SMOOTHED PARTICLE HYDRODYNAMICS

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11th Apr 2023

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COMPUTATIONAL ASTROPHYSICS

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Content

01 - Introduction

- (1977) Gingold & Monaghan / Lucy
  Application field: Astrophysics

- A mesh-free particle method:
  Discretize functions and
  differential operators

Mon. Not. R. astr. Soc. (1977) 181, 375–389

Smoothed particle hydrodynamics: theory and application to non-spherical stars

R. A. Gingold and J. J. Monaghan* *Institute of Astronomy, Madingley Road, Cambridge, CB3 0HA*

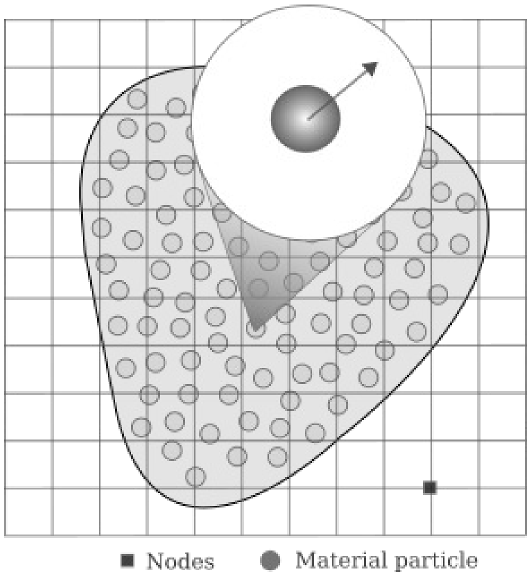
Received 1977 May 5, in original form February 2

THE ASTRONOMICAL JOURNAL VOLUME 82, NUMBER 12 DECEMBER 1977

A numerical approach to the testing of the fission hypothesis

L. B. Lucy[a)]
Institute of Astronomy, Cambridge, United Kingdom
European Southern Observatory, Geneva, Switzerland
(Received 12 August 1977; revised 16 September 1977)

- Interpretation of SPH sample
  Math.: Coefficients (App.)
  Phys.: Particles (Properties)



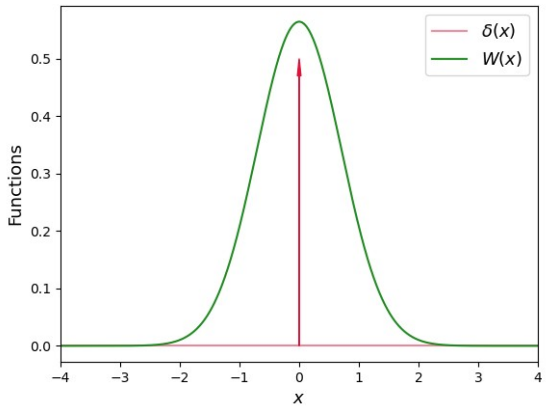
02 – Formulation of SPH

1. Kernel Approximation

Integral representation of a function:

The dirac-δ identity $\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}, \int_\Omega \delta(x)dx = 1$

$$A(\mathbf{r}) = \int_\Omega A(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')d\mathbf{r}'$$



Kernel approximation

Introduce a Smoothing function

$$A(\mathbf{r}) = \int_\Omega A(\mathbf{r}')W(\mathbf{r} - \mathbf{r}', h)d\mathbf{r}'$$

