Chapter 12: Nonholonomic and Underactuated Systems

Overview

Every Robot system is subject to a variety of motion constraints, but not all of these can be expressed as configuration constraints. A familiar example of such a system is a car. At low speeds, the rear wheels of the car roll freely in the direction they are pointing, but they prevent slipping motion in the perpendicular direction. This constraint implies that the car cannot translate directly to the side. We know by experience, however, that this velocity constraint does not imply a constraint on configurations; the car can reach any position and orientation in the obstacle-free plane. In fact, the prevented sideways translation can be approximated by parallel-parking maneuvers.

This no-slip constraint is a *nonholonomic constraint*, a constraint on the velocity. In addition to rolling without slipping, conservation of angular momentum is a common source of nonholonomic constraints in mechanical systems.

If, instead of viewing the car as a system subject to a motion constraint, we considered the fact that there are only two inputs (speed and steering angle) to control the car's three degrees of freedom, we might call the system *underactuated*. Under- actuated systems have fewer controls than degrees of freedom. For second-order mechanical systems, such as those described in the previous chapter, underactuation implies equality constraints on the possible accelerations of the system.

In this section we study motion planning for systems that are underactuated or subject to motion constraints. Our first task is to determine if the constraints actually limit the reachable states of the robot system. This is a controllability question. The next problem is to construct algorithms that find motion plans that satisfy the motion constraints. A last problem, not addressed in this chapter, is feedback stabilization of the motion plans during execution.

We begin in section 12.1 by providing some background information on vector fields and their Lie (pronounced "lee") algebras. Section 12.2 defines the class of control systems we will consider. Section 12.3 describes different controllability notions and tests for these nonlinear systems. Section 12.4 specializes the discussion to second-order mechanical systems. Finally, section 12.5 describes a number of methods for motion planning for nonholonomic and underactuated systems.

12.1 Preliminaries

First we must decide how generally to define the state spaces of the robotic systems we will consider. For example, we could treat a very general case, allowing the state space of the system to be any smooth manifold. This would allow us to study, e.g., the motion of a spherical pendulum. The configuration space of this system is the sphere S^2. Or we could limit our treatment to systems evolving on Lie groups, particularly matrix Lie groups. This would allow us to model the orientation of a satellite as a point in $SO(3)$.

In this chapter, we restrict our attention even further to systems evolving on vector spaces $\mathcal{M} = \mathbb{R}^n$. This allows us to get to the main results as quickly as possible. Also, any n-dimensional manifold is locally "similar" (diffeomorphic) to $\mathbb{R}^n$, so, equipped with a proper set of local coordinates, any n-dimensional manifold can be treated locally as $\mathbb{R}^n$. By making this simplification, we require the use of a local coordinate system in our computations, and we may lose information about the global structure of the space. As examples, the true configuration space of a 2R robot arm is the torus $T_2 = S_1 \times S_1$, which is doughnut-shaped while $\mathbb{R}^2$ is not; and a global representation of the orientation of a satellite is $SO(3)$, which is different from a local representation using three Euler angles ($(\mathbb{R}^3)$). See figure 12.1 for another example.

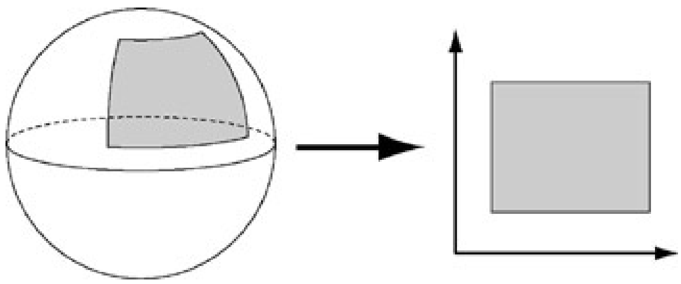

Figure 12.1: Latitude and longitude coordinates allow us to treat a patch of the sphere S^2 as a section of the plane $\mathbb{R}^2$
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Although we focus on vector state spaces, most of the ideas in this chapter generalize immediately to general manifolds.

In this chapter, $q \in Q$ denotes the configuration of the system and

$x \in \mathcal{M}$ denotes the state of the system. If the system is kinematic, then the state is simply the configuration $(\mathcal{M} = \mathcal{Q})$, and the controls are velocities. If the system is a second-order mechanical system, then *x* includes both configurations *q* and velocities $\dot{q}$, and the controls are forces (accelerations). The dimension of the configuration space $\mathcal{Q}$ is $n_{\mathcal{Q}}$, and the dimension of the state space $\mathcal{M}$ is *n*.

We will carry two examples throughout the chapter: a unicycle, a kinematic system; and a model of a planar spacecraft, a second-order mechanical system. We will treat all systems uniformly, as systems with state *x* on a state space $\mathcal{M}$. Only in section 12.4 and subsection 12.5.7 will we specialize our study to second-order mechanical systems such as the spacecraft model.

EXAMPLE 12.1.1: Unicycle example.

The unicycle is a wheel that rolls upright on a horizontal plane (figure 12.2). The configuration of the wheel is $q = [q_1, q_2, q_3]^T$, describing the contact point of the wheel on the plane (q_1, q_2) and the steering angle q_3 of the wheel. (We could also include the rolling angle of the wheel, i.e., the location of the air nozzle on the tire, in the description of the configuration, but we will ignore this for now.) The system is kinematic, so $x = [x_1, x_2, x_3]^T = q$, $\mathcal{M} = \mathcal{Q} = \mathbb{R}^3$, and $n_{\mathcal{Q}} = n = 3$. (Since we are dealing with local coordinates, we are ignoring the fact that the global structure of the space is $\mathbb{R}^2 \times S^1$. This will not affect the equations of motion, but requires the use of *mod*2π arithmetic on the third coordinate.) The controls are the rolling speed of the wheel and the rate of change of the steering angle. Sideways translation of the wheel is prevented by the no-slip constraint imposed by the wheel. This example is sometimes known as the rolling penny, or the pizza cutter, and it is similar to a model for a car.

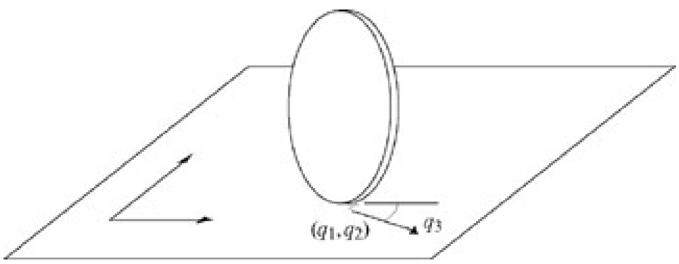

Figure 12.2: The unicycle system. The position of the point of contact is given by (q_1, q_2), and the heading direction is given by q_3.

