14th Edition

ENGINEERING MECHANICS STATICS AND DYNAMICS

A. K. TAYAL

UMESH

ENGINEERING MECHANICS STATICS AND DYNAMICS

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Preface to the Fourteenth Edition

It gives me a great pleasure to present the Fourteenth Edition of this book on Engineering Mechanics. I express my gratitude for the wide acceptability of the book by the academics as well as student community and it gives me deep sense of satisfaction. In this new edition, a new chapter on 'Shear Force and Bending Moment' has been added. This topic is included to meet the current requirements of some universities.

I again, request for the suggestions and comments for the improvement of the book.

Delhi
January, 2011

Dr. A.K. TAYAL

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CHAPTER 1 Introduction

1.1 ENGINEERING MECHANICS

It is the science which deals with the physical state of rest or motion of bodies under the action of forces. Depending upon the nature of the body involved, it can be further divided into *mechanics of rigid bodies, mechanics of deformable bodies* (also called strength of materials) and the *mechanics of fluids.*

In this book, we shall deal with the mechanics of *rigid bodies*. Rigid bodies are those *bodies which do not deform under the action of applied forces.* The mechanics of rigid bodies is studied in two parts, *statics* and *dynamics. Statics deals with bodies at rest and dynamics with bodies in motion.*

1.2 IDEALIZATION OF BODIES

Matter is made up of atoms and molecules. But the real picture of matter as atoms and molecules is too complex to deal with. So to study the average measurable behaviour of bodies, we assume that *the matter is continuously distributed. Such a description of matter is called a continuum. A continuum can be rigid or deformable depending upon the assumptions we make.*

Rigid Body. The physical bodies deform, although slightly, under the action of loads or external forces. But in many situations this deformation is negligibly small to affect the results. So, the assumption of a rigid body shall mean that the body does not deform or the distance between any two points of the body does not change under the action of a applied force.

Particle. It is defined as an object whose mass is concentrated at a point. This assumption is made when the size of a body is negligible and is irrelevant to the description of the motion of the body.

1.3 BASIC CONCEPTS

The study of mechanics involves the concepts of *space, time, mass* and *force.*

(*i*) Concept of space is essential to fix the position of a point. To fully define the position of a point in space we shall need to define some frame of reference and coordinate system.

(*ii*) Concept of time is essential to relate the sequence of events, for example, starting and stopping of the motion of a body.

(*iii*) Concept of mass is essential to distinguish between the behaviour of the two bodies under the action of an identical force.

(*iv*) Concept of a force is essential as an agency which changes or tends to change the state of rest or of uniform motion of a body.

But in order to describe the state of rest or motion of a body, some *datum* or *reference* is required. *A body can be said to be at rest or in motion only with respect to some reference frame. This reference*, preferably, should be fixed in space. *As it is doubtful to locate any fixed reference in the universe, so the earth surface is usually employed as a reference frame.* Such a reference serves as an inertial frame. *A truly inertial frame is one which moves at constant velocity.*

Finally, we also need to define the *units* which are to be used to measure length, time, mass and force and any other physical quantities which may be involved in this study.

1.4 FUNDAMENTAL PRINCIPLES

The elementary mechanics rests on a few fundamental principles based on experimental observations. These are,

1. Newton's Three Laws of Motion
2. Newton's Law of Gravitation.
3. The Parallelogram Law for the Addition of Forces.
4. The Principle of Transmissibility of a Force.

1. (*a*) **Newton's First Law.** Everybody continues in a state of rest or of uniform motion in a straight line unless it is compelled to change that state by a force imposed on the body. First law thus helps us to define a force.

(*b*) **Newton's Second Law.** The acceleration of a given particle is proportional to the impressed force and takes place in the direction of the straight line in which the force is impressed.

$$F = ma. \qquad ...(1.1)$$

This law helps us to measure a force quantitatively.

(*c*) **Newton's Third Law.** To every action there is equal and opposite reaction. Which means, that the forces of action and reaction between two bodies are equal in magnitude but opposite in direction.

2. Newton's Law of Gravitation. Two particles are attracted towards each other along the line connecting them with a force whose magnitude is proportional to the product of their masses and inversely proportional to the square of the distance between them.

$$F = G\,\frac{m_1 m_2}{r^2} \qquad ...(1.2)$$

where r is the distance between the particles and

G is the universal constant called constant of gravitation.

The force of attraction exerted by the earth on a particle lying on its surface is governed by this law.

If a particle of mass m lies on the surface of the earth of mass M and radius R, the force exerted by the earth is equal to weight W of the particle. Therefore,

$$F = W = mg = \frac{GmM}{R^2}$$

or

$$g = \frac{GM}{R^2} \qquad ...(1.3)$$

where, g is acceleration due to gravity.

3. The Parallelogram Law. If two forces acting at a point are represented in magnitude and direction by the adjacent sides of a parallelogram then the diagonal of the parallelogram passing through their point of intersection represents the resultant in both magnitude and direction. This implies that a force can be represented by a straight line with an arrow in the manner of a vector.

4. Principle of Transmissibility of a Force. It states that the condition of equilibrium or of motion of rigid body will remain unchanged if the point of application of a force acting on the rigid body is trasmitted to act at any other point along its line of action.

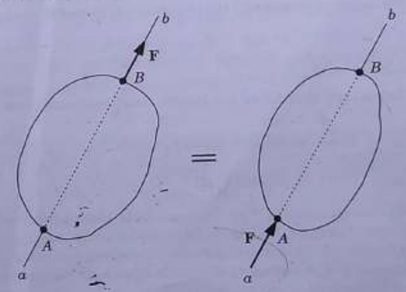

Fig. 1.1

A force **F** acting on the rigid body at point B, having the line of action ab, can be replaced by the same force **F** but acting at the point A provided this new point A lies along the line of action ab of the force (Fig. 1.1)

In other words, the force **F** acting at point A can be transmitted to act at any other point B along its line of action without changing its effect on the rigid body.

Next consider a prismatic bar AB which is acted upon by two equal and opposite coaxial forces $\mathbf{P}_1$ and $\mathbf{P}_2$ as shown in Fig. 1.2 (*a*).

(a) Pull (b) Push (c) No force

Fig. 1.2

Using the principle of transmissibility, the force $\mathbf{P}_1$ acting at point A can be transmitted to act at point B and the force $\mathbf{P}_2$ acting at point B can be transmitted to act at A. This new system of forces [Fig. 1.2 (*b*)] acting on the bar, as obtained by the principle of transmissibility, does not change the condition of equilibrium of the body.

But from the point of view of internal forces, the state of *tension* (*pull*) has been changed to a state of *compression* (*push*). That is, the tendency of *elongating* the bar has been changed to that of *shorting* the bar.

Further, the two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ can be transmitted to the middle point C of the bar [Fig. 1.2 (*c*)] such that there is no internal force developed in the bar.

It is, therefore, clear that the principle of transmissibility can be used to discuss the condition of equilibrium or motion of a rigid body and to determine the external forces acting on the rigid body. *But it should not be used to determine the internal forces and deformations of the body.*

The fundamental principles discussed above shall be used in the following chapters to develope the equations expressing mathematically the conditions of rest or of motion of a body or a system of bodies.

1.5 SYSTEMS OF UNITS

In mechanics four important quantities are *length*, *time*, *mass* and *force*. These are related as,

$$F = ma$$

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

$$\text{Force} = (\text{Mass}) \times \frac{\text{Length}}{(\text{Time})^2}$$

The units of all the four quantities, therefore, cannot be chosen arbitrarily. The units of three quantities can be defined arbitrarily (but are to be internationally accepted) in terms of base units. Then the unit of the fourth quantity is derived using the above relationship to form a consistent system of units. For example, if the units of length, time and mass are chosen as base units, the unit of force can be derived.

The different systems of units prevalent are:

1. Centimetre-Gram-Second System (C.G.S.)
2. Foot-Pound-Second System (F.P.S.)
3. Metre-Kilogram-Second System (M.K.S.)
4. International System of Units (SI Units.)

SI Units. SI is the abbreviation for 'The System International d'Unites' (also called The International System of Units) and is the modern version of the metric system. SI was formally recognised by the Eleventh General Conference of Weight and Measures in 1960. This system

is now being adopted throughout the world. In India also, a statutory decision was taken in 1956 to change over to the metric system in conformity with the SI concepts.

The International System of units *consists of 7 base units, 2 supplementary units and a number of derived units*. In this system, the units of length, mass and time form the base units and are,

metre (m) for length

kilogram (kg) for mass

second (s) for time.

The unit of force is Newton and it is a derived unit. A newton force is defined as the force which can impart an acceleration of 1 m/s^2 to a mass of 1 kilogram.

$$1\text{ N} = (1\text{ kg})\,(1\text{ m/s}^2) = \frac{1\text{ kg m}}{\text{s}^2}$$

Weight W of a body of mass m is a force and should be measured in newton.

$$W = mg$$

It follows that the weight of a body of mass of 1 kg is

$$W = (1\text{ kg}).\left(\frac{9.81\text{ m}}{\text{s}^2}\right) = 9.81\text{ N}$$

The units for all forms of energy is joule (J) and the unit for power is watt (W).

SI units are *absolute system* of *units* and so, they are independent of the location where measurements are made. They are also *coherent*, *rational* and *comprehensive*.

Principal SI units used in mechanices are given in Table 1.1.

Table 1.1 *Principal SI Units Used in Mechanics*

Physical Quantity	*Unit*	*Symbol*
Acceleration	metre/second^2	m/s^2
Angle	radian	rad
Angular velocity	radian/second	rad/s
Angular acceleration	radian/second^2	rad/s^2
Area	square metre	m^2
Density	kilogram/metre^3	kg/m^3
Energy	joule	J (Nm)
Force	newton	$\text{N}\left(\frac{\text{kg m}}{\text{s}^2}\right)$
Frequency	hertz	Hz (s^{-1})
Length	metre	m
Mass	kilogram	kg
Moment of a Force	newton metre	Nm
Power	watt	W (J/s)

(Conted..)

Physical Quantity	Unit	Symbol
Pressure	pascal	Pa (N/m^2)
Stress	pascal	Pa (N/m^2)
Time	second	s
Torque	newton metre	Nm
Velocity	metre/second	m/s
Volume	cubic metre	m^3
Work	joule	J (Nm)

Multiples and submultiples of the SI units are obtained through the use of prefixes defined in Table 1.2.

TABLE 1.2 *SI Prefixes*

Multiplication Factor	Prefix	Symbol Used
1 000 000 000 000 = 10^{12}	tera	T
1 000 000 000 = 10^{9}	giga	G
1 000 000 = 10^{6}	mega	M
1 000 = 10^{3}	kilo	k
100 = 10^{2}	hecto	h
10 = 10^{1}	deca	da
0.1 = 10^{-1}	deci	d
0.01 = 10^{-2}	centi	c
0.001 = 10^{-3}	milli	m
0.000 001 = 10^{-6}	micro	μ
0.000000001 = 10^{-9}	nano	n
0.000 000 000 001 = 10^{-12}	pico	p

2 CHAPTER Concurrent Forces in a Plane

2.1 FORCE

It is defined as an agency which changes or tends to change the position of rest or of uniform motion of a body. The force, therefore, has the capacity to impart motion to a body. From the point of view of mechanics, a force can also produce *push, pull or twist* in a body.

A force may be exerted by some actual physical contact like, pushing a lawn-mower. A force can also be exerted without any physical contact as, '*the action at a distance*'. In the case of the gravitational force or the magnetic force acting on a body, there is no physical contact between the bodies. In this sense *we can classify forces as contacting and non-contact type of forces.*

To define a force completely we have to specify its magnitude, direction and point of application. The magnitude of a force is measured in terms of newton (N).

Force is a vector quantity and rules of vector addition and subtraction are applicable to the addition and subtraction of forces as explained in the following pages.

2.2 SCALAR AND VECTOR

We are familiar with the quantities like mass, time, volume and energy which can be completely defined by stating their magnitudes and which do not have any direction. They can be added and subtracted according to the law of algebra. Such quantities are called *scalar quantities.*

On the other hand, quantities like displacement, velocity, acceleration momentum and force possess both; magnitude as well as direction. To define these quantities we have to specify their (*a*) *magnitude,* (*b*) *direction* and (*c*) *point of action. Such quantities, which possess magnitude and direction and can be added according to the parallelogram law, are termed as vector quantities.*

A vector is represented in the printed text by a bold faced letter* such as **P** and in hand writing by placing a bar above the letter as $\bar{P}$. The magnitude of a vector **P** is represented by an ordinary letter *P*. As force is a vector quantity, it shall follow the laws of vector algebra.

Graphically a force (like a vector) is represented by the segment of a straight line. The straight line represents the line of action of the force **F** and its length, the magnitude of the force. The direction (or sense) of the force is indicated by placing an arrow head on this straight line (Fig. 2.1).

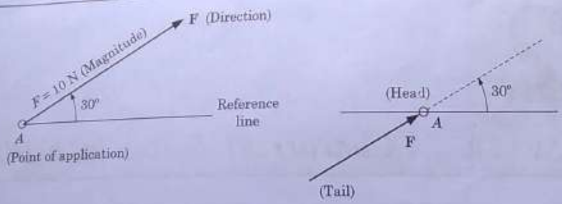

Fig. 2.1

Either the head or the tail of the vector may be used to indicate the point of application of a force. But all the forces involved must be represented consistently.

2.3 ADDITION OF TWO FORCES: PARALLELOGRAM LAW

Before discussing the parallelogram law, let us clarify the concept of *equal forces* and *equivalent forces*.

Two forces are said to be *equal* if they have the same magnitude and direction even if their points of application are not the same.

Two forces are said to be *equivalent* if (in some sense) they produce the some effect on a rigid body. To clarify the point let us consider two 25 paise coins and one 50 paise coin. They are not equal in size, shape and weight yet are equivalent in their buying capacity. Interestingly, they are not equivalent in their capacity to operate a 50 paise coin operated public telephone. *Equivalence is thus based on some specific effect.*

Most of the time in mechanics, we are concerned with the forces having an equivalent effect on a rigid body rather than the equal forces. *The resultant of two forces (or their sum) acting on a body, in this sense, is a equivalent force*

Parallelogram Law. It was mentioned earlier that two forces add according to the parallelogram law. This law can be stated as *"If two force acting at a point are represented in magnitude and direction by the adjacent sides of a parallelogram then the diagonal passing through their point of intersection represents the resultant in both magnitude and direction."*

The sum of the two forces **P** and **Q** acting at the point A, with the included angle θ, can be obtained by constructing a parallelogram such that the forces **P** and **Q** represent the two adjacent sides of the parallelogram as shown in Fig. 2.2.

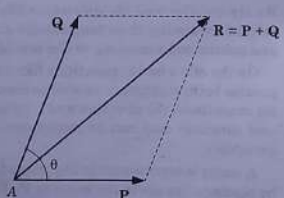

Fig. 2.2

* Vector quantities have not been distinguished by bold faced letter in the solved examples of this book.

The diagonal that passes through the point A represents the sum or the resultant **R** of the forces **P** and **Q**

$$\mathbf{R} = \mathbf{P} + \mathbf{Q} \qquad ...(2.1)$$

The sum or the resultant of **P** and **Q** is independent of the order in which they are added.

$$\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P} \qquad ...(2.2)$$

Law of Triangle of Forces. Instead of constructing the parallelogram the sum of the resultant of the two forces can be determined by the triangle law.

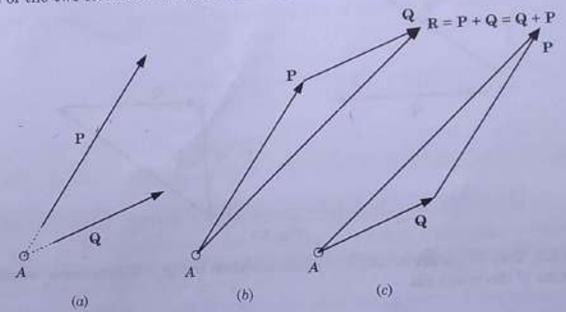

Fig. 2.3

This can be stated as *"If two forces acting at a point are represented by two sides of a triangle taken in order, then their sum or resultant is represented by the third side taken in an opposite order."*

The sum or resultant of two forces **P** and **Q** acting at point A [Fig. 2.3 (a)] can be obtained by constructing a triangle such that the forces **P** and **Q** are represented by the two sides of this triangle taken in an order. The closing side [Fig. 2.3 (b)] taken in an opposite order then represents the sum or the resultant **R** of the forces **P** and **Q**.

The forces **P** and **Q** can be added in any order as shown in Fig. 2.3 (b) and (c) as,

$$\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P} \qquad ....(2.3)$$

The magnitude of the resultant **R** can be determined graphically by measuring the length of vector **R** of the force triangle.

The magnitude of the resultant **R** can also be determined trigonometrically if the included angle β between the forces **P** and **Q** is known (Fig. 2.4) using the relation

$$R^2 = P^2 + Q^2 - 2\,PQ\cos\beta \qquad ...(2.3)$$

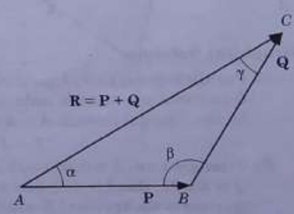

Fig. 2.4

The remaining angles can be computed using the law of sines as,

$$\frac{P}{\sin\gamma} = \frac{Q}{\sin\alpha} = \frac{R}{\sin\beta} \qquad ...(2.4)$$

Subtraction. The difference **P** – **Q** can be found by adding to the force **P** a force equal and opposite to the force **Q** as shown in Fig. 2.5.

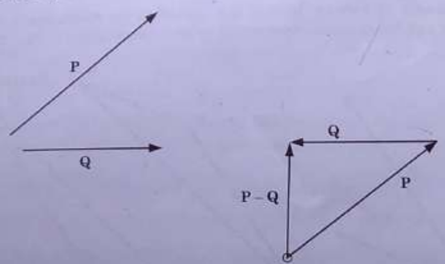

Fig. 2.5

Example 2.1. Two forces are acting at a point as shown in Fig. 2.6. Determine the magnitude and direction of the resultant.

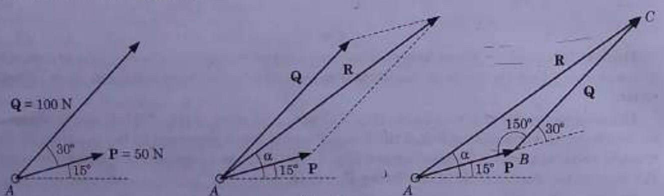

Fig. 2.6

Graphical Solution

(a) **Parallelogram Law.** A parallelogram with sides equal and parallel to the forces P and Q is drawn to a suitable scale. The magnitude and direction of the resultant R is given by the diagonal through A and is measured to be

$$R = 145.5 \text{ N}, \ \alpha = 35° \quad Ans.$$

(b) **Triangle Law.** A triangle with its adjacent sides equal and parallel to the forces P and Q is drawn, (head to tail) to a suitable scale. The closing side taken in opposite order represents the resultant R which is measured to be

$$R = 145.5 \text{ N}, \ \alpha = 35° \quad Ans.$$

Trigonometric Solution

Instead of drawing the triangle to scale and measuring the resultant R we can determine it by applying law of cosines to the triangle ABC.

$$R^2 = P^2 + Q^2 - 2\,PQ\cos\beta$$
$$R^2 = 50^2 + 100^2 - 2(50)(100)\cos 150°$$
$$R^2 = 21160$$
$$R = 145.46 \text{ N} \quad Ans.$$

The direction α can be determined by applying law of sines to the ΔABC

$$\frac{\sin A}{Q} = \frac{\sin 150°}{R}$$

$$\sin A = \frac{\sin 150°}{145.46} \times 100 \qquad \angle A = \angle CAB$$

$$\angle A = 20.10°$$

Hence $\alpha = 15° + 20.10°$

or $\alpha = 35.1°$ *Ans.*

2.4 CONCEPT OF THE RESULTANT OF SEVERAL FORCES

When a number of forces acting on a rigid body are replaced by a single force which has the same effect on the rigid body as that of all the forces acting together then this single force is called the resultant of several forces.

Before we discuss the methods to determine the resultant of several forces let us explain the various systems of forces.

Point Force. Here we assume that a finite force is transmitted through an infinitesimal area or a point. Such a force is called a point force. The forces which are distributed over an area shall be discussed later.

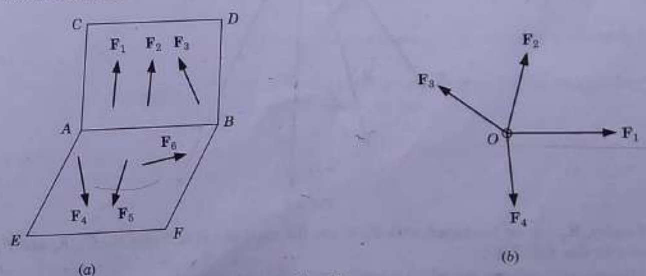

Fig. 2.7

Coplanar Forces. When a number of forces lie in the plane they are said to be coplanar forces. In Fig. 2.7 (*a*), forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ are contained in the plane *ABCD* and are coplanar forces. Similarly forces $\mathbf{F}_4$, $\mathbf{F}_5$ and $\mathbf{F}_6$ are contained in the place *ABEF* and are coplanar forces. But forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, $\mathbf{F}_5$ and $\mathbf{F}_6$ taken together are non-coplanar forces.

In this book we shall be concerned mainly with the coplanar forces. The coplanar forces can be :

Concurrent Forces. Are those forces whose lines of action pass through a common point as shown in Fig. 2.7 (*b*).

Parallel Forces. A set of forces whose lines of action are parallel to each other are called parallel forces.

General System of Forces. If several forces acting in a plane are such that they do not intersect in one point and are not parallel then they represent a general system of forces.

In this chapter we shall be discussing about the concurrent forces.

2.5 RESULTANT OF SEVERAL CONCURRENT COPLANAR FORCES: POLYGON LAW

If several forces are acting at a point and are coplanar their resultant can be found by the repeated use of parallelogram law. To find the resultant of the coplanar and concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ acting at *O* we can begin with any two forces say $\mathbf{F}_1$ and $\mathbf{F}_2$ and apply parallelogram law to obtain their resultant $\mathbf{R}_1$. Next the force $\mathbf{R}_1$ can be combined with $\mathbf{F}_3$ to obtain the resultant $\mathbf{R}_2$.

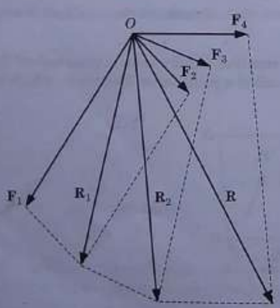

Fig. 2.8

Finally, $\mathbf{R}_2$ can be combined with $\mathbf{F}_4$ to get the resultant $\mathbf{R}$ of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ as shown in the Fig. 2.8.

Alternatively, polygon law which is equivalent to the repeated application of parallelogram law can be applied determine the resultant of a number of concurrent coplanar forces.

The Law of Polygon of Forces. It may be stated as "*If a number of coplanar forces are acting at a point such that they can be represented in magnitude and direction by the sides of a polygon taken in an order, their resultant is represented in both magnitude and direction by the closing side of the polygon taken in the opposite order.*"

The resultant **R** does not depend upon the order in which the forces are chosen to draw the polygon.

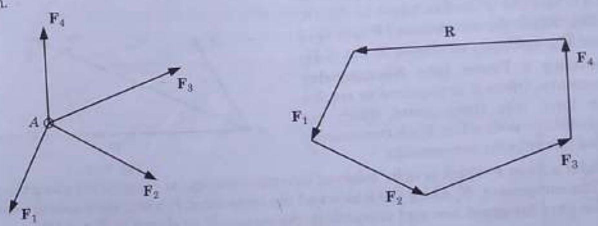

Fig. 2.9

2.6 RESOLUTION OF A FORCE INTO COMPONENTS

We have learnt to determine the resultant of two or more forces acting on a particle. Let us now consider the reverse problem. That is, how to replace a single force **F** acting on a particle by two or more forces which (together) have the same effect on the particle as the force **F**? These forces are called the components of the original force **F** and the process is called resolving the force **F** into components.

Theoretically, a force can be resolved into an infinite number of sets of components of forces.

In practice, a situation requiring the resolution of a force **F** into *two* components often arises. Two such cases are discussed below :

(*a*) One component **P** is known is magnitude and direction and the other component **Q** is to be determined.

The second component **Q** of force **F** can be obtained graphically either by completing the parallelogram or by completing the triangle as shown in Fig. 2.10.

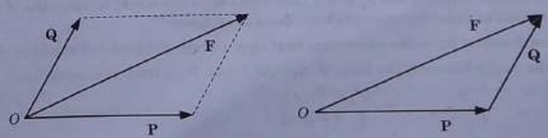

Fig. 2.10

(b) When the lines of action of both the components are known but their magnitudes P and Q are to be determined.

The magnitudes P and Q of the two components can be determined by applying the parallelogram law. From the head of the vector F, draw two lines parallel to the given lines of action (Oa and Ob) of the two forces as shown. This shall define the components **P** and **Q** of the force **F** graphically as shown in Fig. 2.11.

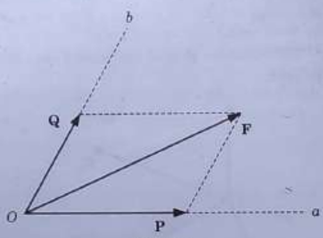

Fig. 2.11

Resolving a Force into Rectangular Components. Often it is required to resolve a given force into components which are perpendicular to each other. Such components are called rectangular components.

Consider a force **F** which is to be resolved into two rectangular components along the x- and y-axis. The component, F_x along the x-axis and the component F_y along the y-axis are obtained using the parallelogram law and completing the rectangle as shown in Fig. 2.12.

The x and y axes are generally horizontal and vertical. But any other directions can also be chosen. For example, a vertical force **F** can be resolved into components tangential and perpendicular to an inclined plane as shown in Fig. 2.13.

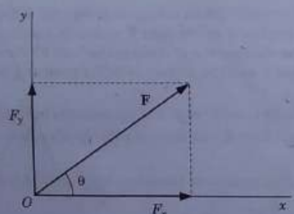

Fig. 2.12

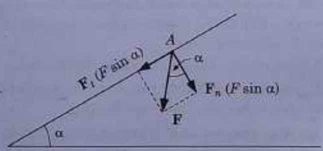

Fig. 2.13

The rectangular components F_x and F_y are also called the scalar components of the force **F** as the directions of these components have been predefined.

These components have the advantage that they can be manipulated algebraically.

If θ be the angle between the force **F** and the x-axis then from trigonometry,

$$F_x = F\cos\theta$$

$$F_y = F\sin\theta$$

$$\tan\theta = \frac{F_y}{F_x} \text{ and } F = \sqrt{F_x^2 + F_y^2} \qquad \text{...(2.5)}$$

2.7 RESULTANT OF A SEVERAL CONCURRENT COPLANAR FORCES BY SUMMING RECTANGULAR COMPONENTS (METHOD OF PROJECTIONS)

The resultant of two forces can be found graphically by using parallelogram law or the law of triangle of forces. For more than two forces, repeated use of the parallelogram law or the law of polygon of forces can give us the resultant. Now let us discuss the analytical method.

Consider a number of coplanar forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ acting on a particle at O [Fig. 2.14(a)].

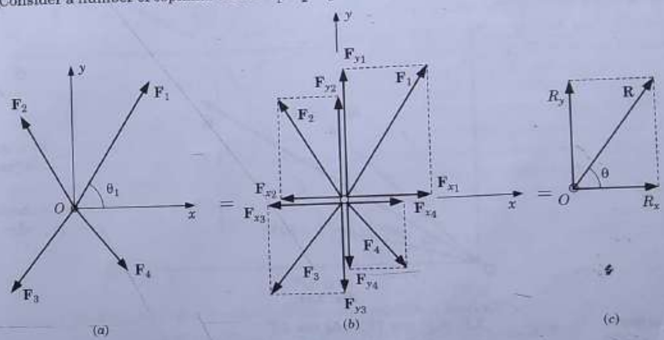

Fig. 2.14

Each force acting at O can be replaced by its rectangular components F_{x1} F_{y1}; F_{x2}, F_{y2} etc. These components of forces produce the same effect on the particle as the forces themselves.

Now the horizontal components can be added into a single force R_x by using the law of parallelogram of forces. This simply reduces to an algebraic sum of the horizontal components F_{x1}, F_{x2},...etc. as they lie along the same line

$$R_x = F_{x1} + F_{x2} + F_{x3} + F_{x4}$$

Similarly, vertical components F_{y1}, F_{y2} ... etc. can be added into a single force R_y, such that,

$$R_y = F_{y1} + F_{y2} + F_{y3} + F_{y4}$$

Therefore,

$$R_x = \Sigma F_x \text{ and } R_y = \Sigma F_y \qquad \text{...(2.6)}$$

Now, these two perpendicular forces R_x and R_y can be added vectorially to determine the resultant **R** of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$.

$$R = \sqrt{R_x^2 + R_y^2}$$

and

$$\tan\theta = \frac{R_y}{R_x} \qquad \text{...(2.7)}$$

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If the force $\mathbf{F}_1$ makes an angle θ_1 with the x-axis then its components are,

$$F_{x1} = F_1 \cos\theta_1$$
$$F_{y1} = F_1 \sin\theta_1.$$

If θ_2, θ_3 and θ_4 be the angles that the forces $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ make with the x-axis, their components can be similarly determined.

Example 2.2. Two forces are acting at a point as shown in the figure. Determine the magnitude and direction of the resultant choosing the x- and y-axes as shown and resolving the forces P and Q along these axes.

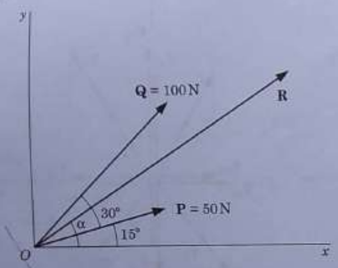


Fig. 2.15

Solution:

$$\Sigma F_x = P\cos 15° + Q\cos 45°$$
$$\Sigma F_x = 50 \times 0.966 + 100 \times 0.707$$
$$\Sigma F_x = 119 \text{ N}$$
$$\Sigma F_y = P\sin 15° + Q\sin 45°$$
$$\Sigma F_y = 50 \times 0.259 + 100 \times 0.707$$
$$\Sigma F_y = 83.64 \text{ N}.$$

If the resultant R has components R_x and R_y then,

$$R_x = \Sigma F_x = 119 \text{ N}$$
$$R_y = \Sigma F_y = 83.64 \text{ N}$$
$$R = \sqrt{R_x^2 + R_y^2}$$
$$R = \sqrt{119^2 + 83.64^2}$$
$$R = 145.46 \text{ N} \quad \textit{Ans.}$$
$$\tan\alpha = \frac{R_y}{R_x} = \frac{83.64}{119.0}$$
$$\alpha = 35.10° \ \textit{Ans.}$$

where α is the angle the resultant R makes with x-axis.

Example 2.3. Find the magnitude and direction of the resultant R of four concurrent forces acting as shown in Fig. 2.16.

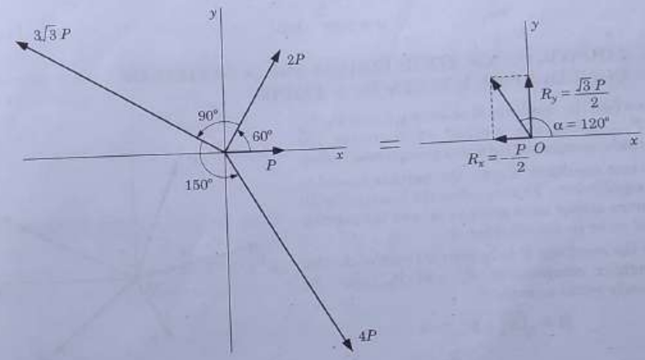


Fig. 2.16

Solution: Choosing the x and y axes as shown and resolving forces,

$$\Sigma F_x = (P)\cos 0° + (2P)\cos 60° + (3\sqrt{3}P)\cos 150° + (4P)\cos 300°$$
$$\Sigma F_x = P + 2P\left(\frac{1}{2}\right) - 3\sqrt{3}P\left(\frac{\sqrt{3}}{2}\right) + 4P\left(\frac{1}{2}\right)$$
$$\Sigma F_x = -\frac{P}{2}$$
$$\Sigma F_y = (P)\sin 0° + (2P)\sin 60° + (3\sqrt{3}P)\sin 150° + (4P)\sin 300°$$
$$\Sigma F_y = 0 + 2P\left(\frac{\sqrt{3}}{2}\right) + 3\sqrt{3}P\left(\frac{1}{2}\right) - 4P\left(\frac{\sqrt{3}}{2}\right)$$
$$\Sigma F_y = \frac{\sqrt{3}}{2}P$$

If the resultant of these forces is R having components R_x and R_y,

$$R_x = \Sigma F_x = -\frac{P}{2} \quad \text{and} \quad R_y = \Sigma F_y = \frac{\sqrt{3}}{2}P$$
$$R = \sqrt{R_x^2 + R_y^2} = P\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$R = P \quad \textit{Ans.}$$

$$\tan\alpha = \frac{R_y}{R_x} = \frac{+\frac{\sqrt{3}P}{2}}{-\frac{P}{2}} = \tan^{-1}(-\sqrt{3})$$

$$\alpha = 120° \quad Ans.$$

2.8 EQUATIONS OF EQUILIBRIUM FOR A SYSTEM OF CONCURRENT FORCES IN A PLANE

We can find the resultant **R** of several forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, F_4... using the method of summing the rectangular components of forces as explained earlier.

If their resultant is zero, the particle is said to be in equilibrium. *That is, when the resultant of all the forces acting on a particle is zero the particle is said to be in equilibrium.*

For the resultant **R** to be zero, its each of the two rectangular components R_x and R_y, must be separately equal to zero. If,

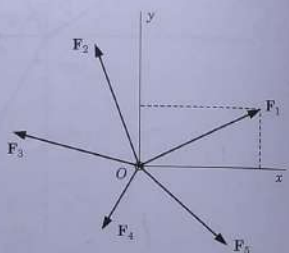


Fig. 2.17

$$R = \sqrt{R_x^2 + R_y^2} = 0$$

then, $R_x = 0$

$R_y = 0$

Therefore $R_x = F_{x1} + F_{x2} + F_{x3} + \ldots = 0$

or $\Sigma F_x = 0$ ...(2.8)

and $R_y = F_{y1} + F_{y2} + F_{y3} + \ldots = 0$

or $\Sigma F_y = 0$

where, F_{x1}, F_{x2} ...are the components of the forces F_1, F_2 ...along the x-axis

and, F_{y1}, F_{y2} ...are the components of the forces F_1, F_2 ... along the y-axis.

Equations $\Sigma F_x = 0$ ...(2.8)

$\Sigma F_y = 0$ ...(2.9)

are called the *equations of equilibrium*. If a number of *concurrent* forces lying in a plane are in equilibrium these equations are to be satisfied.

Notes :

1. The x and the y axes can be arbitrarily chosen through the point of concurrency. For example, in the case of a body resting on an inclined plane, x and y axes can be chosen along and normal to the inclined plane.
2. The two equations of equilibrium can be solved to find a maximum of two unknowns. Determination of a force in magnitude and direction or determination of the magnitudes of two forces whose directions are known, amounts to solving for two unknowns.

Example 2.4 Two ropes are tied together at C. If the maximum permissible tension in each rope is 3.5 kN, what is the maximum force P that can be applied and in what direction?

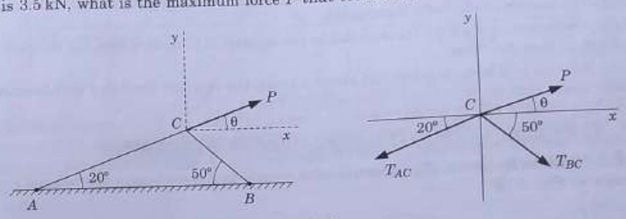


Fig. 2.18

Solution: Consider the equilibrium of the point C,

$\Sigma F_x = 0$: $P\cos\theta + T_{BC}\cos 50° - T_{AC}\cos 20° = 0$

$\Sigma F_y = 0$: $P\sin\theta - T_{BC}\sin 50° - T_{AC}\sin 20° = 0$

where, T_{AC} and T_{BC} are tension in the strings AC and BC

$P\cos\theta = T_{AC}\cos 20° - T_{BC}\cos 50°$ ...(i)

$P\sin\theta = T_{AC}\sin 20° + T_{BC}\sin 50°$ ...(ii)

Dividing (ii) by (i)

$$\frac{P\sin\theta}{P\cos\theta} = \tan\theta = \frac{T_{AC}\sin 20° + T_{BC}\sin 50°}{T_{AC}\cos 20° - T_{BC}\cos 50°}$$

But $T_{AC} = T_{BC} = 3.5$ kN

$$\tan\theta = \frac{0.766 + 0.342}{0.94 - 0.643} = 3.703$$

$$\theta = 75° \quad Ans.$$

Substituting $\theta = 75°$ and $T_{AC} = T_{BC} = 3.5$ kN in (i)

$$P = \frac{3.5(0.94 - 0.643)}{\cos 75°} = \frac{1.04}{0.259} = 4.0 \text{ kN}$$

$$P = 4.0 \text{ kN} \quad Ans.$$

2.9 CONSTRAINT, ACTION AND REACTION

A body is not always free to move in all directions. *The restriction to the free motion of a body in any direction is called a constraint.*

A ball resting on a frictionless horizontal plane is free to move in a horizontal direction. But, it cannot move vertically downward. So, a horizontal plane is a support which restricts or constraints the motion of the ball vertically downwards. The ball presses the plane downward with a force equal to its own weight and the plane in turn must react on the ball or exert an equal upward force on the ball. This is expected according to Newton's third law.

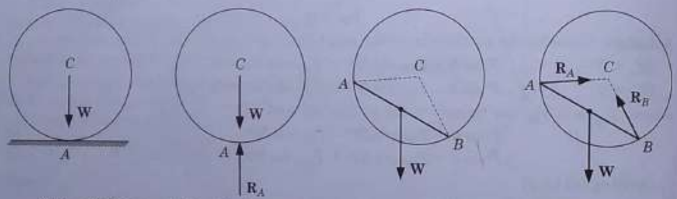
In general, the action of a constrained body on any support induces an equal and opposite reaction from the support. This reaction will be induced in a direction in which the support restricts the motion of the body it supports.

If the linear motion of a body is restricted by the support in some direction, the reaction shall be a *force* in that direction.

If the rotation of a body is restricted about a point, the reaction shall be a *couple* acting on the body about that point.

2.10 TYPES OF SUPPORTS AND SUPPORT REACTIONS

1. Frictionless Support. The reaction acts normal to the surface at the point of contact as shown in Fig. 2.19.



Sphere Resting on a Horizontal Plane *A Rod Resting Inside a Sphere*

Fig. 2.19

2. Roller and Knife Edge Supports. The roller and the knife edge restrict the motion normal to the surface of the beam *AB*. So, reactions $\mathbf{R}_A$ and $\mathbf{R}_B$ shall act normal to the surface at the points of contact *A* and *B* as shown in Fig. 2.20.

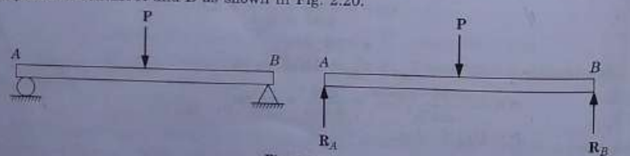


Fig. 2.20

3. Hinged Support. The hinge restricts the motion of the end *A* of the beam *AB* both in the horizontal as well as vertical directions. Thus there are two *independent* reactions X_A and Y_A acting on the beam at *A*.

These two rectangular components can be combined into a single force or reaction $\mathbf{R}_A$. Therefore, *the reaction at the hinge can be represented by a single force* $\mathbf{R}_A$ *in an unknown direction or by the components* X_A and Y_A. The reaction at a hinge whether represented by a single force or by its two rectangular components, involves two unknowns (one direction one magnitude or two magnitudes). This is shown in Fig. 2.21.

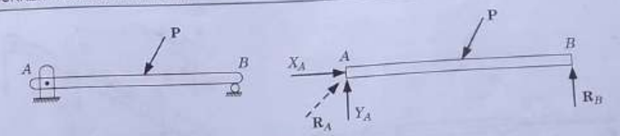


Fig. 2.21

4. Built-in-Support. If the end *A* of a beam *AB* is embedded in the concrete, it restricts the motion of the end *A in* the horizontal and the vertical directions. It also restricts the rotation of the beam *AB* about the point *A*. The reactions X_A and Y_A, therefore, shall be exerted both in the horizontal and the vertical directions accompanied by a reaction couple M_A as shown in Fig. 2.22.

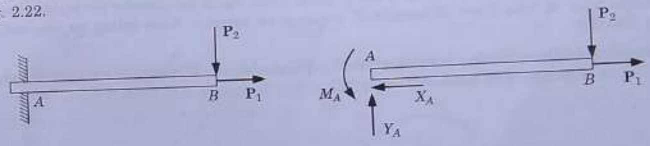


Fig. 2.22

2.11 FREE-BODY DIAGRAM

To clearly identify the various forces acting on a body in equilibrium we have to draw its free-body diagram. Only then, we can write the equations of the equilibrium of the body. This concept should be thoroughly mastered before attempting any further study of the subject.

To draw the free-body diagram of a body we remove all the supports (like wall, floor, hinge or any other body) and replace them by the reactions which these supports exert on the body.

Before discussing this concept of free-body diagram further, let us consider two types of forces that act on a body. They are : *external forces* and *internal forces*.

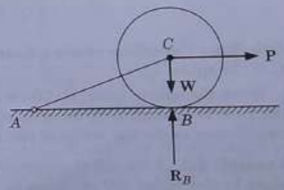
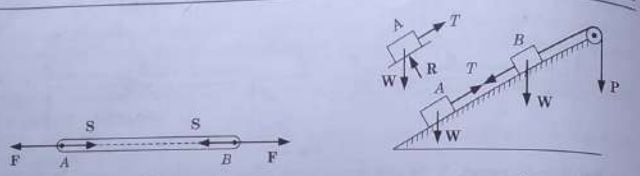


Fig. 2.23

External Forces. These are forces which act on a body or a system of bodies from outside. For example, in the case of the roller shown in the Fig. 2.23, (*i*) Weight of roller **W**, (*ii*) Applied force **P** and (*iii*) The reaction $\mathbf{R}_B$ at the point of contact, are the external forces acting on the roller.

Internal Forces. Are those forces which hold together the particles of a body. And if, more than one body is involved, it may be the force that holds the two bodies together.

If we try to pull a bar by applying two equal and opposite forces **F** then an internal force **S** comes into play to hold the body together [Fig. 2.24 (*a*)].



(a) (b)

Fig. 2.24

If two bodies A and B connected by a string are held on an inclined plane by a force $\mathbf{P}$, then the force of tension $\mathbf{T}$ in the string is the internal force. But if we consider the equilibrium of a single body A, this force of tension $\mathbf{T}$ becomes an external force acting on the body A [Fig. 2.24 (b)].

Example 2.5 Draw the free-body diagram of a sphere of weight W resting on a frictionless plane surface as shown in Fig. 2.25.

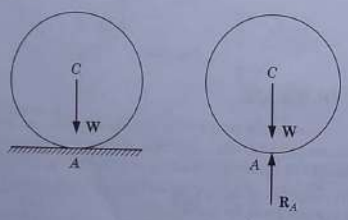


Fig. 2.25

Solution: The sphere exerts a downward force $\mathbf{W}$ on the surface acting through the centroid C of the sphere.

When the sphere is isolated from the surface, a reaction $\mathbf{R}_A$ which is equal and opposite to the force $\mathbf{W}$ is exerted on the sphere by the surface at the point of contact A. The sphere is in equilibrium under the action of two equal and opposite forces $\mathbf{W}$ and $\mathbf{R}_A$ which are collinear.

Example 2.6 *A bar AB of weight W is hinged at A to the wall and is supported in a vertical plane by the string BD as shown in Fig. 2.26. Draw the free-body diagram of the bar.*

Solution: The forces acting on the bar when isolated from the support are : (i) a force $\mathbf{W}$ equal to its own weight and acting vertically downward (ii) the pull $\mathbf{T}$ of the string along BD and (iii) the reaction $\mathbf{R}_A$ at the hinge in an unknown direction. Since the bar AB is in equilibrium under the action of three non-parallel forces, so they must pass through the point E. It thus fixes the unknown direction of the reaction $\mathbf{R}_A$ as shown in Fig. 2.26.

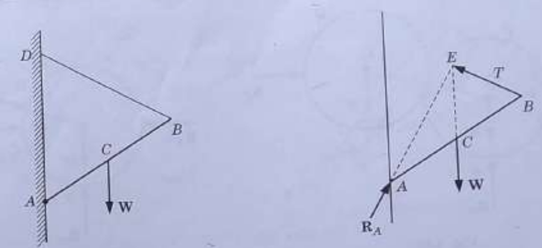


Fig. 2.26

Example 2.7 Two similar sphere P and Q (Fig. 2.27) each of weight W rest inside a hollow cylinder which is resting on a horizontal plane. Draw the free-body diagrams of :

(a) Both the spheres taken together.
(b) The sphere P.
(c) The sphere Q.

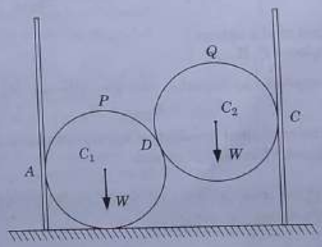


Fig. 2.27

Solution: Let $\mathbf{R}_A$, $\mathbf{R}_B$ and $\mathbf{R}_C$ be the reactions at the points A, B and C *of* the cylinder and the plane *on* the spheres.

(a) **Free-body diagram of spheres P and Q.**
Note that the reaction at the point of contact D does not appear in the free-body diagram, it being an internal force between the two spheres (Fig. 2.28).

(b) **Free-body diagram of sphere P.**
$\mathbf{R}_{D(Q)}$ is the reaction of the sphere Q on the sphere P at the point of contact D, acting in the direction normal to the surface. That is, along C_1C_2 (Fig. 2.29).

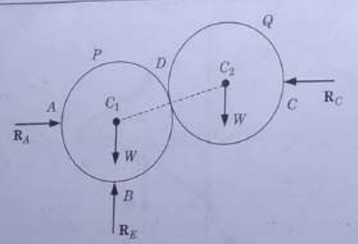


Fig. 2.28

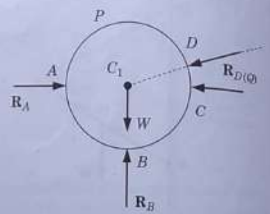


Fig. 2.29

(c) **Free-body diagram of sphere Q.**

$\mathbf{R}_{D(P)}$ is the reaction of the sphere P on the sphere Q acting at the point of contact D (Fig. 2.30).

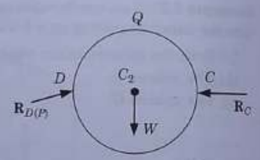


Fig. 2.30

It should be noted that,

$$\left\{\begin{array}{l}\text{The reaction of the sphere}\\ P \text{ on the sphere } Q, \mathbf{R}_{D(P)}\end{array}\right\}$$

and

$$\left\{\begin{array}{l}\text{The reaction of the sphere}\\ Q \text{ on the sphere } P, \mathbf{R}_{D(Q)}\end{array}\right\}$$

are equal in magnitude, opposite in direction and are collinear. So,

$$R_{D(P)} = R_{D(Q)} = R_D$$

They do not appear in the combined free-body diagram of the two spheres as they form a pair of equal, opposite and collinear forces that cancel.

2.12 (*A*)EQUILIBRIUM OF A BODY SUBJECTED TO TWO FORCES (TWO FORCE BODY)

If a rigid body is subjected to forces acting only at the two points, it is called a two force body. The equilibrium of a rigid body acted upon by only two forces, is of considerable interest and is being separately discussed.

Consider a rigid body in shape of a L-shaped plate acted upon by two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ at the ends A and B respectively as shown in Fig. 2.31 (*a*).

Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction and have the same line of action.

So, the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ as shown in the Fig. 2.31 (*a*) cannot keep the body in equilibrium when $\mathbf{F}_1 = \mathbf{F}_2$.

Whereas, forces, $\mathbf{F}_1$ and $\mathbf{F}_2$ as shown in the Fig. 2.31 (*b*) can keep the body in equilibrium when $\mathbf{F}_1 = \mathbf{F}_2$.

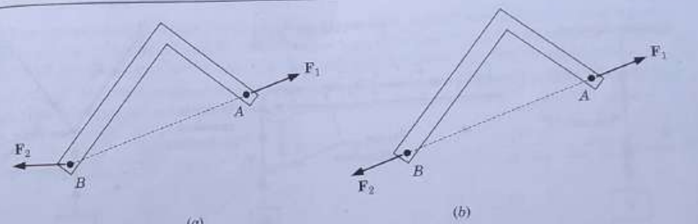


Fig. 2.31

A prismatic bar AB, with two forces acting at its ends A and B can be in equilibrium if, two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal in magnitude opposite in direction and are collinear with the centre line of the bar as shown in the Fig. 2.32.

Fig. 2.32

Example 2.8 A body ABC of negligible weight is hinged at A with a force F acting at its end C. Determine the angle θ which this force should make with the horizontal to keep the edge AB of the body vertical.

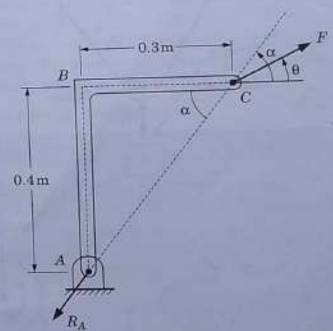


Fig. 2.33

Solution: The body ABC is a two force body. At the point C force F is acting at angle θ with the horizontal and at the point A the reaction R_A of the hinge is acting in an unknown direction. These two forces should be equal, opposite and collinear for the body to be in equilibrium. To be collinear both forces should act along the line joining points A and C such that the edge AB remains vertical.

Therefore, $\theta = \alpha$

$$\tan\alpha = \frac{0.4}{0.3} = 3.333$$

$$\alpha = 53.13° \quad \textit{Ans.}$$

Tension in the String, Rope, Belt, Cable and Chain. Consider a weight W [Fig. 2.34 (*a*)] supported at the end of a string attached to a fixed support. Tension in the string is an internal force marked as shown.

In a *continuous* string, rope, belt, cable and chain passing over a pulley etc., the tension remains same throughout provided,

1. String, rope, cable etc., are assumed to be inextensible and massless.
2. Pulleys etc., are assumed to be massless.
3. Frictionless conditions are assumed to exist.

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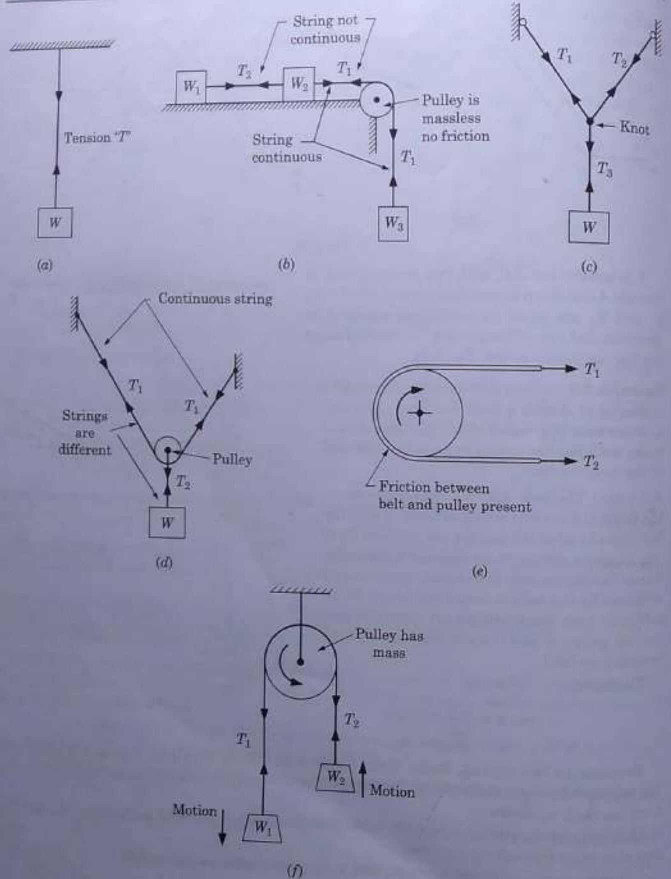


Fig. 2.34 Tension in the string

Representation of Axial Forces in Bar. Consider a bar AB with two equal and opposite forces acting at its ends A and B. If this bar is in equilibrium, these forces must be collinear with the geometric axis of the bar, and *this bar is a two force member in equilibrium.*

In Fig. 2.35 (*a*) the external forces acting in this fashion are trying to pull or elongate the bar. The bar here is said to be under *tensile forces.*

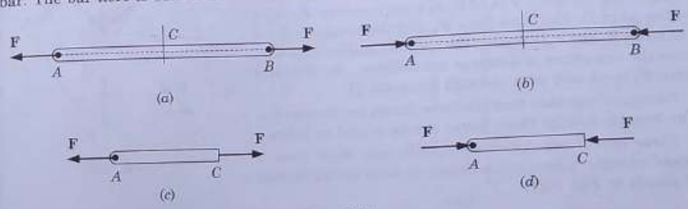
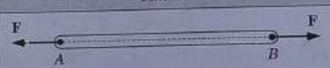
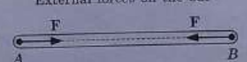


Fig. 2.35

In Fig. 2.35 (*b*) the external forces acting this fashion are trying to push or shorten the bar. The bar here is said to be under *compressive forces.*

Now let us cut the bar AB at C. The two portions of the bar AC and CB are now in equilibrium because of the internal resisting force **F** coming into play. This internal force is equal and opposite to the external force **F** and is collinear with it. This is shown in Fig. 2.35 (*c*) and (*d*).

Table 2.1 The Representation of the Internal Force in the Bar

Tension	*Compression*
 External forces on the bar	 External forces on the bar
 Internal forces in the bar (tensile)	 Internal forces in the bar (compressive)
 A portion of the bar in equilibrium	 A portion of the bar in equilibrium

2.12 (*B*) EQUILIBRIUM OF A BODY SUBJECTED TO THREE FORCES

When a body is acted upon by three coplanar forces it can be in equilibrium if either the lines of action of the three forces intersect at one point (that is concurrent) or they are parallel.

To prove the above statement let us consider a rigid body with three non-parallel froces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, acting at points A, B and C respectively (Fig. 2.36).

Let the lines of action of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ intersect at D. Transmit the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ to act through the point D. Replace now the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ by their resultant $\mathbf{R}$ acting at the point D. The third force $\mathbf{F}_3$ and the resultant $\mathbf{R}$ (of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$) can keep the body in equilibrium if they have the same lines of action or are collinear. So, the third force $\mathbf{F}_3$ must also pass through the point D.

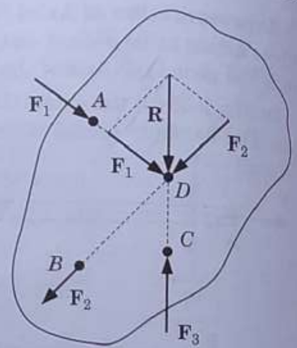


Fig. 2.36

So we can conclude that the three forces are concurrent. The triangle law for these forces can be stated as below,

Three concurrent forces in equilibrium must form a closed triangle of force when drawn in head to tail fashion as shown in Fig. 2.37.

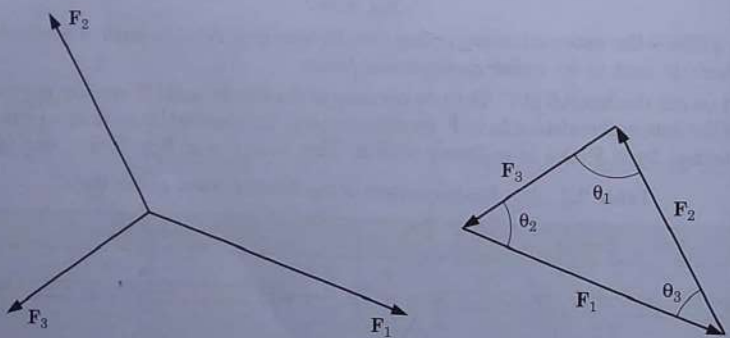


Fig. 2.37

Consider the triangle of force shown. Law of sines can be conveniently used to solve the problem. Thus,

$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}\,. \qquad ...(2.10)$$

If a body is in equilibrium under the action of three non-parallel forces, above method often simplifies the solution rather than using the equilibrium equations.

Example 2.9 Two bars AB and AC are hinged together at C. Their other ends are hinged to a vertical wall at A and B as shown. Find the axial forces in the bar.

Solution: Axial forces include in the bars AC and BC are internal forces. But, when we consider the free-body diagram of the point C these become the external forces acting at the point C. Let these be S_{AC} and S_{BC}. *It is important to note here that each bar is acted upon by only two forces at its ends.* One of these forces is the component of 1000 N and the other is the reaction at the hinge. *So each bar is a two force member.*

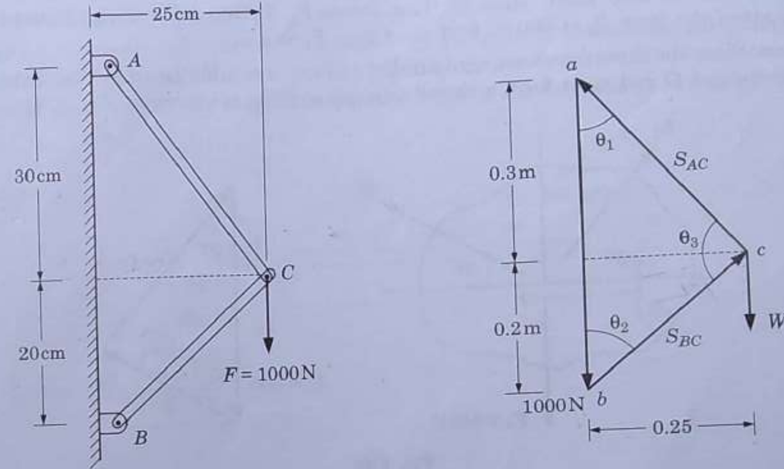


Fig. 2.38

Free-body of the point C is as shown in Fig. 2.38. As the point C is in equilibrium under the action of three forces they must be concurrent and should form a closed triangle of force abc similar to the triangle ABC.

$$\frac{S_{AC}}{AC} = \frac{S_{BC}}{BC} = \frac{1000}{AB} \qquad \text{where,} \quad AC = \sqrt{(0.3)^2 + (0.25)^2}$$

$$\frac{S_{AC}}{0.3905} = \frac{S_{BC}}{0.3206} = \frac{1000}{0.5} \qquad AC = 0.3905 \text{ m}$$

$$S_{AC} = \frac{0.3905 \times 1000}{0.5} = 780 \text{ N} \quad \textit{Ans.} \qquad BC = \sqrt{(0.20)^2 + (0.25)^2}$$

$$S_{BC} = \frac{0.3206 \times 1000}{0.5} = 640 \text{ N} \quad \textit{Ans.} \qquad BC = 0.3206 \text{ m}$$

We can also apply sine law to the force triangle abc,

$$\frac{S_{AC}}{\sin\theta_2} = \frac{S_{BC}}{\sin\theta_1} = \frac{1000}{\sin\theta_3} \qquad \text{where,} \quad \sin\theta_1 = \frac{0.25}{0.3905} = 0.64$$

$$\frac{S_{AC}}{0.78} = \frac{S_{BC}}{0.64} = \frac{1000}{0.999} \qquad \theta_1 = 39.87°$$

$$S_{AC} = 780 \text{ N} \quad \textit{Ans.} \qquad \sin\theta_2 = \frac{0.25}{0.3206} = 0.78$$

$$S_{BC} = 640 \text{ N} \quad \textit{Ans.} \qquad \theta_2 = 51.26°$$

$$\theta_3 = 180 - (\theta_1 + \theta_2)$$

$$\theta_3 = 89.87°$$

$$\sin\theta_3 = 0.9999$$

Example 2.10 A body acted upon by three forces F_1, F_2 and F_3 is in equilibrium. If the magnitude of the force F_3 is 500 N, find the forces F_1 and F_2.

Solution: Since the three forces are non-parallel and are in equilibrium they must intersect at a common point O and must form a closed triangle of force as shown in Fig. 2.39.

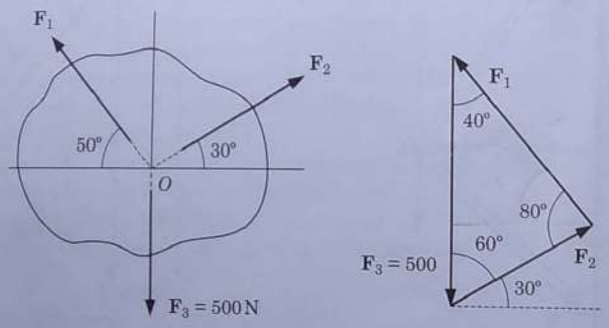


Fig. 2.39

Applying law of sines; $\dfrac{F_3}{\sin 80^\circ} = \dfrac{F_1}{\sin 60^\circ} = \dfrac{F_2}{\sin 40^\circ}$

$$F_1 = \frac{500 \times \sin 60^\circ}{\sin 80^\circ} = \frac{500 \times 0.866}{0.985}$$

$$F_1 = 439.6 \text{ N} \quad \textit{Ans.}$$

$$F_2 = \frac{500 \times \sin 40^\circ}{\sin 80^\circ} = \frac{500 \times 0.643}{0.985}$$

$$F_2 = 326.4 \text{ N} \quad \textit{Ans.}$$

Alternative Method. Let us solve this problem by writing the equations of equilibrium

$\Sigma F_x = 0$:

$$F_2 \cos 30^\circ - F_1 \cos 50^\circ = 0 \quad \ldots(i)$$

$\Sigma F_y = 0$:

$$-F_3 + F_2 \sin 30^\circ + F_1 \sin 50^\circ = 0 \quad \ldots(ii)$$

or

$$0.866\, F_2 - 0.643\, F_1 = 0$$

$$0.5\, F_2 + 0.766\, F_1 = 500$$

Solving the two equations simultaneously, we get

$F_1 = 439.6$ N *Ans.*

$F_2 = 326.4$ N *Ans.*

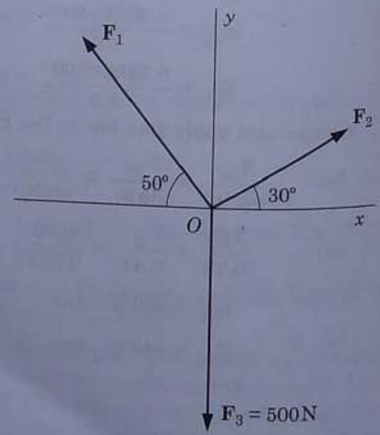


Fig. 2.40

Reaction at the Hinge

Example 2.11 A force of $P = 5000$ N is applied at the centre C of the beam AB of length 5 m. Find the reactions at the hinge and roller supports.

Solution: The reaction, R_B at the roller acts normal to the beam as shown in Fig. 2.41.

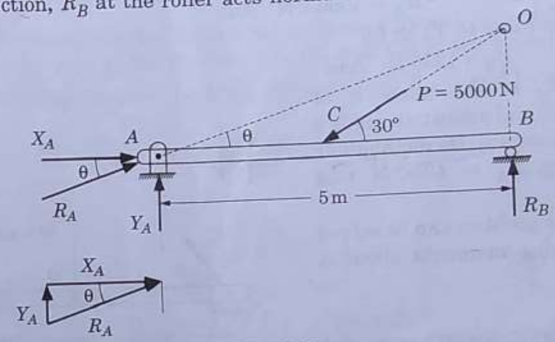


Fig. 2.41

The hinge is capable of resisting the motion of the end A in vertical as well as in horizontal directions. Therefore, in general, the reaction at the hinge consists of two components, X_A and Y_A, acting in horizontal and vertical directions respectively.

The reaction at the hinge can also be indicated by a single resultant reaction R_A acting in an unknown direction.

Let us first consider the reaction as a single force R_A to be acting at A. The three forces (R_A, R_B and P) acting on the beam are in equilibrium and so must pass through a common point O. This fixes the direction θ of the reaction R_A.

$$\frac{OB}{CB} = \tan 30^\circ \quad \text{or} \quad OB = CB \tan 30^\circ = 2.5 \times 0.577$$

$$OB = 1.443 \text{ m}$$

$$\frac{OB}{AB} = \tan\theta \quad \text{or} \quad \tan\theta = \frac{1.443}{5.0} = 0.2886\ ;\ \theta = 16.1^\circ \quad \ldots(i)$$

Also, $\quad \tan\theta = \dfrac{Y_A}{X_A} \quad \ldots(ii)$

Hence $\quad \dfrac{Y_A}{X_A} = 0.2886 \quad \ldots(iii)$

Let us now write the equations of equilibrium in terms of rectangular components.

$\Sigma F_x = 0$: $\quad X_A - 5000 \cos 30^\circ = 0 \quad \ldots(iv)$

$$X_A = 5000 \times 0.866$$

$$X_A = 4330 \text{ N} \quad \textit{Ans.}$$

$\Sigma F_y = 0$: $\quad Y_A + R_B - 5000 \sin 30^\circ = 0 \quad \ldots(v)$

From (*iii*) and (*iv*),

$$Y_A = 0.2886 \times 4330 \text{ N}$$
$$Y_A = 1250 \text{ N} \quad \textit{Ans.}$$
$$R_A = \sqrt{(X_A)^2 + (Y_A)^2} = \sqrt{(4330)^2 + (1250)^2}$$
$$R_A = 4506.8 \text{ N} \quad \textit{Ans.}$$

Substituting the value of Y_A in (*v*)

$$R_B = 1250 \text{ N} \quad \textit{Ans.}$$

The reaction R_A at the hinge is of magnitude 4506.8 N and acts at an angle θ = 16.1° with the horizontal. Its rectangular components being X_A = 4330 N and Y_A = 1250 N.

Alternatively, the problem can be solved more easily by taking moments about A (Ref. Art. 2.13)

$\Sigma M_A = 0$:

$$R_B \times 5 - 5000\ (2.5 \sin 30°) = 0 \quad ...(vi)$$

(Sum of the moments about A is zero).

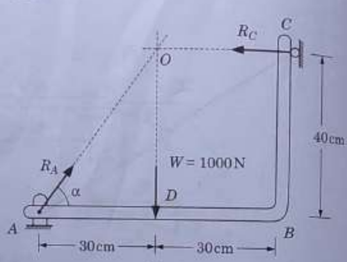


Fig. 2.42

The problem thus can be solved using equations (*iv*), (*v*) and (*vi*) without the need of fixing geometrically the direction θ of the reaction R_A.

Example 2.12 A corner plate ABC is hinged to a fixed support at A and rests on a roller at C. If a force of W = 1000 N is acting as shown find the reactions at the supports.

Solution: Consider the free-body diagram of the plate, the forces acting on the plate are,

1. force W = 1000 N, acting vertical downwards.
2. reaction R_C acting normal to surface of contact, that is, acting horizontally
3. reaction at the hinge R_A.

As the plate is in equilibrium under the action of three forces, the forces must pass through a point as shown in Fig. 2.42.

Forces W and R_C intersect at point O therefore, the reaction R_A at the hinge must pass through O.

The three forces must form a closed triangle which is similar to triangle AOD. Therefore,

$$\frac{W}{OD} = \frac{R_C}{AD} = \frac{R_A}{AO} \qquad (AO = \sqrt{0.4^2 + 0.3^2})$$

$$\frac{W}{0.4} = \frac{R_C}{0.3} = \frac{R_A}{0.5} \qquad (AO = 0.5 \text{ m})$$

$$R_A = \frac{1000 \times 5}{4} = 1200 \text{ N} \quad \textit{Ans.}$$

$$R_C = \frac{1000 \times 3}{4} = 750 \text{ N} \quad \textit{Ans.}$$

Angle α that R_C makes with the horizontal is

$$\tan \alpha = \frac{0.4}{0.3}$$

or

$$\alpha = 53.1° \quad \textit{Ans.}$$

Example 2.13 Two equal loads of 2500 N are supported by a flexible string $ABCD$ at points B and C. Find the tensions in the portions AB, BC and CD of the string.

Solution: Let the tension in the portions AB, BC and CD be T_1, T_2 and T_3 respectively as shown in Fig. 2.43 (*a*).

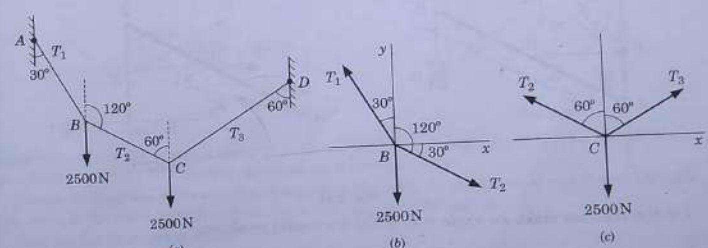


Fig. 2.43

Consider the free-body diagram of point B. Writing the equations of equilibrium.

$\Sigma F_x = 0$:

$$-T_1 \sin 30° + T_2 \cos 30° = 0$$

$$T_1 = \frac{T_2 \cos 30°}{\sin 30°} \quad ...(i)$$

$\Sigma F_y = 0$:

$$T_1 \cos 30° - 2500 - T_2 \sin 30° = 0 \quad ...(ii)$$

Substituting for T_1

$$T_2 \left(\frac{\cos 30°}{\sin 30°}\right) \cos 30° - T_2 \sin 30° = 2500$$

$$T_2 = 2500 \text{ N} \quad \textit{Ans.}$$

$$T_1 = \frac{T_2 \cos 30°}{\sin 30°} = \frac{2500 \times 0.866}{0.5}$$

$$T_1 = 4330 \text{ N} \quad \textit{Ans.}$$

Consider the free-body diagram of point C. Writing the equation of equilibrium

$\Sigma F_x = 0$:

$$-T_2 \cos 30° + T_3 \cos 30° = 0$$
$$T_3 = T_2$$
$$T_3 = 2500 \text{ N} \quad \textit{Ans.}$$

This problem involves the equilibrium of three concurrent forces and can also be solved using the law of triangle of forces.

Example 2.14 A rigid prismatic weightless bar AB is supported in a vertical plane by a hinge at the end A and by a horizontal string attached to the bar at D as shown. The end B of the bar carries a load W. Find the tension in the string and the direction of the reaction at the hinge in terms of W and θ.

Solution: The free-body diagram of the bar AB is as shown. The load W acts at B vertically downwards, tension T acts at D horizontally and reaction at the hinge acts along AE. As the three forces keep the bar in equilibrium they must be concurrent. W and T intersect at E so the reaction R_A must pass through E (three force body) as shown in Fig. 2.44.

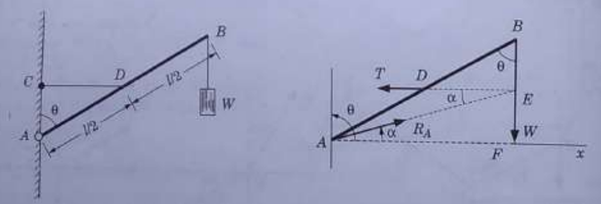


Fig. 2.44

Let the reaction make an angle α with the horizontal as shown,

$$BE = \frac{l}{2}\cos\theta = EF$$

$$\tan\alpha = \frac{EF}{AF} = \frac{\frac{l}{2}\cos\theta}{l\sin\theta} = \frac{\cot\theta}{2}$$

$$\alpha = \tan^{-1}\left(\frac{\cot\theta}{2}\right)$$

Consider the equilibrium of the bar AB

$\Sigma F_x = 0$: $\quad R_A\cos\alpha - T = 0$ or $R_A\cos\alpha = T$ ...(*i*)

$\Sigma F_y = 0$: $\quad R_A\sin\alpha - W = 0$ or $R_A\sin\alpha = W$ ...(*ii*)

Dividing (*ii*) by (*i*)

$$\tan\alpha = \frac{W}{T}$$

$$T = \frac{W}{\tan\alpha}$$

using,

$$\tan\alpha = \frac{\cot\theta}{2}$$

$$T = \frac{2W}{\cot\theta} = 2W\tan\theta$$

$$T = 2W\tan\theta \quad \textit{Ans.}$$

Example 2.15 A uniform wheel of 60.0 cm diameter and weighing 1000 N rests against a rectangular block 15 cm high lying on a horizontal plane as shown in the Fig. 2.45. It is to be pulled over this block by a horizontal force P applied to the end of a string wound round the circumference of the wheel. Find the force P when the wheel is just about to roll over the block.

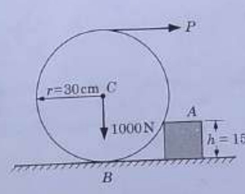


Fig. 2.45

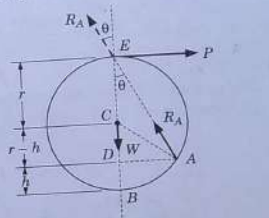


Fig. 2.46

Solution: When the wheel is about to roll over the block, it just lifts of the horizontal plane and loses contact at B. Therefore, reaction at B becomes zero.

Free-body diagram of the wheel is as shown in Fig. 2.46. As the forces W and P pass through the point E the third force (reaction R_A) must also pass through E.

Writing the equation of equilibrium

$\Sigma F_x = 0$: $\quad P - R_A\sin\theta = 0$ or $R_A\sin\theta = P$ ...(*i*)

$\Sigma F_y = 0$: $\quad R_A\cos\theta - W = 0$ or $R_A\cos\theta = W$ ...(*ii*)

Dividing (*i*) by (*ii*)

$$\tan\theta = \frac{P}{W}$$

In triangle ADE,

$$\tan\theta = \frac{DA}{DE} \qquad DE = 2r - h$$

$$DA = \sqrt{CA^2 - CD^2}$$

$$\tan\theta = \frac{\sqrt{r^2-(r-h)^2}}{(2r-h)} = \frac{\sqrt{2rh-h^2}}{(2r-h)} \qquad DA = \sqrt{r^2-(r-h)^2}$$

$$\tan\theta = \frac{\sqrt{h}\,\sqrt{(2r-h)}}{(2r-h)} = \frac{\sqrt{h}}{\sqrt{2r-h}} = \frac{\sqrt{0.15}}{\sqrt{0.6-0.15}}$$

$$\tan\theta = 0.577$$

But, $\quad \dfrac{P}{W} = \tan\theta$

Therefore, $\quad P = W\tan\theta = 1000 \times 0.577$

$$P = 577\text{ N} \quad \textit{Ans.}$$

Example 2.16 An octagonal frame with radial members consists of hinged bars as shown in Fig. 2.47. In the radial member OA a tensile force of 1000 N is produced by means of turn buckle T. Determine the axial forces in the bars AB and OB.

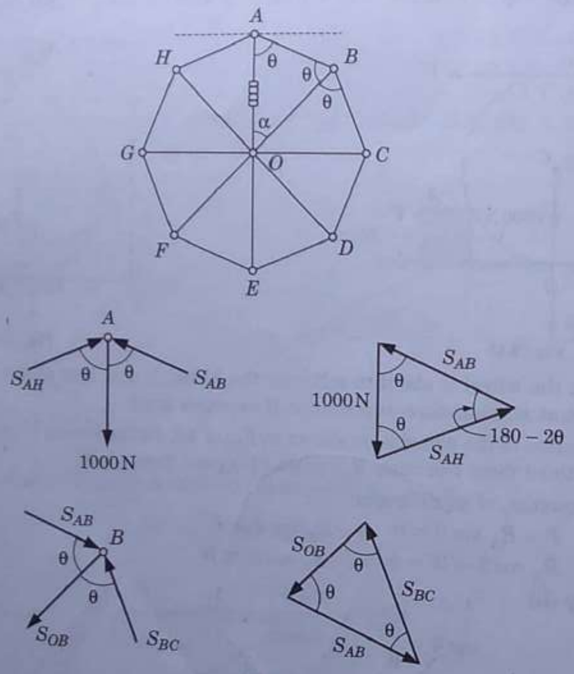


Fig. 2.47

Solution:

$$\angle\alpha = \frac{360^\circ}{8} = 45^\circ$$

$$\angle\theta = \frac{180^\circ - 45^\circ}{2} = 67.5^\circ$$

The free-body diagram of the joint A is as shown, where S_{AH} and S_{AB} are the axial forces in the bars AH and AB. Three forces in equilibrium should form a closed triangle. Using the law of sines,

$$\frac{S_{AB}}{\sin\theta} = \frac{S_{AH}}{\sin\theta} = \frac{1000}{\sin(180-2\theta)}$$

$$S_{AB} = S_{AH} = 1000 \times \frac{\sin\theta}{\sin 45^\circ}$$

$$S_{AB} = S_{AH} = \frac{1000 \times 0.924}{0.707}$$

$$S_{AB} = S_{AH} = 1306.8 \text{ N (Comp.)} \quad \textit{Ans.}$$

Consider the free-body diagram of the joint B. Three forces S_{AB}, S_{OB} and S_{BC} are in equilibrium and should form closed triangle.

Therefore,
$$\frac{S_{OB}}{\sin(180-2\theta)} = \frac{S_{BC}}{\sin\theta} = \frac{S_{AB}}{\sin\theta}$$

$$S_{OB} = \frac{S_{AB}}{\sin\theta}\sin(180-2\theta) = \frac{1306.8}{\sin 67.5^\circ}\sin 45^\circ$$

$$S_{OB} = 1000 \text{ N (Tension)} \quad \textit{Ans.}$$

Example 2.17 A smooth circular cylinder of weight W and radius r rests in a V-shaped groove whose sides are inclined at angle α and β to the horizontal as shown. Find the reactions R_A and R_B at the points of contact. Given $\alpha = 25^\circ$, $\beta = 65^\circ$, $W = 500$ N.

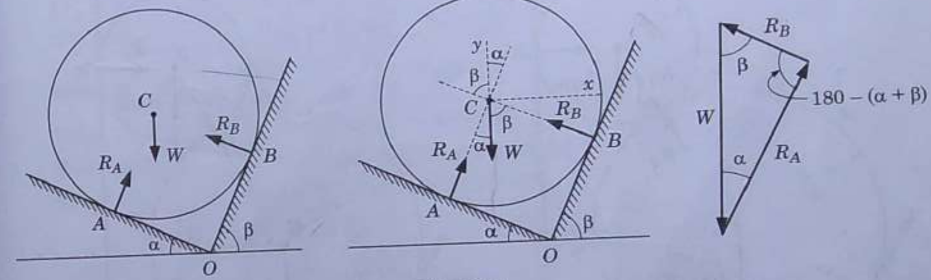


Fig. 2.48

Solution: Consider the equilibrium of the sphere, whose free-body diagram is as shown in Fig. 2.48.

R_A and R_B are the reactions at the points A and B (of the planes on the sphere) and act normal to the planes OA and OB. These, when extended pass through the centre C of the sphere. Weight W acts vertically through C.

Now we have a set of three concurrent forces R_A, R_B and W acting on the sphere. These forces must form a closed triangle as shown. Applying the law of sines

$$\frac{W}{\sin(180-(\alpha+\beta))} = \frac{R_A}{\sin\beta} = \frac{R_B}{\sin\alpha}$$

$$R_A = \frac{W\sin\beta}{\sin(\alpha+\beta)} \quad \text{and} \quad R_B = \frac{W\sin\alpha}{\sin(\alpha+\beta)}$$

Alternative Method. Writing the equations of equilibrium

$\Sigma F_x = 0:$ $\quad R_A \sin\alpha - R_B \sin\beta = 0$

$\Sigma F_y = 0:$ $\quad R_A \cos\alpha + R_B \cos\beta - W = 0$

After solving for R_A and R_B we get the same results as obtained earlier

Substituting, $\alpha = 25°$, $\beta = 65°$, $W = 500$ N

$$R_A = \frac{500 \sin 65°}{\sin 90°}, \quad R_B = \frac{500 \sin 25°}{\sin 90°}$$

$$R_A = 453.2 \text{ N} \qquad R_B = 211.3 \text{ N} \quad \textit{Ans.}$$

Example 2.18 Two cylinders A and B rest in a horizontal channel as shown in Fig. 2.49. The cylinder A has a weight of 1000 N and radius of 9.0 cm. The cylinder B has a weight of 400 N and a radius of 5.0 cm. The channel is 18.0 cm wide at the bottom with one side vertical. The other side is inclined at an angle 60° with the horizontal. Find the reactions at points L, N and P.

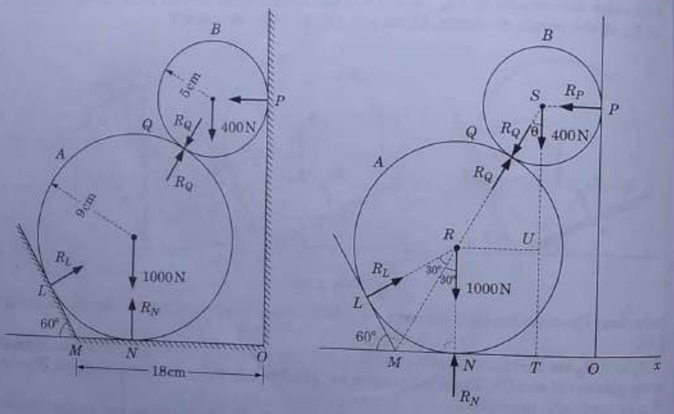


Fig. 2.49

Solution. Consider the free-body diagram of the cylinder B. Various forces acting are:

(*i*) Weight of the cylinder $B = 400$ N, acting downward

(*ii*) Reaction of the channel wall at the point $P = R_P$

(*iii*) Reaction of the cylinder A on the cylinder $B = R_Q$ acting along the line RS, R and S being the centres of the cylinders.

Consider the free-body diagram of the cylinder A. Various forces acting are:

(*i*) Weight of the cylinder $A = 1000$ N acting downward

(*ii*) Reaction of the cylinder B on the cylinder $A = R_Q$ acting along the line RS, R and S being the centres of the cylinders

(*iii*) Reaction of the bottom surface at the point N on the cylinder $= R_N$

(*iv*) Reaction of the channel wall at L on the cylinder $= R_L$.

Except for the reaction R_Q, we know the directions of all other forces. Let θ be the angle that the force R_Q makes with vertical then

$$\sin\theta = \frac{RU}{RS} \qquad RS = 9 + 5 = 14 \text{ cm}$$

$$RU = NT = MO - MN - TO$$

$$\sin\theta = \frac{7.8}{14} = 0.5571 \qquad RU = 18 - MN - 5$$

$$\theta = 33.86° \qquad MN = RN \tan 30° = 9 \times \frac{1}{\sqrt{3}} = 5.2 \text{ cm}$$

$$RU = 18 - 5.2 - 5 = 7.8 \text{ cm}$$

Writing the equations of equilibrium for cylinder B

$$\Sigma F_x = 0: \qquad R_Q \sin 33.86° - R_P = 0 \qquad \text{...}(i)$$

$$\Sigma F_y = 0: \qquad R_Q \cos 33.86° - 400 = 0 \qquad \text{...}(ii)$$

From (*ii*), $R_Q = \dfrac{400}{\cos 33.86}$, $\qquad R_Q = 481.9$ N

From (*i*), $R_P = R_Q \sin 33.86°$, $\qquad R_P = 268.5$ N *Ans.*

Writing the equations of equilibrium for the cylinder A.

$$\Sigma F_x = 0: \qquad R_L \sin 60° - R_Q \sin 33.86° = 0$$

$$R_L = \frac{R_Q \sin 33.86°}{\sin 60°} = \frac{481.9 \times 0.5571}{0.866}$$

$$R_L = 310 \text{ N} \quad \textit{Ans.}$$

$$\Sigma F_y = 0: \qquad R_N - 1000 - R_Q \cos 33.86° + R_L \cos 60° = 0$$

$$R_N = 1000 + 481.9 \times 0.830 - 310 \times 0.5$$

$$R_N = 1248.2 \text{ N} \quad \textit{Ans.}$$

Example 2.19 Two rollers of weights P and Q are connected by a flexible string AB. The rollers rest on two mutually perpendicular planes DE and EF as shown in Fig. 2.50.

Find the tension in the string and the angle θ that it makes with the horizontal when the system is in equilibrium.

Given $\quad P = 50$ N, $\quad Q = 100$ N and $\alpha = 30°$

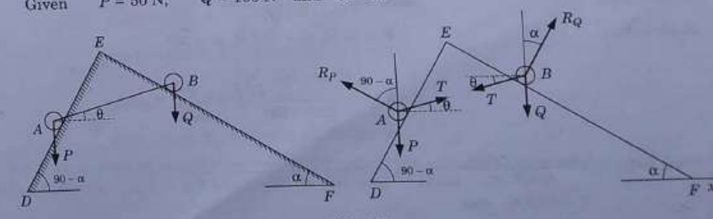


Fig. 2.50

Solution: The free-body diagram of each roller is shown separately in Fig. 2.50. The x and the y axes are along the horizontal and vertical directions.

Let T be the tension of the string. R_P and R_Q be the reactions of the planes on the rollers acting normal to the planes DE and DF respectively. P and Q being weights acting vertically downward. Considering the equilibrium of the roller P.

$\Sigma F_x = 0:$ $\quad T\cos\theta - R_P \sin(90-\alpha) = 0$

$$T\cos\theta = R_P\cos\alpha \qquad \text{...}(i)$$

$\Sigma F_y = 0:$ $\quad R_P\cos(90-\alpha) - P + T\sin\theta = 0$

$$R_P\sin\alpha - P + T\sin\theta = 0 \qquad \text{...}(ii)$$

Considering the equilibrium of the roller Q

$\Sigma F_x = 0:$ $\quad R_Q\sin\alpha - T\cos\theta = 0 \qquad \text{...}(iii)$

$\Sigma F_y = 0:$ $\quad R_Q\cos\alpha - Q - T\sin\theta = 0 \qquad \text{...}(iv)$

The four equations obtained can be solved for four unknowns θ, T, R_P and R_Q

From (*i*)
$$R_P = \frac{T\cos\theta}{\cos\alpha}$$

Substituting in (*ii*)
$$\frac{T\cos\theta}{\cos\alpha}\sin\alpha - P + T\sin\theta = 0$$
$$P = T(\sin\theta + \cos\theta\tan\alpha) \qquad \text{...}(v)$$

From (*iii*)
$$R_Q = \frac{T\cos\alpha}{\sin\alpha}$$

Substituting in (*iv*)
$$\frac{T\cos\theta}{\sin\alpha} - Q - T\sin\theta = 0$$
$$Q = T(\cos\theta\cot\alpha - \sin\theta) \qquad \text{...}(vi)$$

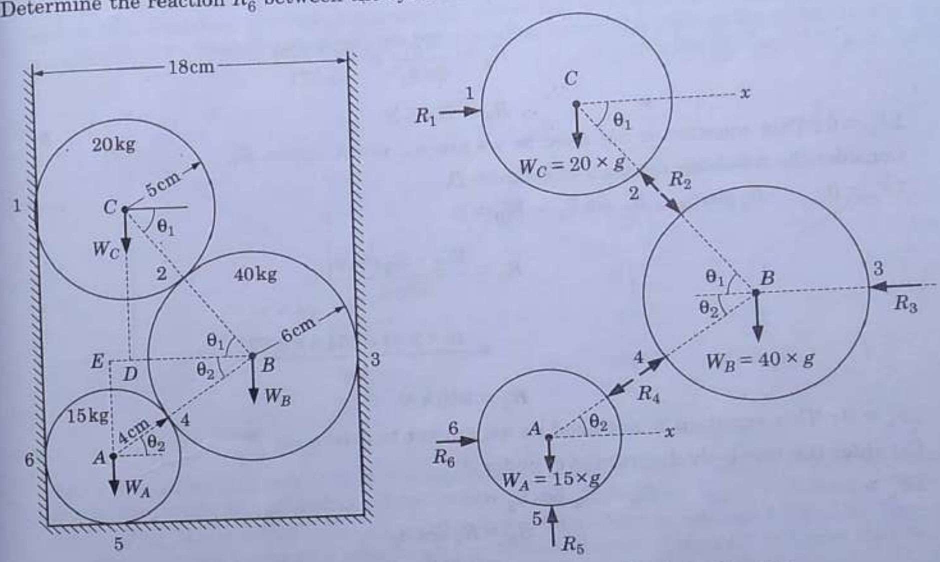
Dividing (*v*) by (*vi*)
$$\frac{Q}{P} = \frac{\cos\theta\cos\alpha - \sin\theta}{\cos\theta\tan\alpha + \sin\theta}$$

Rearranging,
$$\tan\theta = \frac{(P\cot\alpha - Q\tan\alpha)}{(P+Q)}$$

Substituting,
$$P = 50\text{ N},\ Q = 100\text{ N},\ \alpha = 30°$$
$$\tan\theta = \frac{\left(50\sqrt{3} - 100\frac{1}{\sqrt{3}}\right)}{(100+60)} = \frac{1}{3\sqrt{3}}$$
$$\theta = 10.9°$$

From (*vi*)
$$T = \frac{Q}{(\cos\theta\cot\alpha - \sin\theta)} = \frac{100}{(\cos 10.9°\cot 30° - \sin 10.9°)}$$
$$T = 66.2\text{ N} \quad \textit{Ans.}$$

Example 2.20 Three cylinders are piled up in a rectangular channel as shown in Fig. 2.51 (*a*). Determine the reaction R_6 between the cylinder A and the vertical wall of the channel.



(*a*) (*b*) Free-body diagrams

Fig. 2.51

Solution: The forces involved are the weights W_A, W_B, W_C of the three cylinders and reaction R_1, R_2, R_3, R_4, R_5 and R_6 at the various points of contact. Further, R_1, R_3 and R_6 act horizontally, R_5 acts vertically, R_2 acts along the line joining the centres of cylinders B and C and reaction R_4 acts along the line joining the centres of cylinders A and B. So, we have to determine the inclination of the lines BC and AB to enable us to resolve the reactions acting along these lines [Fig. 2.51 (*b*)].

$$\cos\theta_1 = \frac{BD}{BC} = \frac{18-5-6}{6+5} = \frac{7}{11}$$
$$\cos\theta_1 = 0.636,\ \theta_1 = 50.5°,\ \sin\theta_1 = 0.771$$
$$\cos\theta_2 = \frac{BE}{BA} = \frac{18-4-6}{(4+6)} = \frac{8}{10}$$
$$\cos\theta_2 = 0.8,\ \theta_2 = 36.87°,\ \sin\theta_2 = 0.6$$

We have to start by considering the equilibrium of the cylinder C as it involves only two unknown forces.

Consider the free-body diagram of cylinder C.

$\Sigma F_y = 0$: $$R_2 \sin\theta_1 - W_C = 0$$

$$R_2 = \frac{W_C}{\sin\theta_1} = \frac{20 \times 9.81}{0.771}$$

$$R_2 = 254.5\ \text{N}$$

$\Sigma F_x = 0$: This equation is not used as we are not to determine R_1.

Consider the free-body diagram of cylinder B.

$\Sigma F_y = 0$: $$R_4 \sin\theta_2 - R_2 \sin\theta_1 - W_B = 0$$

$$R_4 = \frac{W_B + R_2 \sin\theta_1}{\sin\theta_2}$$

$$= \frac{40 \times 9.81 + 254.5 \times 0.771}{0.6}$$

$$R_4 = 980.8\ \text{N}$$

$\Sigma F_x = 0$: This equation is not used as we are not to determine R_3.

Consider the free-body diagram of cylinder A.

$\Sigma F_x = 0$: $$R_6 - R_4 \cos\theta_2 = 0$$

$$R_6 = R_4 \cos\theta_2$$

$$= 980.8 \times 0.8$$

$$R_6 = 784.6\ \text{N}$$

$\Sigma F_y = 0$: This equation is not used as we are not to determine R_5.

PROBLEMS

2.1. Find the resultant of coplanar concurrent forces acting at the point O. [49 N, $\theta = -26°$]

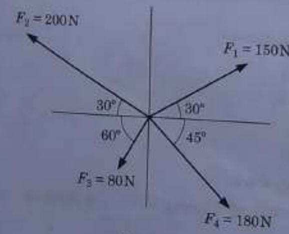


Fig. P.2.1

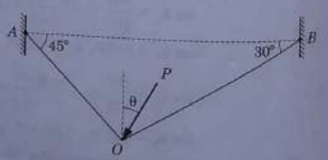


Fig. P.2.2

2.2. A force P is applied at O to the string AOB. If the tension in each part of the string is 50 N, find the magnitude and direction of force P for equilibrium conditions. [61 N, $\theta = 7.5°$]

2.3. A 300 N force is applied at C. Determine (*a*) the value of angle θ for which the larger of the cable tension is as small as possible (*b*) the corresponding values of tension in the cables AC and BC. [$T_{AC} = 173.2$ N, $T_{BC} = 173$ N, $\theta = 60°$]

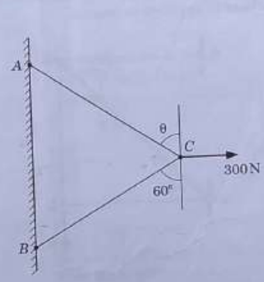


Fig. P.2.3

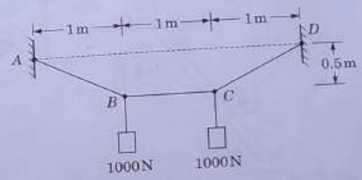


Fig. P.2.4

2.4. Two equal weights each of 1000 N is supported by a flexible string as shown. Find the tensions in the portion AB, BC and CD of the string. [$T_{AB} = 2236$ N, $T_{BC} = 2000$ N, $T_{CD} = 2236$ N]

2.5. A string ACB of length l carries a small pulley C from which is suspended a load W. The string hangs between two vertical walls which are distance d apart. The end A is higher than the end B by height h. Find the position of equilibrium as defined by the angle α. Given,

$$d = \frac{l}{2},\ h = \frac{l}{4}$$ [$\alpha = 60°$]

[**Hint:** Tensions in the portions AC and BC are equal].

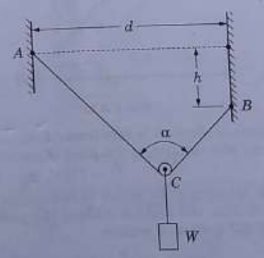


Fig. P.2.5

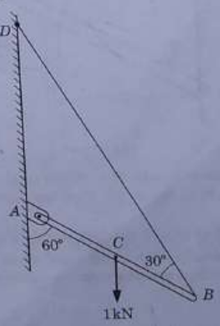


Fig. P.2.6

2.6. A bar AB of weight 1 kN is hinged to a vertical wall at A and supported at the end B by a cable BD. Find the tension in the cable and the magnitude and direction of reaction at the hinge.
[T = 866 N, R_A = 500 N acting at angle θ = 30° with the horizontal]

2.7. A weightless bar AB is hinged at A and is acted upon by two equal forces as shown. Determine the angle θ which the bar will make with the vertical in equilibrium position. [θ = 45°]

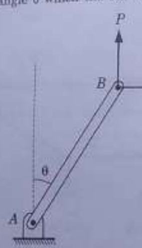


Fig. P.2.7

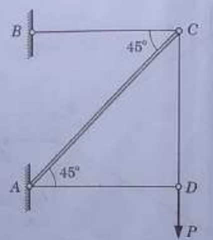


Fig. P.2.8

2.8. Determine the axial forces produced in the hinged bars as shown in Fig. P.2.8 due to the load P acting at D.
[$S_{AD} = 0$, $S_{CD} = P(T)$, $S_{AB} = \sqrt{2}\,P(C)$, $S_{BC} = P(T)$]

2.9. A bar of negligible weight is hinged to a wall at A and supported by a string as shown in the Fig. P.2.9. If a vertical load W acts at B find the tension in the string and axial force in the bar.
[$T = P \tan\theta$, $S = P \sec\theta$ (C)]

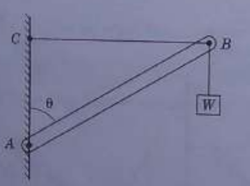


Fig. P.2.9

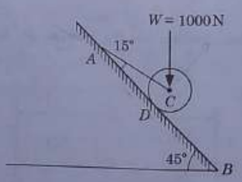


Fig. P.2.10

2.10. A roller of weight W = 1000 N rests on a smooth inclined plane. It is kept from rolling down the plane by a string AC. Find the tension in the string and reaction at the point of contact D.
[T = 733 N, R_D = 897 N]

2.11. A ball of weight W resting upon a smooth plane is attached at its centre to two strings which pass over smooth pulleys and carry loads P and Q. Find the angle θ defining the position of equilibrium and the reaction between the ball and the horizontal surface.

$$\left[\theta = \sin^{-1}\left(\frac{P}{Q}\right),\ R = W - \sqrt{Q^2 - P^2}\right]$$

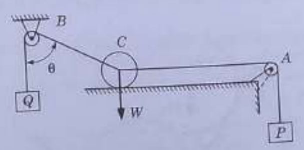


Fig. P.2.11

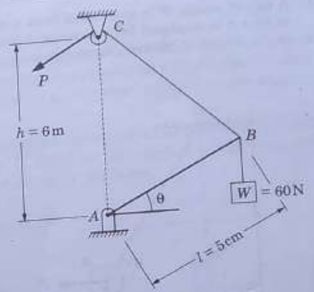


Fig. P.2.12

2.12. A prismatic bar AB of length l = 5 m and of negligible weight is hinged at A and supported at B by a string that passes over a pulley C. A vertical load of 60 kN applied at the end B of the bar is supported by a force P applied to the string. Find the axial force in the bar and the limiting value of the tension T when the bar approaches vertical position. Distance between the hinge and the pulley h = 6 m.
[S_{AB} = 50 kN (C), T = 10 kN]

2.13. A bar AB of weight 100 N is hinged at A and is pulled by a cable attached at B by a force F. Find the force F and the magnitude and direction of the reaction at A if the bar is in equilibrium position as shown in Fig. P.2.13.
[F = 70.72 N, R_A = 136.6 N, 60°]

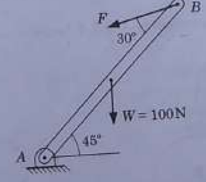


Fig. P.2.13

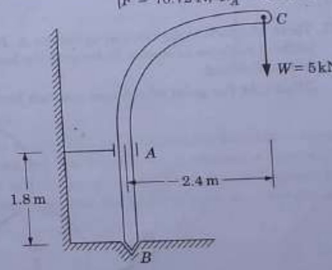


Fig. P.2.14

2.14. A crane is pivoted at the end B and is supported by a guide at A. Determine the reaction produced at A and B by a vertical load W = 5 kN applied at C.
[R_A = 6.665 kN, R_B = 8.33 kN]

2.15. A ball of weight W is suspended from a string of length l and is pulled by a horizontal force P so that it is displaced by a distance d from the vertical position. Find the tension in the string in the displaced position, force P required and the angle θ.

$$\left[P = W \cot\theta, T = W \text{ cosec } \theta \text{ where, } \theta = \cos^{-1}\frac{d}{l}\right]$$

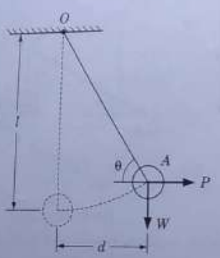


Fig. P.2.15

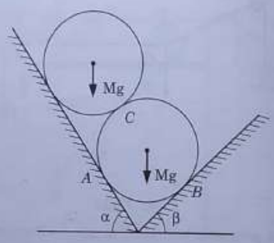


Fig. P.2.16

2.16. A sphere of mass M rests in a V-groove whose sides are inclined at angles α and β to the horizontal. Another identical sphere of same mass M rests on the first sphere and in contact with the side inclined at angle α. Find the reaction R_B on the lower sphere at the point B.

$$\left[R_B = \frac{2\, Mg \sin\alpha}{\sin(\alpha+\beta)}\right]$$

2.17. Three identical right circular cylinders A, B and C, each of weight W, are arranged on smooth inclined surfaces as shown. Determine the least value of angle θ that will prevent the arrangement from collapsing.

[**Hint :** At the point of collapse reaction between the cylinders A and B becomes zero]. [10.9°]

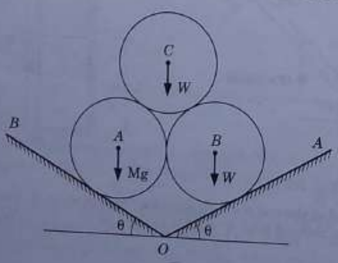


Fig. P.2.17

2.13 MOMENT OF A FORCE

A force can rotate a nut when applied by a wrench or can open a door while the door rotates on its hinges. A force thus, can produce a rotary motion besides producing a translatory motion. The measure of this turning effect produced by a force on a body can be called as the moment of force.

Moment of a Force about an Axis. The moment of a force about an axis through a point, or, for short, the moment of a force about a point, is equal to the product of the force and perpendicular distance of the point from the line of action of the force.

$$M_O = F \times d \qquad ...(2.11)$$

The point O is called the *moment centre* and the distance d is called the *arm of the force*.

The unit of the moment = Unit of the force × unit of the distance.

The unit of moment is newton-metre (N-m).

The moment of a force about a point is a vector which is directed perpendicular to the plane containing the moment centre and the force. But usually we are concerned with the direction of its rotational tendency, that is, whether an applied force tends to rotate a body clockwise or anticlockwise.

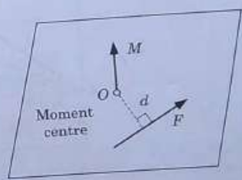


Fig. 2.52(a)

Fig. 2.52

In the Fig. 2.52 (b) the force $\mathbf{F}_1$ has a tendency to produce an anticlockwise rotation about the moment centre. In the Fig. 2.52 (c) on the other hand, the force $\mathbf{F}_2$ has a tendency to produce a clockwise rotation. As matter of convention, an anticlocking moment is taken to be positive and a clockwise moment as negative. *While adding moments, the sense of each moment should be taken into account.*

2.14 THEOREM OF VARIGNON

It states *"The moment of a force about an axis is equal to the sum of the moments of its components about the same axis."*

Consider a force $\mathbf{F}$ acting at a point A and having component $\mathbf{F}_1$ and $\mathbf{F}_2$ in any two directions.

Let us choose any point O, lying in the plane of the forces, as a moment centre.

Attach at A two rectangular axes such that the y-axis is along the line AO and the x-axis is perpendicular to it, as shown in Fig. 2.53.

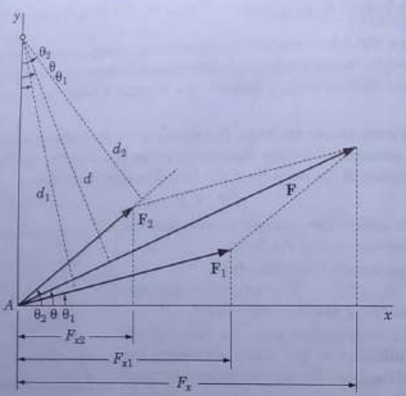


Fig. 2.53

Moment of the force **F** about O,

$$Fd = F(OA \cos \theta) = OA\,(F \cos \theta)$$

$$Fd = OA\,F_x \qquad \text{...(2.12) (a)}$$

Moment of the force $\mathbf{F}_1$ about O,

$$F_1 d_1 = F_1(OA \cos \theta_1) = OA\,(F_1 \cos \theta_1)$$

$$F_1 d_1 = OA\,F_{x1} \qquad \text{...(2.12) (b)}$$

Moment of the force $\mathbf{F}_2$ about O,

$$F_2 d_2 = F_2(OA \cos \theta_2) = OA\,(F_2 \cos \theta_2)$$

$$F_2 d_2 = OA\,F_{x2} \qquad \text{...(2.12) (c)}$$

Adding 2.12 (*b*) and 2.12 (*c*)

$$F_1 d_1 + F_2 d_2 = OA\,(F_{x1} + F_{x2}) \qquad \text{...(2.12) (d)}$$

But,

$$F_x = F_{x1} + F_{x2} \quad \text{as,}$$

The sum of the x-components of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ = x-components of the resultant **F**

Therefore,

$$OA(F_x) = OA(F_{x1} + F_{x2})$$

So from 2.12 (*a*) and 2.12 (*d*)

$$Fd = F_1 d_1 + F_2 d_2 \qquad \text{...(2.13)}$$

By successive application of the above method, the theorem of Varignon can be extended to a system of several forces and their resultant.

2.15 EQUATIONS OF EQUILIBRIUM

System of Coplanar Concurrent Forces. Suppose that the algebraic sum of the moments of a system of coplanar concurrent forces is zero. Two possibilities that exist are,

(*a*) The resultant of the system of forces is zero and the forces are in equilibrium.

(*b*) The moment centre lies on the line of action of the resultant.

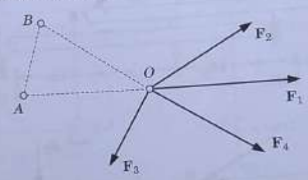


Fig. 2.54

The clarify the situation, let us assume that a number of concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$ be acting at a point O (Fig. 2.54). Let the algebraic sum of the moments of these forces about any point A be zero, that is,

$$\Sigma M_A = 0 \text{ then,}$$

either (*a*) the resultant of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$ ... is zero or (*b*) the point A lies on the line of action of the resultant. Which implies that the resultant lies along the line OA.

Let us now choose any other point B and find the sum of the moments of these forces about this point B.

Suppose it is also zero, that is.

$$\Sigma M_B = 0$$

Then, either (*a*) the resultant force is zero or (*b*) line of action of the resultant lies along OB.

Since the resultant cannot have two lines of actions, that is, along OA as well as OB, so that resultant must be zero.

Therefore,

$$\left.\begin{array}{ll} \Sigma M_A = 0 & \text{...(2.14)} \\ \Sigma M_B = 0 & \text{...(2.15)} \end{array}\right\} \text{Moment Equations of Equilibrium}$$

then the resultant of the system of concurrent forces is zero or the system is in equilibrium. Provided, the points A and B (or the two moment centres) do not lie on the straight line passing through the point of concurrence of the force system.

Earlier, we derived the following equations of equilibrium,

$$\Sigma F_x = 0 \qquad \text{...(2.8)}$$

$$\Sigma F_y = 0 \qquad \text{...(2.9)}$$

Both sets of equations are *equivalent* and the choice depends on the problem at hand.

Sometimes the moment equations of equilibrium can offer certain advantage by way of eliminating an unknown reaction or a force provided the moment centre is chosen to lie on the line of action of that force.

Example 2.21 Two beams AB and CD are arranged and supported as shown. Find the reaction at D due to force of 1000 N acting at B as shown in Fig. 2.55 (a).

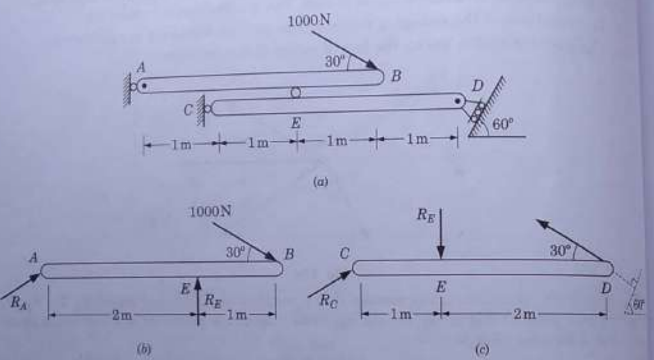


Fig. 2.55

Solution: Consider the free-body diagram of the beam AB, where R_A is the reaction at A and R_E is the reaction at E.

Taking moment about A,

$\Sigma M_A = 0$:

$$R_E \times 2 - 1000\,(3 \sin 30°) = 0$$

$$R_E = \frac{3000 \times 0.5}{2} = 750 \text{ N}$$

Consider the free-body diagram of the beam CD, where R_C is the reaction at C and R_D is the reaction at D.

Taking moment about C,

$\Sigma M_C = 0$:

$$R_D\,(3 \sin 30°) - R_E\,(1) = 0$$

$$R_D\,(3 \sin 30°) - 750\,(1) = 0$$

$$R_D = \frac{750}{3 \sin 30°}$$

$$= \frac{750}{3 \times \frac{1}{2}}$$

$$R_D = 500 \text{ N} \quad \textit{Ans.}$$

Example 2.22 A square block of wood of mass M is hinged at A and rests on a roller at B. It is pulled by means of a string attached at D and inclined at an angle 30° with the horizontal. Determine the force P which should be applied to the string to just lift the block off the roller.

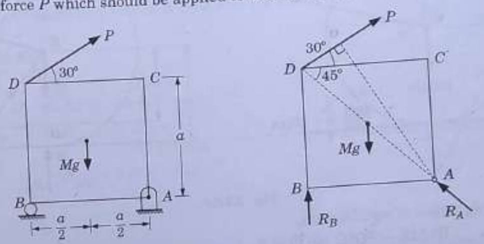


Fig. 2.56

Solution: The free-body diagram of the block is as shown in Fig. 2.56.

Various forces acting are :

(i) Force P (ii) weight of the block Mg (iii) reaction R_B (iv) reactions at the hinge R_A. When the block is at the point of being lifted off the roller B, it shall no longer be in contact with the roller so $R_B = 0$.

The reaction R_A at the hinge can be eliminated by writing the equation of equilibrium as,

$\Sigma M_A = 0$:

$$Mg\left(\frac{a}{2}\right) - P(AD \sin 75°) = 0$$

But

$$AD = \sqrt{AC^2 + CD^2} = \sqrt{a^2 + a^2} = a\sqrt{2}$$

$$Mg\left(\frac{a}{2}\right) - P(\sqrt{2}\,a \sin 75°) = 0$$

$$Mg\left(\frac{a}{2}\right) - P(\sqrt{2}\,a) \times 0.966 = 0$$

$$P = 0.374\ Mg \quad \textit{Ans.}$$

Example 2.23 A uniform wheel of 60 cm diameter weighing 1000 N rests against a rectangular obstacle 15 cm high. Find the least force required which when acting through the centre of the wheel will just turn the wheel over the corner of the block. Also, find the angle θ which this least force shall make with AC.

Solution: The forces acting on the wheel are :

(i) Weight of the wheel $W = 1000$ N (ii) force P and (iii) reaction R_A at the point A. These form a concurrent system of forces passing through the point C as shown in the Fig. 2.57 (a).

But, just when the wheel is about to turn, the reaction R_B at the point B becomes zero.

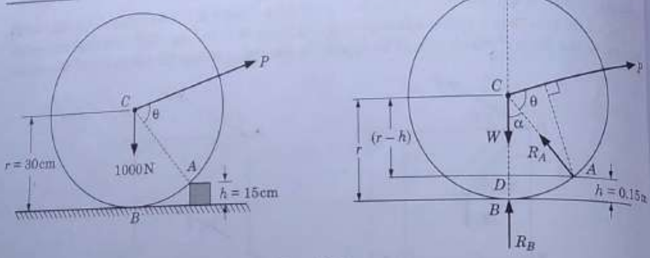


Fig. 2.57(a)

Writing the equation of equilibrium,

$\Sigma M_A = 0$: $\quad W(AD) - P(AC \sin\theta) = 0$

$$P = \frac{W(AD)}{AC \sin\theta}$$

$$AD = \sqrt{AC^2 - CD^2} \text{ and } CD = r - h$$

Therefore,

$$AD = \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2}$$

Substituting for AD,

$$P = \frac{W\sqrt{2rh - h^2}}{r \sin\theta}$$

P is minimum when $\sin\theta = 1$

or $\theta = 90°$ *Ans.*

That is, when P acts at right angles to CA.

$$P_{min} = \frac{100}{0.3} \times \sqrt{2 \times 0.3 \times 0.15 - 0.15 \times 0.15}$$

$$P_{min} = 866 \text{ N} \quad \textit{Ans.}$$

Graphical Solution

Let us first evaluate the angle α defining the direction of R_A

$$\sin\alpha = \frac{AD}{AC} = \frac{\sqrt{2rh - h^2}}{r}$$

$$\sin\alpha = \frac{\sqrt{2 \times 0.3 \times 0.15 - 0.15 \times 0.15}}{0.3}$$

$$\sin\alpha = 0.866 \quad \alpha = 60°$$

Draw a vector $\overline{ab}$ to represent $W = 1000$ N acting vertically downward. From a, draw a line ac parallel to the reaction R_A. The least force P, therefore, is represented by $\overline{bd}$, obtained by dropping perpendicular from b on the line ac.

Applying sine law,

$$\frac{W}{\sin 90°} = \frac{P}{\sin\alpha}$$

$$P_{min} = W \sin\alpha$$
$$= 1000 \times 0.866$$
$$P_{min} = 866 \text{ N} \quad \textit{Ans.}$$

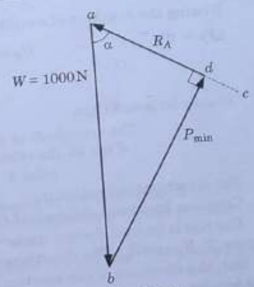


Fig. 2.57(b)

Example 2.24 A cylinder of weight W and radius r is supported in horizontal position against a vertical wall by a bar AB of negligible weight. The bar is hinged to the wall at A and supported at B by a horizontal rope BC. Find the value of the angle θ that the bar should make with the wall so that the tension in the rope is minimum. Assume frictionless conditions.

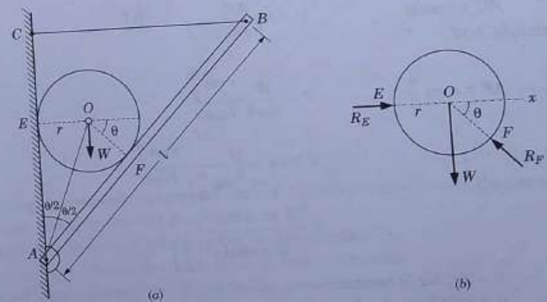


Fig. 2.58

Solution: Consider the free-body diagram of the cylinder, [Fig. 2.58 (b)].

The reaction R_E at E, the reaction R_F at F and the weight W, form three concurrent forces in equilibrium passing through the centre O of the cylinder

Writing the equation of equilibrium

$\Sigma Fy = 0:$ $\quad R_F \sin\theta - W = 0$

$$R_F = \frac{W}{\sin\theta} \qquad \ldots(i)$$

It may be noted that,

The magnitude of the reaction of bar on the cylinder at the point F = The magnitude of the reaction of the cylinder on the bar at point F $= R_F$

But directions are opposite.

Consider free-body diagram of the bar AB.

The bar is in equilibrium under the action of three forces, T, R_F and R_A. The directions of T, R_F are known.

But, the direction of the reaction at the hinge, R_A is not known.

Three forces in equilibrium should be concurrent, but determination of the point of concurrency is difficult so let us take moments about A. This should also eliminate the reaction R_A.

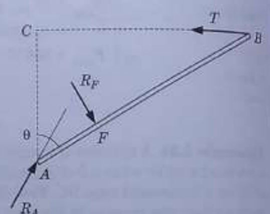


Fig. 2.58 (c)

$\Sigma M_A = 0:$ $\quad T \cdot AC - R_F \cdot AF = 0$

From triangle ABC,

$$AC = l\cos\theta \qquad T = R_F \frac{AF}{AC}$$

From triangle AOF,

$$AF = r\cot\frac{\theta}{2} \qquad T = \frac{W}{\sin\theta} \cdot \frac{r\cot\frac{\theta}{2}}{l\cos\theta}$$

$$T = \frac{W}{2\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)} \cdot \frac{r\cot\frac{\theta}{2}}{l\cos\theta}$$

$$T = \frac{Wr}{2l\cos\theta\sin^2\left(\frac{\theta}{2}\right)} \qquad \ldots(ii)$$

When $\quad \theta = 0°$ (Bar is vertical)

or $\quad \theta = 90°$ (Bar is horizontal).

T becomes infinitely large. So, θ must lie between these two values

For tension T to be minimum,

$$\frac{dT}{d\theta} = 0 \quad \text{or} \quad \frac{d\left(\frac{W}{2l\cos\theta\sin^2\theta/2}\right)}{d\theta} = 0$$

or $$\sin\theta\left(4\sin^2\frac{\theta}{2} - 1\right) = 0$$

or $$\sin\frac{\theta}{2} = \frac{1}{2},\ \theta = 60° \quad \textit{Ans.}$$

Substituting for θ in (ii)

$$T_{\min} = \frac{4\,Wr}{l} \quad \textit{Ans.}$$

Example 2.24 A uniform rod AB of negligible weight is hinged at the end A and supported at end B by a string as shown. Find the value of angle θ corresponding to the position of equilibrium of the bar if $Q = P/2$.

Solution: For the convenience of understanding the geometry of the problem, the forces have been indicated without removing the supports.

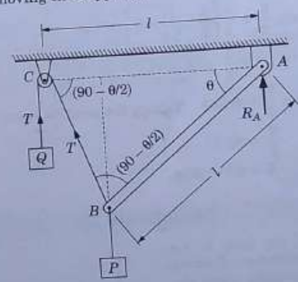


Fig. 2.59

Considering the free-body diagram of the bar AB, the forces acting on the bar are,

(a) Tension $T = Q$ at the end B acting along BC.

(b) Weight P at the end B acting vertically downwards.

(c) Reaction R_A acting at the hinge in an unknown direction.

We can eliminate the reaction R_A by taking moment about A.

So, writing the equation of equilibrium of the bar

$$\Sigma M_A = 0: \quad Pl\cos\theta - Ql\sin\left(90 - \frac{\theta}{2}\right) = 0 \qquad \left\{\begin{array}{l}\text{Triangle } ABC \text{ is isosceles}\\ \angle ABC = \angle ACB\\ = \frac{(180-\theta)}{2}\end{array}\right\}$$

$$P\cos\theta - Q\cos\frac{\theta}{2} = 0$$

$$P\left(2\cos^2\frac{\theta}{2}-1\right)-Q\cos\frac{\theta}{2} = 0$$

$$2P\cos^2\frac{\theta}{2} - P - Q\cos\frac{\theta}{2} = 0$$

$$2P\cos^2\frac{\theta}{2} - Q\cos\frac{\theta}{2} - P = 0$$

Roots of the equation are,

$$\cos\frac{\theta}{2} = \frac{Q \pm \sqrt{Q^2+8P^2}}{4P}$$

$$\cos\frac{\theta}{2} = \frac{1}{4}\left(\frac{Q}{P} \pm \sqrt{\frac{Q^2}{P^2}+8}\right)$$

Substituting, $\dfrac{Q}{P} = \dfrac{1}{2}$

$$\cos\frac{\theta}{2} = \frac{1}{4}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4}+8}\right)$$

$$= \frac{1}{8}\ (\pm\ 5.744)$$

$$\cos\frac{\theta}{2} = 0.843 \quad \text{Taking the positive value}$$

$$\frac{\theta}{2} = 32.5^\circ$$

or $\theta = 65^\circ$ *Ans.*

PROBLEMS

2.18. Three points A, B and C are lying in x-y plane. The moment of a certain force F acting in x-y plane is 180 N-m clockwise about the origin O and 90 N-m anticlockwise around the point A. If its moment about point B is zero, determine the magnitude and direction of the force F and its moment about point C.
[F = 75 N, θ = 36.87; M_C = 120 N-m]

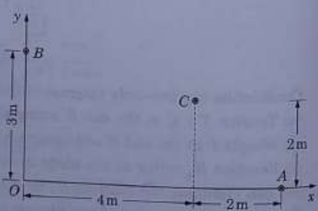


Fig. P.2.18

2.19. A force P = 450 N is applied at the point B of a weightless plate hinged at E. Determine (*a*) the moment of the force P about E (*b*) the horizontal force that must be applied at F for the block to be in equilibrium (*c*) the smallest force required at F to keep the block in equilibrium. [88.8 N-m (clockwise), 395 N, 280 N, 45°]

2.20. A rigid bar AB is supported in a vertical plane by a hinge at the end A and by a horizontal string attached to the bar as shown. The end B of the bar carries a load of W newtons. Find the tensile force in the string. Neglect the weight of the bar. [$T = 2\ W \tan\theta$]

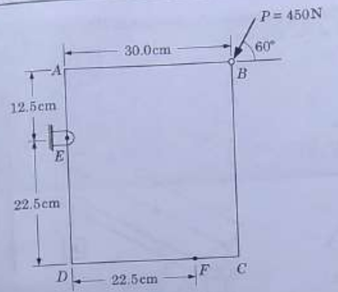


Fig. P.2.19

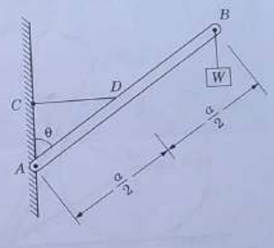


Fig. P.2.20

2.21. Determine the forces exerted on the cylinder at A and B by the spanner wrench due to a vertical force of 250 N applied to the handle. Neglect friction at B. [R_B = 1250 N, R_A = 1275 N, θ = 11.3°]
[**Hint :** Reaction at B acts normal to the cylinder.]

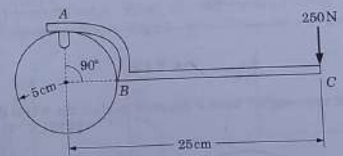


Fig. P.2.21

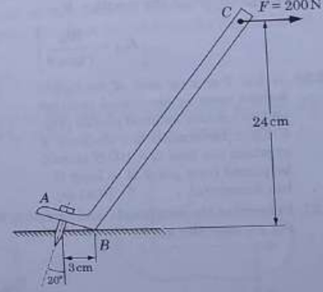


Fig. P.2.22

2.22. Determine the magnitude of the pull exerted on the nail A if a horizontal force of 200 N is applied to handle of a nail-puller as shown. [1702 N]

2.23. A drum full of oil of density 1 kg/litre is resting against a foot step 10 cm high. Find the force required to be applied at the top of the drum to just turn it over the foot step. The drum is of 50 cm diameter, 1.5 m long and of negligible weight. [1444.6 N]

2.24. A smooth cylinder of radius r = 10.0 cm resting on a horizontal surface supports a bar AB of length 30.0 cm which is hinged at A. The weight of the bar is 50 N. The cylinder is kept from rolling away by a string AO of length 20.0 cm. Assuming all surfaces to be frictionless, find the tension in the string. [T = 21.6 N]

[**Hint :** Consider the equilibrium of bar AB and determine the reaction at D by taking moments about A. Then, consider the equilibrium of the cylinder].

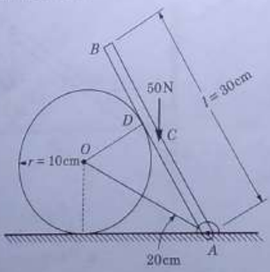


Fig. P.2.24

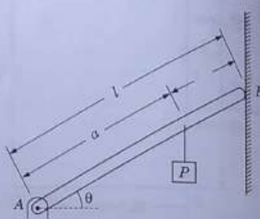


Fig. P.2.25

2.25. A prismatic bar AB of negligible weight is hinged at A with the end B resting against a vertical wall. It supports a vertical load P as shown. Find the reaction R_B.

$$\left[R_B = \frac{Wa}{l\tan\theta}\right]$$

2.26. A bar 2 m long and of negligible weight rests in horizontal position on two smooth inclined planes (Fig. P.2.26). Determine the distance x at which the load $Q = 100$ N should be placed from point B to keep the bar horizontal. [0.81 m]

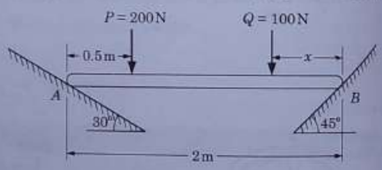


Fig. P.2.26

2.27. Determine the magnitude and direction of the smallest force P required to start the wheel (Fig. P.2.27) over the block. [5.73 kN, 71.4°]

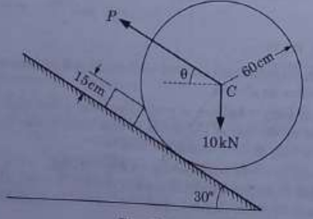


Fig. P.2.27

3 CHAPTER Parallel Forces in a Plane

3.1 PARALLEL FORCES

In the earlier chapters, we dealt with the forces whose lines of action intersect at a point or with a system of concurrent forces. For such a system of forces the parallelogram law could be directly applied to find their resultant.

We can also have a set of forces whose lines of action are parallel to each other. Obviously, a parallelogram can not be drawn to find the resultant of two parallel forces. Therefore, the parallelogram law is to be applied in an indirect manner. But first let us describe the different types of parallel forces.

Types of Parallel Forces

Like Parallel Forces : When the two parallel forces act in the same direction they are termed as like parallel forces [Fig. 3.1 (*a*)]. These forces can be equal or unequal in magnitude.

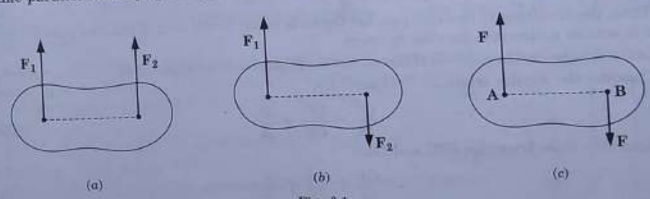


Fig. 3.1

Unlike Unequal Parallel Forces : When the two parallel forces act in the opposite directions and are unequal in magnitude [Fig. 3.1 (*b*)].

Unlike Equal Parallel Forces : When the two parallel forces act in opposite directions and are equal in magnitude as shown in [Fig. 3.1 (*c*)].

3.2 RESULTANT OF TWO PARALLEL FORCES ACTING IN THE SAME DIRECTION

Consider two parallel forces **P** and **Q** acting at the points A and B of the body. Join the points A and B shown in Fig. 3.2.

Superimpose two equal and opposite forces at the points A and B each equal to **S**.

Let $\mathbf{P}_1$ be the resultant of **P** and **S** as determined by parallelogram law.

Similarly, $\mathbf{Q}_1$ be the resultant of the forces **Q** and **S**.

By this procedure, we have replaced the two parallel forces **P** and **Q** by two equivalent non-parallel forces $\mathbf{P}_1$ and $\mathbf{Q}_1$.

Let us transmit the points of application of the forces $\mathbf{P}_1$ and $\mathbf{Q}_1$ to the point O where they are intersecting.

Next, resolve the forces $\mathbf{P}_1$ and $\mathbf{Q}_1$ acting at O, into the original components acting at the same point O. That is, into forces **P** and **Q** and forces **Q** and **S**.

The components **S** at O, of $\mathbf{P}_1$ and $\mathbf{Q}_1$ cancel, being collinear and acting in opposite directions.

The other components **P** and **Q**, are acting along the same line OI and in the same direction, so they add up. Resultant, therefore, is given by,

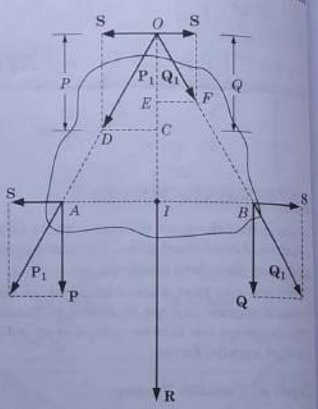


Fig. 3.2

$$\mathbf{R} = \mathbf{P} + \mathbf{Q} \qquad ...(3.1)$$

Thus, the resultant of two like parallel forces is equal to the sum of the two parallel forces and it acts in a direction parallel to them.

Let us determine the position of line of action of this resultant **R** (point I).

Consider the similar triangles OAI and ODC

$$\frac{OI}{IA} = \frac{OC}{CD} = \frac{P}{S} \qquad ...(3.2)$$

Similarly, from triangles OBI and OEF

$$\frac{OI}{IB} = \frac{OE}{EF} = \frac{Q}{S} \qquad ...(3.3)$$

From (3.2) and (3.3)

$$\frac{IB}{IA} = \frac{P}{Q} \qquad ...(3.4)$$

Stated in words, *the resultant of two like parallel forces acts parallel to them and its position is such that it divides the distance between their points of application in the ratio which is inversely proportional to their magnitudes.*

Analytically, the resultant of two or more parallel forces can be determined by the method of summing up their x and y components. The position of the line of action of the resultant then can be determined using the theorem of Varignon.

3.3 RESULTANT OF TWO UNEQUAL PARALLEL FORCES ACTING IN OPPOSITE DIRECTIONS

Consider two unequal parallel forces **P** and **Q** acting at the points A and B of a body. The force **P** is larger than **Q** (Fig. 3.3).

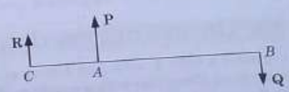


Fig. 3.3

If **R** be the resultant of **P** and **Q** acting at the point C, then,

$$\mathbf{R} = \mathbf{P} - \mathbf{Q}$$

The point C lies outside of AB and on the same side as the larger force **P**. The position of point C is such that

$$\frac{P}{Q} = \frac{BC}{AC}$$

Thus, in the case of two unlike parallel forces the resultant lies outside the line joining the points of action of the two forces and on the same side as the larger force.

The above statement can be proved by using the law of parallelogram of forces and the principle of superposition as was done in the case of like parallel forces.

3.4 TWO EQUAL PARALLEL FORCES ACTING IN OPPOSITE DIRECTIONS : COUPLE

A system of two equal parallel forces acting in opposite directions cannot be replaced by a single force. In such a case, the two forces form a *couple* which has a tendency to rotate the body. The perpendicular distance between the lines of action of the two forces is termed as the *arm* of the couple.

Moment of a Couple. The rotational tendency of the couple is measured by its moment. The moment of a couple is the product of the either one of the forces forming the couple and the arm of the couple.

It has the same units as the moment of a force, N-m.

Moment of the couple = $F \times d$ ...(3.5)

Tendency to rotate in anticlockwise direction is assumed positive and in clockwise as negative.

To summarize :

1. The algebraic sum of the forces forming a couple is zero.
2. The algebraic sum of the moments of the two forces forming a couple is independent of the position of the moment centre chosen as shown below (Fig. 3.4)

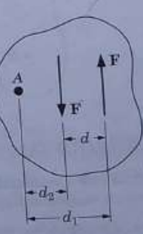


Fig. 3.4

Taking moments about any point A,

$$M = Fd_1 - Fd_2$$

$$\text{or, } M = Fd$$

3. To add two couples take the algebraic sum of their moments.

From above, we can deduce that two couples acting in a plane can be in equilibrium if their moments are equal in magnitude and opposite in direction.

3.5 THE RESOLUTION OF A FORCE INTO A FORCE AND A COUPLE

Consider a force **F** acting on a body at the point A. This is to be replaced by a force acting at some point B together with a couple as shown in Fig. 3.5. Introduce two equal and opposite forces at B, each of magnitude **F** and acting parallel to the force at A.

From the principle of superposition, the second system of forces is equivalent to the single force acting at A.

Of the three equal forces, consider the two forces acting in opposite directions at points A and B. They form a couple of moment

$$M = F \times d$$

Thus, the original force **F** acting at the point A can be replaced by a force **F** applied at another point B, together with a couple of magnitude $F \times d$. The distance d being the perpendicular distance between the lines of action of the forces at A and B.

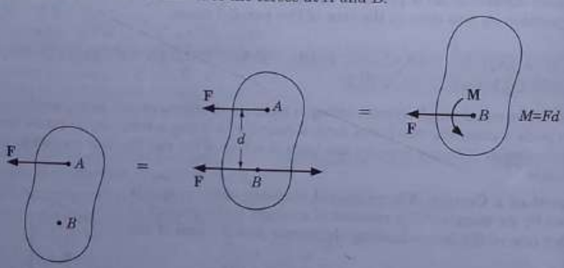


Fig. 3.5

3.6 EQUIVALENT SYSTEM OF FORCES

Let us now summarize the various *operations* by which one system of forces can be transformed into an equivalent system of forces :

1. By replacing two forces acting at a point by their resultant.
2. By resolving a force into two components.
3. By cancelling two equal and opposite forces acting at a point.
4. By attaching two equal and opposite forces at a point.
5. By transmitting a force along its line of action.

3.7 GENERAL CASE OF PARALLEL FORCES IN A PLANE

Consider a number of coplanar parallel forces $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, $\mathbf{Y}_4$, acting at points A, B, C, D of a body as shown in Fig. 3.6.

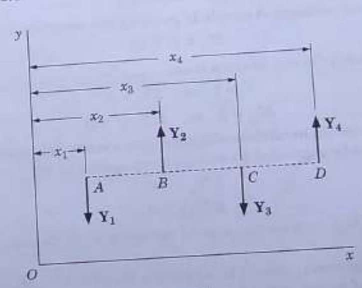


Fig. 3.6

Choose the coordinate axes such that the y-axis is parallel to these forces and the x-axis perpendicular to them. The distance of the forces $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, $\mathbf{Y}_4$, from the origin along the x-axis be x_1, x_2, x_3, x_4.

Let the resultant **Y** of these forces lie at distance x from the y-axis.

Resolve the forces along the y-axis

$$\mathbf{Y} = Y_1 + Y_2 + Y_3 + Y_4$$

Or,

$$\mathbf{Y} = \Sigma Y_i \qquad ...(3.6)$$

Let the sum of the moments of all these forces about the origin O be M_O,

$$M_O = Y_1x_1 + Y_2x_2 + Y_3x_3 + Y_4x_4$$

$$M_O = \Sigma(Y_i x_i) \qquad ...(3.7)$$

Now three possibilities exist.

Case I. The system of forces reduces to a single resultant force **Y**.

The magnitude of this resultant is given by the equation 3.6 as derived earlier

$$Y = \Sigma Y_i \qquad ...(3.6)$$

Let the resultant lie at a distance x from the y-axis. Equating the sum of the moments of all the forces about O to the moment of the resultant Y about the same point.

$$M_O = Yx$$

$$x = \frac{M_O}{Y} = \frac{\Sigma(Y_i x_i)}{Y} = \frac{\Sigma(Y_i x_i)}{\Sigma Y_i} \qquad ...(3.8)$$

Case II. The system of forces reduces to a couple.

The couple is formed by two equal and opposite parallel forces. Therefore, the resultant of all these forces must be zero,

$$Y = \Sigma Y_i = 0 \quad \text{...(3.9)}$$

The magnitude of the moment of couple is given by the equation 3.7 derived earlier,

$$M_O = \Sigma(Y_i x_i) \quad \text{...(3.10)}$$

Case III. The system is in equilibrium then,

$$Y = 0 \quad \text{or,} \quad \Sigma Y_i = 0 \quad \text{...(3.11)}$$

and

$$M_O = 0 \quad \text{or,} \quad \Sigma(Y_i x_i) = 0 \quad \text{...(3.12)}$$

Above equations express the conditions of equilibrium of a set of parallel forces. The conditions of equilibrium can also be expressed by two moment equations as,

$$\Sigma(M_A)_i = 0 \quad \text{...(3.13)}$$

$$\Sigma(M_B)_i = 0 \quad \text{...(3.14)}$$

Where $\Sigma(M_A)_i$ and $\Sigma(M_B)_i$ express the sum of moments of all the forces about the points A and B respectively.

The points A and B, however, should be so chosen that the line AB joining these points is not parallel to the line of action of the forces.

EXAMPLES

Example 3.1 Two men support a weightless wooden beam AB with a weight of 1000 N hanging from the beam as shown in Fig. 3.7. Find the load shared by the each man.

Solution: Let the loads supported by men at the ends A and B be P and Q respectively. Considering the equilibrium of the bar

$\Sigma F_y = 0: \quad P + Q - W = 0$

(P, Q are also the reactions acting on the beam)

$$P + Q = 1000 \quad \text{...}(i)$$

$\Sigma M_A = 0:$

$$Q\,(1.0 \cos 60°) - 1000\,(0.6 \cos 60°) = 0 \quad \text{...}(ii)$$

$Q = 600$ N *Ans.*

From (*i*), $P = 400$ N *Ans.*

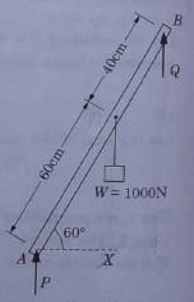


Fig. 3.7

Example 3.2 A beam ABC is supported and loaded as shown in Fig. 3.8. Find the reactions at the supports.

Solution: Free-body diagram of the beam ABC is as shown.

Writing the equation of equilibrium

$\Sigma F_y = 0:$

$$R_B - R_A - 1000 = 0$$

$$R_B - R_A = 1000 \quad \text{...}(i)$$

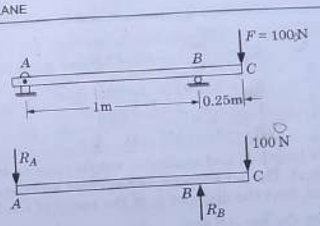


Fig. 3.8

$\Sigma M_A = 0: \quad R_B(1) - 1000\,(1.25) = 0 \quad \text{...(ii)}$

$R_B = 1250$ N (up). *Ans.*

From (*i*), $R_A = 250$ N (down). *Ans.*

Example 3.3 A beam is supported and loaded as shown in Fig. 3.9. Find the reactions at A and B.

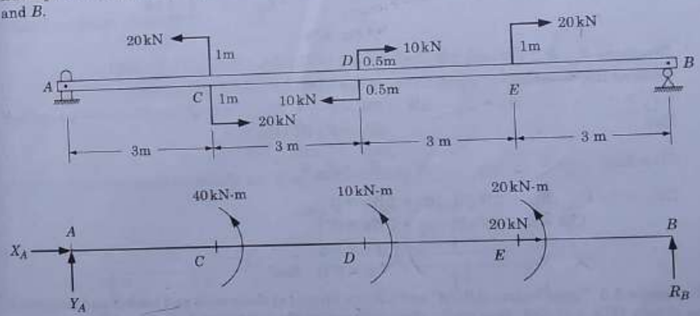


Fig. 3.9

Solution: Beam is hinged at A (X_A, Y_A) and roller supported at B (R_B). Forces about C form a couple of magnitude $M_C = 20\,(1 + 1) = 40$ kN-m acting anticlockwise and forces about D forms a couple of magnitude $M_D = 10\,(0.5 + 0.5) = 10$ kN-m acting clockwise. A force of 20 kN is acting at a distance of 1 m from E. It is transferred to the point E as a force of 20 kN and a couple of magnitude $20\,(1) = 20$ kN-m acting clockwise.

$\Sigma F_x = 0: \quad X_A + 20 = 0, \; X_A = -20$ kN *Ans.* (change the assumed direction)

$\Sigma F_y = 0: \quad Y_A + R_B = 0$

$\Sigma M_A = 0: \quad 40 - 10 - 20 + R_B(12) = 0$

$$R_B = -\frac{10}{12} = -0.833 \text{ kN}$$

$R_B = 0.833$ kN (down). *Ans.*

$Y_A = -R_B = -(-0.833)$

$Y_A = 0.833$ kN (up). *Ans.*

Example 3.4 A bar AB of length $2l$ and negligible weight rests on two roller supports C and D placed at a distance l apart. The bar supports two vertical loads as shown. For the reactions at the supports to be equal, find the distance x, of the end A of the bar, from the support C.

Solution: Forces acting on the bar AB are:

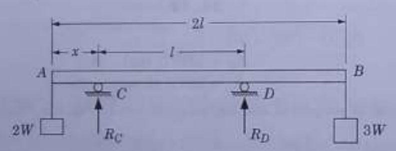


Fig. 3.10

Reactions R_C, R_D; $2W$ and $3W$

Writing the equations of equilibrium of the bar

$\Sigma F_y = 0$: $\quad R_C + R_D - 2W - 3W = 0$

But, $\quad R_C = R_D$ (Given)

Therefore, $\quad R_C = R_D = \dfrac{5W}{2}$

$\Sigma M_C = 0$: $\quad R_D l - 3W(2l - x) + 2Wx = 0$

$$(2.5\,W)l - 3W(2l - x) + 2Wx = 0$$

$$5x = 3.5l$$

$$x = 0.7l \quad \textit{Ans.}$$

Example 3.5 Three beams AB, BC and CD are hinged at their ends and loaded and supported as shown [Fig. 3.11 (a)]. Determine the reactions at the points at A, E, F and D.

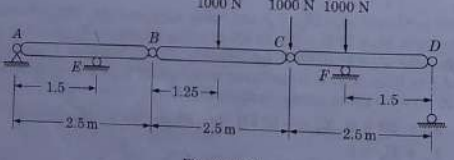


Fig. 3.11 (a)

Solution: Consider the free-body diagram of the beam CD.

The reactions at D, F and C act vertically.

Taking moments about C

$\Sigma M_C = 0$: $\quad R_F \times 1 - 1000 \times 1 - R_D \times 2.5 = 0$

$$R_F - 2.5R_D = 1000 \quad ...(i)$$

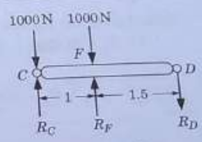


Fig. 3.11(b)

Consider the combined free-body diagram of the beam BC and CD.

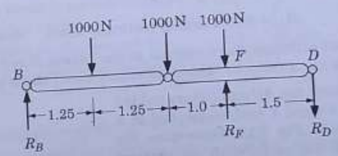


Fig. 3.11 (c)

Taking moments about B

$\Sigma M_B = 0$:

$$-1000 \times 1.25 - 1000 \times 2.5 - 1000 \times 3.5 + R_F(3.5) - R_D(5.0) = 0 \quad ...(ii)$$

$$3.5R_F - 5R_D = 7250$$

Solving equations (i) and (ii) for R_F and R_D

$R_F = 3500$ N. *Ans.*

$R_D = 1000$ N. *Ans.*

Consider the combined free-body diagram of beams AB, BC and CD.

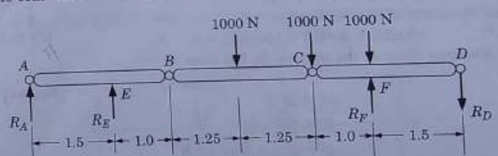


Fig. 3.11 (d)

Taking moments about A.

$\Sigma M_A = 0$:

$$R_E(1.5) - 1000(3.75) - 1000(5) - 1000(6) + R_F(6) - R_D(7.5) = 0$$

$$1.5R_E + 6R_F - 7.5R_D = 14750$$

Substituting for R_F and R_D

$$1.5\,R_E + 6(3500) - 7.5(1000) = 14750$$

$$R_E = 833.0 \text{ N} \quad \textit{Ans.}$$

$$\Sigma F_y = 0: \; R_A + R_E - 1000 - 1000 - 1000 + R_F - R_D = 0$$

$$R_A = R_D - R_E - R_F + 3000$$

$$R_A = 1000 - 3500 - 833 + 3000$$

$$R_A = -\,333 \text{ N}$$

Or $\qquad R_A = 333$ N acting vertically down. *Ans.*

It may noted here that although the reactions at the hinges B and C appeared in the free body diagrams but were eliminated by taking moments about these points.

Example 3.6 Two prismatic bar AB and CD are welded together in the form of a rigid T and suspended in a vertical plane as shown in Fig. 3.12. Determine the angle θ that the bar AB will make with the vertical when a load of 100 N is applied at the end D. Two bars are identical and each weighs 50 N.

Solution: Consider the free-body diagram of both the bars together.

Forces acting on the bars are :

(*a*) reaction R_A at the hinge A.
(*b*) weight of the bar AB = 50 N.
(*c*) weight of the bar CD = 50 N.
(*d*) force of 100 N acting at D.

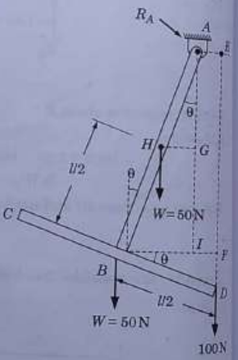


Fig. 3.12

Let the length of each bar be l.

Writing the equation of equilibrium,

$\Sigma M_A = 0: \qquad 50(HG) + 50(BI) - 100(AE) = 0$

(By taking moments about A, the reaction R_A has been eliminated.)

$$AE = IF$$

$$IF = BF - BI$$

$$IF = l/2 \cos\theta - l \sin\theta$$

$$50(l/2 \sin\theta) + 50(l \sin\theta) - 100(l/2 \cos\theta - l \sin\theta) = 0$$

$$(5/2)\sin\theta + 5\sin\theta - 5\cos\theta + 10\sin\theta = 0$$

$$17.5 \sin\theta = 5\cos\theta$$

$$\tan\theta = \frac{5}{17.5} = 0.2857$$

$$\theta = 15.86^\circ. \quad \textit{Ans.}$$

Example 3.7 A three wheeler scooter of weight 2 kN and a driver weighing 0.5 kN are shown schematically in Fig. 3.13, where G and D denote the positions of C.G. of the scooter and location of the driver respectively.

Find the reactions at the wheels A, B and C of the scooter for the equilibrium conditions on a level road.

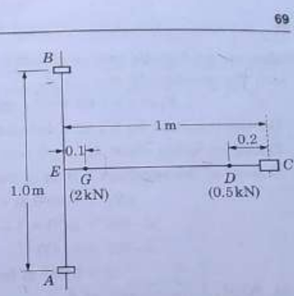


Fig. 3.13

Solution: (*The problem involves vertical parallel forces which do not lie in a plane*).

The various forces acting on the scooter are :

Weight of the scooter 2 kN, weight of the driver 0.5 kN and the reactions R_A, R_B and R_C at the wheels A, B and C.

First consider the equilibrium of the portion EC

Taking moments about point E $\qquad$ (Forces acting at E are eliminated)

$$\Sigma M_E = 0: \quad R_C \times 1 - 0.5(0.8) - 2 \times (0.1) = 0$$

$$R_C = 0.6 \text{ kN}. \quad \textit{Ans.}$$

As the wheel A and B are symmetrically placed, therefore,

$$R_A = R_B$$

Consider the equilibrium of the entire scooter.

$$\Sigma F_y = 0: \quad R_A + R_B + R_C = 2 + 0.5 \quad = 2.5$$

$$R_C = 0.6 \qquad R_B = R_A$$

$$2R_A = 1.9, \qquad R_A = 0.95 \text{ kN} \quad \textit{Ans.}$$

As, $\qquad R_B = R_A$, so $R_B = 0.95$ kN $\quad$ *Ans.*

Example 3.8 A rigid bar is subjected to a system of parallel forces as shown in Fig. 3.14 (*a*).

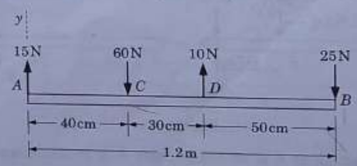


Fig. 3.14 (*a*)

Reduce this system to

(*a*) a single force
(*b*) a single force-moment system at A
(*c*) a single force-moment system at B.

Solution: (*a*) A single force or resultant.

Let the resultant be R_Y,

$$R_Y = 15 - 60 + 10 - 25 = -60 \text{ N} \quad Ans.$$

Let the resultant R_y act at a distance x from A.

Taking moments about A,

Moment of the resultant = Sum of the moments of its components

$$xR_Y = -60(0.4) + 10(0.7) - 25(1.2)$$
$$x(-60) = 24.0 + 7 - 30.0$$
$$x(-60) = (-47)$$
$$x = 0.783 \text{ m from } A \quad Ans.$$

(*b*) Single force-moment at A.

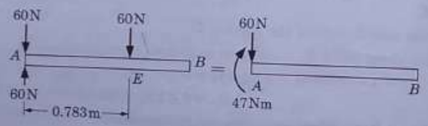


Fig. 3.14 (*b*)

If the force of 60 N acting at E is moved to the point A, it is accompanied by a moment [Fig. 3.14 (*b*)],

$$M_A = -60 \times 0.783$$
$$M_B = -47.0 \text{ N-m} \quad Ans.$$

(*c*) Single force-moment at B.

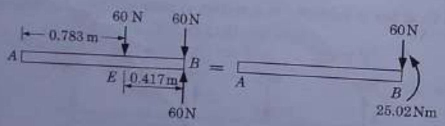


Fig. 3.14 (*c*)

If the force of 60 N acting at E is moved to point A, it is accompanied by a moment [Fig. 3.14 (*c*)],

$$M_B = 60 \times 0.417$$
$$M_B = 25.0.2 \text{ N-m} \quad Ans.$$

Example 3.9 A vertical pole is anchored in a cement foundation. Three wires are anchored to the pole as shown. If the reaction at the point A consists of an upward vertical force of 5000 N and a moment of 10000 N-m as shown, find the tensions in the wire (Fig. 3.15).

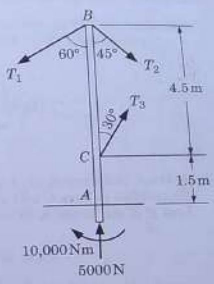


Fig. 3.15

Solution: Let the tensions in the wire be T_1, T_2 and T_3. Consider the equilibrium of the pole.

Taking moments about B.

$$\Sigma M_B = 0: \quad T_3(4.5 \sin 30°) - 10000 = 0$$

$$T_3 = \frac{10000}{4.5 \times 0.5}$$

$$T_3 = 4444.4 \text{ N} \quad Ans.$$

$$\Sigma F_x = 0: \quad T_3 \sin 30° + T_2 \sin 45° - T_1 \sin 60° = 0$$

Substituting for T_3 in the above equation

$$4444.4(0.5) + 0.707\, T_2 - 0.866\, T_1 = 0$$

Or,

$$-0.707\, T_2 + 0.866\, T_1 = 2222.2 \qquad ...(i)$$

$$\Sigma F_y = 0: \; 5000 - T_1 \cos 60° - T_2 \cos 45° + T_3 \cos 60° = 0$$

Substituting for T_3,

$$5000 - 0.5T_1 - 0.707\, T_2 + 4444.4 \times 0.866 = 0$$

$$0.707\, T_2 + 0.5T_1 = 8848.8 \qquad ...(ii)$$

Solving (*i*) and (*ii*) simultaneously

$$T_1 = 8104.6 \text{ N}$$
$$T_2 = 6784.2 \text{ N} \quad Ans.$$

PROBLEMS

3.1. A uniform beam AB of weight $W = 100$ N rests on two roller supports C and D as shown. If a force of 250 N is applied to the end B, find the range of the values of force F for which beam will remain in equilibrium. [25.0 N ≤ F ≤ 650 N]

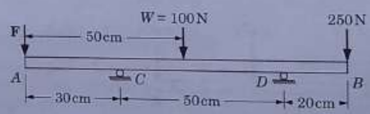


Fig. P.3.1

3.2. A rigid bar is subjected to a system of parallel forces as shown in Fig. P.3.2. Reduce the given system of forces to (*a*) a single force or resultant (*b*) force-moment system at A (*c*) force-moment system at B.

[(*a*) 15 kN, at a distance 1.67 m from A
(*b*) 15 kN, 25 kN-m (clockwise)
(*c*) 15 kN, 12.5 kN-m (anticlockwise)]

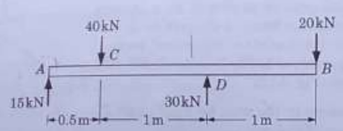


Fig. P.3.2

3.3. A beam AB hinged at A and is supported at B by a vertical chord which passes over two frictionless pulleys C and D. If the pulley D carries a vertical load Q, find the position x of the load P if the beam is to remain in equilibrium in the horizontal position.

$$\left[x = \frac{Ql}{2P}\right]$$

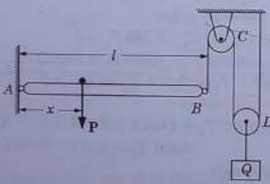


Fig. P.3.3

3.4. Determine the reaction at the fixed end A of a cantilever for the loading shown in Fig. P.3.4.
[1400 N, 2000 N-m Anticlockwise]

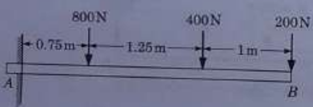


Fig. P.3.4

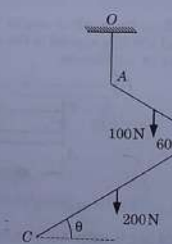


Fig. P.3.5

3.5. Two bars AB and CD of lengths 1 m and 2 m and of weights 100 N and 200 N respectively are rigidly joined at B suspended by a string AO as shown in Fig. P.3.5. Find the inclination θ of bar BC to the horizontal when the system is in equilibrium.
[θ = 19.1°]

3.6. A 500 N force is applied to the point A of a L-shaped plate. Find the equivalent force-couple system at B.
[500 N, 60°, 308 N-m clockwise]

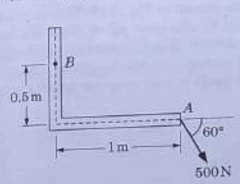


Fig. P.3.6

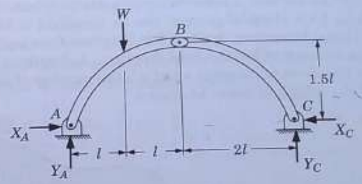


Fig. P.3.7

3.7. A three hinged arch is loaded as shown. Find the reactions X_A, Y_A, X_C and Y_C at the hinges.

$$\begin{bmatrix} X_A = 0.333\overrightarrow{W} & X_C = 0.333\overleftarrow{W} \\ Y_A = 0.75\,W\uparrow & Y_C = 0.25\,W\uparrow \end{bmatrix}$$

3.8. For a system of levers shown in Fig. P.3.8, find the load P required to maintain the system in equilibrium with the bar AB in the horizontal position.
[40 N]

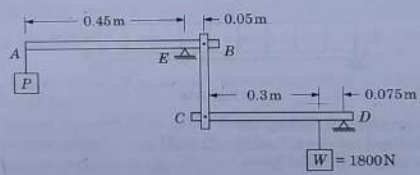


Fig. P.3.8

3.8 DISTRIBUTED FORCE IN A PLANE

We have so far discussed the equilibrium of bodies under the action of concentrated forces, that is the forces which act at a point. *Distributed forces and loads are another type of forces and loads which act over a length, area or volume of a body.* For example, a cable suspended between the two supports at its ends can support a load which is continuously distributed along its length. The water pressure acting on the sides and the bottom of a container filled with water, is a force which is distributed over the area.

When dealing with the equilibrium of a body under the action of a distributed force, *we have to replace such distributed force by a single concentrated force. This concentrated force should represent the resultant of the distributed force.* The method to determine the magnitude, direction and the line of action of single concentrated force which can replace different types of distributed loads shall be discussed in this section.

Distributed Load on Beam. A concentrated load as mentioned earlier is represented by a single line with an arrow head indicating the point of application of the load.

Consider now, a beam carrying some distributed load as shown in Fig. 3.16. This distributed load, for a physical picture, can be imagined to be caused by the sand piled along the length of beam to varying depths. The diagram showing the variation of the load along the length of the beam is called the *load diagram*. A distributed load acting over the length of a beam can be imagined to be made up of a large number of parallel loads acting over the small elements of the length of beam as shown.

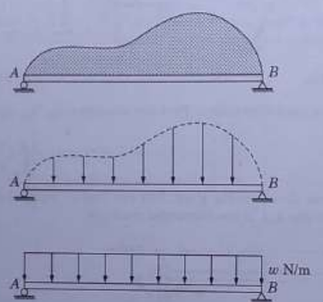


Fig. 3.16

In this way, the problem of finding the resultant of distributed load reduces to finding the resultant of a system of parallel forces.

A distributed load is indicated by a series of lines with arrows. Its magnitude is given in terms of load per unit length (N/m) of the beam. The most common type of load is a *uniformly distributed load* when the load per unit length of the beam is constant.

Let us consider a beam of length L supporting a distributed load as shown in Fig. 3.17.

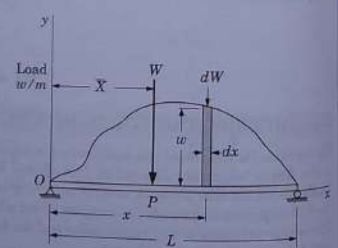


Fig. 3.17

Consider an element of the beam of length dx situated at a distance x from O. If load on the element is w/m, then the total load acting on the element is

$$dW = wdx$$

The total load acting on the beam

Or,

$$W = \int_0^L wdx \qquad ...(3.15)$$

The total load acting on the beam = Area under the load diagram.

A single concentrated load W thus represents the resultant of the given distributed load.

To find the line of acting of the resultant W, take the moments about O.

$$W\bar{x} = \int_0^L (wdx)x$$

$$\bar{x} = \frac{\int_0^L (wdx)x}{W} \qquad ...(3.16)$$

where $\bar{x} = OP$, that is, the line of action of the resultant intersects OX at a distance $\bar{x}$ from O.

Above expression shows that the resultant W acts through the centroid of the area of the load diagram. *The centroid of the load diagram, however, should not be confused with the centroid of the beam.*

3.9 HYDROSTATIC PRESSURE: FORCES ON SUBMERGED SURFACES

In a fluid at rest, the intensity of pressure at a point is same in all directions and is given by,

$$p = wh = \rho gh \qquad ...(3.17)$$

where p = pressure (N/m^2). Pressure is a force acting over a unit area in a direction normal to the area

h = vertical distance of the point from the free surface

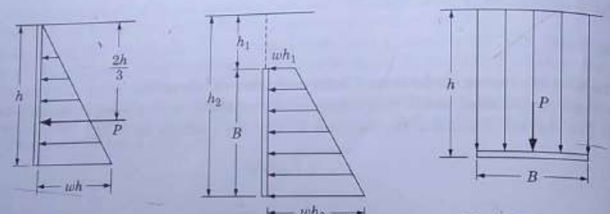
w = specific weight of the fluid (N/m^3)

ρ = density of the fluid (kg/m^3).

From the above formula, we find that *the pressure varies linearly with the depth*. Consider a rectangular float plate of length L and width B immersed in water with its length normal to the plane of Fig. 3.19. The total resultant force P acting over a unit length of this flat plate, due to water pressure, is equal to the area of the load diagram and it passes through the load diagram. For the simplest case, where the pressure distribution is triangular, the resultant passes through the centroid of this triangle which is at a distance of 2/3 h from the free surface as shown in Fig. 3.19.

Fig. 3.18

The distribution of the pressure together with the resultant force P, acting per unit length of the plate, in the different positions of the plate, is shown in Fig. 3.19.



$P = \frac{(wh)h}{2}$ per unit length $(B = h)$

$P = \frac{(wh_1 + wh_2)}{2} B$ per unit length

$P = (whB)$ per unit length

Fig. 3.19

Example 3.10 A beam supports a distributed load as shown (Fig. 3.20). Determine (*a*) the resultant of this distributed load (*b*) the reaction at the supports *A* and *B*.

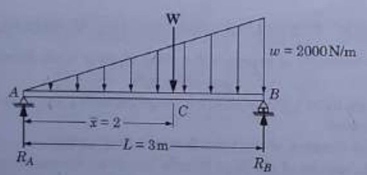


Fig. 3.20

Solution: Area under the load curve is the resultant of the distributed load

$$= \frac{w \times L}{2} = \frac{2000 \times 3}{2}$$

$$W = 3000 \text{ N.}$$

This resultant passes through the centroid of the triangular load diagram.

Therefore, $AC = 2/3 \quad AB = 2/3 \times 3 = 2$ m

$$AC = 2 \text{ m}$$

Let R_A and R_B be the reactions at the supports. Consider the equilibrium of the entire beam as free-body.

$\Sigma M_A = 0$:

$$R_B \times 3 - M\bar{x} = 0$$

$$R_B \times 3 - 3000 \times 2 = 0$$

$$R_B = 2000 \text{ N} \quad \textit{Ans.}$$

$\Sigma F_y = 0$:

$$R_A + R_B = 3000$$

$$R_A = 3000 - 2000$$

$$R_A = 1000 \text{ N} \quad \textit{Ans.}$$

Example 3.11 A beam supports a load distributed parabolically over its length. Determine the resultant of this distributed load and its line of action (Fig. 3.21).

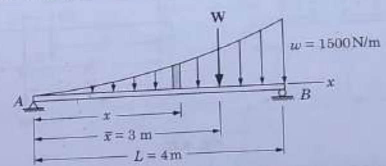


Fig. 3.21

Solution: The equation of the parabolic load is of the form

$$x^2 = kw$$

where k is an unknown constant

At $x = 4$, $w = 1500$ N/m,

Substituting in (*i*) $\quad k = \frac{x^2}{w} = \frac{4 \times 4}{1500} = \frac{16}{1500}$

Resultant of the distributed load = W = Area of the diagram

$$W = \int_0^L w\,dx = \int_0^L \frac{x^2}{k}\,dx = \left| \frac{x^3}{3k} \right|_0^L$$

$$W = \frac{L^3}{3k}$$

Substituting for k $\quad W = \frac{4 \times 4 \times 4}{3 \times \frac{16}{1500}} = 2000 \text{ N} \quad \textit{Ans.}$

The line of action is given by

$$\bar{x} = \frac{\int_0^L wx\,dx}{W} = \frac{\int_0^L wx\,dx}{W} = \frac{L^4}{4kW}$$

$$\bar{x} = \frac{4 \times 4 \times 4 \times 4}{4 \times \frac{16}{1500} \times 2000}$$

$$\bar{x} = 3 \text{ m} \quad \textit{Ans.}$$

Example 3.12 A beam supports a load distributed as shown (Fig. 3.22). Find the resultant of the distributed load and its line of action.

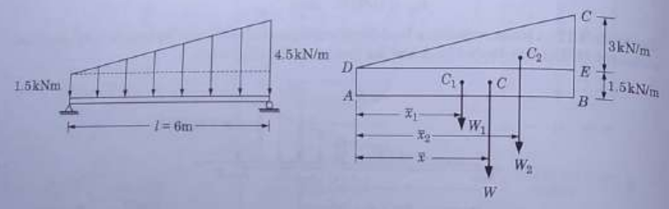


Fig. 3.22

Solution. Let us divide the area of the load diagram *ABCD* into a rectangle and triangle as shown in Fig. 3.22.

Area of load diagram $ABED = 1.5 \times 6 = 9.0$ kN

$$W_1 = 9.0 \text{ kN}$$

Centroid of *ABED* lies at,

$$\bar{x}_1 = \frac{6}{2} = 3 \text{ m}$$

Area of the load diagram $DCE = \frac{3 \times 6}{2}$

$$W_2 = 9 \text{ kN}$$

Centroid of *DCE* lies at,

$$\bar{x}_2 = \frac{2}{3} \times 6 = 4 \text{ m}.$$

The resultant W of the two concentrated loads, W_1 and W_2 is the resultant of the distributed load. Therefore,

$$W = W_1 + W_2 = 9 + 9 = 18 \text{ kN} \quad \textit{Ans.}$$

and it lies at,

$$\bar{x} = \frac{W_1\bar{x}_1 + W_2\bar{x}_2}{W} = \frac{9 \times 3 + 9 \times 4}{18} = 3.5\text{m} \quad \textit{Ans.}$$

Example 3.13 A beam is hinged at *A* and roller supported *B*. It is acted upon by loads as shown in Fig. 3.23. Find reactions at *A* and *B*.

Solution: Let X_A and Y_A be the reactions at the hinge *A* and R_B at the roller support *B*.

Resultant of uniformly distributed force of 5 kN/m acting between $EB = 5 \times 4 = 20$ kN and acts at a distance of 2 m from *B*.

Resultant of distributed load between $BC = 1/2\,(10 \times 3) = 15$ kN and acts through the centroid of triangular load diagram, at a distance of $1/3 \times 3 = 1$ m from *B*.

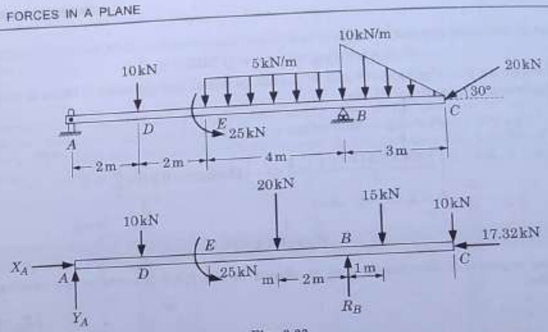


Fig. 3.23

Force of 20 kN at *C* is resolved into components 17.32 kN and 10 kN.

$$\Sigma F_x = 0: \quad X_A - 17.32 = 0$$

$$X_A = 17.32 \text{ kN} \quad \textit{Ans.}$$

$$\Sigma F_y = 0: \quad Y_A - 10 - 20 + R_B - 15 - 10 = 0$$

$$Y_A + R_B = 55$$

$$\Sigma M_A = 0: \quad 10\,(2) + 25 - 20\,(6) + R_B\,(8) - 15\,(9) - 10\,(11) = 0$$

$$R_B = 45 \text{ kN} \quad \textit{Ans.}$$

$$Y_A + 45 = 55$$

$$Y_A = 10 \text{ kN} \quad \textit{Ans.}$$

Example 3.14 A concrete dam has rectangular cross-section of height h and width b and is subjected to a water pressure on one side. Determine the minimum width b of the dam if the dam is not to overturn about the point *B* when $h = 4$ m. Assume density of water = 1000 kg/m^3 and the density of concrete = 2400 kg/m^3.

Solution: Consider a unit length of the dam normal to the plane of the figure.

The total resultant force due to water pressure per unit length of the dam

$$P = \left(\frac{wh}{2}\right)h$$

$$P = \frac{\rho g h^2}{2} = \frac{1000 \times 9.81 \times 4^2}{2}$$

It acts through the centroid of the triangular load

$$\bar{x} = \frac{2}{3},\ h = \frac{2}{3} \times 4 = \frac{8}{3} \text{m}$$

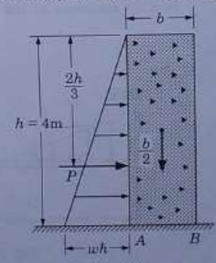


Fig. 3.24

Weight of the dam per unit length

$$W (h \times b \times 1)\rho g = (4 \times b \times 1)\ 2400 \times 9.81 \text{ N}$$

Moments about B of the forces P and W should be equal and opposite if there is no resultant turning moment acting about B. So,

$$P \times \frac{1}{3}h - W\frac{b}{2} = 0$$

$$\left(\frac{1000\times 9.81\times 4\times 4}{2}\right)\times\frac{1}{3}\times 4-(4b\times 2400\times 9.81)\times\frac{b}{2} = 0$$

$$b = 1.491 \text{ m} \quad \textit{Ans.}$$

PROBLEMS

3.9. A beam supports a distributed load as shown in Fig. P.3.9. Determine the reactions at the supports.

$$\begin{bmatrix} R_A = 0.25\text{ kN} \\ R_B = 1.75\text{ kN} \end{bmatrix}$$

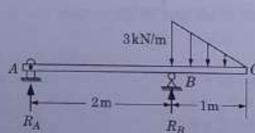


Fig. P.3.9

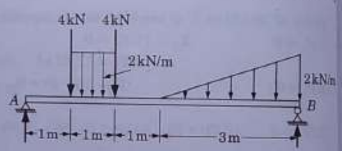


Fig. P.3.10

3.10. Determine the reactions R_A and R_B for the beam supported and loaded as shown in Fig. P. 3.10.

$$\begin{bmatrix} R_A = 8\text{ kN} \\ R_B = 5\text{ kN} \end{bmatrix}$$

3.11. A cantilever supports a distributed load as shown. Determine (*a*) the resultant of this distributed load (*b*) reactions at the supports.

$$\begin{bmatrix} (a)\ 1.2\text{ kN} \\ (b)\ R_A = 1.2\text{ kN} \\ M_A = 0.8\text{ N·m} \end{bmatrix}$$

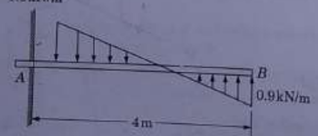


Fig. P. 3.11

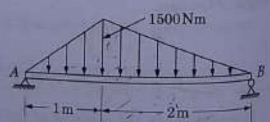


Fig. P.3.12

3.12. A simply supported beam carries a distributed load as shown in Fig. P.3.12. Find the reactions at the supports.

$$\begin{bmatrix} R_A = 1250\text{ N} \\ R_B = 1000\text{ N} \end{bmatrix}$$

3.13. A partition 3 m long divides a storage tank. On one side of this partition is petrol of density 780 kg/m^3, stored to a depth of 1.8 m; and on the other side is oil of density 880 kg/m^3 to a depth of 0.9 m. Determine the total resultant force exerted on the partition and position of its line of action. [26.7 kN, 0.755 m from A]

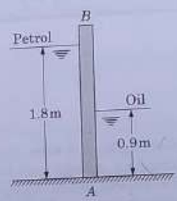


Fig. P. 3.13

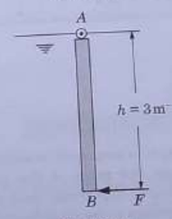


Fig. P.3.14

3.14. A plane rectangular plate of height 3 m and length 2 m is hinged at A and is subjected to water pressure from one side. How much force F must be applied at point B (the middle of the bottom edge) of the plate, to keep it in vertical position? [F = 58.86 kN]

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4 CHAPTER

Centroid, Centre of Mass and Centre of Gravity

4.1 INTRODUCTION

Very often it is required to define a point such that the length of a wire, the area of a plate, the volume, the mass or the gravitational forces acting on a body *may be assumed to be concentrated at that point. Such points are often called as central points.* A problem of the above nature involves some *distributed quantity*. In the earlier chapter we discussed the method to replace a distributed force by a single concentrated force. Similar method shall be used now to determine the central points. Some commonly used central points are :

Centroid of the length of a curve.

Centroid of the area of a surface.

Centroid of the volume of a body.

Mass centre of the mass of a body.

Centre of gravity of the gravitational forces acting on a body.

Let us clarify the difference between the centre of gravity (C.G.) of a body and its centre of mass (C.M.).

Centre of Gravity. Centre of gravity of a body is a point through which the resultant of the distributed gravity forces acts irrespective of the orientation of the body.

Centre of Mass. It is the point where the entire mass of a body may be assumed to be concentrated.

The centre of mass (C.M.) and the centre of gravity (C.G.) of a body are different only when the gravitational field is not *uniform and parallel*. For most practical purposes they are assumed to be the same.

4.2 CENTRE OF GRAVITY OF A BODY : DETERMINATION BY THE METHOD OF MOMENTS

Consider a body of mass M. Let this body be composed of 'n' number of masses $\Delta M_1, \Delta M_2, \Delta M_3$ ... ΔM_n, distributed within the body such that

$$M = \Delta M_1 + \Delta M_2 + \ldots \Delta M_n$$

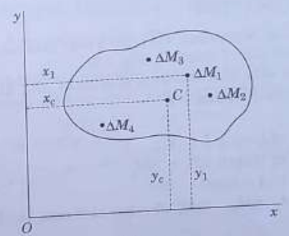


Fig. 4.1

The distance of these masses with respect to the axes be,

$$(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n).$$

Mathematically speaking, it can be stated that the mass M of the body has been divided into 'n' elements of masses.

Let the C.G. of the whole mass M lie at a distance (x_c, y_c) with respect to the reference axes.

Let us assume that the gravitational field is uniform and parallel.

Gravitational force acting on the mass $\Delta M_1 = \Delta M_1 g$. Similarly, we can find the gravitational forces acting the masses $\Delta M_2, \Delta M_3 \ldots \Delta M_n$.

To find the resultant of parallel forces $\Delta M_1 g, \Delta M_2 g, \Delta M_3 g \ldots \Delta M_n g$ we apply the principle of moments.

