Appendix to "Integrated nested Laplace approximations for extended latent Gaussian models with application to the Naomi HIV model"

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1 Modified Naomi model description

1.1 Notation

1.1.1 Indexing

Consider some country, and let x refer to a district located within the Spectrum region R_x. We restrict our attention to the time period $t = T_1$ of the most recent national household survey with HIV testing. Let $s \in \{\text{F}, \text{M}\}$ be sex, and $a \in \{\text{0-5}, \text{5-10}, \ldots, \text{75-80}, \text{80+}\}$ be five-year age bands. As short-hand, we write $a = l$ to refer to the age band with lower bound l. We index the following quantities by district, sex, age-band: population size $N_{x,s,a}$, HIV prevalence $\rho_{x,s,a}$, ART coverage $\alpha_{x,s,a}$ and annual HIV incidence rate $\lambda_{x,s,a}$. Our model operates at this level of granularity, although sometimes the data observed is at an aggregate level. We use $\{\cdot\}$ to refer to a generic aggregate set over which an observation is made, such as $\{a\} = \{\text{15-19}, \ldots, \text{45-49}\}$ for adults.

1.1.2 Structured random effects

Consider some generic random effect u. We use $u \sim \text{ICAR}(\sigma)$ to refer to the Besag intrinsic conditional auto-regressive model (ICAR) (Besag, York, and Mollié 1991) with marginal standard deviation $\sigma > 0$. For the reparameterised Besag-York-Mollie model (BYM2) (Simpson et al. 2017) with marginal standard deviation $\sigma > 0$ we write $u \sim \text{BYM2}(\sigma, \phi)$, where u is comprised of a spatially structured ICAR component with proportion $\phi \in (0, 1)$ and spatially unstructured IID component with proportion $1 - \phi$ We follow recommendations of Freni-Sterrantino, Ventrucci, and Rue (2018) on scaling, disconnected adjacency graph components, and islands. The first order auto-regressive model is specified by $u \sim \text{AR1}(\sigma, \phi)$.

1.2 Process specification

1.2.1 HIV prevalence

We model HIV prevalence by

$$\text{logit}(\rho_{x,s,a}) = \beta_0^\rho + \beta_S^{\rho,s=\text{M}} + u_a^\rho + u_a^{\rho,s=\text{M}} + u_x^\rho + u_x^{\rho,s=\text{M}} + u_x^{\rho,a<15} + \eta_{R_x,s,a}^\rho$$

where β_0^ρ is an intercept term, $\beta_s^{\rho,s=\text{M}}$ is the difference in logit prevalence for men compared to women, $u_a^\rho \sim \text{AR1}(\sigma_A^\rho, \phi_A^\rho)$ are age random effects for women, $u_a^{\rho,s=\text{M}}$ are age random effects for the difference in logit prevalence for men compared to women age a, $u_x^\rho \sim \text{BYM2}(\sigma_X^\rho, \phi_X^\rho)$ are spatial random effects for women, $u_x^{\rho,s=\text{M}} \sim \text{BYM2}(\sigma_{XS}^\rho, \phi_{XS}^\rho)$ are spatial random effects for the difference in logit prevalence for men compared to women in district x, $u_x^{\rho,a<15} \sim \text{ICAR}(0, \sigma_{XA}^\rho)$ are spatial random effects for the ratio of paediatric prevalence to adult women prevalence, and $\eta_{R_x,s,a}^\rho$ are fixed offsets specifying assumed odds ratios for prevalence outside the age ranges for which data are available. We use the prior distributions $\mathcal{N}(0, 5)$ for the fixed effects β_0^ρ and $\beta_s^{\rho,s=\text{M}}$, $\mathcal{N}^+(0, 2.5)$ for all standard deviation terms, $\mathcal{U}(-1, 1)$ for all AR1 correlation parameters, and $\text{Beta}(0.5, 0.5)$ for all BYM2 proportion parameters.

