Image compression using discrete Weyl-Heisenberg transform

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***Abstract*—This article proposes a new approach to raster image compression, based on the use of the two-dimensional real discrete Weyl-Heisenberg transform (DWHT). This discrete transform is orthogonal and is based on the optimal Weyl-Heisenberg signal basis, which has the best time-frequency localization. The indicated properties are ensured by choosing the optimal forming function of the basis and the best ratio of its parameters. In addition, to assess the potential possibilities of using the discrete Weyl-Heisenberg transform in compression problems, the main criteria for compression efficiency were formulated and DWHT was compared with other well-known orthogonal transforms – discrete cosine transform (DCT) and discrete Hartley transform (DHT). It is experimentally shown that the proposed method based on discrete Weyl-Heisenberg transform has much better compression characteristics. The paper also presents the results of comparing three compression methods (DHT, DCT and DWHT) in the form of corresponding tables and figures of the restored images.**

Keywords*—imaging, compression, cosine transform, Hartley transform, Weyl-Heisenberg transform, good localization, optimal basis

I. INTRODUCTION

Currently, the most important role int digital signal processing is played by discrete orthogonal transforms, which are actively used in various problems of digital filtering and spectral analysis. Meanwhile, the mathematical tools of discrete orthogonal transforms find their application in the field of data compression for the subsequent economy storage or transmission of information. An example is the discrete cosine transform (DCT) [1-2], which has gained wide popularity and served as the basis for the development of information compression algorithms such as JPEG, MPEG, MP3, etc.

Today, significant technological progress achieved in the development of new algorithms for the transmission and processing of information makes research of increasingly complex methods for obtaining frequency and time-frequency characteristics of signals especially relevant. The synthesis of the basis, which allows you to functionally separate the signal in the time-frequency domain into specific fragments, and then analyze the spectral features of the signal inside them, is a difficult task. However, exactly with the help of such bases it is possible to consider the non-stationary features of the signal and obtain greater compression efficiency.

The general theory of constructing well-localized basis systems and the corresponding spectral decompositions is presented in [3-5]. But the most important examples of such systems are the Weyl-Heisenberg bases (WH bases) [6-7], obtained by uniform shifts in time and frequency of one or a whole family of phase-shifted functions. In these works, it is shown that the Weyl-Heisenberg basis, constructed on the basis of an arbitrary forming impulse, will not be optimal, since the time-frequency localization of basis functions may be unacceptable. That is why, of special interest in the study is the Weyl-Heisenberg basis, which is based on the optimal time-frequency properties of the Gaussian. It is known that the time and frequency shifts of the Gaussian function represent the Gabor basis [3], which is not orthogonal, and the computational algorithms for spectral decomposition and inverse reconstruction constructed on its basis turn out to be unstable and difficult to implement [4]. At the same time, in [6-7], a synthesis of computationally efficient algorithms for the formation of large-dimension orthogonal WH bases for which the basis functions are localized to a Gaussian is described. Moreover, the synthesis procedure and its subsequent application are focused on processing finite discrete signal implementations.

Note that the classical method for the synthesis of discrete bases [3-5] assumes that the input signals are infinite sequences (real or complex). It determines the use of the appropriate Z-transforms, convolutions and decompositions on an infinite discrete time interval. As a result, the obtained structure of WH bases cannot be directly used for the practical processing of final implementations of signals and images. Their additional modification and refinement are needed to translate the analytical description and the corresponding fast algorithms to a finite interval.

In [6-7], an algebraic approach to the synthesis of optimal signal WH bases is used, which initially assumes a finite duration of the processed signals. The algebraic method of time and spectral transforms on a finite interval with group operations of addition and subtraction modulo is used. Therefore, the basis functions synthesized on their basis have a circulant structure of time and frequency shifts, consistent with a finite processing interval, and the prerequisites are created for constructing effective computational algorithms with use of polyphase circulant decompositions and fast finite spectral transforms. Thus, the well-known mathematical principle is implemented – the optimal processing algorithms should be synthesized in Euclidean spaces that are consistent with the structure of the processed signals.

This article proposes and explores a new approach to raster image compression based on the use of an orthogonal WH basis specially optimized for the task of processing real images. To do this, we construct the two-dimensional real discrete Weyl-Heisenberg transform (DWHT), which has the property of orthogonality and the best time-frequency localization. These properties are ensured by choosing the optimal forming function of the basis and the best ratio of its parameters. In order to assess the potential of the DWHT in the image compression problem, compression efficiency criteria are formulated. And they are based on the calculation of the compression ratio, the calculation of the norm of the difference between the original and compressed images and visual comparison. As an alternative, two other known discrete orthogonal transforms are used for comparison according to the indicated criteria: discrete cosine transform (DCT, [1-2]) and discrete Hartley transform (DHT, [8]).