The moment of the resultant of all the forces about the y-axis = The sum of the moments of all the forces about y-axis

$$Mg(x_c) = (\Delta M_1 g)x_1 + (\Delta M_2 g)x_2 + \ldots (\Delta M_n)x_n$$

$$x_c = \frac{(\Delta M_1)x_1 + (\Delta M_2)x_2 + \ldots(\Delta M_n)x_n}{M}$$

where; $M = \Delta M_1 + \Delta M_2 + \ldots \Delta M_n$

Above equation can be expressed as

$$M = \Sigma(\Delta M_i).$$

The numerator can also be expressed in a similar manner.

Therefore, $$x_c = \frac{\Sigma(\Delta M_i x_i)}{\Sigma(\Delta M)_i} \quad \ldots(4.1)$$

Similarly, $$y_c = \frac{\Sigma(\Delta M_i y_i)}{\Sigma(\Delta M)_i} \quad \ldots(4.2)$$

The summation sign 'Σ' means that all the elements of the masses are to be considered.

4.3 CONCEPT OF CENTROID

One dimensional Body (Line Segment). Consider a body in the shape of a curved homogen wire of uniform cross-section and of length L (Fig. 4.2).

Divide the length of this wire into elements of lengths ΔL_1, ΔL_2 ... ΔL_n.

Let the uniform area of cross-section = A

Density of the wire = ρ

The mass M of the wire of length $L = AL\rho$

The mass of an element of length $\Delta L_1 = \Delta M_1$

ΔM_1 = volume × density = $A(\Delta L_1)\rho$

Similarly, the masses ΔM_2, ΔM_3 ... ΔM_n of other elements can be determined.

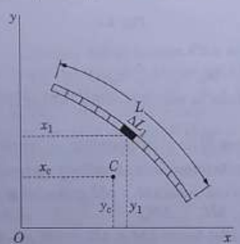


Fig. 4.2

Let the distances of the centres of these lengths with respect to the axes be,

$$(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$$

Applying the principle of moments,

$$x_c = \frac{\Sigma(\Delta M_i x_i)}{(\Sigma \Delta M)_i}$$

$$x_c = \frac{(A\Delta L_1\rho)x_1 + (A\Delta L_2\rho)x_2 + (A\Delta L_3\rho)x_3 + \dots (A\Delta L_m\rho)x_n}{A\Delta L_1\rho + A\Delta L_2\rho + A\Delta L_3\rho + \dots A\Delta L_n\rho}$$

$$x_c = \frac{A\rho(\Delta L_1 x_1 + \Delta L_2 x_2 + \Delta L_n x_n)}{A\rho(\Delta L_1 + \Delta L_2 + \dots \Delta L_n)}$$

$$x_c = \frac{\Sigma(\Delta L_i x_i)}{\Sigma(\Delta L)_i} \qquad \dots(4.3$$

Similarly, $$y_c = \frac{\Sigma \Delta L_i y_i}{\Sigma(\Delta L)_i} \qquad \dots(4.$$

Because the density ρ and the area of cross-section A are constant over the entire length of the wire, the coordinates of C.G. of the wire become the coordinates of the centroid of the wire generally called as the *centroid of a line segment.*

4.4 CENTROID TWO DIMENSIONAL BODY

Now consider the case of a homogenous plate or lamina of uniform thickness t and density ρ and total area A.

Divide the area of the plate into elements of areas ΔA_1, ΔA_2 ... ΔA_n.

The distances of the centres of these areas with respect to the axes be

$$(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$$

Mass of the plate $\qquad M = (At)\rho$

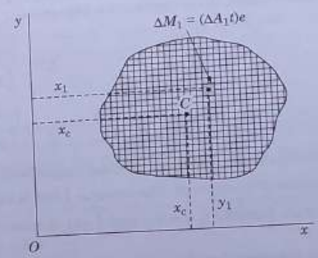


Fig. 4.3

Mass ΔM_1 of the element

$$\Delta A_1 = (A_1 t)\rho$$

Using, $$x_c = \frac{\Sigma \Delta M_i x_i}{\Sigma(\Delta M)_i}$$

$$x_c = \frac{(\Delta A_1 t\rho)x_1 + (\Delta A_2 t\rho)x_2 + \dots (\Delta A_n t\rho)x_n}{\Delta A_1 t\rho + \Delta A_2 t\rho + \dots (\Delta A_n t\rho)}$$

$$x_c = \frac{t\rho(\Delta A_1 x_1 + \Delta A_2 x_2 + \dots \Delta A_n x_n}{t\rho(\Delta A_1 + \Delta A_2 + \dots \Delta A_n)}$$

$$x_c = \frac{\Sigma(\Delta A_i x_i)}{\Sigma(\Delta A)_i} \qquad \dots(4.5)$$

Similarly, $$y_c = \frac{\Sigma(\Delta A_i y_i)}{\Sigma(\Delta A)_i} \qquad \dots(4.6)$$

x_c and y_c are the coordinates of the centroid of the plate; generally called as *the coordinates of the centroid of an area.*

Again, as the density and thickness of the plate are constant over the entire area so, the coordinates of the C.G. become the coordinates of the centroid of the area.

Generally the term centroid is used for the centre of a gravity of a geometrical figure and the term centre of gravity is used when referring to actual physical bodies.

4.5 DETERMINATION OF CENTROID AND CENTRE OF GRAVITY: INTEGRATION METHOD

By considering a plane figure to be made up of a number of small elements of length or area we cannot generate the true shape of the figure. To truly generate the shape of a figure, we have to make the size of these elements very very small and their number very large.

For example, the circumference of a circle can be generated only by a polygon having a large number of very very small sides.

Mathematically speaking, when the terms ΔL or ΔA occurring in the expressions of C.G. and centroid become infinitesimally small, these expressions (equations 4.1 to 4.6) can be written as

$$\left.\begin{array}{lll} x_c = \dfrac{\int x dL}{\int dL} & x_c = \dfrac{\int x dA}{\int dA} & x_c = \dfrac{\int x dm}{\int dm} \\[2ex] y_c = \dfrac{\int y dL}{\int dL} & y_c = \dfrac{\int y dA}{\int dA} & y_c = \dfrac{\int y dm}{\int dm} \end{array}\right] \quad \ldots(4.7)$$

where, dL, dA and dm denote the length, area and mass respectively of a differential element chosen and (x, y) the coordinates of its centroid. The integral $\int x dA$ is known as the *first moment of area with respect to the y-axis*. Similarly, the integral $\int y dA$ denotes the first moment of area with respect to the x-axis.

Above expressions can be integrated to determine the coordinates of centroid and C.G. as explained in the following section. Also, some important points regarding the choice of a differential element for setting up the integrals and choosing the axes of reference are discussed.

Integration Method

(*a*) **Choice of Differential Element.** In the determination of the coordinates of the centroid by intergration method, a proper choice of the differential element can considerably ease and simplify the setting up and the valuation of the integrals of the type.

$$x_c = \frac{\int x dA}{\int dA},\ y_c = \frac{\int y dA}{\int dA}$$

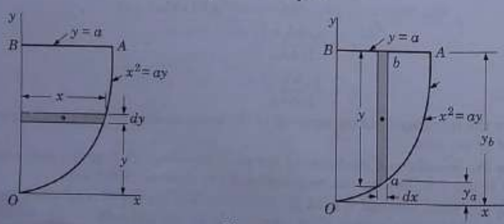


Fig. 4.4

Consider an area OAB bounded by the curve $x^2 = ay$ and the straight line $y = a$ as shown in Fig. 4.4.

We can consider either a horizontal differential element (strip) or a vertical differential element (strip).

For a horizontal strip

Area of the differential element $dA = x\, dy$

The position of its centroid is $\left(\dfrac{x}{2}, y\right)$

For a vertical strip

Area of the differential element $dA = y\, dx$ or $dA = (y_b - y_a)dx$

The position of its centroid is

$$\left(x, \frac{y_a + y_b}{2}\right)$$

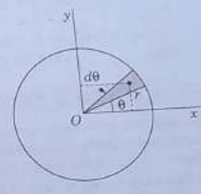


Fig. 4.5

It may be noted that the point a lies on the curve OA and the point b on the straight line AB. It can be appreciated that of the two elements considered, the horizontal differential element is a better choice for evaluating the integral in the present situation.

Triangular Element

A triangular element can also be choosen in certain situations as shown in Fig. 4.5.

Area of the element $dA = \dfrac{(rd\theta)r}{2}$

The position of its centroid is

$$= \left(\frac{2}{3} r\cos\theta, \frac{2}{3} r\sin\theta\right)$$

(*b*) **Choice of the Axes of Reference.** The determination of the position of the centroid or C.G., involves taking the moments of lengths, areas or masses with respect to some axes. So, some frame of reference or coordinates axes are to be choosen. *Although, the location of the centroid or C.G. does not depend upon the reference axes chosen, but their proper choice can considerably simplify the calculations.* Axis of symmetry of a figure (provided it exists) if chosen as a reference axis can simplify calculations.

Because, if a figure or a curve has some axis of symmetry then the centroid shall lies on this axis of symmetry.

Consider a figure having the shape of a dumbell as shown (Fig. 4.6).

The figure is symmetrical about the x as well as the y-axis.

Let us discuss the symmetry about the x-axis : Consider any element of area ΔA_1 at a distance y_1 above the x-axis. Because of the symmetry, there will be a similar element of area ΔA_1 situated at a distance y_1 below the x-axis. If the

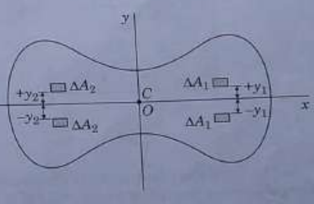


Fig. 4.6

sum of the moments of all the elements of the area above the x-axis are taken then, it will cancel with the sum of the moments of similarly elements of the area lying below the x-axis.

Thus, $y_c = 0$, or the centroid lies on the x-axis.

Similar argument is valid regarding the symmetry about the y-axis.

Thus, $x_c = 0$, or the centroid lies on the y-axis.

4.6 CENTROID OF A COMPOSITE PLANE FIGURE

A composite area or a curve is one which can be considered to be made up of several pieces or components that represent familiar geometric shapes (e.g., rectangle, circle, triangle, semicircle, ellipse etc.) and for which the positions of individual centroids are known.

The entire area shown in Fig. 4.7 can be considered to the made up of a triangle, rectangle and a semicircle.

To find the centroid of this area, divide the area into different component parts as discussed above (not infinitesimal elements). Then, we can use the equation,

$$x_c = \frac{\Sigma \Delta A_i x_i}{\Sigma \Delta A_i}, \; y_c = \frac{\Sigma \Delta A_i y_i}{\Sigma \Delta A_i}$$

treating each component part as an element of area ΔA_i and the distance of its centroid as (x_1, y_1).

The positions of the centroid of the component parts can be taken as standard results.

For example, for L-Section in Fig. 4.8 (a)

$$x_c = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2}, \; y_c = \frac{A_2 x_2 + A_2 x_2}{A_1 + A_2}$$

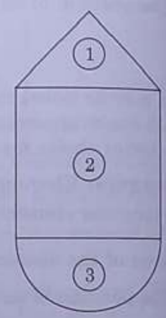
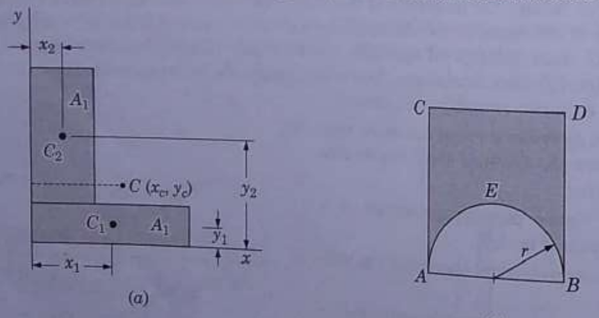


Fig. 4.7

If there is a *hole or void* (a non-existing area) then it may be treated as a *negative area*.

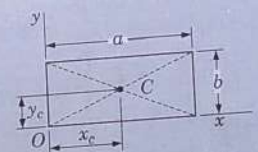
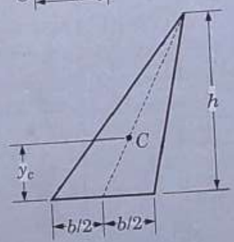
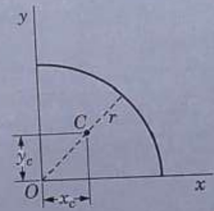
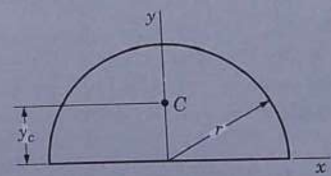
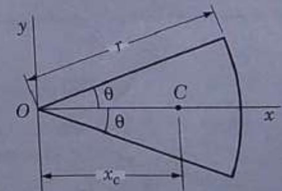


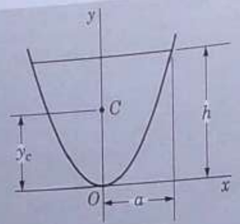
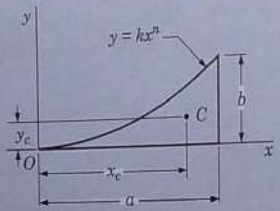
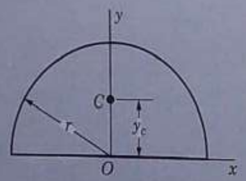
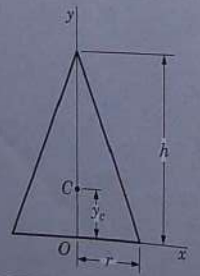
(a) (b)

Fig. 4.8

In Fig. 4.8 (b) hatched area, may be considered to be made up of a rectan-gular area $ABCD$ and a negative area formed by a semicircle of radius r.

Table 4.1 *Centroids of Various Shapes of Areas*

Shape		Area	x_c	y_c
Rectangle		ab	$\frac{a}{2}$	$\frac{b}{2}$
Triangle		$\frac{bh}{2}$	–	$\frac{h}{3}$
Quarter-circle		$\frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
Semi-circle		$\frac{\pi r^2}{2}$	0	$\frac{4r}{3\pi}$
Circular-Sector		θr^2	$\frac{2r\sin\theta}{3\theta}$	0

Parabola		$4\frac{ah}{3}$	0	$\frac{3h}{5}$
General Spander		$\frac{ab}{n+1}$	$\left(\frac{n+1}{n+2}\right)a$	$\left(\frac{n+1}{2n+1}\right)\frac{b}{2}$
Hemisphere		$\frac{2}{3}\pi r^3$	0	$\frac{3}{8}r$
Right circular cone		$\frac{1}{3}\pi r^2 h$	0	$\frac{h}{4}$

EXAMPLES

Example 4.1 Find the centroid of a straight uniform wire of length L.

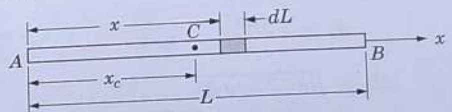


Fig. 4.9

Solution: Choose the x-axis passing through the centre of the wire and along its length.

Divide the length L of the wire into elements of infinitesimal length dL.

Let us choose an element at a distance x from the origin.

$$x_c = \frac{\int x dL}{\int dL}$$

But

$$dL = dx$$

$$x_c = \frac{\int_0^L x\,dx}{\int_0^L dx} = \frac{\left[\frac{x^2}{2}\right]_0^L}{[x]_0^L}$$

$$x_c = \frac{L}{2} \quad \textit{Ans.}$$

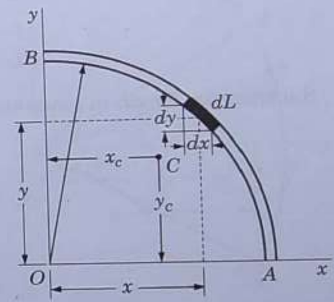


Fig. 4.10

Example 4.2 Find the centroid of a uniform wire bent in the form of a quadrant of the arc of a circle of radius r as shown, in Fig. 4.10.

Solution: Here we have to determine the centroid of the length of a curve, so let us divide the length AB into infinitesimal elements of length dL.

Choose the origin of coordinates at the centre of the circle O.

The figure is symmetrical about the line joining the origin to middle point of the length AB. Therefore, we need to determine only one of the coordinates of the centroid.

Let us determine y_c.

Consider an element of length dL at a distance (x, y) from the axes,

$$y_c = \frac{\int y dL}{\int dL}$$

Expressing dL in terms of x and y

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

Equation of the circle is given by

$$x^2 + y^2 = r^2$$

Differentiating

$$2x\,dx + 2y\,dy = 0$$
$$2x\,dx = -2y\,dy$$

Or
$$dy = -\frac{x\,dx}{y}$$

As
$$dL = \sqrt{(dx)^2 + (dy)^2}$$

Therefore,
$$dL = \sqrt{(dx)^2 + \left(\frac{x}{y}dx\right)^2}$$

$$dL = \sqrt{\frac{y^2 + x^2}{y^2}}\,dx$$

$$dL = \frac{r}{y}dx$$

Substituting for dL in equation (*i*)

$$y_c = \frac{\int_0^r y.\frac{r}{y}dx}{\int dL} = \frac{\int_0^r r\,dx}{\left(\frac{2\pi r}{4}\right)} = \frac{r[x]_0^r}{\frac{\pi r}{2}}$$

$$y_c = \frac{r^2}{\frac{\pi r}{2}} \text{ or } y_c = \frac{2r}{\pi} \quad Ans.$$

Because of symmetry,

$$x_c = y_c = \frac{2r}{\pi} \quad Ans.$$

Now let us solve the same problem by using polar coordinates.

Integral to be evaluated is

$$y_c = \frac{\int y\,dL}{\int dL}$$

Converting into polar coordinates, see Fig. 4.11

$$y = r \sin\theta$$
$$dL = r\,d\theta$$

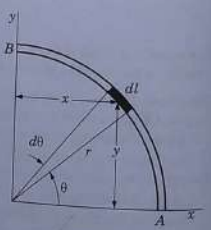


Fig. 4.11

$$y_c = \frac{\int_0^{\pi/2} (r\sin\theta)(r\,d\theta)}{\int_0^{\pi/2} r\,d\theta}$$

$$y_c = \frac{\int_0^{\pi/2} r^2 \sin\theta\,d\theta}{\int_0^{\pi/2} r\,d\theta} = \frac{r^2[-\cos\theta]_0^{\pi/2}}{[\theta]_0^{\pi/2}}$$

$$y_c = \frac{r^2}{\frac{r\pi}{2}} \text{ or } y_c = \frac{2r}{\pi} \quad Ans.$$

Note here that working in polar coordinates considerably simplifies the evaluation of the integral *because the polar coordinates best match the boundary of the figure.*

Example 4.3 Determine the coordinates of the centroid of a lamina in the shape of a circular sector of radius r and central angle 2α.

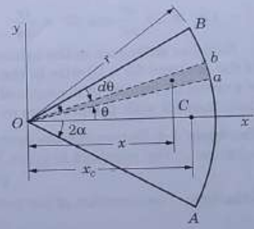


Fig. 4.12

Solution: Reference axes are as shown in Fig. 4.12. Note here that the figure is symmetrical about the x-axis therefore,

$$y_c = 0$$

Divide the area into infinitesimal triangular elements each of altitude r and base $r\,d\theta$ as shown in Fig. 4.12. Area of the triangular element $Oab = 1/2(r\,d\theta)r$. Centroid of the triangular area Oab lies on the line joining O to the mid-point of ab and at a distance $2/3r$ from O.

Distance x of the centroid from the y-axis $= 2/3r \cos\theta$

$$x_c = \frac{\int x\,dA}{\int dA} = \frac{2\int_0^{\alpha}\left(\frac{2}{3}r\cos\theta\right)\left(\frac{r^2 d\theta}{2}\right)}{2\int_0^{\alpha}\frac{r^2 d\theta}{2}}$$

$$x_c = \frac{\frac{r^3}{3}\int_0^{\alpha}\cos\theta\,d\theta}{\frac{r^2}{2}\int_0^{\alpha}d\theta} = \frac{2r[\sin\theta]_0^{\alpha}}{3[\theta]_0^{\alpha}}$$

$$x_c = \frac{2r\sin\alpha}{3\alpha} \quad Ans.$$

Centroid of a quadrant of a circle can be determined if we substitute

$$2\alpha = \frac{\pi}{2} \text{ or } \alpha = \frac{\pi}{4}$$

$$x_c = \frac{2r\sin\frac{\pi}{4}}{3\left(\frac{\pi}{4}\right)}$$

$$x_c = \sqrt{2}\,\frac{4r}{3\pi} \quad \text{(Distance } OC\text{)}$$

Further, the position of centroid with respect to the radii OA and OB (as shown in Fig. 4.13) is

$$Oa' = \sqrt{2}\,\frac{4r}{3\pi}\cos 45° = \frac{4r}{3\pi}$$

$$Ob' = \sqrt{2}\,\frac{4r}{3\pi}\cos 45° = \frac{4r}{3\pi}$$

Note the difference in the values of the coordinates of the centroid for a (*i*) wire in the shape of quadrant of a circle (*ii*) lamina of the same shape, by comparing these results with the previous example.

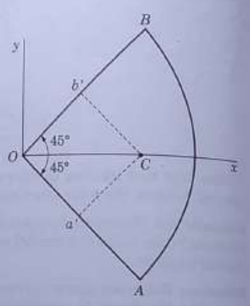


Fig. 4.13

Example 4.4 Determine the centroid of the parabolic spandrel as shown in Fig. 4.14 (*a*). The equation of the parabola is given by

$$y = kx^2. \qquad \dots(i)$$

Solution: The equation of the parabola is

$$y = kx^2$$

The value of the constant k is determined by substituting the coordinates (a, b) of a point on the curve in the equation (*i*)

$$b = ka^2 \quad \text{or} \quad k = \frac{b}{a^2}$$

The equation of the curve can be written as

$$y = \frac{b}{a^2}x^2 \quad \text{or} \quad x = \frac{a}{\sqrt{b}}\sqrt{y}$$

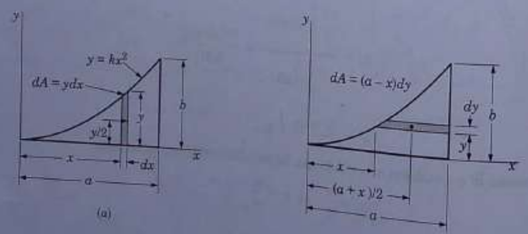


Fig. 4.14

Solution: **By Vertical Strip:** Consider a vertical differential element (or strip) of height y and width dx.

Area of the strip $dA_s = y\,dx$

Distance of the centroid of the strip from the y-axis $= x$

$$x_c = \frac{\int x dA}{\int dA} = \frac{\int x.ydx}{\int ydx} = \frac{\int_0^a x\left(\frac{b}{a^2}x^2\right)dx}{\int_0^a \frac{b}{a^2}x^2 dx}$$

$$x_c = \frac{\left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a}{\left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a} \qquad x_c = \frac{3}{4}a \quad \textit{Ans.}$$

$$y_c = \frac{\int ydA}{\int dA}$$

Distance of the centroid of the strip from the x-axis $= y/2$

$$y_c = \frac{\int \frac{y}{2}ydx}{\int ydx} = \frac{\int_0^a \frac{1}{2}y^2 dx}{\int_0^a ydx} = \frac{\int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx}{\int_0^a \frac{b}{a^2}x^2 dx}$$

$$y_c = \frac{\left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a}{\left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a} \qquad y_c = \frac{3}{10}b \quad \textit{Ans.}$$

Solution: **By Horizontal Strip.** Consider a horizontal differential element (strip) as shown in Fig. 4.14 (*b*).

Area of the strip $dA = (a - x)dy$.

Distance of the centroid of the element from the y-axis $= (a + x)/2$

$$x_c = \frac{\int x dA}{\int dA} = \frac{\int_0^b \left(\frac{a+x}{2}\right)(a-x)dy}{\int_0^b (a-x)dy}$$

$$x_c = \frac{\int_0^b \left(\frac{a^2 - x^2}{2}\right) dy}{\int_0^b (a-x) dy} = \frac{\frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b} y\right) dy}{\int_0^b \left(a - \frac{a}{\sqrt{b}}\sqrt{y}\right) dy}$$

$$x_c = \frac{3 \times a^2 b}{4 \times ab} \quad x_c = \frac{3}{4}a \quad \textit{Ans.}$$

$$y_c = \frac{\int y dA}{\int dA} = \frac{\int_0^b y(a-x) dy}{\int_0^b (a-x) dy} = \frac{\int_0^b y\left(a - \frac{a}{\sqrt{b}}\sqrt{y}\right) dy}{\int_0^b \left(a - \frac{a}{\sqrt{b}}\sqrt{y}\right) dy}$$

$$y_c = \frac{3ab^2}{10 \times ab}$$

$$y_c = \frac{3b}{10} \quad \textit{Ans.}$$

Example 4.5 Determine the coordinates of the centroid of the shaded area formed by the intersection of a straight line and a parabola as shown in Fig. 4.15. The equation of the parabola is given by $y = x^2/a$ and of straight line by $y = x$.

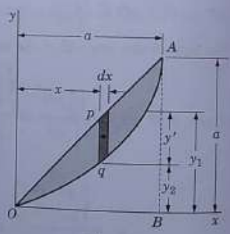


Fig. 4.15

Solution: The straight line and the parabola

$$y = x \qquad \text{...(i)}$$

$$y = \frac{x^2}{a} \qquad \text{...(ii)}$$

intersect at the points O and A.

Solving (*i*) and (*ii*) simultaneously the coordinates of the points O and A are, O (0, 0) and A (a, a).

Divide the area into infinitesimal vertical strips. Consider a strip of height y' and width dx.

Area dA of the strip $= (pq)\, dx$

$$dA = y' dx$$

$$dA = (y_1 - y_2)\, dx$$

where y_1 is the coordinate of the point p which lies on the straight line OA and y_2 is the coordinate of the point q which lies on the parabola OA.

Using the equations (*i*) and (*ii*), y_1 and y_2 can be expressed in terms of x as

$$y_1 = x,\; y_2 = \frac{x^2}{a}$$

Area of the vertical strip,

$$dA = \left(x - \frac{x^2}{a}\right) dx.$$

Distance of the centroid of the strip from the y-axis $= x$

$$x_c = \frac{\int x dA}{\int dA} = \frac{\int_0^a x\left(x - \frac{x^2}{a}\right) dx}{\int_0^a \left(x - \frac{x^2}{a}\right) dx}$$

$$x_c = \frac{\int_0^a \left(x^2 - \frac{x^3}{a}\right) dx}{\int_0^a \left(x - \frac{x^2}{a}\right) dx}$$

$$x_c = \frac{\left[\frac{x^3}{3} - \frac{x^4}{4a}\right]_0^a}{\left[\frac{x^2}{2} - \frac{x^3}{3a}\right]_0^a},\quad x_c = \frac{a}{2} \quad \textit{Ans.}$$

$$y_c = \frac{\int y dA}{\int dA}$$

where,

$$dA = \text{area of the strip} = (y_1 - y_2) dx$$

$$y = \frac{y_1 + y}{2} = \text{the distance of the centroid of the strip from the } x\text{-axis}$$

$$y_c = \frac{\int \left(\frac{y_1 + y}{2}\right)(y_1 - y_2) dx}{\int (y_1 - y_2) dx}$$

Using equations (*i*) and (*ii*) y_1 and y_2 can be expressed in terms of x as

$$y_1 = x,\; y_2 = \frac{x^2}{a}$$

$$y_c = \frac{\int_0^a \frac{1}{2}\left(x - \frac{x^2}{a}\right)\left(x + \frac{x^2}{a}\right) dx}{\int_0^a \left(x - \frac{x^2}{a}\right) dx}$$

$$y_c = \frac{\frac{1}{2}\int_0^a \left(x^2 - \frac{x^4}{a^2}\right) dx}{\int_0^a \left(x - \frac{x^2}{a}\right) dx} = \frac{\frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5a^2}\right]_0^a}{\left[\frac{x^2}{2} - \frac{x^3}{3a}\right]_0^a},$$

$$y_c = \frac{2a}{5} \quad \textit{Ans.}$$

Alternative Method. The problem can be solved by treating the shaded area as a composite area. That is,

Shaded Area = Area of the triangle OAB – Area under the curve

$$y = x^2/a.$$

Area of triangle OAB and the location of its C.G. is listed. Area under the curve $y = x^2/a$ is the area of the spandrel ($n = 2$, $a = b = a$). The area and the location of its C.G. is also listed. Centroid of the composite area now can be determined.

Example 4.6 Locate the centroid of the shaded area obtained by removing a semicircle of diameter a from a quadrant of a circle of radius a.

Solution: To determine the centroid (x_c, y_c) of the shaded area let us consider that the shaded area is obtained by subtracting the area of the semicircle of radius $a/2$ from the area of the quadrant of circle of radius a. In this sense, therefore, the area to be subtracted is treated to be as negative area.

Reference axes are as shown in the Fig. 4.16. Different areas and coordinates of their centroids are tabulated below:

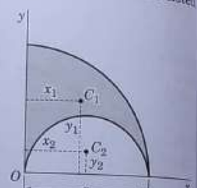


Fig. 4.16

Figure	*Area*	*x-coordinate of the centroid*	*y-coordinate of the centroid*
Quadrant of circle of radius = a	$A_1 = \frac{\pi}{4}a^2$	$x_1 = \frac{4a}{3\pi}$	$y_1 = \frac{4a}{3\pi}$
Semicircle of radius = $a/2$	$A = \frac{\pi(a/2)^2}{2}$ $A_2 = -\frac{\pi a^2}{8}$	$x_2 = \frac{a}{2}$	$y_2 = \frac{4}{3\pi}\left(\frac{a}{2}\right) = \frac{2a}{3\pi}$

Centroid of the composite area

$$x_c = \frac{A_1x_1 + A_2x_2}{A_1 + A_2}$$

$$x_c = \frac{\frac{\pi}{4}a^2\left(\frac{4a}{3\pi}\right) - \frac{\pi a^2}{8}\left(\frac{a}{2}\right)}{\frac{\pi a^2}{4} - \frac{\pi a^2}{8}}$$

$$x_c = \frac{\frac{a}{3\pi} - \frac{a}{16}}{\frac{1}{8}} = 8a\left(\frac{1}{3\pi} - \frac{1}{16}\right)$$

$x_c = 0.349a$ *Ans.*

$$y_c = \frac{A_1y_1 + A_2y_2}{A_1 + A_2}$$

$$y_c = \frac{\frac{\pi}{4}a^2\left(\frac{4a}{3\pi}\right) - \frac{\pi a^2}{8}\left(\frac{2a}{3\pi}\right)}{\frac{\pi a^2}{4} - \frac{\pi a^2}{8}} = \frac{\frac{a}{3\pi} - \frac{a}{12\pi}}{\frac{1}{8}} = \frac{2a}{\pi}$$

$y_c = 0.636a$ *Ans.*

Example 4.7 A square hole is punched out of a circular lamina as shown. The diagonal of the square which is punched out is equal to the radius of circle. Find the centroid of the remaining lamina.

Solution: Let the radius of the circle be = a.

As the diagonal of the square is a, side of the square $= \frac{a}{\sqrt{2}}$.

Choosing the reference axes as shown in Fig. 4.17, it is seen that figure is symmetrical about x-axis therefore, $y_c = 0$.

Different areas and coordinates of their centroids are tabulated below:

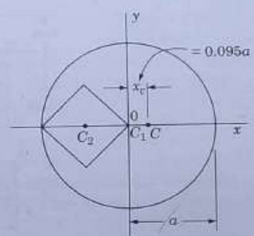


Fig. 4.17

Figure	*Area*	*x-coordinate of the centroid*	*y-coordinate of the centroid*
Circle of radius a	$A_1 = \pi a^2$	$x_1 = 0$	$y_1 = 0$
Square of side	$A_2 = -\frac{a}{\sqrt{2}}\cdot\frac{a}{\sqrt{2}}$ $= \frac{a}{\sqrt{2}}$	$x_2 = -\frac{a}{2}$ $A_2 = -\frac{a^2}{2}$	$y_2 = 0$

$$x_c = \frac{A_1x_1 + A_2x_2}{A_1 - A_2}$$

$$x_c = \frac{\pi a^2(0) + \left(-\frac{a^2}{2}\right)\left(-\frac{a}{2}\right)}{\pi a^2 - \frac{a^2}{2}} = \frac{\frac{+a^3}{4}}{\pi a^2 - \frac{a^2}{2}}$$

(–ve Area and –ve coordiante)

$$x_c = \frac{a}{4(\pi - 0.5)}$$

$x_c = +\ 0.095a$ *Ans.*

Example 4.8 Determine the centroid of the cross-sectional area of an unequal I-section.

Solution: The figure is symmetrical about the vertical line which is also chosen as the y-axis. x-axis is choosen to coincide with the bottom edge AB of Fig. 4.18.

The figure can be considered to be made up of three reactangles, $ABCD$, $EFGH$ and $IJKL$.

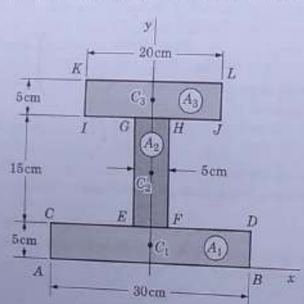


Fig. 4.18

From symmetry, centroid of the total figure lies on y-axis.

Therefore,

Figure	*Area* (cm^2)	*x-coordinate of the centroid* (cm)	*y-coordinate of the centroid* (cm)
Rectangle $ABCD$	$A_1 = 5 \times 30 = 150$	0	$y_1 = 5/2 = 2.5$ cm
Rectangle $EFGH$	$A_2 = 5 \times 15 = 75$	0	$y_2 = 5 + 15/2 = 12.5$ cm
Rectangle $IJKL$	$A_3 = 20 \times 5 = 100$	0	$5 + 15 + 5/2 = 22.5$ cm

$$y_c = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3} = \frac{(150 \times 2.5) + (75 \times 12.5) + (100 \times 22.5)}{150 + 75 + 100}$$

$$y_c = \frac{3562.5}{325}$$

$y_c = 10.96$ cm. *Ans.*

Example 4.9 Find the centroid of the cross-sectional area of a Z-section as shown in Fig. 4.19.

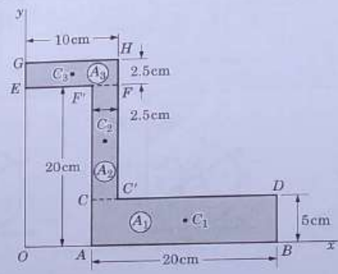


Fig. 4.19

Solution: Since the figure has no axis of symmetry so any convenient axes can be chosen as the axes of reference. Choosing one of the axes along the bottom edge AB and the other along the vertical edge GE,

Figure	*Area* (cm^2)	*x-coordinate of the centroid* (cm)	*y-coordinate of the centroid* (cm)
Rectangle $ABCD$	$A_1 = 5 \times 20 = 100$	$x_1 = 10 - 2.5 + 20/2$, $x_1 = 17.5$	$y_1 = 5/2 = 2.5$
Rectangle $CC'FF'$	$A_2 = 2.5 \times 15 = 37.5$	$x_2 = 10 - 2.5/2$ $x_2 = 8.75$	$y_2 = 5 + 15/2$ $y_2 = 12.5$
Rectangle $EFGH$	$A_3 = 10 \times 2.5$ $A_3 = 25$	$x_3 = 10/2 = 5.0$	$y_3 = 20 + 2.5/2 = 21.25$

$$x_c = \frac{A_1 x_1 + A_2 x_2 + A_3 x_3}{A_1 + A_2 + A_3}$$

$$x_c = \frac{(100 \times 7.5) + (37.5 \times 8.75) + (25 \times 5)}{100 + 37.5 + 25} = \frac{2203.125}{162.5}$$

$x_c = 13.557$ cm. *Ans.*

$$y_c = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3}$$

$$y_c = \frac{(100 \times 2.5) + (37.5 \times 12.5) + (25 \times 21.25)}{(100 + 37.5 + 25)}$$

$$y_c = \frac{1250}{162.5}, \quad y_c = 7.69 \text{ cm. } Ans.$$

Example 4.10 A body is made up of a hemisphere and a cone each of radius r as shown [in] Fig. 4.20 (a). The hemispherical surface of the body rests on a horizontal plane.

What should be the greatest height of the cone so that the combined body of the hemisphe[re] and the cone may stand upright?

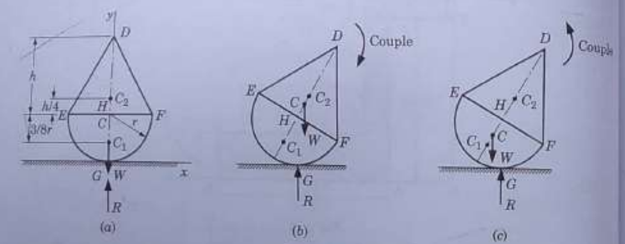


Fig. 4.20

Solution: Let the positions of the centroids of the hemisphere and the cone be C_1 and C_2 respectively. The point H be the centre of the common base EF. The combined weight of th[e] cone and hemisphere be W.

Tilt the body slightly to the right as shown in Fig. 4.20 (b).

The force acting on the body are :

1. Combined weight W of the body acting vertically downwards through the point C, th[e] combined C.G. of the cone and the hemisphere.
2. The reaction R of the plane on the body acting vertically upward at the point of conta[ct] G.

If the point C lies above H, then the body will have a tendency to tilt further due to the coupl[e] acting on the body formed by the forces R and W as shown in Fig. 4.20 (b).

On the other hand, if point C lies below H, then the body will have a tendency to return back to the vertical position due to the couple acting on the body formed by the forces R and W as shown in Fig. 4.20 (c).

In the limiting case, the height of the cone should be such that the position of the combined C.G. of the cone and the hemisphere (C) should coincide with the midpoint H of the base EF.

Let the radius of hemisphere = r

The height of the cone = h

Choosing the axes of reference as shown, let us determine the position of the C.G. of the body.

Note here that the y-axis is the axis of symmetry therefore, $x_c = 0$

Figure	*Volume*	*x-coordinate of the centroid*	*y-coordinate of the centroid*
Hemisphere	$V_1 = 2/3\pi r^3$	0	$y_1 = r - 3r/8$ $y_1 = 5r/8$
Cone	$V_2 = \pi/3r^2h$	0	$y_2 = r + h/4$

$$y_c = \frac{V_1y_1 + V_2y_2}{V_1 + V_2}$$

$$y_c = \frac{\left(\frac{2\pi}{3}r^3\right)\left(\frac{5r}{8}\right) + \left(\frac{\pi}{3}r^2h\right)\left(r + \frac{h}{4}\right)}{\frac{2\pi}{3}r^3 + \frac{\pi}{3}r^2h}$$

But $\quad y_c = r$ (As discussed earlier)

$$r\left(\frac{1}{3}h + \frac{2}{3}r\right) = \frac{5}{12}r^2 + \frac{h}{3}r + \frac{h^2}{12}$$

$$h^2 = 3r^2$$

$$h = \sqrt{3}r$$

$$h = 1.732r \quad \textit{Ans.}$$

4.7 THEOREMS OF PAPPUS AND GULDINUS

There are two theorems for determining the surface area and volume generated by revolving respectively a plane curve or a plane area about a non-intersecting axis lying in its plane.

Theorem I: The area of the surface generated by a revolving a plane curve about a non-intersecting axis in the plane of the curve is equal to the product of (i) the length of the curve and (ii) the distance travelled by the centroid G of the curve during the revolution.

$$A = L\,(\bar{x}\theta)$$

where, A = surface area generated

L = length of the curve

and $\bar{x}\theta$ = the distance travelled by the centroid G of the curve during rotation through angle θ radian about the axis of rotation (y-axis) for one complete rotation $\theta = 2\pi$

To illustrate the above theorem consider a straight line AB of length H at a distance r from y-axis be rotated about the same axis (Fig. 4.21).

It is obvious that a cylinder is generated for one complete rotation.

$$\bar{x} = r, \quad L = H, \quad \theta = 2\pi$$

Area of the surface of cylinder $A = H\,(r\,2\pi) = 2\pi rH$

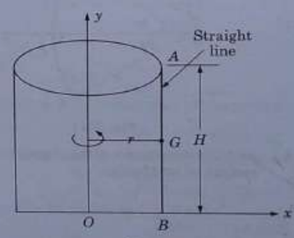


Fig. 4.21

Theorem II: The volume of the solid generated by revolving a plane area about a non-intersecting axis in its plane is equal to (*i*) the product of the area and (*ii*) the length of the path travelled by the centroid G of the area during revolution

$$V = A\,(\bar{x}\theta)$$

where, V = volume generated

A = Generating area

$\bar{x}\theta$ = the distance travelled by the centroid G of the area during rotation through angle θ radian about the axis of rotation (y-axis). For one complete rotation $\theta = 2\pi$

To illustrate the above, consider a rectangular shaded area $ABCO$ (Fig. 4.22) rotated about y-axis.

Area of $ABCD$, $A = r \times H$

The centroid G is at a distance $r/2$ from y-axis

$$\bar{x} = \frac{r}{2}$$

$$\theta = 2\pi$$

$$\text{Volume of cylinder generated} = rH\left(\frac{r}{2}2\pi\right) = \pi r^2 H.$$

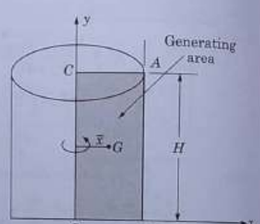


Fig. 4.22

PROBLEMS

4.1. Find the coordinates of the centroid of the length of a circular arc of radius r and central angle 2α (Fig. P.4.1).

$$\left[\frac{r\sin\alpha}{\alpha}, 0\right]$$

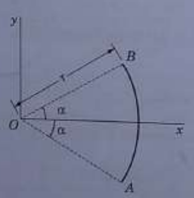


Fig. P.4.1

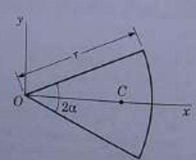


Fig. P.4.2

4.2. Find the coordinates of the centroid C of a circular sector of central angle 2α and radius r, by the method of integration.

$$\left[\frac{2r\sin\alpha}{3\alpha}, 0\right]$$

4.3. Determine by direct integration the coordinates of the centroid of the shaded area formed by the intersection of a straight line and the curve $y = kx^2$.

[1/2α, 2/5 b]

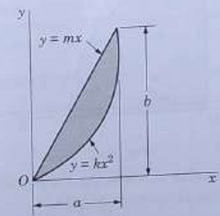


Fig. P.4.3

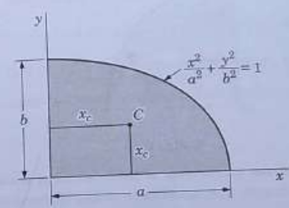


Fig. P.4.4

4.4. Determine by direct integration the coordinates of the centroid of the shaded area formed by the quarter of an ellipse as shown in Fig. P.4.4.

$$\left[x_c = \frac{4a}{3\pi}, y_c = \frac{4b}{3\pi}\right]$$

4.5. Find the coordinates of the centroid of the area left after removing a square area from plate as shown in Fig. P.4.5.

[5/12a, 5/12a]

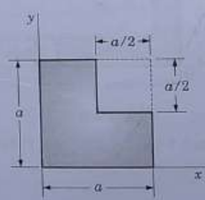


Fig. P.4.5

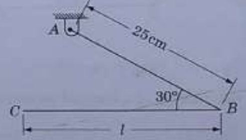


Fig. P.4.6

4.6. A homogenous wire ABC is bent and attached to the hinge at A. Find the length l of the portion BC, for which the portion BC of wire is horizontal.

[53.43 cm]

4.7. Find the coordinates of the centroid of the area obtained after removing a semicircle of radius 10 cm from a quadrant of a circle of radius 20 cm.

[6.98 cm, 12.72 cm]

4.8. Determine the coordinates of the centroid of the shaded area as shown in Fig. P.4.8.

[26.3 cm, 23.6 cm]

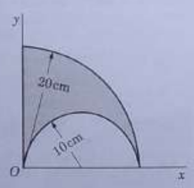


Fig. P.4.7

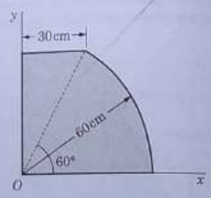


Fig. P.4.8

4.9. Find the coordinates of the centroid of semicircular area as shown in Fig. P.4.9. [0, 11.2 cm]

4.10. Determine the coordinates of the centroid of a L-section as shown in Fig. P.4.10. [2.61 cm, 3.15 cm]

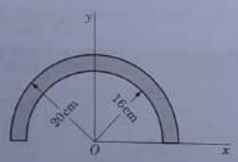


Fig. P.4.9

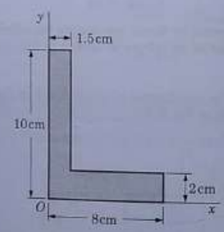


Fig. P.4.10

4.11. Find the centroid of a channel section as shown in Fig. P.4.11. [1.8 cm, 5.0 cm]

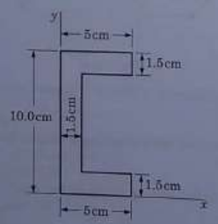


Fig. P.4.11

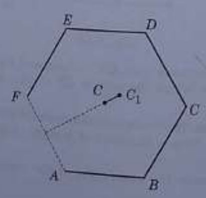


Fig. P.4.12

4.12. A rod of length 50 cm is bent so as to form the five sides of a regular hexagon $ABCDEF$, as shown in Fig. P.4.12. Locate the centroid C_1 of the figure and determine the distance AC_1. [AC_1 = 11.53 cm]

[**Hint.** Consider the centroid of the complete hexagon and the centroid of the side FA to determine the centroid of the five sides of the hexagon].

4.13. A plane lamina $ABCD$ is hung freely form point D. Find the angle made by DB with the vertical. [θ = 29.62°]

[**Hint.** Find the location of the centroid 'G' line DG will be a vertical line, or $\Sigma M_D = 0$]

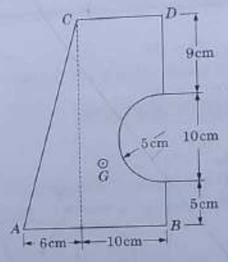


Fig. P.4.13

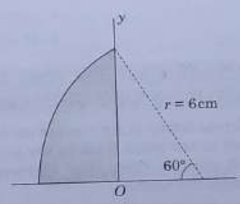


Fig. P.4.14

4.14. Determine the centroid of the **shaded area** which is bounded by straight lines and a circular arc as shown in Fig. P.4.14.

$\begin{bmatrix} x = 1.23 \text{ cm} \\ y = 2.04 \text{ cm} \end{bmatrix}$

5

CHAPTER

General Case of Forces in a Plane

5.1 GENERAL CASE OF FORCES ACTING IN A PLANE: EQUATIONS OF EQUILIBRIUM

If several forces acting in a plane are such that they do not intersect in one point (that is, are not concurrent) and are not parallel then, they represent a general system of coplanar forces.

Consider a body acted by several coplanar forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, ...

Let us choose some convenient coordinate axes x-y.

Let $F_{x1}, F_{x2}, F_{x3}, \ldots$ represent the components of these forces along the x-axis and

$F_{y1}, F_{y2}, F_{y3}, \ldots$ represent their components along the y-axis.

Now let us study the *three possibilities* that exits :

Case I. Let the system of forces reduce to a single resultant force **R** whose components (along the x and y axes) are F_x and F_y respectively.

The sum of the components of all the forces along the x-axis = the component along the x-axis of their resultant.

Therefore, $F_x = F_{x1} + F_{x2} + F_{x3} + \ldots$

$$F_x = \Sigma(F_x)_i \qquad \ldots(5.1)$$

For the same reason,

$$F_y = F_{y1} + F_{y2} + F_{y3} + \ldots$$

$$F_y = \Sigma(F_y)_i \qquad \ldots(5.2)$$

The magnitude of the resultant

$$R = \sqrt{F_x^2 + F_y^2} \qquad \ldots(5.3)$$

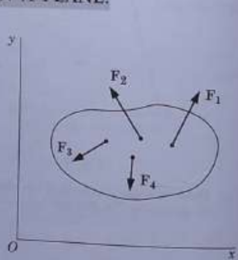


Fig. 5.1

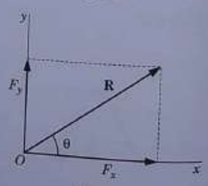


Fig. 5.2

and its direction

$$\theta = \tan^{-1}\frac{F_y}{F_x} \qquad \ldots(5.4)$$

The line of action of the resultant can be determined using the Varignon's theorem.

Choose the origin O as the moment centre.

Let, $(M_0)_1, (M_0)_2, (M_0)_3 \ldots$ be the moments of the given forces with respect to the origin. M_0 be the sum of the moments of all the forces about the same point.

The moment of a force about a point = The sum of the moments of its components about the same point

Therefore, $M_0 = (M_0)_1 + (M_0)_2 + (M_0)_3 + \ldots$

If the resultant lies at a distance d from the origin then,

$$M_0 = R \times d$$

$$d = \frac{M_0}{R} = \frac{\Sigma(M_0)_i}{\sqrt{F_x^2 + F_y^2}} \qquad \ldots(5.5)$$

Case II. The system of force reduce to a single couple.

As a couple is a system of two equal unlike parallel forces whose resultant is zero, therefore,

$$\Sigma(F_x)_i = 0 \qquad \ldots(5.6)$$

$$\Sigma(F_y)_i = 0 \qquad \ldots(5.7)$$

The moment M_0 of the resultant is given by

$$M_0 = \Sigma(M_0)_i \qquad \ldots(5.8)$$

Case III. The system is in equilibrium.

Then

$$\left.\begin{aligned} \Sigma(F_x)_i &= 0 \quad \ldots(5.9) \\ \Sigma(F_y)_i &= 0 \quad \ldots(5.10) \end{aligned}\right] \text{Resultant force is zero}$$

$$\Sigma(M_0)_i = 0 \quad \ldots(5.11) \quad \text{Resultant couple is zero}$$

Above three equations express the conditions for the equilibrium of a body when acted upon by a general system of forces.

The conditions of equilibrium can alternatively be expressed by *Three Moment Equations as,*

$$\Sigma(M_A)_i = 0 \qquad \ldots(5.12)$$

$$\Sigma(M_B)_i = 0 \qquad \ldots(5.13)$$

$$\Sigma(M_C)_i = 0 \qquad \ldots(5.14)$$

where, $\Sigma(M_A)_i$, $\Sigma(M_B)_i$, $\Sigma(M_C)_i$ represent the algebraic sum of the moments of the forces acting on the body about the points A, B and C respectively. *But the points A, B and C should not lie on a straight line.*

Comments

(*a*) The three equations of equilibrium can determine only three unknowns such as,

(*i*) the magnitude of three forces whose directions are known.

(*ii*) the magnitude and direction of one force and the magnitude of the second force.

(b) In the case of a system of concurrent forces and in the case of a system of parallel forces, *only two of the above equations of equilibrium become available.* Therefore, in these cases only two unknowns can be determined.

Example 5.1 The force $3F$, $7F$ and $5F$ act simultaneously along the three sides AB, BC and CA on an equilateral triangle ABC. Find the magnitude, direction and position of the resultant force.

Solution: Coordinate axes x-y choosen are as shown in Fig. 5.3.

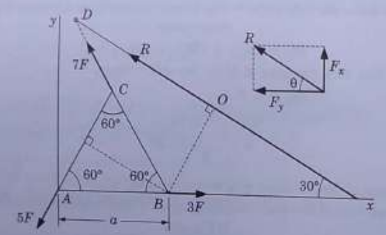


Fig. 5.3

Let the resultant R have components F_x and F_y.

Resolving forces along the x-axis

$$F_x = 3F - 7F\cos 60° - 5F\cos 60° = 3F - 7F \times 0.5 - 5F \times 0.5$$
$$F_x = -3F$$

Resolving forces along the y-axis

$$F_y = 0 + 7F\cos 60° - 5F\sin 60° = F\sqrt{3}/2(7-5)$$
$$F_y = 1.732F$$
$$R = \sqrt{F_x^2 + F_y^2} = \sqrt{(-3F)^2 + (1.732F)^2}$$
$$R = 3.464F \quad \textit{Ans.}$$
$$\tan\theta = \frac{F_y}{F_x} = \frac{1.732F}{-3F} = -0.577$$
$$\theta = -30° \quad \textit{Ans.}$$

To find the position of the resultant R, equate the moment of the resultant R and the sum of the moments of force $3F$, $7F$ and $5F$ about the point B.

$$R \times BO = 5F \times (a \sin 60°)$$
$$BO = \frac{5F(a \times 0.866)}{3.464F} \text{ or } BO = 1.25a$$

Let R intersect BC produced at D.

In triangle BOD, $\quad \dfrac{BO}{BD} = \sin 30°$

$$BD = \frac{1.259}{0.5} \text{ or } BD = 2.5a$$
$$CD = BD - BC \text{ or } CD = 1.5a$$

The resultant intersects BC produced at D such that $CD = 1.5a$ *Ans.*

Example 5.2 A vertical mast PQ of a crane is pivoted at P and is supported by a guide Q. Find the reactions at P and Q due to the loads acting as shown in Fig. 5.4.

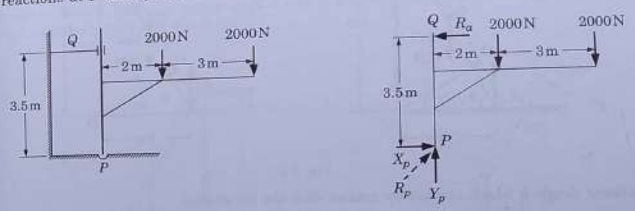


Fig. 5.4

Solution: The free-body diagram of the mast is as shown.

The reaction R_Q at the guide acts horizontally and the reaction R_P at the pivot has components X_P and Y_P.

Writing the equations of equilibrium

$\Sigma F_x = 0:$ $\quad X_P - R_Q = 0$

$$X_P = R_Q$$

Taking moments about P

$\Sigma M_P = 0:$ $\quad R_Q(3.5) - 2000(2) - 2000(5) = 0$

$$R_Q = \frac{14000}{3.5},\ R_Q = 4000 \text{ N} \quad \textit{Ans.}$$
$$X_P = R_Q = 4000 \text{ N}$$

$\Sigma F_y = 0:$ $\quad Y_P - 2000 - 2000 = 0$

$$Y_P = 4000 \text{ N.}$$

Combining the components X_P and Y_P,

$$R_P = \sqrt{X_P^2 + Y_P^2} \text{ or } R_P = \sqrt{(4000)^2 + (4000)^2}$$
$$R_P = 5656.8 \text{ N} \quad \textit{Ans.}$$

Example 5.3 A man weighing 75 N stands on the middle rung of a 25 N ladder resting on a smooth floor and against a wall. The ladder is prevented from slipping by a string *OD*. Find the tension in the string and reactions at *A* and *B* as shown in Fig. 5.5.

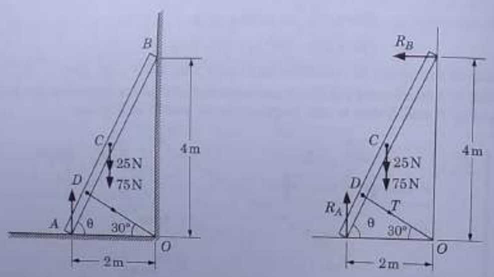


Fig. 5.5

Solution: Angle θ which the ladder makes with the horizontal

$$\tan\theta = 4/2 = 2,\ \theta = 63.43°$$

Consider the free-body diagram of the ladder. The various forces acting on the ladder are:

(i) Weight of the ladder and weight of the man = 75 + 25 = 100 N
(ii) Tension in the string T
(iii) Reaction R_B
(iv) Reaction R_A.

Equation of equilibrium are,

$\Sigma F_x = 0$: $\quad T\cos 30° - R_B = 0 \qquad$ ...(i)

$\Sigma F_y = 0$: $\quad R_A - (75 + 25) - T\sin 30° = 0$

$$R_A - T\sin 30° = 100 \qquad \text{...(ii)}$$

Taking moments about O,

$\Sigma M_0 = 0 : R_B \times 4 + 75 \times 1 + 25 \times 1 - R_A \times 2 = 0$

$$2R_A - 4R_B = 100 \qquad \text{...(iii)}$$

From (i) $\qquad T = \dfrac{R_B}{\cos 30°}$

Substituting in (ii)

$$R_A - \frac{R_B}{\cos 30°}\sin 30° = 100$$

$$R_A - 0.577\, R_B = 100 \qquad \text{...(iv)}$$

Solving equation (iv) and (iii) simultaneously gives,

$R_B = 35.13$ N *Ans.*

$R_A = 120.26$ N *Ans.*

From (i)

$$T = \frac{R_B}{\cos 30°} = \frac{35.13}{0.866} = 40.56$$

$T = 40.56$ N *Ans.*

Example 5.4 A man weighing 100 N stands on the middle rung of a ladder whose weight can be neglected. The end *A* rests on the ground against a stop and the end *B* rests on the corner of a wall as shown. Find the reactions at *A* and *B*. Neglect friction between the ladder and the ground and the ladder and the wall.

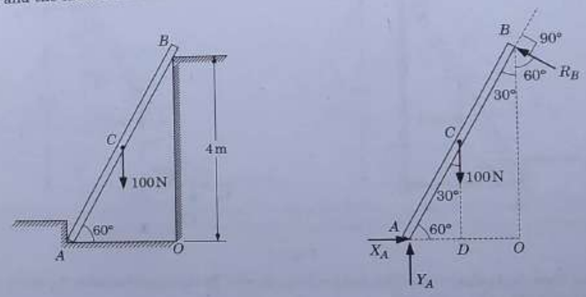


Fig. 5.6

Solution: Considering the free-body of the ladder. The various forces acting on the ladder are,

(i) Weight of the man acting at $C = 100$ N
(ii) Reaction of the ground at A, having components X_A and Y_A
(iii) Reaction of the corner of the wall on the ladder acting normal to the ladder (just like a knife edge support).

Writing the equations of equilibrium,

$\Sigma F_x = 0$: $\quad X_A - R_B \sin 60° = 0 \qquad$ ...(i)

$\Sigma F_y = 0$: $\quad Y_A - W - R_B \cos 60° = 0 \qquad$ ...(ii)

Taking moments about A, $\qquad$ where, $AB = OB \sec 30°$

$\Sigma M_A = 0$: $\quad R_B(AB) - 100(AD) = 0 \qquad AB = 4 \times 1.154 = 4.616$ m

$R_B = \dfrac{100 \times 1.154}{4.616} \qquad AD = CD \times \tan 30°$

$AD = 2 \times 0.577$

$R_B = 25$ N *Ans.* $\qquad AD = 1.154$ m

From (i) $X_A = R_B \sin 60°$
$= 25 \times 0.866$
$X_A = 21.65$ N *Ans.*

From (ii) $Y_A = W + R_B \cos 60°$
$= 100 + 25 \times 0.5$
$Y_A = 112.5$ N *Ans.*

Example 5.5 If the end B of the ladder of the previous example rests against a wall (instead of the corner of a wall) find the reactions at A and B (Fig. 5.7).

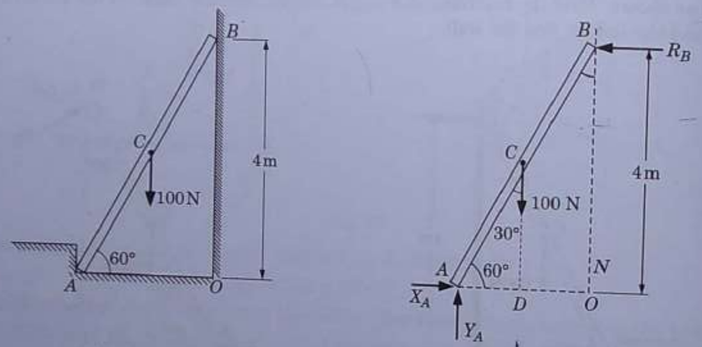


Fig. 5.7

Solution: Free-body diagram of the ladder is as shown. In this case reaction R_B acting on the ladder is normal to the wall at B.

Considering the equilibrium of the ladder,

$\Sigma F_x = 0$: $X_A - R_B = 0$ ...(i)

$\Sigma F_y = 0$: $Y_A - 100\text{ N} = 0$ ...(ii)

or $Y_A = 100$ N *Ans.*

Taking moments about A,

$\Sigma M_A = 0 : 100 \times AD - R_B \times OB = 0$

$AD = CD \tan 30°$
$AD = 1.154$ m

$$R_B = \frac{100 \times 1.154}{4}$$

$R_B = 28.85$ N *Ans.*

From (i) $X_A = R_B = 28.85$, $X_A = 28.85$ N *Ans.*

Example 5.6 A hollow right circular cylinder of diameter 9 cm and height $h = 12$ cm is open at both ends and rests on a smooth horizontal plane. A uniform bar of length $l = 24$ cm and weight 10 N is placed within the cylinder and rests as shown. Find the reactions at the points A and D.

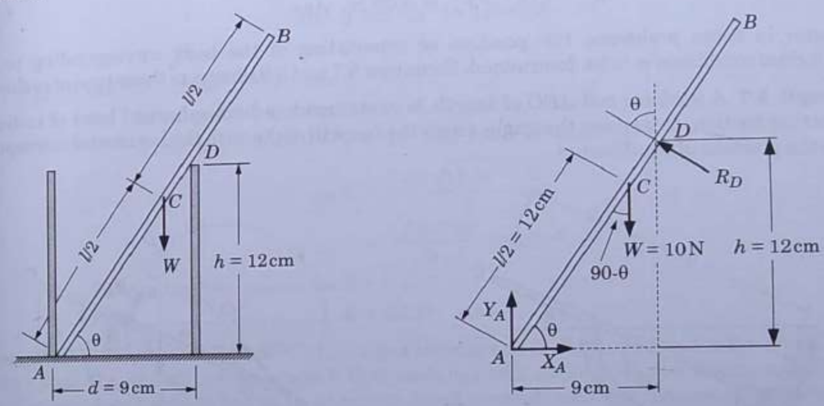


Fig. 5.8

Solution: Consider the free-body diagram of the bar AB. Forces acting are, weight of the bar $W = 10$ N, reaction R_D acting normal to the bar at D, and the reaction at A having components X_A, Y_A

$$l = 24\text{ cm}, \tan\theta = \frac{12}{9} = 1.333,$$

$$\theta = 53.13°,$$

$$AD = \sqrt{9^2 + 12^2} = 15\text{ cm} = 0.15\text{ m}$$

Consider the equilibrium of the bar.

$\Sigma F_x = 0$: $X_A - R_D \sin\theta = 0$
$X_A = R_D \sin 53.13°$
$X_A = 0.799\, R_D$ ...(i)

$\Sigma F_y = 0$: $Y_A + R_D \cos\theta - W = 0$
$Y_A = W - R_D \cos 53.13° = W - 0.6\, R_D$ ...(ii)

$\Sigma M_A = 0$: $R_D \times AD - W \cdot \frac{l}{2}\cos\theta = 0$

$$R_D = \frac{10 \times 0.12 \cos 53.13°}{0.15} = 4.8\text{ N} \quad Ans.$$

Substituting for R_D in (i) and (ii)

$$X_A = 0.799\,R_D = 0.799 \times 4.8 = 3.835 \text{ N}$$
$$Y_A = W - 0.6\,R_D = 10 - 0.6 \times 4.8 = 7.12 \text{ N}$$
$$R_A = \sqrt{X_A^2 + Y_A^2} = 8.09 \text{ N} \quad \textit{Ans.}$$

Note: In some problems, the position or orientation of the body corresponding to t equilibrium conditions is to be determined. Examples 5.7 and 5.8 illustrate these type of proble

Example 5.7 A uniform rod *ABC* of length $3r$ rests inside a hemispherical bowl of radiu Neglecting friction, determine the angle which the bar will make with the horizontal correspo ing to the position of equilibrium.

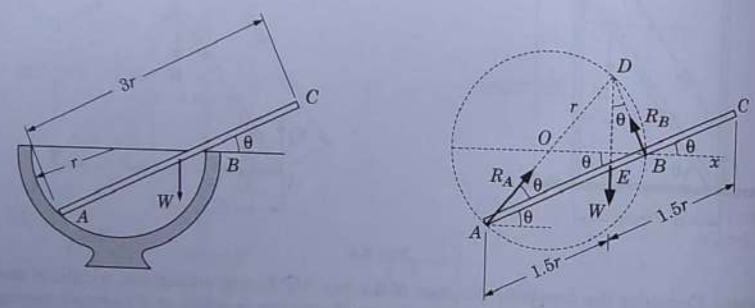


Fig. 5.9

Solution: Free-body diagram of the rod *ABC* is as shown in Fig. 5.9. The forces involved ar

Weight of the rod *W* acting through *E* vertically downward.

Reaction R_A acting along the normal to hemisphere at *A*, therefore, passing through the cent *O*.

Reaction R_B acting along the normal to the bar at *B*.

These three forces being in equilibrium, they must pass through a common point *D*.

The point *D* must lie on the circle as shown, because $\angle ABD = 90°$ (Angle subtended by diameter, at any point lying on the circumference of the circle is a right angle).

By the geometry of the figure,

$$\angle CBx = \angle OBA = \angle OAB = \angle EDB = \theta$$

From triangle *DEB*, $EB = BD \tan\theta$

From triangle *ADB*, $BD = AD \sin\theta = 2r \sin\theta$

Therefore, $EB = 2r \sin\theta \tan\theta$ ...(i)

Also, $AB = 2r \cos\theta$

and, $EB = AB - AE = 2r\cos\theta - 1.5r$ ...(ii)

Equating (i) and (ii)

$$2r \sin\theta \tan\theta = 2r\cos\theta - 1.5r$$
$$2r \sin\theta \frac{\sin\theta}{\cos\theta} = 2r\cos\theta - 1.5r$$
$$2\sin^2\theta = 2\cos^2\theta - 1.5\cos\theta$$
$$2\sin^2\theta - 2\cos^2\theta + 1.5\cos\theta = 0$$
$$2(1-\cos^2\theta) - 2\cos^2\theta + 1.5\cos\theta = 0$$
$$2 - 4\cos^2\theta + 1.5\cos\theta = 0$$
$$4\cos^2\theta - 1.5\cos\theta - 2 = 0. \quad \text{The roots of which are,}$$
$$\cos\theta = \frac{1.5 \pm \sqrt{(1.5)^2 + 32}}{8}$$
$$= \frac{1.5 \pm 5.85}{8} = 0.919$$

Taking the +ve value, $\cos\theta = 0.919$

$$\theta = 23.2° \quad \textit{Ans.}$$

Example 5.8 Two bars *AB* and *BC* are hinged together at *B* and supported in a vertical plane by two pegs *D* and *E*. Find the angle θ that each bar will make with the horizontal when in equilibrium. Assume each bar to be of length *l* and mass *M* and frictionless conditions.

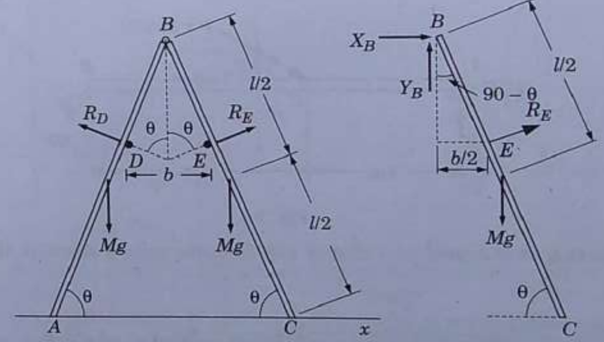


Fig. 5.10

Solution: First, consider the free-body diagram of bars *AB* and *BC*. The forces acting are : Weights of the bars (*Mg*), reactions R_D and R_E of the pegs on the bars acting normal to the bars as shown in Fig. 5.10.

Writing the equations of equilibrium

$\Sigma F_x = 0$: $\quad R_E \sin\theta - R_D \sin\theta = 0$

$$R_E = R_D$$

$\Sigma F_y = 0:$ $\quad R_E \cos\theta + R_D \cos\theta - 2\,Mg = 0$

$$R_D = R_E = \frac{Mg}{\cos\theta}$$

Now, consider the free-body diagram of the bar BC

Taking moments about B

$\Sigma M_B = 0:$ $$R_E\left(\frac{b/2}{\cos\theta}\right) - Mg\left(\frac{l}{2}\cos\theta\right) = 0$$

$$R_E\frac{b}{2}\frac{1}{\cos\theta} - Mg\frac{l}{2}\cos\theta = 0$$

Substituting for R_E, $\dfrac{Mg}{\cos\theta}\dfrac{b}{2}\dfrac{1}{\cos\theta} - Mg\dfrac{l}{2}\cos\theta = 0$

$$\cos^3\theta = \frac{b}{l},\ \cos\theta = \sqrt[3]{\frac{b}{l}} \quad \textit{Ans.}$$

PROBLEMS

5.1. A beam AB is hinged at the end A and roller supported at the end B. Determine the reactio[ns] X_A and Y_A of the hinged support and the reaction R_B at the end B and its direction.
[X_A = 4330 N, Y_A = 1500 N, R_B = 2000...]

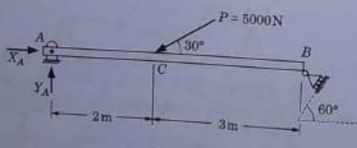


Fig. P.5.1

5.2. Find the reaction at D caused by a load of 1200 N in the system shown in the Fig. P. 5.2.

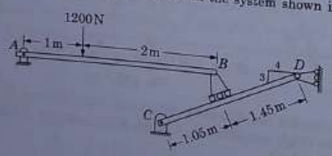


Fig. P.5.2

5.3. Find the maximum weight that can be lifted by crane weighing 100 kN without tipping over wheel B as shown in Fig. P.5.3. Calculate the corresponding reactions at the supports C and D.
[W = 200 kN; C_x = 34.64 kN, D_x = 0; C_y = 124 kN, D_y = 196 kN]

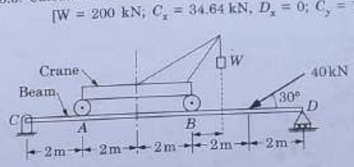


Fig. P.5.3

5.4. A plate $ABCD$ of 15 × 30 cm size is subjected to four forces P, 2.5P, 2P, 2.5P acting at the point B, E, C and F respectively. Find the magnitude, direction and position of the resultant R.
[R = 2.236F, θ = 63.4°, Resultant intersects BA at a distance of 13.25 cm from B and intersect BC produced at a distance of 26.5 cm from B]

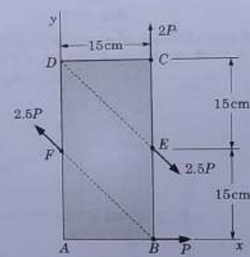


Fig. P.5.4

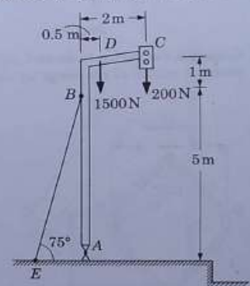


Fig. P.5.5

5.5. A traffic light signal pole is supported by a cable BE and is hinged at A. Find the tension in the cable if the weight 1500 N of pole acts through point D and weight 200 N of the signal light acts through point C.
[T = 888.6 N, R_A = 2568 N]

5.6. A uniform bar AB of weight W and length $8l$ is hinged at A and carries a smooth ring D of weight $2W$. The ring is held in position by the string DE such that $ED = EA = l$. Find the distance AD when the system is in equilibrium.
[$AD = l$]

[**Hint.** Consider the *f.b.d.* of the ring and then the *f.b.d.* of the bar and ring combined].

5.7. A uniform ladder AB of length $2l$ and weight W rests on ground and against a vertical wall. It is kept from slipping by a horizontal string of length 'a' tied to its end A and to the wall at C. Find the tension in the string AC.

If a man of weight $2W$ climbs three- fourth of the way up the ladder, find the tension in the string.

$$\left[\frac{Wa}{2\sqrt{4l^2 - a}};\ \frac{2Wa}{\sqrt{4l^2 - a^2}}\right]$$

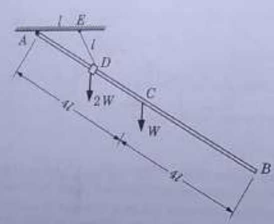


Fig. P.5.6

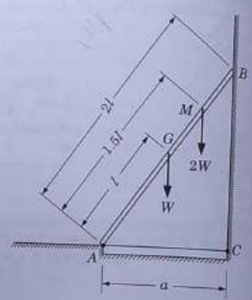


Fig. P.5.7

5.8. Calculate the pressure exerted against the sides of a wooden cube of weight $W = 100$ N by the points A and B of the tongs as shown. Assume $a = 10$ cm, $b = 20$ cm, $c = 20$ cm, $d = 40$ cm.

$[R_A = R_B = 95 \text{ N}]$

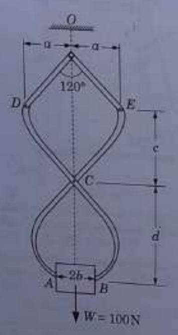


Fig. P.5.8

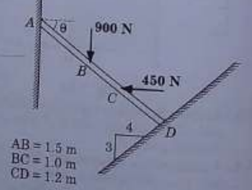


Fig. P.5.9

5.9. A bar of negligible weight and of length 3.7 m is acted upon by a vertical force of 900 N and a horizontal force of 450 N as shown in Fig. P.5.9. Determine the equilibrium position of the bar as defined by the angle θ that it makes with the horizontal.

$[\theta = 28.66°]$

5.10. A hollow cylinder of radius r is open at both ends and rests on a smooth horizontal plane. Two spheres having weights W_1 and W_2 and radii r_1 and r_2 respectively are placed inside the cylinder as shown. Find the minimum weight W of the cylinder in order that it will not tip over. Neglect friction.

$$\left[W = W_1 \frac{(2r - r_1 - r_2)}{r}\right]$$

[**Hint.** Find the reaction forces acting on the cylinder and then equate the moments about point A.]

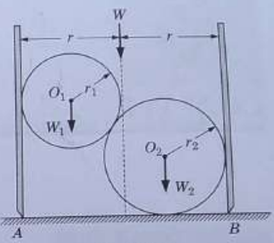


Fig. P. 5.10

5.11. A prismatic bar AB of weight W is resting against a smooth vertical wall at A and is supported on a small roller at the point D. If a vertical force F is applied at the end B, find the position of equilibrium as defined by the angle θ.

$$\left[\cos\theta = \sqrt[3]{\frac{2(F+W)b}{(2F+W)l}}\right]$$

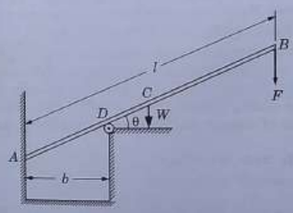


Fig. P.5.11

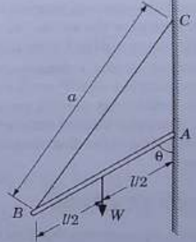


Fig. P.5.12

5.12. A uniform bar AB of length l and weight W is hinged to the wall at its end A and supported by means of a string of length a which is tied to the wall as shown. Find the angle θ that the bar will make with the wall corresponding to the position of equilibrium.

$$\left[\theta = \cos^{-1}\sqrt{\frac{a^2 - l^2}{3l^2}}\right]$$

5.13. A boom of negligible weight carries a load W (Figure P.5.13). Show that the compression in the boom is constant for all values of θ. Also find the limiting value of T as θ approaches 90°.

$[T = W(l - h)/h]$

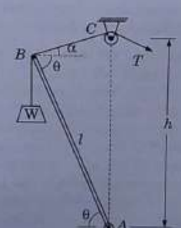


Fig. P. 5.13

6

CHAPTER

Frictio[n]

6.1 INTRODUCTION

So far we assumed the surfaces to be smooth. With this assumption there is no resistance t[o] sliding and the reaction exerted by a surface on a body supported by it, would act in a directio[n] which is normal to the surface at the point of contact. Such an assumption was aimed to simplif[y] the calculations without seriously effecting the final results in certain situations.

In some situations, resistance to sliding between two contiguous surfaces can be of importan[ce] in determining the equilibrium conditions and this resistance cannot be neglected. For exampl[e] in the case of a body resting on an inclined plane. Whenever the surfaces of two bodies are i[n] contact, there is a limited amount of resistance to sliding between them which is called frictio[n]. *The friction is a force distribution at the surfaces of contact and acts tangential to the surfa[ce] of contact.*

6.2 DRY FRICTION

The friction between dry surfaces in contact is called dry friction. It is also called *Coulomb frictio[n]*. *The major cause of such friction is believed to be the interlocking of microscopic protuberanc[es] (that is, minute projections on the surfaces) which oppose the relative motion.* Suc[h] protuberances are always present howsoever smooth the surfaces may be. Another type of frictio[n] is fluid friction. The friction between two surfaces in the presence of fluid is called *fluid frictio[n]*.

Limiting Friction. Consider a body of mass m having the corresponding weight W restin[g] on a horizontal surface. Let a continuously increasing force **P** be applied to this body as show[n] in Fig. 6.1 (*a*). This force **P** will be opposed or resisted by the frictional force **F**. The force **F** shal[l] go on increasing to balance the increasingly applied force **P** and the body shall remains at rest. Then, there comes a limit beyond which the frictional force **F** cannot increase and the body begins to move. The frictional force at this instant is called the limiting friction. *The limiting friction, therefore, is the maximum frictional force exerted at the time of impending motion, that is, when the motion is about to begin.* Once the body begins to move there is a decrease in the frictional effect from the maximum effect observed under the static conditions. The frictional force between two moving surfaces is called *kinetic or dynamic friction.*

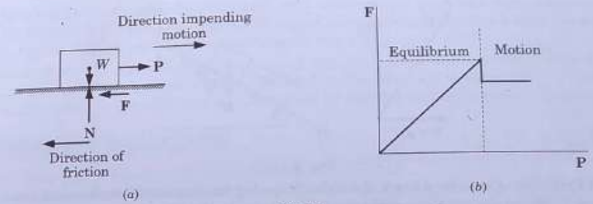


Fig. 6.1

6.3 LAWS OF DRY FRICTION

1. The total friction that can be developed is independent of the magnitude of the area in contact.
2. The total friction that can be developed is proportional to the normal force transmitted across the surface of contact.
3. For low velocities, total amount of frictional force that can be developed is practically independent of velocity. But, the force necessary to start the motion is greater than that necessary to maintain the motion.

2nd Law can be expressed mathematically as,

$$F \propto N$$

$$F = \mu N \qquad \text{...(6.1)}$$

where **F** is the force required to start sliding or is equal to the limiting friction.
N is the normal reaction between the surfaces.
μ is the coefficient of static friction.

The equation $F = \mu N$ *is valid only at the time of impending motion*. After the motion starts 'μ' is called the coefficient of kinetic friction.

Angle of Friction and Resultant Reaction. The normal reaction **N** and the limiting frictional force **F** ($= \mu N$) acting at the surface of contact can be combined into a single resultant **R** called the resultant reaction at the point of contact. The angle which this resultant **R** makes with the normal reaction **N** is called the angle of friction and is denoted by ϕ.

where, $\tan \phi = \dfrac{F}{N}$

$\tan \phi = \dfrac{\mu N}{N} = \mu$ (F is the limiting friction)

Or, $\tan \phi = \mu \qquad \text{...(6.2)}$

For physical understanding, consider a body resting on a plane OA [Fig. 6.2 (*a*)]. This plane OA is slowly raised from the horizontal position to incline it with the horizontal. The angle of inclination of this plane with the horizontal, when the body begins to slide down, is called the angle of friction.

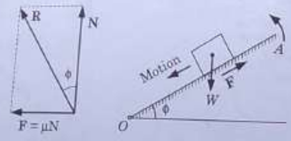


Fig. 6.2 (a)

Cone of Friction. Consider a block of weight W resting on a horizontal surface and acted upon by a force P. When we consider coplanar forces, in order for the *motion not to occur* in any direction (⇄) as shown in Figure 6.2 (*b*), resultant R must lie within the angle ABC, where ϕ is the angle of friction.

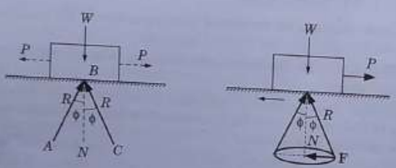


Fig. 6.2 (b)

Next, consider the situation when the direction of force P is gradually changed through 360°. For the motion not to occur, resultant reaction R must be contained within the cone generated by revolving line AB about the normal BN. The inverted cone so formed with semi-central angle equal to the angle of friction ϕ is called the cone of friction. Further, for the motion to occur the resultant R will lie on the surface of the cone.

Notes:

1. The sense of friction is always opposite to the sense of impending motion.
2. At the point of contact, the normal reaction **N** and frictional force μ**N** can also be represented by a single resultant reaction **R**. It lies between the forces **N** and μ**N** and it inclined at an angle, ϕ to the direction of normal reaction **N**. For analytical approach it is often much simpler to deal with the normal reaction and frictional force as two separate forces rather than a single resultant reaction **R**.

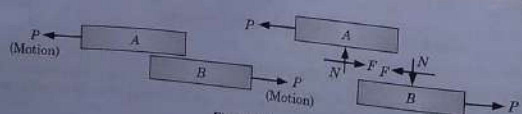


Fig. 6.2 (c)

3. When two bodies are involved and both show a tendency to move with respect to each other the normal reaction as well as the frictional force acting on the two bodies are equal in magnitude but opposite in sense as shown in Fig. 6.2 (c).

Moment equation of equilibrium can be used only when the line of action of forces involved are known. When frictional force is present its line of action is known *only* inthe case of a line contact or point contact, e.g. in the case of a ladder resting on ground against a wall:

6.4 ROLLING RESISTANCE

When a body (say a wheel) rolls on the ground, the point of the wheel in contact with the ground at any instant, has no relative motion with respect to the ground. Thus a large amount of frictional force is eliminated. But, in practice the resistance to the rolling motion exists. It is mainly due to the deformation of the surface upon which the wheel rolls. Therefore, the contact between the wheel and the ground is not limited to a single point but extends over an area.

The distance 'a' as shown in Fig. 6.3 is referred as, *"the forward length of deformation" and is defined as the coefficient of rolling resistance.*

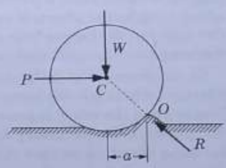


Fig. 6.3

6.5 FORCE OF FRICTION ON A WHEEL

Consider a block of weight W resting on a horizontal surface Fig. 6.4 (*a*). If it were to move, it can do so by sliding over the surface and in the normal circumstances it is incapable of rolling. The friction force shall act opposite to the direction of motion as shown.

Consider next a wheel of weight W and radius r resting on a horizontal surface. In fact, we can consider any other body which is capable of sliding as well as rolling like; a cylinder, a sphere or a disc. The sliding or slipping of the wheels of an automobile, when locked by braking, is a well known experience. *This wheel can be made to move forward by applying a force or by applying a torque.* The situation in the two cases is very different as discussed below.

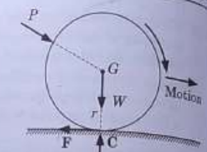
(*a*) A body that can only slide.

(*b*) A body that can slide and roll.

Fig. 6.4

When a Force is applied. When a force **P** is applied to the wheel as shown in Fig. 6.4 (*c*), the basic tendency of the applied force is to move the wheel forward. The frictional force **F** coming into play, opposes this forward translatory motion. It acts from right to left at the point of contact C as shown.

This frictional force **F** shall exert a turning moment equal to $(F \times r)$ about the centre G of the body thus making the wheel roll clockwise. Whether the wheel will slide, roll, or slide as well roll, shall depend upon the amount of friction present. If the body rolls without sliding then the frictional force **F** is less than μ**R**. The equation $F = \mu R$, is valid only when the body is at the point of sliding or slipping.

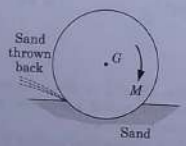


(c)
Friction force **F** is acting opposite to the direction of motion.

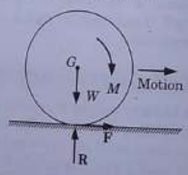
Fig. 6.4

When acted upon by a Torque. This situation can be represented if the wheel is powered as in the case of an automobile wheel. Imagine the wheels of a car sunk in the sand Fig. 6.4 (*d*). The wheel under torque from the engine shall keep spinning, throwing back the sand, without moving forward.

So, we observe that in the absence of friction the wheel keeps rotating without moving forward. When sufficient friction is provided, the wheel moves forward. That is when the wheel is powered, the basic tendency is to rotate the wheel and it is the friction that provides the necessary forward force to make it move. Thus friction in this case acts in the same direction as that of the motion of the wheel as shown in Fig. 6.4 (*e*).



(*d*)
In the absence of friction, the wheel just spins.



(*e*)
The frictional force **F** makes the wheel move forward and acts in the same direction as the motion of the wheel.

Fig. 6.4

Example 6.1 A wooden block rests on a horizontal plane. Determine the force required to (*a*) pull it (*b*) push it. Assume the mass μ of the block to be 5 kg and the coefficient of friction μ = 0.4.

Solution: (*a*) Pull : Let P_1 be the force required to just pull it. Free-body diagram is as shown in Fig. 6.5 (*a*).

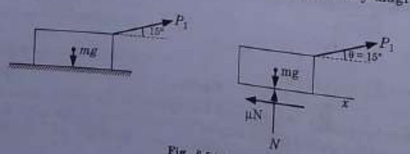


Fig. 6.5 (*a*)

Writing the equations of equilibrium

$$\begin{cases}\theta = 15° \\ m = 5\text{ kg}\end{cases}$$

$\Sigma F_x = 0:$
$$P_1 \cos\theta - \mu N = 0$$
$$0.966 P_1 - 0.4\text{ N} = 0 \quad ...(i)$$

$\Sigma F_y = 0:$
$$P_1 \sin\theta - mg + N = 0$$
$$0.259 P_1 - 5g + N = 0 \quad ...(ii)$$

Eliminating N between (*i*) and (*ii*) and solving for P_1

$$0.966 P_1 - 0.4(5g - 0.259 P_1) = 0$$
$$P_1 = \frac{0.4 \times 5 \times 9.81}{(0.966 + 0.259 \times 0.4)}$$
$$P_1 = 18.44\text{ N} \quad \textit{Ans.}$$

(*b*) Push : Let P_2 be the force required.

Writing the equations of equilibrium

$\Sigma F_x = 0:$ $\quad \mu N - P_2 \cos 15° = 0$
$$0.4\,N - 0.966 P_2 = 0 \quad ...(i)$$

$\Sigma F_y = 0:$ $\quad N - mg - P_2 \sin 15° = 0$
$$N - 5 \times g - 0.259 P_2 = 0 \quad ...(ii)$$

Eliminating N and solving for P_2

$$P_2 = \frac{0.4 \times 5 \times 9.81}{(0.966 - 0.259 \times 0.4)}$$
$$P_2 = 23.0\text{ N} \quad \textit{Ans.}$$

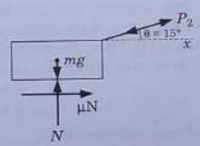


Fig. 6.5 (*b*)

Example 6.2 A rectangular block of mass M rests on a floor. The coefficient of friction between the block and the floor is μ. What is the highest position for a horizontal force P that would permit it to just move the block without tipping.

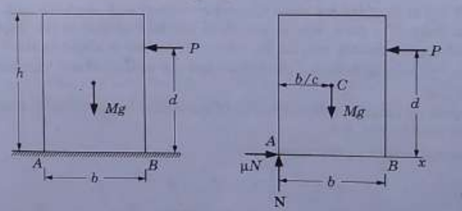


Fig. 6.6

Solution: As tipping occurs, the block will just be lifted off the floor and shall maintain a contact with the floor only at the edge A. So, the normal reaction N and the frictional force μN shall act at A.

Free-body diagram is as shown in Fig. 6.6.

Writing the equation of equilibrium,

$\Sigma F_x = 0$: $\qquad \mu N - P = 0$ ...(i)

$\Sigma F_y = 0$: $\qquad N - mg = 0$ ...(ii)

Taking moment about A,

$$\Sigma M_A = 0: \qquad P(d) - Mg\left(\frac{b}{2}\right) = 0 \quad \text{...(iii)}$$

From (i) and (ii) $N = Mg$ and $P = \mu Mg$

Substituting in (ii)

$$\mu Mgd = Mg\frac{b}{2}$$

Or

$$d = \frac{b}{2\mu} \quad Ans.$$

The problem can also be solved by combining the normal reaction N and the frictional force μN into a single resultant reaction R. Then, the forces acting on the block reduce to a system of three concurrent forces in equilibrium as shown in Fig. 6.7.

$$\phi = \text{angle of friction}$$

$$\tan\phi = \mu$$

From triangle AOD,

$$\tan\phi = \frac{AD}{OD} = \frac{b/2}{d}$$

$$\mu = \frac{b}{2d}$$

$$d = \frac{b}{2\mu} \quad Ans.$$

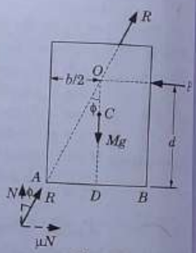


Fig. 6.7

Example 6.3 A 7.0 m long ladder rests against a vertical wall, with which it makes an angle of 45°, and on a floor. If a man, whose weight is one half of that of the ladder, climbs it, at what distance along the ladder will he be, when the ladder is about to slip?

The coefficients of friction between the ladder and the wall is 1/3 and that between the ladder and the floor is 1/2.

Solution: Suppose the man climbs a length l of the ladder before slipping impends. Free-body diagram is as shown in Fig. 6.8.

Writing the equations of equilibrium,

$$\Sigma F_x = 0: \qquad \frac{1}{2}N_A - N_B = 0 \quad \text{...(i)}$$

$$\Sigma F_y = 0: \qquad N_A + \frac{N_B}{3} - W - \frac{W}{2} = 0 \quad \text{...(ii)}$$

Taking moments about A,

$\Sigma M_A = 0$:

$$N_B(7\sin 45°) + \frac{1}{3}N_B(7\cos 45°) - W\left(\frac{7}{2}\cos 45°\right) - \frac{W}{2}(l\cos 45°) = 0 \quad \text{...(iii)}$$

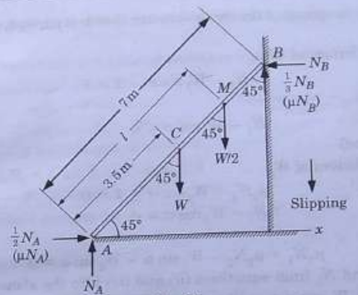


Fig. 6.8

From (i) $N_A = 2N_B$, substituting in (ii)

$$2N_B + \frac{N_B}{3} - \frac{3W}{2} = 0$$

$$N_B = \frac{9}{14}W, \; N_A = \frac{9}{7}W$$

Substituting for N_A and N_B in (iii)

$$\frac{9}{14}W(7\sin 45°) + \frac{1}{3}\left(\frac{9}{14}W\right)(7\cos 45°) - W\left(\frac{7}{2}\cos 45°\right) - \frac{W}{2}(l\cos 45°) = 0$$

$$\frac{9}{14}\times 7 + \frac{1}{3}\cdot\frac{9}{14}\times 7 - \frac{7}{2} = \frac{l}{2}$$

$$l = 5 \text{ m} \quad Ans.$$

Example 6.4 Two blocks of weight $W_1 = 50$ N and $W_2 = 50$ N rest on a rough inclined plane and connected by a string as shown in Fig. 6.9. The coefficients of friction between the inclined plane and W_1 and W_2 are $\mu_1 = 0.3$ and $\mu_2 = 0.2$ respectively. Find the inclination of the plane for which slipping will impend.

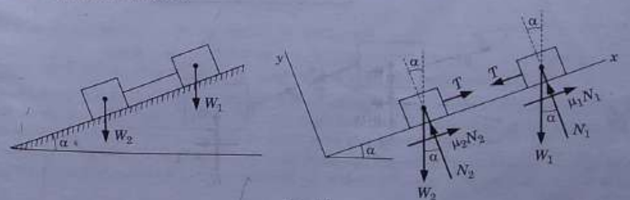


Fig. 6.9

Solution: Free-body diagrams of the two blocks are shown separately, where T is the tension in the string.

Consider the equilibrium of W_1,

$\Sigma F_x = 0:$ $\qquad \mu_1 N_1 - W_1 \sin\alpha - T = 0$ ...(i)

(Along the plane)

$\Sigma F_y = 0:$ $\qquad N_1 - W_1 \cos\alpha = 0$ ...(ii)

(Normal to the plane)

Consider the equilibrium of W_2

$\Sigma F_x = 0:$ $\qquad \mu_2 N_2 - W_2 \sin\alpha + T = 0$ ...(iii)

$\Sigma F_y = 0:$ $\qquad N_2 - W_2 \cos\alpha = 0$ ...(iv)

Add (i) and (iii)

$$\mu_1 N_1 + \mu_2 N_2 - W_1 \sin\alpha - W_2 \sin\alpha = 0 \quad ...(v)$$

Substitute for N_1 and N_2 from equations (ii) and (iv) into the above equation to get,

$$\mu_1 W_1 \cos\alpha + \mu_2 W_2 \cos\alpha - W_1 \sin\alpha - W_2 \sin\alpha = 0$$

$$\tan\alpha = \frac{(\mu_1 W_1 + \mu_2 W_2)}{(W_1 + W_2)}$$

Substituting

$$\mu_1 = 0.3\,\mu_2 = 0.2$$

$$W_1 = 50\text{ N},\ W_2 = 50\text{ N}$$

$$\tan\alpha = \frac{(0.3\times 50 + 0.2\times 50)}{(50+50)}$$

$$\tan\alpha = 0.25,\ \alpha = 14° \quad \textit{Ans.}$$

Example 6.5 Two blocks of weight W_1 and W_2 are connected by a string and rest on a horizontal plane as shown in Fig. 6.10. Find the magnitude and direction of the least force P that should be applied to the upper block to induce sliding. The coefficient of friction for each block is to be taken as μ.

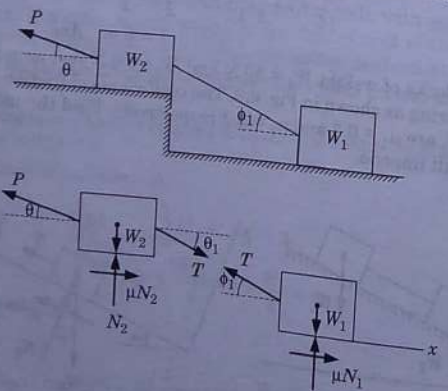


Fig. 6.10

Solution: Let the force P make and angle θ with the horizontal. Note that angles θ_1 and θ are not known.

Free-body diagram of each block is shown separately in Fig. 6.10. Let T be the tension in string.

Writing the equations of equilibrium of block W_1

$\Sigma F_x = 0:$ $\qquad \mu N_1 - T\cos\theta_1 = 0$ ...(i)

$\Sigma F_y = 0:$ $\qquad T\sin\theta_1 + N_1 - W_1 = 0$ ...(ii)

Writing the equations of equilibrium of block W_2

$\Sigma F_x = 0:$ $\qquad \mu N_2 + T\cos\theta_1 - P\cos\theta = 0$ ...(iii)

$\Sigma F_y = 0:$ $\qquad N_2 - W_2 + P\sin\theta - T\sin\theta_1 = 0$ ...(iv)

There are four equations and six unknowns (N_1, N_2, T, P, θ_1, θ).

Let us eliminate T, N_1, N_2 and θ_1

From (i) and (ii) eliminating N_1,

$$T = \frac{W_1\mu}{(\mu\sin\theta_1 + \cos\theta_1)} \quad ...(v)$$

From (iii) and (iv) eliminating N_2,

$$T = \frac{P(\cos\theta + \mu\sin\theta) - \mu W_2}{(\mu\sin\theta_1 + \cos\theta_1)} \quad ...(vi)$$

Eliminating T from (v) and (vi),

$$\frac{W_1\mu}{(\mu\sin\theta_1 + \cos\theta_1)} = \frac{P(\cos\theta + \mu\sin\theta) - \mu W_2}{(\mu\sin\theta_1 + \cos\theta_1)}$$

$$P = \frac{\mu(W_1 + W_2)}{(\cos\theta + \mu\sin\theta)}$$

But

$$\mu = \tan\phi = \frac{\sin\phi}{\cos\phi}$$

$$P = \frac{\dfrac{\sin\phi}{\cos\phi}(W_1 + W_2)}{\left(\cos\theta + \dfrac{\sin\phi}{\cos\phi}\sin\theta\right)}$$

$$P = \frac{(W_1 + W_2)\sin\phi}{\cos(\theta - \phi)}$$

For P to be minimum $\cos(\theta - \phi)$ should be maximum

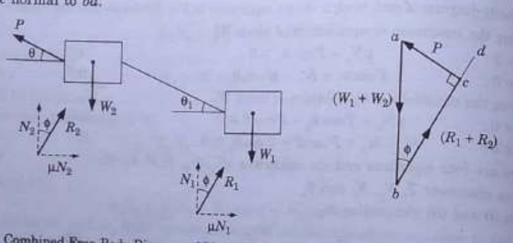
$$\cos(\theta - \phi) = 1,\ (\theta - \phi) = 0,\ \theta = \phi$$

$P_{min} = (W_1 + W_2)\sin\theta$ and act at an angle $\theta = \phi$ with the horizontal. *Ans.*

Alternative Method. In this problem, it is advantageous to combine the normal reaction N and the friction force μN into a single resultant reaction R. This resultant reaction R lies between N and μN and is inclined at an angle ϕ to the normal reaction N.

Free-body of both the blocks combined is as shown in Fig. 6.11. *Note that the tension in the string T is an internal force between the blocks so, it does not appear in the free-body diagram.*

To draw the triangle of force, first draw the vector $\overline{ab}$ to represent $(W_1 + W_2)$. From b dr bd parallel to the resultant reactions R_1 and R_2. To determine the minimum force P, from drawn ac normal to bd.



Combined Free-Body Diagram of Block W_1 and W_2 Force Triangle

Fig. 6.11

The vector ca represents the least force P.

From triangle abc, $P = (W_1 + W_2) \sin \phi$ *Ans.*

Example 6.6 A diesel locomotive of weight W is at rest, find the reactions at the points of contac A and B.

When it is pulling a train the draw bar pull P is just equal to the total friction at the points of contact A and B. Find the new magnitudes of the vertical reactions at A and B.

Solution: When at rest pull $P = 0$ (Fig. 6.12).

Writing the equations of equilibrium,

$\Sigma F_y = 0$: $R_A + R_B - W = 0$ ...(i)

Taking moments about A,

$\Sigma M_A = 0$: $R_B(2a) - W(a) = 0$ ...(ii)

Solving (i) and (ii)

$$R_A = R_B = \frac{W}{2} \quad Ans.$$

Fig. 6.12

Fig. 6.13

When Pulling the Train (Fig. 6.13).

Let the friction force be f at each point of contact. Writing the equations of equilibrium,

$\Sigma F_x = 0$: $P - f - f = 0$ ...(i)

$\Sigma F_y = 0$: $R_A + R_B - W = 0$ ...(ii)

$\Sigma M_A = 0$: $R_B(2a) - W(a) - P(b) = 0$ ...(iii)

$$R_B = \frac{Pb + Wa}{2a}, \; R_B = \frac{Wa + Pb}{2a} \quad Ans.$$

Substituting in (ii)

$$R_A = \frac{Wa - Pb}{2a} \quad Ans.$$

Example 6.7 A four wheel drive car shown in Fig. 6.14, has a mass of 2000 kg with passengers. The roadway is inclined at an angle θ with the horizontal. If the coefficient of static friction between the tyres and the road is 0.3, what is the maximum inclination θ that the car can climb?

Solution: In the case of powered vehicles, the frictional force provides the forward tractive force. So the frictional force acts in the direction of motion.

Assume the forces to be coplanar and θ to be the maximum angle of inclination that the car can climb.

Free-body diagram is as shown in Fig. 6.14.

Writing the equation of equilibrium

$\Sigma F_x = 0$:

$\mu N_A + \mu N_B - mg \sin \theta = 0$

(Along the inclined plane)

Or

$0.3(N_A + N_B) = 2000\, g \sin \theta$ ...(i)

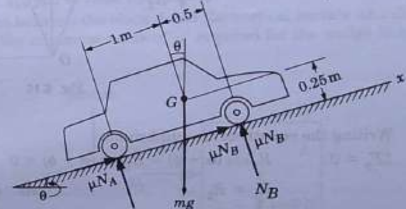


Fig. 6.14

$\Sigma F_y = 0$:

$N_A + N_B - mg \cos \theta = 0$

$$N_A + N_B = 2000\, g \cos \theta \quad \text{...(ii)}$$

Dividing (i) by (ii)

$\tan \theta = 0.3$, $\theta = \tan^{-1} 0.3$

$\theta = 16.7°$ *Ans.*

Example 6.8 A steel wedge of angle 2α is used for splitting wood by applying a force P. After the wedge is driven into the wood, determine the relation between the angle of wedge 2α and angle of friction ϕ so that it may not slip out.

Solution: When the Wedge is being driven, free-body diagram of the wedge is shown in Fig. 6.15. Various forces acting on the wedge are :

(i) external force P

(ii) resultant reaction R_1 acting on the wedge, inclined at an angle ϕ with the normal to the face OA.

(iii) resultant reaction R_2 acting on the wedge, inclined at an angle ϕ with the normal to the face OB.

Note carefully the direction of the friction force and the fact that R_1 is the resultant of normal reaction N_1 and frictional force μN_1. Similarly R_2 is the resultant of N_2 and μN_2.

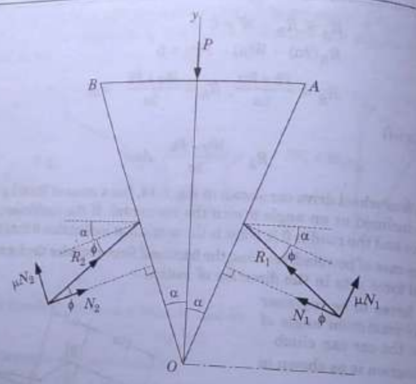


Fig. 6.15

Writing the equations of equilibrium,

$\Sigma F_x = 0$: $\quad R_2 \cos(\alpha + \phi) - R_1 \cos(\alpha + \phi) = 0$

$R_1 = R_2$

So, $\quad N_1 = N_2 \qquad As \quad R_1 = \sqrt{N_1^2 + (\mu N_1)^2}$

$\Sigma F_y = 0$: $\quad R_1 \sin(\alpha + \phi) + R_2 \sin(\alpha + \phi) - P = 0$

But, $\quad R_1 = R_2$

Therefore, $\quad P = 2R_1 \sin(\alpha + \phi) \quad$ *Ans.*

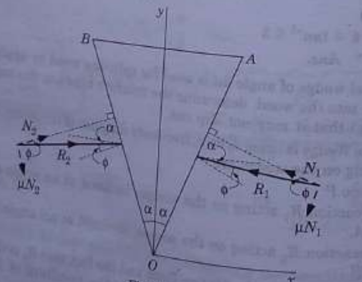


Fig. 6.16

The force P can be determined if R_1 is known or if the normal reaction N_1 of the wood on the wedge and the coefficient of friction ($\mu = \tan\phi$) are known.

When the wedge has been driven and the Force P is *removed* :

The wedge now tries to slip out, so the direction of friction is *reversed*. As no force is acting $P = 0$.

Writing the equations of equilibrium,

$\Sigma F_x = 0$: $\quad R_2 \cos(\alpha - \phi) - R_1 \cos(\alpha - \phi) = 0$

$R_1 = R_2$

$\Sigma F_y = 0$: $\quad R_1 \sin(\alpha - \phi) + R_2 \sin(\alpha - \phi) = 0$

$2R_1 \sin(\alpha - \phi) = 0$

Or $\quad \alpha = \phi \quad$ *Ans.*

Example 6.9 (*a*) Block *A* weighing 1000 N is to be raised by means of a 15° wedge *B* weighing 500 N. Assuming the coefficient of friction between all contact surfaces to be 0.2, determine what minimum horizontal force *P* should be applied to raise the block.

(*b*) Assuming that there is no friction between the block *A* and the vertical surface and the wedge is of negligible weight, what is the minimum value of 'μ' required for the wedge to be self-locking?

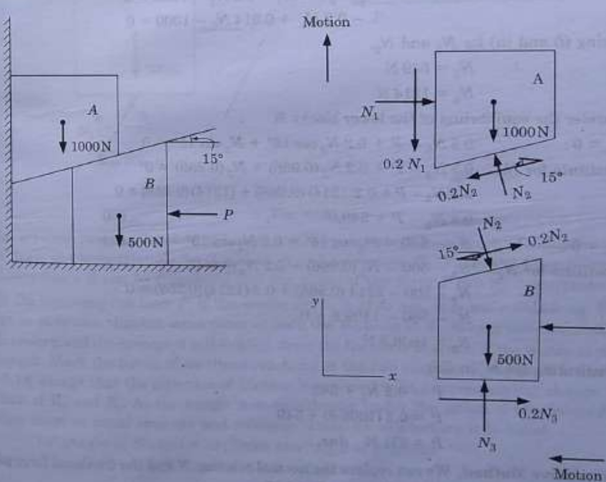


Fig. 6.17

Solution: When the upper block A is at the point of moving up, the lower block B (wedge) is at the point of moving from right to left. This shall help in marking the proper directions of frictional forces.

Free-body diagrams of the blocks A and B are shown separately in the Fig. 6.17.

Note that the magnitude of normal reaction (N_2) and the frictional force ($0.2\ N_2$) acting on the block A and on the block B are the equal but are acting in opposite directions. *They are related like action and reaction*

Consider the equilibrium of the upper block A,

$\Sigma F_x = 0$: $\quad N_1 - 0.2\ N_2 \cos 15° - N_2 \sin 15° = 0 \qquad \begin{cases} \cos 15° = 0.966 \\ \sin 15° = 0.259 \end{cases}$

$$N_1 - 0.2\ N_2(0.966) - N_2(0.259) = 0$$
$$N_1 - 0.1932\ N_2 - 0.259\ N_2 = 0$$

or $\quad N_1 = 0.4522\ N_2 \qquad$ ...(*i*)

$\Sigma F_y = 0$: $\quad N_2 \cos 15° - 0.2\ N_2 \sin 15° - 0.2\ N_1 - 1000 = 0$

$$N_2\ (0.966) - 0.2\ N_2\ (0.259) - 0.2\ N_1 - 1000 = 0$$
$$-\ 0.2\ N_1 + 0.914\ N_2 - 1000 = 0 \qquad \text{...}(ii)$$

Solving (*i*) and (*ii*) for N_1 and N_2,

$$N_1 = 549\ \text{N}$$
$$N_2 = 1214\ \text{N}$$

Consider the equilibrium of the lower blocks B,

$\Sigma F_x = 0$: $\quad 0.2\ N_3 - P + 0.2\ N_2 \cos 15° + N_2 \sin 15° = 0$

Substitute for N_2, $\quad 0.2\ N_3 - P + 0.2\ N_2\ (0.966) + N_2\ (0.259) = 0$

$$0.2\ N_3 - P + 0.2(1214)\ (0.966) + (1214)(0.259) = 0$$
$$0.2\ N_3 - P + 549.0 \qquad \text{...}(iii)$$

$\Sigma F_y = 0$: $\quad N_3 - 500 - N_2 \cos 15° + 0.2\ N_2 \sin 15° = 0 \qquad$ ...(*iv*)

Substitute for N_2, $\quad N_3 - 500 - N_2\ (0.966) + 0.2\ N_2\ (0.259) = 0$

$$N_3 - 500 - 1214\ (0.966) + 0.2\ (1214)(0.259) = 0$$
$$N_3 - 500 - 1109.8 = 0$$
$$N_3 = 1609.8\ \text{N}$$

Substituting for N_3 in (*iii*)

$$P = 0.2\ N_3 + 549$$
$$P = 0.2\ (1609.8) + 549$$
$$P = 871\ \text{N} \quad \textit{Ans.}$$

Alternative Method. We can replace the normal reaction N and the frictional force μN by a single resultant reaction R as shown in Fig. 6.18.

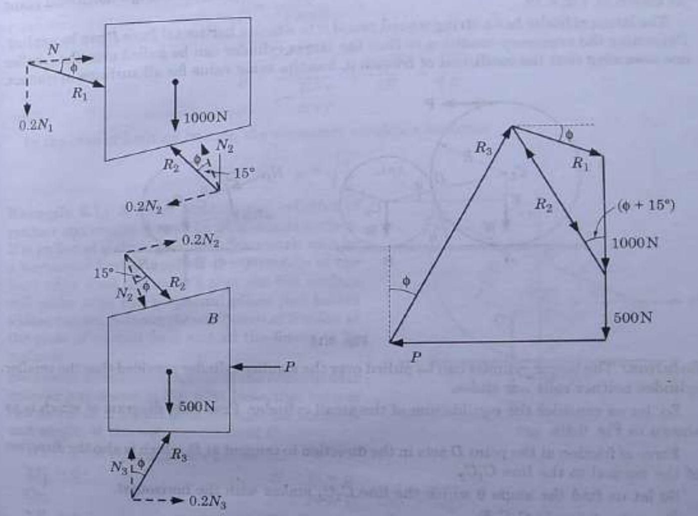


Fig. 6.18

The solution now can be obtained more easily by drawing the polygon of forces as shown. First draw the force triangle for the block A and determine R_2. Then, the force polygon for the lower block B can be completed and the value of the force P can be determined.

(*b*) On removing the force P, if the wedge tends to slide out, it is *not self-locking*. For the wedge to slide out, slipping must occur at both the surfaces of the wedge simultaneously.

To understand the concept of self-locking, draw the free body diagram of the wedge as it tends to slip out. Mark the forces of resultant reactions at the two surfaces of wedge. It is similar to Fig. 6.18, except that the direction of friction is reversed with a corresponding change in the position of R_2 and R_3. As the wedge is in equilibrium under the action of two forces R_2 and R_3 they must be equal opposite and collinear. Thus, for self locking it is found,

$$15° \text{ (Angle of Wedge)} < 2\phi \text{ (Twice the Angle of Friction)}$$

For, $\quad \phi = 7.5°,\ \mu = 0.1316 \quad \textit{Ans.}$

Example 6.10 Two heavy circular cylinders of radii R and r rest on a rough horizontal plane as shown in Fig. 6.19.

The larger cylinder has a string wound round it to which a horizontal force P can be applied. Determine the necessary condition so that the larger cylinder can be pulled over the smaller one assuming that the coefficient of friction μ, has the same value for all surfaces in contact.

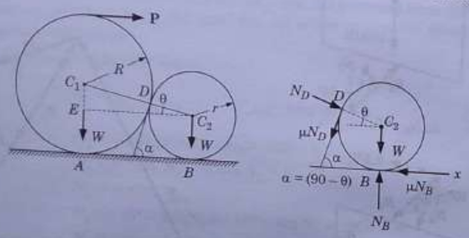


Fig. 6.19

Solution: The larger cylinder can be pulled over the smaller cylinder provided that the smaller cylinder neither rolls nor slides.

So, let us consider the equilibrium of the small cylinder. Free-body diagram of which is as shown in Fig. 6.19.

Force of friction at the point D acts in the direction to tangent at D, which is also the direction of the normal to the line C_1C_2.

So let us find the angle θ which the line C_1C_2 makes with the horizontal.

From the triangle C_1C_2E

$$\sin\theta = \frac{R-r}{R+r}, \ \cos\theta = \sqrt{1-\sin^2\theta} \text{ or } \cos\theta = \frac{2\sqrt{Rr}}{R+r}$$

Therefore, the angle α that the tangent at D makes with the horizontal is

$$\alpha = (90 - \theta).$$

Writing the equations of equilibrium of the small cylinder,

$\Sigma F_x = 0:$ $\quad -\mu N_B - \mu N_D \cos(90-\theta) + N_D\cos\theta = 0 \quad$ ...(i)

Taking moments about C_2

$\Sigma M_{C2} = 0:$ $\quad \mu N_D(r) - \mu N_B(r) = 0$

$$N_D = N_B \quad \text{...(ii)}$$

Substituting (ii) in (i)

$$-\mu N_D - \mu N_D\sin\theta + N_D\cos\theta = 0$$

$$-\mu - \mu\sin\theta + \cos\theta = 0$$

$$\mu(1+\sin\theta) = \cos\theta$$

$$\mu = \frac{\cos\theta}{1+\sin\theta}$$

$$\mu = \frac{\dfrac{2\sqrt{Rr}}{R+r}}{1+\dfrac{R-r}{R+r}} = \frac{2\sqrt{Rr}}{2R} = \sqrt{\frac{r}{R}}.$$

In the case of limiting friction the necessary condition becomes

$$\mu > \sqrt{\frac{r}{R}} \quad \textit{Ans.}$$

Example 6.11 A short semicircular cylinder of radius r and weight W rests on a horizontal surface. If is pulled at right angles to its geometric axis by a horizontal force P applied at the middle of the front edge. Find the angle θ that the flat surface will make with the horizontal plane just before sliding begins. Assume the coefficient of friction at the point of contact as μ and all the forces to be coplanar.

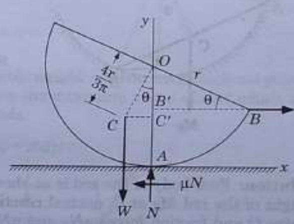


Fig. 6.20

Solution: Free-body diagram of the semicircular cylinder is as shown in Fig. 6.20. Note, that *normal reaction N at A would pass through the centre O* and weight W through the centroid C.

Writing the equations of equilibrium

$\Sigma F_y = 0:$ $\quad N - W = 0$

Or, $\quad N = W \quad$ ...(i)

$\Sigma F_x = 0:$ $\quad P - \mu N = 0$

Or $\quad P = \mu N \quad$ ...(ii)

Taking moments about A,

$\Sigma M_A = 0:$

$$W(CC') - P(AB') = 0$$

$$W(OC\sin\theta) - P(AO - B'O) = 0$$

$$W\left(\frac{4r}{3\pi}\sin\theta\right) - P(r - r\sin\theta) = 0 \quad \text{...(iii)}$$

Using (i) and (ii) to eliminate P

$$W\frac{4r}{3\pi}\sin\theta - \mu W_r(r - r\sin\theta) = 0$$

$$\left(\frac{4r}{3\pi} + \mu r\right)\sin\theta = \mu r$$

$$\sin\theta = \frac{3\mu\pi}{(4+3\pi\mu)} \quad \textit{Ans.}$$

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Example 6.12 The two ends of a circular rod AB of mass M are supported by a circular ring in vertical plane as shown in Fig. 6.21. The length of the rod is such so as to subtend an angle of 90° at the centre of the ring. The coefficients of friction at the points of contact A and B are μ each. What is the greatest angle of inclination θ that the rod can make with the horizontal in the condition of equilibrium?

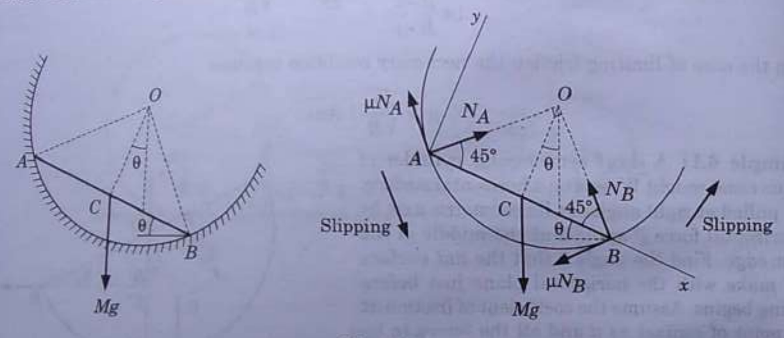


Fig. 6.21

Solution: Free-body of the rod is as shown in Fig. 6.21. Various forces acting on the rod are: weight of the rod Mg at C, normal reaction N_A and N_B at the ends A and B passing through centre O and friction forces μN_A and μN_B at ends A and B.

Writing the equations of equilibrium,

Resolving the forces along x-axis (along the rod)

$\Sigma F_x = 0$: $+ Mg\cos(90 - \theta) - N_B\cos 45° + N_A\cos 45° - \mu N_B\cos 45° - \mu N_A\cos 45° = 0$

Or $\qquad \sqrt{2}\, mg\sin\theta = (N_B - N_A) + \mu(N_A + N_B)$ ...(*i*)

Resolving the forces along y-axis (perpendicular to the rod)

$\Sigma F_y = 0$: $- Mg\sin(90 - \theta) + N_B\sin 45° - \mu N_B\sin 45° + \mu N_A\sin 45° = 0$

Or, $\qquad \sqrt{2}\, mg\cos\theta = (N_B + N_A) + \mu(N_A - N_B) + N_A\sin 45°$ ...(*ii*)

Taking moments about C,

$\Sigma M_C = 0$: $N_B CD\sin 45° - \mu N_B CB\sin 45° - N_A AC\sin 45° - \mu N_A AC\sin 45° = 0$

$$\mu N_A + N_A + \mu N_B - N_B = 0 \quad ...(iii)$$

We have three equations involving the unknowns N_A, N_B and θ. Eliminate N_A and N_B and solve for θ.

Dividing (*i*) by (*ii*)

$$\tan\theta = \frac{(N_B - N_A) + \mu(N_A + N_B)}{(N_B + N_A) + \mu(N_A - N_B)} \quad ...(iv)$$

From (*iii*), we have

$$N_A = \frac{N_B(1-\mu)}{(1+\mu)}$$

Substituting this value of N_A in (*iv*)

$$\tan\theta = \frac{\left[N_B - \dfrac{N_B(1-\mu)}{1+\mu}\right] + \mu\left[\dfrac{N_B(1-\mu)}{1+\mu} + N_B\right]}{\left[N_B + \dfrac{N_B(1-\mu)}{1+\mu}\right] + \mu\left[\dfrac{N_B(1-\mu)}{1+\mu} - N_B\right]}$$

$$\tan\theta = \frac{(1+\mu-1+\mu)+\mu(1-\mu+1+\mu)}{(1+\mu+1-\mu)+\mu(1-\mu-1-\mu)}$$

$$\tan\theta = \frac{2\mu + 2\mu}{2 - 2\mu^2} = \frac{2\mu}{1-\mu^2}$$

Substituting for $\qquad \mu = \tan\phi$

$$\tan\theta = \frac{2\tan\phi}{1-\tan^2\phi} = \tan^2\phi$$

$$\theta = 2\phi \quad \textit{Ans.}$$

Example 6.13 A horizontal bar 1 m long and of negligible weight rests on rough inclined planes [Fig. 6.22 (*a*)]. If the angle of friction is 15°, determine the minimum value of x at which the load Q = 200 N may be applied before slipping impends.

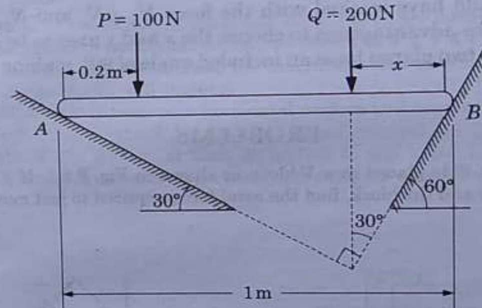


Fig. 6.22 (*a*)

Solution: Let us assume that the end B slips down the 60° plane so the end A shall move up the 30° plane.

Forces acting on the bar are,

(*i*) force P = 100 N, (*ii*) force Q = 200 N, (*iii*) normal reaction N_a and frictional force μN_a acting on the end A and (*iv*) reaction N_B and frictional force μN_b acting on the end B.

We choose to replace N_a and μN_a by a single force R_a and N_b and μN_b by R_b. Further x-axis is choosen to be along the bar AB.

Free-body diagram of the bar AB is as shown [Fig. 6.22 (*b*)].

Equations of equilibrium are,

$\Sigma F_y = 0$: $\qquad R_a\cos 45° - R_b\cos 45° = 0$

$\qquad \therefore \qquad R_a = R_b$ ...(*i*)

$\Sigma F_x = 0$: $\qquad R_a\sin 45° + R_b\sin 45° - 100 - 200 = 0$ ...(*ii*)

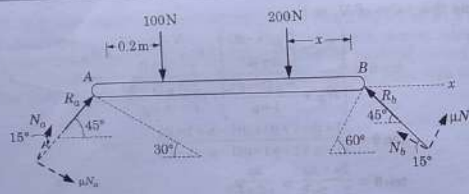


Fig. 6.22 (b)

Using (i), $2R_a \sin 45° = 300$

$$R_a = 212\text{ N and } R_b = 212\text{ N}$$

$\Sigma M_B = 0$: $200(x) + 100(1 - 0.2) - R_a(1 \sin 45°) = 0$ ...(iii)

$$200x - 100 - 20 - 212 \times 0.707 = 0$$

$$200x = 69.9$$

$$x = 0.35\text{ m}$$

Alternatively, we could have worked with the force N_a, μN_a and N_b, μN_b.

In that case it could be advantageous to choose the x and y axes to lie along the 60° and 30° planes respectively. The two planes have an included angle of 90°, making such a choice possible in this problem.

PROBLEMS

6.1. A cylinder of weight W is placed in a V-block as shown in Fig. P.6.1. If ϕ be the angle of friction between the cylinder and the block, find the axial force required to just move the cylinder. Assume $\phi < \theta$.
$$\left[P = \frac{W \tan\phi}{\cos\theta}\right]$$

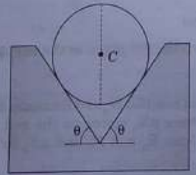


Fig. P.6.1

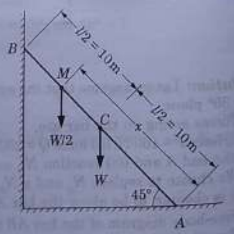


Fig. P.6.2

6.2. A uniform ladder AB of length $l = 20$ m and weight W is supported by the horizontal floor at A and by a vertical wall at B. It makes an angle 45° with the horizontal. If a man, whose weight is one-half that of the ladder, ascends the ladder, how much length x of the ladder he shall climb before the ladder slips.

If a boy now stands on the end A the ladder, what must be his least weight w so that the man may go on the top of the ladder? Assume,

μ between the ladder and the wall = 1/3

μ between the ladder and the floor = 1/2 [x = 14.3 m, w = 0.25 W]

6.3. Determine the magnitude and direction of the smallest force P which will cause the block to move up the inclined plane (Fig. P.6.3). The angle of friction between the block and the plane is ϕ.
$$\begin{bmatrix} P = W\sin(\theta + \phi) \\ \alpha = \theta + \phi \end{bmatrix}$$

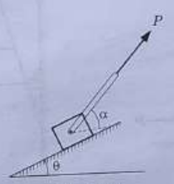


Fig. P.6.3

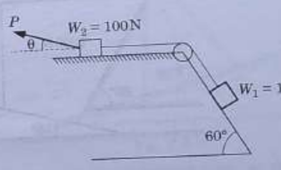


Fig. P.6.4

6.4. Two rectangular blocks of weight, W_1 = 150 N and W_2 = 100 N, are connected by a string and rest on an inclined plane and on a horizontal surface as shown in Fig. P.4.4. the coefficient of friction for all contiguous surfaces is μ = 0.2. Find the magnitude and direction of the least force P at which the motion of the blocks will impend. [P = 161.7 N, θ = 11.31°]

6.5. A block of weight W_1 = 1000 N rests on a horizontal surface and supports on its top another block of weight W_2 = 250 N. The weight W_2 is attached by an inclined string AB to the vertical wall. Find the magnitude of the horizontal force P applied to the lower block to cause slipping to impend. The coefficient of friction for all contacting surfaces may be assumed to μ = 0.3. [422.5 N]

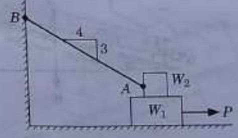


Fig. P.6.5

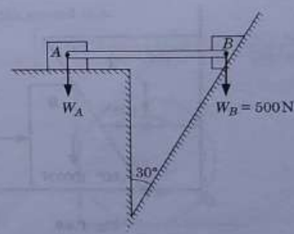


Fig. P.6.6

6.6. Two blocks are connected by a horizontal link AB and rest on two planes as shown in Fig. P.6.6. What is the smallest weight W_A of the block A for which the equilibrium can exist? Assume the coefficient of friction for the block A and the horizontal surface to be 0.4 and the angle of friction for the block B on the inclined plane is ϕ = 20°. [1050 N]

6.7. A block weighing 1000 N is to be raised against a surface inclined at 60° to the horizontal by means of a 15° wedge (Fig. P.6.7). Find the horizontal force P which will just start the block to move if the coefficient of friction between all the surfaces of contact be 0.2. Assume the wedge to be of negligible weight. [595 N]

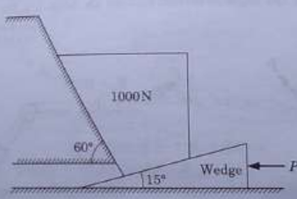


Fig. P.6.7

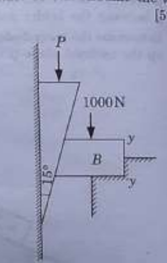


Fig. P.6.8

6.8. A 15° wedge of negligible weight is to be driven to tighten a body B which is supporting a vertical load of 1000 N. If the coefficient of friction for all contacting surfaces be 0.25, find the minimum force P required to drive the wedge (Fig. P.6.8). [232 N]

[**Hint.** Assume the reaction of the surface y-y as zero.]

6.9. Two blocks A and B are resting against a wall and the floor as shown in Fig. P.6.9. Find the value of the horizontal force P applied to the lower block that will hold the system in equilibrium. Coefficients of friction are : 0.25 at the floor, 0.3 at the wall and 0.2 between the blocks. [$P_{minimum}$ = 81 N]

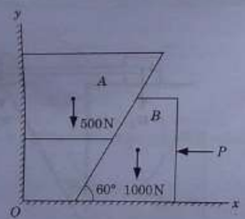


Fig. P.6.9

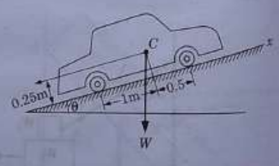


Fig. P.6.10

6.10. A rear wheel drive car of mass 2000 kg is shown in Fig. P.6.10. The road is inclined at an angle θ with the horizontal. If the coefficient of friction between the types and the road is 0.3, what is the maximum inclination θ that the car can climb? [θ = 11°]

[**Hint.** Friction acts only on the rear wheels.]

6.11. Two blocks A and B of weights 250 N and 150 N rest on a plane which is slowly raised from the horizontal position to an angle θ. Find the maximum angle that can be reached before bodies slip down the incline. Assume,

μ between block B and the plane = 0.2

μ between block A and the plane = 0.3.

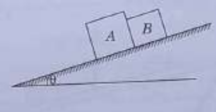


Fig. P.6.11

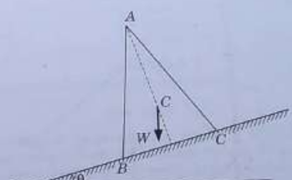


Fig. P.6.12

6.12. A right cone of the height h and the base of radius r rests on an inclined plane as shown in Fig. P.6.12. The coefficient of friction between the plane and the cone is μ.

If the angle of the inclination of the plane is slowly increased till the cone is just at the point of toppling over find the relation between μ, r and h. $\left[\mu = \frac{4r}{h}\right]$

[**Hint.** At the point of toppling, the weight of the cone acting through C will pass through the point B.]

6.13. The distance between the centres of the wheels of a bicycle is $2a$ and its centre of gravity is at a height h above the ground and at a distance x in front of its middle point.

Find the greatest slope of the incline on which the bicycle can rest without slipping if (a) the rear wheel is braked (b) the front wheel is braked.

The coefficient of friction between the tyre and the ground is μ.

$$\left[\theta = \tan^{-1}\frac{\mu(a-x)}{(\mu h + 2a)},\ \theta = \tan^{-1}\frac{\mu(a+x)}{(2a - \mu h)}\right]$$

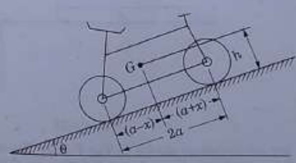


Fig. P.6.13

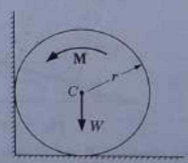


Fig. P.6.14

6.14. A homogenous cylinder of weight W rests on a horizontal floor in contact with a wall. If the coefficient of friction for all contact surfaces be μ, determine the couple M acting on the cylinder which will start counter clockwise rotation (Fig. P.6.14). $\left[M = \frac{r\mu W}{1+\mu^2}(1+\mu)\right]$

6.15. A square uniform lamina $ABCD$ of weight W is placed vertically with its end A of the edge AB on the floor and the end B against a vertical wall. If μ_a and μ_b be the coefficients of friction at the ground and the wall, show that when the lamina is in limiting equilibrium the inclination of the edge AB with the horizontal is,

$$\left[\tan\theta = \frac{1-\mu_a\mu_b}{1+2\mu_a+\mu_a\mu_b}\right]$$

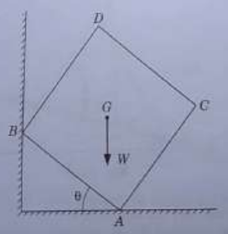


Fig. P.6.15

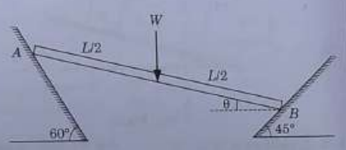


Fig. P.6.16

6.16. A uniform plank of weight W and length L is placed with its ends in contact with inclined planes as shown (Fig. P.6.16). Angle of friction is 15°. Determine the maximum value of angle θ at which slipping impends. [θ = 36.2°]

6.17. A block shown in Fig. P.6.17 weighing 1000 N is resting on a rough horizontal plane. The plane is gradually lifted to increase the angle θ. Determine whether sliding of block or overturning about A will occur first and the angle at which it occurs. Assume $\mu = 0.3$. [Sliding occurs first]

[**Hint.** Determine the location of centroid C.]

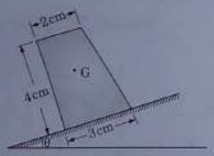


Fig. P.6.17

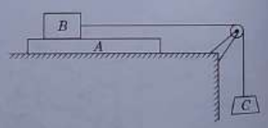


Fig. P.6.18

6.18. Three blocks A, B and C each of mass m are connected as shown in Fig. P.6.18. The coefficient of friction between A and B is 0.3 and that between A and the surface is 0.1. How far does A move when B has moved 1 m relative to A? [0.4 m]

6.19. A block of mass 200 kg is to be raised upwards by simultaneously pushing two identical wedges B and C under it. Each wedge weighs 200 N and the wedge angle is 15°. If the coefficient of friction at all surfaces in contact is 0.3, find the minimum value of forces P required for doing the job. [9 kN]

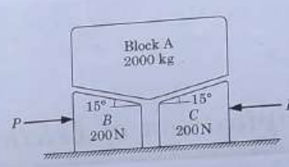


Fig. P.6.19

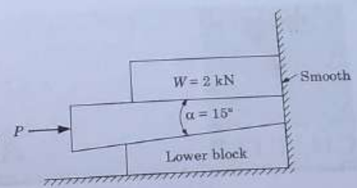


Fig. P.6.20

6.20. Determine the horizontal force P to be applied to the wedge so as to raise the weight W of 2 kN. Assume the wedge to be of negligible weight and the contact surfaces between weight W and vertical wall are smooth. The coefficient of friction between weight W and wedge is 0.25 and that between wedge and lower block is also 0.25. Is the system self locking one? [1.61 kN, self locking]

[**Hint.** Slippage must occur at both surfaces when force P is removed, otherwise wedge is self locking. For self locking, $\alpha < 2\phi$ and can be shown by solving the problem with the direction of friction reversed.]

7

CHAPTER

Application of Friction

7.1 BELT AND ROPE DRIVES

Flexible members such as belts and ropes are frequently used to transmit power from one shaft to another. These devices are termed as *non-positive drives* because of the possibility of the slip, occurring between the belt and the pulley.

Fig. 7.1 shows an arrangement to transmit power from shaft A of an electric motor to the shaft B of a flour mill. Two pulleys are keyed to the shafts and a belt is laid tightly over the pulleys to create the necessary friction. When the pulley on the shaft A (called driver) rotates, it carries with it the belt by virtue of the friction between the rim of the pulley and the belt, which in turn, rotates the pulley on the shaft B (called driven pulley).

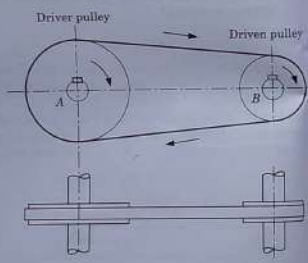


Fig. 7.1

The various types of belts in use are, (*a*) flat belt, (*b*) V-belt and (*c*) circular belt or rope. To accommodate a V-belt or a rope, a groove is provided on the rim of the pulley as shown in the Fig. 7.2.

Flat belt (*a*) V-belt (*b*) Circular belt (*c*)

Fig. 7.2

To transmit a large power a single belt or a rope may not be sufficient. So, a number of belts or ropes in parallel can be used for this purpose.

Speeds of the Shafts Connected by a Belt

Let D_1 be the diameter of the pulley on the shaft A.

N_1 be the speed of the shaft A in r.p.m.

D_2 be the diameter of the shaft B in r.p.m.

N_2 be the speed of the shaft B in r.p.m.

Length of the belt that runs over the pulley D_1 per minute = $(\pi D_1) N_1$

Length of the belt that runs over the pulley D_2 per minute = $(\pi D_2) N_2$

If there is no slip.

$$\pi D_1 N_1 = \pi D_2 N_2$$

$$\frac{N_2}{N_1} = \frac{D_1}{D_2} \qquad ...(7.1)$$

7.2 TYPES OF BELT DRIVES

The belt over the two pulleys can be laid in any of the two arrangements called (*a*) open belt drive and (*b*) crossed belt drive.

Open Belt Drive : Length of the Belt

In this arrangement both the pulleys rotate in the same direction.

It can be seen from Fig. 7.3 that only a part of the belt is in contact with the pulley. The angle subtended at the centre of the pulley by the portion of the belt in contact with it is called the *angle of contact or the angle of lap*.

r_1 and r_2 be the radii of the two pulleys such that $r_1 > r_2$.

d be the distance between the centres O_1 and O_2 of the pulleys,

L be the length of the belt required.

From the geometry of the figure,

$$\angle O_1AB = \angle O_2BA = 90^\circ$$

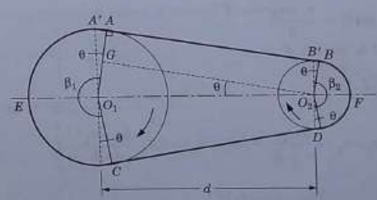


Fig. 7.3

Draw O_2G parallel to AB.

Let $\angle GO_2O_1 = \theta$ radians

then $\angle A'O_1A = \theta$

$\angle B'O_2B = \theta$

and $AB = GO_2$

Length of the belt $L = 2(Arc\ EA + AB + Arc\ BF)$

$$Arc\ EA = \left(\frac{\pi}{2}+\theta\right)r_1,\ Arc\ BF = \left(\frac{\pi}{2}-\theta\right)r_2$$

$$AB = GO_2$$

From ΔO_1GO_2

$$GO_2 = \sqrt{d^2-(r_1-r_2)^2}$$

$$AB = GO_2 = \sqrt{d^2-(r_1-r_2)^2} = d\sqrt{1-\left(\frac{r_1-r_2}{d}\right)^2}$$

$$AB = d\left[1-\left(\frac{r_1-r_2}{d}\right)^2\right]^{1/2}$$

Evaluating the above expression using Binomial Theorem,

$$AB = d\left[1-\frac{1}{2}\left(\frac{r_1-r_2}{d}\right)^2\right] = \left[d-\frac{(r_1-r_2)^2}{2d}\right]$$

Length of the belt $$L = 2\left[r_1\left(\frac{\pi}{2}+\theta\right)+\left(d-\frac{(r_1-r_2)^2}{2d}\right)+r_2\left(\frac{\pi}{2}-\theta\right)\right]$$

$$L = \left[\pi(r_1+r_2)+2\theta(r_1-r_2)+2d-\frac{(r_1-r_2)^2}{d}\right]$$

From ΔO_1GO_2,

$$\sin\theta = \frac{r_1-r_2}{d}$$

For small values of θ, $\sin\theta = \theta$

$$\theta = \frac{r_1-r_2}{d}$$

$$L = \left[\pi(r_1+r_2)+\frac{2(r_1-r_2)}{d}(r_1-r_2)+2d-\frac{(r_1-r_2)^2}{d}\right]$$

$$L = \left[\pi(r_1+r_2)+2d+\frac{(r_1-r_2)^2}{d}\right] \quad \text{... (7.2 a)}$$

Angle of lap for the larger pulley $r_1 = \beta_1 = 180° + 2\theta$.

Angle of lap for the smaller pulley $r_2 = \beta_2 = 180° - 2\theta$.

The larger pulley has a greater angle of lap as seen from the above.

But the smaller of the two angles of lap is to be used for determining the driving tensions T_1 and T_2. (explained later).

Crossed Belt Drive: Length of the Belt

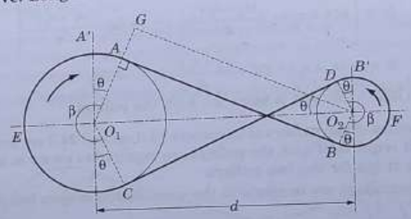


Fig. 7.4

In this arrangement the two pulleys rotate in the opposite directions but *both the pulleys have the same angle of lap.*

Using the same notations as used in the case of open drive, length of the belt is,

$$L = 2[Arc\ EA + AB + Arc\ BF]$$

Draw, O_2G parallel to AB

$$AB = GO_2$$

From ΔO_1GO_2,

$$GO_2 = \sqrt{d^2-(r_1+r_2)^2} = d\left[1-\left(\frac{r_1+r_2}{d}\right)^2\right]^{1/2}$$

Expanding the above expression using Binomial Theorem,

$$GO_2 = d\left[1-\frac{1}{2}\left(\frac{r_1+r_2}{d}\right)^2\right]$$

$$AB = GO_2 = d-\frac{(r_1+r_2)^2}{2d}$$

Length of the belt $$L = 2\left[r_1\left(\frac{\pi}{2}+\theta\right)+\left(d-\frac{(r_1+r_2)^2}{2d}\right)+r_2\left(\frac{\pi}{2}+\theta\right)\right]$$

$$L = \left[\pi(r_1+r_2)+2\theta(r_1+r_2)+2d-\frac{(r_1+r_2)^2}{d}\right]$$

But $\sin\theta = \frac{r_1 + r_2}{d}$

For small values of θ

$$\sin\theta \approx \theta = \frac{r_1 + r_2}{d}$$

Substituting for θ

$$L = \left[\pi(r_1 + r_2) + 2\frac{(r_1 + r_2)}{d}(r_1 + r_2) + 2d - \frac{(r_1 + r_2)^2}{d}\right]$$

$$L = \pi(r_1 + r_2) + 2d + \frac{(r_1 + r_2)^2}{d} \qquad ...(7.2\ b)$$

Angle of lap $\beta = (180° + 2\theta)$ and is same for both the pulleys.

Example 7.1 The centre of two pulleys of diameters 12.0 cm and 24.0 cm are 30 cm apart. Find the length of the belt required if both the pulleys are required to rotate in the same direction. Find also the angles of lap for the two pulleys.

Solution: As the two pulleys are to rotate in the same direction open belt drive is to be used.

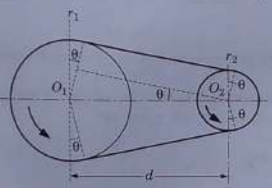


Fig. 7.5

$$r_1 = \frac{24}{2}\text{ cm} = 0.12\text{ m}, \quad r_2 = \frac{12}{2}\text{ cm} = 0.06\text{ m}$$

$$d = 30\text{ cm} = 0.3\text{ m}$$

$$\sin\theta = \frac{r_1 - r_2}{d} = \frac{0.12 - 0.06}{0.3} = 0.2$$

$$\theta = 11.54°$$

Angle of lap:

β_1, for larger pulley = 180 + 2(11.54) = 203.08°

β_2, for small pulley = 180 − 2(11.54) = 156.92° *Ans.*

Length of the belt :

$$L = \left[\pi(r_1 + r_2) + 2d + \frac{(r_1 - r_2)^2}{d}\right]$$

Substituting values

$$L = \left[\pi(0.12 + 0.06) + 0.3 \times 2 + \frac{(0.12 - 0.06)^2}{0.3}\right]$$

$$L = 0.5655 + 0.6 + 0.012$$

$$L = 1.177\text{ m}$$

$L = 117.7$ cm *Ans.*

Example 7.2 The centres of two pulleys of diameters 60 cm and 30 cm are 3.5 m apart. Find the length of the belt required if the two pulleys are to rotate in the opposite directions. Find the angles of lap for the two pulleys.

Solution: As the two pulleys are to rotate in the opposite directions cross belt drive is to be used (Fig. 7.6).

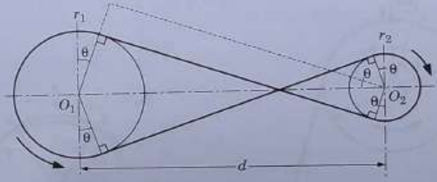


Fig. 7.6

$$r_1 = \frac{60}{2}\text{ cm} = 0.3\text{ m}, \quad r_2 = \frac{30}{2}\text{ cm} = 0.15\text{ m}$$

$$d = 3.5\text{ m}$$

$$\sin\theta = \frac{r_1 + r_2}{d} = \frac{0.3 + 0.15}{3.5} = 0.128$$

$$\theta = 7.4°$$

Angle of lap : For both the pulleys is same

$$\beta = 180° + 2\theta$$

$\beta = 180° + 2(7.4) = 194.8°$ *Ans.*

Length of the Belt :

$$L = \left[\pi(r_1 + r_2) + 2d + \frac{(r_1 + r_2)^2}{d}\right]$$

Substituting values,

$$L = \left[\pi(0.3 + 0.15) + 2(3.5)\frac{(0.3 + 0.15)^2}{3.5}\right]$$

$$L = 1.414 + 7.0 + 0.0578$$

$L = 8.471$ m *Ans.*

7.3 BELT FRICTION : RATIO OF TENSIONS

Consider pulley or radius r driven in the clockwise direction by a flat belt as shown in Fig. 7.7 (a).

Let T_1 = tension on the tight side
(Tension on the side of the belt leaving the pulley)

T_2 = tension on the slack side
(Tension on the side of the belt approaching the pulley)

β = the angle of contact or lap subtended by the portion of the belt AB in contact with the pulley.

Consider the equilibrium of an element ab of the belt, located at an angle θ from the point tangency at A. Let the ab subtend an angle $d\theta$ at the centre O.

Free-body diagram of element ab is as shown in Fig. 7.7 (b).

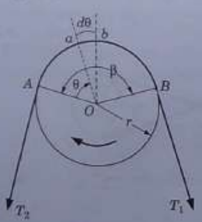


Fig. 7.7 (a)

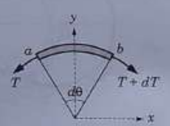


Fig. 7.7 (b)

Let the tension at a be $= T$
and the tension at b be $= T + dT$ (when the belt is just about to slip)

Normal reaction $= dN$

Friction force $dF = \mu dN$

where, μ = coefficient of friction between the pulley and the belt.

Equations of equilibrium are,

$$\Sigma F_x = 0: \quad T\cos\frac{d\theta}{2} + dF - (T + dT)\cos\frac{d\theta}{2} = 0 \qquad \text{...(7.3)}$$

$$\Sigma F_y = 0: \quad dN - T\sin\frac{d\theta}{2} - (T + dT)\sin\frac{d\theta}{2} = 0 \qquad \text{...(7.4)}$$

As angle $d\theta$ is very small, substitute

$$\cos\frac{d\theta}{2} = 1, \ \sin\frac{d\theta}{2} \approx \frac{d\theta}{2}$$

and $dF = \mu dN$ in the above equations

$$T + \mu dN - (T + dT) = 0$$

Or,

$$dT - \mu dN = 0 \qquad \text{...(7.5)}$$

and

$$dN - T\frac{d\theta}{2} - (T + dT)\frac{d\theta}{2} = 0$$

Or,

$$dN - T\,d\theta = 0 \qquad \text{... (7.6)}$$

after neglecting the term $\left(dT\frac{d\theta}{2}\right)$, being small in magnitude.

Eliminating dN from (7.5) and (7.6), we get

$$dT = \mu T\,d\theta$$

Or,

$$\frac{dT}{T} = \mu d\theta \qquad \text{...(7.7)}$$

Integrate the above expression for the entire length AB of the belt in contact with the pulley.

Using the conditions,

at A, $\theta = 0$ and $T = T_2$

and at B, $\theta = \beta$ and $T = T_1$

$$\int_{T_2}^{T_1}\frac{dT}{T} = \int_0^{\beta}\mu d\theta$$

$$[\log_e T]_{T_2}^{T_1} = \mu\beta$$

$$\log_e\frac{T_1}{T_2} = \ln\frac{T_1}{T_2} = \mu\beta$$

Or,

$$\frac{T_1}{T_2} = e^{\mu\beta} \qquad \text{...(7.8)}$$

The above expression in terms of logarithm to the base 10 can be written as

$$2.3\log_{10}\frac{T_1}{T_2} = \mu\beta \qquad \text{...(7.9)}$$

The value of $e = 2.718$ can be used for the numerical calculations.

The above results are applicable in the case of,

(i) a flat belt passing over a pulley

(ii) a band brake

(iii) a rope wound round a circular cylindrical surface.

Ratio of Tensions : Rope and V-Belt Drive

In the case of a rope or a V-belt drive, the rim of the pulley is grooved to accommodate the rope or the V-belt. The ratio of tensions is determined as follows :

Let 2α be the angle of the groove.

The other notations used are the same as that used in the case of the flat belt.

Free-body diagram for an element ab of the belt, subtending an angle $d\theta$, is similar to the flat belt and is as shown in Fig. 7.8 (c).

But the normal reaction and the frictional force are acting in a slightly different manner than in the case of the flat belt. This can be understood by considering a cross-section of the rope or the V-belt as shown in Fig. 7.8 (a) and (b).

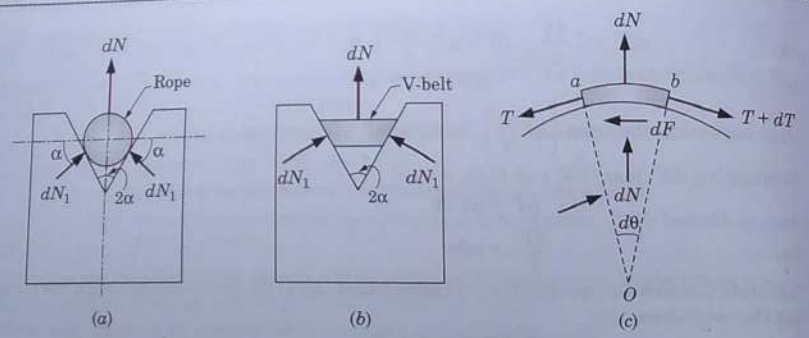


Fig. 7.8

where, dN_1 represents the normal reactions exerted by the sides of the groove on the belt or the rope

and dN is the resultant of the normal reactions dN_1 acting on the belt

$$dN = dN_1 \sin \alpha + dN_1 \sin \alpha = 2\, dN_1 \sin \alpha$$

Or,
$$dN_1 = \frac{dN}{2 \sin \alpha} \quad \text{...(7.10)}$$

The friction force
$$dF = \mu dN_1 + \mu dN_1$$
$$dF = 2\mu dN_1$$
$$dF = 2\mu \frac{dN}{2 \sin \alpha} = \mu dN \operatorname{cosec} \alpha$$
$$dF = (\mu \operatorname{cosec} \alpha)\, dN \quad \text{...(7.11)}$$

In the equations (7.3) and (7.4) derived in the case of the flat belt we substitute,

$$dF = (\mu \operatorname{cosec} \alpha)\, dN \quad \text{...(7.12)}$$

and can obtain the following results for ropes and V-belts :

$$\frac{T_1}{T_2} = e^{(\mu \operatorname{cosec} \alpha)\beta}$$

or
$$\log_e \frac{T_1}{T_2} = (\mu \operatorname{cosec} \alpha)\, \beta \quad \text{...(7.13)}$$

or
$$2.3 \log_{10} \frac{T_1}{T_2} = (\mu \operatorname{cosec} \alpha)\, \beta \quad \text{...(7.14)}$$

7.4 CENTRIFUGAL TENSION

A belt running over a pulley experiences a centrifugal force similar to what a body experiences while moving in a circular path. This centrifugal force changes the values of the tensions on the two sides of the belt *but it comes into play only when the belt is moving.*

Consider a small element ab of the belt subtending an angle $d\theta$ at the centre of the pulley.

Let

F_c = centrifugal force acting on the element ab

w = weight of the belt per unit length

v = linear velocity of the belt

r = radius of the pulley

T_c = additional tension due to the centrifugal force acting on the belt at A and B.

This is also the tension in the element ab.

Length of the element $ab = rd\theta$

Weight of the element $= w\,(rd\theta)$

Centrifugal force $= \dfrac{mv^2}{r}$

Centrifugal force acting on the element ab of the belt

$$= \frac{(wrd\theta)}{g} \frac{v^2}{r}$$

or
$$F_c = \frac{wv^2}{g}\, d\theta$$

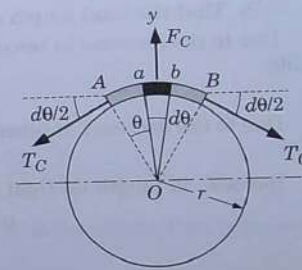


Fig. 7.9

Centrifugal force and centrifugal tensions acting on the element ab must be in equilibrium.

Resolving the forces,

$\Sigma F_x = 0$:
$$T_c \cos \frac{d\theta}{2} - T_c \cos \frac{d\theta}{2} = 0 \quad \text{...(7.15)}$$

$\Sigma F_y = 0$:
$$F_c - 2T_c \sin \frac{d\theta}{2} = 0$$

or
$$\frac{wv^2}{g}\, d\theta - 2T_c \sin \frac{d\theta}{2} = 0 \quad \text{...(7.16)}$$

For small values of angle $d\theta$

$$\sin \frac{d\theta}{2} \approx \frac{d\theta}{2}$$

Substituting in (7.16)
$$\frac{wv^2}{g} d\theta - 2T_c \frac{d\theta}{2} = 0$$

$$T_c = \frac{wv^2}{g} \quad \text{...(7.17)}$$

The following points may be noted about the equation 7.16.

1. Centrifugal tension is the additional tension in the belt due to the centrifugal force. So when the belt is in motion;

   The tension on the tight side = $T_1 + T_c$

   The tension on the slack side = $T_2 + T_c$

2. Centrifugal tension T_c depends only on the weight of the belt per unit length and speed of the belt, so the expression for T_c is same for a flat belt, a V-belt and a rope.

7.5 INITIAL TENSION IN THE BELT

When the belt is laid on the pulleys and is stationary, the tensions on the two sides of the belt are same. This tension T_0 is called the initial tension in the belt. Some initial tension has to be given to the belt so that it grips the pulleys. As the belt comes into motion, tensions on the two sides become different. They are called driving tensions and are then denoted by T_1 and T_2.

Also, the length of the belt changes due to the applied force. Let,

α = Coefficient of increase in length of belt per unit force.

Let us assume,

(i) That the material of the belt is perfectly elastic

(ii) That the total length of the belt remains unchanged.

Due to the increase in tension from T_0 to T_1, the increase in length of the belt on the tight side

$$= \alpha(T_1 - T_0).$$

Due to the decrease in tension from T_0 to T_2, decrease in length of the belt on the slack side

$$= \alpha(T_0 - T_2).$$

Increase in length on tight side = Decrease in length on the slack side

$$\alpha(T_1 - T_0) = \alpha(T_0 - T_2)$$

$$T_0 = \frac{T_1 + T_2}{2} \qquad ...(7.18)$$

7.6 POWER TRANSMITTED BY BELTS

Let the pulley A drive the pulley B and T_1 and T_2 be driving tensions (Fig. 7.10). Turning moment acting on the pulley $B = (T_1 - T_2)r_2$ N-m

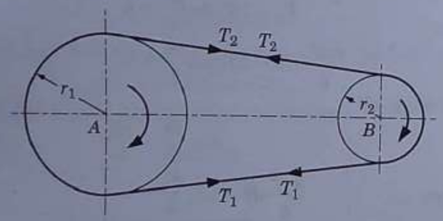


Fig. 7.10

If ω_2 is the angular velocity of the pulley B then,

Work done per second = $(T_1 - T_2)r_2\omega_2$ watt

Work done per second by the belt = $(T_1 - T_2)v$ watt ...(7.19)

where, v is the velocity of the belt. Note here that the centrifugal tension does not affect the power transmitted as,

$$(T_1 + T_c) - (T_2 + T_c) = T_1 - T_2$$

Condition for the Transmission of Maximum Power

Power transmitted by the belt

$$P = (T_1 - T_2)v$$

If $$\frac{T_1}{T_2} = K$$

then $$P = T_1\left(1 - \frac{1}{K}\right)v$$

Maximum permissible tension in the belt

$$T_m = T_1 + T_c$$

where, $$T_c = \frac{w}{g}v^2 \quad \text{(centrifugal tension)}$$

Therefore, $$T_1 = T_m - T_c$$

$$P = (T_m - T_c)\left(1 - \frac{1}{K}\right)v = \left(T_m - \frac{wv^2}{g}\right)\left(1 - \frac{1}{K}\right)v$$

To determine the condition for transmitting maximum power P, differentiate P with respect to v and set it to zero.

$$\frac{dP}{dv} = \left(1 - \frac{1}{K}\right)\left(T_m - \frac{3wv^2}{g}\right) = 0$$

$$T_m = 3\frac{wv^2}{g} = 3T_c$$

$$T_m = 3T_c$$

or $$T_c = \frac{T_m}{3} \qquad ...(7.20)$$

The power transmitted is maximum when the centrifugal tension in the belt is one-third the maximum permissible tension T_m in the belt.

$$T_c = \frac{T_m}{3}$$

But $$T_m = T_1 + T_c$$

$$T_1 = T_m - T_c = T_m - \frac{T_m}{3} = \frac{2}{3}T_m$$

$$P_{max} = T_1\left(1 - \frac{1}{K}\right)v$$

$$P_{max} = \frac{2}{3}T_m\left(1 - \frac{1}{K}\right)v \text{ watts.} \qquad ...(7.21)$$

The maximum tension that can be permitted in a belt shall depend upon the mechanical strength of the belt.

If σ_m = Maximum permissible stress allowed in the belt

then $T_m = \sigma_m \times$ (The area of cross-section of the belt). ...(7.22)

Summary of Results:

Driving Tensions are T_1 and T_2

Initial Tension $T_0 = \dfrac{T_1 + T_2}{2}$

Centrifugal Tension $T_c = \dfrac{w}{g} v^2$.

where w = weight of the belt per unit length.

This formula is applicable to a flat belt, a V-belt and a rope. Maximum Permissible Tension in the belt

$$T_m = T_1 + T_c$$

Based on mechanical strength,

$$T_m = \sigma_{max} \times \text{(Area of cross-section of the belt)}$$

where, σ_{max} = Maximum permissible stress in the belt

Ratio of driving tensions

For rope/V-belt	*For flat belt*
$\dfrac{T_1}{T_2} = e^{(\mu \operatorname{cosec} \alpha)\beta}$	$\dfrac{T_1}{T_2} = e^{\mu\beta}$
or $\log_e \dfrac{T_1}{T_2} = (\mu \operatorname{cosec} \alpha)\beta$	or $\log_e \dfrac{T_1}{T_2} = \mu\beta$
or $2.3 \log_{10} \dfrac{T_1}{T_2} = (\mu \operatorname{cosec} \alpha)\beta$	or $2.3 \log_{10} \dfrac{T_1}{T_2} = \mu\beta$

β = Angle of lap or angle of contact *in radians.*

2α = Angle of groove *in degrees.*

When two pulleys connected by a rope or belt are of unequal diameters the *minimum value of the angle of lap 'β' should be used in the formula for driving tensions. Because, the slip will occur first on the pulley with the minimum angle of lap.*

Power transmitted by the belt

$$P = (T_1 - T_2)v \text{ watts}$$

Condition for transmitting maximum power is

$$T_c = \frac{T_m}{3}$$

or $$T_1 = \frac{2}{3} T_m$$

Example 7.3 A rotating drum of radius r braked by a device as shown in Fig. 7.11. Calculate the braking moment with respect to the point C. Assume the kinetic friction between the drum and the brake shoe to be μ.

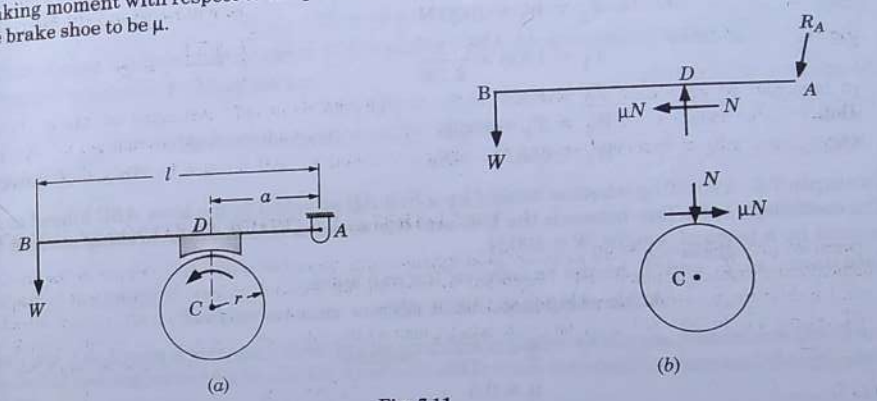


Fig. 7.11

Solution: Consider the free-body diagram of the lever AB. R_A is the reaction at the hinge, N is the normal reaction and μN is the frictional force.

Taking moments about A,

$\Sigma M_A = 0$: $\quad W(l) - N(a) = 0$

Or, $\quad N = \dfrac{Wl}{a}$

Frictional force $\quad \mu N = \mu\left(\dfrac{Wl}{a}\right)$

Taking moment of this frictional force with respect to C gives braking moment

$$M = \left(\frac{\mu Wl}{a}\right) r. \quad \textit{Ans.}$$

Example 7.4 A belt supports two weights W_1 and W_2 over a pulley as shown. If W_1 = 1000 N, find the minimum weight W_2 to keep W_1 in equilibrium. Assume that the pulley is locked and $\mu = 0.25$.

Solution: Let the tensions in the belt be T_1 and T_2 as shown. Since the weight W_2 just checks the tendency of weight W_1 to move down, tension on the side of W_1 is larger.

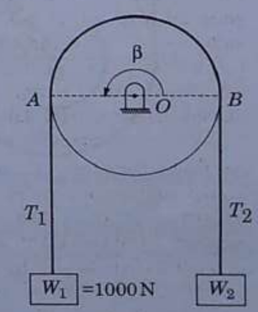


Fig. 7.12

That is, $\quad T_1 > T_2$

$\mu = 0.25, \ \beta = \pi,$

$W_1 = 1000$ N

Using $\quad \dfrac{T_1}{T_2} = e^{\mu\beta}$

$= 2.718^{\pi(0.25)}$

$$\frac{T_1}{T_2} = 2.193.$$

But $\quad T_1 = W_1 = 1000 \text{ N}$

$$T_2 = 1000 \times \frac{1}{2.193}$$

$$T_2 = 456 \text{ N}$$

But $\quad W_2 = T_2 = 456 \text{ N}$

Or, $\quad W_2 = 456 \text{ N. } \textit{Ans.}$

Example 7.5 A rotating wheel is braked by a belt AB attached to the lever ABC hinged at B. The coefficient of friction between the belt and the wheel is 0.5. Find the braking moment M exerted by a vertical weight $W = 100$ N.

Solution: Let T_1 and T_2 be the tension on the two sides.

$$\text{Angle of lap } \beta = 180° + 45°$$

$$\beta = \frac{225 \times \pi}{180} \text{ radians}$$

$$\mu = 0.5$$

$$\frac{T_1}{T_2} = e^{0.5 \cdot \frac{225 \cdot \pi}{180}}$$

$$\frac{T_1}{T_2} = 7.12 \quad ...(i)$$

Considering the free-body diagram of the lever and taking moments about B.

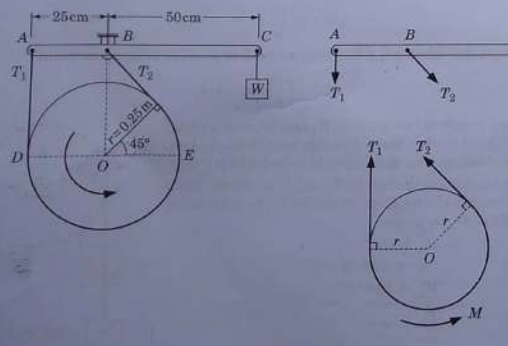


Fig. 7.13

$\Sigma M_B = 0:$

$$T_1 \times (0.25) - W(0.5) = 0$$

$$T_1 = 2W$$

Or, $\quad T_1 = (2 \times 100) \text{ N} \quad ...(ii)$

Considering the free-body diagram of the pulley and taking moments about O.

Applied moment = Braking moment

$$M = (T_1 r - T_2 r)$$

$$\text{Braking moment} = r(T_1 - T_2)$$

Using (i) and (ii)

$$\text{Braking moment} = rT_1\left(1 - \frac{1}{7.12}\right)$$

$$= 0.25(2 \times 100)\left(1 - \frac{1}{7.12}\right)$$

$$\text{Braking moment} = 42.98 \text{ N-m} \quad \textit{Ans.}$$

Example 7.6 A rope is wrapped three turns around a cylinder as shown. Determine the force required to just support a weight of 1 kN. Coefficient of friction between the rope and the cylinder is 0.30.

Solution: Angle of lap $\beta = 2\pi \times 3 = 6\pi$ radian for 3 turns of the rope.

The rope is wound on a surface with no groove so,

$$2\alpha = 180°$$

$$\alpha = 90°$$

Or, $\quad \text{cosec } \alpha = 1$

Using, $\quad \frac{T_1}{T_2} = e^{(\mu \text{ cosec } \alpha)\beta}$

where T_1 is the larger tension in that section of the rope which is about to slip.

That is, $\quad T_1 = W = 1 \text{ kN} = 1000 \text{ N.}$

$$T_2 = F$$

Therefore, $\quad \frac{1000}{F} = e^{(0.30 \times 1 \times 6\pi)}$

$$= 2.718^{(0.3 \times 1 \times 6\pi)}$$

$$\frac{1000}{F} = 285.5$$

$$F = \frac{1000}{285.5} = 3.5 \text{ N} \quad \textit{Ans.}$$

It can be seen here that a weight of 1000 N can be checked from slipping down by applying a very small force (3.5 N) because of the presence of frictional force.

Example 7.7 Find the number of ropes required to transmit 50 kW. The maximum permissible tension in the rope is 1500 N, velocity of the rope is 10 m/s and the weight of the rope is 4 N/m. Assume the angle of contact as 180° and pulley groove angle as 60°.

Solution: $T_{max} = 1500 \text{ N}, \beta = 180° = \pi$ radian

$$2\alpha = 60°, \alpha = 30°, \mu = 0.2, w = 4 \text{ N/m}, v = 10 \text{ m/s}$$

Centrifugal tension $T_c = \frac{w}{g}v^2 = \frac{4 \times (10)^2}{9.81} = 40.8 \text{ N}$

$$T_1 = T_{max} - T_c = 1500 - 40.8 = 1459.20 \text{ N}$$

Using $\frac{T_1}{T_2} = e^{(\mu \operatorname{cosec} \alpha)\beta}$

$$\frac{T_1}{T_2} = e^{(0.2 \operatorname{cosec} 30°)\pi} = 2.718^{1.257} = 3.514$$

$$\frac{T_1}{T_2} = 3.514,\ T_2 = \frac{T_1}{3.514} = \frac{1459.2}{3.514} = 415.4 \text{ N}$$

$$T_2 = 415.4 \text{ N}$$

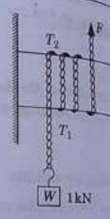


Fig. 7.14

Power transmitted by a single rope

$$P = (T_1 - T_2)v = (1459.2 - 415.4)10.0$$

$$P = 10.44 \text{ kW}$$

Number of ropes required to transmit 50 kW

$$= \frac{50.0}{10.44} = 4.79$$

Whole number of ropes = 5. *Ans.*

Example 7.8 Find the power transmitted by a cross belt drive connecting two pulleys of 45.0 cm and 20.0 cm diameters which are 1.95 m apart. The maximum permissible tension in the belt is 1 kN, coefficient of friction is 0.20 and the speed of larger pulley is 100 r.p.m.

Solution: $\mu = 0.2,\ r_1 = \frac{45}{2} \text{ cm} = 0.225 \text{ m}$

$$r_2 = \frac{20}{2} \text{ cm} = 0.1 \text{ m}$$

Let us calculate the angle of lap.

For cross-drive, angle of lap for both the pulleys is the same.

$$\beta = 180 + 2\theta$$

where $\theta = \sin^{-1}\left(\frac{r_1 + r_2}{d}\right) = \sin^{-1}\left(\frac{0.225 + 0.10}{1.95}\right)$

Or, $\theta = 9.6°$

$$\beta = 180° + 2(9.6) = 199.2°$$

$$\beta = 199.2 \times \frac{\pi}{180} = 3.477 \text{ radian}$$

$$T_{max} = T_1 + T_c = T_1 = 1 \text{ kN} = 1000 \text{ N}$$

$$\frac{T_1}{T_2} = e^{\mu\beta} = e^{0.2 \times 3.477}$$

(As the data for calculating T_c is not given, so we assume $T_c = 0$)

$$\frac{T_1}{T_2} = 2.718^{(0.2 \times 3.477)}$$

$$\frac{T_1}{T_2} = 2.004,\quad T_1 = 1000 \text{ N}$$

$$T_2 = \frac{1000}{2.004} = 499 \text{ N}$$

$$\text{Power} = (T_1 - T_2)v$$

Velocity of the belt $= v = \omega r = \frac{2\pi N}{60} \times r$

The values of N and r must refer to the *same* pulley.

Speed of the 45 cm diameter pulley is 100 r.p.m.

Therefore, $v = \frac{2\pi \times 100}{60} \times 0.225 = 2.36 \text{ m/s}$

$$P = (1000 - 449)2.36$$

$$P = 1.81 \text{ kW}. \quad \textit{Ans.}$$

Example 7.9 A belt 100 m wide and 8.0 mm thick is transmitting power at a belt speed of 1600 m/minute. The angle of lap for the smaller pulley is 165° and the coefficient of friction is 0.3. The maximum permissible stress in the belt is 2 MN/m^2 and the mass of the belt is 0.9 kg/m. Find the power transmitted and the initial tension in the belt.

Find the maximum power that can be transmitted and the corresponding belt speed.

Solution: Area of cross-section of the belt

$$= \left(\frac{100}{1000} \times \frac{8}{1000}\right) \text{m}^2$$

$$\mu = 0.3,\quad \sigma_{max} = 2 \text{ MN/m}^2$$

$$v = 1600 \text{ m/min.},\ v = 26.67 \text{ m/s}$$

$$m = \frac{w}{g} = 0.9 \text{ kg/m}$$

$$\beta = 165° = \frac{6 \times \pi}{180} \text{ radian}$$

$$\frac{T_1}{T_2} = e^{\mu\beta} = 2.718^{\left(0.3 \times \frac{165 \times \pi}{180}\right)}$$

$$\frac{T_1}{T_2} = 2.372$$

$$T_c = \frac{w}{g}v_2 = mv^2$$

$$T_c = 0.9 \times (26.67)^2 = 640 \text{ N}$$

$$T_{max} = \sigma_{max} \times (\text{Area of cros-section of the belt})$$

$$T_{max} = 2 \times 10\left(\frac{100}{1000} \times \frac{8}{1000}\right) = 1600 \text{ N}$$

$$T_1 = T_{max} - T_c = 1600 - 640$$

$$T_1 = 960 \text{ N}$$

$$T_2 = \frac{T_1}{2.373} = \frac{960}{2.372}$$

$$T_2 = 404.72 \text{ N}$$

$$\text{Power transmitted} = (T_1 - T_2)v$$

$$= (960 - 404.72)26.67$$

$$= 14809.1 \text{ watt}$$

$$P = 14.81 \text{ kW}$$

$$\text{Initial Tension} = \frac{T_1 + T_2}{2} = \frac{960 + 404.72}{2}$$

$$T_0 = 682.4 \text{ N} \quad \textit{Ans.}$$

To Transmit Maximum Power

$$T_c = \frac{T_{max}}{3} = \frac{1600}{3} \text{ N}$$

Also,

$$T_c = mv^2 = 0.9(v)^2$$

Equating,

$$v = \sqrt{\frac{1600}{3 \times 0.9}} = 24.34 \text{ m/s}$$

$$T_1 = T_{max} - T_c = 1600 - \frac{1600}{3} = 1066.67 \text{ N}$$

$$T_2 = \frac{T_1}{2.372} = \frac{1066.67}{2.372} = 449.7 \text{ N}$$

Maximum Power is Transmitted at 24.34 m/s of the belt speed

$$P_{max} = (1066.67 - 449.7)24.34$$

$$P_{max} = 14.99 \text{ kW}. \quad \textit{Ans.}$$

7.7 FRICTION IN A SQUARE THREADED SCREW

Consider a screw jack consisting of square threaded screw and a base. The screw carries a load W which can be raised or lowered by turning a handle of length l with a force F.

Let, the mean radius of the thread $= r$

and the pitch of the thread $= p$

The *pitch* is the axial distance measured between two consecutive threads.

The *lead* is the axial distance through which the screw advances in one turn.

Lead and pitch are equal in the case of a single threaded screw.

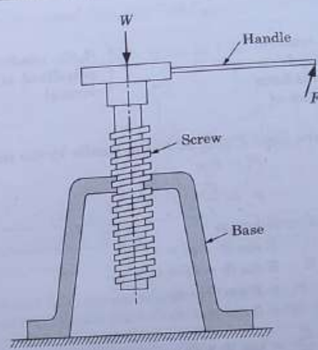


Fig. 7.15

To analyse it mathematically, let us unwrap the threads on the nut (fixed in the base) and treat it as an inclined plane of inclination

$$\tan\theta = \frac{p}{2\pi r} \qquad ...(7.23)$$

The screw moving upon this inclined plane can be treated as a body having a weight equal to load W supported by the screw (Fig. 7.15).

