Date:

Partial Differentiation

Ex 3(a)

1) a) $f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$\cancel{\frac{\delta f}{\delta x} = 2ax + 2hy + 2g}$

$$\frac{\delta f}{\delta x} = \frac{\delta}{\delta x}(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$

$$= 2ax + 2hy + 2g$$

$$= 2(ax + hy + g)$$

$$\frac{\delta f}{\delta y} = \frac{\delta}{\delta y}(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$

$$= 2hx + 2by + 2f$$

$$= 2(by + hx + f)$$

b) $f(x,y) = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$\frac{\delta f}{\delta x} = \frac{1}{1+(x^2+y^2/x+y)^2}\left[\frac{(x+y)(2x) - (x^2+y)^2}{(x+y)^2}\right]$$

$$= \frac{\cancel{(x+y)^2}}{(x+y)^2+(x^2+y^2)^2} \times \frac{x^2+2xy-y^2}{\cancel{(x+y)^2}}$$

$$= \frac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)^2}$$

$$\frac{\delta f}{\delta y} = \frac{1}{1+(x^2+y^2/x+y)^2}\left[\frac{(x+y)(2y) - (x^2+y^2)}{(x+y)^2}\right.$$

$$= \frac{\cancel{(x+y)^2}}{(x+y)^2+(x^2+y^2)^2} \times \frac{y^2+2xy-x^2}{\cancel{(x+y)^2}}$$

$$= \frac{y^2+2xy-x^2}{(x+y)^2+(x^2+y^2)^2}$$

c) $f(x,y) = y^x$

$f = y^x$

$\log f = x \log y$

diff wrt x,

$$\frac{1}{f}\frac{\delta f}{\delta x} = \log y$$

$$\frac{\delta f}{\delta x} = y^x \log y$$

$\log f = x \log y$

diff wrt y,

$$\frac{1}{f}\frac{\delta f}{\delta y} = \frac{x}{y}$$

$$\frac{\delta f}{\delta y} = \frac{xy^x}{y} = xy^{x-1}$$

2) a) $u = c\tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\delta u}{\delta x} = \frac{c}{1+x^2/y^2}\left[\frac{\cancel{y}}{y^{\cancel{2}}}\right]$$

$$= \frac{cy^{\cancel{2}}}{x^2+y^2} \times \frac{1}{\cancel{y}}$$

$$= \frac{cy}{x^2+y^2}$$

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta y}\left(\frac{cy}{x^2+y^2}\right)$$

$$= \frac{c(x^2+y^2) - cy(2y)}{(x^2+y^2)^2}$$

$$= \frac{c(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\delta u}{\delta y} = \frac{c}{1+x^2/y^2}\left[\frac{-x}{y^2}\right]$$

$$= \frac{-cxy^{\cancel{2}}}{x^2+y^2} \times \frac{1}{\cancel{y^2}}$$

$$= \frac{-cx}{x^2+y^2}$$

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta x}\left(\frac{-cx}{x^2+y^2}\right)$$

$$= \frac{-c(x^2+y^2) + cx(2x)}{(x^2+y^2)^2}$$

$$= \frac{c(x^2-y^2)}{(x^2+y^2)^2}$$

$$\therefore \frac{\delta^2 u}{\delta x\,\delta y} = \frac{\delta u^2}{\delta y\,\delta x}$$

Date:

b) $u = \log(x^2+y^2)$

$$\frac{\delta u}{\delta x} = \frac{2x}{x^2+y^2}$$

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta y}\left(\frac{2x}{x^2+y^2}\right)$$

$$= -\frac{2x(2y)}{(x^2+y^2)^2}$$

$$= \frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{\delta u}{\delta y} = \frac{2y}{x^2+y^2}$$

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta x}\left(\frac{2y}{x^2+y^2}\right)$$

$$= -\frac{2y(2x)}{(x^2+y^2)^2}$$

$$= \frac{-4xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\delta^2 u}{\delta x \delta y} = \frac{\delta^2 u}{\delta y \delta x}$$

c) $u = x\cos y + y\cos x$

$$\frac{\delta u}{\delta x} = \cos y - y\sin x$$

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta y}(\cos y - y\sin x)$$

$$= -\sin y - \sin x$$

$$= -(\sin x + \sin y)$$

Page No.
Date:

$$\frac{\delta u}{\delta y} = -x\sin y + \cos x$$

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta x}(-x\sin y + \cos x)$$

$$= -\sin y - \sin x$$

$$= -(\sin x + \sin y)$$

$$\therefore \frac{\delta^2 u}{\delta x \delta y} = \frac{\delta^2 u}{\delta y \delta x}$$

3) $$z = \frac{x^2y^2}{(x+y)}$$

$$\frac{\delta z}{\delta x} = \frac{(x+y)(2xy^2) - (x^2y^2)}{(x+y)^2}$$

$$= \frac{x^2y^2 + 2xy^3}{(x+y)^2}$$

$$\frac{\delta^2 z}{\delta x^2} = \frac{(x+y)^2[2xy^2 + 2y^3] - (x^2y^2 + 2xy^3) \times 2(x+y)}{(x+y)^4}$$

$$= \frac{2y^2}{(x+y)} - \frac{2xy^2(x+2y)}{(x+y)^3}$$

$$x\frac{\delta^2 z}{\delta x^2} = \frac{2xy^2}{(x+y)} - \frac{2x^2y^2(x+2y)}{(x+y)^3}$$

$$\frac{\delta}{\delta y}\left(\frac{\delta z}{\delta x}\right) = \frac{\delta}{\delta y}\left[\frac{x^2y^2 + 2xy^3}{(x+y)^2}\right]$$

$$= \frac{(x+y)^2[2x^2y + 6xy^2] - (x^2y^2 + 2xy^3) \times 2(x+y)}{(x+y)^4}$$

$$= \frac{2xy(x+3y)}{(x+y)^2}$$

Date:

$$\frac{\delta z}{\delta y} = \frac{(x+y)(2x^2y) - (x^2y^2)}{(x+y)^2}$$

$$= \frac{2x^3y + x^2y^2}{(x+y)^2}$$

$$\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta y}\right) = \frac{(x+y)^2[6x^2y + 2xy^2] - (2x^3y + x^2y^2) \times 2(x+y)}{(x+y)^4}$$

$$= \frac{2xy(3x+y)}{(x+y)^2} - \frac{2x^2y(2x+y)}{(x+y)^3}$$

$$y\frac{\delta^2 z}{\delta x \delta y} = \frac{2xy^2(3x+y)}{(x+y)^2} - \frac{2x^2y^2(2x+y)}{(x+y)^3}$$

$$\therefore x\frac{\delta^2 z}{\delta x^2} + y\frac{\delta^2 z}{\delta x \delta y} = \frac{2xy^2}{(x+y)} + \frac{2xy^2(3x+y)}{(x+y)^2} - \frac{2x^2y^2(x+2y)}{(x+y)^3} - \frac{2x^2y^2(2x+y)}{(x+y)^3}$$

$$\Rightarrow \frac{2xy^2}{(x+y)}\left[1 + \frac{3x+y}{x+y}\right] - \frac{2x^2y^2}{(x+y)^3}[x+2y+2x+y]$$

$$\Rightarrow \frac{2xy^2[4x+2y]}{(x+y)^2} - \frac{2x^2y^2 \times 3(x+y)}{(x+y)^3}$$

$$\Rightarrow \frac{4xy^2(2x+y)}{(x+y)^2} - \frac{\cancel{8}6x^2y^2}{(x+y)^2}$$

$$\Rightarrow \frac{8x^2y^2 + 4xy^3 - 6x^2y^2}{(x+y)^2}$$

$$\Rightarrow \frac{2x^2y^2 + 4xy^3}{(x+y)^2}$$

$$\Rightarrow \frac{2[x^2y^2 + 2xy^3]}{(x+y)^2}$$

$$\Rightarrow 2\frac{\delta z}{\delta x}$$

Scanned by CamScanner

4) $u = e^{xyz}$

$\log u = xyz$

diff wrt x,

$\frac{1}{u}\frac{\delta u}{\delta x} = yz \Rightarrow \frac{\delta u}{\delta x} = uyz$

diff wrt y,

$\frac{1}{u}\frac{\delta u}{\delta y} = xz \rightarrow \frac{\delta u}{\delta y} = uxz$

diff wrt z,

$\frac{1}{u}\frac{\delta u}{\delta z} = xy \Rightarrow \frac{\delta u}{\delta z} = uxy$

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta z}\right) = \frac{\delta}{\delta y}(uxy) = \frac{\delta}{\delta y}(e^{xyz}xy)$$

$$= xe^{xyz} + xy\,e^{xyz}(xz)$$

$$= e^{xyz}(x + x^2yz)$$

$$\frac{\delta}{\delta x}\left(\frac{\delta^2 u}{\delta y\,\delta z}\right) = \frac{\delta}{\delta x}[e^{xyz}(x + x^2yz)]$$

$$= e^{xyz}[1 + 2xyz] + (x + x^2yz)e^{xyz}(yz)$$

$$\therefore \frac{\delta^3 u}{\delta x\,\delta y\,\delta z} = e^{xyz}[1 + 2xyz + xyz + x^2y^2z^2]$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

Scanned by CamScanner

Page No.
Date:

5) $z = f(x+ay) + \phi(x-ay)$

$$\frac{\delta z}{\delta x} = f'(x+ay) + \phi'(x-ay)$$

$$\frac{\delta^2 z}{\delta x^2} = f''(x+ay) + \phi''(x-ay)$$

$$\frac{\delta z}{\delta y} = af'(x+ay) + a\phi'(x-ay)$$

$$\frac{\delta^2 z}{\delta y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay)$$

$$\frac{\delta^2 z}{\delta y^2} = a^2[f''(x+ay) + \phi''(x-ay)]$$

$$\frac{\delta^2 z}{\delta y^2} = a^2\frac{\delta^2 z}{\delta x^2}$$

6) $r^2 = x^2 + y^2 + z^2$

diff wrt x,

$$\cancel{2}r\frac{\delta r}{\delta x} = \cancel{2}x \Rightarrow \frac{\delta r}{\delta x} = \frac{x}{r}$$

diff wrt y,

$$\cancel{2}r\frac{\delta r}{\delta y} = \cancel{2}y \Rightarrow \frac{\delta r}{\delta y} = \frac{y}{r}$$

diff wrt z,

$$\cancel{2}r\frac{\delta r}{\delta z} = \cancel{2}z \Rightarrow \frac{\delta r}{\delta z} = \frac{z}{r}$$

Page No.
Date:

$$V = r^m$$

$$\frac{\delta V}{\delta x} = m r^{m-1} \frac{\delta r}{\delta x} = m r^{m-1}\left(\frac{x}{r}\right) = m x r^{m-2}$$

$$\frac{\delta^2 V}{\delta x^2} = m r^{m-2} + m(m-2) x\, r^{m-3} \cdot \frac{\delta r}{\delta x}$$

$$= m r^{m-2} + m(m-2) x^2 \frac{r^{m-2}}{r^2}$$

$$\frac{\delta V}{\delta y} = m r^{m-1} \frac{\delta r}{\delta y} = m r^{m-1}\left(\frac{y}{r}\right) = m y r^{m-2}$$

$$\frac{\delta^2 V}{\delta y^2} = m r^{m-2} + m(m-2) y\, r^{m-3} \frac{\delta r}{\delta y}$$

$$= m r^{m-2} + m(m-2) y^2 \frac{r^{m-2}}{r^2}$$

$$\frac{\delta V}{\delta z} = m r^{m-1} \frac{\delta r}{\delta z} = m r^{m-1}\left(\frac{z}{r}\right) = m z r^{m-2}$$

$$\frac{\delta^2 V}{\delta z^2} = m r^{m-2} + m(m-2) z\, r^{m-3} \frac{\delta r}{\delta z}$$

$$= m r^{m-2} + m(m-2) z^2 \frac{r^{m-2}}{r^2}$$

$$\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V}{\delta z^2} = 3 m r^{m-2} + m(m-2) \frac{r^{m-2}}{\cancel{r^2}} \cancel{(x^2+y^2+z^2)}$$

$$= r^{m-2}[3m + m^2 - 2m]$$

$$= r^{m-2}[m^2 + m]$$

$$= m(m+1) r^{m-2}$$

Date:

7) $z(x+y) = x^2+y^2$

$$z = \frac{x^2+y^2}{x+y}$$

$$\frac{\delta z}{\delta x} = \frac{(x+y)(2x) - (x^2+y^2)}{(x+y)^2}$$

$$= \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\delta z}{\delta y} = \frac{(x+y)(2y) - (x^2+y^2)}{(x+y)^2}$$

$$= \frac{y^2+2xy-x^2}{(x+y)^2}$$

$$\frac{\delta z}{\delta y} - \frac{\delta z}{\delta x} = \frac{y^2+\cancel{2xy}-x^2-x^2-\cancel{2xy}+y^2}{(x+y)^2}$$

$$= \frac{2(y^2-x^2)}{(x+y)^2}$$

$$= \frac{2(y-x)}{y+x}$$

$$\left(\frac{\delta z}{\delta y} - \frac{\delta z}{\delta x}\right)^2 = \frac{4(y-x)^2}{(y+x)^2}$$

$$4\left(1 - \frac{\delta z}{\delta y} - \frac{\delta z}{\delta x}\right) = 4\left[1 - \left(\frac{\cancel{x^2}+2xy-\cancel{y^2}+\cancel{y^2}+2xy-\cancel{x^2}}{(x+y)^2}\right)\right]$$

$$= 4\left[1 - \frac{4xy}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2-4xy}{(x+y)^2}\right]$$

$$= \frac{4(y-x)^2}{(y+x)^2}$$

$$\therefore \left(\frac{\delta z}{\delta x} - \frac{\delta z}{\delta y}\right)^2 = 4\left(1 - \frac{\delta z}{\delta x} - \frac{\delta z}{\delta y}\right)$$

8) $z = xy\, f\left(\frac{x}{y}\right)$

$$\frac{\delta z}{\delta x} = y\left[x f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) + f\left(\frac{x}{y}\right)\right]$$

$$= y\left[\frac{x}{y} f'\left(\frac{x}{y}\right) + f\left(\frac{x}{y}\right)\right.$$

$$x\frac{\delta z}{\delta x} = x^2 f'\left(\frac{x}{y}\right) + yx\, f\left(\frac{x}{y}\right)$$

$$\frac{\delta z}{\delta y} = x\left[\cancel{y} f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + f\left(\frac{x}{y}\right)\right]$$

$$= x\left[-\frac{x}{y} f'\left(\frac{x}{y}\right) + f\left(\frac{x}{y}\right)\right.$$

$$y\frac{\delta z}{\delta y} = -x^2 f'\left(\frac{x}{y}\right) + xy\, f\left(\frac{x}{y}\right)$$

$$x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = \cancel{x^2 f'\left(\frac{x}{y}\right)} + 2xy\, f\left(\frac{x}{y}\right) - \cancel{x^2 f'\left(\frac{x}{y}\right)}$$

$$= 2xy\, f\left(\frac{x}{y}\right)$$

$$= 2z$$

9) $f(x,y) = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\delta f}{\delta x} = \frac{2x}{x^2+y^2} + \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} + \frac{\cancel{x^2}}{x^2+y^2}\left(\frac{-y}{\cancel{x^2}}\right)$$

$$= \frac{2x-y}{x^2+y^2}$$

$$\frac{\delta f}{\delta y} = \frac{2y}{x^2+y^2} + \frac{1}{1+y^2/x^2}\left(\frac{x}{x^2}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{\cancel{x^2}}{x^2+y^2}\left(\frac{x}{\cancel{x^2}}\right)$$

$$= \frac{2y+x}{x^2+y^2}$$

$$\frac{\delta^2 f}{\delta x^2} = \frac{(x^2+y^2)(2) - (2x-y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{2(y^2+2xy-x^2)}{(x^2+y^2)^2}$$

$$\frac{\delta^2 f}{\delta y^2} = \frac{(x^2+y^2)(2) - (2y+x)(2y)}{(x^2+y^2)^2}$$

$$= \frac{2(x^2-2xy-y^2)}{(x^2+y^2)^2}$$

$$\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} = \frac{2}{(x^2+y^2)^2}\left[\cancel{y^2}+\cancel{2xy}-\cancel{x^2}+\cancel{x^2}-\cancel{2xy}-\cancel{y^2}\right]$$

$$= 0$$

Scanned by CamScanner

10) ~~Sol~~ $u = Ae^{-gx}\sin(nt-gx)$

$$\frac{\delta u}{\delta t} = nAe^{-gx}\cos(nt-gx)$$

$$\frac{\delta u}{\delta x} = -gAe^{-gx}\cos(nt-gx) - gAe^{-gx}\sin(nt-gx)$$

$$= -gAe^{-gx}[\sin(nt-gx)+\cos(nt-gx)]$$

$$\frac{\delta^2 u}{\delta x^2} = g^2Ae^{-gx}[\sin(nt-gx)+\cos(nt-gx)]$$
$$-gAe^{-gx}[-g\cos(nt-gx)+g\sin(nt-gx)]$$
$$= g^2Ae^{-gx}[\sin(nt-gx)+\cos(nt-gx)]$$
$$+g^2Ae^{-gx}[\cos(nt-gx)-\sin(nt-gx)]$$
$$= g^2Ae^{-gx}\times 2\cos(nt-gx)$$

$$\therefore \frac{\delta u}{\delta t} = \mu\frac{\delta^2 u}{\delta x^2}$$

$$n\cancel{A}\cancel{e^{-gx}}\cancel{\cos(nt-gx)} = 2g^2\cancel{A}\cancel{e^{-gx}}\cancel{\cos(nt-gx)}\times\mu$$

$$n = 2\mu g^2$$

$$g^2 = \frac{n}{2\mu}$$

$$g = \sqrt{\frac{n}{2\mu}}$$

11) $f(x,y) = \frac{1}{\sqrt{y}}e^{-(x-a)^2/4y}$

Page No.
Date:

Ex 3(b)

1) a) $f(x,y) = ax^2 + 2hxy + by^2$

$$\frac{\delta f}{\delta x} = 2ax + 2hy$$

$$x\frac{\delta f}{\delta x} = 2(ax^2 + hxy)$$

$$\frac{\delta f}{\delta y} = 2hx + 2by$$

$$y\frac{\delta f}{\delta y} = 2(hxy + by^2)$$

$$x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} = 2(ax^2 + hxy + hxy + by^2)$$

$$= 2(ax^2 + 2hxy + by^2)$$

$$= 2f(x,y)$$

b) $f(x,y) = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

$$\frac{\delta f}{\delta x} = \frac{\cancel{(x^{1/5} + y^{1/5})}(x^{-3/4})}{4(x^{1/5} + y^{1/5})^{\cancel{2}}}$$

$$x\frac{\delta f}{\delta x} = \frac{x^{1/4}}{4(x^{1/5} + y^{1/5})}$$

$$\frac{\delta f}{\delta y} = \frac{\cancel{(x^{1/5} + y^{1/5})}(y^{-3/4})}{4(x^{1/5} + y^{1/5})^{\cancel{2}}}$$

$$y\frac{\delta f}{\delta y} = \frac{y^{1/4}}{4(x^{1/5} + y^{1/5})}$$

Date:

$$x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} = \frac{x^{1/4}}{4(x^{1/5}+y^{1/5})} + \frac{y^{1/4}}{4(x^{1/5}+y^{1/5})}$$

$$= \frac{1}{4}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$$= \frac{1}{4} f(x,y)$$

c) $f(x,y,z) = (x^2+y^2+z^2)^{1/2}$

$$\frac{\delta f}{\delta x} = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}(\cancel{2}x)$$

$$x\frac{\delta f}{\delta x} = x^2(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\delta f}{\delta y} = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}(\cancel{2}y)$$

$$y\frac{\delta f}{\delta y} = y^2(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\delta f}{\delta z} = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}(\cancel{2}z)$$

$$z\frac{\delta f}{\delta z} = z^2(x^2+y^2+z^2)^{-1/2}$$

$$x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} + z\frac{\delta f}{\delta z} = x^2(x^2+y^2+z^2)^{-1/2} + y^2(x^2+y^2+z^2)^{-1/2} + z^2(x^2+y^2+z^2)^{-1/2}$$

$$= (x^2+y^2+z^2)^{-1/2}(x^2+y^2+z^2)$$

$$= (x^2+y^2+z^2)^{1/2}$$

$$= f(x,y,z)$$

$\cancel{\tan u = \frac{x^2+y^2}{x+y}}$

$\cancel{\text{Let } z = \tan u}$

$\cancel{\frac{\delta z}{\delta x} \quad \frac{\delta u}{\delta x}}$

$\cancel{\tan u = z = \frac{x^2+y^2}{x+y}}$

~~z is a homogenous eqⁿ of degree 1.~~

$\cancel{\frac{\delta z}{\delta x} = \sec^2 u \frac{\delta u}{\delta x}}$

$\cancel{x\frac{\delta z}{\delta x} = x\sec^2 u \frac{\delta u}{\delta x}}$

2) $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$\tan u = \frac{x^2+y^2}{x+y}$$

Let $z = \tan u = \frac{x^2+y^2}{x+y}$

$\therefore$ z is a homogenous eqⁿ with degree 1

$$\therefore x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = z \qquad -① $$

$$z = \tan u$$

$$x\frac{\delta z}{\delta x} = x\sec^2 u \frac{\delta u}{\delta x} \qquad -②$$

P.T.O

$$y\frac{\delta z}{\delta y} = y\sec^2 u\frac{\delta u}{\delta y} \quad -③$$

from ①, ② & ③,

$$x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = \tan u$$

$$x\sec^2 u\frac{\delta u}{\delta x} + y\sec^2 u\frac{\delta y}{\delta y} = \tan u$$

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \sin u\cos u$$

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \frac{1}{2}\sin 2u$$

4)

3) $$u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

$$\sin u = \frac{x^2+y^2}{x+y}$$

Let $z = \sin u = \frac{x^2+y^2}{x+y}$

z is a homogenous equation of degree 1

$$\therefore x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = z \quad -①$$

$$z = \sin u$$

$$\frac{\delta z}{\delta x} = \cos u\frac{\delta u}{\delta x}$$

$$x\frac{\delta z}{\delta x} = x\cos u\frac{\delta u}{\delta x} \quad -②$$

$$\frac{\delta z}{\delta y} = \cos u\frac{\delta u}{\delta y}$$

Page No.
Date:

$$y\frac{\delta z}{\delta y} = y\cos u\frac{\delta u}{\delta y} \quad -③$$

from ①, ② & ③,

$$x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = \sin u$$

$$x\cos u\frac{\delta u}{\delta x} + y\cos u\frac{\delta u}{\delta y} = \sin u$$

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \tan u$$

4) $$u = \sin^{-1}\left[\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)\right]^{1/2}$$

$$\sin u = \left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2}$$

$$= x^{-1/12}\left[\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}\right]^{1/2}$$

let $z = \sin u = x^{-1/12}\left[\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}\right]^{1/2}$

z is a homogenous eqn of degree $-1/12$

$$\therefore x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = \frac{-z}{12} \quad -①$$

$z = \sin u$

$$x\frac{\delta z}{\delta x} = x\cos u\frac{\delta u}{\delta x} \quad -②$$

$$y\frac{\delta z}{\delta y} = y\cos u\frac{\delta u}{\delta y} \quad -③$$

from ①, ② & ③,

P.T.O

$$\frac{x\delta z}{\delta x} + \frac{y\delta z}{\delta y} = \frac{-1}{12}\sin u$$

$$\frac{x\cos u\,\delta u}{\delta x} + \frac{y\cos u\,\delta u}{\delta y} = \frac{-1}{12}\sin u$$

$$\therefore \frac{x\delta u}{\delta x} + \frac{y\delta u}{\delta y} = \frac{-1}{12}\tan u \quad — (4)$$

Also, diff (4) wrt x,

$$\frac{x\delta^2 u}{\delta x^2} + \frac{\delta u}{\delta x} + \frac{y\delta^2 u}{\delta x\delta y} = \frac{-1}{12}\sec^2 u\frac{\delta u}{\delta x}$$

Multiply by x,

$$\frac{x^2\delta^2 u}{\delta x^2} + \frac{x\delta u}{\delta x} + \frac{xy\delta^2 u}{\delta x\delta y} = \frac{-1}{12}\sec^2 u\frac{x\delta u}{\delta x} \quad — (5)$$

Page No.
Date:

Diff ④ wrt y,

$$x\frac{\delta^2 u}{\delta y \delta x} + y^2\frac{\delta^2 u}{\delta y^2} + \frac{\delta u}{\delta y} = -\frac{1}{12}\sec^2 u \frac{\delta u}{\delta y}$$

Multiply by y,

$$xy\frac{\delta^2 u}{\delta y \delta x} + y^2\frac{\delta^2 u}{\delta y^2} + y\frac{\delta u}{\delta y} = -\frac{1}{12}\sec^2 u \, y\frac{\delta u}{\delta y} \quad -⑥$$

⑤ + ⑥

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} + \left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right)$$
$$= -\frac{1}{12}\sec^2 u \left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right)$$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta y \delta x} + y^2\frac{\delta^2 u}{\delta y^2} - \frac{1}{12}\tan u = \frac{1}{144}\tan u \sec^2 u$$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta y \delta x} + y^2\frac{\delta^2 u}{\delta y^2} = \frac{1}{144}\tan u\left(12 + \sec^2 x\right)$$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = \frac{1}{144}\tan u\left(13 + \tan^2 x\right)$$

5) Let $u = x^n f(y/x)$

$$\frac{\delta u}{\delta x} = nx^{n-1} f(y/x) + x^n f'(y/x)\left(-\frac{y}{x^2}\right)$$

$$x\frac{\delta u}{\delta x} = nx^n f(y/x) - yx^{n-1} f'(y/x) \quad -①$$

$$\frac{\delta u}{\delta y} = x^n f'(y/x)\left(\frac{1}{x}\right)$$

$$y\frac{\delta u}{\delta y} = yx^{n-1} f'(y/x) \quad -②$$

P.T.O.

Date:

① + ②,

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nx^n f(y/x) + \cancel{yx^{n-1} f'(y/x)} - \cancel{yx^{n-1} f'(y/x)}$$

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nu \quad -③$$

diff ③ wrt x,

$$x\frac{\delta^2 u}{\delta x^2} + \frac{\delta u}{\delta x} + y\frac{\delta^2 u}{\delta x \delta y} = n\frac{\delta u}{\delta x}$$

Multiply by x,

$$x^2\frac{\delta^2 u}{\delta x^2} + xy\frac{\delta^2 u}{\delta x \delta y} + x\frac{\delta u}{\delta x} = nx\frac{\delta u}{\delta x} \quad -④$$

diff ③ wrt y,

$$x\frac{\delta^2 u}{\delta x \delta y} + y\frac{\delta^2 u}{\delta y^2} + \frac{\delta u}{\delta y} = n\frac{\delta u}{\delta y}$$

Multiply by y,

$$xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} + y\frac{\delta u}{\delta y} = ny\frac{\delta u}{\delta y} \quad -⑤$$

④ + ⑤,

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} + \left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right) = n\left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right)$$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} + nu = n^2 u$$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = n^2 u - nu$$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = n(n-1)u$$

Page No.
Date:

6) $u = \log\left(\frac{x^2+y^2}{x+y}\right)$

$e^u = x\left[\frac{1+(y/x)^2}{1+(y/x)}\right]$

Let $z = e^u = x\left[\frac{1+(y/x)^2}{1+(y/x)}\right]$

z is a homogenous eqⁿ of degree 1.

$\therefore x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = z$ — ①

$z = e^u$

$x\frac{\delta z}{\delta x} = xe^u\frac{\delta u}{\delta x}$ — ②

$y\frac{\delta z}{\delta y} = ye^u\frac{\delta u}{\delta y}$ — ③

from ①, ② & ③,

$x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = e^u$

$x\cancel{e^u}\frac{\delta u}{\delta x} + y\cancel{e^u}\frac{\delta u}{\delta y} = \cancel{e^u}$

$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 1$

7) $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Let $\sin^{-1}\left(\frac{x}{y}\right) = a$

$\tan^{-1}\left(\frac{y}{x}\right) = b$

$$u = a + b$$

$$x\frac{\partial u}{\partial x} = x\frac{\partial a}{\partial x} + x\frac{\partial b}{\partial x}$$

$$y\frac{\partial u}{\partial y} = y\frac{\partial a}{\partial y} + y\frac{\partial b}{\partial y}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial a}{\partial x} + y\frac{\partial a}{\partial y} + x\frac{\partial b}{\partial x} + y\frac{\partial b}{\partial y}$$

$$a = \sin^{-1}(x/y)$$

$$\sin a = x/y$$

Let $p = \sin a = x/y$

p is a homogenous eqn of degree 0

$$\therefore x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} = 0 \quad \text{--- (1)}$$

$$p = \sin a$$

$$x\frac{\partial p}{\partial x} = x\cos a\frac{\partial a}{\partial x} \quad \text{--- (2)}$$

$$y\frac{\partial p}{\partial y} = y\cos a\frac{\partial a}{\partial y} \quad \text{--- (3)}$$

from (1), (2) & (3),

$$x\cos a\frac{\partial a}{\partial x} + y\cos a\frac{\partial a}{\partial y} = 0$$

$$x\frac{\partial a}{\partial x} + y\frac{\partial a}{\partial y} = 0$$

$$b = \tan^{-1}(y/x)$$

$$\tan b = y/x$$

Let $q = \tan b = y/x$

q is a homogenous eq^n of degree 0

$$\therefore x\frac{\delta q}{\delta x} + y\frac{\delta q}{\delta y} = 0 \quad — ④$$

$$q = \tan b$$

$$x\frac{\delta q}{\delta x} = \frac{\sec^2 b\,\delta b}{\delta x} \quad — ⑤$$

$$y\frac{\delta q}{\delta y} = \frac{\sec^2 b\,\delta b}{\delta y} \quad — ⑥$$

from ④, ⑤ & ⑥,

$$\frac{x\sec^2 b\,\delta b}{\delta x} + \frac{y\sec^2 b\,\delta b}{\delta y} = 0$$

$$x\frac{\delta b}{\delta x} + y\frac{\delta b}{\delta y} = 0$$

$$\therefore x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = x\frac{\delta a}{\delta x} + y\frac{\delta a}{\delta y} + x\frac{\delta b}{\delta x} + y\frac{\delta b}{\delta y}$$

$$= 0$$

8) $u(x,t) = k\sin(p\lambda t + \phi)\sin\lambda x$

$$\frac{\delta u}{\delta t} = kp\lambda \sin\lambda x\cos(p\lambda t + \phi)$$

$$\frac{\delta^2 u}{\delta t^2} = -kp^2\lambda^2 \sin\lambda x\sin(p\lambda t + \phi)$$

$$= -p^2\lambda^2 u(x,t) \quad — ①$$

$$\frac{\delta u}{\delta x} = k\lambda\sin(p\lambda t + \phi)\cos\lambda x$$

$$\frac{\delta^2 u}{\delta x^2} = -k\lambda^2\sin(p\lambda t + \phi)\sin\lambda x$$

