

# The Linear Slicing Method for Equal Sums of Like Powers: Modular and Geometric Constraints

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## Abstract

We study the classical Diophantine equation

$$a^k + b^k = c^k + d^k,$$

with non-negative integer variables  $a, b, c, d \in \mathbb{Z}_{\geq 0}$  and real exponent  $k > 1$ , under the additional linear constraint

$$(c + d) - (a + b) = h, \quad h \in \mathbb{Z}.$$

We view this as a “linear slicing” of the surface of equal sums of two  $k$ th powers by the planes  $a + b = S$  and  $c + d = S + h$ , and we analyze the geometry and arithmetic of these slices.

On the central slice  $h = 0$ , we show, using strict convexity, that there are no non-trivial solutions even over the reals: if  $a, b, c, d \geq 0$  and  $a + b = c + d$ , then  $a^k + b^k = c^k + d^k$  forces  $\{a, b\} = \{c, d\}$ . For integer  $k \geq 2$  we also prove a quantitative separation theorem on the hyperplane  $a + b = c + d$ , showing that distinct unordered pairs with the same sum  $S$  produce values of  $a^k + b^k$  that are separated by  $\gg_k S^{k-2}$ .

Our main new input for shifted slices  $h \neq 0$  is a *modular divisibility obstruction (MDO)*: for every integer  $k \geq 2$  define

$$\mathcal{P}_k := \{p \text{ prime} : (p-1) \mid (k-1)\}, \quad M_k := \prod_{p \in \mathcal{P}_k} p.$$

Then any integer solution with shift  $h$  must satisfy  $M_k \mid h$ ; this modulus  $M_k$  is the *maximal squarefree* modulus for which  $x^k \equiv x \pmod{M_k}$  holds for all integers  $x$ . As immediate corollaries, if  $k$  is prime then  $k \mid h$ , and for fixed  $k$  the necessary condition  $M_k \mid h$  leaves only a  $1/M_k$  fraction of shifts. In particular, this creates a divisibility filter that eliminates, for instance, 99.96% of all linear slices for  $k = 13$  (since  $M_{13} = 2730$ ). Combining MDO with our convexity analysis yields an exclusion zone principle. While this implies a *combined bound* on slice size  $\min\{S, S+h\} \geq 2|h|/(k-1)$ , we show that a global overlap argument provides a strictly stronger constraint. Finally, along any fixed slice  $(S, h)$  we prove an *asymptotic dominance* bound  $k \leq \max\{S, S+h\} \log 2$ , beyond which no integer solutions can occur.

We also formulate the central-slice uniqueness and separation results for arbitrary strictly convex functions, and we discuss how these structural restrictions fit with probabilistic spacing heuristics and with the Bombieri–Lang philosophy.

**Keywords:** Diophantine equations, equal sums of like powers, linear slicing method, modular divisibility obstruction, strict convexity, exclusion zone.

**MSC (2020):** 11D41 (Primary); 11A07; 26A51; 11J25.

## 1 Introduction

The Diophantine equation

$$a^k + b^k = c^k + d^k, \quad a, b, c, d \in \mathbb{Z}_{\geq 0}, \quad (1)$$

has a long history. For  $k = 2$  there are classical parametrizations of all integer solutions. For  $k = 3$  and  $k = 4$  there exist non-trivial parametrizations as well, leading to famous examples such as

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

and

$$635318657 = 59^4 + 158^4 = 133^4 + 134^4,$$

which are the first taxicab numbers of orders 3 and 4, respectively; see for instance the survey of Lander–Parkin–Selfridge [1].

For  $k \geq 5$ , however, the situation is dramatically different. As of December 2025, no non-trivial integer solutions to (1) are known in this range, and the existence of such solutions remains a difficult and largely open problem; see, for example, Browning [2] and the references therein, as well as Guy's collection of unsolved problems [3].

In this paper we impose the linear constraint

$$(c + d) - (a + b) = h, \quad h \in \mathbb{Z}, \quad (2)$$

and regard (1) as a family of equations indexed by the *shift parameter*  $h$ . Introducing

$$S := a + b, \quad S + h := c + d,$$

we compare the functions

$$f_S(x) := x^k + (S - x)^k, \quad f_{S+h}(y) := y^k + (S + h - y)^k,$$

along the *linear slices*  $a + b = S$  and  $c + d = S + h$ ; we refer to this viewpoint as the *linear slicing method*.

The central slice  $h = 0$  corresponds to the hyperplane  $a + b = c + d$ . There, strict convexity immediately forces uniqueness up to permutation, and in the integer case we obtain a quantitative separation of order  $S^{k-2}$  between distinct values of  $a^k + b^k$  along  $a + b = S$ .

