - \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} a-c & 0 \\ b-d & 0 \end{pmatrix} \in S$

and $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} ac & 0 \\ bc & 0 \end{pmatrix} \in S$

Further, let $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in S$, $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in M_2(\mathbb{R})$,

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} pa+qb & 0 \\ ar+sb & 0 \end{pmatrix} \in S$$

Hence S is a left ideal.

Since $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap & aq \\ bq & bq \end{pmatrix} \notin S$,

S is not a right ideal.

b) Let D be a finite integral domain. Since every integral domain is a commutative ring with unity, it is enough to prove only that every non-zero element of D has an (multiplicative) inverse in D.

So let a be a non-zero element of D.

Consider the set $S = \{ab; b \in D\}$

Since D is closed with respect to multiplication, $S \subset D$.

Now, if $b \neq c$, then $ab \neq ac$ because otherwise $ab = ac$,

then $a(b-c) = 0$ $\therefore b - c = 0$ as D has no divisor of zero and hence $b = c$. Thus the elements of S are distinct but as D is finite, S will have as many elements as D has viz., S = D. So these exists an element a' such that $aa' = e$ where e is the unity of D. Clearly a' is the inverse of a, as $aa' = a'a = e$ by the commutativity of D. Since a is arbitrary, every non-zero element has an inverse. Hence D is a field.

8. a) Show that $S = \{6x;\ x \in \mathbb{Z}\}$ is an ideal $\mathbb{Z}$. **[MODEL QUESTION]**
b) Prove that the ring of intergers is not a field.
c) Prove that in a field *F* the equations *a. x = b, y.a = b* have unique solution.

Answer:

a) Here $S = \{6x;\ x \in \mathbb{Z}\}$

Clearly, S is a subring of $\mathbb{Z}$ since $6m - 6n = 6(m-n) \in 6\mathbb{Z}$

and $6m \cdot 6n = 6(6mn) \in 6\mathbb{Z}$.

Next, if $p \in \mathbb{Z}$, $s \in S$, then $s = 6m$ for some $m \in \mathbb{Z}$.

Now, $ps = p6m = 6pm \in S$ and $rp = 6mp = 6mp \in S$.

Hence S is an ideal of $\mathbb{Z}$.

b) $(\mathbb{Z}, +, \circ)$ is not a field

Since every non-zero integer does not have an inverse in $\mathbb{Z}$.

c) We have the equation $ax = b$, $a \neq 0$.

Since $a \neq 0$, a has an inverse, say a^{-1}.

So $a^{-1}(ax) = a^{-1}b$

or, $\left(a^{-1}a\right)x = a^{-1}b$

or, $ex = a^{-1}b$

or, $x = a^{-1}b$.

Thus existence of a solution is proved.

For uniqueness, assume that x^* and $\bar{x}$ are two solutions of the above equation.

Then $ax^* = b$ and $a\bar{x} = b$

So $ax^* = a\bar{x}$

or, $a\left(x^* - \bar{x}\right) = 0$

$\therefore$ $x^* - \bar{x} = 0$ as $a \neq 0$ and F has no divisor of zero.

or, $x^* = \bar{x}$.

Thus the solution is unique.

Argument for the other equation $xa = b$ is very much similar.

GRAPH AND TREE

☞ Chapter at a Glance

Both trees and graphs are two well known mostly used data structures in algorithms.

Tree Data Structure

In Computer science, a tree is a widely used Abstract Data Structure (ADT). It can be defined recursively as a collection of nodes, where each node contains a value, starting with root node and list of references to other nodes (children), with the constraint, that no reference from it, is called leaf node.

Graph Data Structure

In Computer science, graph also an abstract data structure (ADT), that is mean to implement undirected graph and directed graph concepts of mathematics especially the field of graph theory. Graph is a mathematical representation of a set of objects and relationships or links between objects. We represent objects as nodes or vertices in graph and relations between vertices as edges or arcs. So, we can define that graph is set of vertices V and set of edges E. These edges may be directed or undirected.

Now let's see what are the differences between graph and tree in tabular form.

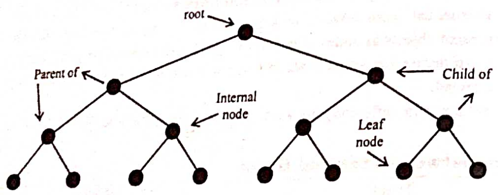
Difference between Tree and Graph

Trees	Graph
1. A tree is a special kind of graph that there are never multiple paths exist. There is always one way to get from A to B.	1. A graph is a system that has multiple ways to get from any point A to any other point B.
2. Tree must be connected.	2. Graph may not be connected.
3. Since it connected we can reach from one particular node to all other nodes. This kind of searching is called traversal.	3. Traversal always not applicable on graphs. Because graphs may not be connected.
4. Tree contains no loops, no circuits.	4. Graph may contain self-loops, loops.
5. There must be a root node in tree.	5. There is no such kind of root node in graphs
6. We do traversal on trees. That mean from a point we go to each and every node of tree.	6. We do searching on graphs. That means starting from any node try to find a particular node which we need.
7. pre-order, in-order, post-order are some kind of traversals in trees.	7. Breath first search, Depth first search, are some kind of searching algorithms in graphs.

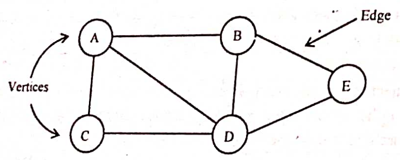
Trees	Graph
	8. Graphs are cyclic or acyclic.
8. Trees are directed acyclic graphs.	9. Graph is network model.
9. Tree is a hierarchical model structure.	10. But all graphs are not trees.
10. All trees are graphs.	11. We differ the graphs like directed graphs and undirected graphs.
11. Based on different properties trees can be classified as Binary tree, Binary search tree, AVL trees, Heaps.	
12. If tree have "n" vertices then it must have exactly "n-1" edges only.	12. In graphs number of edges doesn't depend on the number of vertices.
13. Main use of trees is for sorting and traversing.	13. Main use of graphs is coloring and job scheduling.
14. Less in complexity compared to graphs.	14. High complexity than trees due to loops.

Example

Tree:

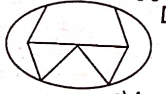


Graph:



Multiple Choice Type Questions

1. What is the chromatic number of the following graph with 7 vertices? **[WBUT 2012(ODD), 2016(EVEN)]**



a) 6 b) 5 c) 4 d) 3

Answer: (d)

2. A graph consisting of simply one circuit, with $n \geq 3$ and n odd, is **[WBUT 2013(EVEN)]**

a) 2-chromatic b) 3-chromatic
c) 4-chromatic d) none of these

Answer: (b)

3. A complete bipartite graph $K_{m,n}$ is a tree when **[WBUT 2013(EVEN)]**

a) m=1, n=2 b) m=2, n=2 c) m=2, n=3 d) none of these

Answer: (a)

4. Let G be a connected simple graph with 8 vertices such that no vertex in G has degree <4. Let n be the number of edges in G. Then which one of the following statements must be FALSE? **[WBUT 2013(ODD)]**

a) $n=15$ b) $n=16$ c) $n=17$ d) $n=19$

Answer: (b)

5. The total number of ways in which a null graph with 5 vertices can be properly coloured is **[WBUT 2013(ODD), 2015(EVEN)]**

a) 225 b) 243 c) 125 d) none of these

Answer: (d)

6. A tree has 21 vertices, then $\chi(G)$ is **[WBUT 2014(ODD), 2017(ODD)]**

a) 20 b) 10 c) 40 d) none of these

Answer: (d)

