VICTORIA UNIVERSITY OF WELLINGTON
Te Whare Wananga o te Upoko o te Ika a Maui

School of Mathematics and Statistics
Te Kura Matai Tatauranga

PO Box 600
Wellington
New Zealand

Tel: +64 4 463 5341
Fax: +64 4 463 5045
Internet: sms-office@vuw.ac.nz

Graph Algorithms with Hostile Partners

Matthew Askes

Supervisor: Rod Downey

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Abstract

A short description of the project goes here.

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Introduction

Chapter 2

Dominating sets

We begin by listing some definitions.

Definition. The Dominating set, D, of a graph $G = (E, V)$ is any subset of V such that every vertex in V is adjacent to at least one vertex in D.

Definition. The Dominating number, $\gamma(G)$, of a graph $G = (E, V)$ is the size of the smallest dominating set of G.

Definition. Independent set, maximum independent set, independence number $\alpha(G)$

2.1 min size dominating set

Lemma 2.1. Let G be a graph.

$$\gamma(G) \geq \alpha(G)$$

Proof. Let X be a minimum dominating set in some graph $G = (V, E)$. By definition of dominating set vertex in V is adjacent to at least one vertex in □

Recall that $\chi(G)$ is the chromatic number of the graph G.

Theorem 2.2 (Willis 2011 3.1). For any graph $G = (V, E)$ [4]

$$\alpha(G) \leq \frac{|V|}{\chi(G)}$$

Recall that $\Delta(G)$ is the maximum degree of any vertex in G.

Theorem 2.3 (Balakrishnan 2012 10.3.2). [2] For any graph G with n vertices,

$$\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$$

It is obvious that in the case when $\gamma(G)$ is known that $\gamma(G) > \gamma_g(G)$.

Theorem 2.4. (Ore 1962) [3] For any graph G with n vertices,

$$\gamma(G) \leq \frac{n}{2}$$

Theorem 2.5. Let G be a graph. If x is a tight upper bound for the domination number, $\gamma(G)$, then

$$\gamma_g(G) \geq x$$

Proof. Let G be a graph where $\gamma(G) = x$. Thus for G we are unable to find a dominating set with $< x$ vertices. Therefore there cannot be a winning strategy for Alice with $< x$ vertices. Therefore $\gamma_g(G) \geq x$ □

Theorem 2.6. Let G be a graph with n vertices, such that $n \geq 4$. Then,

$$\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

Proof. By combination of theorems 2.4 and 2.5 we get $\gamma_g(G) \geq \lfloor \frac{n}{2} \rfloor$ □

Thereom 2.6 is also proved in Alona, Baloghc, Bollobas, and Szabo 2002 [1].
The trivial upper bound is n.

Theorem 2.7. Let G be a graph with n vertices. Then,

$$\gamma_g(G) \leq \left\lceil \frac{2n}{3} \right\rceil$$

Proof. For any graph, G, it suffices to show we have a winning strategy for a spanning tree of G. This is because a dominating set on a spanning tree in a dominating set in the graph. Thus, let T be a spanning tree of G. The winning strategy for Alice is the greedy strategy as follows. D is the current dominating set in T i.e. neighbours of all selected vertices.

1. Pick any vertex, v, not in D with a maximal number of neighbours not in D. That is maximise the set $\{x : x \in N(v) \bigwedge v \notin D\}$.

2. repeat until you have a dominating set.

worst case path graph requires twice the minimum of the path graph???
with no opponent this will give n/3 thus at worst with the opponent it will take 2n/3
At worst Alice will add two vertices to □

Theorem 2.8. Given p players then,

$$\gamma_g(G) \leq p\gamma(G)$$

Bibliography

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[3] Ore, O. Theory of Graphs (COLLOQUIUM PUBLICATIONS (AMER MATHEMATICAL SOC)). American Mathematical Society, dec 1962.

[4] Willis, W. Bounds for the independence number of a graph.