Deterministic Bayesian inference methods for the Naomi model

HIV Inference Lab Group Meeting

Adam Howes

Imperial College London

April 2023

Bayes

- As a statistical modeller, the bulk of our job is in constructing a generative model for data y using parameters ϑ
- This is the joint distribution $p(y, \vartheta) = p(y \mid \vartheta) p(\vartheta)$
- Then we want to compute the posterior $p(\vartheta \mid y)$ which is **just**[1]

$$p(\vartheta \mid y) = \frac{p(y, \vartheta)}{p(y)} = \frac{p(y \mid \vartheta) p(\vartheta)}{p(y)}$$

- The central problem of Bayesian inference is that we can't compute $p(y)$

$$p(y) = \int p(y, \vartheta) \mathrm{d}\vartheta$$

[1] I've bolded this with sarcasm in mind

How might you do it?

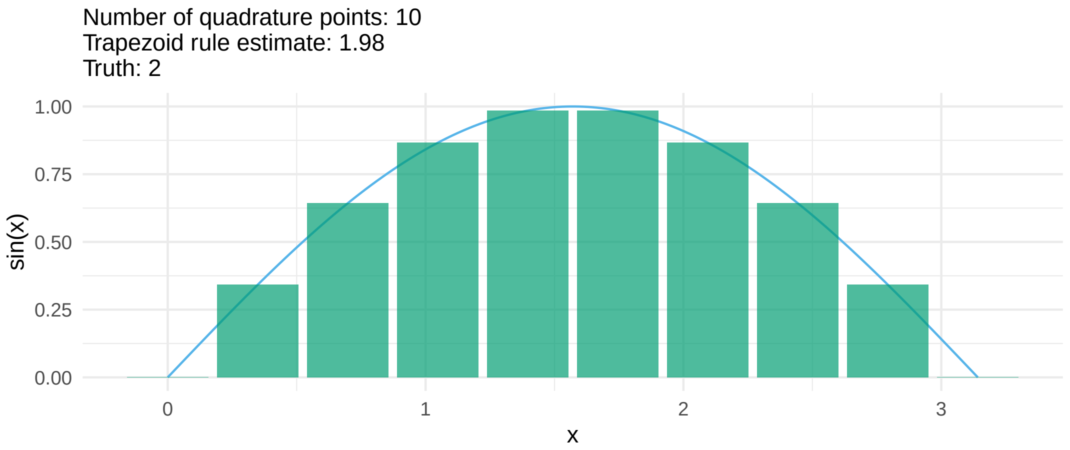
- If you want to integrate something deterministically, you could use numerical integration methods
- Pick nodes $\vartheta \in \mathcal{Q} \subset \Theta$ and weights $\omega : \Theta \to \mathbb{R}$ then compute the sum

$$\tilde{p}(y) = \sum_{\vartheta \in \mathcal{Q}} p(y, \vartheta)\omega(\vartheta)$$

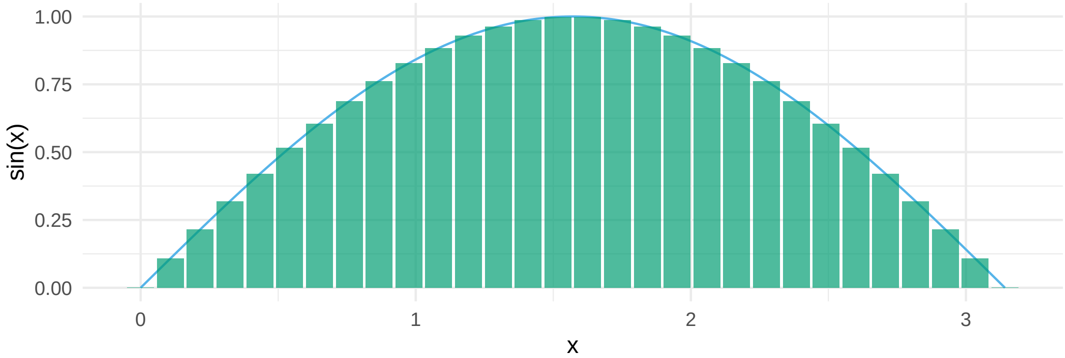
Naive quadrature example

- Remember how integration is introduced? (Rienmann sums)
- Try computing $\int_0^\pi \sin(x)\mathrm{d}x = 2$ using trapezoid rule

