Partial Differentiation

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x.$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y.$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}.$$

$$\boxed{f_{xy} = f_{yx}}$$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

$$\lim_{\substack{x \to a \\ y \to b}} \text{ or } \lim_{(x,y) \to (a,b)} = f(a, b)$$

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Q. Show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

(1) $u = \log(x^2+y^2)$

(2) $u = x\cos y + y\cos x$

Sol. $u = \log(x^2+y^2)$

$\frac{\partial u}{\partial x} = \cancel{\log} \frac{2x}{x^2+y^2}$.

$\frac{\partial^2 u}{\partial x \partial y} \Rightarrow \frac{-2x}{(x^2+y^2)^2}(2y)$

$= -\frac{4xy}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} \Rightarrow \frac{-4xy}{(x^2+y^2)^2}$

(2). $u = x\cos y + y\cos x$.

$\frac{\partial u}{\partial x} = \checkmark$.

Q. $u = e^{xyz}$. Show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$

$\boxed{\text{(move from right to left)}}$

Sol. $u = e^{xyz}$

$\frac{\partial u}{\partial z} = xye^{xyz}$.

$\frac{\partial u}{\partial y \partial z}$.

Q: If $r^2 = x^2+y^2+z^2$, $v = r^m$.

Prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = m(m+1)r^{m-2}$

Sol

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \times \frac{\partial r}{\partial x}$$

Illy for y and z → don't solve! just write Illy

$$\frac{\partial v}{\partial x} = m r^{m-1} \frac{\partial r}{\partial x}$$

$$\cancel{2} r \frac{\partial r}{\partial x} = \cancel{2} x + \cancel{y^2+z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial v}{\partial x} = m r^{m-2}(x)$$

$$\frac{\partial^2 v}{\partial x^2} = m r^{m-2} + m x(m-2) r^{m-3} \frac{\partial r}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = m r^{m-2} + m x^2 (m-2) r^{m-4}$$

$$\frac{\partial^2 v}{\partial y^2} = m r^{m-2} + m y^2 (m-2) r^{m-4}$$

$$\frac{\partial^2 v}{\partial z^2} = m r^{m-2} + m z^2 (m-2) r^{m-4}$$

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Q. $z(x+y) = x^2+y^2$

Show that :→ $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Solution:→

$$z = \frac{x^2+y^2}{x+y}$$

$$\frac{\partial z}{\partial x} \Rightarrow \frac{(x+y)(2x) - (x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \boxed{\frac{x^2-y^2+2xy}{(x+y)^2}}$$

$$\frac{\partial z}{\partial y} = \boxed{\frac{y^2-x^2+2xy}{(x+y)^2}}$$

$$\frac{\partial^2 z}{\partial x^2} \Rightarrow \frac{(x+y)^2\left[2x+2y\right] - \left[(x^2-y^2+2xy)(2)(x+y)\right]}{(x+y)^4}$$

$$\cancel{\frac{\partial^2 z}{\partial x^2} \Rightarrow \frac{2(x+y)^3 - 2\left[(x+y)(x^2-y^2+2xy)\right]}{(x+y)^4}}$$

$$\cancel{\frac{\partial^2 z}{\partial y^2} \Rightarrow \frac{2(x+y)^3 - 2\left[(x+y)(y^2-x^2+2xy)\right]}{(x+y)^4}}$$

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{x^2-y^2+2xy}{(x+y)^2} - \frac{y^2-x^2+2xy}{(x+y)^2}$$

$$\Rightarrow \frac{2x^2-2y^2}{(x+y)^2}$$

$$= \boxed{\frac{2(x-y)}{(x+y)}}.$$

$$4\left(1 - \left[\frac{x^2-y^2+2xy}{(x+y)^2} + \frac{y^2-x^2+2xy}{(x+y)^2}\right]\right)$$

$$\Rightarrow \boxed{4\left(1 - \frac{4xy}{(x+y)^2}\right)}$$

Q: $z = f(x+ay) + \phi(x-ay)$

Prove that :> $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

Q: $f(x,y) = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

*Q: If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, Prove that

Then Prove that

$$u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z)$$

Solution: $u_x^2 = \left(\frac{\partial u}{\partial x}\right)^2$

$$\frac{(a^2+u)(2x) - x^2\left(\frac{\partial u}{\partial x}\right)}{(a^2+u)^2} - \frac{y^2\left(\frac{\partial u}{\partial x}\right)}{(b^2+u)^2} - \frac{z^2\left(\frac{\partial u}{\partial x}\right)}{(c^2+u)^2} = 0.$$

