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**Graph Algorithms with Hostile
Partners**

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Abstract

A short description of the project goes here.

Contents

1	Introduction	1
2	Dominating sets	3
2.1	min size dominating set	3
3	Colouring	5
3.1	Activation Strategy	6
3.1.1	Summery of activation strategy	6
3.1.2	Upper bound using activation strategy for p players	8
3.1.3	Refined Activation Strategy	8
3.1.4	Online colouring	8

Figures

3.1	5
3.2	6

Chapter 1

Introduction

A note on notation.

Let \mathcal{C} be a family of graphs, and $\varphi_g(G)$ is some game number on a graph $G \in \mathcal{C}$. Then $\varphi_g(\mathcal{C})$ is the smallest k such that Alice has a winning strategy on any element in \mathcal{C} in less than k turns.

Chapter 2

Dominating sets

We begin by listing some definitions.

Definition 2.1. The Dominating set, D , of a graph $G = (E, V)$ is any subset of V such that every vertex in V is adjacent to at least one vertex in D .

Definition 2.2. The Dominating number, $\gamma(G)$, of a graph, $G = (E, V)$, is the size of the smallest dominating set of G .

Definition 2.3. Independent set, maximum independent set, independence number $\alpha(G)$

2.1 min size dominating set

Lemma 2.4. *Let G be a graph.*

$$\gamma(G) \geq \alpha(G)$$

Proof. Let X be a minimum dominating set in some graph $G = (V, E)$. By definition of dominating set vertex in V is adjacent to at least one vertex in X . \square

Recall that $\chi(G)$ is the chromatic number of the graph G .

Theorem 2.5 (Willis 3.1 [6]). *For any graph $G = (V, E)$*

$$\alpha(G) \leq \frac{|V|}{\chi(G)}$$

Recall that $\Delta(G)$ is the maximum degree of any vertex in G .

Theorem 2.6 (Balakrishnan 10.3.2 [2]). *For any graph G with n vertices,*

$$\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$$

The trivial lower bound for $\gamma_g(G)$ when $\gamma(G)$ is known is $\gamma(G) > \gamma_g(G)$. This is because there is no dominating set smaller than $\gamma(G)$.

Theorem 2.7 (Ore 1962 [5]). *For any graph G with n vertices,*

$$\gamma(G) \leq \frac{n}{2}$$

Theorem 2.8. *Let G be a graph. If x is a tight upper bound for the domination number, $\gamma(G)$, then*

$$x \leq \gamma_g(G)$$

Proof. Let G be a graph where $\gamma(G) = x$. Thus for G we are unable to find a dominating set with $< x$ vertices. Therefore there cannot be a winning strategy for Alice with $< x$ vertices. Therefore $\gamma_g(G) \geq x$ \square

Theorem 2.9. *Let G be a graph with n vertices, such that $n \geq 4$. Then,*

$$\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

Proof. By combination of theorems 2.7 and 2.8 we get $\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$ \square

Theorem 2.9 is also proved in Alona, Balogh, Bollobas, and Szabo 2002 [1].

The trivial upper bound is n . This is because the set of vertices $V(G)$ is a dominating set.

Theorem 2.10. *Let G be a graph with n vertices. Then,*

$$\gamma_g(G) \leq \left\lceil \frac{2n}{3} \right\rceil$$

Proof. A dominating set on a spanning tree is a dominating set in the parent graph. Thus for any graph, G , it suffices to show we have a winning strategy for a spanning tree of G . Let T be a spanning tree of G . The winning strategy for Alice is the greedy strategy as follows.

Let D be the current dominating set in T i.e. neighbours of all selected vertices.

1. Pick any vertex, v , not in D with a maximal number of neighbours not in D . That is maximise the set $\{x : x \in N(v) \wedge v \notin D\}$.
2. repeat until you have a dominating set.

worst case path graph requires twice the minimum of the path graph???

with no opponent this will give $n/3$ thus at worst with the opponent it will take $2n/3$

At worst Alice will add two vertices to \square

Theorem 2.11. *Given p players then,*

$$\gamma_{gp}(G) \geq p\gamma(G)$$

$$\gamma_{gp}(G) \leq p\gamma_{g2}(G) \leq p \left\lceil \frac{2n}{3} \right\rceil$$

Chapter 3

Colouring

Definition 3.1. Let G be a graph. Then, $\chi_g(G)$ is the smallest number of colours need to guarantee that Alice has a winning strategy on G .

Definition 3.2. We extend the colouring game to have p players. The game chromatic number for p players on some graph G is $\chi_g(G; p)$. Note: $\chi_g(G) = \chi_g(G; 2)$.

Theorem 3.3. Let \mathcal{T} be the family of trees. If we have $p \geq 2$ players then,

$$\chi_g(\mathcal{T}; p) \geq p + 2$$

The following proof is an extended version of the proof of Theorem 5.4 in [3, Bodlaender 1990]

Proof. It suffices to show that there is a tree in where Bob has a winning strategy with $p + 1$ colours.

Consider the graph G as defined in figure 3.1.

