Chapter 2 Solutions

Exercise 2.1.
The first point placed has n degrees of freedom, the next one has one constraint so $n-1$ degrees of freedom, the next has two constraints, etc. So $n+(n-1)+(n-2)+\ldots 1 = n(n+1)/2$. (Get this by summing the outermost pair in the sequence, $n+1 = n+1$, then the pair $(n-1)+2 = n+1$, etc., and observe that there are $n/2$ such pairs.) n of these freedoms are the linear freedoms of placing the first point; the other $n(n-1)/2$ are rotational freedoms. After choosing the first point, the next point is on the sphere S^{n-1}, the next is on S^{n-2}, etc., so the topology of the space is $\mathbb{R}^n \times S^{n-1} \times S^{n-2} \times \ldots \times S^1$.

Exercise 2.2.
(a) The shoulder is a spherical joint (four dof), the elbow has one dof, the wrist has two dof, and between the elbow and the wrist there is one more dof (rotation of the forearm about the axis of the forearm). Therefore the arm has seven dof.
(b) Placing the palm at a fixed position and orientation in space puts six constraints on the arm (the six dof of a rigid body). Keeping the center of the shoulder joint stationary, there is only one dof left: the arc of a circle on which the tip of the elbow can lie. This is one dof, so the arm must have started with seven dof before six constraints were placed on it.

Exercise 2.3.
Treat the shoulder as a spherical joint (three dof) between the torso and the upper arm bone (humerus), and assume the carpal bones just beyond the wrist joint form a rigid body. Then the closed-chain linkage of the forearm between the humerus and the carpal bones, which includes only the radius and the ulna as links, must have four dof, since our solution in the previous exercise tells us that the arm has seven dof.
We know that each of the radius and the ulna must have at least one joint at the proximal (closer to the torso) and distal (closer to the hand) ends of forearm, so there are at least four joints between the humerus and carpal bones. There could be as many as six: three at the elbow (humeroradial, humeroulnar, and proximal radioulnar) and three at the wrist (radiocarpal, ulnocarpal, and distal radioulnar). Without knowing more about the anatomy of the arm, we cannot say for sure.
If we assume the maximum number of joints, six, in the forearm closed chain, then the arm has $J = 7$ joints (the three-dof S joint at the shoulder and the six forearm joints mentioned above) and $N = 5$ links (the torso "ground," the humerus, the ulna, the radius, and the carpal bones). By Grübler's formula,

$$7 = 6(N-1-J) + \sum_{i=1}^{7} f_i = -18 + \text{freedoms of the six forearm joints.}$$

Therefore the six forearm joints must have a total of 25 freedoms. These joints, averaging more than four freedoms each, are not standard joints we have studied. They are stabilized by a complex of ligaments joining the bones.
If we assume the minimum number of joints, four, in the forearm closed chain, then the arm has $J = 5$ joints and $N = 5$ links. By Grübler's formula,

$$7 = 6(N-1-J) + \sum_{i=1}^{5} f_i = -6 + 3 + \text{freedoms of the four forearm joints.}$$

Therefore there must be a total of 10 freedoms at the four forearm joints. These could potentially be joints we have studied, such as two universal joints at the elbow (four dof) and two spherical joints at the wrist (six dof).
The problem is to show correct general reasoning, not to demonstrate a detailed understanding of arm anatomy!

Exercise 2.4.
Once the hands firmly grip the steering wheel, each arm has $n-6$ dof if the wheel is stationary. The mobility

of the wheel adds one dof, though, so the total number of degrees of freedom is $2n - 11$.

Exercise 2.5.

$$\begin{aligned} N &= 6 \text{ (links)} + 1 \text{ (ground)} = 7 \\ J &= 5 \text{ (R joints)} + 2 \text{ (S joints)} = 7 \\ \sum f_i &= 5 \times 1 + 2 \times 3 = 11. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 5.$$

Exercise 2.6.

(a) The wheeled mobile base can be regarded as a rolling coin with C-space $\mathbb{R}^2 \times T^2$. The C-space of a 6R robot arm can be written $S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 = T^6$. The C-space of the wheeled mobile arm therefore can be written $\mathbb{R}^2 \times T^2 \times T^6 = \mathbb{R}^2 \times T^8$.

(b) For this problem, the last link of the 6R robot can be regarded as connected to ground by a revolute joint:

$$\begin{aligned} N &= 6 \text{ (links)} + 1 \text{ (ground)} = 7 \\ J &= 7 \text{ (R joints)} \\ \sum f_i &= 7 \times 1 = 7. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 1.$$

(c) The second identical 6R robot is grasping the last link (i.e., the refrigerator door) of the original 6R robot. In this case,

$$\begin{aligned} N &= 11 \text{ (links)} + 1 \text{ (ground)} = 12 \\ J &= 13 \text{ (R joints)} \\ \sum f_i &= 13 \times 1 = 13. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 1.$$

Exercise 2.7.

(a)

$$\begin{aligned} N &= 6 \text{ (links)} + 1 \text{ (object)} + 1 \text{ (ground)} = 8 \\ J &= 3 \text{ (R joints)} + 6 \text{ (S joints)} = 9 \\ \sum f_i &= 3 \times 1 + 6 \times 3 = 21. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 9.$$

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(b) Consider open chain arms as 7-dof joint connecting object and ground. Then,

$$\begin{aligned} N &= 1\text{ (object)} + 1\text{ (ground)} = 2 \\ J &= n\text{ (open chain arm)} \\ \sum f_i &= n \times 7 = 7n. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = (n + 6).$$

(c) Each of the n 7-dof open chains is replaced by the 6-dof open chains. So,

$$\begin{aligned} N &= 1\text{ (object)} + 1\text{ (ground)} = 2 \\ J &= n\text{ (open chain arm)} \\ \sum f_i &= n \times 6 = 6n. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 6.$$

Exercise 2.8.
Set the degrees of freedom of each open chain leg to α. Then

$$\begin{aligned} N &= 1\text{ (object)} + 1\text{ (ground)} = 2 \\ J &= n\text{ (open chain arms)} \\ \sum f_i &= n \times \alpha = \alpha n. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 6 + (\alpha - 6)n = 6.$$

Therefore, the total degrees of freedom is six regardless of the number of open chain legs.

