5107
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# Discrete Mathematics

Q.) Show Venn diagram for the following

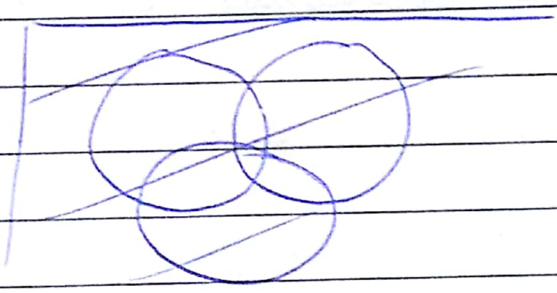
$A - ((B \cap C) \cup D)$

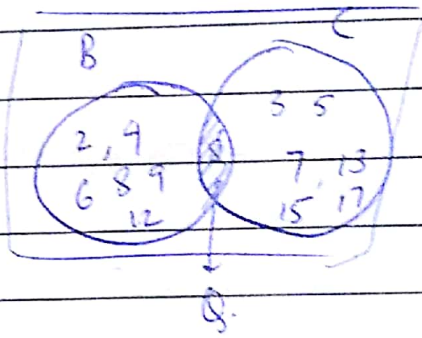
$A = \{1, 3, 5, 7, 8, 9, 10, 11, 13, 15\}$

$B = \{2, 4, 6, 8, 9, 12\}$

$C = \{3, 5, 7, 8, 13, 15, 17\}$

$D = \{1, 5, 7, 10, 11\}$



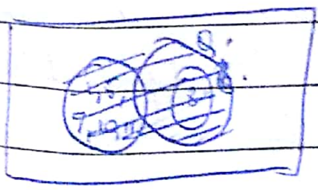



$B \cap C = \{8\}$

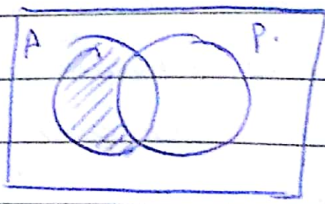
$P = (B \cap C) \cup D \Rightarrow \{1, 5, 7, 8, 10, 11\}$

$A - P = \{3, 9, 13, 15\}$

P =

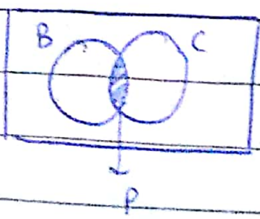


A-P.

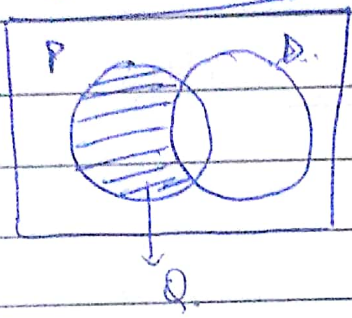



b) $A - ((B \cap C) - D)$ (No data)

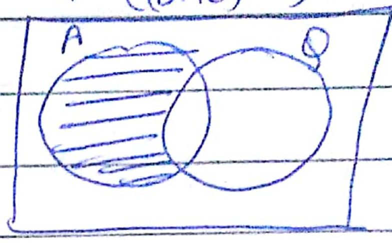
$B \cap C$



$(B \cap C) - D$




$A - ((B \cap C) - D)$.




⇒ Empty set : represented by $\phi$ or $\{\}$

NOTE: $\{\phi\}$ is a singleton set.

⇒ Singleton set : set with just 1 element.

⇒ Power set : set containing all possible sub-sets

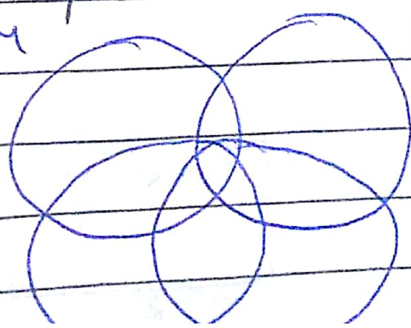
Power set allways contains $\phi$ & that set.

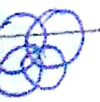
Power set of $\phi$ = $\{\phi\}$.

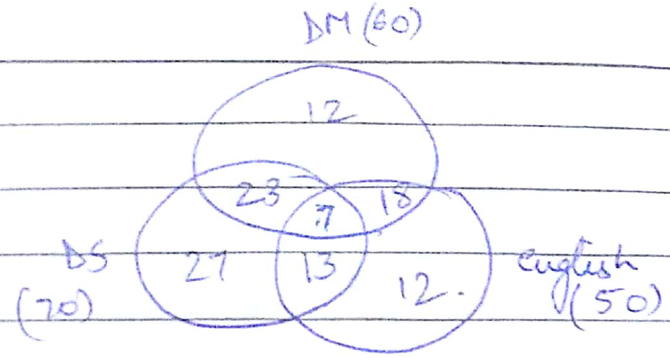
Q. Power set of $\{\phi, \{\phi\}\}$ ?

$\{\phi, \{\phi\}, \{\phi, \{\phi\}\}, \{\{\phi\}\}\}$.

Q.) Suppose there are 130 students in a class. Out of them 60 opted Discrete mathematics, 70 opted DS, 50 opted english, 30 opted both DM & DS, 20 opted DS & english, 25 opted english & DM. Find number of student not opted any thing. (7 opted all 3).

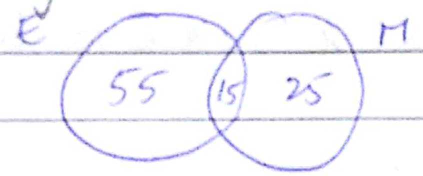




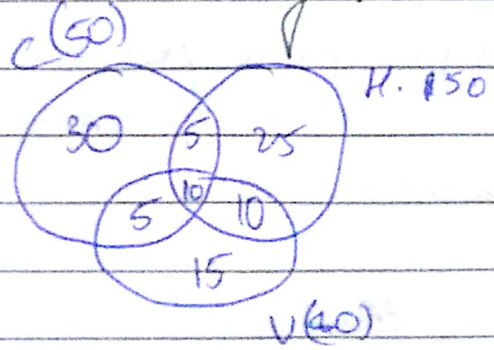

18

Q. In a class 40% of student enrolled for maths and 70% enrolled for eco. If 15% enrolled for both, what % of students of class did not enroll in anything




5 %

Q. Among a group of students, 50 played cricket, 15 played hockey, 40 played volleyball, 15 played cricket & hockey, 20 played hockey & volleyball, 15 played cricket & volley-ball, 10 played all 3. If every student played atleast 1 game, find number of students?