Let the coefficient friction between the screw and base $= \mu$

Angle of friction $\phi = \tan^{-1}\mu$

Raising of Load. When the screw has a tendency to move up friction acts downwards as shown in the free-body diagram (Fig. 7.16)

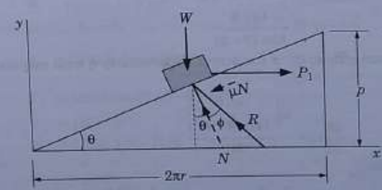


Fig. 7.16

Forces acting are :

(i) Load W
(ii) N is the normal reaction
(iii) μN is the frictional force
(iv) P_1 is the horizontal force acting at the surface of the screw

R, the resultant reaction (of N and μN) is inclined at angle $(\theta + \phi)$ with the vertical

Force P_1 is related to the force F applied to the handle by the relation

$$Fl = P_1 r$$

$$P_1 = \frac{Fl}{r} \qquad ...(7.24)$$

Writing the equations of equilibrium

$\Sigma F_x = 0$: $\quad P_1 - R\sin(\theta - \phi) = 0$

$\Sigma F_y = 0$: $\quad W - R\cos(\theta + \phi) = 0$

$$P_1 = R\sin(\theta + \phi)$$
$$W = R\cos(\theta + \phi)$$

Eliminating R,
$$\frac{P_1}{W} = \tan(\theta + \phi)$$
$$P_1 = W\tan(\theta + \phi)$$
$$P_1 = W\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \qquad ...(7.25)$$

where
$$P_1 = \frac{Fl}{r}$$

$$\text{Efficiency of the screw} = \frac{\text{Ideal effort (when friction is zero)}}{\text{Actual effort}}$$

$$P_{1(\text{Ideal})} = W\left(\frac{\tan\theta + 0}{1 - \tan\theta(0)}\right) \text{ as } \mu = 0 \text{ so, } \tan\phi = 0$$

$$P_{1(\text{Ideal})} = W\tan\theta$$
$$P_{1(\text{Actual})} = W\tan(\theta + \phi)$$
$$\eta = \frac{\tan\theta}{\tan(\theta + \phi)} \qquad ...(7.26)$$

To find the maximum efficiency of the screw, differentiate η with respect to θ and set it equal to zero

$$\frac{d\eta}{d\theta} = \frac{d\left(\dfrac{\tan\theta}{\tan(\theta+\phi)}\right)}{d\theta} = 0$$

which on solving gives,

$$\theta = \frac{\pi}{4} - \frac{\phi}{2} \text{ and } \eta_{max} = \frac{1 - \sin\phi}{1 + \sin\phi} \qquad ...(7.27)$$

θ is also known as the helix angle of the thread.

Lowering of Load. It can be proved that the effort P_2 required to lower the load W (Fig. 7.17) is given by

$$P_{2(\text{Actual})} = W\tan(\theta - \phi) \qquad ...(7.28)$$

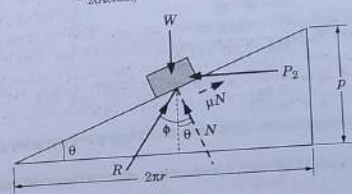


Fig. 7.17

Self Locking. When the load W on the screw remains in place even after the effort is removed (that is when effort is zero) it is called a self locking machine. It can be shown that for the screw to be self locking

$$\phi > \theta \qquad ...(7.29)$$

Example 7.10 A screw jack has square threads of mean diameter of 10.0 cm and a pitch of 1.25 cm. Determine the force that must be applied to the end of 50 cm lever (*i*) to raise (*ii*) to lower a weight of 50 kN. Find the efficiency of the jack. Is it self locking? Assume $\mu = 0.20$.

Solution. $\quad p = 1.25$ cm, $\quad d = 10.0$ cm, $\quad r = 5.0$ cm,
$l = 50$ cm, $\quad W = 50$ kN, $\quad \mu = 0.2$ cm

Helix angle
$$\tan\theta = \frac{P}{2\pi r} = \frac{1.25}{2\pi \times 5.0} = 0.039$$
$$\theta = 2.28°$$

Angle of friction $\phi = \tan^{-1}\mu = \tan^{-1} 0.2$
$$\phi = 11.31°.$$

To raise the Load

Effort required, $\quad P_1 = [W\tan(\theta + \phi)]$

Taking the leverage due to handle into account, force F_1 required,

$$F_1 = [W\tan(\theta + \phi)]\frac{r}{l}$$
$$F_1 = 5000\tan(2.28° + 11.31°) \times \frac{0.05}{0.5}$$
$$F_1 = 1205 \text{ N}$$
$$F_1 = 1.205 \text{ kN} \quad \textit{Ans.}$$

To lower the load

$$F_2 = [W\tan(\theta - \phi)]\frac{r}{l}$$
$$F_2 = [50000\tan(2.28° - 11.31°)]\frac{0.05}{0.5}$$

$$F_2 = -794.5 \text{ N} = -0.794 \text{ kN} \quad \textit{Ans.}$$

(Force required is in opposite sense)

$$\eta = \frac{\tan\theta}{\tan(\theta+\phi)}$$

$$\eta = \frac{\tan 2.28^\circ}{\tan(2.28^\circ+11.31^\circ)}$$

$$\eta = \frac{0.0398}{0.241} = 0.1646$$

$$\eta = 16.46\%.$$

The screw is self locking. As $\theta = 2.28^\circ$, $\phi = 11.31^\circ$ so $\phi > \theta$ satisfying the condition for a self locking machine.

The efficiency of the jack is 16.46% which is less than 50%, verifying further that it is a self locking machine.

7.8 DISC AND BEARING FRICTION

Power can be transmitted between two shafts A and B by virtue of the friction present between the two discs mounted on these shafts as shown in Fig. 7.18.

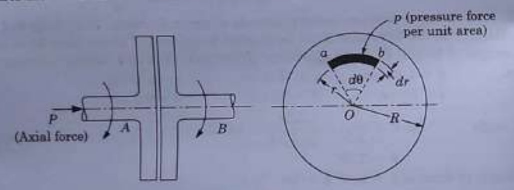


Fig. 7.18

Consider that each disc is of radius R, of area A and be under an axial force of P.

Consider an element ab of the disc at a distance r from the centre and of length $r\,(d\theta)$ having an area,

$$dA = r\,dr\,d\theta$$

Let p be the force acting on the unit area of this element.

Total force acting on the element $= pdA$

Frictional force acting on the element.

$$dF = \mu(pdA)$$

$$dF = \mu(p \,.\, r\,dr\,d\theta)$$

Moment caused by the frictional force dF acting on the area dA, about the axis of the disc.

$$dM = (\mu\, p\,r\,dr\,d\theta)r$$

$$dM = \mu\, p\,r^2 dr\,d\theta$$

The moment caused by the entire frictional force at the contact surface,

$$M = \int_0^{2\pi}\int_0^{R} \mu\, p r^2 d\theta\, dr$$

$$M = \int_0^{R}\int_0^{2\pi} \mu\, p r^2 d\theta\, dr \qquad \left\{\begin{array}{l}\text{First evaluating the integral with} \\ \text{respect to } \theta\end{array}\right.$$

$$M = 2\pi\mu \int_0^R p r^2 dr \qquad \text{...(7.30)}$$

The integration with respect to r can be carried out if we know how 'p' varies with respect to radius r.

There are two possible assumptions :

(a) Pressure p is constant or uniform

$$p = \text{constant} = \frac{P}{\pi R^2} \qquad \text{...(7.31)}$$

Therefore, equation (7.30) can be written as

$$M = 2\pi\, p\mu \int_0^R r^2 dr = \frac{p 2\pi\mu R^3}{3} \qquad \text{...(7.32)}$$

Substituting for p and expressing in terms of total force $P = p\pi R^2$

$$M = \frac{2}{3}\mu P R, \qquad \text{...(7.33)}$$

(b) Pressure p varies inversely with respect to the radius r

$$p \,\alpha \frac{1}{r} \text{ or } p = \frac{k}{r} \qquad \text{...(7.34)}$$

where k is a constant

Total pressure

$$P = \int_0^R 2\pi r p\, dr = \int_0^R 2\pi r \frac{k}{r} dr = 2\pi k R$$

$$P = 2\pi k R, \text{ or } k = \frac{P}{2\pi R} \qquad \text{...(7.35)}$$

From equation (7.34)

$$p = \left(\frac{P}{2\pi R}\right)\frac{1}{r}$$

Substituting for p in equation (7.30) gives,

$$M = 2\pi\mu \int_0^R \left(\frac{P}{2\pi R r}\right) r^2 dr$$

$$M = \frac{2\pi\mu P}{2\pi R}\int_0^R r\,dr$$

$$M = \frac{\mu P}{R}\frac{R^2}{2} = \frac{\mu PR}{R}$$

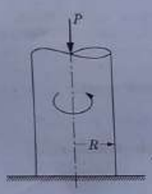
$$M = \frac{\mu PR}{2} \qquad ...(7.36)$$

Bearing Friction. The moment M required to overcome the frictional resistance in the cases of thrust and conical pivot bearings can be calculated by the same method as discussed for the disc clutch. The results are as,

Thrust Bearing

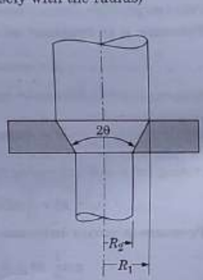
$$M = \frac{2}{3}\mu PR \qquad \text{[If pressure is uniform]} \qquad ...(7.37)$$

$$M = \frac{\mu PR}{2} \qquad \text{(If pressure varies inversely with the radius)} \qquad ...(7.38)$$



Thrust Bearing

Fig. 7.19



Conical Bearing

Fig. 7.20

Conical Pivot Bearing

$$M = \frac{2}{3}\frac{\mu P}{\sin\theta}\left[\frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}\right] \qquad \text{(Pressure is uniform)} \qquad ...(7.39)$$

Example 7.11 The end of a vertical shaft of 30 cm diameter and weighing 20 kN rests on a flat surface. If the coefficient of friction between the surfaces is 0.12 find the frictional torque assuming constant pressure.

Solution: Frictional Torque

$$T = M = \frac{2}{3}PR\mu$$

$$M = \frac{2}{3} \times (20 \times 1000) \times \left(\frac{15}{100}\right) \times (0.12)$$

$$M = 240 \text{ N-m} \quad \textit{Ans.}$$

PROBLEMS

7.1. A rope is looped around two pulleys as shown in Fig. P.7.1. Determine force P required to hold the weight of 500 N. [87.0 N]

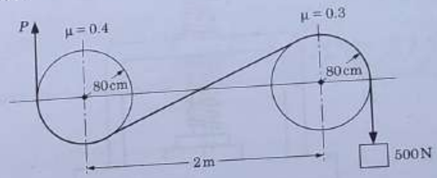


Fig. P. 7.1

7.2. What is the maximum weight W that a force $P = 500$ N will hold up if the coefficient of friction between the drum and the shoe is 0.5? [555.5 N]

7.3. Two parallel shafts 6 metres apart are fitted with 30 cm and 40 cm diameter pulleys which are connected by means of a cross belt. If the direction of rotation of the driven pulley is to be reversed by changing over to an open belt drive, determine the length of the belt that must be reduced. [8.0 cm]

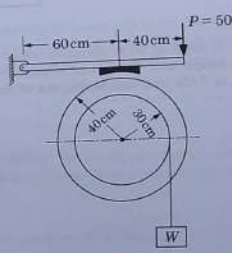


Fig. P.7.2

7.4. Two pulleys one 450 mm diameter and the other 200 mm diameter are mounted on parallel shafts 1.95 m apart. The pulleys are connected by a cross belt. Find the power that can be transmitted by the belt when the larger pulley rotates at 200 r.p.m. and if the maximum permissible tension in the belt is 1 kN. Assume the coefficient of friction between the belt and the pulley to be 0.25. Also determine the initial tension in the belt. [2.736 kW, 709.6 N]

7.5. Power is to be transmitted between two shafts by a V-belt whose mass is 0.9 kg/metre length. The angle of lap is 170° and the groove angle is 45°. Assuming the coefficient of friction between the belt and the pulley to be 0.17, find (*a*) velocity of the belt for maximum power and (*b*) power transmitted at this velocity. [28.54 m/s, 30.66 kW]

7.6. 86 kW of power is to be transmitted between two equal pulleys of 30.0 cm diameter and rotating at 1500 rpm by using an open V-belt drive. The angle of the groove is 30°, the density of the belt material is 1.2 Mg/m^3, the safe stress in the belt is 7 MN/m^2 and the coefficient of friction is 0.12. Determine the cross-sectional area of the belt that is required. [750 mm^2]

7.7. A screw jack has square threads of mean diameter 6 cm. of helix angle 10° and coefficient of friction 0.3. Determine the force that must be applied to the end of 60 cm lever to (*a*) raise (*b*) lower a weight of 3000 N. [75 N, 18 N]

7.8. A wooden block is to be compressed by a force of 18 kN by a device shown in Fig. P.7.8. The screw used has double start square threads of mean diameter 10 mm with a pitch of 2 mm and coefficient of friction 0.30. Find the force required to be applied at the end of the lever 25 cm long. [160 N]

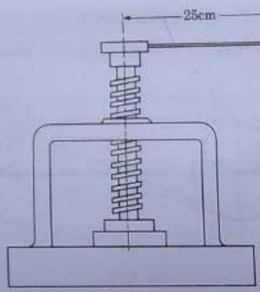


Fig. P. 7.8

7.9. A shaft of 5 cm diameter rests in a conical bearing of cone angle 60°. Calculate the frictional torque and the power required to rotate the shaft at 1000 r.p.m. if the axial load on the shaft is 5 kN and the coefficient of friction is 0.3. Assume the normal pressure to be uniform.

[50 Nm, 5.23 kW]

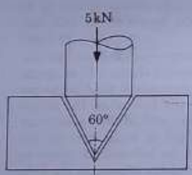


Fig. P. 7.9

8
CHAPTER
Simple Lifting Machines

8.1 INTRODUCTION

Most of us are familiar with a screw jack which enables a person to raise a heavy car while changing a tyre. A crow bar for moving heavy objects and a simple pulley which makes it easier to draw the water out of a well are some other devices which make our task easier. These devices are called simple lifting machines.

8.2 SIMPLE MACHINES AND DEFINITIONS

Simple Machine. The devices which enable us to multiply forces or to change the direction of the applied force or to gain in speed are termed as simple machines. Some commonly used simple machines are:

1. Lever
2. Inclined plane
3. Wedge
4. A pulley and a system of pulleys
5. Wheel and axle
6. Screw jack
7. Winch crab.

Compound Machine. A compound machine is a machine which is made up of a number of simple machines.

Load/Resistance. A machine is used to lift a weight or to overcome some resistance. The load lifted or the resistance overcome is denoted by W and has the unit of force.

Effort. It is the force which is to be applied to the machine to lift the load or to overcome the resistance. It is denoted by P and has the unit of force.

Input. It is the work expanded on the machine and is measured by the product of the effort and the distance through which it moves. It has the unit of N-m.

Output. It is the useful work done by the machine and is measured by the product of the load lifted by the machine the and the distance through which it moves. It has the same unit as input, that is, N-m.

Mechanical Advantage. It is the ratio of the load lifted to the effort applied, Mechanical Advantage (M.A.) = W/P.

Velocity Ratio. It is the ratio of the distance moved by the effort to the corresponding distance moved by the load.

If, the distance moved by the effort $= D$

and the distance moved by the load $= d$

$$\text{Velocity Ratio (V.R.)} = D/d$$

Efficiency of the Machine. Is defined as,

$$\eta = \frac{\text{Useful work done by the machine}}{\text{Work expanded on the machine}}$$

or

$$\eta = \frac{\text{Output of the machine}}{\text{Input to the machine}}$$

$$\text{Input to the machine} = P \times D \text{ Nm}$$

$$\text{Output of the machine} = W \times d \text{ Nm}$$

$$\text{Efficiency } \eta = \frac{W \times d}{P \times D} = \frac{W/P}{D/d} = \frac{\text{M.A.}}{\text{V.R.}}$$

$$\eta = \frac{\text{M.A.}}{\text{V.R.}} \qquad \ldots(8.1)$$

8.3 IDEAL MACHINE AND FRICTIONAL LOSSES

Ideal Machine. The maximum efficiency of a machine can be equal to 1.0 (or 100%) if there are no losses, thus giving an output equal to the input. Such a machine is termed as an ideal machine.

A large part of the losses occurring in a simple machine, are due to friction. So an ideal machine is also called as a frictionless machine. In most machines, a large resistance is overcome or a large load is lifted by a comparatively small effort, so the mechanical advantage usually is greater than one. Since the efficiency is the ratio of the machine advantage to the velocity ratio, so the velocity ratio would also be greater than one. The above fact can also be stated as "in a simple machine, a small force when applied through a large distance overcomes a large force through a small distance."

The mechanical advantage of any simple machine is always less than the mechanical advantage of its ideal model.

But, the velocity ratio of any machine and the velocity ratio of its ideal model are always equal. The velocity ratio of a machine depends upon its geometrical features and it is assumed that there is no lost motion on the input and the output sides due to slippage etc.

The mechanical advantage of a simple machine being a ratio of forces, can be determined by using the equations of equilibrium.

The velocity ratio of a simple machine is determined by giving a unit displacement to the effort and observing the corresponding displacement of the load.

In the case of an ideal machine $\eta = 1$ and M.A. = V.A. ...(8.2)

Frictional Losses. The efficiency of an actual machine is always less than 1 or the output is less than the input. It is so, because of the losses largely due to friction.

$$\text{Output} = \text{Input} - \text{Losses due to friction.}$$

In an actual machine more effort is required to overcome a given resistance than in the corresponding frictionless or ideal machine.

Let us calculate this extra effort required due to friction in the machine.

Let : P_{ideal} = Ideal effort required to overcome the resistance W

P_{Actual} = Actual effort required to overcome the same resistance W

P_{friction} = Extra effort required or the effort wasted in overcoming friction

$$P_{\text{friction}} = P_{\text{actual}} - P_{\text{ideal}}$$

$$\eta = \frac{\text{M.A.}}{\text{V.R.}}$$

For an ideal machine $\eta = 1$.

So, the effort required to overcome a given resistance W can be calculated as

$$\eta = \frac{\dfrac{W}{P_{\text{ideal}}}}{\text{V.R.}} = 1$$

Therefore,

$$P_{\text{ideal}} = \frac{W}{\text{V.R.}}$$

$$P_{\text{friction}} = P_{\text{actual}} - P_{\text{ideal}}$$

$$P_{\text{friction}} = P_{\text{actual}} - \frac{W}{\text{V.R.}} \qquad \ldots(8.3\ a)$$

Similarly, for a given effort P the actual load lifted (or actual resistance overcome) by a machine will be less than that in the case of the corresponding ideal machine.

W_{actual} = The actual resistance overcome by a given effort P

W_{ideal} = The ideal resistance overcome by the same effort P

Then the loss in load lifted due to the friction is,

$$W_{\text{friction}} = W_{\text{ideal}} - W_{\text{actual}}$$

For an ideal machine

$$\eta = 1 = \frac{\text{M.A.}}{\text{V.R.}} = \frac{\dfrac{W_{\text{ideal}}}{P}}{\text{V.R.}}$$

$$W_{\text{ideal}} = P\,(\text{V.R.}),$$

therefore,

$$W_{\text{friction}} = P\,(\text{V.R.}) - W \qquad \ldots(8.3\ b)$$

8.4 SIMPLE MACHINE : PERFORMANCE

Relation Between Effort and Resistance Overcome. For a simple machine if a graph is plotted between the load lifted W and the effort applied P, the relationship obtained is,

$$P = mW + C \qquad \ldots(8.4)$$

Above equation represents the equation of a straight line. The graph for the actual machine is represented by the line AB as shown in Fig. 8.1.

where

$$m = \text{slope of the line } AB = \frac{\Delta P}{\Delta W}$$

$$C = \text{intercept on the } P\text{-axis}$$

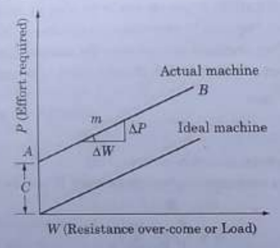


Fig. 8.1

It may be noted that some initial effort (equal to the intercept 'C' on the P-axis) is required before any load can be lifted. This represents the frictional resistance of the machine. The graph for an ideal machine, however, passes through the origin, indicating that no effort is required for the zero load.

Relation Between Efficiency and Resistance Overcome. The variation of efficiency with the load lifted is as shown in Fig. 8.2.

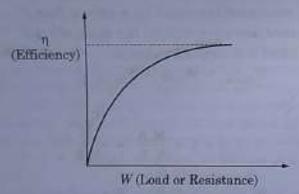


Fig. 8.2

8.5 REVERSIBILITY OF MACHINES AND SELF LOCKING MACHINES

Consider a simple pulley, as shown in Fig. 8.3 where a load W can be lifted by an effort P.

If the effort P is removed ($P = 0$) then, the load W shall move down. Such a machine in which when the effort is removed, the machine is capable of doing work in the reversed direction is called a reversible machine.

On the other hand, a car resting on a screw jack does not come down on the removal of the effort. Such machines are called as non-reversible or self-locking machines. *A non-reversible machine cannot do any work, in the reversed direction, when the effort is removed.*

For a machine to be *reversible* the output of the machine, resulting from the action of the load W, should *at least* be able to overcome the frictional resistance of the machine.

Let W be the resistance overcome through a distance d

and P be the effort applied through a distance D

Losses due to friction = Input – Output = $(P \times D - W \times D)$

When the machine is doing work under the action of the load W alone (i.e., $P = 0$) then

$$W \times d > \text{the frictional losses}$$

Or,
$$W \times d > (P \times D - W \times d)$$
$$2W \times d > P \times D$$
$$\frac{W}{P}\frac{d}{D} > \frac{1}{2}$$
$$\frac{\text{M.A.}}{\text{V.R.}} > \frac{1}{2}$$
$$\eta < \frac{1}{2}$$

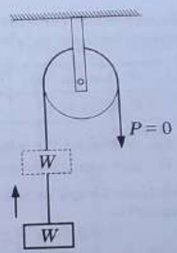


Fig. 8.3

Thus, for a machine to be reversible its efficiency should be more than 50%.

By a similar analysis it can be proved that the efficiency of a self locking machine will be less than 50%.

Example 8.1. In a machine whose velocity ratio is 6.0, and effort of 20 N was able to lift a load of 100 N. Find,

(*a*) Efficiency of the machine.
(*b*) Effort lost in friction.
(*c*) Frictional load.

Solution:

$W_{actual} = 100$ N
$P_{actual} = 20$ N

Mechanical advantage $= \frac{W}{P} = \frac{100}{20} = 5.0$

Velocity ratio $= 6.0$

$$\eta = \frac{\text{M.A.}}{\text{V.R.}} \times 100 = \frac{5}{6} \times 100$$
$$\eta = 83.33\% \quad \textit{Ans.}$$

For a load W of 100 N,

Acutal effort required $= P_{acutal} = 20$ N

Ideal effort required $= P_{ideal} = \frac{W}{\text{V.R.}} = \frac{100}{6.0} = \frac{50}{3}$ N (M.A. = V.R.)

Effort lost in friction $= (W_{actual} - W_{ideal})$

$= 20 - \frac{50}{3} = 3.33$ N *Ans.*

For an effort P of 20 N,

Actual load lifted $W_{actual} = 100$ N

Ideal load lifted $W_{ideal} = P\,(\text{V.R.}) = 20 \times 6 = 120$ N

Frictional load/resistance $= W_{ideal} - W_{actual}$

$= (120 - 100)$ N $= 20$ N *Ans.*

Example 8.2. For a lifting machine having a velocity ratio of 20, the effort required to raise various loads are as tabulated below :

W, (Load in N)	30.0	40.0	50.0	60.0	70.0	80.0	90.0	100.0
P, (Effort in N)	7.0	8.5	10.0	11.5	13.5	14.55	16.0	17.5

(a) determine the relationship between W and P
(b) plot a graph between W and η.

Solution: Graph between W and P is as shown in Fig. 8.4. The equation of the straight lined obtained is

$$P = 0.15\,W + 2.5, \text{ where } m = 0.15 \text{ and } C = 2.5 \quad \textit{Ans.}$$

(b) For plotting the graph between W and η, their values are as tabulated below. The graph obtained is shown in Fig. 8.4.

W (N)	30	40	50	60	70	80	90	100
P (N)	7.0	8.5	10.0	11.5	13.5	14.5	16.0	17.5
$\frac{W}{P}$ = M.A.	4.28	4.70	5.0	5.22	5.19	5.52	5.62	5.71
$\frac{\text{M.A.}}{\text{V.R.}} = \eta$	21.4	23.5	25.0	26.1	25.9	27.6	28.1	28.6

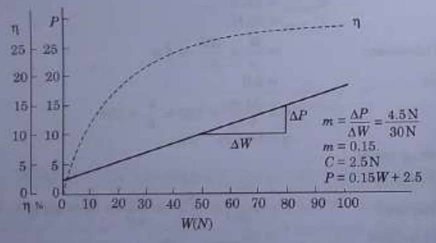


Fig. 8.4

Note here that $\frac{W}{P}$ *(M.A.) changes with load W. But the slope* $m\left(\frac{\Delta W}{\Delta P}\right)$ *of the graph between W and P is constant.*

8.6 PULLEYS AND SYSTEM OF PULLEYS

Pulleys. Pulleys can be used as single units, but to obtain a higher mechanical advantage a system consisting of several pulleys is often used. While dealing with the pulleys the following assumptions are made.

1. The weight of the pulley is negligibly small as compared to the load to be lifted.
2. The friction between the pulley and the rope is negligibly small. The tension therefore, is same through out the rope.

A Single Pulley. Let T be the tension in the string. Considering the free body of the weight.

$$T - W = 0$$
$$W = T$$

But T is also equal to P

Therefore,

$$W = P$$
$$\frac{W}{P} = \text{M.A.} = 1$$

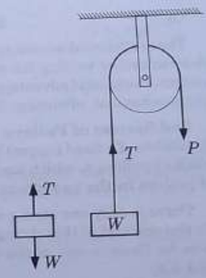


Fig. 8.5

The distance P moved by the load is same as that moved by the effort therefore,

$$\text{V.R.} = 1$$

System of Pulleys. The following three system of pulleys are commonly used :

1. 1st system of pulleys
2. 2nd system of pulleys
3. 3rd system of pulleys.

Ist System of Pulleys. In this system of pulleys there are as many number of strings as the number of pulleys. The first pulley of this system serves to change the direction of the applied force and is a fixed pulley.

Velocity Ratio. Let the rope be pulled down by the effort through a distance y. As there are two segments of the rope supporting the pulley 2, each segment shortens by an equal amount $y/2$. The centre of the pulley 2, therefore, moves up by a vertical distance equal to $y/2$. The upward movement of the pulley 2, moves the centre of the pulley 3 by $\frac{1}{2}\left(\frac{y}{2}\right)$ or $\frac{y}{2^2}$. Similarly, the centre of the pulley 4 moves up by a distance $\frac{y}{2^3}$. Therefore,

$$\text{V.R.} = \frac{\text{Distance moved by the effort}}{\text{Corresponding distance moved by the load}}$$

$$\text{V.R.} = \frac{y}{\frac{y}{2^3}} = 2^3$$

If there are 'n' movable pulleys V.R. = 2^n.

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Mechanical Advantage. Considering the system of pulleys as an ideal machine

$$\eta = \frac{\text{M.A.}}{\text{V.R.}} = 1$$

Or,
$$\text{M.A.} = \text{V.R.} = 2^n \quad ...(8.5)$$

The mechanical advantage of this system of pulleys can also be determined by writing the equations of equilibrium. For an ideal system, mechanical advantage equals velocity ratio, so we determined the mechanical advantage indirectly in this case.

2nd System of Pulleys. It is also known as the pulley block. It consists of a fixed (upper) block of pulleys and a movable (lower) block of pulleys to which load to be lifted is attached. The number of pulleys in the two blocks may or may not be equal.

There is only one string which passes around all the pulleys. To the one end of this string, effort is applied and the other end is to be fixed to one of the pulley blocks as shown in Figs. 8.7 and 8.8.

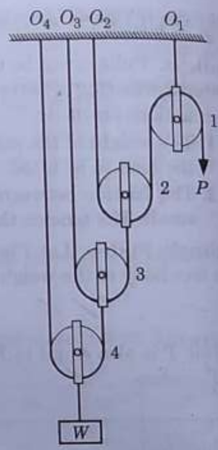
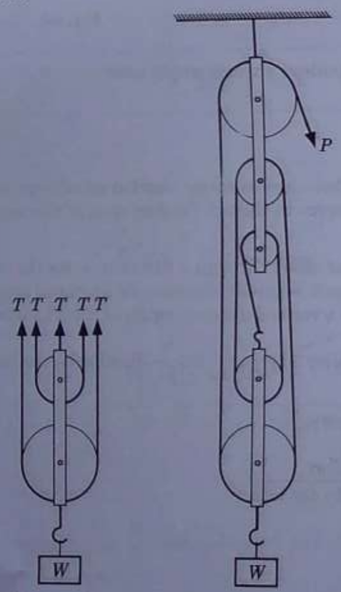
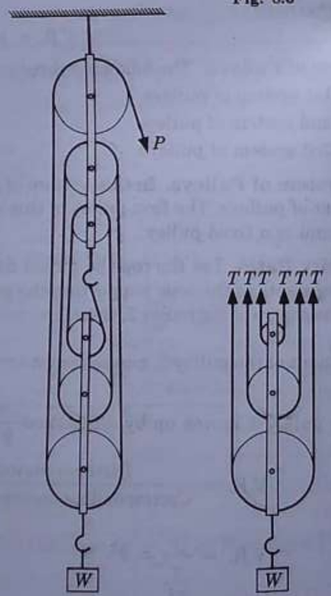


Fig. 8.6



5 segment supporting W

Fig. 8.7



6 segments supporting W

Fig. 8.8

Let n = number of the portions of the string supporting the lower block.

The tension T, in the string remains the same through out and is equal to the effort P.

Considering the equilibrium of the weight W,

$$nT - W = 0$$

$$\text{M.A.} = \frac{W}{P} = n \quad \left[\begin{array}{r} n = 5 \text{ in Fig. 8.7} \\ \text{and } n = 6 \text{ in Fig. 8.8} \end{array}\right]$$

If it is considered to be an ideal machine

$$\eta = 1 = \frac{\text{M.A.}}{\text{V.R.}}$$

Or,
$$\text{V.R.} = \text{M.A.} = n \quad ...(8.6)$$

Alternatively, velocity ratio can be found from geometrical considerations. As the load moves up by a distance y, the effort moves down by a distance ny. The value of the velocity ratio, therefore, is equal to n.

In the above analysis, the weight of the lower block of pulleys was neglected. Let us consider that the weight of the lower pulley block is w.

Writing the equation of equilibrium of the lower pulley block.

$$nT - W - w = 0$$

$$w + W = nT \quad ...(8.7)$$

Note that the velocity ratio of the system remains the same and is equal to n.

Third System of Pulleys. In this system of pulleys there are as many strings as there are number of pulleys as shown in Fig. 8.9. One end of the each string is attached to a common block which supports the load W.

The other ends of the strings are fixed to the centres of the next lower pulleys. The effort is applied to the free end of lower most pulley.

Velocity Ratio. It can be determined by giving a unit vertical displacement to the load and finding the corresponding vertical displacement of the effort. It can be shown that

$$\text{V.R.} = 2^n - 1.$$

where, n = number of pulleys.

Mechanical Advantage. It can be determined by considering the equilibriums of the load W and of the different pulleys. Assuming frictionless conditions, consider the equilibrium of the load W, we get

$$(T_1 + T_2 + T_3 \dots T_n) - W = 0$$

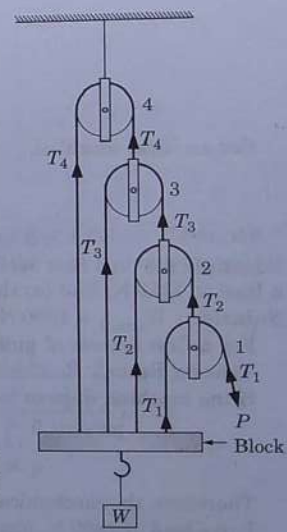


Fig. 8.9

Considering the equilibrium of pulley 1,

$$T_2 - 2T_1 = 0$$

$$T_2 = 2T_1 \qquad \left[\text{Tensions } T_1, T_2, T_3 \text{ and } T_4 \text{ are as shown in Fig. 8.9}\right]$$

As,

$$T_1 = P \text{ so, } T_2 = 2T_1 = 2P$$

Considering the equilibriums of the pulley 2,

$$T_3 - 2T_2 = 0$$

$$T_3 = 2T_2 = 2 \times 2T_1 = 4P$$

Considering the equilibrium of the pulley 3,

$$T_4 - 2T_3 = 0$$

$$T_4 = 2T_3 = 2 \times 2 \times 2 \times P$$

$$T_4 = 2^3P$$

Putting the values of T_1, T_2, T_3 ... in the equation of equilibrium of the load W (n is the number of pulleys).

$$W - (T_1 + T_2 + T_3 + ... + T_n) = 0$$

$$W = P + 2P + 2^2P + 2^3P + ...$$

$$\frac{W}{P} = 1 + 2 + 2^2 + 2^3 + ... 2^n$$

$$\frac{W}{P} = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

$$\text{M.A.} = \frac{W}{P} = 2^n - 1$$

For an ideal machine,

$$\eta = 1 = \frac{\text{M.A.}}{\text{V.R.}}$$

Or,

$$\text{V.R.} = \text{M.A.} = 2^n - 1 \qquad (8.8$$

Example 8.3 In a first system of pulleys there are 4 movable pulleys. If an effort of 100 N lifts a load of 1360 N, find (*a*) the effort wasted in friction, (*b*) the load wasted in friction.

Solution: $W_{actual} = 1360$ N and $P_{actual} = 100$ N

For a first system of pulleys having 4 movable pulleys,

Velocity Ratio V.R. $2^n = 2^4 = 16$

If the machine were to be ideal (frictionless)

$$\eta = 1 \text{ or } 100\%$$

$$\eta = \frac{\text{M.A.}}{\text{V.R.}} = 1 \text{ or, M.A.} = \text{V.R.} = 16$$

Therefore, the mechanical advantage of ideal pulley system = 16

For a load of 1360 N, ideal effort required,

$$P_{ideal} = \frac{W}{16} = \frac{1360}{16} \qquad \left(\text{M.A.} = \frac{W}{P}\right)$$

$$P_{ideal} = 85 \text{ N}$$

$$\text{Effort lost in friction} = P_{actual} - P_{ideal} = 100 - 85$$

$$P_{friction} = 15 \text{ N.} \quad \textit{Ans.}$$

For an effort of 100 N, ideal load lifted,

$$W_{ideal} = 16 \times 100$$

$$W_{ideal} = 1600 \text{ N}$$

$$\text{Load lost in friction} = W_{ideal} - W_{actual} = 1600 - 1360$$

$$W_{friction} = 240 \text{ N} \quad \textit{Ans.}$$

Example 8.4 In a second system of pulleys there are three pulleys in the upper block and two pulleys in the lower block. If the efficiency of the pulley system is 75%, find the effort required to lift a load of 1000 N.

Solution: For the 2nd system of pulleys,

Velocity Ratio V.R. = n

where, n is the number of the segments of the string supporting the lower block.

With 3 pulleys in the upper block and 2 pulleys in the lower block, $n = 5$.

Therefore,

$$\text{V.R.} = 5$$

$$\eta = \frac{\text{M.A.}}{\text{V.R.}}$$

$$\text{M.A.} = \eta \times \text{V.R.}$$

$$\text{M.A.} = \frac{75}{100} \times 5 = 3.75$$

As

$$\text{M.A.} = \frac{W}{P} \text{ and } W = 1000 \text{ N}$$

$$P = \frac{W}{\text{M.A.}} = \frac{1000}{3.75}$$

$$P = 266.67 \text{ N} \quad \textit{Ans.}$$

8.7 WHEEL AND AXLE

It is device to raise heavy loads. It consists of two cylinders A and B of different diameters. The bigger cylinder A is called the wheel, and the smaller cylinder B the axle and they rotate about a common axis (Fig. 8.10).

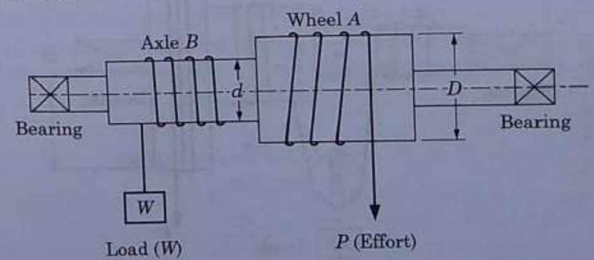


Fig. 8.10

A string is wound round the axle. The one end of this string is fixed to the axle and the other is attached to the load W.

Another string is wound round the wheel. One end of which is fixed to the wheel and to the other, the effort is applied.

The two strings are wound in opposite directions. So, when the effort P pulls the string down it unwinds and turns the wheel. This causes the string on the axle to wind and to lift the load up.

Let
D = Diameter of the wheel
d = Diameter of the axle
W = Load lifted
P = Effort applied

When the wheel and axle turn through one revolution,
distance through which load moves $= \pi d$
distance through which effort moves $= \pi D$

$$\text{V.R.} = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}} = \frac{\pi D}{\pi d}$$

$$\text{V.R.} = \frac{D}{d} \qquad \text{...(8.9)}$$

$$\text{M.A.} = \frac{W}{P}$$

$$\eta = \frac{W/P}{D/d}$$

8.8 DIFFERENTIAL WHEEL AND AXLE

The axle is made up of two cylinders B and C of different diameters. The wheel A and the cylinders B and C turn about a common axis (Fig. 8.11).

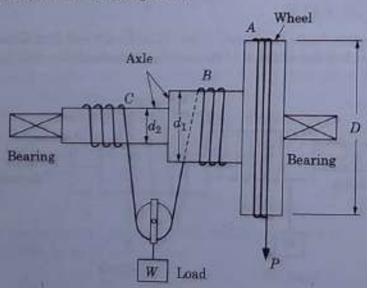


Fig. 8.11

One string is wound round the wheel A and to the one end of which the effort P is applied. The second string is wound round the two cylinders B and C. This string goes around a pulley to which the load W is attached. This string is wound on the two cylinders in such a way that as the cylinders turn, it unwinds on the smaller cylinder and winds at the same time on the larger cylinder, lifting the load W attached to the pulley.

For one revolution of the wheel and axle, the displacement of the effort P is equal to the length of the string that unwinds from the wheel $A = \pi D$.

The length of the string that unwinds from the cylinder $C = \pi d_2$

The length of the string that winds on the cylinder $B - \pi d_1$

The load string shortens by $= \pi d_1 - \pi d_2$

Therefore, the displacement of the load W

$$= \frac{\pi d_1 - \pi d_2}{2}$$

$$\text{V.R.} = \frac{\pi D}{\frac{1}{2}(\pi d_1 - \pi d_2)} = \frac{2D}{d_1 - d_2} \qquad \text{...(8.10)}$$

For a larger velocity ratio, d_1 and d_2 are made nearly equal.

8.9 DIFFERENTIAL PULLEY BLOCK

It consists of a block of two pulleys A and B of diameters D and d respectively, which turn about a common axis through O.

A movable pulley C is attached to the load W. A single string passes around the pulleys as shown. The effort pulls one part of the loop unwinding the string from the pulley A and winding it at the same time on the pulley B. This results in the lengthening of the string on the effort side and shortening of the string on the load side, raising the load W (Fig. 8.12).

In one revolution of the bigger pulley A, the length of string pulled is the displacement of the effort $P = \pi D$

The length of string released from the smaller pulley B in one revolution $= \pi d$

Therefore, the string on the load side shortens by $(\pi D - \pi d)$.

This shortening of the string is divided equally between the two segments of the string supporting the pulley C hence, the displacement of load $= \frac{\pi(D-d)}{2}$

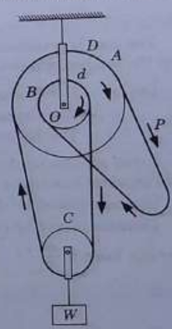


Fig. 8.12

$$\text{V.R.} = \frac{\pi D}{\frac{1}{2}(\pi D - \pi d)}$$

$$= \frac{2D}{(D-d)} \qquad \text{...(8.11)}$$

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8.10 WORM AND WORM WHEEL

If consists of square threaded screw B (called worm) which is engaged to a toothed wheel C (called the worm wheel.)

The effort P is applied to the wheel A which rotates the worm B attached to it (Fig. 8.13).

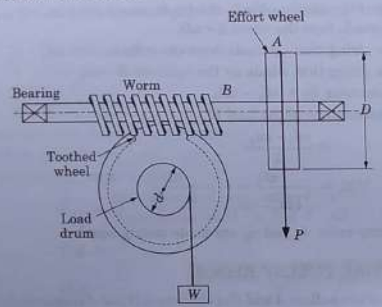

Fig. 8.13

The load is attached to the string wound round the load drum. The load drum is mounted on the toothed wheel C and rotates with it as a single unit.

Let D = Diameter of the effort wheel

d = Diameter of the load drum

T = Number of teeth on the worm wheel

Let us give one revolution to the effort wheel A. This will also rotate the worm by one revolution. If the worm has single start threads, it will move the worm wheel C by one teeth.

Distance moved by the effort in one revolution of the effort wheel = πD

Distance moved by the load in one revolution of the worm, which advances the load drum by one tooth = $\frac{\pi d}{T}$

$$\text{V.R.} = \frac{\pi D}{\pi d/T} = \frac{TD}{d} \quad \text{...(8.12)}$$

For double start threads on the worm, the worm wheel will move by two teeth, hence

$$\text{V.R.} = \frac{\pi D}{2\left(\frac{\pi d}{T}\right)} = \frac{TD}{2d} \quad \text{...(8.13)}$$

8.11 SIMPLE SCREW JACK

It is a lifting device consisting of a square threaded screw which can turn in a fixed nut which generally forms a part of the body of screw jack. The screw has a head on its upper end on which the load W rests [Fig. 8.14(a)].

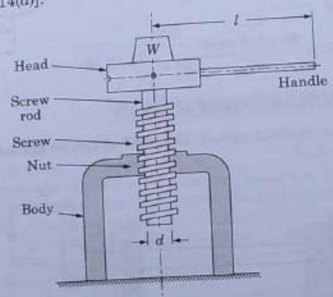

Fig. 8.14 (a)

The effort, to turn the screw, is applied to handle attached to the screw, thereby, raising or lowering the load resting on the head of the screw.

Let P = effort required at the circumference of the screw

F = effort applied at the handle

W = load lifted

p = pitch of the screw

l = length of the handle

Distance moved by the effort (F) in turning the screw, by means of handle, through one revolution = $2\pi l$.

The vertical distance moved by the load = pitch of the screw = p

$$\text{V.R.} = \frac{2\pi l}{p} \quad \text{...(8.14)}$$

If the screw is treated as an inclined plane with friction, the relation between the load lifted W and the effort P required at the circumference of the threads is,

$$P = W \tan(\theta + \phi) \quad \text{(Derived earlier, see chapter 7)}$$

where, θ is helix angle of the thread given by

$$\tan\theta = \frac{p}{\pi d}$$

d is the diameter of mean pitch circle of the screw and $\phi = \tan^{-1}\mu$

Effort F applied at the handle is obtained by taking moments as,

$$F \times l = p \times \frac{d}{2}$$

$$F = \frac{p}{l} \times \frac{d}{2}$$

$$F = \frac{W \tan(\theta + \phi) d}{2l} \qquad \text{...(8.15)}$$

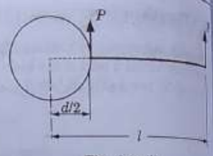

Fig. 8.14 (b)

8.12 SINGLE PURCHASE WINCH CRAB

It consists of two axles or shafts A and B. The shaft A can be turned by applying effort P at a crank of radius R (Fig. 8.15).

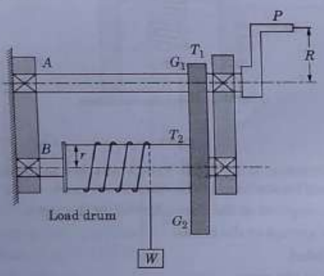

Fig. 8.15

The shaft B has a load drum on which a rope is wound which is attached to the load.

The two shafts are connected by the gears G_1 and G_2. The rotation of the crank turns the shaft B (through gears G_1 and G_2) and winds the rope on the load drum lifting the load W.

Let T_1 = Number of teeth on the gear G_1 (Pinion)
T_2 = Number of teeth on the gear G_2 (Main Gear)
R = Radius of the crank
r = Radius of the load drum
P = Effort applied at the crank
W = Load lifted.

When the crank moves through one revolution the distance moved by the effort = $2\pi R$

For the one revolution of the shaft A, the shaft B and the load drum will make $1 \times \frac{T_1}{T_2}$ revolutions.

The distance moved by the load is equal to the length of the rope wound on the load drum corresponding to $\left(1 \times \frac{T_1}{T_2}\right)$ revolutions,

$$= \frac{T_1}{T_2} \times 2\pi r$$

$$\text{V.R.} = \frac{2\pi R}{\frac{T_1}{T_2} \times 2\pi r} = \frac{R}{r}\frac{T_2}{T_1}$$

$$\text{V.R.} = \frac{R}{r}\frac{T_2}{T_1} \qquad \text{...(8.16)}$$

Example 8.5 A screw jack has square threaded screw of 5 cm diameter and 1 cm pitch. The coefficient of friction at the screw thread is 0.15.

Find the force required at the end of a 70 cm long handle to raise a load of 2000 N. What is the force required if the screw jack is considered to be an ideal machine?

Solution: $W = 2000$ N, $l = 70$ cm, $d = 5$ cm, $p = 1$ cm,

$$\mu = 0.15, \ \phi = \tan^{-1}\mu = 8.53°.$$

Helix angle, $$\tan\theta = \frac{p}{\pi d} = \frac{1}{\pi \times 5} = 0.0637$$

$$\theta = 3.64°$$

Force required (Friction present)

Force P required at the circumference of the screw

$$P = W \tan(\theta + \phi)$$
$$P = 2000 \tan(3.64 + 8.53)$$
$$P = 431.32 \text{ N}$$

Force required at the end of the handle,

$$F \times l = P \times d/2$$

or $$F = \frac{P \times (d/2)}{l}$$
$$= \frac{431.32 \times 0.25}{0.70}$$
$$F = 15.40 \text{ N} \ \textit{Ans.}$$

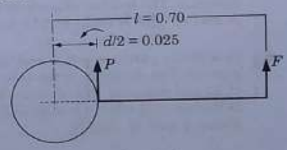

Fig. 8.16

Force required (Ideal Case)

$$\text{V.R.} = \frac{2\pi l}{p} = \frac{2\pi \times 0.7}{0.01} = 439.8$$

$$\eta = \frac{\text{M.A.}}{\text{V.R.}} = 1, \ \text{M.A.} = 439.8$$

[This expression includes the effect due to the handle.

$$\text{M.A.} = \frac{W}{P} = 439.8, \ P = \frac{W}{439.8},$$
$$P = 4.55 \text{ N}$$

Force required at the handle,

$$P = 4.55 \text{ N} \quad \textit{Ans.}$$

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PROBLEMS

8.1. In the case of a lifting machine efforts required to lift loads of 225 N and 330 N were 60 N and 77 N respectively. The velocity ratio of the machine was found to be 20. Find the relation between the load and the effort. Also determine the efficiencies at the loads of 225 N and 330 N.

$$\left[\begin{array}{l} p = 0.162\,W + 23.57 \\ \eta = 18.75\%,\ \ 21.40\% \end{array}\right]$$

8.2. In a first system of pulleys there are 5 movable pulleys. If the efficiency of the machine is 75%, what effort would be required to lift a weight of 4800 N? [200 N]

8.3. In a second system of pulleys there are 5 pulleys in each block. If an effort of 125 N can just lift a load of 1000 N find (*a*) the effort wasted in friction, (*b*) the load wasted in friction. [(*a*) 25 N, (*b*) 250 N]

8.4. There are three pulleys arranged in the third system. Find the effort required to lift a load of 500 N if the efficiency of the pulley system is 80%. [89.3 N]

8.5. A simple screw jack has threads of pitch 5 mm and handle of 45 cm length. Find its velocity ratio. If an effort of 100 N applied at the end of the handle lifts a load of 12,000 N, find the efficiency of the jack. [565.5, 21.22%]

8.6. A screw jack having square threads of mean diameter 6.25 cm and the pitch of 12.5 mm has a coefficient of friction $\mu = 0.05$. Find

(*a*) the tangential force which should be applied at the end of 30 cm long handle to raise a load of 5000 N.

(*b*) whether the jack is selflocking? If not find the torque required to keep the load from descending. [59.34 N, Not self locking, 2.13 N-m]

8.7. In a differential wheel and axle, the diameter of the wheel is 40 cm and that of the axles are 15 cm and 10 cm. The rope used is of 1 cm diameter. Find the load that can be lifted by an effort of 200 N assuming the efficiency of the machine to be 75%. [2460 N]

8.8. In a worm and worm wheel, load drum is of 15 cm diameter and the effort wheel is of 50 cm diameter. The number of teeth on the worm wheel are 25 and the worm has double start threads. A load of 3250 N could just be lifted by an effort of 300 N. Find the efficiency of the machine. [η = 26%]

8.9. In a single purchase winch crab the number of teeth on the pinion is 12 and that on the main gear 48. The diameter of the load drum is 20 cm and the length of the handle is 40 cm. A rope of 10 mm diameter was used to support the load. Determine (*a*) the velocity ratio, (*b*) efficiency of the machine when a load of 1000 N is lifted by an effort of 200 N, (*c*) effort lost in friction while lifting a load of 1000 N. [(*a*) V.R. = 15.238 (*b*) 32.8% (*c*) 134.4 N]

9 CHAPTER

Analysis of Plane Trusses and Frames

9.1 ENGINEERING STRUCTURES

These may be defined as any system of connected members built to support or transfer forces acting on them and to safely withstand these forces.

The engineering structures may be broadly divided into,

(*a*) Trusses (*b*) Frames (*c*) Machines.

Truss. It is a system of uniform bars or members (of circular section, angle section, channel section etc.) joined together at their ends by rivetting or welding and constructed to support loads. The members of a truss are straight members and the loads are applied only at the joints. Every member of a truss is a *two force member* (Fig. 9.1).

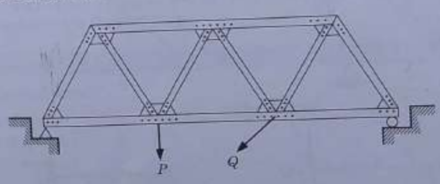
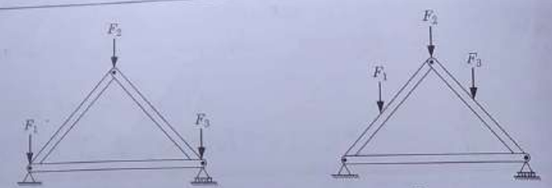


Fig. 9.1

Frame. It is structure consisting or several bars or members pinned together and in which one or more than one of its members is subjected to more than two forces. They are designed to support loads and are stationary structures.

Machine. Machines are structures designed to transmit and modify forces and contain some moving members.

We shall discuss here only the *plane structures*, that is, the structures whose members lie in one plane.



Truss
(Forces are applied only at the joints)

Frame
(Forces may act anywhere on the member)

Fig. 9.2

9.2 RIGID OR PERFECT TRUSS

The terms rigid, with reference to the truss, is used in the sense that the truss is *noncollapsible* when the external supports are removed. The Fig. 9.3 explains the concept.

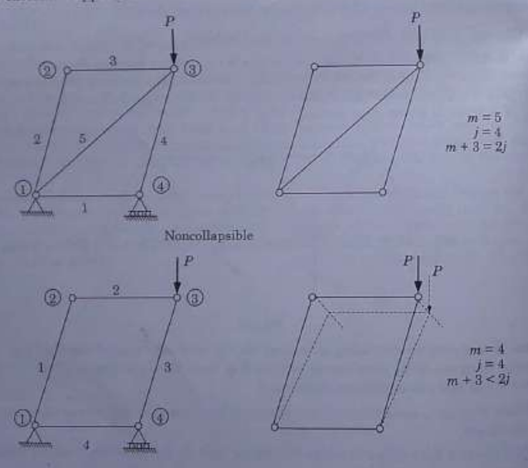


Fig. 9.3

Mathematical Condition for Rigid or Perfect Truss. A truss consists of a number of members which are connected together and form a certain number of joints. For a truss to be rigid or perfect, the relationship between its number of members and the number of joints is,

$$m + 3 = 2j$$

where, m = number of members in the truss
j = number of joints in the truss

if $m + 3 > 2j$, it means that the truss contains more members than required to be just rigid and is over *rigid and statically indeterminate*.

if $m + 3 < 2j$, it means that the truss contains less members than required to be just rigid and is *collapsible or under rigid*.

Statically Determinate. A truss is statically determinate if the equations of static equilibrium alone are sufficient to determine the axial forces in the members without the need of considering their deformations.

Basic Assumptions for the Perfect Truss

1. The joints of a simple truss are assumed to be pin connections and frictionless. The joints, therefore, cannot *resist moments*.
2. The loads on the truss are applied at the joints only.
3. The members of a truss are *straight two force members* with the forces acting collinear with the centre line of the members.
4. The weight of the members are negligibly small unless otherwise mentioned.
5. The truss is statically determinate.

The concept of a two-force member and the representation of axial forces in a member have already been explained in the chapter 2.

9.3 TRUSS: DETERMINATION OF AXIAL FORCES IN THE MEMBERS

The various methods are :

1. Method of joints
2. Method of sections
3. Graphical method.

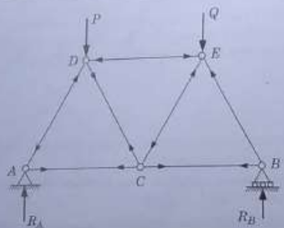
9.3.1 The Method of Joints

In this method the equilibrium of each joint is considered. The procedure is as follows :

Consider the free-body diagram of the entire truss and compute the support reactions using the equations of equilibrium. *Determination of the support reactions may not be necessary in the case of a cantilever type of truss.*

Assume and mark the directions of the axial forces in the members on the diagram as shown in Fig. 9.4 (*a*).

If in the solution the magnitude of a force comes out to be *negative* the assumed direction of the force in the member is simply *reversed*.



Representation of compression in the member EB. Arrows point towards the joints E and B (Push at the joint).

Representation of tension in the member AC. Arrows point away above from the joints A and C (Pull at the joint).

Fig. 9.4 (a)

Choose a joint [Fig. 9.4 (*b*)] and consider its free-body diagram [Fig. 9.5 (c)]. The forces acting on the joint represent a system of concurrent forces in equilibrium. Therefore, only two equations of equilibrium can be written for the each joint and can be solved to determine only two unknown force. *Therefore, start from a joint where not more than two unknown force appear.* For example, we should start from the joint A and not from the joint D [Fig. 9.4 (c)].

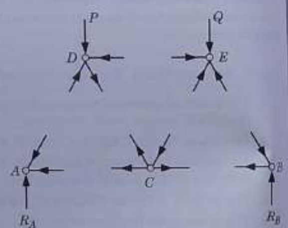


Fig. 9.4 (*b*)

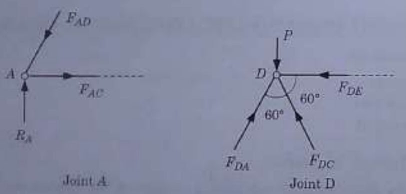


Fig. 9.4 (*c*)

Consider the equilibrium of the remaining joints.

Special Conditions

1. When two members meeting at a joint are not collinear and there is no external force acting at the joint, then the forces in both the members are zero (Fig. 9.5).

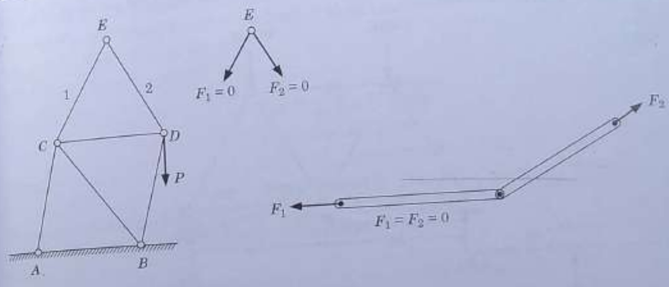


Fig. 9.5

2. When there are three members meeting at a joint, of which two are collinear and third be at an angle and if there is no load at the joint the force in the third member is zero (Fig. 9.6).

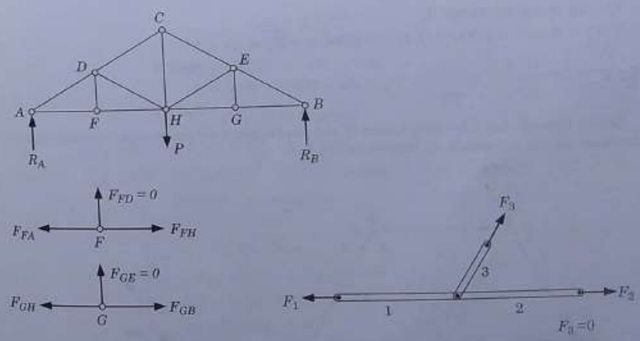


Fig. 9.6

Consider the joint F, force in the member FD is zero. Consider the joint G, force in the member GE is zero.

Example 9.1 Using the method of joints, find the axial forces in all the members of a truss with the loading shown in Fig. 9.7.

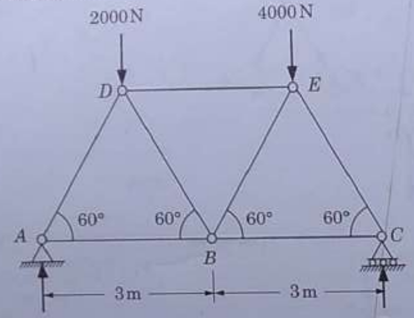


Fig. 9.7

Solution: **Entire Truss.** To determine the support reactions consider the equilibrium of the entire truss.

In general, the reaction at a hinge can have two components acting in the horizontal and the vertical directions. As there is no horizontal external force acting on the truss, so the horizontal component of reaction at A is zero.

Taking moments about A,

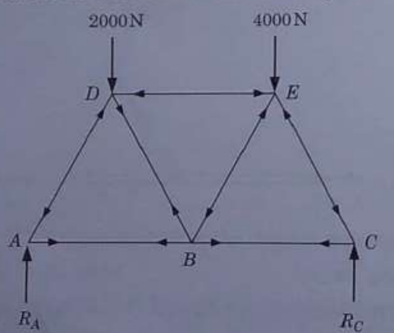
$\Sigma M_A = 0:$ $\quad -2000 \times (1.5) - 4000(4.5) + R_C \times (6) = 0$

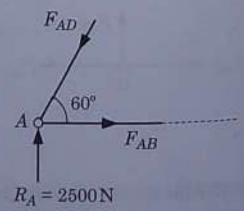
$$R_C = 3500 \text{ N}$$

$\Sigma F_y = 0:$ $\quad R_A + R_C - 2000 - 4000 = 0$

$$R_A = 2500 \text{ N}$$

Before considering the equilibrium of the joints, mark by inspection, the directions of axial forces in all the members as shown in Fig. 9.8.





Joint A

Fig. 9.8

Joint A. Let us begin with the joint A at which there are only two unknown forces. We cannot begin with the joint D because, there are three unknown forces acting at the joint D. Consider the free-body diagram of the joint A. Equations of equilibrium can be written as

$\Sigma F_x = 0:$ $\quad F_{AB} - F_{AD} \cos 60° = 0$ ...(*i*)

$\Sigma F_y = 0:$ $\quad R_A - F_{AD} \sin 60° = 0$ ...(*ii*)

$$F_{AD} = \frac{R_A}{\sin 60°} = \frac{2500}{0.866}$$

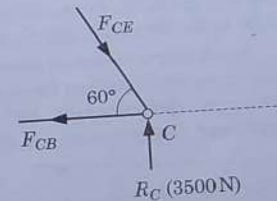
$$F_{AD} = 2887 \text{ N(C)} \quad \textit{Ans.}$$

Using (*i*)

$$F_{AB} = F_{AD} \cos 60° = 2887 \times 0.5$$

$$F_{AB} = 1443 \text{ N(T)} \quad \textit{Ans.}$$

The magnitudes of the forces F_{AB} and F_{AD} are both coming out be positive, therefore, the assumed direction of the forces are correct.



Joint C

Joint C

$\Sigma F_x = 0:$ $\quad R_{CE} \cos 60° - F_{CB} = 0$ ...(*iii*)

$\Sigma F_y = 0:$ $\quad R_C - R_{CE} \sin 60° = 0$ ...(*iv*)

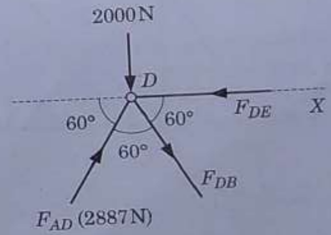
$$F_{CE} = \frac{R_C}{\sin 60°} = \frac{3500}{0.866}$$

$$F_{CE} = 4041 \text{ N(C)} \quad \textit{Ans.}$$

From (*iii*)

$$F_{CB} = \frac{F_{CE}}{\cos 60°} = \frac{4041}{0.5}$$

$$F_{CB} = 2020.5 \text{ N (T)} \quad \textit{Ans.}$$



Joint D

Joint D

$$F_{AD} = 2887 \text{ N (known)}$$

$\Sigma F_x = 0:$ $\quad F_{DB} \cos 60° + F_{AD} \cos 60° - F_{DE} = 0$ ...(*v*)

$\Sigma F_y = 0:$ $\quad F_{AD} \sin 60° - F_{DB} \sin 60° - 2000 = 0$ ...(*vi*)

$$F_{DB} = \frac{2887 \times 0.866 - 2000}{0.866}$$

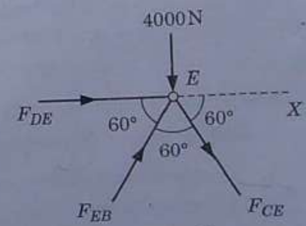
$$F_{DB} = 577 \text{ N(T)} \quad \textit{Ans.}$$

From (*v*)

$$F_{DE} = F_{DB} \cos 60° + F_{AD} \cos 60°$$

$$F_{DE} = 577 \times 0.5 + 2887 \times 0.5$$

$$F_{DE} = 1732 \text{ N(C)} \quad \textit{Ans.}$$



Joint E

Joint E

$$F_{DE} = 1732 \text{ N}$$

$$F_{CE} = 4041 \text{ N (known)}$$

$$\Sigma F_x = 0: \quad F_{DE} + F_{EB} \cos 60^\circ - F_{CE} \cos 60^\circ = 0$$

Or

$$F_{EB} \cos 60^\circ = F_{CE} \cos 60^\circ - F_{DE}$$

Or

$$F_{EB} = \frac{4041 \times 0.5 - 1732}{0.5}$$

$$F_{EB} = 577 \text{ N(C)} \quad \textit{Ans.}$$

There is no need to consider the equilibrium of the joint B as all the forces have been determined.

9.3.2 The Method of Sections

In this method, the equilibrium of a portion of the truss is considered which is obtained by cutting the truss by some imaginary section.

Consider a truss as shown in Fig. 9.9. Cut the truss into two separate portions by passing an imaginary section through those members in which forces are to be determined. The section mn cuts the members EF, BF and BC and *the internal forces in these members become the external forces acting on the two portions of the truss* as shown in Fig. 9.10.

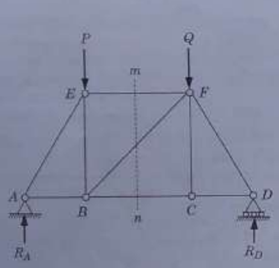


Fig. 9.9

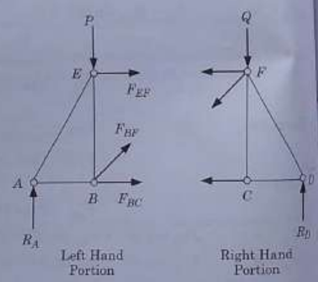


Fig. 9.10

The equilibrium of the entire truss implies that every part of the truss would also be in equilibrium. Therefore, three equations of equilibrium

$$\Sigma F_x = 0, \quad \Sigma F_y = 0 \text{ and } \Sigma M = 0$$

can be written for any one portion of the truss and can be solved to determine the three unknowns.

Following points should be noted while using the method of sections:

1. The section should be passed through the members and not through the joints.
2. A section should divide the truss into two clearly separate and unconnected portions.

3. A section should cut only three members since only three unknowns can be determined from the three equations of equilibrium. However, in special cases more than three members may be cut by a section.
4. When using the moment equation, the moment can be taken about any convenient point which may or may not lie on the section under consideration.

Example 9.2 Find the axial force in the member DE of the truss using the method of sections.

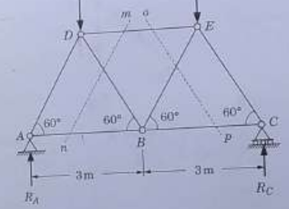


Fig. 9.11

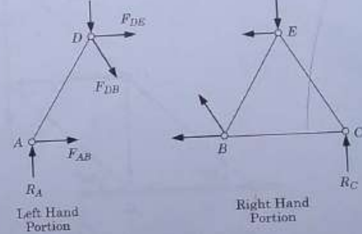


Fig. 9.12

Solution: First determine the reactions at the supports by considering the entire truss as a free-body. They are determined to be $R_A = 2.5$ kN and $R_C = 3.5$ kN. To determine the force in the member DE pass a section cutting the member DE and any two other members of the truss so as to divide the truss into two separate portions. The total number of members cut should not exceed three.

There can be more than one way to pass a section (mn or op).

Consider the truss as cut by the section mn. The two portions of the truss are as shown in Fig. 9.12.

Assume and mark the directions of the forces in the cut members. The forces in the cut members can be assumed to act away from the joints. But, the directions of the axial forces assumed in a member in the two portions of the truss must be consistent with the principle of action and reaction. For example, if the force in the member DE at the joint D is shown to act from left to right then at the joint E it must be shown to act from right to left (i.e. tension).

Consider now the equilibrium of the left hand portion of the truss. The three unknown forces acting on the portion of the truss are

$$F_{DE}, F_{DB} \text{ and } F_{AB}$$

Write the equations of equilibrium.

Taking moments about B,

$$\Sigma M_B = 0: \quad 2000\,(3 \sin 30^\circ) - R_A(3) + F_{DE}(3 \cos 30^\circ) = 0$$

$$2000 \times 3 \times 0.5 - 3R_A + F_{DE}(3 \times 0.866) = 0$$

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$$F_{DE} = \frac{1500}{0.866}$$

$$F_{DE} = 1732 \text{ N(T)} \quad \textit{Ans.}$$

It may be noted that the moment centre B chosen above, does not lie on the section of the truss under consideration.

Example 9.3 Determine the axial forces in the bars of a plane truss loaded as shown in Fig. 9.13.

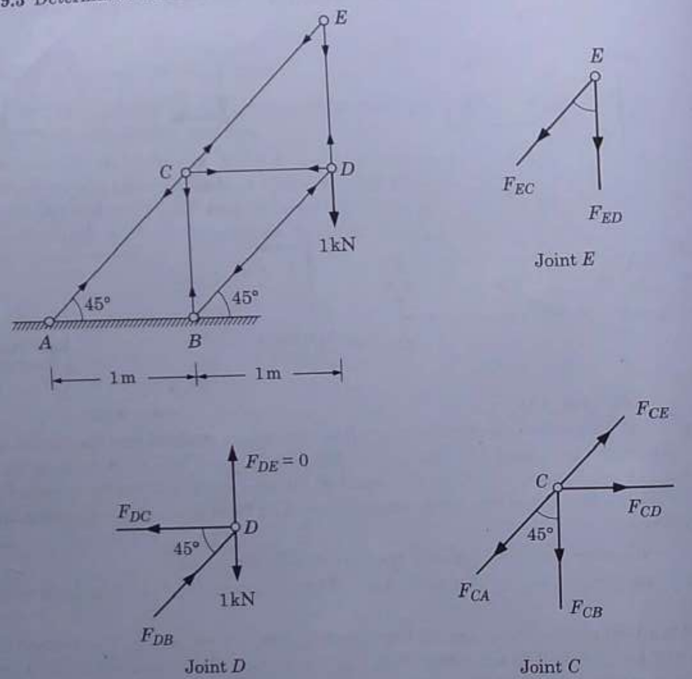


Fig. 9.13

Solution: As the truss is hinged to the foundation, it is not necessary to find the support reactions.

Let us now mark by inspection the axial forces in all the members as shown in Fig. 9.13.

Now consider the equilibrium of the various joints.

Joint E

The joint E is in equilibrium under two forces F_{EC} and F_{ED} which are non-collinear. Hence they must be zero.

$$F_{EC} = F_{ED} = 0$$

Joint D

Consider the equilibrium of the joint D.

$\Sigma F_x = 0:$ $\quad F_{DB} \cos 45° - F_{DC} = 0$

$$F_{DC} = \frac{F_{DB}}{\sqrt{2}}$$

$\Sigma F_y = 0:$ $\quad F_{DB} \sin 45° - 1 = 0$

$$F_{DB} = \frac{1}{\sin 45°} = \sqrt{2}$$

$$F_{DB} = \sqrt{2} \text{ kN(C)}$$

$$F_{DC} = \frac{F_{BD}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} \text{Joint } E$$

$$F_{DC} = 1 \text{ kN(T)}$$

Joint C

$\Sigma F_x = 0:$ $\quad -F_{CA} \sin 45° + F_{CD} = 0$

$$F_{CA} = \frac{F_{CD}}{\sin 45°} = \frac{1}{1/\sqrt{2}}$$

$$F_{CA} = \sqrt{2} \text{ kN(T)}$$

$\Sigma F_y = 0:$ $\quad -F_{CA} \sin 45° - F_{CB} = 0$

$$F_{CB} = -F_{CA} \sin 45°$$

$$F_{CB} = -\sqrt{2} \times \frac{1}{\sqrt{2}} = -1 \text{ kN}$$

$$F_{CB} = 1 \text{ kN(C)}$$

(As the force F_{CB} in the member CB is negative, reverse the assumed direction of force).

$$\left.\begin{aligned} F_{DC} &= 1 \text{ kN(T)} \\ F_{DB} &= \sqrt{2} \text{ kN(C)} \\ F_{CA} &= \sqrt{2} \text{ kN(T)} \\ F_{CB} &= 1 \text{ kN(C)} \\ F_{EC} &= 0 \\ F_{ED} &= 0 \end{aligned}\right\} \textit{Ans.}$$

Example 9.4 Find the axial forces in the members BC, BG, BF, GC, GF and GE of the truss supported and loaded as shown. Use the method of joints.

Solution: Let us first evaluate the angle θ which may be needed while resolving the forces (Fig. 9.14).

From triangle AFG,

$$\frac{FG}{AF} = \tan 30°$$

$$FG = AF \tan 30°$$

$$FG = \frac{2l}{\sqrt{3}}$$

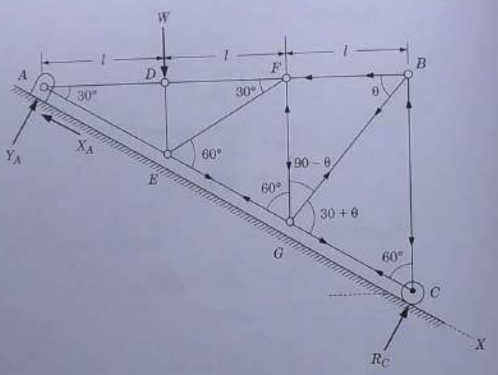


Fig. 9.14

From triangle FBG,

$$\frac{FG}{FB} = \tan \theta$$

$$\theta = \tan^{-1}\frac{FG}{RB}$$

$$\theta = \tan^{-1}\frac{2l}{\sqrt{3}l} = \frac{2}{\sqrt{3}}$$

$$\theta = 49.10°$$

Support Reactions : Consider the equilibrium of the entire truss as a free-body. Taking moments about A.

$\Sigma M_A = 0:$ $\quad - W(AD) + R_C(AC) = 0$

$$- Wl + R_C = \frac{3l}{\cos 30°} = 0$$

$$R_C = \frac{W}{2\sqrt{3}}$$

$\Sigma F_x = 0:$ $\quad W \sin 30° - X_A = 0$

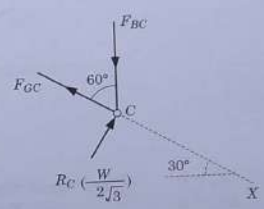
(Along AC) $\quad X_A = \frac{W}{2}$

$\Sigma F_y = 0:$ $\quad Y_A + R_C - W \cos 30° = 0$

(Normal to AC) $\quad Y_A = W \cos 30° - R_C = W\frac{\sqrt{3}}{2} - \frac{W}{2\sqrt{3}}$

$$Y_A = \frac{W}{\sqrt{3}}$$

Joint C



$\Sigma F_x = 0:$ $\quad F_{BC} \cos 60° - F_{GC} = 0$ ...(i)

$\Sigma F_y = 0:$ $\quad F_{BC} \sin 60° - R_C = 0$

$$F_{BC} = \frac{R_C}{\sin 60°} = \frac{W}{2\sqrt{3}}\left(\frac{2}{\sqrt{3}}\right)$$

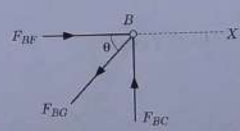
$$F_{BC} = \frac{W}{3}\text{(C)} \quad \textit{Ans.}$$

Substituting for F_{BC} in (i)

$$F_{GC} = F_{BC} \cos 60° = \frac{W}{3}\left(\frac{1}{2}\right)$$

$$F_{GC} = \frac{W}{6}\text{(T)} \quad \textit{Ans.}$$

Joint B



Joint B

$\Sigma F_x = 0:$ $\quad F_{BF} - F_{BG} \cos \theta = 0$ ...(ii)

$\Sigma F_y = 0:$ $\quad - F_{BG} \sin \theta + F_{BC} = 0$

$\theta = 49.10°$

$\sin 49.1° = 0.756$

$\cos 49.1° = 0.655$

$$F_{BG} = \frac{F_{BC}}{\sin \theta}$$

$$= \frac{W}{3}\left(\frac{1}{0.756}\right)$$

$$F_{BG} = 0.441,\ W\text{(T)} \quad \textit{Ans.}$$

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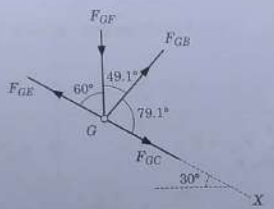
Substituting for F_{BG} in (*ii*)

$$F_{BF} = F_{BG} \cos 49.1° = 0.441\,W \times 0.655$$
$$F_{BF} = 0.289\,W\,(C) \quad Ans.$$

Joint G



Joint G

$$\Sigma F_x = 0: \quad -F_{GE} + F_{GC} + F_{GB} \cos 79.1° + F_{GF} \cos 60° = 0 \qquad ...(iii)$$
$$\Sigma F_y = 0: \quad -F_{GF} \sin 60° + F_{GB} \sin 79.1° = 0$$

Or
$$F_{GF} = \frac{F_{GB} \sin 79.1}{\sin 60°} = \frac{0.441\,W}{0.887}$$
$$F_{GF} = 0.5\,W(C) \quad Ans.$$

Solving (*iii*)
$$F_{GE} = 0.5\,W(T) \quad Ans.$$

Example 9.5 A truss is loaded and supported as shown. Determine the axial forces in the members CE, CG and FG.

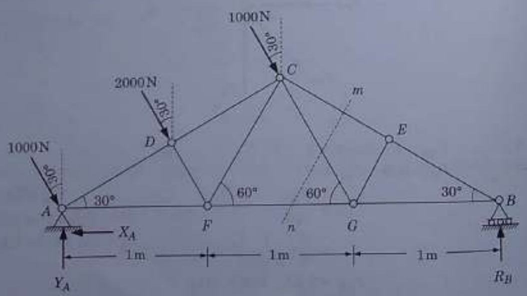


Fig. 9.15

Solution:

Support Reactions : Consider the equilibrium of the entire truss as a free-body.

Taking moments about A,

$$\Sigma M_A = 0: \quad R_B(3) - 2000 \times AD - 1000 \times AC = 0$$
$$3R_B - 2000\,AF \cos 30° - 1000\,AG \cos 30° = 0$$
$$3R_B = 2000 \times \frac{\sqrt{3}}{2} + 1000 \times 2 \times \frac{\sqrt{3}}{2}$$
$$R_B = \frac{2000}{\sqrt{3}}\,\text{N}$$

$$\Sigma F_y = 0: \quad R_B + Y_A - 1000 \cos 30° - 2000 \cos 30° - 1000 \cos 30° = 0$$
$$Y_A = \frac{\sqrt{3}}{2}(4000) - \frac{2000}{\sqrt{3}} = \frac{4000}{\sqrt{3}}$$
$$Y_A = \frac{4000}{\sqrt{3}}\,\text{N}$$

$$\Sigma F_x = 0: \quad -X_A + 1000 \sin 30° + 2000 \sin 30° + 1000 \sin 30° = 0$$
$$X_A = 1000 \times 0.5 + 2000 \times 0.5 + 1000 \times 0.5$$
$$X_A = 2000\,\text{N}$$

Pass a section mn through the truss cutting the members CE, CG and FG. Consider the equilibrium of the right hand portion of the truss as shown in Fig. 9.16.

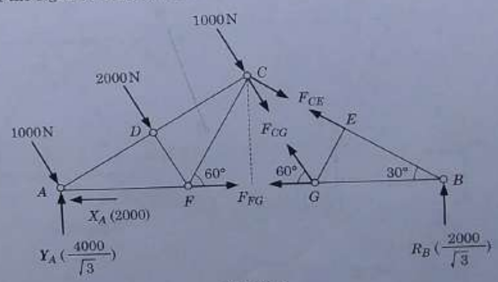


Fig. 9.16

Taking moments about C,

$$\Sigma M_C = 0: \quad -F_{FG} \times 0.5 \tan 60° + R_B(1.5)$$
$$-F_{FG} \times 0.5 \times \sqrt{3} + \frac{2000}{\sqrt{3}} \times 1.5 = 0$$
$$F_{FG} = 2000\,\text{N(T)} \quad Ans.$$

Taking moments about G,

$$\Sigma M_G = 0: \quad F_{CE}(1 \times \sin 30°) + R_B(1) = 0$$

$$F_{CE} = -\frac{2000}{\sqrt{3}} \times \frac{1}{0.5} = -2309 \text{ N}$$

Reverse the sign of the force F_{CE}

$$F_{CE} = 2309 \text{ N(C)} \quad \textit{Ans.}$$

Taking moments about B,

$$\Sigma M_B = 0: \quad F_{CG}(1 \times \sin 60°) = 0$$

$$F_{CG} = 0 \quad \textit{Ans.}$$

Example 9.6 A hexagonal truss formed of 11 bars of 2 m length each. It is hinged at one end and roller supported at the other end. Find the axial forces in the members CD and GB.

Solution: Support Reactions: Consider the equilibrium of the entire truss as a free-body Fig. 9.17.

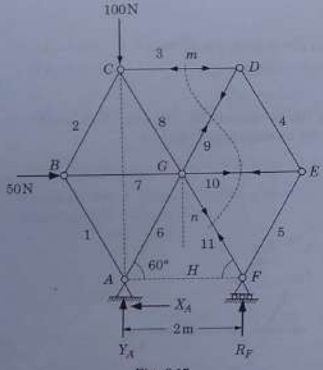


Fig. 9.17

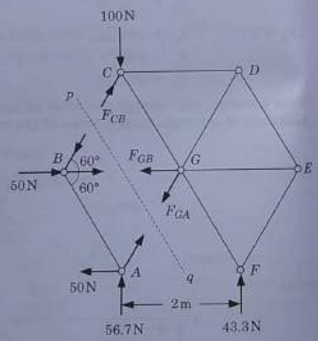


Fig. 9.18

Taking moments about A,

$$\Sigma M_A = 0: \quad R_F(AF) - 50\,(HG) = 0$$

$$HG = 2 \sin 60° = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$R_F = \frac{50 \times \sqrt{3}}{2}$$

$$R_F = 43.3 \text{ N}$$

$$\Sigma F_x = 0: \quad 50 - X_A = 0$$

$$X_A = 50 \text{ N}$$

$$\Sigma F_y = 0: \quad Y_A + R_F - 100 = 0$$

$$Y_A = 100 - 43.3$$

$$Y_A = 56.7 \text{ N}$$

Pass a section mn through the truss cutting the members CD, GD, GE and GF and consider the equilibrium of the right hand portion of the truss. Note that the section mn cuts four members.

Taking moments about G,

$$\Sigma M_G = 0: \quad R_F(1) - F_{CD}(2 \sin 60°) = 0$$

$$43.3 \times 1 - F_{CD} \times 2 \times 0.866 = 0$$

$$F_{CD} = 25.0 \text{ N(C)} \quad \textit{Ans.}$$

Next pass a section pq cutting the members CB, GB and GA.

Consider the equilibrium of the right hand portion of the truss (Fig. 9.18).

Take moments about B

$$\Sigma M_B = 0: \quad -F_{GA}(2 \sin 60°) + 43.3(3) - 100(1) = 0$$

$$-F_{GA}(2 \times 0.866) + 129.9 - 100 = 0$$

$$F_{GA} = 17.26 \text{ N(T)}$$

Take moments about G,

$$\Sigma M_G = 0: \quad -F_{CB}(2 \sin 60°) + 100\,(1) + 43.3\,(1) = 0$$

$$F_{CB} = \frac{143.3}{2 \times 0.866}$$

$$F_{CB} = 82.74 \text{ N(C)}$$

$$\Sigma F_x = 0: \quad F_{CB} \cos 60° - F_{GA} \cos 60° - F_{GB} = 0$$

$$F_{GB} = \cos 60°(F_{CB} - F_{GA})$$

$$F_{GB} = 0.5(82.74 - 17.26)$$

$$F_{GB} = 32.74 \text{ N(T)} \quad \textit{Ans.}$$

Example 9.7 Determine the forces in the bars DC, DH and FH of the truss loaded and supported as shown in Fig. 9.19.

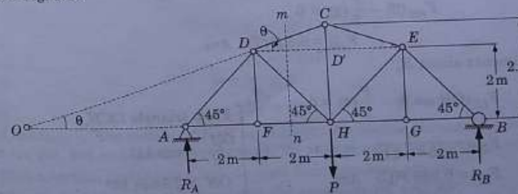


Fig. 9.19

Solution: Support reactions: Consider the entire truss as a free-body and take moments about A,

$$\Sigma M_A = 0: \quad R_B(8) - P(4) = 0$$

$$R_B = \frac{P}{2}$$

$$\Sigma F_y = 0: \quad R_A + R_B = P$$

$$R_A = \frac{P}{2}$$

To find the axial forces in the members DC, DH and FH pass a section mn cutting these members. The two portions of the truss are as shown in Fig. 9.20. Consider the equilibrium of the left hand portion of the truss. Let us first calculate the value of angle θ.

From triangle DCD' (Fig. 9.19),

$$\tan\theta = \frac{CD'}{DD'} = \frac{0.5}{2} = 0.25$$

$$\theta = 14°$$

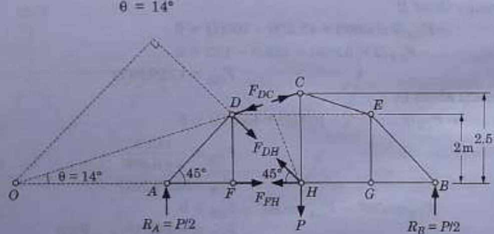


Fig. 9.20

To determine the forces F_{DC}, F_{DH} and F_{FH} in the members we shall use the moment equations.

Taking moments about D,

$$\Sigma M_D = 0: \quad F_{FH}(2) - \frac{P}{2}(2) = 0$$

$$F_{FH} = 0.5\,P\,(\text{T}) \quad \textit{Ans.}$$

Taking moments about H,

$$\Sigma M_H = 0: \quad F_{DC}(OH\sin\theta) - \frac{P}{2}(4) = 0$$

$$F_{DC} = \frac{2P}{10 \times 0.242}$$

$$F_{DC} = 0.826\,P(\text{C}) \quad \textit{Ans.}$$

From triangle OCH,

$$\frac{OH}{CH} = \tan 14°$$

$$OH = 2.5/\tan 14°$$

$$OH = 10 \text{ m}$$

Taking moments about O,

$$\Sigma M_0 = 0: \quad -F_{DH}(OH\sin 45°) + \frac{P}{2}(OH - AH) = 0$$

$$F_{DH}\left(10 \times \frac{1}{\sqrt{2}}\right) = \frac{P}{2} \times (10 - 4)$$

$$F_{DH} = \frac{3P\sqrt{2}}{10}$$

$$F_{DH} = 0.424\,P(\text{T}) \quad \textit{Ans.}$$

Example 9.8 Find the force in the member CF of the truss loaded and supported as shown in Fig. 9.21 (a).

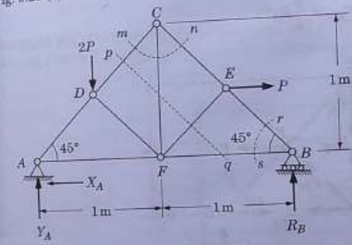


Fig. 9.21 (a)

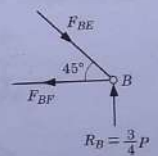


Fig. 9.21 (b)

Solution: Support Reactions : Consider the equilibrium of the entire truss as a free-body.

Taking moments about A,

$$\Sigma M_A = 0: \quad (-2P)(0.5) - P(0.5) + R_B(2) = 0$$

$$R_B = \frac{3}{4}P$$

$$\Sigma F_y = 0: \quad R_B + Y_A - 2P = 0$$

$$Y_A = 2P - \frac{3}{4}P$$

$$Y_A = 1.25P$$

$$\Sigma F_x = 0: \quad P - X_A = 0$$

$$X_A = P$$

To find the force in the member CF let us consider the section through the line mn. Three members are cut, but as these are concurrent, we cannot solve for the three unknowns.

If we take a section through the line pq then four members are cut.

So, first pass a section through rs cutting the members BE and BF.

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Write the equations of equilibrium

$\Sigma F_x = 0$: $\quad -F_{BF} + F_{BE} \cos 45° = 0$ ...(i)

$\Sigma F_y = 0$: $\quad -F_{BE} \sin 45° + \frac{3}{4}P = 0$

$$F_{BE} = \frac{3P}{4 \sin 45°}$$

$$F_{BE} = \frac{3}{2\sqrt{2}}P(\text{C})$$

From (i) $\quad F_{BF} = F_{BE} \cos 45° = \frac{3}{2\sqrt{2}}P\left(\frac{1}{\sqrt{2}}\right)$

$$F_{BF} = \frac{3}{4}P\,(\text{T})$$

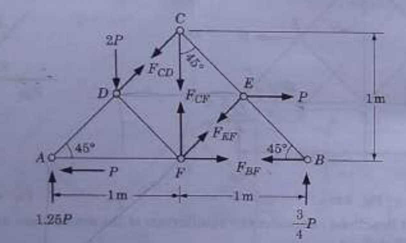


Fig. 9.22

Consider the equilibrium of the right hand portion of the truss as obtained by the section pq and shown in Fig. 9.22.

Taking moments about F,

$\Sigma M_F = 0$: $\quad +F_{CD}\left(1 \times \frac{1}{\sqrt{2}}\right) + \frac{3}{4}P(1) - P\left(\frac{1}{2}\right) = 0$

$$F_{CD} = -\frac{P}{2\sqrt{2}} = 0.354P\,(\text{C})$$

$\sin 45° = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{1}{\sqrt{2}}$

Taking moments about C

$$\Sigma M_C = 0: \frac{3}{4}P \times 1 + P \times \frac{1}{2} - F_{BF}(1) - F_{EF}\left(\frac{1}{2} \times \sqrt{2}\right) = 0$$

Substituting $\quad F_{BF} = \frac{3}{4}P, \quad F_{EF} = \frac{P}{\sqrt{2}}(\text{T})$

Taking moments about A,

$$\Sigma M_A = 0: \frac{3}{4}P(2) - P\left(\frac{1}{2}\right) - F_{CF} \times 1 - F_{EF} \times \left(1 \times \frac{1}{\sqrt{2}}\right) = 0$$

$$F_{EF} = \frac{P}{\sqrt{2}}$$

Substituting,

$$\frac{3}{4} \times 2P - \frac{P}{2} - F_{CF} - \frac{P}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 0$$

$$F_{CF} = \frac{P}{2}(\text{T}) \quad \textit{Ans.}$$

Example 9.9 A cantilever truss is loaded and supported as shown (Fig. 9.23). Find the value of loads P which would produce an axial force of magnitude 3 kN in the member AC.

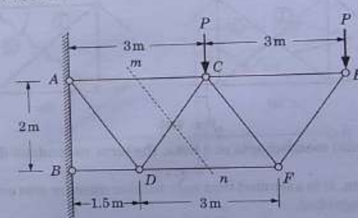


Fig. 9.23

Solution: In this case we need not determine the support reactions.

The force in the member AC can be determined using the method of sections.

Pass a section mn cutting the members AC, DC and DF.

Consider the equilibrium of the right hand portion of the truss as shown in Fig. 9.24.

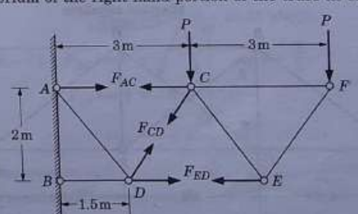


Fig. 9.24

Taking moment about D,

$$\Sigma M_D = 0: \quad F_{AC} \times 2 - P(1.5) - P(4.5) = 0$$

$$F_{AC} = \frac{6P}{2}$$

$$F_{AC} = 3P$$

As the force in the member is 3 kN,

$$F_{AC} = 3 \text{ kN} = 3P$$

$$P = 1 \text{ kN} \quad \textit{Ans.}$$

Counter Diagonals/Crossed Braced Members

Often truss panels are cross braced as shown in Fig. 9.25(a) to pervent collapse. These diagonal members cannot support compression like a cable. Additional diagonal member of this type might be called redundant.

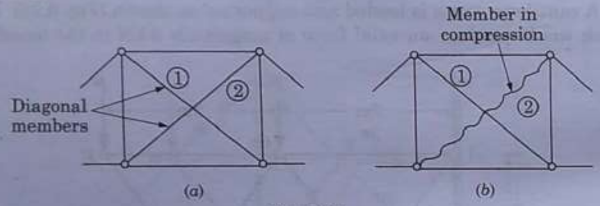


Fig. 9.25

But, as only one diagonal member acts at a time, the term 'redundant' does not apply in such situation.

To solve such a problem, it is assumed that only tension member acts and the other [member 2 in Fig. 9.25 (a)] is disregarded.

If the assumed tension in the member comes out to be +ve on calculation, the choice is correct. If assumed tension comes out to be –ve then the other member is retained and calculations redone. Example 9.10 illustrates the procedure.

Example 9.10 A truss is loaded and supported as shown. Determine the axial forces in the members BD, CE, BE and CD (Fig. 9.26).

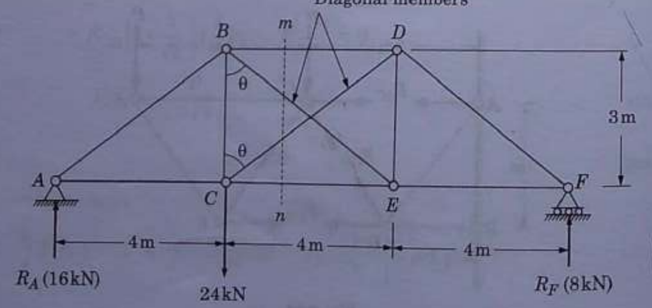


Fig. 9.26 (a)

Solution: From triangle BCD,

$$\tan\theta = \frac{4}{3};\ \cos\theta = \frac{3}{5};\ \sin\theta = \frac{4}{5}$$

Support Reactions : Consider the equilibrium of entire truss.

Taking moments about A,

$$\Sigma M_A = 0: \quad -24(4) + R_F(12) = 0$$

$$R_F = 9 \text{ kN}$$

$$\Sigma F_y = 0: \quad R_A + R_F = 24$$

$$R_A = 24 - 8$$

$$R_A = 16 \text{ kN}.$$

Pass a section mn through the truss. Consider the equilibrium of the left hand portion of the truss as shown in Fig. 9.26 (b).

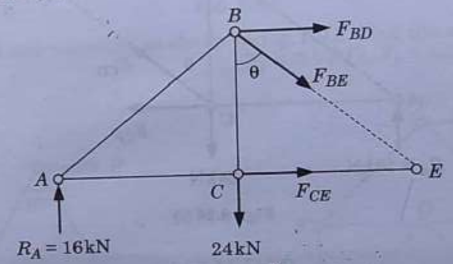


Fig. 9.26 (b)

Assume member BE to be in tension and disregard the member CD.

$$\Sigma M_B = 0: \quad F_{CE}(3) - 16(4) = 0$$

$$F_{CE} = 21.33 \text{ kN(T)}$$

$$\Sigma M_E = 0: \quad -F_{BD}(3) + 24(4) - 16(8) = 0$$

$$F_{BD} = -10.33 = 10.33 \text{ kN(C)}$$

$$\Sigma F_Y = 0: \quad 16 - 24 - F_{BE}\cos\theta = 0 \qquad \cos\theta = \frac{3}{5},$$

$$F_{BE}\cos\theta = -8$$

$$F_{BE} = -8 \times \frac{5}{3} = (-)\ 13.33 \text{ kN}$$

(–)ve sign means compression in the member BE which is not possible

Next, assuming the member CD to be in tension and disregarding the member BE, as shown in Fig. 9.26 (c).

$$\Sigma M_C = 0: \quad -F_{BD}(3) - 16(4) = 0$$
$$F_{BD} = -21.3 \text{ kN}$$
$$F_{BD} = 21.3 \text{ kN(C)} \quad \textit{Ans.}$$
$$\Sigma M_D = 0: F_{CE}(3) + 24(4) - 16(8) = 0$$
$$F_{CE} = 10.66 \text{ kN(T)} \quad \textit{Ans.}$$
$$\Sigma F_y = 0: \quad F_{CD} \cos\theta - 24 + 16 = 0 \qquad \cos\theta = \frac{3}{5}$$
$$F_{CD} = 13.3 \text{ kN(T)} \quad \textit{Ans.}$$

Member CD is found to be in tension. So this choice is correct hence the forces found in member CD, BD and CD are correct.

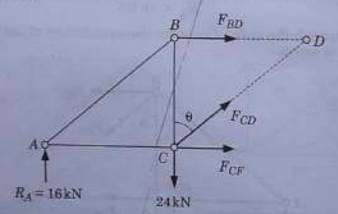


Fig. 9.26 (c)

PROBLEMS

9.1. A truss is loaded and supported as shown Fig. P.9.1. Find by the method of joints the axial forces in each member of the truss.

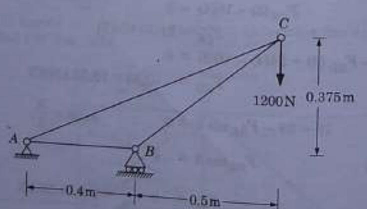


Fig. P.9.1

$$\begin{bmatrix} F_{AB} = 3600\text{N(C)} \\ F_{AC} = 3900\text{N(T)} \\ F_{BC} = 4500\text{N(C)} \end{bmatrix}$$

9.2. A truss is loaded and supported as shown. Find out the members in which the axial forces are zero (Fig. P.9.2). [BC, FG, FI, HI, HK and JK]

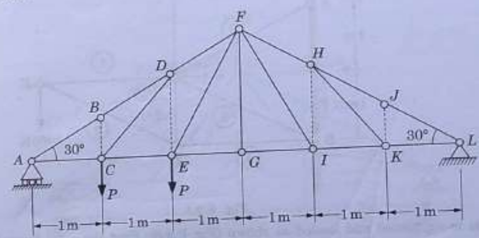
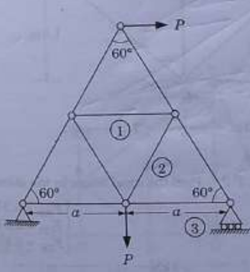


Fig. P.9.2

9.3. A truss is loaded and supported as shown in Fig. P.9.3. Find the axial forces in the members 1, 2 and 3. [$F_1 = 0.5\,P$(C), $F_2 = 0.5\,P$(T) and $F_3 = 0.789\,P$(T)].



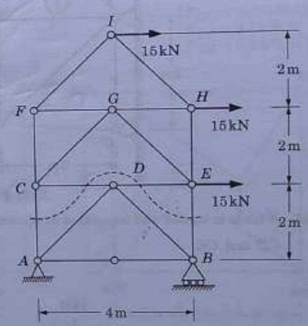


Fig. P.9.3 Fig. P.9.4

9.4. Determine by the method of sections the axial force in the member EB of the truss shown in Fig. P.9.4. [22.5 kN(C)]

9.5. A cantilever truss is loaded as shown in Fig. P.9.5. Find the axial forces in all the members.

$$\begin{bmatrix} F_1 = 2000\text{N(T)} & F_4 = 2000\text{N(C)} \\ F_2 = 2240\text{N(C)} & F_5 = 0 \\ F_3 = 1000\text{N(T)} & F_6 = 2240\text{N(T)} \end{bmatrix}$$

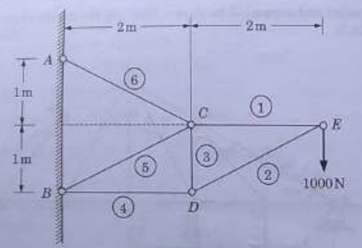


Fig. P.9.5

9.6. A truss is supported and loaded as shown (Fig. P.9.6). Find the axial forces in the members 1 and 2.
[F_1 = 9.14 kN(T), F_2 = 3.86 kN(C)]

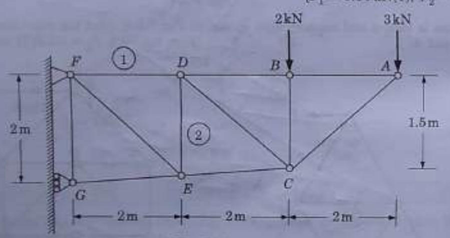


Fig. P.9.6

9.7. A truss is loaded and supported as shown in Fig. P.9.7. Find the axial forces in the member *BD*, *CD* and *CE*.

$$\begin{bmatrix} F_{BD} = 5.2\,\text{kN(C)} \\ F_{CD} = 2.0\,\text{kN(C)} \\ F_{CE} = 6.93\,\text{kN(T)} \end{bmatrix}$$

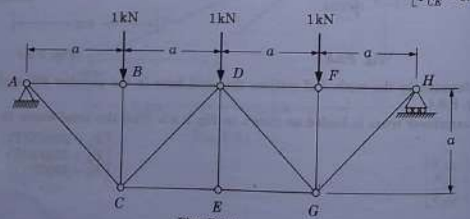


Fig. P.9.7

9.8. A roof truss is supported and loaded as shown in Fig. P.9.8. Find the axial forces in the members *BD* and *CF*.
[F_{BD} = 3.69 N(C), F_{CF} = 2.54(T)]

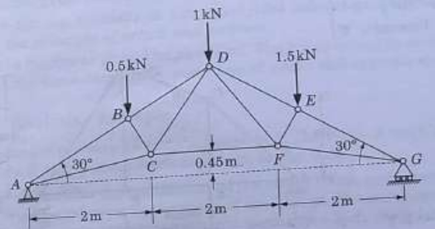


Fig. P.9.8

9.9. A truss is loaded and supported as shown in Fig. P.9.9. Find the axial forces in the members *BD*, *DE* and *EG*.

$$\begin{bmatrix} F_{BD} = 0.47\,\text{kN(C)} \\ F_{DE} = 0.5\,\text{kN(T)} \\ F_{EG} = 0.333\,\text{kN(T)} \end{bmatrix}$$

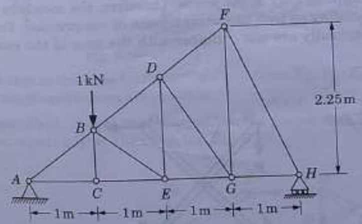


Fig. P.9.9

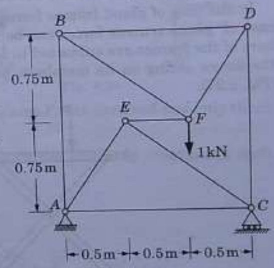


Fig. P.9.10

9.10. A truss is loaded and supported as shown in Fig. P.9.10. Find the axial forces in the members *AB*, *EF* and *CD*.
[F_{AB} = 0.333 kN(C), F_{EF} = 0, F_{CD} = 0.667 kN(C)]

9.11. A plane truss is supported and loaded as shown in Fig. P.9.11. Find the axial force in members CD, EF and CF by the method of sections.

$$\left[\text{Hint}: \begin{array}{c} \Sigma M_C = 0 \\ \Sigma M_F = 0 \\ \text{Determine } '\theta' \end{array}\right] \qquad \left[\begin{array}{l} F_{CD} = 7.454 \text{ kN} \\ F_{EF} = 20 \text{ kN} \\ F_{CF} = 18.86 \text{ kN} \end{array}\right]$$

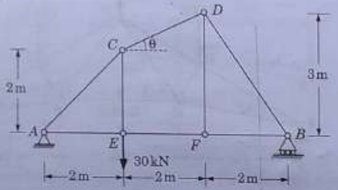


Fig. P.9.11

9.4 FRAMES

Frames are structures consisting of several bars or members pinned together and in which one or more than one of its members are subjected to more than two forces. They are designed to support loads and are usually stationary structures.

In the case of plane frames forces can be applied *anywhere on the members* whereas in the case of plane trusses forces can be applied *only at the joints*. Therefore, the members or the bars of the frames are subjected to *bending as well as simple tension* or *compression*. Further, the forces acting on its members, generally are not collinear with the axes of the members (Fig. 9.27).

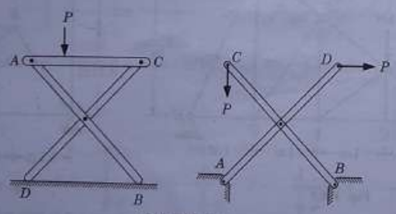


Fig. 9.27 Frames

9.4.1 Method of Analysis

The general problem in the analysis of a frame is to find the magnitudes and directions of the *forces transmitted from one member to another through the connecting pins*, under a given loading. For this we have to consider the conditions of equilibrium of the entire frame together

with the conditions of equilibrium of its members. To consider the equilibrium of the member we have to *dismember the frame*. While representing the forces acting on the members (after dismembering) the following rule should be observed.

The forces acting on the two members at a common joint (after dismembering) should be represented observing the principle of action and reaction.

Two force members in the frame should be identified. The forces acting at each end of such a member should have the same magnitude, same line of action and opposite sense.

9.4.2 Method of Analysis: Example

Consider a frame consisting of two bars AB and CD hinged at E. Their ends A and C are supported at the hinges. The ends B and D carry loads as shown in Fig. 9.28.

Entire Frame. Let us consider the equilibrium of the entire frame and compute whatever support reactions can be computed using the equations of equilibrium. It is convenient to represent the unknown forces by their rectangular components on the free-body diagram as shown in Fig. 9.29.

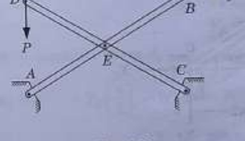


Fig. 9.28

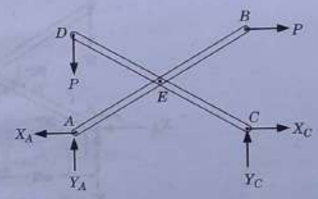


Fig. 9.29

Note that in this case four unknown reactions X_A, Y_A, X_C and Y_C are involved and only three equations of equilibrium are possible.

Dismember. Let us now dismember the frame and draw the free-body diagram of each member separately as shown in the Fig. 9.30.

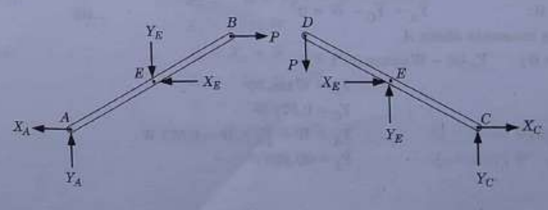


Fig. 9.30

Member AB. To draw the free-body diagram of the member AB, assume the magnitudes and directions of the forces at the interacting joint E (can also be called a common joint). Let them be represented by X_E and Y_E. Write the equation of equilibrium of the member.

Member CD. In the free-body diagram of the member CD, the forces at the joint E common with the member AB, must now be represented observing the *principle of action and reaction*. That is when the horizontal force X_E acting on member AB at E is acting from right to left, the horizontal force acting at E on the member CD must also be of magnitude X_E but should act from left to right. Similar consideration should be given to the vertical force Y_E. Write the equation of equilibrium of the member.

The equation of equilibrium of the entire frame and of the members now can be solved for the desired unknown forces. *If the value of a force comes out to be negative, simply reverse the assumed direction of that force.*

Example 9.11 An X-frame is loaded and supported as shown in Fig. 9.31. Find the horizontal and vertical components of the reaction at A and C.

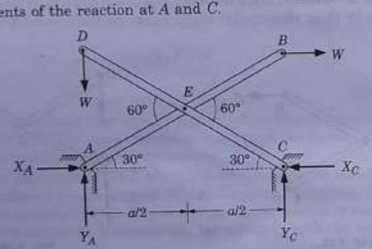


Fig. 9.31

Solution: Entire Frame. Consider the equilibrium of the entire frame as a free-body.

$\Sigma F_x = 0$: $X_A + W - X_C = 0$

$$-X_A + X_C = W \quad \text{...(i)}$$

$\Sigma F_y = 0$: $Y_A + Y_C - W = 0 \quad$ ...(ii)

Taking moments about A,

$\Sigma M_A = 0$: $Y_C(a) - W(a \tan 30°) = 0$

$$Y_C = W \tan 30°$$

$$Y_C = 0.577\ W$$

From (ii) $Y_A = W - Y_C = W - 0.577\ W$

$$Y_A = 00.423\ W.$$

Dismember

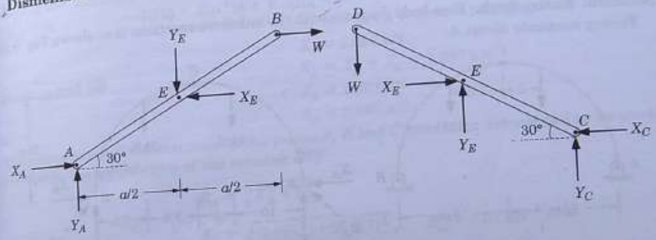


Fig. 9.32

Member AB

$\Sigma F_x = 0$: $X_A + W - X_E = 0 \quad$ ...(iii)

$\Sigma F_y = 0$: $Y_A - Y_E = 0$

$$Y_E = Y_A = 0.423\ W$$

Taking moments about A,

$$\Sigma M_A = 0: \quad -Y_E\left(\frac{a}{2}\right) - W(a \tan 30°) + X_E\left(\frac{a}{2}\tan 30°\right) = 0$$

$$X_E \tan 30° = 2\ W \tan 30° + Y_E$$

$$X_E = \frac{2W \times 0.577 + 0.423\ W}{0.577}$$

$$X_E = 2.733\ W.$$

From (iii) $X_A = -W + X_E = -W + 2.733\ W$

$$X_A = 1.733\ W.$$

From (i) $X_C = X_A + W$

$$X_C = 1.733\ W + W$$

$$X_C = 2.733,$$

$X_A = 1.733\ W$, $\quad X_C = 2.733\ W$

$Y_A = 0.423\ W$, $\quad Y_C = 0.577\ W \quad$ *Ans.*

Example 9.12 A semicircular three-hinged arch is loaded and supported as shown. Find the reactions at the supports A and B.

Solution: **Entire Arch.** Free-body diagram with the unknown reactions is as shown Fig. 9.33.

Taking moments about A.

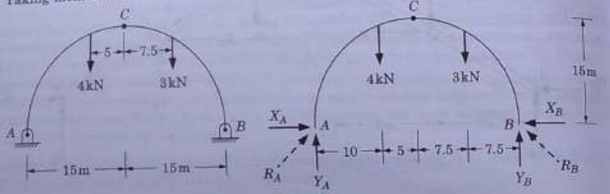


Fig. 9.33

$\Sigma M_A = 0:$ $\quad Y_B(30) - 3(22.5) - 4(10) = 0$

$$Y_B = 3.583 \text{ kN}$$

$\Sigma F_y = 0:$ $\quad Y_A + Y_B - 3 - 4 = 0$

$$Y_A + Y_B = 7$$

$$Y_A = 7 - Y_B = 7 - 3.583, \quad Y_A = 3.417 \text{ kN} \quad ...(i)$$

$\Sigma F_x = 0:$ $\quad X_A - X_B = 0, X_A = X_B$

Dismember

Member AC. Consider the equilibrium of the member AC.

$\Sigma F_x = 0:$ $\quad X_A - X_C = 0$

$$X_A = X_C$$

$\Sigma F_y = 0:$ $\quad Y_A = Y_C - 4 = 0$

Substituting $\quad Y_A = 3.417, \quad Y_C = 4 - Y_A = 4 - 3.417$

$$Y_C = 0.583 \text{ kN.}$$

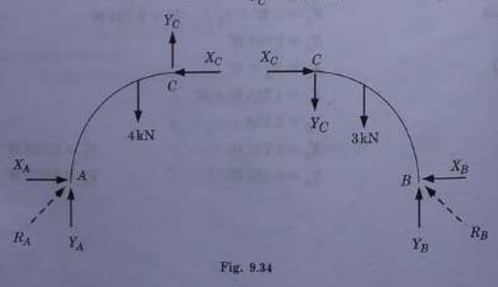


Fig. 9.34

Taking the moments about A

$\Sigma M_A = 0:$ $\quad -4(10) + X_C(15) + Y_C(15) = 0$

$$15X_C = -15Y_C + 40 = -15(0.583) + 40$$

$$X_C = 2.087 \text{ kN}$$

$$X_A = X_B = X_C = 2.087 \text{ kN}$$

From (*i*) and (*ii*)

$$X_A = 2.087 \text{ kN}$$

$$X_B = 2.087 \text{ kN.}$$

The components of the reactions at the points A, B and C have been determined so we need not consider the equilibrium of the member BC.

$$R_A = \sqrt{X_A^2 + Y_A^2}$$

$$R_A = \sqrt{(2.087^2) + (3.417)^2}, \quad R_A = 4.01 \text{ kN} \quad \textit{Ans.}$$

$$R_B = \sqrt{X_B^2 + Y_B^2}$$

$$R_B = \sqrt{(2.087)^2 + (3.583)^2}, \quad R_B = 4.145 \text{ kN} \quad \textit{Ans.}$$

Example 9.13 A frame ABC is loaded and supported as shown in the Fig. 9.35. Find the force in the bar AB.

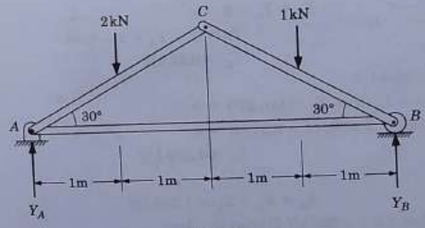


Fig. 9.35

Solution: **Entire Frame.** Consider the entire frame as a free-body. The unknown reactions involved are Y_A and Y_B.

$F_y = 0:$ $\quad Y_A + Y_B - 2 - 1 = 0$

$$Y_A + Y_B = 3$$

Taking moments about A,

$\Sigma M_A = 0:$ $\quad Y_B(4) - 1(3) - 2(1) = 0$

$$Y_B = 1.25 \text{ kN}$$

$$Y_A = 3 - Y_B = 3 - 1.25$$

$$Y_A = 1.75 \text{ kN}$$

Dismember

Member AB. *It is a two-force member* (two forces acting at the ends are equal in magnitude, opposite in sense and are collinear), therefore, for equilibriums

$$X_A = X_B \qquad \ldots(i)$$

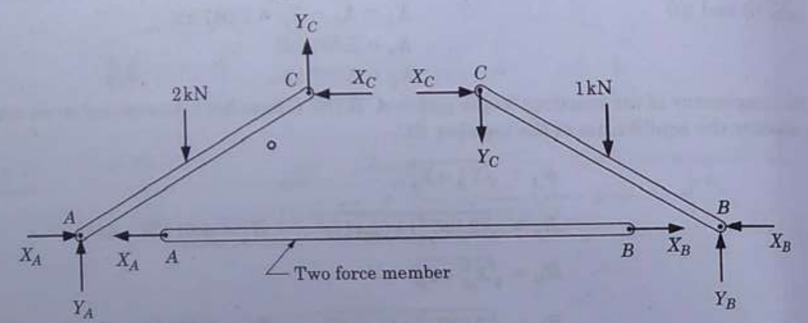


Fig. 9.36

Member AC

$\Sigma F_x = 0$: $\quad X_A - X_C = 0$ or $X_A = X_C$

$\Sigma F_y = 0$: $\quad Y_A + Y_C - 2 = 0$

$$Y_C = 2 - Y_A = 2 - 1.75$$

$$Y_C = 0.25 \text{ kN}$$

Taking moments about A,

$\Sigma M_A = 0: \; -2(1) + Y_C(2) + X_C(2 \tan 30°) = 0$

$$-2 + 0.25 \times 2 + X_C(2 \times 0.5774) = 0$$

$$X_C = 1.299 \text{ kN}$$

From (*i*) and (*ii*)

$$X_A = X_B = X_C = 1.299 \text{ kN}$$

Force in the tie bar AB = 1.299 kN (Tension) *Ans.*

We need not consider the equilibrium of the bar BC as the desired forces have been determined.

Example 9.14 A framed structure as shown in Fig. 9.37 (*a*) supports a load of 1 kN. Find the compressive force in the bar BC and the shear force on the pin at D. The radius of the pulley at E may be assumed to be 0.25 m.

Solution: **Entire Frame.** The external reactions on the frame involve three unknowns X_A, Y_A and Y_B. These reactions are determined by taking the entire frame as a free-body as shown in Fig. 9.37 (*b*).

$\Sigma F_x = 0$: $\quad X_A - 1000 = 0, \quad X_A = 1000$ N

$\Sigma F_y = 0$: $\quad Y_A + Y_B - 1000 = 0, \quad Y_A + Y_B = 1000$

Taking moments about A,

$\Sigma M_A = 0$: $\quad Y_B(2) - 1000(0.5) - 1000(1.25) = 0$

$$Y_B = 875 \text{ N}$$

and

$$Y_A = 125 \text{ N}$$

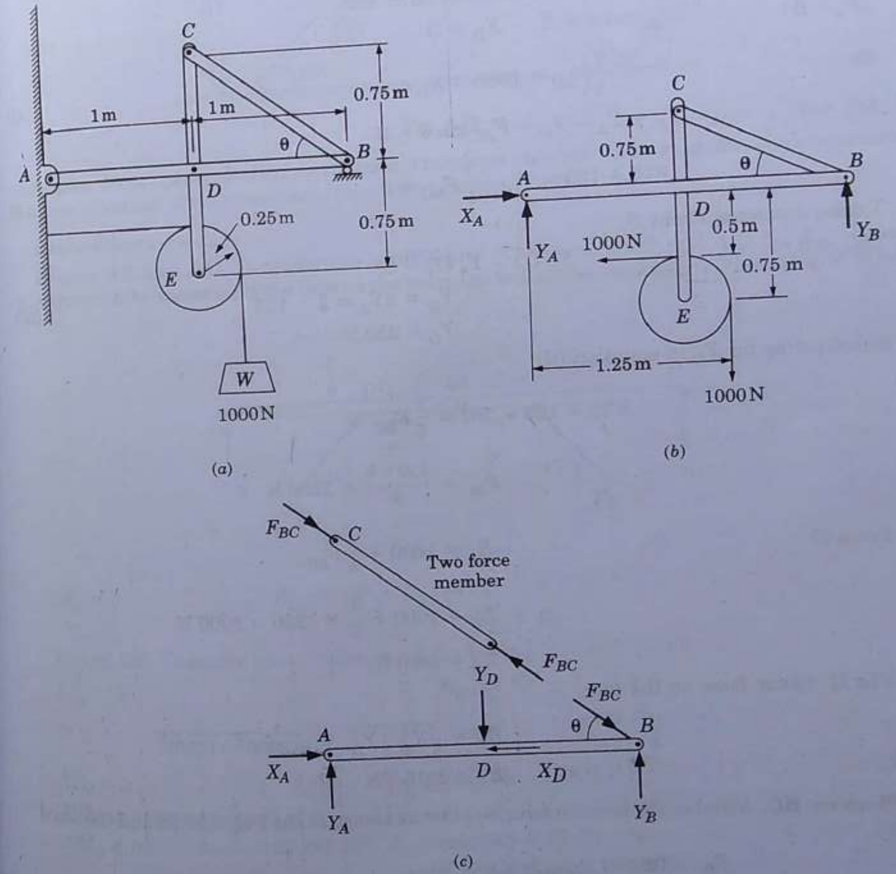


Fig. 9.37

Dismember

Member ADB. The frame is dismembered and the free-body diagram of the member ADB is as shown in Fig. 9.37 (*c*).

Evaluating the angle θ,

$$\tan\theta = \frac{0.75}{1.0} = \frac{3}{4}, \; \sin\theta = \frac{3}{5} \text{ and } \cos\theta = \frac{4}{5}$$

The equation of equilibrium of the member ADB are,

$\Sigma F_x = 0$: $\quad F_{BC}\cos\theta + X_A - X_D = 0$

Or $\quad \frac{4}{5}F_{BC} + 1000 - X_D = 0$ ...(*i*)

$\Sigma F_y = 0$: $\quad Y_B + Y_A - Y_D - F_{BC}\sin\theta = 0$

$$875 + 125 - Y_D - \frac{3}{5}F_{BC} = 0 \quad ...(ii)$$

Taking moments about B,

$\Sigma M_B = 0$: $\quad Y_D(1) - Y_A(2) = 0$

Or $\quad Y_D = 2Y_A = 2 \times 125$ ...(*iii*)

$$Y_D = 250 \text{ N}$$

Substituting for Y_D in equation (*ii*)

$$8.75 + 125 - 250 = \frac{3}{5}F_{BC}$$

$$F_{BC} = \frac{750 \times 5}{3} = 1250 \text{ N}$$

From (*i*) $\quad X_D = 1000 + \frac{4}{5}F_{BC}$

$$X_D = 1000 + \frac{4}{5} \times 1250 = 2000 \text{ N}$$

$$X_D = 2000 \text{ N.}$$

Pin D. Shear force on the pin

$$R_D = \sqrt{X_D^2 + Y_D^2} = \sqrt{(2000^2 + (250)^2}$$

$$R_D = 2015.5 \text{ N} \quad \textit{Ans.}$$

Member BC. Member BC is *a two force member* as shown in the Fig. 9.37 (c) and the axial force

$$F_{BC} = 1250 \text{ N (compressive)} \quad \textit{Ans.}$$

Example 9.15 Two horizontal beams BA and AC hinged together at A and are supported by four bars hinged to the beams as shown in Fig. 9.38. Determine the axial forces in the four bars and the rectangular components of the force acting at the hinge A.

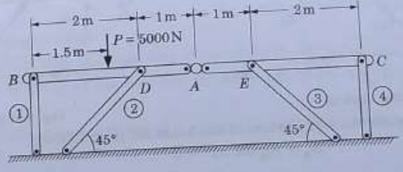


Fig. 9.38

Solution: Note that *each of the four bars is a two force member*. Their axes, therefore, represent the line of action of reactive forces they exert on the beam.

Dismember.

Beams AB and AC. Consider the equilibrium of the beams AB and AC. Y_B, R_D, R_E and Y_C represent the reactions of the bars on the beams as well as the axial force in them [Fig. 9.39 (*a*)].

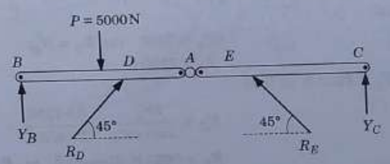


Fig. 9.39 (*a*)

$\Sigma F_x = 0$: $\quad R_D\cos 45° - R_E\cos 45° = 0$

$$R_D = R_E \quad ...(i)$$

Beam AB. Consider the equilibrium of the beam AB [Fig. 9.39 (*b*)].

$\Sigma F_x = 0$: $\quad R_D\cos 45° - X_A = 0$

$$X_A = \frac{R_D}{\sqrt{2}} \quad ...(ii)$$

$\Sigma F_y = 0$: $\quad Y_B + Y_A + R_D\sin 45° - 5000 = 0$ ...(*iii*)

Taking moments about B,

$\Sigma M_B = 0$: $\quad -5000(1.5) + R_D(2\sin 45°) + Y_A(3) = 0$

$$2R_D\sin 45° + 3Y_A = 7500 \quad ...(iv)$$

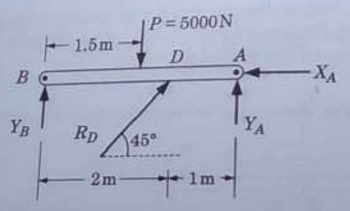


Fig. 9.39 (b)

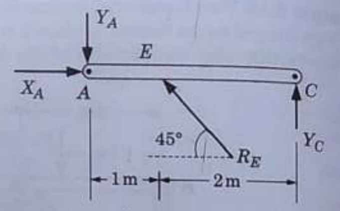


Fig. 9.39 (c)

Beam AC. Consider the equilibrium of the beam AC [Fig. 9.39 (c)].

$\Sigma F_x = 0$: $\quad X_A - R_E \cos 45° = 0$

$$X_A = \frac{R_E}{\sqrt{2}}$$

From (i) $\quad X_A = \frac{R_D}{\sqrt{2}} = \frac{R_E}{\sqrt{2}}$ ...(v)

$\Sigma F_y = 0$: $\quad R_E \sin 45° - Y_A + Y_C = 0$ ...(vi)

Taking moments about C,

$\Sigma M_C = 0$: $\quad -R_E (2 \sin 45°) + Y_A (3) = 0$ ...(vii)

Adding (iv) and (vii)

$$6Y_A = 7500 \quad (\text{as } R_D = R_E)$$

$$Y_A = 1250 \text{ N}$$

Substituting for $Y_A = 1250$ N in (vii)

$$R_E = \frac{3Y_A}{2 \sin 45°} = \frac{3 \times 1250}{2 \times 0.707}$$

Bar 3 → $\quad R_E = 2652$ N (Comp.) $\quad$ as, $R_D = R_E$

Bar 2 → $\quad R_D = 2652$ N (Comp.) $\quad$ *Ans.*

From (ii) $\quad X_A = \frac{R_D}{\sqrt{2}} = \frac{2652}{\sqrt{2}} = 1875.2$ N

$Y_A = 1250$ N

$R_A = \sqrt{(1250)^2 + (1875.2)^2}$

$R_A = 2254$ N(C)

From (iii) $\quad Y_B = 5000 - Y_A - R_D \sin 45°$

$Y_B = 5000 - 1250 - 2652 \times 0.707$

Bar 1 → $\quad Y_B = 1875$ N(C) $\quad$ *Ans.*

From (vi) $\quad Y_C = Y_A - R_E \sin 45°$

$Y_C = 1250 - 2652 \times 0.707$

$Y_C = -625$ N $\quad$ (The assumed direction of the force was compression which is to be reversed)

Bar 4 → $\quad Y_C = +625$ N(T) $\quad$ *Ans.*

Example 9.16 A wire cutter pliers is shown in the Fig. 9.40 (a). Find the force exerted on the wire and the reaction at the hinge E.

***Solution:* Entire Pliers.** Free-body diagram of the pliers is shown in Fig. 9.40 (a).

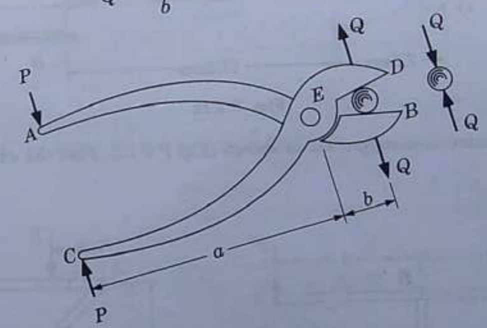
Note that the force Q acting on the wire and the pliers are equal and opposite (action and reaction).

Dismember. Consider the equilibrium of the member AEB. The reaction at the hinge has components X_E and Y_E.

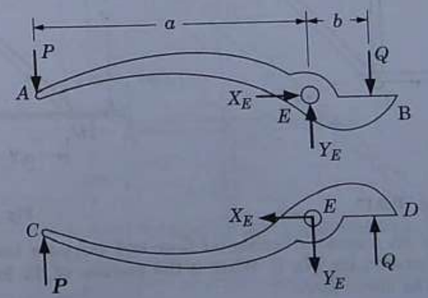
Taking moments about E,

$\Sigma M_E = 0$: $\quad -Q(b) + P(a) = 0$

$$Q = \frac{Pa}{b} \quad \textit{Ans.}$$



(a)



(b)

Fig. 9.40

$\Sigma F_x = 0$: $X_E = 0$

$\Sigma F_y = 0$: $P + Q - Y_E = 0$

Or $Y_E = P + Q = P + \frac{Pa}{b} = \frac{Pb + Pa}{b}$

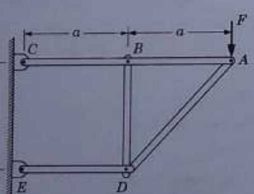
$X_E = 0,\ Y_E = \frac{P(a+b)}{b}$ *Ans.*

PROBLEMS

9.12. A couple of moment M_0 = 550 Nm is applied to the crank *OA* of an engine as shown in the Fig. P.9.12. Determine the force *F* required to be applied to the piston to keep the system in equilibrium. With what force will the piston rub the cylinder? [7.7 kN, 2.2 k

Fig. P.9.12

9.13. For the frame loaded and supported as shown (Fig. P.9.13). Find the axial force in the member *BD*. [2F (T)



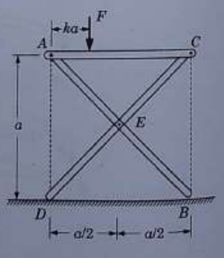


Fig. P.9.13 **Fig. P.9.14**

9.14. A folding table rests on a smooth horizontal floor and carries a load *F* (Fig. P.9.14). Find the value of the shear force on the pin *E*. What is the position of the load *F* corresponding to the maximum value of the shear force.

$\left[\begin{array}{l} F\sqrt{1+(1-2k)^2} \\ \text{Position of load is at } A \text{ or at } C \end{array}\right]$

9.15. A frame consisting of three members is supported and loaded as shown in Fig. P.9.15. Find the rectangular components of the forces transmitted from the members through the connecting pins *C*, *D* and *F*.

$\begin{bmatrix} X_F = 0, & Y_F = +1800\text{N} \\ X_C = 0, & Y_C = +1200\text{N} \\ X_D = 0, & Y_D = +3600\text{N} \end{bmatrix}$

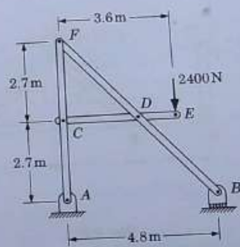


Fig. P.9.15

9.16. The frame shown in the Fig. P.9.16 is supported and loaded as shown. Find the rectangular components of the forces on the pins *E*, *F* and *G*.

$\begin{bmatrix} X_F = 1500\text{N}, & Y_F = 500\text{N} \\ X_E = 1500\text{N}, & Y_E = 1000\text{N} \\ X_G = 1500\text{N}, & Y_G = 500\text{N} \end{bmatrix}$

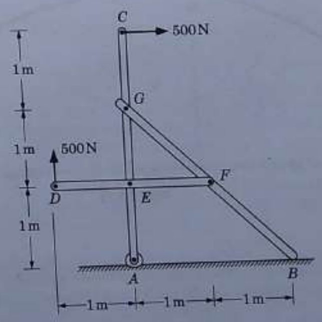


Fig. P.9.16

9.17. Members ABC and CDE are connected by a pin at C and supported by four links 1, 2, 3 and 4. Find the axial force in each link.

$$\left[\begin{array}{ll} F_1 = \dfrac{P}{4}(C), & F_2 = \dfrac{P}{\sqrt{2}}(T) \\ F_3 = \dfrac{P}{\sqrt{2}}(T), & F_4 = \dfrac{P}{4}(T) \end{array}\right]$$

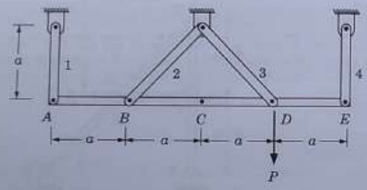


Fig. P.9.17

9.18. An unsymmetrical three hinged arch is shown in Fig. P.9.18. Find the reactions at supports A and B and force on the pin at C.

$$\left[\begin{array}{lll} A_X = 156.25\,\text{kN}; & B_X = 156.25\,\text{kN}; & C_X = 156.25\,\text{kN} \\ A_Y = 187.5\,\text{kN}; & B_Y = 62.5\,\text{kN}; & C_Y = 62.5\,\text{kN} \end{array}\right]$$

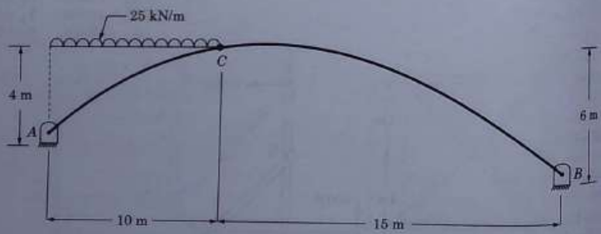


Fig. P.9.18

10 CHAPTER Uniform Flexible Suspension Cables

10.1 CABLES AND LOADING

Cables are flexible members which find many engineering applications such as in transmission lines, suspension bridges, aerial rope ways etc.

A cable is assumed to be perfectly flexible when its resistance to bending is small and may be neglected. Consequently the internal force at any point in the cable is a force of tension directed along the cable. Further, a cable is assumed to be a member of uniform cross-section.

A cable suspended between two supports at its ends (Fig. 10.1) may be subjected to different types of loadings and the shape assumed by the cable, therefore shall depend on the type of loading. Various types of loads are,

Fig. 10.1

1. Concentrated loads
2. Distributed loads

When a cable is subjected to concentrated loads, its own weight is assumed to be negligibly small.

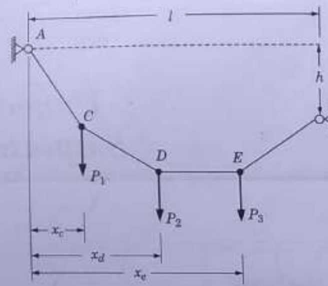
A cable subjected to distributed loads may be loaded in any one of the two ways, as follows :

Cable Uniformly Loaded Per Unit Horizontal Distance. This case is represented by a uniformly distributed load attached to the cable by vertical hangers (Fig. 10.5). The loading is usually much larger than the weight of the cable itself. The cable in this case acquires a parabolic shape. Cables in suspension bridges fall under this category.

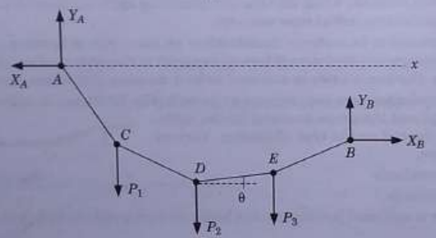
Cable Uniformly Loaded Per Unit Length Along the Cable Itself. This case is represented by a flexible cable or chain freely suspended in the gravity field and subjected only to the action of its own distributed weight (Fig. 10.10). The cable in this case acquires the shape of a catenary. Cables in power transmission lines fall under this category.

10.2 CABLE SUBJECTED TO CONCENTRATED LOADS

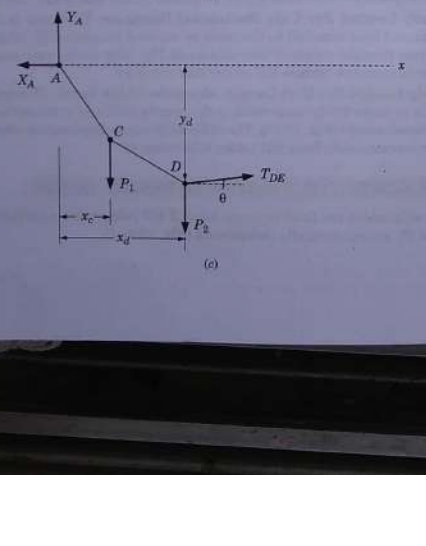
Consider a cable attached to two fixed supports A and B and subjected to a number of concentrated loads P_1, P_2 and P_3 acting vertically downwards [Fig. 10.2(a)].



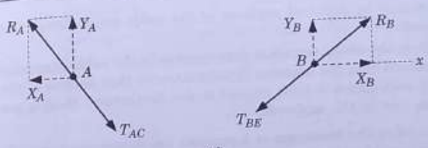
(a)



(b)



(c)



(d)

Fig. 10.2

It is assumed that,

(i) The weight of the cable is negligibly small as compared to the loads supported by it.

(ii) The cable is assumed to be perfectly flexible which means that it offers no resistance to bending and adjusts it shape corresponding to the applied load. Which also implies that the force of tension in the cable is directed along the cable.

In this problem, we are interested in determining

(a) The reaction at the supports A and B which are represented by their rectangular components X_A, Y_A and X_B, Y_B.

(b) Tension in parts of the cable and the maximum tension in the cable.

(c) Sag (or dip) of some point of the cable and the shape assumed by the cable.

Reactions of Supports. Consider the equilibrium of the *entire* cable. The Free-body diagram is as shown [Fig. 10.2 (b)].

Note here that we can write *only three equations* of equilibrium say,

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_B = 0 \qquad \text{...(10.1)}$$

Above three equations shall involve *four unknowns*, X_A, Y_A and X_B, Y_B, so an additional equation is required. For this, we have to consider the equilibrium of *a portion* of the cable. Consider the equilibrium of the portion ACD of the cable [Fig. 10.2(c)]. Forces involved are components X_A, Y_A of the reaction at A, loads P_1, P_2 and tension T_{DE} in the portion DE acting at D.

Problem shall remain unsolved unless we know the sag (Y_d) of the point D. This shall enable us to take moments about D eliminating the force T_{DE}.

Consider next the equilibrium of point A [Fig. 10.2 (d)]. Forces involved are, tension T_{AC} in the portion AC and acting at A and the components of the reaction X_A, Y_A at A. For the point A to be in equilibrium resultant R_A of X_A and Y_A should be equal and collinear with the tension T_{AC} in the portion AC of the cable.

Same is true for the supports B.

Tension of the Cable. Consider again the equilibrium of the portion ACD [Fig. 10.2 (c)].

$$\Sigma F_x = 0 : \qquad T_{DE} \cos \theta - X_A = 0$$

where, θ is the angle which the portion DE of the cable makes with the horizontal.

Hence,

$$T_{DE} \cos \theta = X_A \qquad \text{...(10.2)}$$

Above equation states that horizontal component of the tension in the portion DE of the cable is equal to the horizontal component of the reaction at the support $A(X_A)$. Such a relationship is true for tension in all portions of the cable. Thus, *Although the tensions in different portions of the cable are different but their horizontal components are equal.*

Also, it follows from the above equation that tension in the cable is maximum when $\cos\theta$ is minimum. *Which means that the tensions is maximum in that portion of the cable which has the largest angle of inclination θ with respect to the horizontal. Such a portion of the cable must be adjacent to one of the supports.*

Sag of any Point. After the reactions at a support are determined, sag of any point (say E) can be determined by considering the equilibrium of the portion of the cable between any support (say A) and that point (that is AE). The moment equation, with E as moment centre, shall determine its sag.

To summarize:

1. Horizontal component of tension in any portion of the cable is same and is equal to the horizontal components of the support reactions ($X_A = X_B$).
2. The maximum tension occurs in that portion of the cable which has largest inclination θ with the horizontal. Such a portion is adjacent to one of the supports.
3. To solve a problem, in general, either the sag of any one point of the cable or the horizontal component to the tension in the cable must be known.

Example 10.1 A cable AB supports three vertical loads as shown [Fig. 10.3 (*a*)]. Determine (*i*) the components of reaction at B (*ii*) sag of the point E (*iii*) tension in the portion CA and (*iv*) maximum slope and tension in the cable.

Solution: Let the components of reaction at B be X_B and Y_B and at A be X_A and Y_A.

Free-body of the entire cable is as shown [Fig. 10.3 (*b*)].

$$\Sigma M_A = 0: \quad Y_B(8) - X_B(2) - 400(2) - 600(4) - 400(6) = 0$$

$$8Y_B - 2X_B = 5600 \quad \ldots(i)$$

Since the sag of the point D is known, we consider next the equilibrium of the portion BED of the cable.

Free-body of BED is as shown [Fig. 10.3(*c*)].

$$\Sigma M_D = 0: \quad Y_B(4) - X_B(2.25) - 400(2) = 0$$

$$4Y_B - 2.25\,X_B = 800 \quad \ldots(ii)$$

Solving (*i*) and (*ii*) simultaneously,

$$X_B = 1600\text{ N, and } Y_B = 1100\text{ N.} \quad \textbf{\textit{Ans.}}$$

X_B also represents the horizontal component of the tension in the cable.

Consider the equilibrium of the portion BE of the cable.

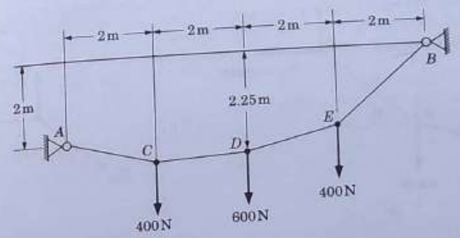
Free-body of BE is as shown [Fig. 10.3 (*d*)].

$$\Sigma M_E = 0: \quad 1100(2) - 1600(y_e) = 0$$

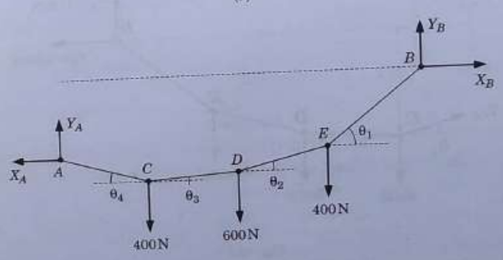
$$y_e = 1.375\text{ m} \quad \textit{Ans.}$$

$$\tan\theta_1 = \frac{1.375}{2},\ \theta_1 = 34.5°$$

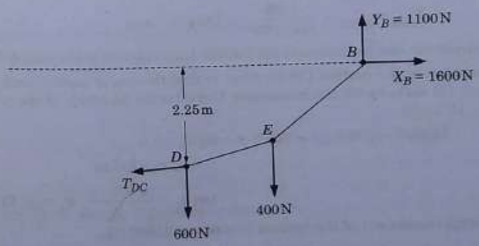
It can be observed that the maximum slope occurs in the position BE of the cable. The maximum tension should also be in this portion (T_{BE}).



(*a*)



(*b*)



(*c*)

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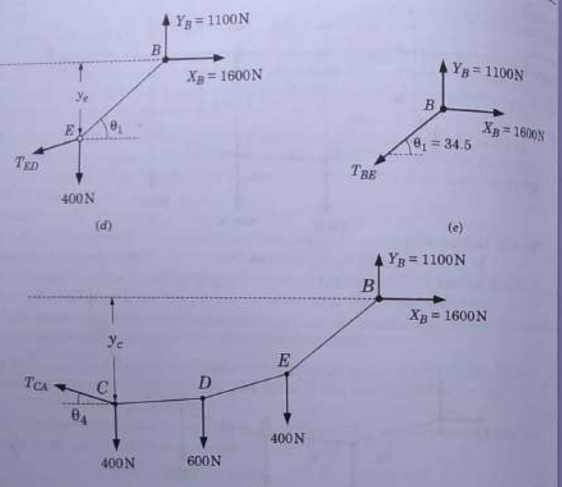


Fig. 10.3

As the horizontal component of the tension in constant and equal to 1600 N

$$T_{max} = T_{BE} = \frac{1600}{\cos 34.50°} = 1941 \text{ N} \quad \textbf{\textit{Ans.}}$$

Above relationship can also be obtained from the free-body diagram of the point B [Fig. 10.3(e)

To find the tension in the portion CA we have to find the sag of point C and the angle θ_4 which the portion CA makes with the horizontal. Consider the free-body of the portion $BEDC$ of the cable [Fig. 10.3 (f)].

$\Sigma M_C = 0$: $\quad 1100(6) - 1600(y_c) - 400 \times 4 - 600(2) = 0$

$$y_c = 2.375 \text{ m}$$

$$\tan\theta_4 = \frac{2.375 - 2}{2}, \; \theta_4 = 10.62°$$

As the horizontal component of the tension is constant (1600 N),

$$T_{CA} = \frac{1600}{\cos\theta_4} = \frac{1600}{\cos 10.62°}$$

$$T_{CA} = 1627.9 \text{ N}$$

Example 10.2. A cable supports three vertical loads as shown [Fig. 10.4 (a)]. Determine (i) the components of the supports reactions (ii) tensions in the portion AC and CD and (iii) maximum tension in the cable if the horizontal component of the tension in the cable is 200 N.

Solution: Consider the equilibrium of the entire cable. Let the components of the support reactions be X_A, Y_A and Y_B, Y_B.

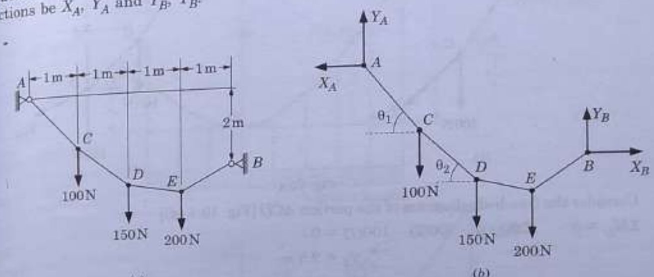


Fig. 10.4

Free-body diagram of entire cable is as shown [Fig. 10.4 (b)].

$\Sigma F_x = 0$: $\quad X_A = X_B =$ Horizontal component of the tension in the cable.

$X_A = X_B = 200$ N

$\Sigma M_A = 0$: $\quad Y_B(4) + 200(2) - 200(3) - 150(2) - 100(1) = 0$

$$Y_B = 150 \text{ N}$$

$\Sigma F_y = 0$: $\quad Y_A + Y_B - 100 - 150 - 200 = 0$

$$Y_A = 300 \text{ N}$$

Consider the free-body diagram of the portion AC [Fig. 10.4(c)]

$\Sigma M_C = 0$: $\quad 200(y_c) - 300(1) = 0$

$$y_c = 1.5 \text{ m}$$

$$\tan\theta_1 = \frac{1.5}{1}, \; \theta_1 = 56.3$$

$$T_{AC}\cos\theta_1 = \text{Horizontal component of the tension in the cable}$$

$$T_{AC} = \frac{200}{\cos 56.3°} = 360 \text{ N}$$

This is also the maximum tension in the cable

$$T_{max} = 360 \text{ N}$$

Alternatively, $\quad T_{max} = \sqrt{(X_A)^2 + (Y_A)^2} = \sqrt{(200)^2 + (300)^2} = 360$ N

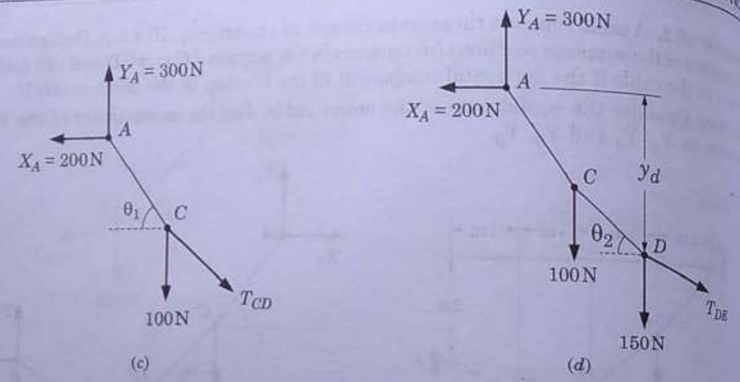


Fig. 10.4

Consider the free-body diagram of the portion ACD [Fig. 10.4 (*d*)]

$\Sigma M_D = 0$: $\quad 200(y_d) - 300(2) - 100(1) = 0$

$$y_d = 2.5 \text{ m}$$

$$\tan\theta_2 = \frac{2.5-1.5}{1} = 1, \; \theta_2 = 45°$$

$$T_{CD} = \frac{200}{\cos 45°} = 282.8 \text{ N}$$

Tension in the other portions can be found similarly.

10.3 CABLE UNIFORMLY LOADED PER UNIT HORIZONTAL DISTANCE (PARABOLIC CABLE)

Consider a flexible cable attached to two fixed supports A and B subjected to a uniformly distributed load w per unit horizontal length as shown in Fig. 10.5. The weight of the cable is negligibly small as compared to the load supported by it. We want to determine the shape of the curve assumed by the cable and tension in the cable.

Let us choose the coordinate axes x and y as shown with the lowest point O of the curve as origin.

To find the tension at any point $P(x, y)$ on cable, consider the equilibrium of the portion OP extending from the lowest point O to the point P. Free-body diagram of the portion OP is as shown in Fig. 10.5.

The various forces acting on it are,

(*i*) The downward load $W = wx$, representing the resultant of the distributed load supported by the portion OP of the cable and acting mid-way between O and P.

(*ii*) Tension T_0 at the lowest point O acting horizontally.

(*iii*) Tension T in the cable at P directed along the tangent to the curve at P. Let this inclined at angle θ with the horizontal.

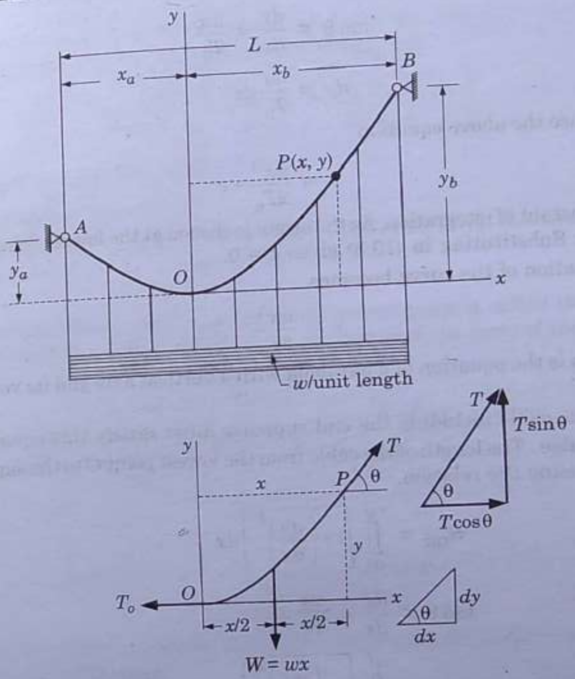


Fig. 10.5

The equations of equilibrium for the portion OP of the cable are

$\Sigma F_x = 0$: $\quad T\cos\theta - T_0 = 0$

Or $\quad T\cos\theta = T_0 \qquad$ ...(10.3)

$\Sigma F_y = 0$: $\quad T\sin\theta - wx = 0$

Or $\quad T\sin\theta = wx \qquad$ ...(10.4)

From (10.3) and (10.4)

$$T = \sqrt{(T_0)^2 + (wx)^2} \qquad ...(10.5)$$

Above equation gives the value of tension at any point of the cable. Tension is minimum equal to T_0 at the lowest point O where $x = 0$. Tension is maximum where x = maximum or at the support B.

Also from (10.3) and (10.4)

$$\tan\theta = \frac{wx}{T_0} \qquad ...(10.6)$$

Since, $\quad \tan\theta = \dfrac{dy}{dx}$

$$\tan\theta = \frac{dy}{dx} = \frac{wx}{T_0}$$

$$dy = \frac{wx}{T_0}dx \qquad ...(10.7)$$

Integrating once the above equation

$$y = \frac{wx^2}{2T_0} + c \qquad ...(10.8)$$

where, c is the constant of integration. As the origin is chosen at the lowest point O, its coordinates are $x = 0$, $y = 0$. Substituting in (10.8) gives $c = 0$.

Hence the equation of the curve becomes

$$y = \frac{wx^2}{2T_0} \qquad ...(10.9)$$

Above equation is the equation of a parabola with a vertical axis and its vertex at the lowest point O.

All points on the cable including the end supports must satisfy this equation.

Length of the Cable : The length of the cable from the lowest point O to the support B (Fig. 10.5) may be obtained using the relation,

$$S_{OB} = \int_0^{x_b}\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)dx$$

Substituting $\tan\theta = \frac{dy}{dx} = \frac{wx}{T_0}$

$$S_{OB} = \int_0^{x_b}\left(\sqrt{1+\left(\frac{wx}{T_0}\right)^2}\right)dx$$

Using binomial theorem to expand the radical

$$S_{OB} = \int_0^{x_b}\left(1+\frac{w^2x^2}{2T_0^2}-\frac{w^4x^4}{8T_0^4}+...\right)dx$$

$$= x_b\left(1+\frac{w^2x_b^2}{6T_0^2}-\frac{w^4x_b^4}{40T_0^4}+...\right)$$

Since, $y_b = \frac{wx_b^2}{2T_0}$ (x_b, y_b; are the coordinates of B)

$$S_{OB} = x_b\left[1+\frac{2}{3}\left(\frac{y_b}{x_b}\right)^2-\frac{2}{5}\left(\frac{y_b}{x_b}\right)^4+...\right] \qquad ...(10.10)$$

In most cases the ratio y_b/x_b is very small, therefore, only first two terms of the series need to be considered.

Similarly the length of the cable from lowest point O to the support A can be calculated by replacing the coordinates of B by the coordinates of $A(x_a, y_b)$.

$$S_{OA} = x_a\left[1+\frac{2}{3}\left(\frac{y_a}{x_a}\right)^2-\frac{2}{5}\left(\frac{y_a}{x_a}\right)^4+...\right] \qquad ...(10.11)$$

$$\text{Total length of the cable} = S_{OA} + S_{OB} \qquad ...(10.12)$$

Supports At The Same Level

It is not necessary that the supports A and B be at the same level. However, if the supports are at the same level,

$$y_a = y_b = h$$

The vertical distance h from the supports to the lowest point is called the sag of the cable and the horizontal distance L between the supports is called the span of the cable (Fig. 10.6).

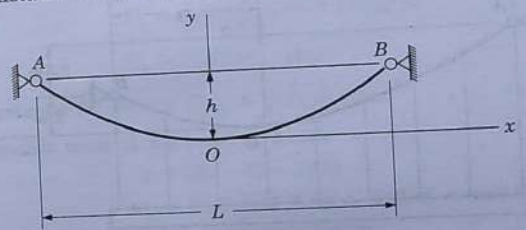


Fig. 10.6

In such a case, the curve (parabola) becomes symmetrical about the y-axis and the results obtained earlier becomes simpler.

$$y = \frac{wx^2}{2T_0}$$

Substituting the coordinates of B ($L/2$, h)

$$h = \frac{w(L/2)^2}{2T_0}$$

$$\text{Sag, } h = \frac{wL^2}{8T_0} \qquad ...(10.13)$$

$$\text{Minimum Tension, } T_0 = \frac{wL^2}{8h} \qquad ...(10.14)$$

Maximum Tension occurs at A and B which are equal,

$$T_{max} = \sqrt{(T_0)^2+(wL/2)^2}$$

$$= \sqrt{\left(\frac{wL^2}{8h}\right)^2+\left(\frac{wL}{2}\right)^2}$$

$$\text{Length of the Cable} = S_{OA} + S_{OB} = 2S_{OB}$$

$$= 2 \times \frac{L}{2}\left[1+\frac{2}{3}\left(\frac{h}{L/2}\right)^2 - \frac{2}{5}\left(\frac{h}{L/2}\right)^4 + \ldots\right]$$

$$= \left(L+\frac{8}{3}\frac{h^2}{L} - \frac{32}{5}\frac{h^4}{L^3} + \ldots\right) \quad \ldots(10.3)$$

Example 10.3. A straight pipe of mass per unit length of 75 kg/m is supported by a cable (Fig. 10.7). The cable is suspended from two points A and B 20 m apart. Determine the location of the lowest point C of the cable and maximum and minimum tension in the cable.

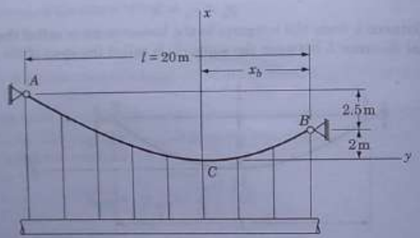


Fig. 10.7

Solution: As the load acting is per unit horizontal distance, it is a parabolic cable.

Equation of the parabolic cable is,

$$y = \frac{wx^2}{2T_0}$$

or

$$T_0 = \frac{wx^2}{2y} \quad \ldots(i) \quad \left\{\begin{array}{l}\text{where, } T_0 \text{ is the tension at the} \\ \text{lowest point } C \text{ and} \\ \qquad w = 75 \times 9.81 \text{ N/m}\end{array}\right.$$

Let the point B be at a horizontal distance x_b from the lowest point C.

The co-ordinates of A and B are $[-(20-x_b), 4.5]$ and $(x_b, 2)$ respectively and should satisfy the equation (i)

$$T_0 = \frac{w(20-x_b)^2}{2 \times 4.5} \quad \ldots(ii)$$

$$T_0 = \frac{w(x_b)^2}{2 \times 2} \quad \ldots(iii)$$

Equating (ii) and (iii)

$$\frac{w(20-x_b)^2}{9.0} = \frac{w(x_b)^2}{4}$$

$$x_b^2 + 32x_b - 320 = 0$$

$$x_b = \frac{32 \pm \sqrt{(32)^2 + 4(320)}}{2}$$

$$x_b = \frac{-32 \pm 48}{2}$$

$$x_b = 8 \text{ m} \qquad \text{(Taking the +ve value)}$$

Thus the lowest point C is situated at a horizontal distance of 8 m from the support B.

$$\text{Maximum Tension } T_0 = \frac{wx^2}{2y}$$

Substituting the coordinates of any point in the above relation say of $B(8, 2)$

$$T_0 = \frac{75 \times 9.81(8)^2}{2 \times 2} = 11772 \text{ N}$$

$$T_0 = 11.77 \text{ kN} \quad \textit{Ans.}$$

$$\text{Maximum Tension} = T_{max} = \sqrt{(T_0)^2 + (wx)^2}$$

The maximum tension occur at the point A, hence

$$x = (20 - 8) = 12 \text{ m}$$

$$w_x = 75 \times 9.81 \times 12 = 8829 \text{ N} = 8.83 \text{ kN}$$

$$T_{max} = \sqrt{(11.77)^2 + (8.83)^2}$$

$$T_{max} = 14.71 \text{ kN} \quad \textbf{\textit{Ans.}}$$

Example 10.4 A cable is attached to a support at A and passes over a small pulley B and supports a load W (Fig. 10.8). The mass of the cable per unit length is 0.5 kg/m. (i) Determine the magnitude of the load W (ii) angle of the cable with the horizontal at B and (iii) the total length of the cable AB. Assume the cable to assume a parabolic shape.

Solution. Treat the load to be uniformly distributed along the horizontal as the cable has been assumed to be parabolic

Weight of the cable per unit length

$$w = mg = 0.5 \times 9.81 = 4.90.5 \text{ N/m}$$

Equation of the cable is

$$y = \frac{wx^2}{2T_0}$$

or

$$T_0 = \frac{wx^2}{2y} \quad \ldots(i)$$

Substituting the coordinates of point B in the above equation

$$T_0 = \frac{(4.905)(15)^2}{2 \times 0.5}$$

$$T_0 = 1103.6 \text{ N}$$

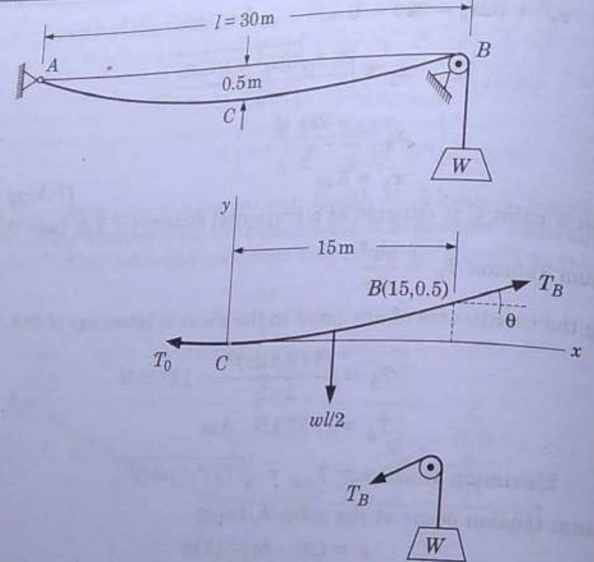


Fig. 10.8

Tension in the cable at the point B,

$$T_B = \sqrt{T_0^2 + (wl/2)^2}$$

$$T_B = \sqrt{(1103.6)^2 + (4.905 \times 15)^2}$$

$$T_B = 1106 \text{ N}$$

As the pulley is frictionless the tension in the cable on each side of the pulley is same, hence

$$W = T_B = 1106 \text{ N}$$

Slope of the cable at B

$$T_B \cos\theta = T_0$$

or

$$\cos\theta = \frac{T_0}{T_B} = \frac{1103.6}{1106} = 0.998$$

$$\theta = 3.8°$$

$\left(\text{Alternatively, } \tan\theta = \frac{wl/2}{T_0}\right)$

Length of the Cable.

Length of cable between C and B

$$S_{CB} = x_b\left[1 + \frac{2}{3}\left(\frac{y_b}{x_b}\right)^2 + \ldots\right] = 15\left[1 + \frac{2}{3}\left(\frac{0.5}{15}\right)^2 + \ldots\right]$$

$$= 15[1 + 0.0007407]$$

$$= 15.0111 \text{ m}$$

Total length of the cable AB

$$S_{AB} = 2 \times 15.0111 \text{ m}$$

$$= 30.022 \text{ m}$$

Example 10.5 A electric cable is attached to two electric poles 30 m apart (Fig. 10.9). If the cable can resist a pull of 400 N and weights 3 N/m determine the smallest value of the sag in the cable assuming it to take a parabolic shape.

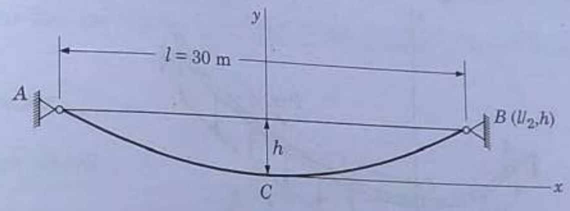


Fig. 10.9

Solution:

$$w = 3 \text{ N/m}$$

$$l = 30 \text{ m}$$

$$\text{Max Pull or Tension} = 400 \text{ N}$$

Equation of the curve is

$$y = \frac{wx^2}{2T_0}$$

or

$$T_0 = \frac{wx^2}{2y}$$

Substituting the co-ordinates of point $B(l/2, h)$ where h is the sag in the cable

$$T_0 = \frac{w(l/2)^2}{2h} = \frac{wl^2}{8h}$$

Maximum pull or tension occurs at B,

$$T_B = T_{max} = \sqrt{T_0^2 + (wl/2)^2}$$

or

$$400 = \sqrt{\left(\frac{wl^2}{8h}\right)^2 + \left(\frac{wl}{2}\right)^2}.$$

$$400 = \frac{wl}{2}\sqrt{\frac{l^2}{(4h)^2} + 1}$$

$$\left(\frac{400 \times 2}{wl}\right)^2 = \frac{l^2}{16h^2} + 1$$

Substituting $l = 30$ m,

$$\left(\frac{400 \times 2}{3 \times 30}\right)^2 = \frac{(30)^2}{16h^2} + 1$$

$$78\,h^2 = 56.25$$

$$h = 0.848 \text{ m} \quad \textbf{\textit{Ans.}}$$

10.4 CABLE UNIFORMLY LOADED PER UNIT LENGTH ALONG THE CABLE ITSELF (CATENARY CABLE)

Consider a cable (Fig. 10.10) loaded uniformly per unit length along the cable itself and supported at A and B. Cables hanging freely under their own weight are loaded in this way. *The cable under consideration is flexible but not weightless.*

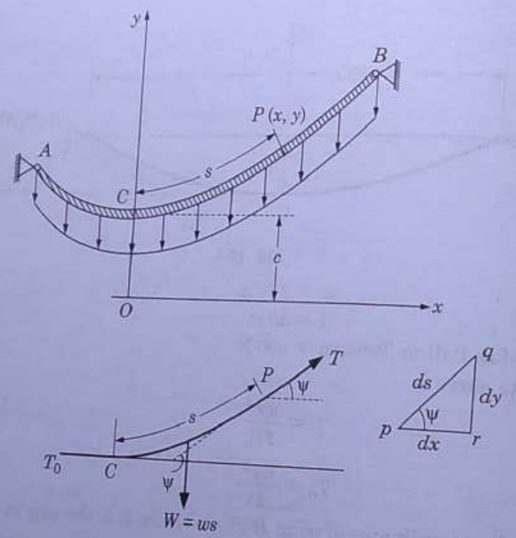


Fig. 10.10

Let w be the load per unit length measured *along* the cable.

Consider the equilibrium of a portion CP of the cable extending from the lowest point C to the point P under consideration and having a length s.

Free-body diagram of this portion is as shown in Fig. 10.10. The various forces acting on it are:

(*i*) The downward load $W = ws$ representing the resultant of the distributed load supported by the portion CP of the cable. The location of W is not known.

(*ii*) Tension T in the cable at P directed along the tangent to the curve at P. Let it be inclined at an angle ψ with the horizontal.

(*iii*) Tension T_0 acting horizontally at the lowest point C.

Equations of equilibrium are :

$\Sigma F_x = 0$:
$$T\cos\psi - T_0 = 0$$
or
$$T\cos\psi = T_0 \qquad ...(10.16)$$
$\Sigma F_y = 0$:
$$T\sin\psi - ws = 0$$
$$T\sin\psi = ws \qquad ...(10.17)$$

From (10.16) and (10.17)
$$T = \sqrt{(T_0)^2 + (ws)^2} \qquad ...(10.18)$$
$$\tan\psi = \frac{ws}{T_0} = \frac{dy}{dx} \qquad ...(10.19)$$

We cannot integrate the equation (10.19) as we do not know the relation between the length s and the co-ordinates x, y of the point P.

Let us assume that the tension T_0 is equal to the weight of c units of length of the cable

or
$$T_0 = wc \qquad ...(10.20)$$

From triangle pqr
$$dx = ds\cos\psi$$

From equation (10.16)
$$\cos\psi = \frac{T_0}{T}$$

So,
$$dx = \frac{T_0}{T}ds$$

As
$$T_0 = wc,\ T = \sqrt{(T_0)^2 + (ws)^2} = \sqrt{(wc)^2 + (ws)^2}$$

Hence,
$$dx = \frac{wc}{\sqrt{(wc)^2 + (ws)^2}}ds = \frac{ds}{\sqrt{1+\left(\frac{s}{c}\right)^2}}$$

Before carrying out the integration, let us select the origin O of the coordinates such that it is at a distance c directly below the point C.

Integrating from $C(o, c)$ to $P(x, y)$
$$x = \int dx = \int_0^s \frac{ds}{\sqrt{1+\left(\frac{s}{c}\right)^2}} = c\left[\sinh^{-1}\frac{s}{c}\right]_0^s$$
$$x = c\sinh^{-1}\frac{s}{c}$$
or
$$s = c\sinh\frac{x}{c} \qquad ...(10.21)$$

Above is the relation in terms of s and c.

To determine the relation in terms of x and y, we can write
$$\frac{dy}{dx} = \tan\psi$$
or
$$dy = dx\tan\psi$$

But,
$$\tan\psi = \frac{ws}{T_0} = \frac{ws}{wc} = \frac{s}{c}$$

Hence,
$$dy = \frac{s}{c}dx$$

Further from equation (10.21)

$$\frac{s}{c} = \sinh\frac{x}{c}$$

Hence,

$$dy = \sinh\frac{x}{c}\,dx \quad \text{...(10.22)}$$

integrating the above equation (10.22) from $C(0, c)$ to $P(x, y)$

$$\int_c^y dy = \int_0^x \sinh\frac{x}{c}\,dx = c\left[\cosh\frac{x}{c}\right]_0^x$$

$$y - c = c\left(\cosh\frac{x}{c} - 1\right)$$

$$y = c\cosh\frac{x}{c} \quad \text{...(10.23)}$$

Above represents the equation of a catenary with vertical axis. Distance 'c' is called the parameter of the catenary. C is the vertex of the catenary.

Various other relations can be derived which are summarised as below :

1. (*a*) Relation between x and y $\quad y = \cosh\frac{x}{c}$

   (*b*) Relation between s and x $\quad s = c\sinh\frac{x}{c}$

   (*c*) Relation between s and y $\quad y^2 = c^2 + s^2$

   (*d*) Relation between s and ψ $\quad s = c\tan\psi$

2. Tension at the lowest point $\quad T_0 = wc$

3. Tension at any point $\quad T = wy$

   (It is proportional to the height of the point above the x-axis.)

4. Span of the catenary when both supports are at the same level

$$= 2c\log(\sec\psi + \tan\psi)$$

where ψ is the angle that the tangent to the curve at the support (A or B) makes with the horizontal.

It can be shown that when the quantity x/c becomes very small or the cable is very light the catenary approximates to a parabola. In that case the relation for the parabolic cable holds good. It may be recalled that in the case of parabolic cable the lowest point itself was chosen as the origin of coordinates.

Example 10.6 A cable of length 200 m and mass 1000 kg is suspended between two points at the same elevation with a sag of 50 m. Find the horizontal distance between the supports and the maximum tension.

Solution: Length of the cable = 200 m

Mass of the cable = 1000 kg

$$w = \frac{1000 \times 9.81}{200} = 49.05 \text{ N/m}$$

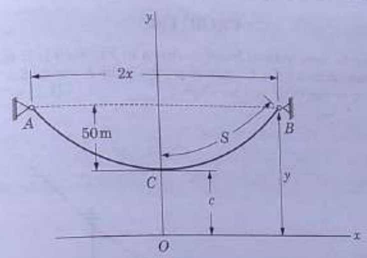


Fig. 10.11

The equation of catenary is

$$y^2 = c^2 + s^2 \quad \text{...(i)}$$

For the point B,

$$s = \frac{200}{2},\ y = c + 50$$

Substituting the value in the equation (*i*)

$$(c + 50)^2 = c^2 + (100)^2$$
$$c^2 + 100c + 2500 = c^2 + 10000$$
$$100c = 7500$$
$$c = 75 \text{ m}$$

$$\text{Maximum tension} = \sqrt{(wc)^2 + (ws)^2}$$
$$c = 75 \text{ m},\ s = 100 \text{ m},\ w = 49.05 \text{ N/m}$$
$$T_{max} = \sqrt{(49.05 \times 75)^2 (49.05 \times 100)^2}$$
$$T_{max} = 6131 \text{ N}$$

To determine the span ($2x$) let us use the equation of catenary

$$y = c\cosh\frac{x}{c}$$

For the point B,

$$x = ?,\ y = c + 50 = 75 + 50 = 125 \text{ m}$$

Substituting in the above equation

$$125 = 75\cosh\frac{x}{75}$$

$$\cosh\frac{x}{75} = \frac{125}{75} = 1.667$$

$$\frac{x}{75} = \cosh^{-1} 1.667$$

$$\frac{x}{75} = 1.1$$

$$x = 75 \times 1.1 = 82.5$$

$$\text{Span} = 2 \times 82.5 = 165 \text{ m}$$

PROBLEMS

10.1. A cable AB supports three vertical loads as shown in Fig. P.10.1. If the point D is 1 m below the support A determine (i) the heights of point C and E above the support A and (ii) the maximum tension in the cable and its slope. [1.11 m 1.16 m, 43.4°, 24.8 kN]

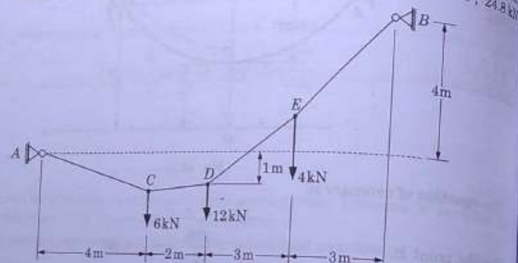


Fig. P.10.1

10.2. A flexible cable is supported as shown in Fig. P.10.2. For a certain value of weight W the sag to span ratio is h/l. Find the sag to span ratio if the weight W is doubled. Assume the cable to be parabolic.

$$\left[\frac{h}{l}=\frac{1}{25}\right]$$

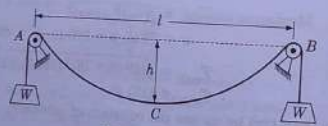


Fig. P.10.2

10.3. A cable carrying a load of 10 kN per metre run of horizontal span is stretched between supports 100 m apart. The supports are at the same level and the sag is 8 m. Find the maximum and minimum tension in the cable. [1640 kN, 1562 kN]

10.4. Show that the length of an endless chain which will hang over a circular cylinder of radius r so as to be in contact with the two-third of the circumference of the cylinder is,

$$r\left\{\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right\}$$

[**Hint :** Span = $2c\log(\sec\psi+\tan\psi)$].

11 CHAPTER Graphical Analysis: Coplanar Forces and Trusses

11.1 INTRODUCTION

The graphical methods of solving the problems of statics are based on the graphical representation of a force by a vector and the law of polygon of forces. The implementation of the procedure shall require the concepts of Bow's notation. Funnicular polygon and Maxwell diagram shall also be explained in this chapter. The graphical methods have the advantage that a person with the limited knowledge of statics can be trained to use the method mechanically, almost like a tool, and can solve the problems easily and efficiently.

A graphical method applicable to a general system of coplanar forces is described and explained with the help of the solved examples. The analysis of simple trusses by graphical method is also included in this chapter.

Example 11.1 Explains the Various Terms and the Procedure.

Example 11.1. A body is being acted upon by four force as shown in Fig. 11.1. Find the magnitude, direction and line of action of a force which should be applied to the body to keep the body in equilibrium.

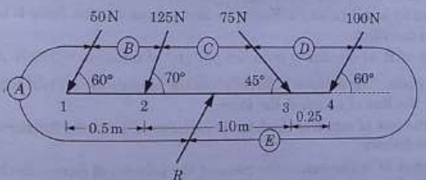


Fig. 11.1

Solution: Draw the sketch of the body to a suitable scale and mark the forces acting on it as shown in Fig. 11.1.

Let the force **R** be required to keep the body in equilibrium. *It may be noted here that* **R** *is not the resultant of the system of forces acting on the body but is their equilibrant.*

Space Diagram (Naming of Spaces and Forces). Bow's notation is used for the designation of the various forces. In this method the spaces between the lines of action of the forces are given some notations. The spaces can be denoted by the letters *A, B, C* ... etc. as shown.

A force then is represented by the letters denoting the two spaces separated by the line of action of the force. That is, by the spaces lying on the two sides of the line of action of the force. Such a diagram is called the space diagram as shown in Fig. 11.1.

The method is quite similar to the method one often uses to point out a person in a crowd. For example if one wants to point out a person (Say Ram) to his friend, he can do so by saying that Ram is the person standing between the man wearing a blue shirt and the man wearing a red shirt.

