COMPUTATION OF LOCAL A-PACKETS IN SAGE

HIRAKU ATOBE

1. Introduction

The local A-packets given by Arthur [1] classify the local factors of automorphic representations appearing in the discrete spectrum of square integrable automorphic forms. The local A-packets are finite sets consisting of unitary representations, but it is difficult combinatorially to give an explicit description of them. In the previous work [2], the author wrote a Sage code[1] for computing examples of local A-packets. It is available at `https://github.com/atobe31/Local-A-packets`. For Sage, see `https://www.sagemath.org`.

In this document, we explain what one can do using this code. First of all, to load the code "packet.sage", type in Sage:

```
sage: load("packet.sage")
```

None

In the rest of this document, we will explain the following commands.

- `LD(x,m)`, see Section 2.2;
- `RD(y,m)`, see Section 2.2;
- `LS(x,m)`, see Section 2.2;
- `RS(y,m)`, see Section 2.2;
- `L_packet(phi,+1)`, see Section 2.3;
- `D(x,m,T)`, see Section 2.4;
- `S(x,m,T)`, see Section 2.4;
- `D00(m,T)`, see Section 2.4;
- `S00(m,T)`, see Section 2.4;
- `D01(m,T)`, see Section 2.4;
- `S01(k,m,T)`, see Section 2.4;
- `AD(m,T)`, see Section 2.5;
- `symbol(E)`, see Section 3.1;
- `nonzero(E,+1)`, see Section 3.1;
- `rep(E)`, see Section 3.1;
- `hat(E)`, see Section 3.2;
- `dual_rep(E)`, see Section 3.2;
- `change(E,i)`, see Section 3.3;
- `orders(E)`, see Section 3.3;
- `eq_class(E)`, see Section 3.4;
- `dim(psi)`, see Section 4.2;

2010 *Mathematics Subject Classification.* Primary 22E50; Secondary 11S37.

Key words and phrases. SageMath; Derivatives; Extended multi-segments; A-packets; Socles.

[1] Our code covers results in [5, 2, 3, 4]. It does not include results in [7].

- `A_packet(psi,+1)`, see Section 4.2;
- `char(E)`, see Section 4.2;
- `par(E)`, see Section 4.2;
- `Is_Arthur(m,T)`, see Section 4.3;
- `soc(s,(a,b),E)`, see Section 5.1;
- `Is_irred(s,(a,b),E)`, see Section 5.2;
- `FRP((a,b),E)`, see Section 5.3.

Acknowledgement. We are grateful to Professor Baiying Liu and his students Alexander Hazeltine and Chi-Heng Lo for pointing out many bugs. Especially, the command `orders(E)` is defined by them. We would like to thank Professor Wee Teck Gan and Professor Atsushi Ichino for their encouragement. The author was supported by JSPS KAKENHI Grant Number 19K14494.

Contents

2. Theory of derivatives

In this section, we explain derivatives in the sense of Jantzen and Mínguez. Fix a non-archimedean local field F of characteristic zero.

2.1. **Representations of $\mathrm{GL}_n(F)$.** First we consider irreducible unipotent representations of $\mathrm{GL}_n(F)$. Here, we say that $\tau \in \mathrm{Irr}(\mathrm{GL}_n(F))$ is *unipotent* if $\tau \hookrightarrow |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_n}$ for $s_1, \dots, s_n \in \mathbb{C}$.

A *segment* is a set

$$[x, y] = \{|\cdot|^x, |\cdot|^{x-1}, \dots, |\cdot|^y\}$$

for $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}_{\geq 0}$. The parabolically induced representation

$$|\cdot|^x \times |\cdot|^{x-1} \times \cdots \times |\cdot|^y$$

has a unique irreducible subrepresentation (resp. quotient) denoted by $\Delta[x, y]$ (resp. $Z[y, x]$). We call $\Delta[x, y]$ a *Steinberg representation*. We denote the set of equivalence classes of irreducible unipotent representation of $\mathrm{GL}_n(F)$ by $\mathrm{Irr}_{\mathrm{unip}}(\mathrm{GL}_n(F))$.

Let $\mathfrak{m} = [x_0, y_0] + \cdots + [x_{r-1}, y_{r-1}]$ be a *multi-segment*, i.e., a multi-set consisting of segments. Suppose that $x_0 + y_0 \leq \cdots \leq x_{r-1} + y_{r-1}$. Then the parabolically induced representation

$$\Delta[x_0, y_0] \times \cdots \times \Delta[x_{r-1}, y_{r-1}]$$

has a unique irreducible subrepresentation denoted by

$$L(\mathfrak{m}) = L(\Delta[x_0, y_0], \dots, \Delta[x_{r-1}, y_{r-1}]).$$

By the Langlands classification, the map $\mathfrak{m} \mapsto L(\mathfrak{m})$ is a bijection between the set of multi-segments and $\cup_{n \geq 0} \mathrm{Irr}_{\mathrm{unip}}(\mathrm{GL}_n(F))$. In this document and our Sage code, we identify $L(\mathfrak{m})$ with $\mathfrak{m}$. For example:

```
sage: m1 = ([-2,-3], [-1,-1]) # An irreducible representation
   of GL_{3}(F).
sage: m1
```

$$([-2, -3], [-1, -1])$$

```
sage: m2 = ([-1/2,-5/2], )  # A Steinberg representation of
   GL_3(F).
sage: m2
```

$$\left(\left[-\frac{1}{2}, -\frac{5}{2}\right]\right)$$

2.2. **Derivatives and socles: The case of $\mathrm{GL}_n(F)$.** In this subsection, the following commands are explained:

- `LD(x,m)`: the highest left $|\cdot|^x$-derivative of $L(\mathfrak{m}) \in \mathrm{Irr}(\mathrm{GL}_n(F))$;
- `RD(y,m)`: the highest right $|\cdot|^y$-derivative of $L(\mathfrak{m}) \in \mathrm{Irr}(\mathrm{GL}_n(F))$;
- `LS(x,m)`: the socle of $|\cdot|^x \times L(\mathfrak{m})$ for $L(\mathfrak{m}) \in \mathrm{Irr}(\mathrm{GL}_n(F))$;
- `RS(y,m)`: the socle of $L(\mathfrak{m}) \times |\cdot|^y$ for $L(\mathfrak{m}) \in \mathrm{Irr}(\mathrm{GL}_n(F))$.

Let $\tau \in \mathrm{Irr}_{\mathrm{unip}}(\mathrm{GL}_n(F))$. We denote by $[\mathrm{Jac}_{(k,n-k)}(\tau)]$ the semisimplification of the Jacquet module of τ along the standard parabolic subgroup of $\mathrm{GL}_n(F)$ with Levi $\mathrm{GL}_k(F) \times \mathrm{GL}_{n-k}(F)$. For $x \in \mathbb{R}$, define the *left* $|\cdot|^x$*-derivative* $L_{|\cdot|^x}(\tau)$ and the *right* $|\cdot|^x$*-derivative* $R_{|\cdot|^x}(\tau)$ by the semisimple representation of $\mathrm{GL}_{n-1}(F)$ satisfying

$$[\mathrm{Jac}_{(1,n-1)}(\tau)] = |\cdot|^x \otimes L_{|\cdot|^x}(\tau) + (\text{others}),$$
$$[\mathrm{Jac}_{(n-1,1)}(\tau)] = R_{|\cdot|^x}(\tau) \otimes |\cdot|^x + (\text{others}).$$

Set
$$L^{(k)}_{|\cdot|^x}(\tau) = \frac{1}{k!}\underbrace{L_{|\cdot|^x}\circ\cdots\circ L_{|\cdot|^x}}_{k}(\tau), \quad R^{(k)}_{|\cdot|^x}(\tau) = \frac{1}{k!}\underbrace{R_{|\cdot|^x}\circ\cdots\circ R_{|\cdot|^x}}_{k}(\tau).$$
If $L^{(k)}_{|\cdot|^x}(\tau) \neq 0$ but $L^{(k+1)}_{|\cdot|^x}(\tau) = 0$, we write $L^{\max}_{|\cdot|^x}(\tau) = L^{(k)}_{|\cdot|^x}(\tau)$ and call it the *highest left* $|\cdot|^x$*-derivative*. Similarly, the *highest right* $|\cdot|^x$*-derivative* $R^{\max}_{|\cdot|^x}(\tau)$ is defined.

