Assignment 6

15-03-2019

(1) For $x, y \in \mathbb{R}$, show that (i) $|xy| = |x||y|$ (ii) (Triangle Inqequality) $|x+y| \leq |x|+|y|$

(2) For $x, y \in \mathbb{R}$, show that $\max\{x, y\} = \dfrac{x + y + |x - y|}{2}$. Identify a similar relation for $\min\{x, y\}$.

(3) Identify the set $\{x \in \mathbb{R} || x - 3| < 5\}$ (with proof).

(4) Prove or disprove: For $x, y \in \mathbb{C}, |xy| = |x||y|$.

(5) For a function $f : X \to, A \subset X$, and $B \subset Y$, define $f^{-1}(B) = \{a \in X | f(a) \in B\}$ and $f(A) = \{b \in Y |$ there exists $a \in X$ such that $b = f(a)\}$.
Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.
   (a) Find $f(A)$ for $A =$ (i) $\left\{1, \frac{1}{2}, \frac{1}{3}, -\frac{1}{2}\right\}$ (ii) $[0, 2]$ (iii) $(1, 2]$ (iv) $[-2, 1)$
   (b) Find $f^{-1}(B)$ for $B =$ (i) $\{4\}$ (ii) $\{1\}$ (iii) $[0, 1]$ (iv) $[-4, 1]$ (v) $(0, 1)$

   What are your answers when (i) $f(x) = x^3$ (ii) $f(x) = \sin(\pi x)$?

(6) Let X be the set of 2×2 matrices with entries in $\mathbb{R}$.
   (a) Find $f^{-1}(\{\mathbf{0}\})$, where $f : X \to X$ is given by
   (i) $f(M) = M^2$ (ii) $f(M) = M^2 - M$ (iii) $f(M) = M - M^T$ (iv) $f(M) = MM^T$.
   In (iv), what is $f^{-1}(\{I\})$?
   (b) Let $f : X \to \mathbb{R}$ be given by $f(M) = \det(M)$. (i) If A is the set of orthogonal matrices, what if $f(A)$? (ii) What if $f^{-1}(\{\mathbf{0}\})$?

(7) For $B = \{0\}, \{1\}, [0, 1], (1, 2]$, find $f^{-1}(B)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by
$f(x, y) =$ (i) x (ii) y (iii) $x^2 + y^2$ (iv) xy.

(8) Let $f : X \to Y$ be a function, $A, A_1, A_2 \subset X; B, B_1, B_2 \subset Y$.
   (a) If $A \subset A_1$, show that $f(A) \subset f(A_1)$. Is the same true under inverse images?
   (b) Show that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. Is the same true for unions and complements?
   (c) For $A_1, A_2 \subset X$, is one of $f(A_1 \cup A_2)$ and $f(A_1) \cup f(A_2)$ contained in the other? Is the containment proper? When does equality hold? Answer these questions for unions and complements.
   (d) What is the relation between A and $f^{-1}(f(A))$? Is the containment proper? When does equality hold? Answer these questions for B and $f(f^{-1}(B))$.

(9) Find the limit of the following sequences if they exist, else prove that the sequence diverges.
   (i) $a_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$.
   (ii) $b_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$.
   (iii) $c_n = n$ for all $n \in \mathbb{N}$.
   (iv) $d_n = (-1)^n$ for all $n \in \mathbb{N}$.