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Mathematics
Unit 1.

* Linear Ordinary Differential Equation of Higher Order

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots = Q \quad \text{— (1)}$$

where $P_1, P_2, P_3 \cdots Q$ are functions of x or constants.

Eq^n (1) can also be written as

$$D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + P_3 D^{n-3} y + \cdots = Q.$$

where $D = \frac{d}{dx}$.

Solution of eq. (1) can be written as

y = complementary factor + Particular Integral

Calculation of complementary factor.

We can write Auxillary equation corresponding to given ODE (i) or (ii) :-

$$m^n y + P_1 m^{n-1} y + P_2 m^{n-2} y + P_3 m^{n-3} y + \cdots = 0 \quad \text{— (iii)}$$

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" LHS " " $\neq 0 \longrightarrow$ CF PI $\rightarrow$

After solving this we get roots of m.

Case 1: When roots of the auxillary eqⁿ maybe real numbers such that

$$m = a, b, c$$

the solution for the ODE is (CFs)

$$y = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx}$$

where c_1, c_2, c_3 are arbitrary const.

Q.) Solve ODE

$$\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0.$$

The given ordinary differential equation is as.

$$\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0.$$

Equation can be expressed as

$$D^3y + 6D^2y + 11Dy + 6y = 0.$$

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The auxillary eq^n corresponding to the question is given as.

$$(m^2 + 6m^2 + 11m + 6)\,y = 0.$$

$$\Rightarrow m^3 + 6m^2 + 11m + 6 = 0.$$

$$\Rightarrow (m+1)(m+2)(m+3) = 0.$$

$$\Rightarrow m = -1, -2, -3.$$

$\Rightarrow$. Hence the solution of given differential eq^n is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}.$$

Write : ① given
② What is procedure
③ Calculation
④ How did you reach there formula
⑤ Final answer.

Case II : If roots of auxillary equation are repeated.

$$m = a, a, a, b, b, c, c, c.$$

then solution is written as.

$$y = c_1 e^{ax}(c_2 x + c_3 x^2 + c_4 x^3)$$
$$+ c_5 e^{bx}(c_6 x + c_7 x^2)$$
$$+ c_8 e^{cx}(c_9 x + c_{10} x^2 + c_{11} x^3)$$

Q2). Solve ODE :-

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$$

The given ODE is

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0 \quad \text{--- (1)}$$

for this, we can write it as :-

$$(D_4 - D_3 - 9D_2 - 11D - 4)y = 0 \quad \text{--- (2)}$$

Auxillary eq^n for this eq^n is :-

$$y(m^4 - m^3 - 9m^2 - 11m - 4) = 0.$$

$$\Rightarrow m^4 - m^3 - 9m^2 - 11m - 4 = 0.$$

$$\Rightarrow (m+1)^3(m-4) = 0.$$

$$\Rightarrow m = -1, -1, -1, 4.$$

Hence the soln of this ODE can be written as.

$$y = c_1 e^{-x}(c_2 x + c_3 x^2 + c_4 x^3) + c_5 e^{4x}.$$

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Case III : If roots of auxiliary eqⁿ are complex number such that

$$m = a + ib$$

solution is given by :-

$$y = c_1 e^{ax}\left(\cos(bx) + i\sin(bx)\right).$$

Q. Solve ODE

$$\frac{d^2y}{dx^2} + 9y = 0.$$

$$\frac{d^2y}{dx^2} + 9y = 0.$$

$$\Rightarrow D_2 y + 9y = 0.$$

$$\Rightarrow (m^2 + 9)y = 0.$$

$$\Rightarrow m = \pm 3i$$

$$\Rightarrow m = 0 + 3i, \quad 0 - 3i.$$

$$\Rightarrow y = c_1\left(\cos(3x) + i\sin(3x)\right) + c_2\left(\cos(3x) - i\sin(3x)\right)$$

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* Calculations for PI.

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} \text{ ----- } = Q \text{ — (i)}$$

or $D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \text{ ----- } = Q$ — (ii)

$\Rightarrow (D^n + P_1 D^{n-1} + P_2 D^{n-2} \text{ ---}) y = Q$ — (iii).

$\Rightarrow \{f(D)\} y = Q$ — (iv).

for Particular Integral

Case I.

Let $\{f(D)\} y = Q$ be an ODE
If ODE is given in particularly the form.

$$f(D) \cdot y = Q = e^{ax}.$$

then,

i) $\{f(D)\}_{D=a} \neq 0$ this method will work.

ii) $\{f(D)\}$ can be factorized

(iii) P. $PI = \frac{1}{f(D)} \cdot Q \cong \frac{e^{ax}}{f(D)}.$

Q.) Solve ODE.

$$(D^2-5D+6)\cdot y = e^{3x}$$

The given ODE is :—

$$(D^2-5D+6)\cdot y = e^{3x}.$$

for CF

auxillary eq^n $\Rightarrow$ $m^2-5m+6=0$

$(m-2)(m-3)=0.$

$m = 2$ or 3.