As in [9, Lemma 2.1], if $\tau \in \mathrm{Irr}_{\mathrm{unip}}(\mathrm{GL}_n(F))$, then $L^{\max}_{|\cdot|^x}(\tau)$ (resp. $R^{\max}_{|\cdot|^x}(\tau)$) is irreducible. On the other hand, by [9, Corollary 4.10], the parabolically induced representation $|\cdot|^x \times \tau$ (resp. $\tau \times |\cdot|^x$) has a unique irreducible subrepresentation, which we write $\mathrm{soc}(|\cdot|^x \times \tau)$ (resp. $\mathrm{soc}(\tau \times |\cdot|^x)$). Moreover, by [9, Theorem 5.11], one can compute them in terms of the Langlands classification. For example:

```
sage: m = ([0,0], [1,-1], [1,0], [1,1])
sage: LD(1,m) # The highest left $|\cdot|^1$-derivative of $L(m)$.
```
$$(2, ([0,-1], [0,0], [0,0], [1,1]))$$
```
sage: RD(0,m) # The highest right $|\cdot|^0$-derivative of $L(m)$.
```
$$(1, ([0,0], [1,-1], [1,1], [1,1]))$$
```
sage: LS(2,m) # The socle of $||^2 \times L(m)$.
```
$$([0,0], [1,0], [1,1], [2,-1])$$
```
sage: RS(0,m) # The socle of $L(m) \times ||^0$.
```
$$([0,0], [0,0], [1,-1], [1,0], [1,1])$$
This means that if $\tau = L(|\cdot|^0, \Delta[1,-1], \Delta[1,0], |\cdot|^1)$, then
$$\begin{aligned} L^{\max}_{|\cdot|^1}(\tau) &= L^{(2)}_{|\cdot|^1}(\tau) = L(\Delta[0,-1], |\cdot|^0, |\cdot|^0, |\cdot|^1), \\ R^{\max}_{|\cdot|^0}(\tau) &= R^{(1)}_{|\cdot|^0}(\tau) = L(|\cdot|^0, \Delta[1,-1], |\cdot|^1, |\cdot|^1), \\ \mathrm{soc}(|\cdot|^2 \times \tau) &\cong L(|\cdot|^0, \Delta[1,0], |\cdot|^1, \Delta[2,-1]), \\ \mathrm{soc}(\tau \times |\cdot|^0) &\cong L(|\cdot|^0, |\cdot|^0, \Delta[1,-1], \Delta[1,0], |\cdot|^1). \end{aligned}$$

2.3. **Representations of classical groups.** In this subsection, the following command is explained:

- `L_packet(phi,1)`: the L-packet associated to $\phi \in \Phi_{\mathrm{temp}}(G_n)$.

Let G_n be a split special orthogonal group $\mathrm{SO}_{2n+1}(F)$ or a symplectic group $\mathrm{Sp}_{2n}(F)$. We denote the set of equivalence classes of irreducible tempered representations of G_n by $\mathrm{Irr}_{\mathrm{temp}}(G_n)$.

By the local Langlands correspondence, we have a surjective map
$$\mathrm{Irr}_{\mathrm{temp}}(G_n) \to \Phi_{\mathrm{temp}}(G_n),$$
where $\Phi(G_n)$ is the set of equivalence classes of L-parameters
$$\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G_n}$$
such that $\phi(W_F)$ is bounded. For $\phi \in \Phi_{\mathrm{temp}}(G_n)$, the fiber Π_ϕ is a finite set and is called the *L-packet* associated to ϕ.

Let S_a denote the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a. We call $\phi \in \Phi_{\mathrm{temp}}(G_n)$ *unipotent* if $\phi|_{W_F} = \mathbf{1}$. Since a unipotent L-parameter ϕ can be regarded as a (self-dual) representation of $\mathrm{SL}_2(\mathbb{C})$, we can write $\phi = \oplus_{i=0}^{t-1} S_{a_i}$. We say that $\phi = \oplus_{i=0}^{t-1} S_{a_i}$ is (unipotent and) *of good parity* if

$$a_i \equiv \begin{cases} 0 \bmod 2 & \text{if } G_n = \mathrm{SO}_{2n+1}(F), \\ 1 \bmod 2 & \text{if } G_n = \mathrm{Sp}_{2n}(F) \end{cases}$$

for $i = 0, \dots, t-1$. In this case, the L-packet Π_ϕ is parametrized by tuple $\varepsilon = (\varepsilon(a_0), \dots, \varepsilon(a_{t-1})) \in \{\pm 1\}^t$ such that

- $a_i = a_j \implies \varepsilon(a_i) = \varepsilon(a_j)$;
- the sign condition $\prod_{i=0}^{t-1} \varepsilon(a_i) = 1$ holds.

If $\pi \in \Pi_\phi$ corresponds to ε, we write $\pi = \pi(\phi, \varepsilon)$. In this document, we call (ϕ, ε) an *enhanced L-parameter*, and identify (ϕ, ε) as

$$([x_0, \varepsilon(a_0)], \dots, [x_{t-1}, \varepsilon(a_{t-1})]),$$

where we set $x_i = \frac{a_i - 1}{2}$.

We call an irreducible representation π of G_n *unipotent* and *of good parity* if π is a unique irreducible subrepresentation of a standard module

$$\Delta[x_0, y_0] \times \cdots \times \Delta[x_{r-1}, y_{r-1}] \rtimes \pi(\phi, \varepsilon),$$

where

- $x_0 + y_0 \leq \cdots \leq x_{r-1} + y_{r-1} < 0$;
- $x_i \in \mathbb{Z}$ if $G_n = \mathrm{Sp}_{2n}(F)$ (resp. $x_i \in (1/2)\mathbb{Z} \setminus \mathbb{Z}$ if $G_n = \mathrm{SO}_{2n+1}(F)$);
- ϕ is unipotent and of good parity.

In this case, we write

$$\pi = L(\Delta[x_0, y_0], \dots, \Delta[x_{r-1}, y_{r-1}]; \pi(\phi, \varepsilon)),$$

and identify π with a pair $(\mathfrak{m}, T)$ where

- $\mathfrak{m} = [x_0, y_0] + \cdots + [x_{r-1}, y_{r-1}]$ is a multi-segment;
- $T = ([\frac{a_0 - 1}{2}, \varepsilon_0], \dots, [\frac{a_{t-1} - 1}{2}, \varepsilon_{t-1}])$ is an enhanced L-parameter.

By considering consistency with A-parameters, we identify an L-parameter $\phi = \oplus_{i=0}^{t-1} S_{a_i}$ with

$$\phi = ((a_0, 1), (a_1, 1), \cdots, (a_{t-1}, 1)).$$

For example:

```
sage: m = ([0,-1], [1,-2]) # A multi-segment.
sage: phi = ((1,1), (3,1), (5,1)) # An $L$-parameter for $\Sp_{8}(F)$.
sage: Pi = L_packet(phi,+1) # An $L$-packet associated to $\phi$.
sage: for T in Pi:
....:     (m,T)
```

$$((\left[0,-1\right],\left[1,-2\right]),(\left[0,1\right],\left[1,1\right],\left[2,1\right]))$$
$$((\left[0,-1\right],\left[1,-2\right]),(\left[0,-1\right],\left[1,-1\right],\left[2,1\right]))$$
$$((\left[0,-1\right],\left[1,-2\right]),(\left[0,-1\right],\left[1,1\right],\left[2,-1\right]))$$
$$((\left[0,-1\right],\left[1,-2\right]),(\left[0,1\right],\left[1,-1\right],\left[2,-1\right]))$$

These are the elements in a non-tempered L-packet for $\mathrm{Sp}_{20}(F)$.

The second variable of `L_packet(phi,e)` is the sign condition, i.e., it requires $\prod_{i=0}^{t-1}\varepsilon(a_i)=e$. Since our group G_n is split, we only need $e=+1$.