Coming back to the example, let us denote the spaces between the lines of action of the various forces as :

Space between the resultant **R** and the force of 50 N by *A*
Space between the force of 50 N and the force of 125 N by *B*
Space between the force of 125 N and the force of 75 N by *C*
Space between the force of 75 N and the force of 100 N by *D*
Space between the force of 100 N and the force of **R** by *E*.

The various forces now shall be represented as :

The force of 50 N by *AB*
The force of 125 N by *BC*
The force of 75 N by *CD*
The force of 100 N by *DE*
The resultant **R** by *EA*

While denoting the spaces in between the forces *one should go around the body in a cyclic order*. This sense of order should be observed when referring to the forces at any time.

Polygon of Forces. To draw the polygon of forces *always start with a known force*. Begin by choosing any point *a*. Draw a vector $\overline{ab}$ to represent the force *AB* of 50 N to some suitable scale. Similarly draw vectors $\overline{bc}$, $\overline{cd}$ and $\overline{de}$ parallel to the forces of 125 N, 75 N and 100 N. The polygon *abcde* is completed by joining *e* and *a*. Vector $\overline{ea}$ now represents the force **R** (equilibrant) both in magnitude and direction.

The magnitude of **R** as measured from the polygon of forces is = 310 N *Ans.*

Funnicular Polygon. The polygon of forces gives the magnitude and direction of force **R** but does not indicate the line of action of the force.

To determine the line of action of the force **R** we have to draw the funnicualar polygon. We have to proceed as follows.

An arbitrary point '*O*' is choosen in the plane of the polygon of forces. Such a point is called *pole*. Join the pole *O* to the *apexes* of the force polygon *a, b, c, d* and *e*.

The connecting lines *oa, ob, oc, od* and *oe* are called the *rays*. These rays are in fact vectors representing forces and can also be named as ray 1, ray 2 etc. for convenience.

Begin with any point *a'* in the space diagram and draw a line *a'b'* parallel to the *oa* and intersecting the line of action of the force *AB* (50 N) at *b'*. From *b'* draw a line *b'c'* parallel to the next ray *ob* and intersecting the line of action of the force *BC* (125 N) at *c'*. Similarly draw lines *c'd'*, *d'e'* and *e'f'* parallel to the rays *oc, od* and *oe*. Extend the lines *a'b'* and *f'e'* to intersect at *g'*.

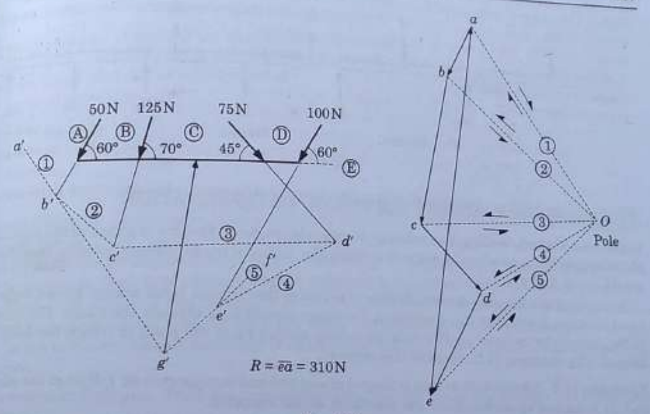


Fig. 11.2

The line of action of the force **R** shall pass through the point *g'*. Draw a line through *g'* and parallel to the side $\overline{ea}$ of the force polygon. The line so obtained determines the line of action of the force **R** acting on the body.

It should be noted here that the apexes of the funnicular polygon lie on the lines of action of the various forces and its sides are parallel to the rays.

11.2 GRAPHICAL CONDITIONS OF EQUILIBRIUM

For any general system of forces in equilibrium, the equations of equilibrium are to be satisfied when using the analytical method.

The corresponding conditions of equilibrium in graphical methods are,

(*a*) A closed polygon of force.
(*b*) A closed funnicular polygon.

The various steps now can be summarized as :

1. Draw the body together with the force acting on it to some convenient scale.
2. Draw the space diagram and represent the forces using Bow's notation. In this connection it may be noted that either the head or the tail of the vector can be used to represent the point of application of a force. But a consistent method of representation is to be used for all the forces in a problem (Fig. 11.3).
3. Draw of polygon of forces starting from a known force. Choose a pole and then drawn the rays.
4. Draw the funnicular polygon with its sides parallel to the rays.
5. Determine the unknown force.

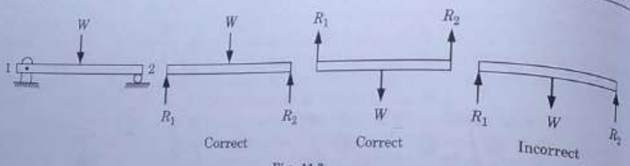


Fig. 11.3

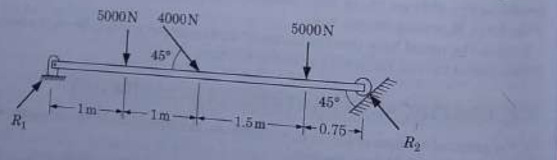
11.3 REACTION AT THE SUPPORTS: DETERMINATION

In the case of a body resting on a roller or a knife edge support, the line of action of the reaction at the support is normal to the support surface. The vertex of the funnicular polygon, therefore, would lie on this normal.

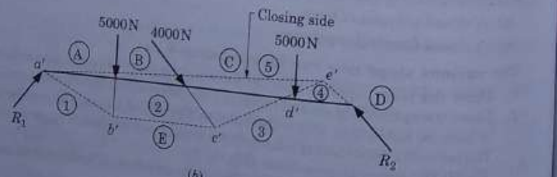
In the case of a hinged support, the line of action of the reaction is not always known before-hand. But, we know that the reaction at the hinge should pass through the hinge. *The vertex of the funnicular polygon in this case, therefore, should lie at the point at which the body is hinged.* The example 11.2 explains this concept.

Example 11.2 A beam 4.25 m long is hinged at one end and is supported on rollers at the other end. It is loaded as shown. Find the reactions at the supports.

Solution: Draw the sketch of the beam to a suitable scale and indicate the forces acting on it as shown in Fig. 11.4 (*a*). The reaction $\mathbf{R}_1$ at the hinged support is unknown both in magnitude and direction.



(*a*)



(*b*)

Fig. 11.4

The reaction $\mathbf{R}_2$ at the roller support acts normal to the support surface which is at an angle of 45° to the beam.

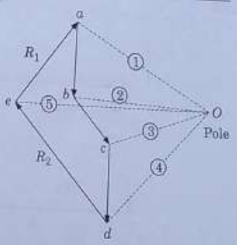


Fig. 11.4 (*b*)

Space Diagram. It denotes the spaces between the lines of action of the various forces as shown in Fig. 11.4 (*b*). The various forces now can be represented as

Force of 5000 N by *AB*
Force of 4000 N by *BC*
Force of 5000 N by *CD*
Reaction $\mathbf{R}_2$ by *DE*
Reaction $\mathbf{R}_1$ by *EA*

Force Polygon. Choose some point *a* and begin with the known forces. Draw vector $\overline{ab}$ representing the force *AB* (5000 N) and parallel to it. Then from *b* draw vector $\overline{bc}$ parallel to the force *BC* (4000 N) and representing it. Then from *c* draw vector $\overline{cd}$ representing the force *CD* (5000 N) and parallel to it. Next, from *d* draw a line parallel to the reactions $\mathbf{R}_2$, as we know only its line of action. The force polygon cannot be *completed* at this stage as the magnitude and the direction of reaction $\mathbf{R}_1$ is yet unknown. Choose any point *O* as pole and draw rays 1, 2, 3 and 4. Ray 5 is not yet known.

Funnicular Polygon. From the incomplete force polygon let us try to draw the funnicualr polygon.

The line of action of the reaction $\mathbf{R}_1$ is not known but it is known that it shall pass through the point *a'* where the beam is hinged. Starting from *a'* draw *a'b'* parallel to ray 1 intersecting the line of action of force *AB* (5000 N) at *b'*. Then draw *b'c'* parallel to the ray 2 and intersecting the line of action of the force *BC* (4000 N) at *c'*, *c'd'* is drawn parallel to the ray 3. Lastly draw *d'c'* parallel to the ray 4 and intersecting the line of action of the reaction $\mathbf{R}_2$ at *e'*.

For a system of forces in equilibrium the funnicular polygon must be a closed one. So join *e'* and *a'* completing the funnicular polygon.

Completion of Force Polygon. The sides of the funnicular polygon are parallel to rays. The side *e'a'* of the funnicular polygon should be parallel to the ray 5. The ray 5 now can be obtained by drawing *oe* parallel to the line *e'a'* and intersecting the line through *d* (and parallel to the reaction $\mathbf{R}_2$) at *e*. Join *e* to *a*.

The vector $\overline{de}$ represents the force *DE* (the reaction $\mathbf{R}_2$) in magnitude and direction. The magnitude is measured, and found to be, $\mathbf{R}_2$ = 9400 N *Ans.*

The vector $\overline{ea}$ represents the force *EA* (the reaction $\mathbf{R}_1$) in magnitude and direction. The magnitude is measured and found to be, $\mathbf{R}_1$ = 7200 N *Ans.*

Example 11.3 A beam is loaded and supported as shown. Find the support reactions.

Solution: Draw the beam to a suitable scale and indicate the forces acting on it. The reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ act normal to the beam as shown in Fig. 11.5 (*b*).

Space Diagram. Denote the spaces between the lines of action of various forces as shown in the figure. The various forces can now be represented as:

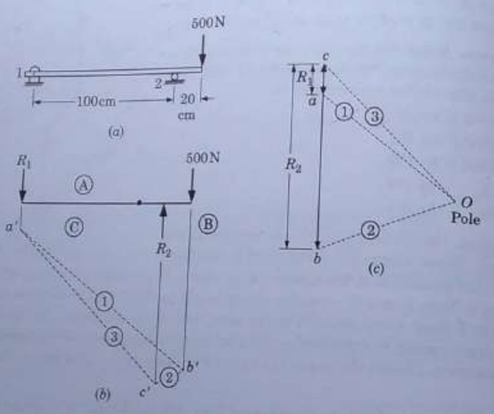


Fig. 11.5

Force of 500 N by AB
Reaction $\mathbf{R}_2$ by BC
Reaction $\mathbf{R}_1$ by CA.

Force Polygon. Draw vector $\overline{ab}$ to represent the force AB (500 N) and parallel to it. Then through b draw a line parallel to the force BC representing the unknown reaction $\mathbf{R}_2$. As we do not known the magnitudes of the reactions $\mathbf{R}_2$ and $\mathbf{R}_1$ the force polygon cannot be completed. Now choose a pole O and draw the rays 1 and 2. Ray 3 is not known yet.

Funnicular Polygon. Draw $a'b'$ parallel to the ray 1 intersecting the line of action of the force AB (500 N) at b'. From b' draw $b'c'$ parallel to the ray 2 and intersecting the line of action of the reaction $\mathbf{R}_2$ at c'. Join c' and a' to close the funnicular polygon.

To complete the force polygon draw from o, line oc parallel to the side $c'a'$ of the funnicular polygon, intersecting the line drawn through b at c. The ray 3 is thus established.

Vector $\overline{bc}$ represents in magnitude and direction the force BC (that is reaction $\mathbf{R}_2$) and is = 600 N acting upward. *Ans.*

Vector $\overline{ca}$ represents in magnitude and direction the force CA (that is reaction $\mathbf{R}_1$) and is = 100 N acting downward. *Ans.*

Example 11.4. A truss is loaded and supported as shown. Find the reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ at the supports as shown in Fig. 11.6 (*a*).

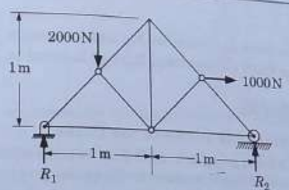


Fig. 11.6 (*a*)

Solution: Draw the truss to suitable scale and indicate the force acting on it. The reaction $\mathbf{R}_1$ at the hinge is unknown both in magnitude and direction. The reaction $\mathbf{R}_2$ is unknown in magnitude and acts normal to the support.

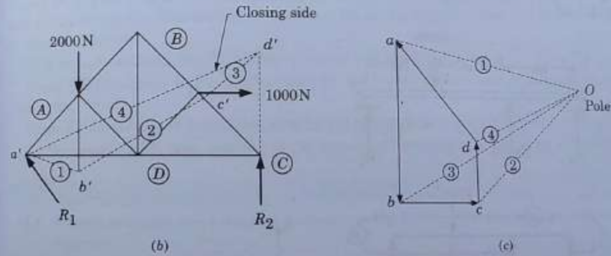


Fig. 11.6

Space Diagram. Denote the spaces between the lines of action of the various forces as shown in Fig. 11.6 (*b*).

Force Polygon. Vectors $\overline{ab}$ and $\overline{bc}$ to represent the forces AB (2000 N) and BC (1000 N) respectively. From c draw cd parallel to the force CD (reaction $\mathbf{R}_2$). The force polygon cannot be completed as the reaction $\mathbf{R}_1$ is unknown both in magnitude and direction. Choose a pole O and draw the rays 1, 2 and 3.

Funnicular Polygon. Let us try to draw the funnicular polygon from the incomplete force polygon. Draw from a' (where the beam is hinged) $a'b'$ parallel to the ray 1 intersecting the line of action of the force AB (2000 N) at b'. Draw $b'c'$ parallel to the ray 2 intersecting the line of action of force BC (1000 N) at c'. From c' draw $c'd'$ parallel to the ray 3 intersecting the line of action of force CD (reaction $\mathbf{R}_2$) at d'. Since the system is in equilibrium the funnicular polygon must be a closed one. Join d' and a'.

Completion of Force Polygon and Determination of the Reactions. The sides of the funnicular polygon are parallel to the rays. Draw the line od parallel to the side $d'a'$ and intersecting the line through c at d. Thus ray 4 is established.

Join d and a.

Vector $\overline{da}$ represents the force DA (reaction $\mathbf{R}_1$) and is equal to 1600 N *Ans.*

Vector $\overline{cd}$ represents the force CD (reaction $\mathbf{R}_2$) and is equal to 750 N *Ans.*

11.4 SPECIAL PROBLEM

When the Support Reactions Lie Between the Loads

The sequence of drawing the force polygon is such that we start with the known forces followed by the unknown forces. The use of Bow's notation in the earlier problems presented no difficulties as the *known and unknown forces were segregated in the space diagram*. But when an unknown force lies in between the known forces, segregation of known and unknown forces may not be possible unless the spaces in between the forces are properly denoted. Let us illustrate this with the help of an example.

Example 11.5 Find the support reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ for the beam loaded and supported as shown in Fig. 11.7 (*a*).

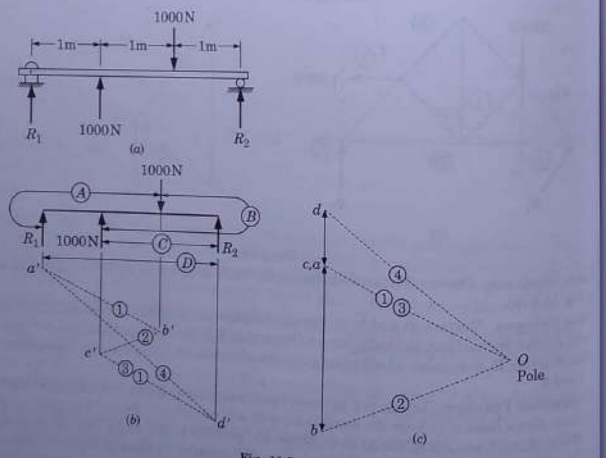


Fig. 11.7

$\mathbf{R}_1 = \overline{da} = 333.3$ N

$\mathbf{R}_2 = \overline{cd} = 333.3$ N

***Solution:* Space Diagram.** First all the known forces should be grouped together in a continuous sequence followed by the unknown forces.

The spaces between the lines of action the various forces are denoted as shown in Fig. 11.7 (*b*).

The various forces are represented as:

Force of 1000 N acting downward by AB

Force of 1000 N acting upward by BC

Reaction $\mathbf{R}_2$ by CD

Reaction $\mathbf{R}_1$ by DA.

Force Polygon: Vector $\overline{ab}$ represents the downward force of 1000 N and vector $\overline{bc}$ represents the upward force of 1000 N. The force polygon is incomplete. Join a, c and b to the pole o to obtain the rays 1, 2 and 3. Note that the points a and c are coincident.

Funnicular Polygon: Draw $a'b'$ parallel to the ray 1 and intersecting the line of action of the force AB (1000 N) at b'. From b' draw $b'c'$ parallel to the ray 2 and intersecting the force BC (1000 N) at c'. From c' draw $c'd'$ parallel to the ray 3 intersecting the force CD (reaction $\mathbf{R}_2$) at d'.

Join d' and a'. The side $d'a'$ of the funnicular polygon must be parallel to some ray.

Force Polygon and Determination of the Reactions. From o draw a line parallel to $d'a'$ and intersecting the line through c and parallel to the force CD (reaction $\mathbf{R}_2$) at d. The ray 4 is thus established.

This completes both the funnicular polygon and the force polygon.

Force CD (Reaction $\mathbf{R}_2$) is represented by the vector $\overline{cd}$ and has a magnitude of 333.3 N (upward). *Ans.*

Force DA (Reaction $\mathbf{R}_1$) is represented by the vector $\overline{da}$ and has a magnitude of 333.3 N *Ans.*

As the reaction $\mathbf{R}_1$ acts downward the assumed direction of $\mathbf{R}_1$ is to be reversed.

PROBLEMS

11.1. A beam is loaded and supported as shown in Fig. P.11.1. Find graphically the reactions at the supports. $\begin{bmatrix} R_1 = 369.7\text{ N} \\ R_2 = 2735\text{ N} \end{bmatrix}$

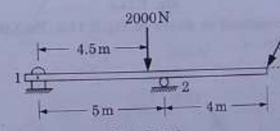


Fig. P.11.1

11.2. A beam is loaded and supported as shown in Fig. P.11.2. Find the reactions R_1 and R_2 at the supports. $\begin{bmatrix} R_1 = 4.8\text{ N} \\ R_2 = 3.5\text{ N} \end{bmatrix}$

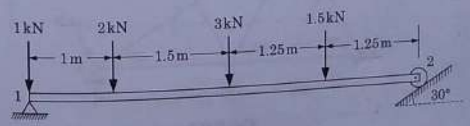


Fig. P.11.2

11.3. A lever is loaded and supported as shown in Fig. P.11.3. Find the support reactions R_1 and R_2.

$\left[\begin{array}{l} R_1 = 864\text{N} \\ R_2 = 768\text{N} \end{array}\right]$

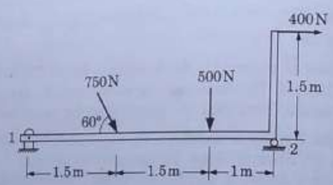


Fig. P.11.3

11.4. A horizontal beam AB is supported by three bars hinged at both the ends as shown in Fig. P.11.4. Find graphically the axial forces in the bars 1, 2 and 3.

$\left[\begin{array}{l} S_1 = 7.10 \text{ kN(C)} \\ S_2 = 5.7 \text{ kN(T)} \\ S_3 = 6.6 \text{ kN(C)} \end{array}\right]$

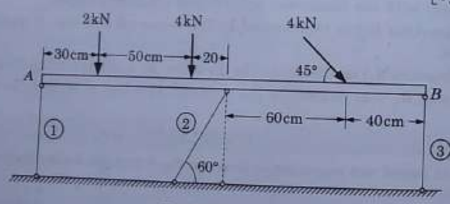


Fig. P.11.4

11.5. A truss is loaded and supported as shown in Fig. P.11.5. Find the reactions R_1 and R_2 at the hinge and roller supports.

$\left[\begin{array}{l} R_1 = 1732\text{N} \\ R_2 = 2646\text{N} \end{array}\right]$

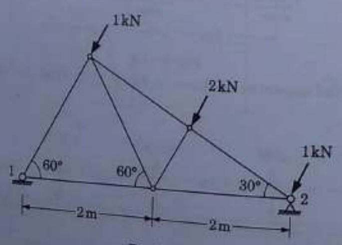


Fig. P.11.5

11.5 GRAPHICAL METHOD OF ANALYSIS OF SIMPLE TRUSSES: MAXWELL DIAGRAM

The graphical method for determining the axial forces in the members of a simple truss is the method of joints, implemented graphically. The method consists of the following steps :

1. Drawing of the truss to a suitable scale and the representation of the forces acting on it.
2. Representation of all the forces (external forces, reactions and the axial forces in the members of the truss) using Bow's notation. In Bow's notation, the spaces between the lines of action of the various forces are denoted by letters A, B etc. A force, then, is represented by the letters denoting the two spaces separated by the line of action of the force.
3. Construction of Maxwell diagram for the truss (also called vector diagram) by considering the equilibrium of the each joint. *But at no time the number of unknown forces at a joint should exceed two.*
4. Determination of the magnitudes and the nature (tension or compression) or forces in the members using the Maxwell diagram and Bow's notation.

The method is illustrated with the help of the solved examples in the following pages. In these examples, the trusses chosen contain relatively fewer members, affording an overall clarity of diagrams and procedure. Three types of problems solved are,

(*a*) a simply supported truss
(*b*) a cantilever truss
(*c*) a truss involving more than two unknown forces at a joint.

Example 11.6 Determine the forces in the members of a truss loaded and supported as shown.

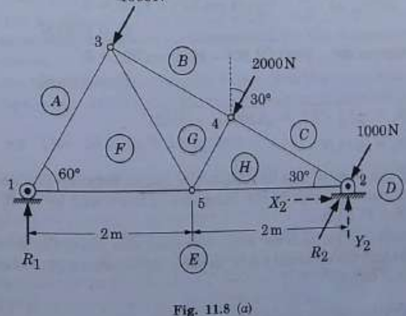


Fig. 11.8 (*a*)

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Solution: Sketch the truss to a suitable scale and mark the forces acting on it.

Support Reactions. The reactions at the supports can be determined either analytically or graphically. Let us determine analytically.

Let the reaction at the roller support be $\mathbf{R}_1$ acting vertically upward and the reaction at the hinge be $\mathbf{R}_2$ having components X_2 and Y_2 as shown in Fig. 11.8 (*a*).

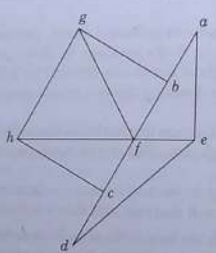

Fig. 11.8

Consider the equilibrium of the entire truss.

Take moments about the point 2,

$$\Sigma M_2 = 0: \quad -R_1(4) + 1000(4\cos 30°) + 2000(2\cos 30°) = 0$$

$$R_1 = 1732 \text{ N}$$

Take moments about the point 1,

$$\Sigma M_1 = 0: \quad 2000(2\cos 30°) + 1000(4\cos 30°) - Y_2(4) = 0$$

$$Y_2 = 1732 \text{ N}$$

$$\Sigma F_x = 0: X_2 - 1000\sin 30° - 2000\sin 30° - 1000\sin 30° = 0$$

$$X_2 = 2000 \text{ N}$$

$$R_2 = \sqrt{X_2^2 + Y_2^2} = \sqrt{(2000)^2 + (1732)^2}$$

$$R_2 = 2646 \text{ N}$$

Representation of the forces. Let us denote the spaces between the lines of actions of the various forces (including the forces in the members of the truss) by *A, B, C, D, E, F, G* and *H* as shown in the Fig. 11.8 (*a*).

Go around the joints in a cyclic order (say clockwise) and name the forces in the members of the truss, reactions and the external loads, in the order in which the letters representing the spaces appear as listed below:

Joint No.	*Member*	*Force in the Member is Denoted by*
1	1 — 3	*AF*
	1 — 5	*FE*
	(Reaction (R_1)	*EA*
2	2 — 4	*HC*
	1000 N	*CD*
	(Reaction R_2)	*DE*
	2 — 5	*EH*
3	(1000 N)	*AB*
	3 — 4	*BG*
	3 — 5	*GF*
	3 — 1	*FA*
4	(2000 N)	*BC*
	4 — 2	*CH*
	4 — 5	*HG*
	4 — 3	*GB*
5	5 — 1	*EF*
	5 — 3	*FG*
	5 — 4	*GH*
	5 — 2	*HE*

Notes:

1. There are 5 joints and 7 members. But some members appear twice in the table above.
2. The force in the member 1 — 3 in named as *AF* while going around the joint 1 but, the force in the same member (3 — 1) is named as *FA* while going around the joint 3, which is perfectly correct.
3. The order of the letters, while naming the forces, is to be *strictly* observed. It shall be useful in determining the nature of forces (whether tension or compression) in the members later.

Construction of The Maxwell Diagram

First, consider the entire truss as a free-body and draw the vectors representing the forces acting on the truss, that is the external loads and the reactions.

Choose a point *a*. Draw a vector $\overline{ab}$ to represent the force *AB* (1000 N). From *b* draw a vector $\overline{bc}$ to represent the force *BC* (2000 N) and from *c* draw a vector $\overline{cd}$ to represent the force *CD* (1000 N). From *a* draw a line parallel to the reaction *EA* (R_1) and measure off vector $\overline{ae}$ to represent the magnitude of the reaction *EA* (1732 N). Join points *e* and *d*. Vector $\overline{de}$ now represents the reaction *DE* (R_2).

Starting from a joint, go around the joints in clockwise order. *But, at no time the joint should contain more than two unknown forces.*

Consider the joint 1 as the free-body. Begin with the known vector $\overline{ea}$ representing the reaction EA (R_1). From a draw af parallel to bar 1 — 3 and ef parallel to the bar 1 — 5 intersecting at the point f. Vector $\overline{af}$ and $\overline{fe}$ thus obtained represent the forces in the bar 1 — 3 and 1 — 5. A new point f is obtained.

Consider the joint 3 as a free-body. Beginning with known vector $\overline{ab}$, go around the joint in a clockwise order.

Draw lines parallel to the members. Vectors $\overline{bg}$ and $\overline{gf}$ so obtained, represent the forces in the members 3 — 4 and 3 — 5. A new point g is obtained.

Next consider the joint 4 as a free-body. Begin with the known vector $\overline{bc}$ and obtain vectors $\overline{ch}$, $\overline{hg}$ and $\overline{gb}$ representing the forces in the members 4 — 2, 4 — 5 and 4 — 3 respectively. A new point h is obtained.

Lastly consider the joint 5 as a free-body. When going around the joint 5, it is found that vector $\overline{ef}$, $\overline{fg}$ and $\overline{gh}$ representing the forces in the members 5 — 1, 5 — 3 and 5 — 4 have already been drawn. Obtain vector $\overline{he}$ representing the force in the member 5 — 2.

We need not consider the equilibrium of the joint 2 as forces in all the members have been determined and the Maxwell diagram completed.

Magnitude and Nature of Forces in the Members

The force in any member is denoted by two capital letters and it is represented in the Maxwell diagram by a vector which can be identified by the corresponding small letters.

The sense of the force in the member can be determined with the help of Bow's notation.

Consider the joint 1. Go around the joint in clockwise manner. The force exerted by the bar 1–5 at the joint 1 is denoted by FE and is represented in the Maxwell diagram by the vector $\overline{fe}$ whose direction is from f to e. That is, it is directed from left to right. Mark an arrow near the joint 1 representing a force acting from left to right.

This force is exerting a pull at the joint 1 and, therefore, is of tensile nature. If we determine the nature of the force in the same member 5–1 at the joint 5, it would be found to act from right to left. The magnitude of the force EF can be determined by measuring the length of the vector $\overline{fe}$.

In a similar manner the nature and the magnitude of forces in the other members can be determined and are tabulated below :

Joint	*Member*	*Magnitude of the force*	*Nature of Force*
1	1 — 3	AF = 2000 N	Compression
	1 — 5	FE = 1000 N	Tension
3	3 — 4	BG = 1700 N	Compression
	3 — 5	GF = 1925 N	Tension
4	4 — 2	CH = 1700 N	Compression
	4 — 5	HG = 2000 N	Compression
5	5 — 2	HE = 3025 N	Compression

Example 11.7 Find the forces in the members of a cantilever truss loaded as shown.

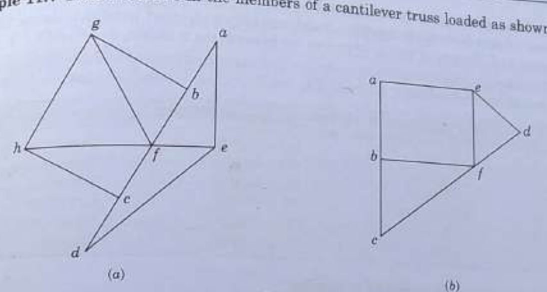

Fig. 11.9

Solution: Sketch the truss to a suitable scale and mark the forces acting on it. [Fig. 11.9 (a)].

Support Reactions. In the case of a cantilever truss it is not necessary to determine the support reactions. Maxwell diagram can be constructed starting from the free end 1.

Representation of the Forces. Denote the spaces between the lines of action of the various forces as shown in Fig. 11.9 (a). The various forces are represented as below :

Joint No.	*Member*	*Force in the Member is Denoted By*
1	2 — 1	FB
	(5000N)	BC
	1 — 3	CF
2	2 — 4	EA
	(5000N)	AB
	2 — 1	BF
	2 — 3	FE
3	3 — 5	CD
	3 — 4	DE
	3 — 2	EF
	3 — 1	FC

Construction of the Maxwell Diagram. Let us start from the free end, that is, from the joint 1. Draw vector $\overline{bc}$ to represent the force BC (5000 N). From c draw vector $\overline{cf}$ parallel to the bar 1 — 3 and from b draw vector $\overline{bf}$ parallel to the bar 1 — 2 to intersect at f. Vectors $\overline{cf}$ and $\overline{fb}$ represent the forces of the bars 1 — 3 and 1 — 2. Also a new point f is obtained.

Consider next, the equilibrium of the joint 2. Draw vector ab to represent the force AB (5000 N). Vector $\overline{bf}$ has already been drawn. From f draw vector $\overline{fe}$ parallel to the bar 2 — 3 and from a draw vector $\overline{ae}$ parallel to the bar 2 — 4 to intersect at e. Vectors $\overline{fe}$ and $\overline{ea}$ represent the forces in the bars 2 — 3 and 2 — 4. Also, a new point e is obtained.

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Lastly consider the equilibrium of the joint 3. Vector $\overline{ef}$ and $\overline{fc}$ have already been drawn. From c draw vector $\overline{cd}$ parallel to the member 3—5 and from e draw vector ed parallel to the member 3—4 to intersect at d. Vectors $\overline{cd}$ and $\overline{de}$ represent the forces in the members 3—5 and 3—4.

The magnitude of the forces in the members can measured from the Maxwell diagram and their sense decided using Bow's notation. The results are as tabulated below :

Joint	*Member*	*Magnitude of the Force*	*Nature of Force*
1	1 — 2	FB = 6800 N	Tension
	1 — 3	CF = 8600 N	Compression
2	2 — 4	EA = 6800 N	Tension
	2 — 3	FE = 5000 N	Compression
3	3 — 5	CD = 12600 N	Compression
	3 — 4	DE = 4100 N	Tension

11.6 METHOD OF SUBSTITUTION

From the two solved examples it is evident that the vector diagram for a joint cannot be drawn if it involves more than two unknown forces. In some trusses the members are so located that there are more than two unknown forces involved at a joint. *Such problems are solved by substituting or replacing some members by an imaginary member so as to reduce the unknown forces at a joint to two only*. After reaching a certain stage of solution the imaginary member is replaced by the actual members and the solution procedure is then continued in the usual manner. This is illustrated by a solved example.

Example 11.8 Find the forces in the members of a truss loaded and supported as shown.

Solution: Fig. 11.10 shows the truss together with the force acting on it.

Support Reaction. Reactions R_1 and R_2, as determined analytically, are

$$R_1 = 7.75 \text{ kN and } R_2 = 8.25 \text{ kN}$$

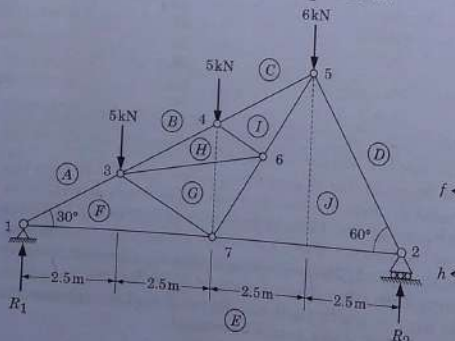

Fig. 11.10 (a)

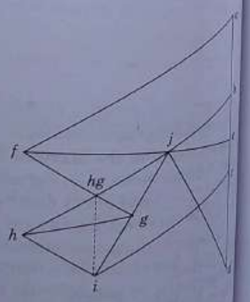

Fig. 11.10 (b)

Representation of the Forces. Denote the spaces between the lines of action of the various forces in Fig. 11.10 (a). The various forces are represented as below :

Joint No. (1)	*Member* (2)	*Force in the Member is Denoted By* (3)
1	1 — 3	AF
	1 — 7	FE
	(Reaction R_1)	EA
2	(Reaction R_2)	DE
	2 — 7	EJ
	2 — 5	JD
3	3 — 1	FA
	(5 kN)	AB
	3 — 4	BH
	3 — 6	HG
	3 — 7	GF
4	4 — 3	HB
	(5 kN)	BC
	5 — 2	CI
	5 — 6	IH
5	5 — 4	IC
	(6 kN)	CD
	5 — 2	DJ
	5 — 6	JI
6	6 — 7	JG
	6 — 3	GH
	6 — 4	HI
	6 — 5	IJ
7	7 — 1	EF
	7 — 3	FG
	7 — 6	GJ
	7 — 2	JE

Maxwell Diagram. Consider the equilibrium of the entire truss as a free-body. Starting from point a, draw vector $\overline{ab}$, $\overline{bc}$ and $\overline{cd}$ to represent the external forces AB (5 kN), BC (5 kN) and CD (6 kN). From d mark off vector $\overline{de}$ to represent the force DE (reaction R_2 = 8.25 kN). The vector $\overline{ea}$ represents the force EA (reaction R_1 = 7.75 kN).

Before we begin with the Maxwell diagram let us understand the difficulties that we are going to face.

Let us start from the joint 1. Vectors $\overline{ea}$, $\overline{af}$ and $\overline{fe}$ represent the forces EA (R_1) AF and FE respectively.

Next, we can consider either the joint 3 or the joint 7. If we consider the joint 3 there are four members involved, out of which, force in the member 3—1 (FA) has already been determined. Thus leaving three unknowns.

If we consider the joint 7, these are again four members are involved. Out of which, force in the member 7 — 1 (EF) is known.

So, which ever joint we consider, there are three unknowns. Let us, therefore, *modify the truss temporarily* so as to have not more than two unknown at the joint 3.

Substitute (or Replace) the two members 3 — 6 and 4 — 6 by one imaginary member between the joints 4 and 7, temporarily as shown in the Fig. 11.10 (a).

After considering the equilibrium of the joint 1, consider the equilibrium of the joint 3 of the *modified truss*. The space notations are also to be modified temporarily till we work with this *imaginary member*. Letters H and G now represent the same space after the members 3 — 6 and 4 — 6 are removed. Let us denote this space by (GH) and refer the forces involved with this space in the corresponding manner.

For the joint 3, vectors $\overline{fa}$ and $\overline{ab}$ have already been drawn. Draw vectors $\overline{bhg}$ and $\overline{fhg}$ parallel to the members 3 — 4 and 3 — 7 to intersect at the point hg.

Next consider the joint 4 of the modified truss. Vectors $\overline{hgb}$ and bc have already been drawn. From point c draw vector $\overline{ci}$ parallel to the member 4 — 5 and from point gh draw vector $\overline{hg}$ parallel to member 4 — 7 to intersect at i.

After establishing the point i in Maxwell diagram return to the original truss.

Reconsider the joint 4 of the original truss now with the *actual* members. Vectors $\overline{bc}$ and $\overline{ci}$ have already been drawn. From i draw vector $\overline{ih}$ parallel to the member 4 — 6 and from b draw vector $\overline{bh}$ parallel to the member 4 — 3 to intersect at h. This completes the vector diagram of the joint 4.

Reconsider the joint 3 now of the original truss. Vectors $\overline{fa}$, $\overline{ab}$ and $\overline{bh}$ have already been drawn. From h draw vector $\overline{hg}$ parallel to the member 3 — 6 and from f draw vector $\overline{fg}$ parallel to the member 3 — 7 to intersect at g. *This completes the vector diagram of the joint* 3.

Disregard the point hg existing in the vector diagram and proceed with the next joints 5, 6 and 7 and draw the vector diagrams in the usual manner completing the Maxwell diagram as shown in Fig. 11.10 (b).

The results are as tabulated below:

Joint No.	*Member*	*Magnitude of the Force*	*Nature of the Force*
1	1 — 3	AF = 15.6 kN	Compression
	1 — 7	FE = 14.0 kN	Tension
3	3 — 4	BH = 15.0 kN	Compression
	3 — 6	HG = 6.0 kN	Tension
	3 — 7	GF = 7.6 kN	Compression
4	4 — 5	CI = 10.6 kN	Compression
	4 — 6	IH = 4.8 kN	Compression
5	5 — 2	DJ = 9.6 kN	Compression
	5 — 6	JI = 8.6 kN	Tension
7	7 — 6	GJ = 9.6 kN	Tension
	7 — 2	JE = 4.9 kN	Tension

PROBLEMS

11.6. Construct a Maxwell diagram for a simply supported truss loaded as shown in Fig. P.11.6 and determine the axial forces in each of the members.

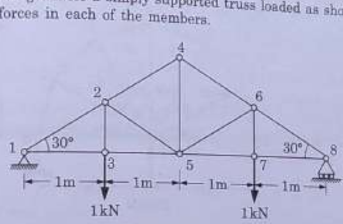

Fig. P.11.6

$$\begin{bmatrix} 1-2=6-8=2\,\text{kN (C)} \\ 1-3=3-5=5-7=7-8=1.73\,\text{kN}\,(T) \\ 2-3=4-5=6-7 \qquad =1\,\text{kN}\,(T) \\ 2-4=2-5=4-6=5-6=1\,\text{kN (C)} \end{bmatrix}$$

11.7. Construct a Maxwell diagram for a cantilever truss supported and loaded as shown in Fig. P.11.7 and determine the axial forces in each of the members.

$$\begin{bmatrix} 1-3=4.7\,\text{kN}\,(T),\ 3-5=3.2\,\text{kN}\,(T),\ 5-7=1.6\,\text{kN}\,(T) \\ 2-4=3.0\,\text{kN}\,(C),\ 4-6=1.5\,\text{kN}\,(C),\ 6-7=1.5\,\text{kN}\,(C) \\ 2-3=1.8\,\text{kN}\,(C),\ 3-4=0.5\,\text{kN}\,(T),\ 4-5=1.6\,\text{kN}\,(C),\ 5-6=0 \end{bmatrix}$$

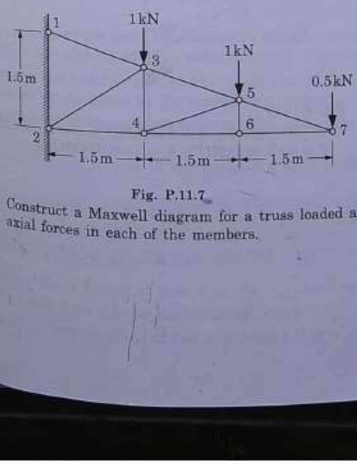

Fig. P.11.7

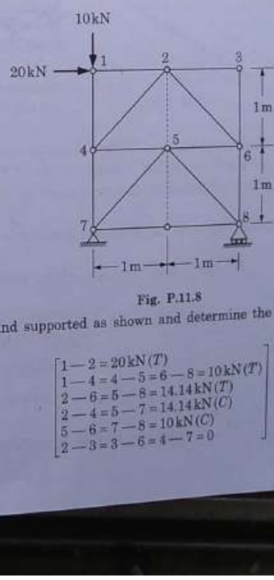

Fig. P.11.8

11.8. Construct a Maxwell diagram for a truss loaded and supported as shown and determine the axial forces in each of the members.

$$\begin{bmatrix} 1-2=20\,\text{kN}\,(T) \\ 1-4=4-5=6-8=10\,\text{kN}\,(T) \\ 2-6=5-8=14.14\,\text{kN}\,(T) \\ 2-4=5-7=14.14\,\text{kN}\,(C) \\ 5-6=7-8=10\,\text{kN}\,(C) \\ 2-3=3-6=4-7=0 \end{bmatrix}$$

11.9. Determine graphically the axial forces in the members of the truss loaded and supported as shown in Fig. P.11.9.

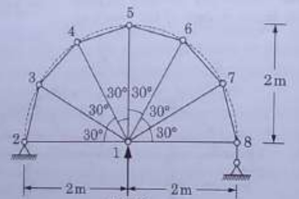

Fig. P.11.9

$$\begin{bmatrix} 1-8=1-2=268\,\text{N}\,(C) \\ 1-3=1-4=1-5=1-6=1-7=536\,\text{N}\,(C) \\ 2-3=3-4=4-5=5-6=6-7=7-8=1035\,\text{N}\,(T) \end{bmatrix}$$

11.10. Construct a Maxwell diagram for a truss supported and loaded as shown in Fig. P.11.10. Determine the axial forces in the members 2 — 3, 3 — 4, 4 — 5, 5 — 6 and 6 — 7 from the diagram.

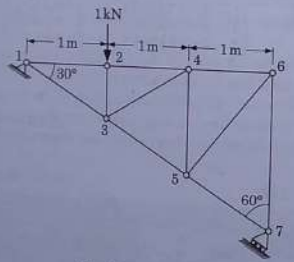

Fig. P.11.10

$$\begin{bmatrix} 2-3=1\,\text{kN}\,(C),\ 3-4=1\,\text{kN}\,(T),\ 4-5=0.5\,\text{kN}\,(C) \\ 5-6=0.422\,\text{kN}\,(T),\ 6-7=0.333\,\text{kN}\,(C) \end{bmatrix}$$

12 CHAPTER Moment of Inertia

12.1 INTRODUCTION

The mass and the surface area of a body are two of its important parameters. But in certain situations the *distribution* of these parameters within the body and their *orientation* with respect to some reference axis can be of as much importance as their absolute values.

Consider a solid cylinder and a hollow cylinder each of radius r, to slide down (without rolling) an inclined plane of angle α, from rest. Both the bodies shall be observed to reach the bottom of the plane at the same time, experiencing the same acceleration due to gravity (equal to $g \sin \alpha$) *irrespective of the mass and the radius.*

Now, let them roll down the same inclined plane without sliding. Which one would reach the bottom first? The answer, although not simple, is; the solid cylinder will reach first, followed by the hollow cylinder.

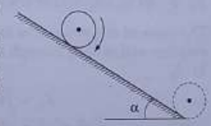

Fig. 12.1

From the above observation we can say that, this phenomena has something to do with the distribution of the mass within the body.

The concept which gives a quantitative estimate of the relative distribution of area and mass of a body with respect to some reference axis is termed as the moment of inertia of the body.

Analogy-wise the role played by the moment of inertia in the rotary motion is similar to the role played by the mass in the translatory motion.

The moment of inertia of an area is called as the area moment of inertia or the *second moment of area.*

The moment of inertia of the mass of a body is called as *the mass moment of inertia.*

12.2 MOMENT OF INERTIA OF AN AREA OF A PLANE FIGURE WITH RESPECT TO AN AXIS IN ITS PLANE (RECTANGULAR MOMENTS OF INERTIA)

Consider a plane figure of area A in the x-y plane as shown in Fig. 12.2.

Divide this area A in to infinitesimal areas.

Let dA be any element of the area situated at a distance (x, y) from the axes.

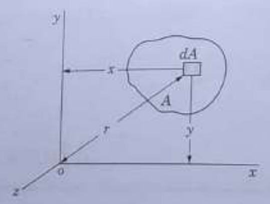

Fig. 12.2

The moment of inertia of the area A with respect to the x-axis (also called the second moment of the area) $= I_x = \int y^2 dA$ ...(12.1)

The moment of inertia of the area A with respect to the y-axis (also called the second moment of the area) $= I_y = \int x^2 dA$ ...(12.2)

Integration should cover the entire area of the figure and its value shall depend upon the shape of the area and its orientation with respect to the axis.

12.3 POLAR MOMENT OF INERTIA

The moment of inertia of an area of a plane figure with respect to an axis perpendicular to the x-y plane and passing through a pole o (z-axis) is called the polar moment of inertia and is denoted by J_0.

$$J_0 = \int r^2 dA$$

As,

$$x^2 + y^2 = r^2 \quad \text{...(12.3)}$$

$$J_0 = \int r^2 dA = \int (x^2 + y^2) dA = I_x + I_y$$

$$J_0 = I_x + I_y \quad \text{...(12.4)}$$

Moment of inertia of an area

= (Area) (Distance)^2 = (Length)^4

Thus, it has the unit of (metre)^4.

The moment of inertia of an area can be determined *with respect to any axis. One commonly used axis is the centroidal axis.* Any axis passing through the centroid of an area is called the centroidal axis. Two of them, are centroidal x-axis and centroidal y-axis.

12.4 RADIUS OF GYRATION OF AN AREA

Consider an area A which has a moment of inertia I_x with respect to the x-axis Fig. 12.3.

Let us imagine this area A to be concentrated into a thin strip parallel to the x-axis. If this area A (concentrated strip), is to have the same moment of inertia (I_x) with respect to the x-axis, the strip should be placed at a distance k_x from the x-axis, as given by the relation

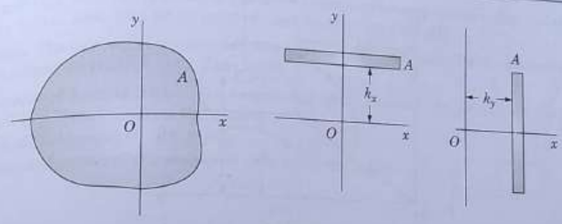

Fig. 12.3

$$I_x = k_x^2 A$$

Defining,

$$k_x = \sqrt{\frac{I_x}{A}} \quad \text{...(12.5)}$$

k_x is known as the radius of gyration of the area with respect to the x-axis and has the unit of length (m).

We can similarly define,

Radius of gyration with respect to the y-axis.

$$k_y = \sqrt{\frac{I_y}{A}} \quad \text{...(12.6)}$$

Radius of gyration with respect to the polar axis,

$$k_0 = \sqrt{\frac{J_0}{A}} \quad \text{...(12.7)}$$

As,

$$J_0 = I_x + I_y$$

$$A (k_0)^2 = A (k_x)^2 + A (k_y)^2$$

we get,

$$k_0^2 = k_x^2 + k_y^2 \quad \text{...(12.8)}$$

12.5 PARALLEL AXIS THEOREM (DISPLACEMENT OF THE AXIS PARALLEL TO ITSELF)

Let x, y be the rectangular coordinate axes through any point O in the plane of figure of area A as shown in Fig. 12.4.

x', y' be the corresponding parallel axes through the centroid C of the area. The axes through the centroid of an area is also called the *centroidal axes.*

The moment of inertia of the area A about the x-axis.

$$I_x = \int (y)^2 dA$$

where, dA is an element of area at a distance y from x-axis.

But

$$y = d_x + y'$$

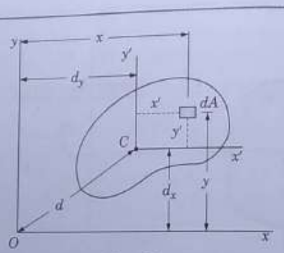

Fig. 12.4

d_x being the perpendicular distance between the axes x and x'

$$I_x = \int (y'+d_x)^2 dA$$

$$I_x = \int (y'^2 + d_x^2 + 2y'd_x) dA$$

$$I_x = \int y'^2 dA + \int d_x^2 dA + 2d_x \int y' dA$$

$$I_x = \int y'^2 dA + Ad_x^2 + 0$$

The terms $\int y' dA$ represents the first moment of the area A about its own centroidal axis x', and is therefore, equal to zero. The term $\int y'^2 dA$ represents the moment of inertia of the area A about the axis x'.

$$I_x = I_{x'} + A(d_x)^2$$

Or $$I_x = \bar{I}_x + A(d_x)^2 \quad \left[\begin{array}{l}\text{The moment of inertia of an area} \\ \text{about its centroidal axis is represented} \\ \text{by } \bar{I}_x \text{ and } \bar{I}_y. \text{ So } I_{x'} = \bar{I}_x, I_{y'} = \bar{I}_y\end{array}\right] \quad ...(12.9)$$

Similarly $$I_y = I_{y'} + A(d_y)^2$$

Or $$I_y = \bar{I}_y + A(d_y)^2 \quad ...(12.10)$$

Thus, we can say that *the moment of inertia of an area with respect to any axis in its plane is equal to the moment of inertia of the area with respect to a parallel centroidal axis plus the product of the area and square of the distance between the two axes.*

Adding (12.9) and (12.10)

$$I_x + I_y = \bar{I}_x + \bar{I}_y + A(d_x^2 + d_y^2)$$

But, $$d_x^2 + d_y^2 = d^2$$

and $$\bar{I}_x + \bar{I}_y = J_c$$

$$I_x + I_y = J_0$$

where J_c is the polar moment of inertia about the centroidal axis.

Therefore, $$J_0 = J_c + Ad^2 \quad ...(12.11)$$

Thus the parallel axis theorem is applicable to the polar moment of inertia also.

Example 12.1 Find the moment of inertia of a rectangular cross-section about its centroidal axes as shown. Also, find its moment of inertia about the base AB.

Solution: The centroid of rectangular area is at C. Centroidal axes x-y are as shown and the area is symmetrical about both these axes.

Moment of Inertia about the centroidal axes. Consider an element of thickness dy situated at a distance y from the x-axis.

Area of the element, $dA = b\,dy$

Moment of inertia of the elemental area about the x-axis

$$dI_x = y^2 (b\,dy)$$

Therefore, $$\bar{I}_x = \int dI_x = \int_{y=-h/2}^{y=+h/2} by^2 dy$$

$$= b\left[\frac{y^3}{3}\right]_{-h/2}^{+h/2}$$

$$\bar{I}_x = \frac{bh^3}{12} \quad Ans.$$

Similarly, we can get $$\bar{I}_y = \frac{hb^3}{12} \quad Ans.$$

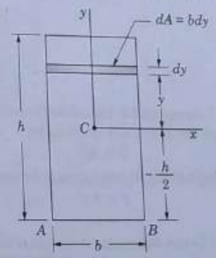

Fig. 12.5

Moments of Inertia about the Base. Using parallel axis theorem,

$$I_{AB} = \bar{I}_x + Ad^2$$

where, d is the perpendicular distance of the centroid C from the base AB.

$$I_{AB} = \frac{bh^3}{12} + (bh)\left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}$$

$$I_{AB} = \frac{bh^3}{3} \quad Ans.$$

Example 12.2 Determine the moment of inertia of a triangle with respect to its base.

Solution: Consider a triangle of base b and height h. Choose x-axis to coincide with the base as shown in Fig. 12.6.

Consider an element of thickness dy situated at a distance y from the x-axis.

Area of this element $dA = l\,dy$

From similar triangles

$$\frac{l}{b} = \frac{h-y}{h} \text{ or } l = b\frac{(h-y)}{h}$$

Moment of inertia of this element about the x-axis

$$dI_x = y^2 dA$$

$$dI_x = y^2\left(b\frac{(h-y)}{h}dy\right)$$

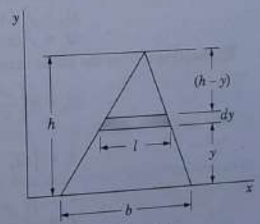

Fig. 12.6

$$I_x = \int_0^h dI_x = \int_0^h y^2 \frac{b(h-y)}{h} dy$$

$$I_x = \frac{b}{h}\int_0^h (y^2 h - y^3) dy$$

$$I_x = \frac{b}{h}\left[\frac{hy^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$I_x = \frac{bh^3}{12} \quad \textit{Ans.}$$

Example 12.3 Calculate the moment of inertia of the shaded area about the x-axis. The equation of the curve OA is given by

$$x = ky^2$$

Solution: Shaded area is bounded by the curves OA and AB whose equations are

$$x = ky^2 \qquad \text{...(i)}$$

$$y = a \qquad \text{...(ii)}$$

Let us determine the value of the constant k in the equation (i). As the point A (a, a) lies on this curve,

$$a = ka^2$$

$$k = \frac{1}{a}$$

equation (i) can be written as

$$x = \frac{1}{a}y^2 \qquad \text{...(iii)}$$

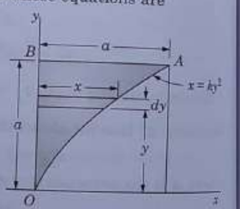

Fig. 12.7

Using a Horizontal Strip. Let us divide the area into horizontal strips of thickness dy.

Consider a strip situated at a distance y from the x-axis.

Area of the elemental strip

$$dA = x\,dy$$

The moment of inertia of this strip about the x-axis

$$dI_x = y^2 dA$$

$$dI_x = y^2 (x\,dy)$$

Eliminating x from the above expression using equation (iii)

$$dI_x = y^2\left(\frac{y^2}{a}\right)dy$$

$$I_x = \int dI_x = \int_0^a \left[\frac{y^4}{a}\right]dy$$

$$I_x = \frac{1}{a}\left[\frac{y^5}{5}\right]_0^a = \frac{a^4}{5} \quad \textit{Ans.}$$

Using Vertical Strip. Let us now divide this area into vertical strips of thickness dx.

Consider a strip situated at a distance x from the y-axis.

Area of this elemental strip

$$dA = y\,dx$$

and

$$y = y_2 - y_1$$

where, the point p lies on the straight line,

$$y = a$$

Therefore, $y_2 = a$

The point q lies on the curve,

$$x = \frac{1}{a}y^2$$

Therefore, $y_1 = \sqrt{xa}$

$$dA = (a - \sqrt{xa})dx$$

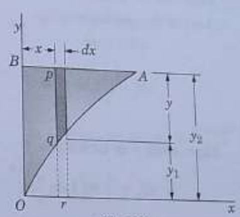

Fig. 12.8

Here we cannot multiply dA by y^2 to get dI_x as, all portions of the elemental strip cannot be assumed to be situated at the same distance y from the x-axis.

Let us, therefore, adopt a different approach.

Moment of inertia of the rectangle pq about the x-axis	=	Moment of inertia of the rectangle pr about the x-axis	−	Moment of inertia of the rectangle qr about the x-axis

$$dI_x = \frac{1}{3}y_2^3 dx - \frac{1}{3}y_1^3 dx$$

$$dI_x = \frac{1}{3}(y_2^3 - y_1^3)dx \qquad \left(\text{For a rectangle } I_x = \frac{1}{3}bh^3\right)$$

Substituting for y_2 and y_1 in terms of x

$$dI_x = \frac{1}{3}(a^3 - (\sqrt{ax})^3)dx$$

$$I_x = \int_0^a \frac{1}{3}(a^3 - \sqrt{ax})^3)dx$$

$$I_x = \frac{1}{3}\left[a^3 x - a^{3/2}\frac{x^{5/2}}{5/2}\right]_0^a$$

$$I_{xx} = \frac{a^4}{5} \quad \textit{Ans.}$$

It can be seen that the choice of a horizontal strip considerably simplifies the calculations in this case.

Example 12.4 Determine the moment of inertia about the x-axis of an area under the sine curve the equation for which is $y = b\sin\frac{\pi x}{a}$.

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Solution: Let us divide the area into vertical strips of thickness dx as shown in Fig. 12.9. Consider an elemental strip situated at a distance x from the y-axis.

Area of the elemental strip

$$dA = y\,dx$$

$$dI_x = \frac{1}{3}(dx)y^3 \qquad \left[\begin{array}{l}\text{Moment of inertia of a}\\ \text{rectangle about its base}\\ = \frac{1}{3}bh^3\end{array}\right]$$

Expressing y in terms of x

$$dI_x = \frac{1}{3}dx\left(b\sin\frac{\pi}{a}x\right)^3$$

$$I_x = \int dI_x = \frac{b^3}{3}\int_{x=0}^{x=a}\left(\sin\frac{\pi}{a}x\right)^3 dx \qquad ...(i)$$

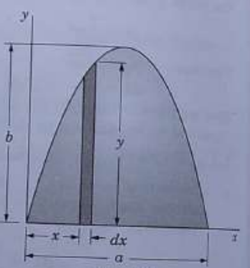

Fig. 12.9

To evaluate the above integral substitute:

$$\frac{\pi x}{a} = w$$

Differentiating, $\frac{\pi}{a}dx = dw$ or $dx = \frac{a}{\pi}dw$

and the limits of integration change to 0 to π.

Equation (*i*) can be written as,

$$I_x = \frac{ab^3}{3\pi}\int_{w=0}^{w=\pi}\sin^3 w\,dw$$

$$I_x = \frac{ab^3}{3\pi}(-1)\left[\frac{\cos w}{3}(2+\sin^2 w)\right]_{w=0}^{w=\pi} \qquad \left(as, \int\sin^3\theta\,d\theta = -\frac{\cos\theta}{3}(2+\sin^2\theta)\right)$$

$$I_x = \frac{4ab^2}{9\pi} \quad Ans.$$

Example 12.5 Determine the moments of inertia of a circular area about the centroidal axes.

Solution: The centroid of a circular area is its centre. Axes x-y passing through C are the centroidal axes.

Method I. Double Integration

Consider an element of area A situated at a radius r and angle θ.

$$dA = (r\,d\theta)dr$$

The centroid of this element lies at a distance $r\sin\theta$ from the x-axis.

Moment of inertia of the elemental area about the x-axis.

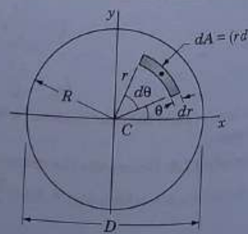

Fig. 12.10

$$dI_x = ((r\,d\theta)dr)\,(r\sin\theta)^2$$

Moment of the circular about the x-axis.

$$\bar{I}_x = \int dI_x = \int(r\,d\theta\,dr)(r^2\sin^2\theta)$$

$$\bar{I}_x = \int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3\sin^2\theta\,d\theta\,dr$$

$$\bar{I}_x = \int_{r=0}^{r=R}\pi r^3 dr \qquad \left(\text{replace, } \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$\bar{I}_x = \frac{\pi R^4}{4} = \frac{\pi D^4}{64} \quad Ans.$$

Because of the symmetry of the circular area,

$$\bar{I}_x = \bar{I}_y \quad Ans.$$

Method II. Single Intergration

Consider an elemental strip at a distance y from the x-axis of thickness dy.

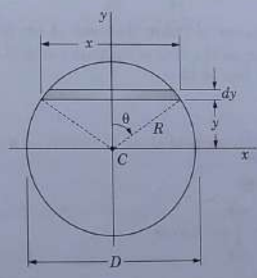

Fig. 12.11

Area of the element $= x\,dy$

but $x = 2(R\sin\theta)$

and $y = R\cos\theta$

differentiating $dy = -R\sin\theta\,d\theta$

$$dA = x\,dy = (2R\sin\theta)(-R\sin\theta\,d\theta)$$

Moment of inertia of the element,

$$dI_x = \int y^2 dA = \int(-2R^2\sin^2\theta\,d\theta)(R\cos\theta)^2$$

$$\bar{I}_x = \int dI_x = \int_{\theta=-\pi}^{\theta=0} -2R^4\cos^2\theta\sin^2\theta\,d\theta$$

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$$\bar{I}_x = -2R^4 \int_{-\pi}^{0} \cos^2\theta \sin^2\theta \, d\theta$$

$$= 2R^4 \int_0^{\pi} \left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{R^4}{2} \int_0^{\pi} (1-\cos^2 2\theta)\, d\theta$$

$$= \frac{R^4}{2} \int_0^{\pi} \left[1-\left(\frac{1+\cos 4\theta}{2}\right)\right] d\theta$$

$$= \frac{R^4}{4} \int_0^{\pi} (1-\cos 4\theta)\, d\theta$$

$$= \frac{R^4}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi}$$

$$\bar{I}_x = \frac{\pi R^4}{4} = \frac{\pi D^4}{64} \quad Ans.$$

Method III. Using the Concept of Polar Moment of Inertia

Consider an element in the shape of a ring of thickness dr and situated at a distance r from the centre C as shown in the Fig. 12.12.

Area of the elemental ring

$$dA = (2\pi r)\, dr$$

Polar moment of inertia of this elem nt about C. (About an axis perpendicular to the plane of the Fig. through C.

$$dJ_c = r^2 (2\pi r\, dr)$$

$$J_c = \int dJ_c = \int_{r=0}^{r=R} (r^2) 2\pi r\, dr$$

$$= \int_{r=0}^{r=R} 2\pi r^3\, dr$$

$$J_c = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$$

For a circular area

$$\bar{I}_x = \bar{I}_y$$

$$J_c = \bar{I}_x + \bar{I}_y = 2\bar{I}_x$$

$$\bar{I}_x = \frac{J_c}{2} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64} \quad Ans.$$

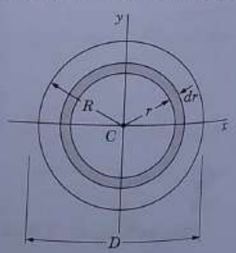


Fig. 12.12

Example 12.6 Determine the moment of inertia of a hollow circular section about its centroidal axes as shown in Fig. 12.13.

Solution: The hollow cross-sectional area can be considered to be made of,

1. a circular area A_1 of radius R_1 having its centroid at C,

   and

2. a removed (or negative) circular area A_2 of radius R_2 having its centroid at C.

The centroid of the hollow circular area lies also at C.

Area A_1

Moment of inertia of the area A_1 about the centroidal x-axis.

$$(\bar{I}_x)_1 = \frac{\pi R_1^4}{4}$$

Area A_2

Moment of inertia of the area A_2 about the centroidal x-axis

$$(\bar{I}_1)_2 = \frac{\pi R_2^4}{4}$$

Moment of inertia of the hollow circular area (composite area) about the centroidal x-axis

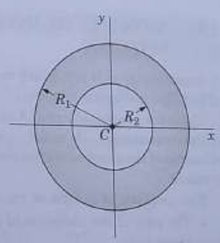


Fig. 12.13

$$\bar{I}_x = (\bar{I}_x)_1 - (\bar{I}_x)_2$$

Or
$$\bar{I}_x = \left(\frac{\pi}{4}R_1^4 - \frac{\pi}{4}R_2^4\right) = \frac{\pi}{4}(R_1^4 - R_2^4) \quad Ans.$$

Example 12.7 The shaded area shown in Fig. 12.14 is equal to 50 cm². Determine its centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$. It is known that the polar moment of inertia of the area about the point A is,

$$J_A = 22.5 \times 10^2 \text{ cm}^4 \quad \text{and} \quad 2\bar{I}_y = \bar{I}_x$$

Solution.

$$A = 50 \text{ cm}^2,\ d = 6 \text{ cm}$$

$$J_A = 22.5 \times 10^2 \text{ cm}^4$$

$$J_A = J_c + Ad^2$$

$$J_c = J_A - Ad^2$$

$$J_c = 22.5 \times 10^2 - 50(6)^2$$

$$= 22.5 \times 10^2 - 18 \times 10^2$$

$$J_c = 4.5 \times 10^2$$

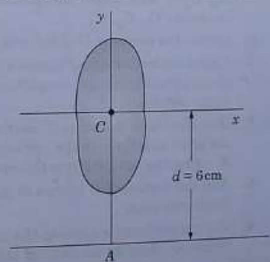


Fig. 12.14

As,
$$\bar{I}_x + \bar{I}_y = \bar{J}_c$$

$$\bar{I}_x + \bar{I}_y = 4.5 \times 10^2 \quad ...(i)$$

$$2\bar{I}_y = \bar{I}_x \quad \text{(given)} \qquad ...(ii)$$

Solving (*i*) and (*ii*) for $\bar{I}_x$ and $\bar{I}_y$

$$3\bar{I}_y = 4.5 \times 10^2$$

$$\bar{I}_y = 1.5 \times 10^2 \text{ cm}^4 \quad \textit{Ans.}$$

$$\bar{I}_x = 3.0 \times 10^2 \text{ cm}^4 \quad \textit{Ans.}$$

12.6 MOMENT OF INERTIA OF A COMPOSITE AREA/HOLLOW SECTION

A composite area is one which can be considered to be made up of several components of areas of familiar geometric shapes.

Consider a *composite* area A as shown in Fig. 12.15. It can be considered to be made up of three *component* areas A_1, A_2 and A_3; a semicircle, a rectangle and a triangle respectively. The moment of inertia of the composite area about an axis is related to the moments of inertia of the component areas as,

The moment of inertia of an area (A) with respect to a given axis
= The sum of the moments of inertia of the component areas (A_1, A_2, A_3) with respect to the same axis.

Quite often it is required to determine the moment of inertia of a composite area with respect to an axis passing through the centroid of the composite area (called the centroidal axis of the composite area). The steps for this are listed below :

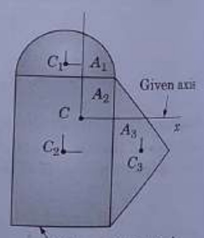
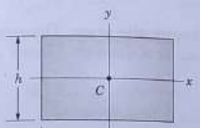
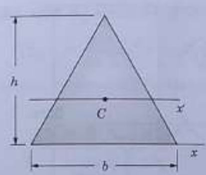
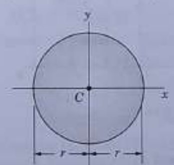
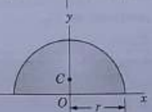
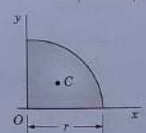
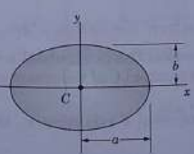


Fig. 12.15

1. Split up the given area A into component areas of familiar shapes. Determine the values of the component areas A_1, A_2, A_3 ... and locate the positions of their individual centroids C_1, C_2, C_3.
2. Locate the centroid C of the composite area.
3. Calculate the moment of inertia of each component area (A_1) about an axis passing through its centroid (C_1) and parallel to the given axis (x).
4. Transfer these moments of inertia of component areas, to the given axis through the centroid C of the composite area A, using the parallel axis theorem.
5. Add the moments of inertia of component areas to obtain the moment of inertia of the composite area.
6. If a composite area consists of a component area which represents a void, hole or an area removed, then its moment of inertia is *negative* and has to be *subtracted*.

Note that the radius of gyration of a composite area is not equal to the sum of the radii of gyration of the component areas.

Table 12.1 *Moments of Inertia of Geometric Shapes (Plane Figures)*

Rectangle		$\bar{I}_x = \frac{1}{12}bh^3$, $\bar{I}_y = \frac{1}{12}b^3h$ $J_c = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$I_x = \frac{1}{12}bh^3$, $I_x' = \frac{1}{36}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_c = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_0 = \frac{1}{4}\pi r^4$
Quarter-circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $\bar{J}_c = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$, $\bar{I}_y = \frac{1}{4}\pi a^3 b$ $J_c = \frac{1}{4}\pi ab(a^2 + b^2)$

Example 12.8 Find the moment of inertia of a plate with a circular hole about its centroidal x-axis as shown in Fig. 12.16.

Solution: **Location of the centroid of the composite area.** The area is symmetrical about the vertical axis y. The centroid, therefore, shall lie of this axis. Choose the reference x-axis as shown in the figure.

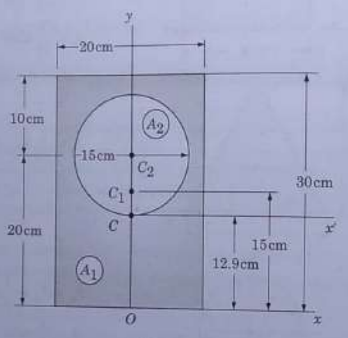


Fig. 12.16

The composite area is up of,

1. Area A_1 of the rectangle having its centroid at C_1.
2. Negative area A_2 of the circle having its centroid at C_2.

Let C be the centroid of the composite area, then

Figure	*Area (cm²)*	*x-coordinate of the centroid (cm)*	*y-coordinate of the centroid (cm)*
Rectangle	$A_1 = 20 \times 30$ $A_1 = 600$	$x_1 = 0$	$y_1 = 15.0$
Circle Removed (Negative)	$A_2 = \pi/4 \times 15^2$ $A_2 = 176.7$	$x_2 = 0$	$y_2 = 20.0$

$$y_c = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2} = \frac{600 \times 15 - 176.7 \times 20}{600 - 176.7}$$

$y_c = 12.9$ cm from the x-axis or from the bottom edge.

Moment of inertia of the composite area about the centroidal x-axis. The axis x' drawn parallel to the x-axis and passing through the centroid C of the composite area is called as the centroidal x-axis of the composite area.

Area A_1 : M.I. of the area A_1 about its centroidal x-axis (that is, the axis through C_1)

$$(\bar{I}_x)_1 = \frac{20 \times (30)^2}{12} = 45000 \text{ cm}^4$$

M.I. of the area A_1 about the centroidal x-axis of the composite area (that is, the x-axis through C),

$$(I'_x) = (\bar{I}_x)_1 + A_1 d^2 \qquad \text{(By parallel-axis theorem)}$$

$$= 45000 + 600\,(2.1)^2 \quad (d = C_1C = 2.1 \text{ cm})$$

$$(I'_x) = 47646 \text{ cm}^4.$$

Area A_2 : M.I. of the area A_2 about its centroidal x-axis (that is, the x-axis through C_2),

$$(\bar{I}_x)_2 = \frac{\pi}{64}(15)^4 = 2485 \text{ cm}^2$$

M.I. of the area A_2 about the centroidal x-axis of the composite area (that is, the x-axis through C),

$$(I'_x)_2 = (\bar{I}_x)_2 + A_2 d^2 \qquad \text{(By Parallel axis theorem)}$$

$$= 2485 + 176.7(7.1)^2 \quad (d = C_2C = 7.1 \text{ cm})$$

$$(I'_x)_2 = 11392 \text{ cm}^4$$

Composite Area : Moment of inertia of the composite area about the centroidal x-axis,

$$I'_x = (I_x') - (I_x')_2$$

$$I'_x = 47646 - 11392$$

$$I'_x = 36254 \text{ cm}^4. \quad \textit{Ans.}$$

Example 12.9 Determine the moment of inertia of the area of T-section as shown in Fig. 12.17 with respect to the centroidal x-axis.

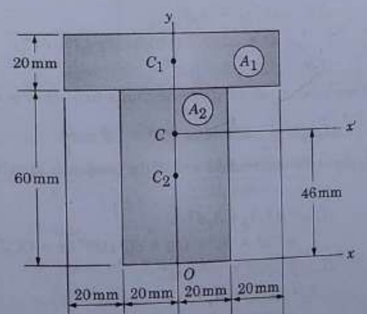


Fig. 12.17

Solution: Let us divide the area into rectangles A_1 and A_2 with their centroids at C_1 and C respectively. The centroid of the composite area be at C.

Location of the centroid C. Place the origin at O and the reference axis be x-y. The centroid would lie on the y-axis being the axis of symmetry.

Figure	*Area* (mm²)	*x-coordinate of the centroid* (mm)	*y-coordinate of the centroid* (mm)
Rectangle A_1	$A_1 = 20 \times 80$ $A_1 = 1600$	$x_1 = 0$	$y_1 = 70$
Rectangle A_2	$A_2 = 60 \times 40$ $A_2 = 2400$	$x_2 = 0$	$y_2 = 30$

$$y_c = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{1600 \times 70 + 2400 \times 30}{1600 + 2400}$$

$$y_c = 46 \text{ mm.}$$

Moment of inertia of the composite area about the centroidal x-axis (that is axis-x' through C).

Area A_1 : M.I. of area A_1 about its centroidal x-axis (that is the axis through C_1),

$$(I_x)_1 = \frac{1}{12}(80)(20)^3 = 53.8 \times 10^3 \text{ mm}^4.$$

M.I. of the area A_1 about the centroidal x-axis of the composite area (that is axis-x' through C).

$$(I_x')_1 = (\bar{I}_x)_1 + A_1 d^2 \qquad \text{(Parallel axis theorem)}$$

$$= 53.8 \times 10^3 + (1600)(24)^2 \; (d = CC_1)$$

$$(I_x')_1 = 975 \times 10^3 \text{ mm}^4.$$

Area A_2 : M.I. of the area A_2 about its centroidal x-axis (that is the axis through C_2)

$$(\bar{I}_x)_2 = \frac{1}{12}(40)(60)^3 = 720 \times 10^3 \text{ mm}^4.$$

M.I. of the area A_2 about the centroidal x-axis of the composite area (that is the axis-x' through C).

$$(I_x')_2 = (\bar{I}_x)_2 + A_2 d^2$$

$$= 720 \times 10^3 + (40 \times 60)(16)^2 \; (d = CC_2)$$

$$(I_x')_2 = 1334 \times 10^3 \text{ mm}^4$$

Composite Area :

$$I_x' = (I_x')_1 + (I_x')_2$$

$$= 975 \times 10^3 + 1334 \times 10^3$$

$$I_x' = 2.31 \times 10^3 \text{ mm}^4. \quad \textit{Ans.}$$

Example 12.10 Find the moments of inertia of the area of the L-section about the centroidal x and y-axis as shown in Fig. 12.18.

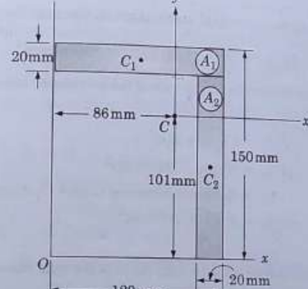


Fig. 12.18

Solution: Let us divide the area into rectangles A_1 and A_2 with their centroids at C_1 and C_2 respectively. The centroid of the composite area be at C.

Location of the centroid C. Place the origin at O and the reference axes be x-y.

Figure	*Area* (mm)²	*x-coordinate of the centroid* (mm)	*y-coordinate of the centroid* (mm)
Rectangle A_1	$A_1 = 120 \times 20$ $A_1 = 2400$	$x_1 = 60$	$y_1 = 140$
Rectangle A_2	$A_2 = 130 \times 20$ $A_2 = 2600$	$x_2 = 110$	$y_2 = 65$

$$x_c = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2} = \frac{2400 \times 60 + 2600 \times 110}{2400 + 2600}$$

$$x_c = 86 \text{ mm}$$

$$y_c = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{200 \times 140 + 2600 \times 65}{2400 + 2600}$$

$$y_c = 101 \text{ mm}$$

Moment of inertia of the composite area about the centroidal x-axis (that is axis-x' through C)

Area A_1:

M.I. of the area A_1 about its centroidal x-axis (that is, the axis through C),

$$(\bar{I}_x)_1 = \frac{1}{12} \times (120)(20)^3 = 80000 \text{ mm}^4$$

M.I. of the area A_1 about the centroidal x-axis of the composite area (that is the axis-x' through C),

$$(I_x')_1 = (\bar{I}_x)_1 + A_1 d^2 \qquad \text{(Parallel axis theorem)}$$

$$(I_x')_1 = 80000 + 2400(39)^2$$

(d = distance between C and C_1 along the y-axis)

$$(I_x')_1 = 3.73 \times 10^6 \text{ mm}^4.$$

Area A_2:

M.I. of the area A_2 about is centroidal x-axis (that is the axis through C_2),

$$(\bar{I}_x)_2 = \frac{1}{12}(20)(130)^2 = 3661666.3 \text{ mm}^4$$

M.I. of the area A_2 about the centroidal x-axis of the composite area (that is the axis-x' through C),

$$(I_x')_2 = (\bar{I}_x)_2 + A_2 d^2 \qquad \text{(Parallel axis theorem)}$$

$$(I_x')_2 = 3661666.3 + 2600(36)^2$$

$$(I_x')_2 = 7.03 \times 10^6 \text{ mm}^4$$

(d = distance between C and C_2 along the y-axis)

Composite Area :

$$I_x' = (I_x')_1 + (I_x')_2$$

$$I_x' = 3.73 \times 10^6 + 7.03 \times 10^6$$

$$I_x' = 0.76 \times 10^6 \text{ mm}^4. \quad \textit{Ans.}$$

Moment of inertia of the composite area about the centroidal y-axis (that is axis-y' through C)

Area A_1:

M.I. of the area A_1 about its centroidal y-axis (that is the axis-y through C_1),

$$(\bar{I}_y)_1 = \frac{1}{12}(20)(120)^3 = 2879949.1 \text{ mm}$$

M.I. of the area A_1 about the centroidal y-axis of the composite area (that is the axis y' through C),

$$(I_y')_1 = (\bar{I}_y)_1 + A_1 d^2 \qquad \text{(Parallel axis theorem)}$$

$$(I_y')_1 = 2879949.1 + 2400(26)^2$$

$$(I_y')_1 = 2879949.1 + 1622400$$

$$(I_y')_1 = 4.5 \times 10^4 \text{ mm}^4$$

(d = distance between C and C_1 along the x-axis)

Area A_2:

M.I. of the area A_2 about its centroidal y-axis (that is the y-axis through C_2),

$$(\bar{I}_y)_2 = \frac{1}{12}(130)(20)^3 = 86666.66 \text{ mm}^4$$

M.I. of the area A_2 about the centroidal y-axis of the composite area (that is the axis-y through C).

$$(I_y')_2 = (\bar{I}_y)_2 + A_2 d^2$$

$$(I_y')_2 = 86666.66 + (2600)(24)^2$$

$$(I_y')_2 = 86666.66 + 1497600$$

$$(I_y')_2 = 1.58 \times 10^6 \text{ mm}^4$$

(d = distance between C and C_2 along the x-axis)

Composite Area :

$$I_y' = (I_y')_1 + (I_y')_2 = (4.5 + 15.8)10^6$$

$$I_y' = 6.08 \times 10^6 \text{ mm}^4. \quad \textit{Ans.}$$

12.7 PRODUCT OF INERTIA

Consider a plane figure of area A in the x-y plane as shown in Fig. 12.19.

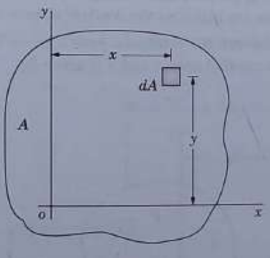


Fig. 12.19

Divide this area into infinitesimal areas. The integral,

$$I_{xy} = \int xy \, dA \qquad \text{...(12.12)}$$

obtained by multiplying each element dA of the area A by its coordinates x and y and the integration extending over the entire area of the plane figure is called the product of inertia (I_{xy}) of the figure with respect to the x and y axes. The following points in the case of product of inertia may be noted.

1. Although the moments of inertia I_x and I_y are always positive for positive areas (A void, hole or area removed is a negative area), the product of inertia I_{xy} for a positive area may be either positive or negative.
2. The product of inertia is zero when either one or both of the x and y axes are axes of symmetry. To understand this concept consider a channel section as shown in Fig. 12.20. The section is symmetrical with respect to the x-axis.

Corresponding to an element dA of coordinates x and y there is a symmetrically placed element dA' of coordinates x and ($-y$) such that the sum of the terms $(x)(-y)\ dA'$ and $(x)\ (y)\ dA$ is zero.

Since the entire area may be considered to be composed of such pair of elements, it follows that the product of inertia I_{xy} for the entire area is zero.

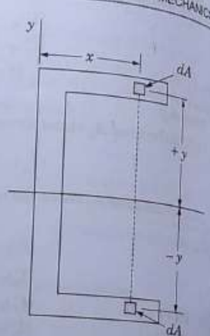


Fig. 12.20

12.8 DISPLACEMENT OF AXES PARALLEL TO THEMSELVES

Parallel axes theorem similar to moments of inertia may also be derived for the product of inertia.

Let x and y be the rectangular coordinate axes through any point O in the plane of the figure A as shown (Fig. 12.21).

x' and y' be the corresponding parallel axes through the centroid C of the area. The axes through the centroid of an area are called as the centroidal axes as mentioned earlier.

Let the product of inertia through the centroidal axes x', y' be known and denoted by $\bar{I}_{xy}$. The product of inertia for the parallel axes x and y is given by,

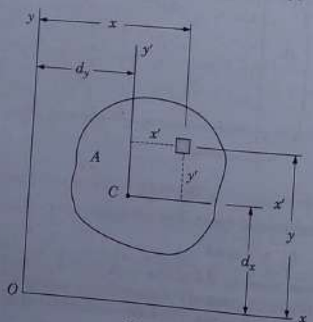


Fig. 12.21

$$I_{xy} = \bar{I}_{xy} + A(d_x)\,(d_y)$$

where d_x and d_y are the coordinates of the centroid with respect to the new axes x and y. For the proof of the above consider any element dA.

$$x = x' + d_x,\ y = y' + d_y$$

$$I_{xy} = \int xy\, dA = \int (x'+d_x)(y'+d_y)\, dA$$

$$I_{xy} = \int x'y'dA + d_x \int y'dA + d_y \int x'dA + \int d_x d_y\, dA$$

Second and third terms of the above equation vanish because they represent first moments of area about the centroidal axes. Hence,

$$I_{xy} = \bar{I}_{xy} + A(d_x)(d_y) \qquad ...(12.13)$$

12.9 ROTATION OF AXIS: PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA*

It can be shown that the product of inertia I_{xy} during rotation of the axes changes its sign and becomes negative. From the above fact it can be concluded that there must be certain directions of the axes for which the product of inertia is zero. The axes taken in these directions are called the principal axes of the area. The two principal axes are perpendicular to each other and are such that the product of inertia of the given area with respect to these axes is zero.

Consider a plane figure of area A. Let the moments of inertia I_x, I_y and the product of inertia I_{xy} with respect to the axes x and y passing through any point O within or outside the area be known.

Let the axes be rotated anticlockwise about O by an angle θ to new position x' and y'. It can be shown that the moments of inertia I'_x and I'_y and the product of inertia $I_{x'y'}$ about the axes x' and y' are given by the equations,

$$I'_x = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \qquad ...(12.14)$$

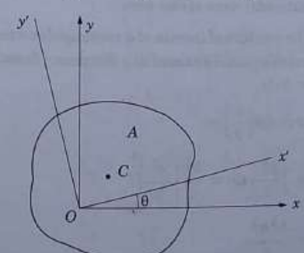


Fig. 12.22

$$I_y' = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \qquad ...(12.15)$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \qquad ...(12.16)$$

Axes x' and y' corresponding to a zero value of product of inertia can be obtained by setting equation 12.16 to zero. Let that angle be denoted by θ_m.

$$0 = \frac{I_x - I_y}{2} \cos 2\theta_m + I_{xy} \sin 2\theta_m$$

Or
$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = \frac{2I_{xy}}{I_y - I_x} \qquad ...(12.17)$$

Above expression can be used to determine the directions of the principal axes through O. This equation defines two values of $2\theta_m$ which are 180° apart. Thus the value of θ_m are 90° apart. The moments of inertia about these axes are called principal moments of inertia and are given by,

$$I_{max} = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \qquad ...(12.18)$$

$$I_{min} = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \qquad ...(12.19)$$

The principal axes also represent the two axes for which moments of inertia are maximum and minimum. If an area possesses an axis of symmetry through point O, this axis must be a principal axis of the area about point O.

The equations 12.14 to 12.19 are valid for any point located inside or outside the given area. The point O can be so chosen so as to coincide with the centroid C of the area. Any axes passing through C is the centroidal axes and the two principal axes of the area through its centroid C are called as the principal centroidal axes of the area.

Example 12.11 Determine the product of inertia of a rectangular area about the x and y axis.

Solution: Consider an element of area dA situated at a distance y from the x axis [Fig. 12.23 (a)]

Areas of the element = $b\,dy$

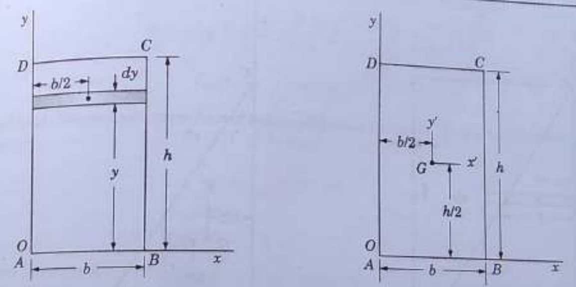
$$dI_{xy} = (b\,dy)\left(\frac{b}{2}\right)(y)$$

$$I_{xy} = \int \frac{b^2}{2} y\,dy = \frac{b^2}{2}\left[\frac{y^2}{2}\right]_0^h$$

$$I_{xy} = \frac{b^2h^2}{4}$$

Alternatively, we can use parallel axes theorem [Fig. 12.23 (b)]

$$I_{xy} = \bar{I}_{x'y'} + A\left(\frac{h}{2}\right)\left(\frac{b}{2}\right)$$



(a) (b)

Fig. 12.23

$$I_{xy} = 0 + (bh)\frac{hb}{4}$$

($I_{x'y'} = 0$, x' and y' being the axes of symmetry)

$$I_{xy} = \frac{b^2h^2}{4}$$

Example 12.12 Determine the product of inertia of a right angle triangle (*i*) with respect to x and y axis (*ii*) with respect to the centroidal axes parallel to x and y axes.

Solution: Consider an element of area dA situated at a distance y from the x-axis [Fig. 12.24 (a)]

$$dA = l\,dy$$

$$\frac{l}{b} = \frac{h - y}{h}$$

Or
$$l = \left(\frac{h - y}{h}\right) b$$

$$I_{xy} = \int dI_{xy} = \int (l\,dy)\left(\frac{l}{2}\right)(y)$$

$$I_{xy} = \int \left(\frac{h - y}{h}\right) b \left(\frac{h - y}{h}\right) \frac{b}{2} y\,dy$$

$$I_{xy} = \frac{b^2}{2h^2}\left[\frac{h^2y^2}{2} + \frac{y^4}{4} - \frac{2hy^3}{3}\right]_0^h$$

$$I_{xy} = \frac{b^2h^2}{24}$$

Area of the triangle = $\frac{1}{2}bh$ and the coordinates of the centroid G are $(b/3, h/3)$.

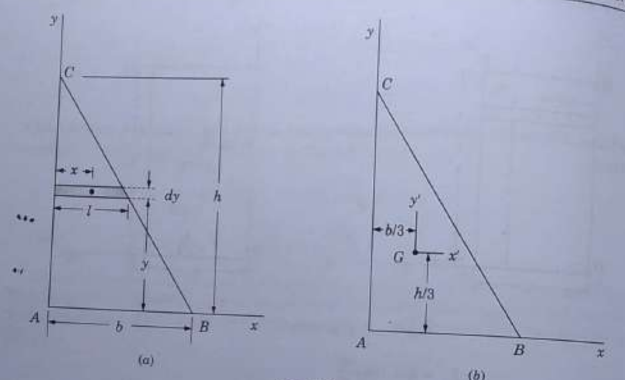


Fig. 12.24

Applying parallel axes theorem [Fig. 12.24(b)]

$$dI_{xy} = \bar{I}''_{xy} I_{x'y'} + \left(\frac{1}{2}bh\right)\left(\frac{b}{3}\right)\left(\frac{h}{3}\right)$$

$$\frac{b^2h^2}{24} = \bar{I}_{x'y'} + \frac{1}{18}b^2h^2$$

$$\bar{I}_{xy}'' = (-)\frac{1}{72}b^2h^2$$

Example 12.13 Find the product of inertia of a quarter of circle with respect to the x and y axes.

Solution: Consider an element of area dA situated at a distance y from the x-axis (Fig. 12.25).

$$dA = x\,dy$$

$$dI_{xy} = (x\,dy)\left(\frac{x}{2}\right)(y)$$

$$= \frac{x^2}{2}y\,dy$$

Using the equation of circle

$$x^2 + y^2 = r^2$$

$$x^2 = r^2 - y^2$$

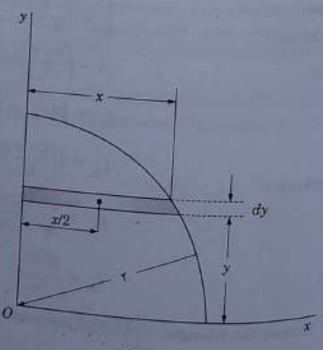


Fig. 12.25

$$I_{xy} = \int_0^r \frac{(r^2 - y^2)y}{2}dy$$

$$I_{xy} = \frac{r^4}{8}.$$

Example 12.14 Determine the product of inertia of the L-section (Fig. 12.26) with respect to the x and y axes.

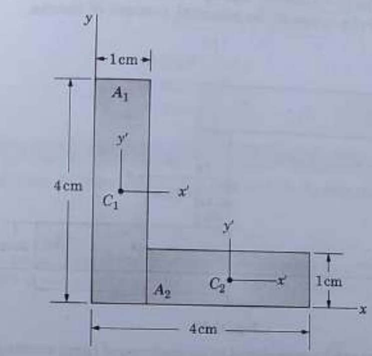


Fig. 12.26

Solution: Consider the area to be composed of two rectangles of areas A_1 and A_2.

Rectangle A_1. Using parallel axis theorem,

$$I_{xy} = I''_{xy} + A_1\left(\frac{1}{2}\right)\left(\frac{4}{2}\right)$$

$$I_{xy} = 0 + 4 \times 1 \times \frac{1}{2} \times \frac{4}{2} \quad (x' \text{ and } y' \text{ are the axes of symmetry})$$

$$I_{xy} = 4\text{ cm}^4.$$

Rectangle A_2. Using parallel axis theorem,

$$I_{xy} = I''_{xy} + A_2\left(1 + \frac{3}{2}\right)\left(\frac{1}{2}\right)$$

$$I_{xy} = 0 + 3 \times 1 \times \frac{5}{2} \times \frac{1}{2} \quad (x' \text{ and } y' \text{ are axes of symmetry})$$

$$I_{xy} = \frac{15}{4}\text{cm}^4$$

Product of inertia of the total area,

$$I_{xy} = 4+\frac{15}{4} = \frac{31}{4}\text{ cm}^4$$

$$I_{xy} = 7\frac{3}{4}\text{ cm}^4.$$

Example 12.15 For a *Z*-section shown (Fig. 12.27) the moments of inertia with respect to the *x* and *y* axes are given to be $I_x = 1548\text{ cm}^4$ and $I_y = 2668\text{ cm}^4$. Determine the principal axes of the section about *O* and the values of the principal moment of inertia.

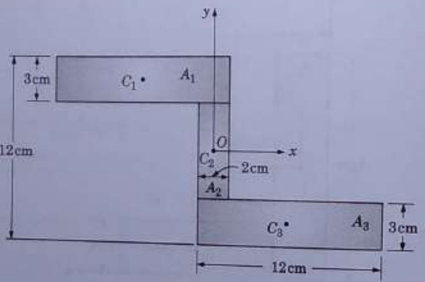


Fig. 12.27

Solution: Area of *Z*-section can be considered to be made up of three rectangles A_1, A_2 and A_3 with their centroids at C_1, C_2 and C_3 respectively. Note that C_2 and *O* are coincident points.

Area (cm)^2	*Distance of the centroid from x and y axes* (cm)	
$A_1 = 12 \times 3 = 36$	$\bar{x}_1 = -5.0$	$y_1 = +4.5$
$A_2 = 2 \times 6 = 12$	$\bar{x}_2 = 0$	$y_2 = 0$
$A_3 = 12 \times 3 = 36$	$\bar{x}_3 = +5$	$y_3 = -4.5$

Product of inertia of the total area,

$$I_{xy} = [0 + 36(-5.0)(4.5)] + [0 + 0] + [0 + 36(5.0)(-4.5)]$$

$$I_{xy} = -1620\text{ cm}^4$$

(Using parallel axis theorem and the concept that product of inertia vanishes if any one of the axes is the axis of symmetry).

$$I_x = 1548\text{ cm}^4, \quad I_y = 2668\text{ cm}^4 \text{ (given)}$$

For finding the directions of the principal axes,

$$\tan 2\theta_m = \frac{2I_{xy}}{I_y - I_x} = \frac{-2\times 1620}{2668-1548}$$

$$\tan 2\theta_m = -2.893$$

$$2\theta_m = -70.93^\circ,\ \theta_m = 35.46^\circ$$

$$\theta_m = -35.46^\circ, \text{ and } +54.54^\circ$$

$$I_{\max,\min} = \frac{I_x+I_y}{2} \pm \sqrt{\left(\frac{I_x-I_y}{2}\right)^2 + (I_{xy})^2}$$

$$= \frac{1548+2668}{2} \pm \sqrt{\left(\frac{1548-2668}{2}\right)^2 + (1620)^2}$$

$$= 2108 \pm 1714$$

$$I_{\max} = 3822\text{ cm}^4$$

$$I_{\min} = 394\text{ cm}^4$$

12.10 MOMENT OF INERTIA OF A MASS (RIGID BODY)

Consider a body of mass *m*. The moment of inertia of the body with respect the axis *AA'* is defined by integral

$$I = \int r^2\, dm \qquad \text{...(12.20)}$$

where, *dm* is the mass of an element of the body situated at a distance *r* from the axis *AA'* and the integration is extended over the entire volume of the body.

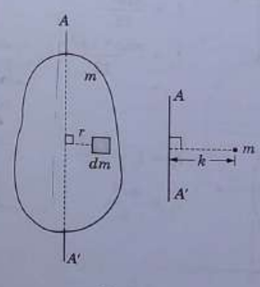


Fig. 12.28

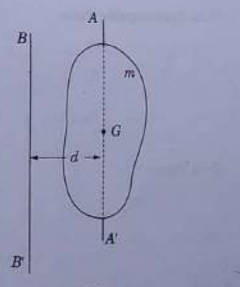


Fig. 12.29

From the above definition, it follows that the moment of inertia of a body has the dimension of mass × (length)^2 or

kilogram (metre)^2 = kgm^2.

The radius of gyration *k* of body with respect to the axis *AA'* is given by the relation

$$I = k^2 m$$

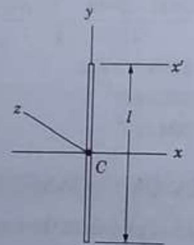
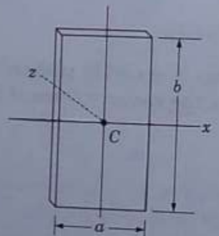
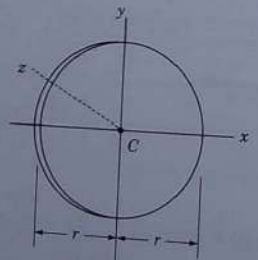
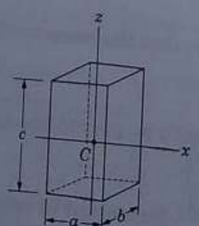
Or

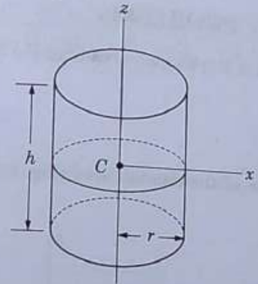
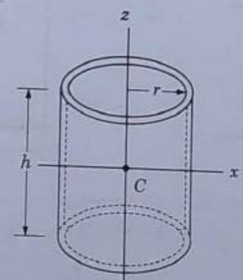
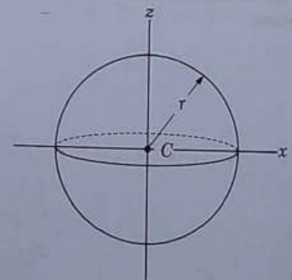
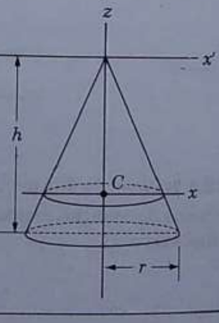
$$k = \sqrt{\frac{I}{m}} \text{ (expressed in metres)} \qquad \text{...(12.20)}$$

Parallel Axis Theorem. Consider a body of mass m. Let the moment of inertia of the body with respect an axis AA' passing through the centre of gravity G of the body $\bar{I}$. Then the moment of the inertia I of the body with respect to the axis BB' which is parallel to the centroidal axis AA' and is at a distance d from it is given by

$$I = \bar{I} + md^2 \qquad ...(12.21)$$

Table 12.2 *Mass Moments of Inertia of Various Bodies*

Body	Figure	Moments of inertia
Slender Rod		$I_x = \frac{ml^2}{12}$, $I_x' = \frac{ml^2}{3}$ $I_y = 0$ $I_z = I_x = \frac{ml^2}{12}$
Thin Rectangular Plate		$I_x = \frac{mb^2}{12}$ $I_y = \frac{ma^2}{12}$ $I_z = \frac{m(a^2+b^2)}{12}$
Thin Disc		$I_x = I_y = \frac{mr^2}{4}$ $I_z = \frac{mr^2}{2}$
Rectangular Prism		$I_x = \frac{m(b^2+c^2)}{12}$ $I_z = \frac{m(a^2+b^2)}{12}$

Body	Figure	Moments of inertia
Circular cylinder		$I_x = \frac{m(3r^2+h^2)}{12}$ $I_z = \frac{1}{2}mr^2$
Cylindrical shell (Hoop)		$I_x = \frac{m}{12}(6r^2+h^2)$ $I_z = mr^2$
Sphere		$I_x = \frac{2}{5}mr^2$ $I_z = \frac{2}{5}mr^2$
Cone		$I_x = \frac{3}{80}m(4r^2+h^2)$ $I_x' = \frac{2}{5}m\left(\frac{r^3}{4}+h^2\right)$ $I_z = \frac{3}{10}mr^2$

PROBLEMS

12.1. Determine the moments of inertia of the shaded area (Fig. P.12.1) with respect to the x-axis and y-axis by direct intergration

$$\left[I_x = \frac{2ab^3}{15},\ I_y = \frac{2a^3b}{7}\right]$$

12.2. Determine the moments of inertia of the shaded area (Fig. P.12.2) about the x-axis and the y-axis.

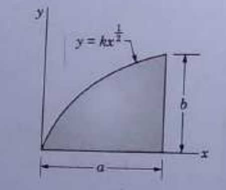


Fig. P.12.1

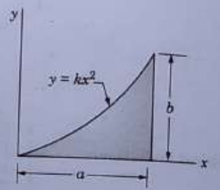


Fig. P.12.2

$$\left[I_x = \frac{ab^3}{21},\ I_y = \frac{a^3b}{5}\right]$$

12.3. Determine the moments of inertia of the shaded area (Fig. P. 12.3) with respect to the x-axis and the y-axis by direct integration.

$$\left[I_x = \frac{ab^3}{28},\ I_y = \frac{a^3b}{20}\right]$$

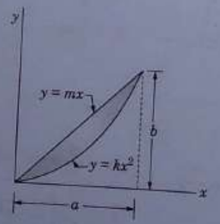


Fig. P.12.3

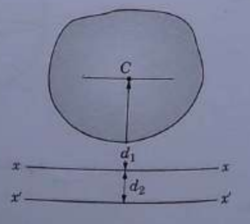


Fig. P.12.4

12.4. The shaded area shown in Fig. P.12.4 is equal to 125 cm^2. If I_{xx} = 35000 cm^4, $I^{"}_{xx}$ = 70000 cm^4 and d_2 = 7.5 cm, determine the distance d_1 and the moment of inertia of the area with respect to the centroidal axis parallel to the axis x-x.

$$\begin{bmatrix} d_1 = 14.91\text{ cm} \\ I_x = 7211.5\text{ cm}^4 \end{bmatrix}$$

12.5. Determine the moments of inertia of the shaded area (Fig. P.12.5) with respect to the centroidal axes parallel and perpendicular to the side AB.

$$[\bar{I}_x = \bar{I}_y = 2.08 \times 10^2\text{ cm}^4]$$

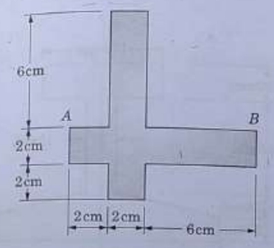


Fig. P.12.5

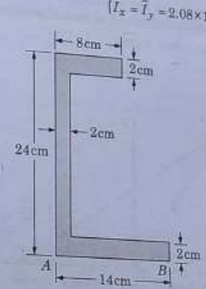


Fig. P.12.6

12.6. Determine the moments of inertia of the cross-section of an iron beam with respect to the centroidal axes parallel and perpendicular to the side AB.

$$\begin{bmatrix} \bar{I}_x = 6463\text{ cm}^4 \\ \bar{I}_y = 1152.3\text{ cm}^4 \end{bmatrix}$$

12.7. Find the moments of inertia of the cross-section of an iron beam (Fig. P.12.7) with respect to the centroidal axes.

$$\begin{bmatrix} \bar{I}_x = 2885\text{ cm}^4 \\ \bar{I}_y = 838\text{ cm}^4 \end{bmatrix}$$

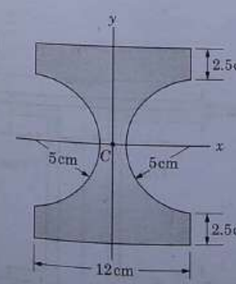


Fig. P.12.7

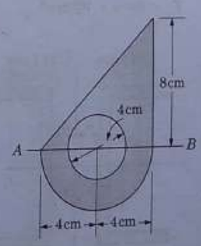


Fig. P.12.8

12.8. Find the moment of inertia of the shaded area (Fig. P.12.8) with respect to the centroidal axes parallel to AB.

$[\bar{I}_{AB} = 429.3\ \text{cm}^4]$

12.9. Determine the coordinates of centroid of the shaded area shown in Fig. P.12.9. Calculate the moment of inertia of thus shaded area w.r.t. the centroidal set of axes parallel respectively to the x and y axes.

$[C(1, 2),\ \bar{I}_x = 36\ \text{cm}^4,\ \bar{I}_y = 63.0\ \text{cm}^4]$

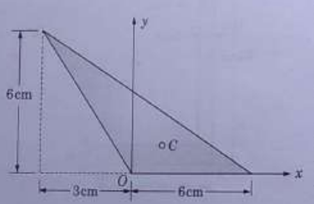


Fig. P.12.9

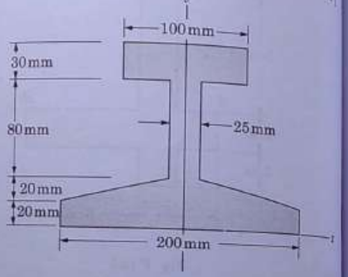


Fig. P.12.10

12.10. Determine the moments of inertia of I-section as shown in Fig. 12.10 about its centroidal axes parallel to the x and y axes.

$$\begin{bmatrix} \bar{I}_x = 31543447\ \text{mm}^4 \\ \bar{I}_y = 19745122\ \text{mm}^4 \end{bmatrix}$$

12.11. Find the value of 'd' for the combined section such that $I_{xx} = I_{yy}$. Given that the two sections are identical and each has,

Area = $4947\ \text{mm}^2$

$I_{xx} = 9312.6 \times 10^4\ \text{mm}^4$

$I_{yy} = 394.6 \times 10^4\ \text{mm}^4$

[110.1 mm]

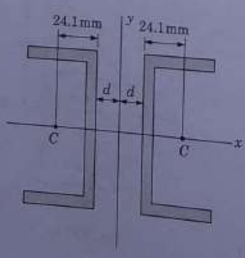


Fig. P.12.11

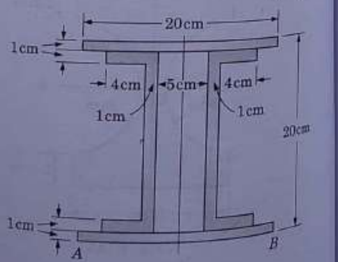


Fig. P.12.12

12.12. Find the moment of inertia of a built up section as shown in Fig. P.12.12 with respect to the centroidal axes parallel to AB.

$[\bar{I}_x = 5743.0\ \text{cm}^4]$

12.13. For a composite area shown in Fig. P.12.13 find the relation between b and r so that x and y will be the principal axes.

$[b = 2r]$

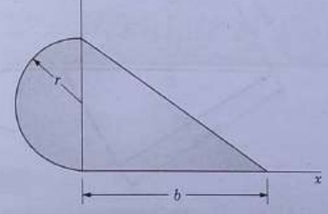


Fig. P.12.13

12.14. Find the product of inertia of the area shown (Fig. P.12.14) with respect to the centroidal x and y axes.

Also find the angle θ defining the directions of principal axes through the centroid and the principal moments of inertia.

$$\begin{bmatrix} 0.8 \times 10^2\ \text{cm}^4, \theta = -25.7^\circ \\ 2.28 \times 10^2\ \text{cm}^4, 0.23 \times 10^2\ \text{cm}^4 \end{bmatrix}$$

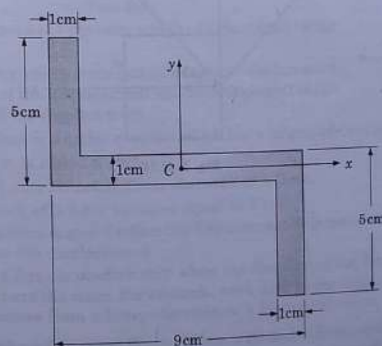


Fig. P.12.14

12.15. Find the product of inertia $\bar{I}_{xy}$ of the rectangle shown in Fig. P.12.15.

$[\bar{I}_{xy} = -141 \text{ cm}^4]$

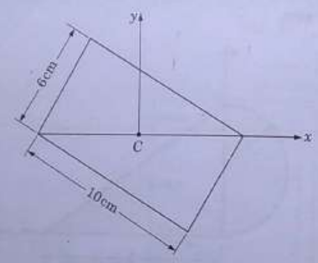


Fig. P.12.15

12.16. Find the moments of inertia and product of inertia of the square of side a as shown in the Fig. P.12.16 with respect to the x' and y' axes.

$$\left[I_x' = I_y' = \frac{5a^4}{24},\ I''_{xy} = -\frac{a^4}{8}\right]$$

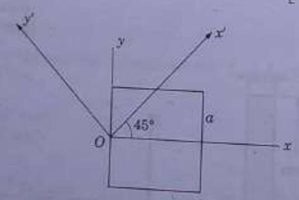


Fig. P.12.16

13 CHAPTER Principle of Virtual Work

13.1 INTRODUCTION

In the earlier chapter, the problems involving the equilibrium of a rigid body and a system of rigid bodies (when acted upon by several forces) were solved using the equations of equilibrium.

In this chapter, we shall discuss another method of expressing the conditions of equilibrium which is based on the principle of virtual work. Before discussing this method let us define a few terms.

Work of a Force. If a particle is subjected to a force F and the particle is displaced by an infinitesimal displacement ds then

The work of the force F during the displacement ds.

$$= (F \cos \theta) ds$$

Where θ is the angle between the force and the displacement vector (Fig. 13.1)

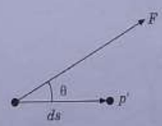


Fig. 13.1

Thus, the work done by a force during an infinitesimal displacement is equal to the product of the displacement and the component of the force in the direction of the displacement.

The work done by a force is a scalar quantity which has a magnitude and sign but no direction.

The unit of work Nm is called a joule (J).

The following points may be noted about the work of a force.

(i) If $\theta = 0$, the work of a force becomes equal to $F(ds)$.

(ii) Work done by a force is zero if either the displacement ds is zero or the force acts normal (i.e., $\theta = 90°$) to the displacement.

(iii) Work done by a force is positive only when the direction of the force and the direction of displacement are the same. For example, work done by the force of gravity is negative when a body moves from a lower elevation to a higher elevation.

Virtual Displacement and Virtual Work. When a system of forces acting on a body are in equilibrium then the displacement of the body would be zero and no work is possible. But, an imaginary infinitesimal displacement can be assumed to be given to the body in equilibrium. Such a displacement is called as the virtual displacement. The resulting work done by the forces acting on the body during the virtual displacement is called as the virtual work.

The notations generally used for the virtual displacements are, δx or δy for the linear displacements and $\delta\theta$ for the angular displacement.

13.2 PRINCIPLE OF VIRTUAL WORK

The principle of virtual work states that "*if a rigid body is in equilibrium, the total work of the external forces acting on the rigid body is zero for any virtual displacement of the body consistent with the geometrical conditions of the body.*"

Before illustrating the above principle, let us enumerate *the force which do no work* due to the reasons discussed earlier.

Force which do no work are :

1. The weight of a body when its centre of gravity moves in a horizontal direction.
2. The reaction at a frictionless hinge when the body rotates about the hinge.
3. The reaction at a frictionless surface when the body moves along the surface.
4. The friction force acting on a wheel when it rolls without slipping.
5. The internal forces of the nature of action and reaction (tension in the string and axial forces in the bars).

13.3 APPLICATION ON THE PRINCIPLE OF VIRTUAL WORK

Consider a lever AB hinged at C with the forces P and Q acting as shown.

Fig. 13.2

We can obtain the relationship between the forces P and Q by writing the equation of equilibrium

$\Sigma M_C = 0$:

$$Qa - Pb = 0$$

$$Qa = Pb$$

Let us now apply the principle of virtual work.

Let us give an infinitesimal angular displacement $\delta\theta$ to the lever about the hinge C as shown in the Fig. 13.3.

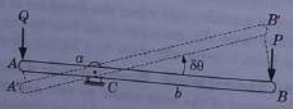


Fig. 13.3

The end A shall move to A' and B' such that,

$$AA' = a\,\delta\theta$$

$$BB' = b\,\delta\theta$$

The virtual displacement of ends A and B are $a\,\delta\theta$ and $b\,\delta\theta$ respectively. *These displacements can be treated to be normal to the lever AB (or as rectangular displacements) for very small values of $\delta\theta$.*

If can be seen that ends A and B of the lever can move in no other way because, the lever is hinged at C. Also, the displacements of the ends A and B are not independent of each other but are related. *Thus, the virtual displacements of the ends A and B are consistent with the geometrical condition of the body.*

The forces doing work are P and Q ; reaction at the frictionless hinge C does no work. Applying the principle of virtual work, we can write,

$$-P(b\,\delta\theta) + Q(a\,\delta\theta) = 0 \qquad \text{(The directions of the force } P \text{ and of}$$

$$Qa = Pb \qquad \text{the displacement at } B \text{ are opposite)}$$

So we obtained the same relation as obtained earlier by using the equation of equilibrium.

Fully Constrained Body. In the earlier example, the lever was movable about the hinge C, that is, it was partially constrained. It was, therefore, possible to give a virtual angular displacement $\delta\theta$ to the lever.

Let us now consider the situation when the lever AB is pin jointed to the bar AD, the end D of which is hinged (Fig. 13.4).

Here the system is completely *constrained or is immovable*. It is, therefore, not possible to give any virtual displacement to the lever.

To solve such problems, we remove one of the constraints or supports and replace it by a suitable force acting at the support point.

Let us remove the bar AD and replace it by a force F_{AD} acting on the lever at A and normal to it.

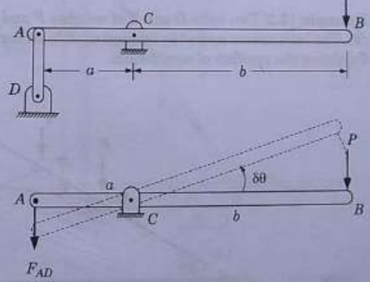


Fig. 13.4

Let us now give a virtual angular displacement $\delta\theta$ to the lever about the hinge C. Equation of virtual work now can be written as

$$F_{AD}(a\,\delta\theta) - P(b\,\delta\theta) = 0$$

$$F_{AD} = \frac{Pb}{a}$$

The force F_{AD}, represents the axial force in the bar AD due to the load P. Thus, *In the case of fully constrained bodies we can make the system movable by removing one of the supports and replacing it by a suitable force.*

While writing the equation of virtual work there may be a choice of giving either a virtual angular displacement ($\delta\theta$), or a linear displacement (δx or δy). Give virtual displacement in one direction at a time so that the unknown force to be determined can do virtual work and it may appear in the equation of virtual work.

Example 13.1 A weight W of 1000 N is to be raised by a system of pulleys as shown in Fig. 13.5. Using the principle of virtual work find the value of the force P which can hold the system in equilibrium.

Solution: The forces which can do work are the load $W = 1000$ N and the force P.

For a virtual downward displacement δy of the force P, the corresponding virtual displacement of the load W is $\frac{\delta y}{2}$ upwards.

The principle of virtual work gives.

$$-W(\delta y/2) + P\delta y = 0$$

$$\frac{1000}{2} - P = 0$$

$$P = 500 \text{ N} \quad Ans.$$

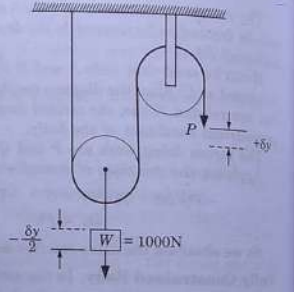


Fig. 13.5

Example 13.2 Two balls D and E of weights P and Q can slide freely along the bars AC and BC. The balls are connected by an inextensible string DE (Fig. 13.6). Find the value of the angle θ defining the position of equilibrium.

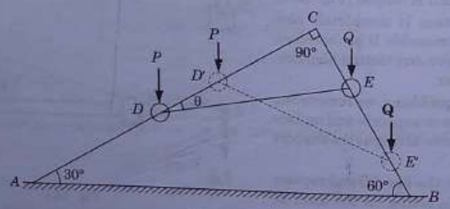


Fig. 13.6

* The –ve sign of δ_{DC} obtained does not mean that work done is negative. It simply indicates that as θ increases distance DC decreases or for +ve $\delta\theta$, δ_{DC} is –ve. Take the absolute value of δ_{DC}. Observe that the work done is positive when the directions of force and displacement are same.

Solution: Let us give to the string DE a virtual angular displacement $\delta\theta$. Forces that do work are P and Q. Let us calculate their virtual linear displacement.

From triangle DCE,

$$DC = l\cos\theta$$

$$CE = l\sin\theta$$

$$\delta_{DC} = -l\sin\theta\,\delta\theta$$

$$\delta_{CE} = l\cos\theta\,\delta\theta$$

where, l is the length of the string DE.

The principle of virtual work gives,

$$-(P\sin 30°)\,\delta_{DC} + (Q\sin 60°)\,\delta_{CE} = 0$$

[Point D moves up and point E down]

$$-P\left(\frac{1}{2}\right)l\sin\theta\,\delta\theta + \frac{Q\sqrt{3}}{2}l\cos\theta\,\delta\theta = 0$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{Q\sqrt{3}}{P} \quad Ans.$$

Example 13.3 A ladder AB of length l and weight W stands in a vertical plane supported by smooth surfaces at A and B. Using the principle of virtual work, find the magnitude of the horizontal force P to be applied at the end A if the ladder is to be in equilibrium.

Solution: Choose the coordinate system with origin at O. Give a small virtual displacement δx to the end A, the corresponding virtual displacements of the end B is δy and of the point C is $\frac{\delta y}{2}$. Also, the angle of inclination θ of the ladder shall change by $\delta\theta$.

The forces doing the work are the weight of the ladder W and the force P. The reactions at A and B do no work.

From the triangle ABO,

$$x = l\cos\theta, \quad y = l\cos\theta$$

Differentiating

$$\delta x = l(-\sin\theta\, d\theta)$$

$$\delta y = l(\cos\theta\, d\theta)$$

The principle of virtual work gives,

$$P\,\delta x = W\frac{\delta y}{2} = 0$$

$$P(l\sin\theta\,\delta\theta) - W\left(\frac{l}{2}\cos\theta\, d\theta\right) = 0$$

$$P = \frac{W}{2}\cot\theta \quad Ans.$$

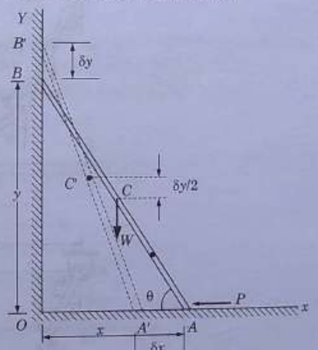


Fig. 13.7

* As θ increases x decreases and y increases or for a +ve $\delta\theta$, dx is –ve and δy is +ve.

Example 13.4 Two uniform rods each of length l and weight W are connected as shown (Fig. 13.8). Using the method of virtual work determine θ_1 and θ_2 corresponding to the equilibrium of the bars.

Solution: Choose a coordinate system with origin at A as shown. We can write,

$$y_1 = \frac{l}{2}\cos\theta_1 \text{ and } y_2 = \left(l\cos\theta_1 + \frac{l}{2}\cos\theta_2\right)$$

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Differentiating

$$\delta y_1 = -\frac{l}{2}(\sin\theta_1\,\delta\theta_1)$$

and

$$\delta y_2 = -l\left(\sin\theta_1\,\delta\theta_1 + \frac{1}{2}\sin\theta_2\,\delta\theta_2\right)$$

$$x = l(\sin\theta_1 + \sin\theta_2)$$

$$\delta x = l(\cos\theta_1\,\delta\theta_1 + \cos\theta_2\,\delta\theta_2)$$

The only forces doing work are the weights of the bars (W) and the force P. Give a vertical angular displacement $\delta\theta_1$ to bar AB keeping θ_2 to be the same.

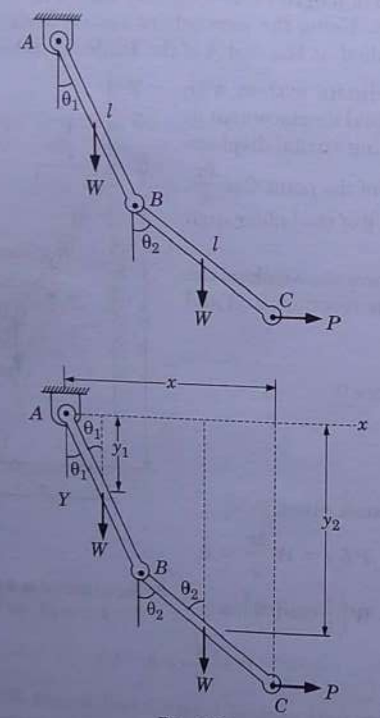

Fig. 13.8

Applying the principle of virtual work,

$$P\delta x + W\delta y_1 + W\delta y_2 = 0,$$

$$P(l\cos\theta_1\,\delta\theta_1) + W\left(-\frac{l}{2}\sin\theta_1\delta\theta_1\right) + W(-l\sin\theta_1\,\delta\theta_1) = 0, \quad (\delta\theta_2 = 0)$$

$$P\cos\theta_1 - \frac{3}{2}W\sin\theta_1 = 0$$

Or

$$\frac{3}{2}W\sin\theta_1 = P\cos\theta_1$$

$$\tan\theta_1 = \frac{2P}{3W} \quad Ans.$$

Next give a virtual angular displacement $\delta\theta_2$ to the bar BC keeping θ_1 to be the same. Applying the principle of virtual work.

$$P\delta x + W\,\delta y_1 + W\,\delta y_2 = 0, \quad (\delta\theta_1 = 0)$$

$$P(l\cos\theta_2\,\delta\theta_2) + W(0) + \left(-\frac{l}{2}\sin\theta_2\delta\theta_2\right) = 0$$

$$P\cos\theta_2 = \frac{W\sin\theta_2}{2}$$

$$\tan\theta_2 = \frac{2P}{W} \quad Ans.$$

Example 13.5 Two beams AC and CD are hinged at C at as shown in Fig. 13.9. Using the principle of virtual work find the reaction at B for the position x of the load P.

Solution: Choose the coordinate system with origin at A as shown in Fig. 13.9.

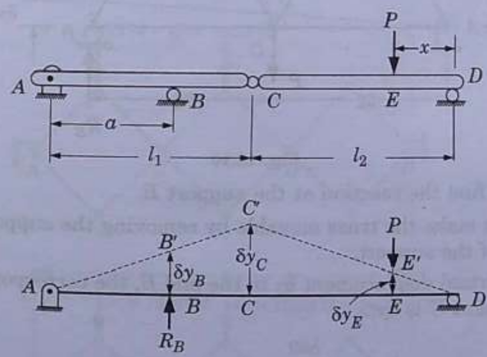

Fig. 13.9

The beam is completely constrained and immovable. Let us remove the support at B and replace it by a vertically reaction R_B.

Let us now give a virtual displacement δy_C to the hinge C. The corresponding virtual displacements of the points B and E then can be obtained. From similar triangles ABB' and ACC',

$$\frac{\delta y_C}{l_1} = \frac{\delta y_B}{a}$$

Or

$$\delta y_B = \frac{a}{l_1}\delta y_C$$

Similarly,

$$\delta y_E = \frac{x}{l_2}\delta y_C$$

The forces doing work are P and R_B.

Applying the principle of virtual work,

$$R_B(\delta y_B) - P(\delta y_E) = 0$$

$$R_B\left(\frac{a}{l_1}\delta y_C\right) - P\left(\frac{x}{l_2}\delta y_C\right) = 0$$

$$R_B = \frac{l_1}{l_2}\frac{x}{a}P \quad Ans.$$

Example 13.6 Using the principle of virtual work find the axial force in the member DE of the simple truss loaded and supported as shown in Fig. 13.10.

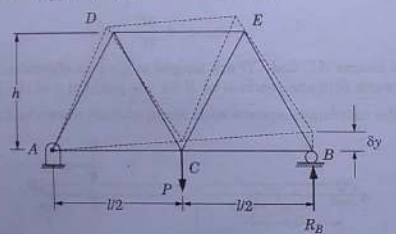

Fig. 13.10

Solution: Let us first find the reaction at the support B.

Reaction at B. Let us make the truss movable by removing the support at B and replacing it by the reaction R_B of the support.

Let us now give a virtual displacement δy to the end B, the corresponding displacement of the load P acting at point C is $\delta y/2$.

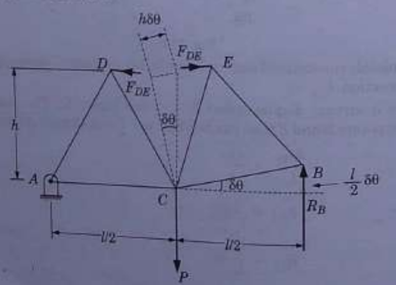

Fig. 13.11

Writing the equation of virtual work

$$-P\left(\frac{\delta y}{2}\right) + R_B(\delta y) = 0$$

$$R_B = \frac{P}{2} \quad Ans.$$

To find the force in the member DE, besides making the system movable, we have to cut the member DE and replace it by the axial force in that member so that the force in the member DE can do work. Next, give a virtual angular displacement $\delta\theta$ about the hinge C. The virtual work equation is,

$$-F_{DE}(h\delta\theta) + R_B\left(\frac{l}{2}\delta\theta\right) = 0$$

As,

$$R_B = \frac{P}{2}$$

$$F_{DE} = \frac{Pl}{4h}\text{(compression)}. \quad Ans.$$

Example 13.7 Determine the horizontal and vertical components of the reactions at A and B of the frame which is made up of three squares with hinged joints as shown in Fig. 13.12.

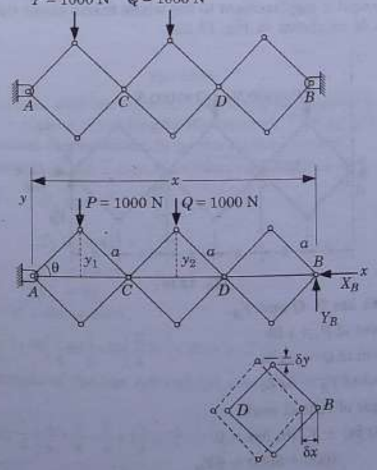

Fig. 13.12

Solution: The system is completely constrained and is immovable. Let us make it movable by removing the support at B and replacing it by reaction X_B and Y_B. Let the side of each square be 'a'.

Choosing the coordinate system with origin at A, we can write,

$$y_1 = y_2 = y = a \sin\theta$$

$$\delta y_1 = \delta y_2 = \delta y = a\cos\theta\,\delta\theta \qquad \theta = 45°$$

$$x = 6a\cos\theta \qquad \text{For } +\delta\theta,\ \delta x \text{ is } -\text{ve and}$$

$$\delta x = -6a\sin\theta\,\delta\theta \qquad \delta y \text{ is } +\text{ve}$$

Let us give a virtual displacement δx to the end B. The corresponding virtual diplacement of the forces P and Q would be δy.

Forces that do work are P, Q and X_B.

Applying the principle of virtual work,

$$P\,\delta y_1 + Q\delta y_2 + X_B\,\delta x = 0 \qquad \text{(work of } Y_B = 0\text{)}$$

$$1000\,(2a\cos\theta\,\delta\theta) - X_B(6a\sin\theta\,\delta\theta) = 0$$

$$X_B = \frac{2\times 1000}{6}\cdot\frac{\cos\theta}{\sin\theta} = \frac{2000}{6} = 333.3\text{ N} \quad \textit{Ans.}$$

Now give a virtual angular displacement to the whole frame about the end A such that line AB turns by an angle $\delta\phi$ as shown in Fig. 13.13.

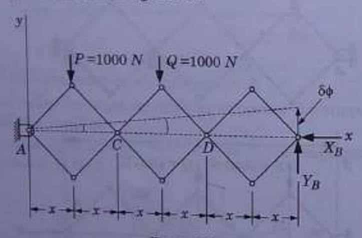

Fig. 13.13

The force doing work are P, Q and Y_B.

Vertical displacement of $P = x\,\delta\phi$

Vertical displacement of $Q = 3x\,\delta\phi$

Vertical displacement of $Y_B = 6x\,\delta\phi$

Applying the principle of virtual work,

$$-P\,(x\,\delta\phi) - Q(3x\,\delta\phi) + Y_B\,(6x\,\delta\phi) = 0$$

$$1000 + 3000 = 6Y_B$$

$$Y_B = \frac{4000}{6} = 666.7\text{ N} \quad \textit{Ans.}$$

Similarly, by removing the support at A and replacing it by the reactions X_A and Y_A we can obtain,

$$\left.\begin{aligned} X_A &= 333.3\text{N} \\ \text{and} \qquad Y_A &= 1333.3\text{N} \end{aligned}\right\} \quad \textit{Ans.}$$

Example 13.8 A hexagonal frame is made up of six bars of identical lengths and cross-sections. Each bar has a weight of W newton. The rod AB is fixed in a horizontal position. A string joins the mid-points of the bars AB and DE. Using the principle of virtual work, find the tension in the string (Fig. 13.14).

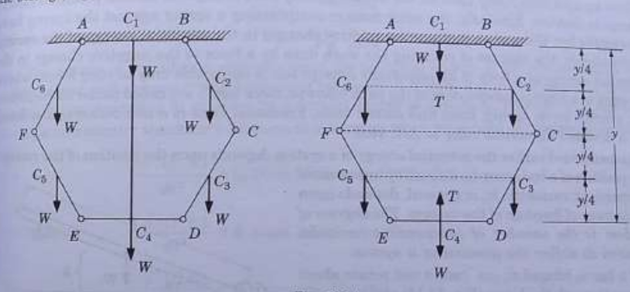

Fig. 13.14

Solution: The system of bars with the connecting string, is an immovable system. Let us make the system movable by replacing the string by the tensions T acting at the points C_1 and C_4.

Let us now give a virtual displacement δy to the bottom bar ED.

The virtual displacements of the C.G.'s of various bars are,

$$C_2 \text{ and } C_6 = \frac{\delta y}{4}$$

$$C_3 \text{ and } C_5 = \frac{3}{4}\delta y$$

$$C_1 \text{ is fixed}$$

$$C_4 = \delta y$$

Writing the equation of virtual work,

$$W\left(\frac{\delta y}{4}\right) + W\left(\frac{\delta y}{4}\right) + W\left(\frac{3}{4}\delta y\right) + W\left(\frac{3}{4}\delta y\right) + W(\delta y) - T(\delta y) = 0$$

(Work done by the weight of the bar AB and the tension T acting C_1 is zero as the bar is fixed).

$$W\left(\frac{1}{4} + \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + 1\right) = T$$

$$\frac{12}{4}W = T$$

$$T = 3\,W \quad \textit{Ans.}$$

13.4 POTENTIAL ENERGY AND EQUILIBRIUM

The application of the principle of virtual work becomes simpler when the potential energy of a system is known.

The potential energy (P.E.) of a body is the energy possessed by the body by virtue of its position. For example, to raise a body of mass m from some fixed reference plane to a height h above this plane, it is necessary to do work equal to mgh *against the gravity force*. This work done against gravity force is stored as potential energy of the body. It may be noted that as the work is done against the gravity force so, the work of gravity force is negative but the potential energy change is positive. Similarly, the work done in compressing a spring against the spring force is negative but there is an increase (or positive change) in the potential energy of the spring. Alternatively, the concept of equating the work done by a force to the negative change in the potential energy of a body is an important concept but is applicable only to such forces where the work of a force is independent of the path followed. Such forces are called *conservative forces* e.g., gravity force, spring, force and elastic force. Frictional force is a non-conservation force. For a detailed discussion refer to Art. 16.6.

As mentioned earlier the potential energy of a system depends upon the position of the system. The position of a body may be defined by one or more independent variables. It, in general, depends upon the degrees of freedom of the system. *The degrees of freedom is the member of independent variables required to define the position of a system.*

If a bar is hinged at one end, it can rotate about an axis through the hinge (Fig. 13.15) and possesses one degree of freedom. The angle θ that the bar makes with the reference line shall define its position.

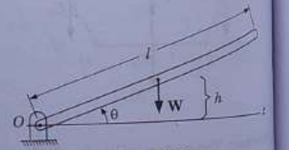

Fig. 13.15

$$\text{P.E.} = Wh = W\left(\frac{l}{2}\sin\theta\right)$$

Degrees of freedom depends upon the kinematic constrains imposed (restrictions to motion) on the system. A body allowed to translate in a plane has two degrees of freedom or its coordinates x, y shall defined its position with respect to the reference axis.

Now that we have understood that the potential energy and the work of a force are related (when conservative forces are involved) we can state the principle of virtual work in terms of potential energy for systems possessing single degree of freedom θ as follows :

If a system is in equilibrium the derivative of its total potential energy is zero.

$$\frac{d(\text{P.E.})}{d\theta} = 0.$$

13.5 STABILITY OF EQUILIBRIUM: STABLE, UNSTABLE AND NEUTRAL

When a system is disturbed from its position of equilibrium by the slightest force it produces a small displacement from equilibrium position. If the system returns to the original position as soon as the disturbing force is removed the equilibrium is said to be a stable one [Fig. 13.16 (a)].

On the removal of the disturbing force if the system moves away from the original position it is called to be in unstable equilibrium [Fig. 13.16 (b)].

On the removal of the disturbing force if the system neither returns nor moves away from its original position it is called to be in neutral equilibrium [Fig. 13.16 (c)].

Stable (a) Unstable (b) Neutral (c)

Fig. 13.16

The nature of equilibrium depends upon the potential energy of the system which can be determined with respect to any convenient reference. A system is in stable, unstable or neutral equilibrium according to whether its potential energy is minimum, maximum or constant.

Mathematically speaking a function is minimum when its second derivative is positive

$$\frac{d^2(\text{P.E.})}{d\theta^2} > 0, \text{ (+ve) Minimum, Stable}$$

Also,

$$\frac{d^2(\text{P.E.})}{d\theta^2} < 0, \text{ (−ve), Maximum, Unstable}$$

$$\frac{d^2(\text{P.E.})}{d\theta^2} = 0, \text{ Constant, Neutral}$$

To Summarize:

1. The positions of equilibrium of a system can be obtained by,

$$\frac{d(\text{P.E.})}{d\theta} = 0$$

2. The stability of equilibrium of a system depends upon whether

$$\frac{d^2(\text{P.E.})}{d\theta} \text{ is } \underset{\text{(Stable)}}{\text{+ve,}} \quad \underset{\text{(Unstable)}}{\text{−ve}}$$

3. A system may has several positions of equilibrium but all of them may not be stable.
4. If a system has several degrees of freedom, its potential energy depends upon several variables and the analysis become more involved. It may be remembered that a body with more than one degree of freedom may be stable for some motion and unstable for others.

Example 13.9 Two uniform rods each of length l and weight W are attached to two gears of equal radius (Fig. 13.17). Determine the positions of equilibrium of the system and state whether the equilibrium is stable, unstable or neural.

Solution: Choose O_1 as the origin of coordinates. The position of the system is defined by the variable θ.

Potential Energy,

$$\text{P.E.} = -W\frac{l}{2}\sin\theta - W\left(2r + \frac{l}{2}\cos\theta\right)$$

Equilibrium Position

$$\frac{d(\text{P.E.})}{d\theta} = -\frac{Wl}{2}(\cos\theta + 0 - \sin\theta) = 0$$

Or

$$\cos\theta - \sin\theta = 0$$
$$\tan\theta = 1$$

Solving, $\theta = 45°$ and $-135°$

Stability,

$$\frac{d^2(\text{P.E.})}{d\theta^2} = -\frac{Wl}{2}(-\sin\theta - \cos\theta)$$
$$= \frac{Wl}{2}(\sin\theta + \cos\theta)$$

For $\theta = 45°$

$$\frac{d^2(\text{P.E.})}{d\theta^2} = 0.707 + 0.707$$
$$= 1.414 > 0 \text{ Stable}$$

For $\theta = -135°$

$$\frac{d^2(\text{P.E.})}{d\theta^2} = -1.414 < 0 \text{ Unstable}$$

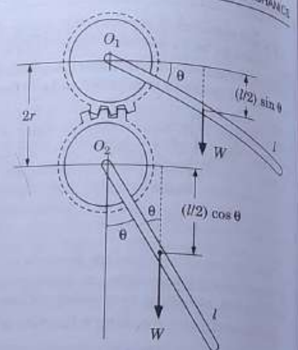

Fig. 13.17

Note that of the two equilibrium positions only one is stable.

Example 13.10 A hemi-spherical cup of radius r and having its centre of gravity at C rests on the top of a spherical surface of radius R (13.18). Establish the criterion of stability of the cup in the position shown assuming that there is sufficient friction to prevent slipping.

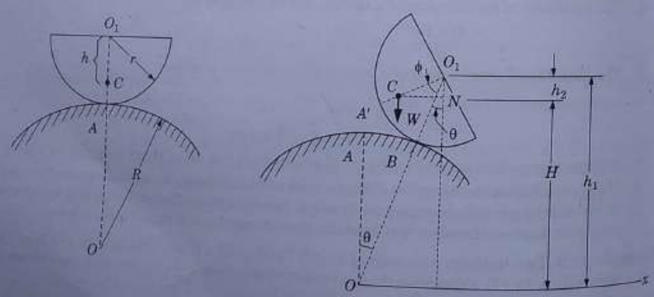

Fig. 13.18

Solution: Choose O as the origin of coordinates. Let us give a small angular displacement to the cup so that the point A moves to A' and B is the new point of contact between the cup and the spherical surface.

For rolling without slipping,

$$R\theta = r\phi$$
$$\phi = \frac{R\theta}{r} \qquad \text{(Angles are in radian)}$$

Potential Energy,

$$\text{P.E.} = WH$$

where, H is the elevation of C above O.

$$H = h_1 - h_2$$
$$H = (r + R)\cos\theta - h\cos(\theta + \phi)$$
$$H = (r + R)\cos\theta - h\cos\left(1 + \frac{R}{r}\right)\theta$$
$$\text{P.E.} = W\left[(r + R)\cos\theta - h\cos\left(1 + \frac{R}{r}\right)\theta\right]$$

Equilibrium Position,

$$\frac{d(\text{P.E.})}{d\theta} = W\left[-(r+R)\sin\theta + h\left(1 + \frac{R}{r}\right)\sin\left(1 + \frac{R}{r}\right)\theta\right] = 0$$

Above equation is satisfied for $\theta = 0$

Stability :

$$\frac{d^2(\text{P.E.})}{d\theta^2} = W\left[-(r+R)\cos\theta + h\left(1 + \frac{R}{r}\right)^2\cos\left(1 + \frac{R}{r}\right)6\right]$$

For $\theta = 0$,

$$\frac{d^2(\text{P.E.})}{d\theta^2} = W\left[-(r+R) + h\left(1 + \frac{R}{r}\right)^2\right]$$

For stability,

$$W\left[-(r+R) + h\left(1 + \frac{R}{r}\right)^2\right] > 0$$
$$h\left(1 + \frac{R}{r}\right)^2 > r + R$$
$$h\left(1 + \frac{R}{r}\right)^2 > r\left(1 + \frac{R}{r}\right)$$
$$h\left(1 + \frac{R}{r}\right) > r$$
$$\frac{R}{r} > \frac{r}{h} - 1$$
$$R > r\left(\frac{r}{h} - 1\right) \quad \textit{Ans.}$$

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Example 13.11 A heavy uniform cube of side a is balanced on a cylindrical surface of radius r. Assuming sufficient friction to prevent slipping, find the relation between a and r consistent with the stability (Fig. 13.19).

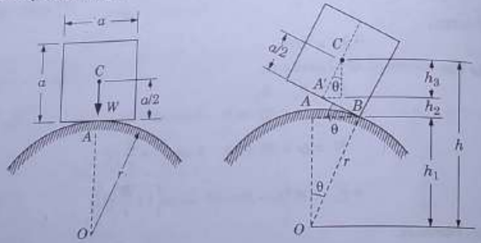

Fig. 13.19

Solution: Choose O as the origin of coordinates. Let us give a small angular displacement to the cube so that point A moves to A' and B be the new point of contact between cube and the supporting cylinderical surface.

For no slip conditions,

$$A'B = \text{arc } AB = r\theta$$

Height of the C.G. (C) in the displaced position above O,

$$h = h_1 + h_2 + h_3$$

$$h = r\cos\theta + r\theta\sin\theta + \frac{a}{2}\cos\theta$$

$$h = \left(r + \frac{a}{2}\right)\cos\theta + r\theta\sin\theta$$

Potential Energy,

$$\text{P.E.} = W\left[\left(r+\frac{a}{2}\right)\cos\theta + r\theta\sin\theta\right]$$

Equilibrium Position,

$$\frac{d(\text{P.E.})}{d\theta} = W\left[-\left(r+\frac{a}{2}\right)\sin\theta + r(\sin\theta + \theta\cos\theta)\right] = 0$$

$$-\left(r+\frac{a}{2}\right)\sin\theta + r\sin\theta + r\theta\cos\theta = 0$$

$$-r\sin\theta - \frac{a}{2}\sin\theta + r\sin\theta + r\theta\cos\theta = 0$$

$$r\theta\cos\theta - \frac{a}{2}\sin\theta = 0$$

Above equation is satisfied for $\theta = 0$

For stability,

$$\frac{d^2(\text{P.E.})}{d\theta^2} = r(-\theta\sin\theta + \cos\theta) - \frac{a}{2}\cos\theta$$

$$= -r\theta\sin\theta + r\cos\theta - \frac{a}{2}\cos\theta$$

Substituting $\theta = 0$

$$\frac{d^2(\text{P.E.})}{d\theta^2} = r - a/2 > 0 \text{ or } r > a/2$$

Example 13.12 A uniform body is made of a hemisphere of radius R and a right circular cone of height H which have a common base. Determine the largest value of the height of the cone consistent with the stability of body in the vertical position (Fig. 13.20).

Solution: Potential Energy,

Let us give an angular displacement θ to the body.

$$\text{P.E.} = W[R + (h - R)\cos\theta]$$

where, W_1 and W_2 and the weights of the hemisphere and cone respectively and C_1 and C_2 are their centres of gravity. C is the C.G. of the combined body at a height h from the base.

$$\frac{d(\text{P.E.})}{d\theta} = W[(-h - R)\sin\theta] = 0$$

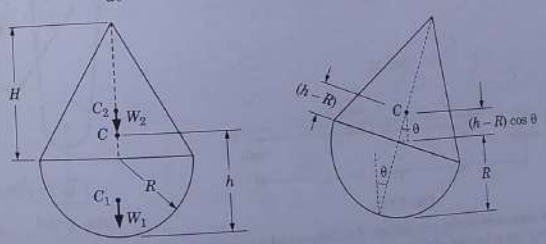

Fig. 13.20

Above equation is satisfied for $\theta = 0$ or in vertical position of the body.

Stability,

$$\frac{d^2(\text{P.E.})}{d\theta^2} = -W[(h - R)\cos\theta]$$

For stability when $\theta = 0$

$$\frac{d^2(\text{P.E.})}{d\theta^2} = -(h - R) > 0$$

Or $$R > h$$

h being the height of the combined C.G. of the body,

$$R > \frac{\frac{2\pi}{3}R^3\left(\frac{5R}{8}\right)+\left(\frac{\pi}{3}R^2H\right)\left(R+\frac{H}{4}\right)}{\frac{2\pi}{3}R^3+\frac{\pi}{3}R^2H}$$

$$R\left(\frac{2\pi}{3}R^3+\frac{\pi}{3}R^2H\right) > \frac{2\pi}{3}R^3\left(\frac{5R}{8}\right)+\left(\frac{\pi}{3}R^2H\right)\left(R+\frac{H}{4}\right)$$

$$3R^2 > H^2$$

$$\sqrt{3}R > H$$

Or maximum height $= \sqrt{3}R$.

PROBLEMS

13.1. A beam is supported and loaded as shown in Fig. P.13.1. Find the components of the reactions at A and B. [X_A = 4000 N, Y_A = 1225 N, Y_B = 1225 N]

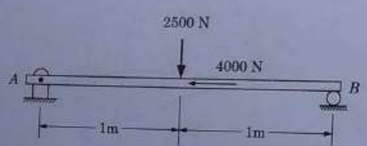

Fig. P.13.1

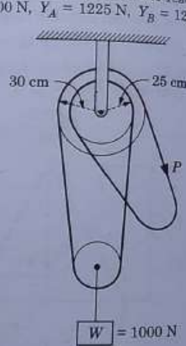

Fig. P.13.2

13.2. The diameters of the two pulleys of a differential pulley block are 30 cm and 25 cm respectively. Using the principle of virtual work calculate the value of the effort P required to lift a load W of 1000 N. [166.67 N]

13.3. A frame is made of hinged bars arranged to form three identical rhombuses as shown in Fig. P.13.3. Find the relation between the forces P and Q for the bars to be in equilibrium. [$3P = Q$]

Fig. P.13.3

13.4. A uniform rod of length $2l$ rests in equilibrium against a smooth vertical wall and upon a smooth peg P at a distance 'a' from the wall (Fig. P.13.4). Show that, in the position of equilibrium, the rod is inclined to the wall an angle $\theta = \sin^{-1}(a/l)^{1/3}$.

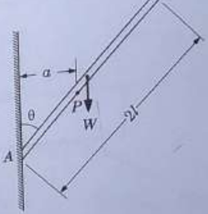

Fig. P.13.4

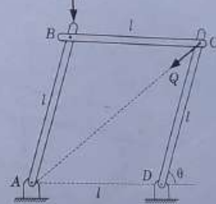

Fig. P.13.5

13.5. Three links each of length l are connected as shown in Fig. P.13.5. Find the relation between the forces P and Q to maintain equilibrium. The line of action of the force Q passes through A. $\left[Q = \frac{P\cos Q}{\sin\theta/2}\right]$

13.6. Two beams AB and CD are arranged and supported as shown in Fig. P.13.6. Find the reaction R_E if a load W of 1000 N acts as shown. [937.5 N]

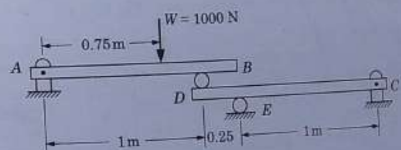

Fig. P.13.6

13.7. Find the expression for the angle θ and tension T in the spring corresponding to position of equilibrium of the mechanism shown in Fig. P.13.7. The unstretched length of the spring is h and the stiffness of the spring is k. $\left[\theta = \sin^{-1}\left(\frac{P+2kh}{4kl}\right), T = \frac{P}{2}\right]$

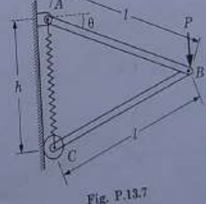

Fig. P.13.7

13.8. A bar AB of weight W rests on two mutually perpendicular planes. Using the principle of virtual work find the angle θ defining the position of equilibrium. The centre of gravity G of the bar is at distances a and b from ends A and B.

$$\left[\textbf{Hint}: \text{Height of } G = \text{Height of } A + (\text{Height of } B - \text{Height of } A)\frac{a}{a+b}\right]$$

$$\left[\tan\theta = \frac{a}{b}\tan\alpha\right]$$

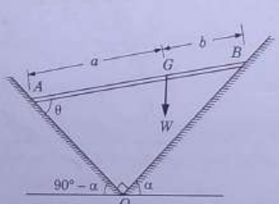

Fig. P.13.8

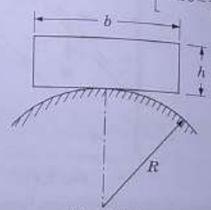

Fig. P.13.9

13.9. A uniform rectangular plank of height h and base width b rests on a cylinderical surface of radius R. Assuming sufficient friction to prevent slipping, find the relation between R and h consistent with the stability (Fig. P.13.9)

$[R > h/2]$

13.10. A uniform body is made of a hemisphere of radius R and a right circular cylinder of height H which have a common base. Determine the largest value of the height of the cylinder consistent with the stability of the body in vertical position (Fig. P.13.10)

$$\left[H = \frac{R}{\sqrt{2}}\right]$$

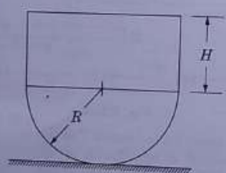

Fig. P.13.10

14 CHAPTER Rectilinear Motion of a Particle

PART A: KINEMATICS

14.1 INTRODUCTION TO DYNAMICS

In statics we considered the bodies at rest. Now we shall begin with the study of dynamics. *Dynamics is the part of mechanics that deals with the analysis of bodies in motion.* While the study of statics is very old science, dynamics is a comparatively new one. The first significant contribution to dynamics was made by Galileo (1564-1642). Newton later (1642-1727), formulated his fundamental laws of motion.

For convenience, dynamics is divided into two branches called *kinematics* and *kinetics*.

Kinematics is the study of the relationships between displacement, velocity, acceleration and time of a given motion, without considering the forces that cause the motion.

Kinetics is the study of the relationships between the forces acting on a body, the mass of the body and the motion of the body. Kinetics therefore, can be used to predict the motion caused by a given forces or to determine the forces required to produce a prescribed motion.

The science of dynamics is based on the natural laws governing the motion of a particle. *The term particle is a convenient idealization of the physical objects which need not be small in size. In this idealization, the mass of the body is assumed to be concentrated at a point and the motion of the body is considered as the motion of an entire unit neglecting any rotation about its own mass centre.* In case where such rotation is not negligible, then the body cannot be considered as a particle.

Types of Motion : When a particle moves in space it describes a curve, called path. This path can be straight or curved.

(i) **Rectilinear Motion.** When a particle moves along a path which is a straight line, it is called rectilinear motion.

(ii) **Curvilinear Motion.** When a particle moves along a curved path it is called curvilinear motion. *If the curved path lies in a plane it is called plane curvilinear motion.*

In this chapter, we shall discuss the rectilinear motion of a particle. The kinematics and kinetics of which are separated into parts A and B.

14.2 RECTILINEAR MOTION : DISPLACEMENT, VELOCITY AND ACCELERATION

Let us consider the motion of a particle along a straight line and explain the above terms.

Displacement. A particle in rectilinear motion, at any instant of time will occupy a certain position on the straight line. To define this position P of the particle, we have to choose some convenient reference point O, called the origin. The distance x of the particle at any time t, is called the displacement of the particle at that time. The displacement is assumed to be positive to the right of the origin and negative to the left (Fig. 14.1).

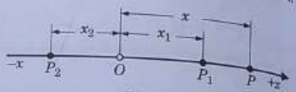

Fig. 14.1

Distance Travelled. The distance travelled by a particle, however, is different that its displacement from the origin. For example, if a particle moves from O to positions P_1 and then to position P_2, its displacement at the position P_2 is $-x_2$ from the origin but, the distance travelled by the particle is $2x_1 + x_2$ (Fig. 14.1).

Average Velocity. Let the position P occupied by a particle at any time t be at a distance x from the origin O.

Let its position P' at time $(t + \Delta t)$ be at a distance $(x + \Delta x)$ from O.

Average velocity of the particle over the time in travel Δt,

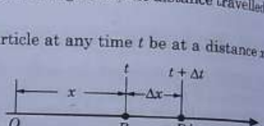

Fig. 14.2

$$v_{\text{average}} = \frac{\Delta x}{\Delta t} \quad \text{...(14.1)}$$

Instantaneous Velocity. It is the velocity at a particular instant of time. It can be obtained from the average velocity by choosing the time interval Δt and the displacement Δx very small.

Instantaneous Velocity, (or velocity)

$$v = \underset{\Delta t \to 0}{\text{limit}} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad \text{...(14.2)}$$

The velocity v is positive if the displacement x is increasing and the particle is moving in the positive direction. The unit of velocity is metre per second (m/s).

Average Acceleration. Let v be the velocity of the particle at any time t. If the velocity becomes $(v + \Delta v)$ at a later time $(t + \Delta t)$ then,

$$\text{Average acceleration} = \frac{\Delta y}{\Delta t} \quad \text{...(14.3)}$$

Instantaneous Acceleration. It is the acceleration of a particle at a particular instant of time and can be calculated by choosing the time interval Δt and the velocity Δv very very small.

$$\text{Acceleration } a = \underset{\Delta t \to 0}{\text{limit}} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \quad \text{...(14.4)}$$

Acceleration is positive if the velocity is increasing. A positive value of acceleration means that the particle is either moving further in the positive direction or is slowing down in the negative direction. The unit of acceleration is metre per second (m/s^2).

$$a = \frac{dv}{dt} \quad \text{as, } v = \frac{dx}{dt}$$

so,

$$a = \frac{d^2x}{dt^2} \quad \text{...(14.5)}$$

Also,

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} \quad \text{and as, } \frac{dx}{dt} = v$$

so,

$$a = v\frac{dv}{dx} \quad \text{...(14.6)}$$

Uniform Motion. A particle is said to have a uniform motion when its acceleration is zero and its velocity is constant.

Uniformly Accelerated Motion. A particle moving with a constant acceleration is said to be in uniformly accelerated motion.

It may be mentioned that velocity and acceleration of a particle are *vector quantities* possessing both the magnitude and direction. But as we are presently dealing with the motion along a straight line we can, therefore, specify these quantities by algebraic number with plus or minus signs.

14.3 GRAPHICAL REPRESENTATIONS

If the position (displacement), velocity and acceleration of a particle are plotted with respect to time, they are known as,

Displacement-Time Diagram (x-t curve)

Velocity-Time Diagram (v-t curve)

Acceleration-Time Diagram (a-t curve)

Displacement-Time Diagram. It is obtained if the position (displacement) of the particle is plotted as function of time (Fig. 14.3).

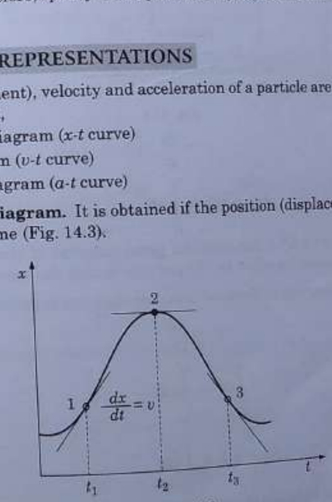

Fig. 14.3

As $v = \frac{dx}{dt}$, we can deduce that the slope of the x-t curve at any instant gives the velocity of the particle at that instant. In Fig. 14.3,

the particle at time t_1 has positive velocity.
at time t_2 has zero velocity
at time t_3 has a negative velocity.

Velocity-Time Diagram is obtained if the velocity of the particle is plotted as a function of time as shown in Fig. 14.4.

As, $a = \frac{dv}{dt}$, the slope of the v-t curve at any instant gives the acceleration of the particle (Fig. 14.4).

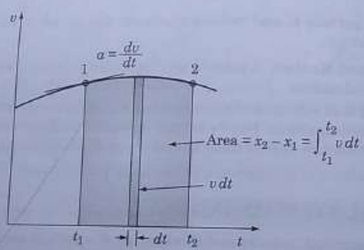

Fig. 14.4

Also as, $$v = \frac{dx}{dt}, \int dx = \int v\,dt$$

Or $$(x_2 - x_1) = \int_{t_1}^{t_2} v\,dt \qquad \text{...(14.7)}$$

The area under the v-t curve, between the given interval of time, gives the change in displacement or the distance travelled by the particle during the same interval.

Accelerations-Time Diagram is obtained if the acceleration is plotted as a function of time.

As, $$a = \frac{dv}{dt}$$

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt$$

$$(v_2 - v_1) = \int_{t_1}^{t_2} a\,dt \qquad \text{...(14.8)}$$

Area under the a-t curve, between the given interval of time, gives the change in velocity of the particle during the same interval.

It may be mentioned here that if the displacement-time curve is a polynomial of degree n, the velocity-time curve will be a polynomial of degree $(n - 1)$ and acceleration-time curve will be a polynomial of degree $(n - 2)$.

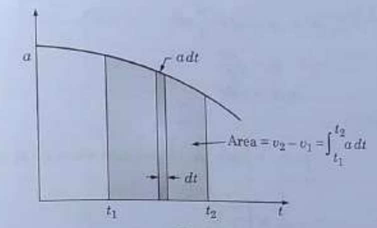

Fig. 14.5

For example if,

x-t curve is second degree curve
v-t curve will be linear curve
and a-t curve will be constant

For illustration see solved Example 14.12.

14.4 MOTION WITH UNIFORM ACCELERATION

Let us derive some useful expressions for the case of motion with uniform acceleration that is,

$$a = \text{constant}$$

As, $$a = \frac{dv}{dt}$$

$$dv = a\,dt$$

Integrating both sides

$$\int_u^v dv = \int_0^t a\,dt$$

$$v - u = at$$

$$v = u + at \qquad \text{...(14.9)}$$

where, u is the initial velocity when $t = 0$ and v is the velocity at any time t.

As, $$v = \frac{dx}{dt}$$

$$dx = v\,dt$$

Substituting, $$v = u + at$$

$$dx = (u + at)\,dt$$

Integrating both sides,

$$\int_0^x dx = \int_0^t (u + at)dt, \text{ assuming that at } t = 0, x = 0$$

$$x = ut + \frac{1}{2}at^2 \qquad ...(14.10)$$

As,
$$a = v\frac{dv}{dx}$$
$$a\,dx = v\,dv$$

Integrating both sides,

$$\int_0^x a\,dx = \int_u^v v\,dv \quad \text{Assuming that when } x = 0, \text{ velocity} = u$$

$$ax = \left[\frac{v^2}{2}\right]_u^v$$
$$2ax = v^2 - u^2$$

Or
$$v^2 - u^2 = 2ax \qquad ...(14.11)$$

Above relations can be used if the *acceleration* of the particle is *constant*.

14.5 MOTION WITH VARIABLE ACCELERATION

We considered the motion of a particle under constant acceleration. The motion under gravity is a special case of this type of motion. In actual practice the acceleration of a particle may not be constant but may vary from instant to instant. The equations derived for uniformly accelerated motion are not applicable to the motion with variable acceleration. The four quantities, displacement, velocity, acceleration and time are related as,

$$v = \frac{dx}{dt}$$
$$a = \frac{dv}{dt}$$
$$a = \frac{d^2x}{dt^2} \qquad ...(14.12)$$
$$a = \frac{v\,dv}{dx}$$

When one or more of the above quantities are specified, the others can be obtained by the process of differentiation or integration, using the above relationships.

In problems requiring integration, the constants of integration can be determined from the specified conditions of motion.

One such condition can be that the displacement of the particle is zero at time $t = 0$. The method for solving the problems of variable acceleration is further explained with the help of solved examples in this chapter.

Displacement, Velocity and Acceleration of Connected Bodies

Example 14.1. A system of two pulleys is shown in Fig. 14.6. Find the relations connecting the displacements, velocities and accelerations of the blocks A and B.

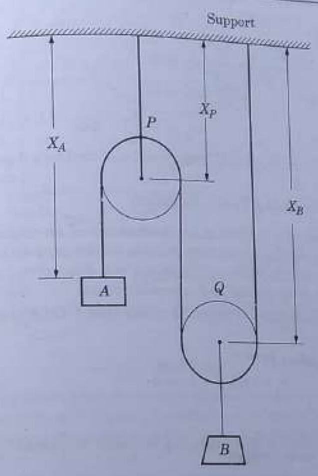

Fig. 14.6

Solution: Let us measure the distances with respect to the support.

As the length of the rope l, going around the pulleys, and the distance of the pulley P from the support is constant, therefore,

$$(x_A - x_P) + (x_B - x_P) + x_B = l$$
$$x_A + 2x_B - 2x_P = l$$
$$x_A + 2x_B = l - 2x_P \qquad ...(i)$$

Giving an increment Δx_A to the block A (that is, lower the block A by Δx_A)

$$\Delta x_A + 2\Delta x_B = 0$$
$$\Delta x_B = -\frac{\Delta x_A}{2} \qquad ...(ii)$$

The displacement of the block B is half the displacement of the block A and is in opposite direction.

Differentiating equation (*i*) w.r.t. time

$$\frac{dx_A}{dt} + \frac{2dx_B}{dt} = 0$$
$$v_A + 2v_B = 0 \qquad ...(iii)$$
$$v_B = -\frac{v_A}{2} \quad Ans.$$

Differentiating again,

$$\frac{d^2x_A}{dt^2}+\frac{2d^2x_B}{dt^2}=0$$

$$(a_A + 2a_B = 0)$$

$$a_B = -\frac{a_A}{2} \quad Ans. \qquad ...(iii)$$

The acceleration of the block B is half the acceleration of the block A and is in opposite direction.

To Determine Displacement-Time Relation

Example 14.2. The crank of a reciprocating mechanism of an engine rotates at an angular velocity of ω rad/s. Find the displacement time relation for the motion of the piston P if the length of the connecting rod is l and the radius of the crank is r. Assuming $l >> r$, find the maximum velocity and acceleration of the piston and where it occurs.

Solution: Consider the position OA of the crank at any time t. Let θ be angle turned with respect to Ox such that $\theta = \omega t$.

The coordinate of the piston P

$$x = r\cos\theta + l\cos\phi. \qquad ...(i)$$

From triangle OAP,

$$\frac{r}{\sin\phi} = \frac{l}{\sin\theta} \text{ or } \sin\phi = \frac{r}{l}\sin\theta = \frac{r}{l}\sin\omega t$$

and

$$\cos\phi = \sqrt{1-\sin^2\phi}$$

$$x = r\cos\omega t + l\sqrt{1-\frac{r^2}{l^2}\sin^2\omega t} \qquad ...(ii)$$

If $l >> r$ equation (*ii*) can be written as,

$x = r\cos\omega t + l$

$v = \dot{x} = -r\omega\sin\omega t$, v is max when,

$\sin\omega t = 1$,

$\theta = 90°, 270°$

$a = \ddot{x} = -r\omega^2\cos\omega t$, acceleration is max when,

$\cos\omega t = 1$

$\theta = 0°, 180°$

A graphic plot of the displacement, velocity and acceleration is shown in Fig. 14.8.

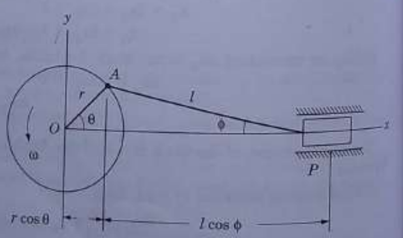

Fig. 14.7

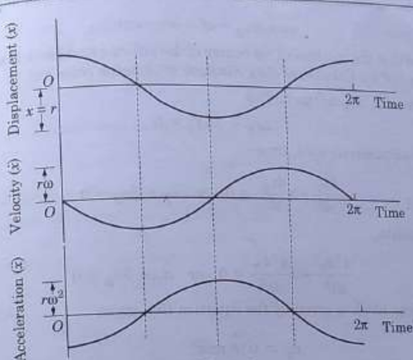

Fig. 14.8

Example 14.3. A trolley resting on a horizontal plane starts from rest and is moved to the right with a constant acceleration of 0.18 m/s^2. Determine (*i*) acceleration of the block B connected to the trolley (*ii*) velocities of the trolley and the block after a time of 4 seconds and the distance moved by each of them.

Solution: The length of the string $abcde$ connecting the trolley T and the block B is constant. This is to be expressed in terms of the distance of the connected bodies (*e.g.*, trolley and the block) from some fixed points (*e.g.*, b, d).

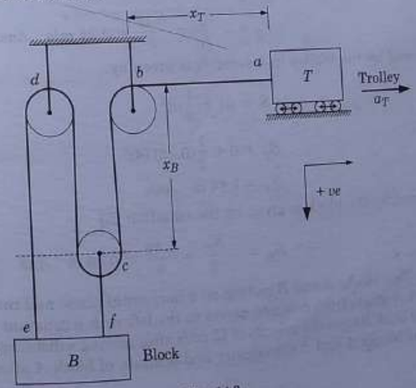

Fig. 14.9

$$x_T + 3x_B + cf = \text{constant} \quad ...(i)$$

It may be noted that the motions of the centre of the pulley c and block B are identical, hence expressed in terms of x_B. Otherwise also, distance cf remains constant.

Giving an increment Δx_T to the trolley

$$\Delta x_T + 3\Delta X_B = 0 \quad ...(ii)$$

Differentiating equation (*i*) w.r.t. time

$$\frac{dx_T}{dt} + 3\frac{dx_B}{dt} = 0 \quad \text{or} \quad v_T + 3v_B = 0 \quad ...(iii)$$

Differentiating again,

$$\frac{d^2x_T}{dt^2} + 3\frac{d^2x_B}{dt^2} = 0 \quad \text{or} \quad a_T + 3a_B = 0 \quad ...(iv)$$

Acceleration of the block is given by the equation (*iv*), given

$$a_T = 0.1\vec{8} \text{ m/s}^2$$

$$a_B = -\frac{a_T}{3} = \frac{0.18}{3} = 0.06 \overset{\uparrow}{\text{m}}/\text{s}^2 \quad \textit{Ans.}$$

(–ve sign indicates direction)

Velocity of the trolley 4 seconds after starting from rest is given by,

$$v = u + at$$

$$v_T = 0 + 0.18 \times 4 = 0.72 \text{ m/s} \quad \textit{Ans.}$$

Velocity the block is given by the equation (*iii*)

$$v_B = -\frac{v_T}{3} = \frac{0.72}{3} = 0.24 \overset{\uparrow}{\text{m}}/\text{s} \quad \textit{Ans.}$$

Distance moved by the trolley in 4 seconds is given by,

$$S = ut + \frac{1}{2}at^2$$

$$S_T = 0 + \frac{1}{2}(0.18)(4)^2$$

$$S_T = 1.44 \text{ m} \quad \textit{Ans.}$$

Distance moved by the block is given by the equation (*ii*)

$$S_B = -\frac{S_T}{3} = \frac{1.44}{3} = 0.48 \text{ m} \quad \textit{Ans.}$$

Example 14.4. Two blocks A and B resting on a horizontal plane and connected by a cord as shown. The block A starts from rest and moves to the left with a constant acceleration. It was observed that the block B attains a velocity of 12 cm/s after moving a distance of 24 cm. Determine (*i*) acceleration of blocks A and B (*ii*) velocity and position of block A after 5 seconds.

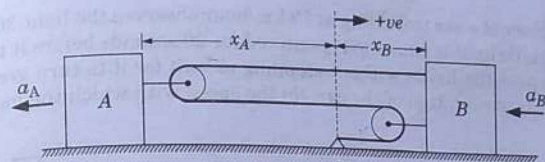

Fig. 14.10

Solution: The length of the cored connecting the blocks A and B can be expressed in terms of distances x_A and x_B of the blocks A and B from a fixed point O as,

$$3x_B + 2x_A = \text{constant} \quad ...(i)$$

Giving an increment Δx_A to the block A

$$3\Delta x_B + 2\Delta x_A = 0 \quad ...(ii)$$

Differentiating (*i*)

$$3\frac{dx_B}{dt} + 2\frac{dx_A}{dt} = 0$$

$$3v_B + 2v_A = 0 \quad ...(iii)$$

Differentiating again

$$3\frac{d^2x_B}{dt^2} + 2\frac{d^2x_A}{dt^2} = 0$$

$$3a_B + 2a_A = 0 \quad ...(iv)$$

For block B, using the equation

$$v^2 - u^2 = 2as$$

$$u = 0,\ v = 12 \text{ cm/s},\ s = 24 \text{ cm}$$

$$a_B = \frac{(12)^2}{2 \times 24} = 3 \text{ cm/s}^2 \quad \textit{Ans.}$$

Using equation (*iv*) for block A

$$a_A = -\frac{3}{2}a_B = \frac{3}{2} \times 3 = 4.5 \overset{\leftarrow}{\text{cm}}/\text{s}^2 \quad \textit{Ans.}$$

Velocity of the block A after 5 seconds

$$v = u + at$$

$$u = 0,\ a = 4.5 \text{ cm/s}^2$$

$$v_A = 0 + 4.5 \times 5 = 22.5 \text{ m/s} \quad \textit{Ans.}$$

Position of the block A after 5 seconds

$$S = ut + \frac{1}{2}at^2$$

$$u = 0,\ a_A = 4.5 \text{ m/s}$$

$$S_A = 0 + \frac{1}{2} \times 4.5\,(5)^2 = 56.25 \text{ m}$$

Example 14.5. Driver of a car travelling at 72 km/hour observes the light 300 m ahead of him turning red. The traffic light is timed to remain red for 20 seconds before it turns green. If the motorist wishes to pass the lights without stopping to wait for it to turn green, determine (*a*) the required uniform acceleration of the car, (*b*) the speed with which the motorist crosses the traffic light.

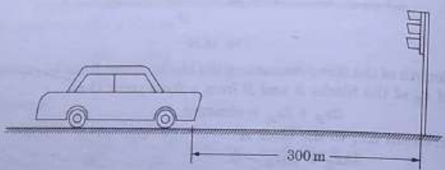

Fig. 14.11

Solution: $72 \text{ km/hour} = \frac{72\times1000}{60\times60} = 20 \text{ m/s}$

$u = 20$ m/s,

$t = 20$ s,

$s = 300$ m

Using, $s = ut + \frac{1}{2}at^2$

$300 = 20 \times 20 + \frac{1}{2}a\,(20)^2$

$a = -0.5 \text{ m/s}^2$ (Deceleration) *Ans.*

Using $v = u + at$

$v = 20 - 0.5 \times 20$

$v = 10$ m/s

Or speed $= 36$ km/hour *Ans.*

Example 14.6. A stone is dropped from the top of a tower 50 m high. At the same time another stone is thrown up from the foot of the tower with a velocity of 25 m/s. At what distance from the top and after how much time the two stones cross each other?

Solution: The condition for the two stones to cross each other is that the sum of the distances S_1 and S_2 travelled by the two stones at the time of crossing should be,

$S_1 + S_2 = 50$ m

Using $s = ut = \frac{1}{2}gt^2$ $S = ut + \frac{1}{2}at^2$

First stone, ~~od~~ for S_1

$u = 0$, $a = g$

$S_1 = \frac{1}{2}gt^2$ $S = ut$

Second stone,

$S_2 = 25t - \frac{1}{2}gt^2$

$S_1 + S_2 = 50$

$= \frac{1}{2}gt^2 + \left(25t - \frac{1}{2}gt^2\right)$

$25t = 50$

$t = 2$ second *Ans.*

$S_1 = \frac{1}{2}gt^2 = \frac{1}{2} \times 9.81 \times 4.$

$S_1 = 19.6$ m from the top *Ans.*

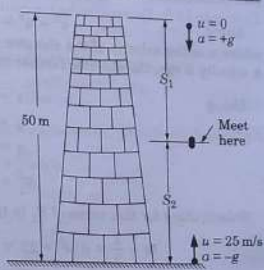

Fig. 14.12

Example 14.7. An open platform elevator is moving down a mine shaft with an acceleration of 0.5 m/s². After the elevator has travelled 25 m (from the top of the shaft) a stone is dropped from the top of the shaft. Determine

(*a*) the time taken the stone to hit the elevator,

(*b*) the distance travelled by the elevator at the time of impact.

Solution: When the stone hits the elevator, both the elevator and the stone must be at the same distance from the top of the shaft.

Stone : Let the time taken by the stone to hit the elevator be $= t$ distance S, travelled by the stone in the time t is obtained as

$u = 0$

$a = 9.81 \text{ m/s}^2$

Using $s = ut + \frac{1}{2}at^2$

$$S = \frac{1}{2} \times 9.81 \times t^2 \quad \text{...}(i)$$

Elevator : Let the distance travelled by the elevator in the time t (same as taken by the stone) be $= S_1$.

But the elevator was 25 m from the top of the shaft when the stone was released.

So the distance of the elevator from the of top of the shaft after the time $t = S_1 + 25$

For the stone to hit the elevator

$$S = S_1 + 25 = \frac{1}{2} \times 9.81\,t^2 \quad \text{...}(ii)$$

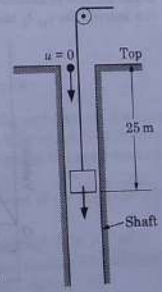

Fig. 14.13

The elevator started from rest, under an acceleration of 0.5 m/s² and covered a distance of 25 m. It further travelled a distance S_1 in the time t with the same acceleration.

$$u = 0,\ s = 25 \text{ m},\ a = 0.5 \text{ m/s}^2$$

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Using $v^2 - u^2 = 2as$, $v^2 = \sqrt{2\times0.5\times25}$, $v = 5$ m/s

where v is the velocity of the elevator when it was at a distance of 25 m from the top. With a velocity 5 m/s, the elevator further travels a distance S_1.

Using $$s = ut + \frac{1}{2}at^2$$

$$s = S_1,\ u = \text{velocity when at 25 m} = 5 \text{ m/s}$$
$$a = 0.5 \text{ m/s}^2$$

$$S_1 = 5t + \frac{1}{2} \times 0.5t^2$$

Substituting for this value of S_1 in (*ii*)

$$5t + \frac{1}{2} \times 0.5t^2 + 25 = \frac{1}{2} \times 9.81 \times t^2$$
$$4.655\,t^2 - 5t - 25 = 0$$

Solving for t, $$t = 2.917 \text{ s} \quad \textit{Ans.}$$

Example 14.8. A train starts from rest and increases its speed from zero to v m/s with a constant acceleration of a_1 m/s^2, runs at this speed for some time and finally comes to rest with a constant deceleration a_2 m/s^2.

If the total distance travelled is x metres, find the total time t required for this journey.

Solution: The velocity time graph of the journey is as shown in the Fig. 14.14. The portion OA represents acceleration from zero to velocity v. AB is the portion of constant velocity v and BC is the portion of deceleration from velocity v to zero. The time taken for the three portions of the journey be t_1, t_2 and t_3 respectively.