7. The chromatic polynomial of Peterson graph is **[WBUT 2014(ODD), 2017(ODD)]**

a) 3 b) 4 c) 5 d) none of these

Answer: (d)

8. The number of ways that a tree with 5 vertices can be coloured by 4 colours is **[WBUT 2014(ODD), 2017(ODD)]**

a) 324 b) 350 c) 20 d) none of these

Answer: (a)

9. Which of the following statement is false **[WBUT 2015(EVEN)]**

a) K_n is always n-partite b) K_n is always m-partite for some $m < n$
c) K_n is n-regular d) K_n is not bipartite for any n

Answer: (c)

10. The graph whose chromatic polynomial is $\lambda(\lambda-1)^5$ is [WBUT 2015(EVEN)]
a) a tree having 6 vertices
b) K_6
c) K_5
d) a tree having 5 vertices
Answer: (a)

11. The number of ways a null graph having 4 vertices can be properly coloured with 5 colour is [WBUT 2015(ODD)]
a) 256
b) 1024
c) 625
d) 125
Answer: (c)

12. A two chromatic graph is [WBUT 2015(ODD)]
a) a tree
b) bi partite graph
c) a cycle with odd number of vertices
d) none of these
Answer: (d)

13. If for a graph G, $\chi(G)=10$ and $P(G, \lambda)$ represents chromatic polynomial of G the $P(G, \lambda)=0$ for [WBUT 2016(ODD)]
a) $\lambda<10$
b) $\lambda>10$
c) $\lambda=10$
d) none of these
Answer: (b)

14. The degrees of the vertices of graph G are 5, 3, 8 and 6 respectively. Then which one f the following statements cannot be true? [WBUT 2017(EVEN)]
a) $x(G)=3$
b) $x(G)\le 7$
c) $x(G)\ge 6$
d) $x(G)=10$
Answer: (d)

15. The chromatic number of a graph containing a circuit of length 11 is [WBUT 2017(EVEN), 2019(EVEN)]
a) 1
b) 2
c) 3
d) none of these
Answer: (b)

16. The total number of ways in which a null graph with 5 vertices can be properly coloured with 3 colours is [WBUT 2017(ODD)]
a) 125
b) 243
c) 225
d) none of these
Answer: (b)

17. Euler formula for graphs is [WBUT 2018(ODD)]
a) $f=e-v$
b) $f=e+v+2$
c) $f=e-v-2$
d) $f=e-v+2$
Answer: (d)

18. A complete graph is planar if [WBUT 2018(ODD)]
a) $n=4$
b) $n>4$
c) $n\le 4$
d) for all integer values of n

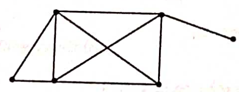
Answer: (a)

19. The number of odd vertices of a simple, connected graph is [WBUT 2019(EVEN)]
a) 5
b) 6
c) even
d) odd
Answer: (c)

20. The maximum number of vertices in a connected graph having 17 edges is [MODEL QUESTION]
a) 18
b) 17
c) 19
d) 12
Answer: (a)

21. In a bipartite graph we cannot find a triangle [MODEL QUESTION]
a) True
b) False
Answer: (a)

22. For the following graph, the value of Σ deg (A) = [MODEL QUESTION]



a) 16
b) 17
c) 18
d) 19
Answer: (b)

23. A simple graph has [MODEL QUESTION]
a) no self loop
b) no parallel edge
c) both (a) and (b)
d) none of these
Answer: (c)

24. Does there exist a simple graph with 5 vertices of the given degrees? 1, 2, 3, 4, 5. [MODEL QUESTION]
a) No
b) Yes
c) Sometime it exists
Answer: (a)

25. Maximum number of edges with n vertices in a completely connected graph is [MODEL QUESTION]
a) $(n-1)$
b) $\frac{n}{2}$
c) $\frac{(n-1)}{2}$
d) $\frac{n(n-1)}{2}$
Answer: (d)

26. Number of elements contained in an incidence matrix of a digraph is [MODEL QUESTION]

a) 1 b) 2 c) 3 d) none of these

Answer: (b)

27. A pseudo graph [MODEL QUESTION]

a) must has loops b) does not have loop
c) must have parallel edges d) none of these

Answer: (c)

28. A simple graph with n vertices has maximum [MODEL QUESTION]

a) $\frac{n(n-1)}{2}$ edges b) $(n-1)$ edges
c) $\frac{n(n+1)}{2}$ edges d) n^2 edges

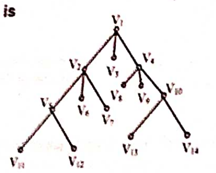
Answer: (a)

29. Chose the correct statement [MODEL QUESTION]

a) Path is an open walk b) Every walk is trail
c) Every trail is a path
d) A vertex cannot appear twice in a walk

Answer: (a)

30. The height of the tree is [MODEL QUESTION]



a) 2 b) 3 c) 0 d) 4

Answer: (b)

31. If a binary tree has 20 pendant vertices, then the number of internal vertices of the tree is [MODEL QUESTION]

a) 20 b) 21 c) 23 d) 19

Answer: (d)

32. The degree of the origin of the longest path in a tree is [MODEL QUESTION]

a) 1 b) 2 c) 3 d) none of these

Answer: (a)

33. A binary tree has exactly [MODEL QUESTION]

a) one root b) two roots
c) three roots d) none of these

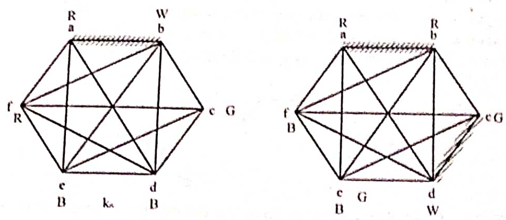
Answer: (a)

Short Answer Type Questions

1. Consider K_6, the complete graph on the six vertices a, b, c, d, e, f. The graph G_1 is obtained from K_6 by deleting the edge ab. The graph G_2 is obtained from G_1 by deleting the edge cd. What are the chromatic numbers of G_1 and G_2?
[WBUT 2012(ODD), 2016(EVEN), 2016(ODD)]

Answer:

Let us draw K_6, G_1 and G_2



The chromatic number of G_1 is 4.

The chromatic number of G_2 is 4.

2. A new flag is to be designed with 6 vertical stripes using 4 colours. In how many ways can this be done so that no 2 adjacent stripes have the same colour?
[WBUT 2012(ODD), 2016(ODD)]

Answer:

______A	C	A
______B	D	B
______A	A	C
______B	B	D
______C	A	A
______D	B	B
$2\times2=4$	$2\times2=4$	$2\times2\times2$

So the required no. of ways $=4+4+8=16$.

3. Show that every bipartite graph is 2-chromatic.
[WBUT 2012(ODD), 2014(ODD), 2016(EVEN), 2017(ODD)]

Answer:

Let G be a bipartite graph.

So, $V(G) = V_1 \cup V_2$ where $V_1 \cap V_2 = \phi$

Colour every vertex in V_1 by one colour and every vertex in V_2 by another colour. Then clearly this colouring is proper.

Thus only two colours suffice to colour G.

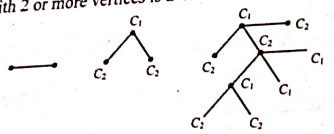
So G is 2-chromatic.

4. Prove that every tree with 2 or more vertices is 2-chromatic.
[WBUT 2013(EVEN), 2013(ODD), 2016(EVEN), 2018(ODD)]

Answer:

Since a tree is connected and has no cycle, if one starts colouring any one vertex with a specific colour and keeps on colouring every alternate vertices with another colour, the entire graph can be coloured with only two colours and since there is one and only one path between any two vertices in a tree, no two vertices will have the same colour.

