for a non-degenerate triangle K and $\beta_1, \beta_2, \beta_3 \in \mathbb{N}$

$$\int_K \lambda_1^{\beta_1} \lambda_2^{\beta_2} \lambda_3^{\beta_3} = 2\,|K| \frac{\beta_1!\beta_2!\beta_3!}{(\beta_1+\beta_2+\beta_3+2)!}$$

As the first step we transform the integral to the refrence triangle $\widehat{K} := \text{convex}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$\int_K \lambda_1^{\beta_1} \lambda_2^{\beta_2} \lambda_3^{\beta_3} = 2\,|K| \int_0^1 \int_0^{1-\xi_2} \xi_1^{\beta_1} \xi_2^{\beta_2} (1-\xi_1-\xi_2)^{\beta_3} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2$$

the barycentric coordinates evaluated on $\widehat{K}$ are the barycentric coordinates on the reference cell, which are $\xi_1, \xi_2, 1-\xi_1-\xi_2$. Using the duffy transformation

$$\int_K f(\xi_1, \xi_2) \mathrm{d}\mathbf{x} = \int_Q f(\widehat{\xi_1}(1-\widehat{\xi_2}), \widehat{\xi_2})(1-\widehat{\xi_2}) \mathrm{d}\widehat{\xi} \tag{1}$$

we transform $\widehat{K}$ to $Q := (0,1)^2$,

$$\begin{aligned}
2\,|K| \int_0^1 \int_0^{1-\xi_2} \xi_1^{\beta_1} \xi_2^{\beta_2} (1-\xi_1-\xi_2)^{\beta_3} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 &= 2\,|K| \int_0^1 \int_0^1 \left(\widehat{\xi_1} \left(1-\widehat{\xi_2}\right)\right)^{\beta_1} \widehat{\xi_2}^{\beta_2} \left(1-\widehat{\xi_1}\left(1-\widehat{\xi_2}\right)-\widehat{\xi_2}\right)^{\beta_3} \left(1-\widehat{\xi_2}\right) \mathrm{d}\widehat{\xi_1} \mathrm{d}\widehat{\xi_2} \\
&= 2\,|K| \int_0^1 \int_0^1 \widehat{\xi_1}^{\beta_1} \left(1-\widehat{\xi_2}\right)^{\beta_1} \widehat{\xi_2}^{\beta_2} \left(1-\widehat{\xi_1}\right)^{\beta_3} \left(1-\widehat{\xi_2}\right)^{\beta_3} \left(1-\widehat{\xi_2}\right) \mathrm{d}\widehat{\xi_1} \mathrm{d}\widehat{\xi_2} \\
&= 2\,|K| \int_0^1 \int_0^1 \widehat{\xi_1}^{\beta_1} \left(1-\widehat{\xi_1}\right)^{\beta_3} \widehat{\xi_2}^{\beta_2} \left(1-\widehat{\xi_2}\right)^{\beta_1+\beta_3+1} \mathrm{d}\widehat{\xi_1} \mathrm{d}\widehat{\xi_2} \\
&= 2\,|K| \int_0^1 \widehat{\xi_1}^{\beta_1} \left(1-\widehat{\xi_1}\right)^{\beta_3} \mathrm{d}\widehat{\xi_1} \int_0^1 \widehat{\xi_2}^{\beta_2} \left(1-\widehat{\xi_2}\right)^{\beta_1+\beta_3+1} \mathrm{d}\widehat{\xi_2}
\end{aligned}$$

with the definition of euler's beta and gamma functions

$$B(\alpha, \beta) := \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \, \mathrm{d}t, \qquad 0 < \alpha, \beta < \infty \tag{2}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{3}$$

$$\Gamma(n) = (n-1)! \tag{4}$$

we get

$$\begin{aligned}
2\,|K| \int_0^1 \widehat{\xi_1}^{\beta_1} \left(1-\widehat{\xi_1}\right)^{\beta_3} \mathrm{d}\widehat{\xi_1} \int_0^1 \widehat{\xi_2}^{\beta_2} \left(1-\widehat{\xi_2}\right)^{\beta_1+\beta_3+1} \mathrm{d}\widehat{\xi_2} &= B(\beta_1+1, \beta_3+1) B(\beta_2+1, \beta_1+\beta_3+2) \\
&= \frac{\Gamma(\beta_1+1)\Gamma(\beta_3+1)}{\Gamma(\beta_1+\beta_3+2)} \frac{\Gamma(\beta_2+1)\Gamma(\beta_1+\beta_3+2)}{\Gamma(\beta_1+\beta_2+\beta_3+3)} \\
&= 2\,|K| \frac{\beta_1!\beta_2!\beta_3!}{(\beta_1+\beta_2+\beta_3+2)!}
\end{aligned}$$