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**Graph Algorithms with Hostile  
Partners**

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**Abstract**

A short description of the project goes here.



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# Chapter 1

## Introduction

A note on notation.

Let  $\mathcal{C}$  be a family of graphs, and  $\varphi_g(G)$  is some game number on a graph  $G \in \mathcal{C}$ . Then  $\varphi_g(\mathcal{C})$  is the smallest  $k$  such that Alice has a winning strategy on any element in  $\mathcal{C}$  in less than  $k$  turns.





## Chapter 2

# Dominating sets

We begin by listing some definitions.

**Definition 2.1.** The Dominating set,  $D$ , of a graph  $G = (E, V)$  is any subset of  $V$  such that every vertex in  $V$  is adjacent to at least one vertex in  $D$ .

**Definition 2.2.** The Dominating number,  $\gamma(G)$ , of a graph,  $G = (E, V)$ , is the size of the smallest dominating set of  $G$ .

**Definition 2.3.** Independent set, maximum independent set, independence number  $\alpha(G)$

### 2.1 min size dominating set

**Lemma 2.4.** *Let  $G$  be a graph.*

$$\gamma(G) \geq \alpha(G)$$

*Proof.* Let  $X$  be a minimum dominating set in some graph  $G = (V, E)$ . By definition of dominating set vertex in  $V$  is adjacent to at least one vertex in  $X$ .  $\square$

Recall that  $\chi(G)$  is the chromatic number of the graph  $G$ .

**Theorem 2.5** (Willis 3.1 [6]). *For any graph  $G = (V, E)$*

$$\alpha(G) \leq \frac{|V|}{\chi(G)}$$

Recall that  $\Delta(G)$  is the maximum degree of any vertex in  $G$ .

**Theorem 2.6** (Balakrishnan 10.3.2 [2]). *For any graph  $G$  with  $n$  vertices,*

$$\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$$

The trivial lower bound for  $\gamma_g(G)$  when  $\gamma(G)$  is known is  $\gamma(G) > \gamma_g(G)$ . This is because there is no dominating set smaller than  $\gamma(G)$ .

**Theorem 2.7** (Ore 1962 [5]). *For any graph  $G$  with  $n$  vertices,*

$$\gamma(G) \leq \frac{n}{2}$$

**Theorem 2.8.** *Let  $G$  be a graph. If  $x$  is a tight upper bound for the domination number,  $\gamma(G)$ , then*

$$x \leq \gamma_g(G)$$

*Proof.* Let  $G$  be a graph where  $\gamma(G) = x$ . Thus for  $G$  we are unable to find a dominating set with  $< x$  vertices. Therefore there cannot be a winning strategy for Alice with  $< x$  vertices. Therefore  $\gamma_g(G) \geq x$   $\square$

**Theorem 2.9.** *Let  $G$  be a graph with  $n$  vertices, such that  $n \geq 4$ . Then,*

$$\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

*Proof.* By combination of theorems 2.7 and 2.8 we get  $\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$   $\square$

Theorem 2.9 is also proved in Alona, Balogh, Bollobas, and Szabo 2002 [1].

The trivial upper bound is  $n$ . This is because the set of vertices  $V(G)$  is a dominating set.

**Theorem 2.10.** *Let  $G$  be a graph with  $n$  vertices. Then,*

$$\gamma_g(G) \leq \left\lceil \frac{2n}{3} \right\rceil$$

*Proof.* A dominating set on a spanning tree is a dominating set in the parent graph. Thus for any graph,  $G$ , it suffices to show we have a winning strategy for a spanning tree of  $G$ . Let  $T$  be a spanning tree of  $G$ . The winning strategy for Alice is the greedy strategy as follows.

Let  $D$  be the current dominating set in  $T$  i.e. neighbours of all selected vertices.

1. Pick any vertex,  $v$ , not in  $D$  with a maximal number of neighbours not in  $D$ . That is maximise the set  $\{x : x \in N(v) \wedge v \notin D\}$ .
2. repeat until you have a dominating set.

worst case path graph requires twice the minimum of the path graph???

with no opponent this will give  $n/3$  thus at worst with the opponent it will take  $2n/3$

At worst Alice will add two vertices to  $\square$

**Theorem 2.11.** *Given  $p$  players then,*

$$\gamma_{gp}(G) \geq p\gamma(G)$$

$$\gamma_{gp}(G) \leq p\gamma_{g2}(G) \leq p \left\lceil \frac{2n}{3} \right\rceil$$

## Chapter 3

# Colouring

**Definition 3.1.** Let  $G$  be a graph. Then,  $\chi_g(G)$  is the smallest number of colours need to guarantee that Alice has a winning strategy on  $G$ .

**Definition 3.2.** We extend the colouring game to have  $p$  players. The game chromatic number for  $p$  players on some graph  $G$  is  $\chi_g(G; p)$ . Note:  $\chi_g(G) = \chi_g(G; 2)$ .