02 – Formulation of SPH

Conditions

Delta function property

$$\lim_{h\to 0} W(x,h) = \delta(x)$$

Normalization property

$$\int_\Omega W(x,h)dx = 1$$

Optionals

Even function

$$W(x,h) = W(-x,h)$$

Positivity defined

$$W(x,h) \geq 0$$

Compact support

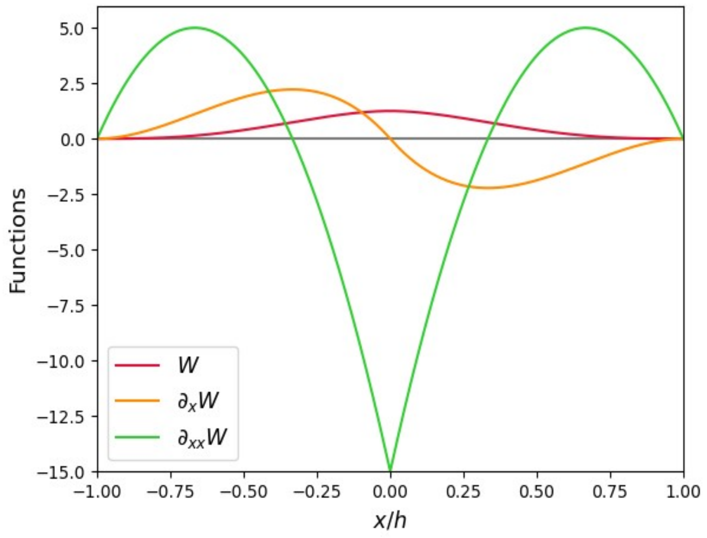
$$W(x,h) = 0 \quad \forall x \notin \Omega$$

Decay: monotonic decreasing

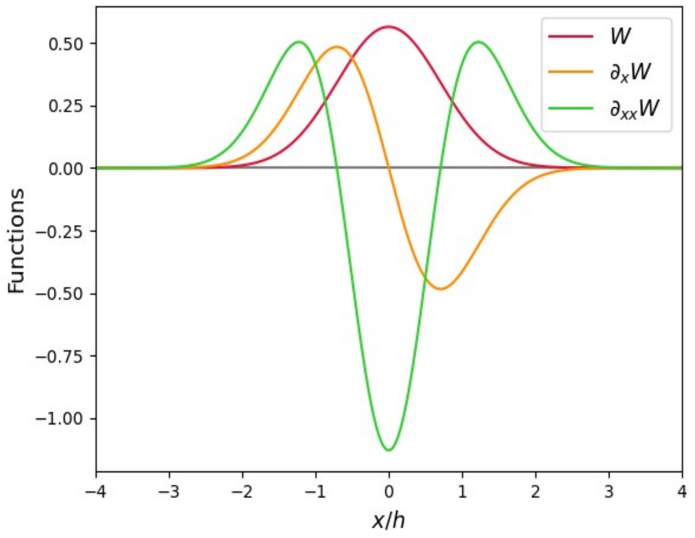
2th order app.

$$f_k(x) = f(x) + \mathcal{O}(h^2)$$

02 – Formulation of SPH



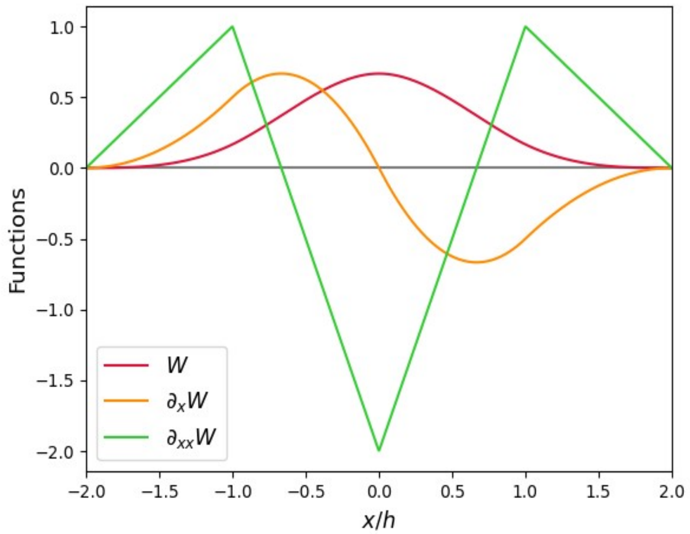
Bell-shaped kernel
Lucy (1977)



Gaussian kernel
Gingold & Monaghan (1977)