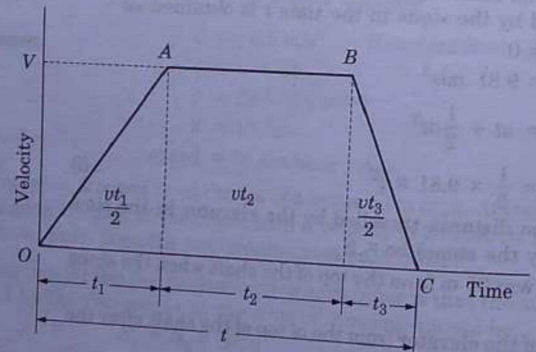

Fig. 14.14

Portion OA : Using,

$$v = u + at$$
$$v = 0 + a_1t_1, \quad \text{initial velocity is zero}$$
$$t_1 = \frac{v}{a_1} \quad \begin{array}{l}\text{acceleration} = a_1\\ \text{time} = t_1\end{array} \quad ...(i)$$

Portion BC : Using,

$$v = u + at$$
$$0 = v - a_2t_3, \quad \text{final velocity is zero}$$
$$t_3 = \frac{v}{a_2} \quad \begin{array}{l}\text{acceleration} = -a_2\\ \text{time} = t_3\end{array} \quad ...(ii)$$

The total distance travelled by the train (x), is given by the area under the velocity-time graph.

$$x = \frac{1}{2}(vt_1) + vt_2 + \frac{1}{2}(vt_3)$$
$$\frac{x}{v} = \frac{t_1}{2} + t_2 + \frac{t_3}{2}$$

Or $$t_2 = \frac{x}{v} - \frac{t_1}{2} - \frac{t_3}{2} \quad ...(iii)$$

The total time of travel

$$t = t_1 + t_2 + t_3$$

Substituting for t_1, t_2 and t_3 using (*i*), (*ii*) and (*iii*)

$$t = \frac{v}{a_1} + \left(\frac{x}{v} - \frac{t_1}{2} - \frac{t_3}{2}\right) + \frac{v}{a_2}$$

Again substituting for t_1 and t_3,

$$t = \frac{v}{a_1} + \frac{x}{v} - \frac{v}{2a_1} - \frac{v}{2a_2} + \frac{v}{a_2}$$
$$t = \frac{x}{v} + \frac{v}{a_1} - \frac{v}{2a_1} + \frac{v}{2a_2}$$
$$t = \frac{x}{v} + \frac{v}{2}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) \quad \textit{Ans.}$$

Example 14.9. A passenger train passes a certain station at 60 km/hr and covers a distance of 12 kilometres with this speed and then stops at the next station 15 kilometres from the first with uniform retardation. A local train starting from the first station covers the same distance in double this time and stops at the next station.

Determine the maximum speed of the local train which covers a part of the distance with uniform acceleration and the rest with uniform retardation.

Solution: The motions of the two trains are shown on velocity-time graph (Fig. 14.15).

Passenger Train : It moves out from a certain station with a constant speed of 60 km/hr (AB) and after travelling a distance of 12 km retards and stops at the next station (BC).

Area under the velocity-time graph represents the distance travelled by the train.

$$= \text{Area } OABC$$
$$\text{Area } OABC = \text{Area } OABB' + \text{Area } BB'C$$
$$\text{Area } OABB' = 12 \text{ km} = v \times t_1 = 60 \times t_1$$
$$t_1 = \frac{12}{60} = \frac{1}{5}\text{hr}$$

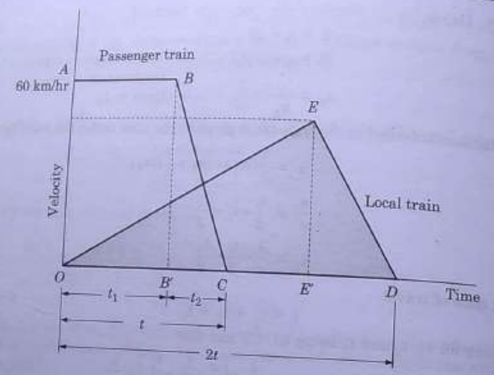

Fig. 14.15

$$\text{Area } BB'C = \frac{vt_2}{2} = 3 \text{ km}$$

$$\frac{60 \times t_2}{2} = 3$$

$$t_2 = \frac{3 \times 2}{60} = \frac{1}{10} \text{ hr}$$

Total time of travel of the passenger train,

$$t = t_1 + t_2 = \frac{1}{5} + \frac{1}{10} = \frac{3}{10} \text{hr}$$

Local Train :

Time of travel of the local train = 2 × time of passenger train

$$= 2 \times \frac{3}{10} = \frac{3}{5} \text{hr}$$

Distance travelled by local train = 15 km

Area

$$OED = \frac{V_{max}}{2} \times 2t$$

$$15 = V_{max} \times t$$

$$15 = V_{max} \times \left(\frac{3}{10}\right)$$

$$V_{max} = \frac{15 \times 10}{3}, \; V_{max} = 50 \text{ km/hr} \quad \textit{Ans.}$$

Distance travelled by a Particle in the *n*th Second

Example 14.10. A particle moving with an acceleration of 10 m/s^2 travels a distance of 50 m during the 5th second of its travel. Find its initial velocity.

Solution: The distance travelled by a particle in time t is given by

$$S = ut + \frac{1}{2}at^2$$

The distance travelled by the particle in the 5th second = The distance travelled in 5 seconds − The distance travelled in 4 seconds

$$S_{0-5} = 5u + \frac{1}{2}a(5)^2 = 5u + \frac{1}{2} \times 10 \times 25 = 5u + 125$$

$$S_{0-4} = 4u + \frac{1}{2}a(4)^2 = 4u + \frac{1}{2} \times 10 \times 16 = 4u + 80$$

$$S_{5\text{th second}} = S_{05} - S_{04} = [(5u + 125) - (4u + 80)]$$

$$50 = u + 45$$

$$u = 5 \text{ m/s} \quad \textit{Ans.}$$

Variable Acceleration

Example 14.11. Motion of particle along a straight line is given by the equation

$$a = t^2 - 2t + 2$$

where a = acceleration in m/s^2

t = time in seconds

After 1 second the distance travelled by the particle and the velocity of the article were found to be 14.75 m and 6.33 m/s. Find the (*i*) distance travelled, (*ii*) velocity and (*iii*) acceleration of the particle after 2 seconds.

Solution: Conditions given are

at, $t = 1$ s, $x = 14.75$ m, and $v = 6.33$ m/s

$$a = t^2 - 2t + 2 \quad ...(i)$$

As

$$a = \frac{dv}{dt} = t^2 - 2t + 2$$

$$v = \int dv = \int a\,dt = \int (t^2 - 2t + 2)dt$$

$$t = \frac{t^3}{3} - \frac{2t^2}{2} + 2t + c_1$$

Let us evaluate the constant of integration c_1 by substituting the given condition that is, when

$$t = 1, \; v = 6.33 \text{ m/s}$$

$$6.33 = \frac{(1)^3}{3} - \frac{2(1)^2}{2} + 2(1) + c_1$$

$$6.33 = \frac{1}{3} - 1 + 2c_1$$

$$c_1 = 5.0$$

$$v = \frac{t^3}{3} - \frac{2t^2}{2} + 2t + 5$$

As $$v = \frac{dx}{dt} = \frac{t^3}{3} - \frac{2t^2}{2} + 2t + 5 \quad ...(ii)$$

$$x = \int dx = \int v\,dt = \int\left(\frac{t^3}{3} - \frac{2t^2}{2} + 2t + 5\right)dt$$

$$x = \frac{t^4}{4\times 3} - \frac{t^3}{3} + \frac{2t^2}{2} + 5t + c_2$$

Evaluating the constant of integration c_2 by substituting the condition, when

$$t = 1,\ x = 14.75 \text{ m}$$

$$14.75 = \frac{(1)^4}{4\times 3} - \frac{(1)^4}{3} + (1)^2 + 5(1) + c_2$$

$$14.75 = \frac{1}{12} - \frac{1}{3} + 1 + 5 + c_2$$

$$c_2 = 9$$

$$x = \frac{t^4}{12} - \frac{t^3}{3} + t^2 + 5t + 9 \quad ...(iii)$$

To find the distance, velocity and acceleration of the particle after 2 seconds we can use the expressions (i), (ii) and (iii).

Distance $$x = \frac{t^4}{12} - \frac{t^3}{3} + t^2 + 5t + 9$$

when $$t = 2$$

$$x = \frac{(2)^4}{12} - \frac{(2)^3}{3} + (2)^2 + 5(2) + 9$$

$$x = 21.67 \text{ m} \quad \textit{Ans.}$$

Velocity $$v = \frac{t^3}{3} - t^2 + 2t + 5$$

when $$t = 2$$

$$v = \frac{(2)^3}{3} - (2)^2 + 2(2) + 5$$

$$v = 7.76 \text{ m/s}$$

Acceleration $$a = t^2 - 2t + 2$$

when $$t = 2,\ a = (2)^2 - 2(2) + 2$$

$$a = 2 \text{ m/s}^2 \quad \textit{Ans.}$$

Example 14.12 Motion of a particle is given by the equation

$$x = t^3 - 3t^2 - 9t + 12 \quad ...(i)$$

Determine the time, position and acceleration of the particle when its velocity becomes zero.

Solution: Differentiating, $$v = \frac{dx}{dt} = 3t^2 - 3\times 2t - 9$$

$$v = 3(t^2 - 2t - 3) \quad ...(ii)$$

Differentiating again, $$a = \frac{dv}{dt} = 3(2t - 2)$$

$$a = 6(t - 1) \quad ...(iii)$$

When the velocity of the particle becomes zero,

$$v = 3(t^2 - 2t - 3) = 0$$

$$(t^2 - 1)(t - 3) = 0$$

Or

Velocity becomes zero at,

$$t = -1 \text{ s, and } t = 3 \text{ s}$$

The value of $t = -1$ s denotes time *before the motion began* and is discarded. So velocity becomes zero at $t = 3$ s *Ans.*

Position of the particle when $t = 3$ s, using (i)

$$x = t^3 - 3t^2 - 9t + 12$$

$$x = (3)^3 - 3(3)^2 - 9(3) + 12$$

$$x = 27 - 27 - 27 + 12$$

$$x = -15 \text{ m} \quad \textit{Ans.}$$

Acceleration of the particle when $t = 3$ s, using equation (iii),

$$a = 6(t - 1) = 6(3 - 1) = 12 \text{ m/s}^2 \quad \textit{Ans.}$$

The graphs of displacement vs. time, velocity vs. time and acceleration vs. time are as shown in Fig. 14.16 and it may be noted that,

(i) displacement vs. time graph is a cubic curve; velocity vs. time graph is a square curve; and acceleration vs. time graph is a straight line.

(ii) particle does not move along any of these curves but along a straight line.

(iii) the point of inflexion on the displacement curve corresponds to zero velocity and the point of inflexion on the velocity curve corresponds to zero acceleration.

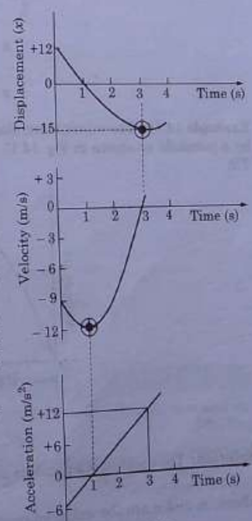

Fig. 14.16

Example 14.13 A particle starts with velocity v_0. Its acceleration and velocity are related by the equation

$$a = -kv$$

where k is some constant

v is the velocity of the particle

a is the acceleration of the particle

Find the displacement time relation.

Solution: $a = -kv$

$$a = \frac{dv}{dt} = -kv$$

$$\frac{dv}{v} = -k\,dt$$

Integrating,
$$\int_{v_0}^{v} \frac{dv}{v} = \int_0^t -k\,dt \quad \text{at } t = 0,\ v = v_0.$$

$$[\ln v]_{v_0}^{v} = [-kt]_0^t$$

$$\ln \frac{v}{v_0} = -kt$$

Or
$$v = v_0 e^{-kt}$$

$$\frac{dx}{dt} = v_0 e^{-kt}$$

$$\int_0^x dx = \int_0^t v_0 e^{-kt}\,dt$$

$$x = \frac{v_0}{(-k)}[e^{-kt}]_0^t = -\frac{v_0}{k}(e^{-kt} - 1), \text{ at } t = 0,\ x = 0$$

$$x = \frac{v_0}{k}(1 - e^{-kt}) \quad \textit{Ans.}$$

Example 14.14. The velocity-time diagram for the rectilinear motion of a particle is represented by a parabola as shown in Fig. 14.17. Find the distance travelled by the particle in the time $T/2$.

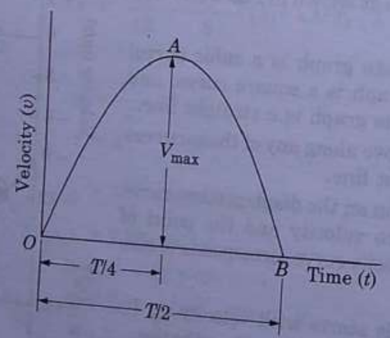

Fig. 14.17

Solution: The equation of a parabola passing through the origin is given by

$$y = mt + nt^2 \quad \text{...}(i)$$

where, m and n are the unknown constants.

If y represents the velocity and t the time then it is also the equation of the velocity-time diagram.

To evaluate the unknown constants substitute the coordinates of the points A and B in equation (i)

$$A\left(\frac{T}{4}, V_{max}\right)$$

$$V_{max} = m\left(\frac{T}{4}\right) + n\left(\frac{T}{4}\right)^2 \quad \text{...}(ii)$$

$$B\left(\frac{T}{2}, 0\right),$$

$$0 = m\left(\frac{T}{2}\right) + n\left(\frac{T}{2}\right)^2 \quad \text{...}(iii)$$

$h = \frac{1}{2}gt^2$

Solving (ii) and (iii),

$$m = \frac{8V_{max}}{T},\ n = -\frac{16V_{max}}{T^2}$$

Distance travelled by the particle can be determined as,

$$v = \frac{dx}{dt}$$

$$x = \int dx = \int v\,dt$$

$$x = \int_0^{T/2} (mt + nt^2)\,dt$$

$$x = \left[\frac{mt^2}{2} + \frac{nt^2}{3}\right]_0^{T/2}$$

$$x = \frac{m}{2}\left(\frac{T}{2}\right)^2 + \frac{n}{3}\left(\frac{T}{2}\right)^3$$

Substituting for the constants m and n,

$$x = \frac{1}{2}\left(8\frac{V_{max}}{T}\right)\frac{T^2}{4} + \frac{1}{3}\left(-\frac{16V_{max}}{T^2}\right)\frac{T^3}{8}$$

$$x = \frac{V_{max}T}{3} \quad \textit{Ans.}$$

PROBLEMS

14.1. A stone is dropped into a well and the sound of splash is heard after 4 seconds. Assuming the velocity of sound to be 350 m/s find the depth of the well. [70.77 m]

14.2. A motorist takes 10 seconds to cover a distance of 20 m and 15 seconds to cover a distance of 40 m. Find the uniform acceleration of the car and the velocity at the end of 15 seconds. [0.267 m/s^2, 4.7 m/s]

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14.3. A stone is dropped from the top of a tower 60 m high. At the same instant, another stone is thrown vertically upwards from the foot of tower to meet the first stone at a height of 18 m. Determine (*a*) the time when the two stones meet, (*b*) the velocity with which the second stone was thrown up.
[2.93 s, 20.5 m/s]

14.4. A cage takes 45 seconds to go down a mine shaft 675 m deep. For the first quarter of the distance the speed is constantly accelerated and during the last quarter of the distance the speed is uniformly retarded. Determine the uniform speed of the cage while travelling the central portion of the shaft if the acceleration and retardation of the cage are to be equal.
[25 m/s]

14.5. A body is thrown vertically up. It was found to travel a distance of 5.0 m during its 3rd second of the travel. Find the initial velocity with which the body was thrown up.
[29.5 m/s]

14.6. A ball is thrown vertically up with a velocity of 30 m/s. Determine the time (*a*) when the ball will be 20 m above the point of projection, (*b*) when its velocity will be 5 m/s, (*c*) when it will return back to its initial position.
[0.7 s, 2.55 s, 6.12 s]

14.7. Water drips at the rate of 5 drops per second from a leaking tap. Determine the vertical distance between two consecutive drops after the lower drop has attained a velocity of 3 m/s.

[**Hint:** First drop attains a velocity of 3 m/s, the second drop travels for a time $\left(0.306 \frac{1}{2}\right)$ seconds].

14.8. The acceleration of a particle is given by the relation $a = \frac{10}{v+1}$ where, a is expressed in m/s^2 and v in m/s. The particle starts with zero initial velocity at position $x = 0$. What is the position of the particle when

$v = 10$ m/s? [38.33 m]

14.9. The velocity-time diagram for the rectilinear motion of a particle is represented by a portion of sine curve as shown in Fig. P.14.9. Find the distance travelled by the particle in the time $\frac{T}{2}$.

$$\left[x = \frac{TV_m}{\pi}\right]$$

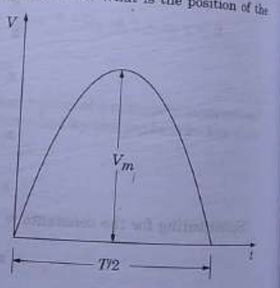


Fig. P.14.9

14.10. A lift moves up with a constant acceleration upto a height of 900 m and 300 m with constant retardation and then comes to rest. Determine : (*a*) acceleration, (*b*) retardation and (*c*) the maximum velocity of the lift if the total time of travel is 30 seconds and the acceleration is one third of the retardation.
[3.55 m/s^2, 10.65 m/s^2, 80 m/s]

14.11. The position of a particle moving along a straight line is defined by the relation $x = t^3 - 9t^2 + 15t + 18$ where x is expressed in metres and t in seconds. Determine the time, position and acceleration of the particle when its velocity become zero.

$$\begin{bmatrix} t = 1, x = 25\,\text{cm}, a = -12\ \text{m/s}^2 \\ t = 5, x = -7\,\text{m}, \quad a = 12\ \text{m/s}^2 \end{bmatrix}$$

14.12. The acceleration of a particle along a straight line is given by the equation,

$$a = 4 - t^2/9$$

If the particle starts with zero velocity from a position $x = 0$, find
(*a*) its velocity after 6 seconds
(*b*) distance travelled in 6 seconds.

[16 m/s, 60 m]

14.13. The acceleration of a particle is given by the relation $a = 90 - 6x^2$ where a is expressed in cm/s^2 and x in centimetres. If the particle starts with zero initial velocity at position $x = 0$, determine (*a*) the velocity when $x = 5$ cm, (*b*) the position where velocity is again zero and (*c*) the position where the velocity is maximum.
[+ 20 cm/s, + 6.71 cm, 3.87 cm]

14.14. The acceleration of a particle moving in a straight line is given by the relation $a\sqrt{v}$. Its displacement and velocity at time $t = 2$ s are $42\frac{2}{3}$ m and 16 m/s respectively. Find the displacement, velocity and acceleration of the particle at time $t = 3$ seconds.

[60.75 m, 20.25 m/s and 4.5 m/s^2]

14.15. A particle starts with an initial velocity of 8 m/s and moves along a straight line. It's acceleration at any time t after start is given by the expression $A - Bt$ where A and B are constants. Determine the equation for displacement if the particle covers a distance 40 m in 5 seconds and stops.

Hint: $\begin{bmatrix} t = 0, & v = 8\,\text{m/s} \\ t = 0 & x = 0 \\ t = 5s & x = 40 \\ t = 5s & v = 0 \end{bmatrix}$ $[x = 8t + 1.6t^2 - 0.32t^3]$

14.16. A particle falls vertically in a medium whose resistance is proportional to the velocity of the particle. Find the velocity and distance travelled by the particle after a time t.

Hint: $\begin{pmatrix} \text{Resistance, } R = kv^2 \\ ma = mg - kv^2 \end{pmatrix}$

$$\begin{pmatrix} v = \frac{g}{k}(1 - e^{-kt}) \\ x = \left(\frac{g}{k}t + \frac{g}{k^2}e^{-kt} - \frac{g}{k^2}\right) \end{pmatrix}$$

14.17. Two blocks P and Q are connected by a string of length 12 m passing over a very small pulley. The arrangement is such that P moves horizontally and Q vertically up. When $x = 4$ m it is found that velocity of P is 3 m/s towards right. Find the velocity of block.

[**Hint:** Length of the string is constant, $l = 12$ m. Expressing displacements x, y of P and Q from same fixed point in terms of l.

$$l = (6 - y) + AB,$$
$$AB^2 = x^2 + 6^2$$
$$l = (6 - y) + \sqrt{x^2 + 6^2}$$

This displacement relationship between P and Q can now be differentiated to get velocity and acceleration relationships]

$[v_q = 1.66$ m/s, $y = 1.21$ m$]$

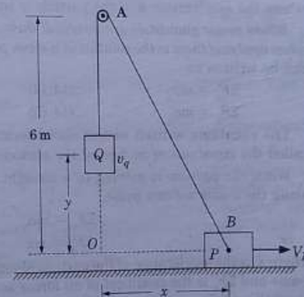
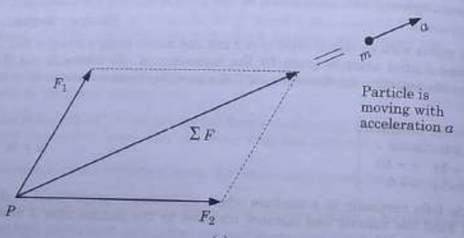


Fig. P.14.17

PART B: KINETICS

14.6 EQUATIONS OF RECTILINEAR MOTION

Consider a particle P of mass m having an acceleration **a** when acted upon by several forces (say $\mathbf{F}_1$ and $\mathbf{F}_2$). Let $\Sigma\mathbf{F}$ be the resultant of these force [Fig. 14.18 (*a*)].



(*a*)

Fig. 14.18

Applying Newton's Second Law

$$\Sigma\mathbf{F} = m\mathbf{a} \qquad ...(14.13)$$

where the acceleration **a** of the particle is in the direction of the resultant force $\Sigma\mathbf{F}$.

Where vector quantities are involved, their x and y components can be considered separately when applying them to the solution of a given problem [Fig. 14.18 (*b*)]. Therefore, above equation can be written as

$$\Sigma F_x = ma_x \qquad ...(14.14)$$

$$\Sigma F_y = ma_y \qquad ...(14.15)$$

The equations written in the above form are called the *equations of motion of the particle.*

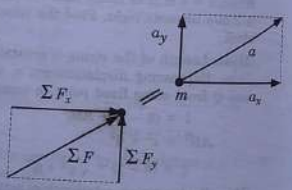
When the particle is moving in a straight line along the x-axis we can write,

$$\Sigma F_x = ma_x$$

or

$$\Sigma F_x = m\ddot{x}$$

where $\ddot{x}$ is the acceleration of the particle along the x-axis and ΣF_x is the resultant of all forces acting along the x-axis.



(*b*)

Fig. 14.18

14.7 EQUATIONS OF DYNAMIC EQUILIBRIUM: D'ALEMBERT'S PRINCIPLE

The equation of motion of the particle P

$$\Sigma\mathbf{F} = m\mathbf{a} \qquad ...(14.16)$$

can be written in the form

$$\Sigma\mathbf{F} - m\mathbf{a} = 0 \qquad ...(14.17)$$

which means that the resultant of the external forces ($\Sigma\mathbf{F}$) and the force ($-m\mathbf{a}$) is zero. The force ($-m\mathbf{a}$) is called inertia force. The inertia of a body can be defined as the resistance to the change in the condition of rest or of uniform motion of a body.

The magnitude of the inertia force is equal to the product of the mass and acceleration of the particle and it acts in a direction *opposite* to the direction of acceleration of the particle.

The equations in the form

$$\Sigma\mathbf{F} - m\mathbf{a} = 0$$

Or

$$\Sigma\mathbf{F} + \underbrace{(-m\mathbf{a})}_{\text{Inertia Force}} = 0 \qquad ...(14.18)$$

or in the component form

$$\Sigma F_x + \underbrace{(-ma_x)}_{\text{Inertia Force}} = 0 \qquad ...(14.19)$$

$$\Sigma F_y + \underbrace{(-ma_y)}_{\text{Inertia Force}} = 0 \qquad ...(14.20)$$

are called *the equations of dynamic equilibrium of the particle.*

So, to write the equation of dynamic equilibrium of a particle add a fictitious force equal to the inertia force to the external forces acting on the particle and equate the sum (resultant) to zero (Fig. 14.19). This concept is known as D'Alembert's Principle. It is very useful concept as moment equation of dynamic equilibrium is easy to conceive and write.

For the simplicity of representation, equations (14.19) and (14.20) can be written as

$$\Sigma F_x = 0 \qquad ...(14.21)$$

$$\Sigma F_y = 0 \qquad ...(14.22)$$

When expressed in the above manner, the terms ΣF_x and ΣF_y are to be *redefined*. That is, ΣF_x and ΣF_y now include the inertia forces also.

Further the equations 14.21 and 14.22 have an appearance similar to the equations of static equilibrium.

It should be clearly understood that the equation of motion of a particle and the equation of dynamic equilibrium of a particle are the two methods of expression which differ only in the concept used and in the manner of writing the equations. The final result, however, shall be the same. Whichever method is followed should be followed consistently and clearly.

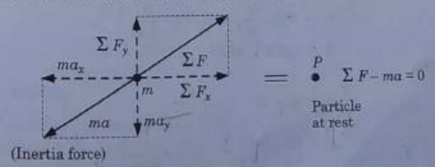


Fig. 14.19

Example 14.15. A force of 250 N acts on a body of mass $m = 100$ kg. Find the acceleration of the body.

Solution: Equation of the Motion : Let the acceleration of the body in the direction of the force be a.

Writing the equation of motion of the body

$$\Sigma F = ma$$
$$250 = 100 \times a$$

(only one force is acting on the body)

$$a = 2.5 \text{ m/s}^2 \quad \textit{Ans.}$$

Equation of Equilibrium : Apply a fictitious force equal to ma to the body, in a direction opposite to the direction of acceleration of the body.

Writing the equation dynamic equilibrium of the body

$$\Sigma F - ma = 0$$
$$250 - 100 \times a = 0$$
$$a = 2.5 \text{ m/s}^2 \quad \textit{Ans.}$$

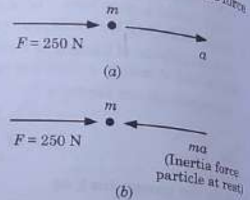


Fig. 14.20

Example 14.16. An elevator has a downward acceleration of 1 m/s². What pressure will be transmitted to the floor of the elevator by a man weighing 500 N travelling in the lift?

Find the pressure if the elevator had an upward acceleration of 1 m/s².

Solution. Pressure R exerted by the floor on the man = Pressure exerted by the man on the floor

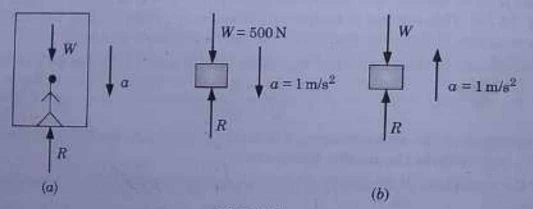


Fig. 14.21

Downward Motion: Writing the equation of motion of the man

$$\Sigma F = ma: \quad \frac{W}{g}a = W - R$$

$$R = W\left(1-\frac{a}{g}\right) = 500\left(1-\frac{1}{9.81}\right)$$

$$R = 449 \text{ N} \quad \textit{Ans.}$$

Upward Motion :

$$\Sigma F = ma: \quad \frac{W}{g}a = R - W$$

$$R = W\left(1+\frac{a}{g}\right) = 500\left(1+\frac{1}{9.81}\right)$$

$$R = 550.97 \text{ N} \quad \textit{Ans.}$$

Example 14.17. An elevator of total weight 5000 N starts to move upwards with a constant acceleration and acquires a velocity 2 m/s after travelling a distance of 2 m. Find the tensile force in the cable during the accelerated motion.

The above elevator while moving up with a velocity of 2 m/s is uniformly decelerated to stop in 2 seconds. Find the pressure at the floor of the elevator under the feet of a man weighing 600 N riding in the elevator.

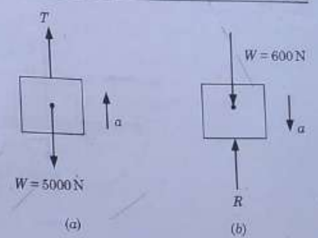


Fig. 14.22

Solution: Motion of Elevator: Acceleration acquired by the elevator after travelling 2 m

$$u = 0, \; v = 2, \text{ m/s}, \; s = 2 \text{ m}$$
$$v^2 - u^2 = 2as$$
$$4 = 2 \times 2 \times a$$
$$a = 1 \text{ m/s}^2.$$

Let T be the tension in the cable.

The equation of motion of the elevator is

$$\Sigma F = ma: \quad \frac{W}{g}a = T - W$$

$$T = W\left(1+\frac{a}{g}\right) = \frac{5000 \times 10.81}{9.81}$$

$$T = 5509.7 \text{ N} \quad \textit{Ans.}$$

Motion of Man

$$u = 2 \text{ m/s}, \quad v = 0, \; t = 2 \text{ s}$$
$$v = u + at, \quad a = -1 \text{ m/s}^2$$

That is, acceleation a is *opposite* to the direction of velocity or is downward.

Let R be the pressure experienced by the man. Equation of motion of the man is,

$$\Sigma F = ma: \quad \frac{w}{g}a = w - R$$

$$R = \left(w - \frac{w}{g}a\right) = w\left(1-\frac{a}{g}\right) = \frac{600 \times 8.81}{9.81}$$

$$R = 538.8 \text{ N} \quad \textit{Ans.}$$

Motion of two Bodies

Example 14.18. Two blocks of masses M_1 and M_2 are connected by a flexible but inextensible string as shown in Fig. 14.23. Assuming the coefficient of friction between block M_1 and the horizontal surface to be μ find the acceleration of the masses and tension in the string. Assume $M_1 = 10$ kg and $M_2 = 5$ kg and $\mu = 0.25$.

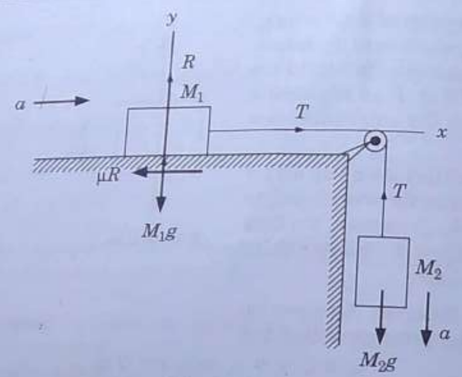


Fig. 14.23

Solution: Let the mass M_2 move down with an acceleration of a m/s^2. The acceleration of M_1 is same as of M_2. Let R be normal reaction and μR be the friction force acting on M_1. T be the tension in the string.

Writing the equations of motion for mass M_1

$\Sigma F_x = ma_x$: $\quad M_1 a = T - \mu R$

$\Sigma F_y = ma_y$: $\quad 0 = R - M_1 g$ ...(*i*)

Writing the equation of motion for the mass M_2 ...(*ii*)

$\Sigma F_y = ma_y$: $\quad M_2 a = M_2 g - T$

Or $\quad T = M_2(g - a)$

From (*i*) and (*ii*) $\quad M_1 a = T - \mu (M_1 g)$ ...(*iii*)

Or $\quad T = M_1(a + \mu g)$ ...(*iv*)

Equating (*ii*) and (*iv*)

$$M_1(a + \mu g) = M_2(g - a)$$

$$M_1 a + M_1 \mu g = M_2 g - M_2 a$$

$$a(M_2 + M_1) = M_2 g - M_1 \mu g$$

Or

$$a = g\,(M_2 - \mu M_1)/(M_1 + M_2) \quad ...(v)$$

Substituting for a in (*iii*)

$$T = M_2(g - a)$$

$$T = M_2\left(g - \frac{g(M_2 - \mu M_1)}{M_1 + M_2}\right)$$

$$T = \frac{M_2 g}{M_1 + M_2}(M_1 + \cancel{M_2} - \cancel{M_2} + \mu M_1)$$

$$T = \frac{M_2 g M_1}{M_1 + M_2}(1 + \mu)$$

$$T = \frac{M_1 M_2 g}{(M_1 + M_2)}(1 + \mu) \quad ...(vi)$$

Substituting, $\quad M_1 = 10$ kg, $M_2 = 5$ kg, $\mu = 0.25$ m

$$a = \frac{g(M_2 - \mu M_1)}{(M_1 + M_2)} = \frac{g[5 - 0.25(10)]}{(10 + 5)}$$

$a = 1.635$ m/s^2 *Ans.*

$$T = \frac{10 \times 5 \times 9.81}{(10 + 5)}(1 + 0.25)$$

$T = 40.875$ N *Ans.*

Example 14.19. A block of mass M_1 resting on an inclined plane is connected of by a string and pulleys to another block of mass M_2 as shown in Fig. 14.24. Find the tension in the string during the motion of the system. Assume the coefficient of friction between the block M_1 and the inclined plane to be 0.2.

Solution: This problem is of the motion of two connected masses involving friction. Two points are to be noted here,

(*i*) Accelerations of the two masses are not same, but related. It can be shown that, Acceleration of mass M_1 = 2 (Acceleration of mass M_2).

(*ii*) In problems involving friction, the direction of motion of the system of bodies are to be clearly acertained and then direction of frictional force should be shown accordingly. A wrong direction of friction force may result in absurd results.

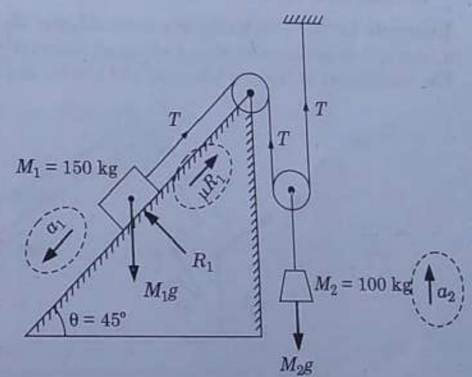


Fig. 14.24

Acceleration of mass $\quad M_1 = a_1$

Acceleration of mass $\quad M_2 = a_2$ and $a_1 = 2a_2$

Forces acting of the mass M_1 are

(*i*) tension in the string

(*ii*) Weight $M_1 g$

(*iii*) Normal reaction R

(*iv*) friction force μR

To understand whether mass M_1 shall move down or up the inclined plane assume static conditions, and $g \cong 10$ m/s^2. For the mass M_1 to move down $M_1 g \sin\theta$ (= 150 × 10 × sin 45° = 1060.5 N) should be more than the sum of tension T and friction force [$\mu R = \mu M_1 g \cos 45° = 0.2 \times 150 \times 10 \times 0.707 = 212.1$(N)]. If the we consider mass M_2 under static conditions we can write $M_2 g(100 \times 10 = 1000\text{ N}) = 2T$ or $T = 500$ N.

$M_2 g \sin\theta$ (1060.5) > T (500) + μR (212.1). So, it is found that the mass M_1 shall move down the plane and mass M_2 shall move up.

Equation of motion of mass M_1

$\Sigma F_x = ma_x$: (Along the plane) $\quad M_1 a_1 = M_1 g \sin 45° - T - \mu R$ ...(i)

$\Sigma F_y = ma_y$: (Normal to the Plane) $\quad 0 = R - M_1 g \cos 45°$

Eliminating R from the above

$$M_1 a_1 = M_1 g \sin 45° - T - \mu M_1 g \cos 45°$$

$$150\, a_1 = 150 \times 9.81 \times 0.707 - T - 0.2 \times 150 \times 9.81 \times 0.707 \quad \text{...(ii)}$$

Equation of motion of mass M_2

$\Sigma F_y = ma_y$: (Vertical direction) $\quad M_2 a_2 = 2T - M_2 g \quad (a_1 = 2a_2)$

$$100(a_1/2) = 2T - 100 \times 9.81 \quad \text{...(iii)}$$

Solving (*ii*) and (*iii*) $T = 539.4$ N *Ans.*

Example 14.20. Two blocks of masses M_1 and M_2 are placed on two incline planes of elevation θ_1 and θ_2 and are connected by a string as shown (Fig. 14.25). Find the acceleration of the masses. The coefficient of friction between the blocks and the planes is μ.

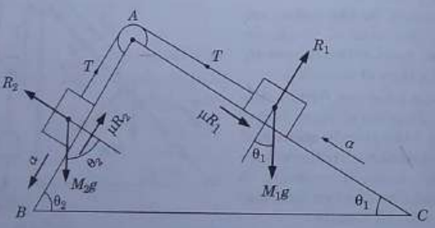


Fig. 14.25

Assume the following numerical data

$$M_1 = 5 \text{ kg},\ \theta_1 = 30°$$

$$M_2 = 10 \text{ kg},\ \theta_2 = 60°\ \mu = 0.33$$

Solution: Let the acceleration of the mass M_2 parallel to the inclined plane be a. The acceleration of M_1 is same in magnitude.

Motion of Mass M_2. Forces acting on the mass M_2 are : weight $M_2 g$, tension T, normal reaction R_2 and friction force μR_2.

Writing the equation of motion

$\Sigma F_x = ma_x$: (Along the plane) $\quad M_2 a = M_2 g \sin \theta_2 - \mu R_2 - T \quad (a_x = a)$

$\Sigma F_y = ma_y$: (Normal to the plane) $\quad 0 = R_2 - M_2 g \cos \theta_2 \quad (a_y = 0)$

Eliminating R_2 from the above

$$M_2 a = M_2 g \sin \theta_2 - \mu(M_2 g \cos \theta_2) - T \quad \text{...(i)}$$

Motion of Mass M_1. Force acting on the mass M_1 are : weight $M_1 g$, tension T, normal reaction R_1 and friction force μR_1.

Writing the equation of motion

$\Sigma F_x = ma_x$: (Along the plane) $\quad M_1 a = T - M_1 g \sin \theta_1 - \mu R_1 \quad (a_x = a)$

$\Sigma F_y = ma_y$: (Normal to the plane) $\quad 0 = R_1 - M_1 g \cos \theta_1 \quad (a_y = 0)$

Eliminating R_1 from the above equations

$$M_1 a = T - M_1 g \sin \theta_1 - \mu(M_1 g \cos \theta_1) \quad \text{...(ii)}$$

Eliminating T from (*i*) and (*ii*)

$$M_2 g \sin \theta_2 - \mu M_2 g \cos \theta_2 - M_2 a = M_1 a + M_1 g \sin \theta_1 + \mu M_1 g \cos \theta_1$$

$$a(M_1 + M_2) = g[M_2 (\sin \theta_2 - \mu \cos \theta_2) - M_2(\sin \theta_1 + \mu \cos \theta_1)$$

$$a = \frac{g}{(M_1 + M_2)}[M_2(\sin \theta_2 - \mu \cos \theta_2) - M_1(\sin \theta_1 + \mu \cos \theta_1]$$

Substituting $\quad M_1 = 5 \text{ kg},\ M_2 = 10 \text{ kg},\ \theta_1 = 30°$

$$\theta_2 = 60° \text{ and } \mu = 0.33$$

$$a = \frac{9.81}{(5+10)}[10(0.866 - 0.33 \times 0.5) - 5(0.5 + 0.33 \times 0.866]$$

$$a = 2.014 \text{ m/s}^2. \quad \textit{Ans.}$$

Example 14.21 Two blocks A and B are held on an inclined plane 5 m apart as shown in Fig. 14.26. The coefficients of friction between the block A and B and the inclined plane are 0.2 and 0.1 respectively. If the blocks begin to slide down the plane simultaneously calculate the time and distance travelled by the each block before collision.

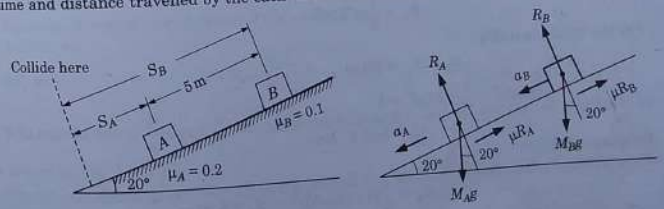


Fig. 14.26

Solution: Let a_A and a_B be the acceleration of the blocks A and B.

Equation of motion of the block A

$\Sigma F_x = ma_x$: (Along the plane) $\qquad M_A a_A = M_A g \sin 20° - \mu_A R_A$

$$M_A a_A = M_A g \sin 20° - 0.2\, R_A \quad (a_x = a_A) \qquad ...(i)$$

or

$\Sigma F_y = ma_y$: (Normal to the plane)

$$0 = R_A - M_A g \cos 20° \quad (a_y = 0) \qquad ...(ii)$$

Solving for a_A

$$M_A a_A = M_A g \sin 20° - 0.2(M_A g \cos 20°)$$
$$a_A = g \sin 20° - 0.2 \times g \cos 20°$$
$$a_A = 9.81 \times 0.342 - 0.2 \times 9.81 \times 0.939$$
$$a_A = 1.510 \text{ m/s}^2$$

Equation of motion of the block B

$\Sigma F_x = ma_x$: $\qquad M_B a_B = M_B g \sin 20° - \mu_B R_B \quad (a_x = a_B)$

or $\qquad M_B a_B = M_B g \sin 20° - 0.1\, R_B$

$\Sigma F_y = ma_y$: $\qquad 0 = R_B - M_B g \cos 20° \quad (a_y = 0) \qquad ...(iii)$

Solving for a_B

$$M_B a_B = M_B g \cos 20° - 0.1\, M_B g \cos 20°$$
$$a_B = g \sin 20° - 0.1 \times g \cos 20°$$
$$a_B = 9.81(0.342 - 0.1 \times 0.939)$$
$$a_B = 2.43 \text{ m/s}^2$$

Let the blocks collide after a time t of release.

Distance S_B travelled by the block B in time t

$$\mu = 0,\ a_B = 2.43 \text{ m/s}^2, \quad (s = S_B)$$

Using,

$$s = ut + \frac{1}{2}at^2$$

$$S_B = \frac{1}{2}(2.43)t^2$$

Distance S_A travelled by the block A in time t,

$$u = 0,\ a = 1.51 \text{ m/s}^2 \quad (s = S_A)$$

$$S_A = \frac{1}{2}(1.51)t^2$$

For the blocks to collide

$$S_B - S_A = 5.0 \text{ m}$$

$$\frac{1}{2}(2.43)t^2 - \frac{1}{2}(1.51)t^2 = 5$$

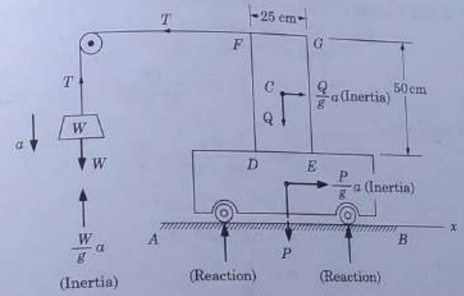
$$t = 3.35 \text{ s} \quad \textit{Ans.}$$

Substituting for t

$$S_A = \frac{1}{2}(1.51)(3.35)^2 \qquad S_A = 8.20 \text{ m}$$

$$S_B = \frac{1}{2}(2.43)(3.35)^2 \qquad S_B = 13.20 \text{ m} \quad \textit{Ans.}$$

Example 14.22. A rectangular block of weight Q rests on a flat car of weight P which may roll along the horizontal plane AB without friction. The car and the block together are to be accelerated by the weight W arranged as shown in Fig. 14.27(a). Assuming that there is sufficient friction between the block and the car to prevent sliding, find the maximum value of the weight W by which the car can be accelerated. What will be this acceleration? Assume $Q = 100$ N and $P = 50$ N.



(a)

Fig. 14.27

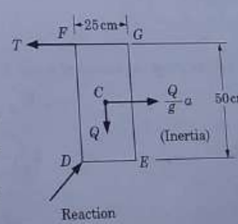
Solution: Let the weight W move downward with a constant acceleration a and the tension in the string be T.

Consider the motion of the weight W.

Add inertia force $\left(\frac{W}{g}a\right)$ acting at the centre of gravity of weight W and in the direction opposite to the acceleration of the weight.

Equation of dynamic equilibrium of the weight W can be written as

$$\Sigma F_y = 0 : \qquad T - W + \frac{W}{g}a = 0 \qquad ...(i)$$



(b)

Fig. 14.27

Add inertia forces $\frac{P}{g}a$ and $\frac{Q}{g}a$ to the forces for acting on the car and the rectangular block as shown in Fig. 14.27(a).

Equation of dynamic equilibrium of the car and the rectangular block taken together can be written as,

$$\Sigma F_x = 0 : \qquad -T + \frac{Q}{g}a + \frac{P}{g}a = 0 \qquad ...(ii)$$

$$\text{or, } Qa + Pa - Tg = 0.$$

The moment equation for the dynamic equilibrium of the rectangular block when the block is about to turn about point D can be written as,

$$\Sigma M_D = 0: \quad T\left(\frac{50}{1000}\right) - Q\left(\frac{25/2}{1000}\right) - \frac{Qa}{g}\left(\frac{50/2}{1000}\right) = 0$$

$$4Tg - Qg - 2Qa = 0 \qquad \text{...(iii)}$$

To solve for a and T, multiply (ii) by 4 and add to (iii)

$$4Qa + 4Pa - Qg - 2Qa = 0$$

$$a(4Q + 4P - 2Q) = Qg$$

$$a = \frac{Qg}{(4P + 2Q)} \qquad \text{...(iv)}$$

Substituting for Q and P,

$$a = \frac{100g}{(200 + 200)} = \frac{g}{4}$$

$$a = \frac{g}{4} = 2.45 \text{ m/s}^2$$

$$a = 2.45 \text{ m/s}^2 \quad \textit{Ans.}$$

From (ii) $\quad T = \left(\frac{Q}{g} + \frac{P}{g}\right)a$

Substituting for a, $\quad T = \frac{(P+Q)}{g}\left(\frac{Qg}{4P + 2Q}\right)$

$$T = \frac{Q(P+Q)}{(4P + 2Q)} \qquad \text{...(v)}$$

Substituting the values of a and T obtained above in equation (i)

$$T + \frac{W}{g}a - W = 0$$

Or $\quad W\left(1 - \frac{a}{g}\right) = T$

$$W\left(1 - \frac{Qg}{(4P + 2Q)}\frac{1}{g}\right) = \frac{Q(P+Q)}{(4P + 2Q)}$$

$$W\frac{(4P+Q)}{(4P + 2Q)} = \frac{Q(P+Q)}{(4P + 2Q)}$$

$$W = \frac{Q(P+Q)}{(4P+Q)} \qquad \text{...(vi)}$$

Substituting the values of P and Q

$$Q = \frac{100(50+100)}{(200+100)} = 50$$

$$W = 50 \text{ N} \quad \textit{Ans.}$$

Example 14.23 A two step pulley supports two weights P and Q as shown (Fig. 14.28). Find the downward acceleration of P. Assume $P = 40$ N, $Q = 60$ N and $r_1 = 2\,r_2$. Neglect friction and inertia of the pulleys.

Solution: Let, the acceleration of weight P on pulley $r_1 = a_p$

The acceleration of weight Q on pulley $r_2 = a_q$

Relation between a_p and a_q:

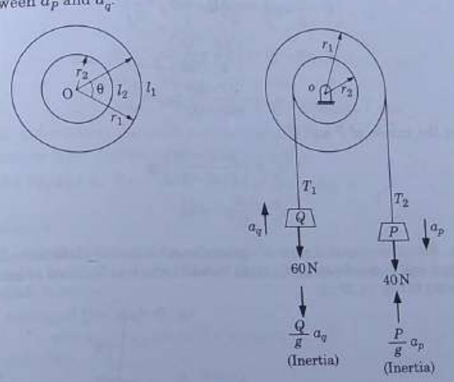


Fig. 14.28

For an angle θ turned by the pulley,

$$\theta = \frac{l_1}{r_1} = \frac{l_2}{r_2}$$

$$l_1 r_2 = l_2 r_1$$

Differentiating twice with respect to time

$$\frac{d^2}{dt^2}(l_1 r_2) = \frac{d^2}{dt^2}(l_2 r_1)$$

$$a_p r_2 = a_q r_1$$

Or $\quad a_q = \frac{a_p r_2}{r_1}$

Consider the dynamic equilibrium of the weights P and Q together.

Taking moments about O,

$$\Sigma M_0 = 0: \quad -r_1\left(P - \frac{P}{g}a_p\right) + r_2\left(Q + \frac{Q}{g}a_q\right) = 0$$

$$\left(P - \frac{P}{g}a_p\right) = \frac{r_2}{r_1}\left(Q + \frac{Q}{g}a_q\right)$$

$$\left(P-\frac{P}{g}a_p\right)=\frac{r_2}{r_1}\left(Q+\frac{Q}{g}a_p\frac{r_2}{r_1}\right)$$

Substituting

$$a_q = a_p\frac{r_2}{r_1} \text{ and } \frac{r_2}{r_1} = \frac{1}{2}$$

$$\left(P-\frac{P}{g}a_p\right) = \frac{1}{2}\left(Q+\frac{Q}{g}a_p\times\frac{1}{2}\right)$$

$$2P - Q = \left(\frac{2P}{g}+\frac{Q}{2g}\right)a_p$$

$$a_p = \frac{(2P-Q)}{(4P+Q)}2g$$

Substituting the values of P and Q

$$a_p = \left(\frac{2\times 40-60}{4\times 40+60}\right)2g$$

$$a_p = \frac{2}{11}g \quad \textit{Ans.}$$

Example 14.24. Smooth wedge A of mass M is placed on a horizontal plane and a block B of mass M starts from rest and slides down its slanting face CD which is inclined at an angle θ to the horizontal as shown in Fig. 14.29 (*a*).

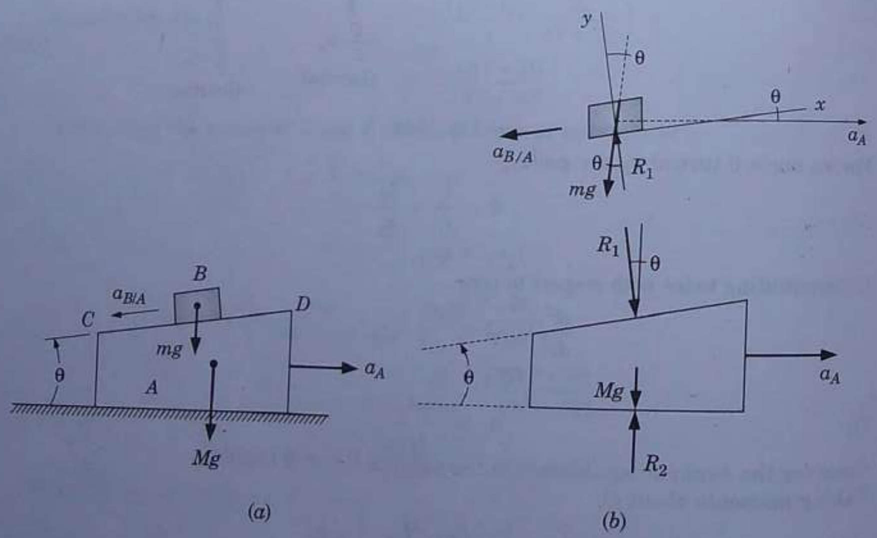


Fig. 14.29

Neglecting friction determine (*a*) the acceleration of the wedge (*b*) the acceleration of the block relative to the wedge. Assume the numerical data as : $m = 6$ kg, $M = 15$ kg and $\theta = 30°$.

Solution: The wedge is A free to side horizontally and the block B slides down the wedge.

Acceleration of the wedge A be $\bar{a}_A$

Acceleration of the block B be $\bar{a}_B$

Acceleration of the block B relative to the wedge A be $\bar{a}_{B/A}$.

$$\bar{a}_{B/A} = \bar{a}_B - \bar{a}_A$$

(Vector Difference)

$$\bar{a}_B = \bar{a}_{B/A} + \bar{a}_A$$

Or

$$(\bar{a}_B)_x = (\bar{a}_{B/A} + \bar{a}_A)_x \text{ in } x\text{-direction} \rightarrow +\text{ve}$$

$$(a_B)_x = (-a_{B/A} + a_A\cos\theta)$$

$$(a_B)_y = (0 - a_A\sin\theta) \text{ in } y\text{-direction} \uparrow +\text{ve}$$

R_1 be the reaction between the block and the wedge.

R_2 be the reaction of the horizontal plane on the wedge.

Motion of the Wedge A. Equation of motion of the wedge is,

$\Sigma F_x = ma$: $\quad Ma_A = R_1 \sin\theta$ ...(*i*)

(Along the horizontal)

Motion of the Block B. Block B experiences an acceleration $a_{B/A}$, with respect to the wedge A, directed down the inclined face of the wedge. Block B also experiences the acceleration of the wedge a_A on which it riding.

Equation of motion of the block B are,

$\Sigma F_x = ma_x$: $\quad m(-a_{B/A} + a_A\cos\theta) = -mg\sin\theta$ ...(*ii*)

(Along the inclined face)

$$a_x = (a_B)_x$$

$\Sigma F_y = Ma_y$: $\quad m(-a_A\sin\theta) = R_1 - mg\cos\theta$ ...(*iii*)

(Normal to the inclined face)

$$[a_y = (a_B)_y = -a_A\sin\theta)$$

Eliminating R_1 from (*i*) and (*iii*),

$$-ma_A\sin\theta = \frac{Ma_A}{\sin\theta} - ma\cos\theta$$

$$a_A(M + m\sin^2\theta) = mg\sin\theta\cos\theta$$

$$a_A = \frac{mg\cos\theta\cos\theta}{M+m\sin^2\theta}$$

From (*iii*)

$$a_{B/A} = a_A\cos\theta + g\sin\theta$$

$$a_{B/A} = \frac{mg\sin\theta\cos\theta}{M+m\sin^2\theta}\cos\theta + g\sin\theta$$

$$a_{B/A} = \frac{g\sin\theta}{M+m\sin^2\theta}(m+M)$$

Substituting

$$m = 6\text{ kg},\ M = 15\text{ kg},\ \theta = 30°$$

$$a_A = 0.157\text{ g},\ a_{B/A} = 0.636\text{ g} \quad \textit{Ans.}$$

Example 14.25. A system of weights connected by string and passing over the two pulley A and B are arranged as shown in Fig. 14.30(*a*). Find the acceleration of each weight. Neglect friction and inertia of pulleys. Given: $P = 30$ N, $Q = 20$ N and $R = 10$ N.

Solution: Let us assume the directions of motion of P, Q and R as below.

Let the weight P move down with an acceleration of a m/s^2. The pulley B then shall move up with the same acceleration a m/s^2.

Let the weight Q move down with an acceleration of a_1 with respect to the pulley B then, the weight R shall move up with the same acceleration with respect to the pulley B. As the pulley B itself is moving up, therefore,

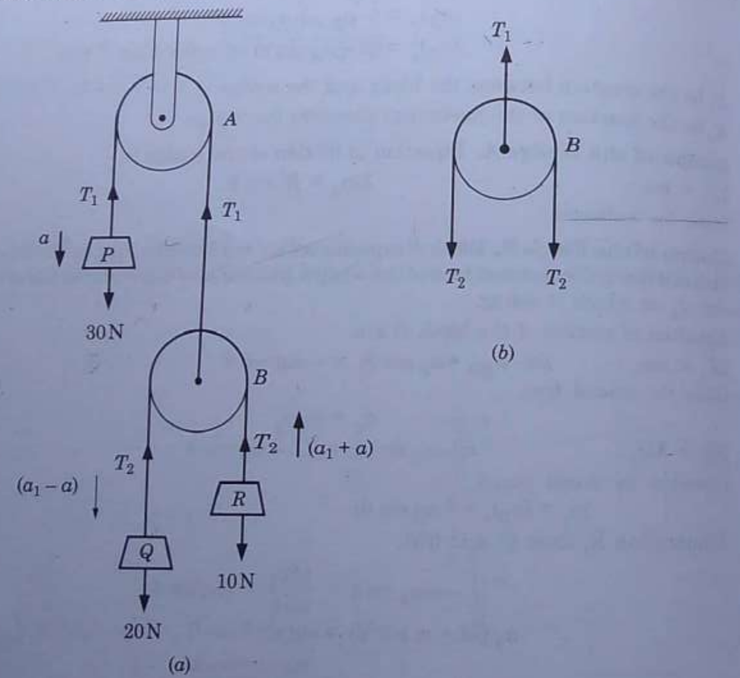


Fig. 14.30

Absolute acceleration of weight $Q = (a_1 - a)$ m/s^2 downward

Absolute acceleration of weight $R = (a_1 + a)$ m/s^2 upward

Let T_1 and T_2 be the tension in the strings. Equations of motion for the weights P, Q and R may be written using,

Mass × Acceleration = The resultant force acting in the direction of acceleration,

Equation of motion of Weight P,

$\Sigma F_y = ma_y$: $$\frac{P}{g}(a) = P - T_1$$

Equation of Motion of the Weight Q,

$\Sigma F_y = ma_y$: $$\frac{Q}{g}(a_1 - a) = Q - T_2$$

Equation of motion of the Weight R,

$\Sigma F_y = ma_y$: $$\frac{R}{g}(a_1 + a) = T_2 - R$$

Next consider the equilibrium of the pulley B neglecting, friction and inertia.

$\Sigma F_y = 0$: $$T_1 - 2T_2 = 0$$
$$T_1 = 2T_2$$

Substituting the numerical values of P, Q and R in above equations and simplifying, we get

$$30a = (30 - T_1)g \quad ...(i)$$
$$20(a_1 - a) = (20 - T_2)g \quad ...(ii)$$
$$10(a_1 + a) = (T_2 - 10)g \quad ...(iii)$$
$$T_1 = 2T_2 \quad ...(iv)$$

We have four equations with four unknowns T_1, T_2, a and a_1 and have to solve for a and a_1.

Substituting, $$T_1 = 2T_2 \text{ in equation (i)} \quad ...(v)$$
$$30a = (30 - 2T_2)g$$

Eliminating T_2 from (*ii*) and (*v*)
$$70a - 40a_1 = -10g \quad ...(vi)$$

Eliminating T_2 from (*ii*) and (*iii*)
$$30a_1 - 10a = 10g \quad ...(vii)$$

Solving (*vi*) and (*vii*) for a and a_1
$$a = \frac{g}{17},\ a_1 = \frac{6}{17}g$$

Acceleration of $$P = a = \frac{g}{17} \downarrow \quad \textit{Ans.}$$

Acceleration of $$Q = a_1 - a = \left(\frac{6}{17} - \frac{1}{17}\right)g = \frac{5}{17}g \downarrow \quad \textit{Ans.}$$

Acceleration of $$R = (a_1 + a) = \left(\frac{6}{17} + \frac{1}{17}\right)g = \frac{7}{17}g \uparrow \quad \textit{Ans.}$$

Example 14.26. A rear wheel drive automobile of mass M is shown in Fig. 14.31. Find the maximum acceleration it can attain on a level road if the coefficient of friction between the types and the road is μ.

Solution: Since the automobile has a rear wheel drive, the friction force F shall act on the rear wheels as shown in Fig. 14.31. The other forces acting on the automobile are the reactions R_A and R_B, force Mg and the inertia force $M\ddot{x}$ where, $\ddot{x}$ is the acceleration of automobile.

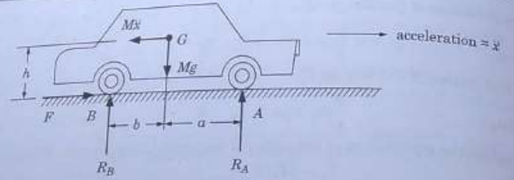


Fig. 14.31

Writing the equations of equilibrium of the automobile

$\Sigma F_x = 0$: $\quad F - M\ddot{x} = 0$

as $\quad F = \mu R_B$

$$\mu R_B - M\ddot{x} = 0$$

$$R_B = \frac{M\ddot{x}}{\mu} \qquad ...(i)$$

Taking moments about A

$\Sigma M_A = 0$: $\quad Mg(a) + M\ddot{x}(h) - R_B(a + b) = 0$

$$R_B = \frac{Mga}{(a+b)} + \frac{Mh\ddot{x}}{(a+b)} \qquad ...(ii)$$

Taking moments about B

$\Sigma M_B = 0$: $\quad R_A(a + b) + M\ddot{x}(h) - Mg(b) = 0$

$$R_A = \frac{Mgb}{(a+b)} - \frac{Mh\ddot{x}}{(a+b)} \qquad ...(iii)$$

Although we do not require the value of R_A but, it has been calculated to show that $R_B > R_A$.

That is, the reaction at the rear wheels is more than that on the front wheels.

To solve for $\ddot{x}$, equate (i) and (ii)

$$\left[\frac{Mga}{(a+b)} + \frac{Mh\ddot{x}}{(a+b)}\right] = \frac{M\ddot{x}}{\mu}$$

$$(ga + h\ddot{x})\mu = (a + b)\ddot{x}$$

$$\ddot{x} = \left(\frac{\mu ag}{a+b-\mu h}\right) \quad \textit{Ans.}$$

Example 14.27. A block of weight W is constrained to slide vertically in frictionless guides. It is in contact with a cam ABC by a weightless rod EF and a follower F. The cam moves horizontally with a constant speed v. Find the greatest speed v that the cam may have without losing contact with the cam follower F. The face of the cam has the shape of a full cosine wave of length l and maximum height h as shown in Fig. 14.32.

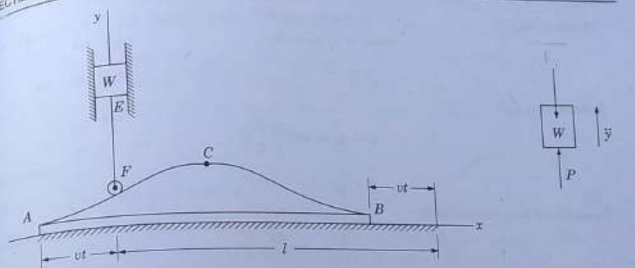


Fig. 14.32

Solution: Let P be the vertical component of the pressure between the cam and the follower transmitted by the rod EF on the weight $W \cdot \ddot{y}$ be the acceleration of the weight W in vertical direction.

Equation of motion of the weight W is

$\Sigma F_y = m\,a_y$: $\quad \frac{W}{g}\ddot{y} = P - W$

$$P = W\left(1 + \frac{\ddot{y}}{g}\right)$$

If the cam follower is not to loose contact with the cam, pressure P must be positive.

For P be positive, $\ddot{y}/g$ should be more than -1.0 or the magnitude of $\ddot{y}$ should be less than g or $\ddot{y} \le g$.

To evaluate $\ddot{y}$ we should determine the equation of the cam.

Let the equation of the cam be

$$y = a + b\cos kx$$

with the coordinate axes passing through the lowest position of the cam as shown.

Let us evaluate the constants a, b and k.

Applying the conditions

when, $\quad x = 0,\ y = 0$ at the beginning ...(i)

$x = l,\ y = 0$ at the end ...(ii)

$x = \frac{l}{2},\ y = h$ at the middle ...(iii)

Substituting (i) $\quad 0 = a + b\cos 0°$

$$b = -a$$

Equation of the cam becomes $\quad y = a(1 - \cos kx)$

Substituting (*ii*)

$$0 = a(1 - \cos kx)$$

$$\cos kl = 1,\ kl = 2\pi$$

$$k = \frac{2\pi}{l}$$

Substituting (*iii*)

$$h = a\left(1 - \cos\frac{2\pi}{l}\frac{l}{2}\right)$$

$$a = \frac{h}{2}$$

Equation of the cam becomes

$$y = \frac{h}{2}\left(1 - \cos\frac{2\pi}{l}x\right)$$

The cam is moving with velocity v, so at any time t,

$$x = vt$$

$$y = \frac{h}{2}\left(1 - \cos\frac{2\pi}{l}vt\right)$$

Above equation is also the vertical displacement-time equation of the cam follower and the block. Differentiating twice

$$y = \frac{h\pi v}{l}\sin\frac{2\pi}{l}vt$$

$$\ddot{y} = \frac{2\pi^2 h v^2}{l^2}\cos\frac{2\pi}{l}vt$$

The maximum negative value of $\ddot{y}$ (in downwad direction) is,

$$(\ddot{y})_{\max} = -\frac{2\pi^2 h v^2}{l^2}$$

when, $\cos\frac{2\pi}{l}vt = -1$

Or when $vt = l/2$

If the cam is not to loose contact, $\ddot{y} \le g$.

Therefore, $\frac{2\pi^2 h v^2}{l^2} \le g$

$$v \le \sqrt{\frac{gl^2}{2\pi^2 h}} \quad Ans.$$

Example 14.28 A particle of mass m falls vertically from rest in a medium whose resistance is proportional to the velocity. Determine the velocity and the distance travelled by the particle after a time '*t*'.

Solution: Equation of motion of the particle is

$\Sigma F = ma$: $mg - R = ma$

The resistance R to the motion of the particle is $R = kv$, where k is a constant

$$mg - kv = ma$$

$$mg - kv = m\frac{dv}{dt} \quad \left(a = \frac{dv}{dt}\right)$$

Or $$\frac{dv}{g - \frac{kv}{m}} = dt$$

Integrating,

$$-\frac{m}{k}\log_e\left(g - \frac{k}{m}v\right) = t + c_1 \quad ...(i)$$

Initial condition of motion, at $t = 0$, velocity $v = 0$

$$-\frac{m}{k}\ln(g) = 0 + c_1$$

$$c_1 = -\frac{m}{k}\ln g$$

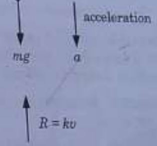


Fig. 14.33

Substituting in equation (*i*)

$$-\frac{m}{k}\ln\left(g - \frac{k}{m}v\right) = t - \frac{m}{k}\ln g$$

$$-\frac{m}{k}\left[\ln\left(g - \frac{k}{m}v\right) - \ln g\right] = t$$

$$\ln\left(\frac{g - \frac{k}{m}v}{g}\right) = -\frac{k}{m}t$$

$$1 - \frac{kv}{mg} = e^{\left(-\frac{k}{m}\right)t}$$

$$\left[1 - e^{\left(-\frac{k}{m}\right)t}\right] = \frac{kv}{mg}$$

Or $$v = \frac{mg}{k}\left[1 - e^{\left(-\frac{k}{m}\right)t}\right]$$

Above equation gives the velocity-time relation. After a very long time $t = \infty$, velocity V attained

$$V = \frac{mg}{k}$$

Further, $$v = \frac{dx}{dt} = \frac{mg}{k}\left[1 - e^{\left(-\frac{k}{m}\right)t}\right]$$

$$dx = \frac{mg}{k}\left[1 - e^{\left(-\frac{k}{m}\right)t}\right]dt$$

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Integrating,
$$x = \frac{mg}{k}\left[t + \frac{m}{k}e^{\left(-\frac{k}{m}\right)t}\right] + c_2$$

At $t = 0$, $x = 0$
$$c_2 = -\frac{m^2 g}{k^2}$$

Substituting in (*ii*),
$$x = \frac{mg}{k}\left[t + \frac{m}{k}e^{\left(-\frac{k}{m}\right)t}\right] - \frac{m^2 g}{k^2}$$

Above equation given displacement-time relation.

It is also possible to obtain a displacement – velocity relation by a similar procedure.

Example 14.29. A car weighing 4000 N is moving at a speed of 100 m/s. The resistance to the car is largely due to air drag which is equal to 0.004 v^2. What distance will it travel before its speed is reduced to 50 m/s?

Solution: Equation of motion of the car is

$\Sigma F = ma$:

$$-0.004\, v^2 = \left(\frac{4000}{g}\right)a$$
$$a = v\frac{dv}{dx}$$
$$-0.004\, v^2 = \left(\frac{4000}{9.81}\right)v\frac{dv}{dx}$$
$$\frac{dv}{v} = -9.81 \times 10^{-6}dx$$

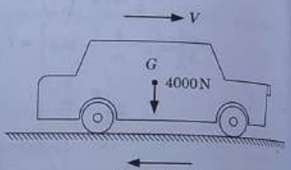

Fig. 14.34

Integrating, $\ln v = -9.81 \times 10^{-6}x + c$

Using, at $x = 0$, $v = 100$ m/s

$$\ln 100 = -9.81 \times 10^{-6} \times (0) + c$$
$$c = \ln 100$$
$$\ln v = -9.81 \times 10^{-6}x + \log 100$$

Distance travelled when $v = 50$ m/s is,
$$x = \frac{\ln 100 - \ln 50}{9.81 \times 10^{-6}}$$
$$x = 70657.2 \text{ m.}$$

Example 14.30 A bar weighing 1 kg/m is bent at right angles into segments $AB = 0.6$ m and $BC = 0.30$ m long. It takes the position as shown in Fig. 14.35 when the frame F to which it is pinned at A is accelerated horizontally. Determine the acceleration and the reaction at A.

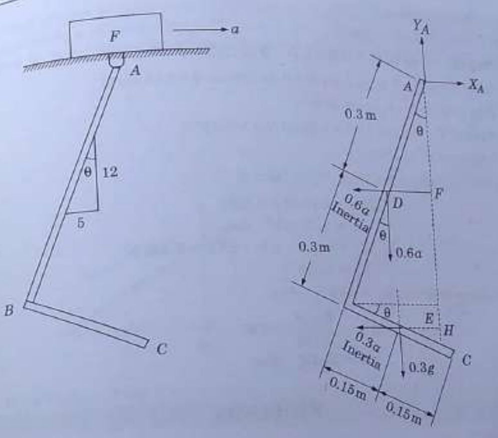

Fig. 14.35

Solution: Let a' be the acceleration of the frame.

$$\tan\theta = \frac{5}{12},\ \sin\theta = \frac{5}{13} = 0.3846,\ \cos\theta = \frac{12}{13} = 0.923$$

Mass of the segment $AB = 0.6 \times 1 = 0.6$ kg acting at D

Mass of the segment $BC = 0.3 \times 1 = 0.3$ kg acting at E

Free body diagram of the bar ABC is shown above. Various forces acting are:

(1) Reactions at pin A, X_A, Y_A

(2) Weight of segment $AB = 0.6\, g$ at D

(3) Inertia force on segment $AB = 0.6a$ at D

(4) Weight of the segment $BC = 0.3g$ at E

(5) Inertia force on segment $BC = 0.3a$ at E

Writing the equations of dynamic equilibrium

$\Sigma F_x = 0$: $X_A - 0.6a - 0.3a = 0$ or $X_A = 0.9a$

$\Sigma F_y = 0$: $Y_A - 0.6g - 0.3g = 0$ $Y_A = 0.9g = 8.829$ N

Note here reaction $X_A = 0.9a$. When the frame is stationary $X_A = 0$

Using moment equation,

$\Sigma M_A = 0$:

$$0.6a\,(AF) - 0.6g\,(DF) + 0.3a\,(AH) - 0.3g\,(IG) = 0$$

$$0.6a\,(0.3\cos\theta) - 0.6g\,(0.3\sin\theta) + 0.3a\,(0.6\cos\theta + 0.15\sin\theta)$$
$$- 0.3g\,(0.6\sin\theta - 0.15\cos\theta) = 0$$

$$0.6a\,(0.277) - 0.6g\,(0.115) + 0.3a\,(0.554 + 0.0577)$$
$$- 0.3g\,(0.231 - 0.138) = 0$$

$$0.166a - 0.069g + 0.1836a - 0.0279g = 0$$

$$0.35a = 0.0969\,g = 0.95$$

$$a = 2.72 \text{ m/s}^2 \quad \textit{Ans.}$$

$$X_A = 0.9a = 0.9 \times 2.72 = 2.448 \text{ N}$$

$$Y_A = 8.829 \text{ N}$$

Total resultant reaction $R_A = 9.15$ N *Ans.*

$$\tan\alpha = \frac{8.829}{2.448} = 3.6$$

$$\alpha = 74.49° \quad \textit{Ans.}$$

PROBLEMS

14.18. A elevator has an upward acceleration of 1 m/s^2 What pressure will be transmitted to the floor of the elevator by man weighing 600 N travelling in the elevator? What pressure will be transmitted if the elevator has an downward acceleration of 2 m/s^2?

Also, find the upward acceleration of the elevator which would cause the man to exert a pressure of 1200 N on the floor. [723.3 N, 477.6 N, 9.81 m/s^2]

14.19. A mass M resting on a smooth table is connected to masses M_1 and M_2 by strings as shown in Fig. P.14.19. Find the acceleration of the systems.

$$\left[\frac{(M_1 - M_2)g}{(M_1 + M_2 + M)}\right]$$

Fig. P.14.19

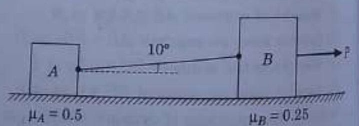

Fig. P.14.20

14.20. Two blocks A and B of masses 5 kg and 20 kg are connected by an inclined string. A horizontal force P of 100 N is applied to the block B as shown in Fig. P. 14.20. Calculate the tension in the string and the acceleration of the system. Assume the coefficient of friction between the plane and the blocks A and B to be 0.5 and 0.25 respectively. [28 N, 1.11 m/s^2]

14.21. A block of mass M_1 lying on an inclined plane of angle θ is pulled up by an another block M_2 connected by a string as shown. The coefficient of friction between the inclined plane and the block M_1 is μ. Find (*a*) the acceleration of the mass M_2 and the tension in the string.

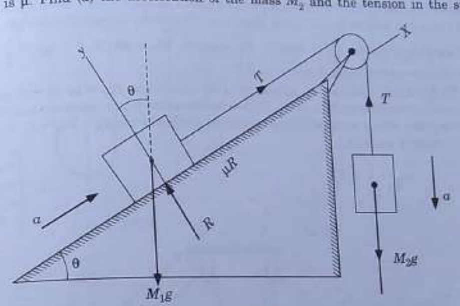

Fig. P.14.21

14.22. Three blocks A, B and C are connected as shown in Fig. P.14.22. Find acceleration of the masses and the tension T_1 and T_2 in the strings. Given, $\mu_1 = 0.2$ and $\mu_2 = 0.25$.

$$\begin{bmatrix} T_1 = 103.8\text{N} \\ T_2 = 32.8\text{N} \\ a = 4.6\text{ m/s}^2 \end{bmatrix}$$

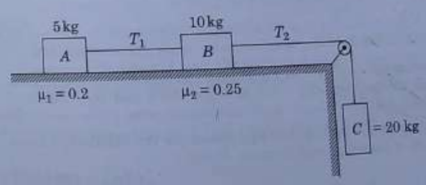

Fig. P.14.22

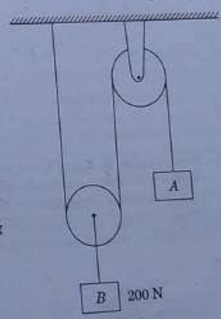

Fig. P.12.23

14.23. In a system of connected bodies (Fig. P. 14.23) the pulleys are frictionless and of negligible weight. Determine the value of weight A required to give 0.6 g acceleration to weight B (*i*) in downward (*ii*) in upward, directions.

[**Hint:** Weight A cannot have acceleration more than 'g' in downward direction.

[(i) 18.18 N (ii) Not possible]

14.24. A system of blocks A, B and C are connected as shown in Fig. P.14.24. Determine the acceleration of each block and tension in the string. Assume pulleys to be of negligible weight and frictionless.

Given, weights of blocks $W_A = 1000$ N, $W_B = 800$ N, $W_C = 400$ N, $\theta = \tan^{-1}\frac{3}{4}$. The coefficient of kinetic friction under block A and C is 0.2.

[**Hint:** To determine whether W_A moves up or down, assume W_B to be at rest and calculate the unbalanced forces acting on W_A and W_C. It shall be found that W_A moves down and W_C up. Kinematic relation between acceleration of blocks can be determined from $x_A + 2x_B + x_C$ = constant, as the length of string is constant.]

$$\begin{bmatrix} a_A = 4.05\ \text{m/s}^2 \\ a_B = 0.58\ \text{m/s}^2 \\ a_C = 1.777\ \text{m/s}^2 \\ T = 376\ \text{N} \end{bmatrix}$$

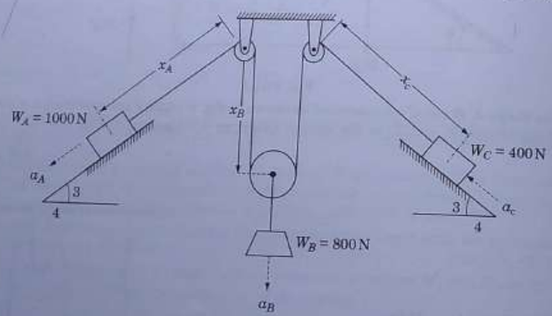

Fig. P.14.24

14.25. Three weights A, B and C are connected as shown in Fig. P.14.25. Determine the acceleration of each weight and tension in the string Given, $W_A = 150$ N, $W_B = 450$ N and $W_C = 300$ N. Length of string is constant.

[**Hint:** $x_A + 2x_B + x_C$ = constant. Assume B to be at rest and determine the unbalanced forces on A and C. It shall be found that A moves up and C moves down]

$$\begin{pmatrix} a_A = 4.05\ \text{m/s}^2 \\ a_B = 0.58\ \text{m/s}^2 \\ a_C = 2.88\ \text{m/s}^2 \\ T = 212\ \text{N} \end{pmatrix}$$

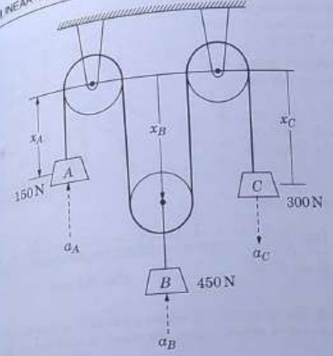

Fig. P.14.25

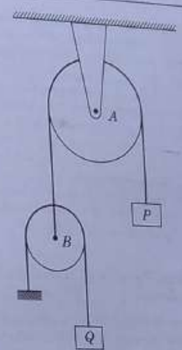

Fig. P.14.26

14.26. Neglecting friction and the inertia of the pulleys find the acceleration of the block Q if the mass of the block P equals the mass of the block Q (Fig. P.14.26)

$$\left[\frac{2g}{5}\right]$$

14.27. Two masses $M_A = 10$ kg and $M_B = 20$ kg are connected by a bar of negligible mass. Find the acceleration of the system when it slides down an inclined plane of inclination 30° as shown in Fig. P.14.27. Also find the force in bar. Assume $\mu_a = 0.15$ and $\mu_b = 0.3$. [0.283 g, 8.53 N (compression)]

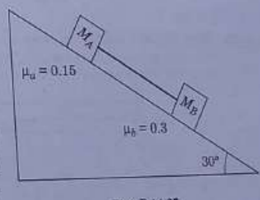

Fig. P.14.27

14.28. A block of mass 30 kg is resting on a horizontal table 1.5 m from its edge. The block A is attached to string whose other end is carrying a body B of mass 3 kg. If the coefficient of friction between the block A and the table is 0.06 find the acceleration of the system and the time required to fall over the edge (Fig. P.14.28). [0.357 m/s^2, 2.9 s]

14.29. A rear wheel drive automobile has a mass of 2000 kg. Find the maximum possible acceleration of this automobile on a level road if the coefficient of friction between the tires and the road is 0.1. [0.665 m/s^2]

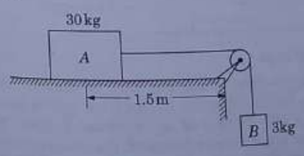

Fig. P.14.28

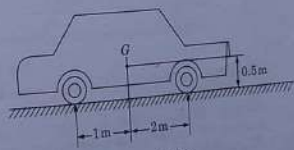

Fig. P.14.29

14.30. A block of mass M, height $2h$ and width $2b$ rests on a flat trolley which moves on a horizontal surface with constant acceleration (Fig. P.14.30). Determine,

(a) The acceleration of the trolley at which the slipping of the block on the trolley will just occur. Assume the coefficient of fiction to be μ.

(b) The acceleration at which the block just tips about the edge A. Assume that there is sufficient friction between the block and the trolley to prevent slipping.

(c) The maximum possible acceleration of the trolley so that the block is not disturbed. Assume $\mu = 0.5$ and $b = 2/3h$.

$$\left[\mu g, \frac{bg}{h}, \frac{g}{2}\right]$$

[**Hint:** For part (c), taken the least value of the acceleration.]

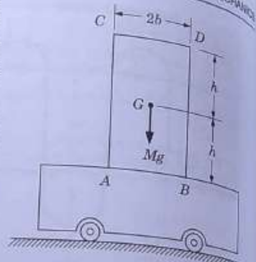

Fig. P.14.30

14.31. A car of weight 40 kN is moving at 100 m/s. The resistance to motion is mainly a due to air drag which is equal to $(0.004)v^2$. What distance will it travel before its speed is reduced to 50 m/s? [70657 m]

$$\left[\textbf{Hint. } \Sigma F = ma : (-004v^2) = \left(\frac{4000}{g}\right)\left(\frac{dv}{dx}v\right). \text{ Integrate to get distance } x\right]$$

14.32. The frame of a machine member accelerates to the right at $0.8g$ m/s^2 as shown in Fig. P.14.32. It carries a uniform bent bar ABC weighing 50 N/m pinned to it at C and braced by the uniform strut DE which weighs 100 N. Determine the components of force at pin D.

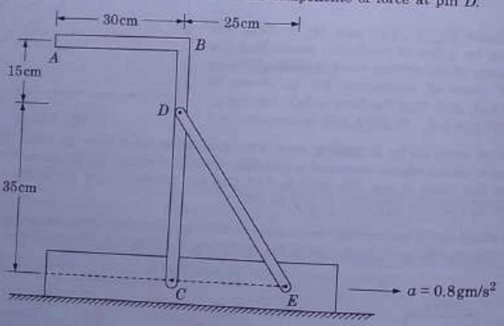

Fig. P.14.32

[**Hint:** The problem is to be solved using the method of analysis used for frames in chapter 9 section 9.5. Each member is placed in dynamic equilibrium by applying inertia forces acting through CG of each part in a direction opposite to that of acceleration. Further, instead of locating CG for ABC, it is more convenient to apply inertia forces on the two segments AB and BC of its length. Gravity forces or weights of members are also to be included. So it reduces to a problem of statics with interia and gravity forces as external forces.]

15 CHAPTER Curvilinear Motion of a Particle

PART A: KINEMATICS

15.1 INTRODUCTION

When a moving particle describes a curved path it said to have a curvilinear motion. If the curved path lies in a plane it is termed as plane curvilinear motion. When the direction of the force acting on a particle varies or when the particle has some initial motion in a direction that does not coincide with the direction of the force acting on the particle, the particle moves in a curved path. For example, an object when thrown horizontally with some initial velocity moves in a curved parabolic path, because the force of gravity acting on the object does not coincide with the initial velocity of the object and the object moves in a curved path.

15.2 POSITION VECTOR, VELOCITY AND ACCELERATION

Position Vector. To define the position of a particle moving along a curved path at any instant, we need to know its coordinates along the x-axis as well as the y-axis.

Alternatively, we can use the concept of the position vector as described below.

Consider the motion of a particle along a curved path as shown in Fig. 15.1. To define the motion of the particle P at any instant t choose a fixed reference axes x-y. Joint the point P to O. The line OP is called the position vector $\mathbf{r}$ of the point P. Since the vector $\mathbf{r}$ is characterized by its magnitude r and direction with reference to axes, it completely defines the position of the particle with position of the particle with respect to these axes at any time t.

Consider now the position P' of the particle, at a later time $t + \Delta t$ as defined by the position $\mathbf{r}'$.

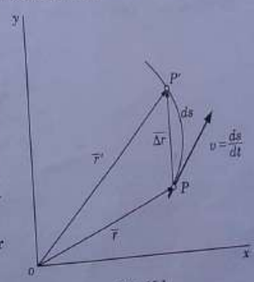

Fig. 15.1

The vector $\Delta\mathbf{r}$ joining P and P' represents the change in the position vector $\mathbf{r}$ during the time interval Δt. This can be verified by applying the triangle law as,

$$\mathbf{r} + \Delta\mathbf{r} = \mathbf{r}'$$

Or

$$\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$$

$\Delta\mathbf{r}$ thus represents a change in the magnitude as well as the change in the direction of the position vector $\mathbf{r}$.

Velocity. Instantaneous velocity of the particle can be defined as

$$\mathbf{v} = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\mathbf{r}}{\Delta t}$$

As Δt and $\Delta\mathbf{r}$ become smaller, the points P and P' get closer and the vector $\mathbf{v}$ obtained at the limit becomes tangent to the part at P. The magnitude of $\Delta\mathbf{r}$ is given by the length of the line segment PP'. But as Δt approaches zero, the length of the line segment PP' approaches the length Δs of the arc PP'.

The magnitude of the velocity v (called speed) is thus obtained as

$$v = \underset{\Delta t \to 0}{\text{Limit}} \frac{PP'}{\Delta t} = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

The speed of the particle can thus be obtained by differentiating with respect to the time the length of the arc described by the particle.

$$v = \frac{ds}{dt} \qquad \text{...(15.1)}$$

15.3 COMPONENTS OF MOTION: RECTANGULAR COMPONENTS

Rectangular Components of Velocity

As the direction of the velocity of a particle in curvilinear motion changes continuously so it is convenient to deal with its components $\mathbf{v}_x$ and $\mathbf{v}_y$.

To obtain the values of $\mathbf{v}_x$ and $\mathbf{v}_y$ let us resolve $\Delta\mathbf{r}$ into components $\overline{PQ}$ and $\overline{QP'}$ parallel to the x and the y axes as shown in Fig. 15.2.

$$\Delta\mathbf{r} = \overline{PQ} + \overline{QP'}$$

$$\mathbf{v} = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\mathbf{r}}{\Delta t}$$

$$= \underset{\Delta t \to 0}{\text{Limit}} \frac{\overline{PQ}}{\Delta t} + \underset{\Delta t \to 0}{\text{Limit}} \frac{\overline{QP'}}{\Delta t}$$

$$\mathbf{v} = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\mathbf{x}}{\Delta t} + \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\mathbf{y}}{\Delta t}$$

$$\mathbf{v} = \mathbf{x}_v + \mathbf{v}_y \quad \text{(vector sum)} \qquad \text{...(15.2a)}$$

as

$$\frac{dx}{dt} = v_x = \dot{x}, \quad \frac{dy}{dt} = v_y = \dot{y}$$

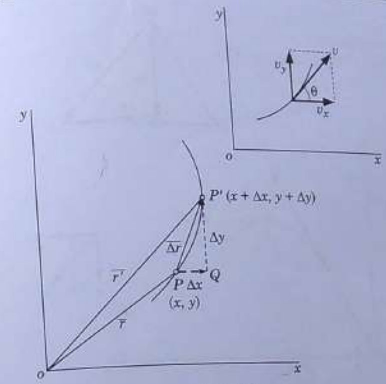

Fig. 15.2

Magnitude,

$$v = \sqrt{v_x^2 + v_y^2} \qquad \text{...(15.2b)}$$

Direction,

$$\theta = \tan^{-1}\frac{v_y}{v_x} \qquad \text{...(15.2c)}$$

Acceleration of the Particle. Consider a particle at the position P having a velocity $\mathbf{v}$ at any time t. After a time Δt its position be defined by P' and its velocity by $\mathbf{v} + \Delta\mathbf{v}$ (Fig. 15.3).

Instantaneous Acceleration of the particle

$$\mathbf{a} = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{dv}{dt} \qquad \text{...(15.3)}$$

It may be emphasized here that in general, the acceleration of the particle at any instant is not tangential to the path of the particle. *That is, the direction of acceleration and velocity may not be the same in a curvilinear motion.*

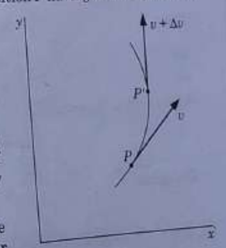

Fig. 15.3

Rectangular Components of Acceleration. Let the acceleration of the particle be denoted by its rectangular components $\mathbf{a}_x$ and $\mathbf{a}_y$ parallel to the x and the y-axis respectively (Fig. 15.4).

Vectors $\overline{pq}$ and $\overline{pr}$ represent the velocities $\mathbf{v}$ and $\mathbf{v}'$ and the vector $\overline{qr}$ the change in the velocity $\Delta\mathbf{v}$ of the particle.

Resolving $\Delta\mathbf{v}$ into components $\Delta\mathbf{v}_x$ and $\Delta\mathbf{v}_y$, we can write

$$\Delta\mathbf{v} = \overline{qr} = \overline{qs} + \overline{sr} \quad \text{(vector sum)}$$

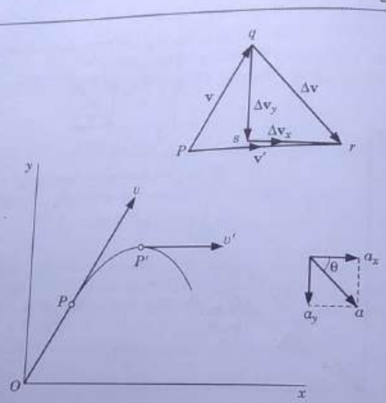

Fig. 15.4

Acceleration $\quad \mathbf{a} = \dfrac{\Delta \mathbf{v}}{\Delta t} = \underset{\Delta t \to 0}{\text{Limit}} \dfrac{\overline{qs}}{\Delta t} + \underset{\Delta t \to 0}{\text{Limit}} \dfrac{\overline{sr}}{\Delta t}$

$$\mathbf{a} = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta v_y}{\Delta t} + \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta v_x}{\Delta t}$$

But $\quad a_x = \underset{\Delta t \to 0}{\text{Limit}} \dfrac{\Delta v_x}{\Delta t}$ and $a_y = \underset{\Delta t \to 0}{\text{Limit}} \dfrac{\Delta v_y}{\Delta t}$

so, $\quad \mathbf{a} = \mathbf{a}_y + \mathbf{a}_x$

or $\quad \mathbf{a} = \mathbf{a}_x + \mathbf{a}_y$ (vector sum) ...(15.4*a*)

Magnitude of the acceleration

$$a = \sqrt{a_x^2 + a_y^2} \qquad ...(15.4b)$$

Direction, $\quad \theta = \tan^{-1} \dfrac{a_y}{a_x}$ ...(15.4*c*)

15.4(*A*) COMPONENTS OF ACCELERATION: NORMAL AND TANGENTIAL

In the case of a particle moving in a curvilinear path, we observed that the velocity of the particle is a vector tangential to the path at any instant. But, in general, the acceleration need not be tangential to the path.

Therefore, sometimes it is convenient to express the acceleration of the particle in component form; one in the direction of the tangent to the path and the other in direction of normal to the

path. These component are called tangential acceleration ($\mathbf{a}_t$) and normal acceleration ($\mathbf{a}_n$) of the particle (15.5).

Consider that the particle has a velocity **v** at a time t and a velocity **v**' at a later time $t + \Delta t$.

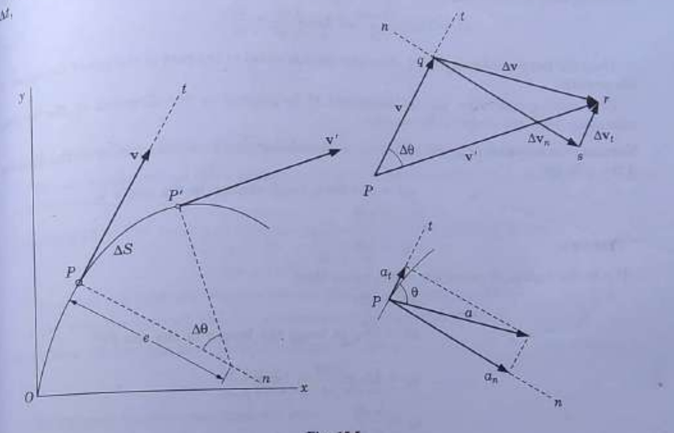

Fig. 15.5

To obtain the change in the velocity of the particle draw $\overline{pq}$ and $\overline{pr}$ representing **v** and **v**' respectively. The closing side $\overline{qr}$ of the triangle represents the change in the velocity $\Delta\mathbf{v}$ in time Δt.

Resolve $\Delta\mathbf{v}$ into components, $\Delta\mathbf{v}_t$ and $\Delta\mathbf{v}_n$, *along the tangent and normal to the path at P.* The axes in these directions be denoted by t and n

$$\Delta\mathbf{v} = \overline{qr} = \overline{qs} + \overline{sr}$$

$$\Delta\mathbf{v} = \Delta\mathbf{v}_n + \Delta\mathbf{v}_t \quad \text{(vector sum)}$$

Acceleration $\quad \mathbf{a} = \underset{\Delta t \to 0}{\text{Limit}} \dfrac{\Delta \mathbf{v}}{\Delta t}$

Or, $\quad \mathbf{a} = \underset{\Delta t \to 0}{\text{Limit}} \dfrac{\Delta \mathbf{v}_t}{\Delta t} + \text{Limit} \dfrac{\Delta \mathbf{v}_n}{\Delta t}$

But $\quad a_t = \underset{\Delta t \to 0}{\text{Limit}} \dfrac{\Delta v_t}{\Delta t}$ and $a_n = \underset{\Delta t \to 0}{\text{Limit}} \dfrac{\Delta \mathbf{v}_n}{\Delta t}$ ...(15.5)

Therefore, $\quad \mathbf{a} = \mathbf{a}_t + \mathbf{a}_n$ (vector sum)

It may be observed that as Δt approaches zero, the point P' coincides with the point P and the directions of a_n and a_t coincide with the direction of tangent and normal to the path at P.

Tangetial Component ($\mathbf{a}_t$). It can be noted that $\overline{sr}$ represents the change in the magnitude of the velocity **v**, therefore,

$$a_t = \underset{\Delta t \to 0}{\text{Limit}} \frac{(v'-v)}{\Delta t} = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \quad ...(15.6)$$

Thus the tangential component of acceleration is equal to the rate of change of the speed of the particle.

Tangential acceleration ($\mathbf{a}_t$) *is considered to be positive in the direction of the tangent coinciding with the sense of the motion.*

Normal Acceleration ($\mathbf{a}_n$). Also it may be observed that $\overline{qs}$ represents a change in the direction of the velocity.

$$qs \approx v\Delta\theta \text{ for a small change in the angle } \theta$$

$$\Delta v_n \approx v\Delta\theta$$

Therefore,
$$a_n = \underset{\Delta t \to 0}{\text{Limit}} \frac{v\Delta\theta}{\Delta t}$$

If ρ is the radius of curvature of the curve then,

$$\Delta s = \rho\Delta\theta$$

$$\Delta\theta = \frac{\Delta s}{\rho}, \ \Delta s \text{ being the length of the arc } PP'$$

$$a_n = \underset{\Delta t \to 0}{\text{Limit}} \frac{v\Delta\theta}{\Delta t} = \underset{\Delta t \to 0}{\text{Limit}} \frac{v}{\rho}\frac{\Delta s}{\Delta t}$$

$$a_n = \frac{v}{\rho}\frac{ds}{dt}$$

But
$$\frac{ds}{dt} = v$$

so,
$$a_n = \frac{v^2}{\rho} \quad ...(15.7)$$

Normal acceleration a_n of a particle at a point is equal to the square of its speed divided by the radius of curvature of the path at that point.

The direction of the normal acceleration is such that it is always directed towards the centre of curvature of the path. This normal acceleration is also called as the centripetal acceleration (centre-seeking acceleration).

It should be noted that the total acceleration **a** of the particle is a vector. This acceleration may be caused due to change in the magnitude of the velocity or change in the direction of the velocity or both.

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_n \quad \text{(vector sum)} \quad ...(15.8a)$$

where,
$$a_t = \frac{dv}{dt}, \text{ and } a_n = \frac{v^2}{\rho}$$

The magnitude
$$a = \sqrt{a_t^2 + a_n^2} \quad ...(15.8b)$$

and the direction
$$\theta = \tan^{-1}\frac{a_n}{a_t} \quad ...(15.8c)$$

To explain the above concepts, consider the motion a particle along a circular path of radius r with a constant speed v.

$$a_n = \frac{v^2}{\rho} = \frac{v^2}{r} (\rho = r)$$

$$a_t = \frac{dv}{dt} = 0 \ (v = \text{constant})$$

$$a = \sqrt{a_t^2 + a_n^2} = a_n$$

$$\theta = \tan^{-1}\frac{v^2/r}{0} = 90°$$

The total acceleration **a** of the particle is equal its normal acceleration a_n and acts in the same direction as a_n.

Example 15.1. The motion of a particle is described by the following equation,

$$x = 2(t+2)^2$$
$$y = 2(t+1)^{-2}$$

Show that the path travelled by the particle is a rectangular hyperbola. Find also, the velocity and the acceleration of the particle at $t = 0$.

Solution: To find the path travelled, eliminate t from

$$x = 2(t+1)^2 \quad ...(i)$$
$$y = 2(t+1)^{-2} \quad ...(ii)$$

Multiplying the two equations, we get

$$xy = 4$$

which represents a rectangular hyperbola.

Displacement-time relation in x-direction is

$$x = 2(t+1)^2$$

The component of the velocity in the x-direction

$$v_x = \frac{dx}{dt} = 2 \times 2(t+1)$$

The component of the acceleration in the x-direction

$$a_x = \frac{d^2x}{dt^2} = 2 \times 2 = 4 \text{ m/s}^2$$

When
$$t = 0$$
$$v_x = 2 \times 2(0+1) = 4 \text{ m/s}$$
$$a_x = 4 \text{ m/s}^2$$

Displacement-time relation in y-direction

$$y = \frac{2}{(t+1)^2}$$

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The component of the velocity in the y-direction

$$v_y = \frac{dy}{dt} = (-2)(t+1)^{-3}$$

The component of the acceleration in the y-direction

$$a_y = \frac{d^2y}{dt^2} = (-4)(-3)(t+1)^{-4}$$

When $t = 0$

$$v_y = 2(-2)(0+1)^{-3} = -4 \text{ m/s}$$

$$a_y = (-4)(-3)(0+1)^{-4} = +12 \text{ m/s}^2$$

Velocity $v = \sqrt{v_x^2 + v_y^2} = \sqrt{(4)^2 + (-4)^2}$

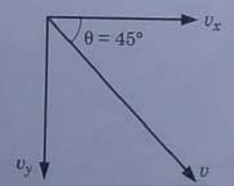
$$\tan\theta = -\frac{4}{4}$$

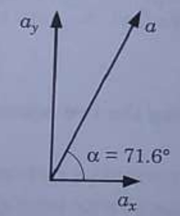
Or $v = 5.66 \text{ m/s},\ \theta = -45°$ *Ans.*

Acceleration $a = \sqrt{a_x^2 + a_y^2} = \sqrt{(4)^2 + (12)^2}$

$$\tan\alpha = \frac{12}{4}$$

Or $a = 12.65 \text{ m/s}^2,\ \alpha = 71.6°$ *Ans.*





Example 15.2. A motorist is travelling on a curved road of radius 20 m at a speed of 72 km/hour. Find the normal and tangential components of acceleration.

If he applies brakes to slow down his car uniformly to a speed of 36 km/hour in 10 seconds, find the normal and the tangential components of deceleration just after the brakes are applied.

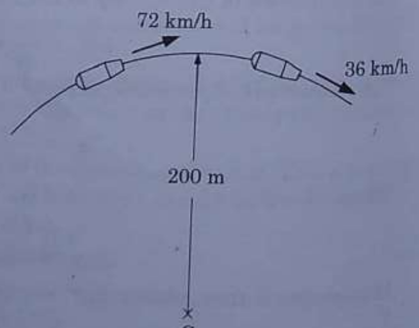


Fig. 15.6

Solution.

$$72 \text{ km/hour} = 20 \text{ m/s}$$

$$36 \text{ km/hour} = 10 \text{ m/s}$$

When travelling at a constant speed of

$$v = 20 \text{ m/s}$$

$$a_n = \frac{v^2}{r} = \frac{20 \times 20}{200}$$

$$a_n = 2 \text{ m/s}^2 \quad \textit{Ans.}$$

$$a_t = \frac{dv}{dt} = 0$$

$$a_t = 0 \quad \textit{Ans.}$$

At the instant when the brakes are just applied, the car has a tangential speed of 72 km/h (20 m/s) and in addition, experiences a tangential deceleration

$$a_t = \frac{dv}{dt} = \frac{\Delta v}{\Delta t} = \frac{20-10}{10} = 1 \text{ m/s}^2 \ (dt = 10 \text{ s})$$

$$a_t = 1 \text{ m/s}^2 \quad \textit{Ans.}$$

And $a_n = \frac{v^2}{r} = \frac{(20)^2}{200}$

$$a_n = 2 \text{ m/s}^2 \quad \textit{Ans.}$$

Example 15.3. A car starts from rest on a curved road of 250 m radius and accelerates at a constant tangential acceleration of 0.6 m/s^2. Determine the distance and the time for which that car will travel before the magnitude of the total acceleration attained by it becomes 0.75 m/s^2.

Solution: Total acceleration,

$$a = 0.75 \text{ m/s}^2$$

$$a_t = 0.6 \text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2}$$

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{(0.75)^2 - (0.6)^2}$$

$$a_n = 0.45 \text{ m/s}^2$$

But $a_n = \frac{v^2}{r} = 0.45 \text{ m/s}^2$

$$v^2 = \sqrt{250 \times 0.45}$$

$$v = 10.6 \text{ m/s}$$

Using, $v = u + at$

$$u = 0,\ v = 10.6,\ a = 0.6 \text{ m/s}^2$$

$$t = \frac{10.6}{0.6},\ t = 17.68 \text{ s} \quad \textit{Ans.}$$

Using, $v^2 - u^2 = 2as$

$$s = \frac{v^2}{2a} = \frac{(10.6)^2}{2 \times 0.6},\ s = 93.63 \text{ m.} \quad \textit{Ans.}$$

Example 15.4. The displacement-time equation for the oscillations of a simple pendulum is given by

$$s = S\cos\left(\sqrt{\frac{g}{l}}\right)t$$

where, S is the maximum displacement of oscillations.

Find (*i*) the maximum velocity (*ii*) the maximum tangential and normal acceleration of the bob.

Solution: It may be recalled here that to describe the displacement-time relation for a curvilinear motion in cartesian coordinates would required two equations.

A single equation of the curvilinear motion of the bob,

$$s = S\cos\left(\sqrt{\frac{g}{l}}\right)t$$

given here, is expressed in the path co-ordinate that is, displacement s measured along the curve and time.

Tangential velocity

$$v = \frac{ds}{dt}$$

$$v = -S\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}\right)t$$

$$v_{max} = \pm S\sqrt{\frac{g}{l}} \quad Ans.$$

$$a_t = \frac{dv}{dt} = -S\sqrt{\frac{g}{l}}\ \sqrt{\frac{g}{l}}\cos\left(\sqrt{\frac{g}{l}}\right)t$$

$$a_t = -S\left(\frac{g}{l}\right)\cos\left(\sqrt{\frac{g}{l}}\right)t$$

$$(a_t)_{max} = \pm S\left(\frac{g}{l}\right) \quad Ans.$$

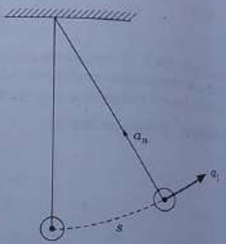


Fig. 15.7

$$a_n = \frac{v^2}{\rho} = \frac{\left(-S\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}\right)t\right)^2}{\rho}$$

Radius curvature ρ = length l of the pendulum.

$$a_n = S^2\,\frac{g}{l^2}\sin\left(\sqrt{\frac{g}{l}}\right)t$$

$$(a_n)_{max} = \frac{S^2 g}{l^2} \quad Ans.$$

Example 15.5. A car enters a curved portion of the road of radius 200 m travelling at a constant speed of 36 km/hour. Determine the components of velocity and acceleration of the car in the x and y directions 15 seconds after it has entered the curved portion of the road.

Also express the velocity and the acceleration of the car in terms of the normal and tangential components.

Solution: Speed of the car

$$v = 36 \text{ km/hour.}$$

$$v = 10 \text{ m/s}$$

Angular velocity

$$\omega = \frac{v}{r} = \frac{10}{200}; \quad \omega = \frac{1}{20} \text{ radian/s}$$

Components in the x and y directions. After a time t the angular position of the car, $\theta = \omega t$.

Coordinates of the car at any time t

$$x = r\cos\theta = r\cos\omega t$$

$$y = r\sin\theta = r\sin\omega t$$

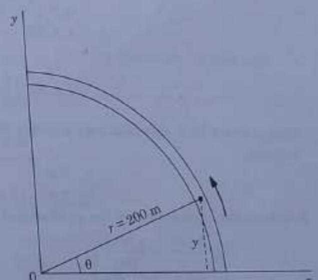


Fig. 15.8

Velocity. Velocity in the x-directions

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(r\cos\omega t)$$

$$v_x = -\omega r\sin\omega t$$

Velocity in the y-direction

$$v_y = \frac{dy}{dx} = \frac{d(r\sin\omega t)}{dt}$$

$$v_y = \omega r\cos\omega t$$

Substituting

$$t = 15 \text{ s}, \ \omega = \frac{1}{20} \text{ rad/s}$$

$$= \frac{1}{20}\times\frac{180}{\pi} \text{ degrees/s}, \ r = 200 \text{ m}$$

$$v_x = -\frac{1}{20}\times 200\left(\sin\left(\frac{1}{20}\times\frac{180}{\pi}\times 15\right)^0\right) = \overleftarrow{6.82} \text{ m/s} \quad Ans.$$

$$v_y = \frac{1}{20}\times 200\left(\cos\left(\frac{1}{20}\times\frac{180}{\pi}\times 15\right)^0\right) = 7.71\uparrow \text{ m/s} \quad Ans.$$

Acceleration. The component of acceleration in x-direction

$$a_x = \frac{dv_x}{dt} = \frac{d}{dx}(-\omega t\sin\omega t)$$

$$a_x = -\omega^2 r\cos\omega t$$

The component of acceleration in y-direction

$$a_y = \frac{dv_y}{dt} = \frac{d}{dy}(\omega r\cos\omega t)$$

$$a_y = -\omega^2 r\sin\omega t$$

Evaluating a_x, a_y at $t = 15$ s

$$a_x = -\left(\frac{1}{20}\right)^2\times 200\left(\cos\left(\frac{1}{20}\times\frac{180}{\pi}\times 15\right)^0\right) = \overleftarrow{0.365} \text{ m/s} \quad Ans.$$

$$a_y = -\left(\frac{1}{20}\right)^2\times 200\left(\sin\left(\frac{1}{20}\times\frac{180}{\pi}\times 15\right)^0\right) = 0.341\downarrow \text{ m/s}^2 \quad Ans.$$

Total acceleration $a = \sqrt{(a_x)^2 + (a_y)^2}$

$$a = \sqrt{(0.365)^2 + (0.341)^2}$$

$$a = 0.5 \text{ m/s}^2 \quad Ans.$$

Direction of acceleration a,

$$\phi = \tan^{-1}\frac{a_y}{a_x} = \frac{0.341}{0.365} = 42.97° \quad Ans.$$

Components in tangential and normal directions.

Velocity

$$v_n = 0$$

$$v_t = v = 10 \text{ m/s}$$

Acceleration. Component of the acceleration in the direction of normal to the path.

$$a_n = \frac{v^2}{r} = \frac{10 \times 10}{200}$$

$$a_n = 0.5 \text{ m/s}^2 \quad Ans.$$

Component of the acceleration in the direction tangent to the path

$$a_t = \frac{dv}{dt} = 0 \quad Ans.$$

Angular position θ of the car after 15 seconds

$$= \omega t = \frac{1}{20} \times \frac{180}{\pi} \times 15$$

$$\theta = 42.97°$$

Note: The components of acceleration are shown in the Fig. 15.9.

The total acceleration a of the car acts along the normal to the path as shown, $\phi = \theta = 42.97°$, proves that resultant of a_x and a_y lies along the normal to the path.

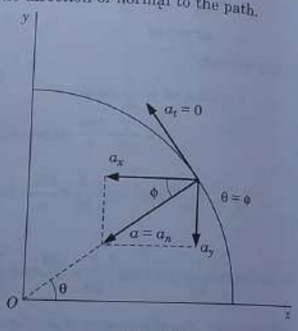


Fig. 15.9

15.4(B) COMPONENTS OF MOTION: RADIAL AND TRANSVERSE COMPONENTS

We saw that the curvilinear motion of a particle can be expressed in terms of rectangular components and also in terms of components directed along the tangent and the normal to the path of the particle.

In certain problems the position of a particle is more conveniently described by its polar coordinates. In that case then it is much simpler to resolve the velocity and acceleration of the particle into components which are parallel and perpendicular to the position vector **r** of the particle. These components are called *radial* and *transverse components*.

Consider a collar P sliding outward along a straight rod OA which itself is rotating about the fixed point O. It is much convenient to define the position of the collar P (Fig. 15.10) at any instant

in terms of its distance r from the point O and the angular position θ of the rod OA with respect to some fixed axis OX. The polar coordinates of the point P thus are (r, θ).

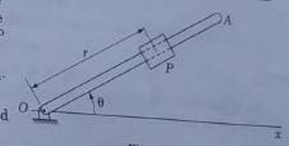


Fig. 15.10

It can be shown that the radial and transverse components of the velocity are,

$v_r = \dot{r}$ (Radial component of velocity is directed along the position vector **r**)

$v_\theta = r\dot{\theta}$ (Transverse component of velocity is directed along the normal to the position vector **r**)

Total velocity **v** [Fig. 15.11(a)].

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_\theta \text{ (vector sum)}$$

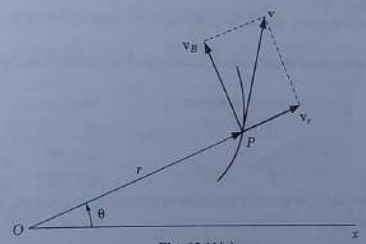


Fig. 15.11(a)

The radial and transverse components of acceleration are,

$a_r = \ddot{r} - r(\dot{\theta})^2$ (Radial component of acceleration is directed along the position vector **r**)

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ (Transverse component of acceleration is directed along the normal to the position vector **r**)

Total acceleration **a** [Fig. 15.11(b)]

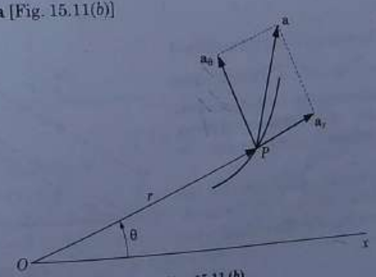


Fig. 15.11 (b)

$$\mathbf{a} = \mathbf{a}_r + \mathbf{a}_\theta \text{ (vector sum)}$$

The components of velocity and acceleration are related as

$$a_r = \dot{v}_r - v_\theta\dot{\theta}$$
$$a_\theta = \dot{v}_\theta + v_r\dot{\theta}$$

From the above relations it can be seen that is general,

a_r is not equal to $\dot{v}_r$.

and

a_θ is not equal $\dot{v}_\theta$.

It should be noted that radial components of velocity and acceleration are taken to be positive in the same sense as of position vector **r**.

Transverse components of velocity and acceleration are taken to be positive if pointing towards the increasing values of θ.

To understand the physical significance of the above results let us assume the following two situations.

(*i*) That, r is of constant length and θ varies. It then reduces to rotation along a circular path.

$$r = \text{constant}$$
$$\dot{r} = \ddot{r} = 0$$
$$v_r = 0, \qquad v_\theta = r\dot{\theta}$$
$$a_r = -r(\dot{\theta})^2 \qquad a_\theta = r\ddot{\theta}$$

(–ve sign indicates that a_r is directed opposite to the sense of position vector **r** or towards O)

It may be recalled that we denoted

$$\dot{\theta} = \omega \text{ and } \ddot{\theta} = \alpha$$

(*ii*) That, only r varies and θ is constant.

It then reduces a rectilinear motion along a fixed direction θ.

$$\theta = \text{constant}$$
$$\dot{\theta} = \ddot{\theta} = 0$$
$$v_r = \dot{r} \qquad v_\theta = 0$$
$$a_r = \ddot{r} \qquad a_\theta = 0$$

Example 15.6. The rotation of rod OA is defined by the relation $\theta = \frac{\pi}{2}(4t - 3t^2)$. A collar P slides along this rod in such a way that its distance from O is given by $r = 1.25t^2 - 0.9t^3$. In these relations, θ is expressed in radians, r in metres and t in seconds.

Determine (*i*) the velocity of the collar (*ii*) the total acceleration of the collar and (*iii*) the acceleration of the collar relative to the rod when t = 1 s.

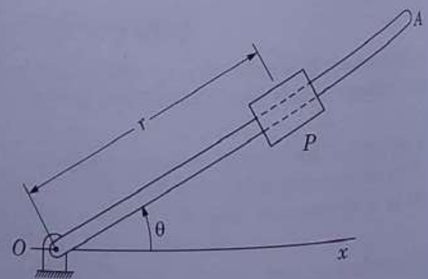


Fig. 15.12 (*a*)

Solution: Equations of motions are

$$\theta = \frac{\pi}{2}(4t - 3t^2)$$
$$r = 1.25t^2 - 0.9t^3$$

and

Evaluating θ, $\dot{\theta}$, $\ddot{\theta}$, $\dot{r}$ and $\ddot{r}$ at t = 1 s

$$\theta = \frac{\pi}{2}(4-3) = \frac{\pi}{2}\text{rad} \qquad r = 1.25 - 0.9 = 0.35 \text{ m}$$
$$\dot{\theta} = \frac{\pi}{2}(4-6t) \qquad \dot{r} = 1.25(2t) - 0.9(3)t^2$$
$$\dot{\theta} = \frac{\pi}{2}(4-6) = \pi \text{ rad/s} \qquad \dot{r} = 2.5 - 2.7 = -0.2 \text{ m/s}$$
$$\ddot{\theta} = \frac{\pi}{2}(0-6) \qquad \ddot{r} = 2.5 - 2.7(2)t$$
$$\ddot{\theta} = -3\pi \text{ rad/s}^2 \qquad \ddot{r} = 2.5 - 5.4 = -2.9 \text{ m/s}^2$$

(*i*) Velocity of collar P

$$v_r = \dot{r} = -0.2 \text{ m/s} \qquad v_\theta = r\dot{\theta} = 0.35 \times (-\pi) = -1.1 \text{ m/s}$$
$$v = \sqrt{(v_r)^2 + (v_\theta)^2} = 1.118 \text{ m/s}$$
$$\tan\alpha = \frac{V_\theta}{V_r} = \frac{-1.1}{-0.2}$$
$$\alpha = 79.70°$$

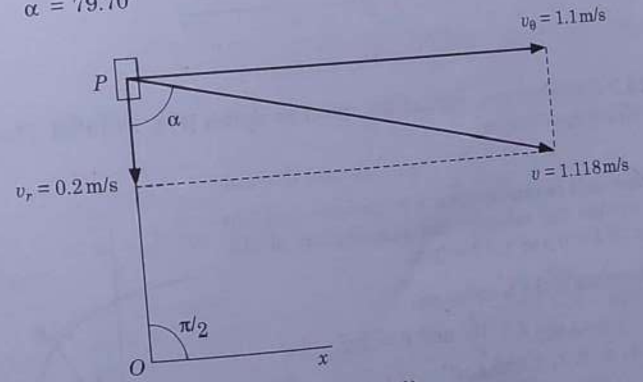


Fig. 15.12 (*b*)

(*ii*) Acceleration of the collar P

$$a_r = \ddot{r} - r(\dot{\theta})^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$a_r = -2.9 - 0.35(-\pi)^2 \qquad a_\theta = 0.35(-3\pi) + 2(-0.2)(-\pi)$$
$$a_r = -6.35 \text{ m/s}^2 \qquad a_\theta = -2.04 \text{ m/s}^2$$
$$a = \sqrt{(a_r)^2 + (a_\theta)^2} = 6.667 \text{ m/s}^2$$

$$\tan\beta = \frac{a_\theta}{a_r} = \frac{-2.04}{-6.35}$$

$$\beta = 17.8°.$$

(*iii*) Acceleration of the collar P relative to the rod.

Motion of the collar with respect to the rod is rectilinear. Hence

$$a_{r/OA} = \ddot{r} = -2.9 \text{ m/s (towards } O)$$

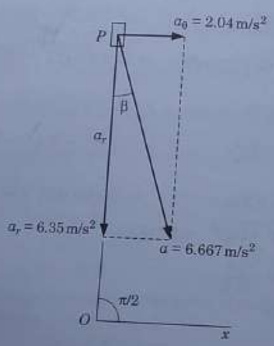


Fig. 15.12 (c)

Example 15.7 A particle moves along the spiral as shown [Fig. 15.13(a)]. The motion of the particle is defined by relations

$$r = 10t \text{ and } \theta = 2\pi t$$

where r is expressed in centimetres, θ in radians and t in seconds. Determine the velocity and acceleration of the particle when (*i*) $t = 0$ and (*ii*) $t = 0.3$s

Solution: Equations of the motions are

$$r = 10t \text{ and } \theta = 2\pi t$$

Evaluating $\dot\theta$, $\ddot\theta$, θ, r, $\dot r$ and $\ddot r$ at

$t = 0$ and $t = 0.3$ s

$\theta = 2\pi t \qquad r = 10t$

$\dot\theta = 2\pi \qquad \dot r = 10$

$\ddot\theta = 0 \qquad \ddot r = 0$

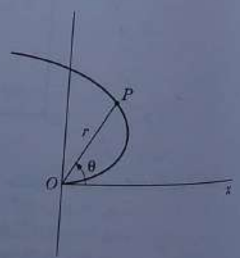


Fig. 15.13 (a)

At, $t = 0$

$\theta = 0 \qquad r = 0$

$\dot\theta = 2\pi$ rad/s $\qquad \dot r = 10$ cm/s

$\ddot\theta = 0 \qquad \ddot r = 0$

At, $t = 0.3$ s

$\theta = 2\pi \times 0.3 = 1.88$ rad $\qquad r = 10 \times 0.3 = 3$ cm

$\dot\theta = 2\pi$ rad/s $\qquad \dot r = 10 = 10$ cm/s

$\ddot\theta = 0 \qquad \ddot r = 0$

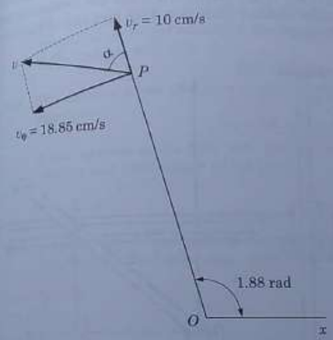


Fig. 15.13 (b)

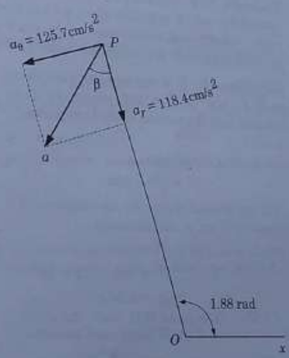


Fig. 15.13 (c)

(*i*) Velocity

$v_r = \dot r \qquad v_\theta = r\dot\theta$

$v_r = 10$ cm/s $\qquad v_\theta = 0$

$$v = \sqrt{(v_r)^2 + (v_\theta)^2} = 10 \text{ cm/s (along } v_r)$$

Acceleration

$a_r = \ddot r - r(\dot\theta)^2 \qquad a_\theta = r\ddot\theta + 2\dot r\dot\theta$

$a_r = 0 - 0(2\pi)^2 \qquad a_\theta = 0(0) + 2(10)(2\pi)$

$a_r = 0 \qquad a_\theta = +125.6$ cm/s^2

$$a = \sqrt{(a_r)^2 + (a_\theta)^2} = 125.6 \text{ cm/s}^2 \text{ (along } a_\theta)$$

(*ii*) Velocity

$v_r = \dot r \qquad v_\theta = r\dot\theta = 3 \times 2\pi$

$v_r = 10$ cm/s $\qquad v_\theta = 18.85$ cm/s

$$v = \sqrt{(v_r)^2 + (v_\theta)^2} = \sqrt{(10)^2 + (18.85)^2}$$

$$v = 21.34 \text{ cm/s}$$

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$$\tan\beta = \frac{a_\theta}{a_r} = \frac{-2.04}{-6.35}$$

$$\beta = 17.8°.$$

(*iii*) Acceleration of the collar P relative to the rod.

Motion of the collar with respect to the rod is rectilinear. Hence

$$a_{r/OA} = \ddot{r} = -2.9 \text{ m/s (towards } O)$$

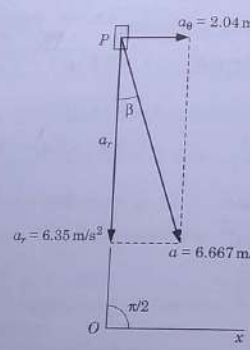


Fig. 15.12 (c)

Example 15.7 A particle moves along the spiral as shown [Fig. 15.13(a)]. The motion of the particle is defined by relations

$$r = 10t \text{ and } \theta = 2\pi t$$

where r is expressed in centimetres, θ in radians and t in seconds. Determine the velocity and acceleration of the particle when (*i*) $t = 0$ and (*ii*) $t = 0.3$s

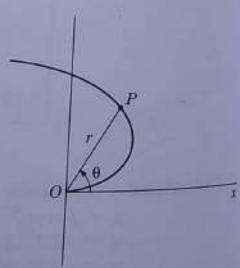


Fig. 15.13 (a)

Solution: Equations of the motions are

$$r = 10t \text{ and } \theta = 2\pi t$$

Evaluating $\dot\theta$, $\ddot\theta$, θ, r, $\dot r$ and $\ddot r$ at

$t = 0$ and	$t = 0.3$ s
$\theta = 2\pi t$	$r = 10t$
$\dot\theta = 2\pi$	$\dot r = 10$
$\ddot\theta = 0$	$\ddot r = 0$

At, $t = 0$

$\theta = 0$	$r = 0$
$\dot\theta = 2\pi$ rad/s	$\dot r = 10$ cm/s
$\ddot\theta = 0$	$\ddot r = 0$

At, $t = 0.3$ s

$\theta = 2\pi \times 0.3 = 1.88$ rad	$r = 10 \times 0.3 = 3$ cm
$\dot\theta = 2\pi$ rad/s	$\dot r = 10 = 10$ cm/s
$\ddot\theta = 0$	$\ddot r = 0$

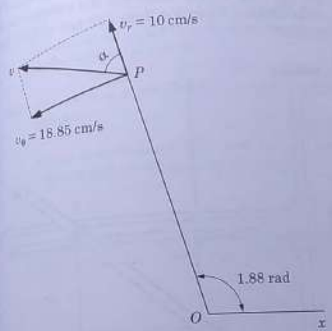


Fig. 15.13 (b)

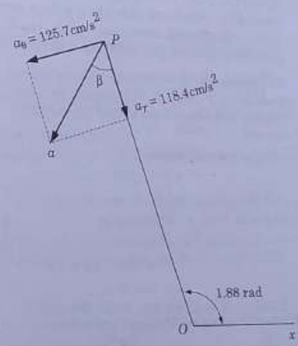


Fig. 15.13 (c)

(*i*) Velocity

$$v_r = \dot r \qquad v_\theta = r\dot\theta$$

$$v_r = 10 \text{ cm/s} \qquad v_\theta = 0$$

$$v = \sqrt{(v_r)^2 + (v_\theta)^2} = 10 \text{ cm/s (along } v_r)$$

Acceleration

$$a_r = \ddot r - r(\dot\theta)^2 \qquad a_\theta = r\ddot\theta + 2\dot r\dot\theta$$

$$a_r = 0 - 0(2\pi)^2 \qquad a_\theta = 0(0) + 2(10)(2\pi)$$

$$a_r = 0 \qquad a_\theta = +\,125.6 \text{ cm/s}^2$$

$$a = \sqrt{(a_r)^2 + (a_\theta)^2} = 125.6 \text{ cm/s}^2 \text{ (along } a_\theta)$$

(*ii*) Velocity

$$v_r = \dot r \qquad v_\theta = r\dot\theta = 3 \times 2\pi$$

$$v_r = 10 \text{ cm/s} \qquad v_\theta = 18.85 \text{ cm/s}$$

$$v = \sqrt{(v_r)^2 + (v_\theta)^2} = \sqrt{(10)^2 + (18.85)^2}$$

$$v = 21.34 \text{ cm/s}$$

Acceleration

$$a_r = \ddot{r} - r(\dot{\theta})^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$a_r = 0 - 3(2\pi)^2 \qquad a_\theta = 3(0) + 2(10)2\pi$$

$$a_r = -118.4 \text{ cm/s}^2 \qquad a_\theta = +125.7 \text{ cm/s}^2$$

$$a = \sqrt{(a_r)^2 + (a_\theta)^2} = \sqrt{(-188.4)^2 + (125.7)^2}$$

$$a = 172.65 \text{ cm/s}^2.$$

Example 15.8. Pivoted link OA carries a pin P whose position is controlled by the horizontal slotted bar which can slide along a fixed vertical bar (Fig. 15.14). What are the x-components of the velocity and acceleration of the pin P at the instant when $y = 5$ cm and the slotted bar is moving upward at a constant velocity of 10 cm/s?

Solution: Pin P is moving in a circular path $x^2 + y^2 = r^2 = 100$ guided by the rotating link OA and the slotted bar moving upward.

Equation of the path is

$$x^2 + y^2 = 100$$

At the instant when the coordinate y of the pin is 5 cm, x coordinate is

$$x^2 + y^2 = 100$$

$$x^2 = 100 - 25 = 75$$

$$x = 8.66 \text{ cm}.$$

Further, it is given that when the coordinate of P are (8.66, 5) its upward acceleration $\frac{dy}{dt} = \dot{y} = 10$ cm/s and $\frac{d^2y}{dt^2} = \ddot{y} = 0$.

Differentiating equation (*i*)

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad ...(ii)$$

$$x\dot{x} + y\dot{y} = 0$$

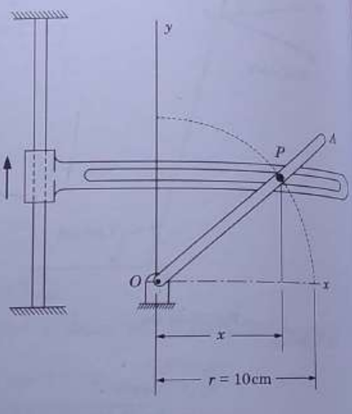


Fig. 15.14

Differentiating again

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = 0$$

$$x\ddot{x} + (\dot{x})^2 + y\ddot{y} + (\dot{y})^2 = 0 \qquad ...(iii)$$

Substituting in equation (*ii*)

$$x = 8.66 \text{ cm}, \qquad y = 5 \text{ cm}$$

$$\dot{y} = 10 \text{ cm/s} \qquad \ddot{y} = 0$$

$$8.66(\dot{x}) + 5(10) = 0$$

$$\dot{x} = -5.77 \text{ cm/s}$$

The x-component of the velocity of the pin = 5.77 cm/s and is directed towards O.

Substituting values in equation (*iii*),

$$8.66(\ddot{x}) + (-5.77)^2 + 5(0) + (10)^2 = 0$$

$$\ddot{x} = -15.40 \text{ cm/s}^2$$

The x-component of the acceleration of the pin = 15.40 cm/s^2 and is directed towards O.

PROBLEMS

15.1. The motion of a particle is described by the following equations

$$x = t^2 + 8t + 4$$

$$y = t^3 + 3t^2 + 8t + 4$$

Determine (*a*) initial velocity of the particle, (*b*) velocity of the particle at $t = 2$s (*c*) acceleration of the particle at $t = 2$s

[(*a*) 11.31 m/s, 45° (*b*) 34.18 m/s, 69.44° (*c*) 18.11/s^2, 83.66°]

15.2. The distance s travelled by a particle moving along a circular path of radius r is given by the equation

$$s = kt^2, \text{ where } k \text{ is a constant.}$$

If the particle starts from rest, find (*a*) tangential velocity and acceleration, (*b*) the normal velocity and acceleration of the particle.

$$\left[(a)\ 2kt, 2k\ \ (b)\ 0, \frac{4k^2t^2}{r}\right]$$

15.3. A ladder AE of length $2l$ has its ends A and B resting against a floor and a wall as shown in Fig. P.15.3. The ladder slips while its ends maintain contact with the floor and the wall.

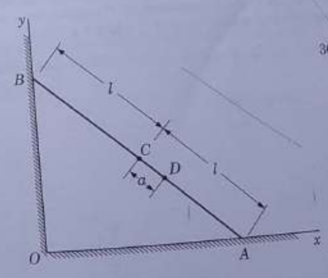


Fig. P.15.3

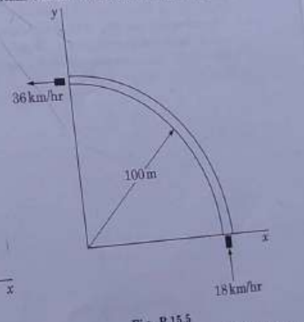


Fig. P.15.5

Show that the mid-point C of the ladder describes a circle of radius l with centre at O. Also show that any other point such as D describes an ellipse.

15.4. A car starts from rest on a curved road of radius 250 m and attains a speed of 18 km/hour at the end of 60 seconds while travelling with a uniform acceleration. Find the tangential and normal accelerations of the car 30 seconds after it started. [0.083 m/s^2, 0.025 m/s^2]

15.5. A car enters a curved section of the road of length equal to the quarter of a circle of radius 100 m at 18 km/hour and leaves at 36 km/hour (Fig. P. 15.5). If the car is travelling with a constant tangential acceleration find the magnitude and direction when (*a*) when it enters the curve, (*b*) when it leaves the curve. [(*a*) 0.346 m/s^2, 43.71(*b*) 1.03 m/s^2, 76.56°]

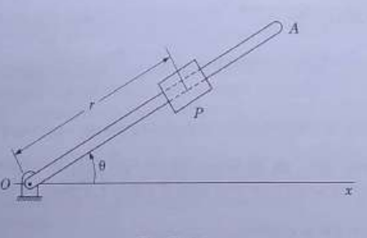


Fig. P.15.6

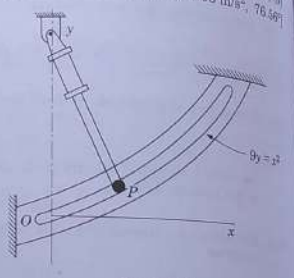


Fig. P.15.7

15.6. The rotation of rod OA is defined by the relation $\theta = 0.3t^2$. A collar P slides along this rod in such a way that its distance from O is given by $r = t^3/3 + 2t$. In these relations θ is expressed in radians, r in centimetres and t in seconds.

Determine (*i*) the velocity, of the collar (*ii*) the total acceleration of the collar when $t = 2s$.

$$\begin{bmatrix} v = 10 \text{ cm/s} \\ a = 19.25 \text{ cm/s}^2 \end{bmatrix}$$

15.7. The telescopic rod (Fig. P.15.7) force the pin P to move along the fixed path $9y = x^2$ when x and y are expressed in centimetres. At any instant t, the x-coordinate of P is given by $x = t^2 - 14t$. Determine the y-components of the velocity and acceleration of P when $t = 15$ s.

$$\begin{bmatrix} v_y = 53.33 \text{ cm/s} \\ a_y = 64.5 \text{ cm/s}^2 \end{bmatrix}$$

PART B: KINETICS

15.5 INTRODUCTION

The equations of motion of a particle relate force, mass and acceleration of the particle. The acceleration of a particle moving in a curvilinear path is a vector which can be resolved into two components mutually perpen-dicular to each other. These can be, a_x and a_y along the directions of the co-ordinate axes x and y respectively.

Or

a_t and a_n along the directions of the tangent and normal to the curve respectively (Fig. 15.15).

The equations of motion, therefore, can be written choosing either set of the components of acceleration as explained in following section.

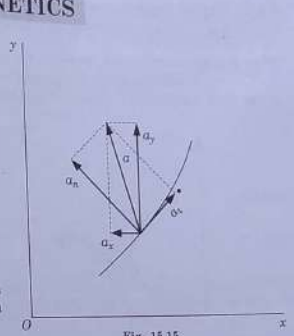


Fig. 15.15

15.6 EQUATIONS OF MOTION: IN RECTANGULAR COMPONENTS

Consider a system of forces acting on a particle P as shown in Fig. 15.16. Resolve the forces acting on the particle in the x and the y directions. Let ΣF_x and ΣF_y represent the sum of the components of the forces in the x and the y directions respectively.

Let a_x and a_y be the components of acceleration in the directions of the x and y respectively.

Applying Newton's second law

$$\Sigma F_x = ma_x \qquad \text{...[15.9(a)]}$$

That is, the sum of the components of the forces acting in the x-direction is equal to the product of the mass of the particle and its acceleration in the x-direction.

Similarly,

$$\Sigma F_y = ma_y \qquad \text{...[15.9(b)]}$$

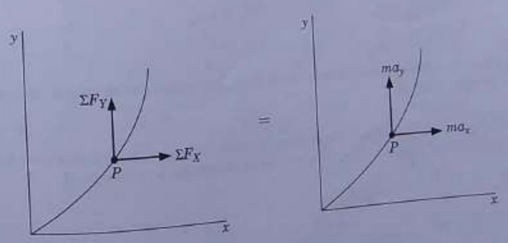


Fig. 15.16

15.7 EQUATIONS OF MOTIONS: IN TANGENTIAL AND NORMAL COMPONENTS

Often it is more convenient to resolve the forces acting on the particle into components F_t and F_n. F_t being the component along the tangent to the path and F_n the component along the normal to the path and directed *towards the centre of curvature of the path* (Fig. 15.17).

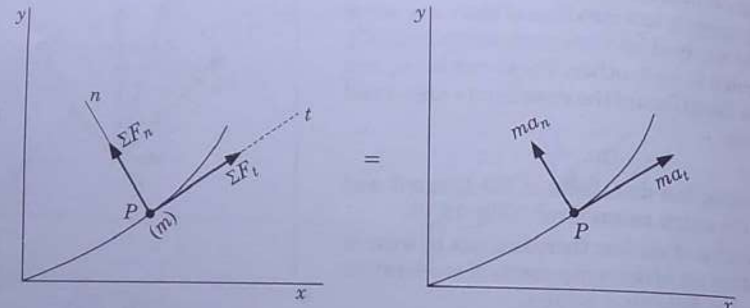


Fig. 15.17

Applying Newton's second law

$$\Sigma F_t = ma_t \qquad ...[15.10(a)]$$

Sum of components of the forces along the tangent to the path = Mass × component of the acceleration along tangent to the path.

Similarly,

$$\Sigma F_n = ma_n \qquad ...[15.10(b)]$$

If a particle is moving in a curved path of radius of curvature ρ and having the velocity v at any instant, we know

$$a_t = \frac{dv}{dt} \qquad a_n = \frac{v^2}{\rho}$$

The equations of motion

$$\Sigma F_t = ma_t \quad \text{and} \quad \Sigma F_n = ma_n$$

become,

$$\Sigma F_t = m\frac{dv}{dt} \qquad \Sigma F_n = \frac{v^2}{\rho}$$

When a particle is moving in a circular path then, ρ = radius r of the circle. If moving with a constant velocity v, then

$$a_t = 0$$

Whether to resolve the forces and the acceleration into rectangular components or normal and tangential components depends upon the nature of the problem to be solved.

15.8 EQUATIONS OF DYNAMIC EQUILIBRIUM (D'ALEMBERT'S PRINCIPLE)

Similar to the case of rectilinear motion of a particle, the curvilinear motion can also be described (using D'Alembert's Principle) by the equations of dynamic equilibrium.

In Rectangular Components. The equations of motion,

$$\Sigma F_x = ma_x$$
$$\Sigma F_y = ma_y$$

When written as

$$\underbrace{\Sigma F_x + (-ma_x)}_{\text{Inertia Force}} = 0 \qquad ...[15.11(a)]$$

$$\underbrace{\Sigma F_y + (-ma_y)}_{\text{Inertia Force}} = 0 \qquad ...[15.11(b)]$$

are called the equation of dynamic equilibrium where, $(-ma_x)$ and $(-ma_y)$ are the inertia forces added to the system of forces acting on the particle, in the directions opposite to the direction of acceleration a_x and a_y as shown in Fig. 15.18.

In Normal Tangential Components. The equations of motion,

$$\Sigma F_t = ma_t$$
$$\Sigma F_n = ma_n$$

When written as

$$\underbrace{\Sigma F_t + (-ma_t)}_{\text{Inertia Force}} = 0 \qquad ...[15.12(a)]$$

$$\underbrace{\Sigma F_n + (-ma_n)}_{\text{Inertia Force}} = 0 \qquad ...[15.12(b)]$$

are called the equations of dynamic equilibrium where, $(-ma_t)$ and $(-ma_n)$ are the inertia forces added to the system of forces acting on the particle in the directions opposite the direction of acceleration a_t and a_n as shown in Fig. 15.19.

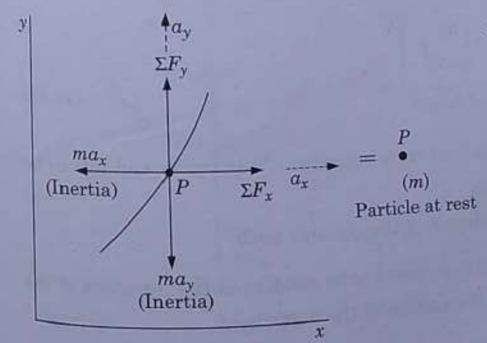


Fig. 15.18

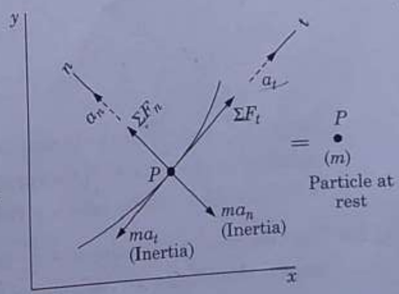


Fig. 15.19

For the simplicity of representation equations 15.11(*a*), 15.11(*b*), 15.12(a) 15.12(*b*) can be written as,

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma F_t = 0$$
$$\Sigma F_n = 0$$

where the terms ΣF_x, ΣF_y; ΣF_t, ΣF_n are to be now *re-defined to include the inertia forces.*

15.9 WORKING CONCEPTS: CURVILINEAR MOTION

Motion of a Particle in a Curved Frictionless Path. A particle may move along a curved path which may lie in vertical plane or in a horizontal plane. These planes are called the planes of motion. The following points may be noted in this connection (Fig. 15.20).

(*i*) The weight of the particle *mg* acts in the plane of motion only when the particle moves in a vertical plane. When the particle moves in a horizontal plane it acts in a plane normal to the plane of motion.

(*ii*) The normal acceleration a_n of the particle is always directed towards the centre of curvature of the path and is given by

$$a_n = \frac{V^2}{\rho}$$

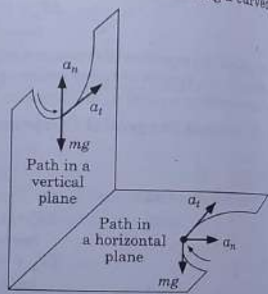


Fig. 15.20

(*iii*) The radius of curvature ρ of the path may not be the same at all the points of the path. In general the radius of curvature at a point *P* is given by,

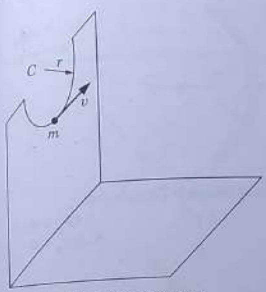
$$\rho_P = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} \quad ...(15.13)$$

where, $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ are to be determined from equation of the curve and are to be evaluated at the point *P* on the curve.

In the case of a circular path ρ = radius *r* of the circular path.

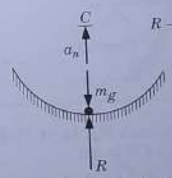
(*iv*) The tangential acceleration $a_t = \frac{dv}{dt}$ is considered to be positive in the direction of the tangent coinciding with the sense of the motion of the particle.

Motion of a Particle with a Constant Velocity in a Circular Path in a Vertical Plane

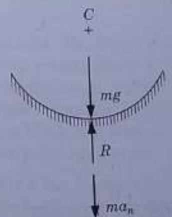


Equation of motion

$$\Sigma F_n = ma_n$$
$$R - mg = ma_n$$

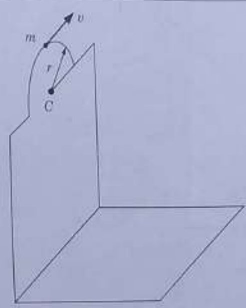


Equation of Dynamic Equilibrium

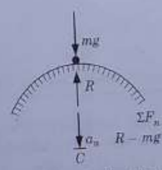


(Inertia) $\Sigma F_n = 0$

$$R - mg - ma_n = 0$$

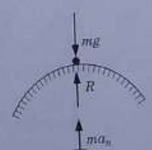


Equation of motion



$$\Sigma F_n = ma_n$$
$$R - mg = ma_n$$

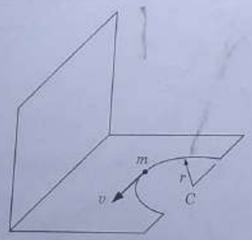
Equation of Dynamic Equilibrium



$$\Sigma F_n = 0$$
$$mg - R - ma_n = 0$$

* Centrifugal Force: Inertia force $(-ma_n)$ is called as the centrifugal force and it acts away from the centre.

Path in a Horizontal Plane



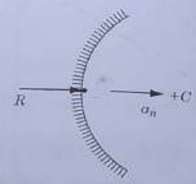
where, R = Reaction

and $a_n = \frac{v^2}{r}$

(Note that a_n is directed towards the centre.)

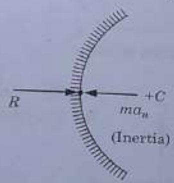
Equation of Motion

$\Sigma F_n = ma_n$
$R = ma_n$



Equation of Dynamic Equilibrium

$\Sigma F_n = 0$
$R - ma_n = 0$



Example 15.9. A small ball of weight W is held by two wires as shown in Fig. 15.21. Find the tension in the wire AB (*i*) before the wire CB cut (*ii*) after the wire CB is cut.

Solution: Before the wire CB is cut the ball is in static equilibrium. The free-body diagram of the ball is as shown in Fig. 15.21.

Writing the equations of equilibrium

$\Sigma F_x = 0$: $\quad T_{CB}\cos 70° - T_{AB}\cos 50° = 0 \quad$ ...(*i*)

$\Sigma F_y = 0$: $\quad T_{CB}\sin 70° + T_{AB}\sin 50° - W = 0 \quad$ ...(*ii*)

Substituting for T_{CB} from (*i*) in (*ii*)

$$\left(\frac{T_{AB}\cos 50°}{\cos 70°}\right)\sin 70° + T_{AB}\sin 50 = W$$

$$T_{AB} = \frac{W}{(2.747 \times 0.643 + 0.766)}$$

$$T_{AB} = 0.395\,W \quad Ans.$$

When the string CB is cut (Fig. 15.22) then the ball is no longer in static equilibrium and a resultant force or an unbalanced force shall act on the ball. Resolving the weight W of the ball along AB.

$$T_{AB} - W\cos 40° = 0$$
$$T_{AB} = W\cos 40°$$
$$T_{AB} = 0.766\,W \quad Ans.$$

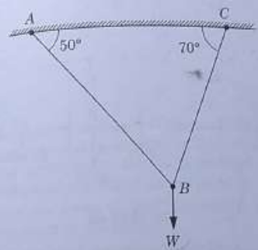


Fig. 15.21

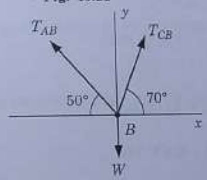


Fig. 15.22

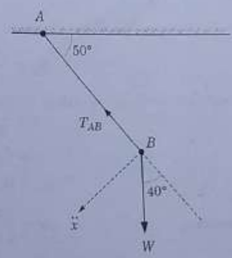


Fig. 15.23

The component $W\sin 40°$, is the unbalanced force acting normal to AB and shall accelerate the ball.

Example 15.10. The frame shown in Fig. 15.24 (*a*) rotates about a vertical axis. The coefficient of friction under block A is 0.40. Determine the co-efficient of friction at block B, if B starts to rise when frame rotates at 40 rpm.

Solution: As the frame rotates, the block A tends to move outwards and friction force acts inwards. Let tension in the string be T.

Consider the free-body diagram of block A

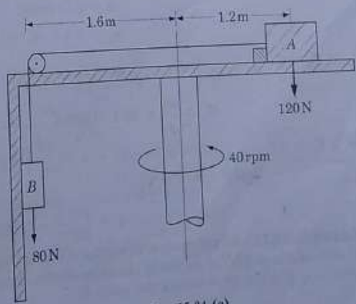


Fig. 15.24 (*a*)

$\Sigma F_y = 0:$ $\quad N - 120 = 0$

$$N = 120\,N$$

$\Sigma F_x = 0:$ $\quad \frac{120}{g}a_n - T - 0.4\,N = 0$

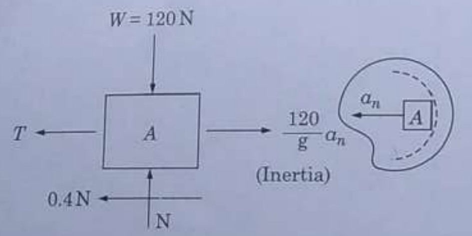


Fig. 15.24 (b)

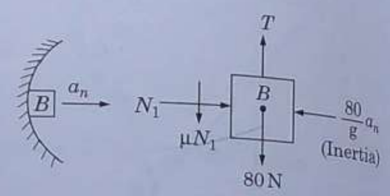


Fig. 15.24 (c)

$$a_n = \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 r,\ n = 40 \text{ rpm}$$

$$\omega = \frac{2\pi n}{60} = \frac{2\pi \times 40}{60} = 4.19 \text{ rad/s}$$

$$r = 1.2 \text{ m}$$

$$a_n = (4.19)^2 \times 1.2 = 21.06 \text{ m/s}^2$$

Substituting $\quad T = \frac{120}{9.81} \times 21.06 - 0.4 \times 120$

$$T = 257.6 - 48 = 209.6 \text{ N}$$

Consider the free body diagram of block B.

$$\omega = 4.19 \text{ rad/s}, \quad r = 1.6 \text{ m}$$

$$a_n = \omega^2 r = (4.19)^2\,(1.6) = 28.1 \text{ m/s}^2$$

$\Sigma F_x = 0:$ $\quad N_1 - \frac{80}{g}a_n = 0$

$$N_1 = \frac{80}{g} \times 28.1 = 229.1$$

$\Sigma F_x = 0:$ $\quad T - 80 - \mu N_1 = 0$

$$209.6 - 80 - \mu\,(229.1) = 0$$

$$\mu = \frac{129.6}{229.1} = 0.566.$$

Example 15.11. An automobile of weight W travels with uniform speed of $v = 100$ km p.h. over a vertical curve ABC of parabolic shape as shown. Determine the total pressure exerted by the wheels of the car as it passes the point B.

Solution: Let us write the equation of motion of the automobile, when at B.

$$\frac{W}{g}a_n = W - R$$

Or $\quad R = W - \frac{W}{g}a_n$

As $\quad a_n = \frac{v^2}{\rho}$

So, $\quad R = W - \frac{W}{g}\frac{v^2}{\rho} \quad \text{...(i)}$

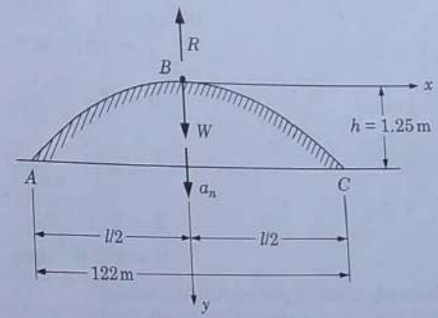


Fig. 15.25

When, v is the speed of the automobile, R the reaction of the surface on the wheels of the automobile and ρ is radius of curvature of the curve at the point B.

Now we have to find the radius of curvature ρ of the curve.

Let the equation of parabola be

$$x^2 = ky \quad \text{...(ii)}$$

where k is a constant to be determined. Point $C(l/2, h)$ lies on the curve so, from (*ii*)

$$\left(\frac{l}{2}\right)^2 = kh, \text{ or } k = \frac{l^2}{4h}$$

Radius of curvature of a curve is given by

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

where d^2y/dx^2 and dy/dx are to be evaluated from the equation of the curve

$$x^2 = ky$$

Or $\quad y = \frac{x^2}{k}$

$$\frac{dy}{dx} = \frac{2x}{k}, \frac{d^2y}{dx^2} = \frac{2}{k}$$

$$\frac{1}{\rho} = \frac{2/k}{\left[1+\left(\frac{2x}{k}\right)^2\right]^{3/2}} \quad \text{...(iii)}$$

To find the value of ρ at the point B substitute the value of its coordinates in the above expression (*iii*)

$$\left(\frac{1}{\rho}\right)_{\text{at } B(0,0)} = \frac{2}{k} = \frac{2}{l^2/4h} = \frac{8h}{l^2}$$

$$\left(\frac{1}{\rho}\right)_{\text{at } B} = \frac{8h}{l^2} = \frac{8 \times 1.25}{(122)^2} = \frac{1}{2592}$$

$$v = 100 \text{ kmph} = 27.77 \text{ m/s}$$

From (*i*)

$$R = W - \frac{W}{g}\frac{v^2}{\rho}$$

$$= W\left(1 - \frac{1}{9.81} \times \frac{(27.77)^2}{2592}\right)$$

$$R = W(1 - .03)$$

$$R = 0.97\ W \quad Ans.$$

Example 15.12. A portion ABC of a railway track consists of a spiral AB of length r and radius of curvature ρ and the tangent BC (Fig. 15.26). The curvature $\rho = r^2/s$, where s is measured from B towards A. A diesel locomotive of weight 10000 N starts from rest at A and increases its speed along AB with a constant tangential acceleration $dv/dt = g/10$. Find the maximum lateral thrust on the outer rail during the motion from A to B and where does it occur?

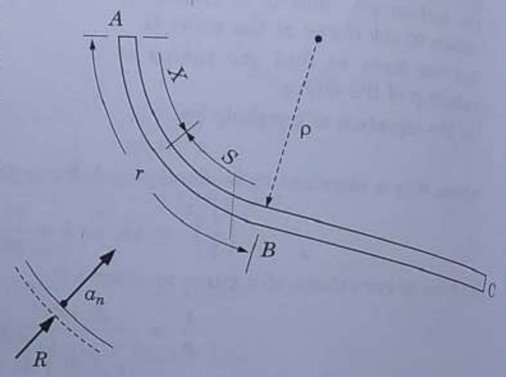


Fig. 15.26

Solution: Let R be the reaction of the rail on the locomotive wheel acting in the plane of motion. Thrust on the rail shall be of the same magnitude but in opposite direction.

Writing the equation of motion of the wheels.

$\Sigma F_x = ma_n$:

$$\frac{W}{g}a_n = R$$

As,

$$a_n = \frac{v^2}{\rho}$$

Therefore,

$$R = \frac{W}{g}\frac{v^2}{\rho} \qquad ...(i)$$

Let us find the value of r and v when the locomotive is at a distance x from A. The radius of curvature $r = r^2/s$ where s is the distance to be measured from B. For a point at a distance x from A,

$$s = (r - x)$$

Therefore,

$$\rho = \frac{r^2}{(r - x)}$$

The locomotive starts from rest and attains an acceleration of $g/10$ after travelling a distance x.

Using

$$v^2 - u^2 = 2\,as$$

where,

$$u = 0,\ a = \frac{g}{10},\ s = x$$

The final velocity v is,

$$v^2 = 2ax = 2\left(\frac{g}{10}\right)x$$

$$v^2 = \frac{gx}{5}$$

Substituting the values of ρ and v in the equation (*i*)

$$R = \frac{W}{g}\left(\frac{\frac{gx}{5}}{\frac{r^2}{(r-x)}}\right)$$

Or

$$R = \frac{Wx}{5}\frac{(r-x)}{r^2}$$

For the maximum value of R,

$$\frac{dR}{dx} = 0$$

$$\frac{d\left(\frac{Wx}{5}\frac{(r-x)}{r^2}\right)}{dx} = 0$$

$$r - 2x = 0$$

Or

$$x = \frac{r}{2} \quad Ans.$$

That is, the reaction R is maximum when the distance travelled is half the length of the spiral

$$R_{\max} = \frac{W}{5}\left(\frac{r}{2}\right)\left(\frac{(r - r/2)}{r^2}\right) = \frac{W}{20}$$

$$W = 10000 \text{ N}$$

$$R_{\max} = \frac{10000}{20}$$

$$R_{\max} = 500 \text{ N} \quad Ans.$$

Thrust on the rail = 500 N and acts in a direction opposite to that of R acting on the wheels.

Example 15.13. A ball of weight 10 N starts from rest from the point O of a smooth vertical track and rolls down under gravity along the tract OAB. Find the reaction exerted on the ball at the point A if the curve OAB is defined by the equation

$$y = h\sin\left(\frac{\pi x}{l}\right)$$

Assume

$$h = \frac{l}{3} = 1 \text{ m.}$$

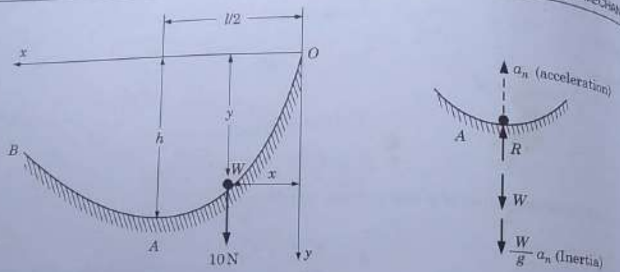


Fig. 15.27

Solution: Let R be the reaction and a_n be the normal acceleration. Writing the equation of dynamic equilibrium of the ball at A.

$$\Sigma F_x = 0: \qquad R - W - \left(\frac{W}{g}\right)a_n = 0$$

$$R = W\left(1 + \frac{1}{g}a_n\right)$$

$$R = W\left(1 + \frac{1}{g}\frac{v^2}{\rho}\right) \qquad \text{...(i)}$$

where, v = velocity of the particle at A

ρ = radius of curvature of the curve at A

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

where d^2y/dx^2 and dy/dx are to be evaluated at A.

Equation of the curve is

$$y = h\sin\frac{\pi x}{l}$$

$$\frac{dy}{dx} = h\left(\frac{\pi}{l}\right)\cos\frac{\pi x}{l}$$

$$\frac{d^2y}{dx^2} = -h\left(\frac{\pi}{l}\right)\left(\frac{\pi}{l}\right)\sin\frac{\pi x}{l}$$

$$\rho = \frac{\left[1 + \left(h\frac{\pi}{l}\cos\frac{\pi x}{l}\right)^2\right]^{3/2}}{(-)h\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}}$$

$$(\rho)_{\text{at } A(l/2,h)} = \frac{1+0}{\frac{(-)h}{l^2}\pi^2} = (-)\frac{l^2}{h\pi^2}$$

$$(\rho)_A = \frac{l^2}{h\pi^2} \text{ (Taking the absolute value)}$$

Applying the principle of conservation of energy between the positions O and A of the ball $v = \sqrt{2gh}$ (h is the vertical distance of point A from O).

$$R = W\left(1 + \frac{1}{g}\frac{v^2}{\rho}\right)$$

Substituting for v and r,

$$R = W\left(1 + \frac{1}{g}\frac{(2gh)}{(l^2/h\pi^2)}\right)$$

Substituting $\qquad h = \frac{l}{3}$

$$R = W\left(1 + \frac{2\pi^2}{9}\right)$$

$$R = W(1 + 2.18) = 10(1 + 2.18)$$

$$R = 31.8 \text{ N} \quad \textit{Ans.}$$

Example 15.14. A weight P attached to the end of a flexible rope of diameter $d = 0.5$ cm. is raised vertically by winching the rope on a reel as shown in Fig. 15.28. The reel it turned uniformly at the rate of 2 revolutions per second. Find the tension in the rope. Neglect the inertia of the rope and the lateral motion of the weight P.

Solution: Let T be the tension in the rope. Writing the equation of motion of the weight P,

$$\Sigma F_x = ma_x: \qquad \frac{P}{g}a = T - P$$

$$T = P + \frac{P}{g}\ddot{x} \qquad \text{...(i)}$$

where $\ddot{x}$ is the acceleration of the weight.

To find $\ddot{x}$ we have to find the relation between the rate of turning of the reel and the length of the rope wound.

Let the initial radius of the reel be R. Radius R of the reel will increase in one revolution to a value $= R + d$.

So, the radius of the reel after a time t second shall become $= (R + 2td)$. Because in t seconds the reel will make 2t revolutions. The length of the rope wound in one revolution over the reel at the time t when its radius is $(R + 2\,td)$

$$= 2\pi(R + 2td)$$

Length of the rope wound per second $\quad$ = length of the rope wound in two revolution

$$= 2(2\pi(R + 2td))$$

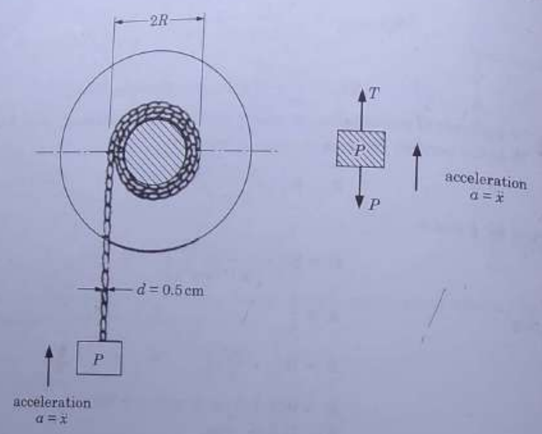


Fig. 15.28

Length of the rope wound per second = velocity v of the weight $P = \dfrac{dx}{dt}$

Acceleration $\quad \ddot{x} = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = \dfrac{d(4\pi(R+2td)}{dt}$

$$\ddot{x} = 4\pi \times 2d$$

Substituting the values,

$$\ddot{x} = 4\pi \times 2 \times 0.5 \times \frac{1}{100}$$

$$\ddot{x} = \frac{4\pi}{100}\,\text{m/s}^2$$

Substituting in equation (*i*)

$$T = W\left(1+\frac{\ddot{x}}{g}\right) = W\left(1+\frac{4\pi}{100\times 9.81}\right)$$

$$T = 1.013\,W \quad Ans.$$

Example 15.14 At what uniform speed of rotation N around the vertical axis AB, the ball P and Q of equal weight shall begin to lift the weight R of the governing device shown in Fig. 15.29? Given $P = Q = 50$ N, $R = 100$ N, $l = 30$ cm. Assume $\theta = 30°$ when the weight R begins to get lifted.

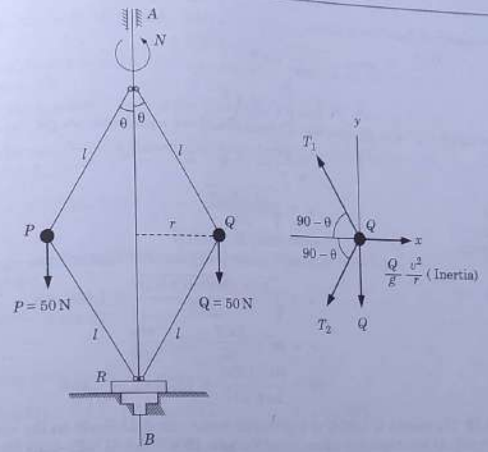


Fig. 15.29 (*a*)

Solution: Let the ball be rotating at a speed of v m/s in a circle of radius r then

$$r = l\sin\theta = l\sin 30°$$

The *shape of the device and the forces acting are symmetrical about the axis AB*. So, the tension in each of the upper rods is same and let it be $= T_1$

Also, the tension in each of the lower rods is same and let it be $= T_2$

The force acting on the weight Q are as shown in Fig. 15.29 (*a*).

Equations of dynamic equilibrium of the weight Q are,

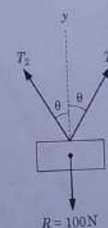


Fig. 15.29 (*b*)

$$\Sigma F_x = 0 : \quad \frac{Q}{g}\left(\frac{v^2}{r}\right) - T_1\cos(90-\theta) - T_2\cos(90-\theta) = 0$$

Or $\quad \dfrac{Q}{g}\dfrac{v^2}{r} - T_1\sin\theta - T_2\sin\theta = 0 \quad$ ...(*i*)

$$\Sigma F_y = 0 : \quad T_1\sin(90-\theta) - T_2\sin(90-\theta) - Q = 0$$

$T_1\cos\theta - T_2\cos\theta - Q = 0 \quad$ ...(*ii*)

Next consider the equilibrium of the weight R when it is just about to be lifted [Fig. 15.29 (*b*)]. In this situation the reaction of the support on the weight R is zero. ...(*iii*)

$$\Sigma F_y = 0 : \quad 2T_2\cos\theta = R$$

$$T_2 = \frac{R}{2\cos\theta}$$

Substituting for T_2 in (*i*) and (*ii*)

$$\frac{Q}{g}\frac{v^2}{r} - T_1 \sin\theta - \frac{R}{2\cos\theta}\sin\theta = 0$$

$$T_1\cos\theta - \frac{R}{2} - Q = 0$$

Substituting $\theta = 30°$ and eliminating T_1 from the above equations, we get

$$\frac{Q}{g}\frac{v^2}{r}\sqrt{3} = Q + R$$

Or $\quad v = \sqrt{\frac{(Q+R)gr}{\sqrt{3}Q}}$

Substituting, $\quad Q = 50\text{N},\ R = 100\text{ N}$

$$r = l\sin 30° = 0.3 \times 0.5 = 0.15 \text{ m}$$

$$v = \sqrt{\frac{(50+100)9.81\times 0.15}{50\sqrt{3}}} = 1.596 \text{ m/s}$$

$$v = \omega r = \frac{2\pi N}{60}r,\ N = \frac{60v}{2\pi r}$$

$$N = \frac{60\times 1.596}{2\pi\times 0.15} = 101.6 \text{ r.p.m.} \quad \textit{Ans.}$$

Example 15.16 The weight $Q = 20$ N of a governor device can slide freely on the vertical shaft *AB* [Fig. 15.30 (*a*)]. At what speed in r.p.m. about the axis *AB* will the 10 N fly-balls lift the sliding weight *Q* free of its support? Assume frictionless conditions and the weights of the bears are negligible.

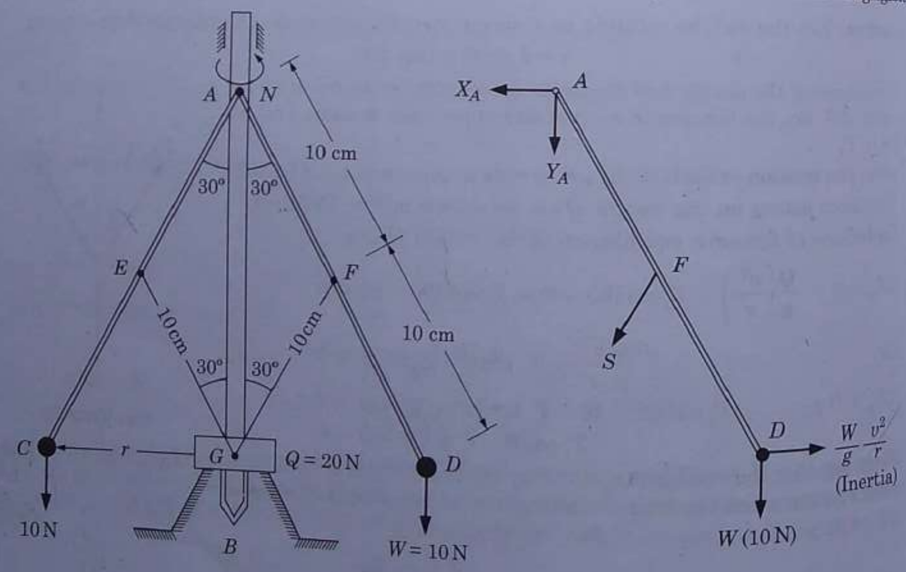


Fig. 15.30 (*a*)

Solution: Let the balls be rotating with a speed of v m/s in a circle of radius r, then

$$r = 20 \sin 30° = 10 \text{ cm}$$

The shape of the device and the forces involved are symmetrical about the vertical axis *AB*.

Therefore, the tension in each of the lower rods *EG* and *FG* is same and let be = *S*.

Consider the equation of dynamic equilibrium of the bar *AD*. Taking moments about *A*,

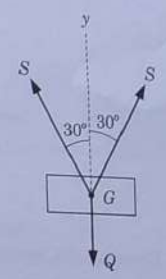


Fig. 15.30 (*b*)

$$\Sigma M_A = 0: \quad -W(0.2\sin 30°) + \frac{W}{g}\left(\frac{v^2}{r}\right)(0.2\cos 30°) - S(0.1\cos 30°) = 0 \quad \text{...(i)}$$

Consider the equilibrium of the weight *Q* when it is about to be lifted off [Fig. 15.30 (*b*)].

$\Sigma F_y = 0$:

$$S\cos 30° + S\cos 30° - Q = 0$$

$$S = \frac{Q}{2\cos 30°} \quad \text{...(ii)}$$

Substituting for *S* in the equation (*i*)

$$-W(0.2\sin 30) + \frac{W}{g}\left(\frac{v^2}{r}\right)(0.2\cos 30°) - \frac{Q}{2\cos 30°}(0.1\cos 30°) = 0$$

$$\frac{W}{g}\frac{v^2}{r}\left(0.2\times\frac{\sqrt{3}}{2}\right) = W(0.2\times 0.5) + 0.05Q$$

Substituting $\quad W = 10\text{ N},\ Q = 20\text{ N},\ r = 0.1\text{ m}$

$$\frac{10}{9.81}\frac{v^2}{0.1}\times\frac{0.2\times\sqrt{3}}{2} = 10\times 0.2\times 0.5 + 0.05\times 20$$

$$v^2 = \frac{2\times 9.81}{\sqrt{3}\times 10}$$

$$v = 1.064 \text{ m/s}$$

$$v = \omega r = \frac{2\pi N}{60}r$$

$$N = \frac{30}{\pi r} = \frac{30\times 1.064}{\pi\times 0.1}$$

$$N = 101.6 \text{ r.p.m.} \quad \textit{Ans.}$$

15.10 MOTION OF VEHICLES: LEVEL AND BANKED ROADS

(a) Motion of a Cyclist Round a Circular Level Road

Consider the motion of a cyclist of weight *W* in a circular path of radius *r*. With a speed of v m/s. The height of his centre of gravity be *h* and the inclination to the vertical be θ as shown in Fig. 15.31.

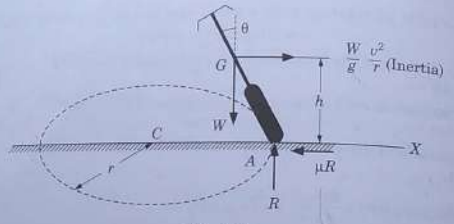


Fig. 15.31

Consider the equilibrium of the cyclist. The various forces acting are :

1. the weight W acting vertically downward through G.
2. the frictional force (μR) acting radially inwards between the tyre and the ground
3. the reaction R
4. inertia force $\frac{W}{g}\left(\frac{v^2}{r}\right)$ (also called centrifugal force) acting radially outwards.

The equations of dynamic equilibrium are,

$\Sigma F_y = 0$: $R - W = 0$

Or $R = W$

$\Sigma F_x = 0$: $\frac{W}{g}\left(\frac{v^2}{r}\right) - \mu R = 0$

Taking moments about the point of contact A,

$\Sigma M_A = 0$: $W(h \tan\theta) - \frac{W}{g}\left(\frac{v^2}{r}\right)(h) = 0$

$$\tan\theta = \frac{v^2}{gr}$$

Or $$v = \sqrt{gr\tan\theta} \quad ...(15.14)$$

It can be seen that the cyclist has to lean inwards to avoid overturning.

Also, it can be observed that the inertia force (or centrifugal force) tends to skid the cyclist away from the centre C. It can be avoided if sufficient frictional force is developed to just balance the centrifugal force.

To avoid skidding

$$\text{Frictional force} \geq \frac{W}{g}\left(\frac{v^2}{r}\right)$$

$$\mu R \geq \frac{W}{g}\left(\frac{v^2}{r}\right)$$

$$\mu W \geq \frac{W}{g}\left(\frac{v^2}{r}\right)$$

$$\mu g r \geq v^2$$

$$v \geq \sqrt{\mu g r}$$

Or $$V_{maximum} = \sqrt{\mu g r} \quad ...(15.15)$$

(b) Motion of a Vehicle on a Level Circular Path. Consider the motion on a vehicle of a level circular path of radius r and having the centre at C. Let,

v be the uniform speed of the vehicle

h be the height of the centre of gravity of the vehicle from the ground

$2b$ be the distance between the wheels.

Magnitude of the Reactions. Consider the equilibrium of the vehicle. The various forces acting are :

1. the weight W acting vertically downwards.
2. the frictional forces F_A and F_B between the types and the ground acting radially inwards.
3. the reactions R_A and R_B.
4. centrifugal force (inertia) $\frac{W}{g}\left(\frac{v^2}{r}\right)$ acting radially outwards.

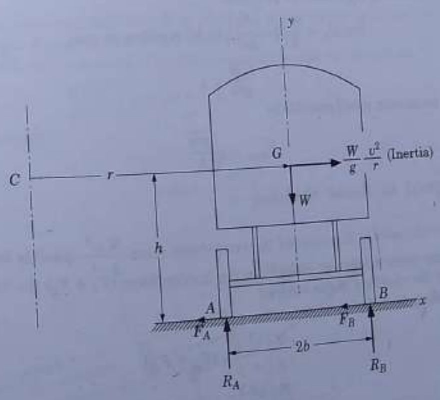


Fig. 15.32

The equations of equilibrium can be written as,

$\Sigma F_x = 0$: $\quad \dfrac{W}{g}\left(\dfrac{v^2}{r}\right) - (F_A + F_B) = 0$

$\Sigma F_y = 0$: $\quad R_A + R_B - W = 0$

Taking moments about A,

$\Sigma M_A = 0$: $\quad R_B(2b) - W(b) - \left(\dfrac{W}{g}\dfrac{v^2}{r}\right)h = 0$

$$R_B = \frac{W}{2}\left(1 + \frac{v^2 h}{grb}\right) \quad ...[15.16\,(a)]$$

From (ii)

$$R_A = W - R_B = W - \frac{W}{2}\left(1 + \frac{v^2 h}{grb}\right)$$

$$R_A = \frac{W}{2}\left(1 - \frac{v^2 h}{grb}\right) \quad ...[15.16\,(b)]$$

Maximum Speed to Avoid Overturning

The reaction R_B at the point B cannot be zero. But the reaction R_A at the point A can become zero when the wheel A is lifted off the ground, that is, when the vehicle is at the point of overturning about B.

To avoid overturning R_A should be positive.

For $R_A = \dfrac{W}{2}\left(1 - \dfrac{v^2 h}{grb}\right)$ to be positive or zero

$$\frac{r^2 h}{grb} \le 1$$

Or maximum speed possible is

$$V_{max} \le \sqrt{\frac{grb}{h}} \quad ...(15.17)$$

Maximum speed to Avoid Skidding

If the speed v of the vehicle is increased the centrifugal force $\dfrac{W}{g}\dfrac{v^2}{r}$ (inertia force) shall also increase. If at any point it becomes more than the friction force $(F_A + F_B)$ the vehicle will slip the ways (called the skidding of the vehicle)

To avoid skidding, therefore,

$$\frac{W}{g}\left(\frac{v^2}{r}\right) \le (F_A + F_B)$$

$$\frac{W}{g}\left(\frac{v^2}{r}\right) \le \mu(R_A + R_B)$$

$$\frac{W}{g}\left(\frac{v^2}{r}\right) \le \mu W$$

$$v \le \sqrt{\mu g r} \quad ...(15.18)$$

(c) Motion of a Vehicle on Banked Circular Path

When a vehicle has to go around a curved road the outer edge of the road is slightly raised with respect to the inner edge and it is called the *banking of road or superelevation* as shown in Fig. 15.33.