2.4. **Derivatives and socles: The case of G_n.** In this subsection, the following commands are explained:

- `D(x,m,T)`: the highest $|\cdot|^x$-derivative of $L(\mathfrak{m};T)\in\mathrm{Irr}(G_n)$ for $x\neq 0$;
- `S(x,m,T)`: the socle of $|\cdot|^x\rtimes L(\mathfrak{m},T)$ for $L(\mathfrak{m},T)\in\mathrm{Irr}(G_n)$ and for $x\neq 0$;
- `D00(m,T)`: the highest $\Delta[0,-1]$-derivative of $L(\mathfrak{m},T)\in\mathrm{Irr}(G_n)$;
- `S00(m,T)`: the socle of $\Delta[0,-1]\rtimes L(\mathfrak{m},T)$ for $L(\mathfrak{m},T)\in\mathrm{Irr}(G_n)$;
- `D01(m,T)`: the highest $Z[0,1]$-derivative of $L(\mathfrak{m},T)\in\mathrm{Irr}(G_n)$;
- `S01(k,m,T)`: the socle of $Z[0,1]^k\rtimes L(\mathfrak{m},T)$ for $L(\mathfrak{m},T)\in\mathrm{Irr}(G_n)$.

Let π be an irreducible unipotent representation of G_n of good parity. We denote by $[\mathrm{Jac}_{P_k}(\pi)]$ the semisimplification of Jacquet module of π along the standard parabolic subgroup P_k of G_n with Levi $\mathrm{GL}_k(F)\times G_{n-k}$. For $x\in\mathbb{R}$, define the *k-th $|\cdot|^x$-derivative* $D^{(k)}_{|\cdot|^x}(\pi)$ by the semisimple representation of G_{n-k} satisfying

$$[\mathrm{Jac}_{P_k}(\pi)]=(\underbrace{|\cdot|^x\times\cdots\times|\cdot|^x}_{k})\otimes D^{(k)}_{|\cdot|^x}(\pi)+(\text{others}).$$

If $D^{(k)}_{|\cdot|^x}(\pi)\neq 0$ but $D^{(k+1)}_{|\cdot|^x}(\pi)=0$, we write $D^{\max}_{|\cdot|^x}(\pi)=D^{(k)}_{|\cdot|^x}(\pi)$ and call it the *highest $|\cdot|^x$-derivative*.

As in [8, Lemma 3.1.3], if $x\neq 0$, then $D^{\max}_{|\cdot|^x}(\pi)$ is irreducible. On the other hand, by [5, Proposition 3.3], when $x\neq 0$, the parabolically induced representation $|\cdot|^x\rtimes\pi$ has a unique irreducible subrepresentation, which we write $\mathrm{soc}(|\cdot|^x\rtimes\pi)$. Explicit formulas for them are given in [5, Proposition 6.1, Theorem 7.1]. For example:

```
sage: m = ([0,-1],) # A Steinberg representation.
sage: T = ([0,+1], [1,-1], [1,-1]) # A tempered representation.
sage: (m,T) # An Irreducible representation $L(m,T)$.
```

$$((\left[0,-1\right]),(\left[0,1\right],\left[1,-1\right],\left[1,-1\right]))$$

```
sage: D(1,m,T) # The highest $|\cdot|^1$-derivative of $L(m,T)$
    .
```

$$(2,(\left[0,-1\right]),(\left[0,1\right],\left[0,1\right],\left[0,1\right]))$$

```
sage: S(1,m,T) # The socle of $|\cdot|^x \rtimes L(m,T)$.
```

$$((),(\left[0,1\right],\left[1,-1\right],\left[1,-1\right],\left[1,-1\right],\left[1,-1\right]))$$

This means that $\pi=L(\Delta[0,-1];\pi(S_1+S_3+S_3,(+,-,-)))$, then

$$D^{\max}_{|\cdot|^1}(\pi)=D^{(2)}_{|\cdot|^1}(\pi)=L(\Delta[0,-1];\pi(S_1+S_1+S_1,(+,+,+))),$$
$$\mathrm{soc}(|\cdot|^1\rtimes\pi)=\pi(S_1+S_3+S_3+S_3+S_3,(+,-,-,-,-)).$$

When $x = 0$, both $D^{\max}_{|\cdot|^0}(\pi)$ and $\mathrm{soc}(|\cdot|^0 \rtimes \pi)$ can be reducible. Instead of them, in [5, Section 3.4], we defined the $\Delta[0,-1]$-*derivative* $D^{(k)}_{\Delta[0,-1]}(\pi)$ and the $Z[0,1]$-*derivative* $D^{(k)}_{Z[0,1]}(\pi)$ by

$$[\mathrm{Jac}_{P_{2k}}(\pi)] = \Delta[0,-1]^k \otimes D^{(k)}_{\Delta[0,-1]}(\pi) + Z[0,1]^k \otimes D^{(k)}_{Z[0,1]}(\pi) + (\text{others}).$$

The highest derivatives $D^{\max}_{\Delta[0,-1]}(\pi)$ and $D^{\max}_{Z[0,1]}(\pi)$ are defined similarly as in the cuspidal case.

Let π be an irreducible unipotent representation of G_n of good parity. We say that π is $|\cdot|^x$-*reduced* if $D^{(1)}_{|\cdot|^x}(\pi) = 0$. By [5, Proposiiton 3.7], if π is $|\cdot|^{-1}$-reduced (resp. $|\cdot|^1$-reduced), then $D^{\max}_{\Delta[0,-1]}(\pi)$ and $\mathrm{soc}(\Delta[0,-1]^r \rtimes \pi)$ (resp. $D^{\max}_{Z[0,1]}(\pi)$ and $\mathrm{soc}(Z[0,1]^r \rtimes \pi)$) are irreducible. Explicit formulas for them are given in [5, Proposition 3.8, Section 8] and [3, Appendix A]. For example:

```
sage: m = ([0,-2],) # A Steinberg representation.
sage: T = ([0,-1], [0,-1], [1,+1]) # A tempered representation.
sage: (m,T) # An irreducible representation $L(m,T)$.
```

$$(([0,-2]),([0,-1],[0,-1],[1,1]))$$

```
sage: # Check that $L(m,T)$ is $|\cdot|^{-1}$-reduced.
sage: D(-1,m,T)[0] == 0
```

True

```
sage: D00(m,T) # The highest $\Delta[0,-1]$-derivative of $L(m,
   T)$.
```

$$(1,([-2,-2]),([0,-1],[0,-1],[1,1]))$$

```
sage: S00(m,T) # The socle of $\Delta[0,-1] \times L(m,T)$.
```

$$(([0,-2],[0,-1]),([0,-1],[0,-1],[1,1]))$$

```
sage: # Check that $L(m,T)$ is $|\cdot|^{1}$-reduced.
sage: D(1,m,T)[0] == 0
```

True

```
sage: D01(m,T) # The highest $Z[0,1]$-derivative of $L(m,T)$.
```

$$(1,([0,-2]),([0,1]))$$

```
sage: S01(1,m,T) # The socle of $Z[0,1]^1 \times L(m,T)$.
```

$$(([0,-2],[0,-1]),([0,1],[0,1],[1,1]))$$

This means that if $\pi = L(\Delta[0,-2], \pi(S_1 + S_1 + S_3, (-,-,+)))$, then π is both $|\cdot|^{-1}$-reduced and $|\cdot|^1$-reduced, and

$$\begin{aligned}
D^{\max}_{\Delta[0,-1]}(\pi) = D^{(1)}_{\Delta[0,-1]}(\pi) &= L(|\cdot|^{-2}, \pi(S_1 + S_1 + S_3, (-,-,+))),\\
\mathrm{soc}(\Delta[0,-1] \rtimes \pi) &= L(\Delta[0,-2], \Delta[0,-1], \pi(S_1 + S_1 + S_3, (-,-,+))),\\
D^{\max}_{Z[0,1]}(\pi) = D^{(1)}_{Z[0,1]}(\pi) &= L(\Delta[0,-2], \pi(S_1,+)),\\
\mathrm{soc}(Z[0,1] \rtimes \pi) &= L(\Delta[0,-2], \Delta[0,-1], \pi(S_1 + S_1 + S_3, (+,+,+))).
\end{aligned}$$

2.5. **Aubert duality.** In this subsection, the following command is explained:

- `AD(m,T)`: the Aubert dual of $L(\mathfrak{m}, T) \in \mathrm{Irr}(G_n)$.