Our main new contribution concerns shifted slices  $h \neq 0$ :

- a *modular divisibility obstruction (MDO)*:  $M_k \mid h$  with  $M_k = \prod_{p: p-1 \mid (k-1)} p$ , the maximal squarefree modulus with  $x^k \equiv x \pmod{M_k}$  for all  $x$ ; this contains parity as the case  $p = 2$  and yields strong filters for many odd  $k$  (e.g.  $k = 13 \Rightarrow M_k = 2730$ );
- for integer  $k \geq 2$ , an *exclusion zone* of radius  $\asymp \sqrt{\min\{S, S + h\}}$  around  $\min\{S, S + h\}/2$  on the smaller-sum slice for each fixed  $h \neq 0$ ;
- an *asymptotic dominance* bound along any fixed slice  $(S, h)$ : writing  $S_0 = \max\{S, S + h\}$  and  $M = \max\{a, b, c, d\}$  for a solution, one must have  $k \leq M \log 2 \leq S_0 \log 2$ .

For integer solutions, combining MDO with the exclusion zone gives the lower bound  $\min\{S, S + h\} \geq 2M_k/(k - 1)$  whenever  $h \neq 0$ .

**Organization of the paper.** Section 2 develops the theory of the central slice  $h = 0$ , including qualitative uniqueness and quantitative separation on  $a + b = c + d$ . Section 3 formulates central-slice uniqueness for general strictly convex functions. Section 4 proves the modular divisibility obstruction (MDO), its maximality, and corollaries (parity as a special case, density, prime exponents), and includes a table of  $M_k$  for odd  $k$ . Section 5 contains the exclusion zone principle and its combination with MDO, yielding  $\min\{S, S + h\} \geq 2M_k/(k - 1)$  for  $h \neq 0$ , together with a separate “overlap” bound coming from the global ranges of  $a^k + b^k$  on different slices. Section 6 establishes the asymptotic dominance bounds for fixed slices. Finally, Section 7 discusses heuristics and broader context.

Throughout,  $\mathbb{N} = \{0, 1, 2, \dots\}$  and, unless explicitly stated otherwise, we take  $a, b, c, d \in \mathbb{Z}_{\geq 0}$  and use  $\log$  to denote the natural logarithm.

## 2 The central slice $h = 0$ : uniqueness and quantitative separation

We begin with the most symmetric slice,  $a + b = c + d$ .

### 2.1 Qualitative uniqueness on the central slice

**Theorem 2.1.** *Let  $k > 1$  be real and let  $S \geq 0$ . Define  $f_S(x) = x^k + (S - x)^k$  on  $[0, S]$ . Then  $f_S$  is strictly decreasing on  $[0, S/2]$  and strictly increasing on  $[S/2, S]$ . In particular, if  $x_1, x_2 \in [0, S]$  and  $f_S(x_1) = f_S(x_2)$ , then  $\{x_1, S - x_1\} = \{x_2, S - x_2\}$ .*

*Proof.* We have  $f'_S(x) = k(x^{k-1} - (S - x)^{k-1})$  and  $f''_S(x) = k(k - 1)(x^{k-2} + (S - x)^{k-2}) > 0$  on  $(0, S)$ , so  $f_S$  is strictly convex and symmetric with  $f'_S(S/2) = 0$ . Hence it is strictly decreasing on  $[0, S/2]$  and strictly increasing on  $[S/2, S]$ . Injectivity on  $[0, S/2]$  gives the conclusion.  $\square$

**Corollary 2.2.** *If  $k > 1$  and  $a, b, c, d \geq 0$  satisfy  $a^k + b^k = c^k + d^k$  and  $a + b = c + d$ , then  $\{a, b\} = \{c, d\}$ .*

**Corollary 2.3.** *If  $k > 1$  and  $a, b, c, d \in \mathbb{Z}_{\geq 0}$  satisfy  $a^k + b^k = c^k + d^k$  and  $a + b = c + d$ , then  $\{a, b\} = \{c, d\}$ .*

**Corollary 2.4.** *Let  $k > m > 0$  be real and  $a, b, c, d \geq 0$  satisfy  $a^m + b^m = c^m + d^m$  and  $a^k + b^k = c^k + d^k$ . Then  $\{a, b\} = \{c, d\}$ .*

## 2.2 Quantitative separation on the central slice

**Lemma 2.5.** *If  $\varphi$  is strictly convex on an interval and  $x_n = x_0 + nh$  lie in that interval, then  $u_n := \varphi(x_n)$  satisfies  $u_{n+1} - u_n$  strictly increasing in  $n$ .*

