CS 228: Tutorial Solutions

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Problem Set 4

Notation: $\mathbb{N} = \{1,\ 2,\ 3, \ldots\}$ denotes the set of positive integers.

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1. Given: $\varphi = \forall x \exists y R(x, y) \wedge \exists y \forall x \neg R(x, y)$.

φ is satisfiable over the following structure: $\mathcal{A} = (\mathbb{N}, R^{\mathcal{A}})$, where $R^{\mathcal{A}} = \{(x, y) \in \mathbb{N}^2 : y = x + 1\}$. Note that $\mathbb{N}$ is infinite and countable.
It is clear that given any x, if we choose $y = x + 1$, then $y \in \mathbb{N}$ and we have that $R(x, y)$.
Also, given $y = 1 \in \mathbb{N}$, and given any $x \in \mathbb{N}$, it not true that $R(x, 1)$. (There is no natural number n such that $n + 1 = 1$.)

Another example could have been to take any infinite countable set X as universe and choose a particular element $x_0 \in X$ and say that $R^{\mathcal{A}} = \{(x, y) \in X^2 : y = x_0\}$. That is, any element is related to x_0 and only to x_0.

2.
(i) $\varphi_B(x, y) = \big(\forall z (P(z, x) \leftrightarrow P(z, y))\big) \wedge \neg F(x)$.
(All parents of x are parents of y and vice-versa and x is not a female.)
(ii) $\varphi_A(x, y) = F(x) \wedge \Bigg(\forall z \big(P(z, y) \to \exists w [P(z, w) \wedge P(w, y)]\big)\Bigg)$
(x is a female and given any parent of x, that parent is the grandparent of y.)
(iii) $\varphi_C(x, y) = \forall g \forall p \big([P(g, p) \wedge P(p, x)] \to [\exists p' (P(g, p') \wedge P(p', y) \wedge \neg p = p')]\big)$.
(Note that I've assumed first cousin.)
(iv) $\varphi_O(x) = \forall p \forall x' (P(p, x) \to x = x')$.
(If p is a parent of x and x' is a human such that p is a parent of x', then x and x' must be the same.)
(v) Guess - $\varphi_H(x, y)$ which says that "x is a husband of y" cannot be defined.

3.
$$\forall n [(\text{Even}(n) \wedge \neg \exists z (\text{Zero}(z) \wedge z + 2 = n)) \to (\exists p \exists q (\text{Prime}(p) \wedge \text{Prime}(q) \wedge n = p + q))].$$

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4. Let $\mathcal{A}$ be a structure of τ. Let $c_{\mathcal{A}}$ denote the fixed element that is assigned to c.
(A1) $\forall x \forall y \forall z [op(x, op(y, z)) = op(op(x, y), z)]$.
(A2) $\forall x (op(x, c_{\mathcal{A}}) = x)$.
(A3) $\forall x \exists y (op(x, y) = c_{\mathcal{A}})$.
(A4) $\forall x \forall y \forall z [op(x, z) = op(y, z) \to x = y]$.

5.
(i) Take the structure $\mathcal{A}$ whose universe just consists of 0. It trivially satisfies ψ.
(ii) Let $\mathcal{A}$ be the structure with universe as $\{0, 1\}$ and $+$ defined as:

$$0 + 0 = 0,\ 0 + 1 = 0,\ 1 + 0 = 0,\ 1 + 1 = 0.$$

This clearly does not satisfy φ_2.
(iii) No, that is not the case. For example, take $\mathcal{A}$ whose universe consists of all 2×2 invertible matrices with real entries with $+$ defined to multiplication and 0 to be the identity element. Then, it is clear that this $\mathcal{A}$ satisfies ψ but not α as there exist invertible matrices which do not commute.
A simpler (but more abstract) is the S_3 group. You may look it up.
(iv) (a) Let $\mathcal{A} = \{0,\ 1\}$ and let $+$ be defined as

$$0 + 0 = 0,\ 0 + 1 = 1,\ 1 + 0 = 1,\ 1 + 1 = 1.$$

This satisfies $\varphi_1 \wedge \varphi_2$ but not ψ.
(b) Let $\mathcal{A} = \{0,\ 1,\ 2\}$ and let $+$ be defined as

$+$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Then, $\mathcal{A}$ satisfies $\varphi_2 \wedge \varphi_3$ but not ψ as $1+(1+2) = 1+1 = 0 \neq 2 = 0+2 = (1+1)+2$.
(c) Let $\mathcal{A}$ be the structure with universe as $\{0,1\}$ and $+$ defined as:

$$0+0=0,\ 0+1=0,\ 1+0=0,\ 1+1=0.$$

This satisfies $\varphi_1 \wedge \varphi_3$ but not ψ.

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7. Let the structure be $\mathcal{G}$ with $u(\mathcal{G}) = \{1,2\}$. And $E^{\mathcal{G}} = \{(1,1),(1,2),(2,1)\}$.
This satisfies the latter but not the former.
It does not satisfy the former as one can take $x = 2$, then no matter what y is, z takes all possible values. In particular, z takes the value 2. As $(2,2) \notin E^{\mathcal{G}}$, we have it that $E(x,z)$ is not true. Thus, the former sentence is not true.
For the latter sentence, choose $x = 1$. For $y = 1$, choose $z = 1$ and for $y = 2$, choose $z = 1$. Thus, we are done.

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6. The above example illustrates the difference.

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8. (a) $\varphi_{45} = (\exists^{\geq 45} x(x = x)) \wedge \neg(\exists^{\geq 46} x(x \neq x))$.

(b) $\forall x_1 \forall x_2 \ldots \forall x_{n-1} \exists x_n (x_n = x_n \wedge x_n \neq x_{n-1} \wedge x_n \neq x_{n-2} \wedge \cdots \wedge x_n \neq x_2 \wedge x_n \neq x_1)$.

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9. By 8. (b), we have shown that $\exists^{\geq n}$ can actually be written as a FO formula. Thus, we can use it freely. So, the answer is simply $(\exists^{\geq n} x(x = x)) \wedge \neg(\exists^{\geq (m+1)} x(x \neq x))$.