~~Self Notes:~~

* ~~Double Integral:~~

~~let $I = \int_{y}^{x}\int_{y} f(x,y)\,dx\,dy$.~~

Gauss Elimination

$$a_1 x + b_1 y + c_1 z = k_1$$
$$a_2 x + b_2 y + c_2 z = k_2$$
$$a_3 x + b_3 y + c_3 z = k_3$$

→ reduce to triangular method.

→ back substitution

$$\rightarrow \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1' \\ 0 & b_2' & c_2' & k_2' \\ 0 & 0 & c_3'' & k_3'' \end{array}\right]$$

$c_3'' z = k_3''$ $\qquad z = \dfrac{k_3''}{c_3''}$

$b_2' y + c_2' z = k_2'$

find.

* But there is a huge error in its answers which is reduced by Gauss Elimination using partial pivoting.

* Guassian Elimination with partial pivoting.

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \end{array}\right]$$

Find biggest absolute value in 1st column and interchange with row 1. (say a_3).

so $R_1 \longleftrightarrow R_3$.

$R_1 = \frac{1}{a_3} R_1$ — so element become 1 / make pivot entry = 1

$$\begin{bmatrix} 1 & b_3/a_3 & c_3/a_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{bmatrix}$$

$R_2 = R_2 - a_2 R_1$

$R_3 \to R_3 - a_1 R_1$

0

$$\begin{bmatrix} 1 & b_3/a_3 & c_3/a_3 \\ 0 & b_2' & c_2' \\ 0 & b_1' & c_1' \end{bmatrix}$$

Now do same with this sub matrix

it will become upper triangular matrix.

Page No.
Date

LU decomposition:

$Ax = b$

↳ decomposed in lower and upper $\triangle$ matrix

$$A = LU$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$A_{n\times n} \Rightarrow$ ~~to~~ $\boxed{n^2+n}$ unknown.
↓
giving n^2 value

we need to reduce n extra entries to 1.

✓ if we take $l_{ii} = 1$ · Doolittle methode
✓ ______ $u_{ii} = 1$ Crout's method.

eg Crout's method.

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{\text{multiply}}$$

compare and find values of $l_{11} \dots l_{33}, u_{12} \dots u_{23}$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = L \quad U$$

Date

$$Ax = b$$

$$\underbrace{LU}x = b$$

$$LZ = b$$

↳ $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Forward substitution

to calc'

$z_1 = b_1 / l_{11}$

z_2 —

z_3 —

$$Ux = Z.$$

↳ Now by back subtition. find x.
as done in guassian eli —

inverse L^{-1} and U^{-1}

$$A = LU$$

$$A^{-1} = (LU)^{-1} = U^{-1} L^{-1}$$

$$Z = L^{-1} b \qquad x = U^{-1} z$$

~~Pro~~

Methods fails when diagonal element are zero.

* **Direct Method**
  → Guassian method
  → LU decompost

→ Iterative Methods are good when matrix are large and sparse.

~~Jacobi~~ and Gauss ~~Seidel~~ method.

$$\boxed{X^{k+1} = P X^{k} + \underset{\substack{\uparrow \\ \text{column matrix}}}{Q}}$$

$$A x = B$$

$$\underset{\substack{\uparrow \\ \text{lower}}}{(L + D + U)} X = B$$

Diagonal matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

→ **Jacobi Method**

$$D x = -(L+U) x + B$$

$$D X^{k+1} = -(L+U) X^{k} + B.$$

$$X^{k+1} = \underbrace{-D^{-1}(L+U)}_{P} X^{k} + \underbrace{B}_{Q.}$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12} x_2^{k} - a_{13} x_3^{k}\right]$$

$\vdots$

x_3^{k+1}.

Start with $x_1^{0} = x_2^{0} = x_3^{0} = 0$

Then Iteratively find $x_1^{1}, x_2^{1}, x_3^{1}$

$$x_1^{n}, \; x_2^{n}, \; x_3^{n+1}$$

both gives same Ans

$$\left[\begin{array}{l} x_1^{n}, \; x_2^{n}, \; x_3^{n+1} \\ x_1^{n+1}, \; x_2^{n+1}, \; x_3^{n+1} \end{array}\right.$$

It is long and calculative.

* Guass - Seidal Method.

$Ax = B$

$(L+D+U)X = B$

$(D+L)x = -Ux + B$

D.

