

# Genus-One Fibrations and the Jacobian of Linear Slices in the Quintic Equal-Sum Problem

Valery Asiryan

[asiryanvalery@gmail.com](mailto:asiryanvalery@gmail.com)

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## Abstract

We study the Diophantine equation

$$a^5 + b^5 = c^5 + d^5, \quad a, b, c, d \in \mathbb{Z}_{\geq 0},$$

under the linear slicing constraint

$$(c+d) - (a+b) = h.$$

First, we give a self-contained proof that any solution must satisfy the necessary congruence  $30 \mid h$ ; this is the  $k = 5$  instance of the *modular divisibility obstruction* (MDO).

Second, we symmetrize the problem by passing to sums and differences  $S = a + b$ ,  $u = b - a$ ,  $T = c + d = S + h$ ,  $v = d - c$  and reduce the quintic equality to a biquadratic equation in  $v$ . Writing  $Z = v^2$ , we obtain an explicit discriminant  $D_Z(S, u)$ , which must be a perfect square for solvability, and we give an exact (and computable) criterion for the existence of an *integer*  $v$  of the required parity  $v \equiv T \pmod{2}$  (equivalently, for the existence of integers  $c, d$  with sum  $T$ ) in terms of  $D_Z$  and integrality/square conditions on  $Z$ . We also record the additional size constraints on  $u$  and  $v$  needed to recover solutions with  $a, b, c, d \in \mathbb{Z}_{\geq 0}$ .

Third, we provide the appropriate geometric interpretation: for nonzero slices  $h \neq 0$  the equation  $Y^2 = D_Z(S, u)$  defines a family of genus-one curves over  $\mathbb{Q}(S)$ , which need not admit a rational section; thus one must pass to the Jacobian fibration to speak about Mordell–Weil rank. On the Jacobian side we isolate a uniform arithmetic obstruction: for every nonzero slice parameter  $h$  the Jacobian  $E_h/\mathbb{Q}(S)$  carries a global rational 2-torsion section and the square class of the quadratic discriminant governing full rational 2-torsion is determined by an explicit quintic factor

$$Q_5(S, h) = S^5 + 4hS^4 + 8h^2S^3 + 8h^3S^2 + 4h^4S + \frac{4}{5}h^5.$$

In particular, full rational 2-torsion would force  $S \cdot Q_5(S, h)$  to be a square in  $\mathbb{Q}$ . By homogeneity this square condition reduces to rational points on a universal genus-two hyperelliptic curve independent of  $h$ ; using a verified MAGMA computation (rank bound 0) we show that this curve has only the affine rational point  $(0, 0)$  and the two points at infinity, hence  $S \cdot Q_5(S, h)$  is never a square for  $S, h \neq 0$ .

Finally, for the first admissible integer slice  $h = 30$  we compute the classical invariants  $(I, J)$  of the associated binary quartic and write down the Jacobian elliptic curve  $E_{30}/\mathbb{Q}(S)$  explicitly. We then exploit the presence of a global rational 2-torsion point and apply the injectivity criterion of Gusić–Tadić for the specialization homomorphism to obtain a computationally verified upper bound

$$\text{rank } E_{30}(\mathbb{Q}(S)) \leq 1.$$

A simple homogeneity normalization in the slice parameter (setting  $x = S/h$  and scaling the Weierstrass coordinates) shows that, for every fixed  $h \neq 0$ , the Jacobian fibration  $E_h/\mathbb{Q}(S)$  is isomorphic over a rational function field to a universal model independent of  $h$ ; in particular, the same generic rank bound holds uniformly for all  $h \in \mathbb{Q}^\times$ . We nevertheless present the specialization computations at  $h = 30$  because it is the first admissible integer slice parameter and yields small coefficients and convenient integer specializations. This restricts the possible structure of any infinite families of *rational points* on the associated Jacobian fibration in any fixed nonzero slice; any corresponding family of *integral* solutions on admissible integer slices would additionally have to satisfy the integrality, parity, and size constraints arising in the reduction.

**Keywords:** Diophantine equations, equal sums of like powers, quintic equation, linear slicing, genus-one fibration, Jacobian of binary quartics, hyperelliptic curves, Chabauty–Coleman method, specialization, computational number theory.

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# 1 Introduction

The existence of nontrivial equalities of the form

$$a^5 + b^5 = c^5 + d^5, \quad a, b, c, d \in \mathbb{Z}_{\geq 0}, \quad (1)$$

with  $\{a, b\} \neq \{c, d\}$  is unknown (see, e.g., [1, 2, 3]). For quartic Diophantine equations, geometric methods can be highly effective; for instance, Elkies [4] produced solutions to the related equation  $A^4 + B^4 + C^4 = D^4$  by exploiting an elliptic fibration on a diagonal quartic K3 surface. In contrast, the quintic surface underlying (1) (which is of general type) remains much less understood from the standpoint of explicit arithmetic constructions and obstructions.

One systematic way to organize a search is to impose a linear constraint on the sums:

$$(c + d) - (a + b) = h, \quad (2)$$

where  $h \in \mathbb{Z}$  is fixed. We refer to (2) as a *linear slice*.

Throughout, we set

$$S := a + b, \quad T := c + d = S + h.$$

We also use the differences

$$u := b - a, \quad v := d - c.$$

Then

$$a = \frac{S - u}{2}, \quad b = \frac{S + u}{2}, \quad c = \frac{T - v}{2}, \quad d = \frac{T + v}{2},$$

so integrality requires  $u \equiv S \pmod{2}$  and  $v \equiv T \pmod{2}$ .

**Nonnegativity constraints.** In the original problem (1) we require  $a, b, c, d \in \mathbb{Z}_{\geq 0}$ . In terms of  $(S, u)$  this is equivalent to the size condition

$$|u| \leq S \quad (\text{together with } u \equiv S \pmod{2}),$$

and similarly, since  $c, d \geq 0$  forces  $T \geq 0$ , we have

$$|v| \leq T \quad (\text{together with } v \equiv T \pmod{2}).$$

Because (1) is symmetric under exchanging  $a$  and  $b$  (and also  $c$  and  $d$ ), one may, when searching for solutions, impose the normalization  $0 \leq u \leq S$  and  $0 \leq v \leq T$  without loss of generality (keeping track of the induced identifications).

**Rational vs. Integral constraints.** Several intermediate steps below (discriminant curves, genus-one fibrations over  $\mathbb{Q}(S)$ , and Jacobian elliptic curves over  $\mathbb{Q}(S)$ ) concern *rational* points and sections over a function field. These geometric objects do not encode the *inequality* constraints  $|u| \leq S$  and  $|v| \leq T$  required to recover solutions of (1) in  $\mathbb{Z}_{\geq 0}$ . Whenever we translate back from  $(S, u, T, v)$  to  $(a, b, c, d) \in \mathbb{Z}_{\geq 0}$ , we therefore impose the nonnegativity (size) constraints, together with the integrality and parity conditions, explicitly.

**Universality vs. Integrality.** The Jacobian fibration associated with a fixed nonzero slice parameter  $h$  is defined over the function field  $\mathbb{Q}(S)$ . At the level of Jacobian geometry (e.g. Mordell–Weil rank over  $\mathbb{Q}(S)$ ), the parameter  $h \neq 0$  can be normalized away by the change of variables  $x = S/h$  and a scaling of Weierstrass coordinates; see Subsection 4.5. In contrast, when one seeks *integral* solutions of (1) on a fixed integer slice (2), the congruence condition  $30 \mid h$  and the integrality/parity/size constraints in  $(S, u, T, v)$  depend on the chosen integer  $h$  and cannot be removed by such a normalization.

