Torsion

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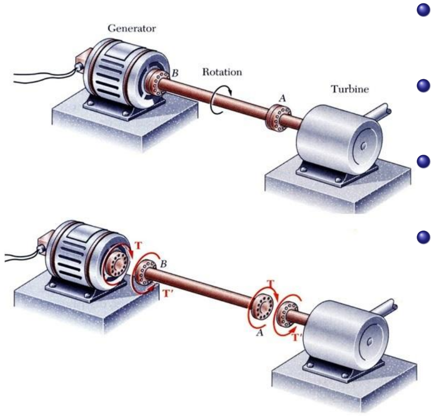
Torsion : Introduction

- When structural members or machine parts of circular (and non-circular) cross-section are subjected to twisting couples or torques they are said to be in *torsion*.
- Torsion is our introduction to the problems of variable stress.
- However our study will be limited to only circular members.
- Examples of torsion:
  1. *Transmission shaft.*
  2. *Springs.*
  3. *Automobile wheels.*

Example



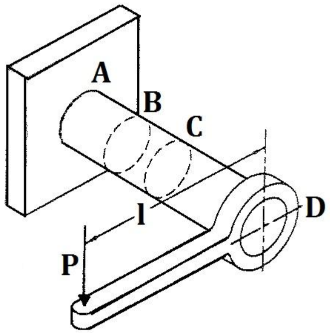
- *Turbine exerts torque T on the shaft.*
- *Shaft transmits the torque to the generator.*
- *Generator creates an equal and opposite torque T'.*
- *To analyze the stresses and strains in a circular member, some assumptions must be made.*

Assumptions

1. The material is homogeneous throughout.
2. A plane section perpendicular to the axis of a circular member remains plane and circular sections remain circular after the application of torque. (No Warping) [1]
3. The shearing strain varies linearly with the distance from the central axis of the circular member.
4. The twist is uniform along the length.
5. The shearing stress is proportional to shearing strain within elastic limit, i.e. Hooke's Law is followed.

[1] For circular shaft only.

Torsion Formula

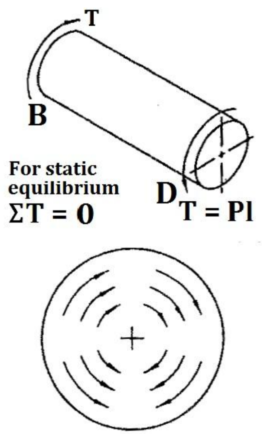


Derivation

- *Consider, a circular shaft AD, fixed at A and is subjected to a torque T, produced due to application of load P at the end of a lever at a distance l.*

$$\therefore T = Pl$$

Torsion Formula



Torque T causing shear stress

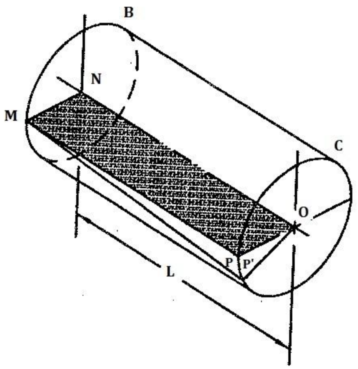
Derivation (contd.)

- *The perpendicular cut section at B shows torque T causing shear stress.*

$$\sum T = 0$$

$$\implies \textit{Applied Torque} = \textit{Resisting Torque}$$

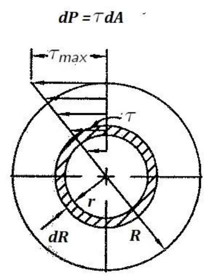
Torsion Formula



Derivation (contd.)

- *Due to the application of torque T deformation occurs in the BC portion of the shaft.*
- *Plane MNOP moves to a new positon MNOP' while remaining plane.*
- *The fiber OP moves to a new position OP', causing uniform twist while remaining straight.*
- *This phenomenon of plane remaining plane and line remaining straight after deformation subsequently indicates that a linear variation of shearing stress or strain from the axis of the circular member is present.*

Torsion Formula



Derivation (contd.)

From figure, shearing stress,

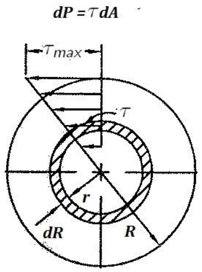
$$\tau = (\frac{r}{c})\tau_{max}$$

where, r = any radius
c = distance from central axis to the outer fiber
The force exerted on elemental area dA,

$$dP = \tau dA$$

$$\implies dP = (\frac{r}{c})\tau_{max} dA$$

Torsion Formula



Derivation (contd.)