1.2.2 ART coverage

We model ART coverage by

$$\text{logit}(\alpha_{x,s,a}) = \beta_0^\alpha + \beta_S^{\alpha,s=\text{M}} + u_a^\alpha + u_a^{\alpha,s=\text{M}} + u_x^\alpha + u_x^{\alpha,s=\text{M}} + u_x^{\alpha,a<15} + \eta_{R_x,s,a}^\alpha$$

with terms and priors analogous to the HIV prevalence model above.

1.2.2.1 HIV incidence rate

We model HIV incidence rate by

$$\log(\lambda_{x,s,a}) = \beta_0^\lambda + \beta_S^{\lambda,s=\text{M}} + \log(\rho_x^{\text{15-49}}) + \log(1 - \omega \cdot \alpha_x^{\text{15-49}}) + u_x^\lambda + \eta_{R_x,s,a}^\lambda$$

where β_0^λ is an intercept term proportional to the average HIV transmission rate for untreated HIV positive adults, $\beta_S^{\lambda,s=\text{M}}$ is the log incidence rate ratio for men compared to women, $\rho_x^{\text{15-49}}$ is the HIV prevalence among

adults 15-49 calculated by

$$\rho_x^{\text{15-49}} = \frac{\sum_{s\in\{\text{F,M}\}}\sum_{a=15}^{45} N_{x,s,a}\cdot\rho_{x,s,a}}{\sum_{s\in\{\text{F,M}\}}\sum_{a=15}^{45} N_{x,s,a}},$$

$\alpha_x^{\text{15-49}}$ is the ART coverage among adults 15-49 calculated by

$$\alpha_x^{\text{15-49}} = \frac{\sum_{s\in\{\text{F,M}\}}\sum_{a=15}^{45} N_{x,s,a}\cdot\rho_{x,s,a}\cdot\alpha_{x,s,a}}{\sum_{s\in\{\text{F,M}\}}\sum_{a=15}^{45} N_{x,s,a}\cdot\rho_{x,s,a}},$$

ω is the average reduction in HIV transmission rate per 1% increase in population ART coverage and is fixed at $\omega = 0.7$ via the EPP model, $u_x^\lambda \sim \mathcal{N}(0,\sigma^\lambda)$ with $\sigma^\lambda \sim \mathcal{N}^+(0,1)$ are IID spatial random effects, and $\eta^\lambda_{R_x,s,a}$ specify log incidence rate ratios by sex and age group calculated from Spectrum model output

1.2.3 ANC testing cascade

For women 15-49 the predicted number of ANC clients $\Psi_{x,a}$ is a log-linear model

$$\log(\Psi_{x,a}) = \log(N_{x,\text{F},a}) + \psi_{R_x,a} + \beta^\psi + u_x^\psi$$

where $N_{x,\text{F},a}$ are the female population sizes, $\psi_{R_x,a}$ are ASFR in Spectrum region R_x at time t, β^ψ are the log rate ratio for the number of ANC clients relative to the predicted fertility, $u_x^\psi \sim \mathcal{N}(0,\sigma^\psi)$ are district random effects. HIV prevalence $\rho_{x,a}^{\text{ANC}}$ and ART coverage $\alpha_{x,a}^{\text{ANC}}$ among pregnant women modelled with logit-linear models

$$\text{logit}(\rho_{x,a}^{\text{ANC}}) = \text{logit}(\rho_{x,F,a}) + \beta^{\rho^{\text{ANC}}} + u_x^{\rho^{\text{ANC}}} + \eta_{R_x,a}^{\rho^{\text{ANC}}} \tag{1}$$

$$\text{logit}(\alpha_{x,a}^{\text{ANC}}) = \text{logit}(\alpha_{x,F,a}) + \beta^{\alpha^{\text{ANC}}} + u_x^{\alpha^{\text{ANC}}} + \eta_{R_x,a}^{\alpha^{\text{ANC}}}, \tag{2}$$

where $\eta_{R_x,a}^{\theta^{\text{ANC}}}$ for $\theta \in \{\rho,\alpha\}$ are offsets for the log fertility rate ratios for HIV positive women compared to HIV negative women and for women on ART to HIV positive women not on ART, calculated from Spectrum model outputs for region R_x, $\beta^{\theta^{\text{ANC}}}$ for $\theta \in \{\rho,\alpha\}$ are the average differences between population and ANC outcomes after removing the offset, and $u_x^{\theta^{\text{ANC}}} \sim \mathcal{N}(0,\sigma_X^{\theta^{\text{ANC}}})$ for $\theta \in \{\rho,\alpha\}$ are district random effects.