Hence every free with 2 or more vertices is 2-chromatic. (Proved)



The same can be seen inductively also. Assuming that a graph with m vertices is 2 colourable addition of a vertex requires only the colour not given to the adjacent vertex. So the status remains unchanged.

5. Prove that the chromatic polynomial for the complete graph K_n having n vertices is $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots\ldots\ldots\ldots(\lambda-n+1)$ and conversely. [WBUT 2013(EVEN)]

Answer:

Let $v_1, v_2, \ldots\ldots, v_n$ denote the n vertices of a complete graph K_n.

Clearly if x is the number of colours, v_1 can be coloured in x ways, v_2 in $x-1$ ways, ... and v_n in $x-n+1$ ways.

Hence the fundamental principle of counting asserts that K_n can be coloured in $x(x-1)(x-2)\cdots\cdots(x-n+1)$ ways with x or fewer colours.

Hence the chromatic polynomial for K_n is $x(x-1)(x-2)\cdots\cdots(x-n+1)$.

6. If G is a graph with n vertices and δ denotes the minimum of the degrees of the vertices in G then show that its chromatic number

$$X(G) \geq \frac{n}{n-\delta}$$

[WBUT 2018(EVEN)]

Answer:

G is a graph with n vertices and δ denotes the min. degree of the vertices of G.

Let $n = 1$ then, $X(G) = 1$ (Chromatic no.)

$$\frac{n}{n-\delta} = \frac{1}{1-0} = 1$$

i.e., $X(G) \geq \frac{n}{n-\delta}$ **(Proved)**

Then, $n = 2$, $X(G) = 2$ (No two vertices can be coloured same)

So, $\frac{n}{n-\delta} = \frac{2}{2-0} = 1$

also, $X(G) \geq \frac{n}{n-\delta}$ **(Proved)**

Now, by induction, let it is true $n = m$ then for $n = m+1$

$\therefore$ $X(G) = \frac{m}{2} + 1$ (If m is even)

So, $\frac{n}{n-\delta} = \frac{m+1}{m+1-\frac{m}{2}-1} = \frac{m+1}{\frac{m}{2}} = \frac{2(m+1)}{m}$

as, it is true for $n = m$

So, $X(G) = \frac{m}{2} \geq \frac{m}{m-\frac{m}{2}}$

We can say $\frac{m}{2} + 1 \geq \frac{2}{m}(m+1)$

$\therefore$ $\boxed{X(G) \geq \frac{n}{n-\delta}}$ for $n = m+1$ (By induction) **(Proved)**

7. Give the definition of an Euler Graph. [MODEL QUESTION]

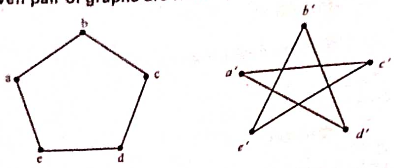
Answer:

Euler Graph:

A circuit of a graph G is called an Eulerian Circuit, if it includes each edge of g exactly once. A graph containing an Eulerian Circuit is called an Eulerian graph. Let G be a graph. A tour of G is a closed walk which includes every edge at least once. A tour of G is called Eulerian if it includes every edge exactly once.

A graph G is called Eulerian if it has a Eulerian tour.
If the graph is Eulerian any tour which crosses each edge only once for the completion of the tour for every edge incident on a vertex, there must one edge for its way out. Thus every vertex must be of even degree.
Conversely, since the graph is connected, every pair of vertices must be connected by a path and each vertex being of even degree it will be possible to travel every edge just once and thus complete the tour. This can be seen by taking simple connected acyclic graphs with 1, 2, 3 or 4 edges.

8. Show that given pair of graphs are isomorphic. **[MODEL QUESTION]**



Answer:
We note first that
i) both graphs have the same no. of vertices.

ii) each vertex of both graphs have the same degree.
Now if we define a mapping ϕ as
$\phi : \wedge \to A_1, B \to C_1, E \to D_1, D \to B_1, C \to E_1$

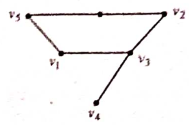
Then evidently every edge of the first graph is carried to the corresponding edge of the second graph.

Hence ϕ is an isomorphism
i.e., the graphs are isomorphic
$f(a) = a,\ f(b) = C_1',\ f(c) = e_1',\ f(d) = b_1',\ f(e) = d'$
Clearly, if bijective and
$f(ea) = a'c', f(bc) = c'e', f(cd) = e'b', f(de) = b'd', f(ea) = d'a'.$

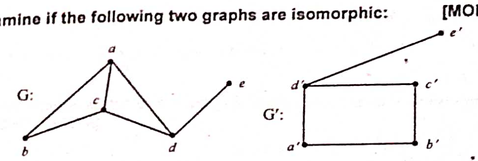
9. Draw the graph represented by the given adjacency matrix: [MODEL QUESTION]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Answer:
Denoting the vertices by v_1, v_2, v_3, v_4 and v_5 we get the graph represented by the given adjacency matrix as



10. Examine if the following two graphs are isomorphic: **[MODEL QUESTION]**

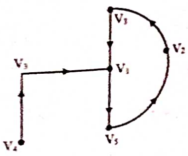


Answer:
The graph G and G' are not isomorphic even though both of the graphs have the same number of vertices. As an isomorphism transforms vertices to vertices their degrees, we observe that the graph G has three ver6tices each of degree 3 viz. a, c, d but the graph G' has only one vertex of degree 3.

11. Draw the directed graph represented by the given adjacency matrix: **[MODEL QUESTION]**

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{vmatrix}$$

Answer:
The graph is



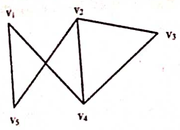
12. Prove that for a graph $G=(V,E)$, were degree of a vertex v is denoted as $\delta(v)$, we can get $\sum_{v\in V}\delta(v)=2|E|$. **[MODEL QUESTION]**

Answer:
The degree of a vertex is the number of edge connecting with the vertex.
As each edges connected with two vertex. So for each edge the there are increase if 1 degree for each of two vertex(if not self loop). For self loop for each edge the there are increase if 2 degree.
So if the graph contains |E| edge then sum of individual degrees of the graph=2|E|
i.e. $\sum_{v\in V}\delta(v)=2|E|$

13. Prove that the number of vertices of odd degree in a graph is always even. **[MODEL QUESTION]**

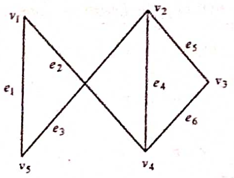
Answer:
Let a graph have n number of vertices $v_1, v_2, \ldots, v_n$ of odd degree and m number of vertices $u_1, u_2, \ldots, u_m$ of even degre. Then by the above theorm, $\Sigma d(v_i) + \Sigma d(u_j)$ is even. This requires that each $\Sigma d(v_i)$ and $\Sigma d(u_j)$ is either odd or even. Since the vertices u_j are of even degree, $\Sigma d(u_j)$ is even. So $\Sigma d(v_i)$ has to be even. This can happen only if the numbr of vertices is even. Hence the proof.