$$\frac{\cancel{(a^2+u)}(2x)}{(a^2+u)^{\cancel{2}}} = \left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right)\frac{\partial u}{\partial x}$$

$$\cancel{\boxed{\left(\frac{2x}{a^2+u}\right) = \frac{\partial u}{\partial x}}}$$

$$\left(\frac{2y}{b^2+u}\right) = \frac{\partial u}{\partial y}.$$

$$\cancel{\left(\frac{2z}{c^2+u}\right) = \frac{\partial u}{\partial z}.}$$

$$4\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$\cancel{= 4\left[\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right]}$$

$$\frac{2x}{a^2+u} = \left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right)\frac{\partial u}{\partial x}.$$

$$\cancel{\frac{2x}{(a^2+u)}}\left[\frac{\frac{2x}{a^2+u}}{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}}\right] = \frac{\partial u}{\partial x}$$

Q. → $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to conduction of heat along a bar without radiation :

if $u = A e^{-gx} \sin(nt - gx)$

where, A, g, n are const.

Prove that $\boxed{g = \sqrt{\frac{n}{2\mu}}}$

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Homogeneous Function :→

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots a_ny^n$$
$$= x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots a_n\left(\frac{y}{x}\right)^n\right]$$

$$\boxed{f(x,y) = x^n\, F(y/x)}$$

Euler's Theo :- If $f(x,y)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \quad [n = \text{degree of eq}^n]$$

Proof :→ Let $f(x,y) = x^n F(y/x)$ be a homogeneous equation of degree n in x, y.

$$f(x,y) = x^n F(y/x)$$
$$\frac{\partial f}{\partial x} = nx^{n-1}F(y/x) + x^n F'(y/x)(-y/x^2)$$
$$\frac{\partial f}{\partial x} = nx^{n-1}F(y/x) - x^{n-2}y\, F'(y/x)$$

$$x\frac{\partial f}{\partial x} = nx^n F(y/x) - x^{n-1}y\, F'(y/x) \quad -(1)$$

Ily, $$y\frac{\partial f}{\partial y} = x^{n-1}y\, F'(y/x) \quad -(2)$$

(1) + (2)

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf}$$

Hence proved

\# $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$

differentiating wrt x.

$$\cancel{x\frac{\partial^2 f}{\partial x^2} + x\frac{\partial f}{\partial x}}$$

$$x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial x \partial y} = n\frac{\partial f}{\partial x}.$$

Multiplying wd x.

$$x^2\frac{\partial^2 f}{\partial x^2} + x\frac{\partial f}{\partial x} + xy\frac{\partial^2 f}{\partial x \partial y} = nx\frac{\partial f}{\partial x} \quad -①$$

llly

$$xy\frac{\partial^2 f}{\partial y \partial x} + y^2\frac{\partial^2 f}{\partial y^2} + y\frac{\partial f}{\partial y} = ny\frac{\partial f}{\partial y}. \quad -②$$

① + ②

$$\boxed{x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + nf = n^2 f.}$$

Q. $f(x,y) = ax^2 + 2bxy + by^2.$

Prove Euler's theorem.

$\cancel{\frac{\partial}{\partial x}\tan y = \sec^2 y \frac{\partial y}{\partial p}}$

Q. $\cancel{\tan u = \frac{x^2+y^2}{x+y}}$ $\quad u = \tan^{-1}\frac{x^2+y^2}{x+y}$.

Prove that $\frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$

Sol^n

$\tan u = \frac{x^2+y^2}{x+y}$.

degree = Numerator ki highest degree − den. ki

= 1.

Putting in Euler's form.

$f = \tan u$.

$$x\frac{\partial \tan u}{\partial x} + y\frac{\partial \tan u}{\partial y} = 1 \cdot \tan u.$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = \tan u$$

$$\frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u}.$$

Q. $u = \sin^{-1}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2}$.

$$\sin u = \left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2}$$

$\left(\frac{1}{3}-\frac{1}{2}\right)\frac{1}{2} \Rightarrow \frac{-1}{12}$

Sol. $\frac{x\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = \frac{-1}{12}\sin u$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{-1}{12}\sin u$$

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Q. (ii) Prove.
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\left(13+\tan^2 u\right)$$

~~x co~~
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{12}\tan u$$

diff w.r.t x, multiply w.r.t x
diff w.r.t y, multiply w.r.t y.
add.