An experimental research shows that the compression method based on the DWHT has improved characteristics for all of the above criteria. This is due to the fact that the DWHT used, unlike DCT and DHT, involves not only the frequency, but also the time domain and has good time-frequency localization. It means that the discrete Weyl-Heisenberg transform provides more accurate fragmentation of the analyzed image in the spectral region for the subsequent screening of non-essential spectral components.

II. Discrete Orthogonal Transforms

The main idea of discrete orthogonal transforms is to change the signal in order to give it a different shape in which it may have an unusual shape, but has useful properties. The main feature of orthogonal transforms is their reversibility, computational stability and simplicity of implementation. It means that the transformed signal, which has changed its shape and appearance, can be easily returned to its original state.

Any discrete orthogonal transform is linear and has a matrix representation. Moreover, the most important property that the transformation matrix should have is the property of unitarity (or orthonormality, if the transform is real), which is written as

$$\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}, \tag{1}$$

where $\mathbf{I}$ – identity matrix, $\mathbf{U}$ – square transform matrix with discrete orthonormal basis functions (vectors) along the columns. Thus, any discrete orthogonal transform is defined by a matrix of basis vectors.

One-dimensional forward and backward orthogonal transforms of the signal vector can be calculated by the formulas

$$\mathbf{b} = \mathbf{U}^*\mathbf{a}, \tag{2}$$

$$\tilde{\mathbf{a}} = \mathbf{U}\mathbf{b}, \tag{3}$$

where $\mathbf{a}$ – column vector of the signal elements, $\mathbf{b}$ – column vector of the spectrum elements, $\tilde{\mathbf{a}}$ – column vector of the restored signal elements.

The formulas of two-dimensional forward and backward orthogonal transforms of the image matrix are written as follows

$$\mathbf{B} = \mathbf{U}^*\mathbf{A}\mathbf{U}, \tag{4}$$

$$\tilde{\mathbf{A}} = \mathbf{U}\mathbf{B}\mathbf{U}^*. \tag{5}$$

where $\mathbf{A}$ – matrix of the image elements, $\mathbf{B}$ – matrix of the image spectrum, $\tilde{\mathbf{A}}$ – matrix of the restored image elements. In particular, $\tilde{\mathbf{A}} = \mathbf{A}$ in the absence of additional compression procedures and the fulfillment of the unitarity condition (1).

One of the first discrete orthogonal transforms, which are widely used in signal processing and filtering, is the discrete Fourier transform. However, as known, the discrete Fourier transform is aimed at processing complex data sequences, while in practice it is often necessary to work with real signals.

In 1942, R. Hartley published an integral transform, closely related to the complex Fourier transform, but it transforming real signals into a real spectrum. Subsequently, this transform was named after the author's last name and became known as the Hartley transform. In 1983, R. Bracewell presented its discrete version and one of the algorithms for its effective computational implementation. In the work [9] the prospects of using the discrete Hartley transform for image processing, including in the field of compression, are noted

The matrix of the discrete orthonormal Hartley transform (DHT) of dimension ($N \times N$) is defined as

$$\mathbf{U}_{DHT}(k,l) = \frac{1}{\sqrt{N}}\operatorname{cas}\left(\frac{2\pi kl}{N}\right),$$
$$k = 0,\dots,N-1,\ l = 0,\dots,N-1,$$

where $\operatorname{cas}(t) = \cos(t) + \sin(t)$ – Hartley kernel.

It is easy to notice that the Hartley transform is different from the Fourier transform of the choice of the kernel. Recall that for the Fourier transform, the kernel function is written as

$$\exp(-jt) = \cos(t) - j\sin(t),$$

where j – imaginary one.

Another important real analogue of the Fourier transform is the discrete cosine transform, which was introduced by N. Ahmed in 1972 [10] and, since in 1973, it has begun to be actively applied in the field of image compression. The matrix of the discrete cosine transform (DCT) of dimension ($N \times N$) is determined according to the following formula

$$\mathbf{U}_{DCT}(k,l) = \begin{cases} \sqrt{\dfrac{1}{N}}, & \text{if } k = 0, \\ \sqrt{\dfrac{2}{N}}\cos\left(\dfrac{\pi(2l+1)k}{2N}\right), & \text{else.} \end{cases}$$
$$k = 0,\dots,N-1,\ l = 0,\dots,N-1.$$

