VICTORIA UNIVERSITY OF WELLINGTON
Te Whare Wānanga o te Ūpoko o te Ika a Māui

School of Mathematics and Statistics
Te Kura Mātai Tatauranga

PO Box 600
Wellington
New Zealand

Tel: +64 4 463 5341
Fax: +64 4 463 5045
Internet: sms-office@vuw.ac.nz

Graph Algorithms with Hostile Partners

Matthew Askes

Supervisor: Rod Downey

Submitted in partial fulfilment of the requirements for
Bachelor of Science with Honours in Mathematics.

Abstract

A short description of the project goes here.

Contents

Figures

Chapter 1

Introduction

Chapter 2

Dominating sets

We begin by listing some definitions.

Definition. The Dominating set, D, of a graph $G = (E, V)$ is any subset of V such that every vertex in V is adjacent to at least one vertex in D.

Definition. The Dominating number, $\gamma(G)$, of a graph $G = (E, V)$ is the size of the smallest dominating set of G.

Definition. Independent set, maximum independent set, independence number $\alpha(G)$

2.1 min size dominating set

Lemma 2.1. *Let G be a graph.*

$$\gamma(G) \geq \alpha(G)$$

Proof. Let X be a minimum dominating set in some graph $G = (V, E)$. By definition of dominating set vertex in V is adjacent to at least one vertex in $\square$

Recall that $\chi(G)$ is the chromatic number of the graph G.

Theorem 2.2 (Willis 2011 3.1 [6])**.** *For any graph $G = (V, E)$*

$$\alpha(G) \leq \frac{|V|}{\chi(G)}$$

Recall that $\Delta(G)$ is the maximum degree of any vertex in G.

Theorem 2.3 (Balakrishnan 2012 10.3.2 [2])**.** *For any graph G with n vertices,*

$$\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$$

It is obvious that in the case when $\gamma(G)$ is known that $\gamma(G) > \gamma_g(G)$.

Theorem 2.4 (Ore 1962 [5])**.** *For any graph G with n vertices,*

$$\gamma(G) \leq \frac{n}{2}$$

Theorem 2.5. *Let G be a graph. If x is a tight upper bound for the domination number, $\gamma(G)$, then*

$$\gamma_g(G) \geq x$$

Proof. Let G be a graph where $\gamma(G) = x$. Thus for G we are unable to find a dominating set with $< x$ vertices. Therefore there cannot be a winning strategy for Alice with $< x$ vertices. Therefore $\gamma_g(G) \geq x$ □

Theorem 2.6. *Let G be a graph with n vertices, such that $n \geq 4$. Then,*

$$\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

Proof. By combination of theorems 2.4 and 2.5 we get $\gamma_g(G) \geq \lfloor \frac{n}{2} \rfloor$ □

Thereom 2.6 is also proved in Alona, Baloghc, Bollobas, and Szabo 2002 [1].
The trivial upper bound is n.

Theorem 2.7. *Let G be a graph with n vertices. Then,*

$$\gamma_g(G) \leq \left\lceil \frac{2n}{3} \right\rceil$$

Proof. A dominating set on a spanning tree in a dominating set in the parent graph. Thus for any graph, G, it suffices to show we have a winning strategy for a spanning tree of G. let T be a spanning tree of G. The winning strategy for Alice is the greedy strategy as follows.

Let D be the current dominating set in T i.e. neighbours of all selected vertices.

1. Pick any vertex, v, not in D with a maximal number of neighbours not in D. That is maximise the set $\{x : x \in N(v) \bigwedge v \notin D\}$.
2. repeat until you have a dominating set.

worst case path graph requires twice the minimum of the path graph???
with no opponent this will give n/3 thus at worst with the opponent it will take 2n/3
At worst Alice will add two vertices to □

Theorem 2.8. *Given p players then,*

$$\gamma_{gp}(G) \geq p\gamma(G)$$

$$\gamma_{gp}(G) \leq p\gamma_{g2}(G) \leq p \left\lceil \frac{2n}{3} \right\rceil$$

Chapter 3

Colouring

Definition. We extend the colouring game to have p players. The game choromatic number for p players and some graph G is $\chi_g(G;p)$. Note: $\chi_g(G) = \chi_g(G;2)$.

