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**Graph Algorithms with Hostile
Partners**

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Abstract

A short description of the project goes here.

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Chapter 1

Introduction

Chapter 2

Dominating sets

We begin by listing some definitions.

Definition. The Dominating set, D , of a graph $G = (E, V)$ is any subset of V such that every vertex in V is adjacent to at least one vertex in D .

Definition. The Dominating number, $\gamma(G)$, of a graph $G = (E, V)$ is the size of the smallest dominating set of G .

Definition. Independent set, maximum independent set, independence number $\alpha(G)$

2.1 min size dominating set

Lemma 2.1. *Let G be a graph.*

$$\gamma(G) \geq \alpha(G)$$

Proof. Let X be a minimum dominating set in some graph $G = (V, E)$. By definition of dominating set vertex in V is adjacent to at least one vertex in X . \square

Recall that $\chi(G)$ is the chromatic number of the graph G .

Theorem 2.2 (Willis 2011 3.1 [6]). *For any graph $G = (V, E)$*

$$\alpha(G) \leq \frac{|V|}{\chi(G)}$$

Recall that $\Delta(G)$ is the maximum degree of any vertex in G .

Theorem 2.3 (Balakrishnan 2012 10.3.2 [2]). *For any graph G with n vertices,*

$$\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$$

It is obvious that in the case when $\gamma(G)$ is known that $\gamma(G) > \gamma_g(G)$.

Theorem 2.4 (Ore 1962 [5]). *For any graph G with n vertices,*

$$\gamma(G) \leq \frac{n}{2}$$

Theorem 2.5. *Let G be a graph. If x is a tight upper bound for the domination number, $\gamma(G)$, then*

$$\gamma_g(G) \geq x$$

Proof. Let G be a graph where $\gamma(G) = x$. Thus for G we are unable to find a dominating set with $< x$ vertices. Therefore there cannot be a winning strategy for Alice with $< x$ vertices. Therefore $\gamma_g(G) \geq x$ \square

Theorem 2.6. *Let G be a graph with n vertices, such that $n \geq 4$. Then,*

$$\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

Proof. By combination of theorems 2.4 and 2.5 we get $\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$ \square

Theorem 2.6 is also proved in Alona, Balogh, Bollobas, and Szabo 2002 [1].
The trivial upper bound is n .

Theorem 2.7. *Let G be a graph with n vertices. Then,*

$$\gamma_g(G) \leq \left\lceil \frac{2n}{3} \right\rceil$$

Proof. A dominating set on a spanning tree is a dominating set in the parent graph. Thus for any graph, G , it suffices to show we have a winning strategy for a spanning tree of G . Let T be a spanning tree of G . The winning strategy for Alice is the greedy strategy as follows.

Let D be the current dominating set in T i.e. neighbours of all selected vertices.

1. Pick any vertex, v , not in D with a maximal number of neighbours not in D . That is maximise the set $\{x : x \in N(v) \wedge v \notin D\}$.
2. repeat until you have a dominating set.

worst case path graph requires twice the minimum of the path graph???

with no opponent this will give $n/3$ thus at worst with the opponent it will take $2n/3$

At worst Alice will add two vertices to \square

Theorem 2.8. *Given p players then,*

$$\gamma_{gp}(G) \geq p\gamma(G)$$

$$\gamma_{gp}(G) \leq p\gamma_2(G) \leq p \left\lceil \frac{2n}{3} \right\rceil$$

Chapter 3

Colouring

Definition. We extend the colouring game to have p players. The game chromatic number for p players and some graph G is $\chi_g(G; p)$. Note: $\chi_g(G) = \chi_g(G; 2)$.

Theorem 3.1. Let T be a tree, if we have $p \geq 2$ players then,

$$\chi_g(T; p) \geq p + 2$$

The following proof is an extended version of the proof of Theorem 5.4 in [3, Bodlaender 1990]

Proof. Consider the graph G as defined in figure 3.1.

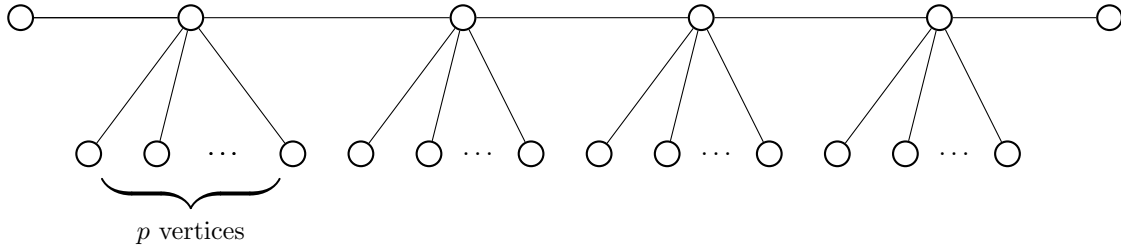


Figure 3.1

We give a strategy for Bob with $p + 1$ colours. Let the colours be $\{c_1, c_2, \dots, c_p, c_{p+1}\}$. On Alice's first move she picks any vertex, v , and colours it. Let the colour of v be c_1 . Bob's first move is to colour any vertex with distance 3 to v . We now have a subgraph in G of the type shown in figure 3.2. We then colour $y_1 \dots y_{p-2}$ with $c_2 \dots c_{p-1}$ respectively.