# Unit 1
## Sets.

Collection (unordered) of distinct elements is sets.

# Subset

$A \subseteq B$ — A is a subset of B.

if $\forall x \in A \Rightarrow x \in B$.

Strict subset / Proper Subset.

$A \subset B$ — A is proper subset of B

if $\exists$ (for some) $x$, $x \in B$ but $x \notin A$.

# Null Set ($\emptyset$ or $\{\}$)

Set that contains no element.

# Singleton Set

set that contains a single element.

# Power set

Set of collection of all possible subsets.

* Operations on Sets

# Union

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

# Intersection

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

# Subtraction

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$

# Compliment

$A'$ or $A^c = \{x \mid x \in \text{Universal set and } x \notin A\}$

# Cartesian product

$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

$N(A \times B) = N(A) \times N(B)$

⇒ No. of elements in a set are called cardinality of set

## Identities

1) Identity Law.

$A \cap U = A$.

$A \cup \phi = A$.

2) Domination Law.

$A \cup U = U$

$A \cap \phi = \phi$

3) Complimentation law.

$(A^c)^c = A$.

4) Commutative Law.

$A \cup B = B \cup A$.

$A \cap B = B \cap A$.

5) Associative Law.

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

6) Distributive Law.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

7) De Morgan's Law

$$\cancel{\overline{A \cup B} = \overline{A \cap B}} \quad \overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

8) Absorption law

$$A \cup (B \cap A) = A.$$

$$A \cap (B \cup A) = A.$$

Q1) Prove that for all sets S and T, $S = (S \cap T) \cup (S - T)$

using distributive law.

$$S = (S \cup (S-T)) \cap (T \cup (S-T))$$
$$= S \cap (T \cup (S-T)).$$

Let $x \in S$.

Case I: $x \in T$.

$\Rightarrow$ $(S \cap T)$ contains $x$.

$\Rightarrow$ S will be a subset of RHS.

Case II: $x \notin T$.

$\Rightarrow$ $(S \cap T) \not\ni x$

However

$x \in (S-T)$

on taking union, $x$ will be there

$\Rightarrow$ S will be subset of RHS. —— 1

Now.

~~Case I~~: $x \in (S \cap T) \cup (S-T)$ — P

Case I: $x \in (S \cap T)$

this means

$x \in S$ & $x \in T$.

$\Rightarrow$ P is subset of S.

Case II: $x \in (S-T)$.

$\Rightarrow$ $x \in S$ but $x \notin T$.

$\Rightarrow$ P is a subset of S. —— 2

from 1 & 2.

$P = S$.

Q2) Prove Absorption law.

$$A \cup (A \cap B) = A.$$

$A \cup (A \cap B)$

using distributive law.

~~$(A \cup A) +$~~

$= (A \cup A) \cap (A \cup B)$

$= A \cap (A \cup B)$

Since $(A \cup B)$ will contain all elements of A.

$\Rightarrow$ on taking intersection with A. we will get A.

$= A$.

Q3) Prove Demorgan's law.

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Let

$x \in \overline{A \cup B}$

$\Rightarrow x \notin (A \cup B)$ i.e. $x \notin A \cup B$.

i.e. $x \notin A$ and $x \notin B$

$\Rightarrow x \in \bar{A}$ and $x \in \bar{B}$

$\Rightarrow x \in \bar{A} \cap \bar{B}$

$\Rightarrow \overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$ —(i)

__/__/__

Now. Let $x \in \overline{A} \cap \overline{B}$

$\Rightarrow x \in \overline{A}$ and $x \in \overline{B}$

$\Rightarrow x \notin A$ and $x \notin B$.

$\Rightarrow x \notin A \cup B$.

$\Rightarrow x \in \overline{A \cup B}$

$\Rightarrow \overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$ —— (2)

from (1) & (2)

LHS = RHS

## * Relations

$R = \{(a,b) \mid R \subseteq A \times B$ where $a \in A$, $b \in B\}$.

## # Properties

1) Reflexive

$\forall a \in A$, $(a,a) \in R$.

(only for $R \subseteq A \times A$)

2). Symmetric

if $a$ is related to $b$.

$\Rightarrow b$ must be related to $a$.

3). ~~Tra~~ Antisymmetric

Irreflexive
$\forall a \in A \mid (a,a) \notin R$.

if a is related to b
and b is related to a.
$\Rightarrow a = b$.

4). Asymmetric
if a is related to b,
b should not be related to a.
i.e. (1,1) is not asymmetric.

5) Transitive
if a is related to b,
b is related to c,
then, a should be related to c.

$A = \{1,2,3\}$
$R = \{(1,2), (2,1)$
$(1,1), (3,1)\}$
NOT Transitive

6) Equivalence

# Equivalence Relation
A relation R is said to be equivalence relation if it follows all 3 properties :- reflexive, symmetric and transitive.

Q1) Let R be drawn over $A \times A$. It hold that
$R = \{(a,b) \mid (a<b)\}$. ~~NEA~~. $A = \{1,2,3,4\}$.
Find the relation R & check its equivalence.
$A = \{1,2,3,4\}$.
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3)$
$(3,4), (4,4)\}$

Q.) $A = \{2, 5, 10, 17 \cdots 122\}$. Find number of all elements $(n(P(A)))$ ?

2 5 10 17
3 5 7.

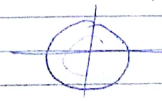
$S_n = 2\ 5\ 10\ 17 \cdots S_n$
$S_n$

$n^2+1 = 122$.
$n = 11$.
$\Rightarrow 2^{11} = n(P(A))$.