As noted above, the aim of this research is to construct a real analogue of the complex WH basis and its application in the field of image compression. The complex matrix of the orthogonal WH basis of dimension ($N \times N$) is determined by the formula [6-7]:

$$\mathbf{U} = \Re\{\mathbf{U}_R\} + j\Re\{\mathbf{U}_I\}, \tag{6}$$

whose elements are calculated according to the equations

$$\mathbf{U}_R(n, lM+k) = g[(n-lM)_N]e^{2\pi j\frac{k}{M}(n-\alpha/2)},$$

«*» – conjugate transpose sign.

$$\mathbf{U}_l(n, lM+k) = jg[(n+\frac{M}{2}-lM)_N]e^{2\pi j\frac{k}{M}(n-\alpha/2)},$$

$$n = 0,...,N-1,\ k = 0,...,M-1,\ l = 0,...,L-1,\ N = LM,$$

where M – number of frequency shifts, L – number of time shifts, α – phase parameter, $g(\cdot)$ – optimized forming WH function of dimension N. Moreover, the matrix $\mathbf{U}$ is unitary, i.e. corresponds to (1).

It should be noted that in image compression problems we are dealing with a real two-dimensional signal, which means that the use of the complex Weyl-Heisenberg matrix is not advisable. By analogy with the Hartley transform [10], we can construct a real version of matrix (6), which is determined by the formula

$$\tilde{\mathbf{U}} = \Re\{\mathbf{U}\} + \Im\{\mathbf{U}\}. \tag{7}$$

This formula (7) can be written in the equivalent form

$$\tilde{\mathbf{U}} = \Re\{\mathbf{U}_R\} + \Re\{\mathbf{U}_I\}, \tag{8}$$

and the corresponding transform formulas for matrix (8) will be similar to (2-5).

Fig. 1 shows the original monochrome square image “lena.jpg” (512 × 512 pixels) and its two-dimensional discrete orthogonal transforms (DHT, DCT and DWHT).

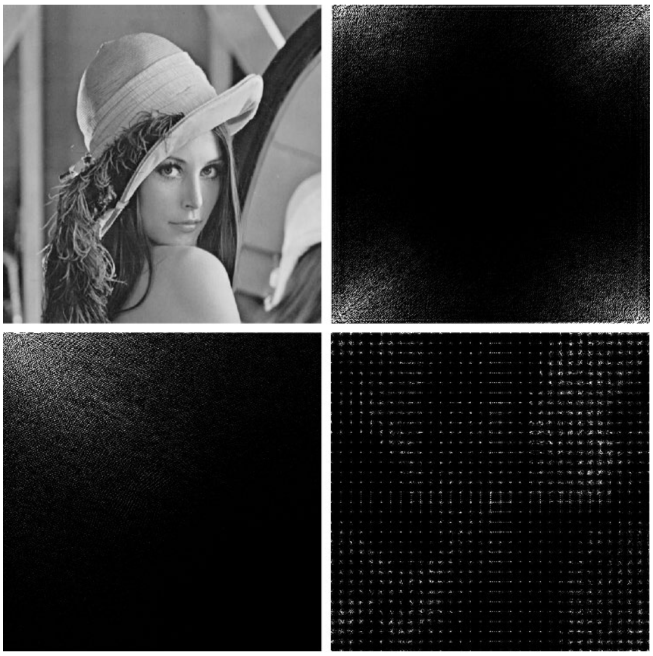

Fig. 1. Original image (top-left), DHT (top-right), DCT (bottom-left) and DWHT (bottom-right).

To assess the difference between the original image and the reconstructed by a discrete orthogonal transform, we calculate the Frobenius norm of the difference between the matrices of the original and reconstructed images:

$$E = \| \mathbf{A} - \tilde{\mathbf{A}} \|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\mathbf{A}_{ij} - \tilde{\mathbf{A}}_{ij}\right)^2} \tag{9}$$

Formula (9) will serve as a criterion for quality losses, that is, the main criterion of the difference between the reconstructed image and the original one.

Table 1 shows the results of reconstruction the original image “lena.jpg” using three discrete orthogonal transforms (DHT, DCT and DWHT).

TABLE I. RESULTS OF IMAGE RECONSTRUCTION

Transform	***DHT***	***DCT***	***DWHT***
Quality losses, E	2.2745e-11	2.6144e-09	2.2792e-09

Very small quality losses during the image reconstruction in this case are caused only by a computational error, since there is no compression.