14. Find the incidence matrix of the graph: **[MODEL QUESTION]**



Answer:
Designating the edges as e_1, e_2, e_3, e_4, e_5 and e_6 as shown in the figure of the given graph G, say, the required incidence matrix is given by

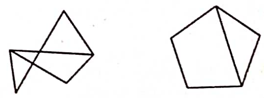
$$I(G)=\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



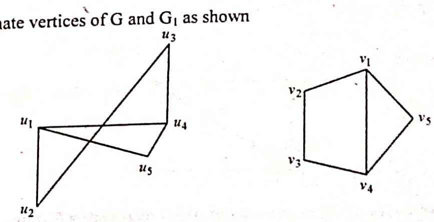
15. Show that there exists no simple graph with five vertices having degrees 4, 4, 4, 2, 2. **[MODEL QUESTION]**

Answer:
A simple graph with 5 vertices can have at most ${}^5C_2$ edges, i.e., 10 edges. So the maximum total degree of the edges will be 20. Here the total degree $=4+4+4+2+2=16$. But for a pentagonal graph, an edge cannot have degree 4 unless there are parallel edges.
Hence the conclusion is that there cannot be a simple graph with specified degrees.

16. Examine whether the graphs *G* and G_1 are isomorphic or not. **[MODEL QUESTION]**



Answer:
Let us designate vertices of G and G_1 as shown



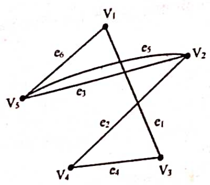
Define a mapping f as
$f(u_i)=v_i$, $i=1,2,3,4,5$
Clearly f is an isomorphism as
$f(u_1u_2)=v_1v_2$, $f(u_2u_3)=v_2v_3$, $f(u_3u_4)=v_3v_4$
and $f(u_1u_5)=v_1v_5$, $f(u_5u_4)=v_5v_4$.

17. Construct the graph or diagraph corresponding to the following incidence matrix: **[MODEL QUESTION]**

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Answer:

The required graph is

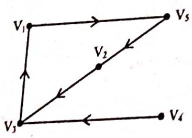


18. Draw the graph represented by the given adjacency matrix: **[MODEL QUESTION]**

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

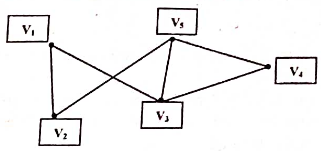
Answer:

The required graph is



Note that the asymmetry of the matrix ensures that it represents a digraph.

19. Find the incidence matrix of the following graph: **[MODEL QUESTION]**



Answer:

The required incidence matrix is

$$\begin{array}{c|cccccc|l} & v_1v_2 & v_1v_3 & v_2v_5 & v_3v_4 & v_3v_5 & v_4v_5 & :\text{Edge} \\ \hline v_1 & 1 & 1 & 0 & 0 & 0 & 0 & \\ v_2 & 1 & 0 & 1 & 0 & 0 & 0 & \\ v_3 & 0 & 1 & 0 & 1 & 1 & 0 & \\ v_4 & 0 & 0 & 0 & 1 & 0 & 1 & \\ v_5 & 0 & 0 & 1 & 0 & 1 & 1 & \\ \text{Vertex} & & & & & & & \end{array}$$

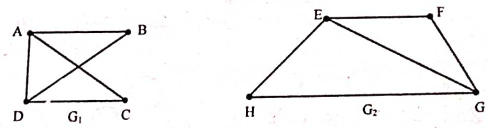
20. Prove that there exists no simple graph with five vertices having degrees 4, 4, 4, 2, 2. **[MODEL QUESTION]**

Answer:

As the graph has 5 vertices and 3 of the vertices have degree 4, each of these three vertices must be joined to all of the remaining vertices but this ensures the degree of the remaining vertices to be at least three. But this contradict the fact that degree of each of the remaining vertices is 2.

Hence no such graph is possible.

21. Find whether the following two graphs are isomorphic or not: **[MODEL QUESTION]**



Answer:

Define φ: $G_1 \rightarrow G_2$ as follows:

$\varphi(A) = E$, $\varphi(D) = G$, $\varphi(B) = F$, $\varphi(C) = H$,

Clearly Q is an isomorphism.

Hence G_1 and G_2 are isomorphic.

22. The minimum number of edges in a connected graph with n vertices is $n-1$. **[MODEL QUESTION]**

Answer:
A minimally connected graph is a tree so we can prove that a tree with n vertices has n-1 edges as follows:
Let T be a tree with n vertices. Let $e \in T$ be an edge whose end vertices are u and v. There is no other path between u and v except e. Therefore, deletion of e produces two trees say T_1 and T_2 (see figure below).

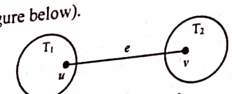
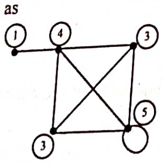


Fig: A tree with n vertices

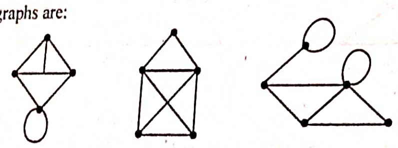
Let the number of vertices of T_1 and T_2 be n_1 and n_2, where $n_1+n_2=n$, as T has n vertices. Obviously, n_1 and n_2 both are less than n. Hence by induction hypothesis, the number of edges of T_1 and T_2 are respectively, n_1-1 and n_2-1.
Thus, $T-e$ contains $(n_1-1)+(n_2-1)=n-2$ edges and hence T has exactly $n-1$ edges.
From this theorem one can state the following result.
The minimum number of edges of a connected graph with n vertices is $n-1$.

23. Draw the three distinct connected graphs which are not isomorphic from the degree sequence {1, 3, 3, 4, 5}. **[MODEL QUESTION]**

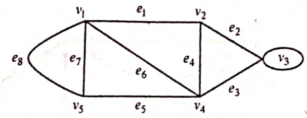
Answer:
The given graph can be drawn as



So, the three graphs are:



24. Construct incidence matrix from the following graph: **[MODEL QUESTION]**



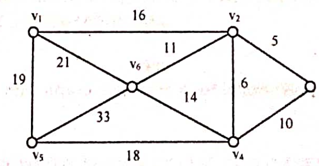
Answer:
The required incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

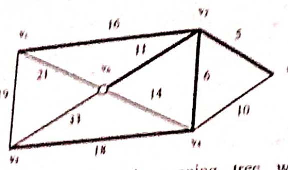
25. Show that a tree of n vertices has $n-1$ edges. **[MODEL QUESTION]**

Answer:
Let T be the tree. The result will be provided by method of introduction on n. Clearly the result is true for $n = 1, 2$.
We assume the result is true for k number of vertices where k<n. In T let a be an edge with end vertices A and B. Since two vertices in a tree are connected by only one path so there is no other path between A and B. So T-a, the graph obtained from T by deleting the edge a, becomes a disconnected graph. Now the graph T-a has two components, say T_1 and T_2. Let T_1 and T_2 has n_1 and n_2 number of vertices. So $n= n_1+ n_2$.If the component T_1 contains a circuit then T would have a circuit which is not possible. So T_1 is a tree. Similarly T_2 is a tree also. So by our hypothesis T_1 and T_2 has n_1-1 and n_2-1 number of edges. Thus T-a contains $(n_1-1) + (n_2-1)= n_1+ n_2-2=n-2$ number of edges. Hence T has n-2+1=n-1 number of edges.

26. Find the minimal spanning tree of the following labelled connected graph by Kruskal's Algorithm. **[MODEL QUESTION]**



Answer:
The minimal spanning tree



The red marked sub-graph is the minimal spanning tree with total weight $= 16 + 11 + 5 + 6 - 18 = 20$.

