All MA 107 Assignments

Aryaman Maithani

Assignment 1

14-01-2019

(1) Let $S \subset \mathbb{R}$ and $a \in \mathbb{R}$. Define what it means to say that

  (a) a is a minimum of the set S.
  (b) the set S has a minimum.

(2) Prove the following statements given that $\mathbb{R}$ is a field. State clearly the axioms you are using.

  (a) For all $a, b, c \in \mathbb{R}$ $a + b = a + c$ implies $b = c$.
  (b) For all $a, b, c \in \mathbb{R}$ and $a \neq 0$, $a \cdot b = a \cdot c$ implies $b = c$
  (c) For all $a \in \mathbb{R}$, $a \cdot 0 = 0$
  (d) If for any $a, b \in \mathbb{R}, a \cdot b = 0$ then $a = 0$ or $b = 0$.
  (e) For all $a \in \mathbb{R}, -a = (-1) \cdot a$
  (f) For all $a, b \in \mathbb{R}, a, b \neq 0$, then $1/(a \cdot b) = (1/a) \cdot (1/b)$

(3) Prove the following statements given that $\mathbb{R}$ is an ordered field. State clearly the axioms you are using.

  (a) For all $a, b \in \mathbb{R}$, if $a < b$ then $(-b) < (-a)$.
  (b) $0 < 1$.
  (c) For all $a, b \in \mathbb{R}$, is $0 < a < b$ then $0 < 1/b < 1/a$.
  (d) $a \in \mathbb{R}, a \neq 0$ then $a^2 > 0$.
  (e) If for some $a, b \in \mathbb{R}, a^2 + b^2 = 0$ then $a = b = 0$.
  (f) For all $a, b \in \mathbb{R}$, if $0 < ab$ then $0 < a, b$ or $a, b < 0$.
  (g) For all $a, b \in \mathbb{R}$, if $ab < 0$ then $a < 0, 0 < b$ or $b < 0, 0 < a$.
  (h) If $a \in \mathbb{R}$ is such that $a \cdot a = a$ then $a = 0$ or $a = 1$.
  (i) For all $a, b \in \mathbb{R}$, if $0 < a < b$ then $a^2 < b^2$
  (j) For all $a, b \in \mathbb{R}$, if $0 < a < b$ then $a < \sqrt{a \cdot b} < b$
  (k) For all $a, b \in \mathbb{R}$, if $a \leq b$ then $a \leq \dfrac{a+b}{2} \leq b$
  (l) For all $a, b \in \mathbb{R}$, if $0 < a < b$ then $\sqrt{a \cdot b} < \dfrac{a+b}{2}$

Notation i) a^2 denotes $a \cdot a$
ii) $\sqrt{x}$ is a number such that $(\sqrt{x})^2 = x$.
As we have not proven that such a number does exist, you may assume for now that it does.

Assignment 2

21-01-2019

(1) Let X be a set, $A, B \subset X, x \in X$. Define (i) $A \cap B$ (ii) $A \cup B$ (iii) $A \subset B$ (iv) $A = B$

(2) Write A as a subset of B, where:

(a) $A = \emptyset$; (i) $B = \mathbb{R}$ (ii) $B = \mathbb{Q}$ (iii) $B = \mathbb{Z}$ (iv) $B = \mathbb{N}$

(b) $B = \mathbb{R}$; (i) $A = (0, 1]$ (ii) $A = \mathbb{Z}$ (iii) $A = \mathbb{Q}$

(c) $B = \mathbb{R}^2$; A is (i) the $X-$axis (ii) the unit circle
(iii) the set of solutions of the equation $x + 2y = 0$

(3) Describe the following sets:

(a) (i) $\{x \in \mathbb{R} : x(x-1)(x-2) > 0\}$ (ii) $\{x \in \mathbb{R} : \cos(2\pi x) = 0\}$ (iii) $\{x \in \mathbb{R} : x^2 = 1\}$

(b) (i) $\left\{(x, y) \in \mathbb{R}^2 : \dfrac{x}{y} + \dfrac{y}{x} \geq 2\right\}$ (ii) $\{(x^2, x) : x \in \mathbb{R}\}$

(c) $A \times B$, where (i) $A = [0, \infty)$ and $B = [2, 3]$. (ii) $A = [3, 4]$ and $B = \mathbb{N}$.