Aubert [6] defined an involution $\pi \mapsto \hat{\pi}$ on $\mathrm{Irr}(G_n)$. In [5, Algorithm 4.1], we established an algorithm to compute $\hat{\pi}$. For example:

```
sage: m = ([0,-2],) # A Steinberg representation.
sage: T = ([0,-1], [0,-1], [1,+1]) # A tempered representation.
sage: (m,T) # An irreducible representation $L(m,T)$.
```

$$(([0,-2]), ([0,-1], [0,-1], [1,1]))$$

```
sage: AD(m,T) # The Aubert dual of $L(m,T)$.
```

$$(([0,-2]), ([0,-1], [0,-1], [1,1]))$$

```
sage: AD(m,T) == (m,T)
```

True

This means that if $\pi = L(\Delta[0,-2], \pi(S_1 + S_1 + S_3, (-,-,+)))$, then $\hat{\pi} \cong \pi$.

3. Extended multi-segments

In this section, we explain the notion of extended multi-segments and related topics.

3.1. **Definition symbols and associated representations.** In this subsection, the following command are explained:

- `symbol(E)`: the symbol associated to an extended multi-segment $\mathcal{E}$;
- `nonzero(E,+1)`: the non-vanishing criterion for $\pi(\mathcal{E})$;
- `rep(E)`: the representation $\pi(\mathcal{E})$ associated to $\mathcal{E}$.

Recall that $G_n = \mathrm{SO}_{2n+1}(F)$ or $G_n = \mathrm{Sp}_{2n}(F)$.

Definition 3.1 ([2, Definition 3.1])**.** *A* unipotent extended multi-segment *for G_n is a weak equivalence class of multi-set of the form*

$$\mathcal{E} = \{([A_i, B_i], l_i, \eta_i)\}_{0 \le i \le t-1},$$

where

- *$[A_i, B_i]$ is a segment;*
- *$A_i \in \mathbb{Z}$ (resp. $A_i \in (1/2)\mathbb{Z} \setminus \mathbb{Z}$) if $G_n = \mathrm{Sp}_{2n}(F)$ (resp. $G_n = \mathrm{SO}_{2n+1}(F)$);*
- *$a_i = A_i + B_i + 1$ and $b_i = A_i - B_i + 1$ are positive integers such that*
$$\sum_{i=0}^{t-1} a_i b_i = \begin{cases} 2n+1 & \text{if } G_n = \mathrm{Sp}_{2n}(F), \\ 2n & \text{if } G_n = \mathrm{SO}_{2n+1}(F); \end{cases}$$
- *if $A_i < A_j$ and $B_i < B_j$, then $i < j$;*
- *$l_i \in \mathbb{Z}$ with $0 \le l_i \le b_i/2$;*
- *$\eta_i \in \{\pm 1\}$; and*
- *a sign condition $\prod_{i=0}^{t-1} (-1)^{[\frac{b_i}{2}] + l_i} \eta_i^{b_i} = 1$ holds.*

Here, $\mathcal{E} = \{([A_i, B_i], l_i, \eta_i)\}_{0 \le i \le t-1}$ and $\mathcal{E}' = \{([A'_i, B'_i], l'_i, \eta'_i)\}_{0 \le i \le t'-1}$ are weak equivalent *if*

- *$t = t'$, $[A_i, B_i] = [A'_i, B'_i]$, $l_i = l'_i$; and*
- *if $l_i = l'_l < b_i/2$, then $\eta_i = \eta'_i$.*

We associate a symbol to an extended multi-segment $\mathcal{E} = \{([A_i, B_i], l_i, \eta_i)\}_{0 \le i \le t-1}$ as follows. When $t = 1$ so that $\mathcal{E} = \{([A, B], l, \eta)\}$, we write

$$\mathcal{E} = \left(\underbrace{\overset{B}{\lhd} \lhd \cdots \overset{B+l-1}{\lhd}}_{l} \; \overset{B+l}{\odot} \odot \cdots \odot \overset{A-l}{\odot} \; \underbrace{\overset{A-l+1}{\rhd} \cdots \rhd \overset{A}{\rhd}}_{l} \right),$$

where $\odot$ is replaced with $\oplus$ and $\ominus$ alternately, starting with $\oplus$ if $\eta = +1$ (resp. $\ominus$ if $\eta = -1$). In general, we put each symbol vertically. For the meaning of "vertically", see the following example.

Example 3.2. (1) *If*

$$\mathcal{E}_1 = \{([2, -1], 1, -1), ([3, 0], 1, +1), ([2, 1], 1, +1)\},$$

then the symbol is

$$\begin{array}{c} \begin{array}{ccccc} -1 & 0 & 1 & 2 & 3 \end{array} \\ \mathcal{E}_1 = \left(\begin{array}{ccccc} \lhd & \ominus & \oplus & \rhd & \\ & \lhd & \oplus & \ominus & \rhd \\ & & \lhd & \rhd & \end{array} \right). \end{array}$$

(2) *If*

$$\mathcal{E}_2 = \left\{([\tfrac{5}{2}, -\tfrac{5}{2}], 2, +1), ([\tfrac{1}{2}, -\tfrac{1}{2}], 1, +1), ([\tfrac{3}{2}, \tfrac{3}{2}], 0, -1)\right\},$$

then

$$\begin{array}{c} \begin{array}{cccccc} -5/2 & -3/2 & -1/2 & 1/2 & 3/2 & 5/2 \end{array} \\ \mathcal{E}_2 = \left(\begin{array}{cccccc} \lhd & \lhd & \oplus & \ominus & \rhd & \rhd \\ & & \lhd & \rhd & & \\ & & & & \ominus & \end{array} \right). \end{array}$$

In [2, Section 3.2], for a unipotent extended multi-segment $\mathcal{E}$ for G_n, we defined a representation $\pi(\mathcal{E})$ of G_n. It is irreducible unipotent of good parity, or zero. By [2, Theorems 3.6, 4.4], we have a combinatorial criterion for $\pi(\mathcal{E}) \neq 0$. Since $\pi(\mathcal{E})$ is defined by using derivatives, one can compute its Langlands data when it is nonzero.

When $\mathcal{E} = \{([A_i, B_i], l_i, \eta_i)\}_{0 \le i \le t-1}$ satisfies that

$$(\mathcal{P}') \qquad B_i < B_j \implies i < j,$$

the representation $\pi(\mathcal{E})$ can be computed by Sage. For example:

```
sage: E1 = (([2,-1],1,-1), ([3,0],1,+1), ([2,1],1,+1)) # An
   extended multi-segment.
sage: symbol(E1) # The symbol associated to $E_1$
[['-1' '0' '1' '2' '3']
 [' <' '-' '+' '>' ' ']
 ['  ' '<' '+' '-' '>']
 ['  ' ' ' '<' '>' ' ']]
sage: nonzero(E1,+1) # Is $\pi(E1) \not= 0$?
True
sage: rep(E1) # The representation associated to E_1.
(([-1,-2],[0,-2],[1,-3]),([0,-1],[1,1],[1,1],[2,-1]))
```

```
sage: E2 = (([5/2,-5/2],2,+1), ([1/2,-1/2],1,+1),                41
    ([3/2,3/2],0,-1)) # An extended multi-segment.
sage: symbol(E2) # The symbol associated to $E_2$.                42
```

```
[['-2.5' '-1.5' '-0.5' '0.5' '1.5' '2.5']
 ['  <  ' '  <  ' '  +  ' '  -  ' '  >  ' '  >  ']
 ['     ' '     ' '  <  ' '  >  ' '     ' '     ']
 ['     ' '     ' '     ' '     ' '  -  ' '     ']]
```

```
sage: nonzero(E2,+1) # Is $\pi(E2) \not= 0$?                      43
```

True

```
sage: rep(E2) # The representation associated to E_2.             44
```

$$\left(\left(\left[-\frac{5}{2},-\frac{5}{2}\right],\left[-\frac{1}{2},-\frac{3}{2}\right]\right),\left(\left[\frac{1}{2},-1\right],\left[\frac{3}{2},-1\right]\right)\right)$$

This means that both $\pi(\mathcal{E}_1)$ and $\pi(\mathcal{E}_2)$ are nonzero, and

$$\pi(\mathcal{E}_1) = L(\Delta[-1,-2], \Delta[0,-2], \Delta[1,-3]; \pi(S_1+S_3+S_3+S_5, (-,+,+,-))),$$
$$\pi(\mathcal{E}_2) = L(|\cdot|^{-\frac{5}{2}}, \Delta[-\tfrac{1}{2},-\tfrac{3}{2}]; \pi(S_2+S_4,(-,-))).$$

Note that the second variable of `nonzero(E,e)` is for the sign condition, i.e., it requires

$$\prod_{i=0}^{t-1}(-1)^{[\frac{b_i}{2}]+l_i}\eta_i^{b_i} = e.$$

If one considers the case where G_n is not quasi-split in the future, one would need `nonzero(E,-1)`.

