Markov Chain

defination

A random process in which the occurrence of future state depends on the immediately preceding state and only on it, is known as Markov chain (or Process)

→ State : A state is a condition or location of an object in the system at particular time.

transition prob

Transition Probability

The probability of moving from one state to another state or remaining in the same state during a single time period is called the Transition probability.

Mathematically,

$$P_{ij} = \underset{\substack{\uparrow \\ \text{probability}}}{P}\left(\text{Next state } S_j \text{ at } \underbrace{t=1}_{\text{can be any value}} \middle| \text{ Initial state } S_i \text{ at } \underbrace{t=0}_{\text{can be any value}}\right)$$

$i \rightarrow$ Initial State $\qquad j \rightarrow$ Next State

Discrete - Time Markov Chains

In this class of models, the system transitions among a discrete set of values at various points in time.

Given figure shows an example system with 4 states.

discreate

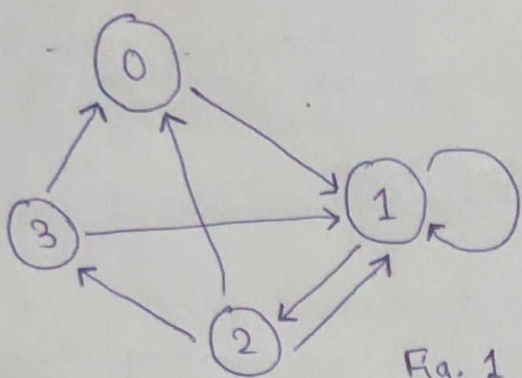

Fig. 1

For discrete-time Markov chains, transitions are assumed to occur at discrete points in time.

Let X_n denote the state of the system at time n, where $n = 0, 1, 2, \dots$

The fundamental assumption that underlies a Markov chain is the Markov property

$$Pr\{X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} = Pr\{X_{n+1} = j \mid X_n = i_n\}$$

Thus the Markov property states that if "present" state of the system (X_n) is known, then the "future" (X_{n+1}) is independent of the "past" $(X_0, \dots, X_{n-1})$.

In other words, in order to characterize the future behaviour (X_{n+1}) of the system, knowing the present (X_n) state is just as good as knowing the present state and the entire past $(X_0, \dots, X_{n-1})$ history. In Markov process, the past becomes irrelevant given the present state.

The conditional probabilities $\Pr\{X_{n+1} = j \mid X_n = i\}$ are called the single-step transition probabilities or just the transition probabilities.

Often these probabilities are assumed to be independent of n, in which case the chain is said to be homogeneous, and the transition probabilities can be written as

$$p_{ij} = \Pr\{X_{n+1} = j \mid X_n = i\}$$

homogenous

Note: Unless stated otherwise, the Markov chains are assumed to be homogeneous.

→ The matrix P formed by the elements p_{ij} is known as the transition matrix.

Example - The transition matrix associated with already drawn Fig. 1 has the form

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & p_{01} & 0 & 0 \\ 0 & p_{11} & p_{12} & 0 \\ p_{20} & p_{21} & 0 & p_{23} \\ p_{30} & p_{31} & 0 & 0 \end{pmatrix} \end{array}$$

where the p_{ij} values denote nonzero entries.

P is a stochastic matrix, meaning that its row sum equal to one ($\sum_j p_{ij} = 1$ for each i), since the transition probabilities out of any state i must sum to 1.

(The columns, however, need not sum to 1)

While time is observed at discrete points, these points do not necessarily need to be evenly spaced in real time. For example, in a queueing system, we ~~next~~ might measure the state of the system whenever a customer arrives to the queue. In this case, X_1 would be the state of the system as seen by the first arriving customer, X_2 would be the state of the system as seen by the second arriving ~~ca~~ customer and so forth.

queue

m-step transition probabilities

For a Markov chain, one may be interested in the m-step transition probabilities, defined as the probability of being in state j exactly m steps after being in state i.

m steps

More precisely, the m-step transition probability for a homogeneous chain is

$$p_{ij}^{(m)} = \Pr\{X_{n+m} = j \mid X_n = i\}$$

which is independent of n.