## 2 A necessary modular obstruction: $30 \mid h$

**Proposition 1.** *If (1) holds for some  $a, b, c, d \in \mathbb{Z}_{\geq 0}$  and  $h = (c + d) - (a + b)$ , then*

$$30 \mid h.$$

*Proof.* Let  $p \in \{2, 3, 5\}$ . Since  $5 \equiv 1 \pmod{p-1}$ , Fermat’s little theorem implies  $x^5 \equiv x \pmod{p}$  for all integers  $x$  (including multiples of  $p$ ). Reducing (1) modulo  $p$  gives

$$a + b \equiv c + d \pmod{p},$$

hence  $p \mid (c + d) - (a + b) = h$ . Therefore  $2 \mid h$ ,  $3 \mid h$ , and  $5 \mid h$ , so  $30 \mid h$ .  $\square$

*Remark 1.* Thus only one residue class modulo 30 is admissible for  $h$ , i.e.  $29/30 \approx 96.67\%$  of linear slices are ruled out by a simple congruence. In the language of prior work [5] this is the  $k = 5$  case of the modular divisibility obstruction (MDO)  $M_k \mid h$ .

In the remainder of the paper, when we discuss integral solutions on a fixed integer slice, we assume  $30 \mid h$ . Before specializing to a particular slice, we isolate a uniform obstruction on the Jacobian side (Subsection 4.2) showing that, for  $h \neq 0$ , the Jacobian cannot have full rational 2-torsion. We then discuss a normalization showing that the Jacobian geometry for any fixed  $h \neq 0$  is universal up to scaling (Subsection 4.5). After this general discussion we concentrate on the first admissible positive slice:

$$h = 30, \quad T = S + 30.$$

### 3 Symmetrization and reduction to a discriminant condition

#### 3.1 Symmetrization of the quintic sums

**Lemma 1.** *Let  $S = a + b$  and  $u = b - a$  with  $u \equiv S \pmod{2}$ . Then*

$$16(a^5 + b^5) = S^5 + 10S^3u^2 + 5Su^4. \quad (3)$$

*Similarly, if  $T = c + d$  and  $v = d - c$  with  $v \equiv T \pmod{2}$ , then*

$$16(c^5 + d^5) = T^5 + 10T^3v^2 + 5Tv^4. \quad (4)$$

*Proof.* Write  $a = (S - u)/2$ ,  $b = (S + u)/2$ . Then

$$(S - u)^5 + (S + u)^5 = 2(S^5 + 10S^3u^2 + 5Su^4),$$

since the odd powers of  $u$  cancel. Dividing by 2 gives (3). The proof of (4) is identical with  $(S, u)$  replaced by  $(T, v)$ .  $\square$

Define

$$\mathcal{L}(S, u) := S^5 + 10S^3u^2 + 5Su^4.$$

Then (1) is equivalent to

$$T^5 + 10T^3v^2 + 5Tv^4 = \mathcal{L}(S, u),$$

with the parity constraints  $u \equiv S \pmod{2}$  and  $v \equiv T \pmod{2}$ .

#### 3.2 A biquadratic equation in $v$

Rearranging the last equality gives a biquadratic equation:

$$5Tv^4 + 10T^3v^2 + (T^5 - \mathcal{L}(S, u)) = 0. \quad (5)$$

Let  $Z = v^2$ . Then (5) becomes a quadratic equation in  $Z$ :

$$5TZ^2 + 10T^3Z + (T^5 - \mathcal{L}(S, u)) = 0. \quad (6)$$

**Proposition 2** (Discriminant formula). *The discriminant of (6) is*

$$D_Z(S, u) = 80T^6 + 20T\mathcal{L}(S, u) = 80T^6 + 20T(S^5 + 10S^3u^2 + 5Su^4). \quad (7)$$

*Proof.* The discriminant of (6) equals

$$(10T^3)^2 - 4 \cdot (5T) \cdot (T^5 - \mathcal{L}(S, u)) = 100T^6 - 20T(T^5 - \mathcal{L}(S, u)),$$

which simplifies to (7).  $\square$

### 3.3 An exact integer-solvability criterion

**Proposition 3** (Exact criterion for an integer  $v$  (and the additional bounds for  $c, d \geq 0$ )). *Fix  $h \in \mathbb{Z}$  and  $S \in \mathbb{Z}_{\geq 0}$  and set  $T = S + h$ . Let  $u \in \mathbb{Z}$  with  $u \equiv S \pmod{2}$ . (When relating back to (1) we additionally impose  $|u| \leq S$ , which is equivalent to  $a, b \in \mathbb{Z}_{\geq 0}$ .) Assume  $T \neq 0$ .*

(a) *There exists an integer  $v$  satisfying (5) and the parity constraint  $v \equiv T \pmod{2}$  (equivalently, there exist integers  $c, d \in \mathbb{Z}$  with  $c + d = T$  and  $v = d - c$ ) if and only if there exists an integer  $Y$  such that:*

1.  $Y^2 = D_Z(S, u)$ ;
2.  $Z := \frac{-10T^3 + Y}{10T}$  is a nonnegative integer;
3.  $Z$  is a perfect square in  $\mathbb{Z}$  (so that  $v = \pm\sqrt{Z} \in \mathbb{Z}$ );
4.  $Z \equiv T \pmod{2}$  (equivalently, the resulting  $v = \pm\sqrt{Z}$  satisfies  $v \equiv T \pmod{2}$ ).

(b) *If in addition  $T \geq 0$ , then one can choose such a  $v$  so that the recovered*

$$c = \frac{T - v}{2}, \quad d = \frac{T + v}{2}$$

*satisfy  $c, d \in \mathbb{Z}_{\geq 0}$  if and only if the conditions in (a) hold and moreover*

5.  $Z \leq T^2$  (equivalently,  $|v| \leq T$ ).

*Proof.* Equation (6) has (rational) roots

$$Z = \frac{-10T^3 \pm \sqrt{D_Z(S, u)}}{10T}.$$

Since  $Y$  may be chosen with either sign once  $Y^2 = D_Z(S, u)$ , it suffices to write

$$Z = \frac{-10T^3 + Y}{10T}.$$

Hence an integer solution  $Z \in \mathbb{Z}_{\geq 0}$  exists if and only if  $D_Z(S, u)$  is a square  $Y^2$  in  $\mathbb{Z}$  and  $Z$  as above is an integer  $\geq 0$ . Finally,  $Z = v^2$  for an integer  $v$  if and only if  $Z$  is itself a perfect square. This proves (a), except for the parity condition.

To recover integer  $c, d$  from  $T$  and  $v$  via  $c = (T - v)/2$  and  $d = (T + v)/2$ , one must have  $v \equiv T \pmod{2}$ . If  $Z = v^2$  is a perfect square, then  $v^2 \equiv v \pmod{2}$ , so the parity condition  $v \equiv T \pmod{2}$  is equivalent to  $Z \equiv T \pmod{2}$ . This yields the stated equivalence in (a).

For (b), assume  $T \geq 0$ . Then  $c, d \in \mathbb{Z}_{\geq 0}$  is equivalent to  $|v| \leq T$  (together with  $v \equiv T \pmod{2}$ ), since  $c, d$  are the half-sum/half-difference of  $T$  and  $v$ . In terms of  $Z = v^2$  this is exactly the bound  $Z \leq T^2$ .  $\square$

*Remark 2.* In the degenerate case  $T = 0$  (equivalently  $S = -h$ ), the reduction above involves division by  $T$ . Under the original nonnegativity constraints  $a, b, c, d \in \mathbb{Z}_{\geq 0}$ , one necessarily has  $c + d = T = 0$ , hence  $c = d = 0$ , and then (1) forces  $a^5 + b^5 = 0$ , so  $a = b = 0$ . Thus  $T = 0$  yields only the trivial solution, and the interesting case is  $T \neq 0$ .