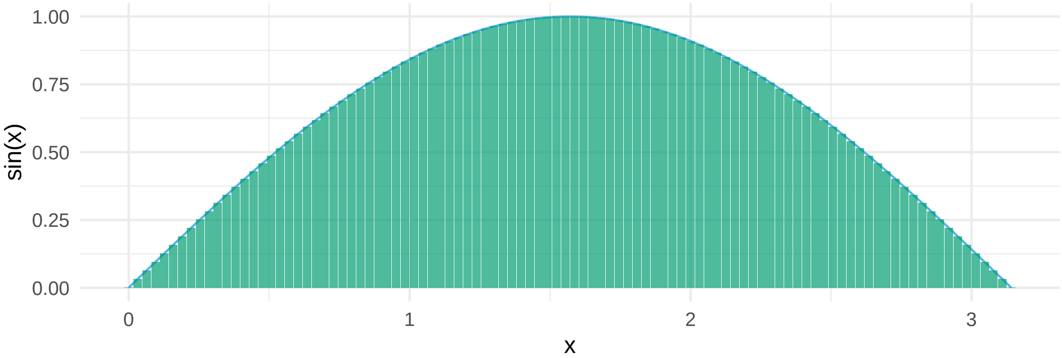
```r
trapezoid_rule <- function(x, spacing) {
  w <- rep(spacing, length(x))
  w[1] <- w[1] / 2
  w[length(x)] <- w[length(x)] / 2
  sum(w * x)
}
```


Number of quadrature points: 10
Trapezoid rule estimate: 1.98
Truth: 2
1.00
0.75
0.50
0.25
0.00
sin(x)
0
1
2
3
x

Number of quadrature points: 30
Trapezoid rule estimate: 1.998
Truth: 2



Number of quadrature points: 100
Trapezoid rule estimate: 2
Truth: 2



Monte Carlo as an example of numerical integration

- Suppose we can sample $\vartheta_i \sim p(y, \vartheta)$ for $i = 1, \ldots, N$
- If we set $\omega(\vartheta_i) = 1/N$ for all i then we get a **Monte Carlo** (MC) estimate

$$\tilde{p}(y) = \frac{1}{N} \sum_i p(y, \vartheta_i)$$

- For complicated models[2] it's not possible to sample directly from $p(y, \vartheta)$, but we can usually sample from a Markov chain which if you squint a bit is good enough (MCMC)

[2] Or not even that complicated

Monte Carlo is fundamentally unsound

- "Monte Carlo ignores information" according to O'Hagan (1987)
- Suppose $N = 3$ and we sample $\vartheta_1, \vartheta_2, \vartheta_3$ with $\vartheta_2 = \vartheta_3$ then our MC estimate is
$$\tilde{p}(y) = \frac{1}{3}\left(p(y, \vartheta_1) + p(y, \vartheta_2) + p(y, \vartheta_3)\right)$$
- This is despite the fact that nothing new about the function has been learned by adding $\{\vartheta_3, p(y, \vartheta_3)\}$

Application to HIV survey sampling

- This is a digression but...
- Say we're running a household survey, and sample the same individual twice
- We didn't learn anything new about HIV by surveying them again!
- This doesn't just bite for nodes or individuals which are exactly the same: an analogous argument can be made if they are close together and we expect their function evaluations to be similar

$\implies$ Bayesian quadrature, Bayesian survey design
For some half-baked thoughts, see `athowes.github.io/fourth-gen/paper.pdf`

Latent variables and hyperparameters

- Quadrature doesn't work very well when $\dim(\vartheta)$ gets even moderately sized
- Previously I had all of the parameters under the symbol ϑ
- What if we split them up as being $\vartheta = (x, \theta)$
- The key part about this is that $\dim(x)$ is big and $\dim(\theta)$ is small

Names for x	Names for θ
Latent variables, random effects, latent field	Hyperparameters, fixed effects

Spatio-temporal statistics

- There is nothing inherently special about spatio-temporal statistics, but we do often end up in similar situations because of the structures of the problems we tackle
- We have observations indexed by space $s \in \mathcal{S}$ and time $t \in \mathcal{T}$
- Usually we associate parameters to spatio-temporal locations as well as observations
- This ends up with us having something like $\{x_{s,t}\}$

What's important about this?