Let $P^{(m)}$ be the matrix formed by the elements $p_{ij}^{(m)}$.

It can be shown that $P^{(m)} = \underbrace{P \cdot P \cdots P}_{m \text{ times}} = P^m$

That is, the matrix of m-step transition probabilities can be obtained by multiplying the single-step matrix P by itself m times.

* The probability of being in any state j at time m can be defined as

$$\pi_j^{(m)} = \Pr\{X_m = j\}$$

We have $\pi_j^{(m)} = \sum_i \pi_i^{(m-1)} p_{ij}$

Matrix notation for this can be written as

$$\pi^{(m)} = \pi^{(m-1)} P$$

pi

Applying this rule recursively, we have

$$\pi^{(m)} = \pi^{(m-1)} P = \pi^{(m-2)} P \cdot P = \cdots = \pi^{(0)} P^m$$

where $\pi^{(0)}$ denotes the initial state distribution.

Example

Consider a Discrete-Time Markov Chain with two possible states 0 and 1 and transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix} \end{array}$$

The n-step transition probabilities are obtained by successive multiplication of P. Such as

$$P^2 = P \cdot P = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{11}{25} & \frac{14}{25} \\ \frac{7}{25} & \frac{18}{25} \end{bmatrix} \end{array}$$

$$P^4 = P^2 \cdot P^2 = \begin{bmatrix} \frac{11}{25} & \frac{14}{25} \\ \frac{7}{25} & \frac{18}{25} \end{bmatrix} \begin{bmatrix} \frac{11}{25} & \frac{14}{25} \\ \frac{7}{25} & \frac{18}{25} \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{219}{625} & \frac{406}{625} \\ \frac{203}{625} & \frac{422}{625} \end{bmatrix} \end{array}$$

For example, $p_{01}^{(4)} = \frac{406}{625}$, meaning that if the system is currently in state 0, then in 4 steps, it will be in state 1 with probability $\frac{406}{625} = 0.6496$
$\approx 65\%$

→ If the system starts in state 0 with probability $\frac{1}{4}$ and state 1 with probability $\frac{3}{4}$. That is

$$\pi^{(0)} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

We have the result $\pi^{(m)} = \pi^{(0)} P^m$

$\therefore$ For $m = 4$,

$$\pi^{(4)} = \pi^{(0)} P^4 = \begin{array}{cc} 0 & 1 \end{array} \left[\begin{array}{cc} \frac{1}{4} & \frac{3}{4} \end{array} \right] \begin{array}{c} \\ 0 \\ \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} \frac{219}{625} & \frac{406}{625} \\ \frac{203}{625} & \frac{422}{625} \end{bmatrix} \end{array}$$

$$= \left[\frac{828}{2500} \quad \frac{1672}{2500} \right]$$

$$= \begin{array}{cc} 0 & 1 \\ [0.3312 & 0.6688] \end{array}$$

pi

$$\frac{1}{4}\left(\frac{219}{625}\right) + \frac{3}{4}\left(\frac{203}{625}\right)$$
$$= \frac{1}{4 \times 625}\left[219 + 3(203)\right]$$
$$= \frac{1}{2500}(828)$$

$$\frac{1}{4}\left(\frac{406}{625}\right) + \frac{3}{4}\left(\frac{422}{625}\right) = \frac{1}{2500}(406 + 1266)$$
$$= \frac{1672}{2500} = 0.6688.$$

Hence, the probability of being in state 1 at time 4 is $0.6688 \sim 66.9\%$.

Chapman- Kolomogorov Equations

We have already defined the one-step transition probabilities p_{ij}. We now define the n-step transition probabilities $p_{ij}^{(n)}$ to be the probability that a process in state i will be in state j after n additional transitions.

That is $p_{ij}^{(n)} = P\{X_{n+m} = j \mid X_m = i\}$ $n \geq 0$ $i, j \geq 0$

n steps

Of course $p_{ij}^{(1)} = p_{ij}$

The Chapman- Kolomogorov equations provide a method for computing these n-step transition probabilities.

