INTRODUCTION

The most frequently used terms 'space', 'time', 'mass' and 'motion' very simple for an ordinary person, perturbed the minds of scientists for the past so many centuries. Answers did come, in various stages, sometimes contradicting to one another. However, scientists always gave up the lesser satisfactory explanations and accepted only experimentally verified ones. The Newtonian principle of relativity* talks about two philosophic al concepts, namely, absolute space, and absolute time. Newton realised that in practical problems one is always concerned with relative motion. Then question comes in mind is that 'relative to what?'. Hence, the idea of absoluteness was abandoned and concept of inertial frame of reference was accepted. In this frame of reference, Newton's laws of motion are valid. Galilean transformation equations were successfuly employed to transform coordinates of a particle from one inertial frame to another inertial frame. Distance between two points is found to be invariant under Galilean transformation. However, this principle of relativity, though held for mechanics, was no longer true for electrodynamics and optics. This has been verified by Michelson-Morley experiment.

Einstein started with a radically new idea that the motion through ether is a meaningless concept, while motion relative to material bodies alone has a physical significance. Einstein formulated his relativity principle, without reference to any absolute frame. According to him, the relativity principle was valid in all frames of reference, in uniform rectilinear motion, relative to one another. Further, the postulated that the relativity principle must be applicable to both the laws of mechanics, as well as laws of electro-magnetism (Maxwell's equations). Thirdly, he considered the velocity of light to be same in all inertial frames. It does not depend upon the velocity of the observer or that of its source. This revolutionary new theory, in 1905 was proposed by Einstein, known as "Special Theory of Relativity."

3.1 ALBERT EINSTEIN : A LUMINOUS STAR IN PHYSICS

The twentieth century has undoubtedly been the most significant for the advance of science, in general, and *Physics*, in particular. It has seen yet another luminous star, in the person of **Albert Einstein,** appear on the firmament among the all-time great that includes *Newton* and *Maxwell.* Few, indeed, if any, have so completely shaken the world of science, since *Newton*, by the sheer boldness and profoundness of their ideas. He literally created an upheavel by the publication, in quick succession, in the year 1905, two epoch-making papers, on the concept of the *photon* and on the *Electrodynamics of moving bodies* respectively, with yet another on *the mathematical analysis of Brownian motion* thrown in, in between. By far the most revolutionary of them is the one on the electrodynamics of moving bodies which demolishes at one stroke some of the most cherished and supposedly infallible laws and concepts and gives the breath-takingly new idea of the relativity of space and time.

Truly it may be said that just as the enunciation of Newton's laws of motion heralded emancipation from the age-old Aristotlean ideas of motion, so also did Einstein's theory of relativity make a proclamation, loud and clear, of emancipation from the crippling bondage to lumineferous ether and the confused notions of absolute space and time.

As to an introduction to the theory of relativity, we can do no better than to listen reverently to what *Einstein* himself says with his characteristic humility. This is quoted in the box 1.

* 'Absolute motion, which is the translation of a body from one absolute space to another absolute space, can never be detected for translatory motion. It can only be perceived in the form of the motion, relative to the other material bodies.'

Box - I

I am anxious to draw attention to the fact that this theory of relativity is not speculative in origin; it owes its invention entirely to the desire to make physical theory fit observed fact as well as possible. We have here no revolutionary act but the natural continuation of a line that can be traced through centuries. The abandonment of a certain concept connected with space, time and motion hitherto treated as fundamental must not be regarded as arbitrary but only conditioned by observed facts. The law of the constant velocity of light in empty space which has been confirmed by the development of electrodynamics and optics and the equal legitimacy of all inertial systems (special theory of relativity) which was proved in a particularly incisive manner by Michelson's famous experiment, between them made it necessary, to begin with, that the concept of time should be made relative, each inertial system being given its own special time .... It is, in general, one of the essential features of the theory of relativity that it is at pains to work out the relations between general concepts and empirical facts more precisely. The fundamental principle here is that the justification for a physical concept lies exclusively in its clear and unambiguous relation to facts that can be experienced.

We shall see in what follows the gradual development of the subject and how Einstein ultimately arrived at the conclusions he did.

3.2 SEARCH FOR A FUNDAMENTAL FRAME OF REFERENCE

It has already been pointed out in the last chapter (2.7) that it was on account of Newton's insistence as to the existence of a *fundamental* or *absolute frame of reference*, called *absolute space*, that the search for it was carried on. This search resulted in the discovery, or rather the invention, of that monstrosity of a medium, *luminiferous ether*, as the following brief account will show.

As we know, **Maxwell** clearly demonstrated, in the year 1864, the inter-relationship between electricity, magnetism and light when, from the known properties of electricity and magnetism, he formulated his celebrated *theory of electromagnetic radiation* and gave the well known *equations of electromagnetic field* which, bear his name and are identical with those that represent a wave phenomenon. He thus established the presence of electromagnetic waves in space, travelling with the speed of light. In other words, he *proved light to be an electromagnetic wave.*

Since waves known hitherto (like sound and water waves) all required a material medium (air and water respectively) for their propagation, it was supposed that there must also be a suitable medium to carry these newly found electromagnetic waves which travelled even through empty space between the stars and the earth. This intangible medium, though no one knew what it actually was, came to be referred to as *luminiferous ether*, pervading all space, empty or otherwise.

Further, to enable it to transmit light, a transverse wave motion, the ether had to be a *rigid solid* and in view of the tremendous velocity of light, it had to have a *large shear modulus* and yet all material objects, like the earth and the planets and stars etc, were to continue in their regular courses through it without encountering the slightest resistance. Utterly incredible as it seems to us today, no one seemed to object to its existence. On the other hand, it was felt that perhaps it was the absolute space or the fundamental frame of reference Newton was looking for and in which (or in a frame of reference fixed relative to it) his laws of motion would hold *perfectly*.

On the face of it, it was such an inviting—and exciting—proposition, and many a brilliant experiment were sought to be devised to establish the existence of this elusive medium, ether, and hence that of an absolute frame of reference. I was but natural that light waves should be used for the purpose, since, after all, ether was little else but just a vehicle for them.

Now, the orbit of the earth around the sun having a radius of 150 million km, its orbital velocity works out to 30 km/sec. So that, even assuming that the sun has no ether drift, the velocity of the earth through the ether must be this orbital velocity of 30 km/sec and the same must, therefore, also be the velocity of the resulting ether drift. Since the velocity of light is 3×10^5 km/sec, it will be easily seen that the velocity of the expected ether drift is just $30/3 \times 10^5 = 10^{-4}$ of the velocity of light. Obviously, therefore, a very sensitive apparatus was called for, if the experiment was to succeed.

Maxwell had actually despaired of any terrestrial methods being sensitive enough for the purpose and started looking for any astronomical evidence that may come his way in support. But **A.A. Michelson**, of the USA, when the learnt about it in the year 1879, accepted the challenge and, along with his colleague **E.W. Morley** (at the *Case Institute of Technology* in Cleveland), succeeded in performing, in the year 1887, one of the most famous experiments in the history of physics, (see next article) which gave a null result, striking at the very root of the ether hypothesis.

3.3 MICHELSON—MORLEY EXPERIMENT

The apparatus used by *Michelson* and *Morley* is known as *interferometer* since it depends upon the principle of interference of light and is shown diagrammatically in Fig. 3.1.

Here, S is a source of monochromatic light, a parallel beam from which falls upon a thin, parallel-sided glass plate P_1, thinly silvered on the back surface. The incident beam is thus split up into two parts at A at right angles to each other and of *equal intensity, viz.*, a reflected one, travelling upwards and a refracted one, travelling along the original direction. The former suffers reflection at mirror M_1 at T along its own path and gets refracted through P_1 on to the telescope T. The latter similarly gets reflected back as it falls normally on mirror M_2 at C and is then reflected downwards from the back surface of P_1 along the same path as the former to enter the telescope. The two mirrors (M_1 and M_2) are heavily silvered on the front face (to avoid multiple reflections) and are arranged at right angles to each other at the same distance D from plate P_1.

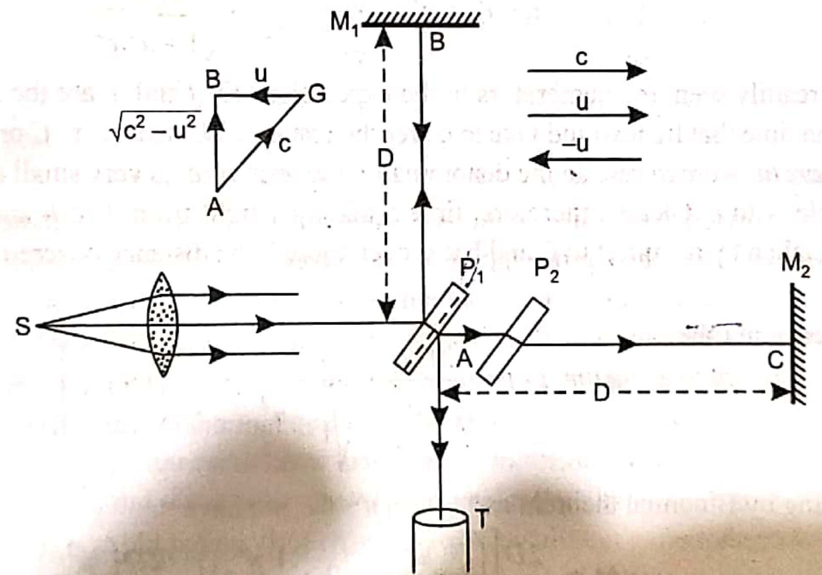

Fig. 3.1

As will be easily seen, the beam reflected upwards to M_1 traverses the thickness of plate P_1 *thrice* whereas the one refracted on to mirror M_2 does so only *once*. To make their paths through glass and air equal, therefore, a *compensating plate* P_2, identical with P_1, is arranged parallel to it, as shown.

If the beam of light be exactly parallel and if distances AB and AC from plate P_1 to the two mirrors respectively be the same (D), the two parts of the beam from M_1 and M_2 arrive at the telescope *in phase* and the field of view appears bright. If, however, their paths differ by an odd number of half wavelengths, they arrive at the telescope *out of phase*, *i.e.*, in opposing phases, and the field of view appears dark.

Since the beam of light incident on P_1 is almost always a slightly divergent one, some of the rays reaching the telescope from M_1 and M_2 are in phase and some out of phase, so that what we observe through the telescope is an *interference pattern of dark and bright fringes.*

Now, if u be the velocity of the apparatus (*i.e.*, of the earth), relative to the ether, *from left to right*, the drift velocity of the ether must be $-u$, *i.e.*, *in the opposite direction, from right to left*. So that, if c be the actual velocity of light its relative velocity with respect to the apparatus, along AC $= (c-u)$ and, therefore, time taken to cover the distance D from A to C, say, $t_1 = D/(c-u)$. And, on the return journey, from C to A, the relative velocity of light $= (c+u)$ and, therefore, time taken to cover the same distance D from C to A, say, $t_2 = D/(c+u)$.

$\therefore$ *Total time taken by light to travel from A to C and back*, say, t, is given by

$$t = t_1 + t_2 = \frac{D}{(c-u)} + \frac{D}{(c+u)} = \frac{2cD}{(c^2-u^2)} = \frac{2D/c}{1-u^2/c^2}.$$

And the beam proceeding upwards from A to B must obviously be moving along the direction AG (as shown in the inset), so that the resultant of its velocity c and that of the ether drift $(-u)$ is along AB. This resultant velocity will thus clearly be $\sqrt{c^2-u^2}$ and the time taken to cover the distance D from A to B thus equal to $D/\sqrt{c^2-u^2} = t_1'$, say. On the return journey too from B to A, the resultant velocity of light will be the same $\sqrt{c^2-u^2}$, so that, again, time taken to cover the distance D from B to $A = t_1' = D/\sqrt{c^2-u^2}$.

$\therefore$ *total time taken by light to travel from A to B and back* say,

$$t' = t_1' + t_1' = 2t_1' = \frac{2D}{\sqrt{c^2-u^2}} = \frac{2D/c}{\sqrt{1-u^2/c^2}}$$

As will be readily seen, the numerators in the expressions for t and t' are the *same, viz.*, $2D/c$, and represent the time that light would take to cover the distance $2D$, from A to C or from A to B and back, *if the apparatus were at rest*. In the denominator, the term u^2/c^2 is very small unless u happens to be comparable with c. Clearly, therefore, time t' taken by light from A to B and back is a little less than time t, taken by it from A to C and back, even though the distance covered is the same D in either case.

This difference in time

$$\Delta t = t - t' = \frac{2D/c}{1-\frac{u^2}{c^2}} - \frac{2D/c}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{2D}{c}\left(1-\frac{u^2}{c^2}\right)^{-1} - \frac{2D}{c}\left(1-\frac{u^2}{c^2}\right)^{\frac{-1}{2}}.$$

Or, expanding by Binomial theorem and taking $u \ll c$, we have

$$\Delta t = \frac{2D}{c}\left[\left(1+\frac{u^2}{c^2}\right)-\left(1+\frac{1}{2}\frac{u^2}{c^2}\right)\right] = \frac{2D}{c}\left(\frac{u^2}{2c^2}\right) = \frac{Du^2}{c^3}.$$

And, therefore, distance covered by light in time Δt

$$= \Delta t \times c = \frac{Du^2}{c_3} c = D\frac{u^2}{c^2},$$

indicating that the *optical path AC* is longer by Du^2/c^2 than the optical path AB or that this is the *path difference introduced* between the two parts of the incident beam (reflected from M_1 and M_2 respectively) *due to motion of the apparatus.*

The apparatus is now turned through 90° so that AB comes into the line of motion and AC, perpendicular to it, so that the optical path AB is now longer than the optical path AC by the same

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$$n\lambda = 2\,Du^2/c^2,$$

whence,
$$n = 2\,Du^2/c^2\lambda.$$

Michelson and Morley had effectively increased distance D to nearly 11 metres (by repeated reflections) and, in order that the apparatus may be rotated without producing any strain, it was mounted on a stone slab, floated in mercury and kept rotating slowly at about 10 rotations per hour.

Thus, with *orbital velocity u of the earth, and hence that of the apparatus,* equal to 18.5 miles or 30 km/sec or 3×10^6 cm/sec, the *velocity of light,* $c = 3 \times 10^{10}$ cm/sec and with the wavelength of light used, $\lambda = 6 \times 10^{-5}$ cm, the expected value of n, in accordance with the expression above, comes to

$$\frac{2 \times 1100 \times 9 \times 10^{12}}{9 \times 10^{20} \times 6 \times 10^{-5}} = 0.37 \approx 0.4,$$

which could be accurately measured, in view of the high sensitivity of the apparatus, capable of measuring a shift as small as one-hundredth of a fringe.

Negative Result

The actual shift of the interference pattern observed, however, was much too small, almost negligible, *indicating little or no relative velocity between the earth and the ether.*

The experiment has since been repeated below the earth's surface as well as at high altitudes at different times of the year and, lately, with highly monochromatic light from a *laser*, but in all cases, the result has been a negative one. Other experiments to discover the ether wind have similarly failed.

It is always hard, however, to discard an idea to which one has grown accustomed. Strenuous efforts were therefore made to explain away the negative result of the experiment. Thus, Michelson himself hazarded the suggestion that the earth dragged along with it the ether in its immediate neighbourhood and there was thus no relative motion between the two. This was, however, easily demolished by *Lodge* who, in the year 1892, measured the velocity of light near rapidly rotating bodies and came to the conclusion that not more than half a per cent of the velocity of light could thus be communicated to the ether.

The same year (1892), *Fitzgerald* and *Lorentz* put forth the suggestion that probably there was interaction between the ether and a material body moving relatively to it, and that, as a result, the body gets shortened in all its dimensions *parallel to the relative velocity,* such that if L_o be the length of the body when at rest, and if it be moving with a speed u parallel to its length, the new length L acquired by it is given by the relation, $L = L_o\sqrt{1 - u^2/c^2}$.

It will be easily seen that if we make use of this suggestion in the experiment discussed above, we shall find that whereas distance AB will remain unchanged, distance AC will get shortened to $D\sqrt{1-u^2/c^2}$. So that, substituting $D\sqrt{1-u^2/c^2}$ for D in the expression for t above, we have

$$t = \frac{(2D/c)\left(\sqrt{1-u^2/c^2}\right)}{1-u^2/c^2} = \frac{2D/c}{\sqrt{1-u^2/c^2}},$$ which is the same as t'.

This contraction hypothesis, therefore, easily explains why the *Michelson-Morley experiment* gave a negative result. It is, however, open to the objection that it has been put forward for the specific purpose of explaining away the difficulty and that it does not follow from the theory.

It was 18 years later, in 1905, that the true and simple explanation for the negative result of Michelson and Morley's experiment was furnished by **Einstein,** namely, that *the velocity of light in space is a universal constant.* In fact, he made this constancy of the velocity of light one of the basic postulates of his special theory of relativity, as we shall presently see.

Example 1. In an Michelson and Morley experiment, the length of the arm of the interferometer was 11 meters, the wavelength of light 5.5. × 10^{-7} meters and the earth's velocity 30 km/sec, calculate the amount of fringe-shift?

Solution. The required fringe shift, *i.e.,* path difference introduced is given by

$$\Delta x = \frac{2Du^2}{c^2}$$

and the corresponding phase difference introduced will be

$$\delta = \frac{2Du^2}{c^2\lambda}$$

Here,

$$l = 1100 \text{ cm}, \lambda = 5.5 \times 10^{-7} \text{ m} = 5.5 \times 10^{-5} \text{ cm}$$

$$u = 30 \times 10^5 \text{ cm/sec}$$

$$c = 3 \times 10^{10} \text{ cm/sec}$$

$$\therefore \quad \text{required fringe shift} = \frac{2 \times 1100 \times (3 \times 10^6)^2}{(3 \times 10^{10})^2 \times 5.5 \times 10^{-5}} = \frac{2}{5} = 0.4$$

Example 2. In the Michelson-Morley experiment, the wavelength of the monochromatic light used is 5000 Angstrom units. What will be the expected fringe-shift on the basis of stationary ether hypothesis of the effective length of each path be 5 meters? (Given velocity of the earth = 3 × 10^4 m/sec and c = 3 × 10^8 m/sec)

Solution. We know that if the fringe shift be n fringes, we have $n = 2\,Du^2/c^2\,\lambda$. Here, $D = 5$ m, $u = 3 \times 10^4$ m/sec, $c = 3 \times 10^8$ m/sec and $\lambda = 5000 \times 10^{-8}$ cm $= 5 \times 10^{-7}$ m

So that,

$$n = \frac{2 \times 5(3 \times 10^4)^2}{(3 \times 10^8)^2 \times 5 \times 10^{-7}} = 0.2 \quad \text{or} \quad 1/5.$$

Thus, *the expected fringe shift is one-fifth of a fringe-width.*

Example 3. A stream of width l is flowing with a velocity v and a person wishes to row straight across it. In what direction should he strike out and how long would it take him to cross the stream? How long would it take him to cover a distance l upstream and back?

Solution. In order to be able to row straight across the stream, the person must strike out at an angle θ upstream with the direction of flow, such that the resultant R of his own velocity V and that of the stream v is perpendicular to the latter.

Then, clearly, $V\cos\theta = v$, whence, $\theta = \cos^{-1} v/V$.

And the resultant velocity $R = V\sin\theta$, perpendicular to the direction of stream-flow.

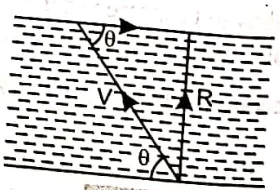


Fig. 3.2

∴ *time taken to row straight across the stream*

$$= \frac{l}{V\sin\theta} = \frac{l}{V\sqrt{1-\cos^2\theta}} = \frac{l}{V\sqrt{1-v^2/V^2}} = \frac{l}{\sqrt{V^2 - v^2}}$$

And, clearly, resultant velocity of the person upstream = $(V + v)$ and his resultant velocity downstream = $(V + v)$.

Hence time taken to cover a distance l upstream

$$= l/(V - v)$$

and *time taken to cover the same distance back, i.e., downstream* $= l/(V + v)$

$\therefore$ *time taken to cover a distance l upstream and back*

$$= \frac{l}{V - v} + \frac{l}{V + v} = \frac{2lv}{V^2 - v^2}.$$

3.4 EINSTEIN'S CONCEPT OF RELATIVITY

When the experiments in search of the ether drift failed, it began to be increasingly realised that there was no such thing as an *absolute* or *privileged* frame of reference and that the basic laws of physics took the same form in all inertial frames of reference. The implications of this Galilian invariance principle were emphasised by the French mathematical physicist *Henri Poincaree* when he stated that '... the laws of physical phenomena [are] the same, whether for an observer fixed or for an observer carried along in a uniform movement of translation, so that we have not and could not have any means of discerning whether or not we are carried along in such a motion'.

This simply means that if we are drifting with uniform speed in a spaceship, with all the windows closed, we shall not be able to say, with the help of any experiments we might choose to perform, whether we are at rest or in motion. If, however, we look out of a window, we shall be able to say merely that we are in motion with respect to the fixed stars but not whether we or the stars are actually in motion.