*Proof.* Strict convexity gives  $\varphi(x_n) < \frac{1}{2}(\varphi(x_{n-1}) + \varphi(x_{n+1}))$ , whence  $u_{n+1} - u_n > u_n - u_{n-1}$ .  $\square$

**Theorem 2.6.** *Let  $k \geq 2$  be integer and let  $a, b, c, d \in \mathbb{Z}_{\geq 0}$  satisfy  $a + b = c + d = S \geq 2$  and  $\{a, b\} \neq \{c, d\}$ . Then*

$$|a^k + b^k - c^k - d^k| \geq k(k-1) \lfloor S/2 \rfloor^{k-2} \geq C_k S^{k-2}, \quad C_k = k(k-1) 3^{2-k}.$$

*Proof.* Let  $S \geq 2$  and put  $n := \lfloor S/2 \rfloor$ . For  $t \in \{0, 1, \dots, S\}$  set  $F(t) := t^k + (S-t)^k$ . Since  $F(t) = F(S-t)$ , every unordered pair  $\{u, S-u\}$  is represented uniquely by  $x := \min\{u, S-u\} \in \{0, 1, \dots, n\}$ , and its value is  $F(x)$ . By Theorem 2.1,  $F$  is strictly decreasing on  $[0, S/2]$ , hence the sequence  $u_t := F(t)$  for  $t = 0, 1, \dots, n$  is strictly decreasing.

Apply Lemma 2.5 to  $f_S$  on the arithmetic progression  $0, 1, \dots, n$ . Then  $\Delta_t := u_{t+1} - u_t$  is strictly increasing in  $t$ , so  $g_t := u_t - u_{t+1} = -(\Delta_t)$  is strictly decreasing in  $t$ . Therefore for any  $0 \leq x < y \leq n$  we have

$$u_x - u_y = \sum_{t=x}^{y-1} g_t \geq g_{n-1} = u_{n-1} - u_n.$$

Thus, for distinct unordered pairs with sum  $S$ ,

$$|a^k + b^k - c^k - d^k| \geq u_{n-1} - u_n.$$

It remains to bound  $u_{n-1} - u_n$  from below.

Case  $S = 2n$  (even). Then  $u_n = 2n^k$  and  $u_{n-1} = (n-1)^k + (n+1)^k$ , hence

$$\begin{aligned} u_{n-1} - u_n &= (n+1)^k + (n-1)^k - 2n^k \\ &= 2 \sum_{\substack{j \text{ even} \\ j \geq 2}} \binom{k}{j} n^{k-j} \\ &\geq k(k-1)n^{k-2}. \end{aligned}$$

Case  $S = 2n + 1$  (odd). Then  $u_n = n^k + (n + 1)^k$  and  $u_{n-1} = (n - 1)^k + (n + 2)^k$ , so

$$u_{n-1} - u_n = (n + 2)^k - (n + 1)^k + (n - 1)^k - n^k = \sum_{j=2}^k \binom{k}{j} n^{k-j} (2^j - 1 + (-1)^j).$$

Here each coefficient  $(2^j - 1 + (-1)^j) \geq 0$  and equals 4 when  $j = 2$ , hence

$$u_{n-1} - u_n \geq 4 \binom{k}{2} n^{k-2} = 2k(k-1)n^{k-2} \geq k(k-1)n^{k-2}.$$

Combining the cases yields

$$|a^k + b^k - c^k - d^k| \geq k(k-1)n^{k-2} = k(k-1) \lfloor S/2 \rfloor^{k-2}.$$

Finally, for  $S \geq 2$  one has  $\lfloor S/2 \rfloor \geq S/3$ , hence  $\lfloor S/2 \rfloor^{k-2} \geq 3^{2-k} S^{k-2}$ , giving the constant  $C_k$ .  $\square$

### 3 A general formulation for strictly convex functions

**Theorem 3.1.** *Let  $I \subset \mathbb{R}$  be an interval and  $\varphi : I \rightarrow \mathbb{R}$  strictly convex. Fix  $S \in \mathbb{R}$  and define  $F(x) = \varphi(x) + \varphi(S - x)$  on  $D = I \cap (S - I)$ . Then  $F$  is strictly decreasing on  $D \cap (-\infty, S/2]$ , strictly increasing on  $D \cap [S/2, \infty)$ , and  $F(x_1) = F(x_2)$  implies  $\{x_1, S - x_1\} = \{x_2, S - x_2\}$ .*

*Proof.*  $F$  is strictly convex by convexity of  $\varphi$  and symmetric:  $F(x) = F(S - x)$ . Hence it has a unique minimum at  $S/2$  and the claimed monotonicity/uniqueness follow. For background, cf. Karamata/majorization [4, 5].  $\square$

### 4 Modular constraints on the shift: the MDO

We derive a universal necessary congruence for the shift  $h$  depending only on  $k$ .