P.T.O

Date:

$$\frac{\delta^2 u}{\delta x^2} = -\delta^2 u(x,t) \quad -②$$

Put ② in ①,

$$\frac{\delta^2 u}{\delta t^2} = -p^2\delta^2 u(x,t)$$

$$= p^2 \left(\frac{\delta^2 u}{\delta x^2}\right)$$

$$\therefore \frac{\delta^2 u}{\delta t^2} = p^2\frac{\delta^2 u}{\delta x^2}$$

9) $$u(x,y,z,t) = \frac{1}{(2a\sqrt{\pi t})^3} e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}$$

$$\frac{\delta u}{\delta t} = \frac{-1}{(2a\sqrt{\pi t})^3} e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}\left[\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{2a^2t^3}\right]$$

$$+ e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t^2}}\left[\frac{-3t^{-1}}{2(2a\sqrt{\pi t})^3}\right]$$

$$= \frac{-e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}}{2t(2a\sqrt{\pi t})^3}\left[3+\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{2a^2t^2}\right] \quad -①$$

$$\frac{\delta u}{\delta x} = \frac{1}{(2a\sqrt{\pi t})^3} e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2} \frac{(x-x_0)}{2a^2t^2}$$

$$= \frac{1}{2a^2t(2a\sqrt{\pi t})^3}\left[(x-x_0)e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}\right]$$

$$\frac{\delta^2 a}{\delta x^2} = \frac{1}{2a^2t(2a\sqrt{\pi t})^3}\left[e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t}\frac{(x-x_0)^2}{2a^2t}\right.$$

$$\left.+ e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}\right]$$

Page No.
Date:

$$\frac{\delta^2 u}{\delta x^2} = \frac{e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}}{2a^2t\,(2a\sqrt{\pi t})^3}\left[1 + \frac{(x-x_0)^2}{2a^2t^2}\right]$$

Similarly,

$$\frac{\delta^2 u}{\delta y^2} = \frac{e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}}{2a^2t\,(2a\sqrt{\pi t})^3}\left[1 + \frac{(y-y_0)^2}{2a^2t^2}\right]$$

$$\frac{\delta^2 u}{\delta z^2} = \frac{e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}}{2a^2t\,(2a\sqrt{\pi t})^3}\left[1 + \frac{(z-z_0)^2}{2a^2t^2}\right]$$

$$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = \frac{e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}}{2a^2t\,(2a\sqrt{\pi t})^3}\left[3 + \frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{2a^2t^2}\right]$$

$$\therefore a^2\left(\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2}\right) = \frac{e^{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2/4a^2t^2}}{2t\,(2a\sqrt{\pi t})^3}\left[3 + \frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{2a^2t^2}\right] \quad \text{— (2)}$$

$$\therefore \frac{\delta u}{\delta t} = -a^2\left(\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2}\right)$$

Ex 3(c)

1) $u = \frac{x}{y}$

$\frac{\delta u}{\delta x} = \frac{y}{y^2} = \frac{1}{y}$

$\frac{\delta u}{\delta y} = \frac{-x}{y^2}$

$x = e^t$

$\frac{dx}{dt} = e^t$

$y = \log t$

$\frac{dy}{dt} = \frac{1}{t}$

$\therefore \frac{du}{dt} = \frac{\delta u}{\delta x} \times \frac{dx}{dt} + \frac{\delta u}{\delta y} \times \frac{dy}{dt}$

$= \frac{1}{y} \times e^t + \left(\frac{-x}{y^2}\right)\frac{1}{t}$

$= \frac{e^t}{\log t} - \frac{e^t}{t(\log t)^2}$

$= \frac{e^t}{\log t}\left(1 - \frac{1}{t\log t}\right)$

2) $u = e^{3x+2y}$

$\frac{\delta u}{\delta x} = 3e^{3x+2y}$

$\frac{\delta u}{\delta y} = 2e^{3x+2y}$

$x = \cos t$

$\frac{dx}{dt} = -\sin t$

Page No.
Date:

$y = t^2$

$\frac{dy}{dt} = 2t$

$\therefore \frac{du}{dt} = \frac{\delta u}{\delta x} \times \frac{dx}{dt} + \frac{\delta u}{\delta y} \times \frac{dy}{dt}$

$= 3e^{3x+2y}(-\sin t) + 2e^{3x+2y}(2t)$

$= e^{3x+2y}[4t - 3\sin t]$

$= e^{3\cos t + 2t^2}(4t - 3\sin t)$

3) $z = x^3 + y^3$

$\frac{\delta z}{\delta x} = 3x^2$

$\frac{\delta z}{\delta y} = 3y^2$

$x = a\cos t$

$\frac{dx}{dt} = -a\sin t$

$y = b\sin t$

$\frac{dy}{dt} = b\cos t$

$\therefore \frac{dz}{dt} = \frac{\delta z}{\delta x} \times \frac{dx}{dt} + \frac{\delta z}{\delta y} \times \frac{dy}{dt}$

$= 3x^2(-a\sin t) + 3y^2(b\cos t)$

$= 3[by^2\cos t - ax^2\sin t]$

$= 3[b^3\sin^2 t\cos t - a^3\cos^2 t\sin t]$

$= 3\sin t\cos t[b^3\sin t - a^3\cos t]$

$z = x^3 + y^3$

$= a^3\cos^3 t + b^3\sin^3 t$

P.T.O

Date:

$$\frac{dz}{dt} = 3a^3\cos^2 t(-\sin t) + 3b^3\sin^2 t\cos t$$

$$= 3\sin t\cos t\,(b^3\sin t - a^3\cos t)$$

4) a) $f(x,y) \rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{\delta f}{\delta x} = \frac{2x}{a^2}$$

$$\frac{\delta f}{\delta y} = \frac{2y}{b^2}$$

$$\therefore \frac{\delta f}{\delta x} + \frac{dy}{dx} \times \frac{\delta f}{\delta y} = 0$$

$$\frac{\cancel{2}x}{a^2} + \frac{\cancel{2}y}{b^2} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-xb^2}{a^2 y}$$

b) $f(x,y) \rightarrow \sin(xy) - e^{xy} - x^2 y = 0$

$$\frac{\delta f}{\delta x} = \cos(xy)[y] - e^{xy}(y) - 2xy$$

$$= y[\cos(xy) - e^{xy} - 2x]$$

$$\frac{\delta f}{\delta y} = \cos(xy)[x] - e^{xy}(x) - x^2$$

$$= x[\cos(xy) - e^{xy} - x]$$

$$\frac{dy}{dx} = \frac{-\delta f/\delta x}{\delta f/\delta y} = \frac{-y[\cos(xy) - e^{xy} - 2x]}{x[\cos(xy) - e^{xy} - x]}$$

$$= \left[\frac{e^{xy} + 2x - \cos(xy)}{\cos(xy) - e^{xy} - x}\right]\frac{y}{x}$$

Page No.

Date:

c) $f(x,y) \rightarrow e^x + e^y = 2xy$

$$\frac{\delta f}{\delta x} = e^x - 2y$$

$$\frac{\delta f}{\delta y} = e^y - 2x$$

$$\therefore \frac{dy}{dx} = \frac{-\delta f / \delta x}{\delta f / \delta y} = \frac{-(e^x - 2y)}{(e^y - 2x)}$$

$$= \frac{e^x - 2y}{2x - e^y}$$

5) a) $f(x,y) \rightarrow ax^2 + 2hxy + by^2 = 1$

$f_x = 2(ax+hy)$; $f_{xx} = 2a$

$f_y = 2(by+hx)$; $f_{yy} = 2b$

$f_{xy} = 2h$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(ax+hy)}{(hx+by)}$$

$$\frac{d}{dx}(f_x) = f_{xx} + \frac{dy}{dx} f_{yx} = \frac{f_y f_{xx} - f_x f_{xy}}{f_y^2}$$

$$\frac{d}{dx}(f_y) = f_{xy} + \frac{dy}{dx} f_{yy} = \frac{f_y f_{xy} - f_x f_{yy}}{f_y^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{f_x}{f_y}\right)$$

$$= -\left[\frac{\cancel{f_y}(f_y f_{xx} - f_x f_{xy})/\cancel{f_y} - f_x (f_y f_{xy} - f_x f_{yy})/f_y}{(f_y)^2}\right]$$

$$= \frac{-1}{(f_y)^2}\left[\frac{(f_y)^2 f_{xx} - f_x f_y f_{xy} - f_x f_y f_{xy} + (f_x)^2 f_{yy}}{f_y}\right]$$

$$= -\left[\frac{(f_x)^2 f_{yy} - 2 f_x f_y f_{xy} + (f_y)^2 f_{xx}}{(f_y)^3}\right]$$

$$= -\left[\frac{4(ax+hy)^2(2b) - 4(ax+hy)(hx+by)(2h) + 4(hx+by)^2(2a)}{8(hx+by)^3}\right]$$

Date:

$$= -\left[\frac{b(ax+hy)^2 - h(ax+hy)(hx+by) + a(hx+by)^2}{(hx+by)^3}\right]$$

b) $f(x,y) \rightarrow x^3 - 3ax^2 + y^3 = 0$

$f_x = 3x^2 - 6ax$; $f_{xx} = 6x - 6a$

$f_y = 3y^2$; $f_{yy} = 6y$

$f_{xy} = 0$

$$\therefore \frac{d^2y}{dx^2} = -\left[\frac{(f_x)^2 f_{yy} - 2f_x f_y f_{xy} + (f_y)^2 f_{xx}}{(f_y)^3}\right]$$

$$= -\left[\frac{(3x^2-6ax)^2 6y + (3y^2)^2(6x-6a)}{216y^6}\right]$$

$$= -\left[\frac{54x^2y(x-2a)^2 + 54y^4(x-a)}{216y^6}\right]$$

$$= -2\left[\frac{x^2(x-2a)^2 + y^3(x-a)}{4y^5}\right]$$

$$= -2\left[\frac{x^2(x^2-4ax+4a^2) + (3ax^2-x^3)(x-a)}{4y^5}\right]$$

$$= -2\left[\frac{\cancel{x^4} - \cancel{4ax^3} + 4a^2x^2 + \cancel{3ax^3} - 3a^2x^2 - \cancel{x^4} + \cancel{ax^3}}{4y^5}\right]$$

$$= \frac{-2a^2x^2}{4y^5}$$

c) $f(x,y) \rightarrow x^4 + y^4 - 4a^2xy = 0$

$f_x = 4x^3 - 4a^2y$; $f_{xx} = 12x^2$

$f_y = 4y^3 - 4a^2x$; $f_{yy} = 12y^2$

$f_{xy} = -4a^2$

P.T.O

Date:

$$\therefore \frac{d^2y}{dx^2} = -\left[\frac{(f_x)^2 f_{yy} - 2f_xf_yf_{xy} + (f_y)^2 f_{xx}}{(f_y)^3}\right]$$

$$= -\left[\frac{\cancel{16}(x^3-a^2y)^2(\cancel{12}^{\,3}y^2) - 2(\cancel{16})(x^3-a^2y)(y^3-a^2x)(-\cancel{4}a^2) + \cancel{16}(y^3-a^2x)(\cancel{12}^{\,3}x^2)}{\cancel{64}(y^3-a^2x)^3}\right]$$

$$= -\left[\frac{3y^2(x^3-a^2y)^2 + 2(x^3-a^2y)(y^3-a^2x)a^2 + 3x^2(y^3-a^2x)^2}{(y^3-a^2x)^3}\right]$$

6) $f(x,y) \rightarrow x^2+y^2+2gx+2fy+c^2=0$

$f_x = 2x+2g$; $f_{xx} = 2$

$f_y = 2y+2f$; $f_{yy} = 2$

$f_{xy} = 0$

$$\therefore \frac{d^2y}{dx^2} = -\left[\frac{(f_x)^2 f_{yy} - 2f_xf_yf_{xy} + (f_y)^2 f_{xx}}{(f_y)^3}\right]$$

$$= -\left[\frac{(2x+2g)^2(2) + (2y+2f)^2(2)}{(2y+2f)^3}\right]$$

$$= -\left[\frac{\cancel{8}(x+g)^2 + \cancel{8}(y+f)^2}{\cancel{8}(y+f)^3}\right]$$

$$= -\left[\frac{x^2+g^2+2gx+y^2+2fy+f^2}{(y+f)^3}\right]$$

$$= -\left[\frac{(x^2+y^2+2gx+2fy)+g^2+f^2}{(y+f)^3}\right]$$

$$= -\left[\frac{-c^2+g^2+f^2}{(y+f)^3}\right]$$

$$= \frac{c^2-f^2-g^2}{(y+f)^3}$$

Ex 3(e)

1)
$$z = \log(u^2+v)$$
$$u = e^{x+y^2}$$
$$v = x^2+y$$

$$\frac{\delta z}{\delta x} = \frac{\delta z}{\delta u} \times \frac{du}{dx} + \frac{\delta z}{\delta v} \times \frac{dv}{dx}$$

$$= \frac{2u}{u^2+v} \times e^{x+y^2} + \frac{1}{u^2+v} \times 2x$$

$$= \frac{2}{u^2+v}\left[ue^{x+y^2} + x\right]$$

$$= \frac{2}{u^2+v}\left[e^{2x+2y^2} + x\right]$$

$$= \frac{2}{e^{2x+2y^2} + x^2+y}\left[e^{2x+2y^2} + x\right]$$

$$\frac{\delta z}{\delta y} = \frac{\delta z}{\delta u} \times \frac{du}{dy} + \frac{\delta z}{\delta v} \times \frac{dv}{dy}$$

$$= \frac{2u}{u^2+v} \times 2ye^{x+y^2} + \frac{1}{u^2+v} \times 1$$

$$= \frac{1}{u^2+v}\left[4ue^{x+y^2} + 1\right]$$

$$= \frac{1}{e^{2x+2y^2}+x^2+y}\left[4e^{2x+2y^2} + 1\right]$$

2)
$$z = f(x,y)$$
$$x = e^u + e^{-v}$$
$$y = e^u - e^{-v}$$

2) $z = f(x,y)$

$x = e^{-u} + e^{v}$

$y = e^{-u} - e^{v}$

$\frac{dx}{du} = -e^{-u}$; $\frac{dx}{dv} = e^{-v}$

$\frac{dy}{du} = -e^{-u}$; $\frac{dy}{dv} = -e^{v}$

$$\frac{\delta z}{\delta u} = \frac{\delta z}{\delta x} \times \frac{dx}{du} + \frac{\delta z}{\delta y} \times \frac{dy}{du}$$

$$= \frac{\delta z}{\delta x}(-e^{-u}) + \frac{\delta z}{\delta y}(-e^{-u})$$

$$= -e^{-u}\frac{\delta z}{\delta x} - e^{-u}\frac{\delta z}{\delta y} \quad \text{— (1)}$$

$$\frac{\delta z}{\delta v} = \frac{\delta z}{\delta x} \times \frac{dx}{dv} + \frac{\delta z}{\delta y} \times \frac{dy}{dv}$$

$$= \frac{\delta z}{\delta x}(e^{-v}) + \frac{\delta z}{\delta y}(-e^{v})$$

$$= e^{-v}\frac{\delta z}{\delta x} - e^{v}\frac{\delta z}{\delta y} \quad -② $$

① - ②,

$$\frac{\delta z}{\delta u} - \frac{\delta z}{\delta v} = -e^{-u}\frac{\delta z}{\delta x} - e^{v}\frac{\delta z}{\delta x} - e^{-u}\frac{\delta z}{\delta y} + e^{v}\frac{\delta z}{\delta y}$$

$$= -\frac{\delta z}{\delta x}(e^{-u} + e^{v}) - \frac{\delta z}{\delta y}(e^{-u} - e^{v})$$

$$= -x\frac{\delta z}{\delta x} - y\frac{\delta z}{\delta y}$$

3) $H = f(y-z, z-x, x-y)$

Let $u = y-z$

$v = z-x$

$\omega = x-y$

$H = f(u, v, \omega)$

$$\frac{\delta H}{\delta x} = \frac{\delta H}{\delta u} \times \frac{du}{dx} + \frac{\delta H}{\delta v} \times \frac{dv}{dx} + \frac{\delta H}{\delta \omega} \times \frac{d\omega}{dx}$$

$$\frac{\delta H}{\delta x} = -\frac{\delta H}{\delta v} + \frac{\delta H}{\delta \omega} \quad — ①$$

$$\frac{\delta H}{\delta y} = \frac{\delta H}{\delta u} \times \frac{du}{dy} + \frac{\delta H}{\delta v} \times \frac{dv}{dy} + \frac{\delta H}{\delta \omega} \times \frac{d\omega}{dy}$$

$$\frac{\delta H}{\delta y} = \frac{\delta H}{\delta u} - \frac{\delta H}{\delta \omega} \quad — ②$$