Q:→ If u is a homogeneous function of x,y of degree n. Prove by using Euler's theo that.
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Q:→ If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$

Then, show that
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

Q: Show that the function
$$u(x,t) = k\sin(p\lambda t + \phi).\sin\lambda x$$
satisfy the eq^n.
$$\frac{\partial^2 u}{\partial t^2} = p^2\left(\frac{\partial^2 u}{\partial x^2}\right)$$

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TOTAL DERIVATIVE :-

$z = f(x, y)$

$x = \phi(t)$

$y = g(t)$

$\cancel{\frac{\partial f}{\partial x} =}$

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}}$$

Q. ~~Find~~ show that $\cancel{\frac{\partial^2 y}{}}$ $\frac{d^2y}{dx^2} = \frac{c - g^2 - f^2}{(y+f)^3}$.

$x^2 + y^2 + 2gx + 2fy + c = 0$

using partial derivative.

Q. If $u = e^{3x+2y}$

where $x = \cos t$, $y = t^2$

find $\frac{du}{dt}$?

using partial derivative.

Sol $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$.

$\Rightarrow 3e^{3x+2y}\cdot(-\sin t) + 2e^{3x+2y}(2t)$

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CHANGE OF VARIABLE :→

$z = f(u, v)$ $\qquad u = g(x, y)$

$\qquad\qquad\qquad v = h(x, y)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

Q: $z = \log(u^2 + v)$

$u = e^{x+y^2}$

$v = x^2 + y$

find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

Sol. $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{2u}{u^2+v}\, e^{x+y^2} + \frac{1}{u^2+v}(2x)$$

Q: $z = f(x, y)$

$x = e^{u} + e^{-v}$

$y = e^{-u} - e^{v}$.

Show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Sol: $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}$$

$$= \frac{\partial z}{\partial x}\cdot e^{u} + \frac{\partial z}{\partial x}(-e^{v})$$

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^{u} - e^{v})}$$

$$\boxed{\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial y}(e^{-u} + e^{v})}$$

Put into above expression.

Q Let $H = f(y-z, z-x, x-y)$

Show that $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$

Sol.

Let $H = f(u, v, w)$

where $u = y-z$

$v = z-x$

$w = x-y$

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial x}.$$

$$= 0 - 1\left(\frac{\partial H}{\partial v}\right) + 1\cdot\left(\frac{\partial H}{\partial w}\right)$$

$$\boxed{\frac{\partial H}{\partial x} = \frac{\partial H}{\partial w} - \frac{\partial H}{\partial v}} \quad -①$$

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial H}{\partial w}\cdot\frac{\partial w}{\partial y}.$$

$$\boxed{\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w}} \quad -②$$

$$\boxed{\frac{\partial H}{\partial z} = -\frac{\partial H}{\partial u} + \frac{\partial H}{\partial v}} \quad -③$$

① + ② + ③

$$\boxed{\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0}$$

Eule

Ho

Q: Let $u = f(r, s)$

$r = x + y \quad , \quad s = x - y$

Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

Sol: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(1)} \quad -(1)$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}} \quad -(2)$$

(1) + (2)

Hence proved.

Q:> $x = r\cos\theta \qquad y = r\sin\theta. \qquad u = f(x, y)$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)u$$

Sol:> $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial u}{\partial y}\left(\frac{y}{r}\right)$

$$\boxed{r\frac{\partial u}{\partial r} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}}$$

RHS $r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)u$

$= \cancel{x\frac{\partial u}{\partial x}} \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - u\right)$

$\Rightarrow x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - u\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - u\right)$

$\Rightarrow \cancel{x^2 \frac{\partial^2 u}{\partial x^2}} + \cancel{xy \frac{\partial^2 u}{\partial x \partial y}} - \cancel{x \frac{\partial u}{\partial x}}$