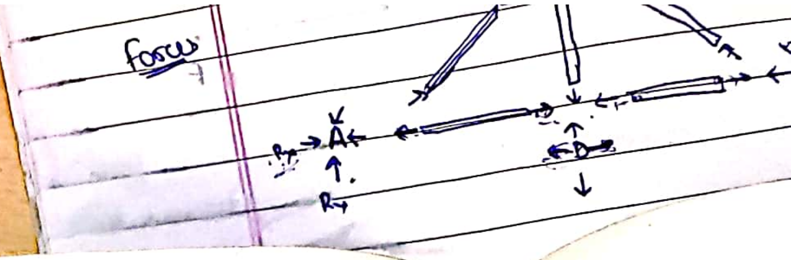
How near, indeed, had Poincare thus come to expounding the theory of relativity and yet how far he actually was form it. For, instead of grasping the implications of the failure of all ether-drift experiments, and building up a new theory on its basis, discarding older notions or space and time, he concerned himself with trying to somehow save the old classical theory by suitable adjustments and modifications in it.

Box - II

The real import of the negative results of the ether-drift experiments was clearly seen and understood by *Einstein.* For, discussing the reciprocal electrodynamic action of a magnet and a conductor where 'the experimentally observable phenomenon depends only on the relative motion of the conductor and the magnet ....',

Einstein says:

'Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relative to the 'light medium', suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that ... the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good*. We will raise this conjecture (the purport of which will hereafter be called the *Principle of Relativity*) to the status of a postulate, and also introduce another postulate which is only apparently irreconcible with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body ... The introduction of a "luminiferous ether" will prove to be superfluous in as much as the view here to be developed will not require an 'absolutely stationary space' provided with special properties, ...'



and revolutionary ideas into a cogent theory which he announced to an unsuspecting world in the year 1905 as his *Special theory of relativity*. And, ten years later, in 1915, followed the second and the more complex and difficult part of it in the form of the *General theory of relativity*. The former deals with problems associated with unaccelerated frames of reference, *i.e.*, those which move with uniform relative velocity with respect to one another and the latter with those associated with accelerated ones.

We shall concern ourselves here mainly with only the **Special theory of relativity** which is not only comparatively simpler but has also produced the most profound effect on the entire field of physics.

3.5 SPECIAL THEORY OF RELATIVITY

The two **basic postulates** on which the theory rests may be stated in a precise form thus:

1. The laws of physics all take the same identical form for all frames of reference in uniform relative motion, *i.e.*, for all the inertial frames of reference.

This, it will be readily seen, is a direct consequence of the absence of an absolute or fixed frame of reference. For, if the laws of physics were to take on different forms in different frames of reference, it would be easily inferred from these differences as to which of them are at rest in space and which in motion. But, as we have seen, such distinction between the state of rest and of uniform motion is precluded by the absence of a universal frame of reference. The first postulate is thus merely a generalised statement of this observed fact.

2. The velocity of light in free space is the same (c) relative to any inertial frame of reference, *i.e.*, it is invariant to transformation from one inertial frame to another and has thus the same value (3.0×10^{10} cm or 3×10^{8} m/sec) for all observers irrespective of their state of motion.

This postulate is clearly a statement of the result of *Michelson—Morley* experiment.

The two implications of the above postulates are immediately obvious:

(*i*) Velocity being *not* invariant to Galilian transformation, it follows that if c be the velocity of light in frame S, that in frame S', moving relative to S with velocity v, must be $c' = c - v$. In accordance with the second postulate of the special theory of relativity, however, we have $c' = c$ since c must always have the same value irrespective of the state of motion of the frames of reference.

The new postulate is thus clearly at variance with Galilian transformation.

(*ii*) Again, suppose a flash of light is emitted from the origin when points O and O' of the two frames of reference S and S' (Fig. 3.3) just cross each other. Suppose further that it is possible for the two observers in the two frames of reference to see the light or rather the *wave front* (a surface of equal phase) spreading out into space. Then, in view of the constancy of the value of c, irrespective of the motion of the two frames of reference, each of the two observers at O and O' respectively will, even after drifting apart, claim to be at the centre of the spreading spherical wave front of light, as shown in Fig. 3.4, thinking that the other has moved away from the centre of the sphere.

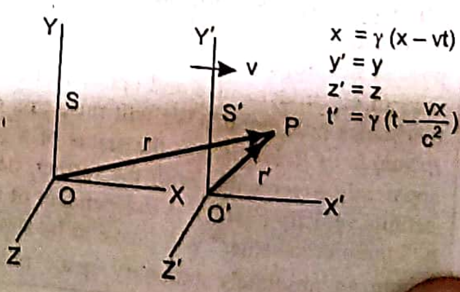


Fig. 3.3

As can be readily seen, this will not be the case if, instead of being in free space, our obser[vers] were to be in boats in water and if a stone were dropped (instead of a flash of light emitted) wh[en] they just crossed each other. The ripple pattern formed will appear to be different to each obser[ver]

d they will be able to say from the very shape of it whether or not they
: in motion or which one of them is in motion and which, stationary.

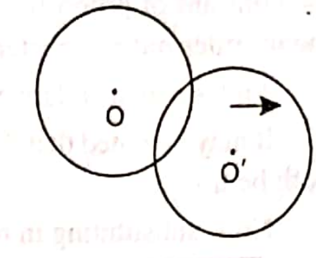


Fig. 3.4

This difference between the two cases, it may be carefully noted, is ie to the fact that *whereas water functions as a frame of reference by elf, space does not*, and, again, *whereas the speed of the waves in water ries with the motion of the observer, the speed of the waves in space e., of light or electromagnetic waves) is quite independent of it.*

6 LORENTZ COORDINATE TRANSFORMATION (OR SIMPLY, LORENTZ TRANSFORMATION)

As just mentioned in §3.5 above, the Galilian transformation is quite at variance with the elocity of light in free space being independent of the motion of the observer or the source of light. /e, therefore, look for a transformation which is consistent with this new concept.

As we have seen under the article referred to, if a flash of light be emitted as the two observers ist cross each other in the two frames of reference (Fig. 3.3), each thinks himself to be at the centre f a uniformly expanding spherical wave front (Fig. 3.4). Clearly, *equation of the wave front, as seen y the observer at O in reference frame S*, is

$$x^2 + y^2 + z^2 = c^2 t^2, \quad ...(i)$$

And *equation of the wave front, as seen by the observer at O' in reference frame S'*, is

$$x'^2 + y'^2 + z'^2 = c'^2 t'^2, \quad ...(ii)$$

where c, the velocity of light is the same in either frame of reference.

In order, therefore, that both the observers may seem to be at the centre of the same expanding vave front, we should have relation (i) = relation (ii).

Now, *Galilian transformation* gives $x' = x - vt$, $y' = y$, $z' = z$ and $t' = t$.

So that, substituting for x', y', z' and t' in relation (ii) above, we obtain

$$x^2 - 2\,xyt + v^2 t^2 + y^2 + z^2 = c^2 t^2, \quad ...(iii)$$

which is surely not the same as relation (i), showing once again, that *the Galilian transformation fails if the constancy of the value of c be assumed.*

This means clearly that in the new transformation we are looking for, the relation $x' = x - vt$ no longer holds good. Since however, the relation is quite in accord with classical mechanics, the new transformation must be such that the x-coordinate, in accordance with it, also reduces itself to this very value, when the relative velocity v is small compared with c.

The simplest relation for the x-coordinate, in accordance with the new transformation, can be

$$x' = \gamma(x - vt) \quad ...(iv)$$

where γ is independent of the coordinates x or t, but may vary with v.

This relation has the merit of being a *linear* one in x and t, providing for a uniformly expanding wave front and admitting of one and only one interpretation in the system S' of an observation made from system S. A quadratic equation, on the other hand, will obviously give two; and a higher order equation, even more than two interpretations, which is simply inadmissible.

In plain and simple language, the equation $x' = \gamma(x - vt)$ will automatically give only one value of x' for a given value of x, without our having to impose any further conditions.

Considering the inverse transformation, connecting measurements in frame S with those made in frame S', we shall have only the sign of v reversed; so that,

$$x = \gamma(x' + vt'), \quad ...(v)$$

the constant of proportionality γ remaining the same in either case, since the first postulate of the theory rules out any preferred frame of reference.

And, since the relative motion of S and S' is confined to the x-direction, we have $y' = y$ and $z' = z$.

It may be noted that t' *can no longer be taken to be equal to* t, or else relations (*iv*) and (*v*) cannot both be true.

Now, substituting in relation (*v*) the value of x', as given by relation (*iv*), we have

$$x = \gamma[\gamma(x - vt) + vt'] = \gamma^2(x - vt) + \gamma\, vt'.$$

or,

$$\gamma\, vt' = x - \gamma^2(x - vt) = x - \gamma^2 x + \gamma^2 vt = \gamma^2 vt + x\,(1 - \gamma^2),$$

whence,

$$t' = \gamma t + x\left(\frac{1-\gamma^2}{\gamma v}\right) \quad ...(vi)$$

In order to determine the value of γ, we note that the reference frame S' moves with relative velocity v in the $+x$-direction with respect to reference frame S and that the flash of light is emitted when the two observers at O and O' are opposite each other, *i.e.*, at the time the flash is emitted, $x = x' = 0$ and also $t = t' = 0$. Since the velocity of light is the same c in the two frames of reference, we have

x-coordinate of the light pulse in frame S, given by $x = ct$. ...(*vii*)

and x'-coordinate of the light pulse in frame S', given by $x' = ct'$ ...(*viii*)

Substituting in relation (*viii*) the values of x' and t' from relations (*iv*) and (*vi*) respectively, we have

$$\gamma(x - vt) = c\gamma t + cx\left(\frac{1-\gamma^2}{\gamma v}\right),$$

whence, solving for x, we obtain $x = ct\left[\dfrac{1 + v/c}{1 - (c/v)(1/\gamma^2 - 1)}\right]$.

Since $x = ct$, as shown in relation (*vii*) above, we have

$$\frac{1 + v/c}{1 - (c/v)(1/\gamma^2 - 1)} = 1. \quad \text{or,} \quad 1 + \frac{v}{c} = 1 - \left(\frac{c}{v}\right)\left(\frac{1}{\gamma^2} - 1\right),$$

whence,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

The factor v/c is quite often represented by β; so that,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

Substituting this value of γ in relations (*iv*) and (*vi*) above, we have

$$x' = \frac{x - vt}{1 - v^2/c^2} = \gamma(x - vt)$$

and

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

And, since, as already mentioned, S' moves along the $+x$ direction relative to S,

$$y' = y$$

and

$$z' = z$$

These values of x', y', z' and t', when substituted in relation (*ii*) above, give relati[...] as it should be if the two observers are to seem to be at the centre of the same spher[...]

This *relativistic transformation* is referred to a **Lorentz transformation,** after *H.A. Lorentz,* a Dutch physicist, because it was he who first showed that this transformation was the only one under which the laws of electricity and magnetism have the same form in all reference frames in relative motion (*i.e.*, in all *inertial* frames), though it was *Einstein,* of course, who showed its wider significance as being the only proper transformation for *all* types of measurements, electromagnetic or otherwise.

Equations (*ix*), (*x*), (*xi*) and (*xii*) are, therefore, referred to as **Lorentz transformation equations, or Lorentz coordinate transformation equation.**

The Galilian and Lorentz (or Relativistic) transformations for an observer in reference frame S', moving with uniform velocity v relative to frame S in the *+x-direction,* are arranged side by side below, so that the differences between the two may become apparent at a glance:

Galilian transformations	**Lorentz (or Relativistic) transformations**
$x' = x - vt$	$x' = \dfrac{x - vt}{\sqrt{1 - v^2/c^2}} = \gamma(x - vt)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$t' = t$	$t' = \dfrac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} = \gamma\left(t - \dfrac{vx}{c^2}\right)$

N.B. If v/c be small, as it most often would be, we may take only the first two terms of the Binomial expansion of $\dfrac{1}{\sqrt{1-v^2/c^2}}$ or $\left(1-\dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}$, namely, $\left(1+\dfrac{1}{2}\dfrac{v^2}{c^2}\right)$ as the value of γ. This approximation is valid enough even if v/c has a value up to about 0.5.

As will be readily seen, when $v \ll c$, *i.e.*, when $v/c \to 0$, we have $\gamma = 1/\sqrt{1-v^2/c^2} \approx 1$ and Lorentz (or Relativistic) transformations reduce to Galilian ones. So that, Galilian or Newtonian physics is just a particular case of Relativistic physics. In fact, v for the earth (18.5 mi/sec) being only 1/10,000 th of the value of c, is negligibly small compared with c and the Galilian or Newtonian physics is thus perfectly valid for almost all our practical purposes, the relativistic effects manifesting themselves only at very high values of v, when they are comparable with c, as in the case of electrons, protons, mesons etc.

We have, in our discussion above, assumed frame S' to be moving with velocity v in the positive direction of x relative to frame S. As seen from frame S', therefore, frame S would appear to be moving in the negative x direction *i.e.*, with velocity $-v$. Consequently, the **Inverse Lorentz transformation,** which converts measurements made in frame S' into those in frame S, may be obtained from Lorentz transformations by simply changing the sign of v from positive to negative. We shall then have

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} = \gamma(x' + vt'), y = y', z = z'$$

and

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} = \gamma\left(t' + \frac{vx'}{c^2}\right).$$

3.7 RESULTS FLOWING FROM LORENTZ TRANSFORMATION

Let us now examine some of the relativistic effects, *i.e.*, effects which are the result of relativistic or Lorentz transformation. Quite a few of them, as we shall see, will appear to be surprisingly new and strange because, in view of the small relative motion between the frames of reference of which we have experience in our daily life, we do not ordinarily come across any perceptible relativistic phenomena.

Let us consider a few important examples:

3.7.1 Length Contraction

The length of a rod, as measured in a frame of reference, *at rest with respect to the observer*, is called its **proper-length** and would obviously be the *same* in all stationary frames of reference. Reference frames that it will be different, depending upon their relative velocities with respect to the observer. Let us see how.

Consider a rod laid along the axis of x in a frame of reference S, at rest with respect to the observer and let x_1 and x_2 be the coordinates of its tow ends which may be noted at leisure, one after the other, the rod being at rest. Then, the *proper length* of the rod, say,

$$L_o = (x_2 - x_1).$$

If the x-coordinates of the ends of the rod in the reference frame S', moving with a uniform velocity v with respect to frame S be x_1' and x_2', *as noted simultaneously at the same instant* t', we have length of the rod in the moving reference frame S' given by $L = (x_2' - x_1')$.

To corelate L_o and L, we note that inverse Lorentz transformation gives

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}}.$$

So that,

$$L_o = x_2 - x_1 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}} - \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} = \frac{L}{\sqrt{1 - v^2/c^2}}$$

whence,

$$L = L_o\sqrt{1 - v^2/c^2},$$

which is clearly *shorter than its proper length* L_o. And, as will be easily seen, the higher the value of v, *i.e.*, the faster the motion of the rod (or of the reference frame carrying it), with respect to the observer, the shorter its length compared with its proper length. So that, *if* $v = c$, *i.e.*, *if the rod be moving with the velocity of light* (*c*), *its length would be zero*.

It follows at once, therefore, that the length of the rod or any other object when $v = 0$, or *when the rod or the object is at rest with respect to the observer, i.e., its proper length, is the largest.*

There will, of course, be no shortening of the length of the rod if it lies along the axis of y or z, perpendicular to the x-direction or the direction of motion of the frame of reference carrying it, though there will, naturally be some shortening if the length of the rod be inclined to the direction of its velocity for, then, it will have a component in the latter direction.

This shortening or contraction in the length of an object along its direction of motion is known as the **Lorentz—Fitzgerald contraction,** since both *Lorentz* and the Irish physicist *G.F. Fitzgerald*, during the 1890's, independently suggested that the null result of Michelson–Morley experiment could be accounted for, if the length of a material object contracted by the fraction $\sqrt{1 - v^2/c^2}$ in its direction of motion with speed v, through ether. Of course, as we know *now*, the contraction is certainly this fraction but for entirely different reasons.

It may be pointed out that there is *reciprocity of the contraction effect, i.e.*, the contraction rks both ways — a length along the x-axis in frame S being contracted for the observer in S' and ength along the x'-axis in frame S' being contracted for the observer in S. For, if the rod be at st along the axis of x' with respect to the moving frame S', its proper length, as measured by the server in S' (at rest with respect to it) will be $L_o = (x'_2 - x'_1)$, where x'_1 and x'_2 are the coordinates of e end-points of the rod along this axis. And if the x-coordinates of the two end-points of the rod frame S, as noted *instantaneously, i.e.*, at the same instant t, by the observer in S', be x_1 and x_2, its ngth, as measured in frame S, is $L = (x_2 - x_1)$.

Since, in accordance with *Lorentz tranformation*,

$$x'_1 = \frac{x_1 - vt}{\sqrt{1-v^2/c^2}} \quad \text{and} \quad x'_2 = \frac{x_2 - vt}{\sqrt{1-v^2/c^2}}.$$

We have
$$L_o = x'_2 - x'_1 = \frac{x_2 - vt}{\sqrt{1-v^2/c^2}} - \frac{x_1 - vt}{\sqrt{1-v^2/c^2}} = \frac{x_2 - x_1}{\sqrt{1-v^2/c^2}}$$
$$= \frac{L}{\sqrt{1-v^2/c^2}}$$

hence,
$$L = L_o\sqrt{1-v^2/c^2}, \text{ the same as before.}$$

hus, *with relative motion between an object and observer, the length of the object, as measured by he latter, always comes out to be shorter than its proper length.*

It is for this reason that if we have two frames of reference in relative motion (one stationary nd the other in motion) along the x-direction, say, a straight line parallel to this direction in one ppears shorter to an observer in the other, — not, so, however, a straight line alone the y or the -direction, perpendicular to the x-direction. Similarly, a square and a circle in one appear to the bserver in the other to be a rectangle and an ellipse respectively (Fig. 3.5), the sides of the square nd the radius of the circle (or their components) in the direction of motion getting shortened.

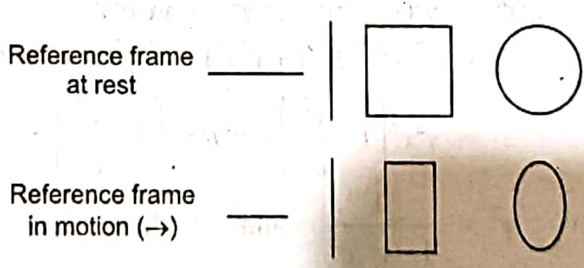


Fig. 3.5

As will be easily seen, however, for small values of v for which $v/c \to 0$, the factor $\sqrt{1-v^2/c^2} = 1$, so that $L = L_o$, a result in accordance with classical mechanics where length is treated as an absolute quantity, unaffected by rest or motion.

This is exactly the reason why at speeds with which we are ordinarily concerned (and which are an infinitesimal fraction of c), the *Lorentz-Fitzgerald* contraction is nil or negligible.

For an interesting example (and evidence) of the contraction. Phenomenon, see worked example at the end of the chapter.

Box - III

Note. It is interesting to point out that even though a fast moving object must undergo Lorentz-Fitzgerald contraction, its length, if observed visually (*i.e.*, directly by the eye) or photographically, would *not* show any such contraction. Strangely enough, this phenomenon was discovered and studied only in the year 1959, more than half a century after the publication of the special theory of relativity. It may, however, be explained thus: The light proceeding from the farthest part of the object leaves the earliest and that from its nearest part, the latest, with that from other parts at times in between these two extremes, to reach the eye or the camera *simultaneously* so as to form a composite image of the object on the retina of the eye or the film of the camera. As can well be imagined, this should actually make the moving object look larger in size in its direction of motion. It so happens, however, that this apparent increase in its length is just offset by the Lorentz-Fitzgerald contraction that it must undergo on account of its relative motion. As a result, it appears to be neither lengthened nor shortened. Thus, if the moving object be a three-dimensional one, like a cube, no change in its shape will come about except a rotation of its orientation, which may approach 180° as $v \to c$. And, in the case of a sphere, there will be no distortion whatsoever.

Example 4. Show that Lorentz transformations are superior to Galilean transformations.

(Kanpur 2002)

Solution. By Lorentz transformation equations are

$$x' = \gamma(x - vt),\ y' = y,\ z' = z,\ t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where,
$$\gamma = \frac{1}{\sqrt{(1 - v^2/c^2)}}$$

If v is very small, than $(v/c) \to 0$ and so $(\gamma \to 1)$.

$$x' = x - vt,\ y' = y,\ z' = z,\ t' = t\ \{t - 0(x)\}$$

This implies, $x' = x - vt,\ y' = y,\ z' = z,\ t' = t$

These are Galilean transformations. These facts prove that Lorentz transformations are superior to Galilean transformations.

Example 5. Show by means of Lorentz transformation equations that
$$x^{2\prime} - c^2 t'^2 = x^2 - c^2 t^2.$$

Solution. As we know, in accordance with Lorentz coordinate transformation,

$$x' = \gamma(x - vt) \text{ and } \quad t' = \gamma(t - vx/c^2)$$

$\therefore$
$$\begin{aligned} x'^2 - c^2 t'^2 &= \gamma^2 (x - vt)^2 - c^2 \gamma^2 (t - vx/c^2)^2 \\ &= \gamma^2 (x^2 - 2xvt + v^2 t^2) - c^2 \gamma^2 (t^2 - 2vxt/c^2 + v^2 x^2/c^4) \\ &= \gamma^2 x^2 \left(1 - \frac{v^2}{c^2}\right) - \gamma^2 t^2 c^2 \left(1 - \frac{v^2}{c^2}\right) = \gamma^2 (x^2 - c^2 t^2)(1 - v^2/c^2) \end{aligned}$$

Now,
$$\gamma^2 = \frac{1}{1 - v^2/c^2} \quad \text{and} \quad \therefore \gamma^2 (1 - v^2/c^2) = 1.$$

So that,
$$x'^2 - c^2 t'^2 = x^2 - c^2 t^2.$$

Example 6. Show that a four dimensional volume element $dx\ dy\ dz\ dt$ is invariant to Lorentz transformation.