$\Rightarrow$ $CF = c_1e^{2x} + c_2e^{3x}.$

for calculation of PI :-

$\{f(D)\}_{D=a} \equiv (D^2-5D+6)_{D=3}.$

$= (9-15+6)$

$= 0.$

$\Rightarrow \{f(D)\}_{D=3} = 0$ hence for PI, Case I method will be used.

Also, $(D^2-5D+6) = (D-3)(D-2)$ can be written.

Hence.

$$PI = \frac{1}{f(D)} \cdot Q = \frac{1}{f(D)} \cdot e^{3x}.$$

$$= \frac{1}{(D-3)(D-2)} \cdot e^{3x} = \frac{1}{(D-3)} \left\{ \frac{e^{3x}}{(D-2)} \right\}$$

valid for every Q

NOTE.

$$\frac{e^{ax}}{D-m} = e^{mx} \int e^{-mx} e^{ax} \, dx. \quad Q.$$

can be general

$$\Rightarrow \frac{1}{D-3} \cdot e^{2x} \int e^{-2x} \cdot e^{3x} \, dx$$

$$\Rightarrow \frac{1}{D-3} \cdot e^{2x} \int e^{x} \, dx.$$

$$\Rightarrow \frac{e^{3x}}{D-3.} = e^{3x} \int e^{-3x} \cdot e^{3x} \cdot dx.$$

$$= e^{3x} \cdot x.$$

$$PI = x \cdot e^{3x}.$$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{2x} + x e^{3x}.$$

Case 1.

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for PI

if i) $\{f(D)\}_{D=a} \neq 0$, then we can procede with this method.

and $PI = \dfrac{Q}{\{f(D)\}_{D=a}} = \dfrac{e^{ax}}{\{f(D)\}_{D=a}}$.

Q. Solve ODE

$$(D^2-5D+6)y = e^{4x}.$$

$CF = c_1 e^{3x} + c_2 e^{2x}$.

Calculation for PI

$$\{f(D)\}_{D=a} = \{f(D)\} = (D^2-5D+6)_{D=4} \neq 0.$$

Hence $PI = \dfrac{1}{\{f(D)\}_{D=a}} Q = \dfrac{e^{4x}}{2}$

$$y = c_1 e^{3x} + c_2 e^{2x} + \frac{e^{4x}}{2}.$$

Case III: Using Binomial Approximation

$\{f(D)\}\, y = Q = x^n.$

$n \in N.$ (not just order)

Q. Solve ODE

$$(D^3 + 3D^2 + 2D)\, y = x^4.$$

Calculation of CF

auxillary eq^n $\Rightarrow$ $\cancel{D^3 + 3D^2}$ $(m^3 + 3m^2 + 2m) = 0$

$m\,(m+1)(m+2) = 0$

$$CF = c_1 e^{-x} + c_2 e^{-2x} + c_3$$

Calculation for PI

$$PI = \frac{Q}{\{f(D)\}} = \frac{x^4}{D^3 + 3D^2 + 2D}.$$

$$= \frac{x^4}{2D\left(1 + \frac{D^2 + 3D}{2}\right)}$$

$$= \frac{1}{2D} \cdot x^4 \cdot \left(1 + \frac{D^2 + 3D}{2}\right)^{-1}$$

using Binomial Expansion

$\frac{1}{D} = \frac{d}{dx}$ dx.

$$= \frac{1}{2D}\left[1 - \left(\frac{D^3 + 3D^2}{2D}\right) + \cdots\right] x^4$$

neglecting higher terms.

$$= \frac{x^4}{2D}\left[1 - \left(\frac{D^2}{2} + \frac{3D}{2}\right)\right]$$

$$\Rightarrow \frac{1}{2D}\left(x^4 - \frac{D^2x^4}{2} - \frac{3Dx^4}{2}\right).$$

$$\Rightarrow \frac{1}{2D}\left(x^4 - \frac{1}{2}\cdot 4\cdot 3x^2 - \frac{3}{2}\cdot 4x^3\right).$$

$\left(\text{since } D = \frac{d}{dx}\right)$

$$PI = \frac{1}{2D}\left(x^4 - 6x^2 - \frac{3x^3}{2}\right).$$

Now. $D = \frac{d}{dx} \Rightarrow \frac{1}{D} = \frac{dx}{d}$

$\Rightarrow$ integrate

$$PI = \int \frac{1}{2}\left[x^4 - 6x^3 - \frac{3}{2}x^4\right] dx.$$

$$= \frac{1}{2}\left(\frac{x^5}{5} - 2x^3 - \frac{3}{2}x^4\right).$$

Hence complete solution

$$\Rightarrow \quad y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{2}\left(\frac{1}{5}x^5 - 2x^3 - \frac{3}{2}x^4\right)$$

Case IV :

when $f(D) \cdot y = Q$.

where - i) $Q = \sin(ax)$; $\cos(ax)$; $\sin(ax+b)$; $\cos(ax+b)$.

ii) $f(D)$ should contain terms of even power of D.

