Inference methods for extended latent Gaussian models

Waterloo SAS Student Seminar Series

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Motivation

- Surveillance of the HIV epidemic in sub-Saharan Africa
- Aim to estimate epidemic indicators useful for monitoring and response, including:
  - Prevalence ρ: the proportion of people who are HIV positive
  - Incidence λ: the proportion of people newly infected
  - Treatment coverage α: the proportion of PLHIV on treatment
- This is a challenging task! Data is noisy, sparse and biased

A simple small-area model for prevalence

- Consider areas $i = 1, \ldots, n$
- Simple random sample survey taken in each area, with sample sizes m_i
- The number of people testing positive is y_i
- Then we can use a binomial logistic regression of the form:

$$\begin{aligned} y_i &\sim \mathrm{Bin}(m_i, \rho_i), \\ \mathrm{logit}(\rho_i) &\sim g(\vartheta_\rho), \quad i = 1, \ldots, n, \end{aligned}$$

- If g is Gaussian then this is a latent Gaussian model in the sense of Rue, Martino, and Chopin (2009)
- One problem with this data is that household surveys are expensive to run, so they only happen rarely

Latent Gaussian models

- Three-stage model

$$
\begin{aligned}
&\text{(Observations)} && \mathbf{y} \sim p(\mathbf{y} \,|\, \mathbf{x}), \\
&\text{(Latent field)} && \mathbf{x} \sim p(\mathbf{x} \,|\, \boldsymbol{\theta}), \\
&\text{(Hyperparameters)} && \boldsymbol{\theta} \sim p(\boldsymbol{\theta}),
\end{aligned}
$$

  where $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$
- Interested in learning both $(\boldsymbol{\theta}, \mathbf{x})$ from data $\mathbf{y}$
- If the middle layer is Gaussian, then it's a latent Gaussian model
- Covers most of the models commonly used in spatiotemporal statistics

Adding ANC surveillance

- Pregnant women attending antenatal care clinics are routinely tested for HIV, to avoid mother-to-child transmission
- This data source is more real-time than household surveys, but it's also more biased, because attendees are unlikely to be as representative of the population
- But perhaps this bias is consistent, in which case we can still make use of the ANC data to improve our model!

Naomi evidence synthesis model

- Combining these three modules is the basis of the Naomi evidence synthesis model
- Used by countries, which provide their own data, to produce HIV estimates in a yearly process supported by UNAIDS
- Can't run long MCMC in this setting, requires fast, accurate, approximations
- Requires something more flexible than `R-INLA`
- Currently using Template Model Builder `TMB` (Kristensen et al. 2015)

Figure 1: A supermodel

Integrated Nested Laplace Approximation

- Rue, Martino, and Chopin (2009) or e.g. Blangiardo and Cameletti (2015)
- Approximate Bayesian inference for latent Gaussian models (LGMs), which are three-stage models with middle layer

$$\text{(Latent field)} \qquad p(\mathbf{x} \mid \boldsymbol{\theta}) = \mathcal{N}(\mathbf{x} \mid \boldsymbol{\mu}(\boldsymbol{\theta}), \boldsymbol{Q}(\boldsymbol{\theta})^{-1}).$$

- `R-INLA` implementation takes advantage of sparsity properties of $\boldsymbol{Q}(\boldsymbol{\theta})$, i.e. if $\mathbf{x}$ is a Gaussian Markov random field (GMRF)

Integrated Nested Laplace Approximation

- Gives approximate posterior marginals $\{\tilde{p}(x_i \mid \mathbf{y})\}_{i=1}^{n}$ and $\{\tilde{p}(\theta_j \mid \mathbf{y})\}_{j=1}^{m}$
- To approximate posterior marginals below requires $\tilde{p}(\boldsymbol{\theta} \mid \mathbf{y})$ and $\tilde{p}(x_i \mid \boldsymbol{\theta}, \mathbf{y})$

$$p(x_i \mid \mathbf{y}) = \int p(x_i, \boldsymbol{\theta} \mid \mathbf{y}) \mathrm{d}\boldsymbol{\theta} = \int p(x_i \mid \boldsymbol{\theta}, \mathbf{y}) p(\boldsymbol{\theta} \mid \mathbf{y}) \mathrm{d}\boldsymbol{\theta}, \quad i = 1, \ldots, n, \tag{1}$$

$$p(\theta_j \mid \mathbf{y}) = \int p(\boldsymbol{\theta} \mid \mathbf{y}) \mathrm{d}\boldsymbol{\theta}_{-j} \quad j = 1, \ldots, m. \tag{2}$$

Integrated Nested Laplace Approximation

1) First Laplace approximate hyperparameter posterior

$$\tilde{p}(\boldsymbol{\theta} \mid \mathbf{y}) \propto \frac{p(\mathbf{y}, \mathbf{x}, \boldsymbol{\theta})}{\tilde{p}_G(\mathbf{x} \mid \boldsymbol{\theta}, \mathbf{y})}\bigg|_{\mathbf{x}=\boldsymbol{\mu}^\star(\boldsymbol{\theta})} \tag{3}$$

which can be marginalised to get $\tilde{p}(\theta_j \mid \mathbf{y})$

- Note here that this involves integrating out a **Gaussian** approximation to the latent field

2) In both (1) and (2) we want to integrate w.r.t. (3), so choose integration points and weights $\{\boldsymbol{\theta}^{(k)}, \Delta^{(k)}\}$