$+ \cancel{y^2 \frac{\partial^2 u}{\partial x^2}} + \cancel{xy \frac{\partial^2 u}{\partial x \partial y}} - \cancel{y \frac{\partial u}{\partial y}}$

$x \frac{\partial}{\partial x}\left(\frac{x \partial u}{\partial x} + y \frac{\partial u}{\partial y} - u \right) + y \frac{\partial}{\partial y}\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} - u \right)$

$\xrightarrow{\text{diff.}}$ $\xrightarrow{\text{diff.}}$

solve normally

$\Rightarrow$

Q.→ $x+y = 2e^{\theta}\cos\phi$ $\quad v = f(x,y)$

$x-y = 2ie^{\theta}\sin\phi$

Prove that

$\cancel{\frac{\partial^2 v}{\partial \phi^2}}$ $\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial \phi^2} = 4xy\left(\frac{\partial^2 v}{\partial x \partial y}\right)$

Sol:

$x = e^{\theta + i\phi}$

$y = e^{\theta - i\phi}$

$v = f(x,y)$

$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta}$

$\boxed{\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot x + y \frac{\partial v}{\partial y}}$

$\boxed{\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}(x) + (y)\frac{\partial}{\partial y}}$

$$\frac{\partial^2 v}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial \theta}\right)$$

$$\frac{\partial^2 v}{\partial \theta^2} = \left(\frac{x\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(\frac{x\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial^2 v}{\partial \theta^2} = \frac{x\partial}{\partial x}\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) + y\frac{\partial}{\partial y}\left(\frac{x\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

$$\boxed{\frac{\partial^2 v}{\partial \theta^2} = \cancel{x}\left(\frac{x\partial v}{\partial x} + \frac{x^2\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + \frac{y\partial v}{\partial y} + y^2\frac{\partial^2 v}{\partial y^2}\right.} \quad (1)$$

Q: Change into polar form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Q: Use the substitution $u = xy$, $v = \frac{x}{y}$ to change the independent variable

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 0 \qquad z = f(u,v)$$

Jacobian Series.

If u and v are 2 differentiable function of the variable x and y then the

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

determinant :- $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the jacobian of u and v wrt x and y denoted by $\frac{\partial(u,v)}{\partial(x,y)}$

(1) $r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1} y/x$.

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \frac{1}{r}.$$

(2)
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r.$$

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Q. $y_1 = \frac{x_2 x_3}{x_1}, \quad y_2 = \frac{x_3 x_1}{x_2}, \quad y_3 = \frac{x_1 x_2}{x_3}$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1 x_2 x_3} \begin{vmatrix} -\frac{x_2 x_3}{x_1} & x_3 & x_2 \\ x_3 & -\frac{x_1 x_3}{x_2} & x_1 \\ x_2 & x_1 & -\frac{x_1 x_2}{x_3} \end{vmatrix}$$

$$= \frac{1}{x_1 x_2 x_3} \begin{vmatrix} -y_1 & x_3 & x_2 \\ x_3 & -y_2 & x_1 \\ x_2 & x_1 & -y_3 \end{vmatrix}$$

$$= \frac{1}{x_1 x_2 x_3} \Big[-y_1(y_2 y_3 - x_1^2) - x_3(-x_3 y_3 - x_1 x_2) + x_2(x_1 x_3 + x_2 y_2) \Big] \quad 2$$

$$= \frac{1}{x_1 x_2 x_3} \Big[-\frac{x_2 x_3}{x_1}\left(\frac{\cancel{x_3} x_1}{\cancel{x_2}} \cdot \frac{x_1 \cancel{x_2}}{\cancel{x_3}} - x_1^2\right) + x_3(x_1 x_2 + x_1 x_2) + x_2(x_1 x_3 + x_1 x_3) \Big] \quad 2$$

Ans = 4

Q:→ $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial z} & \frac{\partial z}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & \cancel{\sin\theta}\ -r\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$= r$ Ans

Properties of Jacobian:→

① If u and v are functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1.$$

Q: If $x = u(1-v)$, $y = uv$.

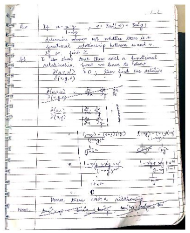
Verify that $\cancel{\partial(u,v)}$ $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1.$

② If u and v are functions of ~~x,y~~ r, s and r, s function x, y; then

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

* ③ If functions u, v, w of three independent variables x, y, z are not independent, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$



Q. If $u = \sin^{-1}x + \sin^{-1}y$

$v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

determine whether there exists a functional relationship between ~~x an~~ u and v. If yes, find so.

