Assignment 9

15-04-2019

(1) Given $a, b \in \mathbb{Z}, a \neq 0$, consider the set $R = \{c \in \mathbb{Z} | \exists q \in \mathbb{Z}(c = b - aq)\}$. Show that $R \cap \mathbb{N} \neq \emptyset$. What can you say about the least element of $R \cap \mathbb{N}$?

(2) Given $a, b \in \mathbb{Z} \setminus \{0\}$, find (a) $\gcd(a, 0)$ (b) $\gcd(a, 1)$ (c) $\gcd(a, a)$. If $a|b$, what is $\gcd(a, b)$? Show that $\gcd(a, b-a) = \gcd(a, b)$, $\gcd(a, a+b) = \gcd(a, b)$, and $\gcd(|a|, |b|) = gcd(a, b)$.

(3) Show that the set of primes in $\mathbb{N}$ is not finite.

(4) Consider the following relations on $\mathcal{A}$, identify whether it is reflexive, symmetric, transitive, or anti-symmetric.

(a) $\mathcal{A} = \mathbb{R}$, and for $a, b \in \mathcal{A}, a$ is related to b if $a \leq b$.
(b) $\mathcal{A} = \mathbb{R}^2$, and for $a = (a_1, a_2), b = (b_1, b_2) \in \mathcal{A}, a$ is related to b if $a_1 \leq b_1$ and $a_2 \leq b_2$. What happens if 'and' is replaced by 'or'?
(c) $\mathcal{A}$ is the set of subsets of a set X, and for $A, B \in \mathcal{A}$, A is related to B if $A \subset B$.
(d) $\mathcal{A}$ is the set of human beings, and for $a, b \in \mathcal{A}$, a is related to b if a is a mother/brother/sibling of b.
(e) $\mathcal{A}$ is the set of students at IITB in your batch, and for $a, b \in \mathcal{A}$, a is related to b if a is in the same hostel as b.
(f) $\mathcal{A} = \mathbb{Z}$, and for $a, b \in \mathcal{A}, a$ is related to b if $12|(b - a)$.
(g) $\mathcal{A} = \mathbb{R}^2$, and for $a, b \in \mathcal{A}, a$ is related to b if a has the same y-coordinate as b.
(h) $\mathcal{A} = \mathbb{R}$, and for $a, b \in \mathcal{A}, a$ is related to b if $b - a$ is an integer.

In each example, pick two unrelated points in $\mathcal{A}$, and identify all points related to each.

(5) Show that every convergent sequence of real numbers is a Cauchy sequence.

(6) Show that every Cauchy sequence of real numbers is bounded.

(7) (a) Show that the series $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$ is a divergent series.

(b) More generally, for $p \in \mathbb{Z}$, show that $\displaystyle\sum_{k=1}^{n} \frac{1}{k^p}$ converges if and only if $p \geq 1$.

(8) Show that if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

The converse need not be true, i.e., if $\displaystyle\lim_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ need not converge.

Give an example of this phenomenon.

(9) Write a_n in terms of n for the following sequences:
1. $\{3, 7, 11, 15, 19, \cdots\}$.
2. $\{2, 3, 4/2, 5/6, 6/24, 7/120, \cdots\}$.
3. $\{x^{3/2}, x^{9/4}, x^{15/8}, x^{21/16}, x^{27/32}, \cdots\}$.
4. $\{1, 3, 15, 105, 945, \cdots\}$.

(10) Find the limits of the following sequences if they exist:

1. $a_n = \dfrac{\sin^2(n\pi/6)}{2^n}$.
2. $a_n = \dfrac{\ln(3+2e^{n^2})}{n^2+1}$.
3. $a_n = n^p,\ p \in \mathbb{R}$.
4. $a_n = \dfrac{n!}{n^n}$.
5. $a_n = \dfrac{\ln(n)}{n}$.
6. $a_n = \ln(n+1) - \ln(n)$.
7. $a_n = \dfrac{(-5)^n}{n!}$.
8. $a_n = \dfrac{1+2n+4n^3}{2+3n^2}$.

(11) Find the limit of the series, i.e., find $\sum_{n=1}^{\infty} a_n$ where

(1) $a_n = \dfrac{1}{2^n}$. (2) $a_n = \dfrac{1}{n(n+1)}$. (3) $a_n = \dfrac{1}{n(\ln n)^p}$ if $p > 1$.

(12) Do the following series $\sum_{n=1}^{\infty} a_n$ converge or diverge?

(1) $a_n = sin(n\pi/2)$. (2) $a_n = \dfrac{\ln(n)}{n}$. (3) $a_n = \dfrac{10^n}{(n+1)4^{2n+1}}$.