EXAMPLE 12.1.2: Planar body with thrusters (PBWT) example.

The body moves in a frictionless, inviscid plane by means of two thrusters fixed to the body (figure 12.3). The mass and inertia of the body (about the center of mass) are unit. The line of action of the thrust u_1 is through the center of mass, and the line of action of the thrust u_2 is perpendicular and a distance d from the center of mass. The configuration is $q = [q_1, q_2, q_3]^T$, describing the location of the center of mass (q_1, q_2) and the angle q_3 of the line of action of the first thruster relative to the world q_1-axis. The system is second-order, so $x = [x_1, x_2, x_3, x_4, x_5, x_6]^T = [q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3]^T$, $\mathcal{M} = \mathbb{R}^6$, $n_Q = 3$, and $n = 2n_Q = 6$. Gravitational acceleration a^g acts in the $-q_2$-direction, and a_g may be zero.

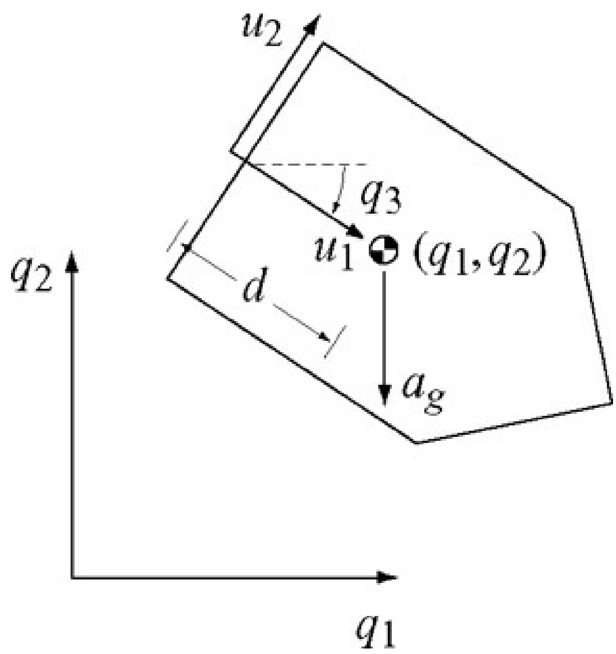

Figure 12.3: The planar body with thrusters (PBWT).

The rest of section 12.1 introduces concepts from differential geometry that will be useful in understanding underactuated systems. For the unicycle, e.g., we will see that its instantaneous motions can be described in terms of two "vector fields" associated with the controls to drive and steer the unicycle. Linear combinations of these two vector fields define a "distribution" describing all possible instantaneous motions of the unicycle. The "integral manifold" describes all the states the system can reach by following vector fields in the distribution. We use the "Lie bracket" to show that two vector fields in the distribution can generate a parallel-parking motion for the unicycle, effectively giving it a sideways motion, meaning that the integral manifold is the entire configuration space—the velocity constraint does not reduce the reachable space.

Section 12.2 describes how a robot system can be expressed as a system of vector fields and controls, and section 12.3 uses the concepts developed in this section to study the set of states reachable by the controls.

12.1.1 Tangent Spaces and Vector Fields

Let $x : \mathbb{R} \rightarrow \mathcal{M}$ be a smooth curve on $\mathcal{M}$ parameterized by *s*. Then dx/ds, evaluated at $x_0 = x(s^0)$, is tangent to the curve at x_0. Call this vector *V*. The vector *V* is a *tangent vector* that is tangent to $\mathcal{M}$ at x_0. The tangent vector *V* lives in $T_{x_0}\mathcal{M}$, the *tangent space* of $\mathcal{M}$ at x_0. This space is an *n*-dimensional vector space $\mathbb{R}^n$ consisting of the tangents of all possible curves passing through x_0 (figure 12.4). The tangent spaces at different points of $\mathcal{M}$ are different spaces.

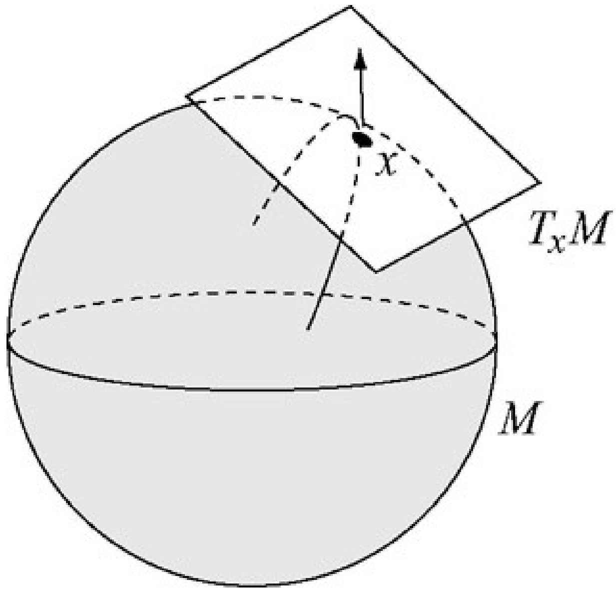

Figure 12.4: A curve on the sphere $\mathcal{M}$, a tangent vector to the curve at *x*, and the tangent space $T_x \mathcal{M}$ it lives in.

The *tangent bundle* of $\mathcal{M}$, written $T\mathcal{M}$, is the 2*n*-dimensional manifold that is the union of tangent spaces at all points in $\mathcal{M}$,

$$T\mathcal{M} = \bigcup_{x \in \mathcal{M}} T_x\mathcal{M}.$$

For the systems we study, $T\mathcal{M} = \mathcal{M} \times \mathbb{R}^n = \mathbb{R}^{2n}$. [1]

A smooth *vector field* $g : \mathcal{M} \to T\mathcal{M}$ is a smooth map from points $x \in \mathcal{M}$ to tangent vectors $g(x) \in T_x\mathcal{M}$. It is possible to define C^k vector fields, but we will assume that all vector fields are infinitely differentiable. (For example, the vector field $g(x) = [x_{21}, \sin x_3, x_1x_2]^T$ is infinitely differentiable, but $[|x_1|, x_2, x_3]_T$ is only C^0.) A picture of the vector field $\frac{1}{2}[-x_2, x_1]^T$ on $\mathbb{R}^2$ is shown in figure 12.5. Tangent vectors are written as column vectors.

