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Graph Algorithms with Hostile Partners

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Abstract

A short description of the project goes here.

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Chapter 1

Introduction

A note on notation.

Let $\mathcal{C}$ be a family of graphs, and $\varphi_g(G)$ is some game number on a graph $G \in \mathcal{C}$. Then $\varphi_g(\mathcal{C})$ is the smallest k such that Alice has a winning strategy on any element in $\mathcal{C}$ in less than k turns.

Chapter 2

Dominating sets

We begin by listing some definitions.

Definition 2.1. The Dominating set, D, of a graph $G = (E, V)$ is any subset of V such that every vertex in V is adjacent to at least one vertex in D.

Definition 2.2. The Dominating number, $\gamma(G)$, of a graph, $G = (E, V)$, is the size of the smallest dominating set of G.

Definition 2.3. Independent set, maximum independent set, independence number $\alpha(G)$

2.1 min size dominating set

Lemma 2.4. *Let G be a graph.*

$$\gamma(G) \geq \alpha(G)$$

Proof. Let X be a minimum dominating set in some graph $G = (V, E)$. By definition of dominating set vertex in V is adjacent to at least one vertex in □

Recall that $\chi(G)$ is the chromatic number of the graph G.

Theorem 2.5 (Willis 3.1 [6])**.** *For any graph $G = (V, E)$*

$$\alpha(G) \leq \frac{|V|}{\chi(G)}$$

Recall that $\Delta(G)$ is the maximum degree of any vertex in G.

Theorem 2.6 (Balakrishnan 10.3.2 [2])**.** *For any graph G with n vertices,*

$$\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$$

The trivial lower bound for $\gamma_g(G)$ when $\gamma(G)$ is known is $\gamma(G) > \gamma_g(G)$. This is because there is no dominating set smaller than $\gamma(G)$.

Theorem 2.7 (Ore 1962 [5])**.** *For any graph G with n vertices,*

$$\gamma(G) \leq \frac{n}{2}$$

Theorem 2.8. *Let G be a graph. If x is a tight upper bound for the domination number, $\gamma(G)$, then*

$$x \leq \gamma_g(G)$$

Proof. Let G be a graph where $\gamma(G) = x$. Thus for G we are unable to find a dominating set with $< x$ vertices. Therefore there cannot be a winning strategy for Alice with $< x$ vertices. Therefore $\gamma_g(G) \geq x$ □

Theorem 2.9. *Let G be a graph with n vertices, such that $n \geq 4$. Then,*

$$\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

Proof. By combination of theorems 2.7 and 2.8 we get $\gamma_g(G) \geq \lfloor \frac{n}{2} \rfloor$ □

Thereom 2.9 is also proved in Alona, Baloghc, Bollobas, and Szabo 2002 [1].

The trivial upper bound is n. This is because the set of vertices $V(G)$ is a dominating set.

Theorem 2.10. *Let G be a graph with n vertices. Then,*

$$\gamma_g(G) \leq \left\lceil \frac{2n}{3} \right\rceil$$

Proof. A dominating set on a spanning tree in a dominating set in the parent graph. Thus for any graph, G, it suffices to show we have a winning strategy for a spanning tree of G. let T be a spanning tree of G. The winning strategy for Alice is the greedy strategy as follows.

Let D be the current dominating set in T i.e. neighbours of all selected vertices.

1. Pick any vertex, v, not in D with a maximal number of neighbours not in D. That is maximise the set $\{x : x \in N(v) \bigwedge v \notin D\}$.

2. repeat until you have a dominating set.

worst case path graph requires twice the minimum of the path graph???

with no opponent this will give n/3 thus at worst with the opponent it will take 2n/3

At worst Alice will add two vertices to □

Theorem 2.11. *Given p players then,*

$$\gamma_{gp}(G) \geq p\gamma(G)$$

$$\gamma_{gp}(G) \leq p\gamma_{g2}(G) \leq p \left\lceil \frac{2n}{3} \right\rceil$$

Chapter 3

Colouring

Definition 3.1. Let G be a graph. Then, $\chi_g(G)$ is the smallest number of colours need to guarantee that Alice has a winning strategy on G.

Definition 3.2. We extend the colouring game to have p players. The game choromatic number for p players on some graph G is $\chi_g(G;p)$. Note: $\chi_g(G) = \chi_g(G;2)$.

Theorem 3.3. *Let $\mathcal{T}$ be the family of trees. If we have $p \geq 2$ players then,*

$$\chi_g(\mathcal{T};p) \geq p+2$$

The following proof is an extended version of the proof of Theorem 5.4 in [3, Bodlaender 1990]

Proof. It suffices to show that there is a tree in where Bob has a winning strategy with $p+1$ colours.

Consider the graph G as defined in figure 3.1.

