Assignment 8

01-04-2019

(1) Let $a, b, c \in \mathbb{Z}$. Prove or disprove:

  (a) If $c|(a+b)$, then $c|a$ or $c|b$.
  (b) If $c|a$ or $c|b$, then $c|ab$.
  (c) If $c|ab$, then $c|a$ or $c|b$.

(2) Let $W = \mathbb{Z} \cap [-100, \infty)$. Show that every non-empty subset of W has a least element.

(3) Let $a_0 = 1$, and for $n \in \mathbb{N}$, let $a_n = \sqrt{2a_{n-1}}$.

  (a) Show that for each $n \in \mathbb{N}, a_{n+1} \geq a_n$. Hint: Use induction on n.
  (b) Show that the set $\{a_n : n \in \mathbb{N}\}$ is bounded above.
  (c) Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is convergent and find its limit.

(4) Prove or disprove: Let (a_n) and (b_n) be two convergent sequences of real numbers with limits a and b respectively. If $a_n < b_n$ for all $n \in \mathbb{N}$, then $a < b$.

(5) Show that the sequence $(p^{1/n})$ converges to 1 for all $p > 0$.
Hint: First prove for $p > 1$ by finding the limit of $a_n = (p^{1/n}) - 1$.

(6) (a) Let $a \in (-1, \infty)$ and $n \in \mathbb{N}$. Show that $(1+a)^n \geq 1 + na$.
  (b) Show that the sequence $a_n = (1 + 1/n)^n$ is strictly increasing.
  (c) Show that the sequence $b_n = (1 + 1/n)^{n+1}$ is strictly decreasing.
  (d) Show that both (a_n) and (b_n) converge to the same limit which lies in $(2, 4)$.

(7) Let $A \subset \mathbb{R}$ be bounded above and $\alpha = \mathrm{lub}(A)$. Show that there is a sequence (a_n) <u>in A</u>, which converges to α.

(8) Let (a_n) be a bounded sequence. Recall the definition of limit inferior and limit superior defined in class. Show that

$$\mathrm{glb}(a_n) \leq \liminf(a_n) \leq \limsup(a_n) \leq \mathrm{lub}(a_n).$$

(9) Let $a, b \in \mathbb{Z}$, $L = \{c \in \mathbb{Z} | \exists k, l \in \mathbb{Z}(c = ka + lb)\}$, and $C = \{c \in \mathbb{N} | c|a \text{ and } c|b\}$.

  (a) Identify L and C when $(a, b) =$ (1) $(2,3)$ (2) $(4,6)$ (3) $(4,8)$.
  (b) Show that if $c \in L$, then for all $n \in \mathbb{Z}, nc \in L$.
  (c) Show that if $c \in C$ and $d|c$, then $d \in C$.
  (d) Prove or disprove: For $a, b \in \mathbb{N}, L \cap C \neq \emptyset$.