02 – Formulation of SPH

$$W(x,h) = \alpha_d \begin{cases} \frac{2}{3} - x^2 + \frac{1}{2}x^3 & 0 \leq x < 1 \\ \frac{1}{6}(2-x)^3 & 1 \leq x < 2 \\ 0 & 2 \leq x \end{cases}$$



Cubic Spline Kernel
Monaghan & Lattanzio (1985)

02 – Formulation of SPH

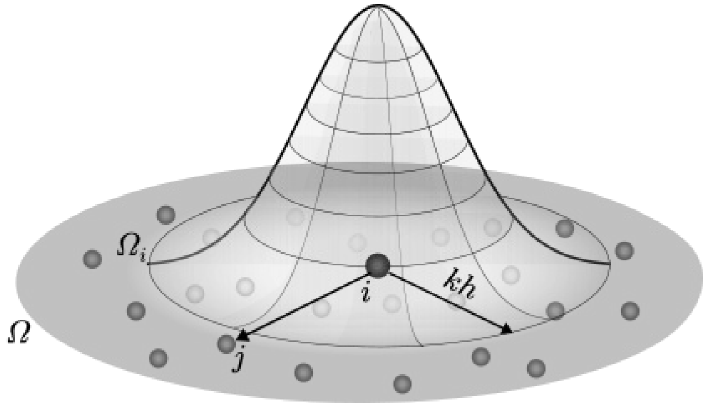
2. Particle Approximation

Discretization by a finite number of particles.

$$A(\mathbf{r}) = \int_\Omega \frac{A(\mathbf{r}')}{\rho(\mathbf{r}')} W(\mathbf{r}-\mathbf{r}', h)\rho(\mathbf{r}')d\mathbf{r}'$$

$$\approx \sum_{j=1}^{N} \frac{m_j}{\rho_j} A_j W(\mathbf{r}-\mathbf{r}_j, h)$$

$$\rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(\mathbf{r}-\mathbf{r}_j, h)$$



Particle approximations using W support domain for particle i.

02 – Formulation of SPH

3. Operator discretization

Direct SPH form:

$$\nabla A_i = \sum_{j=1}^{N} \frac{m_j}{\rho_j} A_j \nabla_i W_{ij}$$

$$\nabla \cdot A_i = \sum_{j=1}^{N} \frac{m_j}{\rho_j} A_j \cdot \nabla_i W_{ij}$$

$$\nabla \times A_i = \sum_{j=1}^{N} \frac{m_j}{\rho_j} A_j \times \nabla_i W_{ij}$$

Symmetric expressions

$$\nabla A_i = \frac{1}{\rho_i} \sum_{j=1}^{N} m_j \, [A_j - A_i] \nabla_i W_{ij}$$

$$\nabla A_i = \sum_{j=1}^{N} \frac{m_j}{\rho_j} \, [A_j - A_i] \nabla_i W_{ij}$$

$$\nabla A_i = \rho_i \sum_{j=1}^{N} m_j \left[\frac{A_j}{\rho_j^2} - \frac{A_i}{\rho_i^2} \right] \nabla_i W_{ij}$$

03 – SPH for Fluid Dynamics

1. SPH Euler equations

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{v}$$
$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla P + \mathbf{g}$$

Gingold & Monaghan (1977), Lucy (1977)

$$\frac{dv_i}{dt} = -\frac{1}{\rho_i}\sum_{j=1}^{N}\frac{m_j}{\rho_j}P_j\nabla_i W_{ij}$$

Does not conserve linear or angular momentum!