Theorem 3.1. *Let T be a tree, if we have $p \geq 2$ players then,*

$$\chi_g(T;p) \geq p+2$$

The following proof is an extended version of the proof of Theorem 5.4 in [3, Bodlaender 1990]

Proof. Consider the graph G as defined in figure 3.1.

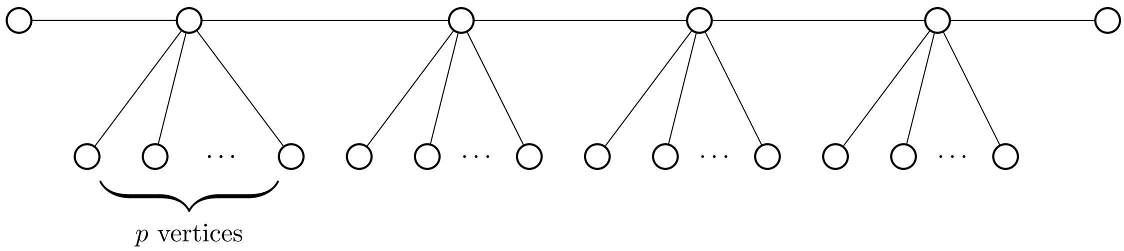

Figure 3.1

We give a strategy for Bob with $p+1$ colours. Let the colours be $\{c_1, c_2, \ldots, c_p, c_{p+1}\}$ On Alice's first move she picks any vertex, v, and colours it. Let the colour of v be c_1. Bobs first move is to colour any vertex with distance 3 to v. We now have a subgraph in G of the type shown in figure 3.2. We then colour $y_1 \ldots y_{p-2}$ with $c_2 \ldots c_{p-1}$ respectively.

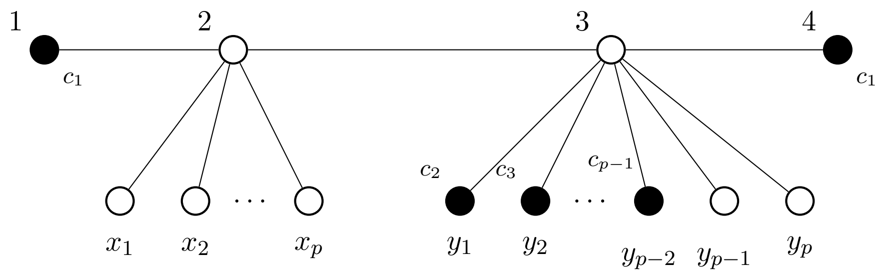

Figure 3.2

We consider three cases.

1. Alice colours 2, x_1, x_2, …, or x_p.

   Bob colours y_{p-1} with c_p and y_p with c_{p+1}. Vertex 3 now has $p+1$ different coloured neighbours and thus Bob wins.

2. Alice colours 3.

   The colour of 3 cannot be one of $c_1 \dots c_{p-1}$. Therefore 3 is either c_p or c_{p+1}. W.l.o.g let the colour of 3 be c_{p+1}. Bob colours $x_1 \dots x_{p-1}$ with $c_2 \dots c_p$ respectively. Vertex 2 now has $p+1$ different coloured neighbours and thus Bob wins.

3. Alice colours y_{p-1} or y_p

   Bob colours 2 with c_p and y_p (or y_{p-1} if Alice coloured y_p) with c_{p+1} . Vertex 2 now has $p+1$ different coloured neighbours and thus Bob wins.

□

Theorem 3.2.

$$\chi_g(G;p) \leq \chi_g(G;2) + p - 2$$

Proof. By induction on the number of vertices, n and the number of players, p.