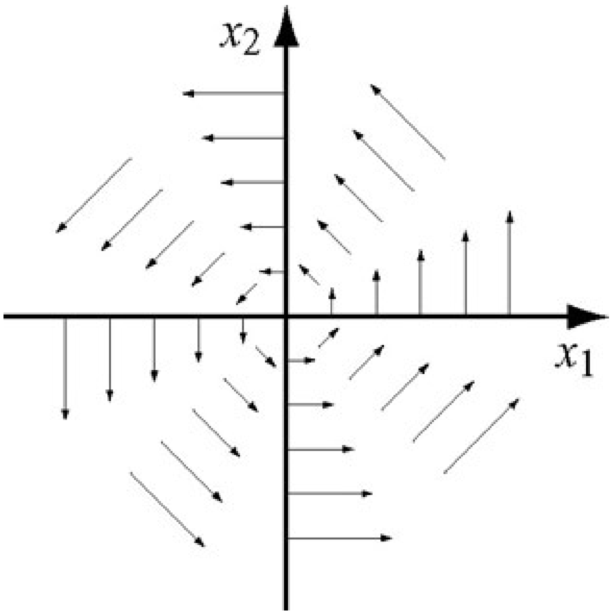

Figure 12.5: The vector field $\frac{1}{2}[-x_2, x_1]^T$

In the case of a kinematic system, $\mathcal{M}$ is the configuration space $\mathcal{Q}$, and $T_{x_0}\mathcal{M} = T_{q_0}\mathcal{Q}$ is the set of all possible velocities of the system at $x_0 = q^0$. In the case of a second-order system, $\mathcal{M}$ is the state space $T\mathcal{Q}$, and $T_{x_0}\mathcal{M} = T_{x_0}T\mathcal{Q}$ is the set of all possible velocities and accelerations of the system at $x_0 = [q_0^T, \dot{q}_0^T]^T$. In this case, however, the state x_0 already specifies the velocity portion $\dot{q}_0$ of the tangent vector $[\dot{q}_0^T, \ddot{q}_0^T]^T$. This implies *drift* in second-order systems, as shown in the PBWT example below.

EXAMPLE 12.1.3: Unicycle (cont.)

A tangent vector for the unicycle is given by $\dot{x} = [\dot{x}_1, \dot{x}_2, \dot{x}_3]^T = [\dot{q}_1, \dot{q}_2, \dot{q}_3]^T$. The unicycle is capable of rolling forward and backward and spinning in place. These two vector fields can be written $g^{uni}{}_1(x) = [\cos x_3, \sin x_3, 0]^T$, rolling forward at unit speed, and $g^{uni}{}_2(x) = [0, 0, 1]_T$, spinning counterclockwise at unit speed. The vector fields can also be written as $g^{uni}{}_1(x) = (\cos x_3)\partial/\partial x_1 + (\sin x_3)\partial/\partial x_2$ and $g^{uni}{}_2(x) = \partial/\partial x_3$, where $\partial/\partial x_1$, $\partial/\partial x_2$, and $\partial/\partial x_3$ are the canonical unit basis vectors of the tangent space, i.e., unit speed tangent vectors along the coordinates (see figure 12.6).

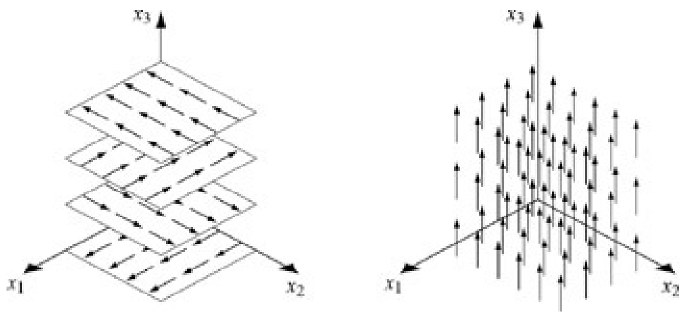

Figure 12.6: The vector fields $g^{\mathrm{uni}}{}_1 = [\cos x_3, \sin x_3, 0]^T$ (shown in constant x_3 layers) and $g^{\mathrm{uni}}{}_2 = [0, 0, 1]_T$.

EXAMPLE 12.1.4: PBWT (cont.)

A tangent vector for the PBWT is given by $\dot{x} = [\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5, \dot{x}_6]^T = [\dot{q}_1, \dot{q}_2, \dot{q}_3, \ddot{q}_1, \ddot{q}_2, \ddot{q}_3]^T$. For this system, we can define three vector fields: the *drift* vector field $g^{pbwt}{}_0(x)$ corresponding to the motion of the body when no thrusters are activated, and the *control* vector fields $g^{pbwt}{}_1(x)$ and $g^{pbwt}{}_2(x)$ corresponding to the acceleration when thrusters 1 and 2 are fired with unit thrust, respectively. Verify that $g^{pbwt}{}_0(x) = [x_4, x_5, x_6, 0, a_g, 0]^T$, $g^{\mathrm{pbwt}}{}_1(x) = [0, 0, 0, \cos x_3, \sin x_3, 0]^T$, and $g_{pbwt2}(x) = [0, 0, 0, -\sin x_3, \cos x_3, -d]$, and write these vector fields in the canonical basis $\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \partial/\partial x_4, \partial/\partial x_5, \partial/\partial x_6\}$. Notice if thruster 1 is fired with thrust u_1, the system follows the vector field $g^{pbwt}{}_0(x) + u_1 g^{pbwt}{}_1(x)$.

Let ϕ^g denote the *flow* of the vector field g, where $\phi^g{}_t(x)$ gives the system state after following the flow ϕ^g from x for a time t. The flow satisfies the equation

$$\frac{d}{dt}\phi_t^g(x) = g\left(\phi_t^g(x)\right).$$

The vector field is *complete* if its flow is defined for all x and t.

The curve $\{\phi_t^g(x) \mid t \in \mathbb{R}\}$ is the *integral curve* of g containing x. The integral curve describes the set of reachable points of $\mathcal{M}$ from x by following the vector field forward and backward in time (figure 12.7). This notion can be generalized to the *integral manifold* of a set of vector fields $\mathcal{G}$, a topic for subsection 12.1.3.

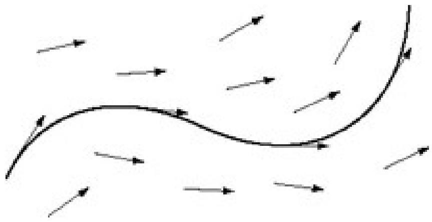

Figure 12.7: An integral curve of a vector field.

12.1.2 Distributions and Constraints

Let $\mathcal{G}$ be a set of vector fields, and let $(\mathcal{G})$ be the linear span of vector fields in $\mathcal{G}$, given by all linear combinations of vector fields in $\mathcal{G}$. At each point $x \in \mathcal{M}$, these vector fields span a linear subspace of $T_x\mathcal{M}$. The set of vector fields $\mathcal{G}$ is said to generate a *distribution* $\mathcal{D} \subseteq T\mathcal{M}$, which is a smooth assignment of a linear subspace of $T_x\mathcal{M}$ for each $x \in \mathcal{M}$. A distribution is *regular* if the dimension of the linear subspace is the same at all x.If the dimension is m, then we say that it is an m-dimensional distribution.