Q. $A \cap B = A \cap C \implies B = C$.
False.
B might have an element x
such that $x \notin A$ & $\notin C$

Q. if $A = \{(x,y) \mid x+2y=3\}$.
$B = \{(x,y) \mid 3x+2y=5\}$
then $A \cap B$.
$A = \{(1,1)\ (3,0) \cdots\}$
$B = \{(1,1), (1/3, 2) \cdots\}$
] (1,1).

Q. if $A = \{(x,y) \mid x^2+y^2 \le 4\}$
$B = \{(x,y) \mid x^2+y^2 \ge a\}$.
then for what value of a, will $A \cap B$ be non empty



$a \in [0, 4]$.

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Q. Classify the following as reflexive, symmetric, transitive, antisymmetric, asymmetric & irreflexive?

i) $R = \{(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$

$A = \{1,2,3\}$

Reflexive; Antisymmetric, transitive

ii) $R = \{(3,3), (6,6), (9,9), (12,12), (6,12), (3,9), (3,6), (3,9), (3,12)\}$ $A = \{3,6,9,12\}$

reflexive, transitive, Antisymmetric

iii) $R = \{(3,12), (3,6), (3,15), (9,6), (6,12), (6,15), (15,12), (12,6), (3,9)\}$ $A = \{3,6,9,12,15\}$

irreflexive ~~asymmetric~~

Q. Let $R \to Z$: by $\{(x,y)\} \in R$ if $x-y=10$.

$R = \{(10,0), (11,1) \dots (\infty, \infty-10)\}$.

asymmetric, irreflexive.

__/__/__

Q.) Check for equivalence.

$A = \{a, b, c, d, e, f\}$.

$R = \{(a,a), (b,b), (a,b), (b,a), (c,c), (d,d), (c,d), (d,d), (c,d), (d,c), (e,e), (f,f), (e,f), (f,e)\}$.

Reflexive ✓

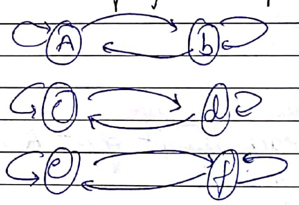
symmetric ✓

transitive ✓

] is equivalence relation.

⇒ Graphical Representation

1. Draw 1 vertex for each element in A.
2. Draw edges for each pair.




for an equivalence Relation we get a complete graph.

* Partition

A partition $\pi$ of a non-empty set A is a collection of non empty subsets of A such that ∀

i) ∀ $s, t \in \pi$

either, $s = t$ or $s \cap T = \emptyset$.

ii) Union of all subsets of $\pi$ = A.

S and t will be equivalence classes.

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eg = A = {1, 3, 5, 7, 9, 11, 13}.
then
$\pi$ = {{1,3}, {5,7,9}, {11,13}}.

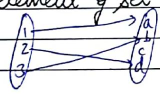
## * Function
Let A and B, be 2 sets. A Function from A to B denoted by :-

$$f : \underset{\text{domain}}{A} \to \underset{\text{codomain}}{B}$$

is a relation from $A \to B$ such that for every $a \in A$, there exists a unique ~~B~~ $b \in B$.

eg $f : N \to N$.
$f(x) = 1 \quad x \in odd.$
$f(x) = x/2 \quad x \in even$

## # Type.

1. One-to-one Mapping injective.
Function for which each element of set A is mapped to a unique element of set B.
eg



2. Onto mapping / Surjective.
~~For~~ Every element in ~~do~~ co-domain must have a pre-image in domain.

3. Bijective
A relation is said to be bijective if it is both surjective ~~or~~ ~~to~~ and injective

4. Many to One. mapping.
If some of the elements in co-domain have more than 1 pre-image.

## * ~~Fund~~ Composition of function
Function composition is a operation that takes 2 function f and g and produces a third function h such that

$$h(x) = g(f(x)).$$

eg if ~~f(x)~~ $f(x) = \{(1,3), (2,1), (3,4), (4,6)\}$.
$g(x) = \{(1,5), (2,3), (3,4), (4,1), (5,3), (6,2)\}$

g(f(x)).

$$gof(x) = \{(1,4), (2,5), (3,1), (4,2)\}.$$

Q. $f : R \to R \Rightarrow f(x) = 2x + 4.$
$g : R \to R \Rightarrow g(x) = x^3$
] Find gof(x) & fog(x)

g(x) $gof(x) = (2x+4)^3$
$fog(x) = 2x^3 + 4$

Q. $f(x) = \sqrt[3]{x-4}$ $\qquad g(x) = x^3 + 4$
$fog(x) = ?$
$fog(x) = x.$

Q. $f(x) = 2x + 4$ $\qquad g(x) = \cancel{\sqrt[3]{x-4}}\ x^3 \cdot h(x) = x^3 + 4$
$fogoh(x) = ?$
~~goh(x)~~ = x.

$$(f \circ g(x))^{-1} = (g^{-1} \circ f^{-1})(x)$$

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$g \circ h(x) = (x^3+3)^3$

$f \circ g \circ h(x) = 2(x^3+3)^3 + 4$

## * Proposition Logic

Proposition is a declarative statement which can have either true value or false value but not both.

eg

| | | |
|---|---|---|
| $2+3=5$ | True | ✓ |
| $2+3=9$ | True | ✓ |
| $x+1=5$ | maybe | ✗ |
| This is a book. | maybe | ✗ |
| This is a wrong statement | ✗ | (Lier's para-dox) |
| Answer this question! | ✗ | not a declarative stmt |

## # Logic Operators

### 1. Conjunction ( ^ or AND)

| p | q | p ^ q |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |
| 0 | 0 | 0. |

$0 \to F$

$T \leftarrow 1$

__/__/____

### 2. Disjunction operator (v - OR)

| p | q | p v q |
|---|---|---|
| F | F | F |
| T | F | T |
| F | T | T |
| T | T | T. |

If both proposition p & q are false, answer is false else answer is True.