3.2. **Duals.** In this subsection, the following commands are explained:

- `hat(E)`: the dual $\hat{\mathcal{E}}$ of $\mathcal{E}$;
- `dual_rep(E)`: the representation $\hat{\pi}(\mathcal{E}) \cong \pi(\hat{\mathcal{E}})$.

In [2, Definition 6.1], for a unipotent extended multi-segment $\mathcal{E}$ for G_n satisfying $(\mathcal{P}')$, we defined another unipotent extended multi-segment $\hat{\mathcal{E}}$ for G_n. By [2, Theorem 6.2], if $\pi(\mathcal{E}) \neq 0$, then its Aubert dual is isomorphic to $\pi(\hat{\mathcal{E}})$. For example:

```
sage: E1                                                          45
```

$$(([2,-1],1,-1),([3,0],1,1),([2,1],1,1))$$

```
sage: hat(E1) # The dual of E1.                                   46
```

$$(([2,-1],2,1),([3,0],1,1),([2,1],0,-1))$$

```
sage: symbol(hat(E1)) # The symbol associated to hat(E1).         47
```

```
[['-1' '0' '1' '2' '3']
 [' <' '<' '>' '>' ' ']
 ['  ' '<' '+' '-' '>']
 ['  ' ' ' '-' '+' ' ']]
```

```
sage: dual_rep(E1) # The Aubert dual of rep(E_1).                 48
```

$$(([-1,-2],[0,-3],[0,-1],[1,-2]),([1,1],[2,1]))$$

```
sage: dual_rep(E1) == AD(rep(E1)[0], rep(E1)[1]) # Check          49
```

True

```
sage: E2                                                          50
```

$$\left(\left(\left[\frac{5}{2},-\frac{5}{2}\right],2,1\right),\left(\left[\frac{1}{2},-\frac{1}{2}\right],1,1\right),\left(\left[\frac{3}{2},\frac{3}{2}\right],0,-1\right)\right)$$

```
sage: hat(E2) # The dual of E2.                                        51
```

$$\left(\left(\left[\frac{3}{2},-\frac{3}{2}\right],1,1\right),\left(\left[\frac{1}{2},\frac{1}{2}\right],0,-1\right),\left(\left[\frac{5}{2},\frac{5}{2}\right],0,1\right)\right)$$

```
sage: symbol(hat(E2)) # The symbol associated to hat(E2).              52
[['-1.5' '-0.5' '0.5' '1.5' '2.5']
 [' < ' ' + ' ' - ' ' > ' '  ']
 ['   ' '   ' ' - ' '   ' '  ']
 ['   ' '   ' '   ' '   ' ' + ']]
sage: dual_rep(E2) # The Aubert dual of rep(E_2).                       53
```

$$\left(\left(\left[-\frac{3}{2},-\frac{3}{2}\right]\right),\left(\left[\frac{1}{2},-1\right],\left[\frac{1}{2},-1\right],\left[\frac{5}{2},1\right]\right)\right)$$

```
sage: dual_rep(E2) == AD(rep(E2)[0], rep(E2)[1]) # Check               54
True
```

Hence we have

$$\hat{\pi}(\mathcal{E}_1)=\pi(\hat{\mathcal{E}}_1)=L(\Delta[-1,-2],\Delta[0,-3],\Delta[0,-1],\Delta[1,-2];\pi(S_3+S_5,(+,+))),$$
$$\hat{\pi}(\mathcal{E}_2)=\pi(\hat{\mathcal{E}}_2)=L(|\cdot|^{-\frac{3}{2}};\pi(S_2+S_2+S_6;(-,-,+))).$$

Remark that the Aubert dual of $\pi(\mathcal{E})$ can be also computed by combining `rep(E)` with the more general command `AD(m,T)`, but it is much slower than `dual_rep(E)`.

3.3. **Changing orders.** In this subsection, the following commands are explained:

- `change(E,i)`: swapping indices i and $i+1$ of $\mathcal{E}$;
- `orders(E)`: the set of $\mathcal{E}'$ given from $\mathcal{E}$ by swapping indices repeatedly.

Let $\mathcal{E}=\{([A_i,B_i],l_i,\eta_i)\}_{0\leq i\leq t-1}$ be an extended multi-segment. By definition, we assume

$$(\mathcal{P}) \qquad A_i<A_j, B_i<B_j \implies i<j.$$

In other words, if $[A_i,B_i]\supset[A_{i+1},B_{i+1}]$ or $[A_i,B_i]\subset[A_{i+1},B_{i+1}]$, one can swap the indices i and $i+1$. If this is done, $l_i,\eta_i,l_{i+1},\eta_{i+1}$ are changed as in [10, Theorem 6.1]. The command `change(E,i)` swaps the indices i and $i+1$ if it can. For example:

```
sage: E1                                                                55
```

$$(([2,-1],1,-1),([3,0],1,1),([2,1],1,1))$$

```
sage: change(E1,0) == E1 # The indices 0 and 1 cannot be                56
   swapped.
True
sage: change(E1,1) # Swap indices 1 and 2.                             57
```

$$(([2,-1],1,-1),([2,1],1,-1),([3,0],1,1))$$

```
sage: symbol(E1)                                                        58
[['-1' '0' '1' '2' '3']
 ['<' '-' '+' '>' ' ']
 [' ' '<' '+' '-' '>']
 [' ' ' ' '<' '>' ' ']]
```

```
sage: symbol(change(E1,1))
```

```
[['-1' '0' '1' '2' '3']
 ['<' '-' '+' '>' ' ']
 [' ' ' ' '<' '>' ' ']
 [' ' '<' '+' '-' '>']]
```

```
sage: E2
```

$$\left(\left(\left[\frac{5}{2},-\frac{5}{2}\right],2,1\right),\left(\left[\frac{1}{2},-\frac{1}{2}\right],1,1\right),\left(\left[\frac{3}{2},\frac{3}{2}\right],0,-1\right)\right)$$

```
sage: change(E2,0) # Swap indices 0 and 1.
```

$$\left(\left(\left[\frac{1}{2},-\frac{1}{2}\right],1,-1\right),\left(\left[\frac{5}{2},-\frac{5}{2}\right],2,1\right),\left(\left[\frac{3}{2},\frac{3}{2}\right],0,-1\right)\right)$$

```
sage: symbol(E2)
```

```
[['-2.5' '-1.5' '-0.5' '0.5' '1.5' '2.5']
 [' < ' ' < ' ' + ' ' - ' ' > ' ' > ']
 ['   ' '   ' ' < ' ' > ' '   ' '   ']
 ['   ' '   ' '   ' '   ' ' _ ' '   ']]
```

```
sage: symbol(change(E2,0))
```

```
[['-2.5' '-1.5' '-0.5' '0.5' '1.5' '2.5']
 ['   ' '   ' ' < ' ' > ' '   ' '   ']
 [' < ' ' < ' ' + ' ' - ' ' > ' ' > ']
 ['   ' '   ' '   ' '   ' ' _ ' '   ']]
```