$(D+L)x^{k+1} = -Ux^{k} + B$

$(x)^{k+1} = -(D+L)^{-1} U x^{k} + (D+L)^{-1} B$

$$x_1^{k+1} = \frac{1}{a_{11}}\left[b_1 - a_{12} x_2^{k} - a_{13} x_3^{k}\right]$$

$$x_2^{k+1} = \frac{1}{a_{22}}\left(b_2 - a_{21} x_1^{k+1} - a_{13} x_3^{k}\right)$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31} x_1^{(k+1)} - a_{32} x_2^{(k+1)}\right.$$

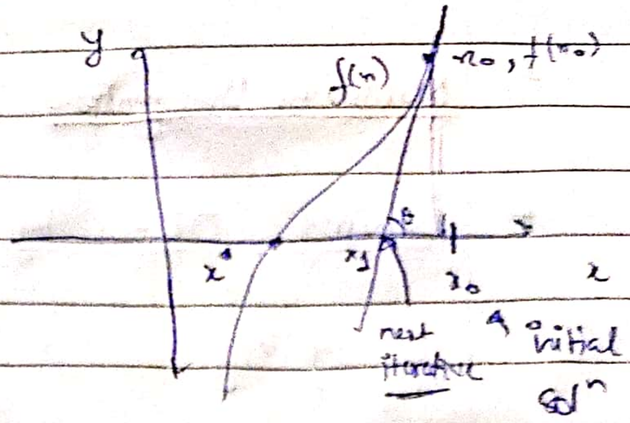
→ Solving non linear equation

* Newton - Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We will find iteratively

$$f'(x_0) = \tan\theta = \frac{f(x_0)}{x_0 - x_1}$$



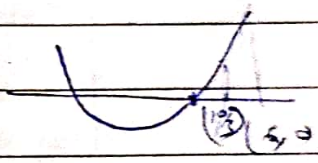
$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad n = 0, 1, 2 \ldots$$

→ If $f'(x_n) = 0$ for some n it fails.

Eg $f(x) = x^2 - 4 = 0$

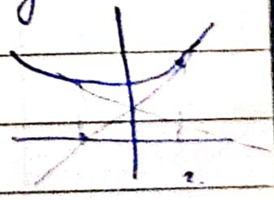


let $x_0 = 6$ $\quad f(x_0) = 32.$

$f'(x) = 2x$ $\quad f'(x_0) = 12$

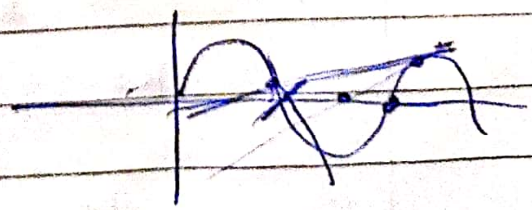
$$x_1 = 6 - \frac{32}{12} = \frac{10}{3}.$$

→ If $f(x)$ has no roots, the values may oscillate.



→ If we take initial ~~root~~ value near to root we may or may not find that root.

eg



*/ Numerical Integration

Solving ODE by R-K Method.

Order 2

$y' = f(x, y)$; $y(x_0) = y_0$

$y_1 = y_0 + k$

$k_1 = h f(x_0, y_0)$

$k_2 = h f(x_0 + h, y_0 + k_1)$

$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[h f(x_0, y_0) + h f(x_0 + h, y_0 + k_1)]$

eg $y' = y$; $y(0) = 1$

$h = 0.01$

$y(0.01) = y(0) + k$

$k = \frac{1}{2}(k_1 + k_2)$

$k_1 = h f(x_0, y_0)$

$= (0.01)(1) = 0.01$

$k_2 = (0.01) f(x_0 + h, y_0 + k_1)$

$= (0.01)(1.01) = 0.0101$

$k = \frac{1}{2}[0.0101 + 0.01]$

$\frac{1}{2}[0.0201]$

$k = 0.01005$

$y(0.01) = y_0 + k$

$1 + 0.01005$

$y(0.01) = 1.01005$

(Q) (2) $\frac{dy}{dx} = y - x$ $\quad y(0) = 1.5$ $\quad$ find $x = 0.2$, 0.4

$y_1 = y_0 + k$ $\quad \underline{h = 0.02}$

~~$k = k_1 + k_2 + k_3$~~

$k_1 = h f(x_0, y_0) = 0.2(1.5 - 0) = 0.30$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) =$

$= (0.2) f\{y - x\}$

$(0.2)\left(1.5 + \frac{0.3}{2} - (0 + 0.1)\right) = 0.310$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.2)\left(1.5 + \frac{0.310}{2} - (0 + 0.1)\right) = 0.3110$

$k_4 = h f(x_0 + h, y_0 + k_3)$

$= (0.2)(1.5 + 0.3110 - (0 + 0.02)) = 0.322$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