Exercise 2.9.
(a) Consider the combination of revolute (R) and prismatic (P) joint as a 2-dof cylindrical (C) joint. Then

$$\begin{aligned} N &= 7\text{ (links)} + 1\text{ (ground)} = 8 \\ J &= 7\text{ (R joints)} + 1\text{ (P joints)} + 2\text{ (C joints)} = 10 \\ \sum f_i &= 7 \times 1 + 1 \times 1 + 2 \times 2 = 12. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 3.$$

(b) Considering all the R and P joints separately,

$$\begin{aligned} N &= 13\text{ (links)} + 1\text{ (ground)} = 14 \\ J &= 16\text{ (R joints)} + 2\text{ (P joints)} = 18 \\ \sum f_i &= 16 \times 1 + 2 \times 1 = 18. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 3.$$

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(c) A fork joint is kinematically equivalent to a C joint, so that

$$\begin{aligned} N &= 7 \text{ (links)} + 1 \text{ (ground)} = 8 \\ J &= 6 \text{ (R joints)} + 2 \text{ (P joints)} + 1 \text{ (C joint)} = 9 \\ \sum f_i &= 6 \times 1 + 2 \times 1 + 1 \times 2 = 10. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 4.$$

(d)

$$\begin{aligned} N &= 5 \text{ (links)} + 1 \text{ (ground)} = 6 \\ J &= 6 \text{ (R joints)} + 1 \text{ (P joints)} = 7 \\ \sum f_i &= 6 \times 1 + 1 \times 1 = 7. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 1.$$

(e)

$$\begin{aligned} N &= 13 \text{ (links)} + 1 \text{ (ground)} = 14 \\ J &= 14 \text{ (R joints)} + 4 \text{ (P joints)} = 18 \\ \sum f_i &= 14 \times 1 + 4 \times 1 = 18. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 3.$$

(f)

$$\begin{aligned} N &= 6 \text{ (links)} + 1 \text{ (ground)} = 7 \\ J &= 8 \text{ (R joints)} + 1 \text{ (P joints)} = 9 \\ \sum f_i &= 8 \times 1 + 1 \times 1 = 9. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 0.$$

Exercise 2.10.

(a)

$$\begin{aligned} N &= 5 \text{ (links)} + 1 \text{ (ground)} = 6 \\ J &= 7 \text{ (R joints)} \\ \sum f_i &= 7 \times 1 = 7. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 1.$$

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(b)

$$\begin{aligned} N &= 5 \text{ (links)} + 1 \text{ (ground)} = 6 \\ J &= 2 \text{ (R joints)} + 4 \text{ (P joints)} = 6 \\ \sum f_i &= 2 \times 1 + 4 \times 1 = 6. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 3.$$

(c)

$$\begin{aligned} N &= 2 \text{ (sliding links)} + 11 \text{ (rod links)} + 1 \text{ (ground)} = 14 \\ J &= 2 \text{ (P joints)} + 10 \text{ (R joints)} + 6 \text{ (overlapping R joints)} = 18 \\ \sum f_i &= 2 \times 1 + 10 \times 1 + 6 \times 1 = 18. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 3.$$

(d)

$$\begin{aligned} N &= 20 \text{ (links)} + 1 \text{ (ground)} = 21 \\ J &= 9 \text{ (P joints)} + 8 \text{ (R joints)} + 10 \text{ (overlapping R joints)} = 27 \\ \sum f_i &= 9 \times 1 + 8 \times 1 + 10 \times 1 = 27. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 6.$$

Exercise 2.11.
(a)

$$\begin{aligned} N &= 5 \text{ (links)} + 1 \text{ (ground)} = 6 \\ J &= 6 \text{ (U joints)} \\ \sum f_i &= 6 \times 2 = 12. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 6.$$

(b)

$$\begin{aligned} N &= 6 \text{ (rods)} + 1 \text{ (plate)} + 1 \text{ (ground)} = 8 \\ J &= 3 \text{ (P joints)} + 3 \text{ (U joints)} + 3 \text{ (S joints)} = 9 \\ \sum f_i &= 3 \times 1 + 3 \times 2 + 3 \times 3 = 18. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 6.$$

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(c)

$$\begin{aligned} N &= 5 \text{ (rods)} + 1 \text{ (plate)} + 1 \text{ (ground)} = 7 \\ J &= 2 \text{ (P joints)} + 3 \text{ (U joints)} + 3 \text{ (S joints)} = 8 \\ \sum f_i &= 2 \times 1 + 3 \times 2 + 3 \times 3 = 17. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 5.$$

(d)

$$\begin{aligned} N &= 7 \text{ (links)} + 1 \text{ (ground)} = 8 \\ J &= 3 \text{ (P joints)} + 6 \text{ (U joints)} = 9 \\ \sum f_i &= 3 \times 1 + 6 \times 2 = 15. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 3.$$

(e)

$$\begin{aligned} N &= 7 \text{ (links)} + 1 \text{ (ground)} = 8 \\ J &= 2 \text{ (R joints)} + 4 \text{ (U joints)} + 2 \text{ (P joints)} = 8 \\ \sum f_i &= 2 \times 1 + 4 \times 2 + 2 \times 1 = 12. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 6.$$

(f)

$$\begin{aligned} N &= 3 \text{ (rods)} + 1 \text{ (plate)} + 1 \text{ (ground)} = 5 \\ J &= 3 \text{ (3 dof joints)} + 3 \text{ (S joints)} = 6 \\ \sum f_i &= 3 \times 3 + 3 \times 3 = 18. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 6.$$

Exercise 2.12.
(a)

$$\begin{aligned} N &= 7 \text{ (links)} + 1 \text{ (ground=legs)} = 8 \\ J &= 3 \text{ (R joints)} + 6 \text{ (U joints)} = 9 \\ \sum f_i &= 3 \times 1 + 6 \times 2 = 15. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 3.$$

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(b)

$$\begin{aligned} N &= 8 \text{ (links)} + 1 \text{ (ground)} = 9 \\ J &= 4 \text{ (R joints)} + 5 \text{ (P joints)} = 9 \\ \sum f_i &= 4 \times 1 + 5 \times 1 = 9. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 3.$$

(c)

$$\begin{aligned} N &= 12 \text{ (links)} + 1 \text{ (plate)} + 1 \text{ (ground)} = 14 \\ J &= 6 \text{ (R joints)} + 6 \text{ (U joints)} + 6 \text{ (S joints)} = 18 \\ \sum f_i &= 6 \times 1 + 6 \times 2 + 6 \times 3 = 36. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 6.$$

(d) The spatial parallel mechanism consists of four RFRRPR serial subchains, where F is a four-bar parallelogram linkage. Each serial subchain with RFRRPR joints can be regarded as ground with a single 6-dof joint:

$$\text{dof} = 1 \text{ (four-bar parallelogram linkage)} + 1 \times 4 \text{ (R joints)} + 1 \text{ (P joint)} = 6.$$

Now apply Grübler's formula to the 4−RFRRPR mechanism:

$$\begin{aligned} N &= 1 \text{ (plate)} + 1 \text{ (ground)} = 2 \\ J &= 4 \text{ (RFRRPR joints)} \\ \sum f_i &= 4 \times 6 = 24. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 6.$$

Exercise 2.13.

$$\begin{aligned} N &= 6 \text{ (legs)} + 1 \text{ (upper platform)} + 1 \text{ (ground)} = 8 \\ J &= 12 \text{ (S joints)} \\ \sum f_i &= 12 \times 3 = 36. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 6.$$

The upper platform can simultaneously translate and rotate about the vertical axis, and also translate horizontally.

Exercise 2.14.

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(a)

$$
\begin{aligned}
N &= 6 \text{ (links)} + 1 \text{ (upper platform)} + 1 \text{ (ground)} = 8 \\
J &= 3 \text{ (P joints)} + 6 \text{ (U joints)} = 9 \\
\sum f_i &= 3 \times 1 + 6 \times 2 = 15.
\end{aligned}
$$

Substituting the above values into the spatial version of Grübler's formula,

$$
\text{dof} = 6(N - 1 - J) + \sum f_i = 3.
$$

(b) If the three P joints are locked, the robot loses three degrees of freedom and thus should become a structure, but clearly the robot can move.