The set of extended multi-segments $\mathcal{E}'$ given from $\mathcal{E}$ by swapping indices repeatedly can be computed by the command `orders(E)`. For example:

```
sage: E1
```

$$(([2,-1],1,-1),([3,0],1,1),([2,1],1,1))$$

```
sage: len(orders(E1)) # Cardinality of orders(E1).
```

$$3$$

```
sage: for E in orders(E1):
....:     symbol(E)
```

```
[['-1' '0' '1' '2' '3']
 ['<' '-' '+' '>' ' ']
 [' ' '<' '+' '-' '>']
 [' ' ' ' '<' '>' ' ']]

[['-1' '0' '1' '2' '3']
 ['<' '-' '+' '>' ' ']
 [' ' ' ' '<' '>' ' ']
 [' ' '<' '+' '-' '>']]

[['-1' '0' '1' '2' '3']
 [' ' ' ' '<' '>' ' ']
 ['<' '-' '+' '>' ' ']
 [' ' '<' '+' '-' '>']]
```

```
sage: E2
```

$$\left(\left(\left[\frac{5}{2},-\frac{5}{2}\right],2,1\right),\left(\left[\frac{1}{2},-\frac{1}{2}\right],1,1\right),\left(\left[\frac{3}{2},\frac{3}{2}\right],0,-1\right)\right)$$

```
sage: len(orders(E2)) # Cardinality of orders(E2).
```

$$3$$

```
sage: for E in orders(E2):
....:     symbol(E)
```

```
[['-2.5' '-1.5' '-0.5' '0.5' '1.5' '2.5']
 [' < ' ' < ' ' + ' ' - ' ' > ' ' > ']
 ['   ' '   ' ' < ' ' > ' '   ' '   ']
 ['   ' '   ' '   ' '   ' ' - ' '   ']]
[['-2.5' '-1.5' '-0.5' '0.5' '1.5' '2.5']
 ['   ' '   ' ' < ' ' > ' '   ' '   ']
 [' < ' ' < ' ' + ' ' - ' ' > ' ' > ']
 ['   ' '   ' '   ' '   ' ' - ' '   ']]
[['-2.5' '-1.5' '-0.5' '0.5' '1.5' '2.5']
 ['   ' '   ' ' < ' ' > ' '   ' '   ']
 ['   ' '   ' '   ' '   ' ' + ' '   ']
 [' < ' ' < ' ' < ' ' > ' ' > ' ' > ']]
```

3.4. **Strongly equivalence classes.** In this subsection, the following command is explained:

- `eq_class(E)`: the strongly equivalence class of $\mathcal{E}$.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two extended multi-segment. Suppose that $\pi(\mathcal{E}) \neq 0$. We say that $\mathcal{E}$ and $\mathcal{E}'$ are *strongly equivalent* if $\pi(\mathcal{E}) \cong \pi(\mathcal{E}')$. By [4, Theorem 3.5], such $\mathcal{E}'$ can be obtained from $\mathcal{E}$ by a finite chain of three operations (C), (UI) and (P). Here, (C) is the changing orders explained in the previous subsection. When $\pi(\mathcal{E}) \neq 0$, the command `eq_class(E)` gives the strongly equivalence class of $\mathcal{E}$ up to the changing orders. For example:

```
sage: symbol(E1)
```

```
[['-1' '0' '1' '2' '3']
 ['<' '-' '+' '>' ' ']
 [' ' '<' '+' '-' '>']
 [' ' ' ' '<' '>' ' ']]
```

```
sage: C1 = eq_class(E1) # The strongly equivalence class of E_1.
sage: len(C1) # Cardinality of C1.
```

$$4$$

```
sage: for E in C1:
....:     symbol(E)
....:     rep(E1) == rep(E)
```

```
[['-1' '0' '1' '2' '3']
 [' <' '-' '+' '>' ' ']
 [' ' '<' '+' '-' '>']
 [' ' ' ' '<' '>' ' ']]
True
[['-1' '0' '1' '2' '3']
 [' <' '-' '+' '>' ' ']
 [' ' '<' '+' '>' ' ']
 [' ' ' ' '<' '-' '>']]
True
[['-2' '-1' '0' '1' '2' '3']
 [' <' ' <' '-' '>' '>' ' ']
 [' ' ' <' '+' '>' ' ' ' ']
 [' ' '  ' '<' '+' '-' '>']
 [' ' '  ' ' ' '<' '>' ' ']]
True
[['-2' '-1' '0' '1' '2' '3']
 [' <' ' <' '-' '>' '>' ' ']
 [' ' ' <' '+' '>' ' ' ' ']
 [' ' '  ' '<' '+' '>' ' ']
 [' ' '  ' ' ' '<' '-' '>']]
True
```

```
sage: symbol(E2)
    [['-2.5' '-1.5' '-0.5' '0.5' '1.5' '2.5']
     ['  < ' '  < ' '  + ' '  - ' '  > ' '  > ']
     ['    ' '    ' '  < ' '  > ' '    ' '    ']
     ['    ' '    ' '    ' '    ' '  _ ' '    ']]
sage: C2 = eq_class(E2) # The strongly equivalence class of E_2
    .
sage: len(C2) # Cardinality of C2.
    1
```

4. Local A-packets

In this section, we construct local A-packets.

4.1. **Overview.** An A-parameter for G_n is a homomorphism

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G_n}$$

such that $\psi(W_F)$ is bounded. We call ψ *unipotent* if $\psi|_{W_F} = \mathbf{1}$. Such an A-parameter is regarded as a (self-dual) representation of $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ so that we can write $\psi =$

$\oplus_{i=0}^{t-1} S_{a_i} \boxtimes S_{b_i}$. In this document, we identify $\psi = \oplus_{i=0}^{t-1} S_{a_i} \boxtimes S_{b_i}$ with

$$\psi = ((a_0, b_0), (a_1, b_1), \dots, (a_{t-1}, b_{t-1})).$$

Note that if $b_i = 1$ for all i, then $\psi = \phi$ is a tempered L-parameter.

We say that $\psi = \oplus_{i=0}^{t-1} S_{a_i} \boxtimes S_{b_i}$ is *of good parity* if

$$a_i + b_i \equiv \begin{cases} 1 \bmod 2 & \text{if } G_n = \mathrm{SO}_{2n+1}(F), \\ 0 \bmod 2 & \text{if } G_n = \mathrm{Sp}_{2n}(F) \end{cases}$$

for all $0 \leq i \leq t-1$. In this case, we define the *enhanced component group* of ψ by $\mathcal{A}_\psi = \oplus_{i=0}^{t-1} (\mathbb{Z}/2\mathbb{Z})\alpha_i$. The *component group* $\mathcal{S}_\psi$ of ψ is defined as the quotient of $\mathcal{A}_\psi$ by the subgroup generated by

- $\alpha_i + \alpha_j$ for $(a_i, b_i) = (a_j, b_j)$; and
- $z_\psi = \alpha_0 + \cdots + \alpha_{t-1}$.

Let $\widehat{\mathcal{A}_\psi}$ and $\widehat{\mathcal{S}_\psi}$ denote the Pontryagin duals of $\mathcal{A}_\psi$ and $\mathcal{S}_\psi$, respectively. Note that $\widehat{\mathcal{S}_\psi} \subset \widehat{\mathcal{A}_\psi}$. For $\varepsilon \in \widehat{\mathcal{A}_\psi}$, we write $\varepsilon(a_i, b_i) = \varepsilon(\alpha_i)$.

Let $\mathrm{Irr}_{\mathrm{unit}}(G_n)$ be the set of equivalence classes of irreducible unitary representation of G_n. By [1, Theorem 1.5,1], for an A-parameter ψ for G_n, Arthur defined a finite subset $\Pi_\psi \subset \mathrm{Irr}_{\mathrm{unit}}(G_n)$ together with a map

$$\Pi_\psi \to \widehat{\mathcal{S}_\psi},\ \pi \mapsto \langle \cdot, \pi \rangle_\psi .$$

4.2. **Construction.** In this subsection, the following commands are explained:

- `dim(psi)`: the dimension of an A-parameter ψ;
- `A_packet(psi,+1)`: the A-packet associated to ψ;
- `char(E)`: the character of the component group associated to $\mathcal{E}$;
- `par(E)`: the A-parameter associated to $\mathcal{E}$.