Let us find the angle of banking required (or superelevation) in case of a car and a locomotive travelling in a circular path of radius r such that,

(a) In the case of a car there is no side frictional force on its wheel, Fig. 15.33 (a).

(b) In the case of a locomotive there is no flange pressure or side thrust on its wheels, Fig. 15.33 (b).

The analysis of both is the same and the forces involved are shown in the Fig. 15.33 (a) and (b).

Forces acting are,

1. the weight W acting vertically downwards through G
2. the reaction R normal to the road and symmetrical with respect to the wheels
3. the centrifugal force $\dfrac{W}{g}\left(\dfrac{v^2}{r}\right)$

If there is no side frictional force [Fig. 15.33 (a)] acting on the car types and in the case of locomotive no side flange pressure acting on the locomotive wheels [Fig. 15.33 (b)], the equations of dynamic equilibrium can be written as,

$\Sigma F_x = 0$: $\quad \dfrac{W}{g}\left(\dfrac{v^2}{r}\right) - R\sin\theta = 0$

(Along the horizontal)

Or $$R\sin\theta = \frac{W}{g}\left(\frac{v^2}{r}\right) \quad ...(15.19)$$

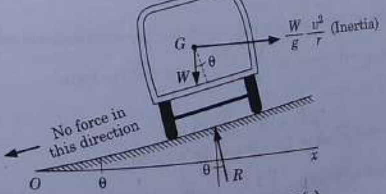


Fig. 15.33(a)

$\Sigma F_y = 0$: $\quad R\cos\theta - W = 0$

Or $\quad R\cos\theta = W$ ...(15.20)

Dividing (15.19) by (15.20)

$$\tan = \frac{v^2}{gr} \qquad ...(15.21)$$

Or

$$v = \sqrt{gr\tan\theta} \qquad ...(15.22)$$

The *angle of banking* is given by the expression (15.21) so that there is no slide friction or side thrust when a vehicle is travelling in circular path of radius r with speed v.

The maximum speed of a vehicle is given by the expression (15.22) so that there is no side friction or no side thrust when the vehicle is travelling in a circular path of radius r, banked at an angle θ.

In the case of railway tracks the term *superelevation* is used quite often. If b is the distance between the rail centres then superelevation e is given by

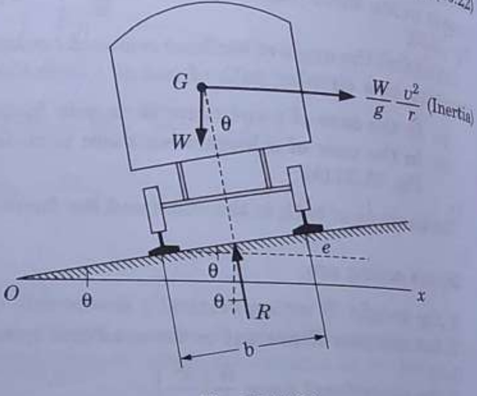


Fig. 15.33(*b*)

$$e = b\sin\theta$$

for, no side thrust,

$$\tan\theta = \frac{v^2}{gr}$$

Angle θ being very small, therefore,

$$\sin\theta = \tan\theta$$

$$e = b\tan\theta = \frac{bv^2}{gr}$$

$$e = \frac{bv^2}{gr} \qquad ...(15.23)$$

Example 15.17. Express the maximum speed with respect to skidding of a car travelling on a banked road in terms of the radius r of the curve, the banking angle θ and the friction angle ϕ between the types and the road.

Solution: Forces acting on the car are shown in Fig. 15-34. R is the total reaction acting normal to the road, F is the total frictional force and $\frac{W}{g}\left(\frac{v^2}{r}\right)$ is the inertia force.

If the car is at the point of skidding

$$F = \mu R \qquad (\mu = \tan\phi) \qquad ...(i)$$

The equation of dynamic equilibrium of the car can be written as,

$\Sigma F_x = 0$:

$$-F - W\sin\theta + \frac{W}{g}\left(\frac{v^2}{r}\right)\cos\theta = 0$$

(Along the track)

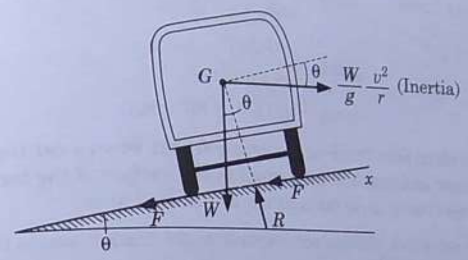


Fig. 15.34

Or

$$F + W\sin\theta - \frac{W}{g}\left(\frac{v^2}{r}\right)\cos\theta = 0 \qquad ...(ii)$$

$\Sigma F_y = 0$:

(Normal to the track)

$$R - W\cos\theta - \frac{W}{g}\left(\frac{v^2}{r}\right)\sin\theta = 0$$

Or

$$R = W\cos\theta + \frac{W}{g}\frac{v^2}{r}\sin\theta \qquad ...(iii)$$

From (*i*) and (*iii*)

$$F = \mu\left(W\cos\theta + \frac{W}{g}\frac{v^2}{r}\sin\theta\right) \qquad ...(iv)$$

Substituting for F in equation (*ii*)

$$\mu\left(W\cos\theta + \frac{W}{g}\frac{v^2}{r}\sin\theta\right) + W\sin\theta - \frac{W}{g}\frac{v^2}{r}\cos\theta = 0$$

Or

$$v^2 = gr\left(\frac{\mu + \tan\theta}{1 - \mu\tan\theta}\right) = gr\left(\frac{\tan\phi + \tan\theta}{1 - \tan\theta\tan\theta}\right)$$

$$v = \sqrt{gr\tan(\theta + \phi)} \quad \textit{Ans.}$$

Example 15.18. A cyclist riding on a level road has to turn a corner of radius 50 m. Find the maximum speed with which he can travel without the fear of skidding.

What is the angle he shall make while negotiating the corner? Assume the coefficient of friction between the tyres and track $\mu = 0.15$.

Solution: Maximum speed with no skidding

$$v = \sqrt{\mu gr}$$

$$v = \sqrt{0.15 \times 9.81 \times 50}$$

$$v = 8.58 \text{ m/s} \quad \textit{Ans.}$$

The angle θ with the vertical

when negotiating the corner $\tan\theta = \frac{v^2}{gr}$

$$\tan\theta = \frac{(8.58)^2}{9.81\times 50}$$

$$\tan\theta = 0.15 = 8.53° \quad \textit{Ans.}$$

Example 15.19. An automobile track is so designed that when a car travels at 100 km/hour, the force between the car and track acts normal to the surface of the track. Find the angle of banking of the track assuming it to be a circle of radius 250 m.

Solution: As the force acting on the car acts normal to the track it means that the side frictional force is zero.

Using the relation derived earlier, the angle of banking is

$$\tan\theta = \frac{v^2}{gr}, \quad v = 100 \text{ km/hour} = 27.78 \text{ m/s}$$

$$\tan\theta = \frac{(27.78)^2}{9.81\times 250}$$

$$\tan\theta = 0.3146 = 17.46° \quad \textit{Ans.}$$

Example 15.20. A vehicle weighing 10000 N is to turn a circular corner of radius 100 m on a level road with a speed of 10 m/s. The height of it C.G. above the road is 1 m and the distance between its wheels is 1.5 m. Find the reactions at the wheels.

At what maximum speed can it travel corner without the fear of overturning?

Solution: $W = 10000$ N, $r = 100$ m $v = 10$ m/s

$h = 1$ m $\quad 2b = 1.5$ m

$$R_A = \frac{W}{2}\left(1 - \frac{v^2 h}{grb}\right)$$

$$R_B = \frac{W}{2}\left(1 + \frac{v^2 h}{grb}\right)$$

Substituting the values

$$R_A = \frac{10000}{2}\left(1 - \frac{(10)^2\times 1}{9.81\times 100\times 1.5}\right)$$

$$R_A = 4660.2 \text{ N} \quad \textit{Ans.}$$

$$R_A = \frac{10000}{2}\left(1 + \frac{(10)^2\times 1}{9.81\times 100\times 1.5}\right)$$

$$R_B = 5339.8 \text{ N} \quad \textit{Ans.}$$

The maximum speed to avoid overturning on the level road

$$v_{max} = \sqrt{\frac{grb}{h}} = \sqrt{\frac{9.81\times 100}{1}\times\frac{1.5}{2}}$$

$$v_{max} = 27.12 \text{ m/s} \quad \textit{Ans.}$$

Example 15.21. A bob weighing 1 N suspended by a cord from the ceiling of a railway carriage was found to make an angle $\theta = 8°$ with vertical when the railway carriage was negotiating a curve of 100 m.

Find the speed of the carriage and the tension in the cord.

Solution: Let T be the tension in the cord. Writing the equations of dynamic equation of the bob,

$$\Sigma F_x = 0: \quad \frac{W}{g}\left(\frac{v^2}{r}\right) - T\sin\theta = 0 \quad \text{...(i)}$$

$$\Sigma F_y = 0: \quad T\cos\theta - W = 0 \quad \text{...(ii)}$$

From (i) and (ii)

$$\tan\theta = \frac{v^2}{gr}$$

or $$v = \sqrt{gr\tan\theta}$$

$$v = \sqrt{9.81\times 100\times \tan 8°}$$

$$v = 11.74 \text{ m/s}$$

or $$v = 42.27 \text{ km/hour} \quad \textit{Ans.}$$

$$T = \frac{W}{\cos\theta} = \frac{1.0}{\cos 8°}$$

$$T = 1.01 \text{ N} \quad \textit{Ans.}$$

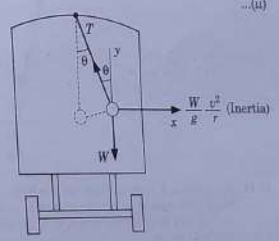


Fig. 15.35

PROBLEMS

15.8. A ball of weight W is attached to bar AB which is hinged to the wall at A. The ball is held in position by a string CB in a vertical plane as shown in Fig. P.15.8. Determine the compressive force in the bar AB just before the string is cut and after the string is cut. [0.653 W, 0.5 W]

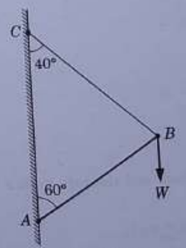


Fig. P.15.8

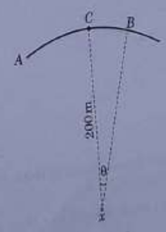


Fig. P.15.9

15.9. A roadway bridge is in the form of a circular arc of radius 200 m. Find the greatest speed with which a vehicle can cross the bridge without just loosing contact at the highest point C of the roadway Fig. P.15.9.

If the speed of the vehicle is reduced to 72 km/hour, at what point B, as defined by the angle θ, the vehicle will just loose contact with the roadway?

[159.4 km/hour, 78.4°]

15.10. A particle is moving with constant angular velocity 'ω' in a horizontal circle inside a smooth hollow conical surface as shown in Fig. P.15.10. Find its distance r from the axis.

$\left[\dfrac{g}{\omega^2}\tan\theta\right]$

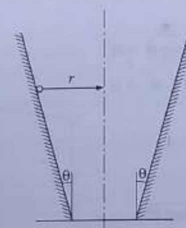


Fig. P.15.10

15.11. A small block rests on turn-table which, starting from rest, is rotated in such a way that the block undergoes a constant tangential acceleration $a_t = 2\ \text{m/s}^2$ (Fig. P.15.11) Determine (*a*) how long it will take for the block to start slipping on the turntable (*b*) the speed v of the block at that instant.

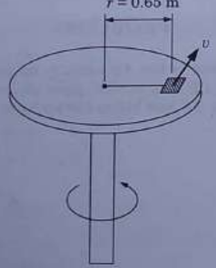


Fig. P.15.11

Assume the coefficient of static friction between the block and the turn-table $\mu = 0.60$.
[1.896 m/s, 0.948 s]

[**Hint** : acceleration of this block $a = \sqrt{a_t^2 + a_n^2}$ and $ma = mg\mu$].

15.12. An automobile is negotiation a curve of radius 100 m. The distance between the centre to centre of the wheels 1.2 m and the height of the centre of gravity of the automobile is 75 cm above the ground. Find the maximum speed in m/s to avoid overturning.

[28 m/s]

15.13. A motor cycle rider travelling at 18 km/hour has to turn a corner. Find the least radius of the curve he should follow for the safe travelling if the coefficient of friction between the tyres and the road is 0.2.

[12.74 m]

15.14. Two wires AB and CB are tied to a sphere at B which revolves at a constant speed v in a horizontal circle as shown in Fig. P.15.14. If the tension is same in both portions of the wires determine the speed v.

[8.63 m/s]

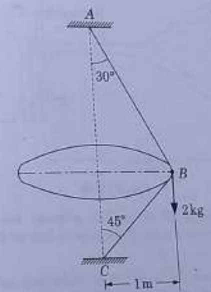


Fig. P.15.14

15.15. A ball of weight W moves with uniform speed v along a vertical cosine curve ABC. Find the pressure exerted by the ball on the path as it passes the point B (Fig. P.15.15).

$\left[W\left(1 + \dfrac{4\pi^2 h v^2}{g l^2}\right)\right]$

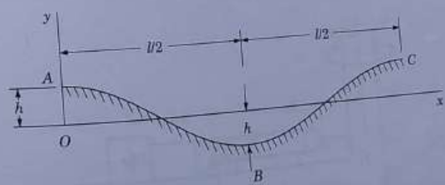


Fig. P.15.15

15.16. The arrangement shown in Fig. P.15.16. rotates about the vertical axis OO at a constant r.p.m. The weight of the vertical bar AB, hinged at C, is 15 N and the weight of the ball at the top is 30 N. When the system is at rest, the initial tension in the spring DE is 100 N. At what speed in r.p.m. will the contact at M be broken?

[47.5 r.p.m.]

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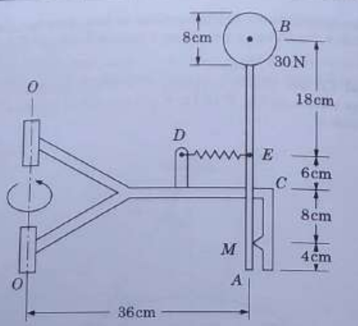

Fig. P.15.16

15.17. A railway track has a radius of 400 m. The distance between the rails is 2 m and the superelevation of the outer rail is 5 cm. If there is no lateral thrust on the rails find the speed of the train. [35.66 km/h]

15.18. A curve in a speed track has a radius of 150 m. There was no lateral friction force exerted on the wheels of a car travelling at 135 km/h. Find the banking angle θ of the track.

If a racing car starts skidding when travelling at the speed of 300 km/h determine the coefficient of static friction between the tyres and the track. [43.7°, 0.68]

15.19. Two blocks having weights and position as shown in Fig. P.15.19 rest on a turn table which rotates about the vertical axis at a constant speed. The coefficient of friction between blocks and the table is 0.2. Neglecting the friction and weight of the pulley calculate the speed at which the blocks will start to slide. What is the tension in the string? $\begin{bmatrix} N = 16.3\,\text{rpm} \\ T = 16.35\,\text{N} \end{bmatrix}$

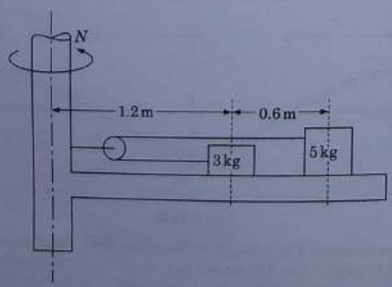

Fig. P.15.19

15.20. A rod ABC supporting a ball of mass 10 kg at its lower end rotates at 20 rpm about a vertical axis through A as shown in Fig. P.15.20. It is fixed in position by the rod BD. Neglecting the masses of rods AC and BD, calculate the force in the rod BD. Is the force compressive or tensile? Calculate the speed in rpm when this force is equal to zero. [39.6 N (comp.), 26.26 rpm]

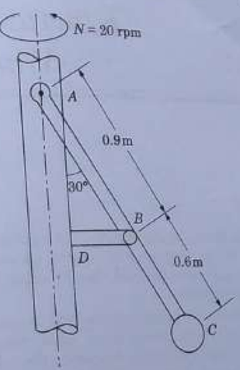

Fig. P.15.20

16

CHAPTER

Kinetics of a Particle: Work and Energy

16.1 INTRODUCTION

In the preceding chapters the problems dealing with the motion of particle were solved by applying Newton's second law $\mathbf{F} = m\mathbf{a}$. If a particle is acted upon by a known force $\mathbf{F}$ we can determine its acceleration a by applying the above law and then by using the principles of kinematics, determine its velocity and displacement.

An alternative method of solving the problems involving the motion of a particle is to apply the method of work and energy which relates force, mass, velocity and displacement. The advantage of this method is that we can directly determine the velocity of the particle without requiring to determine its acceleration. Also as work and energy are scalar, we have to deal with scalar quantities.

16.2 WORK OF A FORCE

If a particle is subjected to a force $\mathbf{F}$ and the particle is displaced by an infinitesimal displacement $\mathbf{ds}$ the work done dU by the force is given by

$$dU = F ds \cos\alpha$$

where α is the angle between the force and the displacement vector [Fig. 16.1 (*a*)].

Thus, the work done by a force during an infinitesimal displacement is equal to the product of the displacement ds and the component of the force $F\cos\alpha$ in the direction of the displacement.

The work done by a force during a finite displacement from position P_1 to P_2 can be obtained by integrating the above equation,

$$U_{1-2} = \int_{S_1}^{S_2} (F\cos\alpha)\, ds$$

where the displacements S_1 and S_2 are measured along the path is shown in Fig. 16.1 (*b*).

Work done by a constant force in rectilinear motion. If a particle moves along a straight line from position* 1 to position 2 under a constant force $\mathbf{F}$

W.D., $$U_{1-2} = \int_{x_1}^{x_2} (F\cos\alpha)dx$$

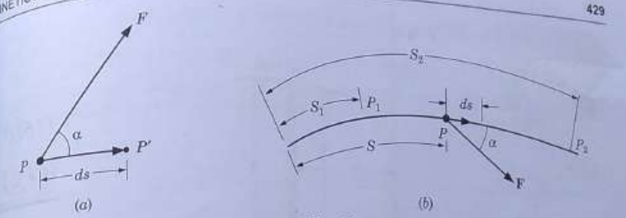

Fig. 16.1

W.D. for a displacement x from the origin,

$$U_{1-2} = (F\cos\alpha)x$$

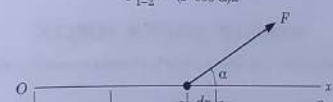

Fig. 16.1 (*c*)

Graphically the work done by a force can be represented by the area under the curve shown in the Fig. 16.2.

Units. Unit of work is newton-metre (Nm) or joule (J). One joule may be defined as the work done by a force of one newton when its point of application moves by one metre along the line of action of the force.

Following points may be noted the work of a force:

1. Work done by a force is zero if either the displacement is zero or the force acts normal to the displacement. For example, gravity force does no work when a body moves horizontally.
2. Work done by a force is positive if the direction of the force and the direction of displacement are the same. For example, work done by force of gravity is positive when a body moves from a higher elevation to a lower elevation. A positive work can be said as the work done by a *force* and a negative work as the work done *against a force*.
3. Work is scalar quantity and has magnitude and a sign but no direction.
4. Work done by a force depends on the path over which the force moves except in the case of *conservative forces*. Force due to gravity, spring force, elastic force are conservative forces, whereas, friction force is a *non-conservative* force. This point shall be explained later.

* Subscripts 1 and 2 denote the two positions of the particle which can also be called as the initial and the final positions of the particle.

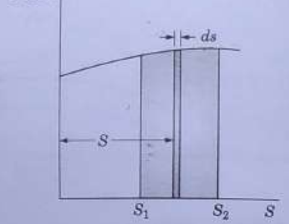

Fig. 16.2

WORK OF VARIOUS FORCES

Work of the Force of Gravity. Consider a particle of mass m moving along a path in a vertical plane x-y (Fig. 16.3).

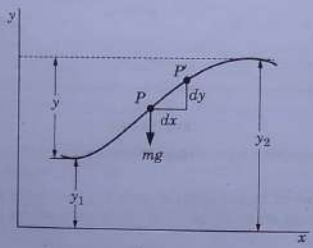

Fig. 16.3

Let the particle P be displaced vertically by dy to a new position P'.

Work done $dU = (-mg)(dy)$ — dy is positive upwards, mg is negative upwards

Work of gravity force from position 1 to position 2

$$U_{1-2} = \int_{y_1}^{y_2} -mg\,dy$$

$$U_{1-2} = -mg(y_2 - y_1)$$

$$U_{1-2} = -mgy \qquad \ldots(16.1)$$

Work of gravity force as calculated above comes out to be negative as the direction of force and the direction of displacement are opposite.

Work done by gravity force is positive when a particle moves from a higher elevation to a lower elevation.

Gravity force does no work when a body moves horizontally.

Work of the Force of Spring. Consider a spring of stiffness k (or spring constant) which is stretched a distance x from its undeformed position as shown in the Fig. 16.4.

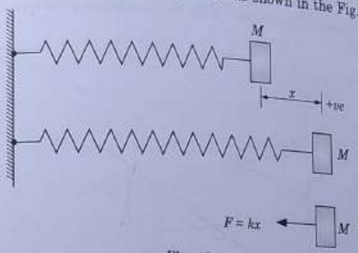

Fig. 16.4

The stiffness or spring constant k of a spring is the force required to deform it by a unit length. Force exerted by the spring on a body of mass M, $F = kx$.

It should be noted here that the force exerted by a spring is not a constant force but it varies linearly with the displacement x from the undeformed position.

Work done on the spring (or against the spring force) during a small displacement dx $\quad dU = -F\,dx$

The force and the displacement are in opposite directions, therefore, work is negative.

Work done on the spring when stretched from a length x_1 to x_2

$$U_{1-2} = \int_1^2 dU = -\int_{x_1}^{x_2} F\,dx$$

Substituting

$$F = kx$$

$$U_{1-2} = -\int_{x_1}^{x_2} (kx)\,dx = -\frac{1}{2}k(x_2^2 - x_1^2)$$

$$U_{1-2} = -\frac{1}{2}k(x_2^2 - x_1^2)$$

Let $kx_1 = F_1$, the force of the spring when stretched a distance x_1 from undeformed position.

$kx_2 = F_2$, the force of the spring when stretched a distance x_2 from the undeformed position.

Then,

$$U_{1-2} = -\frac{1}{2}k(x_2^2 - x_1^2) = -\frac{1}{2}k(x_2 + x_1)(x_2 - x_1)$$

which can be expressed as,

$$U_{1-2} = -\frac{(F_1 + F_2)}{2}(x_2 - x_1) \qquad ...(16.2)$$

$$U_{1-2} = -\frac{1}{2} \text{ (Average Force) (Displacement)}$$

Graphically it is represented by the area of the trapezoid under the force-displacement diagram as shown in Fig. 16.5.

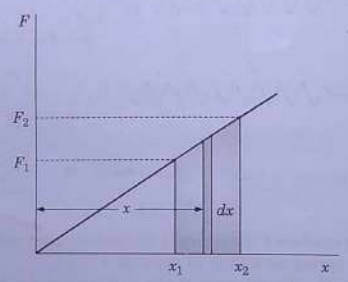

Fig. 16.5

Work done in stretching a spring is negative or the work is done *on* the spring.

Work done when a spring returns towards undeformed position is positive or the work is done *by* the spring.

Work of the Elastic Force. If a prismatic bar of area of cross-section A, length L and elastic constant E is stretched then the work of elastic force can be calculated by treating it as a spring of stiffness k.

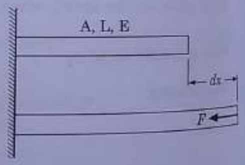

Fig. 16.6

$$k = \frac{AE}{L} \qquad ...(16.3)$$

k is to be calculated using the consistent units of A, E and L.

Example 16.1. A spring of stiffness 1000 N/m is stretched by 10 cm from the undeformed position. Find the work of the spring force. Also, find the work required to stretch it by another 10 cm.

Solution:

$$k = 1000 \text{ N/m}$$
$$x_1 = 10 \text{ cm} = 0.1 \text{ m}$$
$$x_2 = 20 \text{ cm} = 0.2 \text{ m}$$

Work required to stretch the spring by 10 cm from the undeformed position

$$U_{0-10} = -\frac{k}{2}(x_1^2 - x_0^2)$$

$$U_{0-10} = -\frac{k}{2}x_1^2$$

$$U_{0-10} = -\frac{1}{2} \times 1000(0.1)^2$$

$$U_{0-10} = -5 \text{ Nm} \quad \textit{Ans.}$$

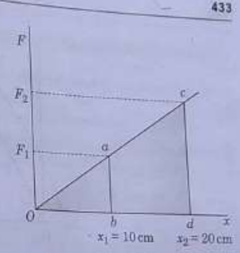

Fig. 16.7

U_{0-10} is the area of the triangle oab.

Work required to stretch from 10 cm to 20 cm

$$U_{10-20} = -\frac{1}{2}k(x_2^2 - x_1^2)$$

$$U_{10-20} = -\frac{1}{2} \times 1000(0.2^2 - 0.1^2)$$

$$U_{10-20} = -15 \text{ Nm} \quad \textit{Ans.}$$

U_{10-20} is the area of the trapezoid $abcd$.

It may be noted here that work required to stretch the spring for the first 10 cm is 5 Nm and for next 10 cm it is 15 Nm.

16.3 ENERGY OF A PARTICLE

In the simplest sense it can be defined as the capacity to do work. Which implies, that energy has the same unit as work and it is a scalar quantity like work. Energy can manifest itself in many forms like mechanical energy, thermal energy, electric energy etc. Here we are mainly concerned with the mechanical energy of a particle, which consists of its potential energy and kinetic energy.

Kinetic Energy of a Particle. It is the energy possessed by a particle by virtue of its motion. If a particle of mass m is moving with velocity v, its kinetic energy T is given by,

$$T = \frac{1}{2}mv^2$$

Unit. Unit of K.E. is

$$\text{kg}\left(\frac{\text{m}}{\text{s}}\right)^2 = \left(\frac{\text{kgm}}{\text{s}^2}\right)\text{m or Nm or J.}$$

16.4 PRINCIPLE OF WORK AND ENERGY

Consider a particle P of mass m acted upon by a force **F** and moving with velocity **v** along a path which can be rectilinear or curved as shown in Fig. 16.8.

At any position P of the particle let its distance from the reference point O along the path be S. Let α be the angle that the force vector makes with the tangent to the path at P. Resolve the force F along the tangent and normal to the path at P.

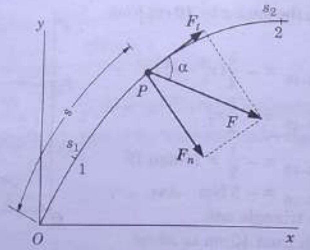

Fig. 16.8

Tangential component of the force, $F_t = F\cos\alpha$

Normal component of the force. $F_n = F\sin\alpha$

The equation of motion in the tangential direction of the particle is,

$$F_t = ma_t$$ where, a_t is the acceleration of the particle in tangential direction

$$F\cos\alpha = ma_t$$

As, $$a_t = \frac{dv}{dt}$$

Therefore, $$F\cos\alpha = m\frac{dv}{dt} = m\frac{dv}{ds}\cdot\frac{ds}{dt}$$

$$\frac{ds}{dt} = v$$

$$F\cos\alpha = mv\frac{dv}{ds}$$

Let v_1 and v_2 be the velocities of the particle at point 1 and 2 and the corresponding distance be S_1 and S_2.

Integrating the above equation,

$$\int_{S_1}^{S_2}(F\cos\alpha)dS = \int_{v_1}^{v_2} mv\,dv$$

$$\int_{S_1}^{S_2}(F\cos\alpha)dS = \left(\frac{mv_2^2}{2} - \frac{mv_1^2}{2}\right)$$

Left hand side of the above equation represents the work (U_{1-2}) of the force **F** exerted on the particle during its displacement from position 1 to 2. Right hand side represents the change in kinetic energy of the particle.

Hence, $$U_{1-2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2}$$

Or $$U_{1-2} = (T_2 - T_1) \qquad \text{...(16.4)}$$

The above expression is a symbolic representation of the Work-Energy Principle which can be stated as follows :

The work done by a force acting on a particle during its displacement is equal to the change in the kinetic energy of the particle during that displacement.

It may be recalled here that work and energy are both scalar quantities.

Example 16.2. A block of weight W slides down an inclined plane from rest from an elevation h, as shown in Fig. 16.9. Find the velocity of the block when it reaches the point B.

Solution:

Position 1. Block is at rest, velocity is zero.

Position 2. Block after travelling a distance x attains velocity v.

Forces Involved : The block is displaced a distance x along AB. The forces acting in the direction of displacement are $W\sin\theta$ and frictional force $\mu R = \mu W\cos\theta$ as shown in Fig. 16.9.

$$\text{K.E. in the position } 1 = T_1 = 0$$

$$\text{K.E. in the position } 2 = T_2 = \frac{W}{g}\frac{v^2}{2}$$

Work done during the displacement x,

$$U_{1-2} = \text{Work of gravity force} - \text{Work done against friction force}$$

$$U_{1-2} = (W\sin\theta)x - (\mu W\cos\theta)x$$

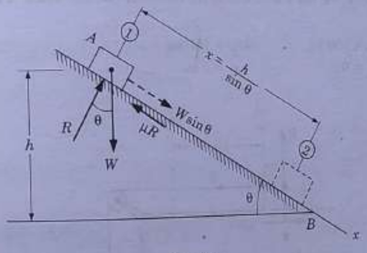

Fig. 16.9

Applying the principle of work-energy,

$$U_{1-2} = T_2 - T_1$$

$$(W\sin\theta - \mu W\cos\theta)x = \frac{W}{g}\frac{v^2}{2}$$

Substituting, $$x = \frac{h}{\sin\theta}$$

$$(W\sin\theta - \mu W\cos\theta)\frac{h}{\sin\theta} = \frac{W}{g}\frac{v^2}{2}$$

$$v = \sqrt{2gh(1-\mu\cot\theta)} \quad \textit{Ans.}$$

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16.5 WORK AND ENERGY PRINCIPLE FOR A SYSTEM OF PARTICLES

The work and energy principle was stated for a single particle in motion. Now let us consider a system of particles in motion, for example, the motion of two bodies connected by a string. To apply the principle of work and energy to a system of particles, we add up the change of kinetic energy of all the particles and equate it to the work of all the forces involved during the displacement of the system of particles.

Symbolically we can write,

$$\Sigma U_{1-2} = \Sigma(T_2 - T_1) \quad ...(16.5)$$

Where, ΣU_{1-2} is the work of all the force acting on the various particles.

Work of Internal Forces. In a problem involving a system of particles, internal forces (like tension in the string) may appear. While calculating the work of all forces, work of internal forces must also be included. In some problems, however, the internal forces may appear as a pair of equal and opposite forces (like action and reaction) *moving through equal distances.* In such a problem therefore, the work of internal forces may become zero.

Example 16.3. Two block A and B of masses 100 kg and 150 kg are connected by a string (Fig. 16.10). If the system is released from rest, find the velocity of the block A after it has moved a distance of 1 m. Assume the coefficient of friction between the block A and B the horizontal plane $\mu = 0.20$.

Solution: $M_A = 100$ kg, $M_B = 150$ kg

$\mu = 0.2$, $x = 1$ m

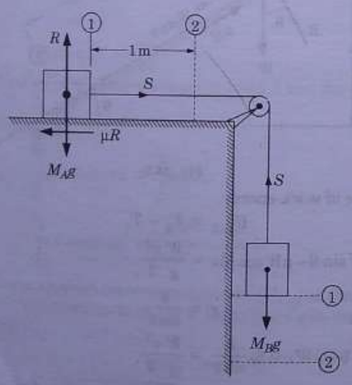

Fig. 16.10

It should be noted that the principle of conservation of energy cannot be applied because a non-conservative force (friction) is involved.

Let S be the tension in the string and v be the velocity of the block A after it has moved a distance $x = 1$ m to the position 2. The velocity of the block B shall also be v.

Block A : Change in K.E.

$$(T_2 - T_1) = \left(\frac{m_A v^2}{2} - 0\right) = \frac{100}{2} v^2 = 50v^2$$

W.D. $= U_{1-2}$ = Work of tension – Work done to overcome friction

$$= S(x) - \mu R(x) = S(x) - (\mu M_A g)x$$

$$U_{1-2} = S(1) - 0.2 \times 100 \times 9.81(1)$$

Block B : Change in K.E.

$$(T_2 - T_1) = \left(\frac{m_B v^2}{2} - 0\right) = \frac{m_B v^2}{2} = \frac{150v^2}{2}$$

$$(T_2 - T_1) = 75v^2$$

W.D. $= U_{1-2}$ = Work of gravity force – Work of tension

(Work done by tension S is negative as the directions of force and displacement are opposite)

$$U_{1-2} = M_B g(x) - S(x)$$

$$U_{1-2} = 150 \times g(1) - S(1)$$

Applying the principle of work and energy to the system of blocks A and B.

$$\Sigma U_{1-2} = \Sigma(T_2 - T_1)$$

$$(S - 0.2 \times 100 \times 9.81) + ((150 \times 9.81) - S) = 50v^2 + 75v^2$$

$$1275.3 = 125v^2$$

$$v = 3.19 \text{ m/s} \quad \textit{Ans.}$$

It may be noted that the work of the internal force of tension S, on the system as a whole, is zero.

16.6 POTENTIAL ENERGY AND CONSERVATIVE FORCES

Potential Energy. The potential energy (P.E.) of a particle is the energy possessed by a particle by virtue of its position.

Consider a particle of mass m which moves from position 1 of elevation y_1 to position 2 of elevation y_2, following the path 1 as shown in Fig. 16.11.

Work of gravity force,

$$U_{1-2} = \int_{y_1}^{y_2} (-mg)\,dy$$

$$\left.\begin{aligned} U_{1-2} &= -mg(y_2 - y_1) \\ \text{Or} \quad U_{1-2} &= -mgy \end{aligned}\right] \quad ...(16.6)$$

U_{1-2} is the work done against the gravity force (hence negative) and represents the increase in potential energy of the particle of mass m. *This can also be understood by saying that the work done against the force of gravity is stored as the potential energy of the particle.*

Next consider the particle to follow a different path 2, between the position 1 and 2. The work done against the gravity force would still be the same and equal to mgy.

So we can conclude that the *work of gravity force is independent of the path followed and depends on the initial and final values of the function mgy. This function is called the potential energy of the particle due to gravity.*

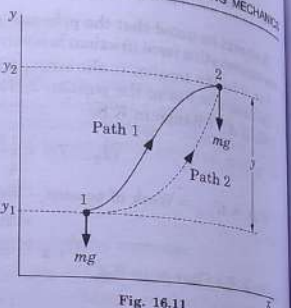

Fig. 16.11

Unit of potential energy : $(\text{kg})\left(\dfrac{m}{s^2}\right)m$ = Nm or joule (J).

The potential energy of a particle shall be represented by the symbol V.

Conservative Forces. If the work of a force in moving a particle between two positions is independent of the path followed by the particle and can be expressed as a change in its potential energy, then such a force is called as a conservative force.

Gravity force is a conservative force whereas, the frictional force is a non-conservative force. The work of friction force depends upon the path followed and this work cannot be expressed as a change in the potential energy so it is a non-conservative force. The various conservative forces are, force due to gravity, spring force and elastic force.

16.7 PRINCIPLE OF CONSERVATION OF ENERGY

Find the principle of work and energy we can write.

$$\text{Work done} = \text{change in the kinetic energy}$$

$$U_{1-2} = T_2 - T_1$$

If a particle moves under the action of a conservative force work done is stored as potential energy

$$U_{1-2} = -(V_2 - V_1)$$

$$\text{Work done} = -\text{(negative change of P.E.)}$$

Combining the above two equations

$$T_2 - T_1 = -(V_2 - V_1)$$

$$T_1 + V_1 = T_2 + V_2$$

$$(\text{K.E.})_1 + (\text{P.E.})_1 = (\text{K.E.})_2 + (\text{P.E.})_2 \quad \ldots(16.7)$$

Which means that sum of the potential energy and the kinetic energy of a particle (or of a system of particles) remains constant during the motion under the action of conservative forces.

Denoting the sum of the potential energy and the kinetic energy by E the above equation can be written as

$$E_1 = E_2 \quad \ldots(16.8)$$

The principle of conservation of energy can be applied to a particle or to system of particles only under the action of conservative forces. *Where frictional force is involved, this principle cannot be applied.*

For example in the case of a simple pendulum of the sum of the P.E. and K.E. remains constant in any position of the pendulum provided the frictional force at the support and the force due to air resistance are negligibly small; both of which are non-conservative forces.

An explanatory note perhaps is required about the expression,

$$U_{1-2} = -(V_2 - V_1) = (V_1 - V_2)$$

For example if a particle moves from a lower elevation to a higher elevation, the potential energy of the particle increases ($V_2 > V_1$) but the work U_{1-2} of the gravity force is negative.

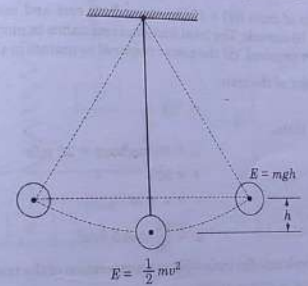

Fig. 16.12

Another important point about the potential energy of a particle is that it has to be determined with respect to some reference which can be chosen arbitrarily as per convenience.

16.8 POWER

It is defined as the time rate at which work is done. Either a small machine or a large machine can be used to do a given amount of work. But a small machine may require more time to do the work than by a large machine. Therefore, in selecting an engine or a machine, rate or doing work or power is much more important criterion than the actual amount of work to be performed.

Average Power. If ΔU is the work done during an interval of time Δt, then

$$\text{Average power} = \frac{\Delta U}{\Delta t}$$

$$\text{Instantaneous power} = \underset{\Delta t \to 0}{\text{Limit}} \frac{U\Delta}{\Delta t} = \frac{dU}{dt}$$

As,

$$dU = (F\cos\alpha)ds$$

and

$$v = \frac{ds}{dt}$$

therefore,
$$\text{power} = \frac{dU}{dt} = \frac{(F\cos\alpha)ds}{dt} = (F\cos\alpha)v$$
$$\text{Power} = (F\cos\alpha)v \qquad ...(16.9c)$$

where, **v** is the velocity of the point where the force **F** is acting and α is the angle between the directions of the force and the velocity. If both are in the same direction, then
$$\text{Power} = F \times v \qquad ...(16.9d)$$

In SI units, power is expressed in Joule/second (J/s) and; this unit is called a watt (W)
$$1\text{ W} = 1\text{ J/s} = \text{Nm/s}$$
$$\text{One metric horse power} = 735.5\text{ watt.}$$

Example 16.4. A train of mass 500×10^3 kg starts from rest and accelerates uniformly to a speed of 90 km/hour in 50 seconds. The total frictional resistance to motion is 15 kN. Determine (*a*) the maximum power required, (*b*) the power required to maintain the speed of 90 km/hour.

Solution: Initial velocity of the train,
$$u = 0$$
Final velocity of the train,
$$v = 90\text{ km/hour} = 25\text{ m/s}$$
Time taken $\quad t = 50$ s

Using $\quad v = u + at$

Acceleration $\quad a = \frac{25}{50} = 0.5\text{ m/s}^2$

The force required to accelerate the train = Mass × acceleration of the train
$$= 500 \times 10^3 \times 0.5 = 250 \times 10^3\text{ N}$$
Frictional force present $= 15 \times 10^3$ N

Total force required = Force required to overcome friction + Force required to accelerate the train

The maximum power is required at
$$t = 50\text{ s}$$
When
$$v = 25\text{ m/s}$$
$$\text{power} = F \cdot v = (250 + 15)10^3 \times 25\ W$$
$$\text{power} = 6.625\text{ MW} \quad \textit{Ans.}$$

At any time after 50 seconds, the force is required only to overcome the frictional resistance of 15×10^3 N.
$$\text{Power required} = Fv = 15 \times 10^3 \times 25\text{ watt}$$
$$\text{Power} = 375\text{ kW} \quad \textit{Ans.}$$

Example 16.5. A weight $W = 50$ N is suspended from a spring of stiffness 10 N/cm and is in equilibrium (Fig. 16.13). Calculate the potential energy of the system with reference to this position, if the weight is pulled down to the floor by a distance of 10 cm. If the weight is then released, what is the maximum height above the floor that the weight W will attain after the release.

Solution: As conservative forces are involved, the principle of conservative of energy can be used
$$W = 50\text{ N}, \quad k = 10\text{ N/cm}$$
Position 1. Weight of 50 N is hanging from the spring in equilibrium so the force exerted by the spring.
$$F_1 = W = 50\text{ N}$$

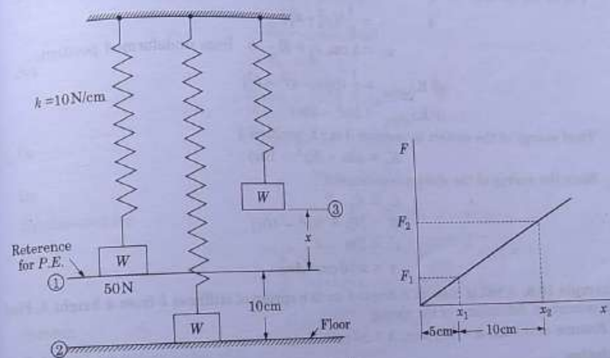

Fig. 16.13

Extension of the spring from the underformed position
$$x_1 = \frac{F_1}{k} = \frac{50}{10} = 5\text{ cm}$$
P.E. of the system of the spring and the weight in position 1 is zero, being the reference position.

Position 2. When pulled by 10 cm to the floor

P.E. of the weight $\quad W = -mgh = -Wh = -50 \times 10$
$$(\text{P.E.})_{gravity} = -500\text{ N cm}$$
P.E. of the spring with respect to the position 1.
$$(\text{P.E.})_{spring} = \frac{1}{2}k\left(x_2^2 - x_1^2\right)$$
$x_2 = 15$ cm, $x_1 = 5$ cm *are the extensions measured from the undeformed position.*
$$(\text{P.E.})_{spring} = \frac{1}{2} \times 10\,(15^2 - 5^2) = 1000\text{ N cm}$$
Total P.E. of the system with respect to the position 1
$$= -500 + 1000$$
Total energy of the system
$$E_2 = +500\text{ N cm} \quad \textit{Ans.}$$

Position 3. When the weight is released let it occupy the position 3 at a height x from the reference position 1 where its K.E. becomes zero. But the energy of the system shall be conserved.

P.E. of the weight w.r.t. position 1

$$= Wh = 50x$$

P.E. of the spring w.r.t. position 1

$$= \frac{1}{2}k(x_3^2 - x_1^2)$$

$$x_1 = 5 \text{ cm}, \ x_3 = (5 - x) \quad \text{from undeformed position.}$$

$$(\text{P.E.})_{spring} = \frac{1}{2}10[(5 - x)^2 - 5^2]$$

$$(\text{P.E.})_{spring} = 5(x^2 - 10x)$$

Total energy of the system in position 3 w.r.t. position 1

$$E_3 = 50x + 5(x^2 - 10x)$$

Since the energy of the system is conserved

$$E_2 = E_3$$

$$500 = 50x + 5(x^2 - 10x)$$

$$x^2 = 100$$

$$x = \pm\ 10 \text{ cm} \quad \textit{Ans.}$$

Example 16.6. A ball of mass m is dropped on to a spring of stiffness k from a height h. Find the maximum deflection δ of the spring.

Assume $m = 5$ kg, $k = 500$ N/m, $h = 10$ cm.

Solution:

Position 1. The ball is at rest at a height h from the top of the spring.

Position 2. The ball touches the spring after falling a height h.

Position 3. The ball and the spring travel a further distance δ and come to rest in this position.

We shall apply the principle of conservation of energy. Work-energy principle is also applicable in this case.

Position 1. P.E. of the ball $V_1 = mgh$

K.E. of the ball $T_1 = 0$

Total energy $E_1 = V_1 + T_1 = mgh$

Position 2. P.E. of the ball $V_2 = 0$

K.E. of the ball $T_2 = \frac{1}{2}mv^2$

Total energy $E_2 = \frac{1}{2}mv^2$

where, v the velocity of the ball in position 2.

Energy of the ball is conserved,

$$E_1 = E_2$$

$$mgh = \frac{1}{2}mv^2$$

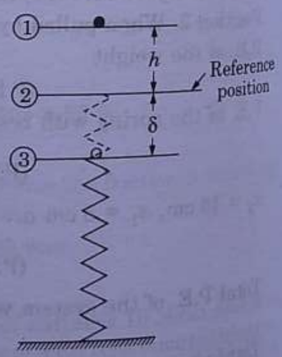

Fig. 16.14

Position 3. K.E. of both the ball and the spring $T_3 = 0$

$$\text{P.E.} = (\text{P.E.})_{spring} + (\text{P.E.})_{ball}$$

$$V_3 = \frac{1}{2}k\delta^2 - mg\delta$$

Total energy of the ball and the spring in position 3,

$$E_3 = \frac{1}{2}k\delta^2 - mg\delta$$

But

$$E_2 = E_3$$

$$\frac{1}{2}mv^2 = \frac{1}{2}k\delta^2 - mg\delta$$

Or

$$mgh = \frac{1}{2}k\delta^2 - mg\delta$$

Or

$$\delta^2 - \frac{2mg\delta}{k} - \frac{2mgh}{k} = 0 \quad ...(i)$$

Substituting for

$$m = 5 \text{ kg}, \ k = 500 \text{ N/m}, \ h = 0.1 \text{ m}$$

$$\delta^2 - \frac{2 \times 5 \times 9.81}{500}\delta - \frac{2 \times 5 \times 9.81}{500} \times 0.1 = 0$$

$$\delta^2 - 0.1962\,\delta - 0.01962 = 0$$

Solving,

$$\delta = \frac{0.1962 \pm 0.3420}{2} \text{ m}$$

Taking the positive value of δ,

$$\delta = 26.91 \text{ cm} \quad \textit{Ans.}$$

Further extension of the Problem:

Weight of the ball $W = mg$

Equation (i) can be written as

$$\delta^2 - \frac{2W}{k}\delta - \frac{2W}{k}h = 0$$

Let

$$\delta_{static} = \frac{W}{k}$$

$$\delta^2 - 2\delta_{static}(\delta) - 2\delta_{static}(h) = 0$$

$$\delta = \frac{2\delta_{static} \pm \sqrt{4\delta_{static}^2 + 4 \times 2h\delta_{static}}}{2}$$

If

$$h = 0 \quad \delta = 2(\delta_{static})$$

That is, if a weight W(corresponding to the mass m of the ball) is placed gently on the spring its deflection is

$$\delta_{static} = \frac{W}{k}.$$

But if a load W is suddenly applied, the deflection of the spring would be twice as large as that obtained by the gradual application of the same load W.

Example 16.7. If the spring of the previous example is placed horizontally (Fig. 16.15) and the same ball now strikes the spring with a velocity equal to that attained by a vertical fall of height h = 10 cm, find the maximum compression of the spring.

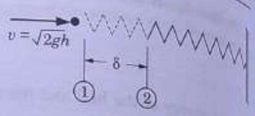

Fig. 16.15

Solution: m = 5 kg, k = 500 N/m, h = 10 cm

Position 1. Velocity of the ball $v = \sqrt{2gh}$

$$\text{K.E. of the ball} = \frac{1}{2}mv^2 = \frac{1}{2}m(\sqrt{2gh})^2$$

$$(\text{K.E.})_{\text{ball}} = mgh$$

$$(\text{P.E.})_{\text{ball}} = 0$$

$$(\text{P.E.})_{\text{spring}} = 0$$

Total energy of the ball and the spring $E_1 = mgh$

Position 2. Let the spring compress by δ

K.E. of the ball = 0

P.E. of the ball = 0

K.E. of the spring = 0

$$\text{P.E. of the spring} = \frac{1}{2}k\delta^2$$

Total energy of the ball and the spring

$$E_2 = \frac{1}{2}k\delta^2$$

Equating $E_1 = E_2$

$$mgh = \frac{1}{2}k\delta^2,$$

$$\delta = \sqrt{2\frac{mgh}{k}}$$

$$\delta = \sqrt{\frac{2\times5\times9.81\times0.1}{50}}$$

$$\delta = 0.140 \text{ m} \quad \textit{Ans.}$$

Example 16.8. A mass m can slide freely up and down a circular steel bar of area of cross-section a, length l and modulus of elastically E. The mass m is allowed to fall through a vertical distance h (Fig. 16.16). Find the maximum elongation of the bar.

Solution: Consider the bar as a spring of stiffness k, where

$$k = \frac{aE}{l}$$

Let us solve this problem using the principle of work and energy.

Let the elongation of the bar be δ.

Position 1.

K.E. of the system $T_1 = 0$

Position 2.

K.E. of the system $T_2 = 0$
(mass comes to rest)

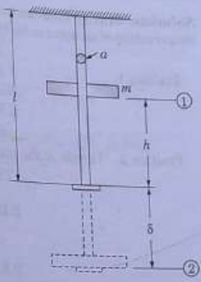

Fig. 16.16

Work done during the displacement from position 1 to position 2 be U_{1-2}.

$$U_{1-2} = mg(h+\delta) \quad - \frac{1}{2}k\delta^2$$

(W.D. by gravity) (W.D. against the elastic force)

$$U_{1-2} = T_2 - T_1$$

$$mg(h+\delta) - \frac{1}{2}k\delta^2 = 0$$

Or
$$mg(h+\delta) = \frac{1}{2}k\delta^2$$

(Or, the work done by gravity force is stored as the elastic energy in the bar.)

Or
$$\delta^2 - \frac{2mg\delta}{k} - \frac{2mgh}{k} = 0$$

Substituting
$$k = \frac{aE}{l}$$

$$\delta^2 - \frac{2mg\delta}{aE/l} - \frac{2mgh}{aE/l} = 0$$

The above equation can be solved to determine δ.

Example 16.9. A collar of mass 5 kg can slide along a vertical bar as shown. The spring attached to the collar is in undefomred state of length 20 cms and stiffness 500 N/m. If the collar is suddenly released find the velocity of the collar if it moves 15 cm down in Fig. 16.17.

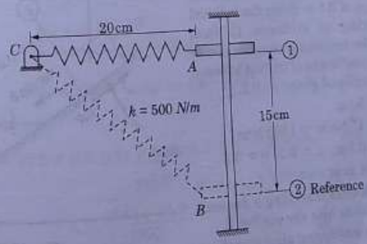

Fig. 16.17

Solution: Gravity force and spring force are both conservative forces so the principle of conservation of energy can be applied.

$$m = 5 \text{ kg}, \; k = 500 \text{ N/m}$$

Position 1.

$$\underset{(\text{mass})}{\text{P.E.}} = \underset{(\text{spring})}{mg(0.15) + 0}$$

$$\text{K.E.} = 0$$

$$E_1 = mg(0.15) = 5 \times 9.81 \times 0.15$$

Position 2. Length of the spring in position 2.

$$CB = \sqrt{20^2 + 15^2} = 25 \text{ cm}$$

$$\text{P.E.} = \underset{(\text{mass})}{0} + \underset{(\text{spring})}{\frac{1}{2}kx^2} \quad x \text{ is extension of spring}$$

$$\text{P.E.} = \frac{1}{2}k(0.25 - 0.20)^2 = \frac{1}{2}k(.05)^2$$

$$\text{K.E.} = \underset{(\text{mass})}{\frac{1}{2}mv^2} + \underset{(\text{spring})}{0}$$

Total Energy,

$$E_2 = \frac{1}{2}k(0.05)^2 + \frac{1}{2}mv^2$$

$$E_2 = \frac{1}{2} \times 500(0.05)^2 + \frac{1}{2} \times 5v^2$$

$$E_1 = E_2$$

$$5 \times 9.81 \times 0.15 = \frac{1}{2} \times 500(0.05)^2 + \frac{1}{2} \times 5\,v^2$$

$$7.36 = 0.625 + 2.5v^2$$

$$v = 16.4 \text{ m/s} \quad \textit{Ans.}$$

Example 16.10. A block of mass 5 kg resting on a 30° inclined plane is released. The block after travelling a distance of 0.5 m along the inclined plane hits a spring of stiffness 15 N/cm (Fig. 16.18). Find the maximum compression of spring. Assume the coefficient of friction between the block and the inclined plane as 0.2.

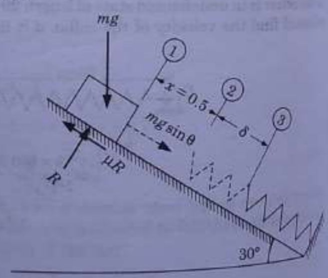

Fig. 16.18

Solution:

$$m = 5 \text{ kg},$$

$$k = 15 \text{ N/cm} = 1500 \text{ N/m}$$

$$x = 0.5 \text{ m}, \; \mu = 0.2, \; \theta = 30°$$

Position 1. Block is at rest.

Position 2. Block after travelling a distance $x = 0.5$ m hits the spring.

Position 3. Block and spring after travelling a distance δ and come to rest.

Applying the principle of work and energy between the positions 1 and 2.

Let the block attain a velocity v when in position 2.

K.E. of the block in position 1, $T_1 = 0$

K.E. of the block in position 2, $T_2 = \frac{mv^2}{2}$

Work done during the displacement from position 1 to 2,

$$U_{1-2} = \text{Work of gravity force} - \text{Work done to over come friction}$$

$$= (mg \sin\theta)x - (\mu R)x \qquad \text{but } R = mg\cos\theta$$

$$= (mg\sin\theta)x - (\mu\, mg\cos\theta)x$$

$$U_{1-2} = T_2 - T_1$$

$$\text{W.D.} = \text{Change in K.E.}$$

$$(mg\sin\theta)x - \mu mg\cos\theta\, x = \frac{mv^2}{2} - 0$$

$$v^2 = 2\,gx(\sin\theta - \mu\cos\theta)$$

Substituting the values,

$$v^2 = 2 \times 9.81 \times 0.5(0.5 - 0.2 \times 0.866)$$

$$v^2 = 3.206, \; v = 1.79 \text{ m/s}$$

Applying the principle of work and energy between the positions 2 and 3.

K.E. of the block in position 2,

$$T_2 = \frac{mv^2}{2}$$

K.E. of the block in position 3,

$$T_3 = 0.$$

Work done between positions 2 and 3.

U_{2-3} = work of gravity force − work done to over come friction − work done on the spring.

$$U_{2-3} = (mg\sin\theta)\delta - (\mu\, mg\cos\theta)\delta - \frac{1}{2}k\delta^2$$

$$U_{2-3} = T_3 - T_2$$

$$\text{W.D.} = \text{change in K.E.}$$

$$(mg\sin\theta)\delta - (\mu\, mg\cos\theta)\delta - \frac{1}{2}k\delta^2 = 0 - \frac{mv^2}{2}$$

Substituting the values.

$$(5 \times 9.81 \times 0.5)\delta - (0.2 \times 5 \times 9.81 \times 0.866)\delta - \frac{1}{2} \times 1500 \times \delta^2 = -\frac{5}{2}(1.79)^2$$

$$750\,\delta^2 - 16.03\,\delta - 8.015 = 0$$

$$\delta = \frac{16.03 \pm 155.9}{1500}$$

Taking the positive value

$$\delta = 0.115 \text{ m or } 11.5 \text{ cm} \quad \textit{Ans.}$$

Example 16.11. If a system of two masses M_1 and M_2 arranged as shown in Fig. 16.19 are released from rest, find the velocity of the mass M_2 after it has fallen a vertical distance of 2 m. Neglect the inertia of the pulleys. Assume $M_1 = M_2 = 10$ kg.

Solution: Kinematics of the system : When the mass M_2 falls through 2 m, the pulley B gets lowered by 1 m and the mass M_1 rises up by 1 m.

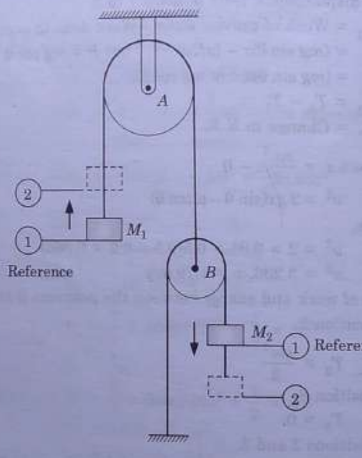

Fig. 16.19

Position 1. Choose position 1 as reference position.

$$\text{K.E.} = 0 \qquad \text{P.E.} = 0$$

Total energy, $E_1 = 0$ (of both masses)

Position 2. If the mass M_2 acquires a velocity v after a fall of 2 m, the velocity of the mass M_1 is $\frac{v}{2}$.

$$\text{P.E.} = M_1 g(1) + (-M_2 g(2))$$
$$\text{P.E.} = 10 \times g - 20\,g = -10\,g$$
$$\text{K.E.} = \frac{1}{2}M_1 v^2 + \frac{1}{2}M_2\left(\frac{v}{2}\right)^2$$
$$\text{K.E.} = \frac{1}{2}\times 10\left(v^2 + \frac{v^2}{4}\right)$$
$$\text{K.E.} = \frac{25}{4}v^2$$

Total energy, $\qquad E_2 = -10\,g + \frac{25}{4}v^2$

Applying the principle of conservation of energy

$$E_1 = E_2$$
$$0 = -10\,g + \frac{25}{4}v^2$$
$$v^2 = \frac{10\times 9.81\times 4}{25}$$
$$v = 3.96 \text{ m/s} \quad \textit{Ans.}$$

Application of the Principle of Conservation of Energy to the Curvilinear Motion.

Example 16.12 A particle of mass m resting on a smooth semicircular cylinder of radius r is prevented from moving by a string AB of length r (Fig. 16.20). If the string is cut, find the value of angle θ defining the position C of the particle when it jumps off the cylindrical surface.

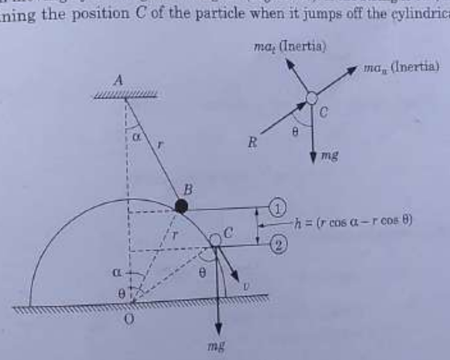

Fig. 16.20

Solution: The initial position of the particle is defined by the angle α.

When the string is cut, particle starts moving on a cylindrical surface of radius r and shall experience normal and tangential acceleration a_n and a_t. Corresponding inertia forces are as shown. Let it jump off at the point C after attaining a velocity v.

The equations of dynamic equilibrium of the particle be written as,

$$R + ma_n - mg\cos\theta = 0 \qquad \text{...(i)}$$

(Sum of the forces along the radius OC is zero)

$$ma_t - mg\sin\theta = 0 \qquad \text{...(ii)}$$

(Sum of forces normal to the radius OC is zero)

Where, $a_n = \dfrac{v^2}{r}$, $a_t = \dfrac{dv}{dt}$, and R = Reaction

It may be noted that equation (*ii*) is not needed in the present analysis.

When the particle jumps off it losses contact and

$$R = 0$$

$$\frac{mv^2}{r} = mg\cos\theta$$

$$\cos\theta = \frac{v^2}{rg} \quad ...(iii)$$

To find the unknown velocity v in the above equation, apply the principle of conservation of energy.

Position 1, P.E. $= 0$ K.E. $= 0$ Total energy, $E_1 = 0$

Position 2, P.E. $= -mg(r\cos\theta - r\cos\theta)$

$$\text{K.E.} = \frac{1}{2}mv^2.$$

$$\text{Total energy, } E_2 = \frac{1}{2}mv^2 - mgr(\cos\alpha - \cos\theta)$$

$$E_1 = E_2$$

$$0 = \frac{1}{2}mv^2 - mgr(\cos\alpha - \cos\theta)$$

$$v = \sqrt{2gr(\cos\alpha - \cos\theta)}$$

Substituting for v in the equation (*i*)

$$\cos\theta = \frac{v^2}{rg} = \frac{2gr(\cos\alpha - \cos\theta)}{rg}$$

$$\cos\theta = 2(\cos\alpha - \cos\theta)$$

$$3\cos\theta = 2\cos\alpha$$

$$\cos\theta = \frac{2}{3}\cos\alpha \quad \textit{Ans.}$$

Example 16.13 Bar AB is hinged at A and to its other end B is attached a mass m (Fig. 16.21). The bar is released from vertical position and swings down. Find the value of angle θ defining its position in the downward swing, at which the axial force in the bar changes from compression to tension.

Solution: Initially the bar is in position 1 and is in compression under a force equal to the weight mg of the mass m.

As the bar is released, the mass m starts moving in a circle of radius l and shall experience normal (a_n) and tangential (a_t) acceleration. Inertia forces are as shown. Let the velocity acquired by the mass m in position 2 be v.

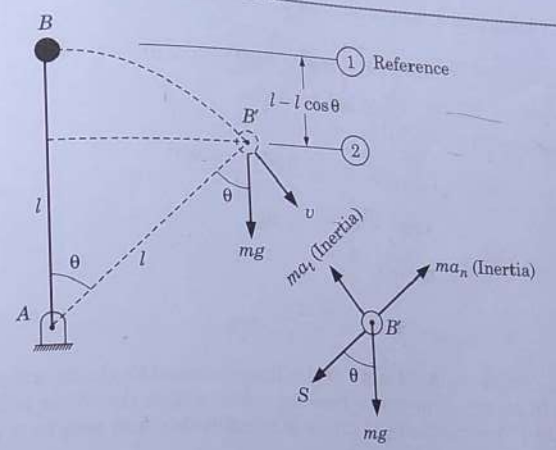

Fig. 16.21

The equation of dynamic equilibrium of the mass can be written as,

$$ma_n - S - mg\cos\theta = 0, \text{(Sum of the forces along the bar is zero)}$$

$$a_n = \frac{v^2}{l}$$

$$S = mg\cos\theta - \frac{mv^2}{l}$$

where, S is the axial force in the bar.1

If the axial force S has to change from compression to tension, it has to become zero at some angle θ.

For $S = 0$, the angular position θ is given by

$$0 = mg\cos\theta - \frac{mv^2}{l}$$

$$\cos\theta = \frac{v^2}{gl} \quad ...(i)$$

The velocity v in the above equation is unknown and can be determined by applying the principle of conservation of energy.

Position 1, P.E. $= 0$, K.E. $= 0$,

Total energy, $E_1 = 0$

Position 2, P.E. $= -mg(l - l\cos\theta)$

$$\text{K.E.} = \frac{1}{2}mv^2$$

$$\text{Total energy, } E_2 = \frac{1}{2}mv^2 - mg(l - l\cos\theta)$$

$$E_1 = E_2$$

$$0 = \frac{1}{2}mv^2 - mgl(1 - \cos\theta)$$

$$v^2 = 2gl(1 - \cos\theta)$$

Substituting for v in (i)

$$\cos\theta = \frac{v^2}{gl} = \frac{2gl(1-\cos\theta)}{gl}$$

$$\cos\theta = 2(1 - \cos\theta)$$

$$\cos\theta = \frac{2}{3}$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \quad Ans.$$

Example 16.14. A flexible chain of length $\pi r/2$ is held on smooth cylinderical surface of radius r as shown on Fig. 16.22(*a*). Calculate the velocity v with which the chain will leave the point B if released from rest. Assume that the chain is inextensible and weighs w per unit length.

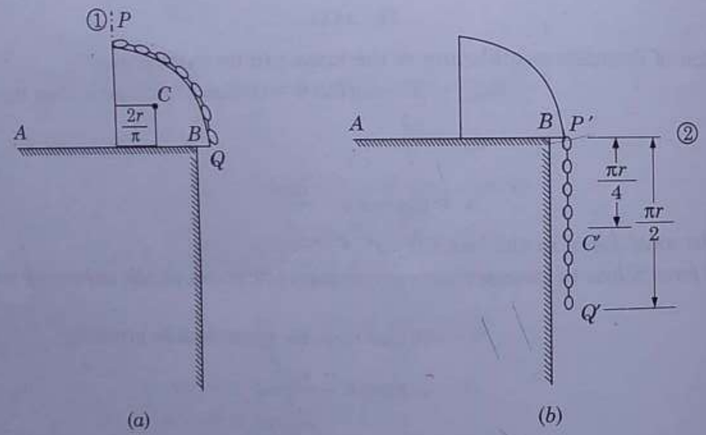

Fig. 16.22

Solution: Let the velocity of the chain when the last link leaves the point B be v. Choose the horizontal surface AB as the reference position.

Position PQ

The chain is in the shape of a quarter of a circle.

K.E. $= 0$ (Entire chain is at rest)

P.E. $=$ P.E. of the entire chain w.r.t. AB.

As the different links of the chain are at the different heights with respect to the reference position AB, we have to find the P.E. of all the links by integration. Which infact is equal to the P.E. of the mass of the entire chain situated at a height equal to the height of the C.G. of the chain in the shape of a quarter of a circle. The chain can be imagined to be interconnected point masses lying along a curve corresponding to the quarter of a circle.

For determine the C.G. refer to Example 4.2

$$\text{P.E.} = \left[\frac{w}{g}\left(\frac{\pi r}{2}\right)\right]g\left(\frac{2r}{\pi}\right) \quad \left(\text{Height of C.G.} = \frac{2r}{\pi}\right)$$

$$\text{P.E.} = wr^2$$

Total energy $\quad E_1 = 0 + wr^2$

Position $P'Q'$

Chain is in vertical position with its C.G. at a vertical distance of $\pi r/4$ from the reference position and moving with velocity v.

$$\text{K.E.} = \frac{1}{2}\left[\frac{w}{g}\left(\frac{\pi r}{2}\right)\right]v^2$$

$$\text{P.E.} = -\left[\frac{w}{g}\left(\frac{\pi r}{2}\right)\right]\left(\frac{\pi r}{4}\right)$$

Total energy $\quad E_2 = \frac{w\pi r v^2}{4g} - \frac{w\pi^2 r^2}{8}$

Applying the principle of conservation of energy

$$E_1 = E_2$$

$$wr^2 = \frac{w\pi r v^2}{4g} - \frac{w\pi^2 r^2}{8}$$

$$\frac{w\pi r v^2}{4g} = wr^2\left(1 + \frac{\pi^2}{8}\right)$$

$$v^2 = 4gr\left(\frac{1}{\pi} + \frac{\pi}{8}\right)$$

$$v = \sqrt{gr\left(\frac{4}{\pi} + \frac{\pi}{2}\right)} \quad Ans.$$

PROBLEMS

16.1. A force of 500 N is acting on a block of mass 50 kg resting on a horizontal surface as shown in Fig. P.16.1. Determine its velocity after the block has travelled a distance of 10 m. Assume the coefficient of friction between the block and the surface to be 0.5. [5 m/s]

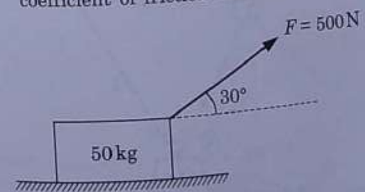

Fig. P.16.1

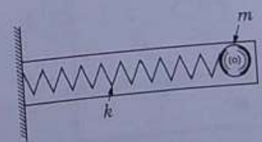

Fig. P.16.2

16.2. A compressed spring as shown in Fig. P.16.2 is used to eject a tennis ball of mass m. If the stiffness of the spring is k and it is initially compressed by an amount δ, find the velocity with which the ball will leave the barrel. Neglect friction,

$$\left[v = \delta\sqrt{\frac{k}{m}}\right]$$

16.3. What velocity must be given to a projectile at the surface of the earth so that it may rise to an infinite height. The radius of the earth may be assumed to be r.

$[\sqrt{2gr}]$

16.4. A car of mass 1500 kg is uniformly accelerated. Its speed increases from 50 kg/hour to 75 km/hour after travelling a distance of 200 m. The resistance to the motion of the car is 0.2% of the weight of the car. Determine (*a*) the maximum power required, (*b*) the power required to maintain a constant speed of 75 km/hour.

$\left[\begin{matrix} 24.88\,\text{kW} \\ 6.13\,\text{kW} \end{matrix}\right]$

16.5. A train weighing 2300 kN moves up an incline 1 in 100. The train starts from rest and moves with a constant acceleration against a frictional resistance of 10 N per kN of the weight of the train. It attains a maximum speed of 30 km/hour in a distance of one kilometre. Determine (*i*) the maximum power required (*ii*) the power required to maintain the speed of 30 km/hour.

[451.24 kW, 383.3 kW]

[Hint:

Total Force Required = Force required to overcome gravitation force + Force required to overcome friction resistance + Force required to accelerate the train]

16.6. Mass M_A = 25 kg rests on a smooth inclined plane of angle 45°. It is connected to another mass M_B = 50 kg by a string as shown in Fig. P.16.6. If the two masses are released from rest, determine their velocities after the mass M_B descends a distance of 0.5 m. [2.05 m/s]

16.7. Two collars A and B each of mass M can slide on a frictionless bar. The collars are initially l metres apart and are connected to a block C of mass 3 M as shown Fig. P.16.7. If the block C is released from rest determine the velocities with which the collars collide.

$[v_A = v_B = \sqrt{0.402gl}]$

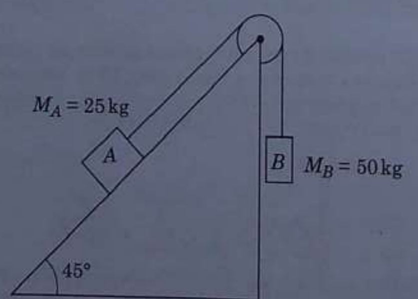

Fig. P.16.6

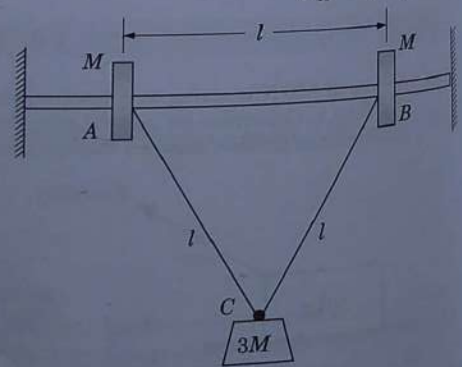

Fig. P.16.7

[Hint: Final velocity of the block C is zero and the collars do no work.]

16.8. A particle of mass m starts from rest at A and slides down a track AB and then enters a vertical loop of radius r at B (Fig. P.16.8). What should be the minimum height h at the starting point of the particle so that it may loop the loop without falling off the track at C? [h = 2.5 r] *Ans.*

$\left[\textbf{Hint} \text{ For the particle not to fall off at } C, \quad \frac{mv^2}{r} = mg.\right]$

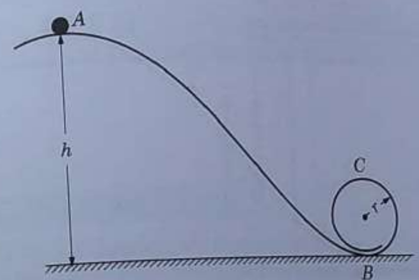

Fig. P.16.8

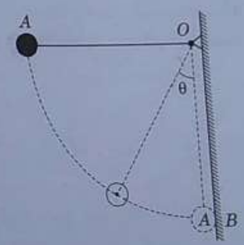

Fig. P.16.9

16.9. A simple pendulum OA when released from rest in the horizontal position falls under gravity and strikes a vertical wall at B (Fig. 16.9). If the coefficient of restitution between the wall and the ball is 0.5 find the angle θ defining the total rebound of the ball. [θ = 41.42°]

16.10. A particle of mass m is attached to the end of an inextensible string of length l as shown in Fig. P.16.10. Determine the minimum initial velocity that must be given to the particle in the lowest position l so that it may swing to the position 2. $[\sqrt{5gl}]$

16.11. A block of mass M resting at A on the surface of a smooth circular cylinder of radius r slides in a vertical plane (Fig. P.16.11). At point B it leaves the cylinder and travels as a projectile hitting the horizontal plane at C. Prove that the distance CD = 1.46r.

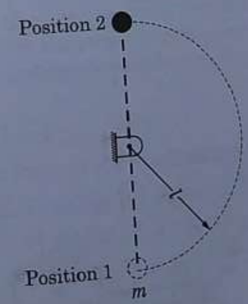

Fig. P.16.10

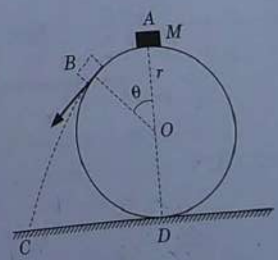

Fig. P.16.11

16.12. A flexible chain PQ of length l is held on a smooth table with a portion h overhanging as shown in Fig. P.16.12. Calculate the velocity with which the chain will leave the table if released from rest. Assume that the chain is inextensible and of weight w per unit length.

$$\left[v = \sqrt{\frac{g}{l}(l^2 - h^2)}\right]$$

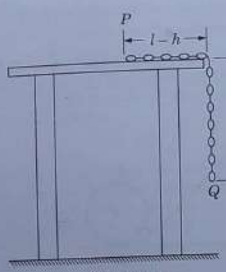

Fig. P.16.12

16.13. A train weighing 1000 kN is accelerated at a constant rate up an inclined plane of 2% grade. The velocity increases from 9 m/s to 18 m/s in a distance of 610 m. The resistance of train is 45 N per kN. Determine the maximum power developed by the engine.

Hint: Power = Force × Velocity

$\Sigma F = ma$: $(P - R - W\sin\theta) = \frac{W}{g}a$

P = Force required, R = Resistance

a = Acceleration $= \frac{v^2 - u^2}{2s}$, $\tan\theta = \frac{2}{100}$

16.14. Determine the constant force P required to give the system of three blocks A, B and C as shown in Fig. P.16.14 a velocity of 3 m/s after moving 4.5 m from rest. The coefficient of friction between blocks and the plane is 0.3. Assume pulleys to be smooth. [P = 1383.4 N]

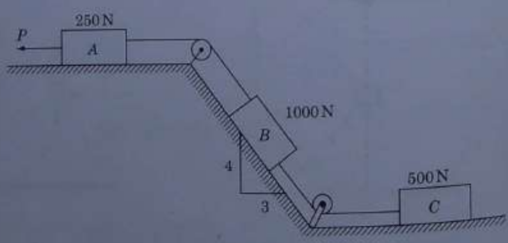

Fig. P.16.14

17 CHAPTER

Kinetics of a Particle: Impulse and Momentum

17.1 INTRODUCTION

In the earlier chapters we discussed the two methods for solving the problems of motion of the particles. These were based on the application of Newton's second law and on the application of the principle of work and energy. In this chapter we shall discuss the third basic method. This method is based on the principle of impulse and momentum and is derived from Newton's second law. The principle relates force, mass, velocity and time and is particularly suitable when large forces act for a very small time. Let us define the terms impulse of a force and momentum.

Impulse of a Force. When a large force acts over a short period of time that force is called an impulsive force.

The impulse of force **F** acting over a time interval for t_1 to t_2 is defined by the integral,

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\,dt \qquad ...(17.1)$$

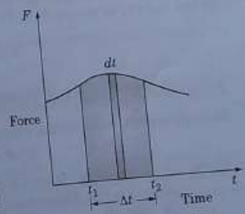

Fig. 17.1

The impulse of a force, therefore, can be visualised as the area under the force vs. time diagram as shown in the Fig. 17.1. When the variation of the force with respect to the time is unknown, the impulse can also be measured as

$$\mathbf{I} = \mathbf{F}_{average} \times \Delta t \qquad ...(17.2)$$

Impulse of a force is a vector quantity and has the unit of Newton second (Ns).

Momentum. Consider the motion of a particle of mass m acted upon by a force **F** (Fig. 17.2).

The equation of motion of the particle in the x and y directions are,

$$F_x = ma_x, \quad \text{and} \quad F_y = ma_y$$

$$\text{Or} \quad F_x = \frac{mdv_x}{dt}, \quad \text{and} \quad F_y = \frac{mdv_y}{dt} \qquad ...(17.3)$$

$$\text{Or} \quad F_x = \frac{d}{dt}(mv_x) \quad \text{and} \quad F_y = \frac{d}{dt}(mv_y)$$

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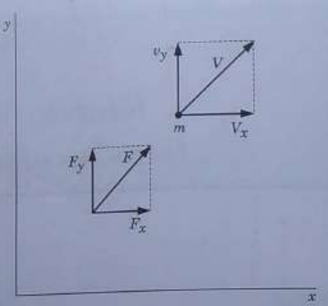

Fig. 17.2

A single equation in the vector form can be written as,

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \qquad ...(17.4)$$

Which states that the force **F** acting on the particle is equal to the rate of change of momentum of the particle.

The vector $m\mathbf{v}$ is called the *momentum* or *the linear momentum*. It has the same direction as the velocity of the particle. The unit of momentum is

$$mv = (\text{kg})\left(\frac{\text{m}}{\text{s}}\right) = (\text{kg})\left(\frac{\text{m}}{\text{s}^2}\right)\text{s} = \text{Ns}$$

17.2 PRINCIPLE OF IMPULSE AND MOMENTUM

Multiplying both sides of the equations (17.3) by dt,

$$F_x dt = d(mv_x) \text{ and } F_y dt = d(mv_y) \qquad ...(17.5)$$

where, $F_x dt$ is the impulse of the force F_x,

$d(mv_x)$ is the differential change in the momentum of the particle in the x-direction in time dt.

$F_y dt$ and $d(mv_y)$ denote similar quantities in the y-direction.

Above equation (17.5) express that the differential change in the momentum of a particle during the time interval dt is equal to the impulse of the force acting during the same interval.

Integrating the equations (17.5) from a time t_1 to a time t_2.

$$\left.\begin{aligned} \int_{t_1}^{t_2} F_x dt &= (mv_x)_2 - (mv_x)_1 \\ \int_{t_1}^{t_2} F_y dt &= (mv_y)_2 - (mv_y)_1 \end{aligned}\right\} \qquad ...(17.6)$$

The above two equation can be combined into a single vector equation as,

$$\int_{t_1}^{t_2} \mathbf{F} dt = m\mathbf{v}_2 - m\mathbf{v}_1 \qquad ...(17.7)$$

If $t_1 = 0$, and $t_2 = t$

we can write

$$m\mathbf{v}_2 - m\mathbf{v}_1 = \int_0^t \mathbf{F} dt \qquad ...(17.8)$$

Final Momentum – Initial Momentum = Impulse of the Force.

The above equation expresses that the total change in the momentum of a particle during a time interval is equal to the impulse of the force acting during the same interval of time.

It may also be noted that the above equation represents a relation between vector quantities for a single particle. In the actual solution of a problem, it should be represented by the two corresponding component equations in the x and y directions.

System of Particles. The principle of impulse and momentum is particularly useful when dealing with a system of particles. For example, a gun firing a bullet. When a problem involves the motion of several particles each particle is to be considered separately.

We can then add vectorially the momentum of all the particles and impulses of all the forces involved and the equation (17.8) can be written as

$$\Sigma m\mathbf{v}_2 - \Sigma m\mathbf{v}_1 = \int_0^t \mathbf{F} dt \qquad ...(17.9)$$

Above equation is also a vector relation which can be replaced by two component equations in the x and the y directions.

Following points should be noted here,

1. When the impulsive forces act on a system, the non-impulsive forces (like, the weight of the particles) generally can be neglected.
2. The internal forces between the particles need not be considered as the sum of the impulses of the internal forces in always zero. Because the internal forces appear in pairs of equal and opposite forces (as action and reaction) acting for the same interval of time, resulting in equal and opposite impulses.

But, the sum of the work done by the internal forces may not be zero because they may not move through the same distances.

17.3 CONSERVATION OF MOMENTUM

It can be observed from the equation (17.9) that *when sum of the impulses due to external forces is zero the momentum of the system remains constant or is conserved,*

$$\Sigma m\mathbf{v}_2 = \Sigma m\mathbf{v}_1 \qquad ...(17.10)$$

Final momentum of the system = Initial momentum of system.

Example 17.1. A ball of mass 100 g is moving towards a bat with a velocity of 25 m/s as shown in Fig. 17.3 (*a*). When hit by a bat the ball attains a velocity of 40 m/s. If the bat and the ball are in contact for a period of 0.015 s determine the average impulse force exerted by the bat on the ball during the impact.

Solution: Let us apply the principle of impulse and momentum to the ball in the *x* and *y* directions.

The component equation in the *x*-direction,

$$(mv_x)_2 - (mv_x)_1 = \int_0^t F_x\,dt$$

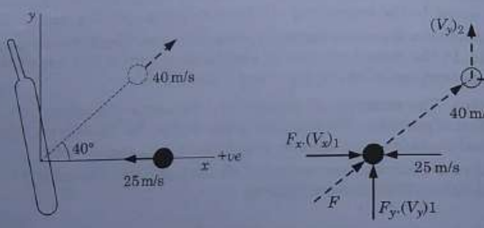

Fig. 17.3

Substituting,

$$m = \frac{100}{1000} = 0.1\text{ kg},\ \Delta t = 0.015\text{ s}$$

$$(v_x)_1 = -25\text{ m/s}$$

$$(v_x)_2 = 40\cos 40° = 30.64\text{ m/s}$$

$$\int_0^t F_x\,dt = (F_x)_{\text{average}}(\Delta t)$$

$$(F_x)_{\text{average}}(\Delta t) = 0.1(30.64) - 0.1(-25)$$

$$(F_x)_{\text{average}} = \frac{5.564}{0.015}$$

$$(F_x)_{\text{average}} = 370.9\text{ N}$$

The equation in the *y*-direction.

$$(mv_y)_2 - (mv_y)_1 = \int_0^t F_y\,dt$$

Substituting,

$$(v_y)_1 = 0,\ \Delta t = 0.015\text{ s}$$

$$(v_y)_2 = 40\sin 40° = 25.72\text{ m/s}$$

$$\int_0^t F_y\,dt = (F_y)_{\text{average}} \times (\Delta t)$$

$$(F_y)_{\text{average}} \times (0.015) = 0.1(25.72) - 0.1(0)$$

$$(F_y) = \frac{2.572}{0.015}$$

$$(F_y)_{\text{average}} = 171.5\text{ N}$$

$$F_{\text{average}} = \sqrt{(F_x)^2_{\text{average}} + (F_y)^2_{\text{average}}}$$

$$= \sqrt{(370.9)^2 + (171.5)^2}$$

$$F_{\text{average}} = 408.6\text{ N} \quad \textit{Ans.}$$

Example 17.2. A gun of mass 3000 kg fires horizontally a shell of mass 50 kg with a velocity of 300 m/s. What is the velocity with which the gun will recoil? Also determine the uniform force required to stop the gun in 0.6 m. In how much time will it stop?

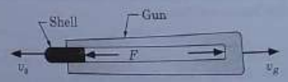

Fig. 17.4

Solution: Mass of the gun $= m_g = 3000$ kg

Mass of the shell $= m_s = 50$ kg

Let

The final velocity of the gun $= v_g$

The final velocity of the shell $= v_s$

and the force due to the gas pressure be F.

The gun and the shell may be considered as a system of two particles. During the short interval of time in which the explosion takes place the force F, acting on the gun and the shell (due to the gas pressure), varies in an unknown manner. But it acts in the manner of action and reaction between the shell and the gun. Their impulses thus, are equal and opposite.

In the absence of any external force, therefore, the momentum of the system is conserved.

Initial velocity of the gun = 0

Initial velocity of the shell = 0

Initial momentum of the gun and the shell = 0 ...(*i*)

Final momentum of the gun and the shell ...(*ii*)

$$= m_g v_g + m_s v_s$$

Equating (*i*) and (*ii*)

$$0 = m_g v_g + m_s v_s$$

$$-m_g v_g = +m_s v_s$$

Which means that the velocities of the shell and the gun are in the opposite directions.

Substituting the values

$$-3000 \times v_g = 50 \times 300$$

$$v_g = -5\text{ m/s} \quad \textit{Ans.}$$

The gun having a velocity of 5 m/s is brought to rest in a distance of 0.6 m.

Using,

$$v^2 - u^2 = 2as$$

$$0 - (5)^2 = 2 \times a \times 0.6$$

$$a = -20.8\text{ m/s}^2$$

Force required to stop the gun $= \text{Mass} \times \text{acceleration}$

$$= 3000 \times 20.8$$
$$= 62400 \text{ N}$$

Time required to stop the gun

$$v = u + at, \qquad t = \frac{5}{20.8}$$
$$t = 0.24 \text{ s} \quad \textit{Ans.}$$

Example 17.3. A man of mass 50 kg stands at the one end of a 5 m long floating boat of mass 250 kg (Fig. 17.5). If the man walks towards the other end of the boat at a steady rate of 1.0 m/s, determine (*a*) the velocity of the boat as observed from the ground (*b*) the distance by which the boat gets shifted.

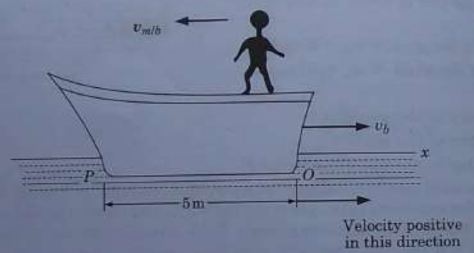

Fig. 17.5

Solution. Consider the man and the boat as a system. As there is no external force acting on it, the momentum of the man and the boat taken together must be conserved.

Let the initial position of the man be at O. Since both the man and the boat are at rest, their initial momentum is zero.

Let the man move from O towards P with a velocity $v_{m/b}$ relative to the boat.

The boat hence shall move in the direction opposite to that of the man.

Absolute velocity of the boat be v_b.

The velocity of the man with respect to the boat

$$v_{m/b} = v_m - v_b$$

where v_m is the absolute velocity of the man.

Absolute velocity of the man

$$v_m = v_{m/b} + v_b$$

given that, $\quad v_{m/b} = -1.0$ m/s (velocity to the left is negative)

Therefore, $\quad v_m = (-1.0 + v_b)$

Final momentum of the man and the boat

$$= m_m v_m + m_b v_b = 50(-1.0 + v_b) + 250(v_b).$$

Since the momentum is conserved

$$0 = 50(-1.0 + v_b) + 250\, v_b$$
$$300\, v_b = 50$$

Absolute velocity of the boat, $v_b = 0.167$ m/s (to the right) *Ans.*

Time taken by the man to move to the other end of the boat

$$= \frac{\text{Distance travelled along the boat}}{\text{Velocity of the man w.r.t. boat}} = \frac{5}{1.0} = 5.0 \text{ s}$$

*The distance travelled by the boat in the same time as the man takes to move to other end of the boat is

$$= \text{velocity of the boat} \times \text{time}$$
$$= 0.167 \times 5$$
$$= 0.835 \text{ m, to the right from } O \quad \textit{Ans.}$$

Example 17.4. A trolley of weight W can move along a horizontal friction less track. Initially the trolley together with a man of weight w standing on it, is moving with a velocity V to the right as shown in Fig. 17.6. Find the increase in the velocity of the trolley if the man runs with a speed of v relative to the floor of the trolley and jumps off to the left.

Solution: Initial velocity of the trolley when the man is standing $= V$

$$\text{Mass of the trolley} = \frac{W}{g}$$

$$\text{Mass of the man} = \frac{w}{g}$$

Initial momentum of the system of the trolley and the man

$$= \frac{W}{g}V + \frac{w}{g}V$$
$$= \frac{V}{g}(W + w)$$

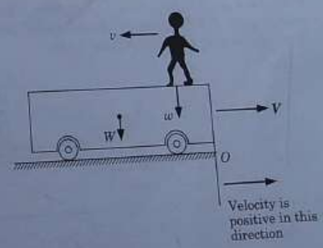

Fig. 17.6

After the man runs,

Let ΔV be the increase in the velocity of the trolley when the man runs and then jumps off.

$$\text{Final velocity of the trolley} = V + \Delta V$$

$$\text{Final momentum of the trolley} = \frac{W}{g}(V + \Delta V)$$

The velocity of man with respect to the trolley $= v$

$$\text{Absolute velocity of the man} = \text{Velocity of man relative to the trolley} + \text{Velocity of the trolley}$$
$$= -v + (V + \Delta V)$$

* The boat and the man form a system of two bodies on which no external force is acting. The motions of the man and boat, therefore, would be such that the location of their combined mass centre remains undisturbed. This can be verified from the answer we get.

$$\text{Final momentum of the man} = \frac{w}{g}[-v + (V + \Delta V)]$$

The momentum of the man and the trolley is conserved as no external force is acting

$$\frac{(W+w)V}{g} = \frac{W}{g}(V+\Delta V)+\frac{w}{g}[-v+(V+\Delta V)]$$

$$WV + wV = WV + W(\Delta V) - wv + wV + w(\Delta V)$$

$$(W + w)\Delta V = wv$$

$$\Delta V = \frac{wv}{(W+w)}$$

In terms of the corresponding masses m and M of the man and the trolley

$$\Delta V = \frac{mv}{(M+m)} \quad \textit{Ans.}$$

It is important to note here that the increase ΔV in the velocity of the trolley is independent of the initial velocity V of the trolley.

Example 17.5. Two men, M_1 of mass 50 kg and M_2 of mass 75 kg, dive off the end of a boat of mass $M = 250$ kg so that their relative velocity with respect to the boat is 4 m/s (Fig. 17.7). If the boat is initially at rest, find its final velocity if (*a*) two men dive simultaneously (*b*) the man of mass 75 kg dives first followed by the man of mass 50 kg (*c*) the man of mass 50 kg dives first followed by the man of mass 75 kg.

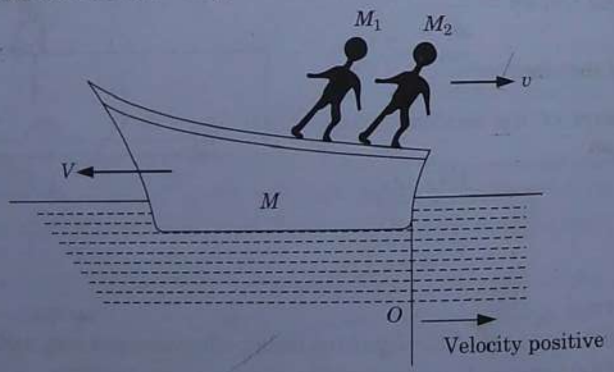

Fig. 17.7

Solution: Mass of the boat $M = 250$ kg

Mass of the man $M_1 = 50$ kg

Mass of the man $M_2 = 75$ kg

Initial velocity of boat and men is zero

Initial momentum of the boat and men = 0

(*a*) Both men jump simultaneously

Using the concept and the formula derived in the earlier example; when a man of mass m with velocity v jumps from a trolley or boat of mass M, the increase in its velocity ΔV is,

$$\Delta V = \frac{mv}{M+m}$$

Here, $m = (50 + 75) = 125$ kg, $v = 4$ m/s

$M = 250$ kg

$$\Delta V = \frac{125\times 4}{(250+125)} = 1.33 \text{ m/s}$$

$$\Delta V = 1.33 \text{ m/s}$$

Final velocity of the boat = 1.33 m/s *Ans.*

(*b*) The man of mass 75 kg jumps first,

$m = 75$ kg

So, $M = (250 + 50) = 300$ kg, (Man of 50 kg remains in the boat)

$v = 4$ m/s

$$\Delta V = \frac{75\times 4}{(300+75)}$$

The man of mass 50 kg jumps next,

$m = 50$ kg

So, $M = 250$ kg (only boat remains)

$v = 4$ m/s

$$\Delta V = \frac{50\times 4}{(250+50)}$$

Final velocity of the boat = 0 + Increase in the velocity

$$= 0 + \frac{75\times 4}{(300+75)} + \frac{50\times 4}{(250+50)}$$

Final velocity of the boat = 1.467 m/s *Ans.*

(*c*) The man of 50 kg jumps first,

$m = 50$ kg

So, $M = (250 + 75) = 325$ kg (Man of 75 kg remains in the boat)

$v = 4$ m/s

$$\Delta V = \frac{50\times 4}{(325+50)}$$

The man of mass 75 kg jumps next,

$m = 75$ kg

So, $M = 250$ kg (only boat remains)

$v = 4$ m/s

$$\Delta V = \frac{75\times 4}{(250+75)}$$

Final velocity of the boat

$$= 0 + \frac{75\times 4}{(325+20)} + \frac{75\times 4}{(250+75)} = 1.456 \text{ m/s} \quad \textit{Ans.}$$

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Example 17.6. A man of mass 70 kg stands in an aluminum canoe of mass 35 kg. He fires a bullet of mass 25 gm horizontally over the bow the canoe to hit a wooden block of mass 2.25 kg resting on a smooth horizontal surface, Fig. 17.8. If the wooden block and the bullet together move with a velocity of the 5 m/s find the velocity of the canoe.

Fig. 17.8

Solution:

$$\text{Mass of the bullet } m = \frac{25}{1000} = 0.025 \text{ kg}$$

Velocity of the bullet $= v$

Mass of the canoe $= 35$ kg

Mass of the man $= 70$ kg

Velocity of the canoe $= V$

Velocity of the block $= 5$ m/s

Consider the bullet and the wooden block as a system.

Initial momentum of the bullet = Final momentum of the bullet and the block.

$$v(0.025) = 5(2.25 + 0.025)$$

$$v = \frac{5 \times 2.275}{0.025}$$

Velocity of the bullet $v = 455$ m/s

Next consider the bullet and the canoe with man.

Momentum of the bullet = Momentum of the canoe and the man.

$$0.025v = (70 + 35)V$$

Substituting for the velocity of the bullet

$$v = 455 \text{ m/s}$$

$$V = \frac{0.025 \times 455}{35}$$

Velocity of the canoe $V = 0.108$ m/s *Ans.*

PROBLEMS

17.1. A block of mass 50 kg resting on a horizontal surface is acted upon by a force **F** which varies as shown in Fig. P.17.1. If the coefficient of friction between the block and the surface is 0.2, find the velocity of the block when (*a*) $t = 5$ seconds (*b*) $t = 10$ seconds. [15.19 m/s, 17.88 m/

17.2. A railway wagon of mass 50,000 kg moving with a velocity of 3.6 km/hour backs into another wagon of mass 30,000 kg that is at rest on a level railway track. After coupling is made find the velocity of the coupled wagons. [2.25 km/hour]

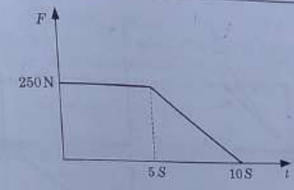

Fig. P.17.1

17.3. The brakes are applied to a car of mass 100 kg travelling at 72 km/hour so as to cause all the wheels to skid.

Determine the time required to stop the car (*a*) on a dry road of $\mu = 0.5$ (*b*) on a wet road of $\mu = 0.1$ [4.08 s, 20.39 s]

17.4. A man of mass 75 kg and a boy of mass 25 kg dive off the end of a boat of the mass 20 kg so that their relative horizontal velocity with respect to the boat is 3 m/s. If initially the boat is at rest find its final velocity if (*a*) the two dive off simultaneously (*b*) the man dives first followed by the boy. [1 m/s, 1.8 m/s]

17.5. A bullet of mass 25 g is fired with a velocity of 500 m/s into a wooden block resting against a rigid vertical wall. If the bullet is brought to rest in 0.5 ms, determine the average impulsive force exerted by the bullet on the block. [25 kN]

17.6. A bomb of mass 1 kg initially at rest explodes and breaks into three pieces of masses 1 : 1 : 3. The two pieces of equal mass fly off in direction 60° to each other with a speed of 30 m/s. What is the velocity of the heavier piece? [17.32 m/s]

[**Hint** : momentum is conserved]

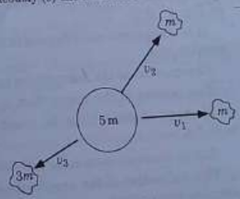

Fig. P.17.6

17.4 ANGULAR MOMENTUM

We know how a force acting on a particle in rectilinear motion is related to the change in momentum of the particle. Let us now consider a particle P of mass m moving along a curvilinear path and derive the corresponding relation. The relation can be stated as,

The rate of change of angular momentum (or the rate of change of moment of momentum) of a particle with respect to any point which lies in the plane of motion of the particle, is equal to the moment of the resultant force acting with respect to the same point.

Consider a particle of mass m moving in a curvilinear path AB. The velocity of the particle P at any instant be $\mathbf{v}$ (Fig. 17.9).

The momentum of the particle = $m\mathbf{v}$.

It is a vector quantity and has the same direction as that of the velocity. The velocity of the particle at any instant is tangential to the path of motion.

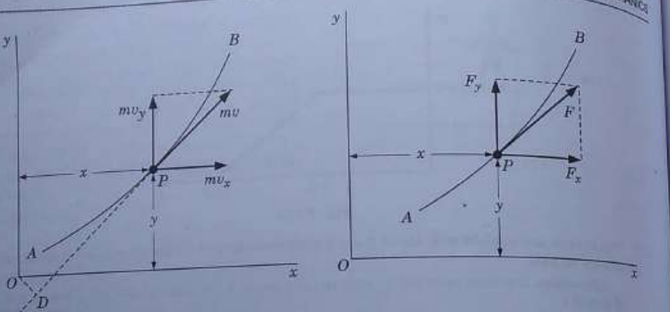

Fig. 17.9

The angular momentum or the moment of momentum of the particle with respect to the origin O is defined as the product of the momentum ($m\mathbf{v}$) and the perpendicular distance OD as shown in Fig. 17.9.

It is often convenient to resolve the momentum and the angular momentum into rectangular components.

Component of the momentum in the x-direction $= mv_x$

Component of the momentum in the y-direction $= mv_y$

Moment of momentum H_0 about O,

$$H_0 = x(mv_y) - y(mv_x) \quad \text{...(17.11)}$$

(Taking clockwise moment as positive)

The derivative of the angular momentum with respect to time or the rate of change of the angular momentum.

$$\frac{dH_0}{dt} = m\frac{d}{dt}(xv_y - yv_x)$$

$$\frac{dH_0}{dt} = \dot{H}_0 = m\left(\frac{dx}{dt}v_y + x\frac{dv_y}{dt} - \frac{dy}{dt}v_x - y\frac{dv_x}{dt}\right) \quad \text{...(17.12)}$$

As,
$$\frac{dx}{dt} = \dot{x} = v_x,\ \frac{dy}{dt} = \dot{y} = v_y$$

$$\frac{dv_x}{dt} = a_x \text{ and } \frac{dv_y}{dt} = a_y$$

(a_x, a_y are acceleration in the x and y directions)

Substituting the above values in equation (17.12)

$$\dot{H}_0 = m(v_x v_y + xa_y - v_y v_x - ya_x) \quad \text{...(17.13)}$$

Or
$$\dot{H}_0 = m(xa_y - ya_x)$$

From Newton's Second Law

$$ma_x = F_x, \quad ma_y = F_y$$

Therefore,
$$\dot{H}_0 = (xF_y - yF_x) \quad \text{...(17.14)}$$

where, F_x and F_y are the rectangular components of the resultant force F acting on the particle.

The right hand side of the above equation represents the moment M_0 of the resultant force acting on the particle about O, hence equation (17.14) can be expressed as,

$$\dot{H}_0 = (xF_y - yF_x) = M_0 \quad \text{...(17.15)}$$

17.5 CONSERVATION OF THE ANGULAR MOMENTUM

We obtained,

$$\frac{dH_0}{dt} = \dot{H}_0 = M_0$$

When the moment M_0 of the resultant force $\mathbf{F}$, acting on the particle about O is zero,

$$\frac{dH_0}{dt} = M_0 = 0 \quad \text{...(17.16)}$$

Integrating with respect to time

$$H_0 = \text{constant} \quad \text{...(17.17)}$$

Stated in words it means that "*the angular momentum of particle about a point O is constant (or is conserved) if, the moment of the resultant force acting on the particle about the same point O is zero*".

An important **corollary** now can be stated as : *the moment of momentum about a point shall be conserved if the resultant force passes through that point.*

The conservation of the moment of momentum or angular momentum in the case of a particle moving under a *central force* is an important example of the above corollary. In the central force motion the only force acting on the particle is directed towards or away from a fixed point so that angular momentum (about this fixed point) is conserved.

Example 17.7. A particle of mass m moves in an x-y plane. The co-ordinates of the particle at any instant are given by

$$x = a\cos\omega t \text{ and } y = b\sin\omega t$$

where, a, b and ω are constants.

Determine the angular momentum of the particle with respect to the origin of the coordinate system.

Solution: Let O be the origin of the or-ordinate system (Fig. 17.10).

Angular momentum of the particle P about O is ...(i)

$$H_0 = (mv_y)x + (mv_x)y$$

$$x = a\cos\omega t \text{ and } y = b\sin\omega t$$

$$\frac{dx}{dt} = \dot{x} = v_x = -a\omega\sin\omega t$$

$$\frac{dy}{dt} = \dot{y} = v_y = b\omega\cos\omega t$$

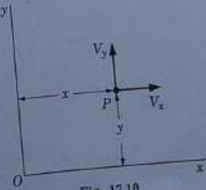

Fig. 17.10

Substituting in the equation (*i*)

$$H_0 = m\,[b\omega \cos \omega t\,(a \cos \omega t) + a\omega \sin \omega t\,(b \sin \omega t)]$$
$$H_0 = ab\omega\, m\,(\cos^2\omega t + \sin^2 \omega t)$$
$$H_0 = ab\omega\, m \quad Ans.$$

Example 17.8 A particle of mass m attached to the end of an inextensible string is rotated with uniform speed v_0 along a circle of radius r_0 over a smooth table top (Fig. 17.11). By pulling the string (which passes through a slot in the top) at the lower end, the radius of the path is reduced to $r_0/2$. Determine the new speed and the tension T in the string. Assume,

$$m = 2\text{ kg},\ r_0 = 1\text{ m},\ v_0 = 20\text{ m/s}$$

Solution: The force acting on the particle are :

(*i*) the gravitational force mg acting vertical downwards

(*ii*) normal reaction R acting upward

(*iii*) tension T in the string acting along the table

The normal reaction and the gravitational force balance each other and the moment of the tension about the point O is zero.

Therefore, the angular momentum about O is conserved.

Let the speed be v_0 at radius r_0 and the speed be v_1 at radius $r_1 = r_0/2$

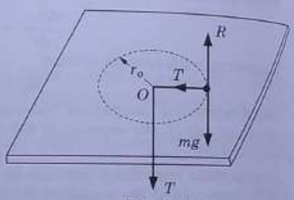

Fig. 17.11

$$(H_0)_1 = (H_0)_2$$
$$(mv_0)r_0 = (mv_1)r_1 = mv_1 r_0/2$$
$$v_1 = 2v_0 = 2 \times 20 = 40\text{ m/s} \quad Ans.$$

$$\text{Tension } T = \frac{mv_1^2}{r} \qquad r = 0.5\text{ m},\ v_1 = 40\text{ m/s}$$

$$T = \frac{2 \times (40)^2}{0.5} = 6400\text{ N} \quad Ans.$$

Example 17.9 A particle of mass m rests on a rough table as shown in Fig. 17.12. It is given a velocity v in a circular path of radius r. Find the time in which the particle will come to rest. The coefficient of friction between the particle and the table is μ.

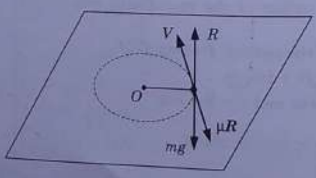

Fig. 17.12

Solution:

$$\dot{H}_0 = M_0$$

$$\begin{pmatrix}\text{Rate of change of moment}\\ \text{of momentum about } O\end{pmatrix} = (\text{Moment of the resultant force about } O)$$

$$\dot{H}_0 = \frac{d}{dt}(mvr)$$

$$M_0 = (\mu R)r, \text{ moment due to frictional force}$$

As, $R = mg$

$$M_0 = (\mu mg)r$$

Therefore,

$$\frac{d}{dt}(mvr) = \mu mgr$$

$$(mr)\frac{dv}{dt} = \mu mgr = (mr)\mu g$$

$$\frac{dv}{dt} = \text{deceleration} = \mu g$$

If the final velocity is to be zero, then the time in which the particle will come to rest

$$= \frac{\text{velocity}}{\text{deceleration}}$$

$$t = \frac{v_0}{\mu g} \quad Ans.$$

PROBLEMS

17.7. A particle of mass m moves in a x-y plane. The coordinates of the particle at any instant are given by,

$$x = a \cos \omega t$$
$$y = b \sin 2\,\omega t$$

where, a, b and ω are constants.

Determine the angular momentum of the particle with respect to the origin of the coordinate system. [$H = 2ab\omega m \cos^3 \omega t$]

17.8. A particle of mass 1 kg is moving with a velocity of 5 m/s are shown in Fig. P.17.8. The co-ordinate of the particle are (3, 2). Find the angular momentum about the origin O. [18 Nms]

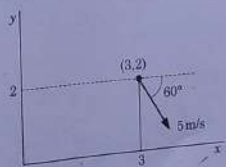

Fig. P.17.8

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17.9. A mass of 5 kg is attached to the end of a bar of length 50 cm and of negligible mass. The is rotating at 60 r.p.m. as shown in Fig. P.17.9. Determine the torque applied to the bar if angular speed increases to 90 r.p.m. in 2 revolutions. [2.45

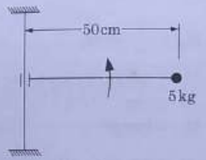
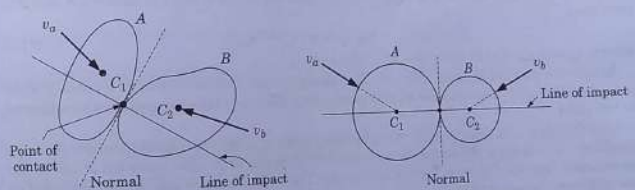


Fig. P.17.9

CHAPTER 18 Impact: Collision of Elastic Bodies

18.1 INTRODUCTION

The phenomenon of collision of two bodies which occurs in a *very small interval of time* and during which the two bodies exert *very large force* on each other, is called an impact.

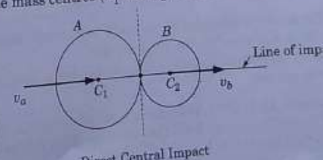


Oblique Non-Central Impact (*a*) Oblique Central Impact (*b*)

Fig. 18.1

Line of Impact. The common normal to the surfaces of two bodies in contact during the impact, is called the line of impact.

Central/Non-Central Impact. When the mass centres (C_1 and C_2) of the colliding bodies lie on the line of the impact, it is called central impact, otherwise it is called non-central or eccentric impact (Fig. 18.1 (*b*), (*a*) respectively).



Direct Central Impact (*c*)

Fig. 18.1

Direct Impact/Indirect (Oblique) Impact. If the velocities of the two bodies before collision are collinear with the line of impact, it is called direct impact, Fig. 18.11 (c) otherwise it is called indirect or oblique impact Fig. 18.1 (*a*) and (*b*).

In the study of the phenomenon of impact an important assumption is made that in comparison with the forces of impact, any other *finite forces* that may act during the impact *are negligible*.

18.2 DIRECT CENTRAL IMPACT

Consider the two spheres A and B of masses m_a and m_b moving in the same direction and along the same straight line with the known velocities v_a and v_b respectively.

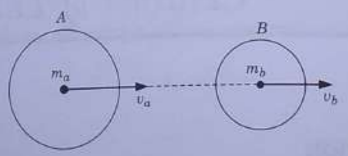


Fig. 18.2

If $v_a > v_b$, the sphere A will strike the sphere B.

Since no external force is acting, the total momentum of the system of spheres A and B is conserved.

$$m_a v_a + m_b v_b = m_a v_a' + m_b v_b' \quad \text{...(18.1)}$$

where, v_a' and v_b' are the velocities after the impact.

As all velocities are directed along the line of impact they can be treated as scalars. For the purpose of fixing the sense of velocity and momentum. They are taken as positive when directed to the right.

Now, it is required to determine the velocities v_a' and v_b' after the impact. A single equation obtained above is not sufficient to determine two unknowns. One more equation or additional information is needed to solve for two unknowns. The other equation can be derived based on the nature of impact and is

$$e = -\frac{(v_b' - v_a')}{v_b - v_a} \quad \text{...(18.2)}$$

where, e is called the *coefficient of restitution* and its value depends upon the nature of impact. In a problem the value e is either given for an impact or is to be determined.

$$e = (-)\frac{\text{velocity of separation}}{\text{velocity of approach}} \quad \text{...(18.3)}$$

For an exact definition of the coefficient of restitution we have to look into the nature of impact which is discussed in the next section.

With these two equations now, we can solve for the unknown velocities v_a' and v_b' if the value of e is known.

Example 18.1. Ball A of mass 1 kg moving with a velocity of 2 m/s, impinges directly on a ball B of mass 2 kg at rest. Find the velocities of the two balls after the impact. Assume the coefficient of restitution $e = 1/2$.

Solution:

$$m_a = 1\text{ kg},\ m_b = 2\text{ kg}$$
$$v_a = 2\text{ m/s},\ v_b = 0$$
$$e = 1/2$$

Principle of conservation of momentum gives

$$m_a v_a + m_b v_b = m_a v_a' + m_b v_b'$$
$$1 \times 2 + 2 \times 0 = 1(v_a') + 2(v_b')$$

Or

$$v_a' + 2v_b' = 2 \quad \text{...(i)}$$

The coefficient of restitution relation gives,

$$e = -\frac{(v_b' - v_a')}{(v_b - v_a)}$$

Or

$$\frac{1}{2} = -\frac{(v_b' - v_a')}{(0-2)}$$

Or

$$v_b' - v_a' = 1 \quad \text{...(ii)}$$

Solving (*i*) and (*ii*) gives

$$v_a' = 0,\ v_b' = 1\text{ m/s} \quad \textit{Ans.}$$

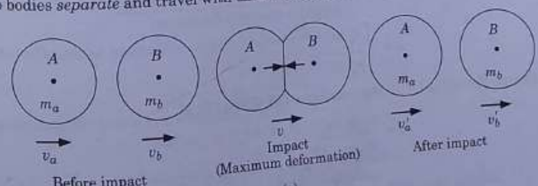
The ball A comes to rest after the impact and the ball B attains the velocity of 1 m/s.

18.3 NATURE OF IMPACT AND THE COEFFICIENT OF RESTITUTION

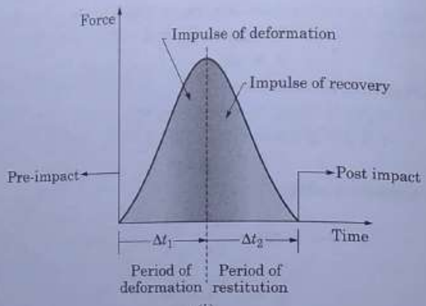
The phenomenon of impact consists of two phases (Fig. 18.3).

The Period of Deformation. Just after the impact the two bodies deform. The time interval from the first contact to the maximum deformation is called the period of deformation. *At the end of this period of deformation, both bodies move with the same velocity v.*

The Period of Restitution. The period of deformation is followed by a *period of restitution (regain or recovery)*. At the end of which the two bodies either regain their original shapes fully or partially, or remain permanently deformed. This depends upon the magnitude of the impact forces and the properties of the materials involved. Also, at the end of the restitution period of the two bodies *separate* and travel with different velocities *except* in the case of *plastic impact*.



(a)



(b)

Fig. 18.3

Consider the body A. It is moving the velocity v_a and collides with another body B. During the deformation period an impulsive force $\mathbf{F}_d$ is exerted by the body B on the body A and changes its velocity to v.

Applying the principle of impulse and momentum

$$m_a v_a - \int F_d\, dt = m_a v \quad ...(18.4)$$

where the integral extends over the period of deformation.

Next consider its motion during the restitution period. Let $\mathbf{F}_r$ be the impulsive force exerted by the body B on the body A changing its velocity from v to v_a'. We can write,

$$m_a v - \int F_r\, dt = m_a v_a' \quad ...(18.5)$$

The force $\mathbf{F}_d$ exerted on the body A during the deformation period is different from the force $\mathbf{F}_r$ exerted during the restitution (recovery) period.

The coefficient of restitution is the ratio of the magnitude of impulses during the restitution period and deformation period.

$$e = \frac{\text{Impulse during restitution or recovery}}{\text{Impulse during deformation}}$$

$$e = \frac{\int F_r\, dt}{\int F_d\, dt} \quad ...(18.6)$$

Substituting the values from the equations (18.4) and (18.5)

$$e = \frac{m_a(v - v_a')}{m_a(v_a - v)}$$

$$e = \frac{(v - v_a')}{(v_a - v)} \quad ...(18.7)$$

By similar analysis for the body B, we can obtain

$$e = \frac{(v - v_b')}{(v_b - v)} = \frac{(v_b' - v)}{(v - v_b)} \quad ...(18.8)$$

Using (18.7) and (18.8)

$$e = \frac{(v - v_a') + (v_b' - v)}{(v_a - v) + (v - v_b)} = \frac{(v_b' - v_a')}{(v_a - v_b)} \quad ...(18.9)$$

Which is same as defined earlier

Or $$e = -\frac{(v_b' - v_a')}{(v_b - v_a)} = (-)\frac{\text{velocity of separation}}{\text{velocity of approach}}$$

The coefficient of restitution is a parameter which indicates the energy loss during an impact and can be determined experimentally.

While using the momentum equation and the coefficient of restitution relation, velocities should be assigned proper sign and should be added and subtracted algebraically.

18.4 IMPORTANT CASES OF IMPACT

1. **Perfectly Elastic Impact** (when $e = 1$)

For example, an impact between two hardened and polished steel balls.

The coefficient of restitution relation gives

$$(-)\frac{(v_b' - v_a')}{(v_b - v_a)} = 1,\ (v_b' - v_a') = -(v_b - v_a)$$

Or, $$v_b' - v_a' = v_b - v_b$$

Also, it can be shown that in the case of a perfectly elastic impact the energy of the system is conserved

Consider an elastic impact between m_a and m_b.

Conservation of momentum gives

$$m_a v_a + m_b v_b = m_a v_a' + m_b v_b'$$

which can be written as

$$m_a(v_a - v_a') = m_b(v_b' - v_b)$$

The coefficient of restitution relation when $e = 1$ is,

$$v_b' - v_a' = -(v_b - v_a)$$

$$v_a + v_a' = v_b + v_b'$$

Multiplying the above two equations,

$$m_a(v_a - v_a')(v_a + v_a') = m_b(v_b' - v_b)(v_b' + v_b)$$

Or $$\frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2 = \frac{1}{2}m_a(v_a')^2 + \frac{1}{2}m_b(v_b')^2$$

K.E. of the two bodies before impact = K.E. of the two bodies after impact.

Which shows that in the case of perfectly elastic impact ($e = 1$), the energy of the system is conserved. *Thus in the case of an elastic impact both momentum and energy are conserved.*

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2. Perfectly Plastic Impact (When $e = 0$). For example, an impact between two putty balls.

The coefficient of restitution relation gives

$$e = 0 = (-)\frac{(v_b' - v_a')}{(v_b - v_a)}$$

Or

$$v_b' = v_a' = v'$$

That is, after a plastic impact, the final velocities of both the bodies become equal (say v') and they move together as one body. As the two bodies are permanently deformed there is no period of restitution or recovery. Note that the kinetic energy of the system is not conserved.

But the total momentum of the system of bodies is conserved and we can write,

$$m_a v_a + m_b v_b = (m_a + m_b)v'$$

The above equation can be solved for v'. The common velocity of the two bodies is,

$$v' = \frac{(m_a v_a + m_b v_b)}{(m_a + m_b)}$$

In any general case of impact when e is not equal to 1, the energy of the system is not conserved.

This can be shown in any given case by comparing the kinetic energy before and after the impact.

Elastic Impact: Important Cases

(a) Impact of Two Equal Masses

$$m_a = m_b = m$$

Conservation of momentum gives

$$m_a v_a + m_b v_b = m_a v_a' + m_b v_b'$$

$$v_a + v_b = v_a' + v_b'$$

The coefficient of restitution relation gives

$$e = 1 = -\frac{v_b' - v_a'}{v_b - v_a}$$

$$v_b - v_a = -(v_b' - v_a')$$

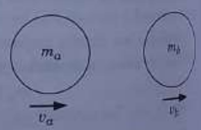

Fig. 18.4(a)

Solving the above two equations,

$$v_a' = v_b$$

$$v_b' = v_a$$

After an elastic impact the two masses exchange velocities.

(b) Impact of Two Bodies. Of the two bodies, one is immovable and of very large mass as compared to the other body

For example, a ball dropped on the floor.

$$m_b = \infty$$

$$v_b = 0$$

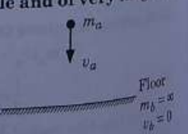

Fig. 18.4(b)

Conservation of momentum gives

$$m_a v_a + m_b v_b = m_a v_a' + m_b v_b'$$

Dividing by m_b

$$\frac{m_a}{m_b}v_a + v_b = \frac{m_a}{m_b}v_a' + v_b'$$

When m_b is very large and v_b is zero

$$v_b' = 0.$$

The coefficient of restitution relation gives

$$e = 1 = (-)\frac{(v_b' - v_a')}{(v_b - v_a)}$$

$$v_b - v_a = -v_b' + v_a'$$

as,

$$v_b = v_b' = 0$$

Therefore,

$$v_a' = -v_a$$

The body m_a would rebound with the same velocity with which it strikes the immovable body.

(c) A Body Strikes Another Body of Equal Mass at Rest

$$m_a = m_b = m, \; v_b = 0$$

Conservation of momentum gives

$$m_a v_a + m_b v_b = m_a v_a' + m_a v_b'$$

Or

$$v_a + 0 = v_a' + v_b'$$

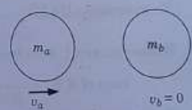

Fig. 18.4(c)

The coefficient of restitution relation gives

$$e = 1 = (-)\frac{(v_b' - v_a')}{(v_b - v_a)}$$

$$v_b - v_a = -v_b' + v_a'$$

Solving the above equations

$$v_a' = 0 \text{ and } v_b' = v_a$$

The striking mass m_a stops after imparting its entire velocity to the mass m_b.

18.5 LOSS OF KINETIC ENERGY DURING IMPACT

Let the masses m_a and m_b moving with velocities v_a and v_b collide with central direct impact. Velocities after the impact be v_a' and v_b'.

Kinetic energy of the two masses before impact

$$= \frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2$$

Kinetic energy of the two masses after impact

$$= \frac{1}{2}m_a (v_a')^2 + \frac{1}{2}m_b (v_b')^2$$

The loss of K.E. during the impact

$$= \frac{1}{2}[(m_a v_a^2 + m_b v_b^2) - (m_a v_a'^2 + m_b v_b'^2)] \qquad ...(18.10)$$

Conservations of momentum gives

$$m_a v_a + m_b v_b = m_a v_a' + m_b v_b' \quad \text{...(18.11)}$$

The coefficient of restitution relation gives

$$e = (-)\frac{v_b' - v_a'}{v_b - v_a}$$

$$v_b' - v_a' = -e(v_b - v_a) \quad \text{...(18.12)}$$

We wish to express the loss of K.E. in terms of the masses of two bodies and their velocities before impact. To eliminate v_a' and v_b' multiply the numerator and denominator of the right hand side of the equation (18.10) by $(m_a + m_b)$

$$\text{Loss of K.E.} = \frac{1}{2(m_a + m_b)}\begin{bmatrix} m_a^2 v_a^2 + m_a m_b v_b^2 + m_a m_b v_a^2 + m_b^2 v_b^2 \\ -m_a^2 v_a'^2 - m_b m_a v_b'^2 - m_a m_b v_a'^2 - m_b^2 v_b'^2 \end{bmatrix} \quad \text{...(18.13)}$$

Or

$$\text{Loss of K.E.} = \frac{1}{2(m_a + m_b)}\begin{bmatrix} \{(m_a v_a + m_b v_b)^2 + m_a m_b (v_a - v_b)^2\} \\ -\{(m_a v_a' + m_b v_b')^2 + m_a b_b (v_a' - v_b')^2\} \end{bmatrix}$$

From equation (18.11)

$$(m_a v_a + m_b v_b)^2 = (m_a v_a' + m_b v_b')^2$$

Hence equation (18.13) becomes

$$\text{Loos of K.E.} = \frac{1}{2(m_a + m_b)}[m_a m_b (v_a - v_b)^2 - m_a m_b (v_a' - v_b')^2]$$

Using equation (18.12),

$$\text{Loss of K.E.} = \frac{1}{2(m_a + m_b)}[m_a m_b (v_a - v_b)^2 - m_a m_b e^2 (v_a - v_b)^2]$$

$$\text{Loss of K.E.} = \frac{m_a m_b}{2(m_a + m_b)}[(v_a - v_b)^2 (1 - e^2)] \quad \text{...(18.14)}$$

Example 18.2 Three spherical balls of mass 2 kg, 6 kg and 12 kg are moving in the same direction with velocities 12 m/s, 4 m/s and 2 m/s respectively. If the ball of mass 2 kg impinges with the ball of mass 6 kg which in turn impinges with the ball of mass 12 kg prove that the balls of masses 2 kg and 6 kg will be brought to rest by the impacts. Assume the balls to be perfectly elastic.

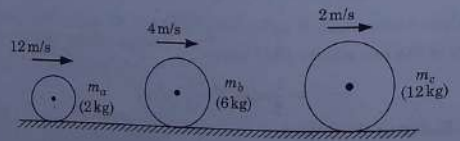

Fig. 18.5

Solution: For perfectly elastic balls $e = 1$

$$m_a = 2\text{ kg},\ m_b = 6\text{ kg},\ m_c = 12\text{ kg}$$

Impact of balls A and B

Conservation of momentum gives

$$m_a v_a + m_b v_b = m_a v_a' + m_b v_b'$$

$$2 \times 12 + 6 \times 4 = 2v_a' + 6v_b'$$

$$2v_a' + 6v_b' = 48 \quad \text{...(i)}$$

and

$$e = -\frac{v_b' - v_a'}{v_b - v_a}$$

$$v_b' - v_a' = e(v_b - v_a)$$

$$v_b' = v_a' = -1(4 - 12) = 8 \quad \text{...(ii)}$$

Solving (*i*) and (*ii*) simultaneously

$$v_a' = 0$$

that is the ball of mass 2 kg is brought to rest.

and $v_b' = 8$ m/s

Impact of Balls B and C

Consider now the impact of the ball B of mass 6 kg moving with the initial velocity of 8 m/s with the ball C of mass 12 kg moving with the velocity of 2 m/s.

Conservation of momentum gives

$$m_b v_b + m_c v_c = m_b v_b' + m_c v_c'$$

$$6 \times 8 + 12 \times 2 = 6v_b' + 12v_c' \quad \text{...(iii)}$$

$$v_b' + 2v_c' = 12$$

also,

$$e = -\frac{(v_c' - v_b')}{(v_c - v_b)}$$

$$v_c' - v_b' = e(v_c - v_b) = -1(2 - 8) \quad \text{...(iv)}$$

Or

$$v_c' - v_b' = 6$$

Solving (*iii*) and (*iv*) simultaneously

$v_b' = 0$, Final velocity of the ball B is zero or it is brought to rest after the second impact.

and $v_c' = 6$ m/s

Example 18.3. A glass ball is dropped on to a smooth horizontal floor from which it bounces to a height of 9 m. On the second bounce it rises to a height of 6 m. From what height the ball was dropped and what is the coefficient of restitution between the glass and the floor?

Solution: Let the ball be dropped from a height h. Using the relation $v^2 - u^2 = 2as$, the velocity of the ball at the time of hitting the floor,

$$v = \sqrt{2gh}$$

Let v' be the velocity of the ball in the upward direction after the first impact. The second body is floor which is of very large mass and at rest. The coefficient of restitution relation gives

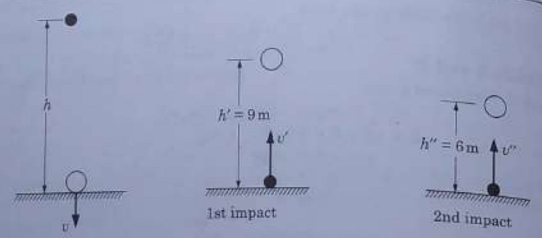

Fig. 18.6

$$e = -\frac{(v'-0)}{(v-0)}$$

$$v' = -e\sqrt{2gh} \qquad \ldots(i)$$

The ball shall move up with a velocity of

$$v' = e\sqrt{2gh}$$

and shall rise to a height h'. Using the relation

$$v^2 - u^2 = 2as$$

$$v' = \sqrt{2gh'} \qquad \ldots(ii)$$

Using equation (*i*)

$$-e\sqrt{2gh} = \sqrt{2gh'}$$

$$h' = e^2 h \qquad \ldots(iii)$$

On the second impact it strikes the floor with a velocity v' and moves up with a velocity v'' after the impact

$$e = -\frac{(v''-0)}{(v'-0)},\quad v'' = -ev' \qquad \ldots(iv)$$

Corresponding to a velocity of v'' the height h'' attained by the ball is given by

$$\sqrt{2gh''} = v''$$

Using equation (*iv*)

$$\sqrt{2gh''} = -ev'$$

From equation (*ii*),

$$v' = \sqrt{2gh'}$$

Therefore,

$$\sqrt{2gh''} = -e\sqrt{2gh'} \qquad \ldots(v)$$

$$h'' = e^2 h'$$

From (*v*),

$$e^2 = \frac{h''}{h'} = \frac{6}{9} = \frac{2}{3}$$

$$e = 0.816 \quad Ans.$$

From (*ii*),

$$h' = e^2 h$$

$$9 = \left(\frac{2}{3}\right)h$$

$$h = 13.5 \text{ m} \quad Ans.$$

18.6 OBLIQUE CENTRAL IMPACT

Let us consider the impact of two spheres A and B. Their initial velocities v_a and v_b are not collinear with the line of impact in this case but their magnitudes and directions (α_1 α_2) are known.

Let the velocities after the impact be v'_a and v'_b. We have to determine the magnitudes as well as the directions of these velocities involving four unknows (v'_a, v'_b, θ_1 and θ_2).

Choose the x-axis along the line of impact and the y-axis along the common tangent to the surfaces in contact as shown in the Fig. 18.7.

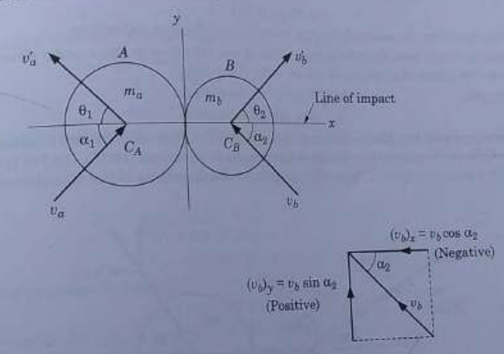

Fig. 18.7

Consider the motion of the bodies in the x and in the y directions assuming that the both bodies move freely before and after the impact.

Motion along the line of impact (x-axis). The component of the *total momentum of the two bodies* along the x-axis remains the same before and after the impact or *is conserved*. Which gives,

$$m_a(v_a)_x + m_b(v_b)_x = m_a(v'_a)_x + m_b(v'_b)x \qquad \ldots(18.15)$$

The coefficient of restitution relation is

$$e = (-)\frac{\text{The component of the velocity of separation after the impact along the x-axis}}{\text{The component of the velocity of approach before the impact along the x-axis}} \qquad \ldots(18.16)$$

$$e = (-)\frac{(v'_b)_x - (v'_a)_x}{(v_b)_x - (v_a)_x}$$

Where,

$(v_a)_x$: The component of the velocity of the body A in the x-direction before impact.

$(v'_a)_x$: The component of the velocity of the body A in the x-direction after impact.

$(v_a)_y$: The component of the velocity of the body A in the y-direction before impact.
$(v'_a)_y$: The component of the velocity of the body A in the y-direction after impact.
Similar notations have been used for the velocities of the body B.

Motion normal to the line of impact (y-axis). If the bodies are assumed to be smooth and frictionless, *the component of the velocity in the y-direction of each body shall remain unchanged or the component of the momentum in y-direction of each body, considered separately, is conserved*

$$(v_a)_y = (v'_a)_y \quad ...(18.17)$$
$$(v_b)_y = (v'_b)_y \quad ...(18.18)$$

The four equations obtained above can be solved for the four unknowns.

It should be noted here that the components of the velocities in the x and y directions can be expressed in terms of angles they make with the x-axis.

For example, $(v_b)_x = v_b \cos \alpha_2$ (negative)

$(v_b)_y = v_b \sin \alpha_2$ (positive).

While solving a problem, the components of the initial and final velocities are to be determined as indicated above then are to be substituted in the above equations (18.15), (18.16), (18.17) and (18.18) with a proper algebraic sign with respect to the directions of the axis choosen.

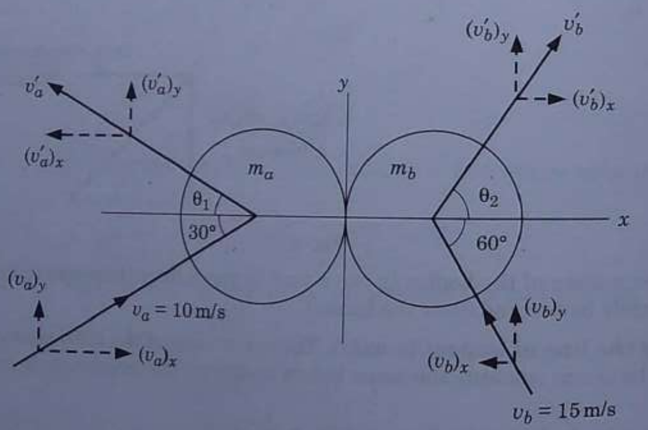

Fig. 18.8

Example 18.4 Two identical frictionless balls strike each other as shown in Fig. 18.8. Assuming $e = 0.90$, determine the magnitude and direction of the velocity of the each ball after the impact.

Solution: Components of the initial velocity of ball A

$$(v_a)_x = v_a \cos 30° = 10 \times \cos 30°$$
$$(v_a)_x = +\ 8.66 \text{ m/s}$$
$$(v_a)_y = v_a \sin 30° = 10 \times \sin 30°$$
$$(v_a)_y = +\ 5 \text{ m/s}$$

Components of the initial velocity of ball B

$$(v_b)_x = -\ v_b \cos 60° = -\ 15 \cos 60°$$
$$(v_b)_x = -\ 7.5 \text{ m/s}$$
$$(v_b)_y = +\ v_b \sin 60° = 15 \cos 60°$$
$$(v_b)_y = +\ 12.99 \text{ m/s}$$
$$m_a = m_b = m$$

Motion in y-direction : Velocity of *each ball* normal to the line of impact remains unchanged

Ball A $\quad (v'_a)_y = (v_a)_y = 5 \text{ m/s} \quad ...(i)$

Ball B $\quad (v'_b)_y = (v_b)_y = 12.99 \text{ m/s} \quad ...(ii)$

Motion in x-direction : Conservation of momentum gives,

$$m_a(v_a)_x + m_b(v_b)_x = m_a(v'_a)_x + m_b(v'_b)_x$$

Or $\quad 8.66 + (-7.5) = (v'_a)_x + m_b(v'_b)_x$

Or $\quad (v'_a)_x + (v'_b) = 1.16 \text{ m/s} \quad ...(iii)$

The coefficient of restitution relation gives

$$e = -\frac{(v'_b)_x - (v'_a)_x}{(v_b)_x - (v_a)_x}$$
$$-e[(v_b)_x - (v_a)_x] = (v'_b)_x - (v'_a)_x$$
$$-0.9\,[(-7.5) - 8.66] = (v'_b)_x - (v'_a)_x$$
$$(v'_b)_x - (v'_a)_x = 14.544 \quad ...(iv)$$

Adding equations (*iii*) and (*iv*)

$$(v'_b)_x = 7.852 \text{ m/s}$$

Substituting for $(v'_b)_x$ in equation (*iii*) and solving

$$(v'_a)_x = -\ 6.692 \text{ m/s}$$

Resultant velocities of balls A and B after impact

$$v'_a = \sqrt{(v'_a)_x^2 + (v'_a)_y^2} \qquad v'_b = \sqrt{(v'_b)_x^2 + (v'_b)_y^2}$$
$$v'_a = \sqrt{(-6.692)^2 + (5.0)^2} \qquad v'_b = \sqrt{(7.852)^2 + (12.99)^2}$$
$$v'_a = 8.35 \text{ m/s} \qquad v'_b = 15.18 \text{ m/s} \quad \textit{Ans.}$$
$$\theta_1 = \tan^{-1}\frac{(v'_a)_y}{(v'_a)_x} \qquad \theta_2 = \tan^{-1}\frac{(v'_b)_y}{(v'_b)_x}$$
$$\theta_1 = \tan^{-1}\frac{5.0}{(-6.692)} \qquad \theta_2 = \tan^{-1}\frac{12.99}{7.852}$$
$$\theta_1 = 36.76° \qquad \theta_2 = 58.84° \quad \textit{Ans.}$$

Example 18.5 A ball is thrown against a wall with a velocity v forming an angle of 30° with the horizontal (Fig. 18.9). Assuming frictionless conditions and $e = 0.50$ determine the magnitude and direction of the velocity of the ball as it rebounds from the wall.

Solution: Choosing the x-axis normal to the wall and the y-axis parallel to wall and resolving the velocity v,

$$v_x = v\cos 30° = 0.866\,v$$
$$v_y = v\sin 30° = 0.5\,v$$

Motion of the ball parallel to the wall (y-axis). Under frictionless condition, the component of the velocity parallel to the wall of the ball is conserved and have

$$v'_y = v_y = 0.5\,v$$

Motion of the ball normal to the wall (x-axis). *As the mass of the wall is infinite, conservation of momentum equation shall not give any useful information.*

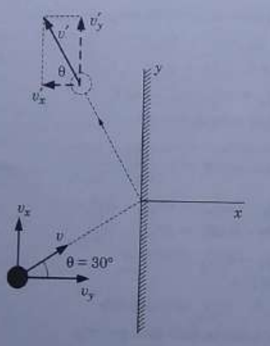

Fig. 18.9

The coefficient of restitution relation gives

$$e = -\frac{(0 - v'_x)}{(0 - v_x)}$$
$$v'_x = 0.5\,v_x$$
$$v'_x = -0.5(0.866\,v)$$
$$v'_x = -(0.433\,v)$$

Resultant velocity v'

$$v' = \sqrt{(v'_x)^2 + (v'_y)^2}$$
$$v' = v\sqrt{(-0.433)^2 + (0.5)^2}$$
$$v' = 0.661\,v \quad \textit{Ans.}$$

$$\theta = \tan^{-1}\frac{v'_y}{v'_x} = \tan^{-1}\frac{0.5v}{0.433v}$$
$$\theta = 49.1° \quad \textit{Ans.}$$

18.7 PROBLEMS INVOLVING ENERGY AND MOMENTUM

We learnt three different method to solve the problems of kinetics. These methods are based on:

1. Newton's second law
2. The principle of work and energy
3. The principle of impulse and momentum.

These methods were explained and discussed in the earlier chapters. A brief summary of these methods is included in this chapter (Table 18.1). In the problems involving an impact, the principle of impulse and momentum is the only practicable method since an impact involves a loss of mechanical energy.

There are some other type of problems which may require the use of both the energy as well as the momentum method.

Consider a pendulum A of mass m and length l, which is released with zero velocity from position A_1. The pendulum is free to swing in a vertical plane and hits a vertical wall at A_2. After the impact (with coefficient of restitution e) with the wall the pendulum swings back through an angle θ to the position A_3, that we wish to determine (Fig. 18.10).

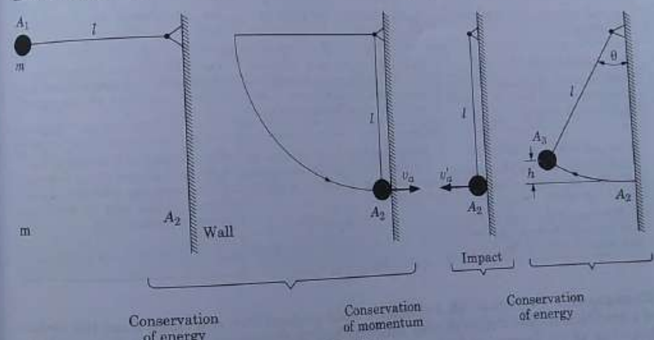

Fig. 18.10

The problem can be explained in three parts as,

1. **Pendulum swings from A_1 to A_2.** The principle of conservation of energy can be used to determine the velocity v_a of the pendulum at A_2.
2. **Pendulum hits the wall.** Using the principle of conservation of momentum and the relation of the coefficient of restitution we can determine the velocity v'_a of the pendulum after the impact.
3. **Pendulum swings from A_2 to A_3.** Applying the principle of conservation of energy we can determine the height h attained by the pendulum and then can determine the angle θ by trigonometry.

TABLE 18.1

Work Energy Principle	Principle of Conservation of Energy	Principle of Impulse Momentum
$\int_{S_1}^{S_2}(F\cos\alpha)dS = \left(\frac{mv_2^2}{2}-\frac{mv_1^2}{2}\right)$ $U_{1-2} = T_2 - T_1$ W.D. by a force acting on a particle is equal to the change in the K.E. of the particle.	$(P.E. + K.E.)_1 = (P.E. + K.E.)_2$ $V_1 + T_1 = V_2 + T_2$ $E_1 = E_2$ Sum of the P.E. and K.E. of a particle remains constant during its motion under the action of conservative force.	$mv_2 - mv_1 = \int_0^t F dt$ The change in the momentum of a particle during a small interval of time (dt) is equal to the impulse of the force acting during the same interval. If $\int F\,dt = 0$, then momentum is conserved.
Comments:		
Applied between two positions of the particle.	Applied between two positions of the particle.	Applied between an interval of time.
It is applicable whether the force involved are conservative or non-conservative. Can be applied if friction is present.	Applicable when only conservative forces (gravtiy, spring and elastic forces) are present. Not applicable when friction is present.	Applicable whether forces are conservative or non-conservative but neglects the effect of non-impulsive forces.
When applied to a system of particles, the work of all the force (whether external or internal) is to be considered. The sum of the work of action and reaction between the particles may or may not be zero.	When applied to a system of particles the P.E. corresponding to the internal forces must be considered.	When applied to a system of particles the internal forces between various particles need not be considered as the sum of their impulses is always zero.

Example 18.6 A spherical ball A of mass m when released from rest slides down the surface of a smooth bowl and strikes another spherical ball B of mass $m/4$ resting at the bottom the bowl (Fig. 18.11). Determine the height h from which the ball A should be released so that after the impact the ball B just leaves the bowl. The coefficient of restitution may be assumed to be 0.8.

Solution: $m_a = m,\ m_b = \frac{m}{4}$

The ball A slides down a height h along a frictionless surface from position 1 to position 2. The velocity v_a of the ball A when it strikes the ball B can be determined using the principle of conservation of energy.

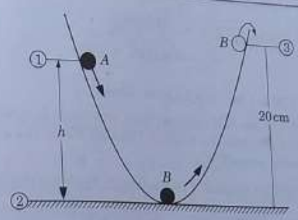

Fig. 18.11

$$mgh = \frac{1}{2}mv_a^2$$

Or $$v_a = \sqrt{2gh}$$

Let, the velocity of the ball A after impact $= v_a'$

the initial velocity of the ball $B = v_b = 0$

the velocity of the ball B after impact $= v_b'$

The ball B after the impact should attain a velocity v_b' just sufficient to rise to a height of 20 cm.

Applying the principle of conservation of energy to the motion of the ball B between position 2 and 3,

$$\underset{\text{(K.E.)}}{\frac{1}{2}\left(\frac{m}{4}\right)(v_b')^2} = \underset{\text{(P.E.)}}{\frac{m}{4}(g)(0.2)}$$

The velocity of the ball B after impact should be

$$v_b' = \sqrt{2g\times 0.2} = \sqrt{0.4g}$$

Applying the principle conservation of momentum to the impact of balls A and B

$$m_a v_a + m_b v_b = m_a v_a' + m_b v_b'$$

$$m(\sqrt{2gh}) + 0 = mv_a' + \frac{m}{4}(\sqrt{0.4g}) \quad \text{...(i)}$$

$$-v_a' + \sqrt{2gh} = 0.945$$

The coefficient of restitution relation gives

$$e = -\frac{v_b' - v_a'}{v_b - v_a}$$

$$e(v_b - v_a) = -v_b' + v_a'$$

$$0.8(0 - \sqrt{2gh}) = -\sqrt{0.4g} + v_a'$$
$$v_a' = 0.8\sqrt{2gh} = 1.981 \quad ...(ii)$$

Solving (i) and (ii)

$$h = 0.104 \text{ m} \quad \text{or} \quad h = 10.4 \text{ cm} \quad Ans.$$

Example 18.7. Sphere A of mass M is released from rest when $\theta_a = 60°$ and strikes the sphere B of mass $3M$ (Fig. 18.12). Calculate the angle θ_b through which the sphere B will swing after the impact. Assuming elastic impact.

Solution: Sphere A moves from position 1 to position 2. Applying the principle of conservation of energy,

$$E_1 = \text{P.E.} + \text{K.E.} = Mg(l - l\cos\theta_a) + 0$$
$$E_2 = \text{P.E.} + \text{K.E.} = 0 + M\frac{v_a^2}{2}$$

where v_a is the velocity acquired by sphere A in position 2.

Equationing E_1 and E_2

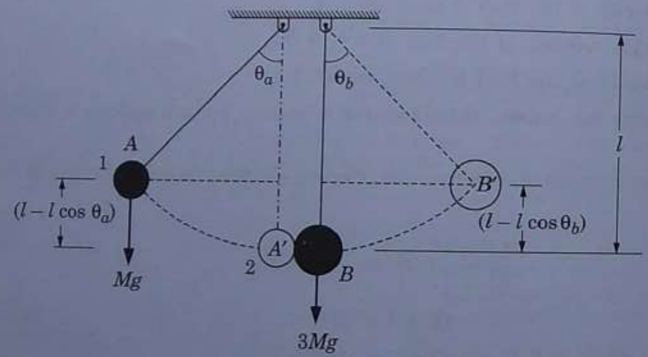

Fig. 18.12

$$Mg(l - l\cos\theta_a) = \frac{Mv_a^2}{2}$$
$$v_a = \sqrt{2g(l - l\cos\theta_a)}$$

Position 2 : Sphere A of mass M moving with velocity v_a strikes the sphere B of mass $3M$ at rest ($v_b = 0$). Let their final velocities be v_a' and v_b'

Conservation of momentum gives,

$$M_a v_a + M_b v_b = M_a v_a' + M_b v_b'$$
$$Mv_a = Mv_a' + 3Mv_b'$$
$$v_a = v_a' + 3v_b' \quad ...(i)$$

The coefficient of restitution relation gives,

$$(v_a' - v_b') = -e(v_a - v_b)$$

For elastic impact,

$$e = 1$$
$$v_a' - v_b' = -v_a \quad ...(ii)$$

Solving (i) and (ii),

$$v_b' = \frac{v_a}{2}$$

Velocity of the sphere B after impact in position 2, is

$$v_b' = \frac{v_a}{2} = \frac{1}{2}\sqrt{2g(l - l\cos\theta_a)}$$

Sphere B from position 2, moves to position 3 till it comes to rest.

Applying principle of conservation of energy to the sphere B.

$$E_2 = \text{P.E.} + \text{K.E.} = 0 + \frac{1}{2}3M(v_b')^2$$
$$E_2 = \frac{1}{2}3M\left(\frac{1}{2}\sqrt{2g(l - l\cos\theta_a)}\right)^2$$
$$E_3 = \text{P.E.} + \text{K.E.} = 3Mg(l - l\cos\theta_b) + 0$$

Equating $\quad E_2 = E_3$

$$\frac{1}{2}3M\left(\frac{1}{2}\sqrt{2gl(1 - \cos\theta_a)}\right)^2 = 3Mgl(1 - \cos\theta_b)$$
$$\frac{1}{4}(1 - \cos\theta_a) = (1 - \cos\theta_b)$$

$\theta_a = 60°$,

$$\frac{1}{4}(1 - 0.5) = (1 - \cos\theta_b)$$
$$\cos\theta_b = 0.875$$
$$\theta_b = 28.95° \quad Ans.$$

Example 18.8. Bullet A of mass 0.01 kg moving with a velocity of 100 m/s hits a bob B of a simple pendulum horizontal (Fig. 18.13). Find the maximum angle through which the pendulum swings when,

(a) The bullet gets embedded in the bob
(b) The bullet rebounds from the surface of the bob with a velocity of 20 m/s
(c) The bullet escapes from the other end of the bob with a velocity of 20 m/s

Assume the mass of the bob to be 1.0 kg and the length of pendulum as 1.0 m.

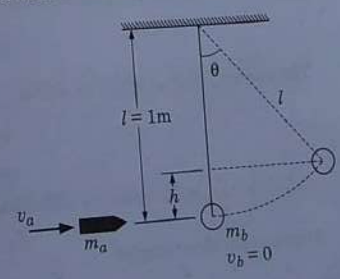

Fig. 18.13

Solution: $\quad m_a = 0.01$ kg $\qquad v_a = 100$ m/s

$m_b = 1.0$ kg $\qquad v_b = 0$

Momentum of the bullet and the bob before impact,

$$= m_a v_a + m_b v_b$$
$$= 0.01 \times 100 + 0$$
$$= 1 \text{ kg m/s} \quad ...(i)$$

(a) Bullet gets embedded in the bob. Let the common velocity be v'.

Momentum after impact of the bullet and the bob

$$= m_a v' + m_b v'$$
$$= v'(0.01 + 1) = 1.01v' \quad \text{...(ii)}$$

As the momentum is conserved, equating (i) and (ii)

$$1 = 1.01v'$$
$$v' = 0.99 \text{ m/s}$$

Applying the principle of conservation of energy, the height h to which bob rises is given by

$$\underset{\text{(K.E.)}}{\frac{1}{2}(m_a + m_b)(v_a')^2} = \underset{\text{(P.E.)}}{(m_a + m_b)gh}$$

$$h = \frac{(v')^2}{2g} \quad \text{...(iii)}$$

$$h = \frac{1}{2} \times \frac{(0.99)^2}{9.81} = 0.05 \text{ m}$$

The angle θ by which the bob swings corresponding to the value of $h = 0.5$ m

$$\cos\theta = \frac{l-h}{l} \qquad \theta = \cos^{-1}\left(\frac{l-h}{l}\right) = \frac{1-0.05}{1.0}$$

$$\theta = 18.2° \quad \textit{Ans.}$$

(b) Let the velocity of the bullet after rebound be v_a'.

$$v_a' = -20 \text{ m/s}$$

Momentum after impact

$$= m_a v_a' + m_b v_b'$$
$$= -0.01 \times 20 + 1 \times v_b' \quad \text{...(iv)}$$

Equating (i) and (iv),

$$1 = -0.2 + v_b' \qquad \text{Or} \qquad v_b' = 1.2 \text{ m/s}$$

The bob after the impact attains a velocity $v_b' = 1.2$ m/s and will rise to a height that can be determined using the principle of conservation of energy.

$$h = \frac{(v'_b)^2}{2g} \qquad h = \frac{(1.2)^2}{2 \times 9.81} = 0.073 \text{ m.}$$

The corresponding angle of swing,

$$\theta = \cos^{-1}\left(\frac{1.0-0.073}{1.0}\right)$$

$$\theta = 22.09° \quad \textit{Ans.}$$

(c) When the bullet pierces and escapes with velocity

$$v_a = 20 \text{ m/s}$$

Momentum after impact

$$= m_a v_a' + m_b v_b'$$
$$= 0.01 \times 20 + 1 \times v_b' \quad \text{...(v)}$$

Equating (i) and (v)

$$1 = 0.2 + v_b'$$
$$v_b' = 0.8 \text{ m/s}$$

The bob after the impact attains a velocity $v_b' = 0.8$ m/s and will rise to a height that can be determined using the principle of conservation of energy.

$$h = \frac{(v_b')^2}{2g}, \qquad h = \frac{(0.8)^2}{2 \times 9.81} = 0.033 \text{ m}$$

The angle of swing corresponding the height

$$\theta = \cos^{-1}\left(\frac{1-0.033}{1}\right)$$

$$\theta = 14.7° \quad \textit{Ans.}$$

Example 18.9. A weight W_a of 50 N falls through a height h of 7.5 cm onto a weight W_b of 50 N which is supported by a spring of stiffness $k = 20$ N/cm (Fig. 18.14). Find the maximum compression of the spring over and above that caused due to the static action of the weight W_b. Assume the impact to be plastic.

Solution:

$$W_a = W_b = 50 \text{ N}$$

Corresponding masses are,

$$m_a = m_b = \frac{50}{g} \text{kg}$$

$$k = 20 \text{ N/cm} = 2 \times 10^3 \text{ N/m}$$

The velocity of the weight W_a just before the impact and after falling from a height 0.075 m can be obtained by using the principle of conservation of energy.

$$v_a = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 0.075}$$
$$v_a = 1.213 \text{ m/s}$$

Weight W_a moving with velocity v_a collides with the weight W_b at rest. After the plastic impact, let both the weights move together with the same velocity v'.

Applying the principle of conservation of momentum.

$$m_a v_a + 0 = (m_a + m_b)v'$$

$$v' = \frac{m_a v_a}{(m_a + m_b)} = \frac{50}{100} \times 1.213$$

$$v' = 0.606 \text{ m/s}$$

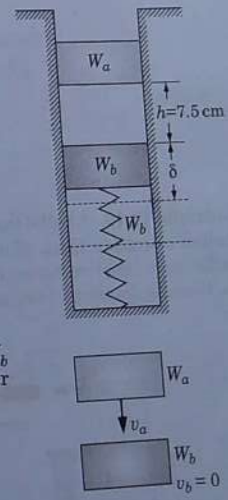

Fig. 18.14

Initial compression or static deflection of the spring due to the weight W_b is,

$$\delta_{st} = \frac{W_b}{k} = \frac{50}{2 \times 10^3}$$

$$\delta_{st} = 0.025 \text{ m}$$

Let both the weights after the impact come to rest after travelling a distance δ. The total compression of the spring from the undeformed position is,

$$\delta t = \delta + \delta_{st}$$

δ, thus represents the compression of the spring over and above that caused by that static action of the weight W_b.

Applying the principle of conservation of energy between the positions just after the impact and when both the weights come to rest,

$$\frac{1}{2}k(\delta_{st})^2 + \frac{(W_a + W_b)}{g}(v')^2 = \frac{1}{2}k(\delta_t)^2 - \frac{(W_a + W_b)}{g}g\delta$$

(P.E. of spring) + (K.E. of weights) = (P.E. of spring) – (P.E. of weights)

$$\frac{1}{2}\times 2\times 10^3(0.025)^2 + \frac{(50+50)}{g}\frac{(0.66)^2}{2} = \frac{1}{2}\times 2\times 10^3(\delta_t)^2 - \frac{(50+50)}{g}g\delta$$

Substituting,

$$\delta = \delta_t - \delta_{st} = \delta_t - 0.025$$

$$1.872 + 0.625 = 100(\delta_t)^2 - 100(\delta_t - 0.025)$$

$$2.497 = 1000\delta_t^2 - 100\delta_t + 2.5$$

$$1000\delta_t^2 - 100\delta_t - 0.003 = 0$$

$$\delta_t^2 - 0.1\delta_t - 0.000003 = 0$$

Solving

$$\delta_t = 0.1 \text{ m} \quad \text{Or} \quad 10 \text{ cm}$$

$$\delta = \delta_t - \delta_{st}$$

$$\delta = 10.0 - 2.5$$

$$\delta = 7.5 \text{ cm}$$

Example 18.10. A bullet travelling horizontally with a velocity of 600 m/s and weighing 0.25 N strikes a wooden block of weight 50 N resting on around horizontal floor (Fig. 18.15). The coefficient of friction between the floor and the block μ = 0.5. Find the distance through which the block is displaced from its initial position.

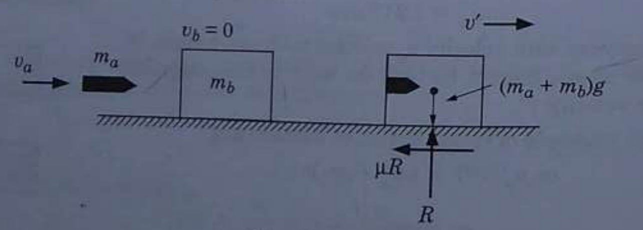

Fig. 18.15

Solution:

Velocity of the bullet before impact $v_a = 600$ m/s

Velocity of the block before impact $v_b = 0$

Mass of the bullet $m_a = \frac{0.25}{g}$ kg

Mass of the block $m_b = \frac{50}{g}$ kg

Let us assume that the bullet after striking the block remains burried in the block and both move with a common velocity v.

Applying the principle of conservation of momentum

$$m_a v_a + m_b v_b = (m_a + m_b)v'$$

$$\frac{0.25}{g}(600) + 0 = \frac{(50+0.25)}{g}v'$$

$$v' = \frac{0.25\times 600}{50.25}$$

$$v' = 2.98 \text{ m/s.}$$

To find the distance travelled by the block apply the principle of work and energy.

K.E. lost by the block with bullet burried = Work done to overcome the frictional force

If s is the distance travelled by the block,

$$(m_a + m_b)\frac{(v')^2}{2} = (\mu R)s$$

where,

$$R = g(m_a + m_b)$$

$$(m_a + m_b)\frac{(v')^2}{2} = \mu g(m_a + m_b)s$$

$$s = \frac{(v')^2}{2g\mu}$$

$$s = \frac{(2.98)^2}{2\times 9.81\times 0.5}$$

$$s = 0.90 \text{ m} \quad \textit{Ans.}$$

Example 18.11. A pile of mass m is driven a distance δ into the ground by the blow of a hammer of mass M falling through a height of h onto the top of the pile (Fig. 18.16). Find the resistance to penetration assuming it to be constant. Assume the impact between the hammer and the pile to be plastic. Given,

$M = 750$ kg, $m = 200$ kg, $h = 1.2$ m, δ = 10 cm.

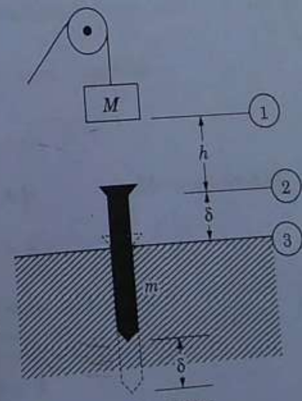

Fig. 18.16

Solution: This problem involves application of the,

1. Conservation of energy
2. Conservation of momentum
3. Principle of work and energy

Between position 1 and 2 of the hammer, principle of conservation of energy gives

$$mgh = \frac{1}{2}mv^2$$

Or

$$v = \sqrt{2gh}$$

where, v is the velocity acquired by the hammer when it just hits the pile.

Position 2. Impact between the hammer and the pile takes place. Since the impact is plastic both hammer and the pile travel with a common velocity v'.

Conservation of momentum gives

$$Mv = (m + M)v'$$

$$v' = \frac{Mv}{(M+m)} = \frac{M\sqrt{2gh}}{(M+m)}$$

After the impact, from position 2 to position 3, both hammer and pile move together with a velocity v' which becomes zero after travelling distance δ.

The principle of conservation of energy cannot be used, as the resistance to penetration is a non-conservative force.

So, the principle of work and energy, is to be applied.

K.E. in position 2, $\quad T_2 = (m + M)\frac{(v')^2}{2}$

Substituting for v', $\quad T_2 = (m + M)\frac{1}{2}\left(\frac{M\sqrt{2gh}}{M+m}\right)^2$

K.E. in position 3 $\quad T_3 = 0$

Work done between position 2 and 3 is due to the resistance of ground and the gravity force.

Let R be the resistance to penetration.

$$U_{2-3} = \underset{\text{(W.D. by the gravity force)}}{(M + m)g\delta} \quad \underset{\text{(W.D. to overcome resistance)}}{-R\delta}$$

$$U_{2-3} = T_3 - T_2$$

$$\text{(W.D.)} = \text{(Change in K.E.)}$$

$$-R\delta + (M + m)g\delta = 0 - (M + m)\frac{1}{2}\left(\frac{M\sqrt{2gh}}{M+m}\right)^2$$

$$R = (M + m)g + \frac{M^2gh}{(M+m)\delta}$$

Substituting values $\quad M = 750$ kg, $m = 200$ kg

$$h = 1.2 \text{ m}, \ \delta = \frac{10}{100} \text{ m} = 0.1 \text{ m}$$

$$R = (750 + 200) \times 9.81 + \frac{(750)^2 \times 9.81 \times 1.2}{(750+200) \times 0.10}$$

$$R = 9319.5 + 69702.6$$

$$R = 79022 \text{ N}$$

$$R = 79 \text{ kN} \quad \textit{Ans.}$$

PROBLEMS

18.1. A sphere of mass 4 kg moving with a velocity of 5 m/s overtakes a sphere of mass 3 kg moving with 4 m/s. If a direct impact takes place find their velocity after the impact. Assume $e = 0.5$. [4.36 m/s, 4.86 m/s]

18.2. A ball of mass 1 kg moving with a velocity of 3 m/s strikes a ball of mass 5 kg moving with a velocity of 0.6 m/s in the same direction. Show that the direction of motion of the first ball is reversed. Find the loss of the kinetic energy. Assume $e = 0.75$. [1.05 Nm]

18.3. A ball drops from the ceiling of a room. After rebounding twice from the floor it reaches a height equal to the half that of the ceiling. Show that the coefficient of restitution is $(1/2)^{1/4}$.

18.4. A ball falls from a height h upon a fixed horizontal plane. If e is the coefficient of restitution show that the total distance travelled by the ball after it has finished rebounding is $\left(\frac{1+e^2}{1-e^2}\right)h$.

18.5. The masses of two balls are in the ratio of 2 : 1 and their respective velocities are in the ratio 1 : 2 but in opposite directions before impact. If the coefficient of restitution is 5/6, prove that after the impact each ball will move back with 5/6th of its original velocity.

18.6. Two smooth identical balls strike each other as shown in Fig. P.18.6. If the value of $e = 0.8$ determine their velocities after the impact. $\left[\begin{array}{l} v_a' = 0.2 \text{ m/s} \rightarrow \\ v_b' = 6.26 \text{ m/s} \ \angle 73.3° \end{array}\right]$

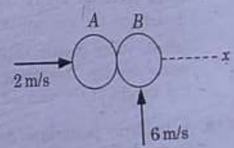

Fig. P.18.6

18.7. Two identical balls each of mass M collide with velocities as shown in Fig. P.18.7. Find the final velocities of the balls after the impact and the loss in kinetic energy. Assume $e = 0.9$.

$\left[\begin{array}{l} v_a' = -2.27 \text{ m/s} \\ v_b' = 2.84 \text{ m/s} \ \angle 60.45° \\ \text{Loss of K.E.} = 1.940 \text{ Nm} \end{array}\right]$

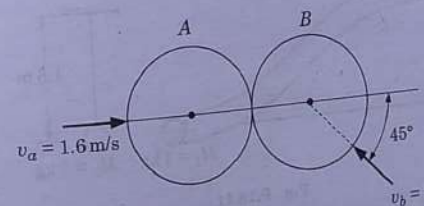

Fig. P.18.7

18.8. A sphere of a mass 2 kg is released from rest and strikes a block of mass 2.5 kg resting on a horizontal surface. How far the block will move after the impact? Assume $e = 0.75$ and the coefficient of friction between the block and the floor is 0.25 (Fig. P.18.8). [2.90 m]

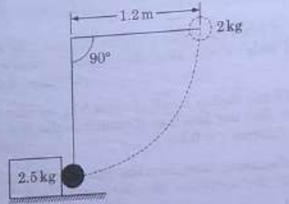

Fig. P.18.8

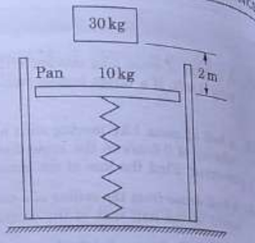

Fig. P.18.9

18.9. A block of mass 30 kg is dropped from a height of 2 m onto the 10 kg pan of a spring scale (Fig. P.18.9). Assuming the impact to be perfectly plastic and the spring stiffness $k = 20$ kN/m, determine the maximum deflection of the pan. [22.5 cm]

[**Hint.** After plastic impact, pan and 30 kg mass attain the same velocity. The spring is in deformed position initially with the weight of the pan acting on it]

18.10. A ballistic pendulum consisting of a block of mass 30 kg suspended from two wires each of length 1.8 m, is used to measure the muzzle velocity of a gun. If the pendulum swings through a horizontal distance $s = 25$ cm when a 40 g bullet is fired into it, determine the muzzle velocity v of the gun (Fig. P.18.10). [439.0 m/s]

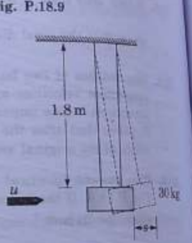

Fig. P.18.10

18.11. A ball M_1 of mass 1 kg traverses a frictionless chute falling through a height of 1.5 m. It then strikes a ball M_2 of mass 2 kg hanging from a cord 1.5 m long. Determine the height to which this ball will rise assuming the impact to be perfectly elastic. [0.666 m]

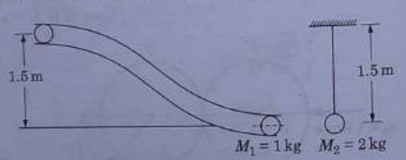

Fig. P.18.11

CHAPTER 19 Relative Motion

19.1 INTRODUCTION

The motion of a particle with respect to a fixed frame of reference is called the absolute motion of the particle. For example, the motion of a train with respect to the platform is termed as the absolute motion of the train.

The motion relative to a set of axes which are moving, is called the relative motion. For example, the motion of a train A with respect to another moving train B is the relative motion of the train A with respect to the train B.

19.2 RELATIVE MOTION BETWEEN TWO PARTICLES: VELOCITY AND ACCELERATION

Consider two particles A and B moving in the same plane x-y (Fig. 19.1).

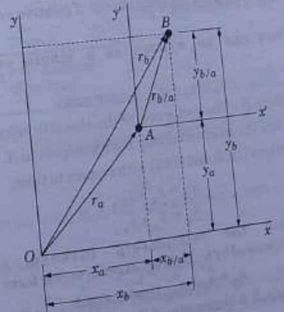

Fig. 19.1

Let x and y be the fixed system of axes centred at O and x'-y' *be the moving system of axes attached to the particle A. The moving system of axes* x'-y' *can translate (but not rotate) with respect to the fixed system of axes x-y,* and remains parallel to it.

The motion of particle B with respect to the fixed axes x-y is called the absolute motion of B and the motion with respect to the moving axis x'-y' attached to the particle A is called the relative motion of B with respect to A.

The vectors $\mathbf{r}_a$ and $\mathbf{r}_b$ define the positions A and B at any instant with respect to the fixed axes x-y.

The vector $\mathbf{r}_{b/a}$ defines the position of B with respect to A.

These position vectors are related by the triangle law.

$$\mathbf{r}_b = \mathbf{r}_a + \mathbf{r}_{b/a} \quad \text{...(19.1)}$$

As, $\mathbf{r} = \mathbf{x} + \mathbf{y}$, we can write the above relation in the terms of the components in the x and y direction as,

$$x_b = x_a + x_{b/a} \quad \text{...(19.2)}$$

$$y_b = y_a + y_{b/a} \quad \text{...(19.3)}$$

where, x_a and y_a are the coordinates of the particle A with respect to the fixed axes x-y. x_b, y_b are the coordinates of particle B with respect to the fixed axes x-y.

And $x_{b/a}$, $y_{b/a}$ are the coordinates of particle B with respect to the moving axes x'-y'.

Differentiating the above equations with respect to time,

$$\dot{x}_b = \dot{x}_a + \dot{x}_{b/a} \quad \text{...(19.4)}$$

$$\dot{y}_b = \dot{y}_a + \dot{y}_{b/a} \quad \text{...(19.5)}$$

$\dot{x}_a, \dot{y}_a, \dot{x}_b, \dot{y}_b$ are the components of the absolute velocities of particles A and B with respect to the fixed axes x-y and $\dot{x}_{b/a}$ and $\dot{y}_{b/a}$ are the components of relative velocity of the particle B with respect to the particle A.

The above equations can be written as a single equation in the vector form as,

$$\mathbf{v}_b = \mathbf{v}_a + \mathbf{v}_{b/a} \text{ (vector sum)} \quad \text{...(19.6)}$$

Absolute velocity of particle B = The vector sum of the absolute velocity of A and the relative velocity B with respect to A.

Differentiating the equations (19.4) and (19.5) with respect to time,

$$\ddot{x}_b = \ddot{x}_a + \ddot{x}_{b/a} \quad \text{...(19.7)}$$

$$\ddot{y}_b = \ddot{y}_a + \ddot{y}_{b/a} \quad \text{...(19.8)}$$

Above equations are the relationship between the components of acceleration. These can be expressed by a single equation in vector form

$$\mathbf{a}_b = \mathbf{a}_a + \mathbf{a}_{b/a} \text{ (vector sum)} \quad \text{...(19.9)}$$

Absolute acceleration of particle B. = Vector sum of the absolute acceleration of A and the relative acceleration of B with respect to A.

19.3 RELATIVE VELOCITY: WORKING CONCEPTS

Consider two trains A and B travelling on two straight and parallel tracks with the speeds of 100 km/hour and 80 km/hour respectively

When moving in the same direction. To an observer on train B, A would appear to move with a speed of (100 – 80) = 20 km/hour with respect to the train B.

We know that displacement, velocity and acceleration are vector quantities which can be represented by a line segment.

Therefore, graphically the above situation can be represented as shown in Fig. 19.2. The relative velocity $\mathbf{v}_{A/B}$ is given by the vector equation,

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B = 100 - 80 = 20 \text{ km/hour}$$

Reative velocity of A w.r.t. B = Absolute velocity of A – Absolute velocity of B.

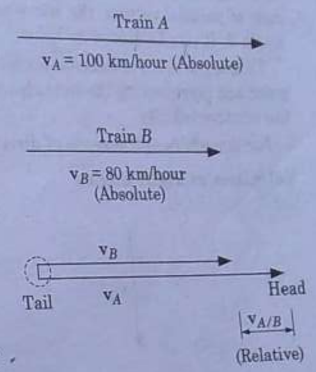

Fig. 19.2

When moving in opposite directions. In this case the graphical representation is as shown in Fig. 19.3 and vectorially it can be expressed as

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$$

$$\mathbf{v}_{A/B} = [100 - (-80)]$$

$$\mathbf{v}_{A/B} = 180 \text{ km/hour.}$$

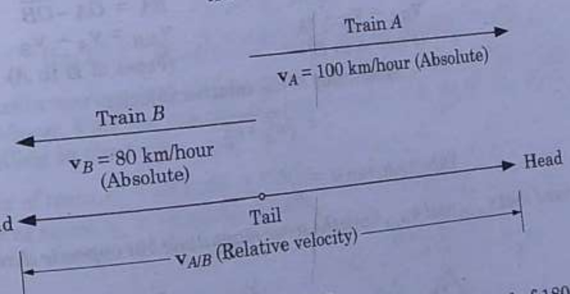

Fig. 19.3

That is, to an observer on train B, A will appear to move with a speed of 180 km/hour with respect to the train B.

To summarize we can say

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$$

Relative velocity of A with respect to B = The vector difference of absolute velocities of A and B.

Determination of the relative velocity of A w.r.t. B is essentially the determination of the vector difference of their velocities $\mathbf{v}_A$ *and* $\mathbf{v}_B$.

To find the vector difference ($\mathbf{v}_A - \mathbf{v}_B$), the two vectors should be placed with their tails joined at a common point. The line joining their heads then, represents the relative velocity. From the case of parallel vectors, the above principle can be extended to any two vectors inclined at an angle θ. It can be stated as below.

To find the relative velocity, place the two velocity vectors with their tails joining at a common point and representing the two sides of a triangle then, the closing side of the triangle represents the relative velocity.

For determining the sense of direction let us discuss the following cases.

Velocities at right angles

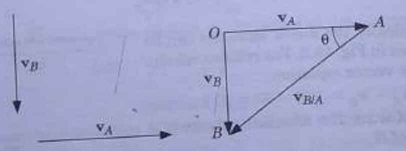

Fig. 19.4

Relative velocity of B w.r.t. A | Relative velocity of A w.r.t. B

$$\overline{OA} + \overline{AB} = \overline{OB} \qquad \overline{OB} + \overline{BA} = \overline{OA}$$

(Triangle law of vectors)

$$\overline{AB} = \overline{OB} - \overline{OA} \qquad \overline{BA} = \overline{OA} - \overline{OB}$$

$$\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A \qquad \mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$$

(From A to B) (From of B to A)

Magnitude of the relative velocity

$$= \sqrt{v_A^2 + v_B^2}$$

$$\text{Direction, } \tan\theta = \frac{v_B}{v_A}$$

It can be noted that $\mathbf{v}_{A/B}$ and $\mathbf{v}_{B/A}$ have the same magnitude but opposite directions (Fig. 19.4).

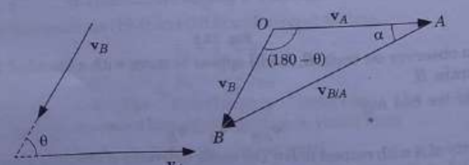

Fig. 19.5 (*a*)

Velocities at angle θ

Relative velocity of B w.r.t. A

$$\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$$

and is represented by the side $\overline{AB}$ of the triangle Fig. 19.5 (*a*). The magnitude of which can be determined using the cosine law as

$$\mathbf{v}_{B/A} = \sqrt{\mathbf{v}_A^2 + \mathbf{v}_B^2 - 2\mathbf{v}_A\mathbf{v}_B\cos(180-\theta)}$$

The angle α that it makes with the direction of velocity v_A can be determined using sine law

$$\frac{\mathbf{v}_{B/A}}{\sin(180-\theta)} = \frac{\mathbf{v}_B}{\sin\alpha}$$

$$\sin\alpha = \frac{\mathbf{v}_B\sin(180-\theta)}{\mathbf{v}_{B/A}}$$

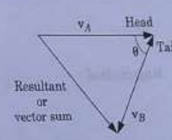

Fig. 19.5 (*b*)

The method of resolution of vectors can also be used to determine the magnitude and direction of the relative velocity.

It may be interesting to recall that to find the resultant of two vectors (or vectors sum), the vectors are placed from head to tail and the closing side of the triangle taken in opposite order represents the resultant [Fig. 19.5 (*b*)].

Alternative Approach. Problems on relative motion can also solved by expressing the absolute and relative velocities of objects as vectors and then following the simple concept that relative velocity of A w.r.t. B is the vector difference of two velocity vectors as,

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$$

$$[(v_{A/B})_x i + (v_{A/B})_y j] = [(v_A)_x i + (v_A)_y j] - [(v_B)_x i + (v_B)_y j]$$

Example 19.1 A train moving at 36 km/hour is hit by a stone thrown at right angles to it with a velocity of 18 km/hour. Find the velocity and the direction with which the stone appears to hit a person travelling in the train.

