Optimization Problems

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4. (a) You have a 10 inch by 15 inch piece of tin which you plan to form into a box (without a top) by cutting a square from each corner and folding up the sides (see Fig. 10). How much should you cut from each corner so the resulting box has the greatest volume?

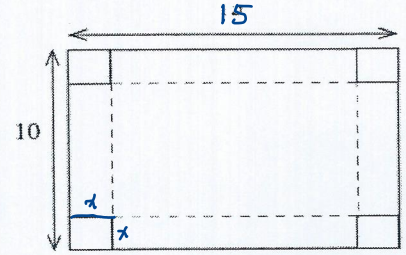


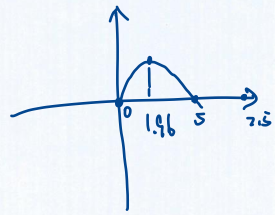
Fig. 10

Volumn = $x(10-2x)(15-2x)$

$x = 0$

$x = 5$

$x = \cancel{7.5}$



$x(150 - 20x - 30x + 4x^2)$

$x(4x^2 - 50x + 150)$

x is edge cut

$f(x) = 4x^3 - 50x^2 + 150x$

$f'(x) = 12x^2 - 100x + 150$

$x\{1.96, 6.37\}$ $x<5$

Acceptable range: $0<x<5$

$x=1.96$ is the global maximum within designated range

$x = 1.96$

You should cut 1.96*1.96 inch from each corner for largest volume.

Example 2: We want to fence a rectangular area in our backyard for a garden. One side of the garden is along the edge of the yard which is already fenced, so we only need to build a new fence along the other 3 sides of the rectangle (Fig. 2). If we have 80 feet of fencing available, what dimensions should the garden have in order to enclose the largest possible area?

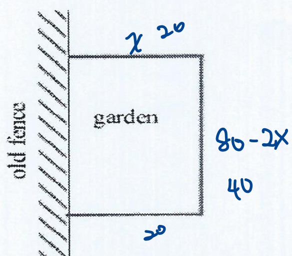


Fig. 2

$f(x) = x(80-2x)$ = Area

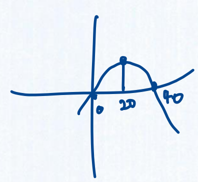
$= -2x^2 + 80x$

$f'(x) = -4x + 80$

$x = 20$

$x=0$

$x=40$



$40 > x > 0$

The function has a global maximum at x=20 in the designated interval.

Therefore, the dimensions should be 20*40 feet

31. (a) The base of a right triangle is 50 and the height is 20 (Fig. 28a). Find the dimensions and area of the rectangle with the greatest area that can be enclosed in the triangle if the base of the rectangle must lie on the base of the triangle.

(b) The base of a right triangle is B and the height is H (Fig. 28b). Find the dimensions and area of the rectangle with the greatest area that can be enclosed in the triangle if the base of the rectangle must lie on the base of the triangle.

(c) State your general conclusion from part (b) in words.

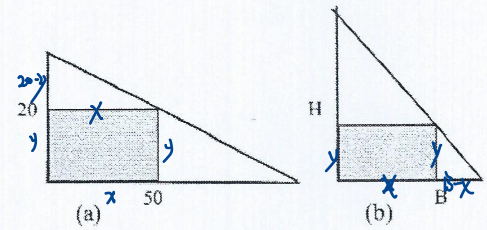


Fig. 28

(a)

$$\frac{20-y}{x} = \frac{20}{50}$$

$$20x = 50(20-y)$$

$$20x = 1000 - 50y$$

$$-50y = 20x - 1000$$

$$y = -\frac{2}{5}x + \frac{1000}{50}$$

$$y = -\frac{2}{5}x + 20$$

Area $f(x) = x(-\frac{2}{5}x+20)$

$$= -\frac{2}{5}x^2 + 20x$$

$$f'(x) = -\frac{4}{5}x + 20$$

$$x = 25$$

$$y = -\frac{2}{5}x + 20 = 10$$

$$\text{Area} = 25 \cdot 10 = 250$$

(a) dimensions 25* 10, area 250

(b) Area $= xy$

$$\frac{B-x}{y} = \frac{B}{H}$$

$$H(B-x) = By$$

$$y = \frac{H(B-x)}{B}$$

$$f(x) = \frac{Hx(B-x)}{B}$$

$$f(x) = \frac{HBx - Hx^2}{B}$$

$$= Hx - \frac{H}{B}x^2$$

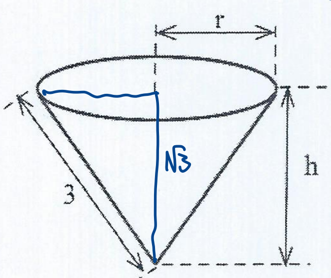
$$f'(x) = -\frac{2H}{B}x + H$$

$$x = \frac{H}{\frac{2H}{B}} = H \cdot \frac{B}{2H} = \frac{B}{2}$$

$$y = \frac{H(B-\frac{B}{2})}{B} = \frac{H\frac{B}{2}}{B} = \frac{HB}{2B} = \frac{H}{2}$$

$$\text{Area} = \frac{B}{2} \cdot \frac{H}{2} = \frac{BH}{4}$$

(c) to obtain the maximum area, the width and height must be half of the width and height of the triangle, respectively.



