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**Graph Algorithms with Hostile  
Partners**

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**Abstract**

A short description of the project goes here.



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# **Chapter 1**

## **Introduction**





## Chapter 2

# Dominating sets

We begin by listing some definitions.

**Definition.** The Dominating set,  $D$ , of a graph  $G = (E, V)$  is any subset of  $V$  such that every vertex in  $V$  is adjacent to at least one vertex in  $D$ .

**Definition.** The Dominating number,  $\gamma(G)$ , of a graph  $G = (E, V)$  is the size of the smallest dominating set of  $G$ .

**Definition.** Independent set, maximum independent set, independence number  $\alpha(G)$

### 2.1 min size dominating set

**Lemma 2.1.** Let  $G$  be a graph.

$$\gamma(G) \geq \alpha(G)$$

*Proof.* Let  $X$  be a minimum dominating set in some graph  $G = (V, E)$ . By definition of dominating set vertex in  $V$  is adjacent to at least one vertex in  $X$ .  $\square$

Recall that  $\chi(G)$  is the chromatic number of the graph  $G$ .

**Theorem 2.2** (Willis 2011 3.1). For any graph  $G = (V, E)$  [4]

$$\alpha(G) \leq \frac{|V|}{\chi(G)}$$

Recall that  $\Delta(G)$  is the maximum degree of any vertex in  $G$ .

**Theorem 2.3** (Balakrishnan 2012 10.3.2). [2] For any graph  $G$  with  $n$  vertices,

$$\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$$

It is obvious that in the case when  $\gamma(G)$  is known that  $\gamma(G) > \gamma_g(G)$ .

**Theorem 2.4.** (Ore 1962) [3] For any graph  $G$  with  $n$  vertices,

$$\gamma(G) \leq \frac{n}{2}$$

**Theorem 2.5.** Let  $G$  be a graph. If  $x$  is a tight upper bound for the domination number,  $\gamma(G)$ , then

$$\gamma_g(G) \geq x$$

*Proof.* Let  $G$  be a graph where  $\gamma(G) = x$ . Thus for  $G$  we are unable to find a dominating set with  $< x$  vertices. Therefore there cannot be a winning strategy for Alice with  $< x$  vertices. Therefore  $\gamma_g(G) \geq x$   $\square$

**Theorem 2.6.** *Let  $G$  be a graph with  $n$  vertices, such that  $n \geq 4$ . Then,*

$$\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

*Proof.* By combination of theorems 2.4 and 2.5 we get  $\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$   $\square$

Theorem 2.6 is also proved in Alona, Balogh, Bollobas, and Szabo 2002 [1].  
The trivial upper bound is  $n$ .

**Theorem 2.7.** *Let  $G$  be a graph with  $n$  vertices. Then,*

$$\gamma_g(G) \leq \left\lceil \frac{2n}{3} \right\rceil$$

*Proof.* A dominating set on a spanning tree is a dominating set in the parent graph. Thus for any graph,  $G$ , it suffices to show we have a winning strategy for a spanning tree of  $G$ . Let  $T$  be a spanning tree of  $G$ . The winning strategy for Alice is the greedy strategy as follows.

Let  $D$  be the current dominating set in  $T$  i.e. neighbours of all selected vertices.

1. Pick any vertex,  $v$ , not in  $D$  with a maximal number of neighbours not in  $D$ . That is maximise the set  $\{x : x \in N(v) \wedge v \notin D\}$ .
2. repeat until you have a dominating set.

worst case path graph requires twice the minimum of the path graph???

with no opponent this will give  $n/3$  thus at worst with the opponent it will take  $2n/3$

At worst Alice will add two vertices to  $\square$

**Theorem 2.8.** *Given  $p$  players then,*

$$\gamma_{gp}(G) \geq p\gamma(G)$$

$$\gamma_{gp}(G) \leq p\gamma_{g2}(G) \leq p \left\lceil \frac{2n}{3} \right\rceil$$

## **Chapter 3**

# **Colouring**

### **3.1 title**



# Bibliography

- [1] ALON, N., BALOGH, J., BOLLOBÁS, B., AND SZABÓ, T. Game domination number. *Discrete Mathematics* 256, 1 (2002), 23 – 33.
- [2] BALAKRISHNAN, R. *A Textbook of Graph Theory (Universitext)*. Springer, sep 2012.
- [3] ORE, O. *Theory of Graphs (COLLOQUIUM PUBLICATIONS (AMER MATHEMATICAL SOC))*. American Mathematical Society, dec 1962.
- [4] WILLIS, W. Bounds for the independence number of a graph. Virginia Commonwealth University.