Solution. This follows straightaway from Lorentz coordinate transformation. For, if the volume element be in a stationary frame S and if a frame S' be moving with a constant velocity relative to S along the x-axis, we have in frame S',

$$dx' = \gamma dx,\ dy' = dy,\ dz' = dz \text{ and } dt' = dt/\gamma$$

$\therefore$
$$dx'\ dy'\ dz'\ dt' = \gamma dx\ dy\ dz\ dt/\gamma = dx\ dy\ dz\ dt.$$

In other words, *the four dimensional volume elements dx dy dz dt is invariant to Lorentz transformation.*

Example 7. The space-time coordinates of two events in frame S are 0, 0, 0, 0 and 5c, 0, 0, respectively. Obtain the space-time interval between them, what should be the velocity of a frame S' relative to S, in which (*i*) the two events may appear to occur simultaneously, (*ii*) the second may appear to occur 2 seconds (*a*) earlier, (*b*) later, than the first even?

Solution. Here, clearly, $x_1 = 0, y_1 = 0, z_1 = 0$ and $t_1 = 0$ and $x_2 = 5c, y_2 = 0, z_2 = 0$ and $t_2 = 4$.

So that, $(x_2 - x_1) = \Delta x = 5c$, $(y_2 - y_1) = \Delta y = 0$, $(z_2 - z_1) = \Delta z = 0$, and $(t_2 - t_1) = \Delta t = 4$

Now, the (*space-time interval*)^2 is given by $S_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2$.

We, therefore, have $S_{12}^2 = (5c)^2 + 0 + 0 - c^2(4)^2 = 25c^2 - 16c^2 = 9c^2$,

whence, the *space-time interval* $S_{12} = \sqrt{9c^2} = 3c$.

Now, imagine a reference frame S' to be moving with velocity v relative to S, along the axis of x. Then, in accordance with Lorentz transformation, the interval $(t_2' - t_1') = \Delta t'$ between the two events in this frame, as it would appear to an observer in S, is given by the relation $\Delta t' = \gamma\left(\Delta t - v\dfrac{\Delta x}{c^2}\right)$

where γ, as we know, is equal to $1/\sqrt{1 - v^2/c^2}$.

Clearly, therefore,

(*i*) *For the two events to appear to occur simultaneously in frame S'*, we should have $\Delta t' = 0$, i.e.,

$$\gamma\left(\Delta t - \frac{v\Delta x}{cz}\right) = 0 \quad \text{or,} \quad \Delta t = \frac{v\Delta x}{c^2}$$

Or, substituting the values of Δt and Δx, we have

$$4 = \frac{v(5c)}{c^2}, \quad \text{whence,} \quad v = \frac{4}{5}c = 0.80\text{ c}.$$

Thus, *the velocity of frame S' must be 0.80 c relative to S, along the axis of x.*

(*ii*) (*a*) *For the second event to appear to occur 2 sec **earlier** than the first in frame S'*, we should have $(t_2' - t_1') = \Delta t' = +2$. So that,

$$2 = \gamma\left(\Delta t - v\frac{\Delta x}{c^2}\right) = \frac{1}{\sqrt{1 - v^2/c^2}}\left(4 - \frac{v(5c)}{c^2}\right) = \frac{1}{\sqrt{1 - v^2/c^2}}\left(4 - 5\frac{v}{c}\right)$$

Squaring and rearranging, we obtain the quadratic equation.

$$29\, v^2 \quad 10\, v$$

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So that, again, $v = 0.94\,c$ or $0.44\,c$.

Here, obviously, the second value is inadmissible, as it gives a positive value of Δt. We, therefore, have $v = 0.94$ c.

Thus, *in this case, the velocity of frame S′ must be 0.94 c relative to S, along the axis of x.*

Example 8. A body travelling at 0.9 c where c is speed of light, is shortened to $(d/d') = \sqrt{1-\frac{v^2}{c^2}}$**, calculate the ratio, between d and d′ in the expression, to the length at rest.**

Solution.

$$\left(\frac{d}{d'}\right) = \sqrt{1-\frac{v^2}{c^2}} = \left[1-\left(0.9\frac{c}{c}\right)^2\right]^{\frac{1}{2}} = 0.436$$

$$= 43.6\% \text{ of the length at rest.}$$

Example 9. Show that the force acting on a particle, as observed by two observers in two inertial frames to reference in the same. (Given $V < C$).

Solution. The force acting on a particle as observed by the observer O in frame S is given by

$$F = \frac{d}{dt}(MV) = M\frac{dV}{dt} = Ma$$

The force acting on a particle as observed by the observer O' in the frame S' is given by

$$F' = \frac{d}{dt}(MV') = M\frac{dV'}{dt} = Ma'$$

Since both are inertial frames of reference, $a = a'$, $F = F'$.

It means, the same force will be observed by the two observers O and O' in two inertial forms of reference S and S'. This means force is invariant to Galilean transformations. It further means, Newton's second law of motion $F = ma$ is valid in all inertial frames of reference, *i.e.* the basic laws of physics are invariant in two inertial frames of reference.

Example 10. What is the length of metre stick moving parallel to its length when its mass is 3/2 of its rest mass. ***(Delhi Uni. 2003)***

Solution. Here

$$m = m_o/\sqrt{1-\frac{v^2}{c^2}} \quad \ldots(i)$$

Therefore,

$$\frac{m}{m_o} = 1/\sqrt{1-\frac{v^2}{c^2}}$$

Similarly,

$$L = L'\sqrt{1-\frac{v^2}{c^2}} \quad \ldots(ii)$$

Dividing (*ii*) by (*i*), we have

$$\frac{L}{m} = \frac{L'}{m_o}\left(1-\frac{v^2}{c^2}\right)$$

$$\therefore \quad L = \frac{m}{m_o}.L'\left(1-\frac{v^2}{c^2}\right)$$

or

$$L = \frac{m}{m_o}L'\left(\frac{m_o}{m}\right)^2 \quad [\because \text{Eq. } (i)]$$

$$= \frac{m_o}{m}.L'$$

But $\frac{m_o}{m} = \frac{2}{3}$ and $L' = 1$ metre,

$$L = \frac{2}{3} \times 1\text{ m} = 0.667\text{ m}$$

Example 11. The length of a rocket ship is 100 metres on the grand. When it is in flight its ıgth observed on the ground is 99 meters calculate the speed?

Solution. By the result of Lorentz contraction, the length will be

$$L' = L\sqrt{1-\frac{v^2}{c^2}}$$

Hence
$$99 = 100\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}$$

$$\left(\frac{99}{100}\right)^2 = 1-\frac{v^2}{c^2} \qquad \text{(as } L' < L\text{)}$$

$$\frac{v^2}{c^2} = \frac{199}{10^4}$$

$$\frac{v}{c} = \frac{\sqrt{199}}{100}$$

or
$$v = \frac{(199)^{1/2}}{100} \times 3 \times 10^8 \text{ m/sec} = 3 \times 10^6 \times (199)^{1/2}$$

$$= 42.3 \times 10^6 \text{ ms}^{-1}$$

Example 12. A rod has a length 100 cm when rod is in a satellite moving with velocity).8*c* relative to laboratory, what is the length of the rod as determined by an observer, (*a*) in the satellite and (*b*) in the laboratory?

Solution. (*a*) The observer in the satellite is at rest-relative to the rod as the rod is in the satellite. So the length of the rod relative to the observer is 100 cm.

(*b*) Now the observer in the laboratory. So that the rod is in motion w.r.to the observer. $v = 0.8c$

$l = 100$ cm

$$L' = L\sqrt{1-\frac{v^2}{c^2}}$$

$l_o = \frac{l_o}{\sqrt{1-\frac{v^2}{c^2}}}$ $\quad l_o = 100\sqrt{1-0.64}$

$= 60$ cm

$$= 100[1-(0.8)^2]^{1/2} = 100 \times 0.6 = 60 \text{ cm}$$

Example 13. Calculate the length of a rod moving with a velocity of 0.8*c* in a direction inclined at 60° to its own length. Proper length of rod is given to be 100 cm.

Solution. Suppose the rod is moving with velocity $v = 0.8c$ along x-axis and the rod as inclined at 60° with x-axis.

Contraction will take place only in x-direction and not in y-direction. Hence $l_x = l\cos 60°$, $l_y = l\sin 60°$.

$$\vec{l} = \hat{i}\,l_x = \hat{j}\,l_y$$

$$l'_x = l_x\sqrt{1-\frac{v^2}{c^2}},\ l'_y = l_y = l\sin 60° = \left(\frac{\sqrt{3}}{3}\right)l,$$

$$l'_x = [(l \cos 60°)\sqrt{(1-(0.64)}] = l\left(\frac{1}{2}\right)(0.6) = 0.3\, l$$

$$l' = [(l'_x)^2 + (l_y)^2]^{\frac{1}{2}} = l[0.09 + (3/4)]^{\frac{1}{2}}$$

$$= 0.916\, l = 0.916 \times 100 = 91.6$$

Here l = length of the rod at rest, and

l' = length of the moving rod.

$\therefore$ Length of the rod in motion = 91.6 cm.

Example 14. Calculate the percentage contraction in the length of a rod in a frame of reference, moving with velocity 0.8c in a direction (a) parallel to its length (b) at an angle of 30° with its length. What is the orientation of the rod in the moving frame of reference in case (b)?

Solution. (*a*) Let L_0 be the length of the rod, placed along the axis of x, in a reference frame S, at rest. Then, its length in a frame S', moving with velocity $0.8c$ relative to S in a direction parallel to its length, *i.e.*, along the axis of x is given by

$$L' = L_0\sqrt{1 - v^2/c^2} = L_0\sqrt{1 - \frac{(0.8c)^2}{c^2}} = L_0\sqrt{1 - 0.64},$$

whence,
$$L' = L_0\sqrt{0.36} = 0.60 L_0$$

$\therefore$ *percentage contraction produced in the length of the rod* $= \dfrac{(L_0 - 0.60\, L_0)}{L_0} \times 100 = 40$

(*b*) In this case, the *component of the length of the rod along its direction of motion.*

$$= L_0 \cos 30° = \frac{\sqrt{3}}{2} L_0.$$

and the *component of its length, perpendicular to this direction* $= L_0 \sin 30° = 0.5\, L_0$.

Only the former component undergoes a change in length and not the latter (being perpendicular to the direction of motion).

If, therefore, L_0'' be the length of the former component in the moving frame S', we have

$$L_0'' = \frac{\sqrt{3}}{2} L_0\sqrt{1 - \frac{(0.8c)^2}{c^2}} = \frac{\sqrt{3}}{2} L_0 \times 0.60 = 0.52\, L_0.$$

Since the other component remains unchanged at $0.5\, L_0$, we have *total length of the rod in frame* S', say. $L'' = \sqrt{(0.52\, L_0)^2 + (0.5\, L_0)^2} = 0.7228\, L_0$.

$\therefore$ *percentage contraction produced in the length of the rod* $= \dfrac{0.7228\, L_0}{L_0} \times 100 = 22.72$

Clearly, if θ be the angle that the length of the rod appears to make with the direction of velocity v of frame S', we have

$$\tan\theta = \frac{\text{length component} \perp v}{\text{length component} \parallel v} = \frac{0.5\, L_0}{0.52\, L_0} = 0.96$$

Or,
$$\theta = \tan^{-1}(0.96) = 43°50'.$$

Thus, *the rod makes an angle of 43°50′ with its direction of motion.*

Example 15. A circular lamina moves with its plane parallel to the $x-y$ plane of a refe[rence] frame S at rest. Assuming its motion to be along the axis of x (or y), calculate the veloc[ity at] which its surface area would appear to be reduced to half to an observer in frame S.

Solution. Taking the motion of the lamina to be along the axis of x of the reference frame S, th velocity v, it is clear that its radius R along this axis will contract to, say, R', whereas the radius pendicular to its *i.e.*, along the axis of y, will remain unaltered. So that, the circular lamina will pear to have assumed an elliptical shape, with its major and minor axes equal to R and R' along the es of y and x respectively. Its surface area will, therefore, appear to be $\pi RR'$. The original surface ea of the circular lamina being πR^2, we have

$$\pi RR' = \frac{1}{2}\pi R^2, \text{ whence, } R' = R/2.$$

But, as we know, R' must be equal to $R\sqrt{1-v^2/c^2}$, where v is the velocity of the circular lamina th respect to the reference frame S.

$$\therefore \quad \frac{R}{2} = R\sqrt{1-v^2/c^2}. \quad \text{Or} \quad \frac{1}{4} = 1 - \frac{v^2}{c^2},$$

hence, $$v^2/c^2 = 3/4. \text{ Or, } v/c = \sqrt{3/2}. \text{ Or, } v = (\sqrt{3/2})c = 0.866c.$$

Thus, *the velocity of the circular lamina relative to frame S must be* 0.866*c*.

Example 16. A rod 1 metre long is moving along its length with a velocity 0.6c. Calculate s length as it appears to an observer (a) on the earth (b) moving with the rod itself.

Solution. Here, obviously, 1 metre is the proper length L_0 of the rod in its own moving frame f reference.

(a) If, therefore, L' be its length, as it appears to an observer in the stationary reference frame f the earth, we have

$$L' = L_0\sqrt{1-v^2/c^2} = 1\sqrt{1-\frac{(0.6\,c)^2}{c^2}} = 1\sqrt{1-0.36}$$
$$= 0.8 \text{ m} = 80 \text{ cm}.$$

Thus, *the observer on the earth will estimate the length of the rod to be 80* cm.

(b) For an observer moving with the rod itself, its length would be the same as its proper length n the moving reference frame, *viz.*, 1 metre, because there is no relative motion between the rod and he observer (*i.e.* $v = 0$).

Example 17. Obtain the volume of a cube, the proper length of each edge of which is L_0 when it is moving with a velocity v along one of its edges.

Solution. Here, clearly, the only change in length, (*i.e.*, a *contraction*) will occur in the particular edge of the cube along which it is moving, the lengths of the other edges, being perpendicular to

$$L/L_0 = 99/100 = \sqrt{1-v^2/c^2} \quad \text{Or,} \quad (99/100)^9 = 1 - v^2/c^2$$

Or, $v^2/c^2 = 1-(99/100)^2 = 1-0.98 = 0.02$, whence, $v^2 = .02c^2$.

Or, $v = 0.1415c$,

i.e., $v = 0.1415 \times 3 \times 10^{10} = 4.245 \times 10^9$ cm/sec.

Thus, the rocket ship should have a velocity 4.245×10^9 cm/sec, with respect to the observer.

3.7.2 Time Dilation

Time intervals too, like length, are affected by relative motion. Thus, to an observer, a clock appears to run slower when in motion, than when at rest with respect to him. So that, the time interval between two events occurring at a given point in space in a frame of reference, moving with respect to the observer, appears to be greater as noted on a clock at rest with respect to him than on an identical clock in the moving frame of reference itself. This is referred to as *time dilation*, meaning *apparent lengthening of time*. Let us deduce this result from *Lorentz transformation*.

Suppose we have two frames of reference, S and S' (Fig. 3.2), the former at rest and the latter moving with a uniform velocity v relative to it, along the $+x$ direction. Suppose further that two identical clocks are carried by the two reference frames at their origins O and O', such that they both show zero time when just against each other, *i.e.*, when their origins just cross each other.

Then, if two events occur at any given point x' in frame S', at times t_1' and t_2', as noted on the clock carried by it and at times t_1 and t_2, *as noted on the clock carried by frame S*, we clearly have time interval between the two events, *as noted on the clock in the moving frame S'* given by $\Delta t' = (t_2' - t_1')$.

And, time interval between the same two events as noted on the clock in the stationary frame S given by $\Delta t = (t_2 - t_1)$.

Now according to inverse Lorentz transformation, we have

$$t_1' = \gamma\left(t_1 - \frac{v_x}{c^2}\right) \quad \text{and} \quad t_2' = \gamma\left(t_2 - \frac{v_x}{c^2}\right)$$

where

$$\gamma = 1/\sqrt{1-(v^2/c^2)}$$

Therefore,

$$\Delta t' = (t_2' - t_1') = \gamma\left(t_2 - \frac{v_x}{c^2}\right) - \gamma\left(t_1 - \frac{v_x}{c^2}\right) = \gamma(t_2 - t_1)$$

or

$$\Delta t' = \gamma . \Delta t$$

Since the factor $1/\sqrt{1-v^2/c^2}$ or γ is greater than 1, we have $\Delta t' > \Delta t$.

Thus, *the time interval $\Delta t'$ between two events occurring at a given point in the moving frame S' appears to be longer, or dilated, by a factor $1/\sqrt{1-v^2/c^2}$ or γ to the observer in the s*

frame S.

The same will happen if frame S' were at rest and S in motion. So that, *like length con time dilation too shows reciprocity effect, i.e.,* works both ways or *is independent of the direc velocity and depends only on its magnitude.* In general, therefore, the interval between two e at the same point in a moving frame appears to be longer by a factor γ to an observer in a station frame.

This interval of time $\Delta t'$ between two events occurring at the same place, measured on th clock, moving with the reference frame S' in which the events occur, and hence at rest with respec to it, is called to *proper time interval* or *the proper time*, the usual symbol for which is τ. Thus, the proper time interval in a moving system is always less than the corresponding time interval, measured on a system at rest.

If follows from the above that the passage of time and hence also all physiological processes that go with it, like pulse and heart beats, and, infact, the process of ageing itself, are slowed down in a fast moving reference frame. Indeed, if the velocity of the moving frame be $v = c$, the passage of time and hence also the process of ageing will stop altogether!

The Twin Paradox

Consider two exactly identical twin brothers. Let one of the twins go to a long space journey at a high speed in a rocket and the other stay behind on the earth. The clock in the moving rocket will appear to go slower than the clock on the surface of earth (as $\Delta t' = \gamma.\Delta t$). Therefore, when he returns back to the earth, he will find himself younger than the twin who behind on the earth stayed.

Incidentally, this also shows that nothing material can really travel faster than light,– a proven fact, having the force of a *fundamental law of nature.*

In case, however, the velocity (v) of the moving frame of reference be small, as is ordinarily the case, so that $v/c \to 0$, we have $\Delta t = \Delta t'$, the time intervals between the two events in both the moving and the stationary systems are the same, – a result in accord with the absolute nature of time in Classical or Newtonian physics.

Most fundamental particles disintegrate spontaneously after a comparatively very brief spell of life, called their *life-time*. In a frame of reference S in which the particle is at rest (*i.e.*, its own frame of reference), the mean life-time is denoted by τ and is called the *proper life-time* of the particle. It is found to depend upon the velocity of the particle. Thus, in a frame of reference S' (say, the laboratory frame) in which it is moving with velocity v relative to S, the life-time gets dilated to

$$\tau' = \gamma T = \frac{\tau}{\sqrt{1 - v^2/c^2}}$$

Further, if N_o be the number of particles that undergo disintegration in the particle's own reference frame, the number of particles the survive up to time t is given by

$$N(t) = N_o e^{t/\tau}$$

Verification of Time Dilation

(*i*) **Indirect verification.** An indirect verification of the time dilation effect is afforded by the fact of $\mu^{\pm}$ *mesons or muons,* for short, produced higher up in the atmosphere, reaching the ground level in the very short span of their life-time.

These elementary particles, having a mass 215 times that of an electron, are produced in the upper reaches of the atmosphere as a result of collision of cosmic rays with the air molecules and own frame of reference. In this much longer life-time, they can easily cover a distance of 0.99c × 34.38 × 10^{-6} = 10.29 km. This at once explains their presence in the laboratory, 10 km below, at the ground level.

(*ii*) **Direct verification.** Directly, the time dilation effect may be verified in the laboratory using a beam of π^+ *mesons or Pions*, for short, (the products of highly energetic nuclear reaction in the upper atmosphere) which are produced artificially in the laboratory by bombarding a suitable target of carbon or beryllium by highly energetic *protons* or *α-particles* from a *synchrocyclotron*. These particles are comparatively heavier than mesons, with a mass 273 times that of an electron but have a much shorter life-time τ of 2.6 × 10^{-8} *sec* in their own frame of reference, after which they disintegrate into a *Meons* and two *neutrinos* each. Though all the $\pi^{\pm}$ *mesons* do not have the same velocity, the faster ones may be assumed to have a velocity of 0.9*c*. So that, their dilated life-time in the laboratory frame of reference should be equal to

$$\gamma\tau = \frac{2.6\times10^{-8}}{\sqrt{1-v^2/c^2}} = \frac{2.6\times10^{-8}}{\sqrt{1-\dfrac{(0.9c)^2}{c^2}}} = 5.96\times10^{-8}\text{ sec, } i.e.,$$

more than twice that in their own frame of reference (in which they are at rest).

To verify whether this is so, we obtain a beam of *pions* in the manner indicated (*i.e.*, by bombarding a ribbon target *T* of *carbon* or *beryllium* (Fig. 3.6) by α-particles or protons from a synchrocyclotron). This beam is narrowed down by passing it through a collimator slit *S*, as shown. The pions in the beam decay into *muons* and *neutrinos* and the muons thus produced are detected by a suitable device D, along side which the beam passes.