$\therefore$ *Torque caused by this force (from the axis of member),*

$$dT = r \times dP = (\frac{r^2}{c})\tau_{max}dA = (\frac{\tau_{max}}{c})r^2 dA$$

$$\implies dT = (\frac{\tau_{max}}{c})r^2 dA$$

Integrating, total moment or torque on the entire cross-section,

$$T = (\frac{\tau_{max}}{c})\int r^2 dA = \frac{\tau_{max}}{c}J$$

Where, $J =$ *Polar moment of inertia* $= \frac{\pi d^4}{32}$

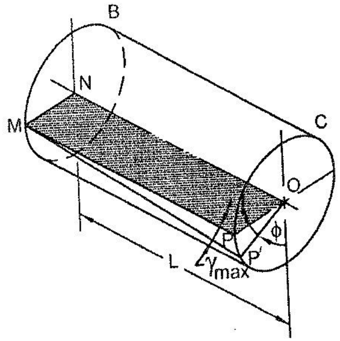
$$\therefore \tau_{max} = \frac{Tc}{J} = \frac{16T}{\pi d^3}$$

Angle of Twist (ϕ or θ)

The angle between OP and OP' is called angle of twist (ϕ). The maximum shearing strain is γ_{max}.
Here, $PP' = c\phi$
Again,
$\tan \gamma_{max} = \gamma_{max} = \frac{PP'}{L}$

$$\implies \gamma_{max} = \frac{c\phi}{L}$$

$$\therefore \phi = \frac{L}{c}\gamma_{max}$$



Angle of Twist (ϕ or θ)

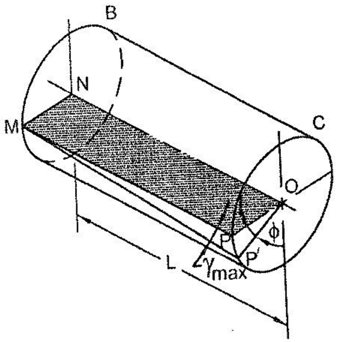
On the surface of the member both shearing stress and shearing strain is maximum. Hence, within proportional limit,

$$G = \frac{\tau_{max}}{\gamma_{max}}$$

$$\implies \gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

$$\implies \frac{c\phi}{L} = \frac{Tc}{JG}$$

$$\therefore \phi = \frac{TL}{JG}$$



Problems

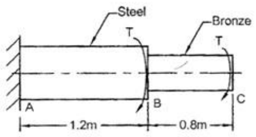


Figure : Problem 1

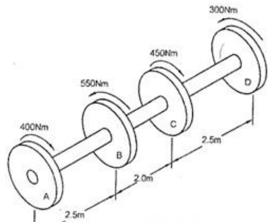


Figure : Problem 2

Problem 1

A compound stepped shaft fabricated from steel and bronze is loaded by two twisting moments as shown in figure. The diameters of the steel and bronze segments are respectively 60 mm and 55 mm. If the allowable shearing stress in steel is 80 MPa, allowable shearing stress in bronze is 40 MPa and angle of twist at the free end is limited to 5 degrees, find the maximum torque T. Modulus of rigidity for steel and bronze are respectively 80 GPa and 35 GPa.

Problem 2

An aluminum shaft of diameter 45 mm is connected with four gears which are subjected to torques as shown in figure. Determine the relative angle of twist of the gear A with respect to the gear D if $G_{Al} = 30$ GPa.

Solution 1

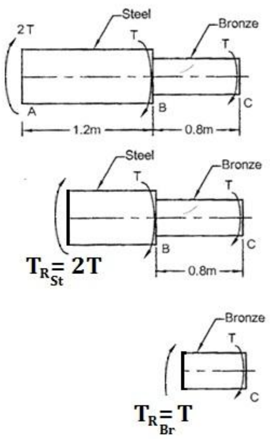


Figure: Free Body Diagrams

Static Equilibrium

If T_R = Resisting Torque, for static equilibrium,

$$\sum T = 0$$

$$\implies T_R - T - T = 0$$

$$\therefore T_R = 2T$$

Shearing Stress in Steel

Resisting force in steel section, $T_{St} = 2T$

Applying Torsion Formula for Steel, $\tau_{St} = \frac{16 T_{St}}{\pi d_{St}^3}$

$$\implies \tau_{allowable} = \frac{16 \times 2T}{\pi \times d_{St}^3} \implies 80 = \frac{16 \times 2T}{\pi \times 60^3}$$