Sol $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$.

$$= \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} + \frac{y(-\cancel{2}x)}{\cancel{2}\sqrt{1-x^2}} & \sqrt{1-x^2} + \frac{x(-\cancel{2}y)}{\cancel{2}\sqrt{1-y^2}} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-x^2}} & \frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-y^2}} \end{vmatrix}$$

$= 0$.

Q:- $u^3 + v^3 = x + y$

$u^2 + v^2 = x^3 + y^3$.

Show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{(y^2 - x^2)}{uv(u-v)}$

If $f_1(x,y,u,v) = 0$

$f_2(x,y,u,v) = 0$.

order or number of variables

where u, v are implicit functions of x, y

then $\frac{\partial(u,v)}{\partial(x,y)} = (-1)^{2} \frac{\partial(f_1,f_2)}{\partial(x,y)} \Big/ \frac{\partial(f_1,f_2)}{\partial(u,v)}$

$$\left[\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}\right]$$

Solution.

$$f_1 = u^3+v^3-x-y$$
$$f_2 = u^2+v^2-x^3-y^3$$

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$$

$$= 3y^2 - 3x^2$$
$$= \boxed{3(y^2-x^2)}$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= 6u^2v - 6uv^2$$
$$= \boxed{6uv(u-v)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} \Rightarrow \frac{3(y^2-x^2)}{6uv(u-v)}$$
$$= \boxed{\frac{(y^2-x^2)}{2uv(u-v)}}$$

Hence proved

Q: $u = x+y+z \quad , \quad u^2 v = y+z \quad , \quad u^3 w = z$

$\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = ?$

Sol

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)}$$

$f_1 = x+y+z-u$

$f_2 = y+z-u^2 v$

$f_3 = z - u^3 w$,

$$= \frac{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} -1 & 0 & 0 \\ -2uv & -u^2 & 0 \\ -2uw & 0 & -u^3 \end{vmatrix}}$$

$$= \frac{1}{u^5} = u^{-5}$$

Taylor's Theorem for function of two variables

If $f(x,y)$ and all its partial derivatives are finite and continuous, then ~~f(x+h~~

~~f(x+h)~~

$$f(x+h,\ y+k)$$

$$= f(x,y) + \left\{ h\frac{\partial}{\partial x}(f(x,y)) + k\frac{\partial}{\partial y} f(x,y) \right\}$$

$$+ \frac{1}{2!}\left\{ h^2 \frac{\partial^2}{\partial x^2} f(x,y) + 2hk \frac{\partial^2}{\partial x \partial y} f(x,y) + k^2 \frac{\partial^2}{\partial y^2} f(x,y) \right\} + \cdots$$

$$- - -$$

$$= f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f(x,y)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f(x,y)$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^3 f(x,y)$$

$$+ \cdots$$

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#. $h = (x-a)$

$k = (y-b)$

$x = a$
$y = b$

Put in above series.

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x} f(a,b) + (y-b)\frac{\partial f}{\partial y} f(a,b) + \frac{1}{2!}\Big[(x-a)^2 \frac{\partial^2 f}{\partial x^2} f(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} f(a,b) + (y-b)^2 \frac{\partial^2 f}{\partial y^2} f(a,b)\Big]$$

Q:→ Expand $e^x \cos y$ in the neighbourhood of the point $(0, \pi/4)$ upto 2^{nd} degree term in x & y.

Sol → ~~$e^x \cos y$~~ → Apply above theo.

Q:→ Explain $\sin(xy)$ in powers of $(x-1), (y-\pi/2)$ upto 2^{nd} degree terms.

Sol: Pt → $(1, \pi/2)$.

Q: Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y-2)$.

_/_/_

Q:→ z is an implicit function of x and y defined by the eqⁿ $z^3 - 2xz + y = 0$ which takes the value $z = 1$ for $x = 1, y = 1$. Expand the function z in increasing powers of $(x-1)$ and $(y-1)$

Sol:→

$f(a,b) = 1$.