### 3. Negation operator ( ~ or ¬ )

| p | ~p |
|---|---|
| T | F |
| F | T. |

### 4. exclusively OR ( ⊕ )

| p | q | p ⊕ q |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F. |

## # Conditional operator.

Let p & q be 2 propositions then conditional operato[r] is represented by $p \to q$ and infered as "if p then q." It will have only false if p is true and q is false

| p | q | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F. |
| F | T | T |
| F | F | T |

⇒ $p \to q$ can be expressed as:

| | | | |
|---|---|---|---|
| if p then q. p implies | if p, q / p only if q | p is sufficient for q / q if p | when q and p / q unless negation p. |

Q. Draw Truth Table for $\sim p \vee q$.

| p | q | $\sim p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

6. Bi-directional Condition $(p \leftrightarrow q)$

Infered as "p if and only if q". Value is true if both proposition contain same value.

equivalent to $(p \to q) \wedge (q \to p)$

| p | q | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T. |

Q. For a real $x$, $f(x) = 5x+1+x^3$. Find preimage of 0.

pre image b/w $-1, 0$.

Q. b) $x^2 - 1$

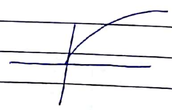
quadratic, many-one

into.

c) $f(x) = \sqrt{x}$.

$y^2 = x$.

one-one, onto.



Q. Write down proposition for following.—

a) You can access the internet from campus only if you are a CS major or not a freshman

(access internet) $\Rightarrow$ ((CS major) $\vee$ ($\sim$(freshman)))

$A \Rightarrow (B \vee (\sim C))$

b) You cannot ride the roller coaster if you are under 4ft tall unless you are older than 16 yr old.

p = ride
q = < 4
r = > 16.

$p \Rightarrow$ ride roller coaster.
$q \Rightarrow$ < 4 ft tall
$r \Rightarrow$ older than 16.

$\boxed{\sim(\sim q \vee r) \to \sim p}$.

$A_2$ $p \to (r \vee (\sim q))$

~~$\sim p \to (q \to r)$~~ $\quad \sim p \to (q \to r)$

| p | q | r | $A_1$ | $A_2$ |
|---|---|---|---|---|
| T | T | T | T. | T. |
| T | T | F | F | F |
| T | F | T | T | T. |
| T | F | F | T. | T. |
| F | T | T | T. | T |
| F | T | F | T. | T |
| F | F | T | T | T |
| F | F | F | T. | T. |

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⇒ for bidirectional.
→ p is neccessary and sufficent for q
→ if p then q, and convercely
→ p if and only if q.

Q. Write proposition using p and q, and logical connective for following
$p$ = it is below freezing
$q$ = it is snowing
1) it is below freezing & snowing
2) it is " " but not snowing
3) it is not below " and it is not snowing
4) it is either snowing or below freezing or both
5) if it is below freezing, it is also snowing
6) either it is below freezing or it is snowing, but it is not snowing if it is below freezing
7) it is below freezing is necessary and suffucent for it to be snowing

1) $p \wedge q$
2). $p \wedge (\sim q)$
(3) $(\sim p) \wedge (\sim q)$
(4) $p \vee q$.
(5) $p \to q$
(6) $(p \oplus q) \wedge (p \to (\sim q))$.
(7) $p \leftrightarrow q$.

* Converse of a Conditional Statement
Let condition statement is $p \to q$,
converse will be $q \to p$.

\# Contrapositive
Let conditional statement be $p \to q$.
Contrapositive statement ⇒ $\sim q \to \sim p$
negate + converse.

\# Inverse
Let conditional statement be $p \to q$,
inverse = $\sim p \to \sim q$.

Q. What are contrapositive converse & inverse of:-
a) Home team wins whenever it is raining.
~~if it is not raining,~~ whenever ~~the home team doesn't~~
contra "if the home team doesn't win, then it is not rai

_/_/_

Converse: If home team wins, it is raining

Inverse: ~~to~~ If it does not rain home team does not win.

Q. Design Truth Table.

1). $(p \to q) \to (q \to p)$

2) $(p \oplus q) \to (p \wedge q)$

3) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg \cancel{q}\, r)$

4) $(p \oplus q) \to (p \oplus \neg q)$

5) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

6) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg p) \wedge p$

1)

| p | q | $p \to q$ | $q \to p$ | A |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | T |
| F | T | T | F | F |
| F | F | T | T | T. |

2)

| p | q | $p \oplus q$ | $p \wedge q$ | A |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | T | F | F |
| F | T | T | F | F |
| F | F | F. | F | T. |

3)

| p | q | r | $p \leftrightarrow q$ | $\sim p \leftrightarrow \sim r$ | A |
|---|---|---|---|---|---|
| T | T | T | T | T | F |
| T | T | F | T | F | T |
| T | F | T | F | T | T |
| T | F | F | F | F | F |
| F | T | T | F | F | F |
| F | T | F | F | T | T |
| F | F | T | T | F | T |
| F | F | F | T | T | F |

4)

5).

6.

#

*

Q

**Precedence**

$\sim, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$

$(p \vee (q \wedge r))$ & $(p \vee q) \wedge (p \vee r)$

| $p$ | $q$ | $r$ | $p \vee (q \wedge r)$ | $(p \vee q) \wedge (p \vee r)$ | A |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | T | T |
| T | F | T | T | T | T |
| T | F | F | T | T | T |
| F | T | T | T | T | T |
| F | T | F | F | F | T |
| F | F | T | F | F | T |
| F | F | F | F | F | T |

Tautology

Hence logically Equivalent.

Q. b) $(p \leftrightarrow p \wedge q) \equiv (p \rightarrow q)$

| $p$ | $q$ | $p \wedge q$ | LHS | RHS | $\equiv$ 'A |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | F | T |
| F | T | F | T | T | T |
| F | F | F | T | T | T |

Hence logically equivalent.

# Logical Equivalence Rules

1) Identity Law.

$A \wedge T \equiv A$

$A \vee F \equiv A$

2) Domination Law.

$A \wedge F \equiv F$

$A \vee T \equiv T.$

3) Idempotence

$P \wedge P = P$

$P \vee P = P.$

4) Double negation Law

$\sim(\sim P) \equiv P.$

5) Commutative Law

$p \wedge q \equiv q \wedge p$

$p \vee q \equiv q \vee p$

6) Associative law.

