Chapter 6 Solutions

Exercise 6.1.
Given x, y and θ, the position of the last joint is determined by $(\bar{x}, \bar{y}) = (x - L_3 \cos\theta, y - L_3 \sin\theta)$. Then the problem boils down to solving the inverse kinematics θ_1 and θ_2 for a planar 2R robot with given end-effector position $(\bar{x}, \bar{y})$, analytic solutions of which are well described in the text. Finally, θ_3 is determined by $\theta_3 = \theta - \theta_1 - \theta_2$.

Exercise 6.2.
By placing the base frame at the intersection of first three axes and tool frame at the intersection of the final two axes, we can think of the given manipulator as an inverted version of the standard elbow manipulator(*e.g.*, a PUMA design with zero shoulder offset), whose base frame and tool frame are switched. The standard elbow manipulator is known to have four inverse kinematic solutions (two corresponding to elbow-up and elbow-down solutions, and two corresponding to the ZYX Euler angle-type wrist, resulting in four inverse kinematic solutions.). Suppose the forward kinematics for the standard elbow manipulator is of the form $e^{[S_1]\theta_1} \ldots e^{[S_6]\theta_6} = T$. It readily follows that $e^{-[S_6]\theta_6} \ldots e^{-[S_1]\theta_1} = T^{-1}$. Note that the forward kinematics of the inverse elbow manipulator can be made to assume exactly this form. Therefore the inverse elbow manipulator also has four inverse kinematic solutions. Intuitively these solutions also correspond to the elbow-up/down and the ZYX Euler angle solutions.

Exercise 6.3.
Given $T' \in SE(3)$, one can obtain the original end-effector pose $T \in SE(3)$ by $T = T' \cdot \mathrm{Trans}(\hat{y}, (2 - \sqrt{2})L)$. Therefore, we can instead solve for

$$T = T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} e^{[S_4]\theta_4} e^{[S_5]\theta_5} e^{[S_6]\theta_6} M,$$

where the S_i are the corresponding joint screws.
It can be easily seen that θ_4, θ_5, θ_6 do not change the position of T; only θ_1, θ_2, θ_3 do. Let the position vector of T be $p = (p_x, p_y, p_z)^T$. Then by following similar reasoning for solving the analytic inverse kinematics of 6R puma type arm, we have two possible solutions for θ_1:

$$\theta_1 = \tan^{-1}(-\frac{p_y}{p_x}),$$

and

$$\theta_1 = \tan^{-1}(-\frac{p_y}{p_x}) + \pi.$$

(in the latter case the original solution for θ_2 is replaced by $\pi - \theta_2$.) Observe that when $p_x = p_y = 0$, the arm is in singluarity configuration, resulting in infinitely many possible solutions for θ_1. Now θ_2 and θ_3 are determined by solving the inverse kinematics for a planar two-link chain. From the law of cosines,

$$\cos\theta_3 = \frac{p_x^2 + p_y^2 + p_z^2 - 2L^2}{2L^2} \triangleq D,$$

holds, and $\theta_3 = \tan^{-1}\left(\pm \frac{\sqrt{1-D^2}}{D} \right)$. Given a solution for θ_3, θ_2 can be obtained as

$$\theta_2 = \tan^{-1}\left(\frac{p_z}{\sqrt{p_x^2 + p_y^2}} \right) - \tan^{-1}\left(\frac{L \sin\theta_3}{L + L\cos\theta_3} \right).$$