1.2.4 ANC attendance

Let $\gamma_{x,x'} \in [0,1]$ be the probability that a person of ART residing in district x recieves ART in district x'. We assume that $\gamma_{x,x'} = 0$ for $x \notin \{x, \text{ne}(x)\}$ such that individuals seek treatment only in their residing district and its neighbours $\text{ne}(x) = \{x' : x' \sim x\}$, where $\sim$ is an adjacency relation and $\sum_{x'\in\{x,\text{ne}(x)\}}\gamma_{x,x'} = 1$.

To model $\gamma_{x,x'}$ for $x \sim x'$ we use a multinomial logistic regression model where $\tilde{\gamma}_{x,x'}$ is the log odds ratio of seeking ART

$$\tilde{\gamma}_{x,x'} = \tilde{\gamma}_0 + u_x^{\tilde{\gamma}} \tag{3}$$

$$\tilde{\gamma}_0 = -4 \tag{4}$$

$$u_x^{\tilde{\gamma}} \sim \mathcal{N}(0,\sigma_X^{\tilde{\gamma}}) \tag{5}$$

where $\tilde{\gamma}_0$ is an intercept specifying the prior mean for the log odds of seeking ART in each neighbouring district compared to the home district, and $u_x^{\tilde{\gamma}}$ are district random effects. We fix $\tilde{\gamma}_{x,x} = 0$ and recover the multinomial probabilities using the softmax

$$\gamma_{x,x'} = \frac{\exp(\tilde{\gamma}_{x,x'})}{\sum_{x^\star\in\{x,\text{ne}(x)\}}\exp(\tilde{\gamma}_{x,x^\star})}$$

Then, given the number of PLHIV on ART $A_{x,s,a} = N_{x,s,a}\cdot\rho_{x,s,a}\cdot\alpha_{x,s,a}$, the number of ART clients who reside in district x and obtain ART in district x' are

$$A_{x,x',s,a} = A_{x,s,a}\cdot\gamma_{x,x'},$$

and the total attending ART facilities in district x' are

$$\tilde{A}_{x',s,a} = \sum_{x \in \{x', \text{ne}(x')\}} A_{x,x',s,a}.$$

1.2.5 Awareness of HIV status

The proportion of HIV positive adults aware of their HIV status is $v_{x,s,a}$, and the proportion of untreated HIV positive adults aware of their HIV status is $\tilde{\nu}_{x,s,a}$ such that

$$v_{x,s,a} = \alpha_{x,s,a} + \frac{\tilde{v}_{R_x,s,a}}{1 - \alpha_{x,s,a}}.$$

1.3 Likelihood specification

1.3.1 Household survey data

Let ν be a household survey occurring at T_1, furnishing weighted observations $\hat{\theta}_{\{x\},\{s\},\{a\},\nu}$ for $\theta \in \{\rho, \alpha, \kappa\}$ with respective Kish effective sample sizes $M^{\hat{\theta}}_{\{x\},\{s\},\{a\},\nu}$, and observed number of cases

$$Y^{\hat{\theta}}_{\{x\},\{s\},\{a\},\nu} = M^{\hat{\theta}}_{\{x\},\{s\},\{a\},\nu} \cdot \hat{\theta}_{\{x\},\{s\},\{a\},\nu}.$$