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

7) Distributive Law.

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

8) De-Morgan's Law.

$\sim(p \wedge q) \equiv \sim p \vee \sim q.$

$\sim(p \vee q) \equiv \sim p \wedge \sim q.$

9) Absorption Law.

$p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

_/_/_

# For conditional operators

1) $p \rightarrow q \cong \sim p \vee q.$

2) $p \rightarrow q \cong \sim q \rightarrow \sim p$

3) $p \vee q \cong \sim p \rightarrow q$ | $p \wedge q \cong \sim(p \rightarrow \sim q)$

4) $\sim(p \rightarrow q) \cong p \wedge \sim q$

5) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

6) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

7) $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

8) $(p \rightarrow r) \vee (q \rightarrow r) = (p \wedge q) \rightarrow r$

9) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

10) $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$

11) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

12) $\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q.$

_/_/_

Q. Show that $\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim q$

$\sim((p \vee (\sim p)) \wedge (p \vee q)).$

$\sim(T \wedge (p \vee q))$

$= \cancel{\sim(p \vee p} \quad \sim((p \wedge p) \vee (p \wedge q))$

$= \sim p \wedge \sim q$

Q. $p \leftrightarrow p \wedge q \equiv p \rightarrow q.$

$\cancel{p \leftrightarrow (p \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow p)}.$

$\cancel{(p \wedge (p \rightarrow q)) \wedge (p \vee p(q \rightarrow p))}.$

$\cancel{\Rightarrow (p \wedge (p \wedge q)) \vee (\sim p \wedge (\sim p \vee \sim q))}.$

$\cancel{\Rightarrow (} p$

$= (p \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow p).$

$= (\sim p \vee (p \wedge q)) \wedge ((\sim p \vee \sim q) \vee p)$

$= (T \wedge (\sim p \wedge q)) \wedge (T)$

$= \underline{(\sim p \wedge q)}.$

__/__/__

Q. Let 2 kinds of ~~inhabited~~ inhabitant in an island ~~there are~~ knights who always tell truth and their opposites knaves who always lie. You encounter 2 pupil A & B. What are A & B if A says B is a knight & B says the 2 of us are of ~~diff~~ opposite ~~the~~ type?

~~A~~ B. ~~p = B is knight~~
~~q = both are opposite type.~~

| A | B | Ans. |
|---|---|---|
| N | N | F |
| N | K | F |
| K | N | F |
| K | K. | T. |

## * ~~Predicates~~ Quantifiers

## # Propositional Satisfiability

A compound proposition is satisfiable if there is a assignment of truth values to its variables that makes it true.

When no such arrangement exists, the compound proposition is false for all arrangement of truth values, it is unsatisfyable.

eg $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$

| p | q | r | A |
|---|---|---|---|
| T | T | T | T |
| T | T | F | F |
| T | F | T | F |
| T | F | F | F |
| F | T | T | F |
| F | T | F | F |
| F | F | T | F |
| F | F | F | T. |

$\textcircled{T} \rightarrow$ satisfiable.

**DUAL**

$\wedge \rightarrow \vee$ | $T \rightarrow F$ | negation DO NOT change.
$\vee \rightarrow \wedge$ | $F \rightarrow T.$ |

__/__/__

Quantifiers are used to optimise predicates.

Types:-

1). Universal Quantifier (forall $\forall$)
2) Existential Quantifier ($\exists$)

eg Some students in class opted DM
$x \rightarrow$ students ~~in class~~ opted DM.
$\exists x : P(x)$ is proposition.

~~eg~~

Variables bounded by some quantifiers $\Rightarrow$ bounded variable
Variables not bounded by quantifier $\Rightarrow$ free variable.

## * Logical Equivalence involving Quantifiers

Q. $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x).$

Let a be a element in Domain.

$\Rightarrow \forall a\,(\underbrace{P(a)}_{\text{TRUE}} \wedge \underbrace{Q(a)}_{\text{TRUE}})$

$\Rightarrow \left.\begin{array}{l}\forall a\ P(a) \rightarrow T \\ \forall a\ Q(a) \rightarrow T.\end{array}\right] T.$

## # Negation of Quantified statement

$\sim (\forall x (P(x))) \equiv \exists x \sim P(x)$

~~Tob~~ That is:-

$\exists \longrightarrow \forall$
$\forall \longrightarrow \exists$

Q. Every student in your class has taken a course in Calculus.
$P(x)$ = x has taken a course in calculus.
~~$Q(x) \Rightarrow$ is student of our class.~~

~~$\forall x \, (Q(x) \rightarrow P(x))$~~

$\sim (\forall x \, P(x)) \Rightarrow$ This is not the case that every student has taken a course in calculus.

$\Rightarrow \exists x \, \neg P(x)$

Q. Express the statement using Predicates & Quantifiers
a) Every student in this class has studied calculus.

$P(x)$ = x has studied calculus.
$\forall x : P(x)$
$Q(x)$ x is a student

$\forall x \, (Q(x) \rightarrow P(x))$.

Q. b) Some students in this class have visited Agra

$P(x) \Rightarrow$ x is student in our class.
$Q(x, ~~y~~) \Rightarrow$ x has visited Agra
~~y = Agra.~~
$\exists x : P(x) \rightarrow Q(x)$

This is false as. if $P(x)$ is ~~true~~ false
$Q(x)$ is false
But ans = TRUE.

NOTE
In case of there exists conditional might fail.

Ans $\exists x : P(x) \wedge Q(x)$.

# Inferences from proposition

- Arguments: Collection of 2 things:- (a) Premises (info)
(b) Conclusion

If you have a voter id, you can vote ] premises.
You have voter id
You can vote $\Rightarrow$ Conclusion

$\rightarrow$ Argument
OR
$P \Rightarrow$ You have voter id
$Q \Rightarrow$ You can vote.

$$\begin{array}{c} P \rightarrow Q \\ P \\ \hline Q \end{array}$$

] Argument form of proposition

# Rules
1) If $(P \rightarrow Q$ and $P)$ is given, you can conclude $Q$.
This Rule is called MODUS PONENS
{ Tautology: $[(P \rightarrow Q) \wedge P] \rightarrow Q$ }

__/__/__

2) If ($p \to q$ and $\neg q$) is given, you can conclude $\neg p$.
This rule is called MODUS TOLLENS.