We show for any p $\chi_g(G_{n+1};p) \leq (\chi_g(G_{n+1};2) + p - 2)$

$$\chi_g(G_n;p) \leq (\chi_g(G_n;2) + p - 2) \qquad \text{from induction} \tag{3.1}$$
$$\chi_g(G_n;p) \leq \chi_g(G_{n+1};p) \tag{3.2}$$
$$(\chi_g(G_n;2) + p - 2) \leq (\chi_g(G_{n+1};2) + p - 2) \tag{3.3}$$

Assume, for a contradiction, $\chi_g(G_{n+1};p) > \chi_g(G_{n+1};2) + p - 2$. Then for $p = 2$ $\chi_g(G_{n+1};2) > \chi_g(G_{n+1};2)+2-2$. This is a contradiction, therefore $\chi_g(G_{n+1};p) \leq \chi_g(G_{n+1};2)+ p - 2$.

Claim: For some n $\chi_g(G_n;p) \implies \chi_g(G_n;p+1)$

By induction hypothesis $\chi_g(G_n;p) \leq \chi_g(G_n;2) + p - 2$

□

Theorem 3.3.

$$\chi_g(G;p) \leq \chi_g(G;p) + 1 \leq \chi_g(G;p+1)$$

$$\chi_g(G;2) + p - 2 \leq \chi_g(G;p+1)$$

L is a linear order, $G = (V, E)$ is a graph, u is a vertex in V, the rank $r(L, G)$ and rank $r(G)$ are defined as:

$$r(u, L, G) = d^+_{G_L}(u) + m(u, L, G)$$
$$r(L, G) = \max_{u \in V} r(u, L, G)$$
$$r(G) = \min_{L \in \Pi(G)} r(L, G)$$

Theorem 3.4 (Theorem 1 in [4])**.** *For any graph $G = (V, E)$ and ordering $L \in \Pi(G)$, if Alice uses the strategy $S(L, G)$ to play the ordering game on G, then the score will be at most $1 + r(L, G)$. In particular, $col_g(G) \leq 1 + r(G)$.*

A proof of $\chi_g(G) \leq 17$ is now a matter of finding an ordering L such that by theorem 3.4 $\chi_g(G) \leq \mathrm{col_g}(G) \leq 1 + r(G) = 17$

Definition (Activation strategy [4])**.** Let $G = (V, E)$ be a graph and L a linear ordering V. We define the activation strategy $S(L, G)$ as follows:

Let U denote the set of unmarked vertices. Alice maintains a subset $A \subset V$ of active vertices. Initially $A = \emptyset$. We activate a vertex x by adding it to A. On her first turn Alice activates and marks the least vertex in the ordering L. Now suppose that Bob has just marked the vertex b. Alice uses algorithm 1 to update A and choose the next vertex.

Algorithm 1 Activation strategy

$x \leftarrow b$
while $x \notin U$ **do**
    $A := A \cup \{x\}$
    $s(x) = \min_L \{u \in N(x) : u < x\} \cap (U \cup \{b\})$
    $x \leftarrow s(x)$
end while
if $x \neq b$ **then**
    choose x
else
    $y \leftarrow \min_L U$
    if $y \neq A$ **then**
        $A \leftarrow A \cup \{y\}$
    end if
    choose y
end if

Bibliography

[1] Alon, N., Balogh, J., Bollobs, B., and Szab, T. Game domination number. *Discrete Mathematics 256*, 1 (2002), 23–33.

[2] Balakrishnan, R. *A Textbook of Graph Theory (Universitext)*. Springer, sep 2012.

[3] Bodlaender, H. L. On the complexity of some coloring games. *Internat. J. Found. Comput. Sci. 2*, 2 (1991), 133–147.

[4] Kierstead, H. A simple competitive graph coloring algorithm. *Journal of Combinatorial Theory, Series B 78*, 1 (2000), 57 – 68.

[5] Ore, O. *Theory of Graphs (COLLOQUIUM PUBLICATIONS (AMER MATHEMATICAL SOC))*. American Mathematical Society, dec 1962.

[6] Willis, W. Bounds for the independence number of a graph. Virginia Commonwealth University.