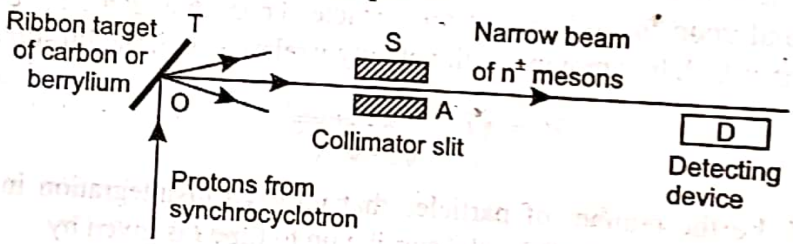

Fig. 3.6

Obviously, the number of *muons* recorded by the detecting device per second is proportional to the number of pions present in the beam in the vicinity of the device, *i.e.*, to the flux-of pions adjacent to it. By observing the recordings of the detecting device at various points along the path *OP* of the beam, therefore, the dilated life-time τ of the moving pions can be obtained.

To obtain the mean life-time or the proper life-time τ of the pions (*i.e.*, their life-time in their own frame of reference, in which they are at rest), they are brought to rest at *A* right at the beginning of their path as they emerge from the slit, by introducing a thick solid slab of a suitable material. The time interval elapsing between the entering of the points in the solid slab and the emerging of muons therefrom, noted electronically, then gives the proper life-time τ of the pions constituting the beam.

It is found that $\tau' = \gamma\tau = \dfrac{\tau}{\sqrt{1-v^2/c^2}}$ and the time dilation effect thus stands verified.

3.7.3. Simultaneity

By *simultaneity* of two events we understand their occurrence at *exactly the same time*. For example, if in a fixed frame of reference, *S*, *A* and *B* are two points distant x_1 and x_2 respect[ively] from the origin *O* and *P*, the carefully determined *mid-point* of *AB*. Then, if a light signal is em[itted] at P, it will arrive at A at exactly the same time as at B, since the velocity of light (c) is the same in all directions. We, therefore, say that the two events, *viz*., the arrival of the light signal at A and at B, occur simultaneously at time t, say, *i.e.*, one event occurs at x_1 at time t and the other at x_2 also at the same time t.

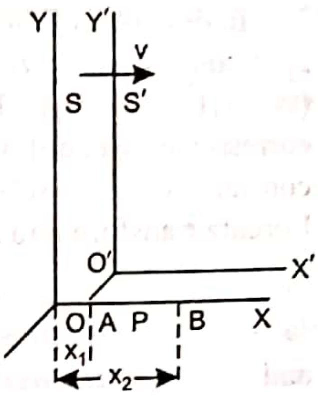

Fig. 3.7.

Let us see if the two events also appear to be simultaneous to an observer in a frame of reference S', moving relative to S, with velocity v along the positive direction of its x-axis. For the observer in S, the corresponding values of t for the events at x_1 and x_2 will, in accordance, with Lorentz transformation, be given respectively by

$$t_1' = \gamma\left(t - \frac{vx_1}{c^2}\right) \quad \text{and} \quad t_2' = \gamma\left(t - \frac{vx_2}{c^2}\right)$$

So that, the time interval between the two events, as observed by the observer in S' will be $t' = (t_2' - t_1') = \gamma v(x_1 - x_2)/c^2$, which is surely *not zero*, indicating that the two events at A and B, which appear to be simultaneous to the observer in S *do not appear so* to the observer in S' but to have a time interval t' between them, the magnitude of which depends upon both, the distance between the two events and the velocity of frame S' and relative to S.

Exactly in the same manner it can be shown that two events that appear to be simultaneous to an observer in frame S' do not appear to be so to the observer in frame S.

As can well be seen, *this failure of simultaneity in one frame of reference in relative motion with another is a direct consequence of the value of c being the same in all frames of reference and in all directions.* So that, if we accept, as we must, the postulate of *invariance of the velocity of light*, we have also to accept the consequence flowing from it, namely, the *relativity of simultaneity*, *i.e.*, the simultaneity of events, separated in space, not being something *absolute* that can be expected or accepted in all frames of reference. It is there in each frame of reference separately but there can be no question of agreement between simultaneity in one frame with that in another in relative motion with it. In short, it is only a *local affair* of each individual frame of reference, whether at rest or in motion.

3.7.4 Invariance of Space-time Interval

It has already been mentioned (§3.3) that to know the true position of a point, we must know its four space-time coordinates, *viz*., the three spatial coordinates x, y, z and the time coordinate t. ... *space-time frame of reference*, or an occurrence there, is, as we have seen before, In case one of the points is not at the origin, and the coordinates of the two points are x_1, y_1, z_1, t_1 and x_2, y_2, z_2, t_2 respectively, the *space-time interval* is obviously $c^2(t_2^2 - t_1^2) - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$. That this is invariant to Lorentz tranformation may be easily seen. For, if the corresponding coordinates of these points in a frame S', moving along the axis of x of frame S with constant velocity v relative to it, be x_1', y_1', z_1', t_1' and x_2', y_2', z_2, t_2 we have, in accordance with Inverse Lorentz transformation,

$$S_{12}'^2 = c^2(t_2^2 - t_1^2) = c^2\gamma^2[(t_2' - t_1') + v(x_2' - x_1')/c^2]^2,$$

$$(x_2 - x_1)^2 = \gamma^2[(x_2' - x_1') + v(t_1' - t_1')]^2, \quad (y_2 - y_1)^2 = (y_2' - y_1')^2$$

and $$(z_2 - z_1)^2 = (z_2' - z_1')^2.$$

$\therefore$ $$c^2(t_2^2 - t_1^2) - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$
$$= c^2\gamma^2[(t_2' - t_1') + v(x_2' - x_1')/c^2]^2 - \gamma^2[(x_2' - x_1') + v(t_2' - t_1')]^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2$$

Or, $$S_{12}'^2 = c^2(t_2' - t_1')^2\,\gamma^2\left(1 - \frac{v^2}{c^2}\right) - (x_2' - x_1')^2\,\gamma^2\left(1 - \frac{v^2}{c^2}\right) - (y_2' - y_1')^2 - (z_2' - z_1')^2$$

Since $$\gamma = 1/\sqrt{1 - v^2/c^2}, \quad \text{or,} \quad \gamma^2 = 1/(1 - v^2/c^2)$$

we have $$\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

$\therefore$ $$S_{12}'^2 = c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2.$$

Thus, $$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1) - (z_2 - z_1)^2 = c^2(t_2' - t_1')^2$$

Or, $$S_{12}^2 = S_{12}'^2, \qquad \text{...}(i)$$

clearly showing that the *space-time interval is invariant to Lorentz transformation.*

It may be noted again that the *space and the time-intervals, taken separately are not invariant.* For, clearly, $(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ is not equal to $(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' + z_1')$ and nor is $(t_2 - t_1) = (t_2' - t_1')$.

Thus whereas in Classical physics, *time* and *space* were taken to be two different things, the Lorentz transformation equations and relation (*i*) above, proclaim clearly that *in the theory of relativity*, time **and space are equivalent** *and can be expressed one in terms of the other.* Thus, it can be seen from equation (*i*) that, *in terms of distance,* 1 *sec* is 3×10^8 *metre*, the distance covered by light in free space in 1 *sec* and, similarly, 1 *metre of time* is $1/3 \times 10^8$ *sec*, the time taken by light to cover a distance of one metre.

Thus, measuring time and space in the same units in a system in which $c = 1$, the Lorentz transformation equations, as also equation (*i*) above are greatly simplified and respectively become

$$x' = (x - vt)/\sqrt{1 - v^2}, \; y' = y, \; z' = z \; \text{ and } t' = (t - vx)/\sqrt{1 - v^2} \qquad \text{...}(ii)$$

and $$(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = (t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2. \qquad \text{...}(iii)$$

Now, if we consider two points (or particles) which, in a given co-ordinate system, have only space but no time (or zero time), we can see from relation (*iii*) that the interval squared would be negative and the interval, therefore, an *imaginary* one. Such an interval is referred to as a *space-like interval*, obviously because the interval, in this case, is more like space than like time. One the other hand, when we have two points or particles, occupying the same place in the coordinate system but having different times, the square of the time being positive and the distance *zero*, the interval squared is positive and hence the interval *real*. Such an interval is called a *time-like interval*.

Incidentally, the invariance of the space-time interval in two frames of reference means, in other ords, the invariance of the value of c in the two frames.

Example 19. A certain young lady on her twenty fifth birthday that it is time to slendrize. he weights 100 kilograms. She has heard that if she moves fast enough, she will appear inner to her stationary friends.

(i) How fast must she move to appear slendrize by a factor of 50%

(ii) At this speed, what will be her mass to be her stationary friends?

(iii) It she maintains her speed untill the day she calls her twenty-nineth birthday, how old will her stationary friends claim she is according to their measurements?

Solution. Given: $m_o = 100$ kg, $L = 50\%$ of $L_o = \frac{50}{100} L_o = \frac{L_o}{2}$ where L_o represents dimensions f the lady at rest and L its length when she is in motion.

$l = l_0 \sqrt{1-\frac{v^2}{c^2}}$

$$\Delta t = 4 \text{ years. For } 29 \text{ years} - 25 \text{ years} = 4 \text{ years}$$

(i) We have, $$L = L_o\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$$ putting the values, we get

$$\frac{L_o}{2} = L_o\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

$\frac{1}{2} = \sqrt{1-\frac{v^2}{c^2}}$

$\frac{1}{4} = 1-\frac{v^2}{c^2}$

Or $$1-\frac{v^2}{c^2} = \frac{1}{4} \quad \text{or} \quad \frac{v^2}{c^2} = \frac{3}{4}$$

$\frac{v^2}{c^2} = \frac{3}{4}$

$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$\boxed{v = \frac{\sqrt{3}}{2} c}$

Or $$V = \frac{\sqrt{3}}{2}c = 0.867\, c$$

$m = 2m_0$
$= 200$ kg

(ii) $$m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{100}{\frac{1}{2}} = 200 \text{ kg}$$

$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$t = 2t_0 = 8$ years

(iii) By time dilation, $$\Delta t' = \frac{\Delta t}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \frac{4}{\frac{1}{2}} = 8$$

$$\frac{v^2}{c^2} = 1 - 0.111 = 0.889 \text{ or } \frac{v}{c} = 0.9429$$

$$v = 0.9429 \times 3 \times 10^8 = 2.83 \times 10^8 \text{ m/sec}$$

Example 21. A man on the moon observes two spaceships coming towards him from opposite directions at speeds 0.8c and 0.7c respectively. What is the relative speed of the two spaceships as measured by an observer on either one? ***(Nagpur Uni. 2005)***

Solution. The two spaceships A as stationary S frame, the moon as the S' frame and the other spaceship B as the object whose speed u_x in the S frame is to be determined.

With respect to S frame, the moon moves with velocity V as shown in Fig. 3.8(*b*)

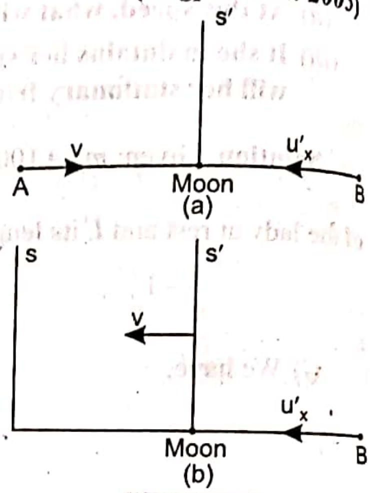

Fig. 3.8

Then, $\quad v = -0.8c,\ u'_x = -0.7c$

Now

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} = \frac{-0.7c - 0.8c}{1 + \frac{(-0.8c)(-0.7c)}{c^2}}$$

$$= \frac{-1.5c}{1.56} = -0.96c$$

$$= -2.88 \times 10^8 \text{ m/sec.}$$

The minus sign shows that the spaceship B approaches A.

Example 22. Two β particles A and B emitted by a radioactive source travel in opposite directions each with a velocity of 0.9c with respect to the source. Find the velocity of B with respect to A.

Solution. The two β-particles A and B move in opposite directions as shown in Fig. 3.9 (*a*). Now we regard the β-particle as stationary S frame, radioactive source R as the S' frame, and β–particle B as the object whose speed u_x in the frame S is to be determined. With respect to S, the radioactive source moves with velocity v as shown in Fig. 3.9 (*b*). Then

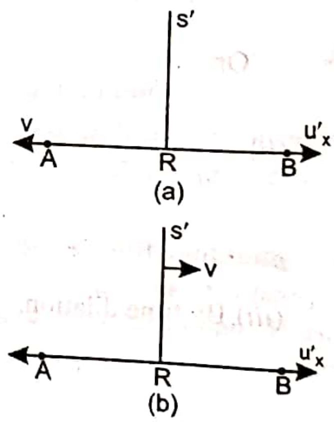

Fig. 3.9

$$v = +0.9c,\ u'_x = +0.9c$$

$$\therefore \quad u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} = \frac{0.9c + 0.9c}{1 + \frac{0.9c \times 0.9c}{c^2}}$$

$$= \frac{1.8c}{1.81} = 2.982 \times 10^8 \text{ m/sec.}$$

Example 23. (*i*) What is mean life of a burst of π^+ mesons, travelling with β = 0.73 ? (The proper mean life time τ is 2.5 × 10⁻⁸ sec.)

(*ii*) What distance is travelled at β = 0.73 during one mean life?

(*iii*) What distance would be travelled without relativistic effects?

Solution. Here, the proper mean life of the π^+ mesons is $\tau = 2.5 \times 10^{-8}$ *sec.*

(*i*) Therefore, their observed mean life when travelling at $\beta = v/c = 0.73$, *i.e.*, at

given by

$$\tau' = \gamma\tau = \frac{2.5 \times 10^{-8}}{\sqrt{1 - v^2/c^2}} = \frac{2.5 \times 10^{-8}}{\sqrt{1 - \left(\frac{0.73c}{c}\right)^2}} = \frac{2.5 \times 10^{-8}}{\sqrt{1 - (0.73)^2}} = \frac{2.5 \times 10^{-8}}{\sqrt{0.4672}}$$

$$= 3.658 \times 10^{-8} \text{ sec.}$$

Scanned by CamScanner

Thus, *the observed mean life of the burst of* π^+ *mesons* is 3.658×10^{-8} *sec.*

(*ii*) Obviously, *distance travelled in one mean life*, at $\beta = 0.73$ or $v = 0.73\,c$

$$= \textit{velocity} \times \textit{observed mean life} = v\tau$$

$$= 0.73 \times 3 \times 10^{10} \times 3.658 \times 10^{-8} = 800.9 \text{ or } 801 \text{ cm.}$$

(*iii*) Distance travelled in one mean life, without relativistic effects, would obviously be

$$= \text{velocity} \times \text{proper mean life} = v\tau$$

$$= 0.73 \times 3 \times 10^{10} \times 2.5 \times 10^{-8} = 547.4 \text{ cm.}$$

Example 24. The life time of mu-mesons is 2.2×10^{-6} sec and their speed $0.998c$, so that they can cover only a distance of $0.998c \times 2.2 \times 10^{-6}$ or 658.6 metres in their entire life time, and yet they are found in profusion at sea level, *i.e.*, at a depth of 10 kilometer from the upper atmosphere where they are produced. How may this be explained on the basis of (*i*) Lorentz-Fitzgerald contraction, (*ii*) time dilation?

Solution. (*i*) Here, 2.2×10^{-8} sec. being the mean life time of the μ-mesons in their own frame of reference, *i.e.*, their *proper life-time*, the distance covered by them during this time in their own frame of reference, say $h_o = 658.6$ m.

The *observed distance* (as seen from *our* reference frame, *i.e.*, the earth) will, therefore, be, say,

$$h = \gamma h_o = h_o\Big/\sqrt{1 - v^2/c^2} = 658.6\Big/\sqrt{1-(0.998c/c)^2}$$

$$= 658.6/0.064 = 10290 \text{ m} = 10.29 \text{ km.}$$

Thus, *the μ-mesons are able to cover a distance of 10 km or more despite their short life-time. This explains their presence at sea level.*

(*ii*) Here, again, the proper life-time of the μ-mesons is $\tau = 2.2 \times 10^{-6}$ *sec* and, therefore, their observed life time $\tau' = \gamma\tau = 2.2 \times 10^{-6}\Big/\sqrt{1 - v^2/c^2}$

$$= 2.2 \times 10^{-6}\Bigg/\sqrt{1 - \left(\frac{0.998c}{c}\right)^2}$$

$$= 2.2 \times 10^{-6}/0.064 = 34.38 \times 10^{-6} \textit{ sec},$$

i.e., their observed life-time is about 16 times their proper life-time and, *in this increased or dilated life-time*, they can cover a distance $0.998c \times 34.38 \times 10^{-6} = 0.998 \times 3 \times 10^6 \times 34.38 \times 10^{-6} = 10290$ *m* = 10.29 km, as in (*i*) above. *Hence their presence at sea level.*

Example 25. A beam of particles of half life 2×10^{-6} sec travels in the laboratory with 0.96 times the speed of light. How much distance does the beam travel before the flux falls to $\frac{1}{2}$ times the initial flux?

Solution. Here, the proper half-life of the beam of particles, $\tau/2 = 2 \times 10^{-6}$ *sec.*

∴ their observed half-life when travelling at velocity $v = 0.96c = \tau'/2$

$= \gamma\tau/2 = 2 \times 10^{-6}\Big/\sqrt{1 - v^2/c^2}$ ...

If 1 per cent of those in the original burst survive to reach the earth's surface, estimate the original height. (In the meson frame of reference, the number of particles which survive to a time t is given by

$$N_{(t)} = N_{(o)}e^{(-t/\tau)}.$$

Solution. We are given the *proper life-time of a μ-meson in its own frame of reference* to be $\tau = 2.6 \times 10^{-6}$ *sec.*

∴ Its *observed life-time in the laboratory frame of reference* is $\tau' = \gamma\tau$.

And, therefore, number of particles that survive up to time t to *reach the laboratory frame of reference* is $N_{(t)} = N_{(o)}e^{-t/\tau'} = N_{(o)}e^{-t/\gamma\tau}$.

Since 1% of the particle in the original burst survive to reach the earths' surface or the laboratory, we have $N_{(t)}/N_{(o)} = e^{-t/\gamma\tau} = 1/100$.

Taking logarithms, we have $\log_e{}^{-t/\gamma\tau} = -\log_e 100 = -2 \times 2.306$.

Or, $t = 4.6052\gamma\tau = 4.6052\tau\Big/\sqrt{1-v^2/c^2} = 4.6052 \times 2 \times 10^{-5}\Big/\sqrt{1-\left(\frac{0.99c}{c}\right)^2}$

Or, $t = 4.6052 \times 2 \times 10^{-5}\Big/\sqrt{1-(0.99)^2} = 4.6052 \times 2 \times 10^{-5}\Big/\sqrt{0.02} = 6.513 \times 10^{-5}$ *sec.*

∴ *estimated original height of the μ-mesons*, say,

$h = v \times t = 0.99 \times 3 \times 10^{10} \times 6.513 \times 10^{-5} = 1.934 \times 10^5 = 2 \times 10^5$ cm very nearly,

Example 27. Show from Lorentz transformation that two events simultaneous ($t_1 = t_2$) at different positions ($x_1 \neq x_2$) in a reference frame S are not, in general, simultaneous in another reference frame S'.

Solution. In accordance with *Inverse Lorentz transformation*, we have

$$t_1 = (t_1' + vx_1'/c^2) = t_2 = \gamma(t_2' + vx_2'/c^2) \qquad ...(i)$$

where v is the velocity of frame S' relative to S and t_1' and t_2', the corresponding values of t_1 and t_2, in frame S'.

In case of the two events are also simultaneous in frame S', t_1' will be equal to t_2'. Let us see if this is so.

From relation (i) above, we have $t_1' = t_2' + \frac{v}{c^2}\left(x_2' - x_1'\right)$ ...(ii)

Now, in accordance with Lorentz transformation, $x_2' = \gamma(x_2 - vt_2)$ and $x_1' = \gamma(x_1 - vt_1)$. Therefore, $(x_2' - x_1') = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1)$.

Since $t_1 = t_2$, we have $(x_2' - x_1') = \gamma(x_2 - x_1)$.

Substituting this value of $(x_2' - x_1')$ in relation (ii) for t_1', above, we have

$$t_1' = t_2' + \frac{\gamma v}{c^2}(x_2 - x_1) \qquad ...(iii)$$

It is thus clear from relation (iii) that with $x_2 \neq x_1$, t_2' cannot be equal to t_1', *i.e., the two events cannot be simultaneous in frame S' also, unless of course, v/c* = [illegible] *there are no relativistic effects at all.*

Example 28. Determine the time (as measured by a clock at rest on rocket) ta[illegible] rocket to reach a distant star and return to earth with a constant velocity $v = \sqrt{0.999}$[illegible] distance of the star is 4 light years. (A light year is defined as the distance travelled b[illegible] beam in vacuum in one year).

Solution. Here, time taken by the rocket for the round trip, to the star and back, as meas[illegible] a clock on the earth, is, say, $t' = \dfrac{(4+4)c}{\sqrt{0.9999}c} = 8$ years.