These equations are $p_{ij}^{(n+m)} = \sum_{k=0}^{\infty} p_{ik}^{(n)} p_{kj}^{(m)}$ — ①

for all $n, m \geq 0$ all i, j

If we let $P^{(n)}$ denotes the matrix of n-step transition probabilities $p_{ij}^{(n)}$ then using equation ①, we can say that

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

where the dot represents matrix multiplication.

Hence, $P^{(n)} = P \cdot P^{(n-1)} = P \cdot P \cdot P^{(n-2)} = \dots = P^n$

Thus $P^{(n)}$ may be calculated by multiplying the matrix P by itself n times.

Classification of States

classification

→ State j is said to be accessible from state i if for some $n \geq 0$, $p_{ij}^{(n)} > 0$. $(i \to j)$

(can also take $n = 0$)

→ Two states i and j accessible to each other are said to communicate and we write $i \leftrightarrow j$. (This means $i \to j$ & $j \to i$.)

Two states that communicate are said to be in the same class.

class

Irreducible (each state is accessible to itself 'OR' say communicate with each other because even if there is no loop even then $p_{ii}^{(0)} > 0$!)

We say that the Markov chain is irreducible if there is only one class — that is, if all states communicate with each other.

irreducible

1. Periodic State (Type of state)

State i is said to have period d if $p_{ii}^{(n)} = 0$ whenever n is not divisible by d and d is the greatest integer with this property.

peroidic state

<u>Note</u>: If $p_{ii}^{(n)} = 0$ for all $n > 0$ then we say that the period of i is infinite.

2 Aperiodic State (Type of state) aperiodic state

A state with period 1 is said to be aperiodic.

For any states i and j define $f_{ij}^{(n)}$ to be the probability that, starting in i, the first transition into j occurs at time n.

That is $f_{ij}^{(0)} = 0$

$$f_{ij}^{(n)} = P\{X_n = j,\ X_k \neq j \text{ for } k = 1, \dots, n-1 \mid X_0 = i\}$$

Let $f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$

Then f_{ij} denotes the probability of ever making a transition into state j, given that the process starts in i.

→ Note that for $i \neq j$, f_{ij} is positive if and only if, j is accessible from i.

recurrent

3. Recurrent State (Type of State)

State j is said to be recurrent if $f_{jj} = 1$

In other words, state j is said to be recurrent if the probability of being revisited from other states is 1.

4. Transient State (Type of State) transient state

A state j is said to be transient, if and only if, the state is not recurrent, i.e., if $f_{jj} < 1$.

Mathematically, this will happen if

$$\lim_{n \to \infty} p_{ij}^{(n)} = 0 \quad \text{for all } i.$$

Examples

1. If $P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \boxed{0} & 0.6 & 0.4 \\ 0 & 1 & 0 \\ 0.6 & 0.4 & \boxed{0} \end{bmatrix} \end{array}$ then $P^2 = \begin{bmatrix} \boxed{0.24} & 0.76 & 0 \\ 0 & 1 & 0 \\ 0 & 0.76 & \boxed{0.24} \end{bmatrix}$

$$P^3 = \begin{bmatrix} \boxed{0} & 0.904 & 0.0960 \\ 0 & 1 & 0 \\ 0.144 & 0.856 & \boxed{0} \end{bmatrix} \qquad P^4 = \begin{bmatrix} \boxed{0.0567} & 0.9424 & 0 \\ 0 & 1 & 0 \\ 0 & 0.9424 & \boxed{0.0576} \end{bmatrix}$$

$$P^5 = \begin{bmatrix} \boxed{0} & 0.97696 & 0.02304 \\ 0 & 1 & 0 \\ 0.03456 & 0.96544 & \boxed{0} \end{bmatrix}$$

The results show that p_{11} and p_{33} are positive for even values for n and zero otherwise (one can confirm this observation by computing P^n for $n>5$).