(4) What is the st $A + 2$, where (i) $A = \mathbb{Z}$ (ii) $A = \{1, 2, 3, 4\}$ (iii) $A = [1, 2)$ (iv) $A = (\infty, 0)$
What is the set $2A$ where A is as above?
What is the set $\frac{\pi}{4}(2\mathbb{Z} + 1)$? Is it related to any of the sets in the previous question?

(5) What is the set $c + \mathbb{Q}$? When does it contain a rational number?
What can you say in the other cases? Answer similar questions about $c\mathbb{Q}$.

(6) Let A and B be non-empty subsets of $\mathbb{R}$, and $c \in \mathbb{R}$
Describe the sets $-A, cA, c + A, A \cap B, A \cup B$ and $A + B$.
How is their lub/glb (if they exist), related to the lub/glb of A and B?

(7) Identify some rational and irrational numbers in $\mathbb{Q} + [0, 1]$? What is this set?

(8) Let $S \in \mathbb{R}$. Is S has a maximum, then S is bounded above and $\max(S) = \text{lub}(S)$.

Note: In (1), the first two sub-parts would require a set description while the last two would be a condition.

Assignment 3

28-01-2019

(1) Identify the set $A \subset \mathbb{N}$ given that: if $x \in A$, then $x+1 \in A$. Justify your answer.

(2) The *absolute value* or *mod* function is defined as follows: $\forall x \in \mathbb{R}$, define

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

Identify the following sets:
(i) $\{x \in \mathbb{R} : |x| = 3\}$ (ii) $\{x \in \mathbb{R} : |x| \leq 3\}$ (iii) $\{x \in \mathbb{R} : |x+5| \leq 3\}$

(3) Find lub(A) and glb(A) in B in the following examples, if they exist. If not, explain why they do not exist.

(a) $A = \mathbb{N}, B = \mathbb{N}$
(b) $A = \{x \in \mathbb{Z} : -1 \leq |x+5| < 8\}, B = \mathbb{Z}$. Does your answer change if $B = \mathbb{R}$?
(c) $A = \{x \in \mathbb{Q} : -1 \leq |x+5| < 8\}, B = \mathbb{Q}$. Does your answer change if $B = \mathbb{R}$?
(d) $A = \{x \in \mathbb{Q} : x \neq 0, 1/x \in \mathbb{N}\}, B = \mathbb{Q}$. Does your answer change if $B = \mathbb{R}$?

(4) Let $A \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that α in an upper bound of S in $\mathbb{R}$. Show that the following statements are equivalent.

(a) α is the least upper bound of A in $\mathbb{R}$.
(b) For every $\epsilon > 0 \in \mathbb{R}$, there exist $a \in A$ such that $\alpha - \epsilon < a \leq \alpha$
(c) For every $t \in \mathbb{R}$ with $t < \alpha$, there exists $a \in A$ such that $t < a \leq \alpha$.

(5) Let $c \in \mathbb{R}; A, B \subset \mathbb{R}$ be bounded and non-empty. State whether the following are true or false. If true, prove it. If false, give a counter-example, and state and prove the corrected version.

(a) If $A \subset B$ then $\text{lub}(A) = \text{lub}(B)$.
(b) $\text{lub}(A \cup B) = \min\{\text{lub}(A), \text{lub}(B)\}$
(c) $\text{lub}(A \cap B)$_____$\{\text{lub}(A), \text{lub}(B)\}$
(d) $\text{lub}(cA) = c\,\text{lub}(A)$
(e) $\text{lub}(A+B) = \text{lub}(A) + \text{lub}(B)$

(6) Let X be a set, $A, B \subset X$. Show that the following are equivalent:
(i) $A \subset B$ (ii) $A = A \cap B$ (iii) $A \subset (A \cap B)$
(iv) $B^c \subset A^c$ (v) $B = A \cup B$ (vi) $(A \cup B) \subset B$

(7) Let $a, b, c, d \in \mathbb{N}$. State the following mathematically and write their negations:

(a) c divides a. (Easier to think of: a is a multiple of c).
(b) c is a common divisor of a and b,
(c) d is a greatest common divisor of a and b.

Assignment 4

04-02-2019

(1) Let X, Y b non-empty sets, $A \subset X, B \subset Y$. Show that $A \times B$ is a subset of $X \subset Y$.