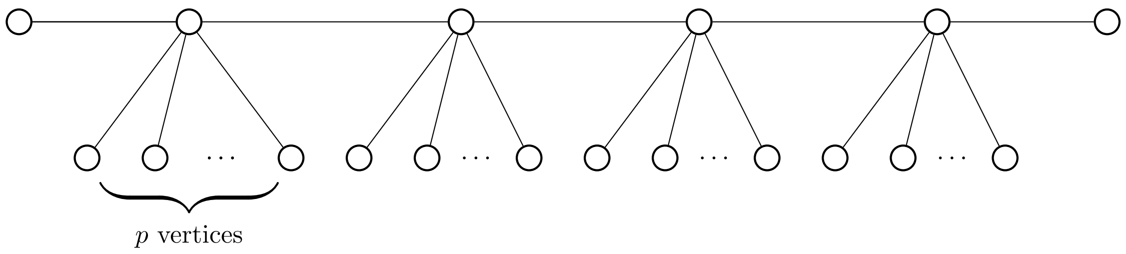

Figure 3.1

We give a strategy for Bob with $p+1$ colours. Let the colours be $\{c_1, c_2, \ldots, c_p, c_{p+1}\}$ On Alice's first move she picks any vertex, v, and colours it. Let the colour of v be c_1. Bobs first move is to colour any vertex with distance 3 to v. We now have a subgraph in G of the type shown in figure 3.2. We then colour $y_1 \ldots y_{p-2}$ with $c_2 \ldots c_{p-1}$ respectively.

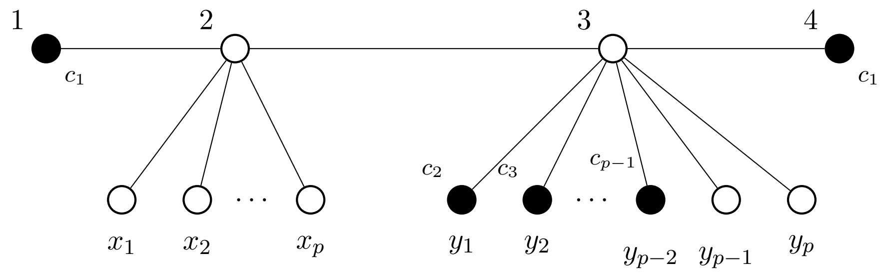

Figure 3.2

We consider three cases.

1. Alice colours 2, x_1, x_2, ..., or x_p.

   Bob colours y_{p-1} with c_p and y_p with c_{p+1}. Vertex 3 now has $p+1$ different coloured neighbours and thus Bob wins.

2. Alice colours 3.

   The colour of 3 cannot be one of $c_1 \dots c_{p-1}$. Therefore 3 is either c_p or c_{p+1}. W.l.o.g let the colour of 3 be c_{p+1}. Bob colours $x_1 \dots x_{p-1}$ with $c_2 \dots c_p$ respectively. Vertex 2 now has $p+1$ different coloured neighbours and thus Bob wins.

3. Alice colours y_{p-1} or y_p

   Bob colours 2 with c_p and y_p (or y_{p-1} if Alice coloured y_p) with c_{p+1} . Vertex 2 now has $p+1$ different coloured neighbours and thus Bob wins.

Therefore we have a winning strategy on G in $\mathcal{T}$ for Bob with $p+1$ colours.

$\square$

3.1 Activation Strategy

3.1.1 Summery of activation strategy

Consider a graph $G(v, E)$ with a linear ordering L on V. U is the set of unmarked vertices, and A the active ones.

Alice starts by marking the least v in L. If u is the last marked vertex by Bob, Alice starts at u activates it and moves to the least unmarked neighbour of u, say w, in L. If w is active Alice markes w, if not Alice repeats this on w until she finds an active vertex.

Definition 3.4 (Kierstead [4])**.** Suppose A, B are disjoint subsets of V. We say that a matching M is a matching from A to B if M saturates A and $B \setminus A$ contains a cover of M.

Let L be a linear order, $G = (V, E)$ be a graph, and u a vertex in V.

For $u \in V(G)$ the matching number $m(u, L, G)$ of u with respect to L in G is defined to be the size of the largest set $Z \subset N^-[u]$ such that there exists a partition $[X, Y]$ of Z and there exist matchings M from $X \subset N^-[u]$ to $V^+(u)$ and N from $Y \subset N^-(u)$ to $V^+[u]$.

Then the rank $r(L, G)$ and rank $r(G)$ are defined as:

$$r(u, L, G) = d^+_{G_L}(u) + m(u, L, G)$$
$$r(L, G) = \max_{u \in V} r(u, L, G)$$
$$r(G) = \min_{L \in \Pi(G)} r(L, G)$$

Definition 3.5 (Activation strategy [4])**.** Let $G = (V, E)$ be a graph and L a linear ordering V. We define the activation strategy $S(L, G)$ as follows:

Let U denote the set of unmarked vertices. Alice maintains a subset $A \subset V$ of active vertices. Initially $A = \emptyset$. We activate a vertex x by adding it to A. On her first turn Alice activates and marks the least vertex in the ordering L. Now suppose that Bob has just marked the vertex b. Alice uses algorithm 1 to update A and choose the next vertex.

