Series 9

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Exercise 1

Let a, b, c be the corners of K. Then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 a + \lambda_2 b + \lambda_3 c = \underbrace{(a-c,\ b-c)}_{=:D} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\det D = 2 \cdot |K|$$

$$\Rightarrow \int_K \lambda_1^{\beta_1} \lambda_2^{\beta_2} \lambda_3^{\beta_3} \, dx = \int_0^1 \int_0^1 \xi_1^{\beta_1} \xi_2^{\beta_2} (1-\xi_1-\xi_2)^{\beta_3} \cdot 2 \cdot |K| \, d\xi_1 \, d\xi_2$$

substitute $u = \xi_1$ $\quad \xi_2 = v(1-u)$ $\Rightarrow d\xi_1 d\xi_2 = 1-u \cdot du\,dv$

$$= \int_0^1 \int_0^1 u^{\beta_1} \cdot v^{\beta_2} (1-u)^{\beta_2} \underbrace{(1-u-v(1-u))}^{\beta_3} \cdot (1-u) 2|K| \, du\,dv$$

$$= 2|K| \int_0^1 \int_0^1 u^{\beta_1} (1-u)^{\beta_2+1} (1-u)^{\beta_3} (1-v)^{\beta_3} \cdot v^{\beta_2} \, du\,dv$$

$$= 2|K| \int_0^1 u^{\beta_1} (1-u)^{\beta_2+\beta_3+1} du \int_0^1 v^{\beta_2} (1-v)^{\beta_3} dv$$

$$= 2|K| \cdot B(\beta_1+1, \beta_2+\beta_3+2) \, B(\beta_2+1, \beta_3+1)$$

$$= 2|K| \cdot \frac{\Gamma(\beta_1+1)\,\Gamma(\beta_2+\beta_3+2)}{\Gamma(\beta_1+\beta_2+\beta_3+3)} \cdot \frac{\Gamma(\beta_2+1)\,\Gamma(\beta_3+1)}{\Gamma(\beta_2+\beta_3+2)}$$

$$= 2|K| \frac{\beta_1!\,\beta_2!\,\beta_3!}{(\beta_1+\beta_2+\beta_3+3)!} \qquad \square$$