27. If T is a binary tree of n vertices then prove that the number of pendant vertices in T is $\frac{n+1}{2}$. **[MODEL QUESTION]**

Answer:
Let p denote the number of pendant vertices of the binary tree T. Then the number of edges in the tree

$$= \frac{1}{2}(p + 3(n - p - 1) + 2)$$

Since there is only one vertex with 2 edges $n - p - 1$ vertices with 3 edges. But for any binary tree with n vertices the number of edges is n − 1 always. Hence

$$n - 1 = \frac{1}{2}\{p + 3(n - p - 1) + 2\}$$

$$\text{or } p = \frac{n+1}{2}$$

Hence the result.

Long Answer Type Questions

1. a) Prove that the chromatic number of complete graph with n vertices (K_n) is n.
[WBUT 2013(ODD), 2016(ODD), 2017(ODD)]

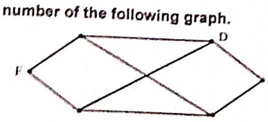
OR,

What is the chromatic polynomial for a complete graph with n vertices?
[WBUT 2016(EVEN)]

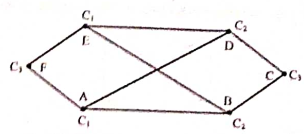
Answer:
Since every complete graph is such that every pair of vertices are joined by an edge, for proper colouring then have to have different colours.
Hence K_n requires n colours for proper colouring.

i.e., $\chi(K_n) = n$.

b) Find the chromatic number of the following graph. **[WBUT 2013(ODD)]**



Answer:
The graph is



Clearly one requires 3 colours as shown in the diagram, since two adjacent vertices have to have different colours.
So the chromatic number of the given graph = 3.

c) Find the chromatic polynomial of a connected graph on three vertices.
[WBUT 2013(ODD)]

Answer:
Here the required chromatic polynomial is

$$C_1\lambda + C_2\frac{\lambda(\lambda-1)}{2!} + C_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$

where $C_1 = C_2 = 0$ and $C_3 = 5!$

i.e., $\frac{5!\lambda(\lambda-1)(\lambda-2)}{3!}$

or, $20\lambda(\lambda-1)(\lambda-2)$; or, $20(\lambda^3 - 3\lambda^2 + 2\lambda)$.

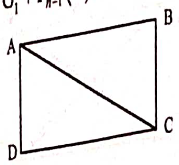
2. a) State decomposition theorem for obtaining chromatic polynomial and find the chromatic polynomial of the following graph using it. **[WBUT 2014(ODD)]**

Answer:
The Decomposition Theorem: Let G be a graph with A and B two non-adjacent vertices. Let G_1 be the graph obtained from G by joining A and B and let G_2 be the

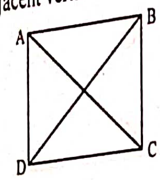
graph obtained from G by fusing the vertices A and B and replacing the parallel edges by single edges, then

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G_1 + P_{n-1}(\lambda) \text{ of } G_2$$

The graph G is



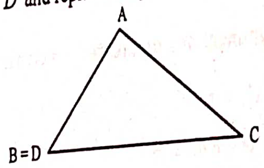
We take B and D as the non-adjacent vertices and by joining B and D we get G_1 as



Clearly G_1 is complete and its chromatic polynomial is given by

$$P_4(\lambda) \text{ of } G_1 = \lambda(\lambda-1)(\lambda-2)(\lambda-3).$$

To get G_2 we fuse B and D and replace the parallel edges by single edges and obtain



Clearly G_2 is complete and $P_3(\lambda)$ of $G_2 = \lambda(\lambda-1)(\lambda-2)$.

Hence the required chromatic polynomial of G is

$$P_4(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda-2)^2.$$

b) Find the closed form of the generating function for numeric function $f_r = \frac{r}{2}(r+1), r > 0$. **[WBUT 2014(ODD)]**

Answer:

Here, $f_r = \frac{r}{2}(r+1)$

The closed form of the generating function for $f_r = \frac{1}{2}\sum_{r=0}^{\infty} r(r+1)x^r$

Now we observe that

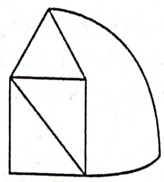
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots \text{to } \infty$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \text{to } \infty$$

$$\frac{+2}{(1-x)^3} = 1\cdot2 + 3\cdot2x + 4\cdot3x^2 + \ldots\ldots \text{to } \infty = \sum_{r=1}^{\infty} r(r+1)x^{r-1}$$

Hence the required closed form is $\frac{2x}{(1-x)^3}$.

3. a) Find the chromatic polynomial of the given graph: **[WBUT 2015(EVEN)]**



Answer:

Let the chromatic polynomial of the graph be

$$P_5(\lambda) = C_1\lambda + C_2\frac{\lambda(\lambda-1)}{2} + C_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!}$$
$$+ C_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + C_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

where C_1, C_2, C_3, C_4, C_5 are constants.

Since the graph has a triangle, so it will require at least three different colors of proper coloring.

Therefore, $C_1 = C_2 = 0$

The given graph has five vertices, thus $C_5 = 5!$.

Moreover, $C_3 = 6$. Because if we choose three vertices v_2, v_3, v_4, then these vertices can be colored by three different colors in $3! = 6$ ways and we have no more choice except v_5 will be the same color as v_2 and v_4 as v_2

Similarly, with 4 colors the vertices v_2, v_3, v_4 can be properly colored in 4.6 = 24 ways.

The forth color can be assigned to v_1 *or* v_5 in 2 ways.

Therefore, $C_4 = 24\cdot2 = 48$

Hence,

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)$$

b) Define Peterson graph and find its chromatic polynomial. [WBUT 2015(EVEN)]

Answer:

In the mathematical field of graph theory, the **Petersen graph** is an undirected graph with 10 vertices and 15 edges. It is a small graph that serves as a useful example and counterexample for many problems in graph theory.

The Petersen graph is the complement of the line graph of K_5. It is also the Kneser graph $KG_{5,2}$; this means that it has one vertex for each 2-element subset of a 5-element set, and two vertices are connected by an edge if and only if the corresponding 2-element subsets are disjoint from each other. As a Kneser graph of the form $KG_{2n-1,n-1}$ it is an example of an odd graph. Geometrically, the Petersen graph is the graph formed by the vertices and edges of the hemi-dodecahedron, that is, adodecahedron with opposite points, lines and faces identified together. The Petersen graph has chromatic number 3, meaning that its vertices can be colored with three colors — but not with two — such that no edge connects vertices of the same color. It has a list colouring with 3 colours, by Brooks' theorem for list colourings. The Petersen graph has chromatic index 4; coloring the edges requires four colors. A proof of this requires checking four cases to demonstrate that no 3-edge-coloring exists. As a connected bridgeless cubic graph with chromatic index four, the Petersen graph is asnark. It is the smallest possible snark, and was the only known snark from 1898 until 1946. The snark theorem, a result conjectured by W. T. Tutte and announced in 2001 by Robertson, Sanders, Seymour, and Thomas, states that every snark has the Petersen graph as a minor.

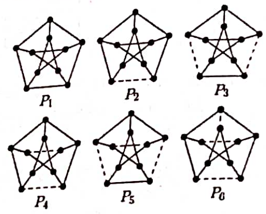


Fig: The six switching-distinct signed Petersen graphs

Additionally, the graph has fractional chromatic index 3, proving that the difference between the chromatic index and fractional chromatic index can be as large as 1. The long-standing Goldberg-Seymour Conjecture proposes that this is the largest gap possible.

The number (a variant of the chromatic index) of the Petersen graph is 5.

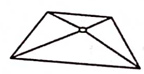
The Petersen graph requires at least three colors in any (possibly improper) coloring that breaks all of its symmetries; that is, its distinguishing number is three. Except for the complete graphs, it is the only Kneser graph whose distinguishing number is not two.