~~$\frac{1}{6}[0.3 + 0.62 + 0.6220 + 0.322)$~~ $= 0.3107$

$y_1 = y_0 + k$

$= 1.5 + 0.3107$

$= 1.8107$

Date

$y = f(x)$ be function
such that $y_0 = f(x_0)$ $y_1 = f(x_1)$ ... $y_n = f(x_n)$
where $x_r = x_0 + rh$

By Newton's forward Diff interpolation.

$$y = f(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{\lfloor 2}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{\lfloor 3}\Delta''' y_0 = f(x_0+rh)$$

where $r = \frac{x - x_0}{h}$ i.e. $x = x_0 + rh$

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx =$$

let $x = x_0 + rh$

$dx = h\,dr$

$$= h\int_0^n f(x_0 + rh)\,dr$$

$$= h\int_0^n y_0 + r\Delta y_0 + \frac{r(r-1)}{\lfloor 2}\Delta y_0^2 \cdots$$

General Newton Cote's formula

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{12}(2n^3 - 3n^2)\Delta^2 y_0 + \frac{1}{24}(n^4 - 4n^3 + 4n^2)\Delta^3 y_0 \cdots\right]$$

$(n+1)$ term

$\boxed{n=1}$

(x_0, y_0) Δy_0 2 pts
(x_1, y_1)

$$\int_{x_0}^{x_1} y\,dx = h\left(y_0 + \frac{\Delta y_0}{2}\right)$$

$$= h\left(y_0 + \frac{y_1 - y_0}{2}\right)$$

$$= \frac{h}{2}\left[y_0 + y_1\right]$$

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}\left[y_1 + y_2\right].$$

$$\int_{x_n}^{x_{n+1}} y\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Page No.
Date

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} + \int_{x_1}^{x_2} \cdots \int_{x_{n-1}}^{x_n}$$

$$= \frac{h}{2}\left[y_0 + y_1 + y_1 + y_2 \cdots y_2 + y_n\right]$$

$$\boxed{\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \cdots y_{n-1})\right]}$$ Trapezoidal rule

$\boxed{\text{If } n=2}$ $\int_{x_0}^{x_2} y\,dx =$ 3 pts $\begin{matrix} y_0 & \Delta y_0 & \Delta^2 y_0 \\ y_1 & \Delta y_1 & \\ y_2 & & \end{matrix}$

$$\int_{x_0}^{x_2} y\,dx = h\left[2y_0 + 2\Delta y_0 + \frac{1}{12}(16-12)\Delta^2 y_0\right]$$

$$\Rightarrow h\left[2y_0 + 2(y_1 - y_0) + \frac{4}{12}\left(\underset{y_1 - y_0}{\Delta y_1} - \underset{y_1 - y_0}{\Delta y_0}\right)\right]$$

$$\Rightarrow h\left[\cancel{2y_0} + 2y_1 - \cancel{2y_0} + \frac{1}{3}\Delta(y_2 - 2y_1 + y_0)\right]$$

$$\Rightarrow \frac{h}{3}\left[6y_1 + y_2 - 2y_1 + y_0\right]$$

$$\Rightarrow \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$\text{llly} \int_{x_n}^{x_{n+2}} y\,dx = \frac{h}{3}\left[y_n + 4y_{n+1} + y_{n+2}\right]$$

$$\int_{x_0}^{x_n} y\,dx = \cancel{\frac{h}{3}\left[y_0 \cdots y_n\right]}$$

Simpson's $\frac{1}{3}$rd rule $\boxed{n=2r}$

$$\boxed{\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 \cdots) + 2(y_2 + y_4 \cdots)\right]}$$

If $n=3$ $\boxed{n=3r}$

Simpson's $\frac{3}{8}$th rule $\Rightarrow$

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3\left[y_1 + y_2 \cdots y_{n-1}\right] + 2(y_3 + y_6 + y_9 \cdots y_{n-3})\right]$$

Page No.
Date

Euler's Method:

$$\boxed{y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})}$$

Given a d.e. $\frac{dy}{dx} = f(x,y)$ $y(x_0) = y_0$

find $y(x_n) = y_n$.

$h = x_n - x_{n-1}$

i.e. $\frac{x_n - x_0}{n} = h$

Euler's Modified Method

$$y_n^{m} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{m-1}) \right]$$

Runge Kutta Method. (4th order)

Given ODE $\frac{dy}{dx} = f(x,y)$ and $y(x_0) = y_0$

To find $y(x_1)$

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4 \right]$$

$k_1 = h f(x_n, y_n)$

$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$k_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$

$k_4 = h f(x_n + h, y_n + k_3)$

Page No.

Date

2nd order

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

$$y = y(x) = ?$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h(x_n + h, y_n + k_1)$$