III. THRESHOLD COMPRESSION

Compressing data by a threshold value is a procedure of equating to zero all those values of the transformed image whose modulus is less than a certain value of T – threshold. This process is lossy compression. Below is the algorithm for compressing the data of the real matrix $\mathbf{B} = (\mathbf{B}_{i,j})$ by the threshold value T.

$$\tilde{\mathbf{B}}_{i,j} = \begin{cases} 0, \text{ if } |\mathbf{B}_{i,j}| < T, \\ \mathbf{B}_{i,j}, \text{ else.} \end{cases}$$

With an optimally selected compression threshold T, the losses will be insignificant, which will eliminate unnecessary information, while preserving the integrity and quality of the reconstructed image. However, in practice it is not always convenient to select the threshold value manually, therefore, we introduce a value such as the compression coefficient

$$K = \left(N_Z / N_T\right) \cdot 100\%,$$

where N_Z – the number of elements of the image spectrum in one or another basis equated to zero, N_T – total number of analyzed elements of this spectrum (for these transforms total number of elements equals $N_T = N^2$).

After zeroing a certain number of the spectrum elements, specified by the compression coefficient K, the reconstructed image becomes a model of the original image and $\mathbf{A} \neq \tilde{\mathbf{A}}$. Moreover, the more correctly chosen basis, the more qualitatively the model displays the original image for a given number of the spectrum elements that are not zeroed $P = N_T - N_Z$. But there is some critical value of $P = P_o$, called the factor dimension of the image, below which the quality of recovery abnormally decreases. The corresponding elements of the reduced spectrum can be considered as the basic factor parameters of the image model for the selected basis. The experiment showed that the factor dimension of the DWHT image model is equal to $P_o = 7865$ that corresponds to a compression coefficient $K = 97\%$ (see Table IV).

To compare the orthogonal transforms presented above (DHT, DCT and DWHT), using the criterion for the quality losses E during image reconstruction, we will establish the same values of the compression coefficient

$$K = K_{DWHT} = K_{DCT} = K_{DHT}. \tag{10}$$

Figures 2-4 show the results of compression of the same image “lena.jpg” using three discrete orthogonal transforms (DHT, DCT and DWHT) for three fixed values of compression coefficient (K = 93%, 95%, 97%). Images are cropped for visual comparison of compression artefacts.

Tables 2-4 shows the detailed results of image compression “lena.jpg” using three orthogonal transforms, by which all three methods can be compared.

Fig. 2. Compressed image using DHT (left), DCT (middle), DWHT (right) for K = 93%.

Fig. 3. Compressed image using DHT (left), DCT (middle), DWHT (right) for K = 95%.

Fig. 4. Compressed image using DHT (left), DCT (middle), DWHT (right) for K = 97%.

TABLE II. COMPARISON OF IMAGE COMPRESSION (K = 93%)

Transform	***DHT***	***DCT***	***DWHT***
Num. of elements, N_T	262144	262144	262144
Num. of zero elements, N_Z	243793	243793	243793
Quality losses, E	528.0564	369.8089	243.9116

TABLE III. COMPARISON OF IMAGE COMPRESSION (K = 95%)

Transform	***DHT***	***DCT***	***DWHT***
Num. of elements, N_T	262144	262144	262144
Num. of zero elements, N_Z	249036	249036	249036
Quality losses, E	626.6192	461.0988	332.5995

TABLE IV. COMPARISON OF IMAGE COMPRESSION (K = 97%)

Transform	***DHT***	***DCT***	***DWHT***
Num. of elements, N_T	262144	262144	262144
Num. of zero elements, N_Z	254279	254279	254279
Quality losses, E	831.1223	631.0928	496.8902

A comparative analysis of visual and numerical characteristics shows that the smallest quality losses during image compression is achieved precisely in the case of DWHT. Moreover, the obtained relative percentage gain

$$\eta = 100\% \cdot (E - E_{DHWT}) / E_{DHWT},$$

is very significant and in the case of K = 97% ($\eta_{DHT} = 67.2\%$ for DHT and $\eta_{DHT} = 27\%$ for DCT, see Table IV).

IV. CONSLUSION

According to the results of research of compression algorithms presented above, it can be concluded that, in contrast with other known discrete orthogonal transforms: cosine (DCT) and Hartley transform (DHT), the Weyl-Heisenberg optimal basis for a fixed compression coefficient shows the lowest quality losses during image reconstruction.

Thus, the use of the Weyl-Heisenberg optimal basis turns out to be a very effective tool in the image compression problem. It is explained by the fact that the image is a non-stationary two-dimensional random process, and the Weyl-Heisenberg transform (DWHT) allows one more correctly to consider these non-stationary features, since it splits it in the time-frequency domain into well-localized fragments for subsequent effective sieving of non-essential spectral components and, as a result, image compression.

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