P.T.O

$$\frac{\delta H}{\delta z} = \frac{\delta H}{\delta u} \times \frac{du}{dz} + \frac{\delta H}{\delta v} \times \frac{dv}{dz} + \frac{\delta H}{\delta \omega} \times \frac{d\omega}{dz}$$

$$\frac{\delta H}{\delta z} = -\frac{\delta H}{\delta u} + \frac{\delta H}{\delta v} \quad — (3)$$

(1) + (2) + (3),

$$\frac{\delta H}{\delta x} + \frac{\delta H}{\delta y} + \frac{\delta H}{\delta z} = \cancel{-\frac{\delta H}{\delta v}} + \cancel{\frac{\delta H}{\delta \omega}} + \cancel{\frac{\delta H}{\delta u}} - \cancel{\frac{\delta H}{\delta \omega}} - \cancel{\frac{\delta H}{\delta u}} + \cancel{\frac{\delta H}{\delta v}}$$

$$\frac{\delta H}{\delta x} + \frac{\delta H}{\delta y} + \frac{\delta H}{\delta z} = 0$$

5) $u = f(r, s)$

$r = x + y$

$s = x - y$

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta r} \times \frac{dr}{dx} + \frac{\delta u}{\delta s} \times \frac{ds}{dx}$$

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta r} + \frac{\delta u}{\delta s} \quad — (1)$$

$$\frac{\delta u}{\delta y} = \frac{\delta u}{\delta r} \times \frac{dr}{dy} + \frac{\delta u}{\delta s} \times \frac{ds}{dy}$$

$$\frac{\delta u}{\delta y} = \frac{\delta u}{\delta r} - \frac{\delta u}{\delta s} \quad — (2)$$

(1) + (2),

$$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} = \frac{\delta u}{\delta r} + \cancel{\frac{\delta u}{\delta s}} + \frac{\delta u}{\delta r} - \cancel{\frac{\delta u}{\delta s}}$$

$$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} = \frac{2\delta u}{\delta r}$$

Scanned by CamScanner

6) $v = f(x,y)$

$x+y = 2e^{\theta}\cos\phi$

$x-y = 2ie^{\theta}\sin\phi$

$\therefore\ x = e^{\theta}(\cos\phi + i\sin\phi) = e^{\theta+i\phi}$

$y = e^{\theta}(\cos\phi - i\sin\phi) = e^{\theta-i\phi}$

$$\frac{\delta v}{\delta\theta} = \frac{\delta v}{\delta x} \times \frac{dx}{d\theta} + \frac{\delta v}{\delta y} \times \frac{dy}{d\theta}$$

$$= e^{\theta+i\phi}\frac{dx}{d\theta} + e^{\theta-i\phi}\frac{dy}{d\theta}$$

$$= x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y}$$

$$\frac{\delta^2 v}{\delta\theta^2} = \frac{\delta}{\delta\theta}\left(\frac{\delta v}{\delta\theta}\right) = x\frac{\delta}{\delta x}\left(x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y}\right) + y\frac{\delta}{\delta y}\left(x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y}\right)$$

$$= x\left(x\frac{\delta^2 v}{\delta x^2} + \frac{\delta v}{\delta x} + y\frac{\delta^2 v}{\delta x\delta y}\right) + y\left(x\frac{\delta^2 v}{\delta y\delta x} + y\frac{\delta^2 v}{\delta y^2} + \frac{\delta v}{\delta y}\right)$$

$$= x^2\frac{\delta^2 v}{\delta x^2} + y^2\frac{\delta^2 v}{\delta y^2} + 2xy\frac{\delta^2 v}{\delta x\delta y} + \left(x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y}\right)$$

P.T.O

Date:

$$\frac{\delta V}{\delta \phi} = \frac{\delta V}{\delta x} \times \frac{dx}{d\phi} + \frac{\delta V}{\delta y} \times \frac{dy}{d\phi}$$

$$= ix\frac{\delta V}{\delta x} - iy\frac{\delta V}{\delta y}$$

$$= i\left(x\frac{\delta V}{\delta x} - y\frac{\delta V}{\delta y}\right)$$

$$\frac{\delta^2 V}{\delta \phi^2} = \frac{\delta}{\delta \phi}\left(\frac{\delta V}{\delta \phi}\right) = ix\frac{\delta}{\delta x}\left(ix\frac{\delta V}{\delta x} - iy\frac{\delta V}{\delta y}\right)$$
$$- iy\frac{\delta}{\delta y}\left(ix\frac{\delta V}{\delta x} - iy\frac{\delta V}{\delta y}\right)$$

$$= i^2x\left(x\frac{\delta^2 V}{\delta x^2} + \frac{\delta V}{\delta x} - y\frac{\delta^2 V}{\delta x\delta y}\right)$$
$$- i^2y\left(x\frac{\delta^2 V}{\delta y\delta x} - y\frac{\delta^2 V}{\delta y^2} - \frac{\delta V}{\delta y}\right)$$

$$= i^2\left(x^2\frac{\delta^2 V}{\delta x^2} + x\frac{\delta V}{\delta x} - xy\frac{\delta^2 V}{\delta x\delta y}\right)$$
$$- i^2\left(xy\frac{\delta^2 V}{\delta x\delta y} - y^2\frac{\delta^2 V}{\delta y^2} - y\frac{\delta V}{\delta y}\right)$$

$$= -x^2\frac{\delta^2 V}{\delta x^2} - y^2\frac{\delta^2 V}{\delta y^2} + 2xy\frac{\delta^2 V}{\delta x\delta y} - \left(x\frac{\delta V}{\delta x} + y\frac{\delta V}{\delta y}\right)$$

$$\therefore \frac{\delta^2 V}{\delta \theta^2} + \frac{\delta^2 V}{\delta \phi^2} = \cancel{x^2\frac{\delta^2 V}{\delta x^2}} + \cancel{y^2\frac{\delta^2 V}{\delta y^2}} + 2xy\frac{\delta^2 V}{\delta x\delta y} + \cancel{\left(x\frac{\delta V}{\delta x} + y\frac{\delta V}{\delta y}\right)}$$
$$\cancel{-x^2\frac{\delta^2 V}{\delta x^2}} \cancel{- y^2\frac{\delta^2 V}{\delta y^2}} + 2xy\frac{\delta^2 V}{\delta x\delta y} \cancel{- \left(x\frac{\delta V}{\delta x} + y\frac{\delta V}{\delta y}\right)}$$

$$\therefore \frac{\delta^2 V}{\delta \theta^2} + \frac{\delta^2 V}{\delta \phi^2} = 4xy\frac{\delta^2 V}{\delta x\delta y}$$

Page No.
Date:

7) $u = f(x, y)$

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\delta u}{\delta r} = \frac{\delta u}{\delta x} \times \frac{dx}{dr} + \frac{\delta u}{\delta y} \times \frac{dy}{dr}$$

$$\frac{\delta u}{\delta r} = \frac{\delta u}{\delta x} \times \cos\theta + \frac{\delta u}{\delta y} \times \sin\theta$$

$$\frac{\delta u}{\delta r} = \frac{x}{r}\frac{\delta u}{\delta x} + \frac{y}{r}\frac{\delta u}{\delta y} \quad \text{--- (1)}$$

$$\cancel{r\frac{\delta u}{\delta r} = x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} \quad \text{--- (2)}}$$

~~Partially differentiate (1) wrt x~~

$$\cancel{r\frac{\delta^2 u}{\delta r^2}\left(\frac{\delta r}{\delta x}\right) + \frac{\delta u}{\delta x}\left(\frac{\delta x}{\delta x}\right) = x\frac{\delta^2 u}{\delta x^2} + \frac{\delta u}{\delta x} + y\frac{\delta^2 u}{\delta x\,\delta y}}$$

Date:

$$\frac{\delta^2 u}{\delta r^2} = \frac{\delta}{\delta r}\left(\frac{\delta u}{\delta r}\right)$$

$$= \frac{x}{r}\frac{\delta}{\delta x}\left(\frac{x}{r}\frac{\delta u}{\delta x} + \frac{y}{r}\frac{\delta u}{\delta y}\right) + \frac{y}{r}\frac{\delta}{\delta y}\left(\frac{x}{r}\frac{\delta u}{\delta x} + \frac{y}{r}\frac{\delta u}{\delta y}\right)$$

$$= \frac{x}{r}\left[\frac{x}{r}\frac{\delta^2 u}{\delta x^2} + \frac{1}{r}\frac{\delta u}{\delta x} + \frac{y}{r}\frac{\delta^2 u}{\delta x \delta y}\right]$$

$$+ \frac{y}{r}\left[\frac{x}{r}\frac{\delta^2 u}{\delta x \delta y} + \frac{y}{r}\frac{\delta^2 u}{\delta y^2} + \frac{1}{r}\frac{\delta u}{\delta y}\right]$$

$$= \frac{x^2}{r^2}\frac{\delta^2 u}{\delta x^2} + \frac{2xy}{r^2}\frac{\delta^2 u}{\delta x \delta y} + \frac{y^2}{r^2}\frac{\delta^2 u}{\delta y^2} + \frac{x}{r^2}\frac{\delta u}{\delta x} + \frac{y}{r^2}\frac{\delta u}{\delta y}$$

$$r^2\frac{\delta^2 u}{\delta r^2} = x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} + r\left(\frac{x}{r}\frac{\delta u}{\delta x} + \frac{y}{r}\frac{\delta u}{\delta y}\right)$$

$$r^2\frac{\delta^2 u}{\delta r^2} = x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} + r\frac{\delta u}{\delta r}$$

$$r^2\frac{\delta^2 u}{\delta r^2} - r\frac{\delta u}{\delta r} = x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2}$$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = r\left[r\frac{\delta}{\delta r}\left\{\frac{\delta}{\delta r}(u)\right\} - \frac{\delta}{\delta r}(u)\right]$$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = r\frac{\delta}{\delta r}\left[r\frac{\delta}{\delta r}(u) - (u)\right]$$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = r\frac{\delta}{\delta r}\left(r\frac{\delta}{\delta r} - 1\right)u$$

Page No.
Date:

Ex3(8) (1) $y_1 = \dfrac{x_2x_3}{x_1}$; $y_2 = \dfrac{x_3x_1}{x_2}$; $y_3 = \dfrac{x_1x_2}{x_3}$

$$\frac{\delta(y_1,y_2,y_3)}{\delta(x_1,x_2,x_3)} = \begin{bmatrix} \delta y_1/\delta x_1 & \delta y_1/\delta x_2 & \delta y_1/\delta x_3 \\ \delta y_2/\delta x_1 & \delta y_2/\delta x_2 & \delta y_2/\delta x_3 \\ \delta y_3/\delta x_1 & \delta y_3/\delta x_2 & \delta y_3/\delta x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -x_2x_3/x_1^2 & x_3/x_1 & x_2/x_1 \\ x_3/x_2 & -x_1x_3/x_2^2 & x_1/x_2 \\ x_2/x_3 & x_1/x_3 & -x_1x_2/x_3^2 \end{bmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{bmatrix} -x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & -x_1x_3 & x_1x_2 \\ x_2x_3 & x_1x_3 & -x_1x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$R_1 = R_1 + R_2$

$R_2 = R_2 + R_3$

$$\frac{\delta(y_1,y_2,y_3)}{\delta(x_1,x_2,x_3)} = \cancel{\begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}} \begin{vmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 4$

~~(8)~~

$\cancel{u^3 + v^3 = x + y}$

$\cancel{\dfrac{\delta u}{\delta x} = \dfrac{1}{3u^2} \;—\; (1)}$; $\cancel{\dfrac{\delta v}{\delta x} = \dfrac{1}{3v^2} \;—\; (2)}$

$\cancel{\dfrac{\delta u}{\delta y} = \dfrac{1}{3u^2} \;—\; (3)}$; $\cancel{\dfrac{\delta v}{\delta y} = \dfrac{1}{3v^2} \;—\; (4)}$

$\cancel{u^2 + v^2 = x^3 + y^3}$

$\cancel{\dfrac{\delta u}{\delta x} = \dfrac{3x^2}{2u} \;—\; (5)}$; $\cancel{\dfrac{\delta v}{\delta x} = \dfrac{3x^2}{2v} \;—\; (6)}$

2)

$$u^3 + v^3 = x + y$$

partially diff. wrt x,

$$3u^2 \frac{\delta u}{\delta x} + 3v^2 \frac{\delta v}{\delta x} = 1 \quad — ①$$

$$u^2 + v^2 = x^3 + y^3$$

partially diff wrt x,

$$2u \frac{\delta u}{\delta x} + 2v \frac{\delta v}{\delta x} = 3x^2 \quad — ②$$

① and ② can be represented as,

$$\begin{bmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{bmatrix} \begin{bmatrix} \delta u/\delta x \\ \delta v/\delta x \end{bmatrix} = \begin{bmatrix} 1 \\ 3x^2 \end{bmatrix}$$

$$\begin{bmatrix} \delta u/\delta x \\ \delta v/\delta x \end{bmatrix} = \begin{bmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3x^2 \end{bmatrix}$$

$$\begin{bmatrix} \delta u/\delta x \\ \delta v/\delta x \end{bmatrix} = \frac{1}{6u^2v - 6uv^2} \begin{bmatrix} 2v & -3v^2 \\ -2u & 3u^2 \end{bmatrix} \begin{bmatrix} 1 \\ 3x^2 \end{bmatrix}$$

$$\frac{\delta u}{\delta x} = \frac{2v - 9v^2x^2}{6u^2v - 6uv^2} = \frac{2 - 9vx^2}{6u(u-v)}$$

$$\frac{\delta v}{\delta x} = \frac{-2u + 9u^2x^2}{6u^2v - 6uv^2} = \frac{9ux^2 - 2}{6v(u-v)}$$

Similarly,

$$\frac{\delta u}{\delta y} = \frac{2 - 9vy^2}{6u(u-v)}$$

$$\frac{\delta v}{\delta y} = \frac{9uy^2 - 2}{6v(u-v)}$$

$$\frac{\delta(u,v)}{\delta(x,y)} = \begin{vmatrix} \delta u/\delta x & \delta u/\delta y \\ \delta v/\delta x & \delta v/\delta y \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{2-9vx^2}{6u(u-v)} & \dfrac{2-9vy^2}{6u(u-v)} \\[2ex] \dfrac{9ux^2-2}{6v(u-v)} & \dfrac{9uy^2-2}{6v(u-v)} \end{vmatrix}$$

$$= \frac{1}{36u^2v^2(u-v)^2} \begin{vmatrix} 2v-9v^2x^2 & 2v-9v^2y^2 \\ 9u^2x^2-2u & 9u^2y^2-2u \end{vmatrix}$$

$$= \frac{1}{36u^2v^2(u-v)^2}\left[\cancel{18}u^2vy^2 - \cancel{4uv} - \cancel{81u^2v^2x^2y^2} + \cancel{18}uv^2x^2 - \cancel{18}u^2vx^2 + \cancel{4uv} + \cancel{81u^2v^2x^2y^2} - \cancel{18}uv^2y^2\right]$$

$$= \frac{1}{2u^2v^2(u-v)^2}\left[u^2vy^2 + uv^2x^2 - u^2vx^2 - uv^2y^2\right]$$

$$= \frac{1}{2uv(u-v)^2}\left[uy^2 + vx^2 - ux^2 - vy^2\right]$$

$$= \frac{1}{2uv(u-v)^{\cancel{2}}}\left[y^2\cancel{(u-v)} - x^2\cancel{(u-v)}\right]$$

$$= \frac{y^2-x^2}{2uv(u-v)}$$

$$\therefore \frac{\delta(u,v)}{\delta(x,y)} = \frac{y^2-x^2}{2uv(u-v)}$$

Page No.
Date:

4) $u = x+y+z$

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta y} = \frac{\delta u}{\delta z} = 1$$

$$u^2 v = y+z$$

$$2uv\frac{\delta u}{\delta x} + u^2\frac{\delta v}{\delta x} = 0 \Rightarrow \frac{\delta v}{\delta x} = \frac{-2v}{u}$$

$$2uv\frac{\delta u}{\delta y} + u^2\frac{\delta v}{\delta y} = 1 \Rightarrow \frac{\delta v}{\delta y} = \frac{1}{u^2} - \frac{2v}{u}$$

$$2uv\frac{\delta u}{\delta z} + u^2\frac{\delta v}{\delta z} = 1 \Rightarrow \frac{\delta v}{\delta z} = \frac{1}{u^2} - \frac{2v}{u}$$

$$u^3 w = z$$

$$3u^2w\frac{\delta u}{\delta x} + u^3\frac{\delta w}{\delta x} = 0 \Rightarrow \frac{\delta w}{\delta x} = \frac{-3w}{u}$$

$$3u^2w\frac{\delta u}{\delta y} + u^3\frac{\delta w}{\delta y} = 0 \Rightarrow \frac{\delta w}{\delta y} = \frac{-3w}{u}$$

$$3u^2w\frac{\delta u}{\delta z} + u^3\frac{\delta w}{\delta z} = 1 \Rightarrow \frac{\delta w}{\delta z} = \frac{1}{u^3} - \frac{3w}{u}$$

$$\frac{\delta(u,v,w)}{\delta(x,y,z)} = \begin{vmatrix} \delta u/\delta x & \delta u/\delta y & \delta u/\delta z \\ \delta v/\delta x & \delta v/\delta y & \delta v/\delta z \\ \delta w/\delta x & \delta w/\delta y & \delta w/\delta z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ \frac{-2v}{u} & \frac{1}{u^2}-\frac{2v}{u} & \frac{1}{u^2}-\frac{2v}{u} \\ \frac{-3w}{u} & \frac{-3w}{u} & \frac{1}{u^3}-\frac{3w}{u} \end{vmatrix}$$

$C_1 = C_1 - C_2$

$C_2 = C_2 - C_3$

P.T.O

$$\frac{\delta(u,v,w)}{\delta(x,y,z)} = \begin{vmatrix} 0 & 0 & 1 \\ -\frac{1}{u^2} & 0 & \frac{1}{u^2} - \frac{2v}{u} \\ 0 & -\frac{1}{u^3} & \frac{1}{u^3} - \frac{3w}{u} \end{vmatrix}$$

$$= 1\left(-\frac{1}{u^2}\right)\left(-\frac{1}{u^3}\right)$$

$$= \frac{1}{u^5}$$

9) $u = \frac{x+y}{1-xy}$

$$\frac{\delta u}{\delta x} = \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\delta u}{\delta y} = \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\delta v}{\delta x} = \frac{1}{1+x^2}$$

$$\frac{\delta v}{\delta y} = \frac{1}{1+y^2}$$

$$\frac{\delta(u,v)}{\delta(x,y)} = \begin{vmatrix} \delta u/\delta x & \delta u/\delta y \\ \delta v/\delta x & \delta v/\delta y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

Scanned by CamScanner

Date:

$$= \cancel{\frac{1}{(1-xy)^2}} - \cancel{\frac{1}{(1-xy)^2}}$$

$$= 0$$

$\therefore$ Functional relationship exists between u and v.

$$v = \tan^{-1}x + \tan^{-1}y$$

$$\tan v = \tan[\tan^{-1}x + \tan^{-1}y]$$

$$\tan v = \frac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\tan(\tan^{-1}y)}$$

$$\tan v = \frac{x+y}{1-xy}$$

$$\therefore u = \tan v$$

10) $u = \sin^{-1}x + \sin^{-1}y$

$$\frac{\delta u}{\delta x} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\delta u}{\delta y} = \frac{1}{\sqrt{1-y^2}}$$

$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\frac{\delta v}{\delta x} = \sqrt{1-y^2} + \frac{y(-x)}{\sqrt{1-x^2}} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$\frac{\delta v}{\delta y} = \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$\frac{\delta(u,v)}{\delta(x,y)} = \begin{vmatrix} \delta u/\delta x & \delta u/\delta y \\ \delta v/\delta x & \delta v/\delta y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \end{vmatrix}$$

$$\frac{\delta(u,v)}{\delta(x,y)} = \cancel{1} - \cancel{\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}} - \cancel{1} + \cancel{\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}}$$

$$= 0$$

∴ relationship exists between u and v.

$$u = \sin^{-1}x + \sin^{-1}y$$

$$\sin u = \sin(\sin^{-1}x + \sin^{-1}y)$$

$$= \sin(\sin^{-1}x)\cos(\sin^{-1}y) + \cos(\sin^{-1}x)\sin(\sin^{-1}y)$$

$$= x\cos(\cos^{-1}\sqrt{1-y^2}) + \cos(\cos^{-1}\sqrt{1-x^2})\,y$$

$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\therefore \quad \sin u = v$$

8)

$$x = u - uv$$

$$\frac{\delta x}{\delta u} = 1 - v$$

$$\frac{\delta x}{\delta v} = -u$$

$$y = uv$$

$$\frac{\delta y}{\delta u} = v \; ; \; \frac{\delta y}{\delta v} = u$$

$$\frac{\delta(x,y)}{\delta(u,v)} = \begin{vmatrix} \delta x/\delta u & \delta x/\delta v \\ \delta y/\delta u & \delta y/\delta v \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u - \cancel{uv} + \cancel{uv}$$

$$= u$$

$x = u - uv$

$x = u - y$

$u = x + y$

$$\frac{\delta u}{\delta x} = 1 \quad ; \quad \frac{\delta u}{\delta y} = 1$$

$y = uv$

$y = (x+y)v$

$$v = \frac{y}{x+y}$$

$$\frac{\delta v}{\delta x} = \frac{-y}{(x+y)^2}$$

$$\frac{\delta v}{\delta y} = \frac{(x+y) - y}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$\frac{\delta(u,v)}{\delta(x,y)} = \begin{vmatrix} \delta u/\delta x & \delta u/\delta y \\ \delta v/\delta x & \delta v/\delta y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

P.T.O

Page No.
Date:

$$\frac{\delta(u,v)}{\delta(x,y)} = \frac{\frac{x+y}{y}}{(x+y)^2}$$

$$= \frac{1}{x+y}$$

$$= \frac{1}{u}$$

$$\frac{\delta(x,y)}{\delta(u,v)} \times \frac{\delta(u,v)}{\delta(x,y)} = u \times \frac{1}{u} = 1$$

6) $x = a\cosh\theta\cos\phi$

$$\frac{\delta x}{\delta\theta} = a\sinh\theta\cos\phi$$

$$\frac{\delta x}{\delta\phi} = -a\cosh\theta\sin\phi$$

$$y = a\sinh\theta\sin\phi$$

$$\frac{\delta y}{\delta\theta} = a\cosh\theta\sin\phi$$

$$\frac{\delta y}{\delta\phi} = \cancel{a\cosh\phi}\; a\sinh\theta\cos\phi$$

$$\frac{\delta(x,y)}{\delta(\theta,\phi)} = \begin{vmatrix} \delta x/\delta\theta & \delta x/\delta\phi \\ \delta y/\delta\theta & \delta y/\delta\phi \end{vmatrix}$$

$$= \begin{vmatrix} a\sinh\theta\cos\phi & -a\cosh\theta\sin\phi \\ a\cosh\theta\sin\phi & a\sinh\theta\cos\phi \end{vmatrix}$$

$$= a^2\sinh^2(\theta)\cos^2\phi + a^2\cosh^2(\theta)\sin^2\phi$$

$$= a^2(-1+\cosh^2(\theta))\cos^2\phi + a^2\cosh^2(\theta)(1-\cos^2\phi)$$

$$= a^2[-\cos^2\phi + \cancel{\cosh^2(\theta)\cos^2\phi} + \cosh^2(\phi) - \cancel{\cosh^2(\theta)\cos^2\phi}]$$

$$= a^2\left[\frac{1+\cosh(2\theta)}{2} - \left(\frac{1+\cos 2\phi}{2}\right)\right]$$

$$= \frac{a^2}{2}\left[1 + \cosh(2\theta) - 1 - \cos 2\phi\right]$$

$$= \frac{a^2}{2}\left[\cosh(2\theta) - \cos 2\phi\right]$$

3) $r^2 = x^2 + y^2 + z^2$

$u = x(1-r^2)^{-1/2}$

$$\frac{\delta u}{\delta x} = (1-r^2)^{-1/2} + x\left(\cancel{-}\frac{1}{\cancel{2}}\right)(1-r^2)^{-3/2}(\cancel{-2x})\left(\frac{x}{\cancel{r}}\right)$$

$$= (1-r^2)^{-1/2} + x^2(1-r^2)^{-1}(1-r^2)^{-1/2}$$

$$= (1-x^2)^{-1/2}[1+u^2]$$

$$\frac{\delta u}{\delta y} = \cancel{-}\frac{1}{\cancel{2}}x(1-r^2)^{-3/2}(\cancel{-2r})\frac{y}{\cancel{r}}$$

$$= x(1-r^2)^{-1/2} \times y(1-r^2)^{-1/2} \times (1-r^2)^{-1/2}$$

$$= (1-r^2)^{-1/2}\,uv$$

$$\frac{\delta u}{\delta z} = (1-r^2)^{-1/2}\,uw$$

$v = y(1-r^2)^{-1/2}$

$$\frac{\delta v}{\delta x} = (1-r^2)^{-1/2}\,uv$$

$$\frac{\delta v}{\delta y} = (1-x^2)^{-1/2}(1+v^2)$$

$$\frac{\delta v}{\delta z} = (1-r^2)\,vw$$

$w = z(1-r^2)^{-1/2}$

$$\frac{\delta w}{\delta x} = (1-r^2)^{-1/2}\,uw$$

Scanned by CamScanner

Date:

$$\frac{\delta w}{\delta y} = (1-x^2)^{-1/2} vw$$

$$\frac{\delta w}{\delta z} = (1-x^2)^{-1/2} (1+w^2)$$

$$\frac{\delta(u,v,w)}{\delta(x,y,z)} = \begin{vmatrix} \delta u/\delta x & \delta u/\delta y & \delta u/\delta z \\ \delta v/\delta x & \delta v/\delta y & \delta v/\delta z \\ \delta w/\delta x & \delta w/\delta y & \delta w/\delta z \end{vmatrix}$$

$$= \begin{vmatrix} (1-x^2)^{-1/2}(1+u^2) & (1-x^2)^{-1/2} uv & (1-x^2)^{-1/2} uw \\ (1-x^2)^{-1/2} uv & (1-x^2)^{-1/2}(1+v^2) & (1-x^2)^{-1/2} vw \\ (1-x^2)^{-1/2} uw & (1-x^2)^{-1/2} vw & (1-x^2)^{-1/2}(1+w^2) \end{vmatrix}$$

$$= (1-x^2)^{-3/2} \begin{vmatrix} 1+u^2 & uv & uw \\ uv & 1+v^2 & vw \\ uw & vw & 1+w^2 \end{vmatrix}$$

$$= \frac{(1-x^2)^{-3/2}}{uvw} \begin{vmatrix} u+u^3 & u^2 v & u^2 w \\ uv^2 & v+v^3 & v^2 w \\ w^2 u & vw^2 & w+w^3 \end{vmatrix}$$

$$= (1-x^2)^{-3/2} \begin{vmatrix} 1+u^2 & u^2 & u^2 \\ v^2 & 1+v^2 & v^2 \\ w^2 & w^2 & 1+w^2 \end{vmatrix}$$

$R_1 = R_1 + R_2 + R_3$

$$\frac{\delta(u,v,w)}{\delta(x,y,z)} = (1-x^2)^{-3/2} \begin{vmatrix} 1+u^2+v^2+w^2 & 1+u^2+v^2+w^2 & 1+u^2+v^2+w^2 \\ v^2 & 1+v^2 & v^2 \\ w^2 & w^2 & 1+w^2 \end{vmatrix}$$

$$= (1-x^2)^{-3/2}(1+u^2+v^2+w^2) \begin{vmatrix} 1 & 1 & 1 \\ v^2 & 1+v^2 & v^2 \\ w^2 & w^2 & 1+w^2 \end{vmatrix}$$

$C_1 = C_1 - C_2$

$C_2 = C_2 - C_3$

Scanned by CamScanner

Page No.

Date:

$$\frac{\delta(u,v,w)}{\delta(x,y,z)} = (1-r^2)^{-3/2}(1+u^2+v^2+w^2)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & v^2 \\ 0 & -1 & 1+w^2 \end{vmatrix}$$

$$= (1-r^2)^{-3/2}(1+u^2+v^2+w^2)$$

$$= (1-r^2)^{-3/2}\left[1+x^2(1-r^2)^{-1}+y^2(1-r^2)^{-1}+z^2(1-r^2)^{-1}\right]$$

$$= (1-r^2)^{-3/2}\left[1+(1-r^2)^{-1}(x^2+y^2+z^2)^{-1}\right]$$

$$= (1-r^2)^{-3/2}\left[1+r^2(1-r^2)^{-1}\right]$$

$$= (1-r^2)^{-3/2}\left[1+\frac{r^2}{1-r^2}\right]$$

$$= (1-r^2)^{-3/2}\left[\frac{1\cancel{-r^2}\cancel{+r^2}}{1-r^2}\right]$$

$$= (1-r^2)^{-3/2}(1-r^2)^{-1}$$

$$= (1-r^2)^{-5/2}$$

Ex 3(g)

1) $f(x,y) = e^{x}\cos y \quad ; \quad (0, \pi/4)$

$$\frac{\delta f}{\delta x} = e^{x}\cos y \Rightarrow \left.\frac{\delta f}{\delta x}\right)_{0,\pi/4} = \frac{1}{\sqrt{2}}$$

$$\frac{\delta^2 f}{\delta x^2} = e^{x}\cos y \Rightarrow \left.\frac{\delta^2 f}{\delta x^2}\right)_{0,\pi/4} = \frac{1}{\sqrt{2}}$$

$$\frac{\delta^2 f}{\delta x \delta y} = -e^{x}\sin y \Rightarrow \left.\frac{\delta^2 f}{\delta x\delta y}\right)_{0,\pi/4} = \frac{-1}{\sqrt{2}}$$

$$\frac{\delta f}{\delta y} = -e^{x}\sin y \Rightarrow \left.\frac{\delta f}{\delta y}\right)_{0,\pi/4} = -\frac{1}{\sqrt{2}}$$

$$\frac{\delta^2 f}{\delta y^2} = -e^{x}\cos y \Rightarrow \left.\frac{\delta^2 f}{\delta y^2}\right)_{0,\pi/4} = -\frac{1}{\sqrt{2}}$$

$$f(x,y) = f[0+(x-0), \pi/4 + (x-\pi/4)]$$

$$= f(0,\pi/4) + (x-0)\left.\frac{\delta f}{\delta x}\right)_{0,\pi/4} + (y-\pi/4)\left.\frac{\delta f}{\delta y}\right)_{0,\pi/4}$$

$$+ \frac{1}{2!}\left[(x-0)^2\frac{\delta^2 f}{\delta x^2} + 2(x-0)(y-\pi/4)\frac{\delta^2 f}{\delta x\delta y} + (y-\pi/4)^2\frac{\delta^2 f}{\delta y^2}\right] + \cdots$$

$$= \frac{1}{\sqrt{2}} + \frac{x}{\sqrt{2}} - (y-\pi/4)\frac{1}{\sqrt{2}} + \frac{1}{2}\left[\frac{x^2}{\sqrt{2}} - 2x(y-\pi/4)\frac{1}{\sqrt{2}} - (y-\pi/4)^2\frac{1}{\sqrt{2}}\right] + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + x - \left(y - \frac{\pi}{4}\right) + \frac{x^2}{2} - x\left(y-\frac{\pi}{4}\right) - \frac{1}{2}\left(y-\frac{\pi}{4}\right)^2\right] + \cdots$$