Consider the two-dimensional regular distribution for the unicycle $\mathcal{D} = \text{span}(\{g_1^{\text{uni}}, g_2^{\text{uni}}\}) = u_1 g_1^{\text{uni}}(x) + u_2 g_2^{\text{uni}}(x), u_2 \in \mathbb{R}$. We might think of this as the "positive" form of the distribution—feasible motions are generated by linear combinations of the vector fields. A "negative" form of the distribution would start with all motions being feasible, then eliminate those that violate motion constraints. For instance, the unicycle distribution could be written

(12.1) $$\mathcal{D}(x) = \{\dot{x} \in T_x\mathcal{M} \mid \omega(x)\dot{x} = 0\}, \quad \omega(x) = [-\sin x_3, \cos x_3, 0], \quad (12.1)$$

where $\mathcal{D}(x)$ is the linear subspace of $T_x\mathcal{M}$ defined by the distribution $\mathcal{D}$. A row vector ω(x) is called a *covector* and lives in the *cotangent space* $T_x^*\mathcal{M} = \mathbb{R}^n$, the dual of $T_x\mathcal{M}$ consisting of all linear functionals of elements of $T_x\mathcal{M}$. In other words, a *covector field* ω pairs with a vector field g to yield a real value, $\omega(x)g(x) \in \mathbb{R}$. This is sometimes called the "natural pairing" of a tangent vector and covector. The canonical basis of covector fields is {dx_1, …,

$dx_n\}$,so that the constraint ω(x) in equation (12.1) can be written as $-\sin x_3 dx_1 + \cos x_3 dx_2$.

A covector field ω is sometimes known as a *one-form*, because it takes a single element of $T_x\mathcal{M}$ and produces a real number, linear in the tangent vector. A *two-form*, as we will see in section 12.4, takes two elements of $T_x\mathcal{M}$ and produces a real number, linear in each of the arguments.

The *cotangent bundle* $T^*\mathcal{M}$ is the union of cotangent spaces $T_x^*\mathcal{M}$ for all $x \in \mathcal{M}$. A set of covector fields $\{\omega_1(x),\ldots, \omega_k(x)\}$ is said to define a *codistribution* $\Omega \subseteq T^*\mathcal{M}$. If the covector fields $\omega_i(x)$, $i = 1 \ldots k$, correspond to motion constraints $\omega_i(x)\dot{x} = 0$ then Ω is called a *constraint* codistribution, and it is said to *annihilate* the distribution $\mathcal{D}$ of feasible motions, and vice versa. Of special interest are velocity constraints of the form

$$f(q, \dot{q}) = 0 \tag{12.2}$$

that cannot be integrated to yield equivalent configuration constraints. Such constraints are called *nonholonomic*. Nonholonomic constraints of the form

$$a(q)\dot{q} = 0$$

are sometimes called *Pfaffian* constraints, as discussed in chapter 10. Pfaffian constraints arise from rolling without slip [e.g., see equation (12.1)] and conservation of angular momentum. In mechanical systems, the covector field $a(q)$ can be interpreted as a generalized force, so $a(q)\dot{q}$ has units of power, and the constraint $a(q)\dot{q} = 0$ is *passive*—it does no work on the system.

In second-order underactuated systems, the underactuation implies the existence of acceleration constraints of the form

$$f(q, \dot{q}, \ddot{q}) = 0.$$

Constraints of this form that cannot be integrated to equivalent velocity constraints are sometimes referred to as "second-order nonholonomic" constraints, but this terminology is not standard.

In general, it is not easy to determine if an acceleration constraint can be integrated to yield an equivalent velocity constraint, or if a velocity constraint can be integrated to yield an equivalent configuration constraint. In the rest of this chapter, we use the "positive" form of the distribution and study the reachable set by vector fields in $\mathcal{G}$.

12.1.3 Lie Brackets

Let $\mathcal{G}$ be a set of vector fields and $\mathcal{D}$ be the distribution defined by $(\mathcal{G})$. We would like to know the reachable set of $\mathcal{M}$ by following vector fields in $\mathcal{D}$. While this is generally difficult globally, it is possible to learn something about the reachable set *locally* by looking at the *Lie brackets* of vector fields in $\mathcal{D}$. Given two vector fields belonging to $\mathcal{D}$, the Lie bracket tells us if infinitesimal motions along these vector fields can be used to locally generate motion in a direction not contained in $\mathcal{D}$. Perhaps the best-known example is the parallel-parking maneuver for a car or, in our case, a unicycle. Direct sideways motion is prohibited by the no-slip constraint, but sideways motion can be approximated by a series of forward-backward and turning maneuvers. The implication of this is that the locally reachable set of $\mathcal{M}$ is not two- dimensional, as the two-dimensional distribution $\mathcal{D}$ might seem to indicate, but fully three-dimensional. The no-slip velocity constraint does not imply a constraint on reachable configurations.

For two vector fields $g_1, g_2 \in \mathcal{G}$, consider the state reached from $x_0 = x(0)$ by first following g_1 for a small time $\epsilon < 1$, then following g_2 for time ϵ, then following $-g_1$ for time ϵ, then following $-g_2$ for time ϵ. This is expressed mathematically as

(12.3) $$x(4\epsilon) = \phi_\epsilon^{-g_2}\left(\phi_\epsilon^{-g_1}\left(\phi_\epsilon^{g_2}\left(\phi_\epsilon^{g_1}(x_0)\right)\right)\right). \tag{12.3}$$

We can take a Taylor series in ϵ to solve the differential equation (12.3) approximately (see, e.g., [330] and problem 29), yielding

(12.4) $$x(4\epsilon) = x_0 + \epsilon^2\left(\frac{\partial g_2}{\partial x} g_1(x_0) - \frac{\partial g_1}{\partial x} g_2(x_0)\right) + O(\epsilon^3), \tag{12.4}$$

where the partial derivatives are evaluated at x_0 and $O(\epsilon^3)$ indicates terms of order ϵ^3, which are dominated by the term of order ϵ^2 when ϵ is small. Note there are no $O(\epsilon)$ terms. The ϵ^2 term represents the approximate net motion of the system, and the term inside the parentheses is the Lie bracket of g_1 and g_2.

The Lie bracket of g_1 and g_2 is written $[g_1, g_2]$ and is given in local coordinates by

(12.5)
$$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2. \tag{12.5}$$

The Lie bracket $[g_1, g_2]$ defines a new vector field, and if it is not contained in $(\mathcal{G})$, then it represents a new motion direction that can be followed approximately. Locally generating motion in this direction is "slower" than following the vector field g_1 or g_2 directly, as the net motion is only $O(\epsilon^2)$ for time $O(\epsilon)$, where $\epsilon < 1$. Again, parallel parking is a well-known example, as approximately generating sideways motion by forward-backward and turning motions is tedious and time-consuming. If $[g_1, g_2] = 0$, then no new motion is created, and the two vector fields are said to *commute*.