iii) $f(D^2) \Rightarrow f(-a^2)$

iv) $D^2 = -a^2$; $D^4 = a^4$, $D^6 = -a^6$ . . . .

v) a and b are const.

then, PI are as :-

$$\frac{\sin(ax)}{f(D^2)} = \frac{\sin ax}{f(-a^2)}$$

$$\frac{\sin(ax+b)}{f(D^2)} = \frac{\sin(ax+b)}{f(-a^2)}$$

$$\frac{\cos(ax)}{f(D^2)} = \frac{\cos ax}{f(-a^2)}$$

$$\frac{\cos(ax+b)}{f(D^2)} = \frac{\cos(ax+b)}{f(-a^2)}$$

Q. Solve

$$(D^2 - 4)\, y = \sin 5x$$

Solu: auxilary eq^n $= (m^2 - 4) \cdot y = 0$

$m = \pm 2$

CF $\Rightarrow c_1 e^{2x} + c_2 e^{-2x}$.

and

$$PI = \frac{1}{f(-D^2)} \sin(ax).$$

$$\Rightarrow \frac{1}{D^2 - 4} \sin(5x).$$

$D^2 \to -a^2$.

$$\Rightarrow \frac{\sin(3x)}{-25-4} = \frac{\sin 3x}{-29}.$$

Case V: If $f(D) \cdot y = Q$ where $Q = x^n \cdot e^{ax}$

where a, n are const.

$$PI \cong \frac{Q}{f(D)} = \frac{1}{f(D)} x^n e^{ax} = e^{ax} \cdot \underbrace{\frac{1}{f(D+a)} \cdot x^n}_{\text{~~rule~~ rule}}$$

Q. Solve

$$(D^2 - 2D + 1)\, y = x^4 e^x.$$

$D = +1$

$\Rightarrow CF = C_1 e^{x}(C_2 x + C_3 x^2)$

For PI

$$PI = \frac{Q}{f(D)} = \frac{1 \times x^2 e^{x}}{D^2 - 2D + 1}$$

$$\Rightarrow e^{x} \cdot \frac{1}{(D+1)^2 - 2(D+1) + 1} \cdot x^2$$

$$\Rightarrow e^{x} \cdot \frac{1}{D^2 + 0} \cdot x^2$$

$$\Rightarrow e^{x} \cdot \frac{1}{D^2} \cdot x^2 \quad \text{i.e.} \quad e^{x} \int\left(\int x^2 \, dx\right) dx$$

$\Rightarrow$ Integerate 2 times

$$\Rightarrow \frac{e^{x} x^4}{12}$$

Case VI When $f(D) \cdot y = Q$ where $Q = x^{n} \cdot \sin(ax)$ where a, n are const.

$$PI = \frac{Q}{f(D)} = \frac{1}{f(D)} \cdot x^{n} \cdot \text{Img}(e^{iax})$$

$$\Rightarrow PI = \text{Img}(e^{iax}) \frac{1}{f(D+ia)} \cdot x^{n}$$

Q. Solve ODE
$(D^2+4)y = x^2 \sin 2x$.

The given question is :-

$$(D^2+4)y = x^2 \sin 2x \quad -(1).$$

auxilary eq^n for the eq^n (1) is

$$(m^2+4).y = 0.$$
$$m^2+4 = 0.$$
$$m = \pm 2i$$

$$\Rightarrow CF = c_1 e^{0.x}(\cos 2x + i \sin 2x) + c_2 e^{0.x}(\cos 2x - i \sin 2x).$$

PI. $\frac{1}{f(D)}.Q = \frac{1}{(D^2+4)} . x^2 \sin 2x$.

$$= \sin 2x . \frac{1}{(D+i2)^2+4} . x^2.$$

$$= \sin 2x . \frac{1}{D^2+4iD} . x^2.$$

$$\cancel{= \sin 2x . \frac{1}{4iD}\left(\frac{1}{(\frac{D}{4i}+1)}\right)} . x^2$$

$$= \sin 2x . \frac{1}{4iD}\left(1 - \frac{D}{4i}\right) . x^2$$

Since main operator is D, take $4i$ common,

$$= \sin 2x \cdot \frac{1}{D}\left(\frac{1}{D}\left(\frac{1}{1+\frac{4i}{D}}\right)\right)\cdot x^2.$$

$$= \sin 2x \cdot \frac{1}{D}\left(\frac{1}{D}\left(1-\frac{4i}{D}-\ldots\right)\right) x^2.$$

$$= \sin 2x \cdot \frac{1}{D}\left(\frac{1}{D}-\frac{4i}{D^2}\right) x^2.$$

$$= \sin 2x \cdot \left(\frac{1}{D^2}-\frac{4i}{D^3}\right) x^2.$$

$$= \sin 2x\left(\frac{1}{3}\,\frac{1}{4}x^4 - 4i\left(\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}x^5\right)\right).$$

Case VII $\quad f(D)\cdot y = Q = x^n \cdot V$

where $V = e^x, \sin(ax), \cos ax.$

* valid for $n=1$ don't know for sec!

$$PI = \frac{1}{f(D)}\cdot(x^n\cdot V) = x^n\cdot\frac{1}{f(D)}\cdot V - \frac{1}{\{f(D)\}^2}\cdot f'(D)\cdot V$$

* **Solution for the second order Linear ODE with variable coefficient**

consider,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R. \quad \text{—(i)}$$

where P, Q, R are function of x.