Let $\mathcal{E} = \{([A_i, B_i], l_i, \eta_i)\}_{0 \leq i \leq t-1}$ be an extended multi-segment for G_n. Then $\mathcal{E}$ gives a unipotent A-parameter of good parity by

$$\psi_{\mathcal{E}} = \oplus_{i=0}^{t-1} S_{A_i+B_i+1} \boxtimes S_{A_i-B_i+1}.$$

By [2, Theorem 3.3], for a unipotent A-parameter ψ of good parity for G_n, its A-packet is given as

$$\Pi_\psi = \{\pi(\mathcal{E}) \mid \psi_{\mathcal{E}} \cong \psi\} \setminus \{0\}.$$

Moreover, by [2, Definition 3.4, Theorem 3.5], $\mathcal{E}$ defines a character $\eta_{\mathcal{E}}$ of $\mathcal{S}_{\psi_{\mathcal{E}}}$ such that

$$\langle \cdot, \pi(\mathcal{E}) \rangle_{\psi_{\mathcal{E}}} = \eta_{\mathcal{E}}.$$

Using these results, A-packets can be constructed in Sage. For example:

```
sage: psi1 = ((2,2), (5,3)) # An $A$-parameter.
sage: dim(psi1) # The dimension of psi1.
19
sage: Pi1 = A_packet(psi1,+1) # The $A$-packet associated to
    psi1.
sage: len(Pi1) # Cardinality of Pi1.
5
```

```
sage: for E in Pi1:
....:     symbol(E)
....:     rep(E)
....:     char(E)
....:     print("")
```

```
[['0' '1' '2' '3']
 ['<' '>' ' ' ' ']
 [' ' '<' '+' '>']]
```

$(([0,-1],[1,-3]),([2,1]))$

$([1,1],((2,2),(5,3)))$

None

```
[['0' '1' '2' '3']
 ['-' '+' ' ' ' ']
 [' ' '+' '-' '+']]
```

$((),([0,-1],[1,1],[1,1],[2,-1],[3,1]))$

$([-1,-1],((2,2),(5,3)))$

None

```
[['0' '1' '2' '3']
 ['<' '>' ' ' ' ']
 [' ' '-' '+' '-']]
```

$(([0,-1]),([1,-1],[2,1],[3,-1]))$

$([1,1],((2,2),(5,3)))$

None

```
[['0' '1' '2' '3']
 ['+' '-' ' ' ' ']
 [' ' '<' '-' '>']]
```

$(([1,-3]),([0,1],[1,-1],[2,-1]))$

$([-1,-1],((2,2),(5,3)))$

None

```
[['0' '1' '2' '3']
 ['-' '+' ' ' ' ']
 [' ' '<' '-' '>']]
```

$(([1,-3]),([0,-1],[1,1],[2,-1]))$

$([-1,-1],((2,2),(5,3)))$

None

```
sage: psi2 = ((1,2), (1,2), (2,1), (2,1)) # An $A$-parameter.
sage: dim(psi2) # The dimension of psi2.
```

8

```
sage: Pi2 = A_packet(psi2,+1) # The $A$-packet associated to
   $psi2$.
sage: len(Pi2) # Cardinality of Pi2.
```

3

```
sage: for E in Pi2:
....:     symbol(E)
....:     rep(E)
....:     char(E)
....:     print("")
```

```
[['-0.5' '0.5']
 ['  + ' ' - ']
 ['  - ' ' + ']
 ['    ' ' + ']
 ['    ' ' + ']]
```

$$\left(\left(\left[-\tfrac{1}{2},-\tfrac{1}{2}\right]\right),\left(\left[\tfrac{1}{2},1\right],\left[\tfrac{1}{2},1\right],\left[\tfrac{1}{2},1\right]\right)\right)$$

$$\left(\left[-1,-1,1,1\right],\left(\left(1,2\right),\left(1,2\right),\left(2,1\right),\left(2,1\right)\right)\right)$$

None

```
[['-0.5' '0.5']
 ['  < ' ' > ']
 ['  < ' ' > ']
 ['    ' ' + ']
 ['    ' ' + ']]
```

$$\left(\left(\left[-\tfrac{1}{2},-\tfrac{1}{2}\right],\left[-\tfrac{1}{2},-\tfrac{1}{2}\right]\right),\left(\left[\tfrac{1}{2},1\right],\left[\tfrac{1}{2},1\right]\right)\right)$$

$$\left(\left[1,1,1,1\right],\left(\left(1,2\right),\left(1,2\right),\left(2,1\right),\left(2,1\right)\right)\right)$$

None

```
[['-0.5' '0.5']
 ['  < ' ' > ']
 ['  < ' ' > ']
 ['    ' ' - ']
 ['    ' ' - ']]
```

$$\left(\left(\left[-\tfrac{1}{2},-\tfrac{1}{2}\right],\left[-\tfrac{1}{2},-\tfrac{1}{2}\right]\right),\left(\left[\tfrac{1}{2},-1\right],\left[\tfrac{1}{2},-1\right]\right)\right)$$

$$\left(\left[1,1,-1,-1\right],\left(\left(1,2\right),\left(1,2\right),\left(2,1\right),\left(2,1\right)\right)\right)$$

None

```
sage: psi3 = ((51,31), (31,45), (13,5)) # An $A$-parameter.
sage: Pi3 = A_packet(psi3,+1) # The $A$-packet associated to
   psi3.
sage: len(Pi3) # Cardinality of Pi3.
```

$$1651$$

Here, `char(E)` $= ([e_0, \ldots, e_{t-1}], ((a_0, b_0), \ldots, (a_{t-1}, b_{t-1})))$ means that $\eta_{\mathcal{E}}(a_i, b_i) = e_i$ for $0 \leq i \leq t-1$. By looking at results for `psi1` and `psi2` in this example, we see that the map $\Pi_\psi \to \widehat{\mathcal{S}_\psi}$ is not necessarily injective nor surjective.

By the command `par(E)`, we can get the A-parameter $\psi_{\mathcal{E}}$ associated to $\mathcal{E}$. Combining the command `eq_class(E)`, we can list all A-parameters ψ such that $\pi(\mathcal{E}) \in \Pi_\psi$. For example:

```
sage: rep(E1)
```

$$(([-1,-2],[0,-2],[1,-3]),([0,-1],[1,1],[1,1],[2,-1]))$$

```
sage: C1 = eq_class(E1)
sage: for E in C1:
....:     par(E)
....:     rep(E1) in [rep(F) for F in A_packet(par(E),+1)]
```

$$((2,4),(4,4),(4,2))$$
True
$$((2,4),(3,3),(5,3))$$
True
$$((1,5),(1,3),(4,4),(4,2))$$
True
$$((1,5),(1,3),(3,3),(5,3))$$
True

4.3. **Determination of Arthur type representations.** In this subsection, the following command is explained:

- `Is_Arthur(m,T)`: Determination whether $L(\mathfrak{m}, T)$ is of Arthur type or not.

Let

$$\pi = L(\Delta[x_0, y_0], \ldots, \Delta[x_{r-1}, y_{r-1}]; \pi([\tfrac{a_0-1}{2}, \varepsilon_0], \ldots, [\tfrac{a_{t-1}-1}{2}, \varepsilon_{t-1}]))$$

be an irreducible unipotent representation of G_n of good parity. In [4, Algorithm 3.3], we gave an algorithm to determine whether π is of Arthur type or not, i.e., there is an A-parameter ψ such that $\pi \in \Pi_\psi$ or not. The command `Is_Arthur(m,T)` tells us the answer, the number of ψ such that $\pi \in \Pi_\psi$, and an extended multi-segment $\mathcal{E}$ such that $\pi \cong \pi(\mathcal{E})$ if it exists. For example:

```
sage: m = ([-2,-3], [-1,-1])
sage: phi = ((1,1), (3,1), (5,1))
sage: Pi = L_packet(phi,+1)
sage: for T in Pi:
....:     (m,T)
....:     Is_Arthur(m,T)
....:     print("")
```

$(([-2,-3],[-1,-1]),([0,1],[1,1],[2,1]))$

$(\text{False}, 0, ())$

None

$(([-2,-3],[-1,-1]),([0,-1],[1,-1],[2,1]))$

$(\text{False}, 0, ())$

None

$(([-2,-3],[-1,-1]),([0,-1],[1,1],[2,-1]))$

$(\text{True}, 2, (([3,-2],2,-1),([1,1],0,-1)))$

None

$(([-2,-3],[-1,-1]),([0,1],[1,-1],[2,-1]))$

$(\text{False}, 0, ())$

None

```
sage: T = Pi[2]
sage: (m,T)
```

$(([-2,-3],[-1,-1]),([0,-1],[1,1],[2,-1]))$

```
sage: E0 = Is_Arthur(m,T)[2]
sage: C0 = eq_class(E0)
sage: for E in C0:
....:     symbol(E)
....:     rep(E) == (m,T)
....:     print("")
```

```
[['-2' '-1' '0' '1' '2' '3']
 [' <' ' <' '-' '+' '>' '>']
 ['  ' '  ' ' ' '_' ' ' ' ']]
```

True

None

```
[['-3' '-2' '-1' '0' '1' '2' '3']
 [' <' ' <' ' <' '-' '>' '>' '>']
 ['  ' ' <' ' <' '+' '>' '>' ' ']
 ['  ' '  ' '  ' ' ' '_' ' ' ' ']]
```

True

None

5. Socles of certain parabolically induced representations

In [3], we studied the socles (i.e., the maximal semisimple subrepresentations) of parabolically induced representations of the form $u(a,b)|\cdot|^s \rtimes \pi_A$, where

- $u(a,b) = \mathrm{soc}(\Delta[\frac{a-1}{2}, -\frac{a-1}{2}]|\cdot|^{-\frac{b-1}{2}} \times \cdots \times \Delta[\frac{a-1}{2}, -\frac{a-1}{2}]|\cdot|^{\frac{b-1}{2}})$ is a (unitary) Speh representation;
- π_A is an irreducible representation of good parity which is of Arthur type;
- $s \in \mathbb{R}$.