That is $p_{11}^{(n)} = 0$, $p_{33}^{(n)} = 0$ whenever n is not divisible by 2

$\therefore$ This means that each of states 1 and 3 has period $d=2$.

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5. Absorbing State (Type of state) absorbing state

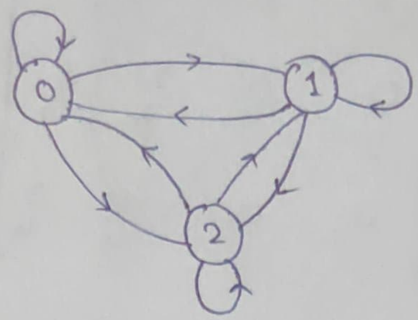
A state j is absorbing if it is certain to return to itself in one transition — that is, $p_{jj} = 1$, $p_{ij} = 0$ $i \neq j$

Every absorbing state is a recurrent state.

Examples for different types of states:

1. $$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ 1 & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 2 & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{array}$$

State-transition diagram is

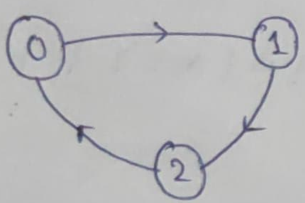


Since all the entries in P are positive and state-transition diagram also indicates that each state can communicate with other states.

$\therefore$ The given Markov chain is irreducible.

irreducible

2.



States 0, 1, 2 are periodic states with period 3.

Because starting from 0, ~~(or 1 or 2~~ we can only reach 0 in 3, 6, ... steps

i.e., if n is not divisible by 3

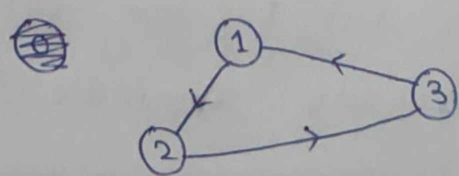
then $p_{00}^{(n)} = 0$

periodic state

Same for states 1 and 2.

3. $$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{array}$$

State-transition diagram is

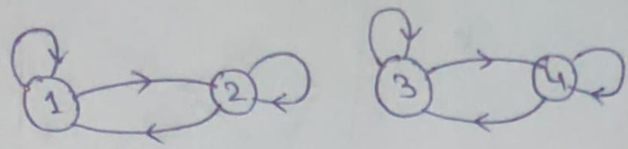


Here, all entries in transition matrix P are not positive but still this will be an irreducible Markov chain because all states can communicate with each other.

4.

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \end{array}$$

State-transition diagram is :



Since states 1, 3 can't communicate with each other

$\therefore$ This Markov chain is not irreducible, that is, this is a reducible Markov chain.

5.

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ P = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \end{array}$$

Period of state $i = d(i)$
$= \lambda(i)$
$= \gcd\{ n \geq 1 \mid p_{ii}^{(n)} > 0 \}$

Here $d(0) = \gcd\{ n \geq 1 \mid p_{00}^{(n)} > 0 \} = \gcd\{1, 2, \ldots\}$
$= 1$

$d(1) = \gcd\{ n \geq 1 \mid p_{11}^{(n)} > 0 \} = \gcd\{2, 3, \ldots\} = 1$

$d(2) = \gcd\{ n \geq 1 \mid p_{22}^{(n)} > 0 \} = \gcd\{1, 2, \ldots\} = 1$

Since $d(i) = 1$ for $i = 0, 1, 2$ thus all these three states are aperiodic states. aperiodic state

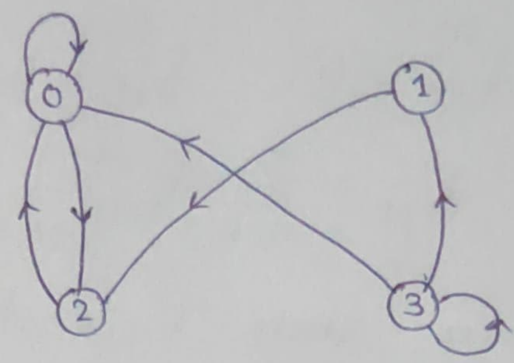
And since all states of this Markov chain are aperiodic hence the Markov chain will be aperiodic.