(2) Find $A \times B \subset \mathbb{R}^2$, where
(i) $A = (0, 1)$, $B = \mathbb{R}$ (ii) $A = \{0, 1\}$, $B = \mathbb{R}$ (iii) $A = \mathbb{N}$, $B = \mathbb{R}$
How do your answers change when (i) A and B are interchanged.
(ii) $B = \mathbb{R}$ is replaced by $B = \mathbb{Z}$? (iii) $B = \mathbb{R}$ is replace by $[0, \infty)$

(3) Prove or disprove: Let $Z \subset X \times Y$. Then there are subsets A and B of X and Y respectively such that $Z = A \times B$.

(4) What is the set $c + \mathbb{Q}$ where $c \in \mathbb{R}$? When does it contain a rational number? What can you say in other cases? Answer similar questions about $c\mathbb{Q}$.

(5) Identify some rational and irrational numbers in $\mathbb{Q} + [0, 1]$? What is this set?

(6) Find the lub and glb of the following sets in $\mathbb{R}$ if they exist. Give an argument supporting your answer.

(a) $\{1/2^q \in \mathbb{R} | q \in \mathbb{Z}\}$
(b) $\{x \in \mathbb{R} | x < 1/n \text{ for some } n \in \mathbb{N}\}$
(c) $\{x \in \mathbb{R} | x > 1/n \ \forall n \in \mathbb{N}\}$

(7) Let $S \subset \mathbb{R}$. If $\text{lub}(S) = a$, then show that $\text{glb}\big(\{-s \in \mathbb{R} | s \in S\}\big) = -a$.

(8) Let $S \subset \mathbb{R}$ be a bounded subset of $\mathbb{R}$. Let $T = \{s^2 \in \mathbb{R} | s \in S\}$. Does it follow that $\text{lub}(T) = (\text{lub}(S))^2$? If yes, prove it. If false, give a counterexample, correct the statement and then prove the corrected statement.

Assignment 5

20-02-2019

(1) Let $x, y \in \mathbb{R}$ be such that $x, y > 0$ and $n \in \mathbb{N}$. Show that if $x^n \leq y^n$, then $x \leq y$.

(2) Show that if $x \in (0,1)$, then $x \notin \mathbb{Z}$.

(3) If $r \in \mathbb{R} \setminus \mathbb{Q}$ and $x \in \mathbb{Q} \setminus \{0\}$, show that rx and $r + x$ are elements of $\mathbb{R} \setminus \mathbb{Q}$.

(4) Show that there is no rational number x such that $x^2 = 3$.

(5) For all $0 < x \in \mathbb{R}$ and $m \in \mathbb{N}$, define $x^{1/m}$ to the unique real number $y > 0$ such that $y^m = x$. Show the following:

   (a) For all $0 < x \in \mathbb{R}, m, n \in \mathbb{N}, (x^m)^{1/n} = (x^{1/n})^m$.
   (b) For $m, n, l, k \in \mathbb{N}$, if $m/n = l/k$, then show that $(x^m)^{1/n} = (x^l)^{1/k}$.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Find $f(A)$ for $A =$
(i) $\{1, 1/2, 1/3, -1/2\}$ (ii) $[0, 2]$ (iii) $(1, 2]$ (iv) $[-2, 1)$ (v) $[-2, -1)$

(7) Is the function $f(x) = x^2$ one-one or onto as a function from
(i) $\mathbb{R}$ to $\mathbb{R}$? (ii) $\mathbb{R}$ to $[0, \infty)$? (iii) $(0, \infty)$ to $(0, \infty)$? (iv) $(0, 1)$ to $(0, 1)$? Can you identify properties of the graph that give the one-one or onto conditions?

(8) Let $f : X \to Y$ and $g : Y \to Z$ be functions.

   (a) If f and g are one-one, show that $g \circ f$ is one-one.
   (b) Is the converse true?
   (c) Answer (a) and (b) with "one-one" being replaced by "onto".

(9) Find a bijection from $(0, 1)$ to A, where $A =$
(i) $(1, 2)$ (ii) $(0, 2)$ (iii) $(1, 3)$ (iv) Can you find a bijection from $(0, 1)$ to $\mathbb{R}$?