If the time taken by the round trip, as measured on the clock carried by the rocket itself be t, clearly t is the *proper time*. So that, we have $t' = \gamma t$, whence,

$$t = t'/\gamma = t'\sqrt{1-v^2/c^2}.$$

Since $v = \sqrt{0.9999}\,c$, we have $v/c = \sqrt{0.9999}$. Or, $v^2/c^2 = 0.9999$.

So that, $$t = 8\sqrt{1-0.9999} = 8\sqrt{0.001} = 8 \times 0.01 = 0.08 \text{ years.}$$

Thus the time for the round trip, as measured on the clock carried by the rocket itself is 0.08 years.

3.7.5 Transformation of Velocity—Velocity Addition

Suppose a particle has a velocity u in a frame of reference S at rest, such that its components along the three coordinate axes are $u_x = d^x/dt$, $u_y = dy/dt$ and $u_z = dz/dt$ and its velocity in a frame of reference S', moving with a uniform velocity v relative to S along the positive direction of the x-axis, is u', with its components $u'_x = dx'/dt$, $u'_y = dy'/dt$, and $u'_z = dz'/dt$ along the three coordinate axes, as measured in S'. Let us see how these component velocities in the two frames of reference are related to each other.

As we know, in accordance with Lorentz transformation,

$$x' = \gamma(x - vt),\ y' = y,\ z' = z \text{ and } t' = \gamma(t - vx/c^2), \text{ where } \gamma = 1/\sqrt{1-v^2/c^2}.$$

So that, $$u'_x = \frac{dx'}{dt} = \frac{\gamma(dx - vdt)}{\gamma(dt - vdx/c^2)} = \frac{dx - vdt}{dt - vdx/c^2}$$

Dividing the numerator and the denominator on the right hand side of the equation by dt, we have

$$u'_x = \frac{dx'}{dt} = \frac{dx/dt - v}{1 - \frac{v}{c^2}\cdot\frac{dx}{dt}}.$$

Or, substituting u_x for dx/dt, we have $u'_x = \dfrac{u_x - v}{1 - vu_x/c^2}$.

And, $$u'_y = \frac{dy'}{dt} = \frac{dy}{\gamma(dt - vdx/c^2)}.$$

Or, dividing the numerator and denominator on the right hand side of the equation by dt, as before, we have

$$u'_y = \frac{dy/dt}{\gamma\left(1 - \frac{v}{c^2}\cdot\frac{dx}{dt}\right)} = \frac{u_y}{\gamma(1 - vu_x/c^2)}. \qquad \left[\because dy/dt = u_y \text{ and } dx/dt = u_x\right.$$

Similarly, $$u'_z = \frac{u_z}{\gamma(1 - vu_x/c^2)},$$

where u'_x, u'_y and u'_z are the components of the resultant velocity u' in frame S'.

By *inverse Lorentz transformation* (*i.e.*, replacing v by $-v$), we obtain

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2},\ u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} \text{ and } u_z = \frac{u'_z}{\gamma(1 + vu'_x/c^2)}.$$

Again, if the velocity of the particle, u', be directed along the x'-axis in frame S', its components u'_y and u'_z in frame S' and hence also u_y and u_z in frame S are each equal to *zero* and we have

$$u = \frac{u' + v}{1 + uv/c^2} \qquad ...(ii)$$

N.B. Quite often it is more convenient to measure and express high velocities as fractions of c rather than in cm or m/sec. So that, we divide the relations for u'_x and u_x, say, by c and put these in the form

$$\frac{u'_x}{c} = \frac{(u_x/c) - (v/c)}{1 - \frac{v}{c} \cdot \frac{u_x}{c}} \text{ and } \frac{u_x}{c} = \frac{(u'_x/c) + (v/c)}{1 + \frac{v}{c} \cdot \frac{u'_x}{c}}.$$

Other relations for u'_y, u'_z, u_y and u'_z may also be similarly expressed.

Now, the velocity u of the particle in frame S may be regarded as the resultant of its velocity u in frame S' and the velocity v of frame S' relative to S. So that, the expressions above give us a method for the addition of high or relativistic velocities (*i.e.*, velocities that approach the value of c) and constitute a **law for the addition of relativistic velocities.** Thus, from relation (*ii*) above, we may formally state the law as follows:

If a particle moves with velocity u' in a reference frame S' and if S' has a velocity v relative to a frame S, the velocity of the particle relative to S is given by $(u' + v)/(1 + u'v/c^2)$.

It may be noted that

(*i*) *If u'_x be very much less than c, so that $u'_x/c \to 0$, the above expressions all reduce to those obtained by Galilian transformation, i.e., $u_x = u'_x + v$, $u_y = u'_y$ and $u_z = u'_z$.*

(*ii*) *For higher values of u'_x, the denominator in the expression for u_x becomes greater than 1 and hence the resultant velocity u_x is smaller than $(u'_x + v)$, the value given by Galilian transformation.*

(*iii*) *If $u'_x = c$, i.e., if the velocity of the particle along the x'-axis in frame S' be equal to that of light (i.e., if the particle be a photon), we have $u_x = \frac{c + v}{1 + vc/c^2} = \frac{c + v}{(c + v)/c} = c$, i.e., the particle has the same velocity c in frame S.*

Thus, *this particular velocity c alone is invariant in the two frames of reference irrespective of their relative velocity v.*

But this result is only to be expected in view of the fact that *Lorentz transformation* itself has been obtained on the basic assumption of the constancy of the value of c in all reference frames, whatever their own velocities.

Incidentally, another interesting result that emerges in that, *under Lorentz transform[ation] no two velocities can add up to more than the value of c.* The same is not true under G[alilian] transformation. The following example will illustrate the point.

Suppose the relative velocity of S' with respect to S, along its x-axis, *i.e.*, $v = 3c/4$ and a par[ticle] in S moves along the $+x'$-direction with velocity $c/2$, *i.e.*, $u'_x = c/2$. Then, in accordance with Galil[ian] transformation, the resultant velocity of the particle, relative to frame S will be

$$u_x = \frac{c}{2} + \frac{3c}{4} = \frac{5c}{4} = 1.25\ c, \text{ i.e., greater than } c.$$

But, under *Lorentz transformation*, it will be

$$\frac{u_x}{c} = \frac{(u'_x/c)+(v/c)}{1+\frac{v}{c}\cdot\frac{u'_x}{c}} = \frac{\left(\frac{1}{2}\right)+\left(\frac{3}{4}\right)}{1+\frac{3}{4}\cdot\frac{1}{2}}$$

$$= \frac{5}{4}\times\frac{8}{11} = \frac{10}{11}, \quad \text{whence, } u_x = \frac{10}{11}c\text{, i.e., less than } c.$$

Example 29. In the laboratory two particles are observed to travel in opposite directions with speed 2.8×10^{10} cm/sec. Deduce the relative speed of the particles.

Solution. Let us call the two particles A and B, with the velocity of $A = +2.8 \times 10^{10}$ cm/sec and that of $B = -2.8 \times 10^{10}$ cm/sec along the axis of x.

Imagining particle B to be in a reference frame S at rest, the laboratory will constitute a reference frame S' in motion, with velocity $-(-2.8 \times 10^{10}) = 2.8 \times 10^{10}$ cm/sec. So that, *the velocity of the moving frame S', i.e.,* $v = -v_x = 2.8 \times 10^{10}$ cm/sec.

The problem thus reduces to determining the velocity of particle A in the moving frame S' of the laboratory as it would appear to an observer in the stationary reference frame S of particle B.

By *Inverse Lorentz transformation,* we have $u_x = \dfrac{u'_x + v}{1+\frac{v}{c^2}u'_x}$.

Here, velocity of particle A in the moving frame S' is $u'_x = 2.8 + 10^{10}$ cm/sec, its velocity in the stationary frame S' of particle B is u_x (to be determined), velocity of the moving frame S', *i.e.*, $v = -v_x$ and $c = 3 \times 10^{10}$ cm/sec.

Substituting these values in the expression of u_x above, we have

$$u_x = \frac{u'_x - v_x}{1-\frac{v_x}{c^2}u'_x} = \frac{2.8\times10^{10}+2.8\times10^{10}}{1-\left(\frac{-2.8\times10^{10}}{c^2}\right)(2.8\times10^{10})} = \frac{5.6\times10^{10}}{1+\left(\frac{2.8\times10^{10}}{3\times10^{10}}\right)^2}$$

$$= \frac{5.6\times10^{10}}{1+(2.8)^{2/9}} = \frac{5.6\times10^{10}}{1.8714} = 2.993 \times 10^{10} \text{ cm/sec.}$$

Thus, the relative velocity of the two particles is 2.993×10^{10} cm/sec.

Example 30. Show that if in the S' frame we have $v'_y = c \sin \phi$ and $v'_x = c \cos \phi$, in the frame S, $v_x^2 + v_y^2 = c$. Frame S' moves with velocity $V\hat{i}$ with respect to the S frame.

Solution. Since frame S' is moving relative to S in the x-direction with velocity V, we have, by *Inverse Lorentz transformation.*

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2}$$

and

$$v_y = \frac{v'_y}{\gamma(1+v'_x V/c)} = \frac{v_y}{(1+y'_x V/c^2)}\sqrt{1-v^2c^2}. \qquad \left[\because \gamma = \frac{1}{\sqrt{1-v^2/c^2}}\right]$$

$\therefore$ putting $v'_q = c\cos\phi$ and $v'_y = c \sin \phi$, we have

$$v_z^2 + v_y^2 = \left(\frac{c\cos\phi + V}{1+\frac{V\cos\phi}{c}}\right)^2 + \frac{(c^2\sin^2\phi)(1-V^2/c^2)}{1+\left(\frac{\gamma\cos\phi}{c}\right)^2}$$

$$= \frac{1}{1+\left(\frac{V\cos\phi}{c}\right)^2}\left(c^2\cos^2\phi + 2cV\cos\phi + V^2 + c^2\sin^2\phi - V^2\sin^2\phi\right)$$

$$= \frac{1}{\left(1+\frac{V\cos\phi}{c}\right)^2}\left[c^2 + 2cV\cos\phi + V^2(1-\sin^2\phi)\right]$$

$$= \frac{1}{\left(1+\frac{V\cos\phi}{c}\right)^2}\left(c^2 + 2cV\cos\phi + V^2\cos^2\phi\right)$$

$$= \frac{c^2}{\left(1+\frac{V\cos\phi}{c}\right)^2}\left(1+\frac{2V\cos\phi}{c}+\frac{V^2\cos^2\phi}{c^2}\right)$$

$$= \frac{c^2}{\left(1+\frac{V\cos\phi}{c}\right)^2}\left(1+\frac{V\cos\phi}{c}\right)^2 = c^2$$

or, $\quad v_x^2 + v_y^2 = c^2.$

Example 31. The velocity of a particle is $6\hat{i}+5\hat{j}+4\hat{k}$ in a frame of references S', moving with velocity 0.8 c along the axis of x, relative to a reference frame S at rest. What is the velocity of the particle in the latter frame?

Solution. Here, clearly, the velocity of the particle in the moving frame S' is $u' = 6\hat{i}+5\hat{j}+4\hat{k}$. So that, we have $u'_x = 6$ m/sec, $u'_y = 5$ m/sec and $u'_z = 4$ m/sec. and v, the velocity of frame S' relative to S is given to be $0.8\,c$, where

$$c = 3\times 10^8 \text{ m/sec.}$$

Now, by *Inverse Lorentz transformation*, we have

$$u_x = \frac{u'_x + v}{1+u'_x v/c^2} = \frac{6+0.8\,c}{1+\frac{6(0.8\,c)}{c^2}} = \frac{6+0.8\,c}{1+4.8/c} = \frac{c(6+0.8\,c)}{c+4.8}$$

$$\approx 0.8\,c = 0.8\times 3\times 10^8 = 2.4\times 10^8 \text{ m/sec.}$$

$$u_y = \frac{u'_y}{\gamma\left(1+\frac{u'_x v}{c^2}\right)} = \frac{u'_y\,(1-y^2/c^2)}{1+\frac{u'_x v}{c^2}} = \frac{5\sqrt{1-(0.8)^2}}{1+\frac{6\times 0.8}{c}} = \frac{5\times 0.6\,c}{c+4.8}$$

$$= 3 \text{ m/sec.}$$

and

$$u_z = \frac{u_z}{\gamma\left(1+\frac{u'_z v}{c^2}\right)} = \frac{u_z(1-v^2/c^2)}{1+\frac{u'_x v}{c^2}} = \frac{4\sqrt{1-(0.8)^2}}{1+\frac{6\times 0.8}{c}} = \frac{4\times 0}{c+}$$

$$\approx 2.4 \text{ m/sec.}$$

$\therefore$ The velocity of the particle in frame S is $(2.4\times 10^8\hat{i}+3\hat{j}+2.4\hat{k})$ m/sec.

Example 32. Two velocities of 0.8c each are inclined to one another at an angle of 30°. btain the value of their resultant.

Solution. Here, we may imagine one of the velocities of $0.8c$ to be along the axis of x and e other inclined to it an angle of 30°. This is equivalent to saying that a frame of reference S', y, is moving along its x'-axis with a velocity of $0.8c$ relative and parallel to a stationary frame S nd a particle is moving with a velocity $0.8c$ inclined to it at an angle of 30°. it is then required to etermine the resultant velocity of the particle as it would appear to an observer in frame S. So that, e have

velocity of frame S', relative to S, along the x'-axis $= v = 0.8c$,

and *velocity of the particle at an angle of 30° with its x'-axis (or the velocity v)* $= u'$.

Resolving u' along the x'-axis (or the velocity v of frame S') and perpendicular to it, *i.e.*, along e y'-axis; we have

component along the y-axis, $u'_x = u' \cos 30° = 0.8\,c\,(\cos 30°)$

$$= 0.8(\sqrt{3}/2\,c) = 0.4\sqrt{3}\,c$$

and *component along the y'-axis,* $u'_y = u' \sin 30° = 0.8\,c\,(\sin 30°)$

$$= 0.8\,(1/2)\,c = 0.4\sqrt{3}\,c.$$

Now, by *Lorentz transformation*, we have the values of these components, as observed in frame S, given by

$$u_x = \frac{u'_x + v}{1 + u'_x\,v/c^2} = \frac{0.4\sqrt{3}\,c + 0.8\,c}{1 + 0.4\sqrt{3}\,c(0.8\,c)/c^2} = \frac{c(0.4\sqrt{3} + 0.8)}{1 + 0.4\sqrt{3(0.8)}}$$

$$= \frac{c(0.4\sqrt{3} + 0.8)}{1 + 0.32\sqrt{3}} = \frac{1.493}{1.554}c = 0.96\,c.$$

and

$$u_y = \frac{u'_y}{\gamma(1 + u'_x\,v/c^2)} = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + u'_x\,v/c^2} = \frac{0.4c\sqrt{1 - (0.8\,c/c)^2}}{\dfrac{1 + 0.4\sqrt{3}c\,(0.8c)}{c^2}}$$

$$= \frac{0.4c\sqrt{1 - 0.64}}{1 + 0.32\sqrt{3}} = \frac{0.4c(0.6)}{1.554} = \frac{0.24}{1.554}c = 0.15c.$$

$\therefore$ resultant velocity u of the particle, as observed in frame $S = \sqrt{u_x^2 + u_y^2}$

$$= \sqrt{(0.96)^2 + (0.15)^2}\,c = \sqrt{0.9442}\,c = 0.97c$$

And the angle that this resultant velocity makes with and component of velocity u of the particle along the y-axis, *i.e.*,

$$u_x = 0.8c \sin 60° = 0.8c\,(\sqrt{3}/2 = 0.4\sqrt{3c}\,)$$

Now, by *Lorentz transformation*, we have the values of these components, as observed by a person in the moving frame S', given by

$$u_x' = \frac{u_z - v}{1 - u_x v/c^2} = \frac{0.4c - 0.4c}{1 - 0.4c(0.4c)/c^2} = 0$$

and

$$u_y' = \frac{u_y}{\gamma(1 - u_x v/c^2)} = \frac{0.4\sqrt{3c}\sqrt{1 - v^2/c^2}}{1 - 0.4c(0.4c)/c^2} = \frac{0.4\sqrt{3c}\sqrt{1 - (0.4c)^2/c^2}}{1 - 0.16}$$

$$= \frac{0.4\sqrt{3}\sqrt{1 - 0.16}}{1 - 0.16}c = \frac{0.4\sqrt{3}\sqrt{0.84}}{0.84}c = 0.4\sqrt{3}/\sqrt{0.84} = 0.756c.$$

$\therefore$ resultant velocity u' as observed by a person in frame S' is given by

$$0i + 0.756\,cj = 0.756\,cj.$$

In other words, the particle will appear to the observer in frame S' to be moving with velocity 0.756c along the axis of y.

3.7.6 Relativistic Doppler Effect

We are already familiar with the *Doppler effect* in Sound, *viz.*, the apparent change in the frequency of sound due to motion of the source of sound and the listener relative to the medium. It bears the name of *Doppler* (an Austrian) who was the first to have worked out the theory in the case of sound waves in the year 1842*. He also called attention to the relevance of the phenomenon in the case of light.

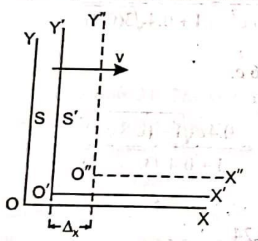

Fig. 3.10

The Doppler effect in light is, however, different from that in sound for the simple reason that, unlike sound, light requires no material medium for its propagation and hence its relative velocity is the same (c) for all observers regardless of their own state of motion. Then again, the velocity of light is so great that any apparent change in the frequency or the wavelength of a light pulse is appreciable only at relativistic speeds, *i.e.*, speeds near about c. Let us then proceed to discuss the case of light.

Imagine two reference frames S and S', where S is stationary and S' in relative motion with respect to S, with a constant velocity v along the axis of x, (Fig. 3.10).

Let two light signals or pulse be emitted from a source placed at the origin O in frame S at time $t = 0$ and $t = T$, where T is the *true time-period* of the light pulses. Suppose the interval between the reception of these pulses by an observer at the origin O' in frame S' is $\Delta t'$. Then, clearly, $\Delta t'$ is the *proper time interval* T' between the two pulses, as measured in frame S' (since they are both observed at the same point O' by an observer at rest with respect to S'). The observer will naturally interpret this interval as the time-period of the pulses received by him, *i.e.*, the apparent time-period of the pulses for the observer in frame S' will be T'.

Since the observer continues to be at O' all the time, the distance $\Delta x'$ covered by him [in frame] S' during the reception of the two pulses is *zero*. Let us obtain the values Δx and Δt in frame [S which] correspond to those of $\Delta x'$ and $\Delta t'$ in frame S'.

* It is interesting to recall that the theory was first tested in Holland, in the year 1845, by musically trained [musicians] who estimated by the car the apparent change in pitch (or frequency) as trumpeters rode ahead in a railroad fla[t car and] again when the two exchanged their positions.

From Inverse Lorentz transformation equation $x = \dfrac{x'+vt'}{\sqrt{1-v^2/c^2}}$

we have

$$\Delta x = \frac{\Delta x'+v\Delta t'}{\sqrt{1-v^2/c^2}},\ \text{because the time-interval here is } \Delta t'.$$

Since $\qquad \Delta x' = 0$, we have $\Delta x = \dfrac{v\Delta t'}{\sqrt{1-v^2/c^2}} = \dfrac{vT'}{\sqrt{1-v^2/c^2}} \qquad [\because \Delta t' = T'.$

Clearly, the second pulse has to cover this much distance more than the first pulse in frame S, along the axis of x, to be able to reach the observer at the origin O', in the moving frame S'.

Similarly, from the Inverse Lorentz transformation equation

$$t = \frac{t'+v_x'/c^2}{\sqrt{1-v^2/c^2}},\quad \text{we have} \quad \Delta t = \frac{\Delta t'+v\Delta x'/c^2}{\sqrt{1-v^2/c^2}}$$

Or, since $\qquad \Delta x' = 0, \quad \Delta t = \dfrac{\Delta t'}{\sqrt{1-v^2/c^2}} = \dfrac{T'}{\sqrt{1-v^2/c^2}}$

This obviously includes both the actual time-period T of the pulses and the time taken ($\Delta x/c$) by the second pulse to cover the extra distance Δx in frame S.

So that, $\qquad \Delta t = T + \Delta x/c.$

Substituting the values of Δt and Δx obtained above, we therefore have

$$\frac{T'}{\sqrt{1-v^2/c^2}} = T + \frac{vT'}{c\sqrt{1-v^2/c^2}},$$

whence, $\qquad T = \dfrac{T'}{\sqrt{1-v^2/c^2}}\left(1-\dfrac{v}{c}\right)$

If ν and ν′ be the *actual* and the *observed* (or *apparent*) frequencies of the light pulses respectively (*i.e.*, their frequencies in frames S and S'), we have $\nu = 1/T$ and $\nu' = 1/T'$. So that,

$$\frac{1}{\nu} = \frac{(1-v/c)}{\nu-\sqrt{1-v^2/c^2}}.\quad \text{Or,}\quad \nu' = \nu-\frac{(1-v/c)}{\sqrt{1-v^2/c^2}}$$

$$= \nu\frac{(1-v/c)}{\sqrt{(1-v/c)(1+v/c)}}$$

Or, $$\nu' = \nu\sqrt{\frac{1-v/c}{1+v/c}} = \nu\sqrt{\frac{1-\beta}{1+\beta}} \qquad \text{...I } [\because v/c = \beta].$$

This is known as *Doppler's formula* and gives the apparent frequency ν′ of the light pulses of actual frequency ν as observed in frame S'.