$\therefore T \approx 1696 Nm$

Solution 1

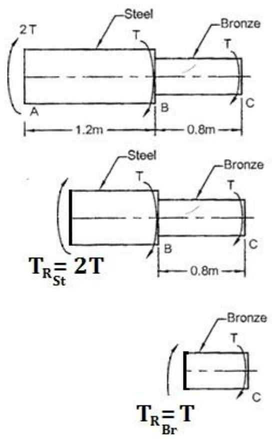


Figure: Free Body Diagrams

Shearing Stress in Bronze

Resisting force in bronze section, $T_{Br} = T$

Applying Torsion Formula for Bronze, $\tau_{Br} = \frac{16 T_{Br}}{\pi d_{Br}^3}$

$$\implies \tau_{allowable} = \frac{16 \times T}{\pi \times d_{Br}^3} \implies 40 = \frac{16 \times T}{\pi \times 55^3}$$

$\therefore T \approx 1307 Nm$

Angle of Twist at free end

$$\phi_{C/A} = \phi_{C/B} + \phi_{B/A} = \overbrace{\frac{TL}{JG}}^{\textbf{CB}} + \overbrace{\frac{TL}{JG}}^{\textbf{BA}}$$

$$\implies \frac{5\pi}{180} = \overbrace{\frac{T \times 0.8 \times 32}{35 \times \pi \times 55^4}}^{\textbf{Bronze}} + \overbrace{\frac{2T \times 1.2 \times 32}{80 \times \pi \times 60^4}}^{\textbf{Steel}}$$

$\therefore T \approx 1780$ Nm

Solution 1

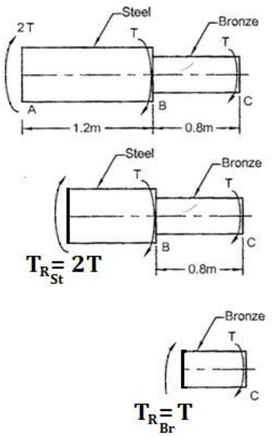


Figure: Free Body Diagrams

Failure

If $T \geq 1696$ Nm then failure will occur in Steel section.
If $T \geq 1307$ Nm then failure will occur in Bronze section.
If $T \geq 1780$ Nm is exceeded then failure will occur in the composite shaft structure due to exceeding strain criterion.
$\therefore$ Failure will began in Bronze section and the maximum torque that can be applied is 1307 Nm. *(Ans.)*

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What will be the answer if you were asked to find out what is the value of torque that will cause the shaft to fail?
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Answer: 1307 Nm
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Solution 2

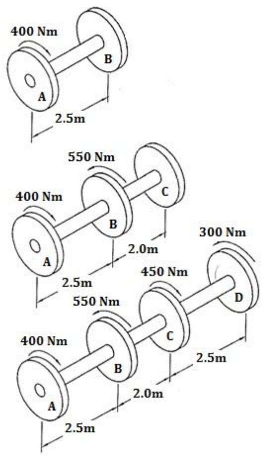

Figure: Free Body Diagrams

Here, $J = \frac{\pi \times d^4}{32} = \frac{\pi \times 45^4}{32} = 40.26 \times 10^4 \ mm^4$
Assuming, clockwise direction to be positive,

$$\phi_{A/D} = \phi_{A/B} + \phi_{B/C} + \phi_{C/D}$$

$$= \overbrace{\frac{TL}{JG}}^{\textbf{AB}} + \overbrace{\frac{TL}{JG}}^{\textbf{BC}} + \overbrace{\frac{TL}{JG}}^{\textbf{CD}}$$

$$= \overbrace{\frac{400 \times 2.5}{40.26 \times 10 \times 30}}^{\textbf{AB}} + \overbrace{\frac{(-150) \times 2}{40.26 \times 10 \times 30}}^{\textbf{BC}}$$

$$+ \overbrace{\frac{300 \times 2.5}{40.26 \times 10 \times 30}}^{\textbf{CD}}$$

$$= 0.12 \text{ rad}$$

$$= 6.88^\circ$$

$\therefore$ The angle of twist of gear A with respect to gear D is 6.88 degrees. (Ans.)
*Always report angle of twist in degrees.