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Step 1: Consider a transformation $y = v \cdot z$ —— ②.

Step 2: Find the value of z from equation (i) if

a) $1 + P + Q = 0$ then $z = e^{x}$

b) $1 - P + Q = 0$ then $z = e^{-x}$

c) $P + Qx = 0$ then $z = x$.

d) $2 + 2Px + Qx^2 = 0$ then $z = x^2$

Step 3: Find value of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ in form of v and z

$$y = v \cdot z.$$

$$= \frac{dy}{dx} = \frac{dv}{dx} \cdot z + v \cdot \frac{dz}{dx}. \quad —— ③.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dv}{dx} \cdot z + v\frac{dz}{dx}\right)$$

$$= \frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} \cdot z + 2 \cdot \frac{dv}{dx} \cdot \frac{dz}{dx} + v \cdot \frac{d^2z}{dx^2}. \quad —— ④.$$

Step 4: Substitute values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ from ② ③ ④ into ①.

$$\Rightarrow \left(\left(z \cdot \frac{d^2v}{dx^2}\right) + 2\frac{dv}{dx} \cdot \frac{dz}{dx} + v\frac{d^2z}{dx^2}\right) + P\left(z\frac{dv}{dx} + v\frac{dz}{dx}\right)$$

$$+ Qvz = R.$$

rearrange we get:-

$$\frac{d^2v}{dx^2} + (\quad)\frac{dv}{dx} = (\quad) \quad -⑥.$$

Step 5: Consider a transformation as:-

$$\left.\begin{aligned} \frac{dv}{dx} &= p \\ \text{then} \quad \frac{d^2v}{dx^2} &= \frac{dp}{dx} \end{aligned}\right\} \quad -⑦$$

Step 6: Using eqⁿ ⑦ we get:-

$$\frac{dp}{dx} + (\quad) p = (\quad) \quad -⑦$$

Step 7 Now this is a homogenous differential eqⁿ.

$$\frac{dp}{dx} + Ap = B$$

$$\text{Soln} = p\,(I.F.) = \int B\cdot(IF)\,dx.$$

$$\Rightarrow \quad p\cdot e^{\int A\,dx} = \int B\cdot e^{\int A\,dx}\,dx$$

Now we get $p = (\qquad)$

$$p = \frac{dv}{dx}.$$

again solve we get:

$$\frac{y}{z} = v = (\qquad)$$

hence

$$y = z \left(\quad \right).$$

Q. Solve.

$$\frac{d^2y}{dx^2} - \left\{ \frac{x}{x-1} \right\} \cdot \frac{dy}{dx} = (x-1) - \frac{y}{x-1} \quad \text{—(i)}$$

$\cancel{P = \frac{x}{x-1} \quad Q = \frac{1}{x-1} \quad R = x-1}$

$\Rightarrow$ $x(x-1)$ $x(x+1)$

We know that standard eq^n of 2nd order as

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R. \quad \text{—(2)}$$

compairing (1) & (2).

$$P = -\frac{x}{x-1} \qquad Q = \frac{1}{x-1} \qquad R = x-1$$

$$\Rightarrow P + Qx = \frac{-x(x-1) + x(x-1)}{x-1}$$

$$= 0.$$

$$\Rightarrow z = x.$$

\# Note that $1 + P + Q = 0$ also satisfies. But you can chose anyone.

Now. Let $y = v \cdot z$.

$$\Rightarrow y = v \cdot x.$$

then we have.

$$\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \quad \text{—(3)}$$

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Also, $$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{d^2v}{dx^2}\cdot x + \frac{dv}{dx}.$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}\cdot x + 2\frac{dv}{dx} \quad - (4)$$

Now transfering (3) & (4) in (1)

$$\Rightarrow \frac{d^2v}{dx^2}\cdot x + 2\frac{dv}{dx} + \left\{\frac{-x}{x-1}\right\}\left(x\frac{dv}{dx} + v\right) + \frac{y}{x-1} = x-1$$

$$\Rightarrow \frac{d^2v}{dx^2}\cdot x + \frac{dv}{dx}\left(2 - \frac{x^2}{x-1}\right) = \underbrace{\cancel{\frac{v\cdot x - y}{x-1}}}_{=0} + (x-1)$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{dv}{dx}\left(\frac{2}{x} - \frac{x}{x-1}\right) = -\frac{1}{x} + 1 \quad - (5)$$

Applying transformation.

$$\frac{dv}{dx} = p.$$

$$\frac{d^2v}{dx^2} = \frac{dp}{dx}.$$

$$\frac{x-1+1}{x-1}$$

$$\Rightarrow \frac{dp}{dx} + p\left(\frac{2}{x} - \frac{x}{x-1}\right) = 1 - \frac{1}{x}.$$

$$1 + \frac{1}{x-1}.$$

$$2\ln x - x - \ln(x-1).$$

$$\ln\frac{x^2}{x-1} \quad x.$$

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this is Homogeneous 1 Degree eq^n.