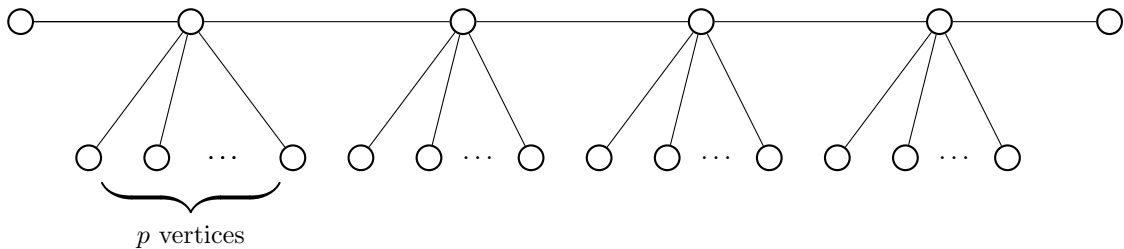


Figure 3.1

We give a strategy for Bob with $p + 1$ colours. Let the colours be $\{c_1, c_2, \dots, c_p, c_{p+1}\}$. On Alice's first move she picks any vertex, v , and colours it. Let the colour of v be c_1 . Bob's first move is to colour any vertex with distance 3 to v . We now have a subgraph in G of the type shown in figure 3.2. We then colour $y_1 \dots y_{p-2}$ with $c_2 \dots c_{p-1}$ respectively.

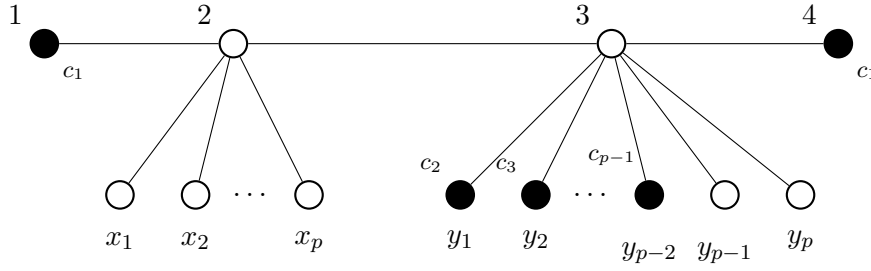


Figure 3.2

We consider three cases.

1. Alice colours 2, x_1, x_2, \dots , or x_p .

Bob colours y_{p-1} with c_p and y_p with c_{p+1} . Vertex 3 now has $p + 1$ different coloured neighbours and thus Bob wins.

2. Alice colours 3.

The colour of 3 cannot be one of $c_1 \dots c_{p-1}$. Therefore 3 is either c_p or c_{p+1} . W.l.o.g let the colour of 3 be c_{p+1} . Bob colours $x_1 \dots x_{p-1}$ with $c_2 \dots c_p$ respectively. Vertex 2 now has $p + 1$ different coloured neighbours and thus Bob wins.

3. Alice colours y_{p-1} or y_p

Bob colours 2 with c_p and y_p (or y_{p-1} if Alice coloured y_p) with c_{p+1} . Vertex 2 now has $p + 1$ different coloured neighbours and thus Bob wins.

Therefore we have a winning strategy on G in \mathcal{T} for Bob with $p + 1$ colours. □

3.1 Activation Strategy

3.1.1 Summery of activation strategy

Consider a graph $G(v, E)$ with a linear ordering L on V . U is the set of unmarked vertices, and A the active ones.

Alice starts by marking the least v in L . If u is the last marked vertex by Bob, Alice starts at u activates it and moves to the least unmarked neighbour of u , say w , in L . If w is active Alice marks w , if not Alice repeats this on w until she finds an active vertex.

Definition 3.4 (Kierstead [4]). Suppose A, B are disjoint subsets of V . We say that a matching M is a matching from A to B if M saturates A and $B \setminus A$ contains a cover of M .

Let L be a linear order, $G = (V, E)$ be a graph, and u a vertex in V .

For $u \in V(G)$ the matching number $m(u, L, G)$ of u with respect to L in G is defined to be the size of the largest set $Z \subset N^-[u]$ such that there exists a partition $[X, Y]$ of Z and there exist matchings M from $X \subset N^-[u]$ to $V^+(u)$ and N from $Y \subset N^-(u)$ to $V^+[u]$.

Then the rank $r(L, G)$ and rank $r(G)$ are defined as:

$$r(u, L, G) = d_{G_L}^+(u) + m(u, L, G)$$

$$r(L, G) = \max_{u \in V} r(u, L, G)$$

$$r(G) = \min_{L \in \Pi(G)} r(L, G)$$

Definition 3.5 (Activation strategy [4]). Let $G = (V, E)$ be a graph and L a linear ordering V . We define the activation strategy $S(L, G)$ as follows:

Let U denote the set of unmarked vertices. Alice maintains a subset $A \subset V$ of active vertices. Initially $A = \emptyset$. We activate a vertex x by adding it to A . On her first turn Alice activates and marks the least vertex in the ordering L . Now suppose that Bob has just marked the vertex b . Alice uses algorithm 1 to update A and choose the next vertex.