**Theorem 3.3.** Let  $\mathcal{T}$  be the family of trees. If we have  $p \geq 2$  players then,

$$\chi_g(\mathcal{T}; p) \geq p + 2$$

The following proof is an extended version of the proof of Theorem 5.4 in [3, Bodlaender 1990]

*Proof.* It suffices to show that there is a tree in where Bob has a winning strategy with  $p + 1$  colours.

Consider the graph  $G$  as defined in figure 3.1.

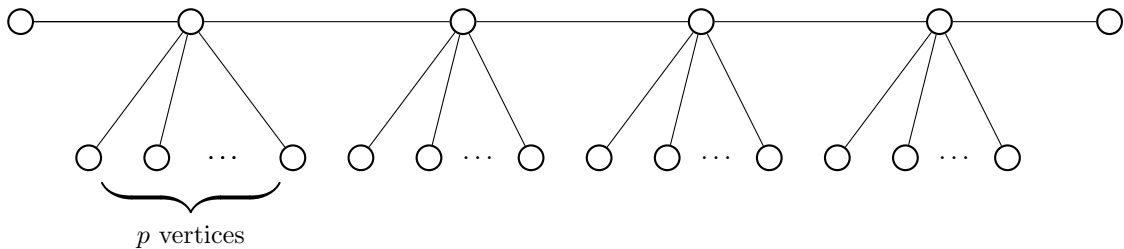


Figure 3.1

We give a strategy for Bob with  $p + 1$  colours. Let the colours be  $\{c_1, c_2, \dots, c_p, c_{p+1}\}$ . On Alice's first move she picks any vertex,  $v$ , and colours it. Let the colour of  $v$  be  $c_1$ . Bob's first move is to colour any vertex with distance 3 to  $v$ . We now have a subgraph in  $G$  of the type shown in figure 3.2. We then colour  $y_1 \dots y_{p-2}$  with  $c_2 \dots c_{p-1}$  respectively.

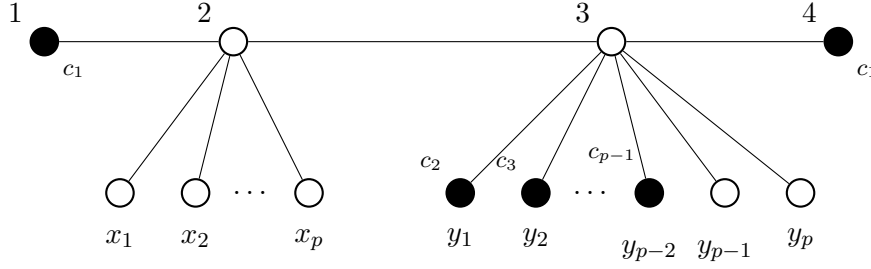


Figure 3.2

We consider three cases.

1. Alice colours 2,  $x_1, x_2, \dots$ , or  $x_p$ .

Bob colours  $y_{p-1}$  with  $c_p$  and  $y_p$  with  $c_{p+1}$ . Vertex 3 now has  $p + 1$  different coloured neighbours and thus Bob wins.

2. Alice colours 3.

The colour of 3 cannot be one of  $c_1 \dots c_{p-1}$ . Therefore 3 is either  $c_p$  or  $c_{p+1}$ . W.l.o.g let the colour of 3 be  $c_{p+1}$ . Bob colours  $x_1 \dots x_{p-1}$  with  $c_2 \dots c_p$  respectively. Vertex 2 now has  $p + 1$  different coloured neighbours and thus Bob wins.

3. Alice colours  $y_{p-1}$  or  $y_p$

Bob colours 2 with  $c_p$  and  $y_p$  (or  $y_{p-1}$  if Alice coloured  $y_p$ ) with  $c_{p+1}$ . Vertex 2 now has  $p + 1$  different coloured neighbours and thus Bob wins.

Therefore we have a winning strategy on  $G$  in  $\mathcal{T}$  for Bob with  $p + 1$  colours. □

## 3.1 Activation Strategy

### 3.1.1 Summery of activation strategy

Consider a graph  $G(v, E)$  with a linear ordering  $L$  on  $V$ .  $U$  is the set of unmarked vertices, and  $A$  the active ones.

Alice starts by marking the least  $v$  in  $L$ . If  $u$  is the last marked vertex by Bob, Alice starts at  $u$  activates it and moves to the least unmarked neighbour of  $u$ , say  $w$ , in  $L$ . If  $w$  is active Alice marks  $w$ , if not Alice repeats this on  $w$  until she finds an active vertex.

**Definition 3.4** (Kierstead [4]). Suppose  $A, B$  are disjoint subsets of  $V$ . We say that a matching  $M$  is a matching from  $A$  to  $B$  if  $M$  saturates  $A$  and  $B \setminus A$  contains a cover of  $M$ .

Let  $L$  be a linear order,  $G = (V, E)$  be a graph, and  $u$  a vertex in  $V$ .

For  $u \in V(G)$  the matching number  $m(u, L, G)$  of  $u$  with respect to  $L$  in  $G$  is defined to be the size of the largest set  $Z \subset N^-[u]$  such that there exists a partition  $[X, Y]$  of  $Z$  and there exist matchings  $M$  from  $X \subset N^-[u]$  to  $V^+(u)$  and  $N$  from  $Y \subset N^-(u)$  to  $V^+[u]$ .

