Exercise Sheet 6: Vector Spaces Cheat Sheet Template

Introduction

Vector spaces are fundamental mathematical structures used to study linear algebraic concepts. This cheat sheet provides a concise overview of vector spaces, including definitions, properties, and examples.

1 Vector Spaces Cheat Sheet

1.1 Definition

A vector space V over a field F is a set of vectors equipped with two operations: vector addition and scalar multiplication, satisfying the following properties:

1. Closure under addition: For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in V$.
2. Closure under scalar multiplication: For all $\mathbf{u} \in V$ and $c \in F$, $c\mathbf{u} \in V$.
3. Commutativity of addition: For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
4. Associativity of addition: For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
5. Identity element of addition: There exists a zero vector $\mathbf{0} \in V$ such that for all $\mathbf{u} \in V$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
6. Inverse elements of addition: For every $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
7. Distributivity of scalar multiplication over field addition: For all $c, d \in F$ and $\mathbf{u} \in V$, $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
8. Distributivity of scalar multiplication over vector addition: For all $c \in F$ and $\mathbf{u}, \mathbf{v} \in V$, $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
9. Compatibility of scalar multiplication with field multiplication: For all $c, d \in F$ and $\mathbf{u} \in V$, $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. Identity element of scalar multiplication: For all $\mathbf{u} \in V$, $1\mathbf{u} = \mathbf{u}$.

1.2 Examples

- R^n, the set of n-dimensional real vectors, is a vector space over R.
- The space of polynomials of degree at most n with coefficients in R, denoted $R[x]_{\leq n}$, is a vector space over R.