21. You have a 6 inch diameter circle of paper which you want to form into a drinking cup by removing a pie–shaped wedge and forming the remaining paper into a cone (Fig. 22). Find the height and top radius of so the volume of the cup is as large as possible.

$$V = \frac{1}{3}\pi r^2 h$$

$$r^2 + h^2 = 9$$

$$r^2 = 9 - h^2$$

$$V = \frac{1}{3}\pi h(9-h^2)$$

$$f(x) = 3\pi h - \frac{1}{3}\pi h^3 \quad \text{Volumn}$$

$$f'(x) = -\pi h^2 + 3\pi = 0$$

$$-\pi h^2 = -3\pi$$

$$h^2 = 3$$

$$h = \sqrt{3}$$

$$9 - 3 = 6$$

$$r = \sqrt{6}$$

Height $\sqrt{3}$
Top radius $\sqrt{6}$

17. Find the dimensions of the rectangle with the largest area if the base must be on the x-axis and its other two corners are on the graph of

(a) $y = 16 - x^2$ on $[-4, 4]$

(b) $x^2 + y^2 = 1$ on $[-1, 1]$

$x(16 - x^2)$ $\quad x(4+x)(4-x)$

Area

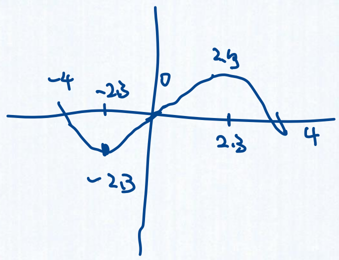
$f(x) = 16x - x^3$

$f'(x) = -3x^2 + 16 = 0$

$x = -2.3, \ 2.3$

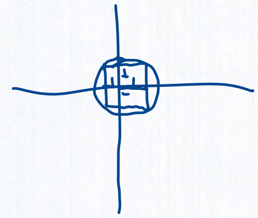
$x = 0$

$x = \pm 4$



$y = 16 - 2.3^2 = 10.71$

Dimensions: $\pm 2.3 * 10.71$



$\frac{dy}{dx}$

$2x + 2y\frac{dy}{dx} = 0$

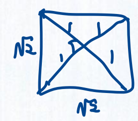
$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$

$y^2 = 1 - x^2$

$= (1+x)(1-x)$

$y = \sqrt{(1+x)(1-x)}$



Dimensions $\sqrt{2} \cdot \sqrt{2}$

$f(x) = x\sqrt{(1-x^2)}$

zeros $x=0$ $\quad x=1$ $\quad x=-1$

$f'(x) = \sqrt{1-x^2} + \frac{1}{2}x(1-x^2)^{-\frac{1}{2}} \cdot -2x$

$= (1-x^2)^{\frac{1}{2}} - x^2(1-x^2)^{-\frac{1}{2}}$

let $(1-x^2)^{\frac{1}{2}} = a$

$0 = a - x^2\frac{1}{a}$

$a = x^2\frac{1}{a}$

$x = (1-x^2)^{\frac{1}{2}}$

$1 = x^2\frac{1}{a^2}$

$= x\frac{1}{a}$

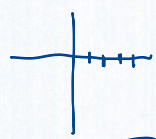
$x = a$

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

16. Find the shortest distance from the point (2,0) to the curve

(a) $y = 3x - 1$ $\quad$ (b) $y = x^2$ $\quad$ (c) $x^2 + y^2 = 1$

? Intuition



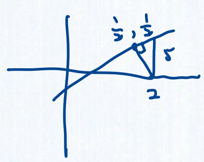
$d = \sqrt{(x+2)^2 + (x^2)^2}$

$y-0$

$2 - 1 = 1$

$d = \sqrt{(2-\frac{1}{2})^2 + (-\frac{1}{2})^2}$

$= \frac{\sqrt{5}}{2}$



$d = \dfrac{(2x - 4 + 4x^3)}{2\sqrt{\ldots}}$

$d = \sqrt{(x-2)^2 + (y-0)^2}$

$x = .825$

$y = .68$

$d = 1.358$

$= \sqrt{(x-2)^2 + (3x-1)^2}$

$= \sqrt{x^2 - 4x + 4 + 9x^2 - 6x + 1}$

$= \sqrt{10x^2 - 10x + 5}$

$d'(x) = \dfrac{1(20x - 10)}{2\sqrt{}}$ $\quad x = \frac{1}{2}$ $\quad y = \frac{3}{2} - 1 = \frac{1}{2}$