Gingold & Monaghan (1982)

$$\frac{dv_i}{dt} = -\sum_{j=1}^{N} m_j\left[\frac{P_i}{\rho_i^2}+\frac{P_j}{\rho_j^2}\right]\nabla_i W_{ij}$$

Writing

$$\nabla_i W_{ij} = \mathbf{r}_{ij}F_{ij}$$

$$\mathbf{f}_{ij} = m_i m_j\left[\frac{P_i}{\rho_i^2}+\frac{P_j}{\rho_j^2}\right]\mathbf{r}_{ij}F_{ij}$$
$$= -\mathbf{f}_{ji}$$

03 – SPH for Fluid Dynamics

2. The energy equation

$$\hat{e} = \frac{1}{2}v^2 + u \qquad \frac{du}{dt} = -\frac{P}{\rho}\nabla \cdot \mathbf{v}$$

$$\frac{d\hat{e}}{dt} = -\frac{1}{\rho}\nabla \cdot (P\mathbf{v})$$

$$\frac{d\hat{e}}{dt} = -\sum_{j=1}^{N} m_j \left[\frac{P_i \mathbf{v}_j}{\rho_i^2} + \frac{P_j \mathbf{v}_i}{\rho_j^2}\right] \cdot \nabla_i W_{ij}$$

Conservation of the total thermokinetic energy!

04 – Examples

1. Toy stars in 1D

(Monaghan & Price, 2004)

Compressibility is retained, but the gravitational force is replaced by a force linear in the coordinates.

$$\Phi = \frac{1}{4}\lambda \sum_i \sum_j m_i m_j (x_i - x_j)^2 = \frac{1}{2}\lambda M \sum_j m_j x_j^2$$

At the CM frame, the motion equation becomes

$$\frac{d^2 x_j}{dt^2} + \Omega^2 x_j = 0 \quad \text{where } \Omega^2 = \lambda M$$

04 – Examples

1. Toy stars in 1D
(Monaghan & Price, 2004)

The Euler equation

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla P + \mathbf{g}$$

The polytropic equation

$$P = k\rho^{1+1/n}$$

In the CM,

$$\frac{d\mathbf{v}_i}{dt} = -\nu\mathbf{v}_i + \sum_{j,j\neq i} m_j \left[\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right] \nabla_i W_{ij} - \lambda x_i$$

The system reaches an equilibrium state.