*Remark 3.* The condition “ $D_Z(S, u)$  is a square” is only the *first* filter: one must also enforce the divisibility and congruence needed for integrality of  $Z$ , the parity constraint  $Z \equiv T \pmod{2}$  (equivalently  $v \equiv T \pmod{2}$ ) to recover integer  $c, d$ , and finally the requirement that  $Z$  be a square.

Moreover, when relating back to the original Diophantine problem (1) with  $a, b, c, d \in \mathbb{Z}_{\geq 0}$ , one must also impose the size constraints  $|u| \leq S$  and (necessarily requiring  $T \geq 0$ )  $|v| \leq T$ , equivalently  $Z \leq T^2$ . These inequalities are essential to ensure nonnegativity of the recovered variables.

### 3.4 Algorithmic consequence

For fixed  $S$  and  $T = S + h$ , an exhaustive search over admissible  $(u, v)$  with  $0 \leq u \leq S$  and  $0 \leq v \leq T$  (and the corresponding parity conditions) is  $O(S^2)$ . Proposition 3 reduces this to scanning only admissible  $u$  values (about  $S/2$  of them if one restricts to  $0 \leq u \leq S$  using the symmetry  $a \leftrightarrow b$ ) and performing a constant number of integer tests (square tests for  $D_Z$  and  $Z$ , the parity test  $Z \equiv T \pmod{2}$ , and in the nonnegative setting also the bound  $Z \leq T^2$ ), i.e.  $O(S)$  work per fixed  $S$ .

## 4 Genus-one curves and Jacobian fibrations for admissible slices

### 4.1 The discriminant curve as a genus-one fibration

Fix  $h$  and write  $T = S + h$ . Over the rational function field  $\mathbb{Q}(S)$ , define the curve

$$\mathcal{C}_h : \quad Y^2 = D_Z(S, u) = 80T^6 + 20T(S^5 + 10S^3u^2 + 5Su^4). \quad (8)$$

For generic  $S$  (i.e. outside a finite set where the quartic in  $u$  degenerates),  $\mathcal{C}_h$  is a smooth genus-one curve over  $\mathbb{Q}(S)$ .

*Remark 4.* The slice  $h = 0$  is degenerate for (8): then  $T = S$  and

$$D_Z(S, u) = 80S^6 + 20S(S^5 + 10S^3u^2 + 5Su^4) = 100S^2(S^2 + u^2)^2$$

is a square in  $\mathbb{Q}(S)[u]$ . Accordingly, in the genus-one/Jacobian discussion we implicitly assume  $h \neq 0$ .

*Remark 5.* A smooth genus-one curve over  $\mathbb{Q}(S)$  need not have a  $\mathbb{Q}(S)$ -rational point. Therefore it need *not* be an elliptic curve in the strict sense (no chosen rational basepoint, hence no group law on the curve itself). Consequently, a computer algebra system failing to find an obvious rational point (or failing to “convert” the model automatically) does *not* imply that no rational points exist and does *not* determine any Mordell–Weil rank. To speak about rank one must pass to the Jacobian fibration.

*Remark 6* (Other admissible slices). The symmetrization and discriminant construction above applies verbatim for any integer  $h$  with  $30 \mid h$ . For each such nonzero  $h$  one obtains a genus-one fibration  $\mathcal{C}_h$  over  $\mathbb{Q}(S)$  and a Jacobian elliptic curve  $E_h/\mathbb{Q}(S)$  defined

by the invariants of the corresponding binary quartic. Subsection 4.2 shows that for every nonzero  $h$  the Jacobian admits a global rational 2-torsion section, but *never* has full rational 2-torsion (uniformly in  $h$ ). Moreover, after the normalization  $x = S/h$  and a scaling of Weierstrass coordinates, the Jacobian fibrations for all fixed  $h \neq 0$  become isomorphic over a rational function field; see Subsection 4.5. In particular, geometric invariants such as the generic Mordell–Weil rank over  $\mathbb{Q}(S)$  are independent of the choice of  $h \neq 0$ .

In the remainder of the paper we restrict to the first nontrivial positive case  $h = 30$ , for which  $h$  is admissible for integral solutions and the coefficients are relatively small, making explicit specialization computations over  $\mathbb{Q}$  convenient. For other admissible integer slices  $h \in 30\mathbb{Z} \setminus \{0\}$ , the Jacobian geometry is the same up to the normalization above, but the translation back to *integral* solutions depends on the chosen  $h$  through the integrality, parity, and size constraints in  $(S, u, T, v)$ .

## 4.2 A uniform discriminant factor and an obstruction to full rational 2-torsion

In the binary quartic model (8) the right-hand side is

$$f_h(u) = a(S, h) u^4 + c(S, h) u^2 + e(S, h),$$

with

$$a(S, h) = 100S(S+h), \quad c(S, h) = 200S^3(S+h), \quad e(S, h) = 80(S+h)^6 + 20(S+h)S^5.$$

Since  $f_h$  has no odd powers of  $u$ , its Jacobian admits a global rational 2-torsion section. To make this explicit, recall the invariants of a binary quartic

$$f(u) = au^4 + bu^3 + cu^2 + du + e : I = 12ae - 3bd + c^2, \quad J = 72ace + 9bcd - 27ad^2 - 27b^2e - 2c^3,$$

and the Jacobian elliptic curve

$$E_{I,J} : \quad y^2 = x^3 - 27Ix - 27J. \tag{9}$$

(See, for example, [6, 7].) In our situation  $b = d = 0$ , so  $I = 12ae + c^2$  and  $J = 72ace - 2c^3$ .

**Lemma 2** (Global 2-torsion). *Let  $h$  be fixed and set  $T = S+h$ . Let  $E_h/\mathbb{Q}(S)$  be the Jacobian of  $\mathcal{C}_h$  written in the form (9). Then the cubic on the right-hand side has a root*

$$e_1(S, h) = -1200S^3T = -1200S^3(S+h) \in \mathbb{Q}(S),$$

hence  $(x, y) = (e_1(S, h), 0)$  is a rational point of order 2 on  $E_h$ .

*Proof.* This follows by a direct substitution of  $x = e_1(S, h)$  into the cubic  $x^3 - 27I(S, h)x - 27J(S, h)$  obtained from  $a(S, h), c(S, h), e(S, h)$ .  $\square$

Shifting the  $x$ -coordinate by  $e_1(S, h)$ , i.e. writing  $x = X + e_1(S, h)$ , puts  $E_h$  into a standard 2-torsion model

$$E_h : \quad y^2 = X(X^2 + A(S, h)X + B(S, h)), \tag{10}$$

where

$$A(S, h) = 3e_1(S, h) = -3600 S^3(S + h), \quad B(S, h) = 3e_1(S, h)^2 - 27I(S, h).$$

The discriminant of the quadratic factor in (10),

$$\Delta_2(S, h) := A(S, h)^2 - 4B(S, h),$$

controls whether the remaining two 2-torsion points are rational: the curve  $E_h$  has *full* rational 2-torsion over the base field if and only if  $\Delta_2(S, h)$  is a square.