Since the velocity of light $c = \nu\lambda = \nu'\lambda'$, we have $\nu = c/\lambda$ and $\nu' = c/\lambda'$, where λ and λ′ are the *wavelengths* of the light pulses corresponding to frequencies ν and ν′ respectively. From relation I, therefore, we have

$$\frac{c}{\lambda'} = \frac{c}{\lambda}\sqrt{\frac{1-v/c}{1+v/c}} = \frac{c}{\lambda}\sqrt{\frac{1-\beta}{1+\beta}}.\quad \text{Or, } \lambda' = \lambda\sqrt{\frac{1+v/c}{1-v/c}} = \lambda\sqrt{\frac{1+\beta}{1-\beta}}, \quad \text{...II}$$

which gives the *apparent* wavelength λ′ of the light pulses, of actual wavelength λ, as observed in frame S'.

In both relations I and II, v is positive or negative according as the observer is receding from, or approaching, the source of light; so that, in the former case, the apparent frequency of the pulse (ν') decreases and its apparent wavelength (λ') increases and in the latter case, the exact opposite happens, *i.e.*, the apparent frequency increases and the apparent wavelength decreases.

N.B. It will be readily seen that if we ignore the second and higher order terms in v and c, we obtain, from I above, the relation $\nu' = \nu(1 - v/c)$ for the apparent frequency. This is a case of *non-relativistic Doppler effect*, the same as in the case of sound.

Now, what we have discussed above is, in point of fact, the *longitudinal Doppler effect*. There is also a *transverse Doppler effect* in the case of light, though it has no non-relativistic counterpart in the case of sound. This effect relates to the frequency observed in a direction at right angles to that of the motion of the source, usually an atom. The apparent frequency is here given by $\nu' = \nu\sqrt{1-v^2/c^2}$, where ν is the actual frequency of the light pulse emitted by an atom in the frame of reference in which it is at rest and ν', the apparent frequency as noted in a frame of reference moving relative to it with velocity v.

Experimental confirmation of Doppler effect. The fact that there is indeed an apparent change in the frequency or the wavelength of light due to Doppler effect has been fully confirmed experimentally by *Ives* and *Stilwell* in the year 1941.

They measured spectroscopically the change of shift in the average wavelength of a particular spectral line emitted by the hydrogen atom. This they succeeded in doing by using a beam of hydrogen atoms in a highly excited state. With the help of a very strong electric field, they accelerated the atoms as molecular (H_2^+) ions and as H_3^+ *ions* to a velocity of about 0.005 c and then examined the spectrum of the atomic hydrogen, produced as a break-up product of these ions.

One particular line in the spectrum was closely studied and its wavelength obtained both in the foreward and backward direction, *i.e.*, in the direction of motion of the atoms as well as in the opposite direction – the former by receiving the spectrum directly on the slit of the spectrometer and the latter, by reflecting the spectrum on to the slit with the help of a plane mirror. Also, the wavelength of the same spectral line was obtained for such of the hydrogen atoms as were at rest, of which there will always be quite a few.

... for the hydrogen atoms at rest be λ_0 and its

Since, the shift in wavelength $(\lambda_{av} - \lambda_0)$, here, depends upon v^2, it is quite independent of the sign of v, *i.e.*, is in the same direction, whether v is positive or negative.

As calculated by this relation, *Ives* and *Stilwell* obtained the value 0.072 Angstrom unit or 0.072×10^{-8} cm, whereas their observed value was found to be 0.074 *Angstrom unit* or 0.074×10^{8} cm.

The excellent agreement between the two values thus confirms the validity of relativistic Doppler effect, as expected.

The recessional red shift. Every element shows its own characteristic lines of definite wavelengths in its spectrum. The spectral lines of all known elements have, therefore, been carefully mapped out to enable us to see at a glance the position and wavelength of the spectral line or lines emitted by any given element.

Now, in the spectrum of the light received from distant stars and galaxies, the characteristic spectral lines of the various known elements present in them are all found to be shifted by various amounts from their normal positions towards the lower frequency or the high wavelength side, namely, the red end of the spectrum.

This shift is attributable to *Doppler effect*, according to which $\lambda' = \lambda\sqrt{(1+\beta)/(1-\beta)} = \lambda\sqrt{(1+v/c)/(1-v/c)}$; so that, for a positive value of velocity v of the source (*i.e.*, with the source receding from the observer), there is an apparent increase, and for a negative value of v (*i.e.*, with the source approaching the observer) there is an apparent decrease, in the wavelength of light received by the observer. In the former case, therefore, there is an apparent shift of the wavelength towards the red end, and in the latter case, towards the violet end of the spectrum. In fact, the velocity of recession or approach of the source may be easily determined from this shift in the wavelength of a spectral line.

The fact that light from the distant stars and galaxies shows this shift of the spectral line towards the red means that they are receding from us. Thus, since the shift of the spectral lines towards the red is due to the recession of the stars and galaxies from us on the earth, the phenomenon is called *recessional red shift* and lends strong support to the theory that the universe is expanding, with all bodies receding from one another. The Doppler effect has thus been instrumental in the detection, for the first time, of the apparent expansion of the universe.

On the basis of a large number of observations made on several galaxies, it is surmised that the relative velocity of a galaxy, at a distance r from us, is given by

$$V = \alpha r,$$

where α is an empirical constant, whose value is found to be near about 3×10^{-18} sec^{-1}.

Interestingly enough, *the reciprocal* of α, *i.e.*, $1/\alpha$, which has the dimensions of time and which is equal to $1/3 \times 10^{-18} = 3 \times 10^{17}$ sec $= 10^{10}$ years, is spoken of as the age of the universe and its product with c, *i.e.*, c/α, which has the dimensions of length or distance and which is equal to $3 \times 10^{17} \times 3 \times 10^{10}$ or 10^{28} years, is similarly spoken of as the *radius of the universe*, though no one yet seems to know for certain what the significance of these terms actually is.

Example 34. A motorist goes through a red light and, when challenged, claims that the colour he actually saw was green ($\lambda = 5.4 \times 10^{-7}$ m) and not red ($\lambda = 6.2 \times 10^{-7}$ m) because of the Doppler effect. What should have been his speed for his claim to be true?

We know that if λ' be the *apparent* or *observed wavelength* of light, λ, the *actual wavelength* and v, the relative speed of the observer with respect to the source of light, we have

$$\lambda' = \lambda\sqrt{\frac{1+\beta}{1-\beta}} = \lambda\sqrt{\frac{1+v/c}{1-v/c}}. \qquad \text{[See 3.7.6]}$$

Since the observer is here approaching the source of light, v is negative

We, therefore, have $$\lambda' = \lambda\sqrt{\frac{1-v/c}{1+v/c}}$$

Or, $\lambda'^2 = \lambda^2\left(\frac{1-v/c}{1+v/c}\right) = \lambda^2(1-v/c)(1+v/c)^{-1} = \lambda^2(1-v/c)^2$, neglecting higher powers of v/c.

Or, $\lambda' = \lambda(1-v/c)$. Or, $\lambda - \lambda' = \frac{v}{c}\lambda$.

Or, substituting the given values, we have

$$(6.2-5.4)\times 10^{-7} = \frac{v(6.2\times 10^{-7})}{3\times 10^8}. \text{ Or, } 0.8\times 3\times 10^8 = 6.2v$$

whence, $$v = \frac{0.8\times 3\times 10^8}{6.2} = \frac{2.4}{6.2}\times 10^8 \text{ m/sec} = \frac{2.4\times 10^8}{6.2\times 1000} = 3871 \text{ km/sec.}$$

Thus, *for the absurd claim of the motorist to be true, his speed should have been* 3871 km/sec.

Example 35. Protons are accelerated through a potential of 20 kV, after which they drift with constant velocity through a region where neutralisation to H atoms and associated light emission takes place. The H emission (λ = 4860.33 *A* for an atom at rest) is observed in the spectrometer. The optical axis of the spectrometer is parallel to the motion of the ions. The spectrum is Doppler-shifted because of the motion of the ions in the direction of observed emission. The apparatus also contains a miror which is palced so as to allow superposition of the spectrum of light emitted in the reverse direction. (1*A* = 10^{-8} cm).

Calculate (*i*) the first order Doppler shift, depending on v/c appropriate to the forward and the backward directions and indicate the appearance of the relevant part of the spectrum on a diagram, (*ii*) the second order (*i.e.*, v^2/c^2) effect.

Solution. In plain and simple language, it simply means that as the high velocity protons get neutralised into hydrogen atoms, the characteristic H_β line of the hydrogen spectrum is emitted. Its position is noted in the spectrometer both when the ions are moving towards and (with the help of the mirror) away from the spectrometer. We are asked (*i*) to determinte the mean *Dopper shift*, using the first power alone of v/c and to represent it on a proper diagram and, (*ii*) to obtain the *Doppler shift* corresponding to the second power of v/c.

(*i*) We have the relation $\lambda' = \lambda\sqrt{\frac{1+\beta}{1-\beta}}$, where λ' is the *observed wavelength* and λ, the *actual wavelenth* emitted and $\beta = v/c$.

So that, $$\lambda' = \lambda(1+\beta)^{\frac{1}{2}}(1-\beta)^{-\frac{1}{2}} = \lambda\left(1+\frac{1}{2}\beta-\frac{1}{2}\beta^2+...\right)\left(1+\frac{1}{2}\beta+\frac{3}{2}\beta^2+...\right)$$

$$= \lambda\left(1+\frac{1}{2}\beta+\frac{3}{8}\beta^2+\frac{1}{2}\beta+\frac{1}{4}\beta^2-\frac{1}{3}\beta^2+...\right)$$

Or, $$\lambda' = \lambda\left(1+\beta+\frac{1}{2}\beta^2+...\right)$$

$\therefore$ restricting ourselves to only the first order terms in β, we have

$$\lambda' - \lambda = \lambda\beta = \lambda v/c.$$

Now, to obtain the value of v, we note that the protons have been accelerated throug[h] of 20 *kV*. If $\therefore$ m_p be the mass of a proton and v, the velocity acquired by it, we have

$$\frac{1}{2}m_p v^2 = 20{,}000\ eV = 20{,}000\times 1.6\times 10^{-12} \text{ ergs, (because } 1\ eV = 1.60\times 10^{-12}\ e[rg]$$

Since $m_p = 1.67\times 10^{-24}$ gm, we have $v^2 = \frac{2\times 20{,}000\times 1.6\times 10^{-12}}{1.67\times 10^{-24}}$

whence, $$v = \sqrt{\frac{2\times 20{,}000\times 1.6\times 10^{-12}}{1.67\times 10^{-24}}} = 1.957\times 10^8 = 2\times 10^8 \text{ cm/sec.}$$

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When the ions are moving forward, towards the spectrometer, v is *negative*; so that,

for foward movement of the ions, $(\lambda' - \lambda) = \dfrac{-\lambda v}{c} = \dfrac{-4861.33 \times 2 \times 10^8}{3 \times 10^{10}} = -32.4$ A.

and *for backward movement of the ions, v being positive, we have*

$$(\lambda' - \lambda) = \frac{\lambda v}{c} = 32.4\ A.$$

Hence in *the first case* (*i.e.*, forward movement), *the wavelenth decreases to* 4861.33 – 32.4 = 4828.93 A and *in the latter case* (*i.e., backward movement*), the wavelength increases to 4861.33 + 32.4 = 4893.73 A. This may be represented diagrammatically, as shown in Fig. 3.10

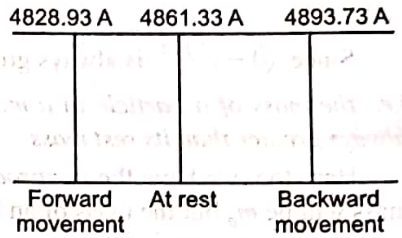

Fig. 3.11

(*ii*) In terms of the second power of β or v/c (or form 3.7.6) we have

$$\lambda' - \lambda = \frac{1}{2}\lambda\beta^2$$

$$= \frac{\frac{1}{2}(4861.33)\,(2 \times 10^8)^2}{(3 \times 10^{10})^2} = 0.108\ \text{A}.$$

Thus, *whether the movement of the ions be forward or backward, there will be an increase in the observed wavelength of the spectral line equal to* 0.108 *A* or 0.108 × 10^8 = 1.08 × 10^7 cm.

3.7.7 The Relativity of Mass—Conservation of Mementum—Force.

The mass of a body too, like length (or space) and time, is dependent on the motion of the body. Its variation with velocity with velocity is, in fact, a direct consequence of the dilation of time, as will be clear from the following.

We know that for all inertial fremes, in relative motion along the axis of x, the displacements Δy and Δz of a particle along the axes of y and z respectively remain unaffected whatever the value of the relative velocity v of the moving reference frame with respect to the stationary one.

The time taken by the particle to traverse this displacement Δy, and hence also its velocity component along the y-axis will, however, depend upon the frame of reference in question (since neither time nor velocity is invariant to Lorentz transformation.)

But, again, the *proper time* $\Delta\tau$ (at noted on a clock carried by the moving frame itself) to cover the displacement Δy will, as we know, be the same in either reference frame, and so will, therefore, be the quantity $\Delta y/\Delta\tau$.

So that, if we take $\Delta y/\Delta\tau$ to be the velocity component of the particle along the axis of y in the moving frame and m_o, its *rest mass* of *proper mass, i.e.*, its mass as taken by an observer at rest with respect to it (*i.e.*, by the observer in the moving frame, in this case), we have *y-component of the momentum of the particle in the moving frame, i.e.*,

$$p_y = m_o \Delta y/d\tau.$$

Now, if Δt be the time taken to cover the displacement Δy, as noted on an identical clock by an observer in the stationary frame, we have by Lorentz transformation, $\Delta\tau = \Delta t\sqrt{1 - v^2/c^2}$, whence,

$$\Delta t/\Delta\tau = 1\big/\sqrt{1 - v^2/c^2}$$

Thus, y-component of the velocity of the particle, as measured in the stationary frame, is $v_y = \Delta y/\Delta t$ and, therefore,

y-component of the momentum of the particle in the stationary fame = mv_y, where m is the *mass of the moving particle as taken by the observer in the stationary frame.*

Since, despite relativistic effects, the basic laws of physics in all frames of reference must have the same mathematical forms, *the low of conservation of momentum must hold good in either frame.* We, therefore, have

$$p_y = mv_y = m_o \frac{\Delta y}{\Delta \tau} = \frac{m_o \Delta y}{dt} \frac{dt}{d\tau} = m_o \frac{\Delta y}{dt} \frac{dt}{d\tau} = m_o v_y \frac{dt}{d\tau},$$

whence, $m = m_o dt/d\tau$.

or, substituting the value of $dt/d\tau$ from above, we have

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}}.$$

Since $\sqrt{1 - v^2/c^2}$ is always greater than, 1, $m > m_o$,

i.e., the mass of a particle in a moving frame, as taken by an observer in a stationary frame, is always greater than its rest mass.

Here too, we have the *reciprocity effect*; so that, for an observer moving with the particle, its mass will be m_o but the mass of an identical particle in the stationary frame will appear to be

$$m = m_o / \sqrt{1 - v^2/c^2}.$$

Alternative, we may arrive at the same result as follows:

Suppose we have a reference frame Σ at rest (Fig. 3.12), in which two identical particles *A* and *B*, (two *protons*, say) are made to under go a symmetrical, glancing collision, moving with equal and opposite velocities, as shown in Fig. 3.12(*a*), such that a line (shown dotted) parallel to the *x* axis bisects the angle between their trajectories.

If the collision be a perfectly elastic one, the *x*-components of the velocities of the two particles remain the same respectively, in magnitude and direction, both before and after the collision and since the particles have equal masses, there is no change of momentum in the *x*-direction. Again, because their *y*-components, after collision are simply reversed in direction, their magnitude remaining unchanged, the change in momentum of particle *A* is equal and opposite to the change in momentum of particle *B* (their initial *y*-components being oppositely directly), so that *the change in momentum along the y-direction too is zero and the momentum of the system thus remains conserved* in accordance with Newtonian physics, under which the masses of the particles remain unaffected by their velocities.

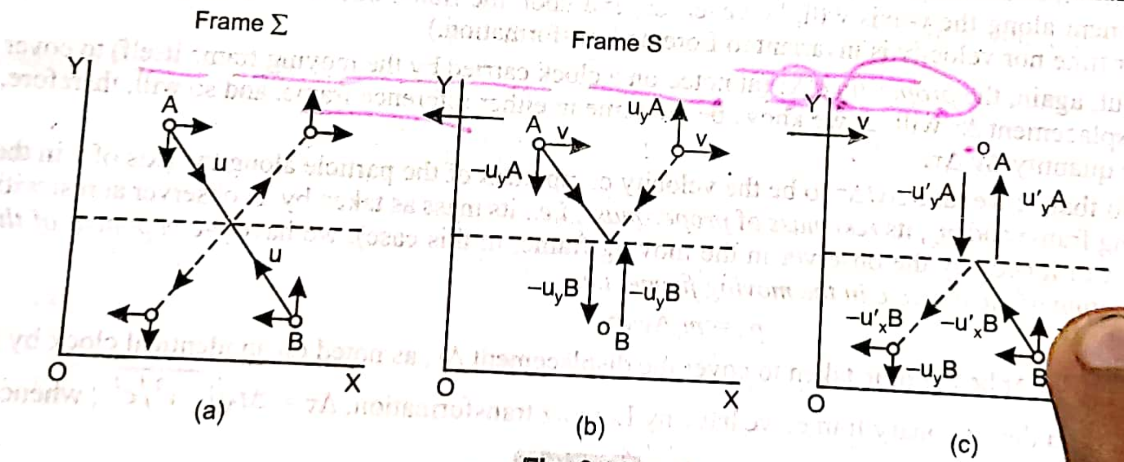

Fig. 3.12

Imagine now that we observe this collision from a frame of reference *S*, moving relative stationary frame Σ with a velocity *equal to the horizontal or the x-component of particle B*; s *B* has now a vertical velocity u_yB, say, its horizontal or *x*-component having been reduced to It, therefore, appears, to move up with velocity u_yB and bounce back after collision with particl with its velocity reversed, as shown in Fig. 3.12 (*b*). Particle *A*, however, has a horizontal compon of velocity *v*, say, and a vertical component v_yA, the former remaining unchanged in magnitude a direction and the latter getting reversed in direction after collision.

... we observe the collision from a frame S' moving with velocity v (equal to that of article A in frame S), relative to frame S, we find that, instead of particle B, particle A now loses s horizontal or x-component of velocity and merely moves down with velocity $-u'_yA$, say, and ounces back after collision with particle B, with its velocity reversed, as shown in Fig. 3.12 (*c*).

In view of the initial symmetry of the glancing collision, the situations in frames S and S' also aturally symmetrical, with particles A and B interchanging their roles. We, therefore, have

$$u'_yA = u_yB. \quad ...(i)$$

In accordance with Lorentz transformation,

$$u_yA = \frac{u'_yA}{\gamma(1 + vu'_xA/c^2)}, \quad \text{[See 3.7.5]}$$

where u'_xA is the horizontal or x-component of A in frame S', which is *zero*, in this case.

So that, $$u_yA = u'_yA/\gamma = u_yB/\gamma \quad \text{[From relation I, above.}$$

Or, $$u_yA = u_yB/\gamma = u_yB\sqrt{1 - v^2/c^2} \quad [\because \gamma = 1/\sqrt{1 - v^2/c^2}$$

showing that the vertical components of the velocities of particles A and B, which are equal in magnitude in the stationary frame Σ, are no longer so in the moving frame. S.

Considering the position in frame S, we find that the x-component of particle A remaining unchanged in magnitude and direction after collision, and particle B having *zero* x-component, *there is no change of momentum along the x-direction,* the only change thus occurring along the y-direction.

Clearly *change in the vertical velocity-component of particle A, on collision*

$$= u_yA - (-u_yA) = 2u_yA,$$

and change in the vertical velocity-component of particle B,

$$= -u_yB - u_yB = -2u_yB.$$

If, therefore, m_o be the mass of each particle (supposed invariant, in accordance with Newtonian physics), we have

change in momentum of the system $= 2m_ou_yA - 2m_ou_yB$,

which is obviously *not equal to zero*, since $u_yA \neq u_yB$, as we have seen above.

In order, therefore, that the low of conservation of momentum should hold, as it must, we must have the masses of particles A and B in the moving frame S different from their *Newtonian or rest masses* in the stationary frame Σ.

Let these be m_A and m_B respectively.