2)

$$f(x,y) = \sin(xy)$$

$$\left.\frac{\delta f}{\delta x}\right)_{1,\pi/2} = \cos(xy)\,[y] = 0$$

$$\left.\frac{\delta f}{\delta y}\right)_{1,\pi/2} = \cos(xy)\,[x] = 0$$

$$\left.\frac{\delta^2 f}{\delta x^2}\right)_{1,\pi/2} = -y^2\sin(xy) = -\frac{\pi^2}{4}$$

$$\left.\frac{\delta^2 f}{\delta y^2}\right)_{1,\pi/2} = -x^2\sin(xy) = -1$$

$$\left.\frac{\delta^2 f}{\delta x\,\delta y}\right)_{1,\pi/2} = -xy\sin(xy) + \cos(xy) = -\frac{\pi}{2}$$

$$\cancel{f(x,y) = f(1,\pi/2)}$$

$$f(x,y) = f[1+(x-1),\ \pi/2 + (y-\pi/2)]$$

$$= f(1,\pi/2) + \left[(x-1)\frac{\delta f}{\delta x} + (y-\pi/2)\frac{\delta f}{\delta y}\right]$$

$$+ \left[(x-1)^2\frac{\delta^2 f}{\delta x^2} + 2(x-1)(y-\pi/2)\frac{\delta^2 f}{\delta x\,\delta y} + (y-\pi/2)^2\frac{\delta^2 f}{\delta y^2}\right]$$

$$+ \cdots$$

$$= 1 + (x-1)^2\left(-\frac{\pi^2}{4}\right) - 2(x-1)(y-\pi/2) - (y-\pi/2)^2\left(\frac{\pi}{2}\right)$$

$$+ \cdots$$

$$= 1 - \frac{\pi^2}{4}(x-1)^2 - 2(x-1)(y-\pi/2) - \frac{\pi}{2}(y-\pi/2)^2 + \cdots$$

3)

$$\cancel{f(x,y) = e^x \sin y}$$

$$\cancel{\left.\frac{\delta f}{\delta x}\right)_{0,0} = e^x \sin y = 0}$$

$$f(x,y) = f[0+(x-0), 0+(y-0)]$$
$$= f(0,0) + \left[(x-0)\frac{\delta f}{\delta x} + (y-0)\frac{\delta f}{\delta y}\right]$$
$$+ \frac{1}{2!}\left[(x-0)^2\frac{\delta^2 f}{\delta x^2} + 2(x-0)(y-0)\frac{\delta^2 f}{\delta x \delta y} + (y-0)^2\frac{\delta^2 f}{\delta y^2}\right] + \cdots$$

3) $f(x,y) = e^x \sin y$

$$\left.\frac{\delta f}{\delta x}\right)_{0,0} = e^x \sin y = 0$$

$$\left.\frac{\delta^2 f}{\delta x^2}\right)_{0,0} = e^x \sin y = 0$$

$$\left.\frac{\delta^3 f}{\delta x^3}\right)_{0,0} = e^x \sin y = 0$$

$$\left.\frac{\delta f}{\delta y}\right)_{0,0} = e^x \cos y = 1$$

$$\left.\frac{\delta^2 f}{\delta y^2}\right)_{0,0} = -e^x \sin y = 0$$

$$\left.\frac{\delta^3 f}{\delta y^3}\right)_{0,0} = -e^x \cos y = -1$$

$$\left.\frac{\delta^2 f}{\delta x \delta y}\right)_{0,0} = e^x \cos y = 1$$

$$\left.\frac{\delta^3 f}{\delta x^2 \delta y}\right)_{0,0} = e^x \cos y = 1$$

$$\left.\frac{\delta^3 f}{\delta x \delta y^2}\right)_{0,0} = -e^x \sin y = 0$$

$$f(x,y) = f[0+(x-0),\ 0+(y-0)]$$

$$= f(0,0) + \left[x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y}\right]$$

$$+ \frac{1}{2!}\left[x^2\frac{\delta^2 f}{\delta x^2} + 2xy\frac{\delta^2 f}{\delta x \delta y} + y^2\frac{\delta^2 f}{\delta y^2}\right]$$

$$+ \frac{1}{3!}\left[x^3\frac{\delta^3 f}{\delta x^3} + 3x^2y\frac{\delta^3 f}{\delta x^2 \delta y} + 3xy^2\frac{\delta^3 f}{\delta x \delta y^2} + y^3\frac{\delta^3 f}{\delta y^3}\right] + \dots$$

$$= y + \frac{1}{2}[2xy] + \frac{1}{3!}[3x^2y - y^3] + \dots$$

$$= y + xy + \frac{3x^2y - y^3}{3!} + \dots$$

4) $f(x,y) = x^2y + 3y - 2$

$$\left.\frac{\delta f}{\delta x}\right)_{1,2} = 2xy = 4$$

$$\left.\frac{\delta f}{\delta y}\right)_{1,2} = x^2 + 3 = 4$$

$$\left.\frac{\delta^2 f}{\delta x^2}\right)_{1,2} = 2y = 4$$

$$\left.\frac{\delta^2 f}{\delta y^2}\right)_{1,2} = 0$$

$$\left.\frac{\delta^2 f}{\delta x \delta y}\right)_{1,2} = 2x = 2$$

$$f(x,y) = f[1+(x-1),\ 2+(y-2)]$$

$$= f(1,2) + \left[(x-1)\frac{\delta f}{\delta x} + (y-2)\frac{\delta f}{\delta y}\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2\frac{\delta^2 f}{\delta x^2} + 2(x-1)(y-2)\frac{\delta^2 f}{\delta x \delta y} + (y-2)^2\frac{\delta^2 f}{\delta y^2}\right]$$

$$+ \dots$$

P.T.O

Date:

$$f(x,y) = 6 + 4(x-1) + 4(y-2) + \frac{1}{2}[4(x-1)^2 + 4(x-1)(y-2)] + \ldots$$

$$= 6 + 4(x-1) + 4(y-2) + 2(x-1)^2 + 2(x-1)(y-2) + \ldots$$

5) $f(x,y,z) = x^2 + y^2 + z^2 + 2xy - yz - 4x - 3y - 8z + 4$

$$\left.\frac{\delta f}{\delta x}\right)_{1,1,1} = 2x + 2y - 4 = 0$$

$$\left.\frac{\delta f}{\delta y}\right)_{1,1,1} = 2y + 2x - 8z - 3 = 0$$

$$\left.\frac{\delta f}{\delta z}\right)_{1,1,1} = 2z - y - 1 = 0$$

$$\left.\frac{\delta^2 f}{\delta x^2}\right)_{1,1,1} = 2$$

$$\left.\frac{\delta^2 f}{\delta y^2}\right)_{1,1,1} = 2$$

$$\left.\frac{\delta^2 f}{\delta z^2}\right)_{1,1,1} = 2$$

$$\left.\frac{\delta^2 f}{\delta x \delta y}\right)_{1,1,1} = 2$$

$$\left.\frac{\delta^2 f}{\delta y \delta z}\right)_{1,1,1} = -1$$

$$\left.\frac{\delta^2 f}{\delta x \delta z}\right)_{1,1,1} = 0$$

$$f(x,y,z) = f[1+(x-1), 1+(y-1), 1+(z-1)]$$

$$= f(1,1,1) + \left[(x-1)\frac{\delta f}{\delta x} + (y-1)\frac{\delta f}{\delta y} + (z-1)\frac{\delta f}{\delta z}\right]$$

$$+ \frac{1}{2}\left[(x-1)^2\frac{\delta^2 f}{\delta x^2} + (y-1)^2\frac{\delta^2 f}{\delta y^2} + (z-1)^2\frac{\delta^2 f}{\delta z^2} + 2(x-1)(y-1)\frac{\delta^2 f}{\delta x \delta y}\right.$$

$$\left. + 2(y-1)(z-1)\frac{\delta^2 f}{\delta z \delta y} + 2(x-1)(z-1)\frac{\delta^2 f}{\delta x \delta z}\right] + \ldots$$

$$= \frac{1}{2}[(x-1)^2(2) + (y-1)^2(2) + (z-1)^2(2) + 2(x-1)(y-1)(2) + 2(y-1)(z-1)(-1)]$$

$$= (x-1)^2 + (y-1)^2 + (z-1)^2 + 2(x-1)(y-1) - (y-1)(z-1) + \ldots$$

6) $z^3 - 2xz + y = 0 \quad ; \quad (1,1,1)$

~~diff~~ partially differentiate wrt x,

$$3z^2\frac{\delta z}{\delta x} - 2x\frac{\delta z}{\delta x} - 2z = 0$$

$$\left.\frac{\delta z}{\delta x}\right)_{1,1,1} = \frac{2z}{3z^2-2x} = 2$$

$$\left.\frac{\delta^2 z}{\delta x^2}\right)_{1,1,1} = \frac{(3z^2-2x)(2\delta z/\delta x) - 2z(6z\,\delta z/\delta x - 2)}{(3z^2-2x)^2}$$

$$= -16$$

partially differentiate wrt y,

$$3z^2\frac{\delta z}{\delta y} - 2x\frac{\delta z}{\delta y} + 1 = 0$$

$$\left.\frac{\delta z}{\delta y}\right)_{1,1,1} = \frac{-1}{3z^2-2x} = -1$$

$$\left.\frac{\delta^2 z}{\delta y^2}\right)_{1,1,1} = \frac{(3z^2-2x)(0) - (-1)(6z\,\delta z/\delta y)}{(3z^2-2x)^2}$$

$$= -6$$

~~partially differentiate wrt z,~~

~~$3z^2 - 2x = 0$~~

$$\frac{\delta^2 z}{\delta x\delta y} = \frac{(3z^2-2x)(2\delta z/\delta y) - (2z)(6z\,\delta z/\delta y)}{(3z^2-2x)^2}$$

$$= -10$$

$$z = f(x,y) = f[1+(x-1), 1+(y-1)]$$

$$= f(1,1) + \left[(x-1)\frac{\delta z}{\delta x} + (y-1)\frac{\delta z}{\delta y}\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2\frac{\delta^2 z}{\delta x^2} + 2(x-1)(y-1)\frac{\delta^2 z}{\delta x \delta y} + (y-1)^2\frac{\delta^2 z}{\delta y^2}\right]$$

$$= 1 + 2(x-1) - (y-1) + \frac{1}{2}\left[-16(x-1)^2 - 20(x-1)(y-1) - 6(y-1)^2\right]$$

$$= 1 + 2(x-1) - (y-1) - 8(x-1)^2 - 10(x-1)(y-1) - 3(y-1)^2 + \ldots$$

Page No.
Date:

Ex 3 (8) 14)

$$T = 400xyz^2 \quad — (1)$$
$$x^2+y^2+z^2 = 1 \quad — (2)$$

$$F \Rightarrow 400xyz^2 + \lambda(x^2+y^2+z^2-1)$$
$$dF = 400[xz^2 dy + yz^2 dx + 2xyz dz] + 2\lambda(x dx + y dy + z dz)$$
$$= (400xz^2 + 2\lambda y)dy + (400yz^2 + 2x\lambda)dx + (800xyz + 2\lambda z) dz$$

$$400xz^2 + 2\lambda y = 0 \quad — (3)$$
$$400yz^2 + 2\lambda x = 0 \quad — (4)$$
$$800xyz + 2\lambda z = 0 \quad — (5)$$

$$x \times (4) + y \times (3) + z \times (5)$$
$$400xyz^2 + 400xyz^2 + 800xyz^2 + 2\lambda(x^2+y^2+z^2) = 0$$
$$1600xyz^2 + 2\lambda = 0$$
$$4T = -2\lambda$$
$$\lambda = -2T \quad — (6)$$

Put (6) in (3), (4), (5)

$$\cancel{400x}\cancel{z^2} + 2y(-2 \times \cancel{400}\cancel{x}y\cancel{z^2}) = 0$$
$$y^2 = 1/4$$
$$y = 1/2$$
$$\cancel{400}\cancel{y}\cancel{z^2} + 2x(-2 \times \cancel{400}x\cancel{y}\cancel{z^2}) = 0$$
$$x^2 = 1/4$$
$$x = 1/2$$
$$\cancel{800}\cancel{xy}\cancel{z} + 2\lambda(-2 \times \cancel{400}\cancel{xy}z^2)\cancel{z} = 0$$
$$z^2 = 1/2$$

$$\therefore T = 400xyz^2$$
$$= 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$
$$= 50 \text{ units.}$$

Date:

15) $P = xyz$ — ①

$x+y+z = a$ — ②

$F = xyz + \lambda(x+y+z-a)$

$dF = yz\,dx + xz\,dy + xy\,dz + \lambda(dx+dy+dz)$

$(yz+\lambda)dx + (xz+\lambda)dy + (xy+\lambda)dz$

$yz+\lambda = 0$ — ③

$xz+\lambda = 0$ — ④

$xy+\lambda = 0$ — ⑤

$x \times ③ + y \times ④ + z \times ⑤$,

$xyz + xyz + xyz + \lambda(x+y+z) = 0$

$3P + a\lambda = 0$

$\lambda = -\frac{3P}{a}$ — ⑥

Put ⑥ in ③, ④ & ⑤,

$\cancel{yz} - \frac{3x\cancel{yz}}{a} = 0$

$x = a/3$

$\cancel{xz} - \frac{3\cancel{x}y\cancel{z}}{a} = 0$

$y = a/3$

$xy - \frac{3}{a}xyz = 0$

$z = a/3$

$\therefore x = y = z = a/3$

Max Product $= \frac{a}{3} \times \frac{a}{3} \times \frac{a}{3} = \frac{a^3}{27}$

Page No.
Date:

1) $f(x,y) = x^3+y^3-3axy$

$\frac{\delta f}{\delta x} = 3x^2 - 3ay = 0 \quad -①$

$\frac{\delta f}{\delta y} = 3y^2 - 3ax = 0 \quad -②$

① − ②,

$3(x^2-y^2) - 3ay + 3ax = 0$

$\cancel{3}(x-y)(x+y) + \cancel{3}a(x-y) = 0$

$(x-y)(x+y+a) = 0$

$x = y$ and $y = -x-a \quad -③$

Put ③ in ①,

$3x^2 - 3ax = 0$

$3x(x-a) = 0 \Rightarrow x = 0, a \rightarrow y = 0, a$

$3x^2 - 3a(-x-a) = 0$

$3x^2 + 3ax + 3a^2 = 0$

$x^2 + ax + a^2 = 0 \rightarrow$ imaginary roots

Stationary points $\Rightarrow$ $(0,0)$ and (a,a)

$\frac{\delta^2 f}{\delta x^2} = r = 6x$

$\frac{\delta^2 f}{\delta y^2} = t = 6y$

$\frac{\delta^2 f}{\delta x \delta y} = s = -3a$

At $(0,0)$,

$rt - s^2 = 36yx - 9a^2 = -9a^2 < 0$

$(0,0)$ is neither max nor min pt.

~~saddle pt.~~

Date:

At (a,a),

$rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

$r = 6a$

$\therefore f_{max} / f_{min} = a^3 + a^3 - 3a^3 = -a^3$

2) $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$\frac{\delta f}{\delta x} = 4x^3 - 4x + 4y = 0$ — ①

$\frac{\delta f}{\delta y} = 4y^3 + 4x - 4y = 0$ — ②

① + ②,

$\cancel{4}(x^3 + y^3) = 0$

$(x+y)(x^2 + y^2 - xy) = 0$

$x = -y$ — ③

Put ③ in ①,

$4x^3 - 4x - 4x = 0$

$4x^3 - 8x = 0$

$4x(x^2 - 2) = 0 \Rightarrow x = 0, \sqrt{2}, -\sqrt{2} \rightarrow y = 0, -\sqrt{2}, \sqrt{2}$

Stationary pts $\rightarrow (0,0), (\sqrt{2}, -\sqrt{2})$ & $(-\sqrt{2}, \sqrt{2})$

$r = \frac{\delta^2 f}{\delta x^2} = 12x^2 - 4$

$s = \frac{\delta^2 f}{\delta x \delta y} = 4$

$t = \frac{\delta^2 f}{\delta y^2} = 12y^2 - 4$

At $(0,0)$,

$rt - s^2 = (-4)(-4) - 16 = 0$

Scanned by CamScanner

(0,0) is neither max. nor min. pt.