Algorithm 1 Activation strategy

$x \leftarrow b$
while $x \notin a$ **do**
  $A := A \cup \{x\}$
  $s(x) = \min_L \{u \in N(x) : u < x\} \cap (U \cup \{b\})$
  $x \leftarrow s(x)$
if $x \neq b$ **then**
  choose x
else
  $y \leftarrow \min_L U$
  if $y \neq A$ **then**
    $A \leftarrow A \cup \{y\}$
  choose y

Theorem 3.6 (Kierstead 1 [4])**.** *For any graph $G = (V, E)$ and linear ordering L on V, if Alice uses the activation strategy $S(L, G)$ to play the ordering game on G, then the score will be at most $1 + r(L, G)$. In particular, $col_g(G) \leq 1 + r(G)$.*

A proof of $\chi_g(G) \leq 18$ is now a matter of finding an ordering L such that by theorem 3.6 $r(G) \leq 17$. See [4].

Recall the definition of pathwidth.

Note that $\chi_g(P) \geq w$

Algorithm 2 Linear order in graph of bounded path width

Require: $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ is the path decomposition of a graph G
Require: L is a set
Ensure: L is a linear order on $V(G)$
$Q \leftarrow \emptyset$ ▷ Q is a FIFO queue
$L \leftarrow \emptyset$ ▷ our linear order for $V(G)$
for all $X_i \in \mathcal{X}$ **do**
  $Q' \leftarrow Q \cup (X_i \setminus Q)$ ▷ Q' is a FIFO queue
  $Q \leftarrow \emptyset$
  for all $x \in Q'$ **do**
    if $x \in \{X_{i+1} \cup \cdots \cup X_n\}$ **then**
      $Q \leftarrow Q \cup \{x\}$
    else
      $L \leftarrow L \cup \{x\}$

The following lemma is effectively the definition of the linear order given by algorithm 2.

Lemma 3.7. *Let $P = (E, V)$ be a graph of bounded pathwidth w and $\{X_1, X_2, \ldots, X_n\}$ be a path decomposition of P of width w.*

Suppose L is a linear ordering on V generated by algorithm 2 and u is a vertex in P. Then there is some i such that $v \in X_i$ and $N^-(v) \subseteq X_i$. Further i is the largest i for which X_i contains v.

Proof. Let X_i be the segment in which v is added to L, that is i is the number of times line 3 has been looped when v is added to L. The algorithm adds a vertex v to L iff there is no $j > i$ such that $v \in X_j$. Thus i is the greatest i such that $v \in X_i$. Therefore by definition of path decomposition all the neighbours of v not in the L are in X_i. Further the neighbours not in L when v is added to L are greater than v in L. Therefore is $N^-(v) \in X_i$. □

Theorem 3.8. *Let P be a graph of bounded pathwidth w. Then,*

$$\chi_g(P) \leq 2w + 2$$

Proof. Suppose $P = (E, V)$ is a graph of bounded pathwidth w and let $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ be a path decomposition of P with width w.

As $\chi_g(P) \leq \mathrm{col_g}(P)$ it suffices to show $\mathrm{col_g}(P) \leq 2w+3$. We do this by invoking theorem 3.6.

Let L be a linear ordering on V generated by algorithm 2. Let v be any vertex in V.

By the definition of path width $d^+(v) \leq w$ and for all $X \in \mathcal{X}$, $|X| \leq w + 1$.

Let $Z \subset N^-[v]$.

By lemma 3.7 $N^-(v) \subset A$ for some $A \in \mathcal{X}$. Therefore$|Z| \leq |N^-(v)| \leq |A| \leq w + 1$. Therefore by definition 3.4 $m(u, L, P) \leq w + 1$ and $r(P) \leq r(L, P) \leq r(u, L, P) \leq 2w + 2$.

Finally by theorem 3.6 $\mathrm{col_g}(P) \leq 2w + 3$

□

3.1.2 Upper bound using activation strategy for p players

The activation strategy finds an upper bound of $\mathrm{col_g}$ (The marking game) and as $\chi_g(G) \leq \mathrm{col_g}$ then we have an upper bound for the χ_g.

We know from [7] at any stage, for some unmarked u $\mathrm{col_g} \leq 2 + \max |A \cup N(u)|$, where A is the set of active vertices.

Claim: for p players $\mathrm{col_g} \leq p + \max |A \cap N(u)|$.

3.1.3 Refined Activation Strategy

The refined activation strategy applies the same basic strategy as above but with two differences.

First, we use a digraph L on G as our ordering. This is only a rough ordering. We partition V into blocks $B_1, B_2, \ldots, B_i$, where if $x \in B_i$, $y \in B_j$ and $i <$ then the an edge xy is in L. The ordering may not be an ordering in each block. However if we ignore what happens in each block we get a linear ordering.

Second when we activate a vertex v if the edge uv fulfils some conditions then reverse the direction of the edge uv in L.

3.1.4 Online colouring

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