To sum up, given the position p, we have 4 possible solutions for $\theta_1, \theta_2, \theta_3$, just like the case for a 6R PUMA-type arm. Now, we are left with determining $\theta_4, \theta_5, \theta_6$. Having the solutions for $\theta_1, \theta_2, \theta_3$, forward kinematics can be manipulated into the form

$$e^{[S_4]\theta_4}e^{[S_5]\theta_5}e^{[S_6]\theta_6} = e^{-[S_3]\theta_3}e^{-[S_2]\theta_2}e^{-[S_1]\theta_1}TM^{-1}$$

where the right-hand side is given. By following the same reasoning as for the 6R PUMA-type arm, this problem boils down to

$$\mathrm{Rot}(\hat{\mathrm{x}}, \theta_4)\mathrm{Rot}(\hat{\mathrm{z}}, \theta_5)\mathrm{Rot}(\hat{\mathrm{y}}, \theta_6) = R,$$

where R is the $SO(3)$ component of $e^{-[S_3]\theta_3}e^{-[S_2]\theta_2}e^{-[S_1]\theta_1}TM^{-1}$. The solutions for $\theta_4, \theta_5, \theta_6$ then correspond to the XZY Euler angles.

Exercise 6.4.
The forward kinematics results in the following relaton:

$$p = e^{[\omega_1]\theta_1}e^{[\omega_2]\theta_2}(0, 1+\theta_3, 0)^T, \tag{6.1}$$

where $\omega_1 = (0,0,1)^T$ and $\omega_2 = (0, \frac{1}{2}, \frac{\sqrt{3}}{2})^T$.
(a) Taking the norm of both sides of Equation (6.1),

$$\theta_3 = \|p\| - 1 = 7.$$

Now we are left with an equation of the form $p = e^{[\omega_2]\theta_2}q$, where $q = (0, 8, 0)^T$.
Figure 6.1 shows a geometric illustration of $p = e^{[\omega_2]\theta_2}q$. Defining vectors p' and q', depicted in Figure 6.1, as $p' = p - (\omega_2^T p)\omega_2 = (0, 3/4, -\sqrt{3}/4)^T$ and $q' = q - (\omega_2^T q)\omega_2 = (-3/4, 3/8, -\sqrt{3}/8)^T$, we can obtain a solution for θ_2 from these two vectors as follows:

$$\begin{aligned}\theta_2 &= \mathrm{atan2}(\omega_2^T(q' \times p'), q' \cdot p') \\ &= 60^\circ.\end{aligned}$$

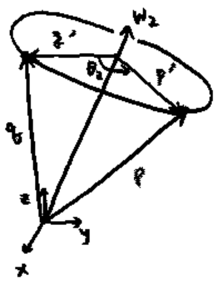

Figure 6.1

(b) Figure 6.2 shows a geometric illustration of Equation (6.1) in general.
Since ω_1 and ω_2 are clearly independent vectors, ω_1, ω_2 and $\omega_1 \times \omega_2$ constitute basis vectors in $\mathbb{R}^3$. Therefore we can represent vector r, depicted in Figure 6.2, with the basis vectors, and coefficients α, β and γ as

$$r = \alpha\omega_1 + \beta\omega_2 + \gamma(\omega_1 \times \omega_2). \tag{6.2}$$

Taking the inner product of both sides of Equation (6.2) with ω_1 and ω_2, and using the relations $\omega_2^T r = \omega_2^T q$ and $\omega_1^T r = \omega_1^T p$, the following two equations in terms of α and β can be obtained:

$$\begin{aligned}\omega_2^T q &= (\omega_2^T\omega_1)\alpha + \beta \\ \omega_1^T p &= \alpha + (\omega_1^T\omega_2)\beta.\end{aligned}$$

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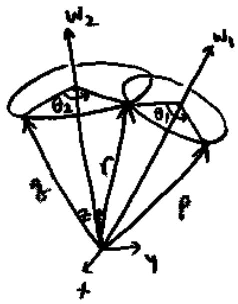

Figure 6.2

Solving for α and β yields $\alpha = -4\sqrt{3}$ and $\beta = 10$. Now, taking the squared norm of Equation (6.2) and solving for γ, we have

$$\gamma^2 = \frac{\|q\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta(\omega_1^T\omega_2)}{\|\omega_1 \times \omega_2\|^2} = 144 = 12^2$$