**Definition 1.** Define

$$Q_5(S, h) := S^5 + 4hS^4 + 8h^2S^3 + 8h^3S^2 + 4h^4S + \frac{4}{5}h^5, \quad (11)$$

and let

$$P(S, h) := S \cdot Q_5(S, h). \quad (12)$$

**Proposition 4** (Explicit discriminant factor). *For every  $h$ , the quadratic discriminant  $\Delta_2(S, h)$  factors as*

$$\Delta_2(S, h) = 12960000 S (S + h)^2 Q_5(S, h). \quad (13)$$

In particular, since  $12960000 = 3600^2$  and  $(S + h)^2$  is already a square, the square class of  $\Delta_2(S, h)$  is controlled by  $P(S, h) = S \cdot Q_5(S, h)$ .

*Proof.* Starting from  $a(S, h), c(S, h), e(S, h)$  one computes  $I(S, h) = 12ae + c^2$  and  $J(S, h) = 72ace - 2c^3$  and writes the Jacobian  $E_h$  in the form (9). The shift  $x = X + e_1(S, h)$  with  $e_1(S, h) = -1200S^3(S + h)$  yields (10). A direct simplification of  $A(S, h)^2 - 4B(S, h)$  then gives (13).  $\square$

Consequently, a necessary condition for full rational 2-torsion on a specialization  $E_{h, S_0}/\mathbb{Q}$  (with  $h \neq 0$  fixed and  $S = S_0 \in \mathbb{Q}^\times$ ) is that  $P(S_0, h)$  be a square in  $\mathbb{Q}$ . We now rule out this square condition uniformly in  $h$ .

### 4.3 Reduction to a universal genus-two curve

Assume  $h \neq 0$  and  $S \neq 0$ . By homogeneity of (12) we may set  $x := S/h$  and rewrite

$$P(S, h) = h^6 \left( x^6 + 4x^5 + 8x^4 + 8x^3 + 4x^2 + \frac{4}{5}x \right).$$

Thus  $P(S, h)$  is a square in  $\mathbb{Q}$  if and only if the genus-two curve

$$y_1^2 = x^6 + 4x^5 + 8x^4 + 8x^3 + 4x^2 + \frac{4}{5}x$$

has a rational point with  $x \neq 0$ . Clearing denominators via  $Y := 5y_1$  gives the *universal* hyperelliptic curve

$$\mathcal{C}_{\text{univ}} : \quad Y^2 = 25x^6 + 100x^5 + 200x^4 + 200x^3 + 100x^2 + 20x, \quad (14)$$

which is independent of  $h$ .

## 4.4 Rational points on the universal curve

**Proposition 5.** *The set  $\mathcal{C}_{\text{univ}}(\mathbb{Q})$  consists of the affine point  $(0, 0)$  and the two points at infinity.*

*Proof.* We compute in MAGMA that the Jacobian  $\text{Jac}(\mathcal{C}_{\text{univ}})$  has rank bound 0 over  $\mathbb{Q}$  and then apply the rank-0 Chabauty–Coleman routine [8, 9] to enumerate all rational points; see Section A.1 for the script and console output.  $\square$

**Theorem 1** (No nontrivial squares). *Let  $S, h \in \mathbb{Q}$  with  $S \neq 0$  and  $h \neq 0$ . Then  $P(S, h) = S \cdot Q_5(S, h)$  is not a square in  $\mathbb{Q}$ . Equivalently, the equation  $y^2 = P(S, h)$  has no solutions with  $y \in \mathbb{Q}$ .*

*Proof.* If  $y^2 = P(S, h)$  has a solution with  $S, h \neq 0$ , then the corresponding ratio  $x = S/h$  yields a rational point on (14) with  $x \neq 0$ . By Proposition 5 no such point exists. Therefore  $P(S, h)$  cannot be a square in  $\mathbb{Q}$ .  $\square$

**Corollary 1.** *Fix  $h \in \mathbb{Q}^\times$ . For every  $S_0 \in \mathbb{Q}^\times$  such that the specialization  $E_{h, S_0}/\mathbb{Q}$  is nonsingular (equivalently, its elliptic discriminant is nonzero), the curve  $E_{h, S_0}$  has exactly one rational point of order 2 (namely the specialization of  $(e_1(S, h), 0)$ ). Equivalently, for every such  $S_0$  the quadratic factor in the 2-torsion model (10) does not split over  $\mathbb{Q}$ .*

*In particular, for every admissible nonzero integer slice parameter  $h \in 30\mathbb{Z} \setminus \{0\}$  and every  $S \in \mathbb{Z}_{>0}$  with nonsingular specialization, the quadratic discriminant  $\Delta_2(S, h)$  is not a square in  $\mathbb{Q}$ .*

## 4.5 Normalization in the slice parameter and universality of the Jacobian fibration

The reduction in Subsection 4.3 already uses the ratio  $x = S/h$  to obtain a universal genus-two curve controlling the square class of  $\Delta_2(S, h)$ . In fact, for every fixed  $h \neq 0$  the entire Jacobian fibration  $E_h/\mathbb{Q}(S)$  is universal up to the same normalization.

*Remark 7* (Universality of the Jacobian geometry for  $h \neq 0$ ). Fix  $h \in \mathbb{Q}^\times$  and set  $T = S + h$ . A direct invariant computation (cf. Proposition 7 below for the slice  $h = 30$ ) shows that the Jacobian of  $\mathcal{C}_h$  admits the Weierstrass model

$$E_h : \quad Y^2 = X^3 - 864000 T^2 S (3T^5 + 2S^5) X - 345600000 T^3 S^4 (9T^5 + S^5), \quad (15)$$

where  $T = S + h$ . This equation is homogeneous in  $(S, h)$ , in the sense that under the substitution  $S = hx$  (so  $T = h(x+1)$ ) the coefficients of  $X$  and the constant term scale as  $h^8$  and  $h^{12}$ , respectively. Thus, after the change of variables

$$x = \frac{S}{h}, \quad X = h^4 X', \quad Y = h^6 Y',$$

the curve (15) becomes the *universal* elliptic curve

$$E_{\text{univ}} : \quad Y'^2 = X'^3 - 864000 x(x+1)^2 (3(x+1)^5 + 2x^5) X' - 345600000 (x+1)^3 x^4 (9(x+1)^5 + x^5) \quad (16)$$

over  $\mathbb{Q}(x)$ , which is independent of  $h$ . Since  $\mathbb{Q}(S) \cong \mathbb{Q}(x)$  for fixed  $h \neq 0$ , it follows that the Jacobian geometries for all fixed  $h \neq 0$  are isomorphic over a rational function field. In particular, the Mordell–Weil rank  $E_h(\mathbb{Q}(S))$  is independent of  $h \in \mathbb{Q}^\times$ , and any bound proved for one convenient value (such as  $h = 30$ ) applies to all  $h \neq 0$ .

**Specializations.** In Section A we verify injective specializations and compute ranks for the slice  $h = 30$  at several *integer* values  $S = S_0$ . Under the normalization above these correspond to rational values  $x = x_0 = S_0/30$  on the universal curve (16) and suffice to bound the *generic* Mordell–Weil rank over the function field (hence for all  $h \neq 0$ ). For a fixed *integer* slice parameter  $h \in 30\mathbb{Z}$ , studying the arithmetic of *integer* specializations  $S = S_0 \in \mathbb{Z}$  may still require separate injectivity checks, since the set of suitable integral parameters can depend on  $h$ .

