SERIES - 9

1. To prove : for any non-generate triangle K and $\beta_1, \beta_2, \beta_3 \in \mathbb{N}_0$,

$$\int_K \lambda_1^{\beta_1} \lambda_2^{\beta_2} \lambda_3^{\beta_3} \, dx = 2|K| \cdot \frac{\beta_1! \, \beta_2! \, \beta_3!}{(\beta_1+\beta_2+\beta_3)!}$$

Sol^n : First, transforming K to the reference triangle $\hat{K}$.

$\hat{K} := \text{convex}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ which leads to.

$$\int_K \lambda_1^{\beta_1} \lambda_2^{\beta_2} \lambda_3^{\beta_3} \, dx = 2|K| \cdot \int_0^1 \int_0^{1-\xi_2} \xi_1^{\beta_1} \xi_2^{\beta_2} (1-\xi_1-\xi_2)^{\beta_3} \, d\xi_1 \, d\xi_2$$

$$= 2|K| \cdot I_1$$

Barycentric coordinates evaluated on $\hat{K}$ are $\xi_1, \xi_2, 1-\xi_1-\xi_2$.

Transforming $\hat{K}$ to $(0,1)^2$ by the Duffy Transformation,

$\xi_1 = \hat{\xi}_1(1-\hat{\xi}_2)$, $\xi_2 = \hat{\xi}_2$, we have,

$$I_1 = \int_0^1 \int_0^1 \hat{\xi}_1^{\beta_1} (1-\hat{\xi}_2)^{\beta_1} \hat{\xi}_2^{\beta_2} \left(1-\hat{\xi}_2-\hat{\xi}_1(1-\hat{\xi}_2)\right)^{\beta_3} (1-\hat{\xi}_2) \, d\hat{\xi}_1 \, d\hat{\xi}_2$$

$$= \int_0^1 \hat{\xi}_1^{\beta_1} (1-\hat{\xi}_1)^{\beta_3} \, d\hat{\xi}_1 \int_0^1 \hat{\xi}_2^{\beta_2} (1-\hat{\xi}_2)^{\beta_1+\beta_3+1} \, d\hat{\xi}_2$$

$$= B(\beta_1+1, \beta_3+1) \; B(\beta_2+1, \beta_1+\beta_3+2)$$

(Using, $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \, dt, \quad 0<\alpha, \beta<\infty$.

$$I_1 = \frac{\Gamma(\beta_1+1)\,\Gamma(\beta_3+1)}{\Gamma(\beta_1+\cancel{\beta_3}+2)} \; \frac{\Gamma(\beta_2+1)\,\Gamma(\cancel{\beta_1+\beta_3+2})}{\Gamma(\beta_1+\beta_2+\beta_3+3)}$$

$$= \frac{\Gamma(\beta_1+1)\,\Gamma(\beta_2+1)\,\Gamma(\beta_3+1)}{\Gamma(\beta_1+\beta_2+\beta_3+3)} = \frac{\beta_1! \, \beta_2! \, \beta_3!}{(\beta_1+\beta_2+\beta_3+2)!}$$