Exercise 2.15.

(a)

$$
\begin{aligned}
N &= 5 \text{ (squares)} + 1 \text{ (ground=square)} = 6 \\
J &= 6 \text{ (R joints)} \\
\sum f_i &= 6 \times 1 = 6.
\end{aligned}
$$

Substituting the above values into the planar version of Grübler's formula,

$$
\text{dof} = 3(N - 1 - J) + \sum f_i = 3.
$$

(b)

$$
\begin{aligned}
N &= 5 \text{ (squares)} + 1 \text{ (ground=square)} = 6 \\
J &= 6 \text{ (R joints)} \\
\sum f_i &= 6 \times 1 = 6.
\end{aligned}
$$

Substituting the above values into the spatial version of Grübler's formula,

$$
\text{dof} = 6(N - 1 - J) + \sum f_i = 0.
$$

However, under the assumption that all the squares are of the same size, the mechanism can move with 1 dof. Grübler's formula is unable to distinguish such cases.

Exercise 2.16.

(a) Since all the links are constrained to move on the surface of a sphere, the planar version (or more accurately, the two-dimensional version) of Grübler's formula must be used. In this case,

$$
\begin{aligned}
N &= 3 \text{ (links)} + 1 \text{ (ground)} = 4 \\
J &= 4 \text{ (R joints)} \\
\sum f_i &= 4 \times 1 = 4. \\
\text{dof} &= 3(N - 1 - J) + \sum f_i = 1.
\end{aligned}
$$

If we had used the spatial (three-dimensional) version of Grübler's formula, we would obtain the result that dof $= -2$, implying the mechanism is incapable of motion.

(b) Since the mechanism has one dof and is constructed of four revolute joints, its C-space is a curve in the four-dimensional torus T^4. Depending on the relative lengths of the links, the curve may be closed, and also have self-intersections.

(c) The workspace is a curve on the sphere as shown in Figure 2.1. At each point on the curve, the orientation of the frame is fixed.

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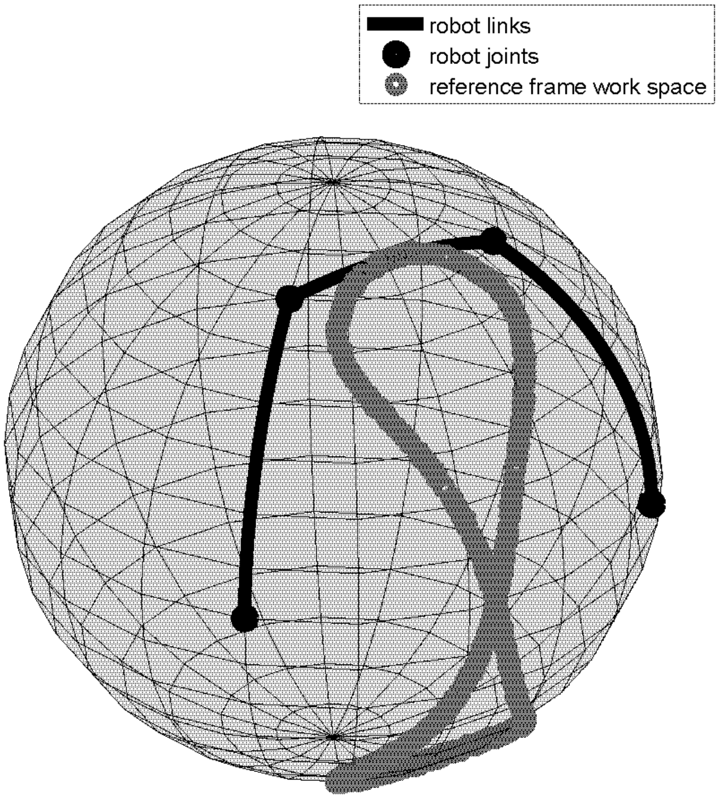

Figure 2.1

Exercise 2.17.

(a) The surgical tool can move freely in the base hole, so there's no joint constraint between the end-effector and the base. In this case,

$$\begin{aligned} N &= 3 \text{ (3 links for leg A)} + 8 \text{ (4 links for each leg B and C)} + 2 \text{ (end-effector and base)} = 13 \\ J &= 4 \text{ (3 R and 1 P joints for leg A)} + 10 \text{ (4 R and 1 U joints for each leg B and C)} = 14 \\ \sum f_i &= 4 \text{ (leg A)} + 2 \times 6 \text{ (leg B and C)} = 16. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 4.$$

(b) The constraint that the surgical tool must pass through point A is equivalent to connecting the tool with a four-dof spherical-prismatic pair: the spherical joint determines the tool orientation, while the prismatic joint determines the displacement along the tool axial direction. Therefore

$$\begin{aligned} N &= 3 + 8 + 2 \text{ (end-effector and base)} = 13 \\ J &= 4 + 10 + 1 \text{ (SP joint between end-effector and base)} = 15 \\ \sum f_i &= 4 + 2 \times 6 + 4 = 20. \end{aligned}$$

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Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 2.$$

(c) Since the axes of all the revolute joints pass through point A, all the links are constrained to move on the two-dimensional sphere and the tool always passes through point A (Refer to exercise 2.16). In this case, the two-dimensional version of Grübler's formula must be used:

$$\begin{aligned} N &= 9 \text{ (3 links for each leg)} + 2 \text{ (end-effector and base)} = 11 \\ J &= 12 \text{ (4 R joints for each leg)} \\ \sum f_i &= 3 \times 4 = 12. \end{aligned}$$

Substituting the above values into the planar version of Grübler's formula,

$$\text{dof} = 3(N - 1 - J) + \sum f_i = 6.$$

Exercise 2.18.

$$\begin{aligned} N &= 6 \text{ (links)} + 1 \text{ (moving platform)} + 1 \text{ (ground)} = 8 \\ J &= 6 \text{ (P joints)} + 3 \text{ (U joints)} = 9 \\ \sum f_i &= 6 \times 1 + 3 \times 2 = 12. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 0.$$

However, this mechanism can move: if the three P joints on the fixed base move identically, the moving platform will move vertically. Therefore, it contradicts the fact that the mechanism has zero degrees of freedom as calculated by Grübler's formula.

Exercise 2.19.
There are $N = 7$ links (ground, torso, two upper arms, two lower arms, and one combined hand-object-hand link) and $J = 8$ joints (three R, four S, and a joint between the box and the table that has three sliding freedoms, two translational and one rotational). By Grübler's formula,

$$\text{dof} = 6(7 - 1 - 8) + \sum_{i=1}^{8} f_i = -12 + 3(1) + 4(3) + 3 = 6.$$

Exercise 2.20.