**Theorem 4.1** (Modular Divisibility Obstruction). *Let  $k \geq 2$  be integer, set*

$$\mathcal{P}_k = \{p \text{ prime} : (p-1) \mid (k-1)\}, \quad M_k = \prod_{p \in \mathcal{P}_k} p.$$

*If  $a, b, c, d \in \mathbb{Z}$  satisfy  $a^k + b^k = c^k + d^k$  and  $h = (c+d) - (a+b)$ , then  $M_k \mid h$ .*

*Proof.* Fix  $p \in \mathcal{P}_k$ , so  $k - 1 = m(p - 1)$  for some  $m \in \mathbb{N}$ . For any integer  $x$ , either  $x \equiv 0 \pmod{p}$  (then  $x^k \equiv x \equiv 0$ ) or  $x \not\equiv 0 \pmod{p}$ , in which case by FLT  $x^{p-1} \equiv 1 \pmod{p}$  and  $x^k = x(x^{p-1})^m \equiv x \pmod{p}$ . Hence  $a^k + b^k \equiv a + b \pmod{p}$  and  $c^k + d^k \equiv c + d \pmod{p}$ , so  $a + b \equiv c + d \pmod{p}$  and thus  $p \mid h$ . Since this holds for each  $p \in \mathcal{P}_k$  and the primes are coprime,  $M_k \mid h$  by CRT [6].  $\square$

**Lemma 4.2** (Maximality on squarefree moduli). *Let  $N$  be squarefree and suppose  $x^k \equiv x \pmod{N}$  holds for all integers  $x$ . Then  $N \mid M_k$ .*

*Proof.* If  $p \mid N$ , then  $x^k \equiv x \pmod{p}$  for all  $x$ . Restricting to units shows  $x^{k-1} \equiv 1$  for all  $x \in (\mathbb{Z}/p\mathbb{Z})^\times$ , so the exponent of  $(\mathbb{Z}/p\mathbb{Z})^\times$  divides  $k-1$ . Since  $(\mathbb{Z}/p\mathbb{Z})^\times$  is cyclic of order  $p-1$ , its exponent equals  $p-1$ . Hence  $p-1 \mid k-1$ , i.e.  $p \in \mathcal{P}_k$ , and therefore  $p \mid M_k$ . As  $N$  is squarefree,  $N \mid M_k$ .  $\square$

**Corollary 4.3** (Parity as a special case). *For every  $k \geq 2$ ,  $2 \in \mathcal{P}_k$  and hence  $2 \mid h$ .*

**Corollary 4.4** (Prime exponents). *If  $k$  is prime, then  $k \in \mathcal{P}_k$  and thus  $k \mid h$ .*

**Corollary 4.5** (Density of admissible shifts). *Among all integers  $h$ , the necessary condition  $M_k \mid h$  selects a subset of asymptotic density  $1/M_k$ .*

*Remark 4.6* (Arithmetic progressions in  $k$  for fixed  $h$ ). Fix  $h \neq 0$  and a prime  $p \nmid h$ . Then any  $k$  with  $(p-1) \mid (k-1)$  is *forbidden* by Theorem 4.1. In particular, if  $3 \nmid h$ , all odd  $k$  are excluded, since  $2 \mid (k-1)$  for odd  $k$ .

*Remark 4.7* (Even exponents). If  $k$  is even, then  $k-1$  is odd and the only prime  $p$  with  $(p-1) \mid (k-1)$  is  $p = 2$ . Hence  $\mathcal{P}_k = \{2\}$  and  $M_k = 2$ ; i.e. MDO coincides with parity. For clarity, our table below lists only *odd* exponents.

## A compact table of $M_k$ for odd exponents

For odd  $k \in \{3, 5, \dots, 19\}$  we record  $\mathcal{P}_k$ , the squarefree modulus  $M_k$ , the density  $1/M_k$ , and the combined lower bound on the slice size (see Proposition 5.5) written as  $2M_k/(k-1)$ .

