Assignment 10

15-04-2019

(1) For $n \in \mathbb{N}$, show that the following operations on $\mathbb{Z}/n\mathbb{Z}$ are well-defined:
For all $a, b \in \mathbb{Z}$, $[a] + [b] = [a + b]$, and $[a][b] = [ab]$.
Prove that $\mathbb{Z}/n\mathbb{Z}$ is a field under these operations if and only if n is a prime.

(2) Show that $\mathcal{M}_2(\mathbb{R})$, the set of all 2×2 matrices with real entries, forms a group under matrix addition, and does not form a group under matrix multiplication.

(3) Let G be a group and $H \subset G$ be non-empty.
Show that H is a subgroup if and only if $ab^{-1} \in H$ for all $a, b \in H$.

(4) Check if H is a subgroup of the given group G.

(a) $G = \mathbb{Z}/12\mathbb{Z}$; $H =$ (i) $\{[1], [11]\}$ (ii) $\{[0], [3], [6], [9]\}$.
Group operation being + as defined above.

(b) $G = \mathcal{M}_2(\mathbb{R})$; $H =$ (i) $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D \,\middle|\, c = 0 \right\}$ (ii) the set of invertible 2×2 matrices.
Group operation being + as defined in the standard manner.

(c) $G = S_4$; $H =$ (i) $\{\mathrm{id}, (12), (34), (12)(34)\}$.
(ii) $\{(123), (134), (143), (132), (234), (243), (124), (142), \mathrm{id}\}$.
Group operation being the standard composition of cycles.

(5) Show that every group of prime order is cyclic.

(6) (a) Find all generators for $\mathbb{Z}/12\mathbb{Z}$, $\mathbb{Z}/9\mathbb{Z}$ and $\mathbb{Z}/13\mathbb{Z}$.
(b) What can you say in general about the generators of $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$ and $n \geq 2$.

(7) Consider the following relations on $\mathcal{A}$, identify whether it is an equivalence relation, a partial order, or a total order.

(a) $\mathcal{A}$ is the set of human beings, and for $a, b \in \mathcal{A}$,
a is related to b if a is a sibling of b.
(b) $\mathcal{A}$ is the set of students at IITB in your batch, and for $a, b \in \mathcal{A}$,
a is related to b if a is in the same hostel as b.
(c) $\mathcal{A} = \mathbb{R}$, and for $a, b \in \mathcal{A}, a$ is related to b if $a \leq b$.
(d) $\mathcal{A} = \mathbb{R}^2$, and for $a = (a_1, a_2), b = (b_1, b_2) \in \mathcal{A}, a$ is related to b if $a_1 \leq b_1$ and $a_2 \leq b_2$.
(e) $\mathcal{A} = \mathbb{R}^2$, and for $a = (a_1, a_2), b = (b_1, b_2) \in \mathcal{A}, a$ is related to b if $a_1 \leq b_1$ or $a_2 \leq b_2$.
(f) $\mathcal{A}$ is the set of subsets of a set X, and for $A, B \in \mathcal{A}$, A is related to B if $A \subset B$.
(g) $\mathcal{A} = \mathbb{Z}$, and for $a, b \in \mathcal{A}, a$ is related to b if $12|(b - a)$.
(h) $\mathcal{A} = \mathbb{R}^2$, and for $a, b \in \mathcal{A}, a$ is related to b if a has the same y-coordinate as b.
(i) $\mathcal{A} = \mathbb{R}$, and for $a, b \in \mathcal{A}, a$ is related to b if $b - a$ is an integer.

In each example of an equivalence relation, pick a point in $a \in \mathcal{A}$, and identify $[a]$.
In each example of a partial (or total) order, pick a point in $\mathcal{A}$, and identify the points related to it.
In the examples in $\mathbb{R}$ or $\mathbb{R}^2$, identify these sets on the number line, or the $xy-$plane.