Chromatic polynomials of six signed Petersen graphs:

$$cp_1(2k+1)=1024k^{10}-2560k^9+3840k^8-4480k^7+3712k^6$$
$$-1792k^5+160k^4+480k^3-336k^2+72k,$$

$$cp_2(2k+1)=1024k^{10}-2560k^9+3840k^8-4480k^7+3968k^6$$
$$-2560k^5+1184k^4-352k^3+48k^2,$$
$$cp_3(2k+1)=1024k^{10}-2560k^9+3840k^8-4480k^7+4096k^6$$
$$-2944k^5+1696k^4-760k^3+236k^2-40k,$$
$$cp_4(2k+1)=1024k^{10}-2560k^9+3840k^8-4480k^7+4224k^6$$
$$-3200k^5+1984k^4-952k^3+308k^2-52k,$$
$$cp_5(2k+1)=1024k^{10}-2560k^9+3840k^8-4480k^7+4096k^6$$
$$-3072k^5+1920k^4-960k^3+320k^2-48k,$$
$$cp_6(2k+1)=1024k^{10}-2560k^9+3840k^8-4480k^7+4480k^6$$
$$-3712k^5+2560k^4-1320k^3+460k^2-90k.$$

4. a) Find the chromatic polynomial for the following graph: [WBUT 2015(ODD), 2018(ODD)]



b) Let $G=(V,E)$ with $|V|=n$ be a connected graph. Let the maximum independent set of G be $\beta(G)$ and the chromatic number of G be $\chi(G)$. Prove that $n\le\beta(G)\chi(G)$. Use this result to show that $\beta(G)\ge\frac{n}{4}$ for a planar graph. [WBUT 2015(ODD)]

c) Prove that the graph consisting of simply one circuit with $n\ge3$ vertices is 2-Chromatic. [WBUT 2015(ODD)]

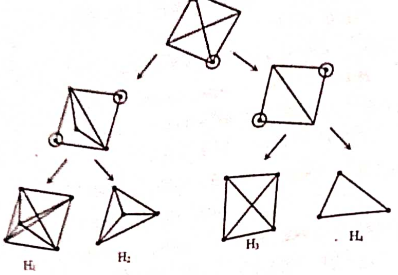
Answer:

a) We use two results here

Result 1: The chromatic polynomial of a complete graph of n vertices is $\lambda(\lambda-1)(\lambda-2)(\lambda-3)\cdots(\lambda-n+1)$

Result 2: If G be a graph with two non-adjacent vertices a and b and if G' denotes the graph obtained by adding an edge between a and b and G'' denotes the simple graph obtained from G by fusing the vertices a and b together and replacing parallel edges by single edges, then

$$P_n^G(\lambda)=P_n^{G'}(\lambda)+P_n^{G''}(\lambda)$$



We how have

The polynomial of H_1 is $\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$

The polynomial of H_2 is $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$

The polynomial of H_3 is $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$

The polynomial of H_4 is $\lambda(\lambda-1)(\lambda-2)$

Hence the required chromatic polynomial of the given graph is

$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)+2\lambda(\lambda-1)(\lambda-2)(\lambda-3)+\lambda(\lambda-1)(\lambda-2)$

or, $\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$

b) 1st part:

Given a k-coloring of G, the vertices being colored with the same color form an independent set. Let, G be a graph with n vertices and c a —coloring of G. We define

$V_i=\{v:c(v)=i\}$

for $i=1,2,\ldots,k$. Each V_i is an independent set. Let $\beta(G)$ be the independence number of

G, we have, $\quad |V_i|\le\beta(G)$.

Since, $\quad n=|V_1|+|V_2|+\ldots+|V_k|\le k.\beta(G)=\chi(G)\beta(G)$,

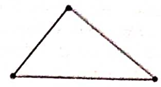
We have, $\quad \chi(G)\ge\dfrac{n}{\beta(G)}$

2nd Part:

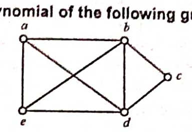
As we know maximum chromatic number of a plannar graph is 4.

So in this case $4\beta(G)\ge n$ or, $\beta(G)\ge\dfrac{n}{4}$.

c) The statement is wrong, as the following is a circuit with 3 vertices which not 2-chromatic.



5. a) Find the chromatic polynomial of the following graph G: [WBUT 2016(ODD)]



Answer:

Let the chromatic polynomial of the graph be

$$P_5(\lambda)=C_1\lambda+C_2\frac{\lambda(\lambda-1)}{2!}+C_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!}+C_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$
$$+C_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

where C_1, C_2, C_3, C_4, C_5 are constants.

Since the graph has a triangle, so it will require at least 3 different colours of proper colouring.

$\therefore\quad C_1=C_2=0$.

The given graph has five vertices, thus $C_5=5!$

Moreover $C_3=3!=6$, because if we choose three vertices b, c, d, then these vertices can be coloured by three different colours in $3!=6$ ways and we have no more choice except the vertex 'a' or 'e' can be coloured by the same colour at 'c'.

Similarly, with 4 colours the vertices b, c, d can be properly coloured in $2\times6+2\times6=24$ ways.

$\therefore\quad C_4=24$.

Hence $P_5(\lambda)=\lambda(\lambda-1)(\lambda-2)+\lambda(\lambda-1)(\lambda-2)(\lambda-3)+\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$

$=\lambda(\lambda-1)(\lambda-2)[1+(\lambda-3)+(\lambda-3)(\lambda-4)]=\lambda(\lambda-1)(\lambda-2)[\lambda^2-6\lambda-10]$

b) Let C_n be a cycle with n vertices. If C_n is a subgraph of a graph G and n is odd, then show that chromatic number of G i.e., $\chi(G)\ge3$. [WBUT 2016(ODD)]

Answer:

We may assume that G is connected. We already know that if G contains an odd cycle then it cannot be coloured by 2 colours. Suppose G contains no odd cycles. Fix any vertex v of G and give it colour 1. Let v_1, v_2, v_3, ..., v_n be the vertices of G which are adjacent to v, colour each of them with colour 2. Next, colour all vertices adjacent to v_i's with colour 1. Continue in this manner, alternatively assigning the colours 1 and 2. Since G is connected (and finite), every vertex of G will get coloured eventually. (In this way, all vertices whose distance from v is even will get colour 1 and those at odd distance from v will get colour 2.) In this colouring, no two adjacent vertices get the same colour. We must ensure, however, that this colouring is done ambiguously, that is, no vertex is coloured with both colours. If this happens for some vertex u, then are two walks from v to u, one of even length and other of odd length. Putting these two walks together and we get a closed walk and thence a circuit of odd length. This circuit may not itself be a cycle. But it can be expressed as a union of disjoint cycles and one of these cycles must be odd, a contradiction. So G is colourable with two colours. Thus we get a lower bound of $\chi(G)$

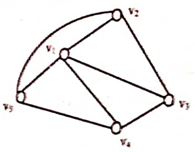
i.e. $\chi(G)\geq 3$.

c) Show that every complete graph is perfect. **[WBUT 2016(ODD)]**

Answer:

Let $G=(A, B, E)$ be a bipartite graph every vertex of which has degree d. Consider an arbitrary subset $S\subseteq A$ and its neighbourhood $N(S)$. Then $|S|d\leq|N(S)|d$, because $|S|d$ is the number of edges between S and $N(S)$ (i.e., the number of edges incident to the vertices of S), which $|N(S)|d$ counts all edges between S and $N(S)$ and possibly some other edges incident to the vertices of $N(S)$. (if such edges exist). Therefore, (1) holds for G and hence there is a matching covering A. This matching is perfect, because $|A|d=|B|d$ (which is the total number of edges counted twice from each side of the graph) implying $|A|=|B|$.