Since $[g_1, g_2]$ is a vector field, we can calculate its Lie bracket with another vector field. A *Lie product of degree k* is a bracket term where the original vector fields appear k times. For instance, $[[[g_1, g_2], g_1], g_2]$isa Lie product of degree 4.

EXAMPLE 12.1.5: Unicycle (cont.)

The rolling and turning vector fields for the unicycle are $g^{uni}{}_1 = [\cos x_3, \sin x_3, 0]^T$ and $g^{uni}{}_2 = [0, 0, 1]^T$, respectively. So

$$
\begin{aligned}
[g_1^{\text{uni}}, g_2^{\text{uni}}] &= \frac{\partial g_2^{\text{uni}}}{\partial x} g_1^{\text{uni}} - \frac{\partial g_1^{\text{uni}}}{\partial x} g_2^{\text{uni}} \\
&= \begin{bmatrix} \frac{\partial g_2^{\text{uni}}}{\partial x_1} & \frac{\partial g_2^{\text{uni}}}{\partial x_2} & \frac{\partial g_2^{\text{uni}}}{\partial x_3} \end{bmatrix} g_1^{\text{uni}} - \begin{bmatrix} \frac{\partial g_1^{\text{uni}}}{\partial x_1} & \frac{\partial g_1^{\text{uni}}}{\partial x_2} & \frac{\partial g_1^{\text{uni}}}{\partial x_3} \end{bmatrix} g_2^{\text{uni}} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Note that the Lie bracket motion is to the side, in the direction prevented by the no-slip constraint (see figure 12.8).

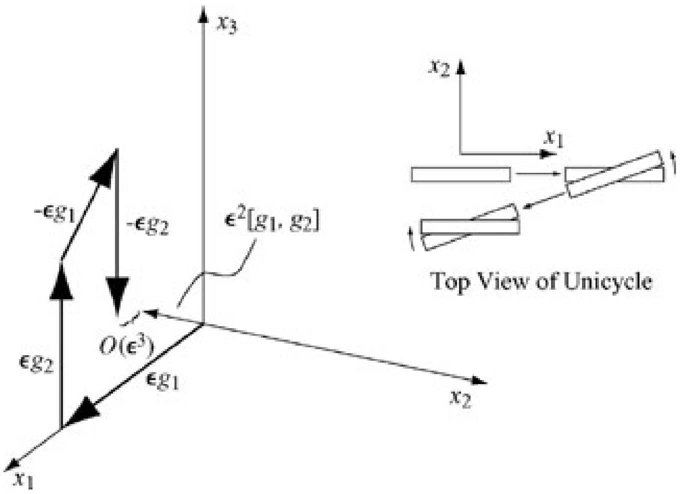

Figure 12.8: Generating a Lie bracket motion for the unicycle, starting from the origin. The net motion is approximately to the side, the Lie bracket direction. It is not exactly to the side, however, due to higher-order terms in ϵ.

The *Lie algebra* of a set of vector fields $\mathcal{G}$, written $\overline{\text{Lie}}(\mathcal{G})$, is the linear span of all Lie products, of all degrees, of vector fields in $\mathcal{G}$. To determine the Lie algebra, define $\mathcal{G}_1 = \mathcal{G}$, and the series

$$\mathcal{G}_{i+1} = \mathcal{G}_i \cup \{[g_j, g_k] \mid \forall g_j \in \mathcal{G}_1, g_k \in \mathcal{G}_i\}.$$

Then $\overline{\text{Lie}}(\mathcal{G})$ is given by the distribution $\text{span}(\mathcal{G}_\infty)$. Forexample, the series for $\mathcal{G} = \{g_1, g_2\}$ begins

$$\mathcal{G}_1 = \{g_1, g_2\}$$
$$\mathcal{G}_2 = \mathcal{G}_1 \cup \{g_3 = [g_1, g_2]\}$$
$$\mathcal{G}_3 = \mathcal{G}_2 \cup \{g_4 = [g_1, g_3], g_5 = [g_2, g_3]\}$$
$$\mathcal{G}_4 = \mathcal{G}_3 \cup \{g_6 = [g_1, g_4], g_7 = [g_1, g_5], g_8 = [g_2, g_4], g_9 = [g_2, g_5]\}$$
$$\vdots$$

The corresponding series $\mathcal{D}_1 = \mathrm{span}(\mathcal{G}_1), \mathcal{D}_2 = \mathrm{span}(\mathcal{G}_2), \cdots$, … is called the *filtration* of the distribution $\mathcal{D}_1$. The filtration is *regular* if each distribution in the filtration is regular. If the filtration is regular, then the dimension of the distribution grows at each step of the construction, or else the construction terminates. (Of course, $\dim(\mathcal{D}_i) \leq n = \dim(\mathcal{M})$ for all *i*.) If the filtration is regular, we are guaranteed a finite value of *k* such that $\mathcal{D}_k = \mathcal{D}_{k+1} = \cdots = \mathcal{D}_\infty$. This distribution is the *involutive closure* $\overline{\mathcal{D}}$ of $\mathcal{D}$, and a distribution $\mathcal{D}$ is *involutive* if $\mathcal{D} = \overline{\mathcal{D}}$.

If the filtration is not regular, then in general there is no way to know a priori a degree *k* at which $\mathcal{D}_k = \overline{\mathcal{D}}$. If there is a degree *k* at which all Lie products become zero, then the Lie algebra is called *nilpotent of order k*.

The *integral manifold* of $\mathcal{D}$ containing x_0 is the set of $\mathcal{M}$ that can be reached from x_0 by vector fields in $\mathcal{D}$, and $\mathcal{D}(x)$ is the tangent space of the integral manifold at *x*.By the well known *Frobenius theorem*,an *m*-dimensional regular distribution $\mathcal{D}$ can be integrated to yield an *m*-dimensional integral manifold if and only if $\mathcal{D}$ is involutive.

If a distribution $\mathcal{D}$ does not have the entire space $\mathcal{M}$ as an integral manifold, then $\mathcal{D}$ is said to generate a *foliation* of $\mathcal{M}$, and each distinct integral manifold is called a *leaf* of the foliation. Consider, e.g., the one-dimensional distribution generated by $g^{\mathrm{uni}}_2 = [0, 0, 1]^T$ (turning motion) for the unicycle (see figure 12.6). The distribution is one-dimensional, regular, and involutive, and the integral manifolds are lines in x_3 (wrapping around at 2π) with fixed position (x_1, x_2). The unicycle is confined to the same leaf of the foliation for all time if it can only follow this vector field. A more interesting example of a foliation is given in example 12.1.7 below.