Then the rank  $r(L, G)$  and rank  $r(G)$  are defined as:

$$\begin{aligned} r(u, L, G) &= d_{G_L}^+(u) + m(u, L, G) \\ r(L, G) &= \max_{u \in V} r(u, L, G) \\ r(G) &= \min_{L \in \Pi(G)} r(L, G) \end{aligned}$$

**Definition 3.5** (Activation strategy [4]). Let  $G = (V, E)$  be a graph and  $L$  a linear ordering  $V$ . We define the activation strategy  $S(L, G)$  as follows:

Let  $U$  denote the set of unmarked vertices. Alice maintains a subset  $A \subset V$  of active vertices. Initially  $A = \emptyset$ . We activate a vertex  $x$  by adding it to  $A$ . On her first turn Alice activates and marks the least vertex in the ordering  $L$ . Now suppose that Bob has just marked the vertex  $b$ . Alice uses algorithm 1 to update  $A$  and choose the next vertex.

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**Algorithm 1** Activation strategy

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```

1:  $x \leftarrow b$ 
2: while  $x \notin A$  do
3:    $A := A \cup \{x\}$ 
4:    $s(x) = \min_L \{u \in N(x) : u < x\} \cap (U \cup \{b\})$ 
5:    $x \leftarrow s(x)$ 
6: end while
7: if  $x \neq b$  then
8:   choose  $x$ 
9: else
10:   $y \leftarrow \min_L U$ 
11:  if  $y \neq A$  then
12:     $A \leftarrow A \cup \{y\}$ 
13:  end if
14:  choose  $y$ 
15: end if

```

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**Theorem 3.6** (Kierstead 1 [4]). *For any graph  $G = (V, E)$  and linear ordering  $L$  on  $V$ , if Alice uses the activation strategy  $S(L, G)$  to play the ordering game on  $G$ , then the score will be at most  $1 + r(L, G)$ . In particular,  $\text{col}_g(G) \leq 1 + r(G)$ .*

A proof of  $\chi_g(G) \leq 18$  is now a matter of finding an ordering  $L$  such that by theorem 3.6  $r(G) \leq 17$ . See [4].

Recall the definition of pathwidth.

Note that  $\chi_g(P) \geq w$

**Theorem 3.7.** *Let  $P$  be a graph of bounded pathwidth  $w$ . Then,*

$$\chi_g \leq 2w + 2$$

*Proof.* Suppose  $P = (E, V)$  is a graph of bounded pathwidth  $w$  and let  $\{X_1, X_2, \dots, X_n\}$  be the path decomposition of  $P$ .

As  $\chi_g(P) \leq \text{col}_g(P)$  it suffices to show  $\text{col}_g(P) \leq 2w + 3$ . We do this by invoking theorem 3.6.

Let  $L$  be any linear ordering on  $V$  such that for any  $a, b \in L$  and  $i, j \in [1, n]$ , if  $a \in X_i, a \notin X_j, b \in X_j$ , and  $i < j$ . then  $a < b$ .

Let  $v$  be any vertex in  $V$ , then by the definition of path width  $d^+(v) \leq w$  and for all  $i \in [1, n]$ ,  $|X_i| \leq w + 1$ .

By definition of path decomposition  $v$  is in some  $X_i$  such that  $N^-(v) \subseteq X_i$ . Let  $Z \subset N^-[v]$ , then  $|Z| \leq |N^-(v)| \leq w + 1$ . Therefore by definition 3.4  $m(u, L, P) \leq w + 1$  and  $r(P) \leq r(L, P) \leq r(u, L, P) \leq 2w + 2$ .

Finally by theorem 3.6  $\text{col}_g(P) \leq 2w + 3$

□

### 3.1.2 Upper bound using activation strategy for p players

The activation strategy finds an upper bound of  $\text{col}_g$  (The marking game) and as  $\chi_g(G) \leq \text{col}_g$  then we have an upper bound for the  $\chi_g$ .

We know from [7] at any stage, for some unmarked  $u$   $\text{col}_g \leq 2 + \max |A \cup N(u)|$ , where  $A$  is the set of active vertices.

Claim: for  $p$  players  $\text{col}_g \leq p + \max |A \cap N(u)|$ .

### 3.1.3 Refined Activation Strategy

The refined activation strategy applies the same basic strategy as above but with two differences.

First, we use a digraph  $L$  on  $G$  as our ordering. This is only a rough ordering. We partition  $V$  into blocks  $B_1, B_2, \dots, B_i$ , where if  $x \in B_i$ ,  $y \in B_j$  and  $i < j$  then the edge  $xy$  is in  $L$ . The ordering may not be an ordering in each block. However if we ignore what happens in each block we get a linear ordering.

Second when we activate a vertex  $v$  if the edge  $uv$  fulfils some conditions then reverse the direction of the edge  $uv$  in  $L$ .

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