CS 228: Tutorial Solutions

Aryaman Maithani

Problem Set 2

1. $\mathcal{P}$ is not consistent.
Proof. For any p, the following are well-formed formulae: $p \vee \neg p$ and $\neg(p \vee \neg p)$. Let A be the former and B be the latter. By hypothesis, $A \to B$ is an axiom. We now derive $\perp$ using this.

1.	$p \vee \neg p \to \neg(p \vee \neg p)$	Axiom
2.	$p \vee \neg p$	LEM
3.	$\neg(p \vee \neg p)$	MP 1, 2
4.	$\perp$	$\perp$ i 2, 3

Thus, by definition, $\mathcal{P}$ is inconsistent.

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2. (a)
(i) To show that $\mathcal{A} = \{\neg, \wedge\}$ is an adequate set of formulae.
Note that $A \vee B \equiv \neg(\neg A \wedge \neg B)$. Thus, we can construct $\vee$ using $\mathcal{A}$.
Also, $A \to B \equiv \neg A \vee B$.
As we had already constructed $\vee$ using $\mathcal{A}$, we are done.

(ii) To show that $\mathcal{B} = \{\neg, \to\}$ is an adequate set of formulae.
Note that $A \vee B \equiv \neg A \to B$. Thus, we can construct $\vee$ using $\mathcal{B}$.
Also, $A \wedge B \equiv \neg(\neg A \vee \neg B)$.
As we had already constructed $\vee$ using $\mathcal{B}$, we are done.

(iii) To show that $\mathcal{C} = \{\to, \perp\}$ is an adequate set of formulae.
Note that $\neg A \equiv A \to \perp$. Thus, we can construct $\neg$ using $\mathcal{C}$.
The result now follows from what we proved earlier as we can construct everything else using $\perp$ and $\neg$.

2. (b)
We shall prove the statement by way of contradiction. Suppose not. That is, C is adequate and $\neg \notin C$ and $\perp \notin C$.
Let us look at the set of all formulae S that contain only a as a propositional atom and connectives from C. By hypothesis, C is adequate, that is, for every formula, there is an equivalent formula with only connectives from C. In particular, an equivalent of the formula $\neg a$ should be in S. Note that $\neg a$ evaluates to F whenever a is T.

Claim: Given any formula $\varphi \in S$, it must evaluate to T whenever a is T. This will show that C is not adequate (as it could not possibly have $\neg a$) and complete our proof.

Proof. We shall prove this by inducting on the number n of connectives in φ.
Base case. $n = 0$. The only such formula is $\varphi = a$. For this, the claim holds trivially.
Inductive hypothesis. Given a formula φ such that it has n or fewer connectives, φ evaluates to T when a is T.
Inductive step. Suppose φ is a formula with $n + 1$ connectives. Then, φ can be written as $\varphi_1 \circ \varphi_2$ where $\circ$ is some connective from C and φ_1, φ_2 are formulae in S with at most n connectives. By inductive hypothesis, φ_1 and φ_2 both evaluate to T whenever a is T. Let us now consider the following partial truth table.

a	φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \to \varphi_2$	$\varphi_2 \to \varphi_1$
T	T	T	T	T	T	T

Thus, it is clear that no matter what $\circ$ is, φ will always evaluate to T whenever a is T.
This completes our proof of the claim, which in turns completes the complete proof.

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3. We show that $\{\downarrow\}$ is adequate by observing the following:
   1. $\neg A \equiv A \downarrow A$
   2. $A \wedge B \equiv \neg(A \downarrow B)$
   3. $A \vee B \equiv \neg(\neg A \wedge \neg B)$

Hence, proved.

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4. We show that $\{\oplus\}$ is not adequate by way of contradiction. Assume that $\{\oplus\}$ were adequate. Then, we should have been able to write $A \wedge B$ in terms of A, B and $\oplus$. We show that this is not possible using the following claim.

Claim: Let φ be any formula made using A, B and $\oplus$. Then, the truth table of φ has an even number of Ts.
Proof. We prove the claim using induction. We shall induct on the number n of $\oplus$s in the formula.
Base case. $n = 0$. The only such formulae are A and B.

A	B	A	B
F	F	F	F
F	T	F	T
T	F	T	F
T	T	T	T

Thus, we have proven the base case.
Inductive hypothesis. Let φ be any formula using A, B and at most n $\oplus$s. Then, φ has an even number of Ts in its truth table.
Inductive step. Let φ be a formula with $n+1$ $\oplus$s. As φ is built using only $\oplus$s, we can write φ as $\varphi_1 \oplus \varphi_2$ for some formulae φ_1 and φ_2 which have at most n $\oplus$s. By induction hypothesis, φ_1 has $2a$ Ts in its truth table and φ_2 has $2b$ Ts for some integers a and b.
Let i denote the number of lines in which a T of φ_1 is paired with an F of φ_2 and let j denote the number of lines in which an F of φ_1 is paired with a T of φ_2. Then, $i + j$ is the total number of Ts in the truth table of φ.
Now note that the remaining $(2a - i)$ Ts of φ_1 were paired with the remaining $(2b - j)$ Ts of φ_2. Thus, $2a - i = 2b - j$. Simple algebra gives us that $i + j = 2(a - b + j)$, which is even, as desired.
This proves our claim.
The claim then completes the proof as $A \wedge B$ has an odd number of T in its truth table and therefore, cannot possibly be written using just $\oplus$.

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5. First we show that consistency implies satisfiability.
If $\mathcal{F}$ is consistent, then $\mathcal{F} \nvdash \perp$, by definition.
As propositional logic is sound and complete, this means that $\mathcal{F} \not\models \perp$.
What the above means is that there is an assignment α such that $\alpha \models \mathcal{F}$ and $\alpha \not\models \perp$. The latter is always true anyway but the existence of an α such that $\alpha \models \mathcal{F}$ shows that $\mathcal{F}$ is satisfiable.

Conversely, assume that $\mathcal{F}$ is satisfiable. Then there exists an assignment α such that $\alpha \models \mathcal{F}$, by definition. As $\perp$ is never true, $\alpha \not\models \perp$. Thus, we have it that $\mathcal{F} \not\models \perp$. (Since there exists at least one assignment such that $\mathcal{F}$ is true but $\perp$ is not.)
As propositional logic is sound and complete, $\mathcal{F} \nvdash \perp$.
Thus, $\mathcal{F}$ is consistent, by definition.

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6. (b) We are given that $\mathcal{F} \vdash \perp$. (Definition of inconsistency.)
Also, $\mathcal{F} = \mathcal{F}_G \cup \{G\}$. Thus, $\mathcal{F}_G,\ G \vdash \perp$.
The above is the same as $\mathcal{F}_G \vdash G \rightarrow \perp$. (Done in class.)
As $G \rightarrow \perp \equiv \neg G$, we get that $\mathcal{F}_G \vdash \neg G$.