Solution: Velocity of train, $v_t = 36$ km/h $= 10$ m/s

Velocity of stone, $v_s = 18$ km/hr $= 5$ m/s

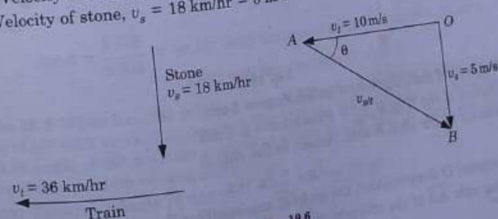

Fig. 19.6

Relative velocity of the stone with respect to the train

$$\bar{v}_{s/t} = \bar{v}_s - \bar{v}_t$$

To represent graphically the above equation, draw from any point O vectors $\overline{OA}$ and $\overline{OB}$ representing the velocities of the train and the stone as shown in the Fig. 19.6. The closing side $\overline{AB}$ of the triangle so formed, represents the relative velocity $v_{s/t}$.

Using triangle law

$$\overline{OA} + \overline{AB} = \overline{OB}$$

$$\overline{AB} = \overline{OB} - \overline{OA}$$

$$\bar{v}_{s/t} = \bar{v}_s - \bar{v}_t$$

Magnitude of

$$v_{s/t} = \sqrt{v_t^2 + v_s^2} = \sqrt{10^2 + 5^2}$$

$$v_{s/t} = 11.2 \text{ m/s} \quad Ans.$$

$$\theta = \tan^{-1}\frac{v_s}{v_t} = \frac{5}{10}$$

$$\theta = 26.5° \quad Ans.$$

Example 19.2 Ship A is moving north-west at a speed of 18 km/hour and the ship B is moving east at a speed 9 km/hour. Find the magnitude and direction of the relative velocity of the ship B with respect to the ship A (Fig. 19.7).

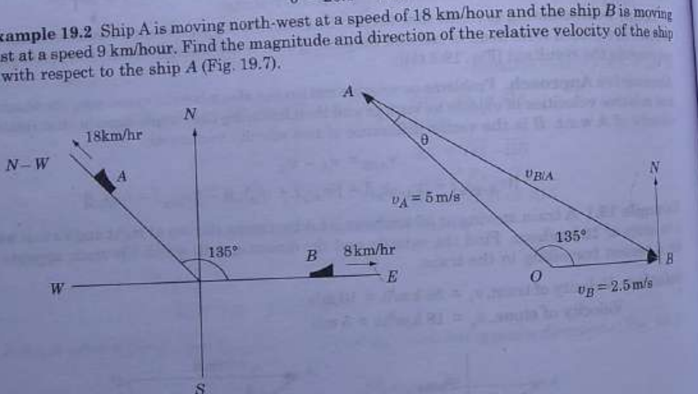

Fig. 19.7

Solution: Speed of the ship A, v_A = 18 km/hour = 5 m/s

Speed of the ship B, v_B = 9 km/hour = 2. 5 m/s

Relative velocity of the ship B with respect to the ship A

$$\bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$

From any point O draw vectors. $\overline{OA}$ and $\overline{OB}$ representing the velocities of the ships A and B. The closing side $\overline{AB}$ of the triangle so formed represents the relative velocity $\bar{v}_{B/A}$.

$$\overline{OA} + \overline{AB} = \overline{OB}$$

$$\overline{AB} = \overline{OB} - \overline{OA}$$

$$\bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$

The magnitude of the relative velocity $v_{B/A}$ can be obtained using cosine law.

$$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos 135°$$

$$AB^2 = 5^2 + (2.5)^2 - 2(5)(2.5) - (-1/\sqrt{2})$$

$$AB = 6.99 \text{ m/s}$$

$$v_{B/A} = 6.99 \text{ m/s.} \quad Ans.$$

Applying sine law,

$$\frac{OB}{\sin\theta} = \frac{AB}{\sin 135°}$$

$$\frac{2.5}{\sin\theta} = \frac{6.99}{0.707}$$

$$\sin\theta = \frac{2.5 \times 0.707}{6.99}$$

$$\theta = 14.65° \quad Ans.$$

Angle θ is determined with respect to the direction of ship A.

Alternative solution

$$\bar{v}_A = (-5\cos 45°)\,i + (5\sin 45°)\,j$$

$$\bar{v}_A = -3.535\,i + 3.535\,j$$

$$\bar{v}_B = 2.5\,i$$

$$\bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$

$$\bar{v}_{B/A} = [2.5\,i] - [-3.535\,i + 3.535\,j]$$

$$\bar{v}_{B/A} = 6.035\,i - 3.535\,j$$

Magnitude of

$$\bar{v}_{B/A} = \sqrt{(6.035)^2 + (3.535)^2}$$

$$\bar{v}_{B/A} = 6.99 \text{ m/s}$$

Direction,

$$\tan\theta = \frac{3.535}{6.035} = 30.35° \text{ with horizontal or } (180° - 135° - 30.35°)$$

$$= 14.65° \text{ with the direction of ship } A.$$

Example 19.3 When a cyclist is riding west at 20 km/hour he finds the rain meeting him at an angle of 45° with vertical as shown in Fig. 19.8 (*a*). When he rides at 12 km/hour he meets the rain at angle of 30° with the vertical as shown in Fig. 19.8 (*b*). Find the magnitude and direction of the absolute velocity of the rain.

Solution: Relative velocity of rain with respect to the cyclist

$$\bar{v}_{R/C} = \bar{v}_R - \bar{v}_C \quad ...(i)$$

Absolute velocity of the rain

$$\bar{v}_R = \bar{v}_{R/C} + \bar{v}_C$$

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It may be noted that the absolute velocity of the rain is same (magnitude and direction) in the two cases.

The absolute velocity of the cyclist is fixed in direction but differs in magnitude in two cases as,

$$v_C = 20 \text{ km/hour}$$
$$v_C' = 12 \text{ km/hour}$$

Let the absolute velocity of the rain make an angle θ with the horizontal.

If the relative velocities of rain in two cases are denoted by $v_{R/C}$ and $v'_{R/C}$ then from equation (i)

$$\bar{v}_R = \bar{v}_{R/C} + \bar{v}_C$$
$$\bar{v}_R = \bar{v}'_{R/C} + \bar{v}'_C$$

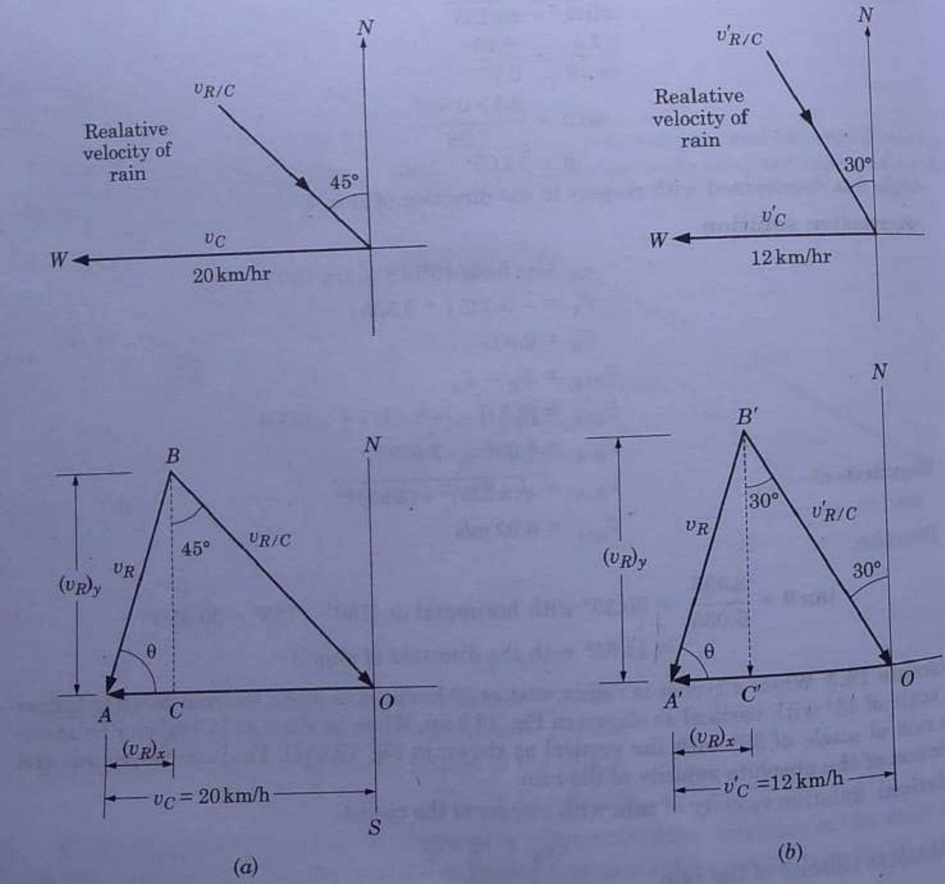

Fig. 19.8

That is,

The absolute velocity of rain = The vector sum of the relative velocity of the rain w.r.t. the cyclist and the absolute velocity of the cyclist.

To find the vectors sum of vectors $\bar{v}_{R/C}$ and $\bar{v}_C$ they are placed head to tail and the closing side BA represents $\bar{v}_R$. Similarly in the second case triangle $OA'B'$ is drawn and the side $B'A$ represents $\bar{v}_R$.

$$\overline{B'A} = \overline{BA} = \bar{v}_R$$

The components $(v_R)_x$ and $(v_R)_y$ of the absolute velocity $\bar{v}_R$ of the rain are,

$$(v_R)_x = CA = C'A'$$
$$(v_R)_y = BC = B'C'$$

And

From triangle OBC,

$$\tan 45° = \frac{20-(v_R)_x}{(v_R)_y}$$

$$20 - (v_R)_x = (v_R)_y \qquad \text{...(ii)}$$

Or

From triangle $OB'C'$

$$\tan 30° = \frac{12-(v_R)_x}{(v_R)_y}$$

$$12 - (v_R)_x = 0.577 \,.\, (v_R)_y \qquad \text{...(iii)}$$

Solving the equations (ii) and (iii)

$$(v_R)_x = 1.09 \text{ km/hour}$$
$$(v_R)_x = 18.91 \text{ km/hour}$$
$$v_R = \sqrt{(v_R)_x^2 + (v_R)_y^2} = \sqrt{(1.09)^2 + (18.91)^2}$$
$$v_R = 18.94 \text{ km/hour} \quad \textit{Ans.}$$
$$\theta = \tan^{-1}\frac{(v_R)_y}{(v_R)_x}$$
$$\theta = \tan^{-1}\frac{18.91}{1.09}$$
$$\theta = 86.7° \quad \textit{Ans.}$$

Alternative solution

When, $\bar{v}_C = -20\,i,\ \bar{v}_{R/C} = \bar{v}_R - \bar{v}_C$

$$\bar{v}_R = (v_R)_x i + (v_R)_y j,\ \bar{v}_{R/C} = [(v_R)_x i + (v_R)_y j] - [-20\,i]$$
$$\bar{v}_{R/C} = [(v_R)_x + 20]\,i + [(v_R)_y]\,j$$

Direction of $\bar{v}_{R/C}$ makes an angle of (90° + 45°) with the horizontal, therefore,

$$-\tan 45° = \frac{(v_R)_y}{[(v_R)_x + 20]} = -1.0 \qquad \text{...(i)}$$

$$(v_R)_x + 20 = -(v_R)_y$$

$$v_C = -12\,i$$

When,

$$\bar{v}'_{R/C} = [(v_R)_x\, i + (v_R)_y\, j] - [-12\, i]$$

$$\bar{v}'_{R/C} = [(v_R)_x + 12]\, i + [(v_R)_y]\, j$$

Direction $\bar{v}'_{R/C}$ makes an angle of $(90° + 30°)$ with the horizontal, therefore,

$$-\tan 30° = \frac{(v_R)_y}{[(v_R)_x + 12]} = -0.577$$

$$(v_R)_x + 12 = -0.577\,(v_R)_y \qquad ...(ii)$$

Solving (*i*) and (*ii*)

$$(v_R)_x = -1.09 \text{ km/hour}$$

$$(v_R)_y = -18.91 \text{ km/hour}$$

$$v_R = 18.94 \text{ km/hour}$$

Example 19.4 Car *A*, when at a distance of 32.5 m north of a crossing was observed to start from rest and travelling towards the crossing with a constant acceleration of 1 m/s^2. At the same instant another car *B* was observed to be just at the crossing and travelling east at a constant speed of 36 km/hour. Find the position, velocity and acceleration of car *A* relative to the car *B*, 5 seconds after the observation [Fig. 19.9 (*a*)].

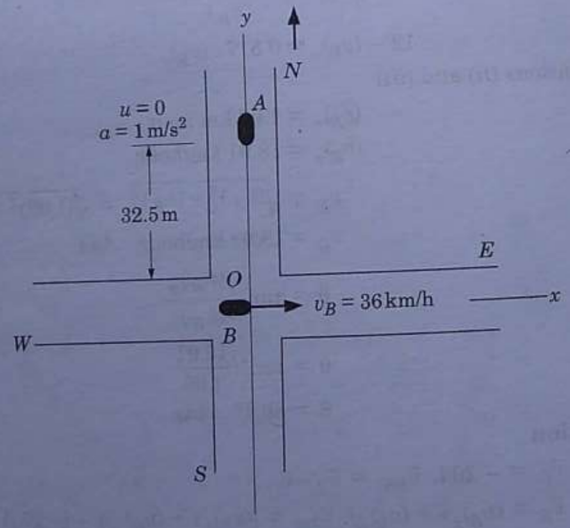

Fig. 19.9 (a)

Solution: The position of the cars *A* and *B* at the time $t = 0$ with respect to the fixed coordinate axes *xoy*, are as shown in Fig. 19.9 (*a*).

Absolution Motion

Car *A* $\quad u = 0,\ a = 1 \text{ m/s}^2,\ t = 5 \text{ s}$

Using $\quad v = u + at$

$$v = 0 + 1 \times 5 = 5 \text{ m/s}$$

$$s = ut + 1/2at^2$$

After 5 seconds,

$$s = 0 + 1/2 \times 1 \times (5)^2 = 12.5 \text{ m}$$

Velocity of car *A* $= 5$ m/s

Acceleration of car *A* $= 1$ m/s^2

Position of car *A* $= y_A = 32.5 - 12.5$

Or $\quad y_A = 20.0$ m from the origin along the *y*-axis.

Car *B*. $\quad$ Velocity $v_B = 36$ km/hour $= 10$ m/s

$$s = ut + 1/2at^2$$

$$s = 10 \times 5 = 50 \text{ m.}$$

After 5 seconds,

Velocity of the car *B*, $\quad v_B = 10$ m/s (constant)

Acceleration of the car *B*, $\quad a_B = 0$

Position of the car *B*, $\quad x_B = 50$ m from the origin along the *x*-axis

Relative Motion of the car A with respect to the car B

Position : Fig. 19.9 (*b*)

Relative position of the car *A* w.r.t. car *B*

$$\bar{r}_{A/B} = \bar{y}_A - \bar{x}_B, \text{ vector } \overline{BA} \text{ represents } \bar{r}_{A/B}$$

$$BA = \sqrt{OA^2 + OB^2} = \sqrt{20^2 + 50^2} = 53.9 \text{ m}$$

$$\theta = \tan^{-1}\frac{20}{50},\ \theta = 21.8° \quad \textit{Ans.}$$

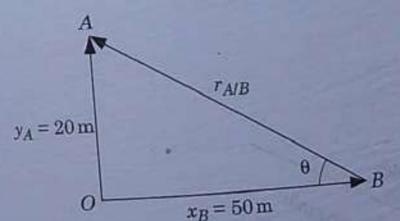

Fig. 19.9 (b)

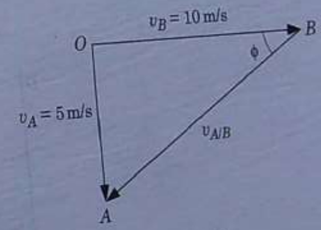

Fig. 19.9 (c)

Velocity : Fig. 19.9 (c)

Relative velocity of the car *A* w.r.t. the car *B*

$$\bar{v}_{A/B} = \bar{v}_A - \bar{v}_B, \text{ vector } \overline{BA} \text{ represents } \bar{v}_{A/B}$$

$$BA = \sqrt{OA^2 + OB^2} = \sqrt{5^2 + 10^2} = 11.2 \text{ m/s}$$

$$v_{A/B} = 11.2 \text{ m/s}$$

$$\phi = \tan^{-1}\frac{5}{10},\ \phi = 26.6° \quad \textit{Ans.}$$

Acceleration: Relative acceleration of car A with respect to the car B,

$$\bar{a}_{A/B} = \bar{a}_A - \bar{a}_B$$
$$\bar{a}_{A/B} = 1 - 0 = 1 \text{ m/s}^2$$
$$a_{A/B} = 1 \text{ m/s}^2 \quad Ans.$$

PROBLEMS

19.1. A railway coach having an ordinary cross seat arrangement is travelling at 14.4 km/hour. A person runs at 5 m/s on the platform. In what direction he must run so that he may enter parallel to the seats? Also find the velocity with which he entered. [36.87° to the train, 3 m/s]

19.2. A cyclist travelling east with a speed of 2 m/s feels the wind to be blowing directly from north. On doubling his speed he feels the wind to blow from north-east. Find the direction and the speed of the wind. [N-W, 2.83 m/s]

19.3. Two ships A and B leave a port at the same time. The ship A is travelling north-west at 32 km/hour and ship B, 40° south of west, at 24 km/hour. Determine (*a*) the speed of the ship B relative to ship A (*b*) at what time they will be 150 km apart? [38.3 km/hour, 3.92 hours]

19.4. Two cars P and Q pass through the intersection at the same instant and travel with velocities of 36 km/hour and 72 km/hour along the roads AB and DC respectively (Fig. P.19.4). Find the velocity of the car Q with respect to the car P and the distance between the two cars after 2 seconds. [26.46 m/s, 52.91 m]

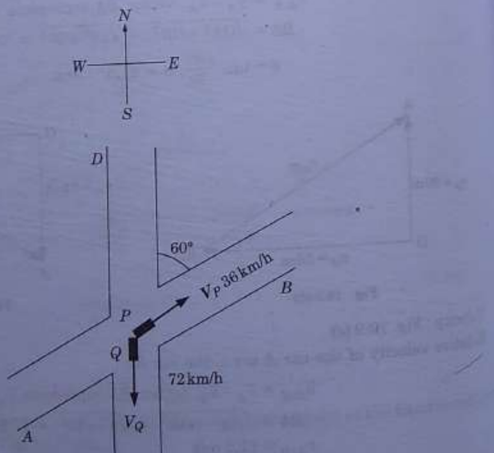

Fig. P.19.4

19.5. A ship sailing east with a speed of 15 km/hour passes a certain point at noon and the second ship sailing north with the same speed passes the same point at 1.30 p.m. At what time they will be closest together and what is the distance between them? [12.45 P.M., 15.90 km]

19.6. Rain drops are falling vertically at a speed of 15 m/s. Wind is blowing horizontally from east to west at a speed of 20 m/s. A man is running towards east with a speed of 8 m/s. Determine the inclination of the umbrella with the horizontal. [28.18° with horizontal]

19.7. Three blocks are connected by pulleys as shown in Fig. P.19.7 and move with constant velocity. Find the velocity of each block given that relative velocity of block A with respect to block C is 30 m/s upwards and that the relative velocity of block B with respect to block A is 20 cm/s downwards.

[$v_a = 12.5$ cm/s ↑
$v_b = 7.5$ cm/s ↓
$v_c = 17.5$ cm/s ↓]

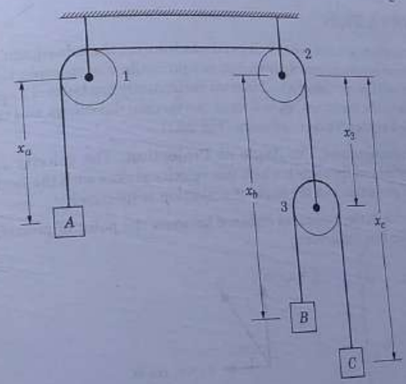

Fig. P.19.7

[**Hint :** $v_{A/C} = v_A - v_C$ = Given ...(i)
$v_{B/A} = v_B - v_A$ = Given ...(ii)
$(x_a + x_3)$ = Constant ...(iii)
$(x_b - x_3) + (x_c - x_3)$ = Constant ...(iv)
[↑ Velocity of block A = Velocity of pulley 3 ↓]

CHAPTER 20

Motion of Projectile

20.1 INTRODUCTION

Any object that is given some initial velocity and which during the subsequent motion is subjected to only the acceleration due to gravity is termed as a projectile. For example, the motion of a bullet fired from a gun and the motion of a ball thrown horizontally are termed as projectile motion. A projectile travels in the horizontal as well as in the vertical directions and traces a curvilinear path. Some related terms are explained below (Fig. 20.1).

Velocity of Projection and The Angle of Projection. The velocity v_0 with which the projectile is projected and the angle α which this velocity makes with the horizontal are known an the velocity of projection and the angle of projection respectively.

Range of a projectile is the horizontal distance between the point of projection and the point where the projectile hits the ground.

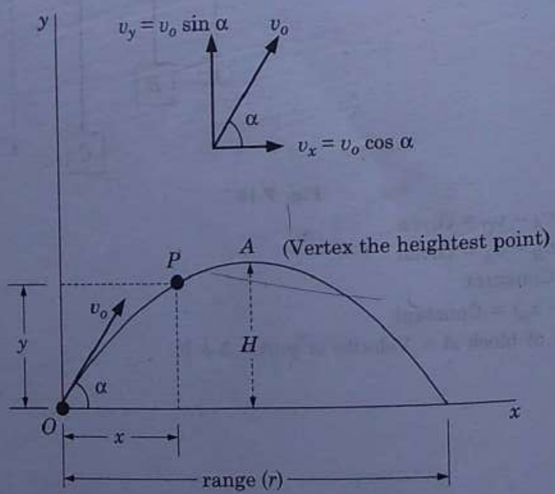

Fig. 20.1

Trajectory is the path followed by the projectile. It would be shown later that a projectile follows a parabolic path.

20.2 EQUATION OF THE PATH: TRAJECTORY

Consider the motion of an object projected from the origin O of the co-ordinate system, with the initial velocity v_0 inclined at an angle α with the horizontal or the x-axis as shown in Fig. 20.1.

The motion of the object P can be studied by resolving its velocity v_0 in the horizontal and the vertical directions as shown.

Horizontal component of the velocity v_0

$$v_x = v_0 \cos\alpha$$

Vertical component of the velocity v_0

$$v_y = v_0 \sin\alpha$$

The horizontal velocity v_x of the object shall remain constant as no acceleration is acting in the horizontal direction ($a = 0$).

The initial velocity in the vertical direction v_y shall go on decreasing because of the constant deceleration due to gravity ($a = -g$).

The object, therefore, is having horizontal and vertical motion simultaneously. The resultant motion would be the vector sum of these two motions and the path followed would be curvilinear.

Let P be the position of the object after a time t. Using the relation $s = ut + \frac{1}{2}at^2$, we can determine,

The distance travelled in the horizontal direction in time t, $$x = (v_0 \cos\alpha)t \quad ...(20.1)$$

The distance travelled in the vertical direction in time t, $$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \quad ...(20.2)$$

The equations (20.1) and (20.2) are the time displacement relations for the projectile. Eliminating the time t, we can obtain a relationship between x and y or the equation of the path of the projectile.

From (20.1)

$$t = \frac{x}{v_0 \cos\alpha}$$

Substituting in (20.2)

$$y = (v_0 \sin\alpha)\left(\frac{x}{v_0 \cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{v_0 \cos\alpha}\right)^2$$

$$y = (\tan\alpha)x - \left(\frac{g}{2v_0^2 \cos^2\alpha}\right)x^2 \quad ...(20.3)$$

The above equation is of the form $y = Ax + Bx^2$ and represents a parabola. Thus the path of a projectile is a parabola.

20.3 EXPRESSIONS FOR TIME OF FLIGHT, HEIGHT, RANGE AND ANGLE OF PROJECTION

Time of Flight. When the projectile P is at heighest point A of the path (vertex), the vertical component of the velocity becomes zero. Using the relation,

$$v = u + at$$

$$0 = v_0 \sin \alpha - gt$$

$$t = \frac{v_0 \sin \alpha}{g}$$

This t is, half the time the projectile has been in the air (from O to A). The total time of flight, is twice this value.

$$\text{Time of flight} = \frac{2v_0 \sin \alpha}{g} \qquad (20.4)$$

The maximum height attained by the projectile can be obtained using the relation,

$$v^2 - u^2 = 2as,$$

$$0 - (v_0 \sin \alpha)^2 = -2gH$$

Maximum height attained H

$$= \frac{v_0^2 \sin^2 \alpha}{2g} \qquad ...(20.5)$$

Range of the Projectile. The range r is the horizontal distance travelled during the time of flight

$$\text{Range} = r = \text{(horizontal velocity) (time of flight)}$$

$$r = (v_0 \cos \alpha)\left(\frac{2v_0 \sin \alpha}{g}\right)$$

$$r = \frac{v_0^2 \sin 2\alpha}{g} \qquad ...(20.6)$$

The Angle of Projection for the Maximum Range of the Projectile

$$\text{Range, } r = \frac{v_0^2 \sin 2\alpha}{g}$$

The range r for a given velocity v_0 is maximum if

$$\sin 2\alpha = 1$$

Or

$$2\alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4} = 45°$$

Corresponding to this value of α, the maximum range is,

$$r_{\text{maximum}} = \frac{v_0^2}{g} \qquad ...(20.7)$$

The Angle of Projection for a Given Range

As, $\sin \theta = \sin(\pi - \theta)$

Therefore,

$$r = \frac{v_0^2 \sin 2\alpha}{g}$$

$$= \frac{v_0^2 \sin(\pi - 2\alpha)}{g}$$

So for a given value of the initial velocity v_0, the same range is obtained for two angles of projection; α and $\frac{(\pi - 2\alpha)}{2}$ or $\left(\frac{\pi}{2} - \alpha\right)$. *These two angles of projection are equally inclined to the direction corresponding to the maximum range (that is, π/4) and are as shown in Fig. 20.2.*

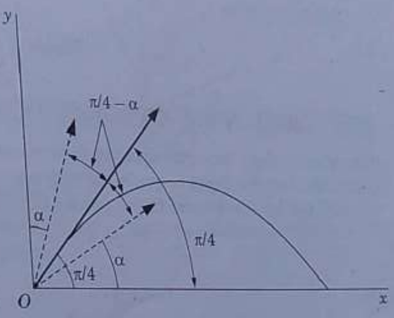

Fig. 20.2

It should, however, be noted that the time of flights corresponding to these two angles of projection α and $\left(\frac{\pi}{2} - \alpha\right)$ would not be the same.

20.4 MOTION OF A PROJECTILE THROWN HORIZONTALLY

Consider an object thrown horizontally with a velocity v_0 from the point A which is at a height h from the horizontal plane Ox. Let it hit the horizontal plane at B after a time t, as shown in Fig. 20.3.

Consider the motion of the projectile along Ox and Oy.

Initial velocity along $Ox = v_0$

Initial velocity along $Oy = 0$

Acceleration along $Ox = 0$

Acceleration along $Oy = g$

Motion along O-y (Vertical Direction). In time t the projectile shall travel a vertical distance h. Using the relation,

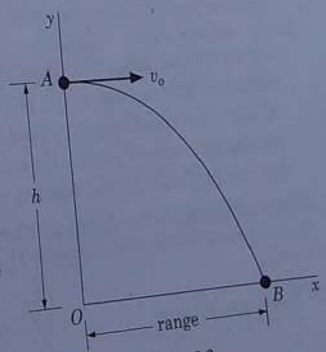

Fig. 20.3

$$s = ut + \frac{1}{2}at^2$$

$$h = 0 + \frac{1}{2}gt^2 \qquad ...(20.8)$$

Time of flight,

$$t = \sqrt{\frac{2h}{g}}$$

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Motion along *O-x* (Horizontal Direction). The range OB or the distance travelled by the projectile along Ox can be obtained using.

$$s = ut + \frac{1}{2}at^2$$

$$\text{Range} = OB = ut + 0$$

Substituting for t,

$$\text{Range} = u\sqrt{\frac{2h}{g}} \qquad \ldots(20.9)$$

20.5 MOTION OF A PROJECTILE UP AN INCLINED PLANE

Consider an object projected up an inclined plane OA which makes an angle β with the horizontal plane OA. Let the object be projected with a velocity v_0 making an angle α with the horizontal Ox. Let OB be the normal to the plane OA (Fig. 20.4).

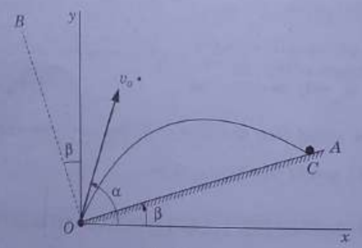


Fig. 20.4

The motion of the projectile can be considered as the vector sum of its motions along and normal to the inclined plane.

Initial velocity along the plane (along OA) = $v_0 \cos(\alpha - \beta)$

Initial velocity normal to the plane (along OB) = $v_0 \sin(\alpha - \beta)$

Acceleration due to gravity along the plane (along OA) = $-g \sin \beta$.

Acceleration due to gravity normal to the plane (along OB) = $-g \cos \beta$

Let the projectile hit the inclined plane OA at C after a time t.

Motion Normal to the Inclined Plane (Along OB). Let the projectile hit the inclined plane after a time t. The distance travelled by the projectile normal to the inclined plane in this time is zero.

Using the relation,
$$s = ut + \frac{1}{2}at^2$$

$$0 = [v_0 \sin(\alpha - \beta)]t - \frac{1}{2}[g \cos \beta]t^2$$

Time of Flight,
$$t = \frac{2v_0 \sin(\alpha - \beta)}{g \cos\beta} \qquad \ldots(20.10)$$

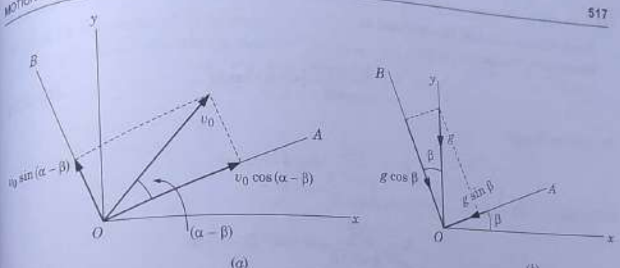


Fig. 20.5

Motion Along the Inclined Plane (Along OA). Range of the projectile is the distance OC travelled by the projectile along the inclined plane.

Using the relation,
$$s = ut + \frac{1}{2}at^2$$

Range along the inclined plane

$$= (v_0 \cos(\alpha - \beta))t - \left(\frac{1}{2}g\sin\beta\right)t^2 \qquad \ldots(20.11)$$

Substituting for t from the equation (20.10)

Range along the plane

$$= v_0 \cos(\alpha - \beta)\left(\frac{2v_0 \sin(\alpha - \beta)}{g\cos\beta}\right) - \frac{1}{2}g\sin\beta\left(\frac{2v_0 \sin(\alpha - \beta)}{g\cos\beta}\right)^2$$

$$= \frac{2v_0^2 \sin(\alpha - \beta)}{g\cos\beta}\left(\cos(\alpha - \beta) - \frac{\sin(\alpha - \beta)\sin\beta}{\cos\beta}\right)$$

$$= \frac{2v_0^2}{g\cos\beta}\sin(\alpha - \beta)\left(\frac{\cos(\alpha - \beta)\cos\beta - \sin(\alpha - \beta)\sin\beta}{\cos\beta}\right)$$

$$= \frac{2v_0^2}{g\cos\beta}\sin(\alpha - \beta)\frac{\cos(\alpha - \beta + \beta)}{\cos\beta}$$

$$= \frac{2v_0^2}{g\cos\beta}\sin(\alpha - \beta)\frac{\cos\alpha}{\cos\beta}$$

Range along the inclined plane $\qquad \ldots[20.12(a)]$

$$= \frac{2v_0^2}{g\cos^2\beta}\sin(\alpha - \beta)\cos\alpha$$

Above relation can also be expressed as,
Range along the inclined plane

$$= \frac{2v_0^2 \cos^2 \alpha}{g \cos\beta}(\tan\alpha - \tan\beta) \quad \text{...[20.12 (b)]}$$

Maximum Range

$$R = \frac{2v_0^2}{g\cos^2\beta}\sin(\alpha-\beta)\cos\alpha$$

Using, $\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

$$R = \frac{v_0^2}{g\cos^2\beta}(\sin(2\alpha-\beta)-\sin\beta)$$

As the angle β of the inclined plane is constant, the maximum value of the range, for a given velocity v_0, is obtained when $\sin(2\alpha - \beta)$ is maximum.

Or
$$\sin(2\alpha-\beta) = 1 = \sin\frac{\pi}{2}$$

$$2\alpha - \beta = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{2}+\frac{\beta}{2} \quad \text{...(20.13)}$$

For determining the value of the maximum range, the above value of α can be substituted in the expression for the range to get,

Maximum range along the inclined plane

$$= \frac{v_0^2}{g\cos^2\beta}(1-\sin\beta)$$

$$R_{max} = \frac{v_0^2}{g(1-\sin^2\beta)}(1-\sin\beta)$$

$$R_{max} = \frac{v_0^2}{g(1+\sin\beta)} \quad \text{...(20.14)}$$

Motion Down the Inclined Plane. The motion down the inclined plane can be studied by replacing β by $(-\beta)$ in the expressions obtained above.

Example 20.1 A projectile is fired with a velocity of 500 m/s at an elevation of 30°. (Fig. 20.6). Find the velocity and the direction of the projectile after 30 second of its firing.

Solution: $v_0 = 500$ m/s, $\alpha = 30°$, $t = 30$ s

Initial velocity in the horizontal direction $v_x = 500 \cos 30° = 433.0$ m/s

Initial velocity in the vertical direction $v_y = 500 \sin 30° = 250.0$ m/s

Motion in the horizontal direction

Horizontal velocity remains constant = 433.0 m/s

So, horizontal velocity after 30 s $v_x' = 433.0$ m/s

Motion in the vertical direction

Let the vertical velocity after a time $t = 30$ s be v_y'. Using the relation $v = u + at$.

$$v_y' = v_y + gt = 250 - 9.81 \times 30$$
$$v_y' = -44.3 \text{ m/s, downwards}$$

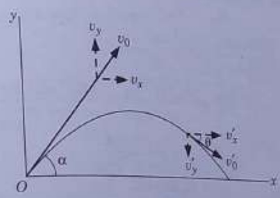


Fig. 20.6

Resultant velocity

$$v_0' = \sqrt{v_x'^2 + v_y'^2}$$

$$v_0' = \sqrt{(433)^2+(44.3)^2}$$

$$v_0' = 225.0 \text{ m/s} \quad \textit{Ans.}$$

$$\tan\theta = \frac{v_y'}{v_x'} = -\frac{44.3}{433.0}$$

$$\theta = 5.84° \quad \textit{Ans.}$$

Example 20.2 Body A is thrown with a velocity of 10 m/s at an angle of 60° to the horizontal. If another body B is thrown at an angle of 45° to the horizontal, find its velocity if it has the same (*a*) horizontal range (*b*) maximum height (*c*) time of flight, as the body A.

Solution: Body A

Initial velocity $v_a = 10$ m/s

Angle of projection $\alpha_A = 60°$

Body B

Initial velocity be $= v_b$

Angle of projection $\alpha_B = 45°$

Velocity v_b for the same range

$$r_a = r_b$$

$$\frac{v_a^2 \sin 2\alpha_A}{g} = \frac{v_b^2 \sin 2\alpha_B}{g}$$

$$v_b^2 = \frac{v_a^2 \sin 2\alpha_A}{\sin 2\alpha_B} = \frac{(10)^2 \times \sin(2\times 60)^\circ}{\sin(2\times 45)^\circ}$$

$$v_b^{\,2} = \frac{100 \times 0.866}{1.0}$$

$$v_b = 9.3 \text{ m/s} \quad \textit{Ans.}$$

Velocity v_b for the same maximum height

$$H_A = H_B$$

$$\frac{v_a^2 \sin^2 \alpha_A}{2g} = \frac{v_b^2 \sin^2 \alpha_B}{2g}$$

$$v_b^{\,2} = \frac{v_a^2 \sin^2 \alpha_A}{\sin^2 \alpha_B} = \frac{(10)^2 \sin^2 60°}{\sin^2 45°}$$

$$v_b^{\,2} = 100 \times \left(\frac{0.866}{0.707}\right)^2$$

$$v_b = 12.25 \text{ m/s.} \quad \textit{Ans.}$$

Velocity v_b for the equal time of flight

$$t_A = t_B$$

$$\frac{2v_a \sin \alpha_A}{g} = \frac{2v_b \sin \alpha_B}{g}$$

$$v_b = \frac{v_a \sin \alpha_A}{\sin \alpha_B} = \frac{10 \times \sin 60°}{\sin 45°}$$

$$v_b = \frac{10 \times 0.866}{0.707}$$

$$v = 12.25 \text{ m/s} \quad \textit{Ans.}$$

Example 20.3 A boy throws a ball so that it may just clear a wall 3.6 m high. The boy is at distance of 4.8 m from the wall. The ball was found to hit the ground at a distance of 3.6 m on the other side of the wall (Fig. 20.7). Find the least velocity with which the ball can be thrown.

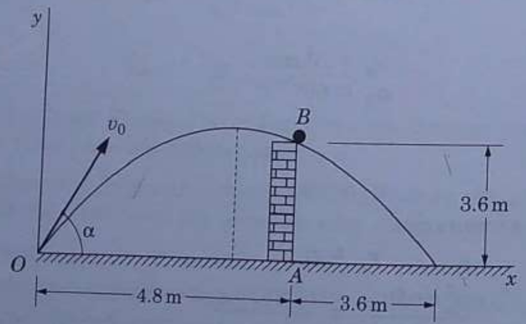


Fig. 20.7

Solution: Let,

The velocity of projection be $= v_0$

The angle of projection be $= \alpha$

Range = 4.8 + 3.6 = 8.4 m

The top of the wall AB must lie on the path of the projectile.

The equation of the path with the point of projection as origin is,

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \qquad ...(i)$$

Point B (4.8, 3.6) lies on this path or curve.

Substituting in (*i*)

$$3.6 = 4.8 \tan \alpha - \frac{g \times (4.8)^2}{2v_0^2 \cos^2 \alpha} \qquad ...(ii)$$

The range of the projectile,

$$r = \frac{v_0^2 \sin 2\alpha}{g} = 8.4 \qquad ...(iii)$$

Equation (*ii*) and (*iii*) involve two unknowns α and v.

From (*iii*)

$$\frac{v_0^2}{g} = \frac{8.4}{\sin 2\alpha}$$

Substituting for $\frac{v_0^2}{g}$ in (*ii*)

$$3.6 = 4.8 \tan \alpha - \frac{(4.8)^2}{2\cos^2 \alpha} \frac{\sin 2\alpha}{8.4}$$

$$3.6 = 4.8 \tan \alpha - \frac{(4.8)^2}{2 \times 8.4} \frac{2 \sin \alpha \cos \alpha}{\cos^2 \alpha}$$

$$3.6 = 4.8 \tan \alpha - 2.74 \tan \alpha$$

$$3.6 = 2.057 \tan \alpha$$

$$\tan \alpha = \frac{3.6}{2.057} = 1.75$$

$$\alpha = 60.2° \quad \textit{Ans.}$$

Substituting in (*iii*) this value of α

$$v_0^{\,2} = g \frac{8.4}{\sin 2\alpha} = \frac{9.81 \times 8.4}{\sin 120.4°} = 95.54$$

$$v_0 = 9.77 \text{ m/s} \quad \textit{Ans.}$$

Example 20.4 A projectile is aimed at a target which lies in the horizontal plane through the point of projection. It falls '*a*' metres short of the target when the angle of projection is α and goes '*b*' metres too far off when the angle of projection is β (Fig. 20.8). Show that the proper angle of projection θ is

$$\theta = \frac{1}{2} \sin^{-1}\left(\frac{a \sin 2\beta + b \sin 2\alpha}{a + b}\right)$$

if the velocity of projection is the same in all the three cases.

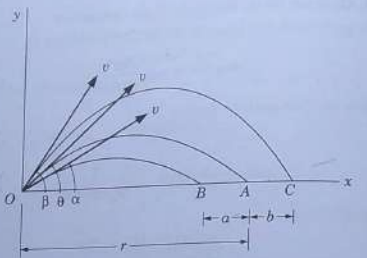


Fig. 20.8

Solution: Let v be the velocity of projection in each case and θ be the correct angle of projection for the desired range r. Then,

$$r - a = \frac{v^2 \sin 2\alpha}{g} \qquad ...(i)$$

$$r + b = \frac{v^2 \sin 2\beta}{g} \qquad ...(ii)$$

$$r = \frac{v^2 \sin 2\theta}{g} \qquad ...(iii)$$

We obtained three equations with three unknowns r, v and θ and we have to determine the value of θ.

To eliminate r subtract (*i*) from (*ii*)

$$a + b = \frac{v^2(\sin 2\beta - \sin 2\alpha)}{g} \qquad ...(iv)$$

Subtracting (*iii*) from (*ii*)

$$b = \frac{v^2(\sin 2\beta - \sin 2\theta)}{g} \qquad ...(v)$$

eliminating v from (*iv*) and (*v*)

$$\frac{v^2}{g} = \frac{a+b}{(\sin 2\beta - \sin 2\alpha)}$$

$$\frac{v^2}{g} = \frac{b}{(\sin 2\beta - \sin 2\theta)}$$

Equating,

$$\frac{a+b}{(\sin 2\beta - \sin 2\alpha)} = \frac{b}{(\sin 2\beta - \sin 2\theta)}$$

$$(a + b) \sin 2\beta - (a + b) \sin 2\theta = b \sin 2\beta - b \sin 2\alpha$$

$$(a + b) \sin 2\theta = (a + b) \sin 2\beta - b \sin 2\beta + b \sin 2\alpha$$

$$(a + b) \sin 2\theta = (a \sin 2\beta + b \sin 2\alpha)$$

$$\sin 2\theta = \frac{a \sin 2\beta + b \sin 2\alpha}{(a+b)}$$

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{a \sin 2\beta + b \sin 2\alpha}{a+b}\right) \quad \textit{Ans.}$$

Example 20.5 Two adjacent guns having the same muzzle velocity of 400 m/s, fire simultaneously at angles of θ_1 and θ_2 for the same target situated at a range of 5000 m. Find the time difference between the two hits (Fig. 20.9).

Solution:

Initial velocity of each gun = 400 m/s

Range of each gun = 5000 m

$$\text{Range } r = \frac{v_0^2}{g}\sin 2\alpha$$

Ist gun

$$\alpha = \theta_1$$

$$r = 5000 = \frac{(400)^2}{9.81}\sin 2\theta_1 \qquad ...(i)$$

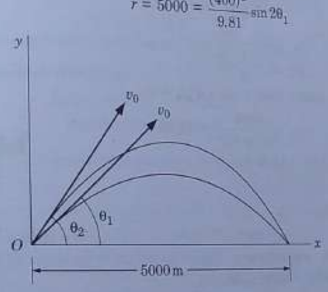


Fig. 20.9

2nd gun

$$\alpha = \theta_2$$

$$r = 5000 = \frac{(400)^2}{9.81}\sin 2\theta_2 \qquad ...(ii)$$

From (*i*) and (*ii*)

$$\sin 2\theta_1 = \sin 2\theta_2$$

Therefore, either

$$2\theta_2 = 2\theta_1 \qquad ...(iii)$$

Or

$$2\theta_2 = (\pi - 2\theta_1)$$

From (*i*)

$$\sin 2\theta_1 = \frac{5000 \times 9.81}{(400)^2} = 0.3065$$

or

$$2\theta_1 = 17.8°$$
$$\theta_1 = 8.9°$$

From (*iii*)

$$2\theta_2 = \pi - 2\theta_1$$
$$= 180° - 17.8°$$
$$\theta_2 = 81.1°$$

To determine the time of flight, consider the horizontal motion of the projectiles.

1st gun

$$\text{time} = t_1$$
$$\text{range} = r = 5000 \text{ m}$$
$$\alpha = \theta_1 = 8.9°$$
$$v_x = v_0 \cos \theta_1$$
$$= 400 \cos 8.9°$$

Using the relation

$$s = ut + \frac{1}{2}at^2$$
$$5000 = (400 \times \cos 8.9°)t_1 + 0$$
$$t_1 = \frac{5000}{400 \cos 8.9°} = 12.65$$
$$t_1 = 12.65 \text{ s}$$

2nd gun

$$\text{time} = t_2$$
$$v_x = 400 \cos 81.1°$$
$$r = 5000 \text{ m}$$
$$5000 = (400 \cos 81.1°)t_2$$
$$t_2 = \frac{5000}{400 \cos 81.1°} = 80.79 \text{ s}$$
$$t_2 = 80.79 \text{ s}$$

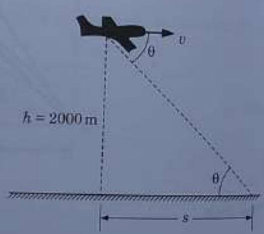


Fig. 20.10

Time difference between the two hits

$$= 80.79 - 12.65$$
$$= 68.14 \text{ s} \quad \textit{Ans.}$$

Example 20.6 The pilot of an aeroplane flying horizontally at a height of 2000 m with a constant speed of 540 km/hour wishes to hit, a target on the ground. At what distance from the target should he release the bomb in order to hit the target? At what angle would the target appear to him from that distance?

Solution: Initial velocity of the bomb is the same as that of the aeroplane that is, 540 km/hour.

Initial velocity of the bomb in the horizontal direction = 540 km/hour or 150 m/s

Initial velocity of the bomb in the vertical direction = 0.0

Motion in the vertical direction

Initial velocity = 0

distance $h = 2000$ m

acceleration $g = 9.81$ m/s^2

Using

$$s = ut + \frac{1}{2}gt^2$$
$$2000 = 0 + \frac{1}{2} \times 9.81t^2$$
$$t = \sqrt{\frac{4000}{9.81}}$$
$$t = 20.2 \text{ s}$$

where, t is the time required to travel down a height 2000 m.

Motion in the horizontal direction

$$\text{Initial velocity} = 150 \text{ m/s}$$
$$\text{Acceleration} = 0.0$$

Using, $s = ut + \frac{1}{2}at^2$, the horizontal distance travelled by the bomb in the time $t = 20.2$ s is

$$s = 150 \times 20.2$$
$$s = 3030 \text{ m} \quad \textit{Ans.}$$

So, the bomb should be released when the aeroplane is at a distance of 3030 m from the target and the angle is

$$\tan \theta = \frac{2000}{3030} = 0.660$$
$$\text{Angle } \theta = 33.4° \quad \textit{Ans.}$$

Example 20.7 A soldier positioned on a hill fires a bullet at an angle of 30° upwards from the horizontal (Fig. 20.11). The target lies 50 m below him and the bullet is fired with a velocity of 100 m/s. Determine,

(*a*) the maximum height to which the bullet will rise above the position of the solider

(*b*) the velocity with which the bullet will hit the target

(*c*) the time required to hit the target.

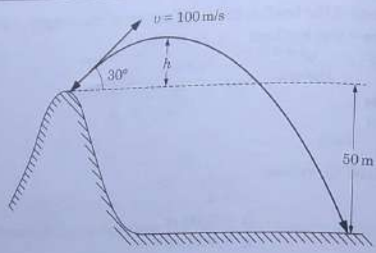


Fig. 20.11

Solution:

Initial velocity in the horizontal direction $v_x = 100 \cos 30° = 86.6$ m/s

Initial velocity in the vertical direction $v_y = 100 \sin 30° = 50$ m/s

(*a*) To find the maximum height attained by the bullet consider the vertical motion of the bullet. Using,

$$v^2 - u^2 = 2as$$
$$u = v_y = 50 \text{ m/s},\ v = 0\ a = -9.81 \text{ m/s}^2$$
$$-(50)^2 = 2(-9.81)h$$
$$h = \frac{(50)^2}{2 \times 9.81}$$
$$h = 127.4 \text{ m} \quad \textit{Ans.}$$

(*b*) The horizontal component of the velocity v of the bullet remains constant.

The horizontal velocity of the bullet when it hits the target $v_x' = 86.6$ m/s

The vertical velocity of the bullet when it hits the target can be found using $v^2 - u^2 = 2as$

$$u = v_y = 50 \text{ m/s},\ v = v_y',$$
$$a = -9.81 \text{ m/s}^2,\ s = -50 \text{ m}$$
$$(v_y')^2 - (50)^2 = -2 \times 9.81 \times 50$$
$$v_y' = \sqrt{50^2 \times 981}$$
$$v_y' = \pm 59 \text{ m/s}$$

The velocity v' with which the bullet hits the target

$$v' = \sqrt{(v_x')^2 + (v_y')^2} = \sqrt{(86.6)^2 + (59)^2}$$
$$v' = 104.8 \text{ m/s} \quad \textit{Ans.}$$

(*c*) The initial upward velocity of the bullet $v_y = 50$ m/s changes to $v_y' = -59$ m/s, when it hits the target (under the acceleration due to gravity in time t). Using,

$$v = u + at$$

Time required to hit the target

$$t = \frac{v - u}{a} = \frac{50 + 59}{9.81}$$
$$t = 11.1 \text{ s} \quad \textit{Ans.}$$

Example 20.8 A hammer weighing 30 N starts sliding from rest from point *A* down a sloping roof which has a coefficient of friction $\mu = 0.18$. find the distance x of the point *G* where the hammer hits the round as shown in Fig. 20.12.

Solution: The hammer after leaving the roof at the point *B* shall have the motion of a projectile. So, let us find the velocity of the hammer at the point *B*.

Motion of the hammer along AB

Equation of motion of the hammer can be written as

$\Sigma F_x = ma_x$: (Along AB) $\quad \frac{W}{g} a = W \sin\theta - \mu R$

$\Sigma F_y = 0$: (Normal to AB) $\quad R - W \cos\theta = 0$

Therefore,

$$a = g(\sin\theta - \mu\cos\theta)$$
$$a = 9.81\,(\sin 30° - 0.18 \times \cos 30°)$$
$$a = 9.81(0.5 - 0.1559)$$
$$a = 3.375 \text{ m/s}^2$$

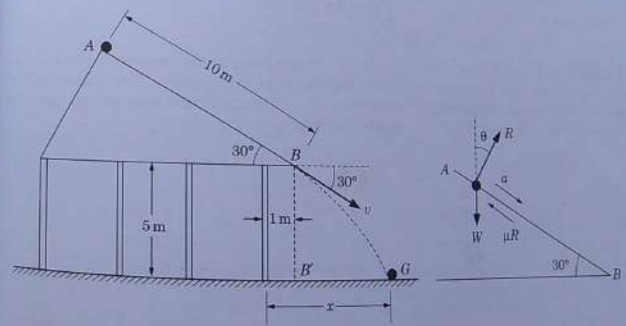


Fig. 20.12

The velocity of the hammer at the point *B*, after travelling a distance of 10 m along the roof with an acceleration = 3.375 m/s² and starting from rest, is given by,

$$v^2 - u^2 = 2as$$
$$v^2 = 2 \times 3.375 \times 10$$
$$v = 8.21 \text{ m/s}$$

The hammer leaves the roof at the point B parallel to the roof with a velocity of 8.21 m/s.

Initial velocity of the hammer in the horizontal direction $= v \cos 30° = 8.21 \times 0.866 = 7.11$ m/s

Initial velocity of the hammer in the vertical direction $= v \sin 30° = 8.21 = \times 0.5 = 4.105$ m/s

Motion in the vertical direction

$$s = 5 \text{ m},\ u = 4.105 \text{ m/s},\ a = 9.81 \text{ m/s}^2$$

Using $s = ut + \frac{1}{2}at^2$

time required to travel a vertical distance $BB' = 5$ m is,

$$5 = 4.105t + \frac{1}{2} \times 9.81t^2$$

$$4.900t^2 + 4.100t - 5 = 0$$

$$t = \frac{-4.10 \pm \sqrt{(4.10)^2 + 4 \times 4.9 \times 5}}{2 \times 4.90}$$

$$t = 0.61 \text{ s}$$

Motion in the horizontal direction

Horizontal distance $B'G$ travelled by the hammer in time $t = 0.61$ s

$$= v \cos 30° \times t = 7.11 \times 0.61$$

$$= 4.34 \text{ m}$$

Therefore, $x = 1 + 4.34$ m

$x = 5.34$ m *Ans.*

Example 20.9 A shell fired from a gun bursts into two parts at the highest point of its path which is at a horizontal distance of 1000 m from the point of firing and 19.6 m above it. Two seconds after the bursting of the shell, one of the parts hits the ground at a point vertically below the point where the shell burst (Fig. 20.13). Determine,

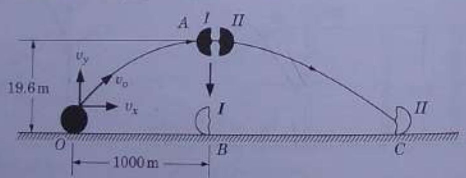


Fig. 20.13

(a) the velocity of the shell just before bursting.

(b) the velocity of the first part immediately after the shell burst.

(c) the velocity of the second part immediately after the shell burst.

(d) the distance between the firing point and the point where the second part of the shell hit the ground.

Solution: The shell is fired from O and travels to the point A as a single body. It bursts at A into two parts. Part I falls from point A to point B, vertically below A. Part II travels to the point C and hits the ground (Fig. 20.13).

Motion of the entire shell from O to A. Let v_0 be the initial velocity of the shell having components v_x and v_y in the horizontal and vertical directions.

The entire shell attains the maximum height at the point A. It means that its vertical velocity becomes zero at A. Using the relation

$$v^2 - u^2 = 2as$$

$$0 - (v_y)^2 = 2(-9.81)(19.6)$$

Initial velocity of the shell in vertical direction

$$v_y = 19.6 \text{ m/s}$$

Time taken by the entire shell to reach the point A

Using, $v = u + at$

$$0 = 19.6 + (-9.81)t$$

$$t = 2.0 \text{ s}$$

Velocity of the shell in horizontal direction remains constant.

Using $s = ut + \frac{1}{2}at^2$

$$1000 = v_x t$$

$$v_x = \frac{1000}{2}$$

Velocity of the shell in the horizontal direction

$$v_x = 500 \text{ m/s}$$

The velocity of the shell at A before bursting is, in horizontal and is

$= 500$ m/s *Ans.*

The velocity of the entire shell, at the highest point A, has no component in the vertical direction. So, the velocities of the two parts of the shell also, cannot have components in the vertical direction.

Velocities of the two parts of the shell after bursting at A

Part I : Part I of the shell has zero velocity both in horizontal and vertical directions after bursting. The velocity in the horizontal direction is zero because it travels from point A to the point B which is vertically below A. The velocity in the vertical direction is zero because the time of travel from O to A is same as its time of travel from A to B, each being 2s.

Part II : Part II has zero velocity in the vertical direction after bursting. Let the horizontal velocity be v_{x2} after bursting.

By conservation of momentum

Momentum of the entire shell = Momentum of part I + Momentum of part II.

As mass of the entire shell is m so the mass of each part is $m/2$.

$$m(500) = \frac{m}{2}(0) + \frac{m}{2}(v_{x2})$$

$$v_{x2} = 1000 \text{ m/s}$$

Velocity of the Part II after bursting

(In horizontal direction) $v_{x2} = 1000$ m/s *Ans.*

The time of travel of the part II from A to C will be the same as the time taken by the part I to travel from A to B as both parts have zero velocity in the vertical direction at the point A.

Distance BC travelled by part II,

$$BC = v_{x2} \times \text{time} = 1000 \times 2$$

$$BC = 2000 \text{ m}$$

Distance from the firing point OC,

$$OC = 1000 + 2000 \text{ m}$$

$$OC = 3000 \text{ m} \quad \textit{Ans.}$$

Example 20.10 A man standing on an incline can fire a short with an initial velocity of 200 m/s at an angle of 60° with the vertical as shown in Fig. 20.14, up as well as down the incline. Find the range BC covered by him if the incline has a rise of 5 in 12.

Solution: The angle of the inclined plane with respect to the horizontal

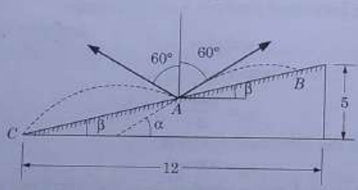


Fig. 20.14

$$\beta = \tan^{-1}\left(\frac{5}{12}\right)$$

$$\beta = 22.62°$$

The angle of projection with respect to the horizontal

$$\alpha = 90° - 60°, \ \alpha = 30°$$

Range AB when the short is fired up the plane

Range $$AB = \frac{2v_0^2}{g\cos^2\beta}\sin(\alpha - \beta)\cos\alpha$$

Substituting the values,

$$AB = \frac{2 \times (200)^2 \sin(30° - 22.62°)\cos 30°}{9.81 \cos^2 22.62^2}$$

$$AB = \frac{2 \times 40000 - 0.128 \times 0.866}{9.81 \times (0.923)^2}$$

$$AB = 1061 \text{ m}$$

Range AC when the short is fired down the plane

$$\alpha = 30°, \ \beta = -22.62°, \ v = 200 \text{ m/s}$$

Range $$AC = \frac{2v_0^2}{g\cos^2\beta}\sin(\alpha - \beta)\cos\alpha$$

Substituting the values,

$$AC = \frac{2 \times (200)^2 \sin(30° + 22.62°)\cos 30°}{9.81 \times \cos^2(-22.62)°}$$

$$AC = \frac{2 \times 40000 \times 0.795 \times 0.866}{9.81 \times (0.923)^2}$$

$$AC = 6590.3 \text{ m}$$

$$BC = 1061 + 6590.3$$

Range $$BC = 7651.3 \text{ m} \quad \textit{Ans.}$$

PROBLEMS

20.1. The horizontal range of a projectile is $4\sqrt{3}$ times its maximum height. Find the angle of projection. [30°]

20.2. By what percentage the maximum range of a projectile is increased if the initial velocity is increased by 10%? [21%]

20.3. A body is projected upwards with a velocity of 30 m/s at an angle of 30° with the horizontal. Determine (*a*) the time of flight (*b*) the range of the body (*c*) the maximum height attained by the body. [(*a*) 3.11 s, (*b*) 80.7 m, (*c*) 11.64 m]

20.4. A helicoper flying horizontally at 80 km/hour and at a height of 500 m intends to bomb a target on the ground. At what distance from the target should the bomb be released in order to hit the target? Also find the magnitude and direction of the velocity with which the bomb hits the target. [222.4 m, 101.4 m/s, 77.4°]

20.5. Find the maximum speed with which a scooter rider should leave a 15° ramp at B in order clear the ditch (Fig. P.20.5) [4.53 m/s]

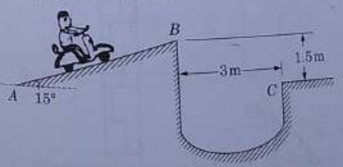


Fig. P.20.5

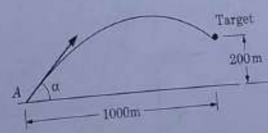


Fig. P.20.6

20.6. A bullet is fired at 125 m/s from a point A and has to hit a target at a horizontal distance of 1000 m from A and situated 200 m higher than A (Fig. P.20.6). Find,

(a) the angle α at which the bullet must be fired in order to strike the target in the minimum time,

(b) the maximum height above A reached by the bullet.

(c) the time of flight. [(a) 32.75° (b) 232.3 m (c) 9.5 s]

20.7. Two adjacent guns having the same muzzle velocity of 350 m/s fire simultaneously at angles of θ_1 and θ_2 for the same target situated at the range of 4200 m. Find the time difference between the hits. [58.2 s]

20.8. A jet of water discharging from a nozzle at A hits a vertical screen 6 m away from it. When the screen is moved 4 m further away, the jet hits the screen again at the same point (Fig. P.20.8). Find the angle α at which the jet is projected. Assume the path of the jet to be a parabola. [46.9°]

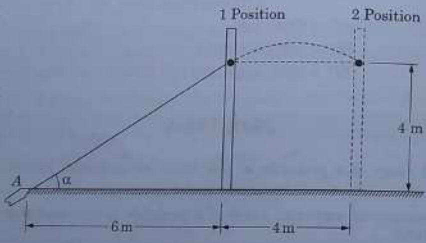


Fig. P.20.8

20.9. A player can throw a ball to a maximum distance of 100 m on a level ground. Find the distance to which he can throw the same ball down a hill from its top whose slope is 15°. What is the best angle at which he should throw the ball? [134.9 m 37.5° with the horizontal]

20.10. A ball of mass 1 kg starts from rest sliding down a frictionless chute from position 1 to position 2. Find the height at which it will hit the wall. [1.67 m]

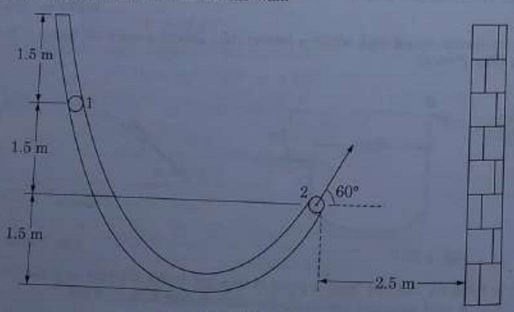


Fig. P.20.10

21 CHAPTER Kinematics of Rigid Body

21.1 INTRODUCTION

A rigid body can perform the various types of motion—from a simple translation to quite complex motions. Here we shall limit our study to the plane motion of a rigid body. *In the plane motion of a rigid body all the particles of the body move in parallel planes.* The motion of the body, therefore, can be represented by the motion in any one of such parallel planes or by a representative plane figure instead of a three dimensional object.

The various types of plane motion can be grouped as follows :

1. **Translation.** A rigid body is said to have a translatory motion if an imaginary straight line drawn on the body remains parallel to its original position, during the motion.

It can be observed that all the particles of the body move along the parallel paths in a translatory motion Fig. 21.1.

If these paths are straight lines, the motion is said to be *rectilinear translation*, if these paths are curved lines, the motion is said to be *curvilinear translation*.

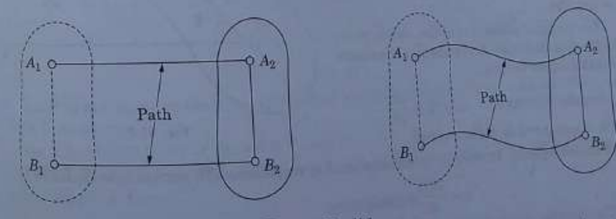


Fig. 21.1

2. **Rotation.** In this case all the particles of a rigid body move in concentric circular paths. The common centre of the circles may be located in the body or outside the body as shown in Fig. 21.2.

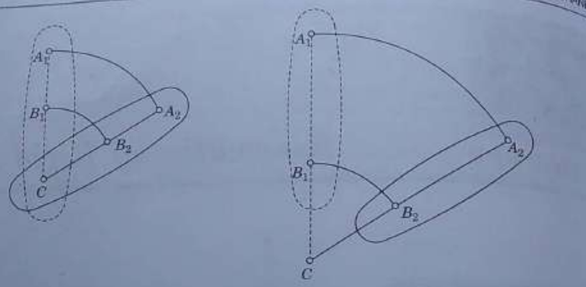


Fig. 21.2

3. General Plane Motion. Any plane motion which is neither a translation nor a rotation is known as a general plane motion.

21.2 ROTATION

Consider a rigid slab which rotates about a fixed axis perpendicular to the plane of the slab and intersecting it at the point O. Let P be a point on the slab Fig. 21.3.

The position of the slab is completely defined by the angle θ which the line OP forms with a fixed direction Ox.

The angle θ is known as the angular coordinate or the slab.

The angular velocity ω of the slab is given by the first derivative of the angular coordinate θ with respect to time.

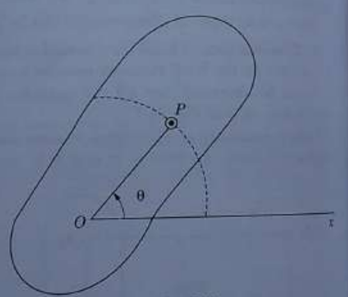


Fig. 21.3

Angular velocity $\quad \omega = \dot{\theta} = \dfrac{d\theta}{dt}$

It is measured in radian/second.

Angular velocity sometimes is also measured in revolutions (N) per minute (R.P.M.).

$$\omega = \frac{2\pi N}{60} \text{ radian/second}$$

The angular acceleration α of the slab is given by the second derivative of the angular coordinate θ with respect to time.

Angular acceleration $\quad \alpha = \ddot{\theta} = \dfrac{d^2\theta}{dt^2}$

It is measured in radian/(second)^2 or rad/s^2.

Uniform Rotation. In this case the angular acceleration is zero and the slab rotates with constant angular velocity.

Uniformly Accelerated Rotation. In this case the angular acceleration α is constant.

The following formulae relating the angular acceleration, angular velocity, angular coordinate and time, can be derived in a manner similar to that used in the case of uniformly accelerated rectilinear motion.

Rectilinear Motion	**Angular Motion**	
$v = u + at$	$\omega = \omega_0 + \alpha t$	
$s = ut + \frac{1}{2}at^2$	$\theta = w_0 t + \frac{1}{2}\alpha t^2$	...(21.1)
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$	

21.3 LINEAR AND ANGULAR VELOCITY, LINEAR AND ANGULAR ACCELERATION IN ROTATION

Consider a rigid slab rotating about a fixed axis perpendicular to the plane of the slab and intersecting it at the point O (Fig. 21.4).

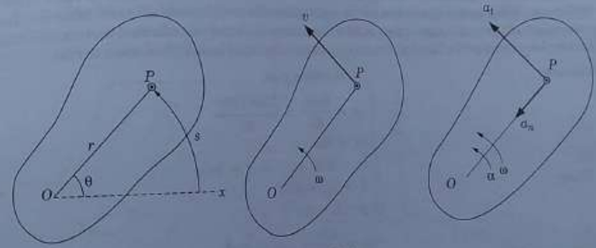


Fig. 21.4

Let P be any point on the rotating slab. As the slab rotates, the point P shall describe a circle with O as centre.

Let r be the distance of the point P from O.

Let the line OP rotate through an angle θ with respect to a reference line Ox describing an arc of length s, then,

$$s = r\theta$$

Differentiating w.r.t. time

$$\frac{ds}{dt} = r\frac{d\theta}{dt} \qquad \text{...(21.2)}$$

$$v = r\omega$$

where, v is the linear velocity of the point P and ω the angular velocity of the slab.

It may be noted here that the angular velocity ω is independent of the choice of the point *P*. But the magnitude of the linear velocity v depends upon the distance of the point *P* from the point *O*.

The tangential and normal components of the linear acceleration of the point *P* by definition are,

$$a_t = \frac{dv}{dt}$$

and

$$a_n = \frac{v^2}{r}$$

As, $v = r\omega$ and angular acceleration $\alpha = \dfrac{d\omega}{dt}$

Therefore,

$$a_t = \frac{d(r\omega)}{dt} = \frac{rd\omega}{dt}$$

$$a_t = \alpha_r \qquad ...(21.3)$$

And

$$a_n = \frac{(r\omega)^2}{r}$$

$$a_n = r\omega^2 \qquad ...(21.4)$$

Example 21.1. A grinding wheel is attached to the shaft of an electric motor of rated speed of 1800 r.p.m. When power is switched on the unit attains the rated speed in 5 s and when the power is switched off the unit coasts to rest in 90 s. Assuming uniformly accelerated motion, determine the number of revolution the unit turns (*a*) to attain the rated speed (*b*) to come to rest.

Solution:

$$N = 1800 \text{ r.p.m.}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1800}{60}$$

$$\omega = 60\pi$$

(*a*)

$$\omega = 60\pi,\ \omega_0 = 0,\ t = 5 \text{ s}$$

Using,

$$\omega = \omega_0 + \alpha t$$

$$60\,\pi = 0 + 5\alpha$$

$$\alpha = \frac{60\pi}{5} = 12\pi \text{ rad/s}^2$$

Using,

$$\omega^2 = \omega_0^{\ 2} + 2\alpha\theta$$

$$(60\,\pi)^2 = 9 + 2(12\pi)\theta$$

$$\theta = \frac{(60\pi)^2}{2 \times 12\pi} \text{ radians}$$

Number of revolutions turned,

$$= \frac{(60\pi)^2}{2 \times 12\pi} \times \frac{1}{2\pi} = 75 \quad \textit{Ans.}$$

(*b*)

$$\omega_0 = 60\pi,\ \omega = 0,\ t = 90 \text{ s}$$

Using

$$\omega = \omega_0 + \alpha t$$

$$0 = 60\pi + \alpha(90)$$

$$\alpha = -\frac{60\pi}{90} \text{ rad/s}^2$$

Using

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$0 = (60\,\alpha)^2 - 2\left(\frac{60\pi}{90}\right)\theta$$

$$\theta = \frac{(60\pi)^2}{\frac{120\pi}{90}} = \frac{3(60\pi)^2}{4\pi}$$

Number of revolutions turned

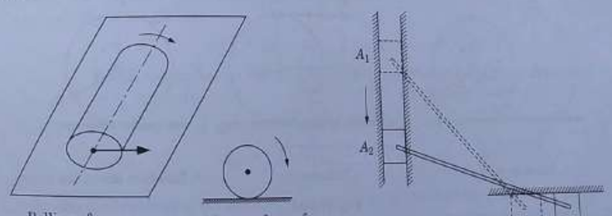
$$= \frac{3(60\pi)^2}{4\pi} \times \frac{1}{2\pi}$$

$$= 1350 \quad \textit{Ans.}$$

21.4 GENERAL PLANE MOTION

Any plane motion which is neither a translation nor a rotation is referred to as a general plane motion. A general plane motion, however, has the characteristic of a plane motion, that is, all particles of the body move in parallel planes.

The rolling of a cylinder, without slipping, on a flat or a curved surface is an example of a general plane motion. Another example is of a bar whose ends slide along the horizontal and vertical tracks (Fig. 21.5).



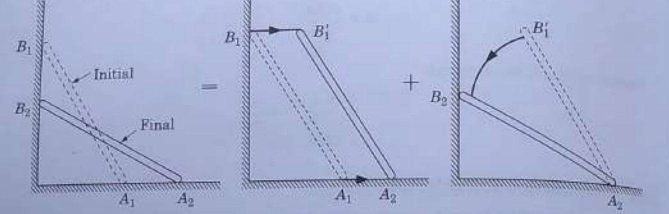
Rolling of a cylinder   Motion of one of the parallel planes

Fig. 21.5

In a plane motion any choosen particles of the rigid body would remain in the same fixed plane during the motion and all other particles would move is planes parallel to this plane. This concept enables us to study the motion of a body by studying the motion in one of these parallel planes and the points lying in that plane. Thus the motion of a cylinder which is rolling without slipping, can be studied by considering the motion of plane circular figure.

Another important aspect of a general plane motion of a rigid body is that it can always be considered as the sum of a plane translation and a rotation about an axis perpendicular to the plane motion.

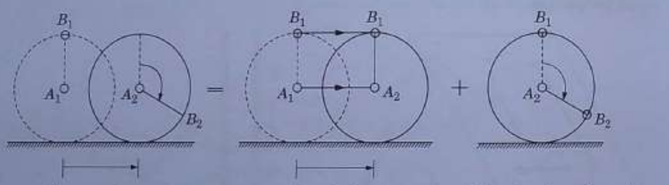
Consider the motion of the bar AB from the initial position A_1B_1 to the position A_2B_2. This general plane motion can be replaced by a motion of translation of the bar from position A_1B_1 to position A_2B_1' together with a rotation about the end A from position A_2B_1' to the final position A_2B_2 as shown in Fig. 21.6 (*a*).



Plane motion = Translation + Rotation about the end A

Fig. 21.6 (*a*)

Fig. 21.6 (*b*) shows the rolling of a cylinder on a horizontal surface. This general plane motion can similarly be replaced by a translation and a rotation as shown in Fig. 21.6 (*b*).



Plane motion = Translation + Rotation about the centre

Fig. 21.6 (*b*)

The above concept of replacing the general plane motion of a rigid body by the sum of a translation and a rotation is an important method to study the motion of rigid bodies.

21.5 ABSOLUTE AND RELATIVE VELOCITY IN PLANE MOTION

Consider a rigid body in general plane motion with respect to the fixed axes x-y. This motion of the rigid body can be completely defined by the motion of a plane figure representing the motion of one of its parallel planes (Fig. 21.7).

Let an imaginary line A_1B_1 in the body be displaced in time Δt to the position A_2B_2. This displacement can be considered to be a sum of a translation from A_1B_1 to A_2B_1' and rotation about A_2 from A_2B_1' to A_2B_2.

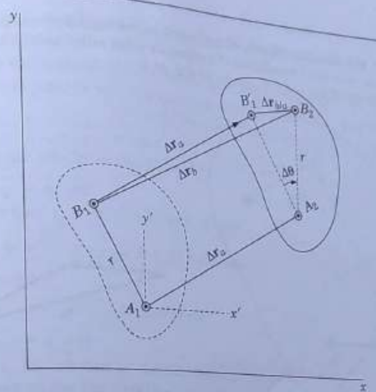


Fig. 21.7

Attach a reference axis x'-y' to the point A_1 such that as the body moves, the axes x'-y' translate, but remain always parallel to the fixed axes x-y.

Applying the triangle law,

$$\Delta \mathbf{r}_b = \Delta \mathbf{r}_a + \Delta \mathbf{r}_{b/a} \quad \text{(vector sum)}$$

Total displacement of A_1B_1 = Translation of A_1B_1 + Displacement due to the rotation of the end B_1 about A_1

Dividing both sides by Δt and in the limit when Δt approaches zero

$$\mathbf{v}_b = \mathbf{v}_a + \mathbf{v}_{b/a}$$

where,

$\mathbf{v}_b$ is the absolute velocity of the point B_1, with respect to the fixed axes x-y

$\mathbf{v}_a$ is the absolute velocity of the point A_1 and corresponds to the translation with respect to the axes x-y

$\mathbf{v}_{b/a}$ is the relative velocity associated with the rotation of the point B_1 with respect to point A_1 and has the magnitude, $\mathbf{v}_{b/a} = r\omega$

r, being the fixed distance of the point B_1 from A_1

ω, is the angular velocity of the body.

If in the above problem, the point A_1 is considered to be a reference point, then such a point is called as a pole and the result can be generalized as,

Velocity of any point P in the rigid body = Vector sum of the velocity of the pole A and the relative velocity of the point P with respect to the pole A

Thus we can say that the velocity of any point in the rigid body in plane motion can be determined if the velocity of translation of a reference point called pole is known together with the angular velocity of the body as shown in Fig. 21.8.

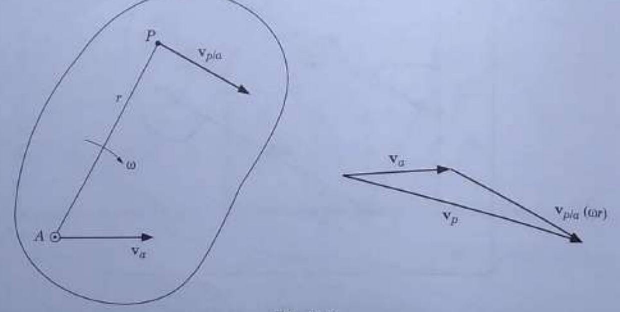


Fig. 21.8

Similarly, the acceleration of a point P is given by

Acceleration of any point P in the rigid body = Vector sum of the acceleration of the pole A and the relative acceleration of point P with respect to the pole A.

$$\mathbf{a}_p = \mathbf{a}_a + \mathbf{a}_{p/a} \qquad ...(21.6)$$

Example 21.2. A cylinder radius 1 m rolls without slipping along a horizontal plane AB. Its centre has a uniform velocity of 20 m/s. Find the velocity of the point E and F on the circumference of the cylinder [Fig. 21.9 (a)].

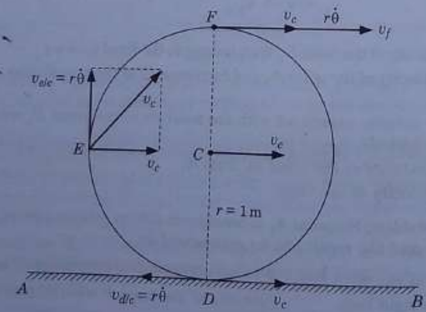


Fig. 21.9 (a)

Solution: As the cylinder rolls without slipping, the velocity of the point D, in contact with the plane AB, must be zero.

As, the motion of the centre C of the cylinder is known, we therefore, choose it as a pole.

Velocity of the point D = Vector sum of the velocity of the pole C and the rotational velocity of the point D w.r.t. the pole C.

$$\overline{v_d} = \overline{v_c} + \overline{v_{d/c}} \quad \text{(vector sum)}$$

If $\dot{\theta}$ be the angular velocity of the cylinder

$$\overline{v_d} = \overline{v_c} - \overline{r\dot{\theta}}$$

But $v_d = 0$ therefore, $v_c = r\dot{\theta}$; $v_c = 20$ m/s (Given)

$$\dot{\theta} = \frac{v_c}{r} = \frac{20}{1} \text{ or } \dot{\theta} = 20 \text{ radians/s}$$

Velocity of the point E

$$\overline{v_e} = \overline{v_c} + \overline{v_{e/c}} \quad \text{(vector sum)}$$

$$\overline{v_e} = \overline{v_c} + \overline{r\dot{\theta}}$$

Vector sum of v_c and $\overline{r\dot{\theta}}$ can be determined by triangle law as shown in the Fig. 21.9 (b).

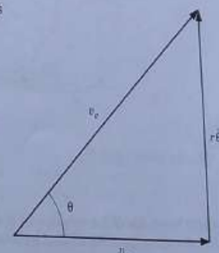


Fig. 21.9 (b)

$$v_e = \sqrt{(v_c)^2 + (r\dot{\theta})^2}$$

but, $r\dot{\theta} = v_c$

$$v_e = \sqrt{(v_c)^2 + (v_c)^2} = \sqrt{2}\, v_c$$

$$v_e = \sqrt{2} \times 20,$$

$$v_e = 28.28 \text{ m/s and } \theta = 45° \quad \textit{Ans.}$$

Velocity of the point F,

$$\overline{v_f} = \overline{v_c} + \overline{v_{f/c}} \quad \text{(vector sum)}$$

$$\overline{v_f} = \overline{v_c} + \overline{r\dot{\theta}} = \overline{v_c} + \overline{v_c} \qquad \left(\begin{array}{l}\overline{v_c} \text{ and } \overline{r\theta} \text{ have the same direction} \\ \text{so it reduces to an algebraic sum}\end{array}\right)$$

$$v_f = 2v_c = 2 \times 20$$

$$v_f = 40 \text{ m/s} \quad \textit{Ans.}$$

To Summarize :

If a cylinder rolls without slipping with a uniform velocity then,

1. The point on the cylinder in contact with the surface is at rest.
2. The point on the top of the cylinder moves with twice the velocity of the centre.
3. The velocity of any point on the circumference is the vector sum of the velocity of the centre of the cylinder and the relative velocity of that point w.r.t. the centre.

Example 21.3. A cylindrical roller is in contact at its top and bottom, with two conveyor belts PQ and RS as shown in Fig. 21.10. If the belts run at the uniform speeds of $v_1 = 3$ m/s and $v_2 = 2$ m/s, find the linear velocity and the angular velocity of the roller. The diameter of the roller may be assumed to be 40 cm.

Solution: Let the linear velocity of the roller be v_c and its angular velocity be $\dot{\theta}$. Choose C as the pole.

Velocity of the point A,

$$\overline{v_a} = \overline{v_c} + \overline{v_{a/c}} \quad \text{...(i)}$$

$$v_a = v_c + r\dot{\theta} \quad \text{...(ii)}$$

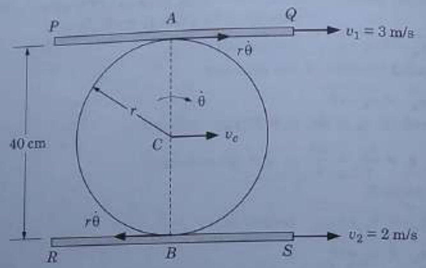


Fig. 21.10

Velocity of the point B

$$v_b = \overline{v_c} + \overline{v_{b/c}} \quad \text{...(iii)}$$

$$v_b = v_a - r\dot{\theta} \quad \text{...(iv)}$$

Right hand side of the equation (*i*) is a vector sum of the v_c and $v_{a/c}$. As these two velocities are in the same direction so that equation (*ii*) reduces to a algebraic sum. Similar argument is true for the equation (*iv*).

$$3 = v_c + r\dot{\theta} \quad \text{...(v)}$$

$$2 = v_c - r\dot{\theta} \quad \text{...(vi)}$$

Solving simultaneously,

$$v_c = 2.5 \text{ m/s}$$

$$\dot{\theta} = 2.5 \text{ radian/s} \quad \textit{Ans.}$$

Example 21.4 A bar AB of length 1 m has its ends A and B constrained to move horizontally and vertically as shown in Fig. 21.11 (*a*). The end A moves with constant velocity of 5 m/s horizontally. Find (*a*) the angular velocity of the bar, (*b*) the velocity of the end B and (*c*) the velocity of the mid point C of the bar at the instant when the axis of the bar makes an angle of 30° with the horizontal axis.

Solution: The velocity of the end A is known, we choose A as the pole.

$$v_a = 5 \text{ m/s}$$

Velocity of the end B

$$\overline{v_b} = \overline{v_a} + \overline{v_{b/a}} \quad \text{(vector sum)} \quad \text{...(i)}$$

$v_{b/a}$ is the relative velocity of the end B w.r.t. A and is due to the rotation of the bar AB around A as centre. The direction of $v_{b/a}$ is normal to the bar at B.

If $\dot{\theta}$ be the angular velocity of bar AB,

$$v_{b/a} = \dot{\theta} l$$

v_a is known both in magnitude and direction, direction of $\overline{v_a}$ is normal to the bar AB at B and the direction of v_b is parallel to the y-axis. The vector diagram of velocities corresponding to the equation (*i*) thus can be drawn as shown in Fig. 21.11 (*b*). The magnitudes of $v_{b/a}$, and v_b can be determined as,

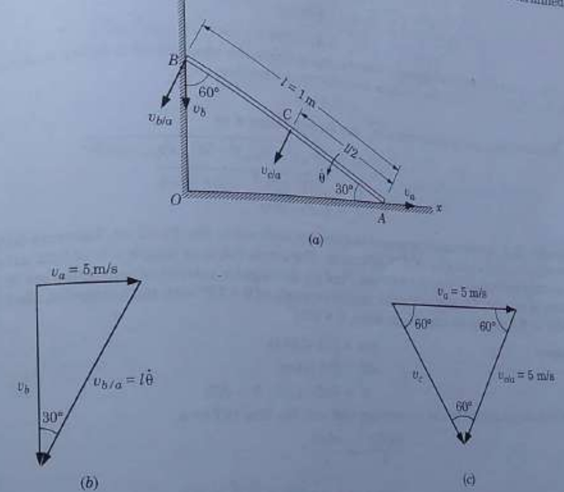


Fig. 21.11

From the vector diagram

$$\frac{v_a}{v_b} = \tan \theta$$

$$v_b = \frac{v_a}{\tan 30°} = \frac{5}{0.5774}$$

$$v_b = 8.66 \text{ m/s} \quad \textit{Ans.}$$

$$\frac{v_a}{v_{b/a}} = \sin 30° = 0.5$$

$$v_{b/a} = \frac{5}{0.5} = 10 \text{ m/s}$$

$$v_{b/a} = l\dot{\theta}, \dot{\theta} = \frac{v_{b/a}}{l} = \frac{10.0}{1}$$

$$\dot{\theta} = 10 \text{ radian/s} \quad \textit{Ans.}$$

Velocity of the point C

$$\overline{v_c} = \overline{v_a} + \overline{v_{c/a}} \quad ...(ii)$$

$$v_{c/a} = \frac{l}{2}\dot{\theta} = \frac{1}{2} \times 10 = 5 \text{ m/s}$$

$$v_a = 5 \text{ m/s}$$

$$\overline{v_c} = \bar{5} + \bar{5} \text{ (vector sum, } v_c \neq \sqrt{5^2 + 5^2}\text{)}$$

The vector diagram of the velocities corresponding to the equation (*ii*) is drawn as shown in the Fig. 21.11 (*c*). Since the triangle is equilateral, therefore,

$$v_c = 5 \text{ m/s}$$

Alternatively, the magnitude of $\overline{v_c}$ can be determine d as,

$$v_c = \sqrt{(v_a)^2 + (v_{c/a})^2 - 2(v_a)(v_{c/a})\cos 60^\circ}$$

$$v_c = \sqrt{(5)^2 + (5)^2 - 2(5 \times 5)0.5}$$

$$v_c = 5 \text{ m/s} \quad Ans.$$

Example 21.5 A reciprocating engine mechanism is shown in Fig. 21.12 (*a*). The crank OA has a constant angular velocity of $N = 300$ r.p.m. The crank OA is of length $r = 12.0$ cm and the connecting rod AB is of length $l = 60$ cm. Find (*a*) the angular velocity of connecting rod (*b*) the velocity of the piston when the crank makes an angle of $\theta = 30°$ with the horizontal. Shall the velocity of the piston be maximum when $\theta = 90°$?

Solution:

$$OA = r = 0.12 \text{ m}$$

$$AB = l = 0.6 \text{ m}$$

$$N = 300 \text{ r.p.m.}, \ \theta = 30°.$$

Let the angle between the connecting rod and the line OB be ϕ.

$$\frac{\sin 30°}{l} = \frac{\sin\phi}{r}$$

$$\sin\phi = \frac{r}{l}\sin 30° = \frac{0.12}{0.6} \times 0.5$$

$$\phi = 5.7°$$

Motion of the Crank OA

Angular velocity

$$\omega_{oa} = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60}$$

$$\omega_{oa} = 31.42 \text{ radian/s}$$

$$v_a = OA \times \omega_{oa} = 0.12 \times 31.42$$

$$v_a = 3.77 \text{ m/s, normal to } OA.$$

Choose the end A of the connecting rod as pole. The velocity of the point B can be written as

$$\overline{v_b} = \overline{v_a} + \overline{v_{b/a}} \quad ...(i)$$

where, $\bar{v}_{b/a}$ is the velocity of the point B with respect to the pole A and acting normal to AB. If ω_{ab} be the angular velocity of the connecting rod AB, then

$$v_{b/a} = (AB)\,\omega_{ab} = l\,\omega_{ab}$$

The velocity v_b of the end B must be horizontal and is equal to the velocity of the piston. The velocity $v_{b/a}$ is known in direction and the velocity v_a is known both in magnitude and direction. The vector diagram of velocities corresponding to the equation (*i*) is drawn as shown in Fig. 21.12 (*c*).

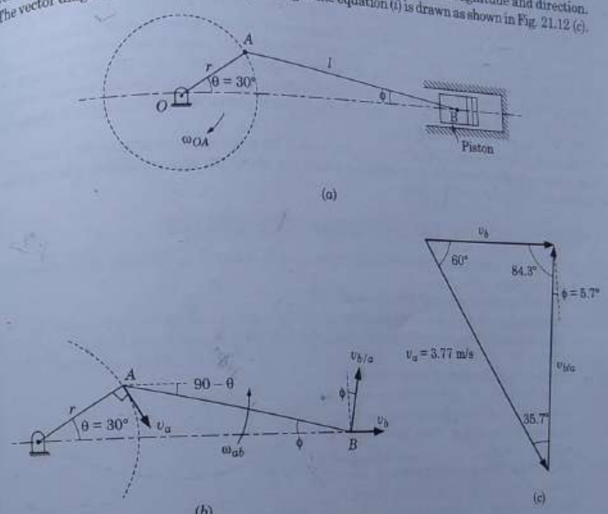


Fig. 21.12

Using sine rule we can write

$$\frac{v_b}{\sin 35.7°} = \frac{v_a}{\sin 84.3°} = \frac{v_{b/a}}{\sin 60°}$$

$$v_a = 3.77 \text{ m/s}$$

$$v_b = \frac{3.77 \times \sin 35.7°}{\sin 84.3°} = 2.21 \text{ m/s} \quad Ans.$$

$$v_{b/a} = \frac{3.77 \times \sin 60°}{\sin 84.3°} = 3.28 \text{ m/s}$$

$$\omega_{ab} = \frac{v_{b/a}}{l} = \frac{3.28}{0.6}$$

$$\omega_{ab} = 5.47 \text{ radian/s} \quad Ans.$$

The velocity of the piston B is maximum for $\theta = 90°$ only when $l >> r$ (Ex. 14.2).

21.6 INSTANTANEOUS CENTRE OF ROTATION IN PLANE MOTION

In the earlier section we learnt that the plane motion of a rigid body can be considered to be a combination of translation of a reference point called pole and a rotation about this pole.

It can also be shown that a rigid body in plane motion, at any given instant of time appears as if rotating about a certain point in the plane of the body. This point which is instantaneously at rest and has zero velocity is called as the *instantaneous centre of rotation*. It should be understood here that the body may seem to be rotating about one point at one instant of time and about another point at the next instant. The instantaneous centre therefore, is changing every instant and is not a fixed point.

Using the above concept, the velocity of *any point* in the body can be determined by assuming *that point* to be rotating, with some angular velocity ω, about the instantaneous centre at the instant.

21.7 LOCATION OF THE INSTANTANEOUS CENTRE

(*i*) When the directions of the velocities of two points A and B in the body are known and are unequal.

Consider two points A and B on the rigid body in plane motion. Let at any instant their velocities be $\mathbf{v}_a$ and $\mathbf{v}_b$ respectively Fig. 21.13.

Draw AC perpendicular to the velocity $\mathbf{v}_a$ at the point A. If this velocity $\mathbf{v}_a$ is the result of rotation about some instantaneous centre then, the centre must lie along AC.

Next draw BD perpendicular to the velocity $\mathbf{v}_b$ at B. By the same argument, the instantaneous centre must lie along BD. Their point of intersection I, therefore, determines the instantaneous centre of rotation of the body at that instant.

The angular velocity of the body ω can be determined as

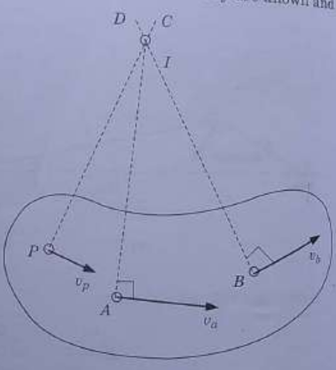


Fig. 21.13

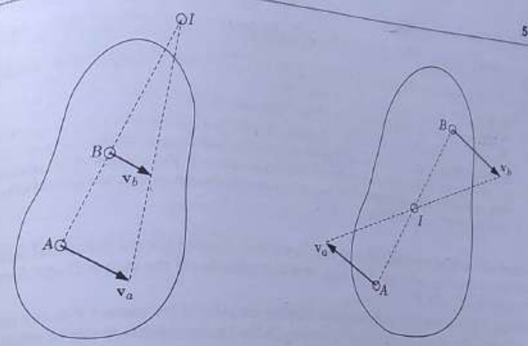
$$\omega = \frac{v_a}{IA}$$

The velocity of any point P is given at that instant by

$$v_p = \omega\, IP$$

(*ii*) When the velocity $\mathbf{v}_a$ and $\mathbf{v}_b$ of the two points in the body are parallel but unequal in magnitude.

The instantaneous centre I can be found by determining the point of intersection of the line AB with the line joining the extremities of the vectors $\mathbf{v}_a$ and $\mathbf{v}_b$ as shown in Fig. 21.14 (*a*) and (*b*).



(*a*) (*b*)

Fig. 21.14

(*iii*) When the velocities $\mathbf{v}_a$ and $\mathbf{v}_b$ of the two points are equal and parallel then the instantaneous centre is at the infinity and all the points of the body have the same velocity as shown in Fig. 21.15.

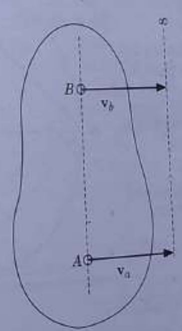


Fig. 21.15

SUMMARY

1. The instantaneous centre is not a fixed point. Its location keeps changing every instant and the path traced by it (locus) is called centrode.
2. The instantaneous centre may lie on or outside the body.
3. *The instantaneous centre is a point identified with the body where the velocity is zero.* Conversely, if a point identified with the rigid body is at rest at any instant, (within or outside the body) it must be the instantaneous centre of rotation of the body. For example, if a circular cylinder rolls without slipping, the point of the contact has zero velocity and is the instantaneous centre of rotation of the cylinder.

Example 21.6 A cylinder of radius 1 m rolls without slipping along a horizontal plane AB. Its centre has a uniform velocity of 20 m/s. Find the velocities of the points D and E on the rim of the cylinder Fig. 21.16.

Solution: When a cylinder rolls without slipping the point of the contact F at any instant has zero velocity. The point of contact F therefore, is the instantaneous centre of rotation of the cylinder.

Velocity of the point D

$$v_d = \omega(FD) \quad \ldots(i)$$

Where, $v_c = \omega\, FC$

Or $\omega = \dfrac{v_c}{r} = \dfrac{20}{1}$

$\omega = 20$ rad/s

and FD is length of the line joining the instantaneous centre F with the point D.

$$FD = \sqrt{r^2 + r^2}$$

$$= \sqrt{2} \times 1 = 1.414 \text{ m}$$

Substituting in (i) $v_d = 20 \times 1.414$

$v_d = 28.28$ m/s

The direction of the velocity v_d is normal to the line FD as shown in Fig. 21.16.

Velocity of the point E

$v_e = \omega(FE)$,

$FE = 2r$

$v_e = 20 \times 2 = 40$ m/s,

The direction of the velocity v_e is normal to the line FE as shown.

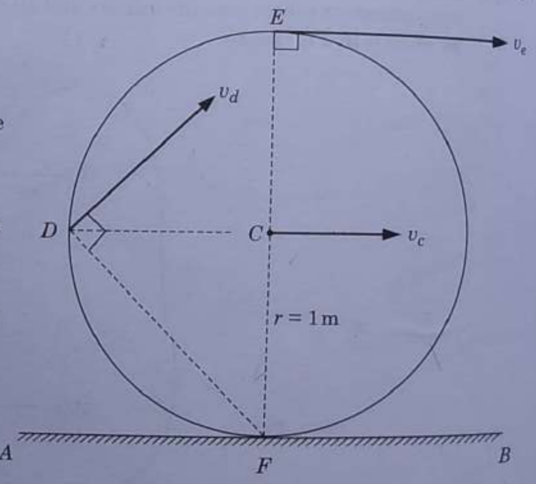


Fig. 21.16

Example 21.7. A roller of radius 5.0 cm rides between two horizontal bars moving in the opposite directions as shown in Fig. 21.17 (a). Calculate the distance 'd' defining the position of the instantaneous centre of rotation of the roller. Assume no slip conditions at the points of contact A and B.

Locate the position of the instantaneous centre when both the bars are moving in the same direction.

Solution: The instantaneous centre I is the point of intersection of the line joining A and B with the line joining the extremities of the velocity vectors $\bar{v}_a$ and $\bar{v}_b$, as shown in Fig. 21.17 (b).

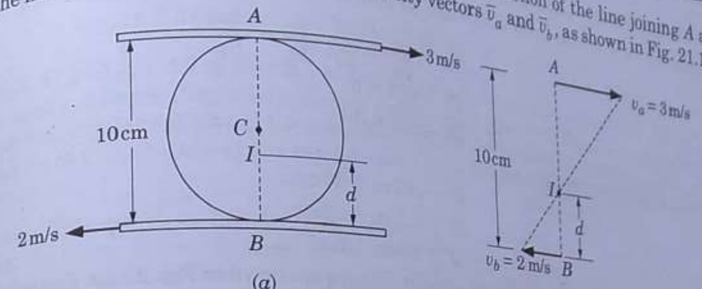


Fig. 21.17

Every point on the roller shall appear to rotate about the instantaneous centre I with an angular velocity ω, therefore,

$$\omega = \frac{v_a}{IA} = \frac{v_b}{IB}$$

$$\frac{3}{IA} = \frac{2}{IB}$$

Or $3IB - 2IA = 0 \quad \ldots(i)$

Also, $IB + IA = AB = 0.1 \quad \ldots(ii)$

Solving, simultaneously (i) and (ii)

$IA = 0.06$ m $= 6$ cm

$IB = 0.04$ m $= 4$ cm

$d = 4$ cm *Ans.*

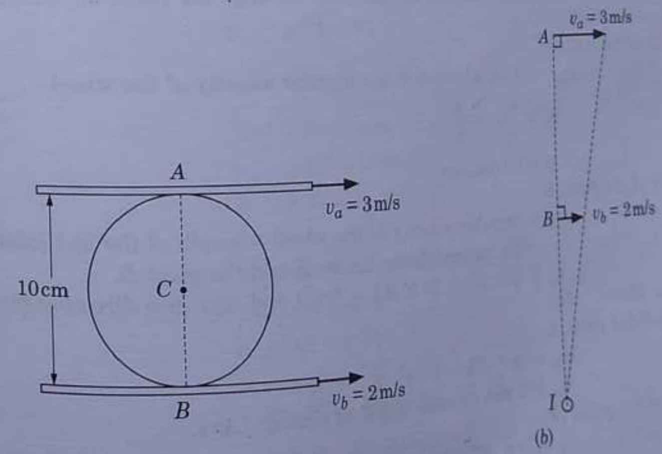


Fig. 21.18

When moving in the same direction.

Location of the instantaneous centre can be determined as shown in Fig. 21.18 (*b*).

$$\omega = \frac{v_a}{IA} = \frac{v_b}{IB}, \quad \frac{3}{IA} = \frac{2}{IB}$$

$$3IB - 2IA = 0$$

Also, $$IA - IB = AB = 0.1 \quad \text{...(iii)}$$

Solving simultaneously (*iii*) and (*iv*) ...(*iv*)

$$IA = 0.3 \text{ m} = 30 \text{ cm}$$

$$IB = 0.2 \text{ m} = 20 \text{ cm}$$

$$d = 20 \text{ cm} \quad \textit{Ans.}$$

Example 21.8 A compound wheel rolls without slipping as shown in Fig. 21.19. The velocity of the centre C is 1 m/s. Find the velocities of the points A, B and D.

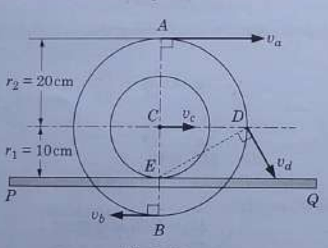


Fig. 21.19

Solution: In the case of a wheel rolling without slipping, the point of contact E is the instantaneous centre.

Velocity of the centre C

$v_c = r_1\omega$, where ω is the angular velocity of the wheel

$$\omega = \frac{v_c}{r_1} = \frac{1}{0.1}$$

$$\omega = 10 \text{ radians/s.}$$

Velocity of the point B

v_b = Angular velocity of the wheel × length of the line joining the instantaneous centre E and the point B.

$v_b = \omega \times EB = 10 \times 0.1 = 1$ m/s and acts in a direction normal to EB as shown. *Ans.*

Velocity of the point A

$$v_a = \omega \times EA = 10 \times 0.3$$

$v_a = 3$ m/s, normal to EA as shown. *Ans.*

Velocity of the point D

$$v_d = \omega \times ED$$

$$ED = \sqrt{CE^2 + CD^2} = \sqrt{(0.1)^2 + (0.2)^2}$$

$$ED = 0.224 \text{ m.}$$

$$v_d = 10 \times 0.224$$

$v_d = 2.24$ m/s normal to ED *Ans.*

Example 21.9 A bar AB of length 1 m has its ends A and B constrained to move horizontal and vertically as shown (Fig. 21.20). The end A moves with constant velocity of 5 m/s horizontally. Find (*a*) the angular velocity of the bar AB (*b*) the velocity of the end B and (*c*) the velocity of the mid point C of the bar at the instant when the axis of the bar makes an angle 30° with the horizontal.

Solution: This problem has been solved earlier. Let us solve it now by the instantaneous centre method.

The velocity v_a of the end A, at any instant, must be horizontal. If we consider the velocity v_a as the result of rotation about an instantaneous centre, this centre must lie on a line perpendicular to the velocity v_a.

Similarly the instantaneous centre must lie on a line normal to the velocity v_b.

The position of the instantaneous centre is thus determined by the point of intersection I of these two lines.

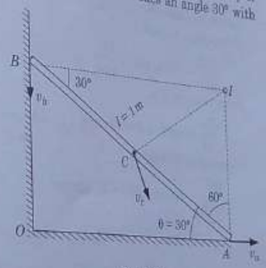


Fig. 21.20

$$IA = BO = l \sin\theta = 1 \times \sin 30°$$

$$IA = 0.5 \text{ m}$$

$$IB = AO = l \cos\theta = 1 \times \cos 30°$$

$$IB = 0.866 \text{ m/s}$$

$$v_a = \omega(IA)$$

$$\omega = \frac{v_a}{IA} = \frac{5.0}{0.5}$$

$\omega = 10$ radians/s *Ans.*

$$v_b = \omega(IB) = 10(0.866)$$

$v_b = 8.66$ m/s *Ans.*

In triangle IAC

$$IA = CA = 0.5 \text{ m and } \angle CAI = 60°$$

Triangle ICA is therefore, equilateral. Hence

$$IC = IA = 0.5 \text{ m}$$

$$v_c = \omega(IC) = 10 \times 0.5$$

$v_c = 5$ m/s *Ans.*

Example 21.10 Bar AB rests at the edge of a wall at some point C with its end A resting on a horizontal floor as shown in Fig. 21.12. If the end A moves with a constant velocity v_a, find the angular velocity of the bar.

Solution: The velocity of point C on the bar, at any instant, must be along the bar and the velocity of the end A must be along the horizontal line OA.

The point of intersection of the normals to these velocities v_c and v_a should locate the instantaneous centre I as shown,

The angular velocity of the bar

$$\omega = \frac{v_a}{IA}$$

Let us evaluate IA

From triangle OCA

$$\frac{OC}{CA} = \sin\theta,\ CA = \frac{OC}{\sin\theta} \quad ...(i)$$

From triangle ICA,

$$\frac{CA}{IA} = \sin\theta,\ CA = IA\sin\theta \quad ...(ii)$$

Equating (i) and (ii)

$$\frac{OC}{\sin\theta} = IA\sin\theta,\ OC = h \quad \text{(given)}$$

$$IA = \frac{h}{\sin^2\theta}$$

$$\omega = \frac{v_a}{IA} = \frac{v_a}{h}\sin^2\theta$$

$$\theta = \frac{v_a\sin^2\theta}{h} \quad \textit{Ans.}$$

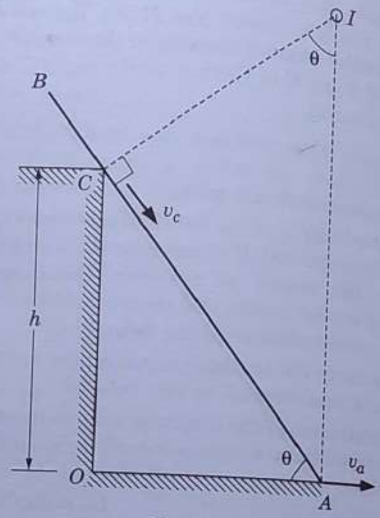


Fig. 21.21

Example 21.11 The end A of a bar AB move with a constant velocity v_a = 2 m/s along the horizontal surface such that it always remains in contact with a disc of radius r = 5 cm which is resting on the horizontal surface as shown (Fig. 21.22). Find the angular velocity of the bar at an instant when the bar makes an angle of $\alpha = 30°$ with the horizontal.

Solution: The velocity of the point C at any instant must be along the rod. So the instantaneous centre must lie on the normal to AB, that is, along the line OC.

Similarly the instantaneous centre of the rod must lie on a line normal to the velocity v_a of the end A. The point of their intersection I locates the instantaneous centre of rotation of the rod.

The angular velocity of the rod AB

$$\omega = \frac{v_a}{IA} \quad ...(i)$$

Let us evaluate IA

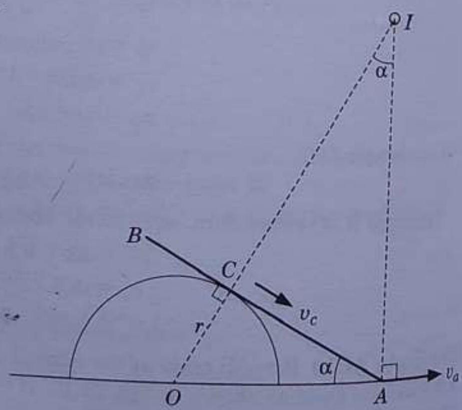


Fig. 21.22

In triangle OAC

$$\frac{OC}{CA} = \sin\alpha,\ CA = \frac{OC}{\tan\alpha} \quad ...(ii)$$

In triangle ICA

$$\frac{CA}{IA} = \sin\alpha,\ IA = \frac{CA}{\sin\alpha} \quad ...(iii)$$

From (ii) and (iii)

$$IA = \frac{OC}{\sin\alpha}\frac{1}{\tan\alpha}$$

$$IA = \frac{OC\cos\alpha}{\sin^2\alpha}$$

Substituting for IA in the equation (i)

$$\omega = \frac{v_a}{IA} = \frac{v_a}{OC}\left(\frac{\sin^2\alpha}{\cos\alpha}\right),\ OC = r$$

$$\omega = \frac{v_a}{r\cos\alpha}\sin^2\alpha$$

At an instant when, $\alpha = 30°$

Substituting $v_a = 2$ m/s, $r = 5$ cm $= 0.05$ m

$$\omega = \frac{2}{0.05}\frac{\sin^2 30°}{\cos 30°} = \frac{2}{0.05}\times\frac{(0.5)^2}{0.866}$$

$$\omega = 11.55 \text{ radian/s} \quad \textit{Ans.}$$

Example 21.12 Solve the Example No. 21.5 by the instantaneous centre method (Fig. 21.23).

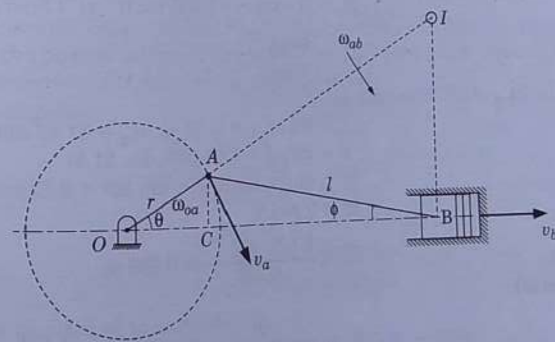


Fig. 21.23

Solution: To locate the instantaneous centre of the connecting rod AB, draw AI normal at A, to the velocity vector v_a (that is along OA) and drawn BI normal to the velocity vector v_b at B. Their point of intersection I locates the instantaneous centre of rotation of the connecting rod AB as shown in Fig. 21.23.

If ω_{ab} be the angular velocity of the connecting rod about I, then

$$v_a = IA(\omega_{ab})$$

$$v_b = IB(\omega_{ab}) \qquad ...(i)$$

$$v_b = \frac{v_a IB}{IA} \qquad ...(ii)$$

Also,

$$v_a = r(\omega_{oa}) \qquad ...(iii)$$

Therefore,

$$v_b = r(\omega_{oa})\,\frac{IB}{IA} \qquad ...(iv)$$

From (i) and (iv) ...(v)

$$r(\omega_{oa}) = IA(\omega_{ab})$$

$$\omega_{ab} = \frac{r(\omega_{oa})}{IA} \qquad ...(vi)$$

$$\omega_{oa} = \frac{2\pi N}{60} = \frac{2\pi\times 300}{60} = 10\pi = 31.42 \text{ radian/s}$$

Let us now evaluate IA and IB

$$BC = l\cos\phi,\ CO = r\cos\theta$$

$$BO = BC + CO = (l\cos\phi + r\cos\theta)$$

In triangle IBO,

$$\frac{IB}{BO} = \tan\theta \text{ or } IB = BO\tan\theta$$

$$IB = (l\cos\phi + r\cos\theta)\tan\theta$$

$$IB = (l\cos\phi\tan\theta + r\sin\theta)$$

$$\frac{OB}{IO} = \cos\theta \text{ or } IO = OB\sec\theta$$

$$IA = IO - r = OB\sec\theta - r$$

$$IA = [(l\cos\phi + r\cos\theta)\sec\theta - r] = l\cos\phi\sec\theta$$

$$IA = \frac{l\cos\phi}{\cos\theta}$$

The determine IA and IB substitute for

$$l = 0.6 \text{ m},\ r = 0.12 \text{ m},\ \omega_{oa} = 31.42 \text{ radian/s}$$

$$\theta = 30°,\ \phi = 5.7° \text{ (See Ex. 21.5)}$$

$$IB = (0.6\times\cos 5.7°\times\tan 30° + 0.12\sin 30°)$$

$$IB = 0.404 \text{ m}$$

$$IA = \frac{0.6\times\cos 5.7°}{\cos 30°} = 0.689 \text{ m.}$$

From equation (v)

$$v_b = r\,\omega_{oa}\frac{IB}{IA} = 0.12\times 31.42\times\frac{0.404}{0.689}$$

$$v_b = 2.21 \text{ m/s} \quad \textit{Ans.}$$

From equation (vi)

$$\omega_{ab} = \frac{r(\omega_{0A})}{IA} = \frac{0.12\times 31.42}{0.689}$$

$$\omega_{ab} = 5.47 \text{ radian/s } \textit{Ans.}$$

Example 21.13 In a four bar mechanism $ABCD$, the bar AB rotates clockwise with an angular velocity of 5 rad/s. Find the angular velocities of the bars BC and CD. When the bar AB makes an angle of 30° with the horizontal, bar CD makes an angle of 60° and the bar BC is horizontal (Fig. 21.24).

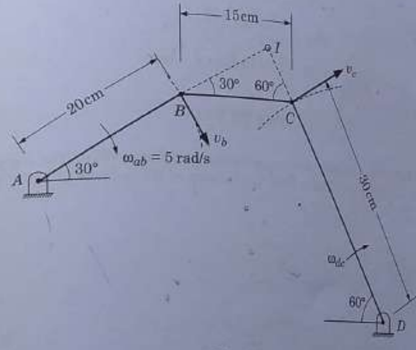


Fig. 21.24

Solution: In the four bar mechanism shown, the bar AB rotates about the fixed end A and the bar DC about the fixed end D.

Therefore, the velocity v_b of the end B is normal to AB and the velocity v_c of the end C is normal to DC as shown in Fig. 21.24.

B and C are also the ends of the bar BC. So the instantaneous centre of rotation I of the bar BC is given by the point of intersection I of the normals drawn to velocities v_b and v_c. That is, the point of intersection I of AB and DC when produced.

Consider triangle BIC

$$\frac{IB}{\sin 60°} = \frac{IC}{\sin 30°} = \frac{BC}{\sin 90°},\ BC = 15 \text{ cm}$$

$$IB = \sin 60°\times BC\times 1$$

$$= 0.866\times 15 = 12.99 \text{ cm}$$

$$IC = \sin 30°\times BC\times 1$$

$$= 0.5\times 15 = 7.5 \text{ cm}$$

Consider the rotation of the bar AB about A

$$v_b = \omega_{ab}(AB) = 5\times(0.2) = 1 \text{ m/s.}$$

Consider the rotation of the bar BC about the instantaneous centre I

$$v_b = \omega_{bc}(IB),\ \omega_{bc} \text{ is the angular velocity of bar } BC$$

$$\omega_{bc} = \frac{v_b}{IB} = \frac{1}{0.1299} = 7.698 \text{ radian/s} \quad \textit{Ans.}$$

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$$v_c = \omega_{bc}(IC) = (7.698)(0.075)$$
$$v_c = 0.577 \text{ m/s}.$$

Consider the rotation of the bar DC about D.

$$v_c = \omega_{dc}(CD),\ \omega_{dc} \text{ is the angular velocity of the bar } CD$$
$$\omega_{dc} = \frac{v_c}{CD} = \frac{0.577}{0.3}$$
$$\omega = 1.94 \text{ radian/s} \quad \textit{Ans.}$$

PROBLEMS

21.1. A wheel of diameter 1 m rolls without slipping on a flat surface. The centre of the wheel is moving with a velocity of 10 m/s. Find the velocity of the point A, B and C (Fig. P.21.1).

$\left(v_a = 10 \text{ m/s},\ v_b = 20 \text{ m/s},\ v_c = 10 \text{ m/s}\right)$

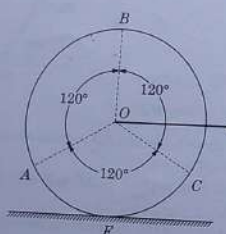

Fig. P.21.1

21.2. A reciprocating engine mechanism is shown in Fig. P.21.2. The crank OA is of length 15 cm and rotating at 600 r.p.m. The connecting rod AB is 70 cm long. Find (a) the angular velocity of the connecting rod (b) the velocity of the piston B (c) the velocity of a point C on the connecting rod at a distance 20 cm from A, when $\theta = 45°$.

$\left(\omega_{ab} = 9.64 \text{ radian/s},\ v_b = 7.67 \text{ m/s},\ v_c = 8.4 \text{ m/s}\right)$

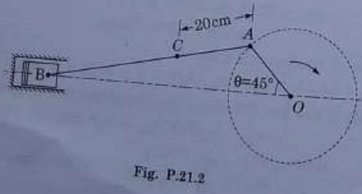

Fig. P.21.2

21.3. A pulley of radius 5 cm and having an angular velocity of 8 rad/s is moving on a horizontal rail with a velocity of 24 cm/s as shown (Fig. P. 21.3). Find the location of the instantaneous centre of rotation of the pulley.

[3 cm below the centre C]

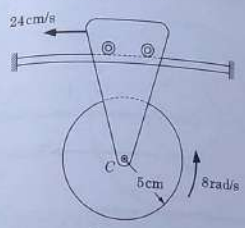

Fig. P.21.3

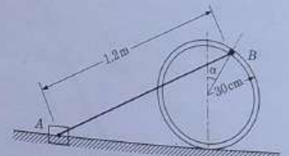

Fig. P.21.4

21.4. A wheel shown in Fig. P.21.4. rolls to the right with a constant velocity of 1.5 m/s. Find (a) the velocity of the block A which slides in a groove cut in the surface of the plane (b) angular velocity of the rod AB when $\alpha = 0°$ and $\alpha = 90°$.

[(a) $v_A = 3$ m/s, $\omega = 0$ (b) $v_A = 1.113$ m/s, $\omega = 1.29$ rad/s]

21.5. A rod AB of length 2 m is in contact with a horizontal surface and an inclined plane as shown in Fig. P.21.5. The end B moves with a constant velocity of 2.4 m/s to the right. Determine the angular velocity of the rod AB and the velocity of the end A when $\theta = 20°$.

[1.356 rad/s, 2.94 m/s 60°]

21.6. In the position shown, the bar O_2B has a constant angular velocity of 3 rad/s counter clockwise. Determine the angular velocity of the bar O_1A (Fig. P.21.6).

[4 radian/s]

21.7. The ends A and B of a straight slender bar of length l are constrained to follow the floor OA and an inclined plane OB. Prove that for the motion of the bar in a vertical plane, the instantaneous centre of rotation describes a circle of radius $\frac{l}{\sin\theta}$ with O as centre. (Fig. P. 21.7).

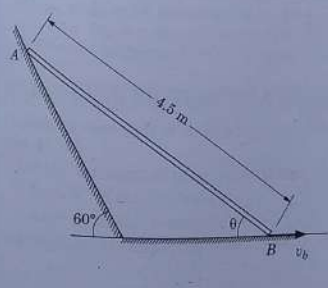

Fig. P.21.5

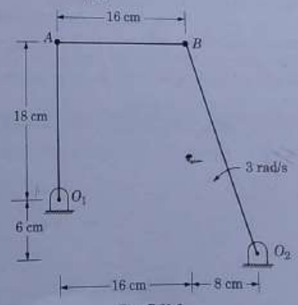

Fig. P.21.6

21.8. A wheel is made to the roll without slipping, towards right, by pulling a string wrapped around a coaxial spool as shown in Fig. P.21.8. Determine with what velocity the string should be pulled so that the centre of the wheel moves with a velocity of 10 m/s? [6 m/s]

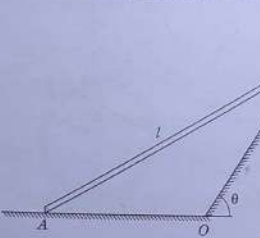

Fig. P.21.7

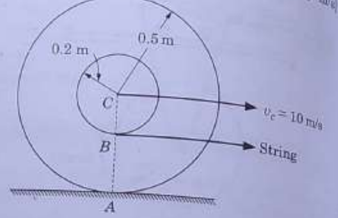

Fig. P.21.8

21.9. In the mechanism shown in Fig. P.21.9 the bar AB has a constant angular velocity of 3 rad/s counter clockwise. Determine the angular velocity of the bars BC and CD.

[$\omega_{BC} = 0$, $\omega_{CD} = 1.6$ rad/s]

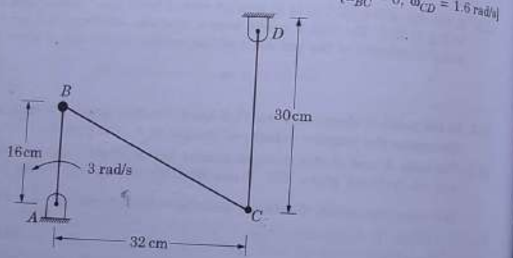

Fig. P.21.9

22 CHAPTER Kinetics of Rigid Body: Force and Acceleration

22.1 INTRODUCTION

In the case of a particle moving in a plane, the relation between the forces acting on the particle and the resulting translatory motion is given by a single vector equation,

$$\Sigma \mathbf{F} = m\mathbf{a} \qquad ...(22.1)$$

The corresponding scalar equations are,

$$\Sigma F_x = ma_x$$
$$\Sigma F_y = ma_y$$

It may be appreciated that the concept of idealizing a body by a particle is an ingenious assumption designed to simplify the analysis.

In the case of a rigid body in plane motion, its rotatory motion is also to be considered. This results in an increase in one equation of motion to account for this rotary motion. The equation for the rotary motion is similar in form to that of translatory motion

$$M = I\alpha \qquad ...(22.2)$$

Comparing it with the equation (22.1) it can be observed that the force has been replaced by the moment of the force, linear acceleration by the angular acceleration and the mass of the body by the moment of inertia of the body about the axis of rotation.

Thus, to describe *the motion of a rigid body in plane motion three equations of the motion are needed.* Further, the motion of a rigid body can be described by the motion of any convenient point located in the rigid body. The mass centre of the body is usually chosen for this purpose.

22.2 PLANE MOTION OF A RIGID BODY: EQUATIONS OF MOTION

Consider a rigid body of mass m in plane motion under the action of the applied external forces (Fig. 22.1).

Let the resultant of these external forces reduce to

a force F_x in the x-direction

a force F_y in the y-direction and

a moment M_G about the mass centre.

Let the mass centre G of the body move parallel to the x-y plane under the action of applied forces.

Consider the motion of any particle P of mass dm situated at a distance r from the axis passing through G and normal to the plane of the motion x-y.

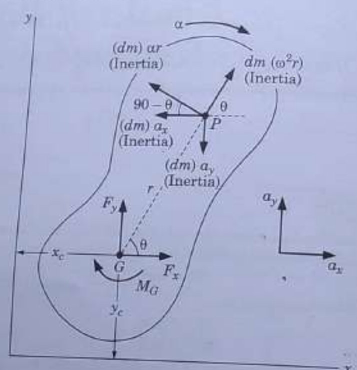

Fig. 22.1

Choose the mass centre G as pole and let its coordinates be x_c, y_c. Let the angle which the line PG makes with the x-axis be θ.

The motion of the point P can considered to be the sum of a motion of translation of the pole G and a rotation of point P about the axis passing through G and normal to the plane of motion.

Let the linear acceleration of the pole G be,

a_x in the x-direction

a_y in the y-direction

These are also the components of linear acceleration on the particle P.

The inertia forces acting on the particle P corresponding to the linar acceleration are :

$(dm)a_x$, acting in the direction opposite to the direction of acceleration a_x

$(dm)a_y$, acting in the direction opposite to the direction of acceleration a_y

Let the angular velocity of the body be ω and the angular acceleration be α.

As the particle P rotates about an axis through G and at a distance r from it, inertia forces acting on it are :

$dm(\omega^2 r)$, acting along GP (Normal component)

$dm\,(\alpha r)$, acting normal to GP (Tangential component)

The inertia forces acting on the entire rigid body can be calculated by adding (or integrating) the inertia forces acting on all the particles of the body.

The equations of dynamic equilibrium of the body can be written as,

$$\Sigma F_x = 0 : F_x - a_x \int dm - \alpha \int r\cos(90-\theta)dm + \omega^2 \int r\cos\theta dm = 0 \quad ...(22.3)$$

$$\Sigma F_y = 0 : F_y - a_y \int dm + \alpha \int r\sin(90-\theta)dm + \omega^2 \int r\sin\theta dm = 0 \quad ...(22.4)$$

$$\Sigma M_G = 0 : M_G + a_y \int r\cos\theta dm - a_x \int r\sin\theta dm - \alpha \int r^2 dm = 0 \quad ...(22.5)$$

But, $\int dm = m$, $\int r\sin\theta dm = 0$, $\int r\cos\theta dm = 0$ and $\int r^2 dm = I_G$

I_G being the moment of inertia of the body about the axis through the mass centre G and normal to the plane of motion x-y.

Therefore, the equations (22.3), (22.4) and (22.5) become,

$$\left.\begin{array}{lcl} \Sigma_x - ma_x = 0 & & F_x = ma_x \\ F_y - ma_y = 0 & \text{or} & F_y = ma_y \\ M_G - I_G\alpha = 0 & & M_G = I_G\alpha \end{array}\right] \begin{array}{l} \text{Equations of Motion} \\ \text{of a Rigid Body} \end{array} \quad \begin{array}{r} ...(22.6) \\ ...(22.7) \\ ...(22.8) \end{array}$$

From the above equations it can be concluded that for a rigid body in plane motion we can write three equations of the motion as,

(i) Two equations for the translatory motion of its mass centre due to the external forces F_x and F_y

$$\left.\begin{array}{l} F_x = ma_x \\ F_y = ma_y \end{array}\right\} \text{Translatory Motion}$$

(ii) One equation for the rotary motion of the body, due to the moment M_G of the external forces about the axis through the mass centre and perpendicular to the plane of motion,

$$M_G = I_G\alpha \} \text{ Rotary motion}$$

F_x, F_y and M_G represent the resultant of the applied external forces.

22.3 RELATION BETWEEN THE TRANSLATORY MOTION AND ROTARY MOTION OF A BODY IN PLANE MOTION

To explain this point, consider the specific case of the plane motion of a right circular cylinder on a horizontal surface as shown in Fig. 22.2.

Let the mass of the cylinder be m, R be the normal reaction and F the force of friction acting on the cylinder. The acceleration of its mass centre G be a_x and angular acceleration α. The cylinder may (i) roll without slipping, (ii) may slip without rolling and (iii) may roll as well as slip.

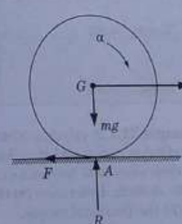

Fig. 22.2

(i) Rolls without slipping. In this case, the forward motion of the mass centre is related to its angular motion. The distance travelled and the angle turned by the body are related as

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$$x = \theta r$$

$$\dot{x} = \dot{\theta} r = \omega r$$

$$\ddot{x} = \ddot{\theta} r = \alpha r, \text{ or } a_x = \alpha r$$

And the frictional force F is such that,

$$F \leq \mu R \qquad \text{(Less then } \mu R)$$

(ii) Slips without rolling. There is no rotary motion involved and

$$F = \mu R$$

(iii) Rolls as well as slips. In this case translatory and rotary motions are independent of each other and

$$F = \mu R$$

Table 22.1 *Analogy between rectilinear motion and rotational motion*

Quantity	*Rectilinear* Symbol	Unit	*Rotational* Symbol	Unit
1. Displacement	x	m	θ	rad
2. Velocity	$v = \frac{dx}{dt}$	m/s	$\omega = \frac{d\theta}{dt}$	rad/s
3. Acceleration	$a = \frac{d^2x}{dt^2}$	m/s^2	$\alpha = \frac{d^2\theta}{dt^2}$	rad/s^2
4. Mass, Moment of Inertia	mass m	kg	Moment of Inertia	kg m^2
5. Effort : Force, Torque	Force $F = m\frac{d^2x}{dt^2} = ma$	N	Torque $T = I\alpha$	Nm
6. Inertia : Force, Couple	$-ma$	N	Couple $= -I\alpha$	Nm
7. Momentum	Linear Momentum $= mv$	Ns	Angular Momentum $= I\omega$	Nms
8. Kinetic Energy	K.E. $= \frac{1}{2}mv^2$	J	K.E. $= \frac{1}{2}I\omega^2$	J
9. Work	$\int F\,dx$	J	$\int T\,d\theta$	J
10. Spring constant	k	N/m	k_t	Nm/rad

Example 21.1 A grinding wheel has a rated speed of 1500 r.p.m. and can be assumed to be a disc of 0.5 m radius and of uniform thickness. It weights 300 N. It is made to turn at 1500 r.p.m in 120 seconds. Determine (a) the number of revolutions that it shall execute before coming to rest (b) the frictional torque.

Solution: N = 1500 r.p.m

$$\omega_0 = \frac{2\pi N}{60} = \frac{2\pi \times 1500}{60} \text{ rad/s}$$

$$\omega = 0 \text{ (Final)}$$

$$t = 120 \text{ s}$$

Using, $\omega = \omega_0 + \alpha t$

$$\alpha = -\frac{\omega_0}{t} = -\frac{2\pi \times 1500}{60 \times 120}$$

Angular deceleration, $\alpha = -1.308$ rad/s^2

Using, $\omega^2 = \omega_0^2 + 2\alpha\theta$

$$0 = \left(\frac{2\pi \times 1500}{60}\right)^2 + 2(-1.308)\theta$$

$$\theta = 9417 \text{ radians}$$

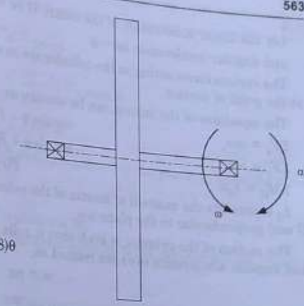

Fig. 22.3

Number of revolution $= \frac{9417}{2\pi}$

Number of revolutions before coming to rest = 1499 *Ans.*

Frictional Torque,

$$M = I_G\alpha, \qquad I_G = \frac{1}{2}mr^2 = \frac{1}{2}\left(\frac{300}{9.81}\right)(0.5)^2$$

$$M = -\frac{1}{2}\left(\frac{300}{9.81}\right)(0.5)^2(1.308)$$

$$M = -5.0 \text{ Nm} \quad \textit{Ans.}$$

Example 22.2. A cylinder of mass m and radius r resting on an inclined plane is released from rest and rolls down the inclined plane without slipping. Determine (a) the acceleration of its centre of mass (b) the maximum angle θ of the inclined plane for which the body will roll without slipping (c) the maximum velocity of the centre of the cylinder after it has rolled a distance of 1 m. Assume the coefficient of static friction $\mu = 0.192$.

What would be the acceleration of the centre of the cylinder if it were to move down the inclined plane under the frictionless conditions?

Solution: The plane motion of the three dimensional cylinder can be studied by the motion of a circular figure of radius r in the x-y plane as shown in Fig. 22.4.

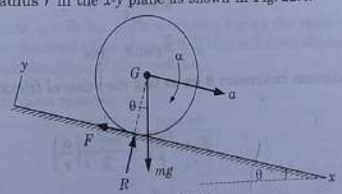

Fig. 22.4

Let the linear acceleration of the centre G be $= a$

and angular acceleration be $= \alpha$

The various forces acting on the cylinder are as shown. F represents the friction force acting at the point of contact.

The equations of the motion can be written as

$$\Sigma F_x = ma_x: \quad mg\sin\theta - F = ma \qquad ...(i)$$

$$\Sigma F_y = ma_y: \quad -mg\cos\theta + R = 0 \qquad ...(ii)$$

$$\Sigma M_G = I_G\alpha: \quad F(r) = I_G\alpha \qquad ...(iii)$$

I_G represents the moment of inertia of the cylinder about an axis through the mass centre G and perpendicular to the plane x-y.

The motion of the cylinder is such that it rolls without slipping. The linear acceleration a and angular acceleration $\alpha(\ddot{\theta})$ are related as,

$$a = r\alpha$$

or

$$\alpha = \frac{a}{r}$$

Substituting for α in (iii)

$$Fr = I_G\left(\frac{a}{r}\right)$$

Eliminating F using (i)

$$(mg\sin\theta - mg)r = I_G\frac{a}{r}$$

$$a\left(m + \frac{I_G}{r^2}\right) = mg\sin\theta$$

$$a = \frac{mg\sin\theta}{\left(m + \frac{I_G}{r^2}\right)} \qquad ...(iv)$$

Substituting,

$$I_G = \frac{mr^2}{2}$$

$$a = \frac{mg\sin\theta}{m + \frac{mr^2}{2r^2}}$$

$$a = \frac{2}{3}g\sin\theta \quad \textit{Ans.}$$

(b) To find the maximum inclination θ let us find the value of friction

From (iii)

$$F = \frac{I_G\alpha}{r} = \frac{mr^2}{2}\frac{1}{r}\left(\frac{a}{r}\right)$$

Substituting

$$a = \frac{2}{3}g\sin\theta$$

$$F = \frac{mr^2}{2}\frac{1}{r}\left(\frac{2}{3}g\sin\theta\right)\frac{1}{r}$$

$$F = \frac{mg\sin\theta}{3}$$

The maximum frictional force possible, for any inclination θ of the plane, is when the body is just about to slide and is equal to μR.

The maximum friction force

$$\mu R = \mu(mg\cos\theta)$$

As long as the frictional force F is less than and value $\mu\, mg\cos\theta$, the cylinder will roll without sliding.

For the cylinder to roll without slipping

$$F < \mu\, mg\cos\theta$$

$$\frac{mg\sin\theta}{3} < \mu\, mg\cos\theta$$

$$\tan\theta < 3\mu = 3 \times 0.192 = 0.576$$

Or

$$\theta \le 30^\circ \quad \textit{Ans.}$$

(c) The maximum value of θ can be 30°. The velocity after rolling a distance of 1 m along the plane can be determined using.

$$v^2 - u^2 = 2as$$

The velocity is maximum when $\theta = 30^\circ$,

$$a = \frac{2}{3}g\sin\theta = \frac{2}{3} \times 9.81 \times \sin 30^\circ$$

$$a = 3.27 \text{ m/s}^2,\ u = 0$$

$$v^2 = 2 \times 3.27 \times 1$$

$$v = 2.56 \text{ m/s}$$

If there is no friction then $F = 0$ so from (i)

$$mg\sin\theta = ma$$

$$a = g\sin\theta = 9.81 \times \sin 30^\circ$$

$$a = 4.905 \text{ m/s}^2 \quad \textit{Ans.}$$

Example 22.3. A sphere, a cylinder and a hoop each having the same mass and radius are released from rest on an inclined plane of angle θ. If they roll down the inclined plane without slipping compare the accelerations of their centres.

Solution: From the previous example

$$a = \frac{mg\sin\theta}{\left(m + \frac{I_G}{r^2}\right)}$$

For a sphere

$$I_G = \frac{2}{5}mr^2$$

$$a = \frac{mg\sin\theta}{\left(m + \frac{2}{5}\frac{mr^2}{r^2}\right)} = \frac{5}{7}g\sin\theta$$

$$a = 0.714\, g \sin\theta$$

For a cylinder $\quad a = \frac{2}{3} g\sin\theta$ (From previous example)

$$a = 0.677\, g \sin\theta$$

For a hoop $\quad I_G = mr^2$

$$a = \frac{mg\sin\theta}{\left(m + \frac{mr^2}{r^2}\right)} = \frac{1}{2} g\sin\theta$$

$$a = 0.5\, g \sin\theta$$

Body	*Moment of inertia*	*Acceleration of the mass centre*
Sphere	$I_G = \frac{2}{5}mr^2$	$0.714\, g \sin\theta$
Cylinder	$I_G = \frac{mr^2}{2}$	$0.667\, g \sin\theta$
Hoop	$I_G = mr^2$	$0.5\, g \sin\theta$

For frictionless condition: The acceleration of the mass centre is same for all the bodies and is equal to $g \sin\theta$.

Example 22.4. A uniform spherical ball of mass m and radius r is projected along a rough horizontal plane with an initial linear velocity of v_0 and zero angular acceleration. The coefficient of friction between the ball and the plane is μ. Determine the time after which the ball will start rolling without sliding. Also, find the linear and angular velocities of ball at the time.

Solution: When the ball is projected, its mass centre G has a velocity v_0 but its linear acceleration and angular acceleration are zero Fig. 22.5.

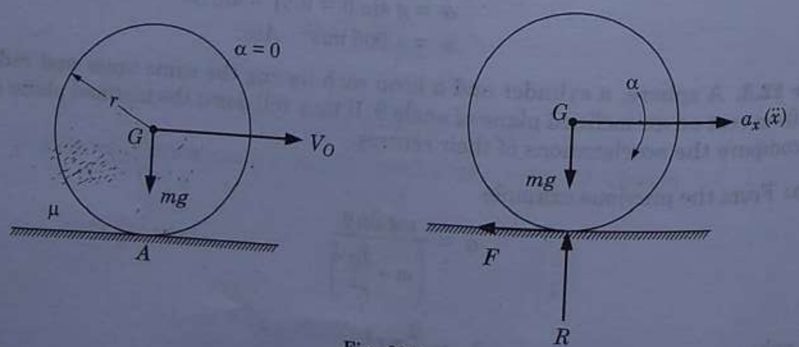

Fig. 22.5

After the ball is projected, let it attain an angular acceleration α and its mass centre a linear acceleration equal to a.

Linear acceleration and angular acceleration are not always related by the relation $a = \alpha r$. This relationship is true only in the case of rolling without sliding.

Forces acting on the ball are shown in Fig. 22.5.

The equations of motion can be written as,

$\Sigma F_x = ma_x$: $\quad -F = ma$

as, $\quad F = \mu R$ (In the case of sliding)

Therefore, $\quad -\mu R = ma$

$\Sigma F_y = ma_y$: $\quad R - mg = 0 \quad$ ...(*i*)

$\Sigma M_G = I_G\alpha$: $\quad Fr = I_G\alpha \quad$ ...(*ii*)

$$(\mu R)r = I_G\alpha$$

From (*ii*) $\quad R = mg \quad$ ...(*iii*)

And $\quad I_G = \frac{2}{5}mr^2$,

Substituting the above values in (*iii*)

$$(\mu\, mg)r = \left(\frac{2}{5}mr^2\right)\alpha$$

$$\alpha = \frac{5\mu g}{2r} \qquad \text{...(iv)}$$

From (*i*) $\quad a = -\frac{\mu R}{m} = -\frac{\mu mg}{m}$

$$a = -\mu g \qquad \text{...(v)}$$

It may be observed from (*iv*) and (*v*) *that α and a are independent of each other*. The direction of acceleration a is found to be opposite to that assumed.

The ball is thus experiencing,

(*i*) A uniform linear deceleration, $a = \mu g$ (being negative). The linear velocity v_0 therefore, continues to decrease,

$$(v = u + at)$$

(*ii*) A uniform angular acceleration $\alpha = 5\,\mu g/2r$. The angular velocity therefore, continues to increase,

$$(\omega = \omega_0 + \alpha t)$$

For the ball to start rolling without sliding, the velocity of the point of contact A of the ball with the plane should become zero or the relations

$$v = r\dot{\theta} = r\omega \text{ and } a = \alpha r = \ddot{\theta} r$$

should be satisfied.

Let it happen after a time t, then,

Using, $\quad v = u + at$, the linear velocity after time t is, $\quad$ ...(*vi*)

$$v = v_0 - (\mu g)t$$

Using, $\omega = \omega_0 + \alpha t$, the angular velocity after time t is,

$$\omega = \left(\frac{5\mu g}{2r}\right)t \qquad ...(vii)$$

At the time when rolling without sliding takes place,

$$v = r\omega$$

Substituting the values using (vi) and (vii)

$$(v_0 - \mu g t) = r\left(\frac{5\mu g}{2r}\right)t$$

$$t = \frac{2v_0}{7\mu g} \quad Ans.$$

Substituting the above value of t in (vi) and (vii)

$$v = v_0 - (\mu g)\frac{2v_0}{7\mu g}, \; v = \frac{5}{7}v_0 \quad Ans.$$

$$\omega = \frac{5\mu g}{2r}\left(\frac{2v_0}{7\mu g}\right), \; \omega = \frac{5}{7}\frac{v_0}{r} \quad Ans.$$

Example 22.5. Two weights, each of 25 N, are suspended from a two-step pulley as shown in Fig. 22.6. Find the acceleration of the weight A. The weight of the pulley is $W = 200$ N and its radius of gyration $i_g = 20$ cm.

Solution: Let the acceleration of the weight A be downward $= \ddot{x}_a$

Acceleration of the weight

$$B = \ddot{x}_b$$

These are related as,

$$\ddot{x}_b = \frac{30}{45}\ddot{x}_a$$

$$\ddot{x}_b = \frac{2}{3}\ddot{x}_a$$

Let the angular acceleration of the pulley be

$$\alpha = \ddot{\theta}$$

Then,

$$\alpha = \frac{\ddot{x}_a}{0.45} = \frac{\ddot{x}_b}{0.3}$$

The equations of motion are :

For the weight A

$$W_a - T_1 = \frac{W_a}{g}\ddot{x}_a$$

Or

$$T_1 = W_a - \frac{W_a}{g}\ddot{x}_a \qquad ...(i)$$

For the weight B

$$T_2 - W_b = \frac{W_b}{g}\ddot{x}_b$$

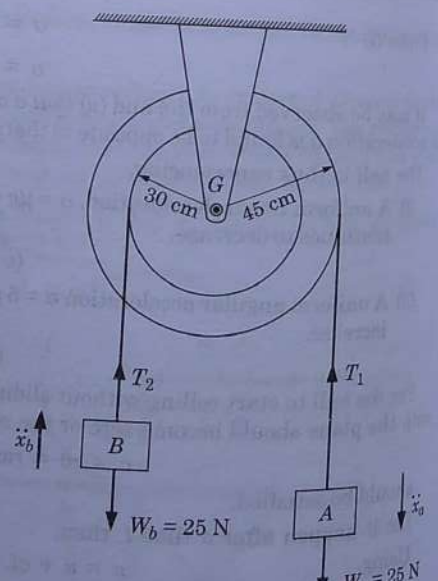

Fig. 22.6

Or

$$T_2 = W_b + \frac{W_b}{g}\ddot{x}_b \qquad ...(ii)$$

For the pulley

$$T_1(0.45) - T_2(0.3) = I_G\alpha = \frac{W}{g}i_g^2\alpha$$

$$0.45T_1 - 0.3T_2 = \frac{W}{g}i_g^2\left(\frac{\ddot{x}_a}{0.45}\right) \qquad ...(iii)$$

Substituting for T_1 and T_2 from (i) and (ii)

$$0.45\left(W_a - \frac{W_a}{g}\ddot{x}_a\right) - 0.3\left(W_b + \frac{W_b\ddot{x}_b}{g}\right) = \frac{W}{g}i_g^2\left(\frac{\ddot{x}_a}{0.45}\right)$$

Substituting

$$\ddot{x}_b = \frac{2}{3}\ddot{x}_a,\; W = 200 \text{ N},$$

$$W_a = W_b = 25 \text{ N},\; i_g = 0.2 \text{ m}$$

$$0.45 \times 25\left(1 - \frac{1}{g}\ddot{x}_a\right) - 0.3 \times 25\left(1 + \frac{1}{g}\frac{2}{3}\ddot{x}_a\right) = \frac{200}{g}(0.2)^2\frac{\ddot{x}_a}{0.45}$$

Solving,

$$\ddot{x}_a = 1.08 \text{ m/s} \quad Ans.$$

Example 22.6. A right circular cylinder of mass m and radius r is suspended from a cord that is wound around its circumference. If the cylinder is allowed to fall freely, find the acceleration of its mass centre G and the tension in the cord (Fig. 22.7).

Solution: Forces acting on the cylinder are (i) tension T in the cord (ii) weight of the cylinder mg

Let the acceleration of the mass centre G be a and its angular acceleration be α.

Equations of motion of the cylinder can be written as,

$\Sigma F_x = ma_x$: No forces acting

$\Sigma F_y = ma_y$: $\quad mg - T = ma \qquad ...(i)$

$\Sigma M_G = I_G\alpha$: $\quad Tr = I_G\alpha$

$$Tr = \frac{mr^2}{2}\alpha \qquad ...(ii)$$

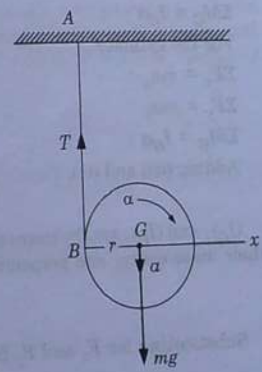

Fig. 22.7

As there is no slip,

$$a = \alpha r,\; \alpha = \frac{a}{r}$$

Eliminating T from (i) and (ii) and substituting for α

$$(mg - ma)r = \frac{mr^2}{2}\frac{a}{r}$$

$$(g - a) = \frac{a}{2}$$

$$a = \frac{2}{3}g \quad Ans.$$

From (i)

$$T = m(g - a) = m\left(g - \frac{2}{3}g\right) = \frac{mg}{3}$$

Example 2.7. A solid cylinder and a thin hoop of equal masses m and radii r are connected by a bar G_1G_2 as shown in Fig. 22.8. The system rolls down the inclined plane without slipping. Find the acceleration of the system and the tension in the bar.

Solution: Forces acting are shown in the Fig. 22.8. T represents the axial forces in the bar and F_1 and F_2 are the friction forces. Let α be the angular acceleration of the each body and a be linear acceleration of their mass centres.

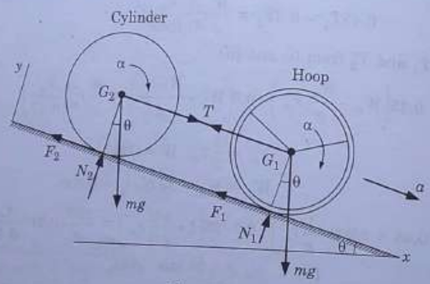

Fig. 22.8

Writing the equations of motion :

For the hoop,

$\Sigma F_x = ma_x$: $$mg \sin\theta - F_1 - T = ma \quad ...(i)$$

$\Sigma F_y = ma_y$: $$N_1 - mg\cos\theta = 0 \quad ...(ii)$$

$\Sigma M_G = I_G\alpha$ $$F_1 r = (I_G)_1\alpha \quad ...(iii)$$

For the cylinder,

$\Sigma F_x = ma_x$: $$mg\sin\theta - F_2 + T = ma \quad ...(iv)$$

$\Sigma F_y = ma_y$: $$N_2 - mg\cos\theta = 0 \quad ...(v)$$

$\Sigma M_G = I_G\alpha$: $$F_2 r = (I_G)_2\alpha \quad ...(vi)$$

Adding (*iii*) and (*vi*),

$$\alpha\{(I_G)_1 + (I_G)_2\} = r(F_1 + F_2)$$

$(I_G)_1$ and $(I_G)_2$ are the moments of inertia of the hoop and the cylinder about the axis through their mass centre and perpendicular to the plane x-y. And are,

$$(I_G)_1 = mr^2, \quad (I_G)_2 = \frac{mr^2}{2}$$

Substituting for F_1 and F_2 from (*i*) and (*iv*),

$$\alpha\left(mr^2 + \frac{mr^2}{2}\right) = r(mg\sin\theta - T - ma + mg\sin\theta + T - ma)$$

$$\alpha\left(\frac{3mr^2}{2}\right) = r(2\,mg\sin\theta - 2\,ma)$$

Substituting for $$\alpha = \frac{a}{r}$$

$$\frac{a}{r}\left(\frac{3mr^2}{2}\right) = r(2\,mg\sin\theta - 2\,ma)$$

$$a = \frac{4}{7}g\sin\theta \quad \textit{Ans.}$$

From (*i*) $$T = mg\sin\theta - F_1 - ma$$

Substituting for $$F_1 = \frac{(I_G)_1\alpha}{r}$$

and for $$\alpha = \frac{a}{r}$$

$$T = mg\sin\theta - \frac{(I_G)_1\alpha}{r} - ma$$

$$T = mg\sin\theta - \frac{mr^2}{r}\frac{a}{r} - ma$$

$$T = mg\sin\theta - ma - ma$$

$$T = m(g\sin\theta - 2a) \quad \textit{Ans.}$$

Substituting, $$a = \frac{4}{7}g\sin\theta$$

$$T = -\frac{mg}{7}\sin\theta$$

(– ve sign means that there is compression in the bar)

Example 22.8. A spool is pulled by a force P applied to the end of a cord wound around the axle of the spool. Find the acceleration of the spool if it rolls without slipping along a horizontal plane. Assume the data as $r_1 = 7.5$ cm, $r_2 = 15$ cm, $\theta = 30°$, $P = 50$ N, $W = 100$ N, *radius of gyration* $i_g = 5.0$ *cm*.

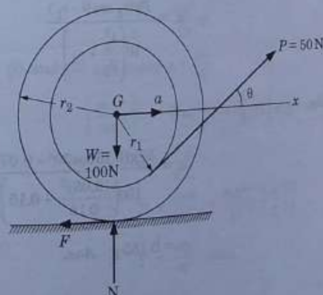

Fig. 22.9

Solution: The forces acting on the spool are,

(*i*) force P (*ii*) weight W (*iii*) reaction N (*iv*) friction force F.

The equations of motion can be written as

$$\Sigma F_x = ma_x : \qquad P\cos\theta - F = \frac{W}{g}a \qquad \text{...}(i)$$

$$\Sigma F_y = ma_y : \qquad N - P\sin\theta = 0 \qquad \text{...}(ii)$$

$$\Sigma M_G = I_G\alpha : \qquad Fr_2 - Pr_1 = \left(\frac{W}{g}i_g^2\right)\alpha \qquad \text{...}(iii)$$

From (*i*)

$$F = P\cos\theta - \frac{W}{g}a$$

Substituting in (*iii*)

$$\left(P\cos\theta - \frac{W}{g}a\right)r_2 - Pr_1 = \left(\frac{W}{g}i_g^2\right)\alpha$$

Relation between α and a is,

$$a = \alpha r_2$$

Or

$$\alpha = \frac{a}{r_2}$$

$$\left(P\cos\theta - \frac{W}{g}a\right)r_2 - Pr_1 = \left(\frac{W}{g}i_g^2\right)\frac{a}{r_2}$$

$$a = \frac{Pg(r_2\cos\theta - r_1)}{W\left(\frac{i_g^2}{r_2} + r_2\right)}$$

Substituting the values,

$$a = \frac{50g(0.15\cos 30^\circ - 0.075)}{100\left(\frac{(0.05)^2}{0.15} + 0.15\right)}$$

$$a = 0.165\,g \quad \textit{Ans.}$$

Example 22.9. A thin uniform bar of mass m and length L is suspended from two vertical inextensible strings. If the right hand string BD is cut find the tension in the left string AC and the angular acceleration of the bar (Fig. 22.10).

Solution: As the string BD is cut, the bar AB starts rotating about the end A. *It should be noted here that the bar does not rotate about its mass centre G.*

Let the angular acceleration of the bar be α then the linear acceleration of the mass centre G is $\alpha\left(\frac{L}{2}\right)$ and its direction would be normal to the bar at G.

Forces acting on the bar are shown in the Fig. 22.10.

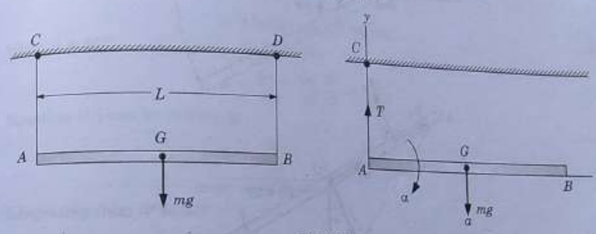

Fig. 22.10

The equations of motion can be written as

$\Sigma F_x = ma_x$: No force

$$\Sigma F_y = ma_y : \qquad mg - T = ma \qquad \text{...}(i)$$

$$\Sigma M_G = I_G\alpha : \qquad T\left(\frac{L}{2}\right) = I_G\alpha$$

As,

$$I_G = \frac{mL^2}{12}$$

Therefore,

$$\frac{TL}{2} = \frac{mL^2}{12}\alpha \qquad \text{...}(ii)$$

Eliminating T from (*i*) and (*ii*) and using $a = \frac{\alpha L}{2}$

$$\left(\frac{mL^2}{12}\right)\alpha = (mg - ma)\frac{L}{2} = \left(mg - m\frac{\alpha L}{2}\right)\frac{L}{2}$$

$$\alpha = \frac{3\,g}{2\,L}$$

$$T = \frac{I_G\alpha}{L/2} = \frac{mL^2}{12}\left(\frac{3\,g}{2\,L}\right)\frac{2}{L}$$

$$T = \frac{mg}{4} \quad \textit{Ans.}$$

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Example 22.10. A thin uniform rod AB of length $L = 1$ m and mass $m = 10$ kg is hinged at the point C which is at a distance of 0.25 m from the end A (Fig. 22.11). The rod is released from the horizontal position. Find (*a*) the angular velocity of the rod when it has rotated through 30° (*b*) the reaction at the hinge.

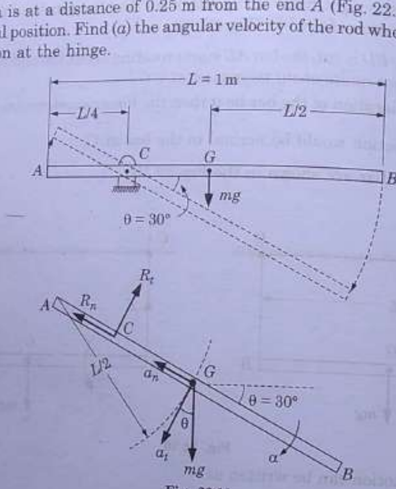


Fig. 22.11

Solution: The bar AB when released rotates about the hinge C. The mass centre G rotates in a circle of radius $L/4$ (0.25 m).

When the bar is at an angle $\theta = 30°$ from the horizontal let its angular velocity be ω angular acceleration be α.

Component of the linear acceleration of the mass centre G, along the bar

$$a_n = \omega^2\left(\frac{L}{4}\right)$$

Component of the linear acceleration of the mass centre G, normal to the bar

$$a_t = \alpha\left(\frac{L}{4}\right)$$

Forces acting on the bar are (*i*) the weight of the bar mg (*ii*) reaction of the hinge on the bar having components R_n and R_t as shown in Fig. 22.11.

Equating of motion of the bar can be written as

$\Sigma F_t = ma_t$: (Normal to the bar)
$$mg\cos\theta - R_t = ma_t = m\left(\frac{\alpha L}{4}\right) \quad \text{...(i)}$$

$\Sigma F_n = ma_n$: (Along the bar)
$$R_n - mg\sin\theta = ma_n = m\left(\frac{\omega^2 L}{4}\right) \quad \text{...(ii)}$$

$\Sigma M_G = I_G\alpha$:
$$R_t\left(\frac{L}{4}\right) = I_G\alpha = \left(\frac{mL^2}{12}\right)\alpha \quad \text{...(iii)}$$

Eliminating R_t from (*i*) and (*iii*) and solving for α

$$m\frac{\alpha L}{12} = mg\cos\theta - \left(\frac{mL^2\alpha}{12}\right)\left(\frac{4}{L}\right)$$

$$\alpha\left(\frac{L}{4}+\frac{L}{3}\right) = g\cos\theta$$

$$\alpha = \frac{12}{7}g\cos\theta$$

$$\alpha = 16.82\cos\theta$$

When $\theta = 30°$,
$$\alpha = 14.56 \text{ rad/s}, \quad \text{...(iv)}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega$$

Equation (*iv*) can be written as

$$16.82\cos\theta = \omega\frac{d\omega}{d\theta}$$

$$\omega\, d\omega = 16.82\cos\theta\, d\theta$$

Integrating from 0° to 30°

$$\int_0^{\omega}\omega\, d\omega = 16.82\int_0^{30°}\cos\theta\, d\theta$$

(Relation $\omega^2 = \omega_0^2 + 2\alpha\theta$ cannot be used as α is not constant)

$$\frac{\omega^2}{2} = [16.82\sin\theta]_0^{30°}$$

$$\omega = \sqrt{2\times 16.82\times 0.5}$$

$$\omega = 4.10 \text{ rad/s} \quad \textit{Ans.}$$

To find the components of the reaction R_t and R_n, substitute for ω and α in the equations (*i*) and (*ii*)

$$R_t = mg\cos\theta - \frac{m\alpha L}{4} = 10\times 9.81\times\cos 30° - \frac{10\times 14.65\times 1}{4}$$

$$R_t = 48.5 \text{ N}$$

$$R_n = \frac{m\omega^2 L}{4} + mg\sin\theta$$

$$= \frac{10\times(4.10)^2\times 1}{4} + 10\times 9.81\sin 30°$$

$$R_n = 91.1 \text{ N}$$

Reaction at the hinge
$$R = \sqrt{R_t^2 + R_n^2}$$

$$= \sqrt{(48.5)^2 + (91.1)^2}$$

$$R = 103.2 \text{ N} \quad \textit{Ans.}$$

22.4 D'ALEMBERT'S PRINCIPLE IN PLANE MOTION

D'Alembert principle can also be applied to study the plane motion of rigid bodies. In this method, the body is brought to a dynamic equilibrium by applying inertia forces and inertia couples. The equation of dynamic equilibrium then can be written for the body.

Inertia forces are to be applied to the mass centre (or C.G.) of the body in a direction opposite to the direction of the acceleration.

Inertia couple can be applied any where in the plane of motion of the body but in a direction opposite to the direction of the angular acceleration of the body.

Equations of dynamic equilibrium can be written as,

$$\Sigma F_x = 0 : \quad X + \underset{\text{(Inertia)}}{(-ma_x)} = 0 \qquad ...(22.9)$$

$$\Sigma F_y = 0 : \quad Y + \underset{\text{(Inertia)}}{(-ma_y)} = 0 \qquad ...(22.10)$$

$$\Sigma M = 0 : \quad M + \underset{\text{(Inertia)}}{(-I\alpha)} = 0. \qquad ...(22.11)$$

X and Y are the external forces acting on the body and M is the moment due to the external forces.

a_x, a_y are the components of the acceleration in the x and the y directions.

α is the angular acceleration and I the moment of inertia of the body about the axis of rotation.

PROBLEMS

22.1. A fly wheel rigidly attached to a shaft of 5 cm radius is placed on two parallel rails which are fixed at an angle $\alpha = 15°$ to the horizontal (Fig. P.22.1). When released from rest, the system rolls through a distance of 0.875 m in 10 seconds. Determine the centroidal radius of gyration of the system. [60 cm]

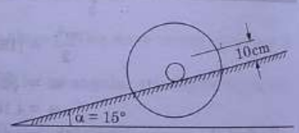


Fig. P.22.2

22.2. Two masses M_1 and M_2 are connected by a string passing over a pulley of mass M and radius of gyration i_g (Fig. P.22.2). Find the acceleration of the system. Are the tensions same on both sides of the string?

$$\left[\frac{(M_1 - M_2)rg}{\frac{Mi_g^2}{r^2} + M_1 r + M_2 r}, \text{ Not same}\right]$$

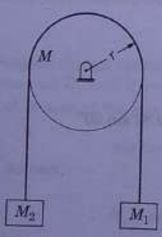


Fig. P.22.2

22.3. A solid homogeneous cylinder A and a pipe B are in contact when they are released from rest from an inclined plane (Fig. P.22.3). If they roll without slipping, determine the distance between them after 2 seconds. [0.85 m]

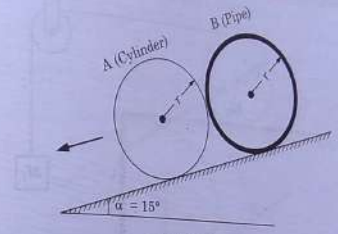


Fig. P.22.3

22.4. A fly wheel of mass of 1000 kg and radius of gyration 30 cm has a block M of mass 15 kg attached to a cord wrapped around its rim of radius r = 40 cm. If the system (Fig. P.22.4) is released from rest find the acceleration of the block M and the speed of the block after it has travelled a distance of 1 m. [0.229 m/s^2, 0.676 m/s]

22.5. Two identical right circular discs A and B of equal weights and radii are arranged in a vertical plane as shown in Fig. P.22.5. Find the acceleration of the centre of falling disc B. [4g/5]

[**Hint** : Acceleration of $C_2 = \alpha r$ + Acceleration of the string].

Fig. P.22.4

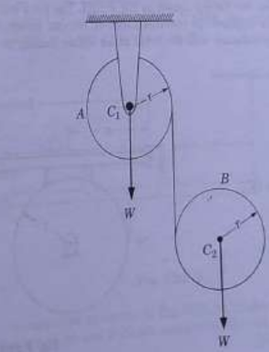


Fig. P.22.5

22.6. A cord passes over a pulley as shown in Fig. P.22.6 carrying a mass M_1 at one end and wrapped around a cylinder of mass M_2 which rolls on a horizontal plane. Determine the acceleration of mass M_1. Assume the pulley to be massless.

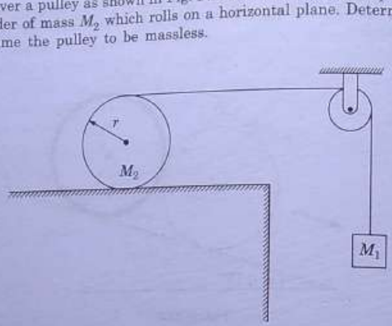


Fig. P.22.6

Hint : $\left[\begin{array}{l}\text{Acceleration of mass } M_1 \text{ is} \\ \text{twice the linear acceleraction} \\ \text{of mass centre of } M_2\end{array}\right]$ $\left[\dfrac{M_1 g}{\left(\dfrac{3}{8}M_2 + M_1\right)}\right]$

22.7. A circular rotor of weight W and radius r which can rotate about its geometric axis is braked by the device shown in Fig. P.22.7. The rotor has an initial angular velocity ω before the brake is applied. If the coefficient of friction between rotor and the brake shoe is μ, how many revolution will the rotor make before coming to rest? $\left[\dfrac{Wa\omega^2 r}{8\pi\mu g b P}\right]$

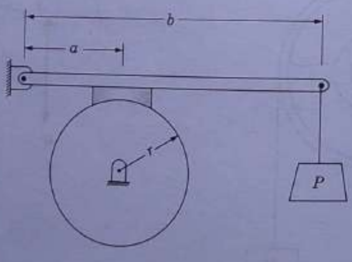


Fig. P.22.7

22.8. A right solid circular cylinder of weight W and radius r is pulled up a 30° inclined plane by a constant force $F = W/2$. The force F is applied to the end of a string wound around the circumference of the cylinder as shown in Fig. P.22.8. Find the acceleration of the centre of mass G of the cylinder assuming no slip at the point of contact A. $\left[\bar{x} = \dfrac{g}{3}\right]$

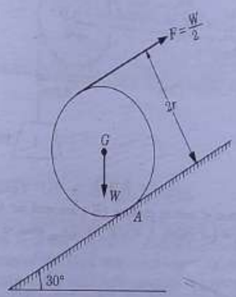


Fig. P.22.8

22.9. A sphere of mass M and radius r is pulled along a horizontal plane by a horizontal force F applied to the end of a string attached to the C.G. of the sphere (Fig. P.22.9). Determine the acceleration of the C.G. of the sphere. Assume it to roll without slipping. [5F/7M]

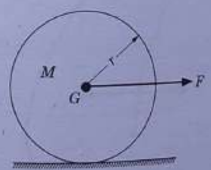


Fig. P.22.9

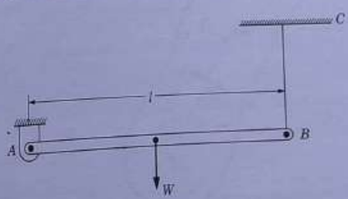


Fig. P.22.10

22.10. A uniform bar of weight W and length l is supported as shown in Fig. P.22.10. If the wire BC suddenly breaks determine (a) the acceleration of the end B (b) the reaction at the pin support. [3/2g, W/4]

22.11. An arrangement consisting of a rod length 1.2 m rigidly fixed to a solid cylinder of diameter of 0.4 m is used for lowering the block A of mass 100 kg. Determine the mass of the rod so that the block will be lowered a distance of 10 m in 5 seconds after the system is released from rest. [22.9 kg]

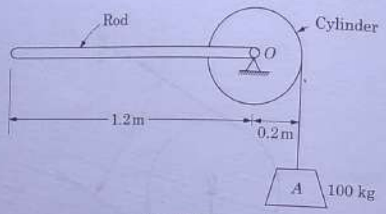


Fig. P.22.11

[**Hint :** Write the equation of motion of block $\Sigma F = ma$ and for the cylinder and the rod $[\Sigma M_0 = I_0 = I_0\alpha$ where, $I_0 = (I_0)_{rod} + (I_0)_{cyn}$ also, $a = 0.2\,\alpha]$

23 CHAPTER

Kinetics of Rigid Body: Work and Energy

23.1 KINETIC ENERGY OF A RIGID BODY

Consider a rigid body in plane motion. Let the body be made up of a large number of particles each of mass Δm. The mass m of the body is the sum of the masses of its particles therefore, $m = \Sigma\Delta m$. The kinetic energy of the rigid body would be equal to the kinetic energy of all its particles. But the plane motion of a rigid body consists of a translatory and a rotary motion. Therefore, *the kinetic energy of a rigid body is the sum of the kinetic energy in translation and the kinetic energy in rotation of all its particles.*

Kinetic Energy in Translation. Let all the particles of the rigid body be moving with the same velocity as the velocity v of its mass centre G.

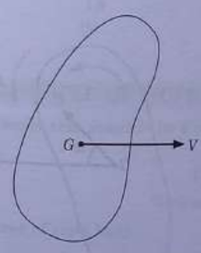


Fig. 23.1

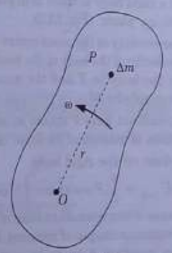


Fig. 23.2

Kinetic energy of particle of mass Δm $= \frac{1}{2}\Delta m v^2$

Kinetic energy of the body $= \frac{1}{2}(\Sigma\Delta m)v^2$

Kinetic energy of the body in translation $= \frac{1}{2}mv^2$

Kinetic Energy in Rotation. Consider a rigid body rotating at an angular velocity ω about a fixed axis intersecting the plane of motion at O as shown (Fig. 23.2).

Consider a particle P of mass Δm situated at a distance r from the axis of rotation.

Velocity of the particle

$$= \omega r$$

Kinetic energy of rotation of the particle

$$= \frac{1}{2}\Delta m(\omega r)^2$$

Kinetic energy of the body is made up of similar particles and is

$$= \frac{1}{2}\omega^2\Sigma(r^2\Delta m)$$

$\Sigma(r^2\Delta m)$ represents the moment of inertia of the whole body about the axis of rotation and is equal to I.

Kinetic energy of rotation of the body $= \frac{1}{2}I\omega^2$

Total kinetic energy of the body = K.E. in translation + K.E. in rotation

Total kinetic energy of the body $= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$ ...(23.1)

23.2 WORK OF THE FORCES ACTING ON A RIGID BODY

Consider a rigid body of mass m in plane motion parallel to x-y plane. (Fig. 23.3).

Let, the velocity of the mass centre G be $\mathbf{v}$, the angular velocity of the body ω, the resultant force acting on the body be $\mathbf{F}$ and the moment of the resultant couple be M.

If the resultant force $\mathbf{F}$ makes an angle α with the direction of motion of its mass centre then,

Work done on the rigid body

$$U_{1-2} = \int(F\cos\alpha)ds + \int M\,d\theta \quad ...(23.2)$$

In the case when a constant force of magnitude $\mathbf{F}$ and a constant couple of moment M move the body from position 1 to position 2 causing a displacement x in the direction of the force and an angular displacement θ, then

$$U_{1-2} = Fx + M\theta \quad ...(23.3)$$

There are some force which do no work during the motion of a rigid body.

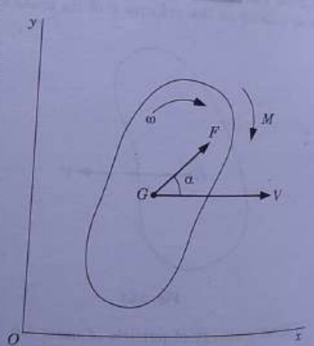


Fig. 23.3

The forces which do no work are:

1. The reaction R of the pin on the body about which the body is rotating. [Fig. 23.4 (a)].
2. The reaction R of the surface on the body when it moves horizontally and the surface is frictionless [Fig. 23.4 (b)].
3. The friction force F acting at the point of contact when the body rolls without sliding on a fixed horizontal or inclined surface [Fig. 23.4(c)].

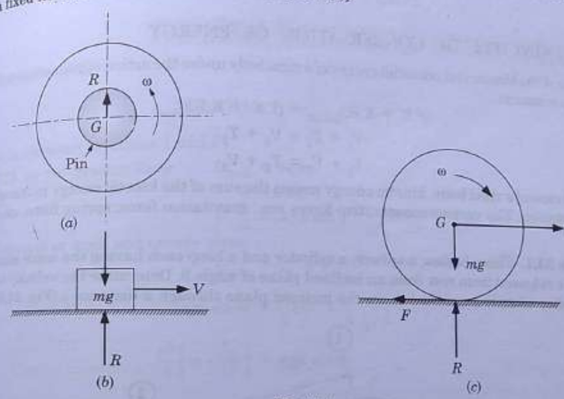


Fig. 23.4

23.3 PRINCIPLE OF WORK AND ENERGY FOR A RIGID BODY

Mathematically this principle for a rigid body can be written in the same form as that for a particle,

$$U_{1-2} = (T_2 - T_1) \quad ...(23.4)$$

Work done = Change in kinetic energy

In the case of rigid body,

$$\text{Work done} = \begin{Bmatrix}\text{Work resulting due to the linear} \\ \text{displacement from initial} \\ \text{position } s_1 \text{ to findal position } s_2\end{Bmatrix} + \begin{Bmatrix}\text{Work resulting due to the} \\ \text{angular displacement from} \\ \text{position } \theta_1 \text{ to position } \theta_2\end{Bmatrix}$$

$$U_{1-2} = \int_{s_1}^{s_2}(F\cos\alpha)ds + \int_{\theta_1}^{\theta_2}Md\theta \quad ...(23.5)$$

$$U_{1-2} = Fs + M\theta \quad ...(23.6)$$

(When force F and moment M are constant.)

$(T_2 - T_1)$ is the change in kinetic energy of the body resulting from translation and rotation between the initial and final positions.

$$\text{Change in K.E.} = (T_2 - T_1)$$

$$= \underbrace{\left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right)}_{\text{Change in K.E. in translation}} + \underbrace{\left(\frac{1}{2}I\omega_2^2 - I\omega_1^2\right)}_{\text{Change in K.E. in rotation}} \quad ...(23.7)$$

23.4 PRINCIPLE OF CONSERVATION OF ENERGY

The sum of the kinetic and potential energy of a rigid body under the action of conservative forces remains constant.

$$(\text{P.E.} + \text{K.E.})_{\text{Initial}} = (\text{P.E.} + \text{K.E.})_{\text{Final}}$$

$$V_1 + T_1 = V_2 + T_2$$

$$T_1 + V_1 = T_2 + V_2 \quad ...(23.8)$$

In the case of a rigid body, kinetic energy means the sum of the kinetic energy in translation and in rotation. The various conservative forces are : gravitation force, spring force, elasticity force

Example 23.1. Three bodies, a sphere, a cylinder and a hoop each having the same mass and radius are released from rest from an inclined plane of angle θ. Determine the velocity of each of the bodies after it has rolled down the inclined plane through a distance s (Fig. 23.5).

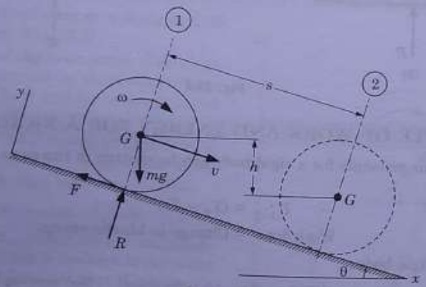


Fig. 23.5

Solution: Consider a body of mass m radius r and moment of inertia I_G. Let it roll from position 1 to position 2.

Forces involved and work done :

(i) gravitational force mg

(ii) friction force F (When a body rolls without sliding, friction force does no work)

(iii) normal reaction R (It does no work as it acts normal to the direction of motion.)

Let the mass centre acquire a linear velocity v when at position 2 and the angular velocity be ω.

Since the body rolls without slipping.

$$v = \omega r$$

Change in K.E. between positions 1 and 2,

$$T_1 = 0$$

$$T_2 = \frac{1}{2}mv^2 + \frac{1}{2}I_G\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}I_G\left(\frac{v}{r}\right)^2$$

Change in K.E.

$$(T_2 - T_1) = \frac{1}{2}mv^2 + \frac{1}{2}I_G\left(\frac{v}{r}\right)^2$$

W.D. between positions 1 and 2

W.D. by the gravity force $\quad U_{1-2} = mgh$

as, $\quad h = s\sin\theta$

so, $\quad U_{1-2} = mg\,s\sin\theta$

Principle of work and energy gives

$$U_{1-2} = T_2 - T_1$$

$$mg(s\sin\theta) = \frac{1}{2}mv^2 + \frac{I_G}{2}\left(\frac{v}{r}\right)^2$$

$$\frac{v^2}{2}\left(m + \frac{I_G}{r^2}\right) = mgs\sin\theta$$

$$v = \sqrt{\frac{2gs\sin\theta}{1 + \frac{I_G}{mr^2}}}$$

Substituting the appropriate values for the moment of inertia, we get

Body	*Moment of inertia* (I_G)	*Velocity*
Sphere	$I_G = \frac{2}{5}mr^2$	$v = 0.845\sqrt{2gs\sin\theta}$
Cylinder	$I_G = \frac{1}{2}mr^2$	$v = 0.816\sqrt{2gs\sin\theta}$
Hoop	$I_G = mr^2$	$v = 0.707\sqrt{2gs\sin\theta}$

Comments on the results

1. A body of larger moment of inertia possesses more K.E. in rotation as compared to the K.E. in translation. The sum of the two kinetic energies, however, remains the same in the three cases.
2. If all the three bodies were to slide down the inclined plane (without rolling) under frictionless conditions, the velocity attained by each body will be the same and equal to $\sqrt{2gs\sin\theta}$.