For HIV prevalence and ART coverage we use binomial working likelihoods as follows

$$Y^{\hat{\rho}}_{\{x\},\{s\},\{a\},\nu} \sim \text{xBin}(M^{\hat{\rho}}_{\{x\},\{s\},\{a\},\nu}, \rho_{\{x\},\{s\},\{a\},T_1}) \tag{6}$$

$$\rho_{\{x\},\{s\},\{a\},t} = \frac{\sum_{x \in \{x\}} \sum_{s \in \{s\}} \sum_{a \in \{a\}} N_{x,s,a} \cdot \rho_{x,s,a}}{\sum_{x \in \{x\}} \sum_{s \in \{s\}} \sum_{a \in \{a\}} N_{x,s,a}}, \tag{7}$$

and

$$Y^{\hat{\alpha}}_{\{x\},\{s\},\{a\},\nu} \sim \text{xBin}(M^{\hat{\alpha}}_{\{x\},\{s\},\{a\},\nu}, \alpha_{\{x\},\{s\},\{a\},T_1}) \tag{8}$$

$$\alpha_{\{x\},\{s\},\{a\},t} = \frac{\sum_{x \in \{x\}} \sum_{s \in \{s\}} \sum_{a \in \{a\}} N_{x,s,a} \cdot \rho_{x,s,a} \cdot \alpha_{x,s,a}}{\sum_{x \in \{x\}} \sum_{s \in \{s\}} \sum_{a \in \{a\}} N_{x,s,a} \cdot \rho_{x,s,a}}. \tag{9}$$

For recent infections we also use a binomial working likelihood

$$Y^{\hat{\kappa}}_{\{x\},\{s\},\{a\},\nu} \sim \text{xBin}(M^{\hat{\kappa}}_{\{x\},\{s\},\{a\},\nu}, \kappa_{\{x\},\{s\},\{a\},T_1}) \tag{10}$$

$$\kappa_{x,s,a} = 1 - \exp(-\lambda_{x,s,a} \cdot \frac{1 - \rho_{x,s,a}}{\rho_{x,s,a}} \cdot (\Omega_T - \beta_T) - \beta_T) \tag{11}$$

$$\Omega_T \sim \mathcal{N}(\Omega_{T_0}, \sigma^{\Omega_T}) \tag{12}$$

$$\beta_T \sim \mathcal{N}(\beta_{T_0}, \sigma^{\beta_T}) \tag{13}$$

where $\kappa_{x,s,a}$ are the predicted proportion recently infected among HIV positive persons, Ω_T is the mean duration of recent infection (MDRI), and β_T is the false recent ratio (FRR). We use an informative prior on Ω_T based on the characteristics of the recent infection testing algorithm (RITA). For PHIA surveys this is $\Omega_{T_0} = 130$ days and $\sigma^{\Omega_T} = 6.12$ days. For PHIA surveys we assume there is no false recency, such that $\beta_{T_0} = 0.0$ and $\sigma^{\beta_T} = 0.0$.

1.3.2 ANC testing data

We include ANC testing data for the year of the most recent survey $Y[T_1]$. Let $W^{\text{ANC}}_{\{x\},Y[t]}$ be the number of ANC clients, $X^{\text{ANC}}_{\{x\},Y[t]}$ the number of those with ascertained status, $Y^{\text{ANC}}_{\{x\},Y[t]}$ the number of those with positive status (either known or tested) and $Z^{\text{ANC}}_{\{x\},Y[t]}$ the number of ANC clients already on ART prior to

first ANC, such that $W^{\text{ANC}}_{x,Y[t]} \geq X^{\text{ANC}}_{x,Y[t]} \geq Y^{\text{ANC}}_{x,Y[t]} \geq Z^{\text{ANC}}_{x,Y[t]}$ When ANC testing data are only available for part of a given year, we denote $M^{\text{ANC}}_{Y[t]} \in \{1, \ldots, 12\}$ the number of months of reported data reflected in counts for year $Y[t]$.