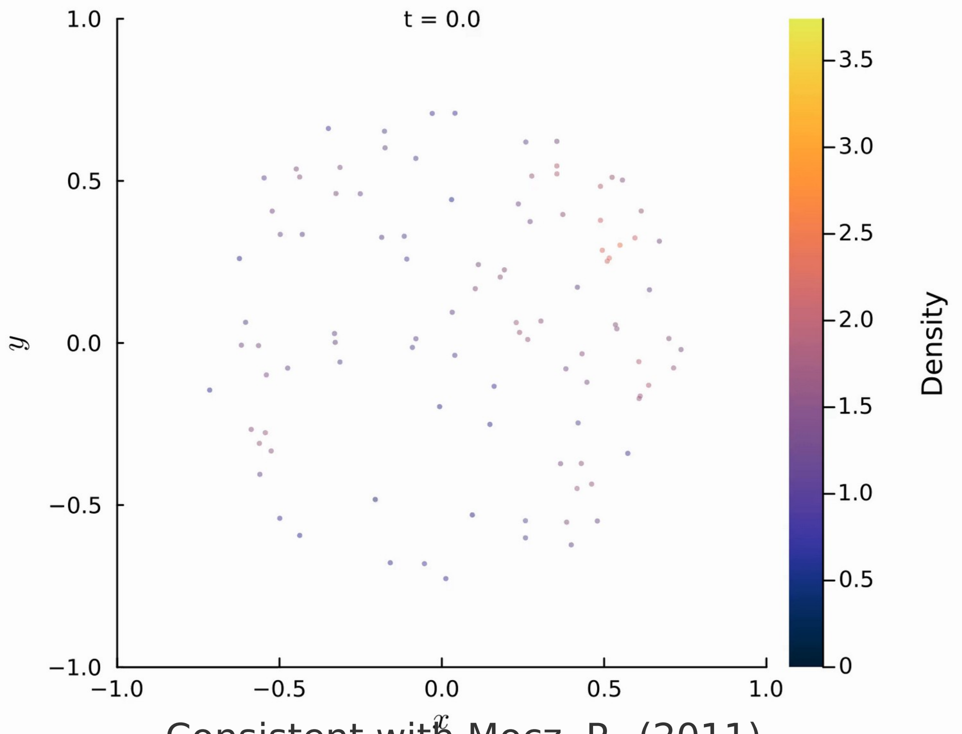
If $n = 1$,

$$\rho(r) = \frac{\lambda}{4k}(R^2 - r^2).$$

04 – Examples

Simulation 1	Typical star collapse into equilibrium
Parameter	**Value**
Number of Particles	N=100
dimension	d=2
Initial time	$t_0 = 0$
Final time	$t_f = 20$
Time step	dt = 0.04
Star mass	M=2
Radius mass	R=0.75
Pressure Constant	k= 0.1
Polytropic index	n=1
Viscosity coefficient	$\nu = 1$
Smoothing lenth	$h = 0.04/\sqrt{N/1000}$
Kernel	Truncated Gaussian Kernel
Initial config.	random inside circle radius R

Table 1. Parameters for the toy star simulation 1

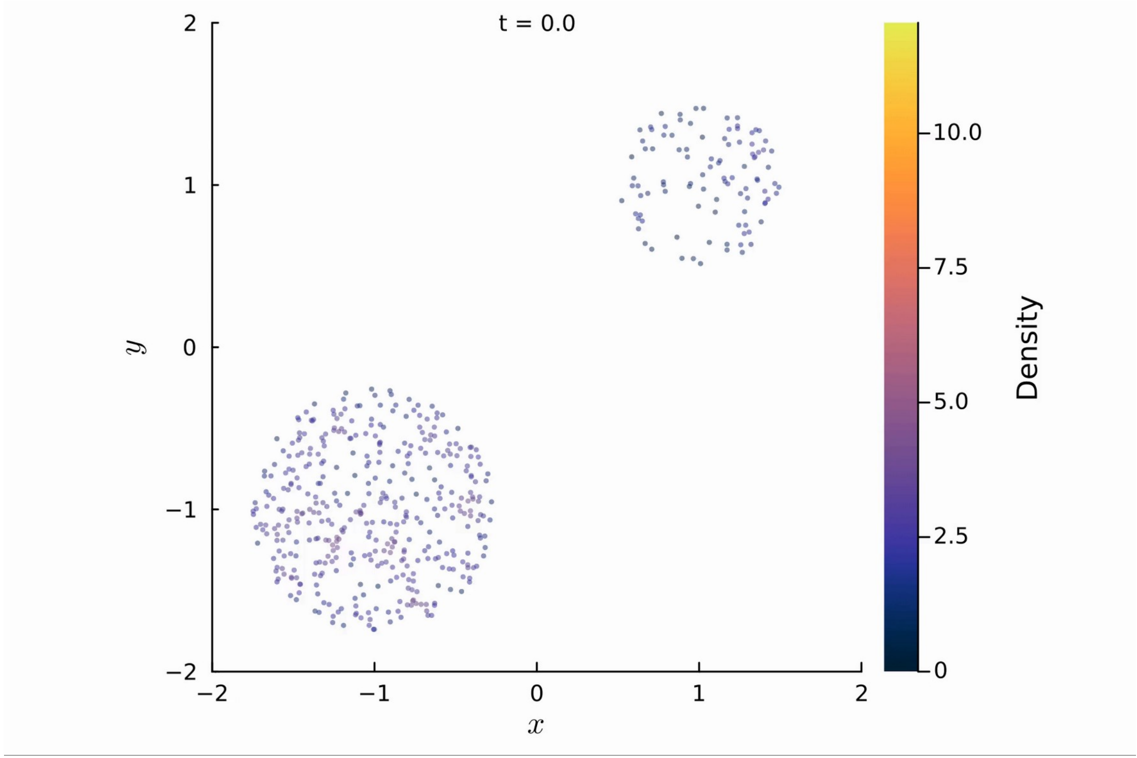


Consistent with Mocz, P., (2011)

