Introduction to Mathematical Concepts (MA 107) - Final Exam
IIT Bombay: 25th April, 2019

Instructions: Show all your work for full marks. Answers without justifications will get a ZERO. You may use only the results PROVED in class. State the results used clearly. **Max. Marks:** 40

(1) Let X and Y be non-empty sets, $A, B \subset X$, and $f : X \to Y$ be a function.
Prove or disprove: $f(A \cup B) = f(A) \cup f(B)$. [3]

(2) Let $a, b \in \mathbb{R}$ be such that $a < b$. Show that $(a, b) \cap \mathbb{Q} \neq \emptyset$.
(You may assume the Archimedean property). [3]

(3) Let $A = \left(-\frac{1}{2}, \frac{1}{2}\right) \subset \mathbb{R}$. Show that $\text{lub}(A) = \frac{1}{2}$. [3]

(4) Let $a \in \mathbb{R}$. Find $\bigcap_{n=1}^{\infty} [a, a + 1/n]$. Justify your steps. [3]

(5) Find the limit of the series $\sum a_n$, where $a_n = \frac{1}{n(n+1)}$ for all n. Justify your steps. [3]

(6) Show that $12\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. [3]

(7) Prove or disprove: There exists a bijection from the set $\mathbb{Q} \cap (0, 1)$ to $\mathbb{Q} \cap [0, 1]$. [3]

(8) Let $a, b \in \mathbb{N}$. [4]

(a) Show there exist unique $q \in \mathbb{N}$ and $r \in \{0, \cdots, a-1\}$, such that $b = aq + r$.
(You may assume existence of q and r, and only need to prove the uniqueness).

(b) Show that $\gcd(a, b) = \gcd(a, r)$.

(9) Let $(a_n), (b_n)$ and (c_n) be sequences of real numbers. Suppose for all $n \in \mathbb{N}$, $a_n \leq b_n \leq c_n$.
If (a_n) and (c_n) converse the same limit $L \in \mathbb{R}$, show that (b_n) converges to L. [4]

(10) Let $a, b, p \in \mathbb{N}$, where p is prime. Show that if $p|(ab)$, then $p|a$ or $p|b$. Justify your steps. [4]

(11) Define a relation on $\mathbb{C}$ as follows: For $a, b \in \mathbb{C}, a \sim b$ if $|a| = |b|$. [5]

(a) Is $i \in \mathbb{C}$ related to $2i$? Why or why not?

(b) Show that $\sim$ is an equivalence relation on $\mathbb{C}$.

(c) Identify the equivalence class of 1 (with justification).

(12) Let G be a group and $f : G \to G$ be defined as $f(a) = a^{-1}$. [5]
Show that f is a (i) well-defined function and (ii) a bijection.

Bonus: (All or nothing)
Prove or disprove: Let $\mathcal{A}$ be a non-empty set with a relation, which is a total order. If $\mathcal{A}$ is in bijection with $\mathbb{N}$, then every non-empty subset of $\mathcal{A}$ has a least order element under the given order. [3]