(a) Referring to the Figure 2.2, when the body is fixed (ground) there are four rigid wings, four rigid legs, four linkages consisting of two links each, and the body (ground), so $N = 17$. There are four wing R joints, four leg R joints, four leg S joints, four leg P joints, and four leg U joints, so $J = 20$. The freedoms of the joints are 1 for the R and P joints, two for the U, three for the U, so $\sum_i f_i = 12(1) + 4(2) + 4(3) = 32$. Grübler's formula gives $6(17 - 1 - 20) + 32 = 8$ dof.

(b) Add six dof for the chassis to get 14 dof.

(c) Keeping a foot at a fixed location adds 3 constraints on that foot, or 12 constraints total, so subtract 12 from 14 (the answer to part (b)) to get 2 dof. Alternatively, by Grübler, add four more S joints at the feet (with 3 dof each) and one more link (ground) compared to the answer in part (a). So $6(18 - 1 - 24) + (32 + 4(3)) = 2$ dof. Note that the wings, of course, can still move with 4 dof, so that means the legs and body only (ignoring the wings) have -2 dof, assuming that none of the constraints are redundant. This means (1) the body and legs cannot move and (2) we even have two constraints on where we can position the legs on the ground.

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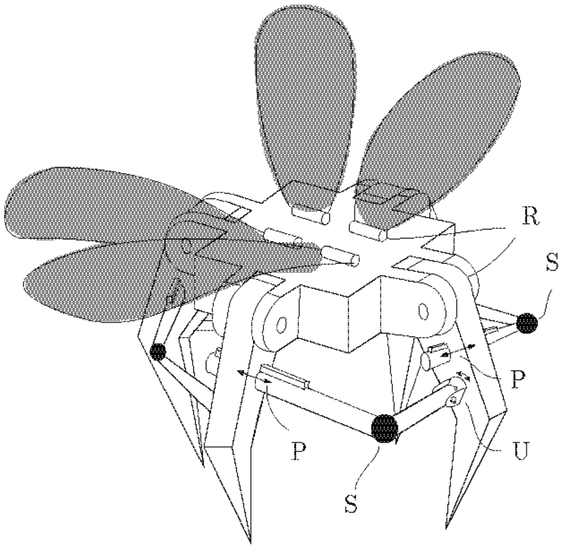

Figure 2.2

Exercise 2.21.

(a) Each of the connections between the body links can be regarded as a three-dof RPR joint. In this case

$$\begin{aligned} N &= 7 \text{ (1 head, 6 body links)} + 1 \text{ (tail=ground)} = 8 \\ J &= 5 \text{ (RPR joints between body links)} + 2 \text{ (R joint between head-body, tail-body)} = 7 \\ \sum f_i &= 5 \times 3 + 2 \times 1 = 17. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 17.$$

(b) Each of the contacts between the body links and the ground can be modelled as a five-dof joint (sliding along two directions, and rotation about three directions). In this case

$$\begin{aligned} N &= 8 \text{ (1 head, 1 tail, 6 body links)} + 1 \text{ (ground)} = 9 \\ J &= 7 \text{ (joints between links)} + 6 \text{ (contacts between body links and ground)} = 13 \\ \sum f_i &= 17 + 6 \times 5 = 47. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 17.$$

(c) Only two of the body links are in contact with the ground. In this case

$$\begin{aligned} N &= 8 \text{ (1 head, 1 tail, 6 body links)} + 1 \text{ (ground)} = 9 \\ J &= 7 \text{ (joints between links)} + 2 \text{ (contacts between body links and ground)} = 9 \\ \sum f_i &= 17 + 2 \times 5 = 27. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 21.$$

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Exercise 2.22.

(a) The palm has six dof and each of the four fingers has four dof, so the hand has 22 dof total. When one finger is in contact with the table, there is one constraint on its position (the equation describing the height of the finger above the table being equal to zero), so the hand has 21 dof. If n fingers are in contact, the hand has $22 - n$ dof.

(b) $26 - n$ dof.

(c) Model the finger contacts as spherical joints, so $N = 14$ (12 finger links, the ellipsoid and the palm ground) and $J = 16$, which includes four U joints, eight R joints, and four S joints (at the finger contacts). By Grübler, dof $= 6(14 - 1 - 16) + 4(2) + 8(1) + 4(3) = 10$.

(d) This question is to see how the student thinks about rolling constraints. A sphere rolling on a surface can achieve any configuration in contact with the surface; the rolling, no-slip nonholonomic velocity constraints do not create any configuration constraint other than that the sphere must remain in contact with the surface. Therefore each finger contact "joint" has five degrees of freedom. Building on (c) above, dof $= -18 + 4(2) + 8(1) + 4(5) = 28$.

Exercise 2.23.

Define the joint variables as shown in Figure 2.3 and let the length of the links be L. The positions of the

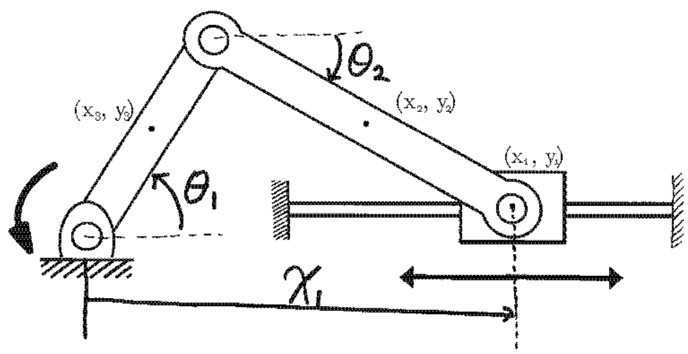

Figure 2.3

centers of each link are denoted (x_3, y_3),(x_2, y_2), (x_1, y_1), respectively (starting from the left in Figure 2.3). Define the eight-dimensional vector $x = (\theta_1, \theta_2, x_1, y_1, x_2, y_2, x_3, y_3)$. The constraint equations $g_i(x) = 0$ of the joints are derived as follows:

$$\begin{aligned}
g_1(x) &= x_1 - L\cos\theta_1 + L\cos\theta_2 \\
g_2(x) &= y_1 - L\sin\theta_1 - L\sin\theta_2 \\
g_3(x) &= x_2 - L\cos\theta_1 + L/2\cos\theta_2 \\
g_4(x) &= y_2 - L\sin\theta_1 - L/2\sin\theta_2 \\
g_5(x) &= x_3 - L/2\cos\theta_1 \\
g_6(x) &= y_3 - L/2\sin\theta_1
\end{aligned}$$

Additionally, there is one more constraint on the slider $g_7(x) = y_1 = 0$. Therefore, there are 7 constraint equations and 8 configuration variables in total. The feasible configuration space of the joint variables can now be determined as

$$C = \{x = (\theta_1, \theta_2, x_1, y_1, x_2, y_2, x_3, y_3)\ |g_i(x) = 0\ (i = 1, \cdots, 7)\}.$$

The configuration space C projected to the space of joint variables $(\theta_1, \theta_2, x_1)$ is given by

$$C_p = \{(\theta_1, \theta_2, x_1)\ |\theta_1 = \theta_2, x_1 = L(\cos\theta_1 + \cos\theta_2)\}.$$

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Exercise 2.24.

