Assignment 7

25-03-2019

Notation: For $c, a \in \mathbb{Z}$, "c is a divisor of a" (or "a is a multiple of c") is denoted by $c|a$.

(1) Let $a, b, c, d \in \mathbb{N}$. State the following mathematically and write their negations:

   (a) c divides a. (Easier to think of: a is a multiple of c).
   (b) c is a common divisor of a and b.
   (c) d is the greatest common divisor of a and b.

(2) Let $a, b, c \in \mathbb{Z}$. Prove the following:

   (a) $c|0$.
   (b) If $a|b$ and $c|a$, then $c|b$.
   (c) If $c|a$ and $c|b$, then $\forall m, n \in \mathbb{Z}, c|(ma + nb)$. In particular, $c|(a + b)$ and $c|(a - b)$.
   (d) Suppose $c|a$. If $a \neq 0$, then $|a| \geq |c|$.
   (e) If $a|c$ and $c|a$, then $a = \pm c$.

(3) Let $S \subset \mathbb{N}$ be such that (1) $1 \in S$ and (2) For $k \in \mathbb{N}$, if $\{1, 2, \cdots, k\} \subset S$, then $k + 1 \in S$. Show that $S = \mathbb{N}$.

(4) Prove the following statement by (i) induction and (ii) well-ordering principle:
Given $n \in \mathbb{N} \setminus \{1\}$, there is a prime number $p \in \mathbb{N}$ such that $p|n$.

(5) Let $(a_n), (b_n)$ be sequences of real numbers which converge and $c \in \mathbb{R}$. Prove the following statements.

   (a) The sequence $(a_n + b_n)$ converges.
   (b) The sequence (ca_n) converges.
   (c) The sequence $(a_n b_n)$ converges.

   If you do not assume that (b_n) converges, what can you say about the convergence in each of the above cases?

(6) Let (a_n) be a sequence of real numbers. If (a_n) is bounded (converges), every subsequence is bounded (converges). If we assume that all the subsequences excluding the original sequence are bounded (convergent), then is the converse true?

(7) Show that every convergent sequence of real numbers is bounded. Is the converse true? Justify your answer.

(8) Let $(a_n), (b_n)$ be sequences of real numbers such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. If they converge to a and b respectively, then, $a \leq b$.

(9) Let (a_n) be a sequence of non-negative real numbers converging to $a \in \mathbb{R}$. Show that $(\sqrt{a_n})$ converges to $\sqrt{a}$.