So calculating integrating factor.

$$\int \frac{2}{x} - \frac{x}{x-1}\, dx.$$

$e^x(f(x)+f'(x))$

$e^x f(x)+c$

$$= 2\log x - x - \log(x-1)$$

$$I.F = e^{\log \frac{x^2}{x-1} - x} = e^{-x} \cdot \frac{x^2}{x-1}.$$

Solution is.

$$\Rightarrow p \cdot \frac{e^{-x} \cdot x^2}{x-1} = \int \frac{\cancel{x-1}}{x} \cdot \frac{x^2}{\cancel{x-1}} \cdot e^{-x} dx$$

calculating RHS.

$$= \int e^{-x} x \, dx.$$

$$= - \int e^{-x}(-x-1+1) . dx.$$

$$\Rightarrow e^{-x}(x-1) + C.$$

$$= p \cdot e^{-x} \cdot \frac{x^2}{x-1} = -e^{-x}(x+1) + C_1.$$

$$= p = -\left(1 - \frac{1}{x^2}\right) + C_1 \cdot e^{x} \cdot \frac{(x-1)}{x^2}$$

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* Solution for Homogeneous Linear Ordinary differen-tial Eq^n.

(Euler - Chauchy Equation)

$$x^n \cdot \frac{d^n y}{dx^n} + P_1 \cdot x^{n-1} \cdot \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \cdot \frac{d^{n-2} y}{dx^{n-2}} \cdots$$

$$\cdots = Q \quad \text{--- (1)}$$

where P_1 P_2 P_3 are constants
and Q is function of x

Soln.

Consider transformation $\quad x = e^z$ --- (2)

$z = \log x$ --- (3)

Differentiating on both the sides of eq^n (3).

$$\frac{dz}{dx} = \frac{1}{x} \quad \text{--- (3).}$$

Also $\quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{x}$

$$\Rightarrow \quad x \frac{dy}{dx} = \frac{dy}{dz}.$$

$$\Rightarrow \quad x \frac{d}{dx} \cong \frac{d}{dz}.$$

(just to make new operator this is rep.)

$$\boxed{x D \cong D_1} \quad \text{--- (5)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$ see previous steps.

$$\frac{d^2y}{dx^2} = \frac{1}{x}\left\{\frac{d}{dx}\left(\frac{dy}{dz}\right)\right\} - \frac{dy}{dz}\cdot\frac{1}{x^2}$$

$$\Rightarrow x^2\cdot\frac{d^2y}{dx^2} = x\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{dy}{dz}$$

$$\Rightarrow x^2\cdot\frac{d^2y}{dx^2} = x\cdot D\cdot(D_1 y) - (D_1(y))$$

using (6)

$$x^2\frac{d^2y}{dx^2} = D_1(D_1 y) - D_1 y$$

$$= D_1(D_1 - 1)y. \quad \text{——— (7)}$$

Similarly we get

$$x^3\frac{d^3y}{dx^3} = D_1(D_1-1)(D_1-2)y.$$

$$x^4\frac{d^4y}{dx^4} = D_1(D_1-1)(D_1-2)(D_1-3)y.$$

$$\vdots$$

$$x^n\frac{d^ny}{dx^n} = D_1(D_1-1)(D_1-2)\cdots\{D_1-(n-1)\}y.$$

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Q. $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x$. — (1)

Solve.

Considering transformation

$$x = e^z \quad \text{— (1)}$$

$$z = \log x. \quad \text{— (2)}$$

$$\frac{dz}{dx} = \frac{1}{x}.$$

Hence.

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx}$$

$$\Rightarrow x\frac{dy}{dz} = \frac{dy}{dz}.$$

$$\Rightarrow x\frac{d}{dz} \cong \frac{d}{dz}. \quad \text{— (4)}$$

Also we can find

$$x^2 \frac{d^2y}{dx^2} = D_1(D_1 - 1)y \quad \text{— (5)}$$

] Shown in Exam.

Substituting values of $x, x\frac{dy}{dx}, x^2\frac{d^2y}{dx^2}$ in eq^n (1)

$= D_1(D_1-1)y - 4D_1y + 6y = e^z.$

$= (D_1^2 - 5D_1 + 6)y = e^z.$

$$\Rightarrow (D_1-2)(D_1-3)y = e^z \quad \text{— (6)}$$

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auxillary eqⁿ for LHS

$$(m_1-2)(m_1-3) = 0.$$

$$m = 2 \text{ or } 3$$

$$CF = c_1 e^{2z} + c_2 e^{3z}$$

$$z = \log x.$$

$$\Rightarrow CF = c_1 x^2 + c_2 x^3.$$

$$PI = \frac{Q}{f(D_1)} = \frac{e^z}{(D_1-2)(D_1-3)}$$

$$\cancel{\Rightarrow \left\{\frac{e^z}{D_1-3}\right\} \cdot \frac{1}{(D_1-2)}} = \frac{e^z}{2} = \frac{x}{2}.$$

$$\cancel{\Rightarrow e^{3z} \int e^{-2z} dx.}$$

$$\Rightarrow c_1 x^2 + c_2 x^3 + \frac{x}{2}$$

Q. i) $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^2$

ii) $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 20y = (x+1)^2$

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* Series Solution for ODE by F.G. Frobenius Method

Consider,

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \quad -(i)$$

where $P_0(x)$, $P_1(x)$ and $P_2(x)$ are functions of x alongwith numbers.