(a) The four-bar linkage is floating in space, so the number of links is 4 since the ground and linkage are not connected. 4 joints connect the adjacent links of the floating linkage. Finally, m in Grübler's formula is set to 6 since it is a spatial mechanism. We therefore have

$$\begin{aligned} N &= 4 \text{ (4 links)} + 1 \text{ (ground)} = 5 \\ J &= 4 \text{ (R joints between links)} \\ \sum f_i &= 4. \end{aligned}$$

Substituting the above values into the spatial version of Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 4.$$

However, the actual dof of the floating system differs from that predicted by Grübler's formula: any single floating system has at least 6 dof. This mismatch will be discussed further in (b).

(b) The degrees of freedom can be calculated by subtracting the number of constraints from the number of variables. First of all, we already know that there are three coordinates for each point and three points on each link. Thus, the total number of variables is 36. The planar four-bar linkage has three types of constraints: rigid body, revolute joint, and planar motion. For rigid body constraints, twelve constraints should be considered. For link 1,

$$\begin{aligned} \|p_A - p_B\| = \text{const.} \quad &\Leftrightarrow \quad \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2} = \text{const.} \\ \|p_B - p_C\| = \text{const.} \quad &\Leftrightarrow \quad \sqrt{(x_B - x_C)^2 + (y_B - y_C)^2 + (z_B - z_C)^2} = \text{const.} \\ \|p_C - p_A\| = \text{const.} \quad &\Leftrightarrow \quad \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2} = \text{const.} \end{aligned}$$

Similarly, constraints on links 2, 3, and 4 can be expressed as above. The four pairs of points are connected by a revolute joint: C with D, F with G, I with J, and L with A. These constraints can be written as follows:

$$\begin{aligned} p_C = p_D \Leftrightarrow \quad (x_C, y_C, z_C) = (x_D, y_D, z_D) \quad &\Leftrightarrow x_C = x_D, y_C = y_D, z_C = z_D \\ p_F = p_G \Leftrightarrow \quad (x_F, y_F, z_F) = (x_G, y_G, z_G) \quad &\Leftrightarrow x_F = x_G, y_F = y_G, z_F = z_G \\ p_I = p_J \Leftrightarrow \quad (x_I, y_I, z_I) = (x_J, y_J, z_J) \quad &\Leftrightarrow x_I = x_J, y_I = y_J, z_I = z_J \\ p_L = p_A \Leftrightarrow \quad (x_L, y_L, z_L) = (x_A, y_A, z_A) \quad &\Leftrightarrow x_L = x_A, y_L = y_A, z_L = z_A. \end{aligned}$$

Finally, only planar motions are admissible: all of the points on the linkage lie on a plane with normal vector $\vec{n} = \frac{\overrightarrow{AC} \times \overrightarrow{AJ}}{\|\overrightarrow{AC} \cdot \overrightarrow{AJ}\|}$, and

$$\begin{aligned} \overrightarrow{AF} \cdot \vec{n} &= 0 \\ \overrightarrow{AB} \cdot \vec{n} &= 0 \\ \overrightarrow{AE} \cdot \vec{n} &= 0 \\ \overrightarrow{AH} \cdot \vec{n} &= 0 \\ \overrightarrow{AK} \cdot \vec{n} &= 0 \end{aligned}$$

The first constraint $\overrightarrow{AF} \cdot \vec{n} = 0$ is to prevent folding between planes DFI and ACJ. The total number of constraints is 29. Therefore, the degrees of freedom of the system is equal to $36 - 29 = 7$ dof. The reason why this differs from the result obtained from Grübler's formula is that the planar four-bar linkage floating in 3-D space undergoes 2-D motion. The spatial version of Grübler's formula should not be applied to planar motion in 3D space.

Exercise 2.25.

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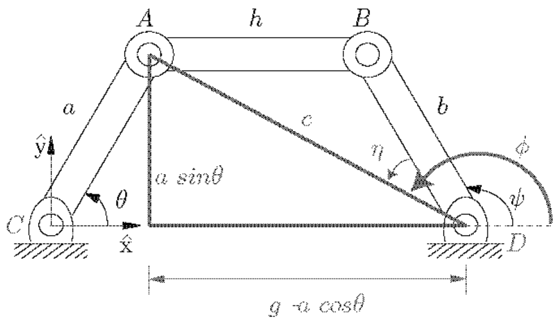

Figure 2.4

(a) Draw a line between joint A and joint D, and set the length of this line as c. In addition, set the angle between BD and AD as η (see Figure 2.4). The following equation can then be derived:

$$\begin{aligned} a^2 + g^2 - 2ag\cos\theta &= c^2 \\ b^2 + c^2 - 2bc\cos\eta &= h^2. \end{aligned}$$

From the above, γ^2 and $\alpha^2 + \beta^2$ can be calculated as follows:

$$\begin{aligned} \gamma^2 &= \left(h^2 - g^2 - b^2 - a^2 + 2ag\cos\theta\right)^2 = \left(h^2 - b^2 - c^2\right)^2 = (-2bc\cos\eta)^2 \\ \therefore \gamma^2 &= 4b^2c^2\cos^2\eta \\ \alpha^2 + \beta^2 &= (2gb - 2ab\cos\theta)^2 + (-2ab\sin\theta)^2 = 4b^2\left(g^2 + a^2 - 2ag\cos\theta\right) \\ \therefore \alpha^2 + \beta^2 &= 4b^2c^2 \end{aligned}$$

$$\gamma^2 = 4b^2c^2\cos^2\eta \le 4b^2c^2 = \alpha^2 + \beta^2.$$

If the constraint $\gamma^2 \le \alpha^2 + \beta^2$ is not satisfied, then $\cos\eta > 1$, which implies that the four-bar linkage is unable to reach a desired output angle ϕ. The maximum value ψ_{max} is determined by the structure, and Figure 2.5 illustrates such a scenario.

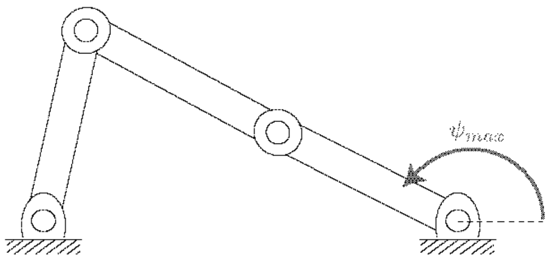

Figure 2.5

(b) Expressing ϕ in terms of a, b, g and θ,

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}\left(\frac{-2ab\sin\theta}{2gb - 2ab\cos\theta}\right) \\ \therefore \phi &= \tan^{-1}\left(-\frac{a\sin\theta}{g - a\cos\theta}\right). \end{aligned}$$

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Therefore, ϕ represents the angle between CD and AD (see Figure 2.4). The relation between ϕ, ψ and η can be obtained from

$$\begin{aligned}\cos(\psi - \phi) &= \frac{\gamma}{\sqrt{\alpha^2 + \beta^2}} = \frac{-2bc\cos\eta}{\sqrt{4b^2c^2}} = -\cos\eta \\ \psi - \phi &= \pm\eta\end{aligned}$$

$$\therefore \psi = \phi \pm \eta.$$

The two possible values of the output angle ψ represent the elbow-up and elbow-down configurations with respect to AD (Figure 2.6).