Algorithm 1 Activation strategy

```

1:  $x \leftarrow b$ 
2: while  $x \notin A$  do
3:    $A := A \cup \{x\}$ 
4:    $s(x) = \min_L \{u \in N(x) : u < x\} \cap (U \cup \{b\})$ 
5:    $x \leftarrow s(x)$ 
6: if  $x \neq b$  then
7:   choose  $x$ 
8: else
9:    $y \leftarrow \min_L U$ 
10:  if  $y \neq A$  then
11:     $A \leftarrow A \cup \{y\}$ 
12:  choose  $y$ 

```

Theorem 3.6 (Kierstead 1 [4]). *For any graph $G = (V, E)$ and linear ordering L on V , if Alice uses the activation strategy $S(L, G)$ to play the ordering game on G , then the score will be at most $1 + r(L, G)$. In particular, $\text{col}_g(G) \leq 1 + r(G)$.*

A proof of $\chi_g(G) \leq 18$ is now a matter of finding an ordering L such that by theorem 3.6 $r(G) \leq 17$. See [4].

Recall the definition of pathwidth.

Note that $\chi_g(P) \geq w$

Algorithm 2 Linear order in graph of bounded path width

Require: $\mathcal{X} = \{X_1, X_2, \dots, X_n\}$ is the path decomposition of a graph G

Require: L is a set

Ensure: L is a linear order on $V(G)$

```

1:  $Q \leftarrow \emptyset$ 
2:  $L \leftarrow \emptyset$ 
3: for all  $X_i \in \mathcal{X}$  do
4:    $Q' \leftarrow Q \cup (X_i \setminus Q)$ 
5:    $Q \leftarrow \emptyset$ 
6:   for all  $x \in Q'$  do
7:     if  $x \in \{X_{i+1} \cup \dots \cup X_n\}$  then
8:        $Q \leftarrow Q \cup \{x\}$ 
9:     else
10:       $L \leftarrow L \cup \{x\}$ 

```

$\triangleright Q$ is a FIFO queue
 \triangleright our linear order for $V(G)$

Lemma 3.7. *Let $P = (E, V)$ be a graph of bounded pathwidth w and $\{X_1, X_2, \dots, X_n\}$ be a path decomposition of P of width w . Suppose L is a linear ordering on V generated by algorithm 2 and u is a vertex in P . Then there is some X_i such that $u \in X_i$ and $N^-(u) \subseteq X_i$.*

Proof. Let X_i be the segment in which v is added to L , that is i is the number of times line 3 has been looped when v is added to L . The algorithm only adds a vertex v to L iff v doesn't appear in any segment X_j such that $j \geq i$. Thus i is the greatest i such that $v \in X_i$. Therefore by definition of path decomposition all the neighbours not in the L are in X_i . Further the neighbours not in L when v is added to L are greater than v in L . That is $N^-(v) \in X_i$. \square

Theorem 3.8. *Let P be a graph of bounded pathwidth w . Then,*

$$\chi_g \leq 2w + 2$$

Proof. Suppose $P = (E, V)$ is a graph of bounded pathwidth w and let $\{X_1, X_2, \dots, X_n\}$ be the path decomposition of P of width w .

As $\chi_g(P) \leq \text{col}_g(P)$ it suffices to show $\text{col}_g(P) \leq 2w + 3$. We do this by invoking theorem 3.6.

Let L be a linear ordering on V generated by algorithm 2. Let v be any vertex in V .

By the definition of path width $d^+(v) \leq w$ and for all $i \in [1, n]$, $|X_i| \leq w + 1$.

Let $Z \subset N^-[v]$.

By lemma 3.7 $N^-(v) \in A$ for some $A \in \mathcal{X}$. Therefore $|Z| \leq |N^-(v)| \leq |A| \leq w + 1$.

Therefore by definition 3.4 $m(u, L, P) \leq w + 1$ and $r(P) \leq r(L, P) \leq r(u, L, P) \leq 2w + 2$.

Finally by theorem 3.6 $\text{col}_g(P) \leq 2w + 3$

\square

3.1.2 Upper bound using activation strategy for p players

The activation strategy finds an upper bound of col_g (The marking game) and as $\chi_g(G) \leq \text{col}_g$ then we have an upper bound for the χ_g .

We know from [7] at any stage, for some unmarked u $\text{col}_g \leq 2 + \max |A \cup N(u)|$, where A is the set of active vertices.

Claim: for p players $\text{col}_g \leq p + \max |A \cap N(u)|$.

3.1.3 Refined Activation Strategy

The refined activation strategy applies the same basic strategy as above but with two differences.

First, we use a digraph L on G as our ordering. This is only a rough ordering. We partition V into blocks B_1, B_2, \dots, B_i , where if $x \in B_i$, $y \in B_j$ and $i < j$ then the edge xy is in L . The ordering may not be an ordering in each block. However if we ignore what happens in each block we get a linear ordering.

Second when we activate a vertex v if the edge uv fulfils some conditions then reverse the direction of the edge uv in L .

3.1.4 Online colouring

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