Analysis of Insertion Sort

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Outline

- Sorting
- Insertion Sort
- Analysis

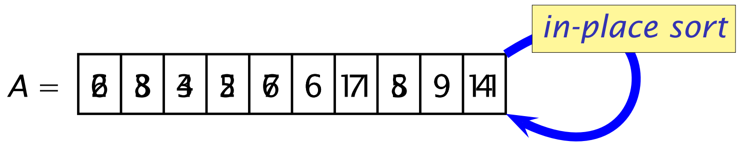
Sorting

- **Input:** a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$

  Output: a sequence $\langle b_1, b_2, \ldots, b_n \rangle$ such that
  - $\langle b_1, b_2, \ldots, b_n \rangle$ is a *permutation* of $\langle a_1, a_2, \ldots, a_n \rangle$
  - $\langle b_1, b_2, \ldots, b_n \rangle$ is *sorted*

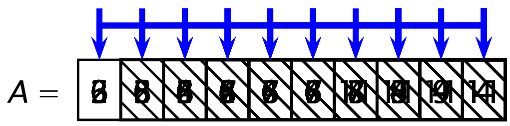
$$b_1 \le b_2 \le \cdots \le b_n$$

- Typically, A is implemented as an array



Insertion Sort

- **Idea:** it is like sorting a hand of cards
  - scan the sequence left to right
  - pick the value at the current position a_j
  - insert it in its correct position in the sequence $\langle a_1, a_2, \ldots a_{j-1} \rangle$ so as to maintain a sorted subsequence $\langle a_1, a_2, \ldots a_j \rangle$



Insertion Sort (2)

```
Insertion-Sort(A)
1  for i = 2 to length(A)
2      j = i
3      while j > 1 and A[j − 1] > A[j]
4          swap A[j] and A[j − 1]
5          j = j − 1
```

- Is Insertion-Sort *correct?*
- What is the time complexity of Insertion-Sort?
- Can we do better?

Complexity of Insertion-Sort

```
Insertion-Sort(A)
1  for i = 2 to length(A)
2      j = i
3      while j > 1 and A[j − 1] > A[j]
4          swap A[j] and A[j − 1]
5          j = j − 1
```

- Outer loop (lines 1–5) runs exactly $n - 1$ times (with $n = length(A)$)
- What about the inner loop (lines 3–5)?
  - best, worst, and average case?

Complexity of Insertion-Sort (2)

```
Insertion-Sort(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
        j = j − 1
```

- **Best case:** the inner loop is *never* executed
  - what case is this?
- **Worst case:** the inner loop is executed exactly $j - 1$ times for every iteration of the outer loop
  - what case is this?

Complexity of Insertion-Sort (3)

- The worst-case complexity is when the inner loop is executed exactly $j - 1$ times, so

$$T(n) = \sum_{j=2}^{n} (j - 1)$$

$T(n)$ is the *arithmetic series* $\sum_{k=1}^{n-1} k$, so

$$T(n) = \frac{n(n-1)}{2}$$

$$\boxed{T(n) = \Theta(n^2)}$$

- Best-case is $T(n) = \Theta(n)$
- Average-case is $T(n) = \Theta(n^2)$

Correctness

- Does Insertion-Sort terminate for all valid inputs?
- If so, does it satisfy the conditions of the sorting problem?
  - A contains a *permutation* of the initial value of A
  - A is *sorted:* $A[1] \leq A[2] \leq \cdots \leq A[\mathit{length}(A)]$
- We want *a formal proof of correctness*
  - does not seem straightforward...

The Logic of Algorithmic Steps

Example:

```
SortTwo(A)
// A must be an array of 2 elements
if A[1] > A[2]
      t = A[1]
      A[1] = A[2]
      A[2] = t
```


Loop Invariants

- We formulate a *loop-invariant* condition *C*
  - *C* must remain true *through* a loop
  - *C* is relevant to the problem definition: we use *C* at the end of a loop to prove the correctness of the result
- Then, we only need to prove that the algorithm terminates

Loop Invariants (2)

- Formulation: this is where we try to be smart
  - *the invariant must reflect the structure of the algorithm*
  - it must be the basis to prove the correctness of the solution
- Proof of validity (i.e., that *C* is indeed a loop invariant): typical *proof by induction*
  - *initialization:* we must prove that
    the invariant C is true before entering the loop
  - *maintenance:* we must prove that
    if *C is true at the beginning of a cycle* ***then*** *it remains true after one cycle*

Loop Invariant for Insertion-Sort

```
Insertion-Sort(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
        j = j − 1
```

- The main idea is to insert $A[i]$ in $A[1 \,..\, i-1]$ so as to maintain a *sorted subsequence* $A[1 \,..\, i]$
- *Invariant:* (outer loop) *the subarray* $A[1 \,..\, i-1]$ *consists of the elements originally in* $A[1 \,..\, i-1]$ *in sorted order*

Loop Invariant for Insertion-Sort (2)

```
Insertion-Sort(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
        j = j − 1
```

- **Initialization:** $j = 2$, so $A[1 \,..\, j-1]$ is the single element $A[1]$
  - $A[1]$ contains the original element in $A[1]$
  - $A[1]$ is trivially sorted

Loop Invariant for Insertion-Sort (3)

```
Insertion-Sort(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
        j = j − 1
```

- **Maintenance:** informally, if $A[1 \,..\, i-1]$ is a permutation of the original $A[1 \,..\, i-1]$ and $A[1 \,..\, i-1]$ is sorted (invariant), then *if* we enter the inner loop:
  - shifts the subarray $A[k \,..\, i-1]$ by one position to the right
  - inserts *key*, which was originally in $A[i]$ at its proper position $1 \le k \le i-1$, in sorted order

Loop Invariant for Insertion-Sort (4)

```
Insertion-Sort(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
        j = j − 1
```

- **Termination:** Insertion-Sort terminates with $i = \mathit{length}(A) + 1$; the invariant states that
  - $A[1 \,..\, i-1]$ is a permutation of the original $A[1 \ldots i-1]$
  - $A[1 \,..\, i-1]$ is sorted

  Given the termination condition, $A[1 \,..\, i-1]$ is the whole A

  So Insertion-Sort is *correct!*

Summary

- You are given a problem P and an algorithm A
  - P formally defines a *correctness* condition
  - assume, for simplicity, that A consists of one loop

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1. Formulate an invariant C
2. **Initialization** (for all valid inputs)
   - prove that C holds right before the first execution of the first instruction of the loop
3. **Management** (for all valid inputs)
   - prove that if C holds right before the first instruction of the loop, then it holds also at the end of the loop
4. **Termination** (for all valid inputs)
   - prove that the loop terminates, with some exit condition X
5. Prove that $X \wedge C \Rightarrow P$, which means that A is correct

Exercise: Analyze Selection-Sort

```
Selection-Sort(A)
n = length(A)
for i = 1 to n − 1
    smallest = i
    for j = i + 1 to n
        if A[j] < A[smallest]
            smallest = j
    swap A[i] and A[smallest]
```

- Correctness?
  - loop invariant?
- Complexity?
  - worst, best, and average case?

Exercise: Analyze Bubblesort

```
Bubblesort(A)
for i = 1 to length(A)
    for j = length(A) downto i + 1
        if A[j] < A[j − 1]
            swap A[j] and A[j − 1]
```

- Correctness?
  - loop invariant?
- Complexity?
  - worst, best, and average case?