1. There might be **a lot** of spatio-temporal locations, so $\dim(x)$ might be pretty big! If you have 100 districts and 10 years, that's already $100 \times 10 = 1000$ parameters
2. Perhaps we're willing to make assumptions about how things vary over space-time[3]

[3] Are there any slides about spatial statistics that don't describe Tobler's first law of geography?

Latent Gaussian models

A latent Gaussian model (LGM) (Rue, Martino, and Chopin 2009) looks along these lines:

$$
\begin{aligned}
&\text{(Observations)} && y \sim p(y \mid x, \theta), \\
&\text{(Latent field)} && x \sim \mathcal{N}(x \mid \mu(\theta), Q(\theta)^{-1}), \\
&\text{(Hyperparameters)} && \theta \sim p(\theta).
\end{aligned}
$$

Laplace approximation

- Remember that we wanted to compute

$$p(y) = \int p(y, \vartheta) \mathrm{d}\vartheta$$

- One trick for doing this is to pretend $p(\vartheta \mid y)$ is Gaussian

> - Mode $\hat{\vartheta} = \arg\max_{\vartheta} \log p(y, \vartheta)$
> - Hessian $H(\hat{\vartheta}) = -\partial_{\vartheta}^2 \log p(y, \vartheta)|_{\vartheta = \hat{\vartheta}}$
> - Gaussian approximation $\implies \tilde{p}_{\mathrm{G}}(\vartheta \mid y) = \mathcal{N}(\vartheta \mid \hat{\vartheta}, H(\hat{\vartheta})^{-1})$

Example

- Consider the following model for $i = 1, \ldots, n$ with fixed a and b

$$y_i \sim \text{Poisson}(\lambda), \quad \lambda \sim \text{Gamma}(a, b).$$

- It's conjugate so we directly know that

$$\lambda \mid y \sim \text{Gamma}(a + \sum_i y_i, b + n)$$

Laplace approximation

- Now
$$p(y) = \frac{p(\vartheta, y)}{p(\vartheta \mid y)} \approx \frac{p(\vartheta, y)}{p_{\mathrm{G}}(\vartheta \mid y)}$$
and we can evaluate RHS where we would like, so let's pick the point at which the Gaussian is most accurate, which is $\hat{\vartheta}$
$$p_{\mathrm{LA}}(y) = \frac{p(\vartheta, y)}{p_{\mathrm{G}}(\vartheta \mid y)}\Big|_{\vartheta=\hat{\vartheta}} = (2\pi)^{\dim(\vartheta)/2} \det(H(\hat{\vartheta}))^{-1/2} p(\hat{\vartheta}, y)$$

Marginal Laplace approximation

- Hey wait a second, is it reasonable to just assume $p(\vartheta \mid y)$ is Gaussian?
- No, not in general. But...

1. We just described a class of models (LGMs) where some subset of the parameters (the latent field x) have a Gaussian prior $\implies$ it's a lot more reasonable to think that they would have a marginal posterior which is close to Gaussian
2. We just talked about how big x is in comparison to θ! $\implies$ most of the work in our integral can be done using a **marginal Laplace** approximation to get rid of x

Dichotomy in statistical inference methods:

1. Ones which aim to be completely general
2. Ones which aim to "exploit" properties of the problem at hand

We are taking approach 2.

