

CS 228: Tutorial Solutions

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Problem Set 2

1. \mathcal{P} is not consistent.

Proof. For any p , the following are well-formed formulae: $p \vee \neg p$ and $\neg(p \vee \neg p)$. Let A be the former and B be the latter. By hypothesis, $A \rightarrow B$ is an axiom. We now derive \perp using this.

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|----|---|----------------|
| 1. | $p \vee \neg p \rightarrow \neg(p \vee \neg p)$ | Axiom |
| 2. | $p \vee \neg p$ | LEM |
| 3. | $\neg(p \vee \neg p)$ | MP 1, 2 |
| 4. | \perp | \perp i 2, 3 |

Thus, by definition, \mathcal{P} is inconsistent.

2. (a)

(i) To show that $\mathcal{A} = \{\neg, \wedge\}$ is an adequate set of formulae.

Note that $A \vee B \equiv \neg(\neg A \wedge \neg B)$. Thus, we can construct \vee using \mathcal{A} .

Also, $A \rightarrow B \equiv \neg A \vee B$.

As we had already constructed \vee using \mathcal{A} , we are done.

(ii) To show that $\mathcal{B} = \{\neg, \rightarrow\}$ is an adequate set of formulae.

Note that $A \vee B \equiv \neg A \rightarrow B$. Thus, we can construct \vee using \mathcal{B} .

Also, $A \wedge B \equiv \neg(\neg A \vee \neg B)$.

As we had already constructed \vee using \mathcal{B} , we are done.

(iii) To show that $\mathcal{C} = \{\rightarrow, \perp\}$ is an adequate set of formulae.

Note that $\neg A \equiv A \rightarrow \perp$. Thus, we can construct \neg using \mathcal{C} .

The result now follows from what we proved earlier as we can construct everything else using \rightarrow and \neg .

2. (b)

We shall prove the statement by way of contradiction. Suppose not. That is, C is adequate and $\neg \notin C$ and $\perp \notin C$.

Let us look at the set of all formulae S that contain only a as a propositional atom and connectives from C . By hypothesis, C is adequate, that is, for every formula, there is an equivalent formula with only connectives from C . In particular, an equivalent of the formula $\neg a$ should be in S . Note that $\neg a$ evaluates to F whenever a is T .

Claim: Given any formula $\varphi \in S$, it must evaluate to T whenever a is T . This will show that C is not adequate (as it could not possibly have $\neg a$) and complete our proof.

Proof. We shall prove this by inducting on the number n of connectives in φ .

Base case. $n = 0$. The only such formula is $\varphi = a$. For this, the claim holds trivially.

Inductive hypothesis. Given a formula φ such that it has n or fewer connectives, φ evaluates to T when a is T .

Inductive step. Suppose φ is a formula with $n + 1$ connectives. Then, φ can be written as $\varphi_1 \circ \varphi_2$ where \circ is some connective from C and φ_1, φ_2 are formulae in S with at most n connectives. By inductive hypothesis, φ_1 and φ_2 both evaluate to T whenever a is T . Let us now consider the following partial truth table.

a	φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_2 \rightarrow \varphi_1$
T	T	T	T	T	T	T

Thus, it is clear that no matter what \circ is, φ will always evaluate to T whenever a is T .

This completes our proof of the claim, which in turns completes the complete proof.

3. We show that $\{\downarrow\}$ is adequate by observing the following:

1. $\neg A \equiv A \downarrow A$
2. $A \wedge B \equiv \neg(A \downarrow B)$
3. $A \vee B \equiv \neg(\neg A \wedge \neg B)$

Hence, proved.

4. We show that $\{\oplus\}$ is not adequate by way of contradiction. Assume that $\{\oplus\}$ were adequate. Then, we should have been able to write $A \wedge B$ in terms of A, B and \oplus . We show that this is not possible using the following claim.

Claim: Let φ be any formula made using A, B and \oplus . Then, the truth table of φ has an even number of T s.

Proof. We prove the claim using induction. We shall induct on the number n of \oplus s in the formula.

Base case. $n = 0$. The only such formulae are A and B .

A	B	A	B
F	F	F	F
F	T	F	T
T	F	T	F
T	T	T	T

Thus, we have proven the base case.

Inductive hypothesis. Let φ be any formula using A, B and at most n \oplus s. Then, φ has an even number of T s in its truth table.

Inductive step. Let φ be a formula with $n + 1$ \oplus s. As φ is built using only \oplus s, we can write φ as $\varphi_1 \oplus \varphi_2$ for some formulae φ_1 and φ_2 which have at most n \oplus s. By induction hypothesis, φ_1 has $2a$ T s in its truth table and φ_2 has $2b$ T s for some integers a and b .

Let i denote the number of lines in which a T of φ_1 is paired with an F of φ_2 and let j denote the number of lines in which an F of φ_1 is paired with a T of φ_2 . Then, $i + j$ is the total number of T s in the truth table of φ . Now note that the remaining $(2a - i)$ T s of φ_1 were paired with the remaining $(2b - j)$ T s of φ_2 . Thus, $2a - i = 2b - j$. Simple algebra gives us that $i + j = 2(a - b + j)$, which is even, as desired.

This proves our claim.

The claim then completes the proof as $A \wedge B$ has an odd number of T in its truth table and therefore, cannot possibly be written using just \oplus .

5. First we show that consistency implies satisfiability.

If \mathcal{F} is consistent, then $\mathcal{F} \not\vdash \perp$, by definition.

As propositional logic is sound and complete, this means that $\mathcal{F} \not\models \perp$.

What the above means is that there is an assignment α such that $\alpha \models \mathcal{F}$ and $\alpha \not\models \perp$. The latter is always true anyway but the existence of an α such that $\alpha \models \mathcal{F}$ shows that \mathcal{F} is satisfiable.

Conversely, assume that \mathcal{F} is satisfiable. Then there exists an assignment α such that $\alpha \models \mathcal{F}$, by definition. As \perp is never true, $\alpha \not\models \perp$. Thus, we have it that $\mathcal{F} \not\models \perp$. (Since there exists at least one assignment such that \mathcal{F} is true but \perp is not.)

As propositional logic is sound and complete, $\mathcal{F} \not\vdash \perp$.

Thus, \mathcal{F} is consistent, by definition.

6. (b) We are given that $\mathcal{F} \vdash \perp$.

(Definition of inconsistency.)

Also, $\mathcal{F} = \mathcal{F}_G \cup \{G\}$. Thus, $\mathcal{F}_G, G \vdash \perp$.

The above is the same as $\mathcal{F}_G \vdash G \rightarrow \perp$.

(Done in class.)

As $G \rightarrow \perp \equiv \neg G$, we get that $\mathcal{F}_G \vdash \neg G$.