$k$	$\mathcal{P}_k = \{p : p-1 \mid k-1\}$	$M_k$	density $1/M_k$	$2M_k/(k-1)$
3	$\{2, 3\}$	6	1/6	6
5	$\{2, 3, 5\}$	30	1/30	15
7	$\{2, 3, 7\}$	42	1/42	14
9	$\{2, 3, 5\}$	30	1/30	60/8
11	$\{2, 3, 11\}$	66	1/66	132/10
13	$\{2, 3, 5, 7, 13\}$	2730	1/2730	5460/12 = 455
15	$\{2, 3\}$	6	1/6	12/14
17	$\{2, 3, 5, 17\}$	510	1/510	1020/16
19	$\{2, 3, 7, 19\}$	798	1/798	1596/18

*Note.* Even  $k$ :  $M_k = 2$  (parity only), hence omitted.

*Remark 4.8* (Rapid growth for larger exponents). While the table lists  $M_k$  for  $k \leq 19$ , the value of  $M_k$  grows very rapidly for exponents with highly composite  $k-1$ . For instance, at  $k = 61$  we have  $60 = 2^2 \cdot 3 \cdot 5$ , leading to

$$\mathcal{P}_{61} = \{2, 3, 5, 7, 11, 13, 31, 61\}, \quad M_{61} = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 61 = 56,786,730.$$

Thus any nonzero shifted solution must satisfy  $|h| \geq M_{61} \approx 5.68 \times 10^7$ , i.e. the nearest admissible slice to  $h = 0$  is almost 57 million away. As a more dramatic illustration, for  $k = 841$  (so  $k-1 = 840$ ) one finds many primes with  $p-1 \mid 840$ , and even a partial product already exceeds  $10^{21}$ , pushing  $|h|$  into the sextillion scale.

*Remark 4.9* (Oscillation of admissible-shift densities). Recall from Corollary 4.5 that for each exponent  $k \geq 2$  the condition  $M_k \mid h$  selects a set of shifts of asymptotic density  $d_k := 1/M_k$ . Since  $M_k \geq 2$  for all  $k$  and  $M_k = 2$  for every even  $k$  (Remark 4.7), we have  $d_k \leq \frac{1}{2}$  for all  $k$  and  $d_k = \frac{1}{2}$  for infinitely many  $k$ , whence

$$\limsup_{k \rightarrow \infty} d_k = \frac{1}{2}.$$

On the other hand, by choosing exponents with  $k-1$  highly divisible (for example, taking  $k-1$  to be a multiple of  $L_n = \text{lcm}(1, 2, \dots, n)$ ), we force  $\mathcal{P}_k$  to contain all primes  $p \leq n+1$ . Thus

$$M_k \geq \prod_{p \leq n+1} p \longrightarrow \infty \quad \text{as } n \rightarrow \infty,$$

so along such a sequence we have  $d_k = 1/M_k \rightarrow 0$ , and hence

$$\liminf_{k \rightarrow \infty} d_k = 0.$$

In the language of real analysis, the sequence  $(d_k)$  therefore oscillates between values arbitrarily close to  $1/2$  and values arbitrarily close to  $0$ .

## 5 Geometry of shifted slices and exclusion zones

Fix real  $k > 1$ . For  $S \geq 0$  set  $f_S(x) = x^k + (S - x)^k$  and

$$V_{\min}(S) := \min_{x \in [0, S]} f_S(x) = 2 \left( \frac{S}{2} \right)^k.$$

**Lemma 5.1.**  *$V_{\min}(S)$  is strictly increasing and strictly convex on  $(0, \infty)$ , with*

$$V_{\min}(S + h) - V_{\min}(S) \geq k \left( \frac{S}{2} \right)^{k-1} h \quad (h > 0).$$

*Proof.* Direct differentiation gives  $V'_{\min}(S) = k(S/2)^{k-1}$  and  $V''_{\min}(S) = \frac{1}{2}k(k-1)(S/2)^{k-2} > 0$ . Apply the mean value theorem and monotonicity of  $V'_{\min}$ .  $\square$

**Lemma 5.2** (Overlap bound for shifted slices). *Let  $k > 1$ , and let  $S > 0$  and  $h \in \mathbb{R}$  be such that  $S + h > 0$ . Suppose there exist real numbers  $a, b, c, d \geq 0$  with*

$$a + b = S, \quad c + d = S + h, \quad a^k + b^k = c^k + d^k.$$

*Then, after possibly interchanging  $(a, b)$  with  $(c, d)$ , we may assume  $h > 0$  and  $S \leq S + h$ , and in this case*

$$S \geq \frac{h}{2^{\frac{k-1}{k}} - 1}.$$

*Equivalently, in symmetric form,*

$$\min\{S, S + h\} \geq \frac{|h|}{2^{\frac{k-1}{k}} - 1}.$$

*Proof.* By symmetry we may assume  $h > 0$  and  $S \leq S + h$ . On the slice  $a + b = S$  the values of  $a^k + b^k$  form the interval