$z = 1$ at $x = 1,\ y = 1$

$$+ (x-1)\frac{\partial z}{\partial x}(1) + (y-1)\frac{\partial z}{\partial y}(1) + \dots$$

MAXIMA & MINIMA FOR FUNCTION OF 2 VARIABLES:

$z = f(x, y)$

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{array}\right\} \text{pt}(x_1, y_1)$$

$$\frac{\partial^2 f}{\partial x^2},\quad \frac{\partial^2 f}{\partial x \partial y},\quad \frac{\partial^2 f}{\partial y^2}$$

denotion $\quad r \quad s \quad t$

Now check:→ $\boxed{rt - s^2}$ → > 0 → $r < 0$ (Maxima), $r > 0$ (Minima)

→ < 0

→ $= 0$ ✗

_/_/__

Q:→ $f(x,y) \to x^3 + y^3 - 3axy$, Find pt. of Maxima, minima.

Sol:

$$\frac{\partial f}{\partial x} \to 3x^2 - 3ay \; ; \quad \frac{\partial f}{\partial y} :\to 3y^2 - 3ax$$

$$\frac{\partial^2 f}{\partial x^2} \to 6x \;, \quad \frac{\partial^2 f}{\partial x \partial y} \to -3a \;, \quad \frac{\partial^2 f}{\partial y^2} :\to 6y$$

$3x^2 - 3ay = 0$.

$x^2 = ay$ $\qquad y^2 = ax$.

$$\left(\frac{y^2}{a}\right)^2 = ay$$

$$\frac{y^4}{a^2} = ay$$

$$y^4 = a^3 y$$

$$y^4 - a^3 y = 0$$

$$y(y^3 - a^3) = 0$$

$$(y-a)(y^2 + a^2 + ay)$$

$\boxed{y = 0, a}$

$x = \frac{y^2}{a}$

$\boxed{(0,0) ; (a,a)}$

$$(6x)(6y) - (-3a)^2$$

$$= 36xy - 9a^2$$

$$\Rightarrow 36a^2 - 9a^2$$

$$= \boxed{27a^2 > 0}$$

$r = 6x$

at (a,a)

$r > 0$.

Minima

Q.7 $f \rightarrow \sin A \cdot \sin B \cdot \sin(A+B)$. Find Maxima, Minima

Ans $\frac{\partial f}{\partial A} \rightarrow \sin B \cdot [\cos A \sin(A+B) + \sin A \cos(A+B)]$

$= [\sin B \sin(2A+B)]$.

$\frac{\partial f}{\partial B} \rightarrow \sin A \sin(A+2B)$

$\sin B \sin(2A+B) = 0$. $\quad \sin A \sin(\cancel{2A+}\ A+2B) = 0$.

$\sin A = 0$

$\sin B = 0$. $\quad 2A+B = \pi$

$\cancel{2A+B = 0}$ $\quad 2A+4B\ 2\pi$

$\cancel{2A+4B = 0}$ $\quad -3B = -\pi$

$-3B = 0$. $\quad B = \pi/3$ $\quad (0,0)$

$B = 0$. $\quad \boxed{(0,0)}$ $\quad \boxed{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)}$

$\frac{\partial^2 f}{\partial A^2} \rightarrow 2 \sin B \cos(2A+B)$ | $\frac{\partial^2 f}{\partial A \partial B} :\rightarrow \sin B \cos(2A+B) + \cos B \sin(2A+B)$

$\boxed{\sin(2A+2B)}$

$\frac{\partial^2 f}{\partial B^2} \rightarrow 2 \sin A \cos(A+2B)$ $\quad$ $\cancel{2 \sin B \cos(2A+B)}$

$rt - s^2$.

$\rightarrow 2 \sin A \cos(A+2B) \cdot 2 \sin B \cos(2A+B) - (\sin(2A+2B))^2$.