6. a) Find the chromatic polynomial of the following graph: **[WBUT 2017(EVEN)]**



Answer:

Since the given graph G has 5 vertices, its chromatic polynomial $f(x)$ is given by

$f(x)=c_1\,{}^xC_1+c_2\,{}^xC_2+c_3\,{}^xC_3+c_4\,{}^xC_4+c_5\,{}^xC_5$

Since G cannot be coloured with one colour, $c_1=0$. Again, since G cannot be coloured with 2 colures, $c_2=0$.

To find c_3 we observe that 3 colours can be assigned to the vertices v_1,v_2,v_3 in 3!. i.e., 6 different ways and then v_4 can be assigned the same colours as of v_2 and v_5 can be assigned the same colours as of v_3.

Thus we are left with no other choice. Hence $c_3=6$.

Next we observe that starting with 4 colours, we choose 3 colures in ${}^4C_3$ ways, i.e., 4 ways and colour the vertices v_1,v_2,v_3 in $4\times3\times2$ ways and the remaining colures can be assigned either to v_4 or to v_5 in 2 ways so $c_4=24\times2=48$

To final c_5, we see that 5 colours can be assigned to 5 vertices in 5!, i.e., 120 ways.

Hence $c_5=120$

Therefore the required chromatic polynomial is

$$f(x)=0\,{}^xC_1+0\,{}^xC_2+6\,{}^xC_3+48\,{}^xC_4+120\,{}^xC_5$$

$$=\frac{6x(x-1)(x-2)}{3!}+48\frac{x(x-1)(x-2)(x-3)(x-4)}{4!}$$

$$+120\frac{x(x-1)(x-2)(x-3)(x-4)(x-5)}{5!}$$

$$=x(x-1)(x-2)+2x(x-1)(x-2)(x-3)(x-4)+x(x-1)(x-2)(x-3)(x-4)(x-5)$$

Since $f(1)=f(2)=0$ but $f(3)=6\neq0$, $\chi(G)=3$

b) What is chromatic polynomial of a graph? Find the chromatic polynomial of a tree having n vertices. **[WBUT 2017(EVEN)]**

Answer:

1st Part:

Chromatic Polynomial: One may be interested in knowing the number of ways a graph of n vertices can be properly coloured by λ-colours. The answer to this question has been given by what is known as chromatic polynomial.

Definition: The chromatic polynomial of a connected graph of n vertices denoted by $P_n(\lambda)$, is a polynomial in λ which gives the number of ways the graph can be properly coloured with at most λ colours.

In fact, this is given by $P_n(\lambda)=\sum_{j=1}^{n}a_j\,{}^{\lambda}C_j$ where a_j denotes the no. of ways of colouring G properly with exactly j colours chosen from λ colours. Such j colours can be chosen from λ colours in ${}^{\lambda}C_j$ ways. Then $a_j\,{}^{\lambda}C_j$ gives the number of ways of colouring G properly with exactly j colours.

Hence $P_n(\lambda)=\sum_{j=1}^{n} a_j\ {}^{\lambda}C_j$

$$=a_1\frac{\lambda}{1!}+a_2\frac{\lambda(\lambda-1)}{2!}+a_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!}+....+a_n\frac{\lambda(\lambda-1)....(\lambda-n+1)}{n!}$$

Note that for determining the chromatic polynomial of a connected graph, the coefficients $a_1,a_2,....,a_n$ need to be calculated from the graph.

For evaluation of $a_1,a_2,.....,a_n$, the following observations will be helpful.

(1) As for a non-null graph i.e., a graph with at least one edge, two colours are required which means $a_1=0$

(2) A complete graph with n vertices requires n colours for proper colouring and this can be done in $n!$ ways i.e., $a_n=n!$

2nd Part:

Let G be a tree with n vertices. We shall prove that the chromatic polynomial $f(x)$ of G is $x(x-1)^{n-1}$.

Clearly the result is true for $n=1$, i.e. with x colours any one vertex graph can be colored in x ways.

If $n=2v$, G is a tree with two vertices joined by an edge. These also the result is true because the no. of ways they can be colours is $x(x-1)$

Let us assume that the result is true for a tree with m vertices.

Now let us take a tree with $m+1$ vertices. We choose a pendant vertex v say. The graph $G-\{v\}$ is a tree with m vertices.

We see that the pendant vertex can be colures in $x-1$ ways as the vertex adjacent to v cannot be given the same colures as of v.

Thus the tree G with $m+1$ vertices can be coloured in $x(x-1)^{m-1}(x-1)$ i.e. $x(x-1)^{(m+1)-1}$ ways.

Thus by induction we see the result is true for n vertices when $n\in N$.

c) Show that a cycle with $n(\geq 3)$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd. **[WBUT 2017(EVEN)]**

Answer:

We note that a cycle with 3 vertices is a triangle. So 3 colours are required to colour it properly. Therefore $\chi(C_3)=3$

A cycle with 4 vertices is a square. So 2 colours are required to colour it, because vertices can be colures alternately by 2 colures. Thus $\chi(C_4)=2$

For a cycle with 5 vertices, we see the first four vertices can be colours by 2 colures and the 5th vertex is to get different colours.

So $\chi(C_5)=3$

The same augment may be applied to any no. of vertices.

Thus we conclude $\chi(C_n)=\begin{cases}2 \text{ if } n \text{ is even}\\ 3 \text{ if } n \text{ is odd}\end{cases}$

7. a) A connected planar graph with n vertices and e edges has $e-n+2$ regions. **[WBUT 2017(ODD)]**

Answer:

Proof: by induction on the number of edges in the graph. Base: If e = 0, the graph consists of a single vertex with a single region surrounding it. So we have 1 − 0 + 1 = 2 which is clearly right. Induction: Suppose the formula works for all graphs with no more than n edges. Let G be a graph with n + 1 edges.

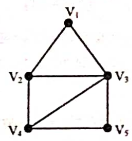
Case 1: G doesn't contain a cycle. So G is a tree and we already know the formula works for trees.

Case 2: G contains at least one cycle. Pick an edge p that's on a cycle. Remove p to create a new graph G′. Since the cycle separates the plane into two regions, the regions to either side of p must be distinct. When we remove the edge p, we merge these two regions. So G′ has one fewer regions than G. Since G′ has n edges, the formula works for G′ by the induction hypothesis. That is v′ − e′ + f′ = 2. But v′ = v, e′ = e − 1, and f′ = f − 1. Substituting, we find that v − (e − 1) + (f − 1) = 2

So

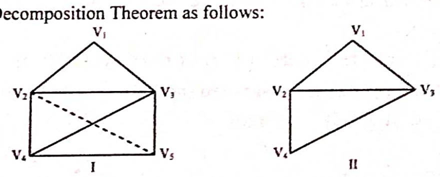
v − e + f = 2

b) Find the chromatic polynomial of the graph given below: **[WBUT 2017(ODD)]**

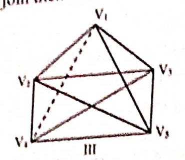


Answer:

We apply the Decomposition Theorem as follows:

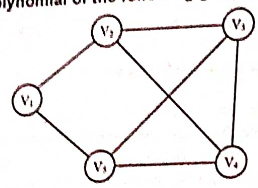


Choosing V_2 and V_5 and join them. Then fuse V_2 and V_5.
We now choose V_1 and V_5 and join then in I and then fuse them



III

8. a) Show that the chromatic polynomial of a cyclic graph of order n is $(\lambda-1)^n+(-1)^n(\lambda-1)$.
b) Find the chromatic polynomial of the following graph: **[WBUT 2018(EVEN)]**



Answer:

a) Chromatic polynomial of a cyclic graph of order n, $(\lambda-1)^n+(-1)^n(\lambda-1)$

$\Rightarrow$ Applying the method of induction,

for $n=2$ its, $(\lambda-1)^2+(-1)^2(\lambda-1)=(\lambda-1)^2+(\lambda-1)=\lambda^2+1-\lambda-1$
$=\lambda^2-\lambda=\lambda(\lambda-1)$ **(Proved)**

Now, for $n=3$, it is true as, $C_3=K_3$ for a cyclic graph.