(10) Show that A is countable, where $A =$
(i) $\{2, 3, 4, 5, \dots\}$ (ii) $\{2, 4, 6, 8, \dots\}$ (iii) $\{1, 3, 5, 7, \dots\}$ (iv) $2\mathbb{Z}$ (v) $2\mathbb{Z}+1$ (vi) $\mathbb{Z}$ (vii) $\mathbb{N}\times\mathbb{N}$

Assignment 6

15-03-2019

(1) For $x, y \in \mathbb{R}$, show that (i) $|xy| = |x||y|$ (ii) (Triangle Inqequality) $|x+y| \leq |x|+|y|$

(2) For $x, y \in \mathbb{R}$, show that $\max\{x, y\} = \dfrac{x+y+|x-y|}{2}$. Identify a similar relation for $\min\{x, y\}$.

(3) Identify the set $\{x \in \mathbb{R} | |x-3| < 5\}$ (with proof).

(4) Prove or disprove: For $x, y \in \mathbb{C}$, $|xy| = |x||y|$.

(5) For a function $f : X \to, A \subset X$, and $B \subset Y$, define $f^{-1}(B) = \{a \in X | f(a) \in B\}$ and $f(A) = \{b \in Y |$ there exists $a \in X$ such that $b = f(a)\}$.
Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

(a) Find $f(A)$ for $A =$ (i) $\left\{1, \frac{1}{2}, \frac{1}{3}, -\frac{1}{2}\right\}$ (ii) $[0, 2]$ (iii) $(1, 2]$ (iv) $[-2, 1)$
(b) Find $f^{-1}(B)$ for $B =$ (i) $\{4\}$ (ii) $\{1\}$ (iii) $[0, 1]$ (iv) $[-4, 1]$ (v) $(0, 1)$

What are your answers when (i) $f(x) = x^3$ (ii) $f(x) = \sin(\pi x)$?

(6) Let X be the set of 2×2 matrices with entries in $\mathbb{R}$.

(a) Find $f^{-1}(\{\mathbf{0}\})$, where $f : X \to X$ is given by
(i) $f(M) = M^2$ (ii) $f(M) = M^2 - M$ (iii) $f(M) = M - M^T$ (iv) $f(M) = MM^T$.
In (iv), what is $f^{-1}(\{I\})$?
(b) Let $f : X \to \mathbb{R}$ be given by $f(M) = \det(M)$. (i) If A is the set of orthogonal matrices, what if $f(A)$? (ii) What if $f^{-1}(\{\mathbf{0}\})$?

(7) For $B = \{0\}, \{1\}, [0, 1], (1, 2]$, find $f^{-1}(B)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by
$f(x, y) =$ (i) x (ii) y (iii) $x^2 + y^2$ (iv) xy.

(8) Let $f : X \to Y$ be a function, $A, A_1, A_2 \subset X; B, B_1, B_2 \subset Y$.

(a) If $A \subset A_1$, show that $f(A) \subset f(A_1)$. Is the same true under inverse images?
(b) Show that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. Is the same true for unions and complements?
(c) For $A_1, A_2 \subset X$, is one of $f(A_1 \cup A_2)$ and $f(A_1) \cup f(A_2)$ contained in the other? Is the containment proper? When does equality hold? Answer these questions for unions and complements.
(d) What is the relation between A and $f^{-1}(f(A))$? Is the containment proper? When does equality hold? Answer these questions for B and $f(f^{-1}(B))$.

(9) Find the limit of the following sequences if they exist, else prove that the sequence diverges.
(i) $a_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. (ii) $b_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$.
(iii) $c_n = n$ for all $n \in \mathbb{N}$. (iv) $d_n = (-1)^n$ for all $n \in \mathbb{N}$.

Assignment 7

25-03-2019

Notation: For $c, a \in \mathbb{Z}$, "c is a divisor of a" (or "a is a multiple of c") is denoted by $c|a$.

(1) Let $a, b, c, d \in \mathbb{N}$. State the following mathematically and write their negations:

(a) c divides a. (Easier to think of: a is a multiple of c).
(b) c is a common divisor of a and b.
(c) d is the greatest common divisor of a and b.

(2) Let $a, b, c \in \mathbb{Z}$. Prove the following:

(a) $c|0$.
(b) If $a|b$ and $c|a$, then $c|b$.
(c) If $c|a$ and $c|b$, then $\forall m, n \in \mathbb{Z}, c|(ma + nb)$. In particular, $c|(a + b)$ and $c|(a - b)$.
(d) Suppose $c|a$. If $a \neq 0$, then $|a| \geq |c|$.
(e) If $a|c$ and $c|a$, then $a = \pm c$.