The existence of integral manifolds smaller than the whole state space $\mathcal{M}$ indicates that the motion constraints actually limit the reachable state space. For example, velocity constraints on a kinematic system might be integrated to yield configuration constraints, indicating that the original constraints are actually holonomic. Similarly, acceleration constraints on a mechanical system might be integrated to yield velocity or even configuration constraints.

Lie brackets satisfy the following properties:

1. Skew-symmetry:

$$[g_1, g_2] = -[g_2, g_1]$$

2. Jacobi identity:

$$[g_1, [g_2, g_3]] + [g_3, [g_1, g_2]] + [g_2, [g_3, g_1]] = 0$$

Taking these properties into account, the *Philip Hall basis* gives a way to choose the smallest number of Lie products that must be considered at each degree *k* to generate a basis for the distribution $\mathcal{D}_k$. See the book by Serre [380] for details.

EXAMPLE 12.1.6: Unicycle (cont.)

From before, we have $g^{uni}_1 = [\cos x_3, \sin x_3, 0]^T$, $g^{uni}_2 = [0, 0, 1]^T$, and $g^{uni}_3 = [g^{uni}_1, g^{uni}_2] = [\sin x_3, -\cos x_3, 0]^T$. The dimension of the distribution defined by $\{g^{uni}_1, g^{uni}_2, g^{uni}_3\}$ is three at all $x \in \mathcal{M}$, implying that the distribution is regular. It is also certainly involutive, since the dimension of $\mathcal{M}$ is three. To see that the three vector fields are indeed linearly independent, we define the 3 × 3 matrix $[g^{uni}_1\ g^{uni}_2\ g^{uni}_3]$ obtained by placing the column vectors side by side. The rank of the matrix is 3 at all $x \in \mathcal{M}$, which can be verified by the determinant

$$\det\left[g_1^{\text{uni}}\ g_2^{\text{uni}}\ g_3^{\text{uni}}\right] = \det\begin{bmatrix} \cos x_3 & 0 & \sin x_3 \\ \sin x_3 & 0 & -\cos x_3 \\ 0 & 1 & 0 \end{bmatrix} = 1.$$

Since the distribution is regular and involutive, it has a three-dimensional integral manifold, which is the entire space $\mathcal{M}$. The distribution $\mathcal{D}_2$ is the involutive closure of $\mathcal{D}_1$. The filtration is regular.

EXAMPLE 12.1.7

Define the vector fields $g_1(x) = [x_1 \cos x_3, x_2 \sin x_3, 0]$ and $g_2 = [0, 0, 1]^T$ on $\mathbb{R}^3$. The vector field g_2 by itself defines a regular one-dimensional involutive distribution. The vector field g_1 does not, however, as it vanishes at $x_1 = x_2 = 0$. The Lie bracket of these vector fields is $[g_1, g_2] = [x_1 \sin x_3, -\cos x_3, 0]^T$, and

$$\det[g_1\ g_2\ [g_1, g_2]] = x_1 x_2.$$

This means that the distribution *span*($\{g_1, g_2, [g_1, g_2]\}$) is rank 3 at points where both x_1 and x_2 are nonzero. It is not regular, as the rank is less at points where either x_1 or x_2 is zero. In fact, it is not hard to see that the integral manifold of this distribution is one-dimensional from points $[0, 0, x_3]^T$, two-dimensional from points $[x_1 \neq 0, 0, x_3]^T$ and $[0, x_2 \neq 0, x_3]^T$, and three-dimensional from all other points. The foliation is pictured in figure 12.9.

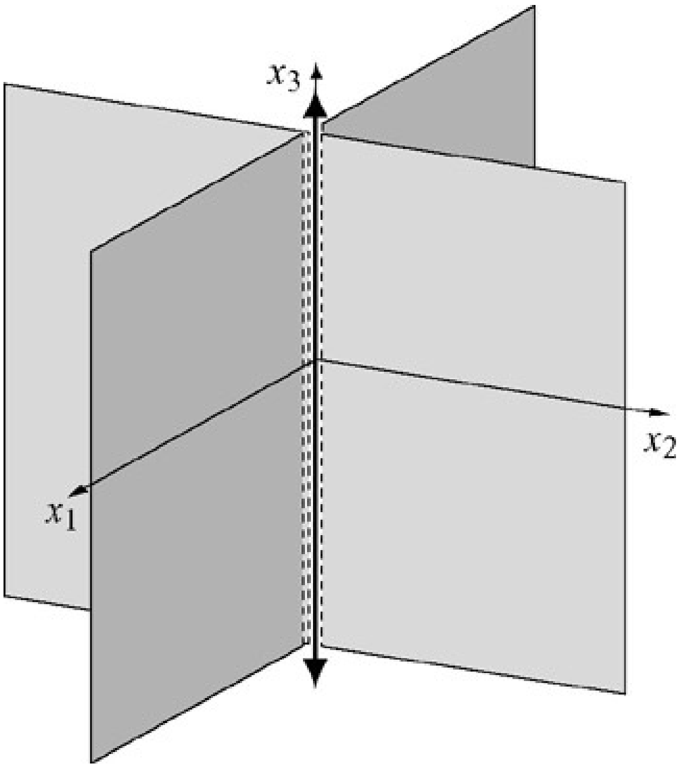

Figure 12.9: The distribution span($\{g_1, g_2, [g_1, g_2]\}$)in example 12.1.7 foliates the state space into nine separate leaves: the line defined by $x_1 = x_2 = 0$, four half-planes, and four three-dimensional quadrants.

EXAMPLE 12.1.8: PBWT (cont.)

As derived previously, we have $g^{pbwt}{}_0 = [x_4, x_5, x_6, 0, a_g, 0]^T$, $g^{pbwt}{}_1 = [0, 0, 0, \cos x_3, \sin x_3, 0]^T$, and $g^{pbwt}{}_2 = [0, 0, 0, -\sin x_3, \cos x_3, -d]^T$. Lie bracket computations show that

$$\begin{aligned}
g_3^{\text{pbwt}} &= [g_0^{\text{pbwt}}, g_1^{\text{pbwt}}] \\
&= [-\cos x_3, -\sin x_3, 0, -x_6 \sin x_3, x_6 \cos x_3, 0]^T \\
g_4^{\text{pbwt}} &= [g_0^{\text{pbwt}}, g_2^{\text{pbwt}}] \\
&= [\sin x_3, -\cos x_3, d, -x_6 \cos x_3, -x_6 \sin x_3, 0]^T \\
g_5^{\text{pbwt}} &= [g_1^{\text{pbwt}}, [g_0^{\text{pbwt}}, g_2^{\text{pbwt}}]] \\
&= [0, 0, 0, d \sin x_3, -d \cos x_3, 0]^T \\
g_6^{\text{pbwt}} &= [g_0^{\text{pbwt}}, [g_1^{\text{pbwt}}, [g_0^{\text{pbwt}}, g_2^{\text{pbwt}}]]] \\
&= [-d \sin x_3, d \cos x_3, 0, d x_6 \cos x_3, d x_6 \sin x_3, 0]^T.
\end{aligned}$$

A computation shows that

$$\det\left[g_1^{\text{pbwt}}\ g_2^{\text{pbwt}}\ g_3^{\text{pbwt}}\ g_4^{\text{pbwt}}\ g_5^{\text{pbwt}}\ g_6^{\text{pbwt}}\right] = d^4.$$

The dimension of the distribution defined by these vector fields is six at all $x \in \mathcal{M}$ (provided $d \neq 0$), so the distribution is both regular and involutive. The integral manifold is the entire space $\mathcal{M}$. The distribution $\mathcal{D}_4$ is the involutive closure of $\mathcal{D}_1$.