If $P_0(x) \neq 0$ for $x = a$, then $x = a$ is known as ordinary point of differential equation (1).

Any point which is not the ordinary point is called singular point of the differential Equation.

Now re-arrange equation as:-

$$= \frac{d^2y}{dx^2} + \left\{\frac{P_1(x)}{P_0(x)}\right\}\cdot\frac{dy}{dx} + \left\{\frac{P_2(x)}{P_0(x)}\right\}y = 0.$$

$$= \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \quad -(2)$$

where $P(x) = \dfrac{P_1(x)}{P_0(x)}$ $\qquad Q(x) = \dfrac{P_2(x)}{P_0(x)}$

If $x = a$, is a regular point or regular-singular point of equation (2) and/or (1) if

Q.

$\lim_{x \to a} (x-a) P(x)$ and $\lim_{x \to a} (x-a)^2 P(x)$ = finite.

passes derivatives of all order in neighbour hood (nbd) of point 'a'

The general solution for ODE given by equation (1) and (2) is of form as if ($x=a$ is a ordinary point)

$$y = a_0 + a_1(a-x) + (a-x)^2 + (a-x)^3 - - - - \quad (3)$$

Again if $x=a$ is a regular point the solution of eq^n (1) and/or (2) may in the following form.

$$y = x^m \left(a_0 + a_1(a-x) + (a-x)^2 + (a-x)^3 - - -\right)$$
$$m \neq 0.$$

$m \in$ number of regular points.

Consider

1) $\frac{d^2y}{dx^2} + 4x(x+2)\frac{dy}{dx} + y = 0.$

2) $\frac{d^2y}{dx} + \frac{5x}{3x(x-2)}\frac{dy}{dx} + y = 0.$

3) $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{1}{x^2}y = 0.$

NOTE: for (1), it has no regular point → May have ~~No~~ soln
for (2), it has $x = 0, 2$ as regular point → soln.
for (3), it has $x = 0$ as regular point. → NO soln

If eq^n has a regular point, it won't have a ordinary point.

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Case I: Solution of ODE when $x=0$ is an ordinary point.

Q.) Solve

$$\frac{d^2y}{dx^2} + xy = 0.$$

All points are ordinary points. $\left(\frac{0}{P(x)}\right)$

Consider possible series soln of eq^n ① as

$$y = a_0 + a_1 x + a_2 x^2 - - - - + a_n x^n + - - \infty. \quad —②$$

Since eq^n ① has no singularity which implies it has no regular points. So possible soln of eq^n ① is expressed by eq^n ②. We have to just find the constants

$a_0, a_1, a_2, a_3, a_4 - - -$

So, differentiating eq^n ② w.r.t x on both sides

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots n a_n x^{n-1}$$

differentiating again:-

$$\frac{d^2y}{dx^2} = 2a_2 + 3\cdot 2\cdot a_3 + 4\cdot 3\cdot a_4 x^2 - - n(n-1)x^{n-2} \quad —③$$

Replace values of y and $\frac{d^2y}{dx^2}$ from eq^n ① & ③ to eq^n ②

$\frac{d^2y}{dx^2} + xy = 0$

$= \Big(2a_2 + 6a_3 x + 12a_4 x^2 \ldots$

$+ x\Big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots$

$= 2a_2 + (6a_3 + a_0)x + (12a_4 + a_1)x^2 \ldots$

$+ \cdots + \Big(n(n-1)a_n x + a_{n-3}\Big)x^{n} \ldots$

for every x, each term ...

$a_2 = 0$

$a_0 = -6a_3 \Rightarrow a_3 = -\frac{a_0}{6}$

$a_4 = -\frac{a_1}{12}$

$\Rightarrow a_n = -\frac{a_{n-3}}{n(n-1)}$

$(n-1)n a_n + a_{n-3}$

Now we have to express $a_0, a_1, a_2, a_3 \ldots$ in terms of a_0 & a_1

$a_3 = -\frac{a_0}{6}$

$a_4 = -\frac{a_1}{12}$

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$$a_7 = \frac{-1}{42} a_4 = + \frac{a_1}{12 \cdot 42}$$

a_0 and a_1 are called undetermined terms.
Substitute $a_0, a_1, a_2 \cdots a_7 \cdots a_n$ in (2)
and we have our soln.

Case II: Solution for ODE when $x=0$ is a ~~Singular~~ regular point.