(Here we use the fact that $\|r\| = \|q\|$). Therefore $\gamma = \pm 12$ and $r = -4\sqrt{3}\omega_1 + 10\omega_2 \pm 12(\omega_1 \times \omega_2) = (\mp 6, 5, \sqrt{3})^T$. Now by following the same reasoning used in (a), from q and r we get θ_2, and from r and p we get θ_3. The two solutions representing elbow up and elbow down configurations are

$$(\theta_1, \theta_2, \theta_3) = (0, 60°, 7), (\text{atan2}(60, -11), -60°, 7).$$

Exercise 6.5.
Among the joint variables, only θ_1 and θ_3 affect p_z:

$$p_z = L + h\theta_1 - \theta_3.$$

A top-down view of this manipulator is shown in Figure 6.3. The angle β can be found from the law of

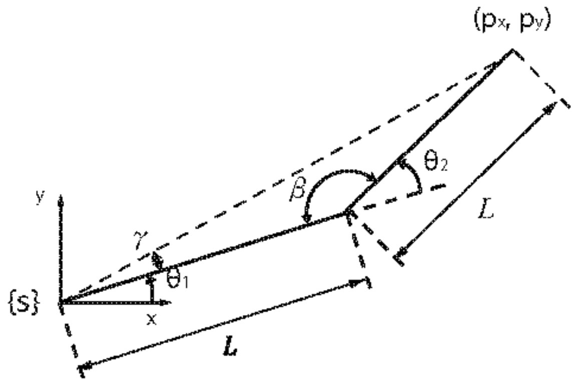

Figure 6.3

cosines:

$$L^2 + L^2 - 2L^2\cos\beta = p_x^2 + p_y^2$$
$$\therefore \beta = \cos^{-1}\left(\frac{p_x^2 + p_y^2 - 2L^2}{2L^2}\right).$$

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From the figure, $\theta_2 = \pi - \beta$. Also, γ can be found by

$$L^2 + p_x^2 + p_y^2 - 2L\sqrt{p_x^2 + p_y^2}\cos\gamma = L^2$$

$$\therefore \gamma = \cos^{-1}\left(\frac{p_x^2 + p_y^2}{2L\sqrt{p_x^2 + p_y^2}}\right).$$

From Figure 6.3, $\tan(\theta_1 + \gamma) = \frac{p_y}{p_x}$. Therefore, $\theta_1 = \tan^{-1}\frac{p_y}{p_x} - \gamma$. Finally for θ_4, considering the orientation of the end-effector frame,

$$\mathrm{Rot}(\hat{z}, \theta_1)\mathrm{Rot}(\hat{z}, \theta_2)\mathrm{Rot}(\hat{z}, \theta_4) = \mathrm{Rot}(\hat{z}, \alpha)$$
$$\theta_1 + \theta_2 + \theta_4 = \alpha.$$

Therefore

$$\theta_4 \quad = \quad \alpha - \theta_1 - \theta_2 = \alpha - \tan^{-1}\frac{p_x}{p_y} + \gamma - \pi + \beta.$$

This is an elbow-down solution; the elbow-up solution can be obtained in the same way:

$$(\theta_1, \theta_2, \theta_3, \theta_4) = \begin{cases} (\tan^{-1}\frac{p_x}{p_y} - \gamma, \pi - \beta, L + h\tan^{-1}\frac{p_y}{p_x} - h\gamma - p_z, \alpha - \tan^{-1}\frac{p_x}{p_y} + \gamma - \pi + \beta) \\ (\tan^{-1}\frac{p_x}{p_y} + \gamma, \pi + \beta, L + h\tan^{-1}\frac{p_y}{p_x} + h\gamma - p_z, \alpha - \tan^{-1}\frac{p_x}{p_y} - \gamma - \pi - \beta) \end{cases}$$

Exercise 6.6.