$$I_k(S) = \left[ 2\left(\frac{S}{2}\right)^k, S^k \right],$$

and on the slice  $c + d = S + h$  the values of  $c^k + d^k$  form

$$I_k(S + h) = \left[ 2\left(\frac{S + h}{2}\right)^k, (S + h)^k \right].$$

If  $a^k + b^k = c^k + d^k$ , then  $I_k(S) \cap I_k(S + h) \neq \emptyset$ , so in particular the left endpoint of  $I_k(S + h)$  cannot exceed the right endpoint of  $I_k(S)$ :

$$2\left(\frac{S + h}{2}\right)^k \leq S^k.$$

Rewriting,

$$2 \cdot \frac{(S + h)^k}{2^k} \leq S^k \iff \frac{(S + h)^k}{2^{k-1}} \leq S^k \iff \frac{S + h}{2^{\frac{k-1}{k}}} \leq S.$$

Thus

$$h \leq (2^{\frac{k-1}{k}} - 1)S,$$

which is the claimed inequality. The symmetric form follows by interchanging  $S$  and  $S + h$  when  $h < 0$ .  $\square$

**Corollary 5.3** (The case  $k = 13$ ). *Let  $k = 13$ , and let  $a, b, c, d \geq 0$  satisfy  $a^{13} + b^{13} = c^{13} + d^{13}$  with  $S = a + b$ ,  $S + h = c + d$  and  $h \neq 0$ . Then*

$$\min\{S, S + h\} \geq C_{13} |h|, \quad C_{13} := \frac{1}{2^{12/13} - 1} \approx 1.115878.$$

*In particular, for integer solutions, Theorem 4.1 implies that  $M_{13} = 2730$  divides  $h$ , so  $|h| \geq M_{13}$  when  $h \neq 0$ , and hence*

$$\min\{S, S + h\} \geq C_{13} M_{13} > 3046,$$

*i.e.  $\min\{S, S + h\} \geq 3047$  on any non-central slice with  $k = 13$ .*

**Theorem 5.4** (Exclusion zone principle). *Let  $k \geq 2$  be integer,  $S > 0$ , and  $h \neq 0$ . Suppose  $a, b, c, d \geq 0$  and  $a^k + b^k = c^k + d^k$ ,  $a + b = S$ ,  $c + d = S + h$ . Let  $\delta = |a - b|/2$ . If  $h > 0$  (so  $S$  is the smaller sum), then*

$$\delta^2 \geq \frac{Sh}{2(k-1)}.$$

*The case  $h < 0$  is symmetric for  $(c, d)$ .*

*Proof.* Let  $X = S/2$ . Since  $a, b \geq 0$ , we have  $0 \leq \delta \leq X$ . We write  $a = X - \delta, b = X + \delta$ . The necessary condition for a solution is  $a^k + b^k \geq \min_{c+d=S+h}(c^k + d^k)$ .

$$(X - \delta)^k + (X + \delta)^k \geq 2 \left( \frac{S + h}{2} \right)^k = 2(X + h/2)^k.$$

We aim to show that this implies  $(k - 1)\delta^2 \geq Xh$ .

We will prove the following inequality, which shows that the  $k$ th power sum is bounded above by the  $k$ th power of its quadratic approximation at the mean  $X$ :

$$\frac{(X - \delta)^k + (X + \delta)^k}{2} \leq \left( X + \frac{k - 1}{2X} \delta^2 \right)^k. \quad (3)$$

If this inequality holds, then combining it with the necessary condition gives:

$$2(X + h/2)^k \leq (X - \delta)^k + (X + \delta)^k \leq 2 \left( X + \frac{k - 1}{2X} \delta^2 \right)^k.$$

Since the function  $t \mapsto t^k$  is strictly increasing for  $t \geq 0$ , taking the  $k$ th root yields:

$$X + h/2 \leq X + \frac{k - 1}{2X} \delta^2,$$

which simplifies to  $Xh \leq (k - 1)\delta^2$ , or  $\delta^2 \geq \frac{Xh}{k - 1} = \frac{Sh}{2(k - 1)}$ , as desired.