(3) Let $S \subset \mathbb{N}$ be such that (1) $1 \in S$ and (2) For $k \in \mathbb{N}$, if $\{1, 2, \cdots, k\} \subset S$, then $k+1 \in S$. Show that $S = \mathbb{N}$.

(4) Prove the following statement by (i) induction and (ii) well-ordering principle:
Given $n \in \mathbb{N} \setminus \{1\}$, there is a prime number $p \in \mathbb{N}$ such that $p|n$.

(5) Let $(a_n), (b_n)$ be sequences of real numbers which converge and $c \in \mathbb{R}$. Prove the following statements.

(a) The sequence $(a_n + b_n)$ converges.
(b) The sequence (ca_n) converges.
(c) The sequence $(a_n b_n)$ converges.

If you do not assume that (b_n) converges, what can you say about the convergence in each of the above cases?

(6) Let (a_n) be a sequence of real numbers. If (a_n) is bounded (converges), every subsequence is bounded (converges). If we assume that all the subsequences excluding the original sequence are bounded (convergent), then is the converse true?

(7) Show that every convergent sequence of real numbers is bounded. Is the converse true? Justify your answer.

(8) Let $(a_n), (b_n)$ be sequences of real numbers such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. If they converge to a and b respectively, then, $a \leq b$.

(9) Let (a_n) be a sequence of non-negative real numbers converging to $a \in \mathbb{R}$. Show that $(\sqrt{a_n})$ converges to $\sqrt{a}$.

Assignment 8

01-04-2019

(1) Let $a, b, c \in \mathbb{Z}$. Prove or disprove:

  (a) If $c|(a+b)$, then $c|a$ or $c|b$.
  (b) If $c|a$ or $c|b$, then $c|ab$.
  (c) If $c|ab$, then $c|a$ or $c|b$.

(2) Let $W = \mathbb{Z} \cap [-100, \infty)$. Show that every non-empty subset of W has a least element.

(3) Let $a_0 = 1$, and for $n \in \mathbb{N}$, let $a_n = \sqrt{2a_{n-1}}$.

  (a) Show that for each $n \in \mathbb{N}, a_{n+1} \geq a_n$. Hint: Use induction on n.
  (b) Show that the set $\{a_n : n \in \mathbb{N}\}$ is bounded above.
  (c) Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is convergent and find its limit.

(4) Prove or disprove: Let (a_n) and (b_n) be two convergent sequences of real numbers with limits a and b respectively. If $a_n < b_n$ for all $n \in \mathbb{N}$, then $a < b$.

(5) Show that the sequence $(p^{1/n})$ converges to 1 for all $p > 0$.
Hint: First prove for $p > 1$ by finding the limit of $a_n = (p^{1/n}) - 1$.

(6) (a) Let $a \in (-1, \infty)$ and $n \in \mathbb{N}$. Show that $(1+a)^n \geq 1 + na$.
  (b) Show that the sequence $a_n = (1 + 1/n)^n$ is strictly increasing.
  (c) Show that the sequence $b_n = (1 + 1/n)^{n+1}$ is strictly decreasing.
  (d) Show that both (a_n) and (b_n) converge to the same limit which lies in $(2, 4)$.

(7) Let $A \subset \mathbb{R}$ be bounded above and $\alpha = \text{lub}(A)$. Show that there is a sequence (a_n) <u>in A</u>, which converges to α.

(8) Let (a_n) be a bounded sequence. Recall the definition of limit inferior and limit superior defined in class. Show that

$$\text{glb}(a_n) \leq \liminf(a_n) \leq \limsup(a_n) \leq \text{lub}(a_n).$$

(9) Let $a, b \in \mathbb{Z}, L = \{c \in \mathbb{Z} | \exists k, l \in \mathbb{Z}(c = ka + lb)\}$, and $C = \{c \in \mathbb{N} | c|a \text{ and } c|b\}$.

  (a) Identify L and C when $(a, b) =$ (1) $(2, 3)$ (2) $(4, 6)$ (3) $(4, 8)$.
  (b) Show that if $c \in L$, then for all $n \in \mathbb{Z}, nc \in L$.
  (c) Show that if $c \in C$ and $d|c$, then $d \in C$.
  (d) Prove or disprove: For $a, b \in \mathbb{N}, L \cap C \neq \emptyset$.