(a) First, consider the position inverse kinematics.
Figure 6.4 on the left is the robot projected to the plane perpendicular to joint axes 4, 5, and 6. Denote

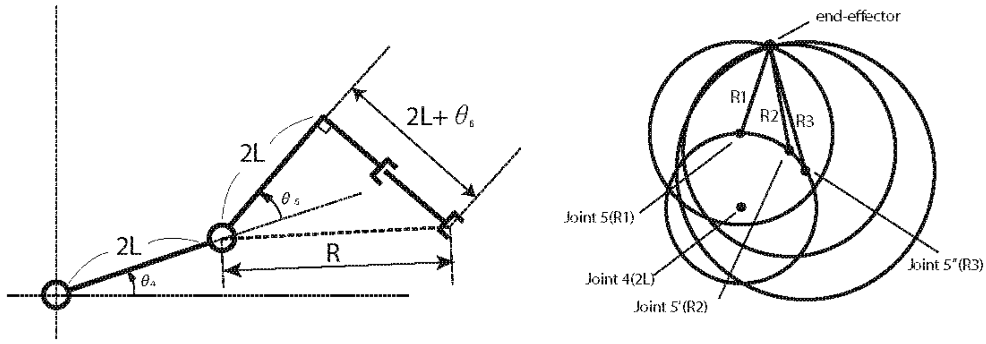

Figure 6.4

the distance between joint 5 and the end-effector by R:

$$R \quad = \quad \sqrt{(2L)^2 + (2L + \theta_6)^2}.$$

Figure 6.4 on the right shows the paths of joint 5 and the end-effector. The radius of the path of joint 5 is $2L$. Let the radii of the three paths be $R1$, $R2$, and $R3$, respectively. We can see that three circles intersect at a point. This means that by changing θ_4, θ_5, and θ_6 simultaneously, the end-effector can be fixed to a stationary point. Therefore, there are an infinite number of position inverse kinematics solutions.

Now let's consider the desired end-effector orientation. Varying θ_1, θ_2 and θ_3, and preserving the desired position, the end-effector frame will rotate about the $\hat{z}_0$ axis. Then varying θ_4, θ_5, θ_6, and preserving the desired position, the end-effector frame will rotate about an axis on the $\hat{x}_0 - \hat{y}_0$ plane. Therefore, two consequent rotations of constant axes determine the end-effector orienation, which means the feasible orientations of the end-effector are of dimension two. Indeed, the end-effector orientation can be determined by,

$$R = e^{[w_1](\theta_1+\theta_2)}e^{[w_4](\theta_4+\theta_5)}R_0 = e^{[\hat{z}_0](\theta_1+\theta_2)}e^{[\hat{x}_0](\theta_4+\theta_5)}R_0.$$

Therefore, unless the given desired end-effector orientation is a valid orientation, no feasible solution exists.

(b) First determine θ_1, θ_2, and θ_3 for point A. Since $\frac{\theta_3}{\sqrt{2}} = -z_A$, θ_3 can be eaily determined as $\theta_3 = -\sqrt{2}z_A$. From Figure 6.5, D and d are determined as

$$\begin{aligned} D &= \sqrt{x_A^2 + y_A^2} \\ d &= L + \frac{\theta_3}{\sqrt{2}} = L - z_A \end{aligned}$$