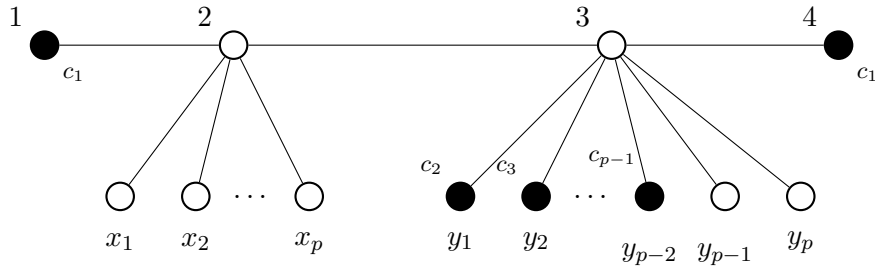


Figure 3.2

We consider three cases.

1. Alice colours 2, x_1, x_2, \dots , or x_p .

Bob colours y_{p-1} with c_p and y_p with c_{p+1} . Vertex 3 now has $p + 1$ different coloured neighbours and thus Bob wins.

2. Alice colours 3.

The colour of 3 cannot be one of $c_1 \dots c_{p-1}$. Therefore 3 is either c_p or c_{p+1} . W.l.o.g let the colour of 3 be c_{p+1} . Bob colours $x_1 \dots x_{p-1}$ with $c_2 \dots c_p$ respectively. Vertex 2 now has $p + 1$ different coloured neighbours and thus Bob wins.

3. Alice colours y_{p-1} or y_p

Bob colours 2 with c_p and y_p (or y_{p-1} if Alice coloured y_p) with c_{p+1} . Vertex 2 now has $p + 1$ different coloured neighbours and thus Bob wins.

□

Theorem 3.2.

$$\chi_g(G; p) \leq \chi_g(G; 2) + p - 2$$

Proof. By induction on the number of vertices, n and the number of players, p .

We show for any p $\chi_g(G_{n+1}; p) \leq (\chi_g(G_{n+1}; 2) + p - 2)$

$$\chi_g(G_n; p) \leq (\chi_g(G_n; 2) + p - 2) \quad \text{from induction} \quad (3.1)$$

$$\chi_g(G_n; p) \leq \chi_g(G_{n+1}; p) \quad (3.2)$$

$$(\chi_g(G_n; 2) + p - 2) \leq (\chi_g(G_{n+1}; 2) + p - 2) \quad (3.3)$$

Assume, for a contradiction, $\chi_g(G_{n+1}; p) > \chi_g(G_{n+1}; 2) + p - 2$. Then for $p = 2$ $\chi_g(G_{n+1}; 2) > \chi_g(G_{n+1}; 2) + 2 - 2$. This is a contradiction, therefore $\chi_g(G_{n+1}; p) \leq \chi_g(G_{n+1}; 2) + p - 2$.

Claim: For some n $\chi_g(G_n; p) \implies \chi_g(G_n; p + 1)$

By induction hypothesis $\chi_g(G_n; p) \leq \chi_g(G_n; 2) + p - 2$

□

Theorem 3.3.

$$\chi_g(G; p) \leq \chi_g(G; p) + 1 \leq \chi_g(G; p + 1)$$

$$\chi_g(G; 2) + p - 2 \leq \chi_g(G; p + 1)$$

L is a linear order, $G = (V, E)$ is a graph, u is a vertex in V , the rank $r(L, G)$ and rank $r(G)$ are defined as:

$$r(u, L, G) = d_{G_L}^+(u) + m(u, L, G)$$

$$r(L, G) = \max_{u \in V} r(u, L, G)$$

$$r(G) = \min_{L \in \Pi(G)} r(L, G)$$

Theorem 3.4 (Theorem 1 in [4]). *For any graph $G = (V, E)$ and ordering $L \in \Pi(G)$, if Alice uses the strategy $S(L, G)$ to play the ordering game on G , then the score will be at most $1 + r(L, G)$. In particular, $\text{col}_g(G) \leq 1 + r(G)$.*

A proof of $\chi_g(G) \leq 17$ is now a matter of finding an ordering L such that by theorem 3.4 $\chi_g(G) \leq \text{col}_g(G) \leq 1 + r(G) = 17$

Definition (Activation strategy [4]). Let $G = (V, E)$ be a graph and L a linear ordering V . We define the activation strategy $S(L, G)$ as follows:

Let U denote the set of unmarked vertices. Alice maintains a subset $A \subset V$ of active vertices. Initially $A = \emptyset$. We activate a vertex x by adding it to A . On her first turn Alice activates and marks the least vertex in the ordering L . Now suppose that Bob has just marked the vertex b . Alice uses algorithm 1 to update A and choose the next vertex.

Algorithm 1 Activation strategy

```

1:  $x \leftarrow b$ 
2: while  $x \notin U$  do
3:    $A := A \cup \{x\}$ 
4:    $s(x) = \min_L \{u \in N(x) : u < x\} \cap (U \cup \{b\})$ 
5:    $x \leftarrow s(x)$ 
6: end while
7: if  $x \neq b$  then
8:   choose  $x$ 
9: else
10:   $y \leftarrow \min_L U$ 
11:  if  $y \neq A$  then
12:     $A \leftarrow A \cup \{y\}$ 
13:  end if
14:  choose  $y$ 
15: end if

```

3.0.1 Summery of activation strategy

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