Consider,

$$P_0(x) \frac{d^n y}{dx^n} + P_1(x) \frac{d^{n-1} y}{dx^{n-1}} + P_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots \phi$$

If the ODE has regular point, then its solution can be given as follows:

$$y = x^m (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \infty) \quad (1)$$

$$m \neq 0$$

$$= y = x^m \sum_{k=0}^{\infty} a_k x^k.$$

$$y = \sum_{k=0}^{\infty} a_k x^{k+m} \quad (2)$$

Differentiate equation w.r.t x

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} (m+k) a_k x^{k+m-1} \quad (3)$$

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further we have :-

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} (m+k)(m+k-1) a_k x^{m+k-2} \quad \text{---} ④$$

Substituting $y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \cdots \frac{d^ny}{dx^n}$ in eq^n ①

$$\begin{pmatrix}\text{Terms with lowest} \\ \text{power of } x\end{pmatrix} + \begin{pmatrix}\text{Terms with 2nd} \\ \text{lowest power of } x\end{pmatrix} + \cdots = 0$$

$$A + B + \cdots \infty = 0.$$

⟹ $(A = 0)$ This eq^n is called as indicial/indigenous eq^n.

we find roots.

Now for the roots of $A = 0$ we have.

Type 1: if $m_1 = m_2$ = constant.

Type 2: if $m_1 \neq m_2$ but $m_1 - m_2 = +1$

Type 3: if $m_1 \neq m_2$ but $m_1 - m_2 \neq 1$.

Now let,

$$B = 0.$$

and we will get recurrence relations in constant terms

$$a_n \cong (a_{n+1}, a_{n+2} \cdots c)$$

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Solve.

Q. $x(x^2+2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 6xy = 0$ — ①.

* This solution has singular point and its possible solu is of form

$$y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 - - - - \infty) \quad — ②$$

$$= \frac{dy}{dx} = mx^{m-1}(a_0 + a_1x + a_2x^2 + a_3x^3 - - - \infty) + x^m(a_1 + 2a_2x + 3a_3x^2 - - -) \quad — ③$$

further differentiating w.r.t 'x' we get:—

didn't consider $x(x^2+2)$

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2}(a_0 + a_1x + a_2x^2 - - -)$$
$$+ mx^{m-1}(a_1 + 2a_2x + 3a_3x^2 - - -)$$
$$+ mx^{m-1}(a_1 + 2a_2x + 3a_3x^2 - - -)$$
$$+ x^m(2a_2 + 3\cdot 2a_3x + \cdots + n(n-1)a_n x^{n-2}) \quad ④$$

Now substituting $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ — ④

and rearranging terms in ascending power of x

$$= (m(m-1)a_0)x^{m-2} + (ma_0 + a_1 m(m-1) + 2ma_1)x^{m-1}$$

for $A = 0$

$m = 1$ or 0.

for $m = 0$ $B = 0$ —— 1.

$y_1 = 0$.

$\underline{\underline{m = 1}}$

$B \Rightarrow (a_0 + 2a_1)$

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Unit 2
Partial Derivatives

Consider $z = f(x,y)$ be a function of 2 independent variable x and y then the partial derivative of z w.r.t x (keeping y const is defined as)

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \left\{ \frac{f(x+\delta x, y) - f(x,y)}{\delta (x)} \right\}.$$

Similarly we have :-

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \left\{ \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \right\}$$

Notation

1) $p = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x} \equiv f_x(x,y) \equiv f_x.$

2) $q = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y} \equiv f_y(x,y) = f_y$

3) $r = \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = \frac{\partial^2 z}{\partial x^2}$

4) $s = \frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \cdot \partial y}.$

5) $t = \frac{\partial q}{\partial y} = \frac{\partial^2 z}{\partial y^2}$

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* Euler's Theorem

If u is a function of 2 independent variable x and y of degree n, then

$$y \cancel{x} \frac{\partial u}{\partial y} + x \frac{\partial u}{\partial x} = n.u.$$

Proof

Let $u = f(x,y) = x^n f\left(\frac{y}{x}\right)$ —(i)

Now differentiating ~~w.r.t~~ eq^n (1) partially w.r.t x

$$p = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^n f\left(\frac{y}{x}\right)\right)$$

$$\Rightarrow \quad p = n \cdot x^n f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\left(\frac{-1}{x^2}\right)$$

$$\Rightarrow \quad p = n x^{n-1} f\left(\frac{y}{x}\right) - x^{n-2} f'\left(\frac{y}{x}\right)$$

$$xp = x\frac{\partial u}{\partial x} = n x^n f\left(\frac{y}{x}\right) - x^{n-1} f'\left(\frac{y}{x}\right) \quad \text{—(2)}$$

further differentiating eq^n (1) partially w.r.t 'y'

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^n \cdot f\left(\frac{y}{x}\right)\right) = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad yq = y\frac{\partial u}{\partial y} = y \cdot x^{n-1} f'\left(\frac{y}{x}\right) \quad \text{—(3)}$$

Adding eq^n 2 & 3

$$xp + yq = \left\{ n x^n f\left(\tfrac{y}{x}\right) - \cancel{x^{n-1} y f'\left(\tfrac{y}{x}\right)} \right\} + \cancel{x^{n-1} y f'\left(\tfrac{y}{x}\right)}$$

$$= n x^n f\left(\tfrac{y}{x}\right) = nu.$$

Hence proved.