In particular, we can determine the irreducibility and the first reducibility points of these induced representations.

5.1. **Socles.** In this subsection, the following command is explained:

- `soc(s,(a,b),E)`: the socle of $u(a,b)|\cdot|^s \rtimes \pi(\mathcal{E})$.

The command `soc(s,(a,b),E)` gives the set of irreducible representations appearing as subrepresentations of $u(a,b)|\cdot|^s \rtimes \pi(\mathcal{E})$. For example:

```
sage: (a,b) = (4,2)
sage: psi = ((2,2), (5,3))
sage: Pi = A_packet(psi,+1)
sage: for E in Pi:
....:     rep(E)
....:     soc(0,(a,b),E)
....:     print("")
```

$(([0,-1], [1,-3]), ([2,1]))$
$[(([0,-1], [1,-3], [1,-2]), ([1,1], [2,1], [2,1])), (([0,-1], [1,-3], [1,-2], [1,-2]), ([2,1]))]$
None
$((), ([0,-1], [1,1], [1,1], [2,-1], [3,1]))$
$[(([1,-2], [1,-2]), ([0,-1], [1,1], [1,1], [2,-1], [3,1]))]$
None
$(([0,-1]), ([1,-1], [2,1], [3,-1]))$
$[(([0,-1], [1,-2], [1,-2]), ([1,-1], [2,1], [3,-1]))]$
None
$(([1,-3]), ([0,1], [1,-1], [2,-1]))$
$[(([1,-3], [1,-2], [1,-2]), ([0,1], [1,-1], [2,-1])), (([1,-3], [1,-2]), ([0,1], [1,-1], [1,-1], [2,-1], [2,-1]))]$
None
$(([1,-3]), ([0,-1], [1,1], [2,-1]))$
$[(([1,-3], [1,-2], [1,-2]), ([0,-1], [1,1], [2,-1]))]$
None

In particular, we see that

$$\begin{aligned}
&u(4,2) \rtimes L(\Delta[0,-1], \Delta[1,-3]; \pi(S_5,+)) \\
&\quad \cong L(\Delta[0,-1], \Delta[1,-3], \Delta[1,-2]; \pi(S_3+S_5+S_5), (+,+,+)) \\
&\quad \oplus L(\Delta[0,-1], \Delta[1,-3], \Delta[1,-2], \Delta[1,-2]; \pi(S_5,+)), \\
&u(4,2) \rtimes L(\Delta[1,-3]; \pi(S_1+S_3+S_5, (+,-,-)))
\end{aligned}$$

$$\cong L(\Delta[1,-3],\Delta[1,-2],\Delta[1,-2];\pi(S_1+S_3+S_5,(+,-,-)))$$
$$\oplus\, L(\Delta[1,-3],\Delta[1,-2];\pi(S_1+S_3+S_3+S_5+S_5,(+,-,-,-,-))).$$

5.2. **Irreducibility.** In this subsection, the following command is explained:

- `Is_irred(s,(a,b),E)`: the irreducibility of $u(a,b)|\cdot|^s \rtimes \pi(\mathcal{E})$.

By [3, Corollary 5.2], one can determine whether $u(a,b)|\cdot|^s \rtimes \pi_A$ is irreducible or not. For example:

```
sage: (a,b) = (4,2)
sage: psi = ((2,2), (5,3))
sage: Pi = A_packet(psi,+1)
sage: E = Pi[2]
sage: rep(E)
```

$$(([0,-1]),([1,-1],[2,1],[3,-1]))$$

```
sage: for s in range(10):
....:     s
....:     Is_irred(s,(a,b),E)
```

0
True
1
True
2
False
3
False
4
False
5
False
6
False
7
True
8
True
9
True

5.3. **First reducibility points.** In this subsection, the following command is explained:

- `FRP((a,b),E)`: the first reducibility point for $u(a,b)|\cdot|^s \rtimes \pi(\mathcal{E})$.

The *first reducibility point for* $u(a,b)|\cdot|^s \rtimes \pi(\mathcal{E})$ is the minimal non-negative real number s_0 such that $u(a,b)|\cdot|^{s_0} \rtimes \pi(\mathcal{E})$ is reducible. It can be computed in Sage by the command `FRP((a,b),E)`. For example:

```
sage: (a,b) = (4,2)
sage: psi = ((2,2), (5,3))
sage: Pi = A_packet(psi,+1)
sage: for E in Pi:
....:     rep(E)
....:     char(E)
....:     FRP((a,b),E)
....:     print("")
```

$(([0,-1],[1,-3]),([2,1]))$
$([1,1],((2,2),(5,3)))$
0
None
$((),([0,-1],[1,1],[1,1],[2,-1],[3,1]))$
$([-1,-1],((2,2),(5,3)))$
1
None
$(([0,-1]),([1,-1],[2,1],[3,-1]))$
$([1,1],((2,2),(5,3)))$
2
None
$(([1,-3]),([0,1],[1,-1],[2,-1]))$
$([-1,-1],((2,2),(5,3)))$
0
None
$(([1,-3]),([0,-1],[1,1],[2,-1]))$
$([-1,-1],((2,2),(5,3)))$
1
None

Especially, even if $\pi,\pi' \in \Pi_\psi$ with $\langle\cdot,\pi\rangle_\psi = \langle\cdot,\pi'\rangle_\psi$, the first reducibility points for $u(a,b)\rtimes\pi$ and for $u(a,b)\rtimes\pi'$ can be different.

References

[1] J. Arthur, *The endoscopic classification of representations. Orthogonal and symplectic groups. American Mathematical Society Colloquium Publications,* **61**. *American Mathematical Society, Providence, RI,* 2013. xviii+590 pp.

[2] H. Atobe, *Construction of local A-packets*. Preprint (2021), arXiv:2012.07232v2.
[3] H. Atobe, *On the socles of certain parabolically induced representations of p-adic classical groups*. Preprint (2021), arXiv:2109.12819v1.
[4] H. Atobe, *The set of local A-packets containing a given representation*. Preprint (2022), arXiv:2201.00952.
[5] H. Atobe and A. Mínguez, *The explicit Zelevinsky–Aubert duality*. Preprint (2020), arXiv:2008.05689v2.
[6] A.-M. Aubert, *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p-adique*. *Trans. Amer. Math. Soc.* **347** (1995), no. 6, 2179–2189 and *Erratum.* ibid. **348** (1996), 4687–4690.
[7] A. Hazeltine, B. Liu, C.-H. Lo, *On the intersection of local Arthur packets for classical groups*. Preprint (2020), arXiv:2201.10539.
[8] C. Jantzen, *Tempered representations for classical p-adic groups*. *Manuscripta Math.* **145** (2014), no. 3-4, 319–387.
[9] E. Lapid and A. Mínguez, *On parabolic induction on inner forms of the general linear group over a non-archimedean local field*. *Selecta Math. (N.S.)* **22** (2016), no. 4, 2347–2400.
[10] B. Xu, *A combinatorial solution to Mœglin's parametrization of Arthur packets for p-adic quasisplit* $\mathrm{Sp}(N)$ *and* $\mathrm{O}(N)$. *J. Inst. Math. Jussieu* **20** (2021), no. 4, 1091–1204.

Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan
Email address: atobe@math.sci.hokudai.ac.jp