From now on we work on the first admissible positive slice  $h = 30$ , so  $T = S + 30$ . This specialization is chosen to keep the coefficients small and to present convenient integer specializations; by Remark 7 the Jacobian geometry for any fixed  $h \neq 0$  is the same up to the normalization  $x = S/h$  and a scaling of Weierstrass coordinates. We write (8) as

$$Y^2 = a(S)u^4 + c(S)u^2 + e(S),$$

where

$$\begin{aligned} a(S) &= 100S(S+30), \\ c(S) &= 200S^3(S+30), \\ e(S) &= 80(S+30)^6 + 20(S+30)S^5. \end{aligned}$$

**Proposition 6.** *Over  $\mathbb{Q}(S)$ , the curve  $\mathcal{C}_{30}$  has no rational points at infinity.*

*Proof.* Consider the smooth projective model of  $\mathcal{C}_{30}$  as a double cover of  $\mathbb{P}_u^1$ . The points at infinity are precisely the points lying above  $u = \infty$ .

Set  $x := 1/u$  and write the affine model near  $x = 0$ . From

$$Y^2 = a(S)u^4 + c(S)u^2 + e(S)$$

we obtain, after multiplying by  $x^4$ ,

$$(Yx^2)^2 = a(S)x^8 + c(S)x^4 + e(S)x^4.$$

Hence the fiber above  $x = 0$  (i.e. above  $u = \infty$ ) consists of two points defined over  $\mathbb{Q}(S)(\sqrt{a(S)})$ ; it is  $\mathbb{Q}(S)$ -rational if and only if  $a(S)$  is a square in  $\mathbb{Q}(S)$ . Since  $a(S) = 100S(S+30) = 10^2 \cdot S(S+30)$  and  $S(S+30)$  is not a square in  $\mathbb{Q}(S)$ , there are no  $\mathbb{Q}(S)$ -rational points above  $u = \infty$ .  $\square$

## 4.6 The Jacobian elliptic curve via invariants

For a binary quartic

$$f(u) = au^4 + bu^3 + cu^2 + du + e,$$

classical invariant theory defines invariants

$$I = 12ae - 3bd + c^2, \quad J = 72ace + 9bcd - 27ad^2 - 27b^2e - 2c^3,$$

and the Jacobian elliptic curve is

$$E_{I,J} : \quad Y^2 = X^3 - 27IX - 27J. \quad (17)$$

See, e.g., standard references on 2-descent via binary quartics such as [6, 7].

In our case  $b = d = 0$  and  $f(u) = a(S)u^4 + c(S)u^2 + e(S)$ , hence

$$I(S) = 12a(S)e(S) + c(S)^2, \quad J(S) = 72a(S)c(S)e(S) - 2c(S)^3.$$

**Proposition 7.** *For  $h = 30$  (so  $T = S + 30$ ) the invariants are*

$$I(S) = 32000 T^2 S(3T^5 + 2S^5), \quad (18)$$

$$J(S) = 12800000 T^3 S^4 (9T^5 + S^5). \quad (19)$$

Consequently, the Jacobian elliptic curve of  $\mathcal{C}_{30}$  is

$$E_{30} : \quad Y^2 = X^3 - 864000 T^2 S(3T^5 + 2S^5) X - 345600000 T^3 S^4 (9T^5 + S^5), \quad (20)$$

where  $T = S + 30$ .

*Proof.* Using  $b = d = 0$ , we have  $I = 12ae + c^2$  and  $J = 72ace - 2c^3$ . Substituting  $a(S) = 100ST$ ,  $c(S) = 200S^3T$ ,  $e(S) = 80T^6 + 20TS^5$  and simplifying gives (18) and (19). Equation (20) follows from (17) by multiplying by 27.  $\square$

*Remark 8.* The computation in Proposition 7 uses only the general coefficients  $a(S, h) = 100S(S + h)$ ,  $c(S, h) = 200S^3(S + h)$  and  $e(S, h) = 80(S + h)^6 + 20(S + h)S^5$  of the binary quartic model (8). Replacing 30 by an arbitrary  $h \in \mathbb{Q}^\times$  yields the general Weierstrass equation (15) and hence the universal normalization described in Remark 7.

*Remark 9* (What specialization theorems do and do not give). If one computes the Mordell–Weil group  $E_{30}(\mathbb{Q}(S))$  (the group of rational sections of the Jacobian fibration), then Silverman’s specialization theorem (see, for example, [10]) implies that any section of infinite order specializes to points of infinite order for all but finitely many specializations  $S = S_0$ . In particular, if  $\text{rank } E_{30}(\mathbb{Q}(S)) > 0$  then  $\text{rank } (E_{30})_{S_0}(\mathbb{Q}) > 0$  for infinitely many (and in fact for “almost all”) integers  $S_0$  of good reduction. However, the converse direction is false in general: knowing  $\text{rank } E_{30}(\mathbb{Q}(S)) = 0$  would not by itself control the set of specializations with positive rank (“rank jumping”).

## 5 A rational 2-torsion point and a 2-torsion model

In this section we exhibit a global rational 2-torsion point on the Jacobian fibration (20) and rewrite it in a standard 2-torsion model.

Let  $F_S(X) = X^3 + a_4(S)X + a_6(S)$  be the cubic on the right-hand side of (20), where

$$a_4(S) = -864000 T^2 S(3T^5 + 2S^5), \quad a_6(S) = -345600000 T^3 S^4 (9T^5 + S^5).$$

**Lemma 3.** *The cubic  $F_S(X)$  has a root*

$$e_1(S) = -1200 S^3 T = -1200 S^3 (S + 30) \in \mathbb{Q}(S).$$

*In particular, the point  $(X, Y) = (e_1(S), 0)$  is a rational 2-torsion point on  $E_{30}$ .*

*Proof.* A direct substitution of  $X = e_1(S)$  into  $F_S(X)$  shows that  $F_S(e_1(S)) = 0$ . This identity was verified symbolically in MAGMA (see Section A). Hence  $(e_1(S), 0)$  is a rational point of order 2 on  $E_{30}/\mathbb{Q}(S)$ .  $\square$

Factoring the cubic gives

$$X^3 + a_4 X + a_6 = (X - e_1)(X^2 + e_1 X + e_1^2 + a_4).$$

If we shift  $X$  by  $e_1$  via  $X = X' + e_1$ , the equation becomes

$$Y^2 = X'(X'^2 + A(S)X' + B(S)),$$

where

$$A(S) = 3e_1(S), \quad B(S) = 3e_1(S)^2 + a_4(S). \quad (21)$$

In these coordinates the 2-torsion point  $(e_1(S), 0)$  moves to  $(X', Y) = (0, 0)$ .

A straightforward but somewhat lengthy computation in MAGMA yields the explicit formulae

$$A(S) = -3600 S^3 T, \quad (22)$$

$$\begin{aligned} B(S) = & -388800000 S^7 - 46656000000 S^6 - 2449440000000 S^5 \\ & - 73483200000000 S^4 - 1322697600000000 S^3 \\ & - 13226976000000000 S^2 - 56687040000000000 S. \end{aligned} \quad (23)$$

Only the factorization of  $B(S)$  and of the associated polynomial

$$\Delta(S) := A(S)^2 - 4B(S),$$

which is (up to powers of  $B(S)$  and a nonzero constant) the nontrivial factor in the discriminant, will be used in the sequel, and not the specific large integer coefficients.

## 6 Injective specialization and an upper bound on the generic rank

### 6.1 The Gusić–Tadić injectivity criterion

For an elliptic curve over a rational function field with a chosen 2-torsion point, Gusić and Tadić [11] provide an explicit criterion for the injectivity of the specialization homomorphism. We recall a slightly specialized version adapted to our setting.