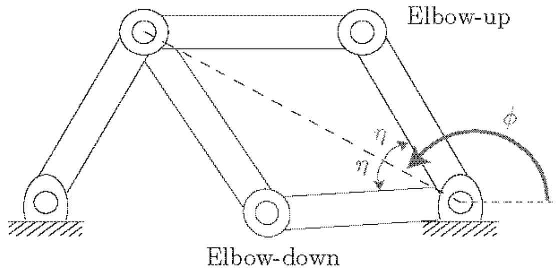

Figure 2.6

(c) By substituting the expressions for a, b, g, h into the equations for α, β, γ, the relation between θ and ψ can be derived as shown in Figure 2.7 (obtained via MATLAB).

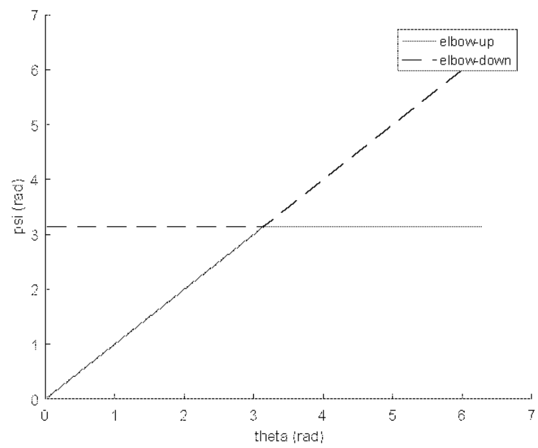

Figure 2.7

(d) Same as 2.25(c) (Figure 2.8).

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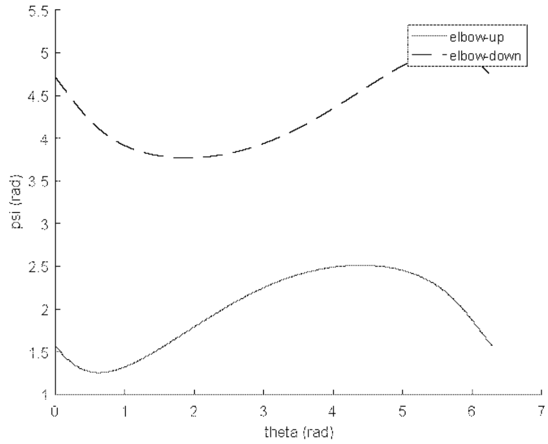

Figure 2.8

(e) Same as 2.25(c) (Figure 2.9).

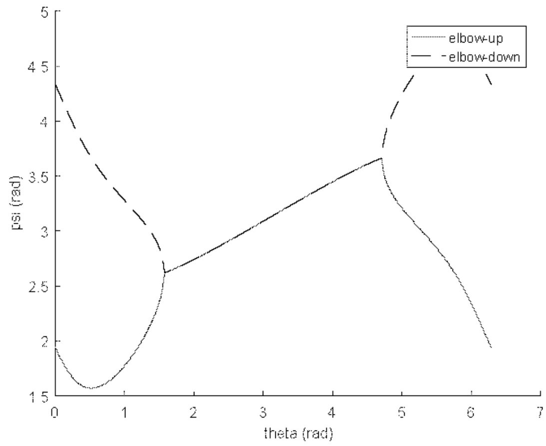

Figure 2.9

Exercise 2.26.

(a) The configuration space is the space of the variables $x_1, y_1, \theta_1, x_2, y_2, \theta_2$ where (x_i, y_i) denotes the center of mass of the i-th link and θ_i denotes the orientation of the link. The constraint equations and corresponding feasible configuration space are given by

$$C \quad = \{q = (x_1, y_1, \theta_1, x_2, y_2, \theta_2) \mid \quad g_i(q) = 0 \ (i = 1, \cdots, 4),$$

where

$$\begin{aligned} g_1(x) &= x_1 - \cos\theta_1 \\ g_2(x) &= y_1 - \sin\theta_1 \\ g_3(x) &= x_2 - (2\cos\theta_1 + \frac{1}{2}\cos(\theta_1 + \theta_2)) \\ g_4(x) &= y_2 - (2\sin\theta_1 + \frac{1}{2}\sin(\theta_1 + \theta_2)). \end{aligned}$$

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(b) The workspace W is the set of all points reachable by the tip as shown in Figure 2.10 (left):

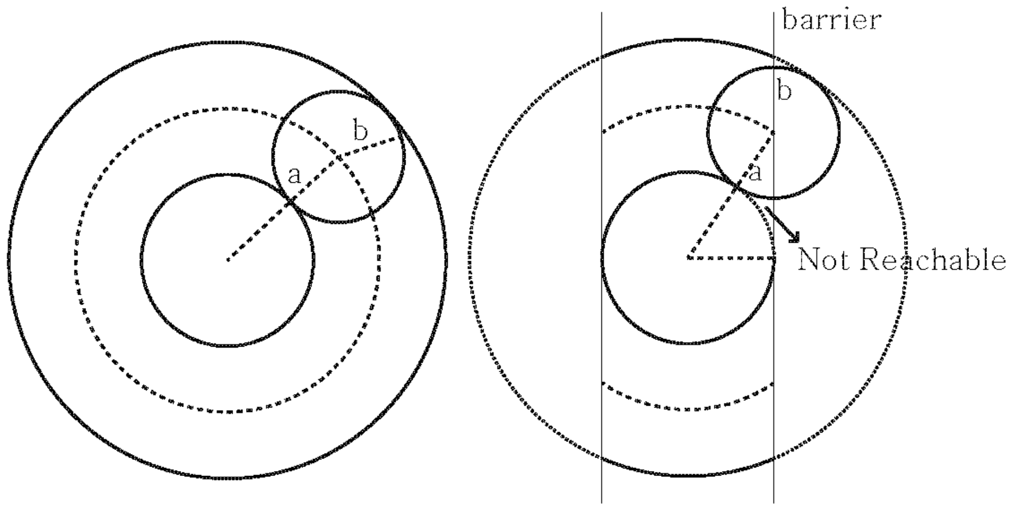

Figure 2.10

$$W = \{(x, y) \mid x = 2\cos\theta_1 + \cos(\theta_1 + \theta_2), y = 2\sin\theta_1 + \sin(\theta_1 + \theta_2), \theta_i \leq 2\pi\}.$$

Corresponding to the given constraint equations, let $a = 2, b = 1$ in Figure 2.10.

(c) Vertical barrier at $x = \pm 1$: exclude the space such that $|x| > 1$ from the original free C-space, whose area is $(a + b)^2\pi - (a - b)^2\pi$. Substituting $a = 2, b = 1$, the area to be excluded is

$$4\int_{a-b}^{a+b} \sqrt{(a + b)^2 - x^2}\; dx.$$

Thus, the left area is now $((a + b)^2\pi - (a - b)^2\pi) - 4\int_{a-b}^{a+b} \sqrt{(a + b)^2 - x^2}\; dx = 8.6323$. The area indicated as "Not Reachable" in Figure 2.10 can be calculated as follows:

$$(\text{triangle} - \text{fan shapes}) = \frac{\sqrt{3}}{2} - \frac{\pi}{4}.$$

As a result we have

$$r = \frac{\text{restricted area}}{\text{original area}} = \frac{8.6323 - 4(\frac{\sqrt{3}}{2} - \frac{\pi}{4})}{8\pi} = 0.3306.$$

Exercise 2.27.