Q. If $u = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{x}{y}\right)$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

$$\frac{\partial u}{\partial x} = \frac{-1}{x\sqrt{x^2-y^2}} \cdot y + \frac{y}{y^2+x^2} \cdot \frac{\partial u}{\partial y}$$

$$xp = \frac{xy}{x^2+y^2} - \frac{\cancel{x}y}{\sqrt{x^2-y^2}}$$

$$q = \frac{1}{\sqrt{x^2-y^2}} - \frac{\cancel{y^2}\,x}{x^2+y^2}$$

$$yq = \frac{y}{\sqrt{x^2-y^2}} - \frac{xy}{x^2+y^2}$$

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$xp + yq = 0.$

Q1 If $u = f(x,y,z)$ be a function of 3 independent variable
P.T.
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

Q2 If $u = f(x,y)$ be a function of 2 independent variable
P.T.
$$x\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Q3. If $z = \frac{(x^2+y^2)}{x+y}$ then show that
$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Q4. If $u = (1-2xy+y^2)^{-1/2}$, then $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3.$

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Q. If $z = \dfrac{x^2+y^2}{x+y}$ then

a) $\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}$ b) $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$

(c) $1 - (b)$.

$$\frac{\partial z}{\partial x} = \cancel{1 \cdot 2x} \; \frac{(x^2+y^2) - 2x(x+y)}{(x+y)^2}$$

$$= \frac{y^2 - x^2 - 2xy}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x^2+y^2) - 2y(x+y)}{(x+y)^2} = \frac{x^2-y^2-2yx}{(x+y)^2}$$

$\Rightarrow$ (a) $= \dfrac{2(x-y)}{x+y}$ (b) $\dfrac{4xy}{(x+y)^2}$ (c) $\left\{\dfrac{x-y}{x+y}\right\}^2$

* Change of Variable

If $z = f(t_1, t_2)$; where $t_1 = \phi_1(x,y)$
$t_2 = \phi_2(x,y)$

then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t_1} \cdot \frac{\partial t_1}{\partial x} + \frac{\partial z}{\partial t_2} \cdot \frac{\partial t_2}{\partial x}$$

similarly for $\dfrac{\partial z}{\partial y}$.

wrong solved

* Higher Order Partial Derivative

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If $z = f(x,y)$
then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\{f(x,y)\} = f_x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} f_x = f_{xx}$$

similarly $\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = f_{xxx}$.

Also, $\frac{\partial}{\partial y}\left\{\frac{\partial^3 z}{\partial x^3}\right\} = \frac{\partial}{\partial y}\{f_{xxx}\} = f_{yxxx}$

$$= \frac{\partial^4 z}{\partial y \partial x^3}.$$

Solve. $u = e^{xy(z+x^2)}$ then

$\frac{\partial^3 u}{\partial x \partial y \partial z}$ (ii) $\frac{\partial^2 u}{\partial y \cdot \partial x}$ (iii) $\frac{\partial^2 u}{\partial x \cdot \partial y}$ (iv) $\frac{\partial^3 u}{\partial x \partial y \partial z}$

$$\frac{\partial u}{\partial x} = e^{xy(z+x^2)} \cdot (yz + 3x^2yz)$$

$$\frac{\partial u}{\partial y} = e^{xy(z+x^2)} \cdot x(z+x^2).$$

$$\frac{\partial u}{\partial z} = e^{xy(z+x^2)} \cdot xy.$$

i) $u^3 \cdot x^2y^2z(1+3x^2)(z+x^2)$

$\frac{x}{f(-a^2)}$ $f(D)y = x \sin ax$ $\frac{\ }{f(-a^2)}$ $f(D^2)$ $\frac{x \sin 3x}{f(D^2+9)}$

$(-a^2)$ $\cancel{9(1+\frac{D^2}{9})}$

2) $= u^2\ xyz(1+3x^2)(z+x^2)$

3)

Q: If $u = \sin(xy) + e^{y+z} + \log(x+z) + \tan^{-1}(x+t)$.

$\frac{\partial^4 u}{\partial t \cdot \partial x \cdot \partial y \cdot \partial z} = ?$

This is order of differentiating

$\frac{\partial u}{\partial t} = \frac{1}{1+(x+t)^2}$

$\cancel{\frac{\partial u}{\partial x} = y\cos(xy) + \frac{1}{x+z} + \frac{1}{1+(x+t)^2}}$

$\frac{\partial^2 u}{\partial t \cdot \partial x} = \frac{-1}{(1+(x+t)^2)^2}(2(x+t)$

$\frac{\partial^3 u}{\partial t \cdot \partial x \cdot \partial y} = 0$.