At $(\sqrt{2}, -\sqrt{2})$,

$$rt - s^2 = (20)(20) - 16 = 384 > 0$$

$$r = 20 > 0$$

$\therefore (\sqrt{2}, -\sqrt{2})$ is min. pt.

$$f_{min} = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(\sqrt{2})(-\sqrt{2}) - 2(-\sqrt{2})^2$$

$$= \cancel{4} + \cancel{4} - 4 - \cancel{8} - 4 = -8$$

At $(-\sqrt{2}, \sqrt{2})$,

$$rt - s^2 = (20)(20) - 16 = 384 > 0$$

$$r = 20 > 0$$

$\therefore (-\sqrt{2}, \sqrt{2})$ is a min. pt.

$$f_{min} = (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(-\sqrt{2})(\sqrt{2}) - 2(\sqrt{2})^2$$

$$= -8$$

3)

$$f(x,y) = x^2y^2 - 5x^2 - 8xy - 5y^2$$

$$\frac{\delta f}{\delta x} = 2xy^2 - 10x - 8y = 0 \quad \text{— (1)}$$

$$\frac{\delta f}{\delta y} = 2x^2y - 8x - 10y = 0 \quad \text{— (2)}$$

(1) − (2),

Date:

① - ②

$2xy^2 - 2x^2y + 2y - 2x = 0$

$\cancel{2}xy(y-x) + \cancel{2}(y-x) = 0$

$(y-x)(xy+1) = 0$

$x = y$ and $x = -1/y$

Case 1 : $x = -1/y$

$2\left(\frac{-1}{\cancel{y}}\right)y^{\cancel{2}} - 10\left(-\frac{1}{y}\right) - 8y = 0$

$-2y + \frac{10}{y} - 8y = 0$

$\cancel{10}y = \frac{\cancel{10}}{y}$

$y^2 = 1$

$y = \pm 1 \rightarrow x = \mp 1$

Case 2 : $x = y$

$2x^3 - 10x - 8x = 0$

$2x^3 - 18x = 0$

$2x(x^2 - 9) = 0$

$2x(x-3)(x+3) = 0$

$x = 0, \pm 3 \rightarrow y = 0, \pm 3$

Stationary pts $\rightarrow$ $(0,0)$, $(\pm 3, \pm 3)$ & $(\mp 1, \pm 1)$

$r = \frac{\delta^2 f}{\delta x^2} = 2y^2 - 10$

$s = \frac{\delta^2 f}{\delta x \delta y} = 4xy - 8$

$t = \frac{\delta^2 f}{\delta y^2} = 2x^2 - 10$

Scanned by CamScanner

At $(0,0)$,

$rt - s^2 = (-10)(-10) - (-8)^2 = 36 > 0$

$r = -10 < 0$

$\therefore$ $(0,0)$ is max. pt.

$f_{max} = 0$

At $(3,3)$,

$rt - s^2 = (8)(8) - (28)^2 = -720 < 0$

$(3,3)$ is saddle point.

At $(-3,-3)$,

$rt - s^2 = (8)(8) - (28)^2 = -720 < 0$

$(-3,-3)$ is saddle pt.

At $(1,-1)$,

$rt - s^2 = (-8)(-8) - (-12)^2 = -80 < 0$

$(1,-1)$ is saddle pt.

At $(-1,1)$,

$rt - s^2 = (-8)(-8) - (-12)^2 = -80 < 0$

$(-1,1)$ is saddle pt.

5) $f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ — (1)

$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 = 0$ — (2)

Page No.
Date:

4) $f(A,B) = \sin A \sin B \sin(A+B)$

$$\frac{\delta f}{\delta A} = \sin B[\cos A \sin(A+B) + \sin A \cos(A+B)]$$

$$= \sin(2A+B)\sin B = 0 \quad — ①$$

$$\frac{\delta f}{\delta B} = \sin A[\cos B \sin(A+B) + \sin B \cos(A+B)]$$

$$= \sin(A+2B)\sin A = 0 \quad — ②$$

① - ②,

$$\frac{1}{2} \times 2\sin(2A+B)\sin B - \frac{1}{2} \times 2\sin(A+2B)\sin A = 0$$

$$\cos 2A - \cancel{\cos 2(A+B)} - \cos 2B + \cancel{\cos 2(A+B)} = 0$$

$$-2\sin(A+B)\sin(A-B) = 0$$

$A+B = 0$ or $A-B = 0$

Case 1 : $A = B$,

$\sin 3A \sin A = 0$

$(3\sin A - 4\sin^3 A)\sin A = 0$

$\sin^2 A(3 - 4\sin^2 A) = 0$

$\sin A = 0, \pm\sqrt{3}/2$

$A = 0, \pm 60°$

$B = 0, \pm 60°$

Case 2: $A = -B$

$\sin^2 A = 0$

$A = 0$

$B = 0$

Stationary pts $\longrightarrow$ $(0,0)$ & $(\pm 60°, \pm 60°)$

Page No.
Date

$r = \frac{\delta^2 f}{\delta A^2} = 2\cos(2A+B)\sin B$

$s = \frac{\delta^2 f}{\delta A \delta B} = \sin(2A+B)\cos B + \cos(2A+B)\sin B$

$= \sin 2(A+B)$

$t = \frac{\delta^2 f}{\delta B^2} = 2\cos(A+2B)\sin A.$

At $(0,0)$,

$rs - t^2 = 0 - 0 = 0$

$(0,0)$ is neither max. nor min. pt.

$\frac{1}{2}$

At $(60^\circ, 60^\circ)$,

$rs - t^2 = \left(-\cancel{2} \times \frac{\sqrt{3}}{\cancel{2}}\right)\left(-\cancel{2} \times \frac{\sqrt{3}}{\cancel{2}}\right) - \left(\frac{-\sqrt{3}}{2}\right)^2 = \frac{9}{4} > 0$

$r = -\sqrt{3} < 0$

$(60^\circ, 60^\circ)$ is max. pt.

$f_{max} = \sin 60^\circ \sin 60^\circ \sin 120^\circ$

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} \times \left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{8}$

At $(-60^\circ, -60^\circ)$,

$rs - t^2 = \left(\cancel{2} \times \frac{\sqrt{3}}{\cancel{2}}\right)\left(\cancel{2} \times \frac{\sqrt{3}}{\cancel{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4} > 0$

$r = \sqrt{3} > 0$

$(-60^\circ, -60^\circ)$ is min pt.

$f_{min} = \sin(-60^\circ)\sin(-60^\circ)\sin(-120^\circ)$

$= -\frac{\sqrt{3}}{2} \times \left(-\frac{\sqrt{3}}{2}\right) \times \left(\frac{-\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{8}$

7) $$P = x^m y^n z^p \quad — ①$$

$$x+y+z = a \quad — ②$$

$$F = x^m y^n z^p + \lambda(x+y+z-a)$$

$$dF = mx^{m-1} y^n z^p dx + nx^m y^{n-1} z^p dy + px^m y^n z^{p-1} dz + \lambda(dx+dy+dz)$$

$$= (mx^{m-1} y^n z^p + \lambda)dx + (nx^m y^{n-1} z^p dy + \lambda) dy + (px^m y^n z^{p-1} + \lambda) dz$$

$$mx^{m-1} y^n z^p + \lambda = 0 \quad — ③$$

$$nx^m y^{n-1} z^p + \lambda = 0 \quad — ④$$

$$px^m y^n z^{p-1} + \lambda = 0 \quad — ⑤$$

$x \times ③ + y \times ④ + z \times ⑤$,

$$P(m+n+p) + \lambda a = 0$$

$$\lambda = \frac{-P(m+n+p)}{a} \quad — ⑥$$

Put ⑥ in ③, ④ & ⑤,

$$m\cancel{x^{m-1}}\cancel{y^n}\cancel{z^p} = \frac{x^m \cancel{y^n}\cancel{z^p}}{a}(m+n+p)$$

$$x = \frac{am}{m+n+p}$$

$$n\cancel{x^m}\cancel{y^{n-1}}\cancel{z^p} = \frac{\cancel{x^m} y \cancel{z^p}}{a}(m+n+p)$$

$$y = \frac{an}{m+n+p}$$

$$p\cancel{x^m}\cancel{y^n} \cancel{z^{p-1}} = \frac{\cancel{x^m}\cancel{y^n} z}{a}(m+n+p)$$

$$z = \frac{ap}{m+n+p}$$

$$P_{max} = \left(\frac{am}{m+n+p}\right)^{m} \left(\frac{an}{m+n+p}\right)^{n} \left(\frac{ap}{m+n+p}\right)^{p}$$

$$= m^m n^n p^p \left(\frac{a}{m+n+p}\right)^{m+n+p}$$

8) $P = x^2 + y^2 + z^2 \quad — ①$

$ax^2 + by^2 + cz^2 = 1 \quad — ②$

$lx + my + nz = 0 \quad — ③$

$$F = x^2+y^2+z^2 + \lambda_1(ax^2+by^2+cz^2-1) + \lambda_2(lx+my+nz)$$

$$dF = 2xdx + 2ydy + 2zdz + \lambda_1(2axdx + 2bydy + 2czdz) + \lambda_2(ldx + mdy + ndz)$$

$$= (2x + 2a\lambda_1 x + l\lambda_2)dx + (2y + 2b\lambda_1 y + m\lambda_2)dy + (2z + 2cz\lambda_1 + n\lambda_2)dz$$

$2x + 2a\lambda_1 x + l\lambda_2 = 0 \quad — ④$

$2y + 2b\lambda_1 y + m\lambda_2 = 0 \quad — ⑤$

$2z + 2cz\lambda_1 + n\lambda_2 = 0 \quad — ⑥$

$x \times ④ + y \times ⑤ + ⑥ \times z$,

$$2P + 2\lambda_1(ax^2+by^2+cz^2) + \lambda_2(lx+my+nz) = 0$$

$$\cancel{2}P + \cancel{2}\lambda_1 = 0$$

$\lambda_1 = -P \quad — ⑦$

Put ⑦ in ④, ⑤ & ⑥,

$$2x - 2aPx + l\lambda_2 = 0$$

$$x = \frac{-l\lambda_2}{2(1-aP)}$$

$$y = \frac{-m\lambda_2}{2(1-bP)}$$

$$z = \frac{-n\lambda_2}{2(1-cP)}$$

Page No.
Date:

Putting ~~values~~ values of x, y, z in (3),

$$\frac{l^2 \cancel{\lambda x}}{\cancel{\lambda}(aP-1)} + \frac{m^2 \cancel{\lambda x}}{\cancel{\lambda}(bP-1)} + \frac{n^2 \cancel{\lambda z}}{\cancel{\lambda}(cP-1)} = 0$$

$$\frac{l^2}{aP-1} + \frac{m^2}{bP-1} + \frac{n^2}{cP-1} = 0$$

Above eq^n gives stationary values of $P = x^2 + y^2 + z^2$

Q

$u = xyz$ — (1)

$x + y + z = 5$ — (2)

$xy + yz + zx = 8$ — (3)

Date:

17) Side of base of square tent $= 2a$

Height $= h$

Height of regular pyramid $= b$.

$$\therefore V = 4a^2h + \frac{4a^2b}{3}$$

$$S = 8ah + 4a\sqrt{a^2+b^2}$$

$$F = S + \lambda V$$

$$= 8ah + 4a\sqrt{a^2+b^2} + \lambda\left(4a^2h + \frac{4a^2b}{3}\right)$$

$$dF = 8a\,dh + 8h\,da + 4\sqrt{a^2+b^2}\,da + \frac{4a^2\,da}{\sqrt{a^2+b^2}} + \frac{4ab\,db}{\sqrt{a^2+b^2}}$$

$$+ \lambda\left(4a^2dh + 8ah\,da + \frac{4a^2\,db}{3} + \frac{8ab\,da}{3}\right)$$

$$= \left(8h + 4\sqrt{a^2+b^2} + \frac{4a^2}{\sqrt{a^2+b^2}} + 8ah\lambda + \frac{8ab\lambda}{3}\right)da + \left(\frac{4ab}{\sqrt{a^2+b^2}} + \frac{4a^2\lambda}{3}\right)db$$

$$+ \left(8a + 4a^2\lambda\right)dh$$

$$8a + 4a^2\lambda = 0$$

$$\lambda = -\frac{2}{a} \quad -①$$

$$\frac{4ab}{\sqrt{a^2+b^2}} + \frac{4a^2\lambda}{3} = 0$$

$$\frac{b}{\sqrt{a^2+b^2}} = -\frac{a}{3}\left(-\frac{2}{a}\right)$$

$$\frac{b}{\sqrt{a^2+b^2}} = \frac{2}{3}$$

$$\frac{b^2}{a^2+b^2} = \frac{4}{9}$$

Page No.
Date:

$$4a^2 + 4b^2 = 9b^2$$
$$4a^2 = 5b^2$$
$$\boxed{a = \frac{\sqrt{5}b}{2}}$$

$$8h + 4\sqrt{a^2+b^2} + \frac{4a^2}{\sqrt{a^2+b^2}} + 8ahf + \frac{8abf}{3} = 0$$

$$8h + 4\sqrt{\frac{5b^2}{4}+b^2} + \frac{4a^2}{\sqrt{5b^2/4+b^2}} + 8ah\left(-\frac{2}{a}\right) + \frac{8ab}{3}\left(-\frac{2}{a}\right) = 0$$

$$8h + 4\left(\frac{3b}{2}\right) + \frac{4a^2}{3b/2} - 16h - \frac{16b}{3} = 0$$

$$8h + 6b + \frac{8}{3b}\left(\frac{5b^2}{4}\right) - 16h - \frac{16b}{3} = 0$$

$$8h + 6b + \frac{10b}{3} - 16h - \frac{16b}{3} = 0$$

$$-8h + 6b - 2b = 0$$
$$8h = 4b$$
$$\boxed{h = \frac{b}{2}}$$

12)
$$r^2 = a^2x^2 + b^2y^2 + c^2z^2 = u \text{ (let)} \quad \text{— (1)}$$
$$x^2 + y^2 + z^2 = 1 \quad \text{— (2)}$$
$$lx + my + nz = 0 \quad \text{— (3)}$$

$$F = a^2x^2 + b^2y^2 + c^2z^2 + \lambda_1(x^2+y^2+z^2-1) + \lambda_2(lx+my+nz)$$
$$dF = 2a^2x\,dx + 2b^2y\,dy + 2c^2z\,dz + \lambda_1(2x\,dx + 2y\,dy + 2z\,dz) + \lambda_2(l\,dx + m\,dy + n\,dz)$$
$$= (2a^2x + 2\lambda_1 x + \lambda_2 l)dx + (2b^2y + 2\lambda_1 y + m\lambda_2)dy + (2c^2z + 2\lambda_1 z + n\lambda_2)dz$$

P.T.O

Page No.
Date:

$2a^2x + 2\lambda_1 x + l\lambda_2 = 0$ — ④

$2b^2y + 2\lambda_1 y + m\lambda_2 = 0$ — ⑤

$2c^2z + 2\lambda_1 z + n\lambda_2 = 0$ — ⑥

$x \times ④ + y \times ⑤ + ⑥ \times z$,

$\cancel{2}u + \cancel{2}\lambda_1 = 0$

$\lambda_1 = -u$ — ⑦

Put ⑦ in ④, ⑤ & ⑥,

$2a^2x - 2ux + l\lambda_2 = 0$

$x = \dfrac{l\lambda_2}{2u - 2a^2}$

$2b^2y - 2uy + m\lambda_2 = 0$

$y = \dfrac{m\lambda_2}{2u - 2b^2}$

$2c^2z - 2uz + n\lambda_2 = 0$

$z = \dfrac{n\lambda_2}{2u - 2c^2}$

Putting values of x, y & z in ③,

$$\frac{l^2\cancel{\lambda_2}}{\cancel{2}u - \cancel{2}a^2} + \frac{m^2\cancel{\lambda_2}}{\cancel{2}u - \cancel{2}b^2} + \frac{n^2\cancel{\lambda_2}}{\cancel{2}u - \cancel{2}c^2} = 0$$

$$\frac{l^2}{u - a^2} + \frac{m^2}{u - b^2} + \frac{n^2}{u - c^2} = 0$$

$$\frac{l^2}{r^2 - a^2} + \frac{m^2}{r^2 - b^2} + \frac{n^2}{r^2 - c^2} = 0$$

Page No.
Date

16)

$$u = d^2 = x^2+y^2+z^2 \quad — (1)$$
$$xyz = a \quad — (2)$$
$$y = bx \quad — (3)$$

$$F = x^2+y^2+z^2 + \lambda_1(xyz-a) + \lambda_2(y-bx)$$
$$dF = 2x\,dx + 2y\,dy + 2z\,dz + \lambda_1(yz\,dx + xz\,dy + xy\,dz) + \lambda_2(dy - b\,dx)$$
$$= (2x + yz\lambda_1 - b\lambda_2)dx + (2y + xz\lambda_1 + \lambda_2)dy + (2z + xy\lambda_1)dz$$

$$2x + yz\lambda_1 - b\lambda_2 = 0 \quad — (4)$$
$$2y + xz\lambda_1 + \lambda_2 = 0 \quad — (5)$$
$$2z + xy\lambda_1 = 0 \quad — (6)$$

from (6),
$$\lambda_1 = \frac{-2z}{xy} \quad — (7)$$

Put (7) in (5),
$$2y - \frac{2\cancel{x}z^2}{\cancel{x}y} + \lambda_2 = 0$$
$$\lambda_2 = \frac{2z^2}{y} - 2y \quad — (8)$$

Put (7) & (8) in (4),
$$2x + \cancel{y}z\left(\frac{-2z}{x\cancel{y}}\right) - b\left(\frac{2z^2}{y} - 2y\right) = 0$$
$$\cancel{2}x - \frac{\cancel{2}z^2}{x} - \frac{\cancel{2}bz^2}{y} + \cancel{2}by = 0$$
$$x - \frac{z^2}{x} - \frac{z^2}{\cancel{y}}\left(\frac{\cancel{y}}{x}\right) + \frac{y^2}{x} = 0$$
$$x - \frac{2z^2}{x} + \frac{y^2}{x} = 0$$

P.T.O.

$$2z^2 = x^2+y^2$$

$$\therefore d^2 = x^2+y^2+z^2 = 3z^2$$

$$2z^2 = x^2+y^2$$
$$2z^2 = x^2+b^2x^2$$
$$2z^2 = x^2(1+b^2) \quad — (9)$$

from (2),

$$xyz = a$$
$$bx^2z = a$$
$$x^2 = \frac{a}{bz} \quad — (10)$$

Put (10) in (9),

$$2z^2 = \frac{a}{bz}(1+b^2)$$

$$z^3 = \frac{a}{2b}(1+b^2)$$

$$z = \left[\frac{a}{2b}(1+b^2)\right]^{1/3}$$

$$\therefore d = \sqrt{3}z$$
$$d = \sqrt{3}\left[\frac{a}{2b}(1+b^2)\right]^{1/3}$$

9)

$$u = xyz \quad — (1)$$
$$x+y+z = 5 \quad — (2)$$
$$xy+yz+xz = 8 \quad — (3)$$

$$F = xyz + \lambda_1(x+y+z-5) + \lambda_2(xy+yz+xz-8)$$

Scanned by CamScanner

$$dF = yz\,dx + xz\,dy + xy\,dz + \lambda_1(dx + dy + dz) + \lambda_2(y\,dx + x\,dy + y\,dz + z\,dy + x\,dz + z\,dx)$$

$$= (yz + \lambda_1 + y\lambda_2 + z\lambda_2)\,dx + (xz + \lambda_1 + x\lambda_2 + z\lambda_2)\,dy + (xy + \lambda_1 + x\lambda_2 + y\lambda_2)\,dz$$

$$yz + \lambda_1 + y\lambda_2 + z\lambda_2 = 0 \quad -④$$

$$xz + \lambda_1 + x\lambda_2 + z\lambda_2 = 0 \quad -⑤$$

$$xy + \lambda_1 + x\lambda_2 + y\lambda_2 = 0 \quad -⑥$$

Case 1:

④ − ⑤,

$$z\cancel{(y-x)} + \lambda_2\cancel{(y-x)} = 0$$

$$\lambda_2 = -z \quad -⑦$$

Also,

$$y\cancel{z} + \lambda_1 - y\cancel{z} - z^2 = 0$$

$$\lambda_1 = z^2 \quad -⑧$$

④ + ⑤ + ⑥,

$$xy + yz + xz + 3\lambda_1 + 2\lambda_2(x+y+z) = 0$$

$$8 + 3\lambda_1 + 10\lambda_2 = 0 \quad -⑨$$

Put ⑦ & ⑧ in ⑨,

$$3z^2 - 10z + 8 = 0$$

$$3z^2 - 6z - 4z + 8 = 0$$

$$3z(z-2) - 4(z-2) = 0$$

$$(z-2)(3z-4) = 0$$

$$z = 2, 4/3$$

Similarly, $y = 2, 4/3$.

from eqn ②,

$$x = 1, 7/3$$

P.T.O.

Page No.
Date:

$\therefore U_{max} = \frac{4}{3} \times \frac{4}{3} \times \frac{7}{3}$

$= \frac{112}{27}$

$U_{min} = 2 \times 2 \times 1$

$= 4$

$\therefore$ Two of the variables are $4/3$ and third is $7/3$ for U_{max}.
Two of the variables are 2 and third is 1 for U_{min}

13) $V = xyz$ — ①

$S = 2(xy + yz + xz)$ — ②

$F = S + \lambda V$

$= 2(xy + yz + xz) + \lambda(xyz - V)$

$dF = 2(xdy + ydx + ydz + zdy + xdz + zdx)$
$+ \lambda(yzdx + xzdy + xydz)$

$= (2y + 2z + \lambda yz)dx + (2x + 2z + \lambda xz)dy$
$+ (2x + 2y + \lambda xy)dz$

$2y + 2z + \lambda yz = 0$ — ③

$2x + 2z + \lambda xz = 0$ — ④

$2x + 2y + \lambda xy = 0$ — ⑤

③ $\times x$ + ④ $\times y$ + ⑤ $\times z$,

$2S + 3\lambda V = 0$

$\lambda = -\frac{2S}{3V}$ — ⑥

Put ⑥ in ③, ④, ⑤,

$$\cancel{2}y + \cancel{2}z = \frac{\cancel{2}S\,\cancel{yz}}{3x\cancel{yz}}$$

$$\cancel{3}xy + \cancel{3}xz = \cancel{2xy} + 2yz + \cancel{2zx}$$

$$xy + xz = 2yz$$

$$\therefore S = 6yz \quad - (7)$$

$$\cancel{2}x + \cancel{2}z = \frac{\cancel{2}S\,\cancel{xz}}{3\cancel{x}y\cancel{z}}$$

$$3xy + \cancel{3}yz = \cancel{2xy} + \cancel{2yz} + 2xz$$

$$xy + yz = 2zx$$

$$\therefore S = 6xz \quad - (8)$$

$$\cancel{2}x + \cancel{2}y = \frac{\cancel{2}S\,\cancel{xy}}{3\cancel{xy}z}$$

$$\cancel{3}xz + \cancel{3}yz = 2xy + \cancel{2yz} + \cancel{2xz}$$

$$xz + yz = 2xy$$

$$\therefore S = 6xy \quad - (9)$$

from (7), (8) & (9),

$$\cancel{6}y\cancel{z} = \cancel{6}x\cancel{z}$$

$$x = y$$

$$\cancel{6x}z = \cancel{6x}y$$

$$y = z$$

$$\therefore x = y = z$$

Smallest surface of given volume V is a cube.

Page No.
Date:

6) $f(x,y) = \frac{x-y}{(x^2+y^2+1)}$

$\frac{\delta f}{\delta x} \Rightarrow \frac{(x^2+y^2+1)-(x-y)(2x)}{(x^2+y^2+1)^2} = 0$

$\Rightarrow \frac{y^2-x^2+1+2xy}{(x^2+y^2+1)^2} = 0$ — ①

$\frac{\delta f}{\delta y} \Rightarrow \frac{-(x^2+y^2+1)-(x-y)(2y)}{(x^2+y^2+1)^2} = 0$

$\Rightarrow \frac{y^2-x^2-1-2xy}{(x^2+y^2+1)^2} = 0$ — ②

① + ②,

$y^2-x^2 = 0$

$(y-x)(y+x) = 0$

$y = x$ and $y = -x$

↳ won't consider as they will yield imaginary values.

$\therefore y = -x$ — ③

Put ③ in ①,

$2x^2 - 1 = 0$

$x = \pm\frac{1}{\sqrt{2}} \Rightarrow y = \mp\frac{1}{\sqrt{2}}$

Stationary pts. $\Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$r = \frac{\delta^2 f}{\delta x^2} = \frac{(x^2+y^2+1)^2(-2x+2y) - (y^2-x^2+1+2xy)\,2(x^2+y^2+1)(2x)}{(x^2+y^2+1)^4}$

$= \frac{2(y-x)}{(x^2+y^2+1)^2} - \frac{4x(y^2-x^2+1+2xy)}{(x^2+y^2+1)^3}$

Page No.
Date:

$$s = \frac{\delta^2 f}{\delta x \delta y} = \frac{(x^2+y^2+1)^2(2y+2x) - (y^2-x^2+1+2xy)2(x^2+y^2+1)(2y)}{(x^2+y^2+1)^4}$$

$$= \frac{2(x+y)}{(x^2+y^2+1)^2} - \frac{4y(y^2-x^2+1+2xy)}{(x^2+y^2+1)^3}$$

$$t = \frac{\delta^2 f}{\delta y^2} = \frac{(x^2+y^2+1)^2(2y-2x) - (y^2-x^2-1-2xy)2(x^2+y^2+1)(2y)}{(x^2+y^2+1)^4}$$

$$= \frac{2(y-x)}{(x^2+y^2+1)^2} - \frac{4y(y^2-x^2-1-2xy)}{(x^2+y^2+1)^3}$$

for $(1/\sqrt{2}, -1/\sqrt{2})$,

$$rt - s^2 = (-1/\sqrt{2})(-1/\sqrt{2}) - (0)^2 = 1/2 > 0$$

$$r = -1/\sqrt{2} < 0$$

$(1/\sqrt{2}, -1/\sqrt{2})$ is a max pt.

$$\therefore f_{max} = \frac{1}{\sqrt{2}}$$

for $(-1/\sqrt{2}, 1/\sqrt{2})$,

$$rt - s^2 = (1/\sqrt{2})(1/\sqrt{2}) - (0)^2 = 1/2 > 0$$

$$r = 1/\sqrt{2} > 0$$

$(-1/\sqrt{2}, 1/\sqrt{2})$ is a min pt.

$$\therefore f_{min} = \frac{-1}{\sqrt{2}}$$

10)

$$u = x^2+y^2+z^2 \quad — (1)$$

$$lx + my + nz = 1 \quad — (2)$$

$$l'x + m'y + n'z = 1 \quad — (3)$$

$$F = x^2+y^2+z^2 + \lambda_1(lx+my+nz-1) + \lambda_2(l'x+m'y+n'z-1)$$

$$dF = 2x\,dx + 2y\,dy + 2z\,dz + \lambda_1(l\,dx + m\,dy + n\,dz) + \lambda_2(l'dx + m'dy + n'dz)$$

$$= (2x + l\lambda_1 + l'\lambda_2)dx + (2y + m\lambda_1 + m'\lambda_2)dy + (2z + n\lambda_1 + n'\lambda_2)dz$$

$$2x + l\lambda_1 + l'\lambda_2 = 0 \quad \text{— (4)}$$

$$2y + m\lambda_1 + m'\lambda_2 = 0 \quad \text{— (5)}$$

$$2z + n\lambda_1 + n'\lambda_2 = 0 \quad \text{— (6)}$$

$x \times (4) + y \times (5) + z \times (6)$,

$$2u + 0\lambda_1 + \lambda_2 = 0 \quad \text{— (7)}$$

$l \times (4) + m \times (5) + n \times (6)$,

$$2 + \lambda_1(l^2 + m^2 + n^2) + \lambda_2(ll' + mm' + nn') = 0$$

$$2 + \lambda_1 \Sigma l^2 + \lambda_2 \Sigma ll' = 0 \quad \text{— (8)}$$

$l' \times (4) + m' \times (5) + n' \times (6)$,

$$2 + \lambda_1(ll' + mm' + nn') + \lambda_2(l'^2 + m'^2 + n'^2) = 0$$

$$2 + \lambda_1 \Sigma ll' + \lambda_2 \Sigma l'^2 = 0 \quad \text{— (9)}$$

Scanned by CamScanner

Page No.
Date

5) $f(x,y) = x^3y^2(1-x-y)$

$$\frac{\delta f}{\delta x} = 3x^2y^2(1-x-y) - x^3y^2 = 0 \quad -①$$

$$\frac{\delta f}{\delta y} = 2yx^3(1-x-y) - x^3y^2 = 0 \quad -②$$

① - ②

$$3x^2y^2(1-x-y) - 2x^3y(1-x-y) = 0$$

$$x^2y(1-x-y)(3y-2x) = 0$$

$$x+y=1 \text{ or } 2x = 3y$$

Case 1: Put $y = 1-x$ in ①

$$-x^3(1-x)^2 = 0$$

$$x = 0, 1 \rightarrow y = 1, 0$$

Case 2: $2x = 3y$

$$3x^2\left(\frac{2x}{3}\right)^2\left(1-x-\frac{2x}{3}\right) - x^3\left(\frac{2x}{3}\right)^2 = 0$$

$$3x^2 \times \frac{4x^2}{9}\left(1-\frac{5x}{3}\right) - x^3 \times \frac{4x^2}{9} = 0$$

$$\frac{4x^4}{9}\left[3-5x-x\right] = 0$$

$$x^4(3-6x) = 0$$

$$x = 0, 1/2 \rightarrow y = 0, 1/3$$

Stationary pts. $\rightarrow$ $(0,0)$, $(0,1)$, $(1,0)$ and $(1/2, 1/3)$

$$r = \frac{\delta^2 f}{\delta x^2} = 6xy^2(1-x-y) - 3x^2y^2 - 3x^2y^2$$

$$= 6xy^2(1-x-y-x) = 6xy^2(1-2x-y)$$

$$s = \frac{\delta^2 f}{\delta x \delta y} = 6x^2y(1-x-y) - 3x^2y^2 - 2x^3y$$

$$= x^2y(6-6x-6y-3y-2x)$$

$$= x^2y(6-8x-9y)$$

$$t = \frac{\delta^2 f}{\delta y^2} = 2x^3(1-x-y) - 2x^3y - 2x^3y$$

$$= 2x^3(1-x-y-y-y)$$

$$= 2x^3(1-x-3y)$$

for (0,0),

$rt - s^2 = 0 - 0 = 0$

$\therefore$ (0,0) is neither max nor min pt.

for (0,1),

$rt - s^2 = 0 - 0 = 0$

$\therefore$ (0,1) is neither max, nor min pt.

for (1,0),

$rt - s^2 = 0 - 0 = 0$

$\therefore$ (1,0) is neither max nor min pt.

for (1/2, 1/3),

$rt - s^2 = (-1/9)(-1/8) - (-1/12)^2 = 1/144 > 0$

$r = -1/9 < 0$

$\therefore$ (1/2, 1/3) is max pt.

$$\therefore f_{max} = \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{8} \times \frac{1}{9} \times \frac{1}{6} = \frac{1}{432}$$