Assignment 9

15-04-2019

(1) Given $a, b \in \mathbb{Z}, a \neq 0$, consider the set $R = \{c \in \mathbb{Z} | \exists q \in \mathbb{Z}(c = b - aq)\}$. Show that $R \cap \mathbb{N} \neq \emptyset$. What can you say about the least element of $R \cap \mathbb{N}$?

(2) Given $a, b \in \mathbb{Z} \setminus \{0\}$, find (a) $\gcd(a, 0)$ (b) $\gcd(a, 1)$ (c) $\gcd(a, a)$. If $a|b$, what is $\gcd(a, b)$? Show that $\gcd(a, b-a) = \gcd(a, b)$, $\gcd(a, a+b) = \gcd(a, b)$, and $\gcd(|a|, |b|) = gcd(a, b)$.

(3) Show that the set of primes in $\mathbb{N}$ is not finite.

(4) Consider the following relations on $\mathcal{A}$, identify whether it is reflexive, symmetric, transitive, or anti-symmetric.

(a) $\mathcal{A} = \mathbb{R}$, and for $a, b \in \mathcal{A}, a$ is related to b if $a \leq b$.
(b) $\mathcal{A} = \mathbb{R}^2$, and for $a = (a_1, a_2), b = (b_1, b_2) \in \mathcal{A}, a$ is related to b if $a_1 \leq b_1$ and $a_2 \leq b_2$. What happens if 'and' is replaced by 'or'?
(c) $\mathcal{A}$ is the set of subsets of a set X, and for $A, B \in \mathcal{A}$, A is related to B if $A \subset B$.
(d) $\mathcal{A}$ is the set of human beings, and for $a, b \in \mathcal{A}$, a is related to b if a is a mother/brother/sibling of b.
(e) $\mathcal{A}$ is the set of students at IITB in your batch, and for $a, b \in \mathcal{A}$, a is related to b if a is in the same hostel as b.
(f) $\mathcal{A} = \mathbb{Z}$, and for $a, b \in \mathcal{A}, a$ is related to b if $12|(b-a)$.
(g) $\mathcal{A} = \mathbb{R}^2$, and for $a, b \in \mathcal{A}, a$ is related to b if a has the same y-coordinate as b.
(h) $\mathcal{A} = \mathbb{R}$, and for $a, b \in \mathcal{A}, a$ is related to b if $b - a$ is an integer.

In each example, pick two unrelated points in $\mathcal{A}$, and identify all points related to each.

(5) Show that every convergent sequence of real numbers is a Cauchy sequence.

(6) Show that every Cauchy sequence of real numbers is bounded.

(7) (a) Show that the series $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$ is a divergent series.

(b) More generally, for $p \in \mathbb{Z}$, show that $\displaystyle\sum_{k=1}^{n} \frac{1}{k^p}$ converges if and only if $p \geq 1$.

(8) Show that if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

The converse need not be true, i.e., if $\displaystyle\lim_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ need not converge.

Give an example of this phenomenon.

(9) Write a_n in terms of n for the following sequences:
1. $\{3, 7, 11, 15, 19, \cdots\}$.
2. $\{2, 3, 4/2, 5/6, 6/24, 7/120, \cdots\}$.
3. $\{x^{3/2}, x^{9/4}, x^{15/8}, x^{21/16}, x^{27/32}, \cdots\}$.
4. $\{1, 3, 15, 105, 945, \cdots\}$.

(10) Find the limits of the following sequences if they exist:

1. $a_n = \dfrac{\sin^2(n\pi/6)}{2^n}$.
2. $a_n = \dfrac{\ln(3+2e^{n^2})}{n^2+1}$.
3. $a_n = n^p,\ p \in \mathbb{R}$.
4. $a_n = \dfrac{n!}{n^n}$.
5. $a_n = \dfrac{\ln(n)}{n}$.
6. $a_n = \ln(n+1) - \ln(n)$.
7. $a_n = \dfrac{(-5)^n}{n!}$.
8. $a_n = \dfrac{1+2n+4n^3}{2+3n^2}$.

(11) Find the limit of the series, i.e., find $\displaystyle\sum_{n=1}^{\infty} a_n$ where

(1) $a_n = \dfrac{1}{2^n}$. (2) $a_n = \dfrac{1}{n(n+1)}$. (3) $a_n = \dfrac{1}{n(\ln n)^p}$ if $p > 1$.

