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Graph Algorithms with Hostile Partners

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Abstract

A short description of the project goes here.

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Introduction

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Dominating sets

We begin by listing some definitions.

Definition. The Dominating set, D, of a graph $G = (E, V)$ is any subset of V such that every vertex in V is adjacent to at least one vertex in D.

Definition. The Dominating number, $\gamma(G)$, of a graph $G = (E, V)$ is the size of the smallest dominating set of G.

Definition. Independent set, maximum independent set, independence number $\alpha(G)$

2.1 min size dominating set

Lemma 2.1. *Let G be a graph.*

$$\gamma(G) \geq \alpha(G)$$

Proof. Let X be a minimum dominating set in some graph $G = (V, E)$. By definition of dominating set vertex in V is adjacent to at least one vertex in $\square$

Recall that $\chi(G)$ is the chromatic number of the graph G.

Theorem 2.2 (Willis 2011 3.1)**.** *For any graph $G = (V, E)$ [5]*

$$\alpha(G) \leq \frac{|V|}{\chi(G)}$$

Recall that $\Delta(G)$ is the maximum degree of any vertex in G.

Theorem 2.3 (Balakrishnan 2012 10.3.2)**.** *[2] For any graph G with n vertices,*

$$\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$$

It is obvious that in the case when $\gamma(G)$ is known that $\gamma(G) > \gamma_g(G)$.

Theorem 2.4. *(Ore 1962) [4] For any graph G with n vertices,*

$$\gamma(G) \leq \frac{n}{2}$$

Theorem 2.5. *Let G be a graph. If x is a tight upper bound for the domination number, $\gamma(G)$, then*

$$\gamma_g(G) \geq x$$

Proof. Let G be a graph where $\gamma(G) = x$. Thus for G we are unable to find a dominating set with $< x$ vertices. Therefore there cannot be a winning strategy for Alice with $< x$ vertices. Therefore $\gamma_g(G) \geq x$ $\square$

Theorem 2.6. *Let G be a graph with n vertices, such that $n \geq 4$. Then,*

$$\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

Proof. By combination of theorems 2.4 and 2.5 we get $\gamma_g(G) \geq \lfloor \frac{n}{2} \rfloor$ $\square$

Thereom 2.6 is also proved in Alona, Baloghc, Bollobas, and Szabo 2002 [1].
The trivial upper bound is n.

Theorem 2.7. *Let G be a graph with n vertices. Then,*

$$\gamma_g(G) \leq \left\lceil \frac{2n}{3} \right\rceil$$

Proof. A dominating set on a spanning tree in a dominating set in the parent graph. Thus for any graph, G, it suffices to show we have a winning strategy for a spanning tree of G. let T be a spanning tree of G. The winning strategy for Alice is the greedy strategy as follows.

Let D be the current dominating set in T i.e. neighbours of all selected vertices.

1. Pick any vertex, v, not in D with a maximal number of neighbours not in D. That is maximise the set $\{x : x \in N(v) \bigwedge v \notin D\}$.
2. repeat until you have a dominating set.

worst case path graph requires twice the minimum of the path graph???
with no opponent this will give n/3 thus at worst with the opponent it will take 2n/3
At worst Alice will add two vertices to $\square$

Theorem 2.8. *Given p players then,*

$$\gamma_{gp}(G) \geq p\gamma(G)$$

$$\gamma_{gp}(G) \leq p\gamma_{g2}(G) \leq p \left\lceil \frac{2n}{3} \right\rceil$$

Chapter 3

Colouring

Theorem 3.1. *Let T be a tree, if we have p players then,*

$$\chi_g(T) \geq p + 2$$

The following proof is an extended version of the proof of Theorem 5.4 in [3, Bodlaender 1990]

Proof. Consider the graph G as defined in figure 3.1.

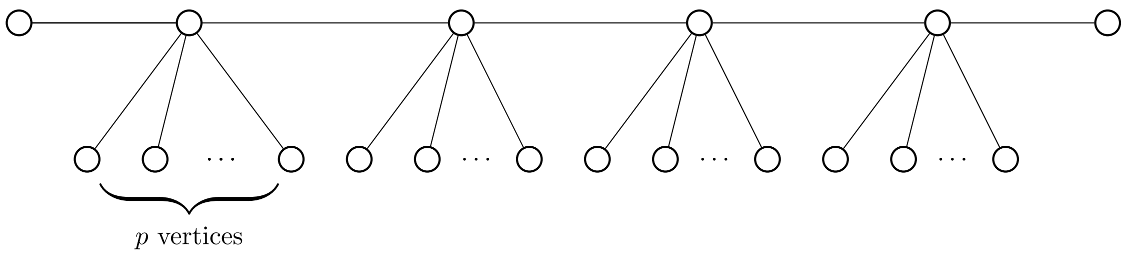

Figure 3.1

We give a strategy for Bob with $p+1$ colours. Let the colours be $\{c_1, c_2, \dots, c_{p+1}\}$ On Alice's first move she picks any vertex, v, and colours it. Let the colour of v be c_1. Bobs first move is to colour any vertex with distance 3 to v. We now have a subgraph in G of the type shown in figure 3.2. We then colour $y_1 \dots y_{p-2}$ with $c_2 \dots c_{p-1}$ respectively.

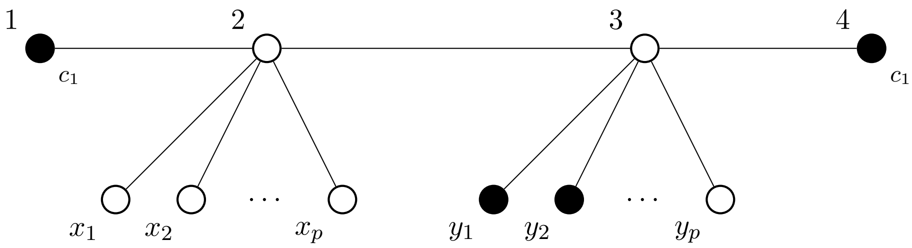

Figure 3.2

We consider two cases.

1. Alice picks 2, x_1, x_2, $\dots$, or x_p.

Bob colours y_{p-1} c_p and y_p c_{p+1}. Vertex 3 now has $p+1$ different coloured neighbours and thus Bob wins.

2. Alice picks 3,y_{P-1}, or y_p

   Bob colours x_1 $\ldots x_{p-1}$ with c_2 $\ldots c_p$ respectively. Vertex 2 now has $p+1$ different coloured neighbours and thus Bob wins.

$\square$

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