It remains to prove (3). We compare the binomial expansions of both sides. The Left Hand Side (LHS) is:

$$\begin{aligned} \text{LHS} &= \frac{1}{2} \sum_{j=0}^k \binom{k}{j} X^{k-j} ((-\delta)^j + \delta^j) \\ &= \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k}{2m} X^{k-2m} \delta^{2m}. \end{aligned}$$

The Right Hand Side (RHS) is:

$$\begin{aligned} \text{RHS} &= \sum_{m=0}^k \binom{k}{m} X^{k-m} \left( \frac{k - 1}{2X} \delta^2 \right)^m \\ &= \sum_{m=0}^k \binom{k}{m} \left( \frac{k - 1}{2} \right)^m X^{k-2m} \delta^{2m}. \end{aligned}$$

We will show that the inequality holds term by term, i.e., the coefficient of  $X^{k-2m}\delta^{2m}$  on the LHS is less than or equal to the corresponding coefficient on the RHS for all  $m \geq 0$ . We need to verify:

$$\binom{k}{2m} \leq \binom{k}{m} \left(\frac{k-1}{2}\right)^m, \quad \text{for } 1 \leq m \leq \lfloor k/2 \rfloor. \quad (4)$$

For  $m = 0$ , both sides are 1. For  $m > \lfloor k/2 \rfloor$ , the LHS coefficient is 0, while RHS is non-negative.

Case  $m = 1$ . LHS is  $\binom{k}{2} = \frac{k(k-1)}{2}$ . RHS is  $\binom{k}{1} \frac{k-1}{2} = \frac{k(k-1)}{2}$ . They are equal.

Case  $m \geq 2$ . We rewrite (4) (assuming  $2m \leq k$ ):

$$\begin{aligned} \frac{k!}{(2m)!(k-2m)!} &\leq \frac{k!}{m!(k-m)!} \left(\frac{k-1}{2}\right)^m \\ \frac{m!(k-m)!}{(2m)!(k-2m)!} &\leq \left(\frac{k-1}{2}\right)^m. \end{aligned}$$

The LHS is a product of  $m$  factors:

$$\text{LHS} = \frac{(k-m)(k-m-1)\cdots(k-2m+1)}{(2m)(2m-1)\cdots(m+1)} = \prod_{j=0}^{m-1} \frac{k-m-j}{2m-j}.$$

Since  $m \geq 2$  and  $0 \leq j \leq m-1$ , we have  $k-m-j \leq k-2$  and  $2m-j \geq m+1 \geq 3$ . Therefore

$$\frac{k-m-j}{2m-j} \leq \frac{k-2}{3} \leq \frac{k-1}{2},$$

so each factor in the product is  $\leq (k-1)/2$ . Hence

$$\prod_{j=0}^{m-1} \frac{k-m-j}{2m-j} \leq \left(\frac{k-1}{2}\right)^m,$$

which proves (4).

We have shown that the expansions satisfy the inequality term by term, with equality for the  $m = 0$  and  $m = 1$  terms. This proves (3) and completes the proof of the theorem.  $\square$

**Proposition 5.5** (Combination with MDO: lower bound on slice size). *Under the hypotheses of Theorem 5.4, one has*

$$\min\{S, S + h\} \geq \frac{2|h|}{k-1}.$$

*If moreover  $h \neq 0$  and  $k \geq 2$  is integer, then by Theorem 4.1  $M_k \mid h$ , hence*

$$\min\{S, S + h\} \geq \frac{2M_k}{k-1}.$$

*Proof.* If  $h > 0$ , then  $\delta \leq S/2$  and Theorem 5.4 gives  $S^2/4 \geq \delta^2 \geq Sh/(2(k-1))$ , hence  $S \geq 2h/(k-1)$ . If  $h < 0$ , apply Theorem 5.4 to the pair  $(c, d)$  with shift  $-h > 0$ : then  $S + h \geq 2|h|/(k-1)$ . Combining,  $\min\{S, S + h\} \geq 2|h|/(k-1)$ . The second bound follows since  $M_k \mid h$  implies  $|h| \geq M_k$  when  $h \neq 0$ .  $\square$

*Remark 5.6* (Relative strength of bounds). It is important to note that the overlap bound (Lemma 5.2) is strictly stronger than the combined bound (Proposition 5.5) for all  $k \geq 2$ . That is,

$$C_k := \frac{1}{2^{\frac{k-1}{k}} - 1} > \frac{2}{k-1}.$$

This inequality is equivalent to  $k-1 > 2(2^{(k-1)/k} - 1)$ , or  $k+1 > 4 \cdot 2^{-1/k}$ . For  $k=2$ ,  $3 > 4 \cdot 2^{-1/2} \approx 2.828$ . For  $k \geq 3$ ,  $k+1 \geq 4$ , while  $4 \cdot 2^{-1/k} < 4$ . While the exclusion zone principle (Theorem 5.4) provides insight into the local geometry near the center of the slice, the global constraint from the overlap of ranges (Lemma 5.2) is dominant.

## 6 Asymptotic dominance on fixed slices

We record a simple growth obstruction for large  $k$  on a fixed slice. Let  $\log$  be natural.

