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Graph Algorithms with Hostile Partners

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Abstract

A short description of the project goes here.

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Dominating sets

We begin by listing some definitions.

Definition. The Dominating set, D, of a graph $G = (E, V)$ is any subset of V such that every vertex in V is adjacent to at least one vertex in D.

Definition. The Dominating number, $\gamma(G)$, of a graph $G = (E, V)$ is the size of the smallest dominating set of G.

Definition. Independent set, maximum independent set, independence number $\alpha(G)$

2.1 min size dominating set

Lemma 2.1. *Let G be a graph.*

$$\gamma(G) \geq \alpha(G)$$

Proof. Let X be a minimum dominating set in some graph $G = (V, E)$. By definition of dominating set vertex in V is adjacent to at least one vertex in $\square$

Recall that $\chi(G)$ is the chromatic number of the graph G.

Theorem 2.2 (Willis 2011 3.1)**.** *For any graph $G = (V, E)$ [6]*

$$\alpha(G) \leq \frac{|V|}{\chi(G)}$$

Recall that $\Delta(G)$ is the maximum degree of any vertex in G.

Theorem 2.3 (Balakrishnan 2012 10.3.2)**.** *[2] For any graph G with n vertices,*

$$\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$$

It is obvious that in the case when $\gamma(G)$ is known that $\gamma(G) > \gamma_g(G)$.

Theorem 2.4. *(Ore 1962) [5] For any graph G with n vertices,*

$$\gamma(G) \leq \frac{n}{2}$$

Theorem 2.5. *Let G be a graph. If x is a tight upper bound for the domination number, $\gamma(G)$, then*

$$\gamma_g(G) \geq x$$

Proof. Let G be a graph where $\gamma(G) = x$. Thus for G we are unable to find a dominating set with $< x$ vertices. Therefore there cannot be a winning strategy for Alice with $< x$ vertices. Therefore $\gamma_g(G) \geq x$ □

Theorem 2.6. *Let G be a graph with n vertices, such that $n \geq 4$. Then,*

$$\gamma_g(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

Proof. By combination of theorems 2.4 and 2.5 we get $\gamma_g(G) \geq \lfloor \frac{n}{2} \rfloor$ □

Thereom 2.6 is also proved in Alona, Baloghc, Bollobas, and Szabo 2002 [1].
The trivial upper bound is n.

Theorem 2.7. *Let G be a graph with n vertices. Then,*

$$\gamma_g(G) \leq \left\lceil \frac{2n}{3} \right\rceil$$

Proof. A dominating set on a spanning tree in a dominating set in the parent graph. Thus for any graph, G, it suffices to show we have a winning strategy for a spanning tree of G. let T be a spanning tree of G. The winning strategy for Alice is the greedy strategy as follows.

Let D be the current dominating set in T i.e. neighbours of all selected vertices.

1. Pick any vertex, v, not in D with a maximal number of neighbours not in D. That is maximise the set $\{x : x \in N(v) \bigwedge v \notin D\}$.
2. repeat until you have a dominating set.

worst case path graph requires twice the minimum of the path graph???
with no opponent this will give n/3 thus at worst with the opponent it will take 2n/3
At worst Alice will add two vertices to □

Theorem 2.8. *Given p players then,*

$$\gamma_{gp}(G) \geq p\gamma(G)$$

$$\gamma_{gp}(G) \leq p\gamma_{g2}(G) \leq p \left\lceil \frac{2n}{3} \right\rceil$$

Chapter 3

Colouring

Definition. We extend the colouring game to have p players. The game choromatic number for p players and some graph G is $\chi_g(G;p)$. Note: $\chi_g(G) = \chi_g(G;2)$.

Theorem 3.1. *Let T be a tree, if we have $p \geq 2$ players then,*

$$\chi_g(T;p) \geq p+2$$

The following proof is an extended version of the proof of Theorem 5.4 in [3, Bodlaender 1990]

Proof. Consider the graph G as defined in figure 3.1.

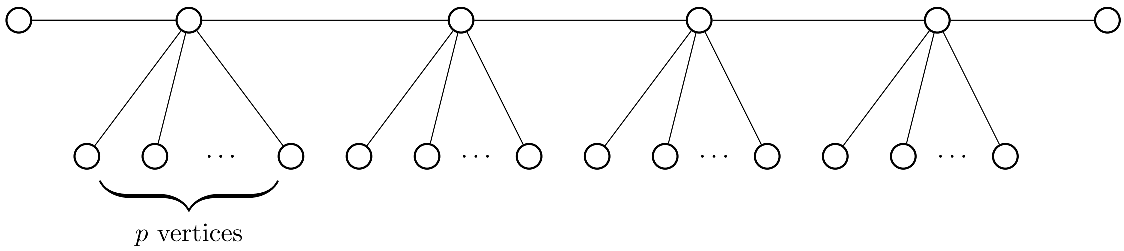

Figure 3.1

We give a strategy for Bob with $p+1$ colours. Let the colours be $\{c_1, c_2, \ldots, c_p, c_{p+1}\}$ On Alice's first move she picks any vertex, v, and colours it. Let the colour of v be c_1. Bobs first move is to colour any vertex with distance 3 to v. We now have a subgraph in G of the type shown in figure 3.2. We then colour $y_1 \ldots y_{p-2}$ with $c_2 \ldots c_{p-1}$ respectively.