(12) Do the following series $\displaystyle\sum_{n=1}^{\infty} a_n$ converge or diverge?

(1) $a_n = sin(n\pi/2)$. (2) $a_n = \dfrac{\ln(n)}{n}$. (3) $a_n = \dfrac{10^n}{(n+1)4^{2n+1}}$.

Assignment 10

15-04-2019

(1) For $n \in \mathbb{N}$, show that the following operations on $\mathbb{Z}/n\mathbb{Z}$ are well-defined:
For all $a, b \in \mathbb{Z}$, $[a] + [b] = [a + b]$, and $[a][b] = [ab]$.
Prove that $\mathbb{Z}/n\mathbb{Z}$ is a field under these operations if and only if n is a prime.

(2) Show that $\mathcal{M}_2(\mathbb{R})$, the set of all 2×2 matrices with real entries, forms a group under matrix addition, and does not form a group under matrix multiplication.

(3) Let G be a group and $H \subset G$ be non-empty.
Show that H is a subgroup if and only if $ab^{-1} \in H$ for all $a, b \in H$.

(4) Check if H is a subgroup of the given group G.

(a) $G = \mathbb{Z}/12\mathbb{Z}$; $H =$ (i) $\{[1], [11]\}$ (ii) $\{[0], [3], [6], [9]\}$.
Group operation being $+$ as defined above.

(b) $G = \mathcal{M}_2(\mathbb{R})$; $H =$ (i) $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D \,\middle|\, c = 0 \right\}$ (ii) the set of invertible 2×2 matrices.
Group operation being $+$ as defined in the standard manner.

(c) $G = S_4$; $H =$ (i) $\{\mathrm{id}, (12), (34), (12)(34)\}$.
(ii) $\{(123), (134), (143), (132), (234), (243), (124), (142), \mathrm{id}\}$.
Group operation being the standard composition of cycles.

(5) Show that every group of prime order is cyclic.

(6) (a) Find all generators for $\mathbb{Z}/12\mathbb{Z}$, $\mathbb{Z}/9\mathbb{Z}$ and $\mathbb{Z}/13\mathbb{Z}$.
(b) What can you say in general about the generators of $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$ and $n \geq 2$.

(7) Consider the following relations on $\mathcal{A}$, identify whether it is an equivalence relation, a partial order, or a total order.

(a) $\mathcal{A}$ is the set of human beings, and for $a, b \in \mathcal{A}$,
a is related to b if a is a sibling of b.
(b) $\mathcal{A}$ is the set of students at IITB in your batch, and for $a, b \in \mathcal{A}$,
a is related to b if a is in the same hostel as b.
(c) $\mathcal{A} = \mathbb{R}$, and for $a, b \in \mathcal{A}, a$ is related to b if $a \leq b$.
(d) $\mathcal{A} = \mathbb{R}^2$, and for $a = (a_1, a_2), b = (b_1, b_2) \in \mathcal{A}, a$ is related to b if $a_1 \leq b_1$ and $a_2 \leq b_2$.
(e) $\mathcal{A} = \mathbb{R}^2$, and for $a = (a_1, a_2), b = (b_1, b_2) \in \mathcal{A}, a$ is related to b if $a_1 \leq b_1$ or $a_2 \leq b_2$.
(f) $\mathcal{A}$ is the set of subsets of a set X, and for $A, B \in \mathcal{A}$, A is related to B if $A \subset B$.
(g) $\mathcal{A} = \mathbb{Z}$, and for $a, b \in \mathcal{A}, a$ is related to b if $12 | (b - a)$.
(h) $\mathcal{A} = \mathbb{R}^2$, and for $a, b \in \mathcal{A}, a$ is related to b if a has the same y-coordinate as b.
(i) $\mathcal{A} = \mathbb{R}$, and for $a, b \in \mathcal{A}, a$ is related to b if $b - a$ is an integer.

In each example of an equivalence relation, pick a point in $a \in \mathcal{A}$, and identify $[a]$.
In each example of a partial (or total) order, pick a point in $\mathcal{A}$, and identify the points related to it.
In the examples in $\mathbb{R}$ or $\mathbb{R}^2$, identify these sets on the number line, or the $xy-$plane.