NOTE : If $i \leftrightarrow j$ then $d(i) = d(j)$

6.

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.8 & 0 & 0.2 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0.3 \end{bmatrix} \end{array}$$

State-transition Diagram:



We have $f_{ij}^{(0)} = 0$

$$f_{ij}^{(n)} = P\{X_n = j,\ X_k \neq j \text{ for } k = 1, \ldots, n-1 \mid X_0 = i\}$$

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

State i is recurrent if $f_{ii} = 1$

State i is transient if $f_{ii} < 1$.

$$f_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)}$$

$f_{00}^{(1)} = 0.8$, $\quad f_{00}^{(2)} = (0.2)(1) = 0.2$

$f_{00}^{(3)}$ not exist or say is 0 because provided

$X_0 = 0$, $\quad X_1 = 0$ for one path

$\quad\quad\quad\quad X_2 = 0$ for other path

i.e, there is no path following which $X_3 = 0$

but $X_1 \neq 0$, $X_2 \neq 0$ given $X_0 = 0$

Similarly we will get, $f_{00}^{(n)} = 0$ for $n \geq 4$.

$$\therefore\ f_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)} = f_{00}^{(1)} + f_{00}^{(2)} = 0.8 + 0.2 = 1$$

Hence, state 0 is recurrent.

$$f_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)}$$

$$f_{11}^{(1)} = 0, \quad f_{11}^{(2)} = 0, \quad f_{11}^{(3)} = 0, \ldots \quad f_{11}^{(n)} = 0$$

$$\therefore \quad f_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)} = 0 < 1$$

Hence state 1 is transient state

<u>Note:</u> Given $i \leftrightarrow j$ (i.e, $i \to j$ & $j \to i$)

Then i is recurrent $\Rightarrow$ j is recurrent

And i is transient $\Rightarrow$ j is transient.

We have $1 \to 0$ but $0 \not\to 1$ i.e, $0 \not\leftrightarrow 1$

We have $0 \leftrightarrow 2$, $0 \not\to 3$, $3 \to 0$ thus $0 \not\leftrightarrow 3$

And $1 \to 2$ but $2 \not\to 1$ thus $1 \not\leftrightarrow 2$ & $1 \not\leftrightarrow 3$

$\Rightarrow$ $0 \leftrightarrow 2$ and 0 is recurrent thus 2 is recurrent.

Now, $$f_{33} = \sum_{n=1}^{\infty} f_{33}^{(n)}$$

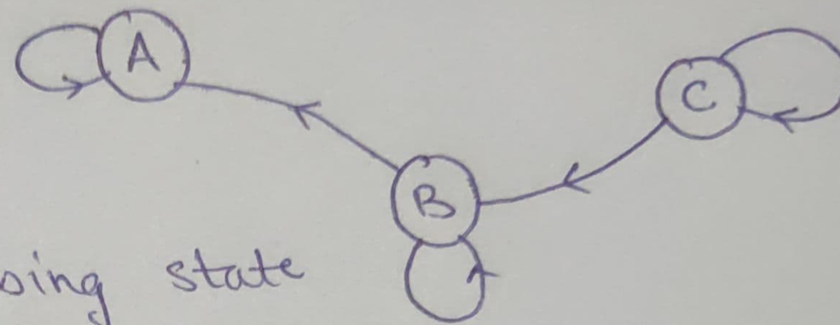
$$f_{33}^{(1)} = 0.3, \quad f_{33}^{(2)} = 0, \quad f_{33}^{(3)} = 0, \ldots, \quad f_{33}^{(n)} = 0, \ldots$$

$$\therefore \quad f_{33} = \sum_{n=1}^{\infty} f_{33}^{(n)} = f_{33}^{(1)} = 0.3 < 1$$

Hence state 3 is a transient state.