Using the law of cosines, α and β can be determined as

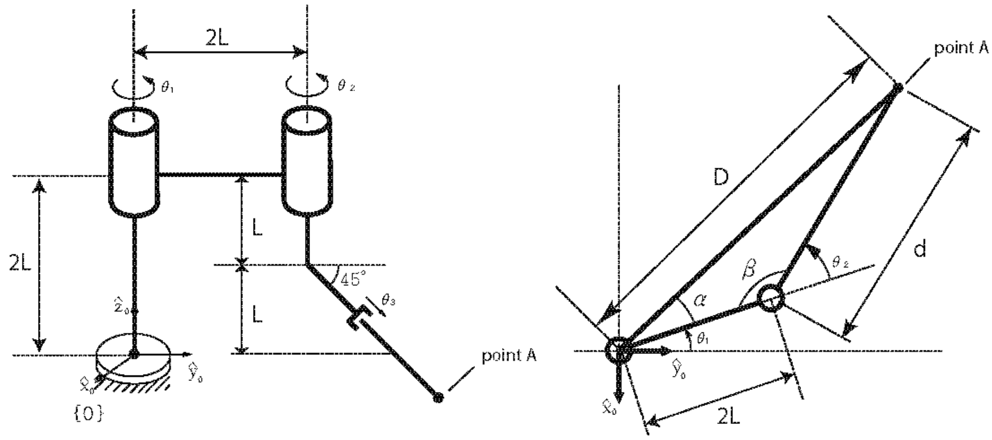

Figure 6.5

$$\begin{aligned} \alpha &= \cos^{-1}\frac{x_A^2 + y_A^2 + 4L^2 - (L - z_A)^2}{4L\sqrt{x_A^2 + y_A^2}} \\ \beta &= \cos^{-1}\frac{4L^2 + (L - z_A)^2 - x_A^2 - y_A^2}{4L(L - z_A)} \end{aligned}$$

For the elbow-down configuration,

$$\begin{aligned} \theta_1 &= \tan^{-1}(-\frac{x_A}{y_A}) - \alpha \\ \theta_2 &= \pi - \beta \end{aligned}$$

For the elbow-up configuration,

$$\begin{aligned} \theta_1 &= \tan^{-1}(-\frac{x_A}{y_A}) + \alpha \\ \theta_2 &= \pi + \beta \end{aligned}$$

Next we determine θ_4 and θ_5 for points A and B. Once θ_1, θ_2, and θ_3 are obtained, we can easily derive the position and orientation of joint 4. Consider a new frame positioned at the center of joint 4 with the same z-axis as frame $\{0\}$ and x-axis in the direction of rotation of joint 4. Then the new coordinates for point B can be determined from the coordinate change. Finally the problem of solving (θ_4, θ_5) boils down to the inverse kinematics problem for a 2R planar robot.

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Exercise 6.7. As shown in Figure 6.6, the slope is mostly horizontal near the initial solution, so the numerical inverse kinematics method jumps over the closer solution and converges to the farther one. The

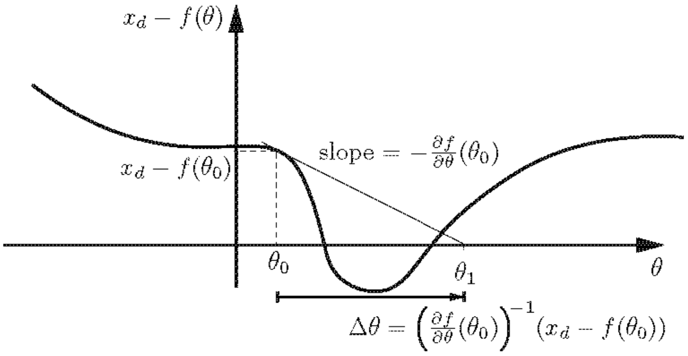

Figure 6.6

basins of attraction for each root depend on the slope of the function $x_d - f(\theta)$ near them. Steeper slopes are more likely to converge to the closer solution.

Exercise 6.8.
The gradient is $[2x, 0; 0, 2y]$. The initial guess and first two iterations are given as: $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (2.5, 5)$, and $(x_3, y_3) = (2.05, 3.4)$. The solution converges to $(x, y) = (2, 3)$ eventually, but $(x, y) = (-2, -3)$, $(x, y) = (-2, 3)$, and $(x, y) = (2, -3)$ are all valid as well. Therefore there are are a total of four solutions.