Marginal Laplace approximation

- What does this look like? Instead of assuming $p(\vartheta \mid y) = p(x, \theta \mid y)$ is Gaussian we assume $p(x \mid \theta, y)$ is

$$\tilde{p}_{\mathrm{G}}(x \mid \theta, y) = \mathcal{N}(x \mid \hat{x}, H(\hat{x}))^{-1}$$

where $\hat{x} = \hat{x}(\theta)$

- Now the marginal Laplace approximation is

$$p_{\mathrm{LA}}(\theta, y) = \frac{p(x, \theta, y)}{\tilde{p}_{\mathrm{G}}(x \mid \theta, y)}|_{x=\hat{x}} = (2\pi)^{\mathrm{dim}(x)/2} \det(H(\hat{x}))^{-1/2} p(\hat{x}, \theta, y)$$

Integrated nested Laplace approximation

- Now we can compute $p_{\text{LA}}(\theta, y)$ but what we really want is still $p(y)$
- But hopefully[4] the dimension of θ is small enough that we can now tackle this with quadrature
- So pick some nodes $\mathcal{Q}$ and a weighting function ω and away we go

$$p(y) \approx \sum_{\theta \in \mathcal{Q}} p_{\text{LA}}(\theta, y)\omega(\theta)$$

- This is the famous integrated nested Laplace approximation (INLA)

[4] Really: hopefully

Naomi evidence synthesis model

- Many people here have worked on Naomi evidence synthesis model
- Used by countries to produce HIV estimates in a yearly process supported by UNAIDS
- Inference for Naomi is currently conducted using $\texttt{TMB}$ by optimising $p_{\texttt{LA}}(\theta, y)$
- Has to be pretty quick to allow for interactive review and development of estimates

Figure 1: A supermodel

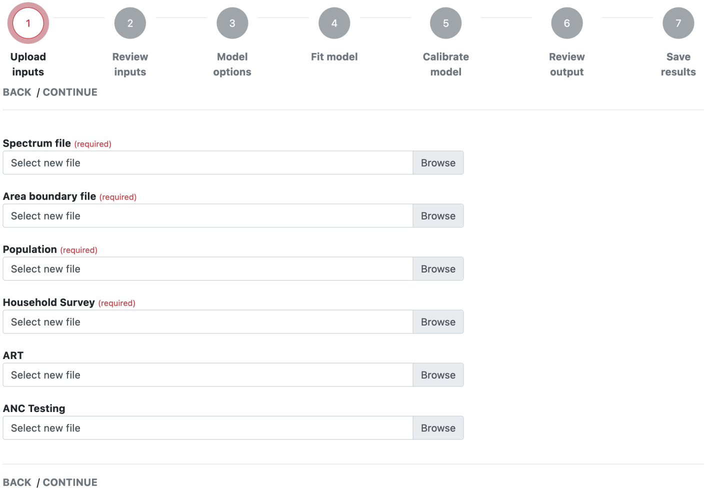


Figure 2: Example of the user interface from https://naomi.unaids.org/

Why do we use TMB

- One answer: lack of other viable options
- Markov chain Monte Carlo (MCMC) takes too long
- Though Naomi is a spatial model with a large Gaussian latent field, it isn't technically a latent Gaussian model, and isn't compatible with the R-INLA implementation of INLA

Idea

- Implement an algorithm similar to INLA which

1. is compatible with Naomi
2. uses `TMB` to perform Laplace approximation

- Does this improves the quality of the inferences we get?

Not a new idea!

> *My main comment is that several aspects of the computational machineery that is presented by Rue and his colleagues* ***could benefit from the use of a numerical technique known as automatic differentiation (AD)*** *. . . By the use of AD one could obtain a system that is automatic from a user's perspective. . . the benefit would be a fast, flexible and easy-to-use system for doing Bayesian analysis in models with Gaussian latent variables*

– Hans J. Skaug (coauthor of `TMB`), RSS discussion of Rue, Martino, and Chopin (2009)

Meanwhile in Canada...