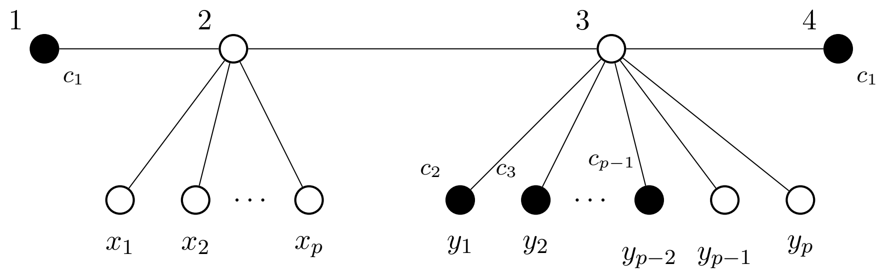

Figure 3.2

We consider three cases.

1. Alice colours 2, x_1, x_2, $\ldots$, or x_p.

   Bob colours y_{p-1} with c_p and y_p with c_{p+1}. Vertex 3 now has $p+1$ different coloured neighbours and thus Bob wins.

2. Alice colours 3.

   The colour of 3 cannot be one of $c_1 \ldots c_{p-1}$. Therefore 3 is either c_p or c_{p+1}. W.l.o.g let the colour of 3 be c_{p+1}. Bob colours $x_1 \ldots x_{p-1}$ with $c_2 \ldots c_p$ respectively. Vertex 2 now has $p+1$ different coloured neighbours and thus Bob wins.

3. Alice colours y_{p-1} or y_p

   Bob colours 2 with c_p and y_p (or y_{p-1} if Alice coloured y_p) with c_{p+1} . Vertex 2 now has $p+1$ different coloured neighbours and thus Bob wins.

□

Theorem 3.2.

$$\chi_g(G;p) \leq \chi_g(G;2) + p - 2$$

Proof. By induction on the number of vertices, n and the number of players, p.

We show for any p $\chi_g(G_{n+1};p) \leq (\chi_g(G_{n+1};2) + p - 2)$

$$\chi_g(G_n;p) \leq (\chi_g(G_n;2) + p - 2) \qquad \text{from induction} \tag{3.1}$$

$$\chi_g(G_n;p) \leq \chi_g(G_{n+1};p) \tag{3.2}$$

$$(\chi_g(G_n;2) + p - 2) \leq (\chi_g(G_{n+1};2) + p - 2) \tag{3.3}$$

Assume, for a contradiction, $\chi_g(G_{n+1};p) > \chi_g(G_{n+1};2) + p - 2$. Then for $p = 2$ $\chi_g(G_{n+1};2) > \chi_g(G_{n+1};2)+2-2$. This is a contradiction, therefore $\chi_g(G_{n+1};p) \leq \chi_g(G_{n+1};2)+p-2$.

Claim: For some n $\chi_g(G_n;p) \implies \chi_g(G_n;p+1)$

By induction hypothesis $\chi_g(G_n;p) \leq \chi_g(G_n;2) + p - 2$

□

Theorem 3.3.

$$\chi_g(G;p) \leq \chi_g(G;p) + 1 \leq \chi_g(G;p+1)$$

$$\chi_g(G;2) + p - 2 \leq \chi_g(G;p+1)$$

L is a linear order, $G = (V,E)$ is a graph, u is a vertex in V, the rank $r(L,G)$ and rank $r(G)$ are defined as:

$$\begin{aligned} r(u,L,G) &= d^+_{G_L}(u) + m(u,L,G) \\ r(L,G) &= \max_{u \in V} r(u,L,G) \\ r(G) &= \min_{L \in \Pi(G)} r(L,G) \end{aligned}$$

Theorem 3.4 (Theorem 1 [4])**.** *For any graph $G = (V,E)$ and ordering $L \in \Pi(G)$, if Alice uses the strategy $S(L,G)$ to play the ordering game on G, then the score will be at most $1 + r(L,G)$. In particular, $col_g(G) \leq 1 + r(G)$.*

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