(a) Let the revolute joint variables be $\theta_1, \theta_2, \theta_3$. Similar to Figure 2.10, the donut-shaped workspace is now moving along the circle of radius $a = 5$. We can derive an analytic expression for the coordinates of the tip (x, y):

$$\begin{aligned} x &= 5\cos\theta_1 + 2\cos(\theta_1 + \theta_2) + \cos(\theta_1 + \theta_2 + \theta_3) \\ y &= 5\sin\theta_1 + 2\sin(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2 + \theta_3). \end{aligned} \tag{2.1}$$

The admissible region of (x, y) is given in Figure 2.11; it is donut-shaped with radii 2 and 8, with total area $64\pi - 4\pi = 60\pi$.

(b) Setting $a = 1, b = 2, c = 5$ in Figure 2.11, it can be seen that the lengths are the reverse of (a). Thus, intuitively one can conjecture that answer is 60π, the same as obtained for (a). To prove this, express

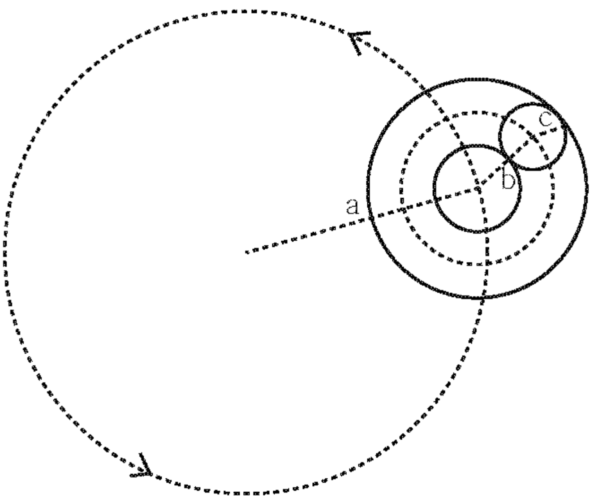

Figure 2.11

the tip coordinates (x, y) as

$$\begin{aligned} x &= \cos\theta_1 + 2\cos(\theta_1+\theta_2) + 5\cos(\theta_1+\theta_2+\theta_3) \\ y &= \sin\theta_1 + 2\sin(\theta_1+\theta_2) + 5\sin(\theta_1+\theta_2+\theta_3). \end{aligned}$$

These equations for the tip coordinates can be transformed into the same form as (a):

$$\begin{aligned} x &= 5\cos\alpha_1 + 2\cos(\alpha_1+\alpha_2) + \cos(\alpha_1+\alpha_2+\alpha_3) \\ y &= 5\sin\alpha_1 + 2\sin(\alpha_1+\alpha_2) + \sin(\alpha_1+\alpha_2+\alpha_3), \end{aligned} \tag{2.2}$$

where $\alpha_1 = \theta_1+\theta_2+\theta_3, \alpha_2 = \theta_1+\theta_2, \alpha_3 = \theta_1$. These α_i range from 0 to 2π, as do the θ_i. Since the equations of the tip coordinates of Equations (2.1) and (2.2) are the same, the area of (b) is also 60π as conjectured.

(c) For problems like this, it is useful to begin with the simplest case (i.e., the two-link planar open chain) and to work your way up to higher degrees of freedom. For the 2R planar chain, based on the above we know that the Cartesian positioning workspace will be an annulus: if the links have length L_1 and L_2 and $L_1 > L_2$, then this annulus will have inner radius $L_1 - L_2$ and outer radius $L_1 + L_2$, and its area is given by $4\pi L_1 L_2$. Clearly increasing the link lengths does enlarge the Cartesian positioning workspace area, so the designer's claim is not entirely incorrect (although one could easily argue that the shape of the workspace—possessing a large hole close to the base—would not be generally useful). Now consider the 3R planar open chain. Assume that the length of the last link is L, which is longer than 5, and the first two links are of lengths 1 and 2 as before. Note that the workspace is independent of the order of the lengths as previously proven. Thus, the equations for the tip coordinates can be derived as follows:

$$\begin{aligned} x &= L\cos\alpha_1 + 2\cos(\alpha_1+\alpha_2) + \cos(\alpha_1+\alpha_2+\alpha_3) \\ y &= L\sin\alpha_1 + 2\sin(\alpha_1+\alpha_2) + \sin(\alpha_1+\alpha_2+\alpha_3), \end{aligned}$$

where $\alpha_1 = \theta_1+\theta_2+\theta_3, \alpha_2 = -\theta_3, \alpha_3 = -\theta_2$. The workspace is again annular with radii $(L-2-1)$ and $(L+2+1)$. Further calculation reveals that the area of the workspace is $\pi((L+3)^2 - (L-3)^2) = 12L\pi$. This linearity can be preserved for arbitrary lengths a, b, c as indicated in Figure 2.11. Here again the Cartesian positioning workspace can be enlarged with increased L, so the designer's claim does not appear to be incorrect. For planar open chains with increasing degrees of freedom, we can expect the situation to be more or less the same.

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However, the notion of workspace volume above is based only on the Cartesian positioning workspace. If one were to take into account the orientation workspace—that is, at any given point in the Cartesian positioning workspace, what is the range of possible orientations achievable by the tip?—then the situation becomes different. The analysis is quite involved and we won't get into it here—some relevant references are "The workspace of a mechanical manipulator," A. Kumar and K. Waldron, ASME J. Mechanical Design, vol. 103, 1981, and "Optimal kinematic design of 6R manipulators," B. Paden and S. Sastry, Int. J. Robotics Research, vol. 7, 1988—but if both the position and orientation workspace are simultaneously considered in the notion of a robot's workspace, then it can be shown that varying the length of the last link does not change this more general notion of a robot's workspace volume. As an analogy, for the 3R planar open chain, at each point in the Cartesian positioning workspace there will be a range of orientations achievable by the tip (some interval on $[0, 2\pi]$); imagine integrating this orientation range over the entire positioning workspace, and using this integral as the total workspace volume.

Exercise 2.28.
The task space and workspace are not the same concept. The task space is the space of configurations as specified by the task itself and independent of the robot. On the other hand, the workspace is the configuration space of the end-effector that the robot can reach, which is primarily determined by the robot's structure and independent of the task.

(a) Writing on a blackboard:
Focusing on what is actually written on the board, the task of drawing is determined by the position of the chalk but not the orientation. Keeping contact with the blackboard, the dimension of the chalk is two (i.e., $(x, y) \in \mathbb{R}^2$). The task space is therefore $\mathbb{R}^2$.

(b) Twirling a baton:
A rigid body has 6 degrees of freedom. Note however that rotation about the central axis of the baton does not change its appearance, so that the task space can be considered to be $\mathbb{S}^2 \times \mathbb{R}^3$. Observe that flipping the baton also does not change the baton's appearance, so that the orientation could actually be a half-sphere, or the real projective space $\mathbb{RP}^2$.