$\therefore$ Deleting an edge from a cycle results in a path which is a true, so, its chromatic polynomial be like $\lambda(\lambda-1)^{n-1}$.

Contracting an edge yields a cycle of length $n-1$ which by the inductive hypothesis has a chromatic polynomial of $(\lambda-1)^{n-1}+(-1)^{n-1}(\lambda-1)$.

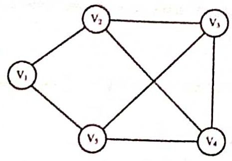
$\therefore$ The difference is —

$$\lambda(\lambda-1)^{n-1}-(\lambda-1)^{n-1}-(-1)^{n-1}(\lambda-1)=(\lambda-1)^n+(-1)^n(\lambda-1) \quad \textbf{(Ans.)}$$

So, by induction, it is proved that chromatic polynomial of a cyclic graph of order n is,

$(\lambda-1)^n+(-1)^n(\lambda-1)$ **(Proved)**

b)



$$P_5(\lambda)=C_1\lambda+C_2\frac{\lambda(\lambda-1)}{2!}+C_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!}+C_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$
$$+C_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

where, C_1, C_2, C_3, C_4, C_5 are constants.

Here, $C_5=5!$, $C_1=0$, $C_2=2!$, $C_3=3!$, $C_4=24$.

$$P_5(\lambda)=\frac{2\lambda(\lambda-1)}{2!}+\frac{6\lambda(\lambda-1)(\lambda-2)}{3!}+\frac{24\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4\times3\times2}$$
$$+\frac{120\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$P_5(\lambda)=\lambda(\lambda-1)+\lambda(\lambda-1)(\lambda-2)+\lambda(\lambda-1)(\lambda-2)(\lambda-3)$$
$$+\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

9. Answer the question very briefly: **[MODEL QUESTION]**

i) Define adjacency matrix of a graph.

Answer:

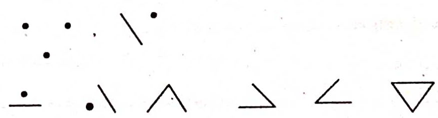
The adjacency matrix A (G) of a graph G is a square matrix $[a_{ij}]$, such that

$a_{ij}=\{r$ if the i^{th} vertex is joined to the j^{th} vertex by r edges.

$a_{ij}=\{0$ if there is no edge joining i^{th} vertex and j^{th} vertex.

ii) Draw all simple graphs of 3 vertices. **[MODEL QUESTION]**

Answer:



10. Show that a simple graph with n vertices and k components has almost $(n-k)(n-k+1)/2$ edges. **[MODEL QUESTION]**

Answer:

Let G be a graph with k components and let its i^{th} component have n_i vertices where $i = 1, 2, \cdots\cdots, k$.

Then the maximum number of edges of G

$$= \sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2} = \frac{1}{2}\sum_{i=1}^{k}(n_i^2 - n_i) = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum n_i$$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{n}{2}$$

Now, $\left\{\sum_{i=1}^{k}(n_i - 1)\right\}^2 = (n-k)^2$

or, $\sum_{i=1}^{k}(n_i - 1)^2 + 2\sum_{\substack{i,j\\ i<j}}^{k}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$

or, $\sum_{i=1}^{k}(n_i^2 - 2n_i + 1) \le n^2 + k^2 - 2nk \left[\because \sum_{\substack{i,j\\ i>j}}(n_i - 1)(n_j - 1) \ge 0\right]$

or, $\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \le n^2 + k^2 - 2nk$

or, $\sum_{i=1}^{k} n_i^2 \le n^2 + k^2 - 2nk + 2n - k$

Hence the maximum number of edges

$$\le \frac{1}{2}\{n^2 + k^2 - 2nk + 2n - k - 2n\} = \frac{(n-k)(n-k+1)}{2}$$

11. a) A graph has 21 edges, 3 vertices each of degree 4 and rest of the vertices are of degree 3. Find out the total number of vertices. **[MODEL QUESTION]**

Answer:

Let n be the no. of vertices

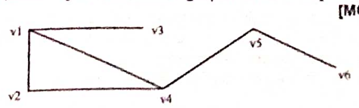
Then we get

$2\times21 = 3\times4 + (n-3)\times3$

or, $42 = 12 + (n-3)\times3$

or, $n = 13$

b) What do you mean by a connected graph? Derive adjacency matrix for the following: **[MODEL QUESTION]**



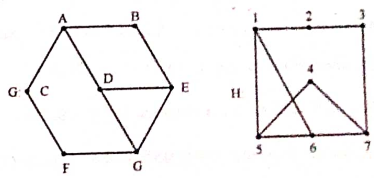
Answer:

Connected graph: A graph is called connected if for every pair of vertices of the graph there exists a path joining them.

The adjacency matrix of the given graph G is

$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array}\begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

12. Examine Graphs H and G are isomorphic or not: **[MODEL QUESTION]**



Answer:

Yes, G is isomorphic to H.

To see this define $\phi: G \to H$ as

$\phi(A) = 1, \phi(B) = 4, \phi(C) = 2, \phi(D) = 5, \phi(E) = 6, \phi(F) = 3, \phi(G) = 7$

Clearly ϕ is bijective and carries edges of G to the corresponding edges of H.

Hence ϕ is an isomorphism of G to H.

13. Write short notes on Hamiltonian Path. **[MODEL QUESTION]**

Answer:

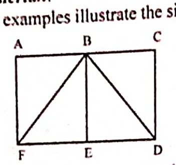
The twin problem which are attributed as the genesis of graph theory apart from the Konigsberg Bridge Problem is the one posed by a famous Irish mathematician Sir

William Rowan Hamilton (1805-1865). As a matter of fact Hamilton made a regular dodecahedron of wood each of whose 20 corners was marked with the name of a city and the problem was to start from any city and find a route along the edges of the dodecahedron that passes through every city exactly once and return to the city of origin. It may be mentioned here that the problem remains still unsolved.

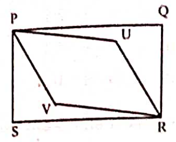
To present the above problem graphically we need the following notions:

Definition:
A cycle in a graph is called ***Hamiltonian*** if it passes every vertex exactly once.
A graph is called Hamiltonian if it admits of a Hamiltonian cycle.
It is obvious that only connected graphs can be Hamiltonian.

Note: also that ***there is as such no relation between Eulerian graphs and Hamiltonian graphs. An Eulerian graph need not be Hamiltonian just as A Hamiltonian graphs may not be Eulerian.***
The following examples illustrate the situation:



Hamiltonian but not Eulerian



Eulerian but not Hamiltonian

A result of importance about Hamiltonian graphs is due to Dirac.

Dirac's Theorem:
A connected graph G with n vertices is Hamiltonian if $n \geq 3$ and $d(v) \geq n$ for each vertex v of G.
Note that the condition stated in the theorem is sufficient only but not necessary.

COMPUTER ARCHITECTURE