Let  $K = \mathbb{Q}(S)$  and consider an elliptic curve

$$E_{30} : \quad Y^2 = X^3 + A(S)X^2 + B(S)X$$

over  $K$  with a rational 2-torsion point at  $(X, Y) = (0, 0)$ . Let  $B(S), \Delta(S)$  denote the polynomials in  $S$  that occur in the Weierstrass model, and write  $B^{\text{sf}}, \Delta^{\text{sf}}$  for their squarefree parts in  $\mathbb{Z}[S]$ . Denote by  $\mathcal{H}$  the set of all nonconstant squarefree divisors (in  $\mathbb{Z}[S]$ ) of either  $B^{\text{sf}}$  or  $\Delta^{\text{sf}}$ .

**Theorem 2** (Gusić–Tadić, specialized form). *Let  $S_0 \in \mathbb{Z}$  be such that the specialized curve  $(E_{30})_{S_0}$  is nonsingular (equivalently, its elliptic discriminant is nonzero). If for every  $g \in \mathcal{H}$  the value  $g(S_0)$  is not a square in  $\mathbb{Q}$ , then the specialization homomorphism*

$$\sigma_{S_0} : E_{30}(\mathbb{Q}(S)) \longrightarrow (E_{30})_{S_0}(\mathbb{Q})$$

is injective.

This follows immediately from the main theorem of [11], since our set  $\mathcal{H}$  contains, in particular, all irreducible factors of  $B$  and of  $A^2 - 4B$ .

## 6.2 Factorization of $B(S)$ and $\Delta(S)$

Using (22) and (23), we compute in MAGMA:

**Proposition 8.** *The polynomials  $B(S)$  and  $\Delta(S) = A(S)^2 - 4B(S)$  factor over  $\mathbb{Z}[S]$  as*

$$B(S) = -388800000 S(S+30)^2 (S^4 + 60S^3 + 1800S^2 + 27000S + 162000), \\ \Delta(S) = 12960000 S(S+30)^2 (S^5 + 120S^4 + 7200S^3 + 216000S^2 + 3240000S + 19440000).$$

Consequently, up to multiplication by rational squares, the nonconstant factors of  $B(S)$  and  $\Delta(S)$  are:

$$S, \quad S+30, \quad Q_4(S) = S^4 + 60S^3 + 1800S^2 + 27000S + 162000,$$

$$Q_5(S) = S^5 + 120S^4 + 7200S^3 + 216000S^2 + 3240000S + 19440000.$$

The set  $\mathcal{H}$  consists precisely of all nonconstant squarefree products of these factors which divide either  $S(S+30)Q_4(S)$  or  $S(S+30)Q_5(S)$ , i.e. the 11 polynomials

$$S, \quad S+30, \quad Q_4, \quad Q_5, \quad S(S+30), \quad SQ_4, \quad (S+30)Q_4, \\ SQ_5, \quad (S+30)Q_5, \quad S(S+30)Q_4, \quad S(S+30)Q_5.$$

*Proof.* This is a straightforward factorization in  $\mathbb{Z}[S]$ , performed in MAGMA; see Section A for the corresponding script and output.  $\square$

## 6.3 Injective specializations and ranks over $\mathbb{Q}$

We now combine Theorem 2 with explicit computations for specialized curves over  $\mathbb{Q}$ .

**Proposition 9.** *There exist integers  $S_0$  with  $1 \leq S_0 \leq 100$  that satisfy the Gusić–Tadić injectivity criterion. In particular, each of the values*

$$S_0 \in \{3, 5, 7, 8, 11, 12, 13, 14, 17, 18, 20, 21\}$$

has this property. For every such  $S_0$  the specialization homomorphism

$$\sigma_{S_0} : E_{30}(\mathbb{Q}(S)) \rightarrow (E_{30})_{S_0}(\mathbb{Q})$$

is injective.

*Proof.* For each nonconstant squarefree divisor  $g(S)$  of  $B^{\text{sf}}$  or  $\Delta^{\text{sf}}$ , we evaluated  $g(S_0)$  for all integers  $1 \leq S_0 \leq 100$  and tested whether  $g(S_0)$  is a square in  $\mathbb{Q}$ . We simultaneously excluded those  $S_0$  for which the elliptic discriminant of  $(E_{30})_{S_0}$  vanishes. The MAGMA script and its output are shown in Section A. In particular, each of the values

$$S_0 \in \{3, 5, 7, 8, 11, 12, 13, 14, 17, 18, 20, 21\}$$

satisfies the hypotheses of Theorem 2, and hence the corresponding specialization homomorphism is injective.  $\square$

*Remark 10.* We do not attempt here to classify all integers  $S_0$  for which the specialization homomorphism is injective; the subset singled out in Proposition 9 already suffices for our application to bounding the generic Mordell–Weil rank.

For these same values  $S_0$  we computed the Mordell–Weil rank of the specialized curves  $(E_{30})_{S_0}/\mathbb{Q}$  using standard MAGMA commands over number fields. We summarize the results in Table 1.

$S_0$	$(E_{30})_{S_0}(\mathbb{Q})_{\text{tors}}$	rank $(E_{30})_{S_0}(\mathbb{Q})$
3	$\mathbb{Z}/2\mathbb{Z}$	2
5	$\mathbb{Z}/2\mathbb{Z}$	2
7	$\mathbb{Z}/2\mathbb{Z}$	4
8	$\mathbb{Z}/2\mathbb{Z}$	3
11	$\mathbb{Z}/2\mathbb{Z}$	3
12	$\mathbb{Z}/2\mathbb{Z}$	1
13	$\mathbb{Z}/2\mathbb{Z}$	3
14	$\mathbb{Z}/2\mathbb{Z}$	1
17	$\mathbb{Z}/2\mathbb{Z}$	1
18	$\mathbb{Z}/2\mathbb{Z}$	2
20	$\mathbb{Z}/2\mathbb{Z}$	3
21	$\mathbb{Z}/2\mathbb{Z}$	2

Table 1: Ranks of the selected injective specializations  $(E_{30})_{S_0}/\mathbb{Q}$  from Proposition 9 (all with  $1 \leq S_0 \leq 100$ ).

## 6.4 An upper bound for the generic Mordell–Weil rank

We now deduce a global constraint on the rank over  $\mathbb{Q}(S)$ .

**Theorem 3.** Fix  $h \in \mathbb{Q}^\times$  and let  $E_h/\mathbb{Q}(S)$  be the Jacobian elliptic curve associated with the slice parameter  $h \neq 0$ . Then

$$\text{rank } E_h(\mathbb{Q}(S)) \leq 1.$$

*Proof.* By Remark 7, the Mordell–Weil rank over the function field is independent of the choice of  $h \neq 0$ . It therefore suffices to prove the stated bound for one convenient value, and we take  $h = 30$ , i.e. the curve  $E_{30}/\mathbb{Q}(S)$ .

Let  $\text{rank } E_{30}(\mathbb{Q}(S)) = r$ . For each  $S_0$  in Proposition 9, the specialization map

$$\sigma_{S_0} : E_{30}(\mathbb{Q}(S)) \hookrightarrow (E_{30})_{S_0}(\mathbb{Q})$$

is an injective group homomorphism. In particular,

$$r \leq \text{rank } (E_{30})_{S_0}(\mathbb{Q})$$

for all such  $S_0$ . By Table 1, the minimal rank among these specializations is  $\text{rank } (E_{30})_{12}(\mathbb{Q}) = \text{rank } (E_{30})_{14}(\mathbb{Q}) = \text{rank } (E_{30})_{17}(\mathbb{Q}) = 1$ . Hence  $r \leq 1$ .  $\square$

*Remark 11.* Theorem 3 does not assert that  $\text{rank } E_h(\mathbb{Q}(S)) = 1$ ; we have not produced a non-torsion section over  $\mathbb{Q}(S)$ , so the possibility  $r = 0$  remains open. The theorem shows that, at the level of the Jacobian fibration, there is at most one independent section of infinite order for any fixed  $h \neq 0$ . Any hypothetical infinite family of rational points on the discriminant fibration that arises from rational sections of the Jacobian must therefore be governed by a very restricted Mordell–Weil group.

