Exercise Sheet 7: Linear Transformations Cheat Sheet Template

Introduction

Linear transformations are fundamental mathematical operations that preserve vector space structures. This cheat sheet provides a concise overview of linear transformations, including definitions, properties, and examples.

1 Linear Transformations Cheat Sheet

1.1 Definition

Let V and W be vector spaces over the same field F. A function $T : V \to W$ is called a linear transformation if it satisfies the following properties:

1. Additivity: For all $\mathbf{u}, \mathbf{v} \in V$, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
2. Homogeneity: For all $\mathbf{u} \in V$ and $c \in F$, $T(c\mathbf{u}) = cT(\mathbf{u})$.

1.2 Properties

- A linear transformation preserves vector space operations: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = cT(\mathbf{u})$.
- The image of a linear transformation is a subspace of the codomain.
- The kernel (null space) of a linear transformation is a subspace of the domain.
- A linear transformation is injective (one-to-one) if and only if its kernel is trivial (contains only the zero vector).
- A linear transformation is surjective (onto) if and only if its image equals its codomain.

1.3 Examples

- Differentiation: The derivative operator $D : R[x] \to R[x]$ is a linear transformation.
- Matrix multiplication: Multiplication by a fixed matrix A defines a linear transformation $T_A : R^n \to R^m$.