7. $P = \begin{array}{c|ccc} & A & B & C \\ A & 1 & 0 & 0 \\ B & 0.3 & 0.7 & 0 \\ C & 0 & 0.2 & 0.8 \end{array}$ irreducible

State-transition diagram is



Here A is an absorbing state as $p_{AA} = 1$ and $p_{AB} = 0 = p_{AC}$

B and C are not absorbing states.

8. $P = \begin{array}{c|ccc} & A & B & C \\ A & 0 & 0 & 1 \\ B & 0 & 1 & 0 \\ C & 1 & 0 & 0 \end{array}$

Here, only state B is an absorbing state.

Stationary Distributions of Markov Chains

A stationary distribution of a Markov Chain is a probability distribution that remains unchanged in the Markov chain as time progresses.

stationary

Typically, it is represented as a row vector π whose entries are probabilities summing to 1, and given transition matrix P, it satisfies

convergence

$$\pi = \pi P$$

In other words, π is invariant by the matrix P.

Ques Consider a Markov chain with three states 1, 2 and 3 and transition matrix

$$\begin{bmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$$

Find the stationary distribution for this chain.

Ans The stationary distribution $(\pi_1, \pi_2, \pi_3) = \pi$ will satisfy $\pi = \pi P$, that is,

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$$

System of equations obtained is -

$$\frac{2}{5}\pi_1 + \frac{1}{5}\pi_2 + \frac{1}{5}\pi_3 = \pi_1$$

$$\frac{2}{5}\pi_1 + \frac{3}{5}\pi_2 + \frac{2}{5}\pi_3 = \pi_2$$

$$\frac{1}{5}\pi_1 + \frac{1}{5}\pi_2 + \frac{2}{5}\pi_3 = \pi_3$$

$$\Rightarrow \quad -\frac{3}{5}\pi_1 + \frac{1}{5}\pi_2 + \frac{1}{5}\pi_3 = 0 \qquad \leftarrow I_0$$

$$\frac{2}{5}\pi_1 - \frac{2}{5}\pi_2 + \frac{2}{5}\pi_3 = 0 \qquad \leftarrow II_0$$

$$\frac{1}{5}\pi_1 + \frac{1}{5}\pi_2 - \frac{3}{5}\pi_3 = 0 \qquad \leftarrow III_0$$

$$\pi_1 - \frac{1}{3}\pi_2 - \frac{1}{3}\pi_3 = 0 \qquad I_1 = -\frac{5}{3} I_0$$

$$-\frac{4}{15}\pi_2 + \frac{8}{15}\pi_3 = 0 \qquad II_1 = II_0 - \frac{2}{5} I_1$$

$$\frac{4}{15}\pi_2 - \frac{8}{15}\pi_3 = 0 \qquad III_1 = III_0 - \frac{1}{5} I_1$$

$$\left[\begin{array}{ccc} \pi_1 & & -1\pi_3 = 0 \\ & \pi_2 - 2\pi_3 & = 0 \\ & & 0 = 0 \end{array}\right.$$

$I_2 = I_1 + \frac{1}{3} II_2$

$II_2 = -\frac{15}{4} II_1$

$III_2 = III_1 - \frac{4}{15} II_2$

$\therefore \pi_1 = \pi_3, \quad \pi_2 = 2\pi_3$

Since the sum of entries of row vector π should be 1 thus $\pi_1 + \pi_2 + \pi_3 = 1$ implies

$$\pi_3 + 2\pi_3 + \pi_3 = 1 \implies 4\pi_3 = 1$$

$$\therefore \pi_3 = \frac{1}{4}$$

and $\pi_1 = \frac{1}{4}, \quad \pi_2 = \frac{1}{2}$

Hence the stationary distribution π for the given Markov chain is $[\pi_1 \;\; \pi_2 \;\; \pi_3] = \left[\frac{1}{4} \;\; \frac{1}{2} \;\; \frac{1}{4}\right]$.

We would like to discuss long-term behaviour of Markov Chains.

Limiting Distribution of Markov Chains

Limiting State probabilities are given by

$$\lim_{n \to \infty} \text{Prob}\{X_n = j \mid X_0 = i\} \qquad i, j = 0, 1, 2, \ldots$$

limting distribution

Limiting distribution means that, what is the probability that the system starting from state i and reaches the state j as a n^{th} - step as $n \to \infty$, and this is also the definition of Limiting State Probabilities.