We now apply the ideas of this section to study controllability of underactuated systems, taking into account the fact that *controls* determine how the system vector fields are followed. It may not be possible to follow arbitrary linear combinations of system vector fields. For example, the drift vector field g^{pbwt}_0 of the PBWT is fundamentally different from the control vector fields g^{pbwt}_1 and g^{pbwt}_2.

[1]We note that if $\mathcal{M}$ is a more general manifold, and it is *parallelizable* (e.g., a Lie group), then $T\mathcal{M} = \mathcal{M} \times \mathbb{R}^n$. The reader should be careful not to generalize improperly, however. For example, $TS^2 \neq S^2 \times \mathbb{R}^2$, for reasons beyond the scope of this chapter. For an intuitive discussion of this issue, see [372].

12.2 Control Systems

Afamily of vector fields $\mathcal{G}$ on a manifold $\mathcal{M}$ is sometimes called a *dynamical polysystem*. The system is *symmetric* if for every $g \in \mathcal{G}$, $-g$ is also in $\mathcal{G}$.

The family of dynamical polysystems we will study are *control affine* nonlinear control systems, written

(12.6)
$$\dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad u \in \mathcal{U} \subset \mathbb{R}^m. \tag{12.6}$$

The vector field g_0 is called the *drift vector field*, defining the natural unforced motion of the system, and the g_i, $i = 1 \ldots m$, are linearly independent *control vector fields*. The control vector u belongs to the control set $\mathcal{U}$, and $u(t)$ is piecewise continuous. If $g_0 = 0$, the system is called *drift-free* or *driftless*. Kinematic systems (such as the unicycle) may be drift-free, but second-order systems (such as the PBWT) are not.

EXAMPLE 12.2.1: Unicycle (cont.)

The control system for the unicycle is written $\dot{x} = g_1^{\text{uni}}(x)u_1 + g_2^{\text{uni}}(x)u_2$, where u_1 is the driving speed and u_2 is the steering control.

EXAMPLE 12.2.2: PBWT (cont.)

The control system for the PBWT is written $\dot{x} = g_0^{\text{pbwt}}(x) + g_1^{\text{pbwt}}(x)u_1 + g_2^{\text{pbwt}}(x)u_2$, where u_1 is the thrust force at thruster 1 and u_2 is the force at thruster 2.

We will consider two classes of control sets:

- $\mathcal{U}_{\pm}$: This class of control sets includes any control set $\mathcal{U}$

containing the origin of $\mathbb{R}^m$ in the interior of its convex hull. In other words, the control set *positively* spans $\mathbb{R}^m$—any point in $\mathbb{R}^m$ can be generated by a positive linear combination of elements of $\mathcal{U}$. An example of such a control set is the cube centered at the origin of $\mathbb{R}^m$, $-1 \leq u_i \leq 1, i = 1, \ldots, m$. Another example consists of only the vertices of this cube.

- $\mathcal{U}_+$: This class of control sets includes $\mathcal{U}_\pm$ as a subset and includes any control set $\mathcal{U}$ that spans $\mathbb{R}^m$ —any point in $\mathbb{R}^m$ can be generated by a linear combination of elements of $\mathcal{U}$. An example of such a control set is the non-negative controls $0 \leq u_i \leq 1$, $i = 1, \ldots, m$.

Examples of the control sets are shown in figure 12.10.

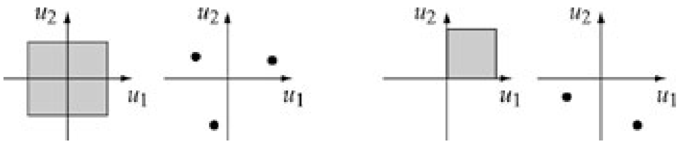

Figure 12.10: For $m = 2$ controls, the two control sets on the left belong to $\mathcal{U}_\pm$ and the two control sets on the right belong to $\mathcal{U}_+$.

The system (12.6) is symmetric if it is drift-free and the control set is symmetric about the origin, e.g., a cube centered at the origin. We will abuse the term slightly and say that a drift-free system is symmetric for any positive-spanning control set $\mathcal{U} \in \mathcal{U}_\pm$, since the controllability properties we discuss in this chapter are the same for any $\mathcal{U} \in \mathcal{U}_\pm$.

If a system has drift but $g_0 \in \text{span}(\{g_1, \ldots, g_m\})$, then we may be able to choose controls $w(x) \in \mathbb{R}^m$ to always cancel the drift, thereby symmetrizing the system by the controls. In this case, the pseudocontrol $u \in \mathcal{U} \in \mathcal{U}_\pm$ can be added on top of the drift-canceling control $w(x)$, so the total control vector is $w(x) + u$, and the system is equivalent to the driftless system

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i, \quad u \in \mathcal{U} \in \mathcal{U}_\pm.$$

As an intuitive example, imagine your motion as you walk on a conveyor. The drift vector field carries you at a constant speed in one direction. You can control your own walking speed, however, to cancel the drift and make progress in the opposite direction.

12.3 Controllability

Let V be a neighborhood of a point $x \in \mathcal{M}$ (i.e., an n-dimensional open set of $\mathcal{M}$ containing x). Let $R^V(x, T)$ indicate the set of reachable points at time T by trajectories remaining inside V and satisfying equation (12.6), and let

$$R^V(x, \leq T) = \bigcup_{0<t\leq T} R^V(x, t).$$

We define the following four versions of nonlinear controllability (see figure 12.11):

- The system is *controllable* from x if, for any $x_{\text{goal}} \in \mathcal{M}$, there exists a $T > 0$ such that $x_{\text{goal}} \in R^{\mathcal{M}}(x, \leq T)$. In other words, any goal state is reachable from x in finite time.
- The system is *accessible* from x if $R^{\mathcal{M}}(x, \leq T)$ contains a full n-dimensional subset of $\mathcal{M}$ for some $T > 0$. See figure 12.11(a).
- The system is *small-time locally accessible* (*STLA*) from x if $R^V(x, \leq T)$ contains a full n-dimensional subset of $\mathcal{M}$ for all neighborhoods V and all $T > 0$. See figure 12.11(b).
- The system is *small-time locally controllable* (*STLC*) from x if $R^V(x, \leq T)$ contains a neighborhood of x for all neighborhoods V and all $T > 0$. See figure 12.11(c).