3) If ($p \to q$ and if $q \to r$) is given, you can conclude $p \to r$
This rule is called HYPOTHETICAL SYLLOGISM

4) If (($p \vee q$) and $\neg p$) is given, you can conclude that $q$
This rule is called DISJUNCTIVE SYLLOGISM

5) If $p$ is given, you can conclude $p \vee q$.
This rule is called ADDITION.

6) If ($p \wedge q$) is given you can conclude $p$
This rule is called SIMPLIFICATION.

7). If ($p$ and $q$) is given you can conclude $p \wedge q$
This rule is called CONJUNCTION

8) If (($p \vee q$) and ($\neg p \vee r$)) is given you can conclude ($q \vee r$)
This rule is called RESOLUTION.

Q Show that the hypothesis
1) a) "It is not sunny this afternoon & it is colder than yesterday"
b) "We will go swimming only if it is sunny"

__/__/__

c) "If we do not go swimming, then we will take a manili trip"
d) "If we take a manili trip, then we will be home by sunset".
Lead to conclusion :- "We will be home by sunset - t."

$p \Rightarrow$ it is sunny this afternoon.
$q \Rightarrow$ it is colder than yesterday.
$r \Rightarrow$ we will go swimming
$s \Rightarrow$ we will take manilli trip
$t \Rightarrow$ we will be home by sunset.

premises

① $\neg p \wedge q$.
② $r \to p$.
③ $\neg r \to s$.
④ $s \to t$.

$t$.

Applying simplification rule on ①

⑥ $\neg p$.

using ⑥ and ② & MODES TOLLENS.

⑦ $\neg r$.

using ⑦ and ③, using MODES PONENS.

⑧ $s$.

using MODES PONENS on ④ & ⑧.

⑨ $t$.

__/__/__

2) a) "If today is tuesday, then I have a test in Maths or Economics"

b) "If my economics professor is sick, then I have no test of Economics"

c) Today is tuesday & my eco. prof. is sick then conclude that I have a test in Mathematics"

p : If today is Tuesday
q : I have a test in Maths
r : " " " " " Eco.
s ⇒ My Eco professor is sick

1) $p \rightarrow (q \vee r)$
2) $s \rightarrow \neg r$
3). $(p \wedge s)$ $q$.

Using simplification in (3).
(4) $p$ (5) $s$.
using MODES PONENS in (2) & (5) & (1) (4)
(6) $\neg r$.
(7) $q \vee r$
using disjunctive Syllogism. (6) & (7).
$q$.

# ~~Resto~~ Resolution Principle.

Represent every premise in form of Disjunction
If it is conjunctive, bifurcate it into a new ~~class~~ clauses and combine them into disjunctive.
i.e. cancel negative & positive & represent rest

__/__/__

in disjunctive.

eg. (1) $p \rightarrow (q \vee r) \longrightarrow c_1 : \sim p \vee (q \vee r)$
(2) $s \rightarrow \neg r \longrightarrow c_2 : \sim s \vee \cancel{\neg r}$
(3) $p \wedge s \longrightarrow c_3 : p$
(4) $q$. $c_4 = s$.

~~cancel~~ $c_1 + c_2 = \sim p \vee q \vee \sim s$.
$c_1 + c_2 + c_3 \Rightarrow q \vee \neg s$.
$c_1 + c_2 + c_3 + c_4 \Rightarrow \boxed{q}$ conclusion

⇒ Rules of Inference for Quantifiers.

| Rules of Inference | Name |
|---|---|
| 1) $\forall x \, P(x)$ <br> $\therefore P(c)$ | Universal Instantiation |
| 2) $P(c)$ for an arbitary $c$ <br> $\therefore \forall x \, P(x)$ | Universal Generalisation |
| 3) $\exists x \, P(x)$ <br> $\therefore P(c)$ | Existential Instantiation |
| 4) $P(c)$ for some arbitary $c$ <br> $\therefore \exists x \, P(x)$ | Existential Generalisation |

Q. Show that the premises
a) "A student in this class has not read the book"

b) "Everyone in this class passed the first exam"
Imply the conclusion
"Someone who passed the first exam has not read the book.

$P(x) \Rightarrow x$ is a student of class ~~who has~~
$Q(x) \Rightarrow x$ has ~~not~~ read the book.
$R(x) \Rightarrow x$ passed the first exam.

$\exists x \; (P(x) \wedge \neg Q(x))$ ①
$\forall x \; (P(x) \rightarrow R(x))$ ②
$\exists x \; R(x) \wedge \neg Q(x)$

$P(c) \wedge R(c)$

In ① using existential instantiation
$P(c) \wedge \neg Q(c)$
$\Rightarrow$ $P(c)$ is true
$\neg Q(c)$ is true.
in ② using universal instantiation
$P(c) \rightarrow R(c)$.
if $P(c)$ is true $\Rightarrow$ $R(c)$ is true
(using MODUS PONENS Rule)

using $R(c)$ true & $\neg Q(c)$ true,
use conjunctive rule.
$\Rightarrow$ $R(c) \wedge \neg Q(c)$
Now using existential generalisation

$\exists x : (R(c) \wedge \neg Q(c))$

Co-set, hash diagram, Lattice.

Q. Validate the following
"All humming birds are richly coloured"
"No large birds lives on honey"
"Birds that do not live on honey are dull in colour"
concl. "Humming Birds are small".

Q. Foreach of these arguments, explain which rules of inferences are used in each step.
a) Ram, a student in this class knows how to write program in Java.
Everyone who knows how to write programs in Java can get a high paying job, therefor someone in this class can get a high paying job.