**Lemma 6.1.** *If  $M \geq 2$  and  $k > M \log 2$ , then  $M^k - (M-1)^k > (M-1)^k$ , hence  $M^k > 2(M-1)^k$ .*

*Proof.*  $\frac{M^k}{(M-1)^k} = \left(1 + \frac{1}{M-1}\right)^k = \exp(k \log(1 + \frac{1}{M-1})) \geq \exp(k/M)$  since  $\log(1+x) \geq x/(1+x)$  with  $x = 1/(M-1)$ . If  $k/M > \log 2$  the ratio exceeds 2.  $\square$

**Theorem 6.2** (Asymptotic dominance on a fixed slice). *Let  $a, b, c, d \in \mathbb{Z}_{\geq 0}$ ,  $h \neq 0$ , and  $k \geq 1$  satisfy  $a^k + b^k = c^k + d^k$  and  $(c + d) - (a + b) = h$ . Let  $S_1 = a + b$ ,  $S_2 = c + d$ ,  $S_0 = \max\{S_1, S_2\}$ , and  $M = \max\{a, b, c, d\}$ . Then  $M \geq 2$  and*

$$k \leq M \log 2 \leq S_0 \log 2.$$

*In particular, fixing  $S$  and  $h$ , there are no solutions with  $a + b = S$ ,  $c + d = S + h$  and  $k > K_0(S, h) := \max\{S, S + h\} \log 2$ .*

*Proof.* Since  $h \neq 0$ , not all of  $a, b, c, d$  lie in  $\{0, 1\}$ ; hence  $M \geq 2$ . If the maximal base  $M$  appeared on both sides, then after reordering we may assume  $a = c = M$ . Cancelling  $M^k$  gives  $b^k = d^k$ , hence  $b = d$ , and therefore  $h = (c + d) - (a + b) = 0$ , a contradiction. Thus for  $h \neq 0$  the value  $M$  appears on exactly one side; say  $M^k + b^k = c^k + d^k$  with  $b, c, d \leq M - 1$ . Then  $M^k - (M - 1)^k \leq |d^k - b^k| \leq (M - 1)^k$ , which fails when  $k > M \log 2$  by Lemma 6.1. Thus  $k \leq M \log 2 \leq S_0 \log 2$ .  $\square$

*Link to MDO.* Thus, for a fixed slice,  $k$  is bounded from above (Theorem 6.2), and for many  $k$  within that range, the slice is forbidden by MDO (Theorem 4.1).

*Remark 6.3.* The bound is crude but explicit and sufficient for our local purposes; typically  $M \approx S_0/2$ , suggesting a heuristic threshold near  $(S_0/2) \log 2$ .

## 7 Concluding remarks

We do not address the global open problem of non-trivial solutions to  $a^k + b^k = c^k + d^k$  for  $k \geq 5$  without linear constraints. Our contribution is structural in the sliced setting (2).

On the central slice  $a + b = c + d$  we have complete uniqueness (Theorem 2.1) and quantitative separation (Theorem 2.6): distinct unordered pairs  $\{a, b\}$  along  $a + b = S$  produce values of  $a^k + b^k$  separated by  $\gg_k S^{k-2}$ . This gives a simple local instance of sparsity heuristics for equal sums of two  $k$ th powers.

For shifted slices  $h \neq 0$ , MDO (Theorem 4.1) produces a *squarefree* modulus  $M_k$  with  $M_k \mid h$ , maximal among such moduli by Lemma 4.2. This contains parity as a special case (Corollary 4.3), yields  $k \mid h$  for prime  $k$ , and gives density  $1/M_k$  of admissible shifts (Corollary 4.5). For many odd  $k$  (e.g.  $k = 13$ ) this drastically thins the set of feasible  $h$ . Combined with the exclusion

zone (Theorem 5.4) we obtain the lower bound  $\min\{S, S+h\} \geq 2M_k/(k-1)$  (Proposition 5.5), which is an explicit structural constraint on the slice size. In addition, the overlap bound of Lemma 5.2 shows that the smaller of the two slice sums must satisfy  $\min\{S, S+h\} \gg_k |h|$ , giving a global geometric restriction based solely on the ranges of  $a^k + b^k$  on the two slices. Finally, along a fixed slice  $(S, h)$ , the dominance bound  $k \leq \max\{S, S+h\} \log 2$  (Theorem 6.2) shows that only finitely many exponents can occur.

From a broader perspective, one may compare these elementary constraints with global spacing heuristics and conjectural arithmetic–geometric uniformity (e.g. Bombieri–Lang), cf. [2, 7, 8, 9].

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