- For low m R-INLA uses a grid-strategy (illustrated in the next slide)
- For larger m this becomes too expensive and R-INLA uses a CCD design is used
- Other approaches, like adaptive Gaussian Hermite quadrature (AGHQ)

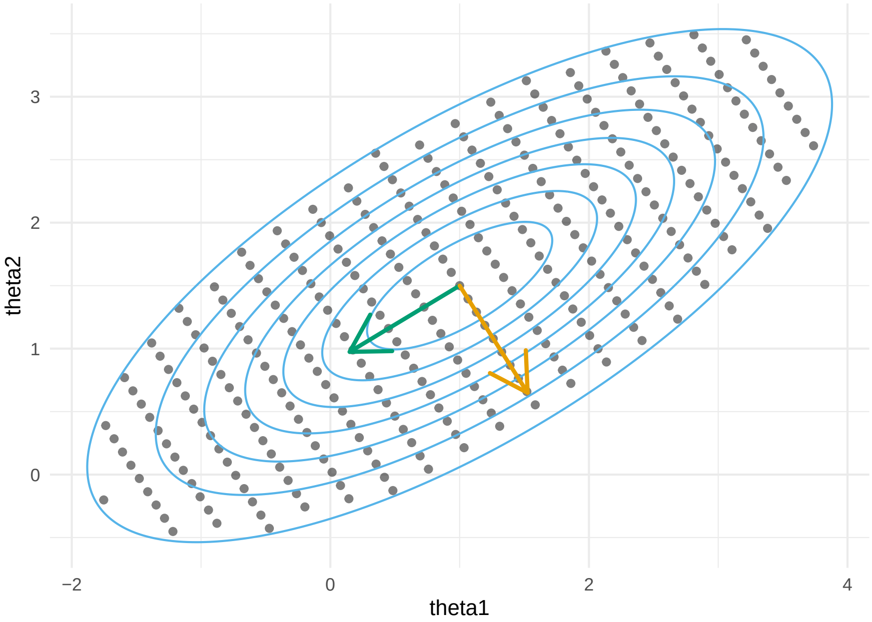

Figure 2: An illustration of the R-INLA grid method for selecting integration points using a

Adaptive Gaussian Hermite Quadrature

- `aghq` R package and vignette (Stringer 2021)
- Gauss-Hermite quadrature is a way of picking nodes and weights, and is based on the theory of polynomial interpolation
- The adaptive part means that it uses the location (mode) and curvature (Hessian) of the target (posterior)
- Use k quadrature points
  - If k is odd then they include the mode
  - If $k = 1$ then it's a Laplace approximation
  - In the vignette $k = 3$ (for each dimension, so 3^m total) is chosen quite often

Integrated Nested Laplace Approximation

3) Choose approximation for $\tilde{p}(x_i \mid \boldsymbol{\theta}, \mathbf{y})$

- Simplest version (Rue and Martino 2007) is to marginalise the $p_G(\mathbf{x} \mid \boldsymbol{\theta}, \mathbf{y})$

$$\tilde{p}(x_i \mid \boldsymbol{\theta}, \mathbf{y}) = \mathcal{N}(x_i \mid \mu_i^\star(\boldsymbol{\theta}), 1/q_i^\star(\boldsymbol{\theta})) \tag{4}$$

- The above is referred to as `method = "gaussian"` in `R-INLA`, and confusingly there are two more complex ones called `"simplified laplace"` and `"laplace"`
- We will discuss ways to do better than this joint Gaussian approximation in the next slide

4) Finally use quadrature to get

$$\tilde{p}(x_i \mid \mathbf{y}) = \sum_{k=1}^{K} \tilde{p}(x_i \mid \boldsymbol{\theta}^{(k)}, \mathbf{y}) \times \tilde{p}(\boldsymbol{\theta}^{(k)} \mid \mathbf{y}) \times \Delta^{(k)} \tag{5}$$

Template Model Builder

- R package which implements the Laplace approximation for latent variable models using AD (via CppAD)
  - For more about AD see e.g. Griewank and Walther (2008)
  - Useful for getting the mode, Hessian
- Write an objective function $f(\mathbf{x}, \boldsymbol{\theta})$ in C++ ("user template")
  - We select $f(\mathbf{x}, \boldsymbol{\theta}) = -\log p(\mathbf{y} \mid \mathbf{x}, \boldsymbol{\theta}) p(\mathbf{x} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta})$

Template Model Builder

```
#include <TMB.hpp>

template <class Type>
Type objective_function<Type>::operator()() {
  // Define data e.g.
  DATA_VECTOR(y);
  // Define parameters e.g.
  PARAMETER(mu);
  // Calculate negative log-likelihood e.g.
  nll = Type(0.0);
  nll -= dnorm(y, mu, 1, true).sum()
  return(nll);
}
```

Template Model Builder

- Performs the Laplace approximation $L_f(\boldsymbol{\theta}) \approx L_f^\star(\boldsymbol{\theta})$ and use R to optimise this with respect to $\boldsymbol{\theta}$ to give $\hat{\boldsymbol{\theta}}$ (the central point in Figure 2)
  - This is done by specifying the `random` argument to be the parameters that you want to integrate out with a Laplace approximation (the latent field)
- MAP estimate of $\mathbf{x}$ conditional on $\hat{\boldsymbol{\theta}}$
- Standard errors calculated using the δ-method (a Gaussian assumption)

References I

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