When a limiting distribution exists, it does not depend on the initial state, $X_0 = i$, so we can write

$$\lim_{n \to \infty} \text{Prob}\{X_n = j\}$$ for all possible values of j

Proposition 4.2.1 Communication is an equivalence relation.

That is (i) $i \leftrightarrow i$ (Reflexive property)

(ii) if $i \leftrightarrow j$ then $j \leftrightarrow i$ (Symmetric property)

reflexive

proposition

(iii) if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$

symmetric

(Transitive property)

transistive

Proof The parts (i) and (ii) follow trivially from the definition of communication.

(i) ~~Two~~ states ~~are~~ j is accessible from i ~~are said to be accessible~~ if $p_{ij}^{(n)} > 0$ $(i \to j)$ for $n \geq 0$

Two states i, j communicate with each other if $i \to j$ and $j \to i$.

So since for each state i, $p_{ii}^{(0)} > 0$

$\Rightarrow i \leftrightarrow i$

$\Rightarrow$ Communication is a reflexive property.

(ii) Given $i \leftrightarrow j$ that is $i \to j$ and $j \to i$

$\therefore$ Using $j \to i$ and $i \to j$ we can say that $j \leftrightarrow i$

Hence communication is a symmetric property.

(iii) Suppose that $i \leftrightarrow j$ and $j \leftrightarrow k$, then there exists m, n such that $p_{ij}^{(m)} > 0$, $p_{jk}^{(n)} > 0$ $(m, n \geq 0)$

Thus $$p_{ik}^{(m+n)} = \sum_{r=0}^{\infty} p_{ir}^{(m)} p_{rk}^{(n)}$$

$$\geq p_{ij}^{(m)} p_{jk}^{(n)} > 0$$

$\Rightarrow p_{ik}^{(m+n)} > 0$ where $m+n \geq 0$

$\therefore i \to k$

Similarly, we can show that there exists an $s \geq 0$ for which $p_{ki}^{(s)} > 0 \Rightarrow k \to i$

Hence $i \leftrightarrow k$

$\therefore$ Communication will ~~f~~ satisfy transitivity.

<u>Proposition 4.2.2</u> If $i \leftrightarrow j$ then $d(i) = d(j)$.

<u>Proof</u> Let m and n be such that $p_{ij}^{m} p_{ji}^{n} > 0$

(This will hold true since $i \leftrightarrow j$)

And suppose that $p_{ii}^{s} > 0$ (here $s \geq 0$)

(i.e, suppose s is the period of i

i.e. $d(i) = s$)

s is greatest such integer $p_{ii}^{(n)} = 0$ $n \neq s$)

proposition

Now $p_{jj}^{n+m} = \sum_{r=0}^{\infty} p_{jr}^{n} p_{rj}^{m} \geq p_{ji}^{n} p_{ij}^{m} > 0$

$\Rightarrow p_{jj}^{n+m} > 0$,

Also $p_{jj}^{n+s+m} \geq p_{ji}^{n} p_{ii}^{s} p_{ij}^{m} > 0$

$\Rightarrow p_{jj}^{n+s+m} > 0$

We know that if $d(j)$ is period of j then

$p_{jj}^{(k)} = 0$ where k is not divisible by $d(j)$ OR

$d(j)$ doesn't divide k

and $p_{jj}^{(t)} > 0$ where t is divisible by $d(j)$

Since here $p_{jj}^{(n+m)} > 0$ and $p_{jj}^{(n+s+m)} > 0$

$\therefore$ $d(j)$ divides both $n+m$ and $n+s+m$.

$\Rightarrow$ $d(j)$ divides s, where $s = (n+s+m) - (n+m)$

$\Rightarrow$ $d(j)$ divides $d(i)$

Similarly we can show that $d(i)$ divides $d(j)$.

$\therefore$ $d(i) = d(j)$

Hence proved.