Then, for the law of conservation of momentum to hold, we must have

$$2m_Au_yA - 2m_Bu_yB = 0. \quad \text{Or,} \quad m_Au_yA = m_Bu_yB.$$

Or, Since $u_yA = u_yB/\gamma$ and, therefore, $u_yB = \gamma u_yA$, we have

Or, $$m_Au_yA = \gamma m_Bu_yA, \text{ whence, } m_A = \gamma m_B.$$

Or, $$m_A = \frac{m_B}{\sqrt{1 - v^2/c^2}}.$$

Now, if in frame S the vertical components u_yA and u_yB be small compared with the horizontal velocity-component v of particle A, we practically have particle B at rest particle A moving relative to it with velocity v. Taking the rest mass of particle, therefore, to be m_oB, we have

$$m_A = \frac{m_oB}{\sqrt{1 - v^2/c^2}}.$$

But, as we know, the two particle being *identical* in every respect, $m_oB = m_oA$, and, therefore,

$$m_A = \frac{m_o A}{\sqrt{1 - v^2/c^2}}.$$

or, representing the rest mass (or inertial mass) of a body, in general, by m_o and its mass (or, *relativistic mass*), when moving with velocity v, by m, we have

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}} = \gamma m_o ,$$

which is the *mass-velocity relation of the Special theory of relativity,* indicating that *mass is no longer the absolute, invariant quantity of Classical physics but depends upon the velocity with which it is moving.*

As in the case of *length* and *time*, so also here, however, this relativistic effect in appreciable only at speeds somewhat comparable with that of light, *e.g.*, those of atomic particles. In case v is small, as it normally is, so that $v/c \to 0$, we have $m = m_o$, *i.e.*, the mass of the body is then the same as its rest mass, a result in accord with the Classical physics.

The first experimental confirmation of the relativity of mass came from *Bucherer*, in the year 1908, when he showed that the value of e/m (*ratio of charge to mass*) for fast moving electrons was smaller than for slower moving ones, due obviously to a higher value of m in the former case, the value of e being always invariant.

Again, to deflect high speed electrons in a synchroton, the magnetic field required is 2000 times as strong as the one that would serve the purpose on the basis of Newton's laws, indicating clearly that the mass of the electrons has, in the synchroton, increased to 2000 times their normal mass, or, in other words, an electron now has the mass of a proton!

The relativistic relationship for mass, together with those for length and time stand fully confirmed and accepted as important basic formulations in atomic physics.

It will be seen from the above discussion that Newton's law of conservation of momentum, $p = m_o v$, where m_o is the rest mass of the body, supposed invariant in all frames of reference, is not invariant to Lorentz transformation and cannot therefore, be acceptable as the correct law at relativistic velocities. What remains constant at these high velocities and is invariant to Lorentz transformation is $\gamma m_o v$ or $m_o v(1 - v^2/c^2)^{-1/2}$. So that, the relativistic law of conservation of momentum is

$$p = \gamma m_o v.$$

At non-relativistic velocities, $v/c \to 0$ and therefore, $\gamma = 1$, so that we have $p = m_o v$, the Newtonian law of conservation of momentum, *i.e.*, at smaller velocities compared with c, the relativistic law of conservation of momentum reduces to the newtonian law. The relativistic law is, therefore, accepted now as the *correct* law of conservation of momentum, invariant in all reference frames, the Newtonian law being a particular case of it.

For most of our ordinary purposes, however, $v << c$ and hence the Newtonian law is good enough. It is only when we deal with high speed particles, like electrons, protons, mesons, etc., that we have to invoke the help of the relativistic law.

If we were to plot the value of momentum p of a particle against velocity v, we should, according to the Newtonian or the Classical theory, obtain a straight line passing through the origin and the momentum should go on increasing with v even if v exceeds c, as shown by the dotted curve in Fig. 3.13.

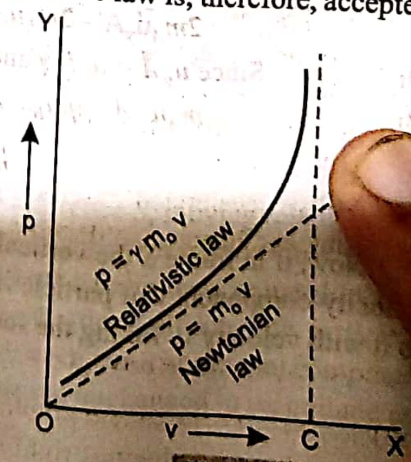

Fig. 3.13

On the other hand, if we plot p against v in accordance with the relativistic law of conservation of momentum, we obtain the full line curve shown, which shows that as $v \to c, p \to \infty$, because the mass then tends to infinity (see next article).

Force. In classical mechanics, as we know, force is defined as the *rate of change of momentum, i.e.*, $F = dp/dt$. This applies equally well to relativistic mechanics. Since *mass* here means relativistic mass, the expression for force, in terms of acceleration works out to be different from the Classical expression $F = ma = mdv/dt$ *in the case of* **linear motion,** *For,*

$$F = dp/dt = \frac{d}{dt}(mv) = \frac{d}{dt}\left(\frac{m_o v}{\sqrt{1-v^2/c^2}}\right) \quad ...(i)$$

Now, in this type of motion, *only the magnitude of the velocity changes and not its direction.* We may, therefore, put the above relation for force in the form

$$F = \frac{d}{dt}\left(\frac{m_o v}{(1-v^2/c^2)^{1/2}}\right) = \frac{m_o(dv/dt)}{(1-v^2/c^2)^{3/2}} \quad ...(ii)$$

Or, since $m_o/(1 - v^2/c^2)^{1/2} = m$, we have

$$F = \frac{m}{(1-v^2/c^2)}\frac{dv}{dt},$$

where dv/dt is, obviously, the rate of change of velocity or the acceleration of the body.

This, it will readily be seen, is different from the Classical relation $F = ma = mdv/dt$.

In the case of **circular motion,** however, the direction of the velocity of the body changes continuously, its magnitude remaining constant. So that, expression I above takes the form

$$F = \frac{m_o}{\sqrt{1-v^2/c^2}} \cdot \frac{dv}{dt} = mdv/dt,$$

where, as we know, dv/dt is the centripetal acceleration (v^2/r), normal to the circular path (of radius r) of the body.

$\therefore$ magnitude of the force is given by

$$F = \frac{m_o}{\sqrt{1-v^2/c^2}} \cdot \frac{v^2}{r} = \frac{mv^2}{r}. \text{ Or, } F = ma,$$

the same relations as in the case of Classical mechanics.

Thus, in the case of circular motion, the Relativistics formula for force is identical with the Classical one ($F = ma$), if only we use relativistic mass m in place of the classical mass.

3.8 ULTIMATE SPEED OF A MATERIAL PARTICLE

In classical mechanics ...

from the value of the rest or inertial mass m_o is quite in-appreciable. Beyond this, however, the mass increases relatively more rapidly and tends *to become infinite as* $v \to c$. As will also be noted from the graph in Fig. 3.13 above, whereas momentum goes on increasing the velocity remains constant indicating that the accelerating force applied only increases the mass and hence the momentum but not velocity.

The idea of infinity mass, however, makes no sense for a number of reasons. Thus, for example (i) it would require an infinity force to accelerate the particle to the speed of light c, (ii) due to Lorentz-Fitzgerald contraction, its length in its direction of motion must be zero, (iii) its volume too, must, therefore, be zero, and (iv) it must exert an infinite gravitational pull on all other bodies in the universe etc., etc.

So that, discarding the notion of an infinite mass, we interpret the above relationship to mean that *no material body can equal or exceed the speed of light in free space, viz.,* 3×10^{10} cm or 3×10^8 m/sec.

Now, it may perhaps be cited in apparent refutation of this statement, that a *photon*, in free space, travels with the free-space speed *c* of light. So, of course, it does, but then a photon is not a material particle since it possesses no rest mass. Also, there are, what are called phase velocities which may even exceed the free-space speed *c* of light, but these are, again, no material particles but only abstract mathematical functions. Finally some of the atomic particles may travel in media like air, glass or water with velocities higher than that of light *in those media* but *by no means higher than its speed in free space.*

Thus, we come to the conclusion that *the ultimate speed for any material particle is the speed of light in free-space, which it can never attain or exceed.*

3.9 EQUIVALENCE OF MASS AND ENERGY

The correct way to state the second law of motion, in accordance with the Special theory of relativity, is to define force as the *time rate of change of momentum, i.e.,* as $F = \frac{d}{dt}(mv)$, since not only does it reduce to the classical second law of motion at low values of velocity but also, taken together with the third law, ensures the validity of the law of conservation of momentum in a closed system even under relativistic conditions.

This is obviously not the same definition as $F = ma$, since

$$\frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = ma + v\frac{dm}{dt},$$

and $dm/dt \neq 0$ if v varies with time.

Now, in mechanics (Newtonian as well as Relativistic), the kinetic energy of a moving body at a given velocity v is equal to the work done in making the body move from rest and attain this velocity. Thus, if F be the component of the force applied to a body in the direction of displacement, the work done or kinetic energy imparted to the body when it is displaced through a distance dS, *i.e.*, $dE_k = F.dS$. Therefore, work done or kinetic energy imparted to it during the whole displacement S until it acquires velocity v is given by

$$K.E. = E_k = \int dE_k = \int_0^S F.ds = \int_0^S \frac{d}{dt}(mv)dS,$$

Now $\quad \frac{d}{dt}(mv)ds = \frac{dS}{dt}d(mv) = vd(mv) \quad [\because dS/dt$

Since the particle starts from rest (*i.e.*, $v = 0$) and finally acquires velocity v, we have

$$K.E. = E_k = \int_0^v vd(mv) = \int_0^v vd\left(\frac{m_o v}{\sqrt{1-v^2/c^2}}\right) \quad \left[\because m = \frac{m_o}{\sqrt{1-v^2/c^2}}\right]$$

Integrating this by part*, we have

$$E_k = \frac{m_o v^2}{\sqrt{1-v^2/c^2}} - m_o \int_0^v \frac{v\,dv}{\sqrt{1-v^2/c^2}}$$

$$= \frac{m_o v^2}{\sqrt{1-v^2/c^2}} + m_o c^2 \left[\sqrt{1-v^2/c^2}\right]_0^v$$

Or,

$$E_k = \frac{m_o c^2}{\sqrt{1-v^2/c^2}} - m_o c^2 = mc^2 - m_o c^2 = (m - m_o)c^2 = \Delta mc^2,$$

where Δm is the relativistic increase in mass with increase in velocity.

Thus, the K.E. of the body is the *product of the increase in its mass and* c^2.

Or, rewriting the equation, we have

$$mc^2 = m_o c^2 + E_k = \frac{m_o c^2}{\sqrt{1-v^2/c^2}} = \gamma m_o c^2. \quad ...(i)$$

Here, $m_o c^2$, the energy due to the rest mass of the body, *i.e.*, its energy when at rest with respect to the observer, is called its rest energy or proper energy E_o** and mc^2, the *total energy E* possessed by the body. So that, we have

total energy $E(= mc^2)$ = rest energy $(= m_o c^2)$ + *kinetic energy* $(E_k)\gamma = m_o c^2$.

This relation may also be obtained straight a way by Binomial expansion of the mass-velocity relation for values of $v/c \to 0$. Thus,

$$m = \frac{m_o}{(1-v^2/c^2)^{1/2}} = m_o(1 - v^2/c^2)^{-1/2}$$

$$= m_o\left(1 + \frac{1}{2}\frac{v^2}{c^2} - ...\right). \text{ Or, } m = m_o + \frac{1}{2} m_o v^2/c^2, \quad ...(ii)$$

whence, $mc^2 = m_o c^2 + \frac{1}{2} mv^2$. Or, $mc^2 = m_o c^2 + E_k = \frac{m_o c^2}{\sqrt{1-v^2/c^2}} = \gamma m_o c^2$.

And $\therefore$ therefore, kinetic energy is given by the relation,

$$E_k = (m - m_o)c^2 = \Delta mc^2.$$

This relativistic expression for *K.E.*, or this *mass-energy equation*, referred to as **Einstein's mass-energy relation**, is by far the most famous of the equations deduced by *Einstein* and stands fully verified and confirmed by experiment.

It will thus be noted that the classical expression $\frac{1}{2}mv^2$ for *K.E.* does give the true K.E. of the body even if we substitute the relativistic values of m and v in it.

On the other hand, the expression $E_k = mc^2 - m_oc^2$ reduces to the classical expression $E_k = \frac{1}{2}mv^2$ for values of $v << c$. For, we have

$$E_k = \frac{m_o}{\sqrt{1-v^2/c^2}} - m_oc^2 = m_oc^2\,(1 - v^2c^2)^{-1/2} - m_oc^2$$

$$= m_oc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} \ldots\ldots\right) - m_oc^2$$

$$= \frac{1}{2}m_ov^2, \quad \begin{cases}\text{The higher terms in } v/c \text{ being} \\ \text{neglible when } v/c << 1.\end{cases}$$

a result in conformity with classical physics.

We thus see that the relativistic expressions for length, *time, mass* and *energy,* all obey the same corresponding principle, *viz.,* that they get reduced to their classical forms for small values of v when $v/c \to 0$. This shows that the former are the really true expressions and the latter, merely approximations to them, valid only when v is small. Since, however, we seldom some across values of v comparable with c, except in atomic physics, the classical expressions, remain true enough for our manifold practical purposes.

Importance of mass-energy equations. It will be seen from relation (*ii*) above *viz.,* $m = m_o + \frac{1}{2}m_ov^2/c^2$, that the total inertial mass of a particle moving with velocity v relative to the observer, is the sum of (*i*) its *rest-mass* m_o, and (*ii*) *an additional mass* (or *inertia*) *equal to* E_k/c^2.

This suggests at once that the addition of energy to a system increases the inertial mass of the system. In plain and simple terms, it means that *mass may appear as energy and energy as mass.* So that, the scope of the law of conservation of energy may be widened into what may be called the *law of conservation of mass-energy*, meaning thereby that, *in any interaction, it is neither mass alone nor energy alone* that *remains conserved but mass, inclusive of energy in terms of mass, of energy inclusive of mass in terms of energy, that is really conserved.*

Here are some illustrative examples:

(*i*) When a *positron* (an elementary particle, having the same mass as an electron and a charge equal and opposite to it, *i.e.,* $+ e$) and an *electron* come together, they *annihilate* one another and an equivalent amount of energy in the form of a pair of γ-*ray photons* (of energy at least 1.02 MeV) appears in their place.

The reverse of this phenomenon of annihilation of matter is that of pair production, where a *photon* of energy at least 1.02 *Mev* gives rise to a *pair of an electron and a positron* in the intense electric field near the nucleus of an atom. Here is what is called *materialisation of energy, i.e., conversion of energy into matter.*

Another example of pair production is the production of a *proton-antiproton pair,* re minimum of energy 1.08 Giga *electron volts.*

The laws of conservation of momentum, mass-energy and of charge hold good in b processes.

(*ii*) When a uranium nucleus is split up, the decrease in its total rest mass appears in th of an equivalent amount of kinetic energy of its fragments. Thus, suppose it is split up into fragments, moving with velocity v, so that the mass of each is no longer its rest mass m_0 but, say which is greater than m_0. The sum of the masses of the two fragments is, ... of the splitting nucleus, or which is greater than their rest mass 2 m_0. The kinetic energy liberated by the splitting nucleus, or the energy due to conversion of $(M - 2\,m_0)$ of matter is thus $E_k = (M - 2\,m_0)c^2$.

As can be easily seen, it is thus possible to calculate the amount of energy that is released due to *fission* in an atomic bomb, since both M and m_0 are known. Distressingly enough, it was for jus suggesting this method of calculating the energy released by an atomic bomb that Eeinstein had been, unthinkingly and uncharitably, dubbed the world over as the 'father of the atomic bomb'.

(*iii*) In the process of *fusion* of nuclei, the total mass of the fusing nuclei is somewhat in excess of the nuclei produced as a result thereof, the difference being converted into energy in accordance with the mass-energy relation. In fact, such conversion of matter is the main source of energy of the sun and the stars, which obtain their enormous energy chiefly from what is called the '*burning*' *of protons into helium gas, i.e., the fusion of protons to form helium atoms*. The change in mass during the formation of one helium atom is given by

$$\Delta m = (\textit{mass of 4 protons} + \textit{mass of 2 electrons}) - (\textit{mass of one helium atom}).$$

So that,

$$\Delta m = (4 \times 1.6725 \times 10^{-24} + 2 \times 0.911 \times 10^{-27}) - (6.647 \times 10^{-24})$$
$$\approx 0.045 \times 10^{-24} \text{ gm.}$$

Hence, *energy released per helium atom produced* $= \Delta mc^2$

$$= 0.045 \times 10^{-24} \times (3 \times 10^{10})^2 \approx 25 \text{ MeV} \qquad [\because 1 \text{ eV} = 1.6 \times 10^{12} \text{ ergs.}]$$

Again, when a number of particles, called nucleons, *i.e.*, protons and neutrons, combine to form a nucleus, there is a loss of mass Δm, *i.e.*, the mass of the nucleus is this much less than that of its constituent protons and neutrons. *The energy-equivalent of this mass-loss, i.e.*, $E = \Delta mc^2$ is called the **binding energy** of the nucleus.

It may be taken as a general rule that *an increase in mass means an increase in energy and vice versa*, though in most everyday cases, these changes in mass or energy are much too small to be discernable. Thus, a gas or a spring should, in accordance with the theory, record a greater mass in the compressed state, the increase in mass, obviously, being the work done to compress it divided by c^2. Similarly, a body when hot (and, therefore, with greater internal energy) should have a somewhat higher mass than when it is cold, and so on.

3.10 TRANSFORMATION OF RELATIVISTIC MOMENTUM AND ENERGY

Consider a particle of rest mass m_0, moving in a reference frame S. Its momentum in this frame is given by $\vec{P} = \gamma m_0\, d\vec{r}/dt$ and its energy by $E = \gamma m_0 c^2$, where $d\vec{r}/dt$ is its velocity.

As we know, an interval of time dt, as noted on a clock in the reference frame S is related to the *proper interval of time* $d\tau$, as noted on a clock carried by the particle itself, as $dt = d\tau/\sqrt{1 - v^2/c^2}$, whence $dt/d\tau = 1/\sqrt{1 - v^2/c^2} = \gamma$.

So that, if p_x, p_y and p_z be the components of the momentum of the particle along the three coordinate axes, we have

$$p_x = \gamma m_0 \frac{dx}{dt} = m_0 \frac{dx}{dt} \cdot \frac{dt}{d\tau} = m_0 \frac{dx}{d\tau} \text{ and, similarly, } p_y = m_0 \frac{dy}{d\tau}$$

$$p_z = m_0 \frac{dz}{d\tau} \text{ and } E = m_0 c^2 \frac{dt}{d\tau} \text{ or } \frac{E}{c^2} = m_0 \frac{dt}{d\tau}.$$

Since both m_0 and τ_0 are Lorentz invariants, the quantities p_x, p_y and p_z get transformed in exactly the same manner as x, y, z and t in a reference from S', moving with velocity v relative to S along the axis of x. And, we therefore have

$$p_x' = \gamma(p_x - vE/c^2),\ p_y' = p_y,\ p_z' = p_z$$

and $$E'/c^2 = \gamma\left(\frac{E}{c^2} - \frac{v}{c^2} p_x\right) \quad \text{or} \quad E' = \gamma(E - vp_x)$$

The inverse Lorentz transformation will thus obviously be

$$p_x = \gamma(p_x' + vE'/c^2), p_y = p_y', p_z = p_z' \text{ and } E = \gamma(E' + vp_x').$$

3.11 CONSERVATION OF RELATIVISTIC ENERGY

The relativistic energy, as we know, is given by the relation $E = mc^2 = \gamma m_0 c^2$, where the value of m, equal to γm_0, has been deduced on the validity of the law of conservation of relativistic momentum. It follows, therefore, that relativistic energy too is subject to a law of conservation similar to that of momentum. Let us, however, deduce the law in a more direct manner as explained below.

Imagine a reference frame S in which two given particles, with momenta $\vec{p}_1$ and $\vec{p}_2$ and energies E_1 and E_2 respectively, undergo a collision. If their respective momenta after the collision be $\vec{p}_3$ and $\vec{p}_4$ and their energies E_3 and E_4, we have, in accordance with the law of conservation of momentum,

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4.$$

Or, if we consider the x-components of their momenta only, we have

$$\vec{p}_{1x} + \vec{p}_{2x} = \vec{p}_{3x} + \vec{p}_{4x}$$

In another frame S' moving with velocity v relative to S along the *axis of x*, the components of the initial and final momenta of the particles along the x'-*axis* will respectively be $p_{1x}', p_{2x}', p_{3x}'$ and p_{4x}' and their energies before and after the collision will be E_1, E_2 and E_1', E_2'.

Since the law of conservation of momentum (being a Lorentz invariant) holds good in this frame too, we have $p_{1x}' + p_{2x}' = p_{3x}' + p_{4x}'$

or, $\gamma(p_{1x} - vE_1/c^2) + \gamma(p_{2x} - vE_2/c^2) = \gamma(p_{3x} - vE_3/c^2) + \gamma(p_{4x} - vE_4/c^2)$.

or, $$(p_{1x} + p_{2x}) - \frac{v}{c^2}.(E_1 + E_2) = (p_{3x} + p_{4x}) - \frac{v}{c^2}(E_3 + E_4).$$

Since, in accordance with the law of conservation of momentum,

$$(p_{1x} + p_{2x}) = (p_{3x} + p_{4x}),$$

we have $$E_1 + E_2 = E_3 + E_4,$$

And since $\gamma = 1/\sqrt{1-v^2/c^2}$, we have $\gamma^2 = 1/(1-v^2/c^2) = c^2/(c^2-v^2)$,

hence, $$\gamma^2v^2 = \gamma^2c^2 - c^2.$$

Substituting this value of γ^2v^2 in the expression for p^2 above, we obtain $p^2 = (\gamma^2c^2 - c^2)m_0^2$, multiplying which by c^2 we get

$$p^2c^2 = m_0^2\gamma^2c^4 - m_0^2c^4. \quad \text{Or,} \quad m_0^2\gamma^2c^4 = p^2c^2 + m_0^2c^4.$$

Now $\gamma m_0c^2 = E$, the total energy of the particle in question. So that,

$$E^2 = p^2c^2 + m_0^2c^4. \quad \text{Or, } E^2 - p^2c^2 = m_0^2c^4,$$

an expression connecting relativistic energy and momentum.

