Exercise Sheet 5: Eigenvalues and Eigenvectors Cheat Sheet Template

Introduction

Eigenvalues and eigenvectors are fundamental concepts in linear algebra, with wide-ranging applications in various fields including physics, engineering, computer science, and statistics.

1 Eigenvalues and Eigenvectors Cheat Sheet

1.1 Definition

Let A be an $n \times n$ matrix. An eigenvector of A is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ, which is called the eigenvalue corresponding to $\mathbf{v}$.

1.2 Finding Eigenvalues

To find the eigenvalues of matrix A, solve the characteristic equation $\det(A - \lambda I) = 0$, where I is the identity matrix of the same size as A.

1.3 Finding Eigenvectors

Once the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are found, the corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ can be obtained by solving the equation $(A - \lambda_i I)\mathbf{v}_i = \mathbf{0}$.

1.4 Example Problem

Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Solution

To find the eigenvalues, we solve the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0.$$

This quadratic equation factors as $(\lambda - 3)(\lambda - 1) = 0$, so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$.

For $\lambda_1 = 3$, solving $(A - 3I)\mathbf{v}_1 = \mathbf{0}$ gives $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 1$, solving $(A - I)\mathbf{v}_2 = \mathbf{0}$ gives $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.