- Alex Stringer in Toronto $\to$ Waterloo had been thinking along similar lines, and made a lot of progress

1. Implementing an algorithm similar to INLA, using a specific quadrature rule $\mathcal{Q}$ called adaptive Gauss-Hermite quadrature (AGHQ) which he and others argue should be the default for this problem
2. Defining a class of models called extended latent Gaussian models (ELGMs) which Naomi fits into

- I will explain what both of these acronyms (AGHQ[5] and ELGM) mean!

[5] Amusingly similar to AGYW: adolescent girls and young women

What's AGHQ?

What's an ELGM?

Why is Naomi an ELGM?

ANC offset from household survey

$$
\begin{aligned}
\text{logit}(\rho_{x,a}^{\text{ANC}}) &= \text{logit}(\rho_{x,F,a}) + \beta^{\rho^{\text{ANC}}} + u_x^{\rho^{\text{ANC}}} + \eta_{R_x,a}^{\rho^{\text{ANC}}}, \\
\text{logit}(\alpha_{x,a}^{\text{ANC}}) &= \text{logit}(\alpha_{x,F,a}) + \beta^{\alpha^{\text{ANC}}} + u_x^{\alpha^{\text{ANC}}} + \eta_{R_x,a}^{\alpha^{\text{ANC}}}.
\end{aligned}
$$

Incidence depends on adult prevalence and coverage

$$\log(\lambda_{x,s,a}) = \beta_0^{\lambda} + \beta_S^{\lambda,s=\text{M}} + \log(\rho_x^{\text{15-49}}) + \log(1 - \omega \cdot \alpha_x^{\text{15-49}}) + u_x^{\lambda} + \eta_{R_x,s,a}^{\lambda}.$$

ART attendance probability product

$$\dot{A}_{x',x,s,a} \sim \mathrm{Bin}(N_{x',s,a}, \pi_{x',x,s,a})$$

where $\pi_{x',x,s,a} = \rho_{x',s,a} \cdot \alpha_{x',s,a} \cdot \gamma_{x',x,s,a}$.

ART attendance multinomial

Aggregation

$$
\begin{tikzcd}
p(\theta, x, y) \arrow[d, "\text{Laplace}"] & \tilde{p}_{\texttt{AQ}}(\theta \mid y) \arrow[r, "\text{Marginalise}"] & \tilde{p}_{\texttt{AQ}}(\theta_j \mid y) \arrow[d, "\text{Spline}"] \\
\tilde{p}_{\texttt{LA}}(\theta, y) \arrow[r, "\text{AGHQ}"] & \tilde{p}_{\texttt{AQ}}(y) \arrow[u, "\text{Normalise}"] & \text{Output}
\end{tikzcd}
$$

Figure 3: Inference for the hyperparameters. Shaped like a snake for no real reason.

$$
\begin{tikzcd}
p(\theta, x, y) \arrow[d, "\text{Laplace}"] & \tilde{p}_{\texttt{AQ}}(\theta \mid y) \arrow[r, "\text{Marginalise}"] & \tilde{p}_{\texttt{AQ}}(\theta_j \mid y) \arrow[d, "\text{Spline}"] \\
\tilde{p}_{\texttt{LA}}(\theta, y) \arrow[r, "\text{AGHQ}"] & \tilde{p}_{\texttt{AQ}}(y) \arrow[u, "\text{Normalise}"'] & \text{Output}
\end{tikzcd}
$$

Figure 4: Inference for the latent field.

Latent field algorithm

Thanks for listening!

- Working on a paper “Fast approximate Bayesian inference for small-area estimation of HIV indicators using the Naomi model” based on this work, joint with Alex Stringer (Waterloo) and my PhD supervisors Seth Flaxman (Oxford) and Jeff Eaton (Imperial)
- Let me know if you’d be up for being an early reader!
- Code for this project is at `athowes.github.io/elgm-inf`

References I

O'Hagan, Anthony. 1987. "Monte Carlo Is Fundamentally Unsound." *The Statistician*, 247–49.

Rue, Håvard, Sara Martino, and Nicolas Chopin. 2009. "Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations." *Journal of the Royal Statistical Society: Series b (Statistical Methodology)* 71 (2): 319–92.