Since m_0 and c are both Lorentz invariants, $m_0^2c^4$ is a constant and therefore, the quantity $E^2 - p^2c^2$ too is a *Lorentz invariant* and its value remains unaltered in any inertial frame of reference.

3.13 MASS VELOCITY, MOMENTUM AND ENERGY OF PARTICLES OF ZERO REST MASS

We have just seen above that *the energy of a particle of rest mass m_0 and momentum p*, is given by

$$E = (p^2c^2 + m_0^2c^4)^{1/2}$$

If, therefore, we have a particle of rest mass zero, like a *photon* or a *nutrino, clearly* $m_0 = 0$ and $\therefore E = pc$, whence, $p = E/c$.

And since, as we have seen, p is also equal to Ev/c^2, we have

$$Ev/c^2 = E/c, \text{ whence, } v = c,$$

i.e., the velocity of a particle of zero rest mass is always c, the speed of light, and is, therefore, invariant and so is its rest mass (zero), *in all frames of reference.*

And, its energy is $$E = mc^2,$$

where m is the mass equivalent of energy E, equal to E/c^2.

To summarise then,

(*i*) *the mass of a particle of zero rest mass, i.e.,* $m = E/c^2$.

(*ii*) *its velocity is c, the same as the speed of light in free space,*

(*iii*) *its momentum* $\vec{p} = mc = E/c$ and

(*iv*) *its energy* $E = mc^2 = pc$.

In case the particle be a *photon* of frequency ν, we have $E = h\nu$, where h is the well known *Planck's constant.*

So that, *for a photon, mass* $m = E/c^2 = h\nu/c^2$ and

momentum $p = E/c = h\nu/c$.

3.14 ATOMIC MASS UNIT

The basic mass unit used in atomic physics is called the *atomic mass unit* (*amu*, for short), which is 1/12[th] of the rest mass of *carbon twelve* (C^{12}), *very nearly equal to the mass of a proton.* Thus, 1 *amu* = 1.66×10^{-27} kg.

Since, energy associated with a rest mass m_0 is $E = m_0c^2$, we have 1 *amu, in terms of energy*

$$= 1.66 \times 10^{-7} \text{ kg} \times (3 \times 10^8)^2 \text{ m}^2/\text{sec}^2 = 1.49 \times 10^{-10} \text{ joules.}$$

$$= 1.49 \times 10^{-10} \times 10^7 = 1.49 \times 10^{-3} \text{ ergs.}$$

And, since 1.60×10^{-19} *joules* = 1 eV (electron volt)*, we have

* It is the amount of energy gained by an electron moving through a potential difference of 1 volt.

$$1 \text{ amu} = \frac{1.49\times10^{-10}}{1.60\times10^{-19}} \text{ eV} = 9.31\times10^8 \text{ eV} = \frac{9.31\times10^8}{10^6}$$

$$= 931 \; MeV \text{ (mega electron volt)*}.$$

Example 36. Find the velocity at which the mass of a particle is double its rest mass.

Solution. We have the relation $m = m_0/\sqrt{1-v^2/c^2}$, where m_0 is the *rest mass* of a particle an *m, its mass at velocity v.*

Here, $m = 2m_0$. So that, $m_0/m = 1/2$.

Putting $v/c = \sin\theta$, we have $\sqrt{1-v^2/c^2} = \sqrt{1-\sin^2\theta} = \cos\theta$.

$\therefore$ $m_0/m = \cos\theta$ and we have $\cos\theta = 1/2$. Or, $\theta = \cos^{-1}(1/2) = 60°$.

$\therefore$ $\sin\theta = \sin 60° = \sqrt{3}/2 = v/c$,

whence, $v = (\sqrt{3}/2)c = (1.732/2)\,c = 0.866\,c = 0.866 \times 3 \times 10^8$

$= 2.598 \times 10^8$ m/sec.

Thus, *the velocity at which the mass of a particle will be double its rest mass is* 2.598×10^8 m/sec.

Example 37. A proton of rest mass 1.67×10^{-24} gm is moving with velocity 0.9 c. Find its mass and momentum.

Solution. We know that if m_0 be the *rest mass* of a particle, in this case, the *proton*, its mass at velocity v is given by

$$m = m_o/\sqrt{1-v^2/c^2}, \text{ whence, } m_0/m = \sqrt{1-v^2/c^2}.$$

Putting $v/c = \sin\theta$, we have $\sin\theta = 0.9\,c/c = 0.9$, whence, $\theta = \sin^{-1}(0.9) = 64°9'$.

So that, $m_0/m = \sqrt{1-\sin^2\theta} = \cos\theta = \cos 64°9' = 0.4360$

$\therefore$ $m = m_0/0.4360 = 1.67 \times 10^{-24}/0.4360 = 3.83 \times 10^{-24}$ gm.

Thus, the *mass of the proton at velocity* 0.9*c will be* 3.83×10^{-24} gm.

And $\therefore$ *momentum of the proton*, $p = mv = m \times 0.9\,c = 3.83 \times 10^{-24} \times 0.9 \times 3 \times 10^{10}$

$= 10.34 \times 10^{-14}$ gm – cm/sec.

Example 38. Calculate the speed of an electron which has kinetic energy 1.02 MeV.

Solution. Hence, we must first know the mass m of the electron at the final speed (v) acquired, before we can determine the latter by the relation $m = m_o/\sqrt{1-v^2/c^2}$.

To determine this mass we note that the kinetic energy associated with a mass m is equal to mc^2.

If, therefore, Δm be the increase in the rest mass m_0 of the electron on acquiring its final speed v, we have Δm = (*K.E. acquired by the electron*)/c^2.

Clearly, *K.E.* acquired by the electron = 1.02×10^6 eV = $1.02 \times 10^6 \times 1.60 \times 10^{-19}$ = 1.632×10^{-13} *joules.*

Since $c = 3 \times 10^8$ m/sec, we have $\Delta m = \dfrac{1.632\times10^{-13}}{(3\times10^8)^2} = 18.13 \times 10^{-31}$ kg.

$\therefore$ *mass of the electron on acquiring its final velocity v, i.e.,* $m = m_0 + \Delta m$

$= (9.11 + 18.13) \times 10^{-31} = 27.24 \times 10^{-21}$ kg.

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*Strictly speaking, 1 *amu* = 931.441 MeV on the C^{12} isotope mass scale and equal to 931.441 MeV on the O^{16} scal

Now, putting $v/c = \sin\theta$, we have $\sqrt{1 - v^2/c^2} = \sqrt{1 - \sin^2\theta} = \cos\theta$.

So that, $m = m_0/\cos\theta$.

Or $\cos\theta = m_0/m = 9.11 \times 10^{-31}/27.24 \times 10^{-31} = 0.3344$.

Or, $\theta = \cos^{-1}(0.3344) = 70°28'$

and, $\sin\theta = \sin 70°28' = 0.9425$

$\therefore \quad \frac{v}{c} = 0.9425,$

whence, $v = 0.9425\,c = 0.9425 \times 3 \times 10^8 = 2.8275 \times 10^8$

$\approx 2.83 \times 10^8$ m/sec.

Thus, *the velocity of the electron,* of *K.E.* 1.02 *MeV*, is 2.83×10^8 m/sec.

Example 39. Given a proton for which $\beta = 0.995$ measured in the laboratory. What are the corresponding relativistic energy and momentum?

Solution. The rest mass of a proton, *i.e.*, $m_0 = 1.67 \times 10^{-24}$ *gm* and $\beta = v/c = 0.995$ or $v = 0.995c$. Hence *its mass at velocity v is given by*

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}} = \frac{1.67 \times 10^{-24}}{\sqrt{1 - (0.995)^2}} \text{ gm.}$$

Hence, its *relativistic energy* $= mc^2 = \dfrac{1.67 \times 10^{-24}}{\sqrt{1 - (0.995)^2}} \times (3 \times 10^{10})^2$

$$= \frac{1.67 \times 9 \times 10^{-4}}{\sqrt{.01}} = 1.503 \times 10^{-2} \text{ ergs.}$$

Now, as we know, 1.60×10^{-12} *ergs* $= 1$ *eV*.

$\therefore$ *relativistic energy of the proton* $= \dfrac{1.503 \times 10^{-2}}{1.60 \times 10^{-12}} = 9.39 \times 10^9\ eV = 9.39$ BeV

$[\because 10^9\ eV = 1\ BeV]$

To determine the momentum (p) of the electron, we use the relation $E^2 - p^2c^2 = m_0^{\,2}c^4$, whence,

$\sqrt{\quad\; ^2 \quad\; ^2\; ^4} \quad\quad p = \sqrt{E^2 - m_0^{\,2}\,c^4}/c.$

Clearly, Δm = *mass of neutron* – (*mass of proton* + *mass of electron*)

$$= 1.6747 \times 10^{-24} - (1.6724 \times 10^{-24} + 9.11 \times 10^{-28})$$

$$= (1.6747 - 1.6733) \times 10^{-24} = 0.0014 \times 10^{-24} \text{ gm.}$$

$\therefore$ *energy released* $= 0.0014 \times 10^{-24} \times (3 \times 10^{10})^2 = 0.0126 \times 10^{-4}$ ergs.

Since 1.60×10^{-13} *ergs* = 1 eV, we have

energy released when a neutron decays into a proton and an electron

$$= \frac{0.0126 \times 10^{-4}}{1.60 \times 10^{-12}} = .007875 \times 10^{8} = 7.875 \times 10^{5}\ eV \approx 0.79 \text{ MeV.}$$

(*ii*) Both the *electron* and the *positron* have the same *rest mass* $m_0 = 9.1 \times 10^{-28}$ *gm*. So that, an electron-positron pair production means the production of a mass $\Delta m = 2m_0 = 2(9.1 \times 10^{-28})$ gm

$\therefore$ *The minimum energy the γ-ray photon must possess for the production of this much mass* $= \Delta mc^2 = 2(9.1 \times 10^{-28})(3 \times 10^{10})^2 = 1.638 \times 10^{-6}$ *ergs.*

$$= \frac{1.638 \times 10^{-6}}{1.60 \times 10^{-12}} = 1.02 \times 10^{-6} \text{ eV.}$$

Thus, *the minimum energy required to produce an electron-positron pair*

$$= 1.02 \times 10^{6} \text{ eV} = 1.02 \text{ MeV.}$$

Example 41.(*a*) Calculate the momentum of a photon whose energy is 1.00×10^{-12} erg.

(*b*) An electron has $\beta = 0.99$. What is its kinetic energy?

Solution. (*a*) We know that the *momentum of a photon, $p = E/c$*, because the velocity of a photon is *c*.

So that,

$$p = \frac{1.00 \times 10^{-12}}{3 \times 10^{10}} = 0.333 \times 10^{-22} = 3.33 \times 10^{-23} \text{ gm-cm/sec.}$$

Or, *the momentum of the photon* $= 3.33 \times 10^{-23}$ gm-cm/sec.

(*b*) *The kinetic energy of a particle, K_E = its total energy – its rest energy*

$$= mc^2 - m_0c^2 = \gamma m_0c^2 - m_0c^2 = m_0c^2(\gamma - 1).$$

Now, m_0 *for an electron* $= 9.11 \times 10^{-28}$ gm, $c = 3 \times 10^{10}$ cm/sec.

and

$$\gamma = (1-\beta^2)^{-1/2} = (1 - v^2/c^2)^{-1/2} = [1 - (0.99)^2]^{-1/2} = (1 - 0.98)^{-1/2}$$

$$= (0.02)^{-1/2} = 7.071.$$

$\therefore$ *kinetic energy of the electron* $= 9.11 \times 10^{-28} \times (3 \times 10^{10})^2 (7.071 - 1)$

$$= 9.71 \times 9 \times 10^{-8} \times 6.071 = 4.977 \times 10^{-6} \text{ ergs.}$$

$$= (4.977 \times 10^{-6})/(1.60 \times 10^{-12}) = 3.11 \times 10^{5}\ eV = 3.11 \text{ MeV.}$$

Example 42. (*i*) show that the momentum of a particle of rest mass m_o and kinetic energy K_E is given by the expression $p = \sqrt{\frac{K_E^2}{c^2} + 2m_oK_E}$.

(*ii*) A nucleus of mass *M* emits a gamma-ray quantum and recoils with velocity $r << c$. Find the wavelength of the gamma-ray quantum and its energy.

Solution. (*i*) We have the relation $E^2 - p^2c^2 = m_o^2c^4$, whence, $E^2 = m_o^2c^4 + p^2c^2$.

or,

$$E = (m_o^2c^4 + p^2c^2)^{\frac{1}{2}}$$

Also, we know that the *total energy E = rest energy + kinetic energy* $= m_oc^2 + K_E$.

Equating the two values of *E*, therefore, we have

$$(m_o^2c^4 + p^2c^2)^{\frac{1}{2}} = m_oc^2 + K_E.$$

Squaring both sides of the equation, we have

$$m_o^2c^4 + p^2c^2 = m_o^2c^4 + 2m_oc^4 + 2m_oc^2k_E + k_{E^2}. \text{ Or, } p^2c^2 = k_{E^2} + 2m_oc^2k_E,$$

whence,
$$p^2 = \frac{k_{E^2}}{c^2} + 2m_ok_E \text{ and,}$$

$$\therefore \quad p = \sqrt{\frac{k_{E^2}}{c^2} + 2m_ok_E}, \textit{ the required relation}$$

(*ii*) Let the *momentum of the* γ-*ray quantum* (*or photon*) be p.

Then, since the momentum of the nucleus $= -Mv$, the negative sign indicating that v is the velocity of recoil (*i.e.*, *in the opposite direction* to that of the γ-ray quantum), we have

Total momentum $= p + (-Mv)$, which must obviously be equal to *zero*, since the initial momentum of the nucleus, before it emitted the γ-ray quantum, was *zero*.

We, therefore, have $p - Mv = 0$. Or, $p = Mv$.

$\therefore$ wavelength of the γ-ray quantum $= h/p = h/mV$.

p + (−mv)

And *its energy*, $E = h\nu = hc/\lambda = hc\dfrac{h}{Mv} = Mvc$.

Example 43. A nucleus of mass m emits a gamma-ray photon of frequency ν_0. Show that the loss of internal energy suffered by the nucleus is not $h\nu_0$ but $h\nu_0(1 + h\nu_0/2mc^2)$.

Solution. Here, clearly, *energy imparted to the photon* $= h\nu_0$ *and, therefore, the momentum imparted to it,* $p = h\nu_0/c$, since the velocity of the photon is c.

The nucleus will naturally lose an equal amount of momentum and recoil with a velocity v, say, thereby losing an additional amount of energy $= \dfrac{1}{2}Mv^2$.

$\therefore$ *Total internal energy lost by the nucleus* $h\nu_o + \dfrac{1}{2}mv^2 = h\nu_o + \dfrac{(mv)^2}{2m}$.

Since mv is the loss of momentum of the nucleus, equal in magnitude to the gain in momentum p of the photon, equal to $h\nu_0/c$, we have

Total internal energy lost by the nucleus $= h\nu_o + \dfrac{p^3}{2m} = h\nu_o + \dfrac{(h\nu_o)^2}{2mc^2} = h\nu_o\left(1 + \dfrac{h\nu_o}{2mc^2}\right)$.

Example 44. The solar constant is the flux of solar energy per sq cm per sec at the distance of the earth from the sun. By measurement it is found that the value of the constant is 1.4×10^6 ergs/sec-cm^2. Show that

(*i*) the total energy generation of the sun is $\approx 4 \times 10^{33}$ ergs/sec; (*ii*) the average rate of energy generation per gm of matter on the sun is ≈ 2 ergs/gm-sec $\approx 6 \times 10^7$ ergs/gm-year; (*iii*) the energy equivalent of 1 gm of hydrogen burned to produce He^4 is $\approx 6 \times 10^{18}$ ergs and (*iv*) if the mass of the sun were one-third hydrogen and the nuclear burning process continued without change, the sun could continue to radiate at its present rate for 3×10^{10} years.

(Radius of the earth's orbit around the sun $= 1.49 \times 10^{13}$ cm; mass of the sun $= 1.99 \times 10^{33}$ gm; mass of a proton $= 1.6725 \times 10^{-24}$ gm; mass of an electron $= 9.11 \times 10^{-23}$ gm; mass of helium atom $= 6.647 \times 10^{-24}$ gm.

Solution. (*i*) If E be the *total energy* emitted by the sun *per second* and if R be the radius of the earth's orbit around the sun, we have *solar constant* $S = E/4\pi R^2 = 1.4 \times 10^6$ *erg*/sec-cm^2(given), whence, $E = 4\pi R^2S = 4\pi(1.49 \times 10^{13})^2\ (1.4 \times 10^6) = 4\pi \times (1.49)^2 \times 1.4 \times 10^{32} = 3.906 \times 10^{33}$ $\approx 4 \times 10^{33}$ ergs/sec.

Thus, the *total energy generation of the sun* $\approx 4 \times 10^{33}$ ergs/sec.

$= 1.99 \times 10^{33}$ gm.

(*ii*) Clearly, total matter on the sun = mass of the sun $= 1.99 \times 10^{33}$ gm.

∴ *Average rate of energy generation per gm of matter on the sun*

$$= \frac{\text{total energy generated per sec}}{\text{mass of the sun}} = \frac{E}{m_s} = \frac{4 \times 10^{33}}{1.99 \times 10^{33}}$$

$$\approx 2 \text{ ergs/gm-sec.}$$

$$= 2 \times 3600 \times 24 \times 365 \approx 6 \times 10^7 \text{ ergs/gm-year.}$$

(*iii*) We know that 4 *protons* and 2 *electrons* constitute a helium atom. So that, *decrease in mass during the formation of a helium atom = mass of 4 protons + mass of 2 electrons-mass of a helium atom** $= 4 \times 1.6725 \times 10^{-24} + 2 \times 9.11 \times 10^{-28} - 6.647 \times 10^{-24} \approx 0.045 \times 10^{-24}$ gm.

This is thus the loss of mass due to the burning of 4 protons or 4 atoms or $4 \times 1.6725 \times 10^{-24}$ gm of hydrogen to form a helium atom (because the mass of a hydrogen atom is practically the mass of its one-proton nucleus, the mass of the single revolving electron being relatively negligible).

∴ *Loss of mass due to burning of 1 gm of hydrogen* $= \dfrac{0.045 \times 10^{-24}}{4 \times 1.6725 \times 10^{-24}} \times 1 = 6.723 \times 10^{-3}$ gm.

And ∴ *Energy equivalent of 1 gm of hydrogen burned* $= 6.723 \times 10^{-3}\, c^2$.

$$= 6 \times 723 \times 10^{-3} \times 9 \times 10^{20} = 6 \times 10^{18} \text{ ergs.}$$

(*iv*) If the mass of the sun were one-third hydrogen, we shall have *mass of hydrogen on the sun* $= \dfrac{1}{3}(\textit{mass of the sun}) = \dfrac{1}{3}(1.99 \times 10^{33}) = 0.667 \times 10^{33}$ gm.

As we have just seen under (*iii*) above, energy produced due to burning of 1 gm of hydrogen $= 6 \times 10^{18}$ ergs.

∴ Energy produced due to burning of the entire hydrogen on the sun

$$= 6 \times 10^{8} \times 0.667 \times 10^{33} = 4 \times 10^{51} \text{ ergs.}$$

Now, the sun generates energy at the rate of 4×10^{33} ergs per sec.

∴ Time taken by the sun to generate the whole amount, *i.e.*, 4×10^{51} ergs *of energy* $= 4 \times 10^{51}/4 \times 10^{33} = 10^{18}$ sec $= 10^{18}/3600 \times 24 \times 365 = 3.17 \times 10^{10} \approx 3 \times 10^{10}$ years.

The sun will thus continue to radiate energy at its present rate for 3×10^{10} years.

Example 45. (*i*) **The binding energy of an electron to the proton (*i.e.*, of hydrogen) is 13.6 eV. Find the loss of mass in the formation of 1 atom of hydrogen.**

(*ii*) **Calculate the binding energy of duetron.**

(mass of proton (m_P) $= 1.6725 \times 10^{-24}$ gm, mass of neutron (m_N) $= 1.6748 \times 10^{-24}$ gm and mass of duetron (m_D) $= 3.3433 \times 10^{-24}$ gm.)

Solution. (*i*) We have *binding energy of hydrogen* $= 13.6\ eV = 13.6 \times 1.60 \times 10^{-12}$

$$\approx 22 \times 10^{-12} \text{ ergs.}$$

∴ *Loss of mass during formation of 1 atom of hydrogen = Binding energy/c^2.* [∵

$$= 22 \times 10^{-12}/9 \times 10^{20} = 2.4 \times 10^{-23} \text{ gm.}$$

(*ii*) Here, *loss of mass on formation of duetron* is given by

$$\Delta m = m_P + m_N - m_D = (1.6725 + 1.6748) \times 10^{-24} - 3.3433$$

$$= 0.004 \times 10^{-24} \text{ gm.}$$

*The mass of the electrons has been taken on the left hand side of the relation to balance the mass of two e… included in the mass of the helium atom.