## 7 Conclusions

We provided a self-contained algebraic reduction of the slice problem for the quintic equal-sum equation and highlighted the exact integrality constraints required when using the discriminant method, including the necessary parity condition  $v \equiv T \pmod{2}$  to recover integer  $c, d$ , as well as the size constraints  $|u| \leq S$  and  $|v| \leq T$  needed to recover solutions with  $a, b, c, d \in \mathbb{Z}_{\geq 0}$ .

We also clarified the geometric interpretation: for every nonzero slice parameter  $h \neq 0$  the discriminant equation defines a family of genus-one curves, which need not possess a rational section, so generic-rank claims require working with the Jacobian fibration.

On the Jacobian side, we isolated a uniform discriminant factor that governs the 2-division field. More precisely, for every slice parameter  $h$  the Jacobian  $E_h/\mathbb{Q}(S)$  admits a global rational 2-torsion section and, after passing to a standard 2-torsion model, the quadratic discriminant factors as

$$\Delta_2(S, h) = 12960000 S (S+h)^2 Q_5(S, h), \quad Q_5(S, h) = S^5 + 4hS^4 + 8h^2S^3 + 8h^3S^2 + 4h^4S + \frac{4}{5}h^5.$$

Since  $12960000$  and  $(S+h)^2$  are squares, full rational 2-torsion would force  $S \cdot Q_5(S, h)$  to be a square. By a homogeneity reduction to a universal genus-two hyperelliptic curve and a verified MAGMA computation (rank bound 0), we proved that no such nontrivial squares occur for  $S, h \neq 0$ . Thus, for every admissible nonzero integer slice parameter  $h \in 30\mathbb{Z} \setminus \{0\}$  and every  $S \neq 0$ , the specialized Jacobian has exactly one rational 2-torsion point for every nonsingular specialization.

For the first admissible positive slice  $h = 30$ , we computed the Jacobian elliptic surface explicitly via the invariants of the associated binary quartic, exhibited a rational 2-torsion

point, and constructed a 2-torsion model. Using the injectivity criterion of Gusić–Tadić and explicit MAGMA computations of specialized ranks over  $\mathbb{Q}$ , we proved that the Mordell–Weil rank of the Jacobian over  $\mathbb{Q}(S)$  satisfies

$$\text{rank } E_h(\mathbb{Q}(S)) \leq 1 \quad \text{for every } h \in \mathbb{Q}^\times,$$

where the uniformity in  $h \neq 0$  follows from the normalization described in Remark 7. Thus the space of rational sections, and hence any infinite family of rational points on the Jacobian fibration that arises from rational sections, is constrained for every nonzero slice. This rank bound concerns the Jacobian fibration; by itself it does not decide the existence (or nonexistence) of integral solutions of (1) on a fixed integer slice, which additionally requires analyzing rational points on the corresponding genus-one torsors and the integrality, parity, and size constraints from the reduction.

As discussed in Remark 6, the same symmetrization and genus-one construction applies to every admissible slice with  $30 \mid h$ , yielding a Jacobian elliptic curve  $E_h/\mathbb{Q}(S)$ . While full rational 2-torsion never occurs for  $h \neq 0$  (Corollary 1), and while the Jacobian geometry for fixed  $h \neq 0$  is universal up to normalization, the arithmetic problem of producing (or ruling out) integral solutions on a given integer slice depends on  $h$  through the integrality and size conditions on  $(S, u, T, v)$  and through the arithmetic of the associated genus-one torsors.

Translating back to the original Diophantine equation, one sees that any putative infinite family of *integer* solutions with  $(c+d) - (a+b) = h$  for a fixed nonzero admissible integer  $h \in 30\mathbb{Z} \setminus \{0\}$  that is obtained via a rational section on the Jacobian side would necessarily be governed by a small Mordell–Weil group, and would be subject to additional square, parity, and size constraints. Determining whether any such solutions exist remains an interesting open problem.

## A Magma scripts and computational details

We record the MAGMA [12] scripts used to support the computations described above. All scripts in this appendix are written for the slice  $h = 30$ ; by Remark 7 this suffices to verify the generic Mordell–Weil rank bound for every fixed  $h \neq 0$ .

### A.1 Universal genus-two curve computation (rank 0)

**Code (rank bound and rational points on  $\mathcal{C}_{\text{univ}}$ ).**

```

Q := Rationals();
P<x> := PolynomialRing(Q);

// Polynomial with integer coefficients (cleared denominators by multiplying by
//   25)
// C_univ: Y^2 = 25x^6 + 100x^5 + 200x^4 + 200x^3 + 100x^2 + 20x
Poly_Int := 25*x^6 + 100*x^5 + 200*x^4 + 200*x^3 + 100*x^2 + 20*x;

// Construct the curve and its Jacobian
C := HyperellipticCurve(Poly_Int);

```

```

J := Jacobian(C);

// Rank bound
print "Calculating Rank Bound...";
rb := RankBound(J);
print "Jacobian Rank Bound:", rb;

// If rank bound is 0, enumerate all rational points via the rank-0 Chabauty
→ routine
if rb eq 0 then
    print "SUCCESS: Rank is 0. Computing ALL rational points via Chabauty0...";
    pts := Chabauty0(J);

    print "Rational Points on Curve (x, Y_new):";
    print pts;
    for pt in pts do
        // Points are printed in projective coordinates (x : Y : z)
        if pt[3] eq 0 then
            print "Point at Infinity found (corresponds to x = infinity in the
                → projective closure).";
        else
            x_val := pt[1]/pt[3];
            if x_val eq 0 then
                print "Point x=0 found. (Corresponds to S=0; in the nonnegative
                    → setting this forces a=b=0.)";
            else
                printf "NON-TRIVIAL POINT FOUND: x = %o\n", x_val;
                print "This would imply a solution with S/h =", x_val;
            end if;
        end if;
    end for;
else
    print "Unexpected Rank > 0. Check calculations.";
end if;

```

### Transcript.

```

Calculating Rank Bound...
Jacobian Rank Bound: 0
SUCCESS: Rank is 0. Computing ALL rational points via Chabauty0...
Rational Points on Curve (x, Y_new):
{@ (0 : 0 : 1), (1 : -5 : 0), (1 : 5 : 0) @}
Point x=0 found. (Corresponds to S=0; in the nonnegative setting this forces
a=b=0.)
Point at Infinity found (corresponds to x = infinity in the projective closure).
Point at Infinity found (corresponds to x = infinity in the projective closure).