$\rightarrow 4 \sin A \sin B \cos(A+2B) \cos(2A+B) - \sin^2 2(A+B)$

$\rightarrow 4\left(\frac{3}{4}\right) \cos^2\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) - \sin^2 2\left(\frac{2\pi}{3}\right)$

$r \rightarrow 2 \sin B \cos(2A+B)$

$2 \frac{\sqrt{3}}{2} (-1)$ $\quad \rightarrow 3 - \sin^2 \frac{4\pi}{3}\left(\pi + \frac{\pi}{3}\right) \frac{3}{4}$

$\rightarrow -\sqrt{3}$ $\quad \rightarrow \cancel{4}\ 9/4 > 0$

Maxima

_/_/_

Q.→ Find the max value of $x^m y^n z^p$ when $(x+y+z)=a$

Ans:→ Method I :- $z = a-x-y$

Put in above eq^n and solve:-

Method II : Lagrange's Method:→

$$f = x^m y^n z^p + \lambda(x+y+z-a)$$

$$\frac{\partial f}{\partial x} \Rightarrow m x^{m-1} y^n z^p + \lambda = 0 \quad -(1)$$

$$\frac{\partial f}{\partial y} = 0 \quad -(2), \quad \frac{\partial f}{\partial z} = 0 \quad -(3)$$

→ equate to zero and find the pt.

$$x(1) + y(2) + z(3).$$

$$\Rightarrow (m+n+p)\, x^m y^n z^p + \lambda a = 0,$$

$$\boxed{\lambda = -\frac{(m+n+p)\, x^m y^n z^p}{a}}$$

$$x = \frac{am}{m+n+p} \;;\; y = \frac{an}{m+n+p} \;;\; z = \frac{ap}{m+n+p}.$$

Now put in $x^m y^n z^p$. and get the ans!

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Q:1 Show that the maximum and minimum values of

$$r^2 = a^2x^2 + b^2y^2 + c^2z^2,$$

$$x^2+y^2+z^2 = 1, \quad lx + my + nz = 0.$$

are given by the eq^n

$$\frac{l^2}{a^2-r^2} + \frac{m^2}{b^2-r^2} + \frac{n^2}{c^2-r^2} = 0.$$

Ans:-

$$f = a^2x^2 + b^2y^2 + c^2z^2 + \lambda(x^2+y^2+z^2-1)$$
$$+ 2\mu\,(lx + my + nz)$$

are le liya 2.

$$\frac{\partial f}{\partial x} \Rightarrow 2a^2x + 2\lambda x + 2\mu l = 0 \quad -(1)$$

$$\frac{\partial f}{\partial y} \Rightarrow 2b^2y + 2\lambda y + 2\mu m = 0 \quad -(2)$$

$$\frac{\partial f}{\partial z} \Rightarrow 2c^2z + 2\lambda z + 2\mu n = 0 \quad -(3)$$

$$x(1) + y(2) + z(3).$$

$$2a^2x^2 + 2\lambda x^2 + 2\mu l x$$
$$+ 2b^2y^2 + 2\lambda y^2 + 2\mu m y$$
$$+ 2c^2z^2 + 2\lambda z^2 + 2\mu n z = 0$$

$$\cancel{2fa} \quad 2r^2 + 2\lambda(1) + 2\mu(0) = 0.$$

$$r^2 + \lambda = 0.$$

$$\lambda \Rightarrow -r^2$$

$$\left[\lambda = -a^2x^2 - b^2y^2 - c^2z^2\right]$$

$$\cancel{2a^2x^2 +} \quad \cancel{a^2x + \lambda x + \mu l = 0}.$$

on putting in above eqn :→

$$x = \frac{-\mu l}{a^2 - r^2}, \quad y = \frac{-\mu m}{b^2 - r^2}, \quad z = -\frac{\mu n}{c^2 - r^2}.$$

Now $lx + my + nz = 0$.

$$-\mu\left(\frac{l^2}{a^2 - r^2} + \frac{m^2}{b^2 - r^2} + \frac{n^2}{c^2 - r^2}\right) = 0.$$

since $\mu \neq 0$.

$$\boxed{\frac{l^2}{a^2 - r^2} + \frac{m^2}{b^2 - r^2} + \frac{n^2}{c^2 - r^2} = 0.}$$

Hence proved.

Q:→ The temperature T at any pt (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the surface of a unit sphere $x^2 + y^2 + z^2 = 1$

Sol $f = 400xyz^2 - \lambda(x^2 + y^2 + z^2 - 1)$

$\boxed{3(i) \to Q17, 19}$ imp*

Q9, Q10, Q11 → wrong que.*

~~Q16 → There's some Mistake (find yourself)~~

3(a, b, c) 3(e), 3(f), 3(g), 3(i)] → Partial ✓