Exercise 6.9.
Solution converged after three iterations using $\epsilon_\omega = 0.001$ rad (or 0.057°) and $\epsilon_v = 10^{-4}$ m.

i	θ_i (in degrees)	(x, y)	$\mathcal{V}_b = (\omega_{zb}, v_{xb}, v_{yb})$	$\lVert\omega_b\rVert$	$\lVert v_b\rVert$
0	$(0.00, 30.00°)$	$(1.866, 0.500)$	$(3.142, 2.145, 3.717)$	3.142	4.292
1	$(121.2°, 52.82°)$	$(-1.512, 0.960)$	$(0.628, -0.545, 0.594)$	0.628	0.806
2	$(90.00°, 128.4°)$	$(-0.783, 0.378)$	$(-0.147, 0.000, -0.147)$	0.147	0.147
3	$(90.00°, 120.00°)$	$(-0.866, 0.5)$	$(0.000, 0.000, 0.000)$	0.000	0.000

Exercise 6.10.

(a) The orientation of the end-effector is determined by $R_{sb}(\theta) = e^{[\hat{z}]\theta_1}e^{[\hat{y}]\theta_2}e^{-[\hat{z}]\theta_3}$. We define a body twist ω_b for $SO(3)$ as $[\omega_b(\theta)] = \log(R_{sb}^{-1}(\theta)R)$. This leads to the following inverse kinematics algorithm, analogous to that of $SE(3)$:
   1) **Initialization**: Given R and inital guess $\theta^0 \in R^3$. Set $i = 0$
   2) Set $[\omega_b] = \log(R_{sb}^{-1}(\theta^i)R)$. While $\lVert\omega_b\rVert > \epsilon_\omega$ for small ϵ_ω:
      - Set $\theta^{i+1} = \theta_i + J_b^\dagger(\theta^i)\omega_b$.
      - Increment i.

(b) The forward kinematics is described in (a), and the corresponding body jacobian is computed as

$$J_b(\theta) = \left[\ e^{[\hat{z}]\theta_3}e^{-[\hat{y}]\theta_2}\hat{z} \ \middle|\ e^{[\hat{z}]\theta_3}\hat{y} \ \middle|\ \hat{z}\ \right] \in \mathbb{R}^{3\times 3}.$$

With the initial guess $\theta^0 = (0, \pi/6, 0)^T$ and desired end-effector frame R as defined in the problem, a single iteration results in $\theta^1 = (23.5505°, 1.5720°, 23.5505°)^T$.

Exercise 6.11.

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(a) The end-effector position vector $p = (x, y, z)^T$ is determined by $p = \mathrm{Rot}(-\hat{z}, \theta_1)\mathrm{Rot}(\hat{\omega}_2, \theta_2)\mathrm{Rot}(-\hat{y}, \theta_3)(0, L, 0)^T$, where each rotaion matrix can be represented in explicit form as

$$\mathrm{Rot}(-\hat{z}, \theta_1) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\omega}_2, \theta_2) = I + [\hat{\omega}_2]\sin\theta_2 + [\hat{\omega}_2]^2(1-\cos\theta_2) = \begin{bmatrix} \cos\theta_2 & \frac{1}{\sqrt{2}}\sin\theta_2 & -\frac{1}{\sqrt{2}}\sin\theta_2 \\ -\frac{1}{\sqrt{2}}\sin\theta_2 & \frac{1}{2}(1+\cos\theta_2) & \frac{1}{2}(1-\cos\theta_2) \\ \frac{1}{\sqrt{2}}\sin\theta_2 & \frac{1}{2}(1-\cos\theta_2) & \frac{1}{2}(1+\cos\theta_2) \end{bmatrix} \text{ and}$$

$$\mathrm{Rot}(-\hat{y}, \theta_3) = \begin{bmatrix} \cos\theta_3 & 0 & -\sin\theta_3 \\ 0 & 1 & 0 \\ \sin\theta_3 & 0 & \cos\theta_3 \end{bmatrix}.$$