$P(x) \Rightarrow$ ~~Ram~~ x is a student of this class
$Q(x)$ x knows how to write a program in Java.
$\exists x \; (P(x) \wedge Q(x))$ — ①

$R(x) \Rightarrow x$ can get high paying job
$\forall x \; Q(x) \longrightarrow R(x)$. — ②

$\exists x \; (P(x) \wedge Q(x))$ — ①
$\forall x \; (Q(x) \longrightarrow R(x))$ — ②
$\exists x \; (P(x) \wedge R(x))$
using universal generalisation $\forall x P(x) \wedge Q(x)$
Using simplification in ①.
$\exists x$ $P(x)$ & $Q(x)$ is true.
using Modus Ponens in ②

as $Q(x)$ is true.
$\Rightarrow R(x)$ is true.

$\Rightarrow \forall x\ P(x) \wedge R(x)$ is true.

—x—

1) $P(Ram) \wedge Q(Ram)$.
2) $\forall x\ P(x) \rightarrow R(x)$.

$\exists x\ P(x) \wedge R(x)$

in ② universal instantiation
$P(c) \rightarrow R(c)$.

Now simplification in (1).
$P(Ram)$ ✓ ③.
$Q(Ram)$ ✓.

Now Modus Ponens in ③ & ②.
$R(Ram)$ ✓ ④

Now perfor conjunction in ③ & ④.
$P(Ram) \wedge R(Ram)$

Existential generalisation.
$\exists x\ P(x) \wedge R(x)$

Q. 1) Somebody in this class enjoys ~~quait~~ whale watching
Every person who enjoys whale watching cares about Ocean pollution.
Therefor there is a person in this class who cares about ocean pollution.

$P(x)$ = x is in class
$Q(x)$ = x likes whale watching
$R(x)$ = x cares about ocean poll.

① $\exists x\ P(x) \wedge Q(x)$
② $\forall x\ Q(x) \rightarrow R(x)$
③ $\exists x\ P(x) \wedge R(x)$.

Existential initiation in ①.
$P(c) \wedge Q(c)$ — ④
using simplification in ④
$P(c)$ — ⑤
$Q(c)$ — ⑥.

Universal instantiation in ②.
$Q(c) \rightarrow R(c)$ — ⑦
using Modes ponens in ⑦ & ⑥.
$R(c)$ — ⑧

Now using conjunction ⑧ & ⑤.
$P(c) \wedge R(c)$
using existential generalisation.
$\exists x\ P(x) \wedge R(x)$

Q. Each of 120 student in class owns a PC
Every one who owns a PC, can use a word processing program
Therefore Ram a student in this class can use a word processing program.

$P(x) \rightarrow$ x is in class.
$Q(x) \rightarrow$ x owns a PC
$R(x) \rightarrow$ x can use word processing program.
$P(Ram)$ (*)

① $\forall x\ P(x) \rightarrow Q(x)$. '→' always with ∀x
② $\forall x\ Q(x) \rightarrow R(x)$.

$P(Ram) \wedge R(Ram)$

universal instantiation in 1 & 2

$P(c) \to Q(c)$. 3

~~simplification in 3.~~

~~$P(x)$ — 4~~

~~$Q(x) \to$ 5.~~

~~Modes ponens in 5 & 2.~~

~~$R(x)$ — 6.~~

~~Now conjunction in 4 & 6.~~

~~$P(c) \wedge R(c)$~~

~~c = Ram.~~

Hypothetical Syllogism

$P(Ram) \to R(Ram)$ 4

using 4 & ★ Modes ponens.

$R(Ram)$ 5.

Now conjunction in ★ & 5.

$P(Raul) \wedge R(Ram)$

* Methods of Proof

Types:

1) Direct Method

$\forall x \, (P(x) \to Q(x))$.

Pick the hypothesis (i.e. $P(x)$) use inference rules and some other rules or theorems to prove ($Q(x)$ is also true)

eg. if $\underbrace{n \text{ is a odd number}}_{P}$, then $\underbrace{n^2 \text{ is odd}}_{Q}$.

to prove $p \to q$

i) assume p is true

$\Rightarrow n$ is odd.

So we can write $n = 2k+1$

$$\Rightarrow n^2 = (2k+1)^2 = 4k^2 + 1 + 4k.$$
$$= 4k \cdot (k+1) + 1$$
$$= 2(2k^2 + 2k) + 1$$
$$= 2k' + 1$$
$$= \text{odd}.$$

Hence $n^2$ = odd.

Hence q is true. if p is true.

$\Rightarrow p \to q$ is true.

2) Indirect method

→ Contrapositive

→ Contradiction

i) Contrapositive

if $p \to q$ then $\sim q \to \sim p$

for proving $p \to q$ we can do,

$\sim q \to \sim p$.

eg if $\underbrace{3n+2 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is odd}}_{q}$.

Direct method

if $3n+2$ is odd.

$\Rightarrow (3n+2) - 2$ is odd.

$\Rightarrow 3n$ is odd.

$\Rightarrow 3 \times n$ is odd.

if since 3 is odd, n is odd

__/__/__

Contrapositive

t.p. : $p \to q$.

enough to prove $\sim q \to \sim p$

$\Rightarrow$ Let us assume $n$ is even

$\Rightarrow$ $n = 2k$.

$3n = 3 \times 2k$.

$= 6k = 2l$.

$3n+2 = 2l+2 = 2(l+1)$

$= 2k' = $ even.

So if $\sim q$ is true, $\sim p$ is also true

i.e. $\sim q \to \sim p$

using contrapositive $p \to q$.

2) Contradiction

eg

$p \Rightarrow 3n+2$ is odd

$q \Rightarrow n$ is odd

t.p. $p \to q$.

Assume $p$ is true & $q$ is false.

$\Rightarrow$ $n = 2k = $ even.

$\Rightarrow$ $3n+2 = 6k+2 = 2l = $ even.

$\Rightarrow$ $p$ is false.

Contradiction

Hence if $p$ is true, $q$ is true.

3) Tautological Method.

if $(p \to q)$ you can prove $q \to p$

__/__/__

eg if $n$ is odd, $n^2$ is odd,

t.p. if $n^2$ is odd $\to$ $n$ is odd.

using contrapositive $(\sim q \to \sim p)$.

i.e. $n$ is even

$n = 2k$.