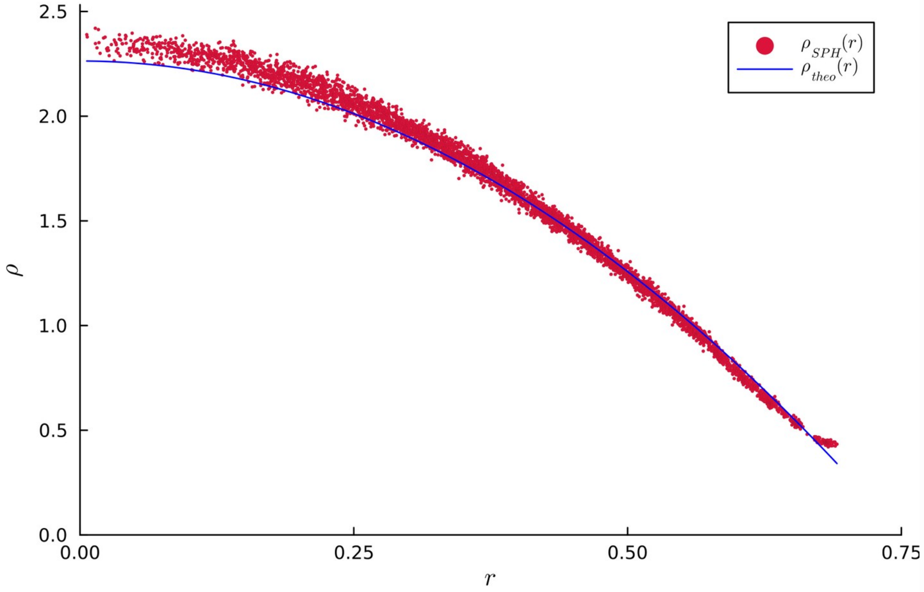
04 – Examples

Simulation 2	soft collision of 2 stars – head on
Parameter	**Value**
Star 1	
Number of particles	$\mathrm{N}_1 = 400$
Star mass	$\mathrm{M}_1 = 4$
Radius mass	$\mathrm{R}_1 = 0.75$
Star 2	
Number of particles	$\mathrm{N}_2 = 100$
Star mass	$\mathrm{M}_2 = 1$
Radius mass	$\mathrm{R}_2 = 0.5$
Total number of particles	$\mathrm{N}{=}\mathrm{N}_1 + N_2$
Smoothing lenth	$\mathrm{h} = 0.04/\sqrt{N/1000}$
Particle mass	$\mathrm{m} = (\mathrm{M}_1 + M_2)/N$

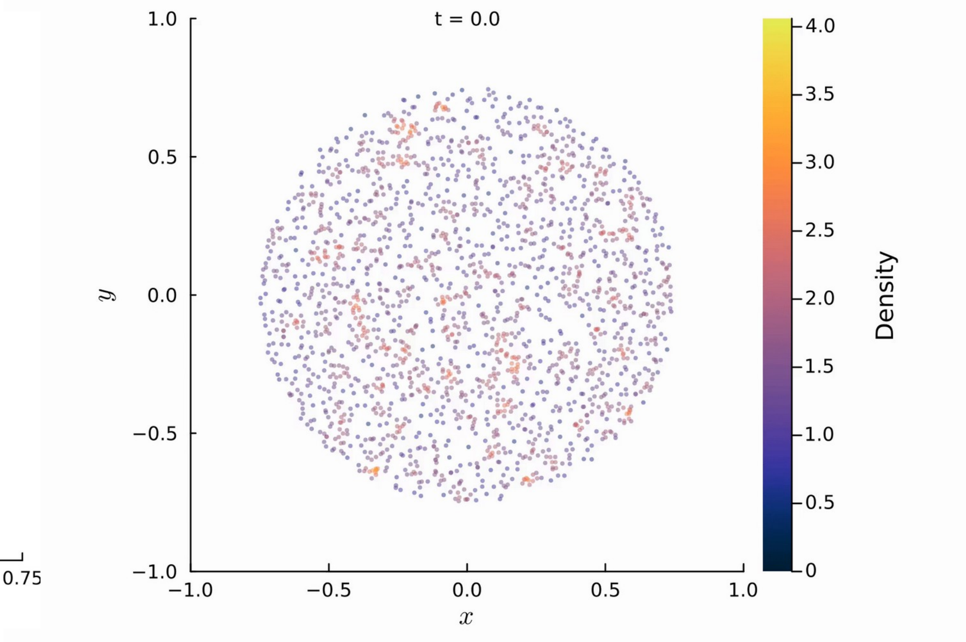
Table 2. Parameters for the toy star simulation 2