The observed number of HIV positive and already on ART among ANC clients at $Y[T_1]$ is modelled by

$$Y^{\text{ANC}}_{\{x\}Y[T_1]} \sim \text{Bin}\left(X^{\text{ANC}}_{\{x\}Y[T_1]}, \rho^{\text{ANC}}_{\{x\},\{15,\ldots45\},t}\right)$$
$$Z^{\text{ANC}}_{\{x\}Y[T_1]} \sim \text{Bin}\left(Y^{\text{ANC}}_{\{x\}Y[T_1]}, \alpha^{\text{ANC}}_{\{x\},\{15,\ldots45\},t}\right)$$

where predicted prevalence and ART coverage are aggregated weighted by the predicted number of pregnant women by age $\Psi_{x,a}$

$$\rho^{\text{ANC}}_{\{x\}\{a\},t} = \frac{\sum_{x\in\{x\}}\sum_{a\in\{a\}} \Psi_{x,a} \cdot \rho^{\text{ANC}}_{x,a}}{\sum_{x\in\{x\}}\sum_{a\in\{a\}} \Psi_{x,a}}$$
$$\alpha^{\text{ANC}}_{\{x\}\{a\},t} = \frac{\sum_{x\in\{x\}}\sum_{a\in\{a\}} \Psi_{x,a} \cdot \rho^{\text{ANC}}_{x,a} \cdot \alpha^{\text{ANC}}_{x,a}}{\sum_{x\in\{x\}}\sum_{a\in\{a\}} \Psi_{x,a} \cdot \rho^{\text{ANC}}_{x,a}}$$

1.3.3 Number receiving ART

Let $\dot{A}_{\{x\},\{s\},\{a\},t}$ be data for the number receiving ART

$$\dot{A}_{\{x\},\{s\},\{a\},t} = \sum_{s\in\{s\}}\sum_{a\in\{a\}}\sum_{x\in\{x\}}\sum_{x\sim x',x=x'} \dot{A}_{x',x,s,a},$$

We model the unobserved numbers of ART clients travelling from x' to x as

$$\dot{A}_{x',x,s,a} \sim \text{Bin}(N_{x',s,a}, \pi_{x',x,s,a})$$

where $\pi_{x',x,s,a} = \rho_{x',s,a} \cdot \alpha_{x',s,a} \cdot \gamma_{x',x,s,a}$. This likelihood is approximated using a normal for the sum of binomials by

$$\dot{A}_{\{x\},\{s\},\{a\},t} \sim \mathcal{N}(\tilde{)}A_{\{x\},\{s\},\{a\},t}, \sigma^{\tilde{A}}_{\{x\},\{s\},\{a\},t})$$

where

$$\tilde{A}_{\{x\},\{s\}\{a\},t} = \sum_{s\in\{s\}}\sum_{a\in\{a\}}\sum_{x\in\{x\}}\sum_{x\sim x',x=x'} N_{x',s,a} \cdot \pi_{x',x,s,a},$$

and

$$\sigma^{\tilde{A}}_{\{x\},\{s\}\{a\},t} = \sqrt{\sum_{s\in\{s\}}\sum_{a\in\{a\}}\sum_{x\in\{x\}}\sum_{x\sim x',x=x'} N_{x',s,a} \cdot \pi_{x',x,s,a} \cdot (1-\pi_{x',x,s,a})}.$$

1.4 Identifiability constaints

If data are missing, some parameters are fixed to default values to help with identifiability. In particular:

- If survey data on ART coverage by age and sex are not available then we set $u^{\alpha}_{a} = 0$ and $u^{\alpha}_{a,s=\text{M}} = 0$ and use the average age/sex pattern of ART coverage from the Spectrum offset $\eta^{\alpha}_{R_x,s,a}$.
- If no ART data (survey or ART programme) are available at T_1 but data on ART coverage among ANC clients are available, the level of ART coverage is not identifiable, but spatial variation is identifiable. In this instance, overall ART coverage is determined by the Spectrum offset, and only area random effects are estimated
$$\text{logit}\left(\alpha_{x,s,a,T_1}\right) = u^{\alpha}_{x} + \eta^{\alpha}_{R_x,s,a}$$
- If survey data on recent HIV infection are not included in the model, then $\beta^{\lambda}_{0} = \beta^{\lambda,s=\text{M}}_{S} = u^{\lambda}_{x} = 0$. The sex ratio for HIV incidence is determined by the sex incidence rate ratio from Spectrum in the same years and the incidence rate in all districts is modelled assuming the same average HIV transmission rate for untreated adults, but varies according to district estimates of HIV prevalence and ART coverage.

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