Exercise Sheet 4: Advanced Linear Algebra Topics Template

Introduction

1 Advanced Linear Algebra Topics

1.1 Example Problem

Let V be a vector space with inner product $\langle \cdot, \cdot \rangle$. Show that the orthogonal complement of a subspace W of V is also a subspace of V.

Solution

Let W be a subspace of V. The orthogonal complement of W, denoted $W^\perp$, is defined as:

$$W^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

To show that $W^\perp$ is a subspace of V, we need to verify three conditions:

1. The zero vector is in $W^\perp$: Since $\langle \mathbf{0}, \mathbf{w} \rangle = 0$ for any $\mathbf{w} \in W$, $\mathbf{0}$ is in $W^\perp$.
2. Closure under addition: Let $\mathbf{v}_1, \mathbf{v}_2 \in W^\perp$. Then, for any $\mathbf{w} \in W$:

   $$\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w} \rangle + \langle \mathbf{v}_2, \mathbf{w} \rangle = 0 + 0 = 0,$$

   so $\mathbf{v}_1 + \mathbf{v}_2$ is in $W^\perp$.
3. Closure under scalar multiplication: Let c be a scalar and $\mathbf{v} \in W^\perp$. Then, for any $\mathbf{w} \in W$:

   $$\langle c\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle = c \cdot 0 = 0,$$

   so $c\mathbf{v}$ is in $W^\perp$.

Since all three conditions are satisfied, $W^\perp$ is a subspace of V.