① SERIES 9 WYCZESANY, STENZEL

We will first manipulate with RHS :

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

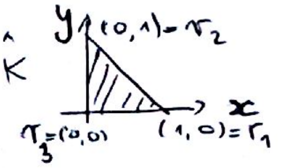
$$\frac{\beta_1!\cdot\beta_2!\cdot\beta_3!}{(\beta_1+\beta_2+\beta_3+2)!}=\frac{\Gamma(\beta_1+1)\cdot\Gamma(\beta_2+1)\cdot\Gamma(\beta_3+1)}{\Gamma(\beta_1+\beta_2+\beta_3+3)}=$$

$$=B(\beta_1+1,\beta_2+1)\cdot\frac{\Gamma(\beta_1+\beta_2+2)\cdot\Gamma(\beta_3+1)}{\Gamma(\beta_1+\beta_2+\beta_3+3)}=$$

$$=B(\beta_1+1,\beta_2+1)\cdot B(\beta_1+\beta_2+2,\beta_3+1)=$$

$$=\int_0^1 t^{\beta_1}(1-t)^{\beta_2}\,dt\cdot\int_0^1 t^{\beta_1+\beta_2+1}(1-t)^{\beta_3}\,dt$$

Now, we consider the case of the reference triangle



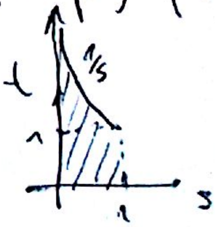
$$x=\lambda_1 r_1+\lambda_2 r_2+\lambda_3 r_3=(\lambda_1,\lambda_2)$$
$$\lambda_3=1-\lambda_1-\lambda_2$$

$$\int_{\hat{K}}\lambda_1^{\beta_1}\lambda_2^{\beta_2}\lambda_3^{\beta_3}\,dx=\int_0^1\left(\int_0^{-x+1}x^{\beta_1}\cdot y^{\beta_2}\cdot(1-x-y)^{\beta_3}\,dy\right)dx=$$

$$=\left\{\begin{cases}p=x+y\\ q=x\end{cases}\Rightarrow\begin{cases}x=q\\ y=p-q\end{cases}\quad D\varphi=\begin{pmatrix}0&1\\1&-1\end{pmatrix}\quad|\det D\varphi|=1\right\}$$

$$=\int_0^1\left(\int_0^p q^{\beta_1}\cdot(p-q)^{\beta_2}(1-p)^{\beta_3}\,dq\right)dp=\int_0^1\left(\int_0^p q^{\beta_1}\cdot p^{\beta_2}\left(1-\frac{q}{p}\right)^{\beta_2}(1-p)^{\beta_3}\,dq\right)dp$$

$$=\left\{\begin{cases}s=p\\ t=\frac{q}{p}\end{cases}\Rightarrow\begin{cases}p=s\\ q=s\cdot t\end{cases}\quad D\varphi=\begin{pmatrix}1&0\\t&s\end{pmatrix}\quad|\det D\varphi|=s\right\}=$$



$$=\int_0^1\int_0^{\frac{1}{s}}s^{\beta_1}\cdot t^{\beta_1}\cdot s^{\beta_2}\cdot(1-t)^{\beta_2}\cdot(1-s)^{\beta_3}\cdot s\,dt\,ds=$$

$$=\int_0^1 s^{\beta_1+\beta_2+1}(1-s)^{\beta_3}\left(\int_0^{1/s}t^{\beta_1}\cdot(1-t)^{\beta_2}\,dt\right)ds$$

And now it would have been the same if the integration over t had different borders of integration : from 0 to 1 , or since

$$\int_0^{1/s} \dots dt = \int_0^1 \dots dt \left(+ \int_1^{1/s} dt \right) = 0.$$

I am not sure where I have made a mistake.

* the factor $2|K|$ comes from a transfor-mation $T: K \longrightarrow \hat{K}$, and is a not too hard calculation

(but I just overslept :( )