```

## A.2 Verification of the 2-torsion root

Code (Jacobian and 2-torsion root).

```
Q := RationalField();
R<S> := PolynomialRing(Q);

T := S + 30;

// a4(S), a6(S) from the Jacobian model
a4 := -864000 * T^2 * S * (3*T^5 + 2*S^5);
a6 := -345600000 * T^3 * S^4 * (9*T^5 + S^5);

// define e1(S)
e1 := -1200 * S^3 * T;

// cubic F_S(X) over R = Q[S]
RX<X> := PolynomialRing(R);
FS := X^3 + a4*X + a6;

// Verify that e1 is a root
F_e1 := Evaluate(FS, e1);

if F_e1 eq 0 then
    print "e1(S) correctly satisfies F_S(e1)=0.";
else
    print "ERROR: e1(S) is not a root!";
    print F_e1;
end if;
```

Transcript.

```
e1(S) correctly satisfies F_S(e1)=0.
```

## A.3 Construction and factorization of $B(S)$ and $\Delta(S)$

Code (construction of  $A, B, \Delta$  and factorization).

```
Z := Integers();
R<S> := PolynomialRing(Z);

T := S + 30;

// a4(S), a6(S)
a4 := -864000 * T^2 * S * (3*T^5 + 2*S^5);
a6 := -345600000 * T^3 * S^4 * (9*T^5 + S^5);

// e1, A(S), B(S), Delta(S)
e1 := -1200 * S^3 * T;
A := 3*e1;
```

```

B := 3*e1^2 + a4;
Delta := A^2 - 4*B;

print "B(S) =", B;
print "Delta(S) =", Delta;

Factorization(B);
Factorization(Delta);

```

### Transcript.

```

B(S) = -388800000*S^7 - 46656000000*S^6 - 2449440000000*S^5 - 73483200000000*S^4
      - 1322697600000000*S^3 - 132269760000000000*S^2 - 56687040000000000*S
Delta(S) = 12960000*S^8 + 2332800000*S^7 + 198288000000*S^6 + 97977600000000*S^5
      + 2939328000000000*S^4 + 52907904000000000*S^3 + 529079040000000000*S^2 +
      2267481600000000000*S
[
  <2, 9>,
  <3, 5>,
  <5, 5>,
  <S, 1>,
  <S + 30, 2>,
  <S^4 + 60*S^3 + 1800*S^2 + 27000*S + 162000, 1>
]
[
  <2, 8>,
  <3, 4>,
  <5, 4>,
  <S, 1>,
  <S + 30, 2>,
  <S^5 + 120*S^4 + 7200*S^3 + 216000*S^2 + 3240000*S + 19440000, 1>
]

```

## A.4 Injective specializations (Gusić–Tadić criterion)

Code (injectivity test for  $1 \leq S_0 \leq 100$ ).

```

Z := Integers();
R<S> := PolynomialRing(Z);

T := S + 30;

// a4(S), a6(S)
a4 := -864000 * T^2 * S * (3*T^5 + 2*S^5);
a6 := -345600000 * T^3 * S^4 * (9*T^5 + S^5);
// e1, A(S), B(S), Delta(S)
e1 := -1200 * S^3 * T;
A := 3*e1;
B := 3*e1^2 + a4;

```

```

Delta := A^2 - 4*B;

// normalisation and radicals
function NormPoly(f)
    g := PrimitivePart(f);
    if LeadingCoefficient(g) lt 0 then g := -g; end if;
    return g;
end function;

function Radical(f)
    g := SquarefreePart(f);
    g := NormPoly(g);
    return g;
end function;

function FactorList(f)
    rad := Radical(f);
    fac := [ NormPoly(ff[1]) : ff in Factorization(rad) | Degree(ff[1]) gt 0 ];
    return fac;
end function;

function SquarefreeDivisors(fac)
    divs := [];
    n := #fac;
    for mask in [1..2^n - 1] do
        g := R!1;
        for i in [1..n] do
            if ((mask div 2^(i-1)) mod 2) eq 1 then
                g *:= fac[i];
            end if;
        end for;
        g := NormPoly(g);
        Append(~divs, g);
    end for;
    return divs;
end function;

// Q-square test
function IsSquareQ(q)
    if q eq 0 then return true; end if;
    if q lt 0 then return false; end if;
    num := Integers()!Numerator(q);
    den := Integers()!Denominator(q);
    return IsSquare(num) and IsSquare(den);
end function;

// squarefree divisors
facB := FactorList(B);

```

```

facD := FactorList(Delta);

divsB := SquarefreeDivisors(facB);
divsD := SquarefreeDivisors(facD);

divs := [];
Skeys := {};
for g in divsB cat divsD do
    key := Sprint(g);
    if not key in Skeys then
        Include(~Skeys, key);
        Append(~divs, g);
    end if;
end for;

print "Total squarefree divisors =", #divs;

// injectivity test
function IsInjective(s0)
    for g in divs do
        v := Evaluate(g, s0);
        if IsSquareQ(Rationals()!v) then
            return false;
        end if;
    end for;
    // good reduction
    E0 := EllipticCurve([Evaluate(a4,s0), Evaluate(a6,s0)]);
    if Discriminant(E0) eq 0 then
        return false;
    end if;

    return true;
end function;

// search for 1 <= S0 <= 100
for s0 in [1..100] do
    if IsInjective(s0) then
        print "Injective specialization at S0 =", s0;
    end if;
end for;

```

**Transcript.**

```

Total squarefree divisors = 11
Injective specialization at S0 = 3
Injective specialization at S0 = 5
Injective specialization at S0 = 7
Injective specialization at S0 = 8
Injective specialization at S0 = 11

```

```

Injective specialization at S0 = 12
Injective specialization at S0 = 13
Injective specialization at S0 = 14
Injective specialization at S0 = 17
Injective specialization at S0 = 18
Injective specialization at S0 = 20
Injective specialization at S0 = 21

```

## A.5 Rank computations for specialized curves

Code (ranks and torsion over  $\mathbb{Q}$ ).

```

Q := Rationals();
R<S> := PolynomialRing(Q);

T := S + 30;

a4 := -864000 * T^2 * S * (3*T^5 + 2*S^5);
a6 := -345600000 * T^3 * S^4 * (9*T^5 + S^5);

Svals := [ 3, 5, 7, 8, 11, 12, 13, 14, 17, 18, 20, 21 ];

Eff := 2;

for s0 in Svals do
    a4_0 := Evaluate(a4, s0);
    a6_0 := Evaluate(a6, s0);

    E0 := EllipticCurve([a4_0, a6_0]);
    E0min := MinimalModel(E0);

    Tor := TorsionSubgroup(E0min);

    lb, ub := RankBounds(E0min : Effort := Eff);
    r0, exact := Rank(E0min : Effort := Eff);

    printf "S0 = %o\n", s0;
    printf "    torsion subgroup      = %o\n", Tor;
    printf "    RankBounds(E0min)     = (%o, %o)\n", lb, ub;
    printf "    Rank(E0min), exact?  = (%o, %o)\n\n", r0, exact;

    // verified exactness checks
    assert lb eq ub;
    assert exact;
    assert r0 eq lb;
end for;

```

Transcript.

```

S0 = 3
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (2, 2)
    Rank(E0min), exact? = (2, true)

S0 = 5
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (2, 2)
    Rank(E0min), exact? = (2, true)

S0 = 7
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (4, 4)
    Rank(E0min), exact? = (4, true)

S0 = 8
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (3, 3)
    Rank(E0min), exact? = (3, true)

S0 = 11
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (3, 3)
    Rank(E0min), exact? = (3, true)

S0 = 12
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (1, 1)
    Rank(E0min), exact? = (1, true)

```

```

S0 = 13
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (3, 3)
    Rank(E0min), exact? = (3, true)

S0 = 14
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (1, 1)
    Rank(E0min), exact? = (1, true)

S0 = 17
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (1, 1)
    Rank(E0min), exact? = (1, true)

S0 = 18
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (2, 2)
    Rank(E0min), exact? = (2, true)

S0 = 20
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (3, 3)
    Rank(E0min), exact? = (3, true)

S0 = 21
    torsion subgroup      = Abelian Group isomorphic to Z/2
Defined on 1 generator
Relations:
2*Tor.1 = 0
    RankBounds(E0min)    = (2, 2)
    Rank(E0min), exact? = (2, true)

```

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