04 – Examples

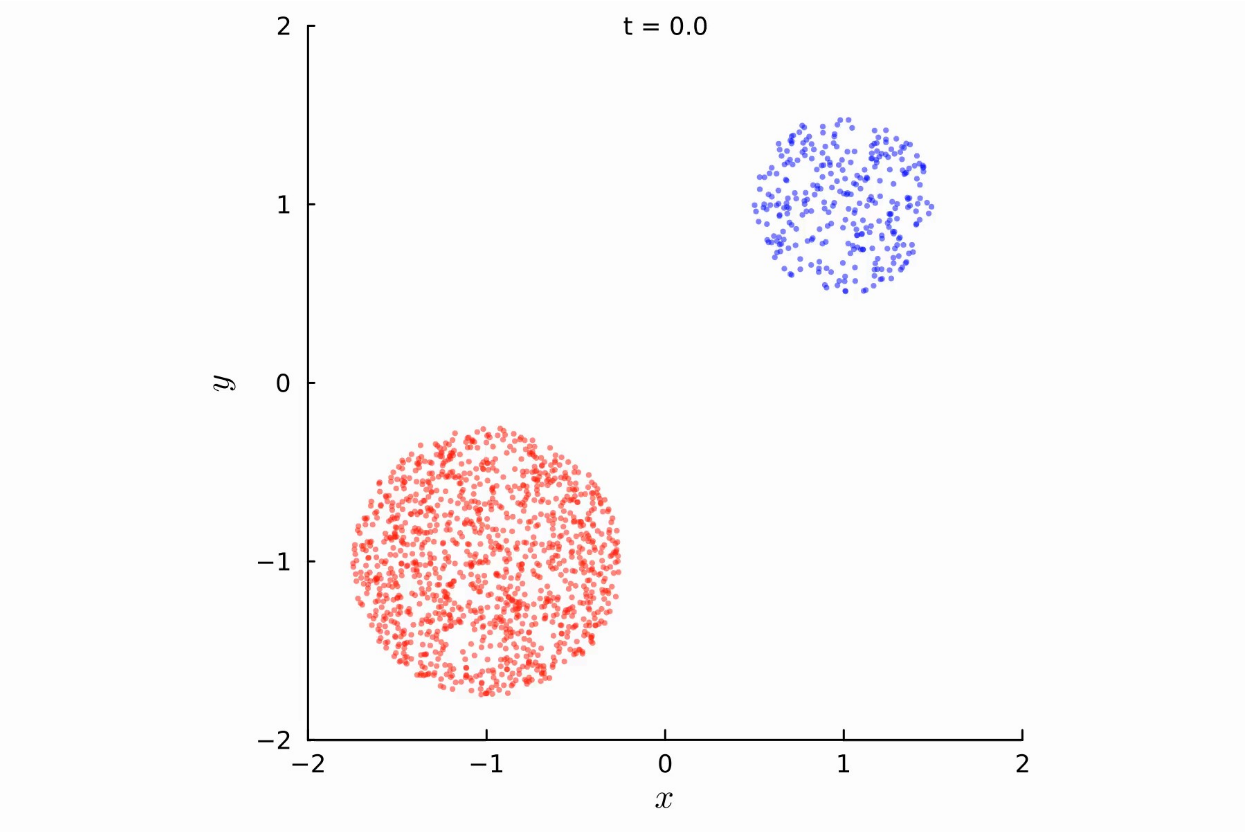


N = 5000



N = 2000

04 – Examples

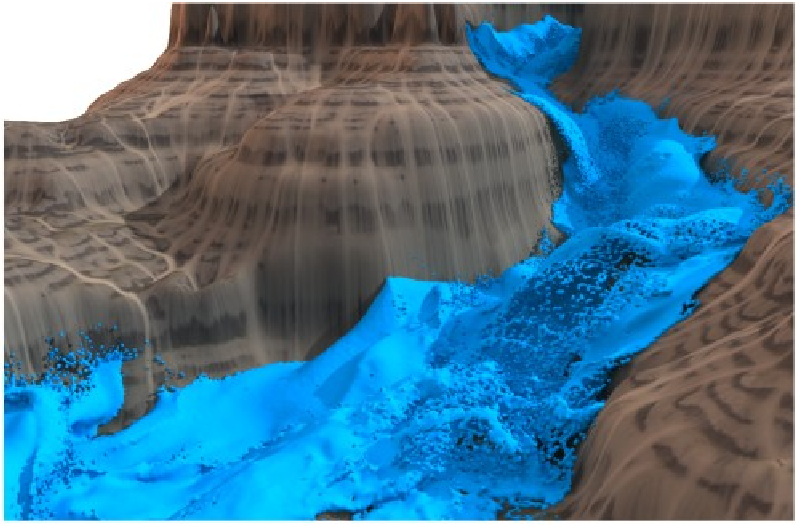


N = 1200

05 – Applications

1. Adventages

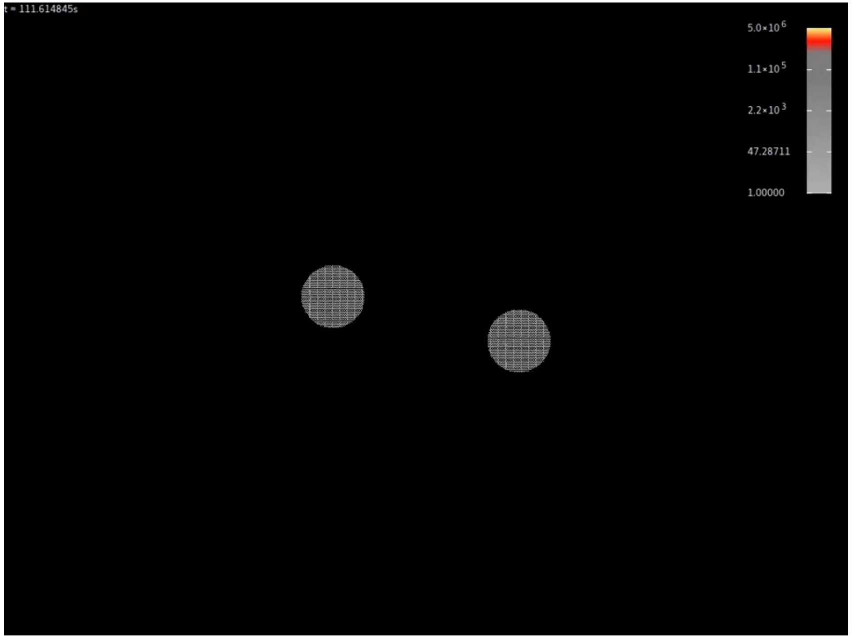
- Conservations laws.
- Interfaces Problems.
- Bridge the gap between the continuum and fragmentation.
- Complex physics are often included easily.
- Computational Advantage: **Parallelization**.



Bender & Koschier (2015)

05 – Applications

2. Astrophysics



Collision of two equal-sized bodies,
Ševeček (2018)

05 – Applications

2. Video Games and Films

The Lord of the Rings: The Return of the King

06 – References

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Thank you!