$n^2 = 4k^2 = 2l$. $=$ even $= \sim p$

Hence if $\sim q \to \sim p$ is true

$\Rightarrow p \to q$.

Hence. using ~~cont~~ tautological method.

if $n$ is odd $\to$ $n^2$ is odd.

* Counting

## Unit 2

~~Exam~~

* **Counting**

# Product Rule

Applicable when a task is bifurcated into a number of subtasks which individually takes different number of attempts and depends on each other.

Suppose that a procedure can be broken down into a sequence of 2 tasks. If their are $n_1$ ways to do first task & for each of these ways there are $n_2$ ways to do the second ~~task~~ task, then their are

$$= n_1 \cdot n_2$$

ways to do procedure.

eg a company with 2 employs Ram, Shyam rents a floor of a building with 12 offices. How many ways are there to assign different offices to these 2 employees.

$^{12}C_2 = \underline{12}\ \underline{11} = 12\times 11$

Now positions can exchange.

$^{12}C_2 \cdot 2! = {}^{12}P_2$.

Q. Chairs of an auditorium are to be labelled with a letter and a positive integer not exceeding 100. Largest number of chairs that can be labled.

$\rightarrow {}^{26}C_1 \cdot {}^{100}C_1 = 2600$.

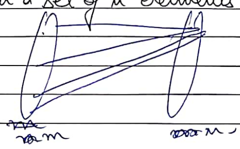
$\frac{4\times 3\cdot 2}{2}$

Q. How many diff. licence plates are available if each plate contains a sequence of 3 letters followed by 3 digits

$\underline{26}\ \underline{26}\ \underline{26}\ \underline{10}\ \underline{10}\ \underline{10}$

$= (260)^3$

Q How many functions are there from a set of m elements with a set of n elements.



$n^m$.

b) what if function are one-one to?

$^nC_m \cdot m!$

# Sum Rule

If a task can be done either one of $n_1$ ways or one of $n_2$ ways where none of the set of $n_1$ ways is same as any of the set of $n_2$ ways. Then number of ways to do the task are

$$= n_1 + n_2.$$

eg A student can select a computer project from one of the 3 list contain 23, 50 and 19 possible projects respectively no repetition in 3 lists. How many possible projects are there to select from

$= \underline{92}$.

Q. Each user on a computer system has a password which 6 to 8 character long, where each character is an upper case or a digit. Each password must contain atleast 1 digit. How many passwords there?

Case I: 1 digit.

$\Rightarrow$ $\cancel{{}^{10}C_1 \, {}^{26}C_5 \cdot 6!} + \cancel{{}^{10}C_1 \, {}^{26}C_6 \cdot 7!} + \cancel{{}^{10}C_1 \, {}^{26}C_7 \cdot 8!}$

Case 2: 2 digit.

$\cancel{{}^{10}C_2 \, {}^{26}C_4 \cdot 6!} + \cancel{{}^{10}C_2 \, {}^{26}C_5 \cdot 7!} + \cancel{{}^{10}C_2 \, {}^{26}C_6 \, 8!}$

Case 3: 3 digit.

$\cancel{{}^{10}C_3 \, {}^{26}C_3 \cdot 6!}$

$(36)^6 \rightarrow$ all possible 6 character.

$(36)^7 \rightarrow$ " " 7 "

$(36)^8 \rightarrow$ " " 8 "

$(26)^6 \rightarrow$ " " 6 " no digit.

$$\Rightarrow \left((36)^6 + (36)^7 + (36)^8\right) - \left((26)^6 + (26)^7 + (26)^8\right)$$

* Inclusion - Exclusion Principle.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Q. How many bit strings of length 8 either starts with 1 bit or end with 2 bits 00.

$\Rightarrow 2^7 + 2^6 - \underbrace{2^5}_{\text{common}} = 160.$

1,000

Q. A computer company receives 350 Applications from computer gadgets for a job planning a new web server. Suppose that 220 of these people majored in CS, 147 majored in business and 51 majored in both. How many of these majored in neither.

$P(T) = P(CS) + P(Business) - P(both) + P(None)$

$350 = 220 + 147 - 51 + x.$

$x = 34$

Q. How many 4 bits string do not have 2 consecutive '1's?

Case 1: 4 zeros

1

Case 2: One 1

4.

Case 3: ~~two~~ 1

1010, 0101, 1001

$\Rightarrow 8.$

Q. Suppose that a branded T-shirt comes in 5 different sizes S, M, L, XL, XXL. Suppose that each size comes in 4 colours, white Red green & black except for XL which comes in Red green & black & XXL which comes in green & black.

How many different shirts shopkeeper needs to stock to have atleast one of all available size & colour.

$\Rightarrow \cancel{5+5+5+3+2} =$

$4+4+4+3+2 =$

_/_/_

## Pigeonhole Principle.

If $k$ is a positive integer and $k+1$ or more objects are placed into $k$ boxs, then there is ~~atle~~ atleast 1 box containing ~~one or~~ 2 or more objects.

Q. How many students must be in a class to gurantee that atleast 2 students recieves the same score on the final exam, if the exam is graded from 0 to 100.

102.

✱

## # Generalised Pigeonhole principle

If ~~k~~ $n$ objects are kept in $k$ boxes, then there is atleast one box containing $\left\lceil \frac{N}{k} \right\rceil$ ($\lceil \cdot \rceil$) objects.

$\lceil \cdot \rceil$ - this is ceilingfunction i.e. $\lceil 3.3 \rceil = 4$.

Q. How many cards must be selected from a standard deck of 52 cards to guarentee that atleast 3 cards of same suit are selected.

$$\left\lceil \frac{N}{4} \right\rceil \geq 3$$

$$\left\lceil \frac{N}{4} \right\rceil = 3 \Rightarrow \frac{N}{4} = 2.$$

$$\boxed{N = 8}.$$

at $N = 8$.

we get 3.

atleast 3.

$N + 1 = \underline{\underline{9}}$.