Example 23.2. A roller of mass $m = 600$ kg and radius $r = 0.25$ m is pushed with a constant force $P = 850$ N on a rough horizontal plane as shown in Fig. 23.6. If the roller starts from rest and rolls without slipping, find the distance required to be rolled if it is to acquire a velocity of 3 m/s.

Solution:

$$m = 600 \text{ kg}$$
$$r = 0.25 \text{ m}$$
$$P = 850 \text{ N}, \; v = 3 \text{ m/s}$$

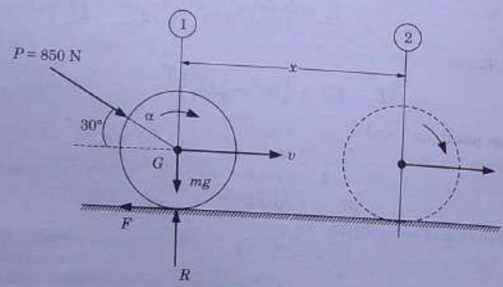


Fig. 23.6

Force involved and work done :

(i) gravity force mg (It does no work as it acts normal to the direction of motion)

(ii) normal reaction R (It does no work)

(iii) friction force F (In the case of a body that rolls without sliding, it does no work)

(iv) external force P.

Change in K.E. between position 1 and 2.

Position 1, $\quad$ K.E. $= 0$, $T_1 = 0$

Position 2, $\quad v = 3$ m/s, $\omega = \dfrac{v}{r}$

$$\text{K.E., } T_2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$T_2 = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mr^2}{2}\right)\left(\frac{v}{r}\right)^2$$

$$T_2 = \frac{3}{4}mv^2$$

Change in K.E.

$$(T_2 - T_1) = \frac{3}{4}mv^2$$

W.D. between positions 1 and 2

$$U_{1-2} = (P\cos\theta)x = (850\cos 30°)x \quad (x \text{ is distance travelled})$$

Principle of work and energy gives

$$U_{1-2} = (T_2 - T_1)$$

$$(850\cos 30°)x = \frac{3}{4}mv^2$$

$$(850 \times 0.866)x = \frac{3}{4}600 \times (3)^2$$

$$x = 5.5 \text{ m} \quad \textit{Ans.}$$

Example 23.3. A uniform bar of mass m and length L hangs from a frictionless hinge. It is released from the horizontal position AB. Find the angular velocity and linear velocity of the centre of mass of the bar when it is in vertical position (Fig. 23.7)

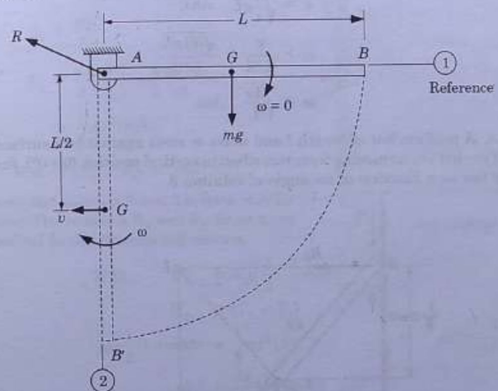


Fig. 23.7

Solution: As only conservative forces are involved, the principle of conservation of energy can be applied.

Position 1. $\quad v = 0$, $\omega = 0$, P.E. $= 0$, K.E. $= 0$

$$E_1 = 0$$

Position 2. Let the linear velocity of the mass centre be v and the angular velocity be ω.

$$\text{P.E.} = -mgh = -\frac{mgL}{2}$$

$$\text{K.E.} = \frac{1}{2}mv^2 + \frac{1}{2}I_G\omega^2, \; \omega = \frac{v}{L/2}, \; I_G = \frac{ML^2}{12}$$

$$\text{K.E.} = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mL^2}{12}\left(\frac{2v}{L}\right)^2$$

Total energy,

$$E_2 = -\frac{1}{2}mgL + \frac{1}{2}mv^2 + \frac{1}{6}mv^2$$

The principle of conservation of energy gives,

$$E_1 = E_2$$

$$0 = -\frac{mgL}{2} + \frac{1}{2}mv^2 + \frac{1}{6}mv^2$$

$$\frac{2}{3}mv^2 = \frac{mgL}{2}$$

$$v^2 = \frac{3}{4}gL$$

$$v = \sqrt{\frac{3}{4}gL} \quad \textit{Ans.}$$

$$\omega = \frac{v}{L/2} = \frac{\sqrt{3/4\,gL}}{L/2}$$

$$\omega = \sqrt{\frac{3g}{L}} \quad \textit{Ans.}$$

Example 23.4. A uniform bar of length l and mass m rests against two surface as shown in Fig. 23.8 (*a*). If the bar starts moving from rest when in vertical position ($\theta = 0°$), find the angular velocity of the bar as a function of its angle of rotation θ.

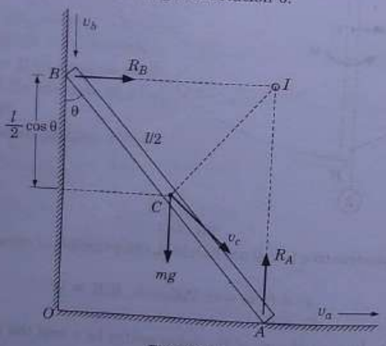


Fig. 23.8(*a*)

Solution: Let the bar make an angle θ with the vertical at any instant. The end A can move along OA and the end B along OB. The instantaneous centre of rotation I of the bar is located as shown in Fig. 23.8 (*a*). From the geometry of the figure it is clear that

$$BC = CA = OC = IC = l/2$$

Let the angular velocity of the bar be $\dot{\theta}$.

The centre of gravity C of the bar moves in a circular path with I as centre.

Its velocity

$$v_c = \dot{\theta}(IC) = \dot{\theta}(l/2)$$

To apply the principle of work and energy let us calculate the change in K.E. and W.D. Kinetic energy in position 1 when the bar is vertical (*i.e.*, $\theta = 0°$) is zero,

$$T_1 = 0$$

Kinetic energy in the position 2 when the bar makes an angle θ with the vertical is,

$$T_2 = \text{K.E. due to translation} + \text{K.E. due to rotation}$$

$$T_2 = \frac{m(v_c)^2}{2} + \frac{1}{2}I\omega^2$$

$$T_2 = \frac{m}{2}\left(\frac{l\dot{\theta}}{2}\right)^2 + \frac{1}{2}I_G(\dot{\theta})^2$$

$$I_G = \frac{ml^2}{12}$$

$$\therefore \quad T_2 = \frac{m}{2}\left(\frac{l\dot{\theta}}{2}\right)^2 + \frac{1}{2}\frac{ml^2}{12}(\dot{\theta})^2$$

$$= \frac{1}{6}ml^2(\dot{\theta})^2$$

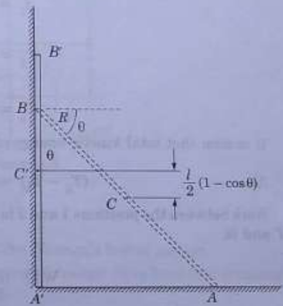


Fig. 23.8 (*b*)

Change in $\text{K.E.} = (T_2 - T_1) = \frac{1}{6}ml^2(\dot{\theta})^2$

Work between the position 1 and 2 is done only by the gravity force. The reaction R_A and R_B do no work as they act normal to the direction of motion.

$$\text{W.D.} \quad U_{1-2} = mg\left(\frac{l}{2} - \frac{l}{2}\cos\theta\right)$$

$$U_{1-2} = (T_2 - T_1)$$

$$mg\frac{l}{2}(1-\cos\theta) = \frac{1}{6}ml^2(\dot{\theta})^2$$

$$(\dot{\theta})^2 = \frac{3g}{l}(1-\cos\theta)$$

$$\dot{\theta} = \sqrt{\frac{3g}{l}(1-\cos\theta)} \quad \textit{Ans.}$$

Alternative Method. The plane motion of a rigid body can be considered to be a combination of translation and rotation. When applying the principle of work and energy, kinetic energy of rigid body thus consists of kinetic energy in translation and in rotation.

Also, a rigid body in plane motion at any given instant of time appears to be rotating about a certain point called instantaneous centre of rotation. Therefore, if kinetic energy of rotation of the body is calculated about its instantaneous centre of rotation (the moment of inertia of the body is to be taken about its instantaneous centre of rotation), then kinetic energy in translation of the body is not to be considered.

In this problems 'I' is the instantaneous centre of rotation of the bar AB.

Kinetic energy in position 1 $T_1 = 0$

kinetic energy in position 2, when bar makes an angle θ with the horizontal is the kinetic energy in rotation of the bar only about its intantaneous centre I and is,

$$T_2 = \frac{1}{2} I_I \omega^2 = \frac{1}{2} I_I (\dot{\theta})^2$$

where, I_I is the moment of interia of the bar about its instantaneous centre I.

$$I_1 = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{1}{3} ml^2$$

$$T_2 = \frac{1}{2}\left(\frac{1}{3} ml^2\right)(\dot{\theta})^2$$

$$T_2 = \frac{1}{6} ml^2 (\dot{\theta})^2$$

It is seen that total kinetic energy of the bar T_2 is same as calculated earlier.

Also $$(T_2 - T_1) = \frac{1}{6} ml^2 (\dot{\theta})^2,$$

Work between the positions 1 and 2 is done only by the gravity force in rotating the bar about 'I' and is,

$$\text{W.D. } U_{1-2} = \int_0^{\theta} M d\theta \qquad \left[\begin{matrix}\text{Analogous} \\ \text{to } \int F\,ds\end{matrix}\right]$$

where, M is the moment of gravity force about I.

$$U_{1-2} = \int_0^{\theta} (mg)\left(\frac{l}{2}\sin\theta\right) d\theta$$

$$= \frac{mgl}{2}[-\cos\theta]_0^{\theta}$$

$$U_{1-2} = \frac{mgl}{2}(1-\cos\theta)$$

$$U_{1-2} = \frac{mgl}{2}(1-\cos\theta)$$

Also work done is same as calculated earlier.

Example 23.5. A circular cylinder of mass m and radius r rolls without slipping on a circular path of radius R (Fig. 23.9). Find the period of oscillations of a cylinder when it is displaced slightly from the equilibrium position.

Solution: Let O be the centre of the circular path of radius R. Let the cylinder be released from rest when in position 1. Let the line OG form an angle θ with the vertical at that instant.

The position 2 is called as the equilibrium position. The centre of mass G of the cylinder shall roll in a circle of radius $(R - r)$ when displaced from the equilibrium position.

If the angular velocity of the line OG be $\dot{\theta}$,

then the linear velocity v of the mass centre G is $= [R - r]\dot{\theta}$

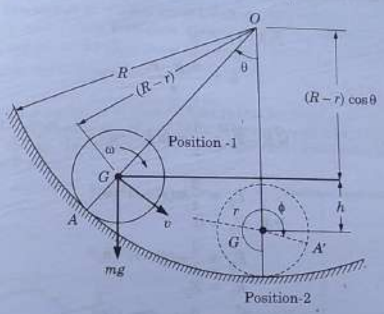


Fig. 23.9

We can solve this problem using two methods:

(*a*) The principle of conservation of energy and (*b*) The Newton's law of motion.

In this chapter we shall use the principle of conservation of energy. Note here that frictional force does *no work* when a body rolls without slipping.

Change in K.E. between position 1 and 2.

Position 1. $$\text{K.E.} = T_1 = 0$$

$$\text{P.E., } V_1 = mgh = mg[(R - r) - (R - r)\cos\theta]$$

$$V_1 = mg(R - r)(1 - \cos\theta)$$

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

For small values of θ, $$\sin\theta \approx \theta$$

Therefore, $$1 - \cos = \frac{\theta^2}{2}$$

$$V_1 = mg(R - r)\frac{\theta^2}{2}$$

Total energy $$E_1 = mg(R - r)\frac{\theta^2}{2}$$

Position 2, $$\text{K.E., } T_2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where, ω *is the angular velocity* ($\dot{\phi}$) *of the cylinder about its mass centre* G and v is the linear velocity of the mass centre and is

$$v = \dot{\theta}(R - r)$$

($\dot{\theta}$ *being the angular velocity of the line OG*)

As the cylinder rolls without slipping

$$v = \omega r$$
$$\dot{\theta}(R-r) = \dot{\phi} r$$
$$\dot{\phi} = \frac{\dot{\theta}(R-r)}{r}$$

$$\text{K.E., } T_2 = \frac{1}{2}m[\dot{\theta}(R-r)]^2 + \frac{1}{2}\left(\frac{mr^2}{2}\right)\left(\frac{\dot{\theta}(R-r)}{r}\right)^2$$
$$T_2 = \frac{3}{4}m(R-r)^2(\dot{\theta})^2$$
$$\text{P.E. } V_2 = 0$$

Total energy, $\quad E_2 = \frac{3}{4}m(R-r)^2(\dot{\theta})^2$

Equating, $\quad E_1 = E_2$

$$mg(R-r)\frac{\theta^2}{2} = \frac{3}{4}(R-r)^2(\dot{\theta})^2$$
$$\frac{3}{2}(R-r)(\dot{\theta})^2 - g\theta^2 = 0$$

Differentiating with respect to time

$$\frac{3}{2}(R-r)2(\dot{\theta})(\ddot{\theta}) - g(2\dot{\theta}\theta) = 0$$
$$\frac{3}{2}(R-r)(\ddot{\theta}) - g\theta = 0$$
$$\ddot{\theta} - \frac{2}{3}\frac{g}{(R-r)}\theta = 0$$

(Minus sign appears, as when θ decreases ϕ increases.)

Comparing the above equation with

$$\ddot{\theta} + p^2\theta = 0$$

We observe that the motion of the cylinder is S.H.M.

Where, $\quad t = \frac{2\pi}{p},\ p^2 = \frac{2}{3}\frac{g}{(R-r)} = 2\pi\sqrt{\frac{3}{2}\frac{(R-r)}{g}}$ *Ans.*

Note. The only force that does work is the gravity force so we can apply the principle of conservation of energy in this problem.

As the line OG rotates through an angle θ about O, the cylinder rotates through an angle ϕ about its mass centre G. Therefore,

$$\theta(R-r) = \phi r$$

Differentiating $\quad \dot{\theta}(R-r) = \dot{\phi} r$

$$\dot{\phi} = \frac{\dot{\theta}(R-r)}{r}$$

which as same as derived earlier.

PROBLEMS

23.1. Determine the kinetic energy of : (*a*) a uniform rigid circular cylinder of mass m and radius r rolling without slipping on a horizontal surface with its mass centre moving with a velocity of v_c.

(*b*) A uniform sphere of the same mass and radius and having the same velocity of the mass centre that of the cylinder.

$$\left[\frac{3}{4}mv_c^2, \frac{7}{10}mv_c^2\right]$$

23.2. A flywheel rigidly attached to a shaft of 4 cm radius is placed on two parallel rails fixed at an angle $\alpha = 15°$ to the horizontal (Fig. P.23.2). When released from rest, it attains a velocity of 16 cm/s after moving a distance of 1.5 along the rails. Determine the controidal radius of gyration of the system. [68.9 cm]

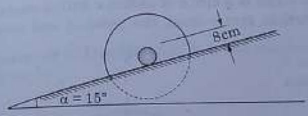


Fig. P.23.2

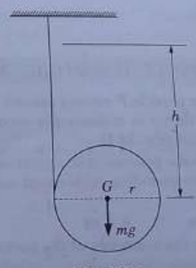


Fig. P.23.3

23.3. A right circular cylinder of mass m and radius r is suspended by a cord wound round its surface. The cylinder is allowed to fall from rest through a vertical distance h. Find the velocity of its centre of gravity (Fig. P.23.3).

$$\left[v_g = \sqrt{\frac{4gh}{3}}\right]$$

23.4. Two identical bars AB and AC of equal mass m and lengths l are hinged at A and rest on a smooth horizontal surface as shown in Fig. P.23.4. The bars are confined to move in a vertical plane and are released from rest in the position shown. Find the angular velocity of the bars when the hinge A strikes the surface. Assume friction-less conditions.

If $l = 1$ m and $\theta = 45°$, with what velocity the hinge A hits the surface?

$$\left[\omega = \sqrt{\frac{3g}{l}\sin\theta}, 4.56 \text{ m/s}\right]$$

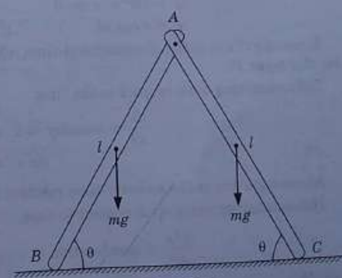


Fig. P.23.4

24

CHAPTER Mechanical Vibrations

24.1 SIMPLE HARMONIC MOTION

Consider a particle P moving around the circumference of a circle of radius r with a constant angular velocity ω radians per second. The particle starts from the point A and moves anticlockwise (Fig. 24.1).

Let P be the position of the particle after a time t and θ be corresponding angle swept out by the radius vector OA.

Therefore, $\theta = \omega t$

Let P' be the projection of the particle P on the x-axis. The x-axis being chosen to lie along the diameter $A'OA$ of the circle. As the particle P moves along the circumference of the circle its projection P' oscillates along the diameter AOA'.

The displacement x of the point P', at any time t, from the mean position O on the x-axis is given by,

$$x = OP' = r\cos\theta$$

$$x = r\cos\omega t \qquad ...(24.1)$$

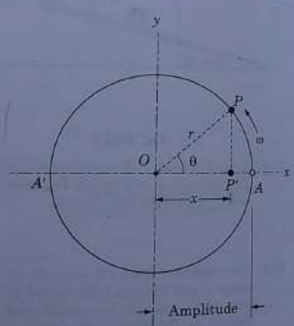


Fig. 24.1

Equating (24.1) is the displacement-time relation for the point P'.

Differentiating with respect to the time

$$\left.\begin{aligned}\frac{dx}{dt} = \text{velocity} = \dot{x} &= -r\omega\sin\omega t\\ \text{or } \dot{x} &= -r\omega\sin\theta\end{aligned}\right] \qquad ...(24.2)$$

Above equation is the velocity-time relation for the point P'.

Differentiating again with respect to time

$$\left.\begin{aligned}\frac{d^2x}{dt^2} = \text{acceleration} = \ddot{x} &= -r\omega^2\cos\omega t\\ \text{or } \ddot{x} &= -r\omega^2\cos\theta\end{aligned}\right] \qquad ...(24.3)$$

Above equation is the acceleration-time relationship for the point P'.

Substituting,

$$r\cos\omega t = x$$

$$\ddot{x} = -\omega^2 x$$

$$\text{or, } \ddot{x} \propto (-x) \qquad ...(24.4)$$

From the above equation it is seen that the acceleration ($\ddot{x}$) *of the point* P' *is proportional to its displacement* x *from the mean position* O *and is directed towards it.* That is, for a positive displacement x, the acceleration $\ddot{x}$ is negative.

Such a type of motion is called a simple harmonic motion (S.H.M.) and is an important type of rectilinear motion which is *repetitive or periodic in nature*. Some terms that are used, are defined below.

Amplitude of Oscillation. It is the maximum displacement of the point P' from the mean position and is equal to the radius r of the circle in this case.

Time Period. The motion of the point P' is periodic and repeats itself after a time period t give by,

$$t = \frac{2\pi}{\omega}$$

Frequency of Oscillation. It is the number of oscillations of the point P' in one second and is given by

$$f = \frac{1}{t} = \frac{\omega}{2\pi}$$

which gives, $\omega = 2\pi f$

The velocity and acceleration of the point P' *for a given angular position* θ are given by the equations (24.2) and (24.3).

The relations for the velocity and acceleration of the point P' *when it is at a given distance* x from the mean position can also be determined as below

$$\text{velocity} = \frac{dx}{dt} = \dot{x} = -r\omega\sin\omega t$$

and, $x = r\cos\omega t$, so, $\cos\omega t = \dfrac{x}{r}$

$$\sin\omega t = \sqrt{1-\cos^2\omega t} = \sqrt{1-\frac{x^2}{r^2}}$$

velocity,

$$\dot{x} = -r\omega\sqrt{1-\frac{x^2}{r^2}}$$

$$\dot{x} = -\omega\sqrt{r^2-x^2} \qquad ...(24.5)$$

acceleration

$$\ddot{x} = -r\omega^2\cos\omega t$$

$$\cos\omega t = \frac{x}{r}$$

$$\ddot{x} = -r\omega^2\frac{x}{r}$$

$$\ddot{x} = -\omega^2 x \qquad ...(24.6)$$

Displacement, velocity and acceleration of the point P' with respect to time (and θ) are plotted as shown (Fig. 24.2).

The velocity of the point P' varies sinusoidally. The magnitude of the velocity is maximum

at $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

that is, at the centre O, and $|v|_{max} = r\omega$

The minimum velocity is,

$$|v|_{min} = 0$$

at $\theta = 0$ and π

that is, at the extremities A and A'.

The acceleration of the point P' has a maximum value of,

$$|\text{acceleration}|_{max} = r\omega^2$$

at the extremities A and A' and

$$|\text{acceleration}|_{min} = 0$$

at the centre.

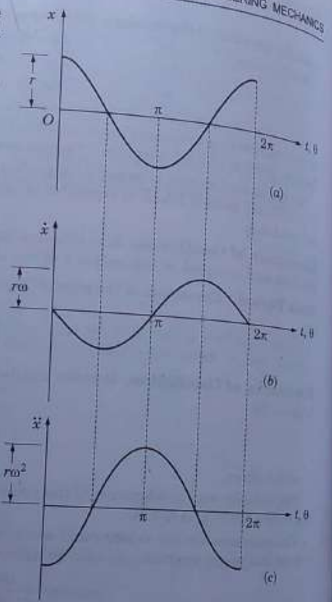


Fig. 24.2

Example 24.1. A particle per-forming S.H.M. has a frequency of 10 oscillations per minute. At a distance of 8 cm from the mean position its velocity is 3/5th of the maximum velocity. Find (*a*) the amplitude of oscillation, (*b*) the maximum acceleration, (*c*) the velocity of the particle when it is a distance of 5 cm from the mean position.

Solution: $f = \frac{10}{60} = \frac{1}{6}$ oscillation/second

$$\omega = 2\pi f = 2\pi \times \frac{1}{6} = \frac{\pi}{3}$$

$$\text{velocity} = \dot{x} = \omega\sqrt{r^2 - x^2}$$

when $x = 8$ cm, velocity $= \frac{3}{5}(v_{max})$

$$\frac{3}{5}(v_{max}) = \omega\sqrt{r^2 - (0.08)^2}$$

$$v_{max} = \omega r$$

$$\frac{3}{5}\omega r = \omega\sqrt{r^2 - (0.08)^2}$$

$$9r^2 = 25r^2 - 25(0.0064)$$

$$16r^2 = 0.16 \text{ or } r = 0.1 \text{ m}$$

Amplitude $r = \frac{1}{10}$ m or $r = 10.0$ cm *Ans.*

Maximum acceleration $\omega^2 r = \left(\frac{\pi}{3}\right)^2 \times 10.0$

$$(\text{acceleration})_{max} = 10.97 \text{ cm/s}^2 \quad \textit{Ans.}$$

Velocity when $x = 5$ cm,

$$x = \omega\sqrt{r^2 - x^2} = \frac{\pi}{3}\sqrt{(0.1)^2 - (0.05)^2}$$

$$\text{velocity} = 9.07 \text{ cm/s} \quad \textit{Ans.}$$

Example 24.2. A particle is performing a S.H.M. When it is at distances of 10.0 cm and 20.0 cm from the mean position its velocities are 1.2 m/s and 0.8 m/s respectively. Find (*a*) the amplitude of oscillations, (*b*) time period of oscillations, (*c*) its maximum velocity and (*d*) its maximum acceleration.

Solution: Velocity at a distance x from the mean position is

$$\dot{x} = \omega\sqrt{r^2 - x^2}$$

where $x = 10 \text{ cm} = 0.1 \text{ m}, \ \dot{x} = 1.2 \text{ m/s}$

and $x = 20 \text{ cm} = 0.2 \text{ cm}, \ \dot{x} = 0.8 \text{ m/s}$

Substituting $1.2 = \omega\sqrt{r^2 - (0.1)^2}$ ...(*i*)

$0.8 = \omega\sqrt{r^2 - (0.2)^2}$ ...(*ii*)

Dividing (*i*) by (*ii*) $\frac{1.2}{0.8} = \frac{\sqrt{r^2 - (0.1)^2}}{\sqrt{r^2 - (0.2)^2}}$

$$\left(\frac{3}{2}\right)^2 = \frac{r^2 - 0.01}{r^2 - 0.04}$$

$$9r^2 - 0.36 = 4r^2 - 0.04$$

$$5r^2 = 0.32, \ r = 0.252 \text{ m} \quad \textit{Ans.}$$

Amplitude of oscillation, $r = 25.2$ cm

Substituting for r in (*i*)

$$1.2 = \omega\sqrt{(0.252)^2 - (0.1)^2}$$

Solving, $\omega = 4.78$ radians/second

Time period $t = \frac{2\pi}{\omega} = \frac{2\pi}{4.78}$

$$t = 6.3 \text{ seconds}$$

$$v_{max} = r\omega = 0.252 \times 4.78$$

$$v_{max} = 1.2 \text{ m/s} \quad \textit{Ans.}$$

Maximum acceleration $r\omega^2 = (0.252)(4.78)^2$

Maximum acceleration $= 5.75 \text{ m/s}^2$ *Ans.*

24.2 FREE VIBRATIONS (WITHOUT DAMPING)

Consider a body of weight W suspended from a spring and constrained to move in a vertical direction. The spring has some unstreched length (without weight W) and it extends by a length δ_{st} when the weight W is suspended (Fig. 24.3).

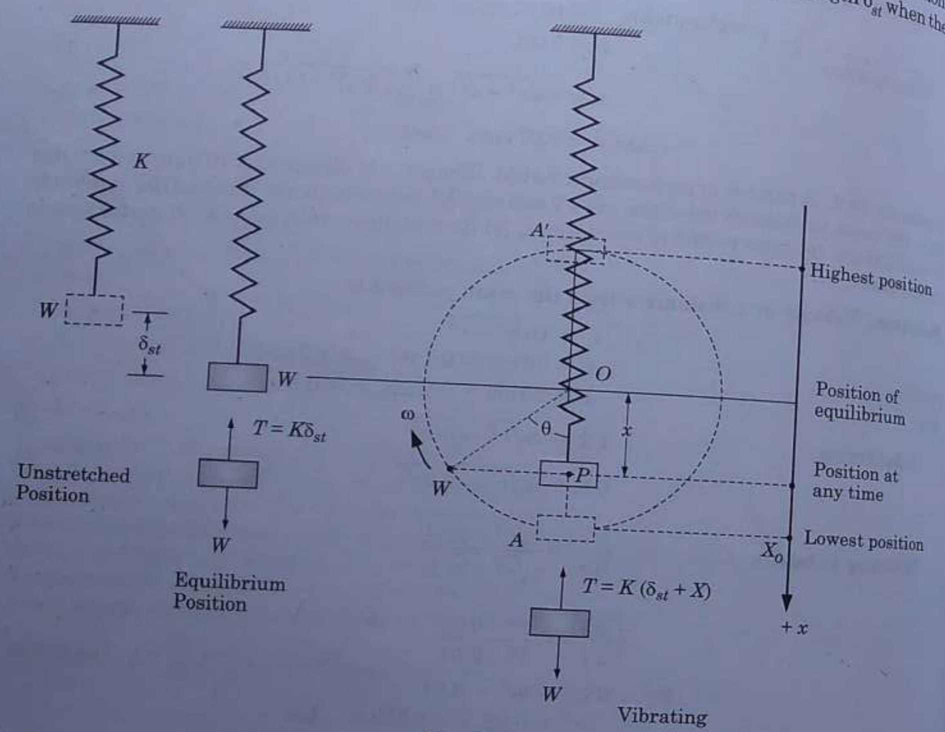


Fig. 24.3

$$\delta_{st} = \frac{w}{k}$$

where, k is the stiffness of the spring or spring constant.

The position O occupied by the weight W is called the position of static equilibrium or the mean position. All displacements of W are measured with respect to this position; taken positive downwards.

If the body is displaced from its position of static equilibrium and then released, the body shall move up and down about its means position or shall be freely vibrating performing S.H.M.

Let P be the position of the body at any time t and x be the corresponding displacement from the mean position.

Tension, T in the spring $= k(\delta_{st} + x)$

Writing the equation of motion of the body

$F = ma$:
$$\frac{W}{g}\ddot{x} = W - k(\delta_{st} + x)$$

As,
$$k\delta_{st} = W$$

$$\frac{W}{g}\ddot{x} = W - W - kx$$

$$\frac{W}{g}\ddot{x} + kx = 0$$

$$\ddot{x} + \left(\frac{kg}{W}\right)x = 0$$

Let
$$\frac{kg}{W} = \omega^2$$

$$\ddot{x} + \omega^2 x = 0$$

$$\ddot{x} = -\omega^2 x \qquad ...(24.7)$$

From the above equation it can be observed that the acceleration of the body is proportional to its displacement (x) from the mean position and is directed toward it, which is a characteristic of the S.H.M. As,

$$\omega^2 = \frac{kg}{W} \text{ or } \omega = \sqrt{\frac{kg}{W}}$$

the time period
$$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{W}{kg}}$$

The significance of the angular velocity ω now can be explained. Imagine a weight W to be rotating with a constant angular velocity $\omega = \sqrt{kg/W}$ in a circle having a radius equal to the amplitude of the vertical vibrations. The vertical oscillations (or S.H.M.) of the body W then can be thought to be the motion of the projection P of this imaginary rotating weight W on the diameter AA'.

The angle $\theta = \omega t$ is measured with respect to the radius vector OA.

Equation (24.7)

$$\ddot{x} + \omega^2 x = 0 \text{ or } \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad ...(24.8)$$

is the acceleration-time relation for the body. To find the displacement time relation we have to solve the above differential equation.

The general solution of the equation

$$\dot{x} + \omega^2 x = 0, \text{ where, } \omega = \sqrt{\frac{kg}{W}}$$

is given by
$$x = C_1 \cos \omega t + C_2 \sin \omega t \qquad ..(24.9)$$

The constants C_1 and C_2 are to be determine from the initial conditions of motion. That is, by the displacement, velocity and acceleration of the body at the time when the motion began ($t = 0$).

Let us assume that the body W was given an initial displacement x_0 from the position of the static equilibrium and then released with zero velocity.

when $\quad t = 0,$

$$x = x_0 \qquad ...(i)$$

and $\quad \dot{x} = 0 \qquad ...(ii)$

Substituting the condition (*i*) in equation (24.9)

$$x_0 = C_1 \cos 0° + C_2 \sin 0°$$

$$C_1 = x_0$$

Differentiating the equation (24.9) and substituting the condition (*ii*)

$$\dot{x} = -C_1\,\omega \sin \omega t + C_2\,\omega \cos \omega t$$

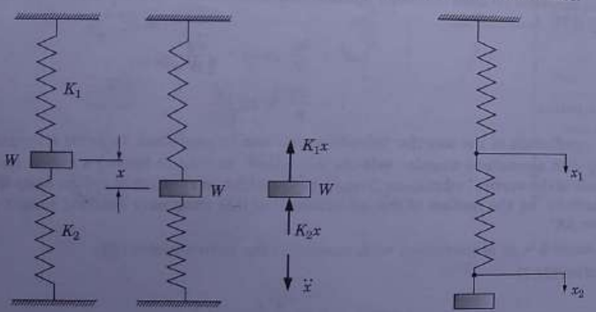
$$0 = -C_1\,\omega \sin 0° + C_2\,\omega \cos 0°$$

$$C_2 = 0.$$

The solution for the assumed initial conditions, therefore, is given by

$$x = x_0 \cos \omega t \qquad ...(24.10)$$

Example 24.3. A weight W is attached to two springs of stiffness K_1 and K_2 connected in two different ways as shown (Fig. 24.4). Find the time period of vibrations in each case.



(*a*) Parallel Arrangement (*b*) Series Arrangement

Fig. 24.4

Solution: Parallel Arrangement : Let the weight W be displaced downwards by a distance x.

The upper spring extends by x and the lower spring shortens by x.

Tension in the upper spring $= K_1x$

Compression in the lower spring $= K_2x$

Total unbalanced force acting on the weight W in the direction of motion, for a downward displacement x,

$$= -(K_1 + K_2)x$$

The differential equation of motion of the weight is

$\Sigma F_x = ma_x$:
$$\frac{W\ddot{x}}{g} = -(K_1 + K_2)x$$

$$\ddot{x} + \frac{(K_1 + K_2)g}{W}x = 0$$

$$\ddot{x} + \omega^2 x = 0$$

or,
$$\omega = \sqrt{\frac{(K_1 + K_2)}{W}g}$$

where,
$$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{W}{g(K_1 + K_2)}}$$

Series Arrangement : Let the weight W be given a downward displacement x. The supper spring extends by x_1 and lower spring by x_2.

The tension in both the springs is same $= T$

$$x = x_1 + x_2$$

But
$$x = \frac{T}{K_1} + \frac{T}{K_2} = T\left(\frac{1}{K_1} + \frac{1}{K_2}\right)$$

$$T = x\left(\frac{K_1K_2}{K_1 + K_2}\right)$$

The unbalanced force acting on the weight W in the direction of motion $= -T$.

Differential equation of motion of weight W is

$\Sigma F_x = ma_x$:
$$\frac{W}{g}\ddot{x} = -T = -x\left(\frac{K_1K_2}{K_1 + K_2}\right)$$

$$\frac{W\ddot{x}}{g} + \frac{K_1K_2}{(K_1 + K_2)}x = 0$$

$$\ddot{x} + \frac{g}{W}\frac{K_1K_2}{(K_1 + K_2)}x = 0$$

$$\ddot{x} + \omega^2 x = 0, \quad \omega^2 = \frac{g(K_1K_2)}{W(K_1 + K_2)}$$

$$t = \frac{2\pi}{\omega}$$

$$t = 2\pi\sqrt{\frac{(K_1 + K_2)W}{(K_1K_2)g}}$$

Equivalent Spring Constant

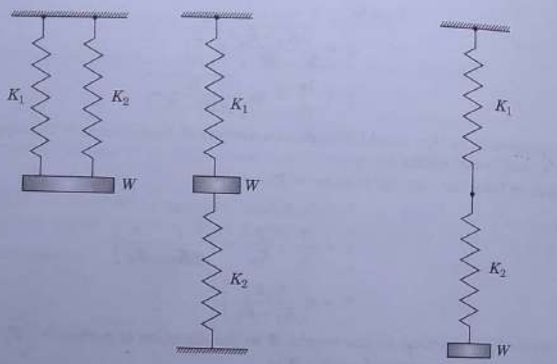
If a combination of springs is replaced by a single spring having an equivalent effect, that single is called as the equivalent spring and its spring constant as the equivalent spring constant.

When two springs are in parallel,

$$K_{\text{equivalent}} = K_1 + K_2$$

When two springs are in series,

$$K_{\text{equivalent}} = \frac{K_1 K_2}{K_1 + K_2}$$



(c) Parallel Arrangement (d) Series Arrangement

Fig. 24.4

Example 24.4 A ball of mass m is attached to a tightly stretched string of length L (Fig. 24.5). Assuming that the tension T in the string remains constant for small lateral displacements of the mass, determine the frequency of oscillation of the system.

Solution: When the ball is displaced by a distance x, the string will be slightly inclined but the tension T shall remain the same.

Unbalanced force acting on the ball in the direction of motion,

$$= -(T\sin\theta_1 + T\sin\theta_2)$$

Since the angles θ_1 and θ_2 are very small

$$\sin\theta_1 \approx \tan\theta_1 = \frac{x}{a}$$

$$\sin\theta_2 = \tan\theta_2 = \frac{x}{L-a}$$

Unbalanced horizontal force

$$= -T\left(\frac{x}{a} + \frac{x}{L-a}\right) = -\frac{xTL}{(L-a)a}$$

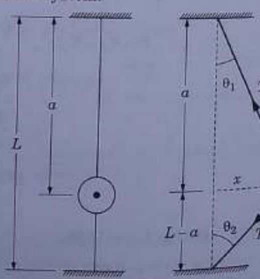


Fig. 24.5

The differential equation of motion of the mass can be written as,

$$\Sigma F_x = ma_x : \qquad m\ddot{x} = -\frac{xTL}{(L-a)a}$$

$$\ddot{x} + \frac{TL}{m(L-a)a}x = 0$$

$$\ddot{x} + \omega^2 x = 0, \ \omega = \sqrt{\frac{TL}{ma(L-a)}}$$

Frequency

$$f = \frac{\omega}{2\pi}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{TL}{ma(L-a)}} \quad \textit{Ans.}$$

Example 24.5. A weight of 50 N suspended from a spring vibrates vertically with an amplitude of 7.5 cm and a frequency of 1 oscillation per second. Find (*a*) the stiffness of the spring, (*b*) the maximum tension induced in the spring, (*c*) the maximum velocity of the weight.

Solution: (*a*)

$$W = 50\text{ N}, f = 1$$

$$x_0 = 7.5\text{ cm} = 0.075\text{ m}$$

$$\omega = 2\pi f = 2\pi \times 1 = 2\pi$$

As

$$\omega = \sqrt{\frac{Kg}{W}}$$

Therefore,

$$2\pi = \sqrt{\frac{K \times 9.81}{50}}$$

Or

$$K = \frac{(2\pi)^2 \times 50}{9.81}$$

$$K = 2.012\text{ N/cm} \quad \textit{Ans.}$$

(*b*) Tension

$$T = K \times \text{Extension}$$

The maximum tension in the spring occurs when the displacement of the weight W is maximum.

$$\left\{\begin{matrix}\text{Total extension of the}\\ \text{spring}\end{matrix}\right\} = \left\{\begin{matrix}\text{Static extension}\\ \text{due to the weight}\\ \text{of 50N}\end{matrix}\right\} + \left\{\begin{matrix}\text{Maximum displacement of}\\ \text{the weight } W \text{ from the}\\ \text{position of static equilibrium}\end{matrix}\right\}$$

$$= \delta_{st} + x_0$$

$$= \left(\frac{50}{2.012} + 7.5\right)\text{cm}$$

$$\text{Maximum tension} = K(\text{extension of the spring})$$

$$= 2.012\left(\frac{50}{2.012} + 7.5\right)$$

$$\text{Maximum tension} = 65.09\text{ N} \quad \textit{Ans.}$$

(*c*)

$$\text{Maximum velocity} = \omega x_0 = 2\pi \times 0.075$$

$$= 0.471\text{ m/s} \quad \textit{Ans.}$$

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Example 24.6. A solid cylinder of diameter d and height h and of uniform density ρ_m floats in a liquid of density ρ_l. Determine the frequency of vertical oscillation of the cylinder when disturbed (Fig. 24.6).

Solution: Let the cylinder be given a vertical downward displacement x causing an increase in the volume of the liquid displaced.

Weight of the volume of the liquid displaced, for a vertical displacement x of the cylinder, represents the unbalanced force (buoyancy force) in the direction of motion and is

$$= \left(\left(\frac{\pi}{4}d^2\right)x\rho_l g\right)$$

The differential equation of motion of the cylinder

$$F = ma: \quad m\ddot{x} = -\left(\frac{\pi}{4}d^2 x\rho_l g\right)$$

where, m is the mass of the cylinder

$$\left(\left(\frac{\pi}{4}d^2 h\right)\rho_m\right)\ddot{x} = -\left(\frac{\pi}{4}d^2 x\rho_l g\right)$$

$$\ddot{x} + \left(\frac{\rho_l}{\rho_m}\frac{g}{h}\right)x = 0$$

$$\ddot{x} + \omega^2 x = 0 \text{ where, } \omega^2 = \frac{\rho_l}{\rho_m}\frac{g}{h}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\rho_l}{\rho_m}\frac{g}{h}} \quad \textit{Ans.}$$

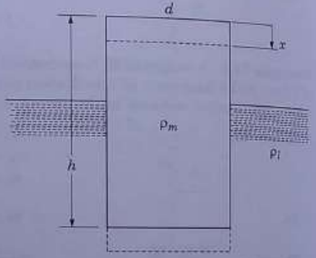


Fig. 24.6

Example 24.7. A solid cylinder of mass m and cross-sectional area A is suspended by a spring of stiffness k and hangs partially in water of density ρ as shown in Fig. 24.7. Find the period for small vertical oscillations of the cylinder. Neglect the inertia of water.

Solution: Let the cylinder be given a downward displacement x.

Increase in the tension of the spring (acting upwards) $= kx$

Increase in the buoyancy force (acting upwards) $= (Ax)\,\rho g$

Total unbalanced force acting in the direction of motion,

$$= -(kx + Ax\,\rho g)$$

Differential equation of motion of the cylinder

$$F = ma: \quad m\ddot{x} = -(kx + Ax\,\rho g)$$

$$\ddot{x} + \left(\frac{k + A\rho g}{m}\right)x = 0$$

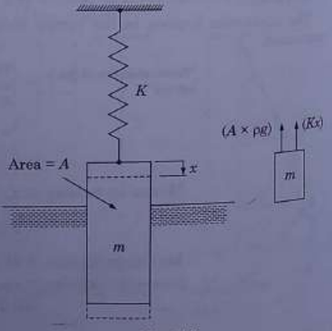


Fig. 24.7

$$\ddot{x} + \omega^2 x = 0, \; \omega^2 = \left(\frac{k + A\rho g}{m}\right)$$

$$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{(A\rho g + k)}}$$

$$t = 2\pi\sqrt{\frac{m}{(A\rho g + k)}} \quad \textit{Ans.}$$

Example 24.8. A uniform bar AB of weight W is placed horizontally on the top edges of two identical discs which rotate with equal angular speeds but in opposite directions as shown in Fig. 24.8. If the centre of gravity G of the bar is displaced from the middle plane OP and then released, shown that it will perform S.H.M. Find also the time period of oscillations in terms of the distance 'a' and the coefficient of friction μ.

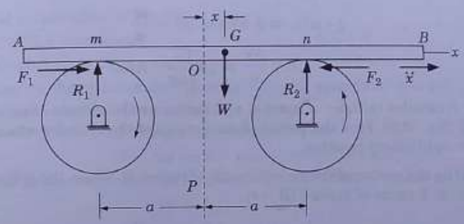


Fig. 24.8

Solution: Choose the middle point between the discs as the origin and let the x-axis be positive to the right.

The forces acting on the bar when its C.G. is displaced a distance x from the middle plane OP are :

(*i*) reactions R_1 and R_2

(*ii*) frictional forces F_1 and F_2 acting at the points of contact m and n

$$F_1 = \mu R_1 \text{ and } F_2 = \mu R_2$$

(*iii*) weight of the bar W

Taking moments about point m

$$\Sigma M_m = 0: \quad R_2(2a) - W(a + x) = 0$$

$$R_2 = \frac{W}{2a}(a + x)$$

Taking moments about point n

$$\Sigma M_n = 0: \quad W(a - x) - R_1(2a) = 0$$

$$R_1 = \frac{W}{2a}(a - x)$$

The unbalanced force acting on the bar in the direction of motion $= -(F_2 - F_1)$.

Differential equation of motion of the bar is

$\Sigma F_x = ma_x$:
$$\frac{W}{g}\ddot{x} = -(F_2 - F_1),\ F_1 = \mu R_1,\ F_2 = \mu R_2$$
$$\frac{W}{g}\ddot{x} = -\left(\mu\frac{W}{g}(a+x) - \mu\frac{W}{2a}(a-x)\right)$$
$$\frac{W}{g}\ddot{x} = -\mu\frac{W}{2a}(2x)$$
$$\frac{W}{g}\ddot{x} + \frac{\mu W}{a}x = 0$$
$$\ddot{x} + \frac{\mu g}{a}x = 0$$
$$\ddot{x} + \omega^2 x = 0 \text{ where, } \omega^2 = \frac{\mu g}{a}$$

Time period,
$$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a}{\mu g}} \quad Ans.$$

Example 24.9. A circular cylinder of mass m and radius r rolls without slipping on a circular path of radius R (Fig. 24.9). Find the period of oscillations of the cylinder when it is displaced slightly from the equilibrium position.

Solution: Let O be the centre of the circular path of radius R. When the cylinder is released from rest it rolls in a circle of radius $(R - r)$.

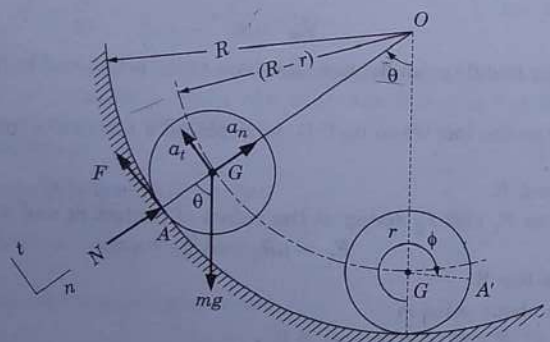


Fig. 24.9

We can define the position of the cylinder by ϕ which is the angle of rotation of the cylinder about its axis of symmetry (through G).

Also, we can define the position of the cylinder by θ which is the angle of rotation of its centre of gravity G about O or the angle of rotation of the line OG.

When the cylinder rolls without slipping θ and ϕ are related as
$$r\phi = (R - r)\theta$$
$$\phi = \frac{(R-r)\theta}{r} \qquad ...(i)$$
[Point A on the circumference of the cylinder will occupy the position A' after rolling to the equilibrium position]

Angular acceleration of the cylinder
$$\alpha = \frac{d^2\phi}{dt^2} = \ddot{\phi} = \frac{(R-r)}{r}\ddot{\theta} \qquad ...(ii)$$

Also the mass centre G is rotating about O with an angular velocity $\dot{\theta}$ in circle of radius $(R - r)$ hence,

The tangential acceleration of the mass centre G (a_t is positive with θ increasing) $\quad a_t = \ddot{\theta}(R - r)$

The normal acceleration of the mass centre $\quad G\ a_n = (\dot{\theta})^2(R - r)$

The force acting on the cylinder are : (*i*) weight mg (*ii*) reaction N and (*iii*) friction force F.

The equations of motion can be written as

$\Sigma F_t = ma_t$: $\quad F - mg\sin\theta = ma_t = m[\ddot{\theta}(R-r)] \qquad ...(iii)$

$\Sigma F_n = ma_n$: $\quad N - mg\cos\theta = ma_n = m[(\dot{\theta})^2(R-r)] \qquad ...(iv)$

As the cylinder is rotating about its own mass centre G with angular acceleration $\alpha = \ddot{\phi}$,

$M = I_G\alpha$: $\quad Fr = I_G\ddot{\phi} \qquad ...(v)$

where,
$$\ddot{\phi} = \frac{\ddot{\theta}(R-r)}{r}$$

But as θ decreases ϕ increases, therefore, equation (v) can be written as
$$Fr = -I_G\frac{\ddot{\theta}(R-r)}{r} \qquad ...(vi)$$

Eliminating F between (*iii*) and (*vi*) and substituting,
$$I_G = \frac{mr^2}{2}$$
$$-\frac{mr^2}{2}\frac{\ddot{\theta}(R-r)}{r^2} - mg\sin\theta = m\ddot{\theta}(R-r)$$

For small values of θ, $\sin\theta \approx \theta$
$$\frac{3}{2}\ddot{\theta}(R-r) + g\theta = 0$$
$$\ddot{\theta} + \frac{2}{3}\frac{g\theta}{(R-r)} = 0 \qquad ...(vii)$$

Comparing with
$$\ddot{\theta} + p^2\theta = 0 \text{ where, } p^2 = \frac{2}{3}\frac{g}{(R-r)}$$
$$t = \frac{2\pi}{p} \text{ or, } t = \sqrt{\frac{3}{2}\frac{(R-r)}{g}} \quad Ans.$$

PROBLEMS

24.1. A body performing S.H.M. has an amplitude of 1 m and the period of oscillation 2.05 s. Find the velocity and acceleration of the body 0.4 second after passing the mean position.
[v = 0.97 m/s, 9.39 m/s²]

24.2. A particle moving with simple harmonic motion has velocities of 8 m/s and 4 m/s when at the distance of 1 m and 2 m from the mean position. Determine (*a*) amplitude, (*b*) period, (*c*) maximum velocity, and (*d*) maximum acceleration of the particle.
[2.236 m, 157 s, 8.94 m/s, 35.78 m/s²]

24.3. A tray of mass m is mounted on three identical springs as in Fig. P.24.3. The period of vibration of the empty tray is 0.5 s. After placing a mass of 1.5 kg on the tray the period was observed to be 0.6 s. Find the mass of the tray and the stiffness of each spring. [3.41 kg, 179.3 N/m]

24.4. A block of mass 50 kg is supported by two springs as shown. The block is pulled 40 mm down from the position of equilibrium and then released. Find the period of vibration, maximum velocity of the block and the maximum acceleration of the block (Fig. P.24.4).
[0.907 s, 0.277 m/s, 1.92 m/s²]

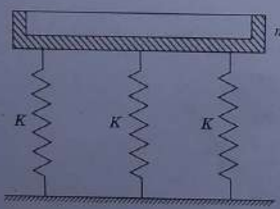


Fig. P.24.3

Fig. P.24.4

24.5. Two springs of stiffness 200 N/m are attached to a ball of weight 5 N as shown in Fig. P.24.5. If the ball is initially displaced by 2.5 cm to the left and released find the period of oscillation of the ball. Find also the ball when it passes through the middle position. [0.224 s, 70 cm/s]

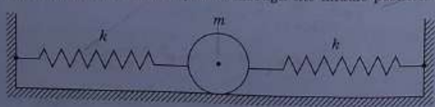
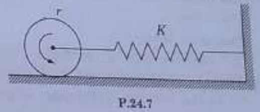


Fig. P.24.5

24.6. A horizontal platform vibrates up and down with S.H.M. of amplitude 2 mm. At what frequency will an object kept on the platform will just loose contact with the platform.
[**Hint** : $mg = m\omega^2 r$]
[11.4 Hz]

24.7. The roller of mass m and radius r is connected by a spring of stiffness k. If the roller is free to roll without slipping find its frequency of oscillations when displaced slightly (Fig. P.24.7).

$$\left[\omega = \sqrt{\frac{2k}{3m}}\right]$$



P.24.7

24.3 PENDULUM MOTION

Pendulums are devices that execute S.H.S. and their time period of oscillations can be varied by suitably changing their physical dimensions.

A body is said to perform S.H.M. if the equation of motion of the body can be expressed in form

$$\frac{d^2x}{dt^2} + p^2x = 0$$

Or

$$\frac{d^2\theta}{dt^2} + p^2\theta = 0$$

where, x or θ is the displacement of the body from the mean position at any time t and p is a constant.

The time period t then can be expressed as

$$t = \frac{2\pi}{p} \text{ and frequency } f = \frac{p}{2\pi}.$$

In a pendulum motion it is assumed that the friction at the support and the air resistance to motion is negligibly small. Then, in general, the various force involved are, (*i*) gravity force, (*ii*) elastic force and (*iii*) spring force.

Above forces are *conservative forces* therefore, the law of conservation of energy can be applied. The equations of motion however, can be obtained using the relationship.

$$\mathbf{F} = m\mathbf{a}$$

The various types of pendulums and pendulum motions are discussed in the following sections.

Simple Pendulum. It consists of a bob or sphere of mass m tied to the one end of an inextensible string of negligible mass, the other end of which is tied to a rigid support.

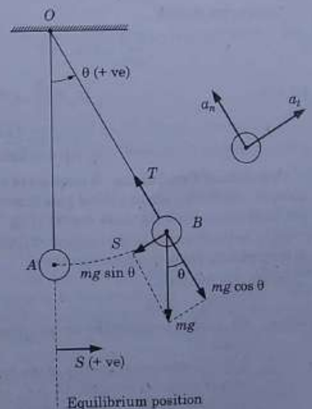


Fig. 24.10

In Fig. 24.10, OA is the equili-brium position and OB is the disturbed position.

The displacement s is measured from the equilibrium position along the arc AB. The mass is moving in circular path and experiences normal as well as tangential accelerations.

The equation of motion of the mass is

$\Sigma F_t = ma_t$: $\qquad -(mg \sin \theta) = ma_t$

(Tangential direction)

The negative sign of the force ($mg \sin \theta$) means that the force is directed towards the equilibrium position whereas the displacement s is positive away from the equilibrium position.

If v be the velocity of the mass of any time t,

$$a_t = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

$$-(mg \sin \theta) = m\frac{d^2s}{dt^2}$$

For small values of θ

$$\sin \theta \approx \theta \approx \frac{s}{l}$$

so,

$$-mg\frac{s}{l} = m\frac{d^2s}{dt^2}$$

$$\frac{d^2s}{dt^2} + \frac{g}{l}s = 0$$

Comparing it with

$$\frac{d^2s}{dt^2} + p^2x = 0$$

$$p^2 = \frac{g}{l},\ t = \frac{2\pi}{p}$$

$$t = 2\pi\sqrt{\frac{l}{g}}.$$

Compound Pendulum. It consists of a rigid body of mass m, oscillating about a fixed axis through a point on the body other than its mass centre (Fig. 24.11).

Let b be the distance of the support O (called the centre of suspension) from the mass centre G.

Using the relation,

$$M_0 = I_0\alpha \qquad ...(i)$$

where, M_0 is the moment of the forces acting on the body about O.

I_0 is the moment of inertia of the body about the axis of rotation through O. (*i.e.*, along z axis)

α is the angular acceleration of the body and is equal to $\ddot{\theta}$

Equation (*i*) can be written as

$$-(mg \sin \theta)b = I_0\ddot{\theta}$$

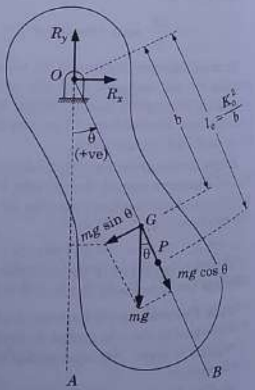


Fig. 24.11

For small angular displacements

$$\sin \theta \simeq \theta$$

$$-\text{mg}\ \theta b \simeq I_0\ddot{\theta}$$

$$\ddot{\theta} + \frac{mgb}{I_0}\theta = 0$$

Comparing it with

$$\ddot{\theta} + p^2\theta = 0$$

$$p^2 = \frac{mgb}{I_0}$$

$$t = \frac{2\pi}{p} \text{ or } t = 2\pi\sqrt{\frac{I_0}{mgb}}$$

Expressing

$$I_0 = mk_0^{\ 2}$$

where k_0 is the radius of gyration with respect to the axis of rotation through O

$$t = 2\pi\sqrt{\frac{mk_0^2}{mgb}} = 2\pi\sqrt{\frac{k_0^2}{gb}} \qquad ...(24.12)$$

Let us compare it with the time period of a simple pendulum

$$t = 2\pi\sqrt{\frac{l}{g}}$$

We can say that the time period of oscillations of compound pendulum is equal to that of a simple pendulum of equivalent length l_e,

$$l_e = \frac{k_0^2}{b} \quad (op = l_e) \qquad ...(24.13)$$

Point P is called the centre of oscillation. This is also called as centre of percussion.

Denoting the radius of gyration of the body (with respect to an axis through its centre of gravity and parallel to the axis of rotation) by k_g, we have

$$k_0^{\ 2} = k_g^{\ 2} + b^2$$

Hence

$$l_e = \frac{k_g^2 + b^2}{b} = \frac{k_g^2}{b} + b \qquad ...(24.14)$$

Torisonal Pendulum. Consider a circular disc attached to the one end of a shaft. The axis OZ of the shaft passes through the centre of the disc and is perpendicular to the plane of the disc. The other end of the shaft is held rigidly. By applying external moment, the body is turned about the vertical axis OZ or is twisted and then released. The body then performs *oscillatory rotation* about the axis OZ and is called *torisonal vibrations* (Fig. 24.12).

It is assumed that the bar or the shaft OA is perfectly elastic but of negligible mass.

Let θ be the angular displacement of any line AB, from the equilibrium position, after a time t.

The equation of motion can be written as

$$M = I\ddot{\theta}$$

where, I is the moment of inertia of the disc with respect to the axis of rotation OZ and

M is the moment, with respect to the axis *OZ* of all external forces acting on the disc.

Since the gravity force coincides with the axis *OZ* it does not contribute to the moment *M*. *Therefore, M is the moment due to the elastic reaction exerted by the shaft on the disc.*

Treating the moment due to elastic reaction in the manner of a torsional spring of stiffness k_t,

$$M = -k_t\theta$$

θ is positive away from the equilibrium position. The minus sign indicates that when a rotation of disc is produced in one direction the twisted shaft exerts on the disc an elastic reaction couple *M* in the opposite direction.

$$-k_t\theta = I\ddot{\theta}$$

Or,

$$I\ddot{\theta} + k_t\theta = 0$$

$$\ddot{\theta} + \frac{k_t\theta}{I} = 0$$

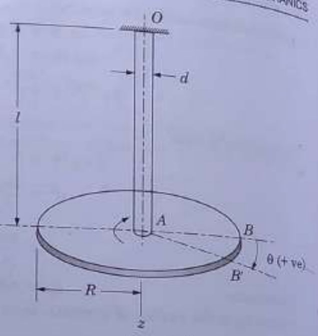


Fig. 24.12

Comparing it with

$$\ddot{\theta} + p^2\theta = 0$$

$$p^2 = \frac{k_t}{I},$$

$$t = \frac{2\pi}{p}$$

$$t = 2\pi\sqrt{\frac{I}{k_t}} \qquad \text{...(24.15)}$$

For a circular shaft of diameter *d*, length *l* and modulus *G* the torsional stiffness k_t can be shown to be.

$$k_t = \frac{\pi d^4 G}{32 l} \qquad \text{...(24.16)}$$

Trifilar Suspension. A disc of radius *r* suspended from three vertical wires of equal length *l* and attached to the disc along its circumference at equal distances, form a trifilar suspension as shown (Fig. 24.13).

Let the mass of the disc be *m* and the weight be *mg*.

Tension in each of the wires in the equilibrium position.

$$T = \frac{mg}{3}$$

Let the disc be given angular twist θ.

Inclination of the wires from the vertical when the disc is twisted by angle θ

$$\phi = \frac{r\theta}{l}$$

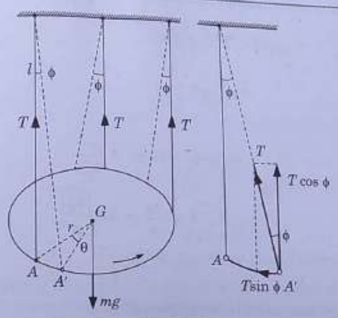


Fig. 24.13

It is to be assumed that for a small angular displacement θ, the tensions in the wires remain unchanged.

Equation of motion is

$$M = I_G\alpha$$
$$= I_G\ddot{\theta} \qquad \text{...(24.17)}$$

Where,

M is the moment due to tensile forces in the wires trying to bring the disc back to the equilibrium position and is given by

$$M = -3(T\sin\phi)r$$

$$M = -3\left(\frac{mg}{3}\sin\phi\right)r$$

For small values of ϕ

$$\sin\phi \approx \phi$$

$$M = -mg\phi r$$

$$M = -mg\frac{r\theta}{l}r$$

$$M = -\frac{mgr^2\theta}{l}$$

Substituting the above value of *M* in (24.17)

$$I_G\ddot{\theta} + \frac{mgr^2\theta}{l} = 0$$

$$I_G = mK_G^2$$

Where, I_G is the moment of inertia of the disc about axis of rotation parallel to the axis through the C.G. of the disc and k_G is the radius of gyration.

$$mk_G^2\ddot{\theta} + \frac{mgr^2}{l}\theta = 0$$

$$\ddot{\theta} + \frac{gr^2}{k_G l}\theta = 0$$

Comparing it with

$$\ddot{\theta} + p^2\theta = 0,\ p^2 = \frac{gr^2}{k_G^2 l}$$

$$t = \frac{2\pi}{p} = 2\pi\sqrt{\frac{k_G^2 l}{gr^2}}$$

$$t = \frac{2\pi k_G}{r}\sqrt{\frac{l}{g}} \qquad \text{...(24.18)}$$

Example 24.10. Find the time period of oscillation of a simple pendulum of length 1 m,

It this pendulum hangs from the ceiling of an elevator moving with

(*a*) an upward acceleration of $g/10$

(*b*) a downward acceleration of $g/10$, find its time period (Fig. 24.14).

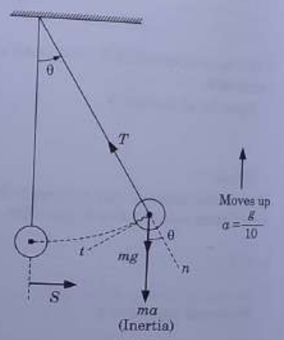


Fig. 24.14

Solution: When the elevator is stationary.

$$t = 2\pi\sqrt{\frac{l}{g}},\ l = 1\text{ m}$$

$$t = 2\pi\sqrt{\frac{l}{g}}$$

$$t = 2.01\ s$$

When the elevator moves up the bob experiences an additional force due to inertia and equal to ma.

The equation of dynamic equilibrium of the bob in the direction of tangent to the path is

$\Sigma F_t = 0$:

$$F_t = ma_t = 0$$

$$F_t = -(mg + ma)\sin\theta$$

where,

$$a_t = \frac{d^2s}{dt^2},\ \sin\theta = \theta = \frac{s}{l}$$

Equation (*i*) becomes

$$m(g+a)\frac{s}{l} + m\frac{d^2s}{dt^2} = 0$$

$$\frac{d^2s}{dt^2} + \frac{(g+a)}{l}s = 0$$

Therefore,

$$t = 2\pi\sqrt{\frac{l}{g+a}} \quad \text{where, } a = \frac{g}{10}$$

$$t = \sqrt{\frac{1}{g + g/10}}$$

$t = 1.912\ s$ (time period decreases) *Ans.*

When moving down

$$a = -\frac{g}{10}$$

$$t = 2\pi\sqrt{\frac{1}{g - g/10}}$$

$t = 2.11\ s$ (time period increases) *Ans.*

Example 24.11. A pendulum having a time period of 1 s in installed in a lift. Determine its time period when (*a*) the lift is moving upwards with an acceleration of $g/10$ (*b*) the lift is moving downward with an acceleration of $g/10$.

Solution: When the lift is stationary,

The time period of the pendulum $t = 1$ s

$$t = 2\pi\sqrt{\frac{l}{g}}$$

$$1 = 2\pi\sqrt{\frac{l}{9.81}}$$

$$l = \frac{1}{4\pi^2} \times 9.81$$

$$= 0.248\text{ m}$$

Length of the pendulum

$$l = 0.248\text{ m}$$

(*a*) Moving up with $a = \frac{g}{10}$

Using the formula derived in the earlier example,

$$t = 2\pi\sqrt{\frac{l}{g+a}},\ t = 2\pi\sqrt{\frac{0.248}{g + g/10}}$$

$$t = 2\pi\sqrt{\frac{10 \times 0.248}{107.91}},\ t = 0.95\text{ s} \quad \textit{Ans.}$$

(*b*) Moving down with $a = -\frac{g}{10}$

$$t = 2\pi\sqrt{\frac{l}{g-a}}$$

$$= 2\pi\sqrt{\frac{0.248}{g - g/10}},\ t = 1.05\text{ s} \quad \textit{Ans.}$$

Example 24.12 A horizontal disc of mass 15 kg, 30 cm in diameter and 3 cm thick is attached at its centre to a 1.0 m long shaft of 1 cm diameter of modulus of rigidity 30×10^9 N/m². Calculate the time period of oscillations of the disc if twisted and released (Fig. 24.15).

Solution: The moment of inertia of the disc about axis of rotation OZ is

$$I = \frac{mR^2}{2}$$

$$I = \frac{15(0.15)}{2} = 0.169 \text{ kg m}^2$$

Stiffness of the shaft

$$k_t = \frac{\pi d^4 G}{32 l}$$

$$k_t = \frac{\pi(0.01)^4 \times 30 \times 10^9}{32 \times 1}$$

$$k_t = 29.45 \text{ N-m/radian}$$

$$t = 2\pi\sqrt{\frac{I}{k_t}}$$

$$t = 2\pi\sqrt{\frac{0.169}{29.45}}$$

$$t = 0.476 \text{ s} \quad Ans.$$

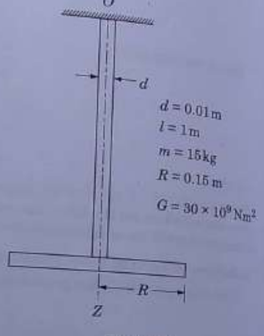


Fig. 24.15

Example 24.13 Two uniform rods each of mass M and length L are welded together to form a T-shaped assembly as shown (Fig. 24.16). Determine the time period of oscillations of the assembly.

Solution: $t = 2\pi\sqrt{\frac{I_A}{(2M)gb}}$

where I_A is the moment of inertia of the assembly of rods about the axis of rotation thorugh A and

b is the distance of the centre of gravity of the assembly from the point of suspension A.

The mass of the assembly

$$= M + M = 2M$$

The distance b is given by

$$b = \frac{\frac{L}{2}M + LM}{M + M} = \frac{3L}{4}$$

$$I_A = \left[\underset{\text{(Rod }AB)}{\frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2}\right] + \left[\underset{\text{(Rod }CD)}{\frac{ML^2}{12} + ML^2}\right]$$

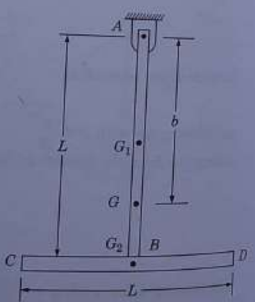


Fig. 24.16

$$I_A = \frac{17}{12}ML^2$$

$$t = 2\pi\sqrt{\frac{\frac{17}{12}ML^2}{(2M)(g)\left(\frac{3L}{4}\right)}}$$

$$t = 2\pi\sqrt{\frac{17\,L}{18\,g}} \quad Ans.$$

PROBLEMS

24.8. A homogeneous circular disc of radius r and M is suspended from a point on its circumference. Determine its period of oscillation. Calculate the time period if $r = 10$ cm.

$$\left[2\pi\sqrt{\frac{3r}{2g}}, 0.777\text{s}\right]$$

24.9. A thin homogeneous wire is bent into the shape of an equilateral triangle of side $a = 250$ mm. Find the period τ for small oscillations if the wire is suspended from a pin at O as shown in Fig. P.24.9. What will be the period if the wire is suspended from a pin located at the mid-point of the side?

[0.933 S, same]

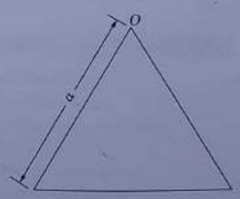


Fig. P.24.9

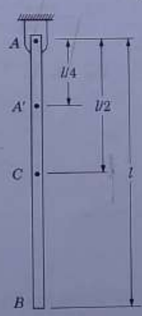


Fig. P.24.10

24.10. A uniform bar of length l is suspended from end A as shown in Fig. P.24.10. Determine the period of small oscillations.

If the bar is then suspended from the point A' at one quarter of its length, find the period of small oscillations.

$$\left[2\pi\sqrt{\frac{2L}{3g}}, 2\pi\sqrt{\frac{7L}{12g}}\right]$$

24.11. A thin circular plate of radius r = 75 cm is suspended from three vertical wires of length l = 60.0 cm each equally spaced around the perimeter of the plate. Find the period of oscillations when the plate is rotated through a small angle about a vertical axis passing through the mass centre and then released (Fig. P.24.11). [1.098 s]

24.12. The period of oscillations about the point A of a connecting rod is observed to be 1.5 s (Fig. P.24.12). The distance of its C.G. from the point A is 45 cm. Determine the centrodial radius of gyration of the connecting rod. [22.1 cm.]

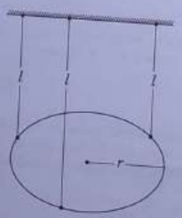


Fig. P.24.11

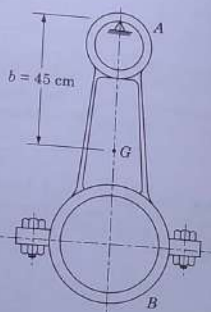


Fig. P.24.12

25 CHAPTER

Shear Force and Bending Moment

25.1 INTRODUCTION

A beam is a structural member whose longitudinal dimension is large compared to its transverse dimension. It is supported along its length and is acted upon by system of loads transverse (at right angles) to its longitudinal axis. It may also be acted upon by some couples.

The effect of loading results in developing shearing force and bending moment at any section of the beam. For designing a beam, information about shear force and bending moment is required. The shearing force and bending moment developed depends upon the combinations of loading and the support conditions of the beam. In this chapter the variation of shear force and bending moment developed along the length of the beam due to different system of loading shall be discussed.

25.2 TYPES OF BEAMS AND LOADING

Types of Beams

1. Cantilever beam
2. Simply supported beam
3. Fixed beam
4. Over-hanging beam
5. Continuous beam

Cantilever Beam. It is a beam which is fixed at one end (A) and free at the other end (B). There is no deflection or rotation at fixed end. (Fig. 25.1)

Simply Supported Beam. A beam supported freely on supports which may be a knife edge or a roller. There is no deflection or displacement of the beam at the ends. (Fig. 25.2)

Fig. 25.1

Fig. 25.2

Fixed Beam. A beam whose both ends are fixed. (Fig. 25.3)

Fig. 25.3

Fig. 25.4

Over Hanging Beam. A beam with one or both ends extended beyond the supports. (Fig. 25.4)

Continuous Beam. A beam with more than two supports is continuous beam. Such a beam may or may not have an overhang (Fig. 24.5).

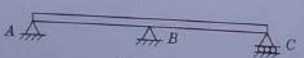


Fig. 25.5

Types of Loads

1. Concentrated load
2. Uniformly distributed load
3. Uniformly varying load
4. A combination of the above loadings

Concentrated Load. A concentrated load is one which is assumed to act at a point. (Fig. 25.6).

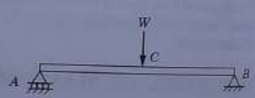


Fig. 25.6

Uniformly Distributed Load. A uniformly distributed load is uniformly or evenly distributed over a part or over the entire length of the beam. The rate of loading is expressed in N/m (Fig. 25.7). For solving numerical problems the total uniformly distributed load is converted into equivalent point load acting at the C.G. of uniformly distributed load.

Fig. 25.7

Uniformly Varying Load. A load whose intensity of loading varies linearly or at constant rate along the length. In triangular load (Fig. 25.8) for example, it increases from zero at one end to some value at other end at a constant rate. A trapezoidal loading (Fig. 25.9) is a combination of uniform and triangular loading.

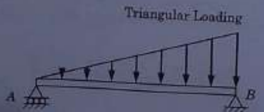


Fig. 25.8

Fig. 25.9

25.3 CONCEPT OF SHEAR FORCE AND BENDING MOMENT

Consider a beam AB (Fig. 25.10(a)) simply supported and carrying a point load W = 1000 N at point C.

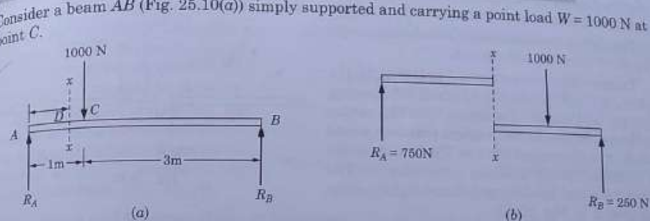


Fig. 25.10

Reactions R_A and R_B at the supports can be determined as,

$\Sigma M_A = 1000 \times 1 - R_B \times 4$

$R_B = 250$ N

$\Sigma F_y = 0 \qquad R_A + R_B - 1000 = 0$

$R_A = 750$ N

Assume a cutting plane x–x, a distance x from A dividing the beam into two segments (Fig. 25.10(b)). On the left portion an unbalanced vertical force (resultant) equal to R_A = 750 N is acting upwards.

Also, on the right portion an equal unbalanced vertical force (resultant), $W - R_B$ = 750 N is acting, but, in opposite direction to that acting on left portion i.e. downward.

This resultant force acting on the section to the left or right of the section is called *shear force*.

Now taking moment of all the external forces about x–x on the left and the right portions.

$$M_{left} = R_A \times x = 750x \text{ Nm (clockwise)}$$
$$M_{right} = -250\,(4 - x) + 1000\,(1 - x)$$
$$= -1000 + 250x + 1000 - 1000x$$
$$M_{right} = -750x \text{ Nm (anticlockwise)}$$

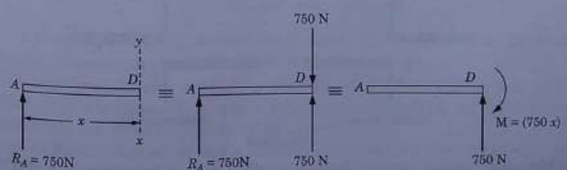


Fig. 25.11

The moment of the *resultant force* about the section acting either to the left or right is called the *bending moment*. Again it may noted that bending moment acting on the left portion and right portion are equal in magnitude but opposite in sign.

For a better understanding of the above concepts consider again the left portion.

Transfer the force acting at A to D (Fig. 25.11), which will be equivalent to a force of 750 N acting upward and couple of moment $750x$ Nm acting clockwise. Similarly, for the right portion it would result in a downward force of 750 N and couple of moment $750x$ Nm acting anticlockwise. The shear force and bending on left and right portion are shown in Fig. 25.12.

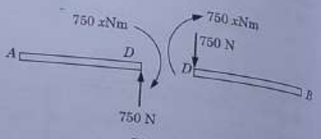


Fig. 25.12

It may be noted that figures shown so far are *not the free body diagrams* of beams or their sections. Infact, in free body diagrams *resisting* forces and *resisting* couples developed in the beam are to be included so as to satisfy the conditions of equilibrium.

25.4 DEFINITION OF SHEAR FORCE AND BENDING MOMENT AND SIGN CONVENTION

Shear Force. Shear force at a section of the beam is the force that is trying to shear-off the section of the beam.

Shear force is obtained by the algebraic summation of all the external forces (loads and reactions) acting normal to the axis of the beam, acting either on left side or right side of the section.

Sign Convention. When the left portion of the section is considered, if the resultant of vertical external forces is acting *upwards*, the shear force on the section is considered *positive*. If the resultant is acting downwards then is considered negative.

When right portion of the section is considered, if the resultant of vertical external forces is acting *downwards* it is considered *positive*. If acting upwards it is considered negative.

For positive shear force the above rule of *sign* produces the effect that tends to move the left hand portion of the beam upwards with respect to right portion. When the shear force is negative, left portion tends to move downward with respect to right portion (Fig. 25.13).

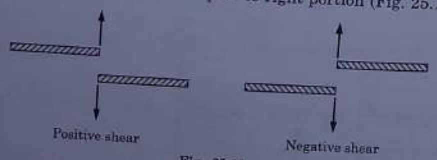


Fig. 25.13

Bending Moment. Bending moment at a section of the beam is the moment that tends to bend the beam. It is obtained by the algebraic summation of moments of all the external force (loads and reactions) about the section acting either to the left or right portion of the section.

Sign Convention. If the left portion of the section is considered the bending moment is considered positive when the sum of all moments of external forces (loads and reactions) is clockwise. When acting anticlockwise it is considered negative.

If the right portion of the section is considered the bending moment is considered positive when the sum of all moments of external forces is anticlockwise. When acting clockwise it is considered negative.

The above *sign* convention for positive bending moment results in the sagging of the beam. For negative bending moment it results in the hogging the beam (Fig. 25.14).

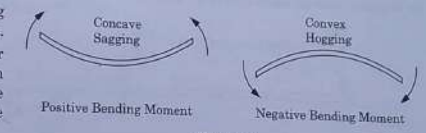


Fig. 25.14

Shear Force and Bending Moment Diagrams. The shear force and bending moment acting on the beam generally vary along the length of the beam. Their variations are normally represented as shear force and bending moment diagrams. The abscissa or x-axis indicates the position of the section.

The ordinate or y-axis indicates the values of shear force or bending moment (positive or negative).

Notes:

1. The curve for bending moment in a portion of the beam is one degree higher than the curve for shear force.
2. The portion in which S.F. is constant. B.M. curve is a straight line.
3. The portion in which S.F. is varying linearly B.M. curve is parabolic.
4. Rate of change of shear force is equal to the rate of loading.
$$\frac{d\,(\text{Shear Force})}{dx} = w$$
5. Shear force is basically the rate of change of bending moment with respect to x.
$$\text{S.F.} = \frac{d\,(\text{B.M.})}{dx}$$
So, at maximum or minimum value of B.M. i.e. when,
$$\frac{d\,(\text{B.M.})}{dx} = 0, \quad \text{S.F.} = 0$$
So, at the point where S.F. is zero in the diagram, B.M. is either maximum or minimum.
6. Point of contraflexure is a point where B.M. changes sign.
7. The shear force changes suddenly at a section where there is a vertical point load.
8. When a beam is subjected to a couple at the section, then B.M. changes suddenly at the section but S.F. remains unchanged at the section.
9. If an inclined load is acting on the beam, it is resolved into two components, horizontal and vertical. Vertical component will contribute to the S.F. and B.M.. Whereas, horizontal component is resisted by hinge as reaction and results is an axial force in the beam.

25.5 SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR STANDARD CASES

Example 25.1. Draw the shear force and bending moment diagrams for a cantilever beam acted by a point load as shown in (Fig. 25.15).

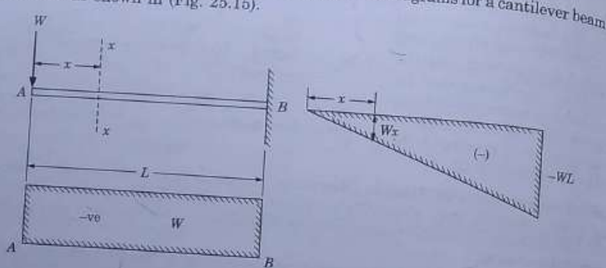


Fig. 25.15

Solution: At the fixed end B, reaction will consist of a force and a couple but it need not be determined.

Consider a section x–x at a distance x from the end A. Consider the left portion, shear force at the section x–x, F_x

$$F_x = -W$$

resultant force W is acting downwards and hence taken as negative.

The shear force is constant at all section as it is not varying with x. So S.F. diagrams is as shown in Fig. 25.15.

Bending moment at section $x - x$,

$$M_x = (-)\, W \cdot x$$

Anticlockwise B.M. on left portion is taken as –ve as it is producing hogging.

Bending moment is varying linearly with x, at $x = L$,

$$M_{x=L} = -WL$$

and B.M. diagram is as shown in Fig. 25.15.

Example 25.2. Draw the shear force and bending moment diagrams for a cantilever beam loaded as shown in Fig. 25.16.

Solution: Intensity of loading is w per unit length (w N/m). Consider a section x–x at a distance x from the end A. Consider the left portion.

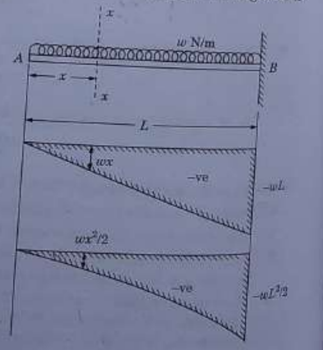


Fig. 25.16

Uniformly distributed load on length x is equivalent to a point load of magnitude wx acting at the C.G. of the section i.e. at a distance of $x/2$ from A.

Shear force $F_x = -wx$ (Acting downward, hence negative)

At A, $x = 0$ $\qquad F_A = 0$

At B, $x = L$ $\qquad F_B = -wL$

The variation of S.F. is shown as in Fig. 25.16.

Bending moment $\qquad M_x = -wx\left(\frac{x}{2}\right)$ Anticlockwise Moment

$$M_x = -\frac{wx^2}{2}$$

At A, $x = 0$ $\qquad M_A = 0$

At B, $x = L$ $\qquad M_B = -\frac{wL^2}{2}$

B.M. diagram is quadratic or parabolic curve as shown in Fig. 25.16.

Example 25.3. A simply supported beam AB is loaded by a point load W at C. Draw the shear force and bending moment diagrams.

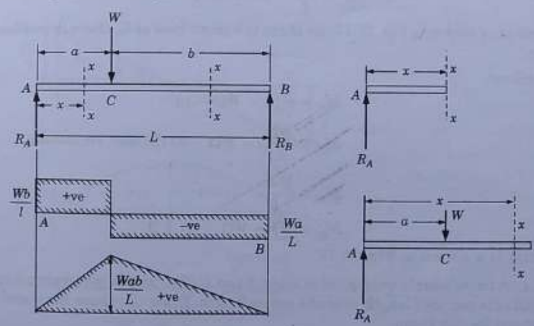


Fig. 25.17

Solution: Let the reactions at supports be R_A and R_B

$\Sigma F_y = 0$ $\qquad -W + R_A + R_B = 0,$

$$R_A + R_B = W$$

$\Sigma M_A = 0$ $\qquad Wa - R_B L = 0$

$$R_B = \frac{Wa}{L},$$

$$R_A = W - \frac{Wa}{L} = \frac{Wb}{L}$$

Consider a section x–x at a distance x from A lying between AC($0 < x < a$).

Shear force $F_x = R_A = +\dfrac{Wb}{L}$ (Acting upward hence positive)

Shear force is constant between A and C.

Bending moment,

$$M_x = R_A x = \frac{Wb}{L}x \text{ (Positive clockwise)}$$

at $x = 0$, $\quad M_A = 0$

$x = a$, $\quad M_C = \dfrac{Wab}{L}$

Consider a section x–x lying between CB ($a < x < L$)

Shear force $F_x = R_A - W = -W + \dfrac{Wb}{L} = -\dfrac{Wa}{L}$ (downward is negative)

$$F_c = -\frac{Wa}{L} \quad \text{(Constant)}$$

S.F. diagram is as shown in Fig. 25.17. As there is a point load at C, there is sudden change in shear force.

Bending moment,

$$M_x = R_A x - W(x - a)$$

$$M_x = \frac{Wb}{L}x - W(x - a) \text{ (Linear variation)}$$

at $x = a$, $\quad M_A = \dfrac{Wba}{L}$ (+ve)

at $x = L$ $\quad M_B = Wb - W(L - a) = 0$

B.M. diagram is a shown in Fig. 25.17.

Example 25.4. A beam simply supported at ends A and B (Fig. 25.18) is carrying a uniformly distributed load of w per unit length over the entire length. Draw the shear force and bending moment diagrams for the beam.

Solution: Reactions at the supported can be determined easily as,

$$R_A = \frac{wL}{2}$$

and $\quad R_B = \dfrac{wL}{2}$

Consider a left section x–x at a distance x from A.

Shear force $F_x = R_A - wx = +\dfrac{wL}{2} - wx$ (Linear variation with x)

at $x = 0$ i.e. at A, $\quad F_A = +\dfrac{wL}{2}$ (Upward, +ve)

at $x = L$ i.e. B, $\quad F_B = \dfrac{wL}{2} - wL = -\dfrac{wL}{2}$ (downward, –ve)

at $x = \dfrac{L}{2}$ at the centre C, $F_C = \dfrac{wL}{2} - \dfrac{wL}{2} = 0$

So shear is upwards (+ve) till $x = L/2$ and then acts downwards (–ve) becoming zero at the centre C of the beam as shown in Fig. 25.18.

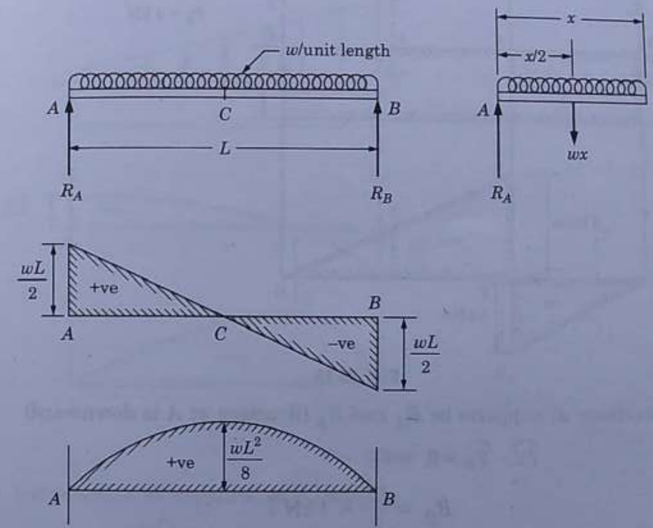


Fig. 25.18

Bending moment. At section x–x bending moment is,

$$M_x = R_A x - wx\frac{x}{2}$$

(+ve) (–ve)

$$= \frac{wL}{2}x - \frac{wx^2}{2} \text{ (Parabolic variation)}$$

At, $\quad x = 0$, i.e. at A $\quad M_A = 0$

$x = L$ i.e. at B $\quad M_B = 0$

$x = \dfrac{L}{2}$ i.e. at C $\quad M_C = \dfrac{wL}{2}\cdot\dfrac{L}{2} - \dfrac{w\left(\frac{L}{2}\right)^2}{2} = \dfrac{wL^2}{8}$

Bending moment varies parabolically and is +ve as it tends to sag the beam (concave upwards).

Example 25.5 A simply supported beam AB of length 3m is hinged at A and roller supported at B. It is subjected to clockwise couple of 12 kNm at a distance of 1 m from the left end A (Fig. 25.19). Draw of S.F. and B.M. diagram.

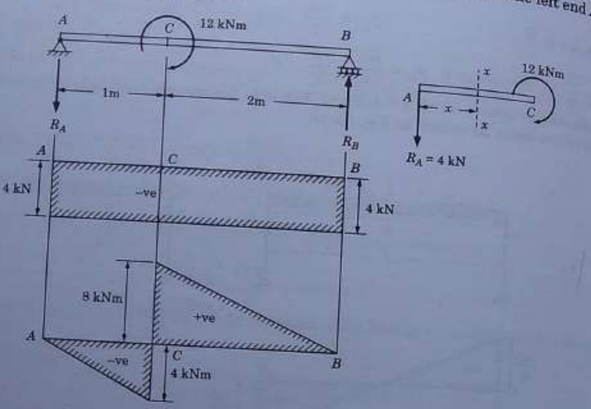


Fig. 25.19

Solution: Let reactions at supports be R_A and R_B (Reaction at A is downward)

$\Sigma M_A = 0$ $$\overset{\frown}{12} - \overset{\frown}{R_B} \times 3 = 0$$

$$R_B = \frac{12}{3} = 4 \text{ kN} \uparrow$$

$\Sigma M_B = 0$ $$\overset{\frown}{R_A \times 3} + \overset{\frown}{12} = 0$$

$$R_A = -\frac{12}{3} = -4 \text{ kN} \downarrow$$

S.F.

Shear force at A, $F_A = -4$ kN (downward)

It remains constant between A and B. S.F. diagram is shown in Fig. 25.19.

B.M.

Take section x–x at distance x from A

$$M_x = R_A \times x \text{ (Anticlockwise)}$$

B.M. at A, $$M_A = 0$$

B.M. at a section just before point C,

$$M_C = -4 \times 1 = -4 \text{ kNm (Anticlockwise –ve)}$$

B.M. at a section after point C

$$M_x = -R_A \cdot x + 12 \text{ kN}$$

at $x = 1$ $$M_C = -4 \times 1 + 12 = +8 \text{ kN}$$

at point B, $x = 3$, $$M_B = 0$$

There is a sudden change in B.M. at C due to couple acting at C (Fig. 25.19).

Example 25.6. Draw the shear force and bending moment diagrams for a simply supported beam carrying a uniformly varying load from zero at one end to w per unit length at the other end (triangular load) as shown in Fig. 25.20.

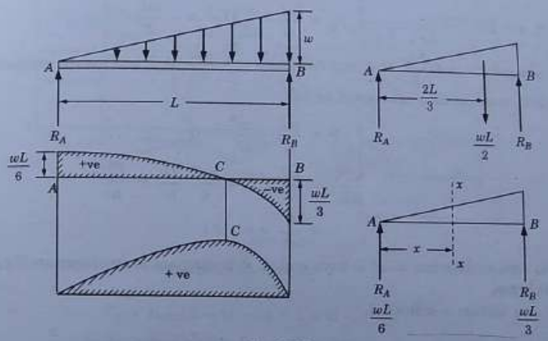


Fig. 25.20

Solution: Let reaction at supports be R_A and R_B

Total load on beam = $\frac{wL}{2}$ (area of load diagram), and is acting at the centroid of load diagram, that is, at distance of $\frac{2}{3}L$ from A.

$\Sigma M_A = 0$ $$-R_B L + \left(\frac{wL}{2}\right)\frac{2}{3}L = 0$$

$$R_B = \frac{wL}{3}$$

$\Sigma F_y = 0$ $$\frac{wL}{2} - R_A + R_B = 0$$

$$R_A = \frac{wL}{6}$$

Load over length $Ax = \left(\frac{w}{L}x\right)\frac{x}{2} = \frac{wx^2}{2L}$

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S.F. Consider any section at distance x from A

$$\text{Shear force } F = R_A - \frac{wx}{L}\cdot\frac{x}{2}$$

$$F = \frac{wL}{6} - \frac{wx^2}{2L}$$

It is seen to be varying parabolically (second degree in x).

At A, $x = 0$
$$F_A = \frac{wL}{6} - 0 = \frac{wL}{6}$$

At B, $x = L$
$$F_B = \frac{wL}{6} - \frac{w}{2L}(L)^2 = -\frac{wL}{3}$$

The shear force is $+\frac{wL}{6}$ at A and decreases to $-\frac{wL}{3}$ at B and varies parabolically.

The S.F. must be zero between A and B

$$F = \frac{wL}{6} - \frac{wx^2}{2L} = 0$$

$$\frac{wx^2}{2L} = \frac{wL}{6},\ x^2 = \frac{wL}{6}\cdot\frac{2L}{w} = \frac{L^2}{3}$$

$$x = \frac{L}{\sqrt{3}} = 0.577\,\text{L}$$

S.F. is zero at distance 0.577 L from end A. S.F. diagram is an shown in Fig. 25.20.

B.M. diagram

B.M. at a section x–x from A,

$$M_x = R_A x - (\text{Load on portion } A\text{–}x) \times \frac{x}{3}$$

$$M_x = \frac{wL}{6}x - \frac{wx^2}{2L}\,\frac{x}{3} = \frac{wL}{6}x - \frac{wx^3}{6L}$$

At $x = 0$,
$$M_A = 0$$

At $x = L$,
$$M_B = \frac{wL}{6}L - \frac{wL^3}{6L} = 0$$

B.M. varies between A and B as three degree curve (cubically)

B.M. is maximum where, S.F. is zero.

Max B.M. occur its $x = \frac{L}{\sqrt{3}}$

$$\text{Maximum B.M.} = \frac{wL}{6}\left(\frac{L}{\sqrt{3}}\right) - \frac{w}{6L}\left(\frac{L}{\sqrt{3}}\right)^3$$

$$\text{Maximum B.M.} = \frac{wL^2}{9\sqrt{3}}$$

B.M. diagram is shown in Fig. 25.20.

Example 25.7. A overhanging beam is loaded and supported as shown in Fig. 25.21. Draw the shear force and bending moment diagrams.

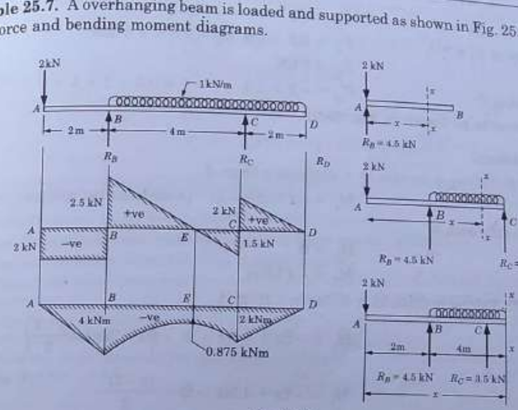

Fig. 25.21

Solution: Reactions at the supports be R_B and R_C

Applying,

$\Sigma R_y = 0$ $\qquad R_B + R_C - 2 - (4 + 2) \times 1 = 0$

$\qquad R_B + R_C = 8\,\text{kN}$ ...(*i*)

$\Sigma M_A = 0$ $\qquad -R_B \times 2 - R_C \times 6 + 4 \times 4 + 2 \times 7 = 0$

$\qquad 2R_B + 6R_C = 30$

$\qquad R_B + 3R_C = 15$ ...(*ii*)

Solving (*i*) and (*ii*) $R_B = 4.5\,\text{kN}$ $\quad R_C = 3.5\,\text{kN}$

Shear Force. Taking a section x–x from A (Between A and B)

$$F_x = -2\,\text{kN} \quad (\text{down force, –ve})$$

Shear force is constant from A to just left of B

$$F_A = -2\,\text{kN}$$

S.F. just left of $B = -2\,\text{kN}$

S.F. just right of $B = -2 + R_B = -2 + 4.5 = +2.5\,\text{kN}$

(It changes because of reaction at B)

Taking section x–x from A (Between A and C)

$$\text{Shear force } F_x = -2 + 4.5 - 1 \times (x - 2)$$
$$= 2.5 - x + 2 = 4.5 - x$$

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Just left of C, $x = 6$, $F_C = 4.5 - 6 = -1.5$ kN

Just right of C, $x = 6$, $F_C = 4.5 - x + R_C = 4.5 - x + 3.5$

$F_C = 2$ kN

S.F. at point D $F_D = -2 + 4.5 - 4 \times 1 + 3.5 - 2 \times 1 = 0$

S.F. diagram is as shown in Fig. 25.21.

Bending moment

Portion AB: Taking a section at a distance x from A

$$M_x = -2 \times x = -2x \quad \text{(Anticlockwise moment -ve)}$$

Varies linearly with x

at $x = 0$, $M_A = 0$

at $x = 2$, $M_B = -4$ kNm

Portion BC: Taking section at a distance x from A

$$M_x = -2x + 4.5(x - 2) - \left\{(x-2)\times 1\times\frac{x-2}{2}\right\}$$

$$M_x = -2x + 4.5(x - 2) - \frac{(x-2)^2}{2}$$

At $x = 2$ $M_B = -4$ kNm

At $x = 6$ $M_C = -12 + 18 - 8 = -2$ kNm

Variation between BC is parabolic.

Portion CD. Taking section at distance x from A

$$M_x = -2x + 4.5\,(x - 2) - \frac{1}{2}(x - 2)^2 + 3.5(x - 6)$$

at $x = 0$ $M_C = -2$ kN

at $x = 8$ $M_D = -2 \times 8 + 4.5(6) - \frac{1}{2} \times (6)^2 + 3.5(2)$

$M_D = 0$

Variation in this portion is again parabolic.

B.M. diagram is as shown in Fig. 25.21.

Note that shear force changes sign from +ve to –ve at E.

Location of E w.r.t. B is

$$\frac{2.5}{BE} = \frac{1.5}{EC},$$

$$\frac{2.5}{BE} = \frac{1.5}{4 - BE}$$

$$BE = 2.5 \text{ m}$$

B.M. at E (Portion BC)

$$M_E = -M_x = -2x + 4.5(x - 2) - \frac{1}{2}(x - 2)^2$$

Putting, $x = 2 + 2.5 = 4.5$ m

$$\therefore \quad M_E = -2 \times 4.5 + 4.5(4.5 - 2) - \frac{1}{2}(4.5 - 2)^2$$

$$M_E = -0.875 \text{ kNm} \quad \text{(BM, when S.F. = 0)}$$

PROBLEMS

25.1. Define a beam. Name and sketch the different types of beams.

25.2. List the different types of loads to which a beam can be subjected.

25.3. Explain the terms, shear force and bending moment.

25.4. Explain the sign conventions that are generally used to plot shear force and bending moment diagrams.

25.5. What do you meant by the point of contraflexure? Is the point of contraflexure different from the point of inflexion?

25.6. Draw the shear force and bending moment diagrams for a cantilever beam loaded as shown in Fig. P. 25.6. [$F_{max} = -12$ kN $M_{max} = -35$ kNm]

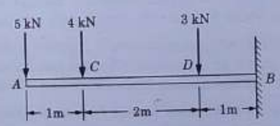

Fig. P. 25.6

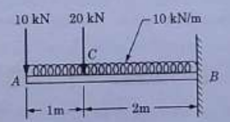

Fig. P. 25.7

25.7. Draw the shear force and bending moment diagram for a cantilever loaded by point loads and uniformly distributed load as shown in Fig. P. 25.7. [$F_{max} = 60$ kN, $M_{max} = -115$ kNm]

25.8. A simply supported beam of length 6 m, carries two point loads as shown in Fig. P. 25.8. Draw shear force and bending moment diagrams for the beam. [$F_{max} = -5$ kN, $M_{max} = 10$ kNm]

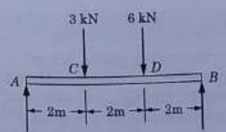

Fig. P. 25.8

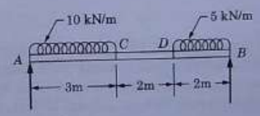

Fig. P. 25.9

25.9. A simply supported beam of length 7 m is carrying uniformly distributed load as shown in Fig. P. 25.9. Draw the shear force and bending moment diagrams for the beam.

[$F_{max} = +25$ kN, $M_{max} = 31.25$]

(Shear force is zero at distance 2.5 m from A)

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25.10. A simply supported beam is loaded and supported as shown in Fig. P. 25.10. Draw the shear force and bending moment diagram.

[$M_{max} = +23$ kNm]

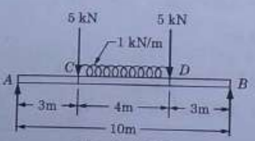

Fig. P. 25.10

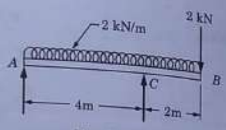

Fig. P. 25.11

25.11. An overhanging beam is loaded and supported as shown in Fig. P. 25.11. Draw the shear force and bending moment diagram for the beam.

[$M_{max} = -8$ kNm, point of contraflexure at distance 2 m from A]

1 APPENDIX

Review of Vectors and Forces in Space

A-1 DEFINITIONS

Scalars. The quantities which can be specified completely by the magnitude only are called scalars or scalar quantities.

Vectors. The quantities possessing magnitude and direction and which add according to the parallelogram law are called vectors or vector quantities.

Free Vector. Vector that can be placed any where in space and moved parallel to itself at will is called a free vector. We do this when we add two vectors head to tail using triangle law.

Sliding Vector. Vector that may be moved or slid along its line of action so that its magnitude, direction and line of action remains the same is called a sliding vector. A force acting on a rigid body can be treated as a sliding vector.

Bound Vector. A bound vector is a vector with a well defined point of application. A bound vector is, therefore, specified by magnitude, direction and its point of application. A vector representing a force acting on a particle can be treated as a bound vector.

Equal Vector. Two vectors are equal if they have the same magnitude and direction.

Equivalent or Equipollent Vectors. Two vectors are said to be equivalent or equipollent if, in a certain sense, they produce the same effect on a rigid body.

A-2 COMPONENTS OF A FORCE

Consider a force **F** acting at the origin O of the system of a rectangular, coordinates x, y and z (Fig. 1). Let it be represented by the vector **OP**. Through P draw planes parallel to the coordinate planes. These planes along with the coordinate planes determine a rectangular parallelopiped or a box. The force **F** is then represented by the diagonal OP of this box and its three components F_x, F_y and F_z by its edges.

The angles θ_x, θ_y and θ_z that the force **F** makes with the x, y and z axes define the direction of the force **F**. It can be shown that,

$$F_x = F\cos\theta_x,\ F_y = \cos\theta_y,\ F_z = F\cos\theta_z \qquad ...(1)$$

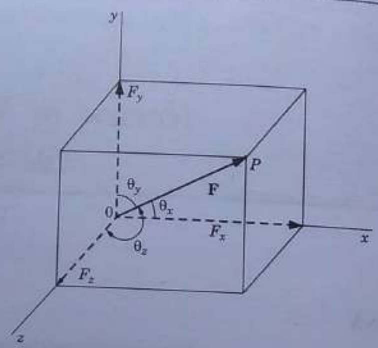

Fig. 1

Applying the Pythagorean theorem it can be proved that

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \quad ...(2)$$

The cosines of θ_x, θ_y and θ_z are known as the direction cosines of the force **F** and are denoted as,

$$l = \cos\theta_x,\ m = \cos\theta_y \text{ and } n = \cos\theta_z$$

It should be noted that *the values of the three angles* θ_x, θ_y *and* θ_z *are not independent* and are related by the identity,

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1 \quad ...(3)$$

Or

$$l^2 + m^2 + n^2 = 1$$

A-3 DEFINING A FORCE BY ITS MAGNITUDE AND TWO POINTS ON ITS LINE OF ACTION

Consider a force **F** defined by the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ a distance d apart (Fig. 2). Let θ_x, θ_y and θ_z be the angles that this force forms with the x, y and z axes. Then,

$$d_x = (x_2 - x_1),\ d_y = (y_2 - y_1),\ d_z = (z_2 - z_1)$$

$$d_x = d\cos\theta_x,\ d_y = d\cos\theta_y,\ d_z = d\cos\theta_z$$

$$d = \sqrt{d_x^2 + d_y^2 + d_z^2}$$

We know

$$F_x = F\cos\theta_x,$$

$$F_y = F\cos\theta_y,$$

$$F_z = F\cos\theta_z$$

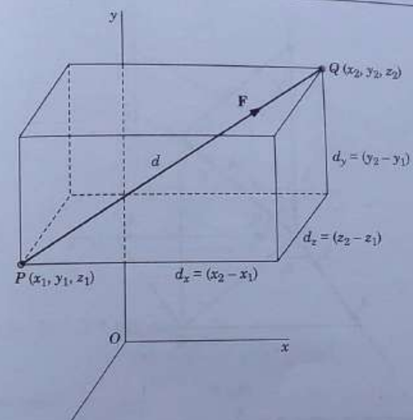

Fig. 2

Therefore,

$$\frac{F_x}{d_x} = \frac{F_y}{d_y} = \frac{F_z}{d_z} = \frac{F}{d}$$

$$\frac{F_x}{(x_2 - x_1)} = \frac{F_y}{(y_2 - y_1)} = \frac{F_z}{(z_2 - z_1)} = \frac{F}{d}$$

A-4 COMPONENTS OF A VECTOR

Consider a set of rectangular axes x, y and z (Fig. 3).

Vector **A** represented by **OP** is to be resolved into three components A_x, A_y and A_z along the rectangular axes (orthogonal components).

Let vector **i**, **j** and **k** be the vectors of *unit length* in the positive x, y and z directions. Then,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

A_x, A_y and A_z are called the scalar components of the vector **A**.

Magnitude of the vector $\mathbf{A} = A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

If the vector **A** forms angle θ_x, θ_y and θ_z with the x, y and z axes respectively

$$\mathbf{A} = (A\cos\theta_x)\mathbf{i} + (A\cos\theta_y)\mathbf{j} + (A\cos\theta_z)\mathbf{k}$$

Unit vector is a vector having a unit magnitude or unit length.

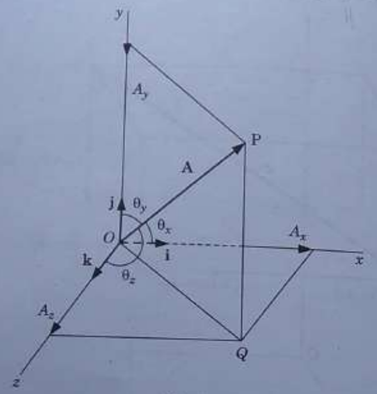

Fig. 3

Unit vector corresponding to vector $\mathbf{A} = \dfrac{\mathbf{A}}{|\mathbf{A}|}$

Or $$\hat{\mathbf{A}} = \frac{A}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Thus it may be noted that any two different unit vectors differ only in direction. Further vectors **i, j, k** are the unit vectors in the x, y and z directions, that is, in the directions of the coordinate axes.

The concept of a unit vector is very useful in determining a vector if its magnitude and direction are known. If F be the magnitude of a force and $\hat{\mathbf{n}}$ be the unit vector in the direction of the force, **F** can be expressed as,

$$\mathbf{F} = F\hat{\mathbf{n}} \quad \text{(See Example 2.)}$$

Position Vector. The position vector of a point P with respect to a point O is the vector **OP**. It is denoted by **r** and is used to specify the position of a point P relative to O (Fig. 4).

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Example 1. A point is located as (– 5, 2, 14) with respect to the origin (0, 0, 0) specify its position.

(*i*) in terms of orthogonal components

(*ii*) in terms of direction cosines and

(*iii*) in terms of its unit vector.

Solution: The components of the vector are,

$$(-5-0) = -5 \text{ along the } x\text{-axis}$$
$$(2-0) = 2 \text{ along the } y\text{-axis}$$
$$(14-0) = 14 \text{ along the } z\text{-axis}$$

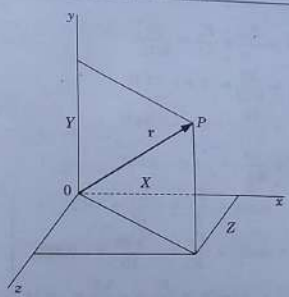

Fig. 4

Position vector $\mathbf{r} = -5\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}$ *Ans.*

Magnitude $$|\mathbf{r}| = \sqrt{(-5)^2 + (2)^2 + (14)^2}$$
$$r = 15$$

Direction consines are,

$$l = \cos\theta_x = -\frac{5}{15} = -0.333$$
$$m = \cos\theta_y = \frac{2}{15} = 0.0133$$
$$n = \cos\theta_z = \frac{14}{15} = 0.933$$
$$\mathbf{r} = (15\, l)\mathbf{i} + (15\, m)\mathbf{j} + (15\, n)\mathbf{k} \text{ Ans.}$$

Unit vector in the direction of **r** is

$$\frac{-5\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}}{|\mathbf{r}|} = \frac{-5\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}}{15}$$
$$= -0.33\mathbf{i} + 0.133\mathbf{j} + 0.933\mathbf{k} \quad \textit{Ans.}$$

Example 2. The coordinates of the initial and terminal points of a force vector **F** = 10 N are (2, 4, 3) and (1, – 5, 2) respectively. Determine the components of the force and its angles with the axes. Specify the force vector.

Solution:

$$d_x = (1-2) = -1$$
$$d_y = (-5-4) = -9$$
$$d_z = (2-3) = -1$$
$$d = \sqrt{d_x^2 + d_y^2 + d_z^2} = \sqrt{(-1)^2 + (-9)^2 + (-1)^2} = 9.11$$
$$\frac{F_x}{d_x} = \frac{F_y}{d_y} = \frac{F_z}{d_z} = \frac{F}{d}$$

$$\frac{F_x}{-1} = \frac{F_y}{-9} = \frac{F_z}{-1} = \frac{10}{9.11}$$

$$F_x = \frac{10}{9.11} \times -1 = -1.1 \text{ N}$$

$$F_y = \frac{10}{9.11} \times -9 = -9.88 \text{ N}$$

$$F_z = \frac{10}{9.11} \times -1 = -1.1 \text{ N}$$

$$\left.\begin{aligned} l &= \cos\theta_x = \frac{F_x}{F} = -\frac{1.1}{10} = -0.11 \\ m &= \cos\theta_y = \frac{F_y}{F} = -\frac{9.88}{10} = -0.988 \\ n &= \cos\theta_z = \frac{F_z}{F} = -\frac{1.1}{10} = -0.11 \end{aligned}\right\} \quad \textit{Ans.}$$

$$\left.\begin{aligned} \theta_x &= 96.3° \\ \theta_y &= 171.09° \\ \theta_z &= 96.3° \end{aligned}\right\} \text{ and } \mathbf{F} = -1.1\mathbf{i} - 9.88\mathbf{j} - 1.1\mathbf{k} \quad \textit{Ans.}$$

Alternative Method. Vector joining the points (2, 4, 3) and (1, – 5, 2) is

$$= (1 - 2)\mathbf{i} + (-5 - 4)\mathbf{j} + (2 - 3)\mathbf{k}$$

$$= -1\mathbf{i} - 9\mathbf{j} - 1\mathbf{k}$$

Unit vector in the direction of vector $(-1\mathbf{i} - 9\mathbf{j} - 1\mathbf{k})$ is

$$\frac{-1\mathbf{i} - 9\mathbf{j} - 1\mathbf{k}}{\sqrt{(1)^2 + (9)^2 + (1)^2}} = \frac{-1\mathbf{i} - 9\mathbf{j} - 1\mathbf{k}}{9.11}$$

$$\mathbf{F} = F\hat{\mathbf{n}}$$

Force vector $\mathbf{F} = 10 \times$ (unit vector in the direction of the force)

$$= \frac{10(-1\mathbf{i} - 9\mathbf{j} - 1\mathbf{k})}{9.11}$$

$$\mathbf{F} = -1.1\mathbf{i} - 9.88\mathbf{j} - 1.1\mathbf{k} \quad \textit{Ans.}$$

Example 3. A vertical pole is guyed by a wire PQ which is anchored by means of a bolt Q as shown (Fig. 5). The tension in the wire is 2500 N. Determine (*i*) the components of the force acting on the bolt (*ii*) the angles θ_x, θ_y and θ_z defining the direction of the force and (*iii*) specify the force vector.

Solution: The line of action of the force **F** acting on the bolt Q passes through Q and P and is directed for Q to P.

Thus the force **F** has the same direction as vector $\overrightarrow{QP}$

Vector
$$\overrightarrow{QP} = (0 - 40)\mathbf{i} + (80 - 0)\mathbf{j} + (0 - (-30))\mathbf{k}$$

$$= -40\mathbf{i} + 80\mathbf{j} + 30\mathbf{k}$$

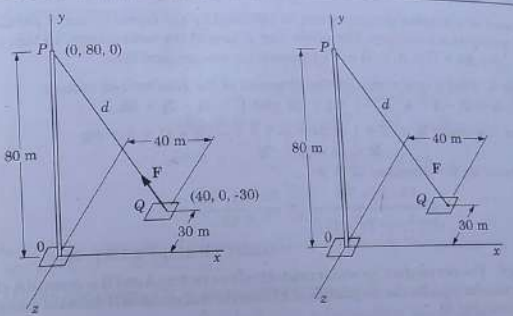

Fig. 5

Unit vector along
$$QP = \frac{-40\mathbf{i} + 80\mathbf{j} + 30\mathbf{k}}{\sqrt{40^2 + 80^2 + 30^2}} = \frac{-40\mathbf{i} + 80\mathbf{j} + 30\mathbf{k}}{94.3}$$

Force vector
$$\mathbf{F} = 2500 \times \text{(unit vector in the direction of } QP)$$

$$= 2500\frac{(-40\mathbf{i} + 80\mathbf{j} + 30\mathbf{k})}{94.3}$$

$$\mathbf{F} = -1600\mathbf{i} + 2120\mathbf{j} + 795\mathbf{k} \quad \textit{Ans.}$$

$$\left.\begin{aligned} \cos\theta_x &= \frac{F_x}{F} = \frac{-1060}{2500} \quad \text{or} \quad \theta_x = 115.1 \\ \cos\theta_y &= \frac{F_y}{F} = \frac{2120}{2500} \quad \text{or} \quad \theta_y = 32.0° \\ \cos\theta_z &= \frac{F_z}{F} = \frac{795}{2500} \quad \text{or} \quad \theta_z = 71.5° \end{aligned}\right\} \textit{Ans.}$$

If we use,

$$\cos\theta_x = \frac{x_2 - x_1}{d} \text{ ... etc., we shall get the same results.}$$

A-5 VECTOR OPERATIONS

Vector Addition. There are two methods to determine the sum of two vectors **A** and **B**. One is to use the parallelogram law as explained in chapter 2. The second method is to add the components.

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

Further,
$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \text{ (Commutative law of addition)}$$

The sum of a number of vectors can be obtained by the repeated use of parallelogram law or by drawing space polygon. The other way is to add the components. In this case,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \text{ (Associative law for addition)}$$

Example 4. Find a unit vector in the direction of the resultant of vectors

$$\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k},\ \mathbf{B} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{C} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

Solution: Resultant $\mathbf{R} = (2 + 1 + 3)\mathbf{i} + (-1 + 1 - 2)\mathbf{j} + (1 + 2 + 4)\mathbf{k}$

$$\mathbf{R} = 6\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$$

Unit vector in the direction of $\mathbf{R}$ is

$$\frac{6\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}}{\sqrt{(6)^2 + (-2)^2 + (7)^2}} = \frac{6\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}}{9.43}$$

$$= (0.636\mathbf{i} - 0.212\mathbf{j} + 0.72\mathbf{k}) \quad Ans.$$

Dot Product. The dot product (or scalar product) of two vectors **A** and **B** is denoted $\mathbf{A} \cdot \mathbf{B}$ (read as A dot **B**) and is a scalar equal to the magnitude of **A** times the magnitude of **B** times the cosine of the angle θ between them (Fig. 6).

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| \cdot |\mathbf{B}| \cos\theta = AB\cos\theta$$

It may noted that,

(*i*) When $\theta = 0$, that is, when vectors **A** and **B** have the same direction,

$$\mathbf{A} \cdot \mathbf{B} = AB$$

(*ii*) When $\theta = 90°$, that is when vectors **A** and **B** are perpendicular,

$$\mathbf{A} \cdot \mathbf{B} = 0$$

(*iii*) $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|$ times (projection of **B** on **A**) (Fig. 6)

$\mathbf{A} \cdot \mathbf{B} = |\mathbf{B}|$ times (projection of **A** on **B**) (Fig. 6)

(*iv*) $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ (commuative law)

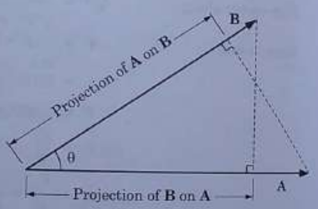

Fig. 6

Scalar product in terms of components

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z \quad \text{where,}$$
$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$
$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

The angle between vectors **A** and **B** is given by

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$$

Example 5. Two vectors **A** and **B** are given as $\mathbf{A} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Determine (*a*) their product (*b*) the angle between them.

Solution:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$
$$\mathbf{A} \cdot \mathbf{B} = [(2) \times (4)] + [(-6) \times (3)] + [(-3)(-1)]$$
$$\mathbf{A} \cdot \mathbf{B} = 8 - 18 + 3$$

$$\mathbf{A} \cdot \mathbf{B} = -7 \quad Ans.$$
$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\ |\mathbf{B}| \cos\theta$$
$$|\mathbf{A}| = A = \sqrt{(2)^2 + (-6)^2 + (-3)^2} = 7$$
$$|\mathbf{B}| = B = \sqrt{(4)^2 + (3)^2 + (-1)^2} = \sqrt{26}$$
$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{-7}{7 \times \sqrt{26}} = -0.196$$
$$\theta = (180 - 78.7)$$
$$\theta = 101.3° \quad Ans.$$

Example 6. Find the projection of vector $A = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ on the line joining the points $P(2, 3, -1)$ and $Q(-2, -4, 3)$.

Solution: Let the line joining PQ be represented by vector **B**.

$$\mathbf{B} = (-2 - 2)\mathbf{i} + (-4 - 3)\mathbf{j} + (3 - (-1))\mathbf{k}$$
$$\mathbf{B} = -4\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$$

As $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$, the projection of vector **A** on vector **B** (that is line PQ) is

$$A\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{B}$$
$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$
$$\mathbf{A} \cdot \mathbf{B} = [4 \times (-4)] + [(-3) \times (-7)] + [1 \times 4]$$
$$\mathbf{A} \cdot \mathbf{B} = 9$$
$$\mathrm{B} = \sqrt{(-4)^2 + (-7)^2 + (4)^2} = 9.0$$
$$A\cos\theta = \frac{9}{9}$$
$$A\cos\theta = 1.0 \quad Ans.$$

Example 7 The point of application of a force $\mathbf{F} = 5\mathbf{i} + 10\mathbf{j} - 15\mathbf{k}$ is displaced from the point $\mathbf{i} + 3\mathbf{k}$ to the point $3\mathbf{i} - \mathbf{j} - 6\mathbf{k}$. Find the work done by the force.

Solution:

$$\text{Work done} = \mathbf{F} \cdot \mathbf{d}$$
$$\mathbf{d} = (3 - 1)\mathbf{i} + (-1 - 0)\mathbf{j} + (-6 - 3)\mathbf{k}$$
$$\mathbf{d} = 2\mathbf{i} - 1\mathbf{j} - 9\mathbf{k}$$
$$\text{Work done} = (5\mathbf{i} + 10\mathbf{j} - 15\mathbf{k}) \cdot (2\mathbf{i} - 1\mathbf{j} - 9\mathbf{k})$$
$$= 10 - 10 + 135$$
$$= 135 \quad Ans.$$

Vector Cross Product. The cross product of vectors **A** and **B**, written as $\mathbf{A} \times \mathbf{B}$ (and read as **A** cross **B**) is such that (*i*) its magnitude of **A** times the magnitude of **B** times the sine of the smaller angle between the two vectors (*ii*) its direction is perpendicular to the plane containing the vectors **A** and **B**. A unit vector in this direction is denoted by $\hat{\mathbf{n}}$. (*iii*) its sense is given by the right hand screw rule (Fig. 7).

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\ |\mathbf{B}| \sin\theta\,\hat{\mathbf{n}}$$

Or $$\mathbf{A} \times \mathbf{B} = AB \sin\theta\,\hat{\mathbf{n}}$$

The magnitude of $\mathbf{A} \times \mathbf{B}$

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$$

In terms of rectangular components

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

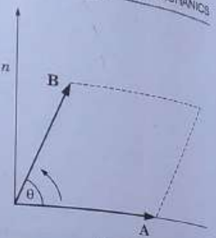

Fig. 7

Or $\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$

Note that, $\mathbf{i} \times \mathbf{i} = 0,\ \mathbf{j} \times \mathbf{j} = 0,\ \mathbf{k} \times \mathbf{k} = 0$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k},\ \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i},\ \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j},\ \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

If, $\theta = 0$ or π $\mathbf{A} \times \mathbf{B} = 0$

Therefore, $\mathbf{A} \times \mathbf{A} = 0$

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A} \quad \text{(Not commutative)}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

Example 8. Determine the cross product of vector $\mathbf{A} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and the angle between them.

Solution:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$$

$$= [(-6)(-1) - (-3)(3)]\mathbf{i} + [(-3)(4) - (2)(-1)]\mathbf{j} + [(2)(3) - (6)(4)]\mathbf{k}$$

$$\mathbf{A} \times \mathbf{B} = 15\mathbf{i} + 10\mathbf{j} + 30\mathbf{k}$$

$\hat{\mathbf{n}}$ (Unit vector parallel to $\mathbf{A} \times \mathbf{B}$) $= \dfrac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$

$$\hat{\mathbf{n}} = \frac{15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}}{\sqrt{15^2 + (-10)^2 + (30)^2}}$$

$$\hat{\mathbf{n}} = \frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$$

Angle between the vector **A** and **B**

$$\sin\theta = \frac{|\mathbf{A} \times \mathbf{B}|}{|\mathbf{A}||\mathbf{B}|} = \frac{|\mathbf{A} \times \mathbf{B}|}{\mathbf{AB}}$$

$$|\mathbf{A} \times \mathbf{B}| = \sqrt{(15)^2 + (-10)^2 + (30)^2} = 35$$

$$|\mathbf{A}| = \sqrt{(2)^2 + (-6)^2 + (-3)^2} = 7$$

$$|\mathbf{B}| = \sqrt{(4)^2 + (3)^2 + (-1)^2} = \sqrt{26}$$

$$\sin\theta = \frac{35}{7\sqrt{26}}$$

$$\theta = 78.7^\circ \text{ or } 101.3^\circ$$

Note. The angle between the vecotrs **A** and **B** comes out to be the same as determined by the dot product method (See Example 5).

A-6 ANGULAR VELOCITY

If a rigid body is rotating with angular velocity ω, the linear velocity of any point P on the body is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

Or $$v = \omega r \sin\theta$$

where, **r** is the position vector of the point P with respect to the origin on the axis of rotation (Fig. 8).

ω is the angular velocity vector. The direction of this vector is along the axis of rotation in the direction of travel of a right handed screw, when turning the way the dody is rotating.

v is linear velocity and is perpendicular to the plane of **r** and ω and in the right hand sense.

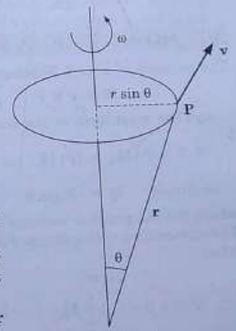

Fig. 8

Example 9. A rigid body is rotating at an angular velocity of 2.0 radian/second about an axis passing through the points $A(0, 1, 2)$ and $B(1, 3, -2)$. Find the linear velocity of a point $P(3, 6, 4)$ on the body (Fig. 9).

Solution: Axis of rotation is given by the vector

$$(1 - 0)\,\mathbf{i} + (3 - 1)\,\mathbf{j} + (-2 - 2)\,\mathbf{k} = \mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}$$

Unit vector along the axis of rotation (and also in the direction of angular velocity ω).

$$\frac{\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}}{\sqrt{(1)^2 + (2)^2 + (-4)^2}} = \frac{\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}}{\sqrt{21}}$$

Vector $$\omega = 2\left(\frac{\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}}{\sqrt{21}}\right)$$

Position vector

$$\mathbf{r} = (3 - 0)\mathbf{i} + (6 - 1)\mathbf{j} + (4 - 2)\mathbf{k}$$

$$\mathbf{r} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{v} = \omega \times \mathbf{r}$$

$$\mathbf{v} = \frac{2}{\sqrt{21}}(1\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \times (3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$$

$$\mathbf{v} = \frac{2}{\sqrt{21}}[(4 + 20)\,\mathbf{i} - (2 + 12)\mathbf{j} + (5 - 6)\mathbf{k}]$$

$$\mathbf{v} = \frac{2}{\sqrt{21}}(24\mathbf{i} - 14\mathbf{j} - \mathbf{k}) \quad \textit{Ans.}$$

Magnitude of

$$\mathbf{v} = \frac{2}{\sqrt{21}}\sqrt{((24)^2 + (-14)^2 + (-1)^2)} = 12.13$$

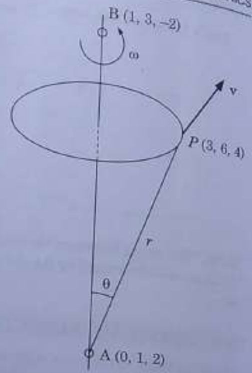

Fig. 9

-7 MOMENT OF A FORCE

he moment of a force **F** about a point O is given by

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

and the magnitude of the moment **M** is

$$|\mathbf{M}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta \quad \begin{pmatrix} \mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \\ \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \end{pmatrix}$$

or simply, $\quad M = rF \sin\theta$

ere r is the position vector of *any point* on the line of action of the force with respect to O. e moment is a vector directed perpendicular to the plane containing the vectors **r** and **F**. [Fig. a)]

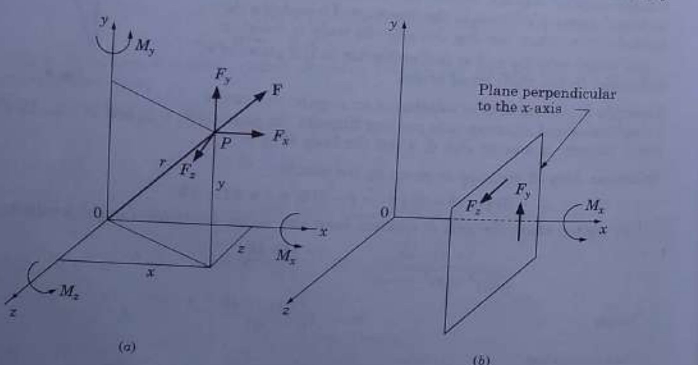

Fig. 10

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\mathbf{M} = (yF_z - zF_y)\mathbf{i} + (zF_x - xF_z)\mathbf{j} + (xF_y - yF_x)\mathbf{k}$$

Or,

$$M_x = yF_z - zF_y \qquad \ldots(i)$$

$$M_y = zF_x - xF_z \qquad \ldots(ii)$$

$$M_z = xF_y - yF_x \qquad \ldots(iii)$$

Equations (*i*), (*ii*) and (*iii*) are the moments about the axes x, y and z respectively. It can be observed that the moment about an axis is due to the forces lying in a plane perpendicular to the axis of the moment [Fig. 10 (*b*)].

Example 10. Find the moment about the point $O(-2, 3, 5)$ of the force $\mathbf{F} = 4\mathbf{i} + 4\mathbf{j} - 1\mathbf{k}$ passing through the points $P(1, -2, 4)$ and $Q(5, 2, 3)$. See Fig. 11.

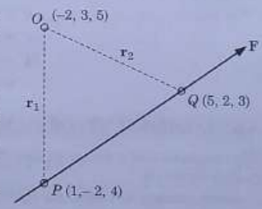

Fig. 11

Solution: $\mathbf{F} = 4\mathbf{i} + 4\mathbf{j} - 1\mathbf{k}$

Moment of the force about O is $\mathbf{r} \times \mathbf{F}$ where, $\mathbf{r}$ is the position vector of any point on the force vector **F** with respect to the point O. We can choose either $\mathbf{r}_1$ or $\mathbf{r}_2$ as the position vector. We shall show that the result is the same.

$$\mathbf{r}_1 = (1 - (-2)\mathbf{i} + (-2) - 3)\mathbf{j} + (4 - 5)\mathbf{k}$$

$$\mathbf{r}_1 = (3\mathbf{i} - 5\mathbf{j} - 1\mathbf{k})$$

$$\text{Moment} = \mathbf{r}_1 \times \mathbf{F}$$

$$= [(-5)(-1) - (-1)(-4)]\mathbf{i} + [(-1 \times 4) - 3(-1)]\mathbf{j} + [3 \times 4 - (-5 \times 4)]\mathbf{k}$$

$$= 9\mathbf{i} - \mathbf{j} + 32\mathbf{k} \quad \textit{Ans.}$$

$$\mathbf{r}_2 = (5 + 2)\mathbf{i} + (2 - 3)\mathbf{j} + (3 - 5)\mathbf{k}$$

$$\mathbf{r}_2 = 7\mathbf{i} - 1\mathbf{j} - 2\mathbf{k}$$

$$\text{Moment} = \mathbf{r}_2 \times \mathbf{F} = [(-1) \times (-1) - (-2)(4)]\mathbf{i} + [(-2)(4) - 7(-1)]\mathbf{j} + [7 \times 4 - (-1)(4)]\mathbf{k}$$

$$= 9\mathbf{i} - \mathbf{j} + 32\mathbf{k} \quad \textit{Ans.}$$

Example 11. A force given by $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ is applied at the point $(1, -1, 2)$. Find the moment of the force **F** about the point $(2, -1, 3)$. See Fig. 12.

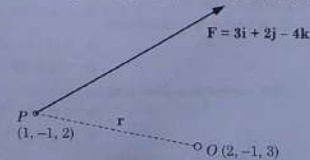

Fig. 12

Solution: $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

The position vector **r** of point P w.r.t. O,

$$\mathbf{r} = (1 - 2)\mathbf{i} + ((-1) - (-1))\mathbf{j} + (2 - 3)\mathbf{k}$$

$$\mathbf{r} = -1\mathbf{i} + 0\mathbf{j} - 1\mathbf{k}$$

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

$$= [0(-4) - (-1)(2)]\mathbf{i} + [(-1)(3) - (-1)(-4)]\mathbf{j} + [(-1)(2) - (0)(3)]\mathbf{k}$$

$$\mathbf{M} = 2\mathbf{i} - 7\mathbf{j} - 2\mathbf{k} \quad \textit{Ans.}$$

Example 12 A force of 22 N act through the point $A(4, -1, 7)$ in the direction of vector $9\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$. Find the moment of the force about the point $O(1, -3, 2)$.

Solution: Unit vector in the direction vector $9\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

$$= \frac{9\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{\sqrt{9^2+6^2+(-2)^2}}$$

$$= \frac{9\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{11}$$

Force

$$\mathbf{F} = 22\times\frac{1}{11}(9\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{F} = (18\mathbf{i} + 12\mathbf{j} - 4\mathbf{k})$$

Position vector $\mathbf{r}$ of the point A on the force with respect to the point O is

$$\mathbf{r} = (4-1)\mathbf{i} + (-1-(-3))\mathbf{j} + (7-2)\mathbf{k}$$

$$\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$$

$$\mathbf{M} = \mathbf{r}\times\mathbf{F} = (3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) \times (18\mathbf{i} + 12\mathbf{j} - 4\mathbf{k})$$

$$= -68\mathbf{i} + 102\mathbf{j} \quad \textit{Ans.}$$

A-8 COMPONENT OF A VECTOR AND MOMENT ABOUT AN AXIS

The component of a vector in any direction is the dot product of the vector with a unit vector in that direction.

Above concept is useful for finding the *Component of a force* in a given direction.

Since moment is also a vector, the above concept is useful is determining the *moment about an axis through* a given moment centre (Note the difference between the moment about a point and the moment about an axis). The moment $\mathbf{M}_{OA}$ about an axis OA passing through a moment centre (O) is dot product of the moment $\mathbf{M}_O$ with a unit vector $\hat{\mathbf{n}}_{OA}$ in the direction of this axis.

$$\mathbf{M}_{OA} = \mathbf{M}_O \times \hat{\mathbf{n}}_{OA}$$

Example 13. A force in x, y and z coordinate is given by $\mathbf{F} = 50\mathbf{i} - 80\mathbf{j} + 30\mathbf{k}$ (N). Determine the components of the force along the directions u, v and w in which w-axis coincides with the z-axis and u and v axis are rotated through 30° in the positive direction about the z-axis (Fig. 13).

Solution: Let us determine the unit vectors in the direction of u, v and w with respect to the x, y and z axis.

Unit vector in u-direction

$$\hat{\mathbf{u}} = (1\cos 30°)\mathbf{i} + (1\cos 60°)\mathbf{j} + (1\cos 90°)\mathbf{k}$$

$$\hat{\mathbf{u}} = 0.866\mathbf{i} + 0.5\mathbf{j}$$

Unit vector in v-direction

$$\hat{\mathbf{v}} = (1\cos 120°)\mathbf{i} + (1\cos 30°)\mathbf{j} + (1\cos 90°)\mathbf{k}$$

$$\hat{\mathbf{v}} = -0.5\mathbf{i} + 0.866\mathbf{j}$$

Unit vector in w-direction.

$$\hat{\mathbf{w}} = (1\cos 90°)\mathbf{i} + (1\cos 90°)\mathbf{j} + (1\cos 0°)\mathbf{k}$$

$$\hat{\mathbf{w}} = \mathbf{k}$$

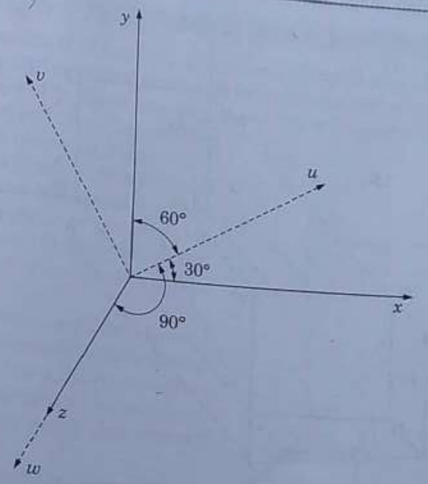

Fig. 13

Component for force $\mathbf{F}$,

along $u = \mathbf{F}\cdot\hat{\mathbf{u}} = (50\mathbf{i} - 80\mathbf{j} + 30\mathbf{k})\cdot(0.866\mathbf{i} + 0.5\mathbf{j}) = 3.3.$ N

along $v = \mathbf{F}\cdot\hat{\mathbf{v}} = (50\mathbf{i} - 80\mathbf{j} + 30\mathbf{k})\cdot(-0.5\,\mathbf{i} + 0.866\mathbf{j}) = -94.28$ N

along $w = \mathbf{F}\cdot\hat{\mathbf{w}} = (50\mathbf{i} - 80\mathbf{j} + 30\mathbf{k})\cdot(\mathbf{k}) = 30$ N

Example 14 A force of magnitude 100 N acts on the side of the block as shown in Fig. 14. Find the

(*i*) Moment of this force about the point E.

(*ii*) Moment of this force about an axis AE.

Solution: Coordinates of the corners of the box are

$A(0, 0, 0)$.	$B(0.5, 0, 0)$,
$C(0.5, 0, 1)$	$D(0, 0, 1)$
$E(0, 0.5, 0)$,	$F(0.5, 0.5, 0)$,
$G(0.5, 0.5, 1)$	$H(0, 0.5, 1)$,

Let us find the force vector $\mathbf{F}$ corresponding to the force of 100 N.

$$\frac{F_x}{X_F - X_C} = \frac{F_y}{Y_F - Y_C} = \frac{F_z}{Z_F - Z_C}$$

$$= \frac{F}{\sqrt{(X_F - X_C)^2 + (Y_F - Y_C)^2 + (Z_F - Z_C)^2}}$$

$$\frac{F_x}{0.5-0.5} = \frac{F_y}{0.5-0} = \frac{F_z}{0-1.0} = \frac{100}{\sqrt{0^2+(0.5)^2+(-1)^2}} = 90$$

$$\mathbf{F} = 0\mathbf{i} + 45\mathbf{j} - 90\mathbf{k}$$

Moment about point E.

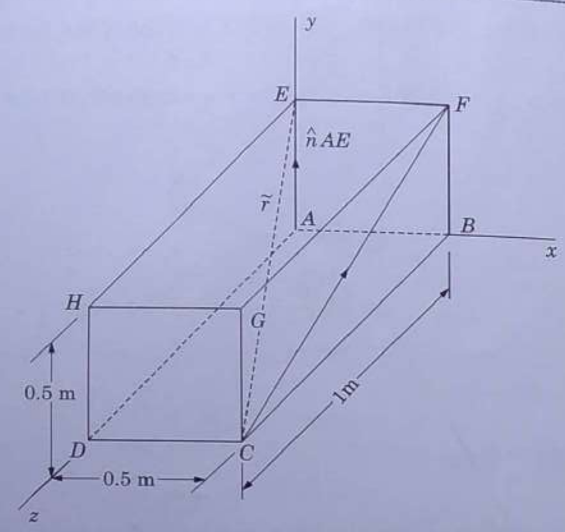

Fig. 14

Take any position vector **r** directed from E on the force **F**. Say $\mathbf{r}_{EC}$,

$$\mathbf{r}_{EC} = (X_C - X_E)\mathbf{i} + (Y_C - Y_E)\mathbf{j} + (Z_C - Z_E)\mathbf{k}$$

$$\mathbf{r}_{EC} = 0.5\mathbf{i} - 0.5\mathbf{j} + 1\mathbf{k}$$

$$\mathbf{M}_E = \mathbf{r}_{EC} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\mathbf{M}_E = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.5 & -0.5 & 1 \\ 0 & 40 & -80 \end{vmatrix}$$

$$\mathbf{M}_E = \mathbf{i}(40 - 40) - \mathbf{j}(40 - 0) + \mathbf{k}(20 - 0)$$

$$\mathbf{M}_E = 0.0\mathbf{i} - 40\mathbf{j} + 20\mathbf{k} \text{ Nm} \quad \textit{Ans.}$$

Moment of the force about the axis AE.

The component of vector in any direction is the dot product of the vector with a unit vector in the desired direction.

Sine moment is a vector, its component about an axis through the moment centre can be found as the dot product of the moment **M** with a unit vector $\hat{\mathbf{n}}$ in the direction of the axis.

Let $\hat{\mathbf{n}}_{AE}$ be the unit vector in the direction of AE.

Unit vector $\hat{\mathbf{n}}_{AE}$ is $0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k}$

Moment about the axis AE, $M_{AE} = M_E \cdot \hat{\mathbf{n}}_{AE}$

$$M_{AE} = (0\mathbf{i} - 40\mathbf{j} + 20\mathbf{k}) \cdot (1\mathbf{j})$$

$$M_{AE} = -40 \text{ Nm} \quad \textit{Ans.}$$

A-9 RESULTANT OF A SYSTEM OF FORCES IN SPACE

While determining the resultant of a system of coplanar forces it was observed that any given system of coplanar forces may be reduced to a single force or a single couple as the case may be. In the case of a spatial system of forces or forces in space this will generally not be possible. It is generally necessary to represent the system of force in space by a force plus a couple. A force-couple system may be reduced to a Single-force if the force and the couple vector are mutually perpendicular, that is, if the force lies in a plane perpendicular to the axis of moment (Example 16). Note that in Fig. 10 (*b*) force F_z and F_y lie in a plane perpendicular to the axis of the moment.

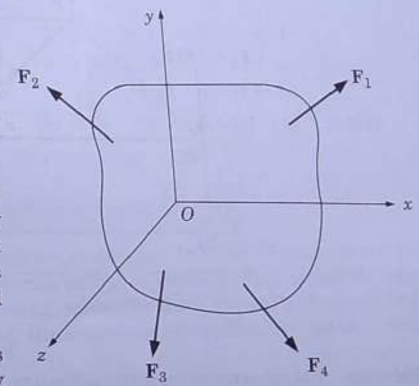

Fig. 15

Consider the problem of replacing a force system acting on a body by a *single resultant force acting through* a desired point plus a *single resultant couple* whose moment is the sum of the moments of the original force system about that desired point.

Let a body (Fig. 15) be acted by several forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$... This force system is to be replaced by a single force acting through the point 'O' plus a couple.

Let **R** be the resultant force acting through 'O' having components R_x, R_y and R_z and $(\mathbf{M}_R)_O$ be the resultant couple which is equal to the sum of the moments of the force system F_1, F_2, F_3, ... about the point 'O'.

Then, $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \ldots$

(Single Vector Equation)

Or $R_x = \Sigma F_x,\ R_y = \Sigma F_y, \text{ and } R_z = \Sigma F_z$

(Three Scalar Equations)

where, $R = \sqrt{(R_x)^2 + (R_y)^2 + (R_z)^2}$

and

$(\mathbf{M}_R)_O = \Sigma \mathbf{M}_O$ {Sum of the moments of all the forces $(F_1, F_2, F_3 \ldots)$ about the point 'O'}

(Single Vector Equation)

Or $(\mathbf{M}_R)_x = \Sigma M_x,\ (\mathbf{M}_R)_y = \Sigma M_y \text{ and } (\mathbf{M}_R)_z = \Sigma M_z$

(Three moment equations about the axes x, y and z)

where,

$$(M_R)_O = \sqrt{(\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2}$$

It can be observed here that the resultant force is independent of the position of the desired point but the resultant couple depends on the position of the point.

Example 15. A rectangular block is acted upon by forces as shown in Fig. 16. Replace this set of forces by an equivalent force-couple system acting at A.

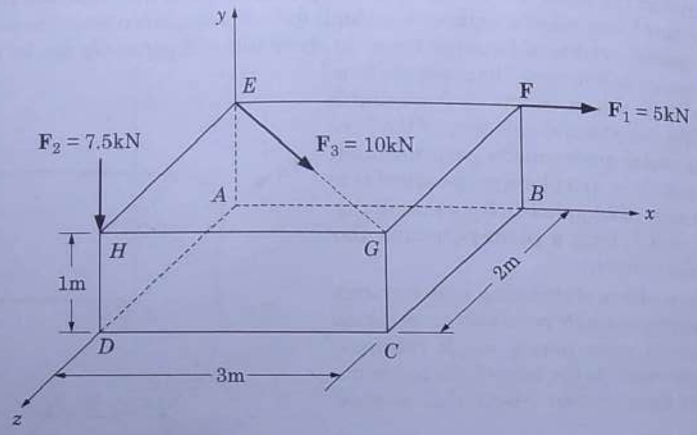

Fig. 16

Solution: Coordinates of various points are

$A(0, 0, 0)$, $E(0, 1, 0)$, $D(0, 0, 2)$, $F(3, 1, 0)$, $G(3, 1, 2)$

Different force vectors can be determined by the method followed in Example 2.

$$\mathbf{F}_1 = \frac{5}{\sqrt{9}}[(3-0)\mathbf{i} + (1-1)\mathbf{j} + (0-0)\mathbf{k}]$$

$$\mathbf{F}_1 = 5\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{F}_2 = \frac{7.5}{\sqrt{1}}[(0-0)\mathbf{i} + (0-1)\mathbf{j} + (1-1)\mathbf{k}]$$

$$\mathbf{F}_2 = 0\mathbf{i} - 7.5\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{F}_3 = \frac{10}{\sqrt{13}}[(3-0)\mathbf{i} + (1-1)\mathbf{j} + (2-0)\mathbf{k}]$$

$$\mathbf{F}_3 = 8.32\mathbf{i} + 0\mathbf{j} + 5.545\mathbf{k}$$

Resultant force **R**,

$$\mathbf{R} = \Sigma F = F_1 + F_2 + F_3$$

$$\mathbf{R} = (5\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + (0\mathbf{i} - 7.5\mathbf{j} + 0\mathbf{k}) + (8.32\mathbf{i} + 0\mathbf{j} + 5.545\mathbf{k})$$

$$\mathbf{R} = 13.32\mathbf{i} - 7.5\mathbf{j} + 5.545\mathbf{k} \quad \text{kN} \quad \textit{Ans.}$$

The resultant couple is found by summing the moments of each force ($\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$) about A. Position vectors from A to convenient points on forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ respectively being,

$$\mathbf{r}_{AE} = (0-0)\mathbf{i} + (1-0)\mathbf{j} + (0-0)\mathbf{k}$$

$$\mathbf{r}_{AE} = 1\mathbf{j}$$

$$\mathbf{r}_{AD} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (2-0)\mathbf{k}$$

$$\mathbf{r} = 2\mathbf{k}$$

$$\mathbf{r}_{AE} = 1\mathbf{j} \qquad \text{(Already calculated)}$$

Moments of all the forces about A,

$$(\mathbf{M}_R)_A = \mathbf{r}_{AE} \times \mathbf{F}_1 + \mathbf{r}_{AD} \times \mathbf{F}_2 + \mathbf{r}_{AE} \times \mathbf{F}_3$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{J} & \mathbf{k} \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{J} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & 7.5 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{J} & \mathbf{k} \\ 0 & 1 & 0 \\ 8.32 & 0 & 5.545 \end{vmatrix}$$

$$= (0\mathbf{i} + 0\mathbf{j} - 5\mathbf{k}) + (15\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + (5.545\mathbf{i} + 0\mathbf{j} - 8.32\mathbf{k})$$

$$(M_R)_A = 20.545\mathbf{i} + 0\mathbf{j} - 13.32\mathbf{k} \quad \text{kN.m} \quad \textit{Ans.}$$

A-10 EQUILIBRIUM OF SPATIAL SYSTEM OF FORCES

Equilibrium of the system of forces acting in a plane was described earlier. We can extend the same concepts to the equilibrium of a system of forces acting in space. Recalling that the term equilibrium is used to express the condition existing, when the force system acting on a body has a resultant force-couple system equal to zero, the equations of equilibrium can be written as follows :

(a) *Equilibrium of concurrent Spatial System of Forces*

$\Sigma F_x = 0$	...(*i*)		$\Sigma M_x = 0$	...(*i*)
$\Sigma F_y = 0$	...(*ii*)	OR	$\Sigma M_y = 0$	...(*ii*)
$\Sigma F_z = 0$	...(*iii*)		$\Sigma M_z = 0$	...(*iii*)
(Force)			(Couple)	

Or a combination of them. But there are only Three Independent conditions of equilibrium and hence not more than *Three* unknown quantities can be determined.

(b) *Equilibrium of Non-concurrent Spatial System of Forces*

$\Sigma F_x = 0$	...(*i*)		$\Sigma M_x = 0$	...(*iv*)
$\Sigma F_y = 0$	...(*ii*)	AND	$\Sigma M_y = 0$	...(*v*)
$\Sigma F_z = 0$	...(*iii*)		$\Sigma M_z = 0$	...(*vi*)
(Resultant Force in zero)			(Resultant couple is zero)	

In this case six equations are available hence not more than Six unknown quantities can be determined.

A-11 TYPES OF SUPPORTS AND SUPPORT REACTIONS (IN THREE DIMENSIONS)

Supports and support reactions for two dimensional problems were discussed in Art. 2.9. It may be recalled that in general, the action of a constrained body on any support induces an equal and opposite reaction from the support. This reaction will be induced in a direction in which the support *restricts* the motion of the body it supports. Based on the above concept are shown below (Fig. 17) some commonly used supports and support reactions.

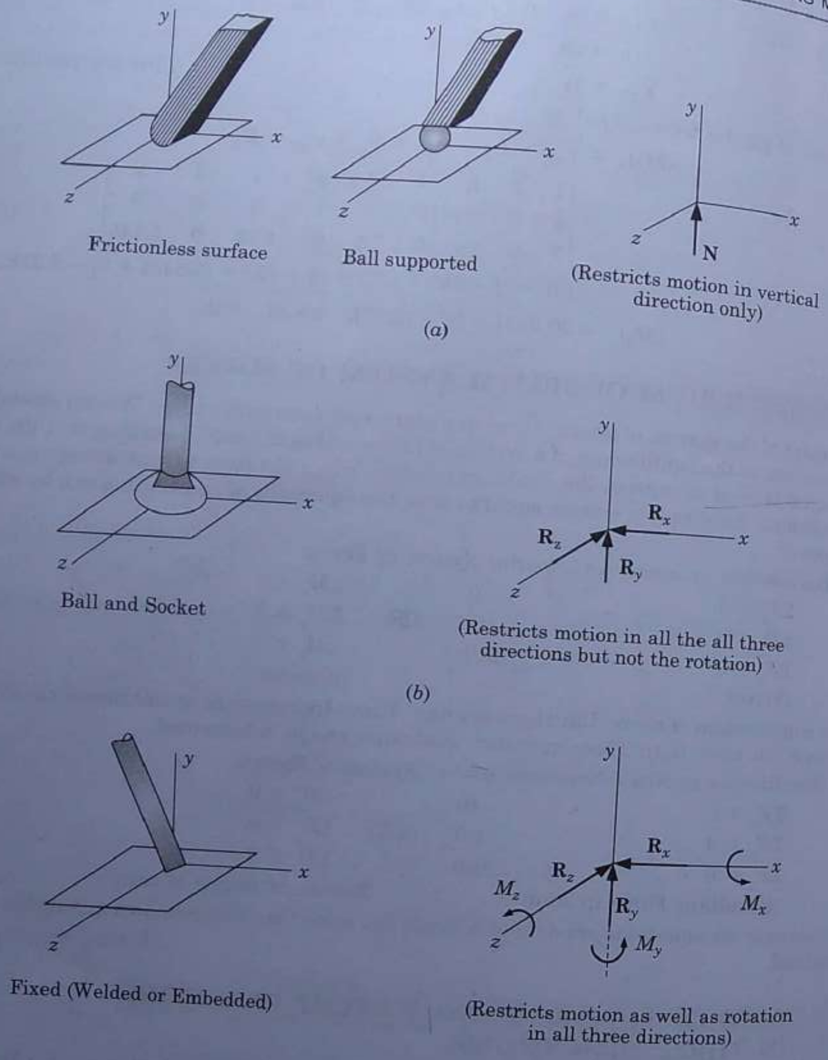

Fig. 17

Example 16. A plate *OABC* is acted upon by a force of 20 kN at *O* and two couple as shown in Fig. 18. Where should the point of application be shifted in the plane of the plate such that the system reduces to a single force.

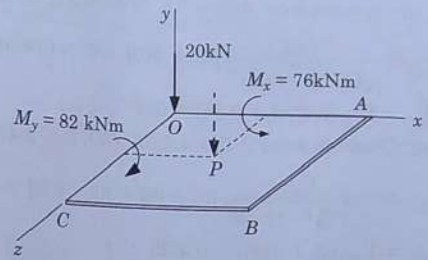

Fig. 18

Solution: The force of 20 kN is such that if shifted to any point P the force will lie in planes perpendicular to x and z axes thereby creating moments about these axes and in sense opposite to the two moments already acting on the plate.

Let force be shifted to P by distance x and z.

$$\left.\begin{aligned}\Sigma M_x &= 0: \quad 76 - 20 \times x = 0\\ \Sigma M_z &= 0: \quad 82 - 20 \times z = 0\end{aligned}\right\} \begin{aligned} x &= 3.80 \text{ m}\\ z &= 4.10 \text{ m}\end{aligned} \quad \textit{Ans.}$$

Example 17. A load W of magnitude 5000 N is supported by three cables. Determine the tension in cables (Fig. 19).

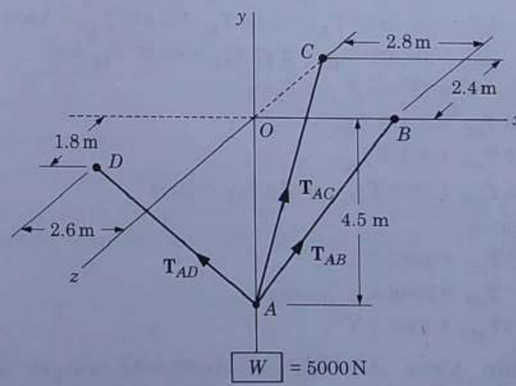

Fig. 19

Solution: This is a problem of concurrent forces in space. Three equations of equilibrium are available which can be solved for three unknown magnitudes of tensions.

$\mathbf{T}_{AB}$, $\mathbf{T}_{AC}$ and $\mathbf{T}_{AD}$ be the tensions in the cables whose directions are known. Coordinates of various points are,

$A(0, 4.5, 0)$, $B(2.8, 0, 0)$, $C(0, 0, -2.4)$, $D(-2.6, 0, 1.8)$

Tension in the cable AB [Directed from $A(0, -4.5, 0)$ to $B(2.8, 0, 0)$].

$$\mathbf{T}_{AB} = \frac{T_{AB}}{\sqrt{(2.8)^2+(4.5)^2}}[(2.8-0)\mathbf{i} + (0-(-4.5))\mathbf{j} + (0-0)\mathbf{k}]$$

$$\mathbf{T}_{AB} = \mathrm{T}_{AB}[0.528\mathbf{i} + 0.85\mathbf{j} + 0\mathbf{k}]$$

Tension in the cable AC (Directed from $A(0, -4.5, 0)$, $C(0, 0, -2.4)$),

$$\mathbf{T}_{AC} = \frac{T_{AC}}{\sqrt{(4.5)^2+(2.4)^2}}[(0-0)\mathbf{i} + (0-4.5)\mathbf{j} + (-2.4-0)\mathbf{k}]$$

$$\mathbf{T}_{AC} = \mathrm{T}_{AC}[0\mathbf{i} + 0.88\mathbf{j} - 0.47\mathbf{k}]$$

Tension in the cable AD (Directed from $A(0, -4.5, 0)$ to $D(-2.6, 0, 1.8)$),

$$\mathbf{T}_{AD} = \frac{T_{AD}}{\sqrt{(2.6)^2+(4.5)^2+(1.8)^2}}(-2.6-0)\mathbf{i}+(0+4.5)\mathbf{j}+(1.8-0)\mathbf{k})$$

$$\mathbf{T}_{AD} = \mathrm{T}_{AD}(-0.473\mathbf{i} + 0.818\mathbf{j} + 0.327\mathbf{k})$$

Weight W is acting vertically downwards

$$W = 5000\,(0\mathbf{i} - 1\mathbf{j} - 0\mathbf{k})$$

$$W = -5000\mathbf{j}.$$

Applying the equations of equilibrium

$$\Sigma F_x = 0 : 0.528\,T_{AB} + 0.0\,T_{AC} - 0.473\,T_{AD} = 0$$

$$\Sigma F_y = 0 : 0.85\,T_{AB} + 0.88\,T_{AC} + 0.818\,T_{AD} - 5000 = 0$$

$$\Sigma F_z = 0 : 0.0\,T_{AB} - 0.47\,T_{AC} + 0.327\,T_{AD} = 0$$

[Note, Now as $\mathbf{F} = (F_x i + F_y j + F_z k)$]

$$0.528\,T_{AB} = 0.472\,T_{AD}$$

$$0.47\,T_{AC} = 0.327\,T_{AD}$$

$$0.85\,T_{AB} + 0.88\,T_{AC} + 0.818\,T_{AD} = 5000$$

Solving above equations

$$\left.\begin{aligned}\mathbf{T}_{AD} &= 2282\text{ N}\\ \mathbf{T}_{AB} &= 2044.3\text{ N}\\ \mathbf{T}_{AC} &= 1587.7\text{ N}\end{aligned}\right\}\ \textit{Ans.}$$

Example 18 A thin plate $ABCDEF$ of negligible weight is supported in x-z plane at points A, B and C such that supports can provide reactions in y-direction only. It is acted upon by forces $P = 5$ kN and $Q = 3$ kN and a couple of 5 kNm as shown. Find the reactions at supports A, B and C.

Solution: Let the support reactions be R_A, R_B and R_C. Applying Moment Equations of Equilibrium

$\Sigma M_x = 0$: $\quad -(R_A \times 1.5) + (5 \times 0.625) + (3 \times 0.5) + 5 = 0$

$$R_A = \frac{9.625}{1.5},\ R_A = 6.417\text{ kN}\quad \textit{Ans.}$$

$\Sigma M_y = 0$: Moment about the axis is due to the forces lying in a plane perpendicular to this axis. There are no forces in the plane perpendicular to y-axis.

$\Sigma M_z = 0$: $(R_C \times 3.5) - (3 \times 3.5) - (5 \times 0.625) = 0$

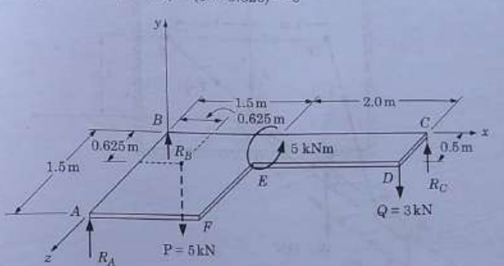

Fig. 20

$$R_C = \frac{13.625}{3.5},\ R_C = 3.893\text{ kN}\quad \textit{Ans.}$$

Applying Force Equations of Equilibrium,

$\Sigma F_x = 0$: No forces in x-direction

$\Sigma F_y = 0 : R_A + R_B + R_C - 5 - 3 = 0$

$6.417 + R_B + 3.893 - 8 = 0$

$R_C = -2.31$ kN *Ans.*

Assumed direction is to be reversed.

$\Sigma F_z = 0$: No force in z-direction.

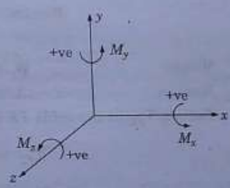

Fig. 20(a)

Example 19 A bar AB weighing 1 kN is held by a ball and socket at A and by two cables EF and DG and is loaded as shown in Fig. 21. Determine the tension in the cables and the reaction at A.

Solution: Various forces acting on the bar AB are,

(*i*) Force $\mathbf{F} = 2$ kN at the end B

(*ii*) Weight of the bar $\mathbf{W} = 1$ kN acting at C

(*iii*) Tension $\mathbf{T}_{CD}$ and $\mathbf{T}_{FE}$ is the cables

(*iv*) Reaction at the ball and socket $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$

It consists of three unknown components A_x, A_y and A_z.

Coordinates of various points are,

$A(0, 0, 0)$, $C(0, 0, 1.25)$, $B(0, 0, 2.5)$

$D(-1, 1, 0)$, $E(1, 1, 0)$, $F(0, 0, 1)$, $G(0, 0, 2)$

Force $\quad \mathbf{F} = 0\mathbf{i} - 2\mathbf{j} + 0\mathbf{k} = -2\mathbf{j}$

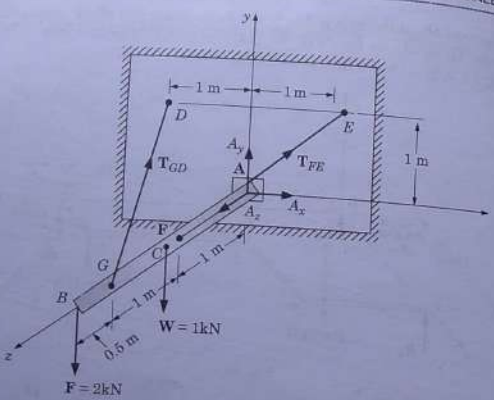

Fig. 21

Weight $\qquad \mathbf{W} = 0\mathbf{i} - 1\mathbf{j} + 0\mathbf{k} = -1\mathbf{j}$

Reaction $\qquad \mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$

Tension $\mathbf{T}_{FE}$ in the cable FE is directed from F to E,

$$\mathbf{T}_{FE} = \frac{T_{FE}}{\sqrt{(1^2+1^2+1^2)}}[(1-0)\mathbf{i} + (1-0)\mathbf{j} + (0-2)\mathbf{k}]$$

$$\mathbf{T}_{FE} = T_{FE}(0.58\mathbf{i} + 0.58\mathbf{j} - 0.58\mathbf{k})$$

Tension $\mathbf{T}_{GD}$ is directed of from G to D.

$$\mathbf{T}_{GD} = \frac{T_{GD}}{\sqrt{(-1)^2+1^2+(-2)^2}}[(1-0)\mathbf{i} + (0-1)\mathbf{j} + (0-2)\mathbf{k}]$$

$$\mathbf{T}_{GD} = \mathbf{T}_{GD}(-0.41\mathbf{i} + 0.41\mathbf{j} - 0.82\mathbf{k})$$

Applying the equations of equilibrium.

$\Sigma F_x = 0 : \quad A_x + 0.58\, T_{FE} - 0.41\, T_{GD} = 0$ ...(i)

$\Sigma F_y = 0 : \quad A_y - 2 - 1 + 0.58\, T_{FE} + 0.41\, T_{GD} = 0$ ...(ii)

$\Sigma F_z = 0 : \quad A_z - 0.58\, T_{FE} - 0.82\, T_{GD} = 0$ ...(iii)

We have five unknowns ($A_x, A_y, A_z, \mathbf{T}_{FE}$ and $\mathbf{T}_{GD}$) and the above three equations are not enough to determine these. So write moment equation,

$$\Sigma \mathbf{M}_A = 0 \text{ (sum of all the forces about 'A' is zero)}$$

$$\Sigma \mathbf{M}_A = \mathbf{r}_{AB} \times \mathbf{F} + \mathbf{r}_{AC} \times \mathbf{W} + \mathbf{r}_{AG} \times \mathbf{T}_{GD} + \mathbf{r}_{AF} \times \mathbf{T}_{FE} + 0 \times \mathbf{A}$$

where, $\mathbf{r}_{AB}, \mathbf{r}_{AC}, \mathbf{r}_{AG}, \mathbf{r}_{AF}$ and zero are the position vector from A to convenient points on forces $\mathbf{F}, \mathbf{W}, \mathbf{T}_{GD}, \mathbf{T}_{FE}$ and $\mathbf{A}$ respectively.

$$\begin{aligned}
\Sigma \mathbf{M}_A &= (2.5\mathbf{k}) \times (-2\mathbf{j}) + (1.25\mathbf{k}) \times (-1\mathbf{j}) + 2\mathbf{k} \times T_{GD}(-0.41\mathbf{i} + 0.41\mathbf{j} - 0.82\mathbf{k}) \\
&\quad + 1\mathbf{k} \times T_{FE}(0.58\mathbf{i} + 0.58\mathbf{j} - 0.58\mathbf{k}) + 0 \times (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) = 0 \\
&= 5\mathbf{i} + 1.25\mathbf{i} + (-0.82\mathbf{j} - 0.82\mathbf{i})\, T_{GD} + (0.58\mathbf{j} - 0.58\mathbf{i})\, T_{FE} = 0 \\
&= \underbrace{(6.25 - 0.82\, T_{GD} - 0.58\, T_{FE})}_{(M_x)}\mathbf{i} + \underbrace{(-0.82\, T_{GD} + 0.58\, T_{FE})}_{(M_y)}\mathbf{j} + \underbrace{0}_{(M_z)}\mathbf{k} = 0 \\
&= M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k} = 0
\end{aligned}$$

Or

$\Sigma M_x = 0 : \quad 6.25 - 0.82\, T_{GD} - 0.58\, T_{FE} = 0$ ...(iv)

$\Sigma M_y = 0 : \quad 0.82\, T_{GD} + 0.58\, T_{FE} = 0$ ...(v)

Solving, we get

$$T_{FE} = 3.8 \text{ kN}, \; T_{FE} = 5.38 \text{ kN}$$

$$A_x = 1.56 \text{ kN}, A_y = -1.67 \text{ kN}, A_z = 6.23 \text{ kN}$$

Alternatively, we could have determined M_x and M_y by taking moments of all the forces about x and y axes respectively. But in this problem calculating perpendicular distances from A to the forces T_{FD} and T_{FE} is somewhat difficult. In Example 18 we calculated directly the components M_x, M_y and M_z rather than getting from **M** because, forces involved were parallel to some axis and it was easier to calculate perpendicular distances.

PROBLEMS

1. The components of a force are $F_x = 300$ N, $F_y = -360$ N and $F_z = 520$ N. Determine the magnitude and direction of the force. [700 N; 64.6°, 120.9°, 42°]
2. A force acts at the origin and forms 110° and 65° with the y and z axes respectively. It is known that the x component of the force is + 30 N. Determine the magniutde of the force and the value of the angle that the force makes with the x-axis. [θ_x = 32.89°; 35.72 N]
3. Determine the two possible values of θ_y for a force **F** if the force forms equal angles with the positive x, y and z axes. [54.70°, 125.3°]
4. Given $A = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find a vector in the direction of **A** but of magnitude 12.0. [$8\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$]
5. Given $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, find $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$. [$-8$, $10\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}$]
6. Find the angle between the vectors $\mathbf{A} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} - 2\mathbf{j}$. [135°]
7. Find the angle between the space diagonals of a cube. [$\cos^{-1} 1/3$]
8. Given $\mathbf{A} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{C} = \mathbf{j} - 5\mathbf{k}$. Find the work done by forces **A** and **B** if a body undergoes the displacement **C**.
9. A force $\mathbf{F} = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k}$ acts at the point (– 5, 2, 1). Find the moment of the force **F** about the origin. [$-9\mathbf{i} - 23\mathbf{j} + \mathbf{k}$]

10. A force of 300 N acts through point $P(1, 6, -5)$ and directed towards $Q(0, 4, 3)$. Find the moment of the force about a point $A(1, 0, -1)$. Assume that distances are in metres.
[$(400\mathbf{i} + 400\mathbf{j} + 600\mathbf{k})$ Nm]

11. Find the shortest distance from the origin to the line passing through $A(-2, 1, 3)$ and $B(4, 5, 0)$.
[3.04 units]

12. A rectangular block is acted upon by forces shown in Fig. P.12. Replace this set of forces by a single resultant force and couple acting at the point O.

$$\left[\begin{array}{l}\mathbf{R} = -39.10\mathbf{i} + 46.47\mathbf{j} - 12.33\mathbf{k} \text{ kN} \\ \mathbf{M} = -19.4\mathbf{i} + 0\mathbf{j} + 59.10\mathbf{k} \text{ kN m}\end{array}\right]$$

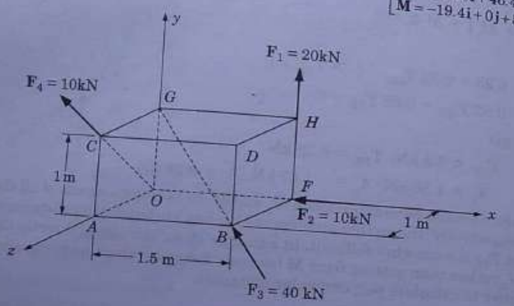

Fig. P.12

Replace the two couples acting on a block as shown in Fig. P.13 by a single equivalent couple.
[$\mathbf{M} = 25.95\mathbf{i} + 39.0\mathbf{j}$ kN m]

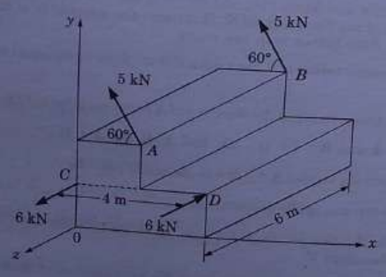

Fig. P.13

14. Determine the forces in the bars AB, AC, and AD when loaded at the joint A by a force, $\mathbf{F} = -30\mathbf{i} - 20\mathbf{j} + 40\mathbf{k}$ (kN) Fig. P.14.

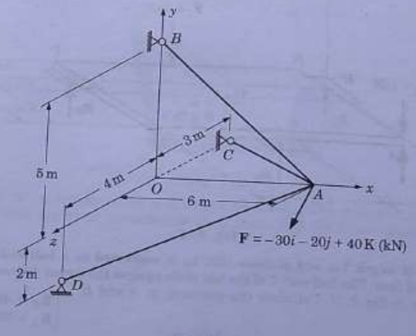

Fig. P.14

15. Replace the forces F_1, F_2 and F_3 acting on the pipe bend (Fig. P.15) by a force couple system at O.
[$\mathbf{R} = 50$ kN, $\mathbf{M} = 0\mathbf{i} - 25\mathbf{j} - 12.54\mathbf{k}$ kNm]

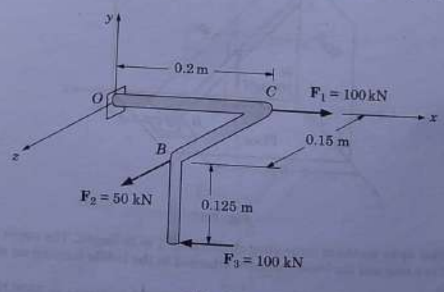

Fig. P.15

16. A uniform rectangular plate $ABCD$ 1 m × 2 m weighing 2 kN rests on three simple supports at, E, F and H. Calculate the reaction at supports for the loading shown in Fig. P.16.
[$R_E = 0.33$ kN, $R_F = 2.835$ kN and $R_G = 3.835$ kN]

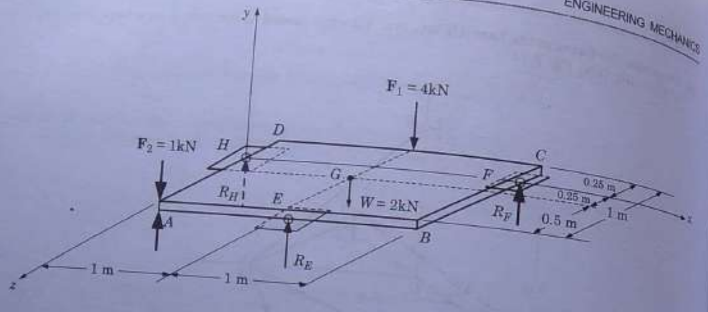

Fig. P.16

17. A uniform bar of length 7 m and of mass 1000 kg is supported by a ball and a socket joint at B in the horizontal floor. The ball end A of the bar rests against the corner formed by smooth vertical walls as shown in Fig. P.17. Calculate the reactions at A and B.

$$\begin{bmatrix} \mathbf{R}_B = -918\mathbf{i} + 981\mathbf{j} - 327\mathbf{k}\ \text{N} \\ \mathbf{R}_A = 981\mathbf{i} + 0\mathbf{j} + 327\mathbf{k}\ \text{N} \end{bmatrix}$$

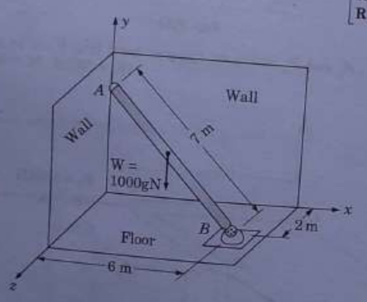

Fig. P.17

18. A laddle is lifted up by means of three sling chains each 1-m in length. The upper ends of the chains are attached to a ring and the lower ends are attached to the laddle forming an equilateral triangle of 1.2 m side.

Find the tension in each chain if the weight of laddel with contents is 4500 N. [2080 N]

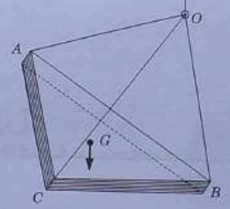

Fig. P.18

19. A weight of 30 kN is supported by a system consisting of two rods and a string as shown in Fig. P.19. The rods are hinge connected at their ends. Find the forces in rod and tension in the string.

$$\begin{bmatrix} F_{AB} = 15\,\text{kN}(C) \\ F_{BC} = 22.52\,\text{kN}(C) \\ T = 40.43\,\text{kN} \end{bmatrix}$$

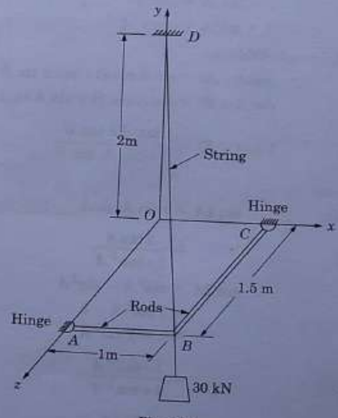

Fig. P.19

2 APPENDIX

Useful Formulae

TRIGONOMETRIC RELATIONS

Pythagoryan Relations

$$\sin^2 A + \cos^2 A = 1$$
$$1 + \tan^2 A = \sec^2 A$$
$$1 + \cot^2 A = \operatorname{cosec}^2 A$$

Angle Sum, Angle Difference Relations

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \ \tan B}$$

Double Angle Relations

$$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

DIFFERENTIATION

$f(x)$	$\frac{d}{dx} f(x)$	$f(x)$	$\frac{d}{dx} f(x)$
a	0	$\sin x$	$\cos x$
x^n	nx^{n-1}	$\cos x$	$-\sin x$
$(ax + b)^n$	$an(ax + b)^{n-1}$	$\tan x$	$\sec^2 x$
		$\cot x$	$-\operatorname{cosec}^2 x$
uv	$u\frac{dv}{dx} + v\frac{du}{dx}$	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
		$\sec x$	$\sec x \tan x$
		$\sin^{-1} x$	$(1 - x^2)^{-1/2}$
		$\cos^{-1} x$	$-(1 - x^2)^{-1/2}$
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$\tan^{-1} x$	$(1 + x^2)^{-1}$
e^x	e^x	$\sinh x$	$\cosh x$
$\log_e x$	$\frac{1}{x}$	$\cosh x$	$\sinh x$
$\log_{10} x$	$\frac{1}{x}\log_{10} e$	$\tanh x$	$\operatorname{sech}^2 x$

INTEGRATION

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
a	ax	a^x	$\frac{a^x}{\log_e a}$
x^n	$\frac{x^{n+1}}{n+1}, n \neq -1$	$\log_e x$	$x \log_e x - x$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)}, n \neq -1$	$\sin ax$	$-\frac{\cos ax}{a}$
$\frac{1}{x}$	$\log_e \lvert x \rvert$	$\cos ax$	$\frac{\sin ax}{a}$
e^{ax}	$\frac{e^{ax}}{a}$	$\tan ax$	$\frac{\log_e \sec ax}{a}$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert$
		$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$

e = Base of natural logarithm

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\dots\frac{1}{n!}$$

$$e = 2.71828$$

$$\log_e N = 2.3 \log_{10} N$$

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