Through straightforward matrix multiplication we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}L\cos\theta_1\sin\theta_2 + \frac{1}{2}L\sin\theta_1(1+\cos\theta_2) \\ -\frac{1}{\sqrt{2}}L\sin\theta_1\sin\theta_2 + \frac{1}{2}L\cos\theta_1(1+\cos\theta_2) \\ \frac{1}{2}L(1-\cos\theta_2) \end{bmatrix}$$

Let, $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a map defined as

$$\begin{bmatrix} x \\ y \end{bmatrix} = f(\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}) = \begin{bmatrix} \frac{1}{\sqrt{2}}L\cos\theta_1\sin\theta_2 + \frac{1}{2}L\sin\theta_1(1+\cos\theta_2) \\ -\frac{1}{\sqrt{2}}L\sin\theta_1\sin\theta_2 + \frac{1}{2}L\cos\theta_1(1+\cos\theta_2) \end{bmatrix}.$$

The Jacobian of f with respect to $\theta = (\theta_1, \theta_2)$ is

$$\frac{\partial f}{\partial \theta} = L\begin{bmatrix} -\frac{1}{\sqrt{2}}\sin\theta_1\sin\theta_2 + \frac{1}{2}\cos\theta_1(1+\cos\theta_2) & \frac{1}{\sqrt{2}}\cos\theta_1\cos\theta_2 - \frac{1}{2}\sin\theta_1\sin\theta_2 \\ -\frac{1}{\sqrt{2}}\cos\theta_1\sin\theta_2 - \frac{1}{2}\sin\theta_1(1+\cos\theta_2) & -\frac{1}{\sqrt{2}}\sin\theta_1\cos\theta_2 - \frac{1}{2}\cos\theta_1\sin\theta_2 \end{bmatrix}.$$

The Newton-Rhapson method is applied via the following iterative update rule for some initial value θ^0:

$$\begin{bmatrix} \theta_1^{k+1} \\ \theta_2^{k+1} \end{bmatrix} = \begin{bmatrix} \theta_1^k \\ \theta_2^k \end{bmatrix} + \begin{bmatrix} \frac{\partial f}{\partial \theta} \end{bmatrix}^{-1}(\begin{bmatrix} x \\ y \end{bmatrix} - f(\begin{bmatrix} \theta_1^k \\ \theta_2^k \end{bmatrix}))$$

Note that given only the position information, θ_3 can be arbitrary.

(b)

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \mathrm{Rot}(-\hat{z}, \theta_1)\mathrm{Rot}(\hat{\omega}_2, \theta_2)\mathrm{Rot}(-\hat{y}, \theta_3)$$

Substituting the explicit forms of the rotation matrices and after straightforward multiplication, we have

$$r_{32} = \frac{1}{2}(1-\cos\theta_2) \tag{6.3}$$

$$r_{12} = \frac{1}{\sqrt{2}}\cos\theta_1\sin\theta_2 + \frac{1}{2}\sin\theta_1(1+\cos\theta_2) \tag{6.4}$$

$$r_{31} = \frac{1}{\sqrt{2}}\cos\theta_3\sin\theta_2 - \frac{1}{2}\sin\theta_3(1+\cos\theta_2) \tag{6.5}$$

From Equation (6.3),

$$\theta_2 = \pm\cos^{-1}(1 - r_{32}).$$

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Let $\rho \sin\phi = \frac{1}{\sqrt{2}}\sin\theta_2$ and $\rho\cos\phi = \frac{1}{2}(1+\cos\theta_2)$ for some positive scalar ρ and angle ϕ determined via

$$\begin{aligned}\rho &= \sqrt{\frac{1}{2}\sin^2\theta_2 + \frac{1}{4}(1+\cos\theta_2)^2} \\ &= \sqrt{1-r_{32}^2} = \sqrt{r_{12}^2+r_{22}^2} = \sqrt{r_{31}^2+r_{33}^2} \\ \phi &= \operatorname{atan2}(\frac{1}{\sqrt{2}}\sin\theta_2, \frac{1}{2}(1+\cos\theta_2))\end{aligned} \tag{6.6}$$