Exercise 2.29.

(a) $\mathbb{R}^2 \times S^1$.
(b) $S^2 \times S^1$ (the chassis position on the sphere and the chassis heading direction).
(c) $\mathbb{R}^2 \times S^1 \times T^3 \times [a, b] = \mathbb{R}^2 \times T^4 \times [a, b]$.
(d) $\mathbb{R}^3 \times S^2 \times S^1 \times T^6 = \mathbb{R}^3 \times S^2 \times T^7$.

Exercise 2.30.
To achieve the desired (x, y) position, change the heading ϕ to point at the goal location. Then roll there. Then change the rolling angle θ by driving the coin around a circle so that the contact point with the plane traces a circle of radius R back to the starting (x, y) position. To change the rolling angle by an amount $\Delta\theta$, R should satisfy $R = \Delta\theta r/(2\pi)$, where r is the radius of the coin. Once the proper (x, y, θ) is achieved, the coin can be rotated to the desired heading angle ϕ.

Exercise 2.31.

(a) Let ϕ_1 be the rotation angle of the left wheel and ϕ_2 be the rotation angle of the right wheel. Let $R_1 = r$ and $R_2 = r$ be the steering radii of the left and right wheel respectively. Furthermore, assume that the distance between the wheels is $2d$. Based on this we have the following equations:

$$\dot{\phi}_1 = \omega_1$$
$$\dot{\phi}_2 = \omega_2$$

$$\dot{x} = (1/2)(\omega_1 + \omega_2) r \cos\theta$$
$$\dot{y} = (1/2)(\omega_1 + \omega_2) r \sin\theta$$
$$\dot{\theta} = (r/2d)(\omega_2 - \omega_1)$$

In vector form we have the equality:

$$\dot{q} = \begin{bmatrix} (1/2) r\cos\theta \\ (1/2) r\sin\theta \\ -(r/2d) \\ 1 \\ 0 \end{bmatrix} \omega_1 + \begin{bmatrix} (1/2) r\cos\theta \\ (1/2) r\sin\theta \\ (r/2d) \\ 0 \\ 1 \end{bmatrix} \omega_2$$

(b) The Pfaffian constraints are dependent on the following differential equations:

$$\dot{x} - (1/2)\dot{\phi}_1 r\cos\theta - (1/2)\dot{\phi}_2 r\cos\theta = 0$$
$$\dot{y} - (1/2)\dot{\phi}_1 r\sin\theta - (1/2)\dot{\phi}_2 r\sin\theta = 0$$
$$\dot{\theta} + (r/2d)\dot{\phi}_1 - (r/2d)\dot{\phi}_2 = 0$$
$$\implies A(q)\dot{q} = \begin{bmatrix} 1 & 0 & 0 & -(1/2) r\cos\theta & -(1/2) r\cos\theta \\ 0 & 1 & 0 & -(1/2) r\sin\theta & -(1/2) r\sin\theta \\ 0 & 0 & 1 & (r/2d) & -(r/2d) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi} 2 \end{bmatrix} = 0$$

(c) The constraint $\dot{\theta} + (r/2d)\dot{\phi}_1 - (r/2d)\dot{\phi}_2 = 0$ is holonomic (integrable) and the other two constraints are nonholonomic.

Exercise 2.32.

(a) Transform the constraint into Pfaffian form:

$$\begin{aligned} A(q)\dot{q} &= 0 \\ \text{where} \quad A(q) &= \begin{bmatrix} 1+\cos q_1 & 1+\cos q_2 & \cos q_1 + \cos q_2 + 4 & 0 \end{bmatrix}. \end{aligned}$$

To check whether $A(q)$ is integrable, assume $A(q) = \frac{\partial g(q)}{\partial q}$:

$$\frac{\partial g(q)}{\partial q_1} = 1 + \cos q_1 \quad \rightarrow g(q) = q_1 + \sin q_1 + h(q_2, q_3, q_4) \tag{2.3}$$
$$\frac{\partial g(q)}{\partial q_2} = 1 + \cos q_2 \quad \rightarrow g(q) = q_1 + \sin q_1 + q_2 + \sin q_2 + h(q_3, q_4) \tag{2.4}$$
$$\frac{\partial g(q)}{\partial q_3} = \cos q_1 + \cos q_2 + 4 \quad \rightarrow \frac{\partial h(q_3, q_4)}{\partial q_3} = \cos q_1 + \cos q_2 + 4 \tag{2.5}$$
$$\frac{\partial g(q)}{\partial q_4} = 0. \tag{2.6}$$

A contradiction occurs in Equation (2.5) since $h(q_3, q_4)$ does not incorporate any q_1 or q_2 term. Therefore, $A(q)$ is not integrable and the velocity constraint is nonholonomic.

(b) Transform the constraint into Pfaffian form:

$$\begin{aligned} A(q)\dot{q} &= 0 \\ \text{where} \quad A(q) &= \begin{bmatrix} -\cos q_2 & 0 & \sin(q_1+q_2) & -\cos(q_1+q_2) \\ 0 & 0 & \sin q_1 & -\cos q_1 \end{bmatrix}. \end{aligned}$$

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To check whether $A(q)$ is integrable, assume $A(q) = \frac{\partial g(q)}{\partial q}$:

$$\frac{\partial g_1(q)}{\partial q_1} = \quad -\cos q_2 \quad \to g_1(q) = -q_1 \cos q_2 + h(q_2, q_3, q_4) \tag{2.7}$$

$$\frac{\partial g_1(q)}{\partial q_2} = \quad 0 \quad \to \frac{h(q_2, q_3, q_4)}{\partial q_2} = -q_1 \sin q_2 \tag{2.8}$$

$$\frac{\partial g_1(q)}{\partial q_3} = \quad \sin(q_1 + q_2) \tag{2.9}$$

$$\frac{\partial g_1(q)}{\partial q_4} = \quad -\cos(q_1 + q_2) \tag{2.10}$$

A contradiction occurs in Equation (2.8) since $h(q_2, q_3, q_4)$ does not incorporate any q_1 terms. Thus, there exists no admissible $g_1(q)$. To check $g_2(q)$,

$$\frac{\partial g_2(q)}{\partial q_1} = \quad 0 \quad \to g_2(q) = h(q_2, q_3, q_4) \tag{2.11}$$

$$\frac{\partial g_2(q)}{\partial q_2} = \quad 0 \quad \to g_2(q) = h(q_3, q_4) \tag{2.12}$$

$$\frac{\partial g_2(q)}{\partial q_3} = \quad \sin q_1 \quad \to \frac{\partial h(q_3, q_4)}{\partial q_3} = \sin q_1 \tag{2.13}$$

$$\frac{\partial g_2(q)}{\partial q_4} = \quad -\cos q_1 \tag{2.14}$$

A contradiction occurs in Equation (2.13) since $h(q_3, q_4)$ does not incorporate any q_1 terms. In conclusion, there exists no $g(q)$ satisfying $A(q) = \frac{\partial g(q)}{\partial q}$ and thus the given constraint is nonholonomic.

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