Equation (6.4) then reduces to $r_{12} = \rho\sin(\theta_1+\phi)$. Also, since $\rho^2 = r_{12}^2 + r_{22}^2$ holds from Equation (6.6), we have $r_{22} = \rho\cos(\theta_1+\phi)$. Therefore,

$$\theta_1 = -\phi + \operatorname{atan2}(r_{12}, r_{22}).$$

Similarly, from Equation (6.5) and (6.6) we have $r_{31} = \rho\sin(\phi-\theta_3)$ and $r_{33} = \rho\cos(\phi-\theta_3)$. Therefore,

$$\theta_3 = \phi - \operatorname{atan2}(r_{31}, r_{33}).$$

Exercise 6.12.
$\theta_d = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6) = (8.86937, 44.5978, -77.1251, 82.793, -0.5554, -32.9868)$

Exercise 6.13.
$\theta_d = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6,\theta_7) = (-8.88147, 5.8112, -16.5589, -35.8052, -15.28, 21.0232, 3.64464)$

Exercise 6.14.
First the condition $\frac{\partial g}{\partial x}(x^*)\dot{x}(0) = 0$ implies $\dot{x}(0) \in \text{Null}(\frac{\partial g}{\partial x}(x^*)) = \text{Range}(\frac{\partial g}{\partial x}(x^*)^T)^{\perp}$. Therefore

$$\dot{x}(0) \perp \text{Range}(\frac{\partial g}{\partial x}(x^*)^T)$$

holds. Meanwhile, the second condition implies that $\nabla f(x^*)$ is perpendicular to $\dot{x}(0)$. Since $\text{Range}(\frac{\partial g}{\partial x}(x^*)^T)$ forms the orthogonal space to arbitrary $\dot{x}(0)$, we have $\nabla f(x^*) \in \text{Range}(\frac{\partial g}{\partial x}(x^*)^T)$ which means there exists some vector $c \in \mathbb{R}^m$ such that $\nabla f(x^*) = \frac{\partial g}{\partial x}(x^*)^T c$. Defining the Lagrange multiplier as $\lambda^* = -c$, we obtain the first-order necesary condition of the form

$$\nabla f(x^*) + \frac{\partial g}{\partial x}(x^*)^T\lambda^* = 0.$$

Exercise 6.15.

(a) The formula can be proven through direct multiplication of $\begin{bmatrix} A & D \\ C & B\end{bmatrix}$ and $\begin{bmatrix} A^{-1}+EG^{-1}F & -EG^{-1} \\ -G^{-1}F & G^{-1}\end{bmatrix}$, and checking whether the resulting four sub-matrices equal those of identity matrix.

(b) The first-order necessary condition can be derived as

$$Qx + c + A^T\lambda = 0,$$

where $\lambda \in \mathbb{R}^m$ is a Lagrange multiplier. The above together with the constraint $Ax = b$ can be integrated into a single matrix equation as

$$\begin{bmatrix} Q & A^T \\ A & 0_{m\times m}\end{bmatrix}\begin{bmatrix} x^* \\ \lambda^*\end{bmatrix} = \begin{bmatrix} -c \\ b\end{bmatrix}.$$

Computing the matrix inverse using the formula in (a), we have the following closed form solutions for optimal x^* and corresponding Lagrange multiplier λ^*:

$$\begin{bmatrix} x^* \\ \lambda^*\end{bmatrix} = \begin{bmatrix} Q^{-1} - Q^{-1}A^T(AQ^{-1}A^T)A^TQ^{-1} & Q^{-1}A^T(AQ^{-1}A^T)^{-1} \\ (AQ^{-1}A^T)^{-1}A^TQ^{-1} & (AQ^{-1}A^T)^{-1}\end{bmatrix}\begin{bmatrix} -c \\ b\end{bmatrix}$$

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