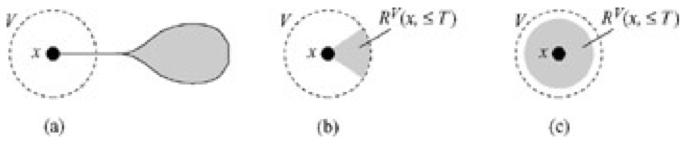

Figure 12.11: Reachable spaces for three systems on $\mathbb{R}^2$. (a) This system is accessible from x, but neither small-time locally accessible (STLA) nor small-time locally controllable (STLC). The reachable set is two-dimensional, but not while confined to the neighborhood V. (b) This system is STLA from x,but not STLC. The reachable set without leaving V does not contain a neighborhood of x. (c) This system is STLC from x.

The phrase "small-time" indicates that the property holds for any time $T > 0$, and "locally" indicates that the property holds for arbitrarily small (but full-dimensional) wiggle room around the initial state. For practical systems, it might take finite time to switch between controls (e.g., putting a car in reverse gear). In this case, we might say a system is locally, but not small-time, controllable. Here we ignore the switch time and retain the standard "small-time locally" terms.

If a property holds for all $x \in \mathcal{M}$, the phrase "from x" can be eliminated. Figure 12.12 shows the implications among the properties. If the vector fields are all analytic, then accessibility implies STLA.

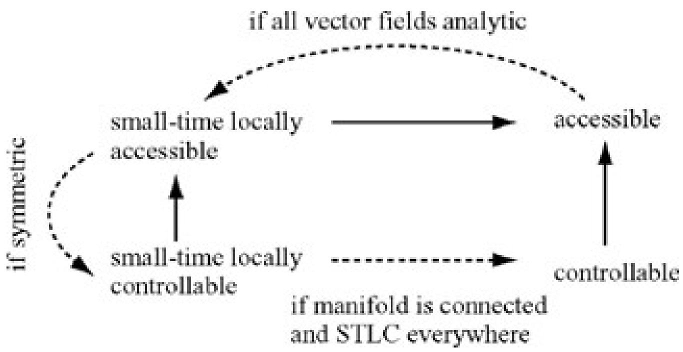

Figure 12.12: Implications among the controllability properties. Dashed arrows are conditional.

Small-time local controllability is of special interest. STLC implies that the system can locally maneuver in any direction, and if the system is STLC at all $x \in \mathcal{M}$, then the system can follow any curve on $\mathcal{M}$ arbitrarily closely. This allows the system to maneuver through cluttered spaces, since any motion of a system with no motion constraints can be approximated by a system that is STLC everywhere. Also, if $\mathcal{M}$ is connected, then the system is controllable if it is STLC everywhere.

STLA and STLC are local concepts that can be established by looking at the behavior of the system in a neighborhood of a state. Accessibility and controllability, on the other hand, are global concepts. As a result, they may depend on things such as the topology of the space and nonlocal behavior of the system vector fields.

Some physical examples of the various properties:

- Imagine setting the minute and hour hands on a watch by turning a knob that can spin in only one direction. The configuration space of the hands is one-dimensional, since the motion of the hour hand is coupled to the motion of the minute hand. Show that this system is accessible, controllable, and STLA on the configuration space, but not STLC.
- Consider the system on $\mathbb{R}^2$ described by the drift vector field $g_0 = [x_2^2, 0]^T$ and the single control vector field $g_1 = [0, 1]^T$, where $u = u_1 \in [-1, 1]$. Show that the system is accessible and STLA from any x but neither controllable nor STLC.
- Consider the system on $\mathbb{R}^2$ described by the drift vector field $g_0 = [x_2, 0]^T$ and the single control vector field $g_1 = [0, 1]^T$, where $u = u_1 \in [-1, 1]$. This is the linear double-integrator $\ddot{q} = u$ written in the first-order form $\dot{x}_1 = x_2, \dot{x}_2 = u$. Convince yourself that the system is STLC only from zero-velocity states [*, 0]^T (see figure 12.13).

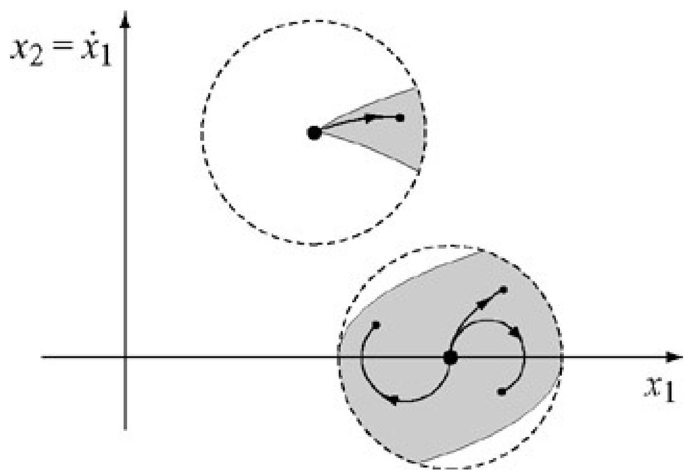

Figure 12.13: Two initial states and neighborhoods for the linear double-integrator with bounded control. The reachable sets from each initial state, by trajectories remaining in the neighborhood, are shaded, and example trajectories are shown. The system is STLC from the initial state where $\dot{x}_1 = x_2 = 0$, but not STLC from the initial state where $\dot{x}_1 = x_2 \neq 0$. Reaching a point left of this initial state (i.e., decreasing the x_1 value) requires $\dot{x}_1$ to become negative—

the x_2 coordinate must leave the neighborhood.

- The unicycle satisfies all the controllability properties if $\mathcal{U} \in \mathcal{U}_{\pm}$.
- Show that the unicycle is accessible, STLA, and controllable in the obstacle-free plane, but not STLC, if $\mathcal{U}$ belongs to the class $\mathcal{U}_{+}$ but not $\mathcal{U}_{\pm}$.
- Any system confined to a *k*-dimensional integral manifold, $k < n$, satisfies none of the controllability properties.

As hinted at in the linear double-integrator example, for second-order systems with velocity variables in the state vector, STLC can only hold at zero velocity. States with nonzero velocity result in drift in the configuration variables that cannot be instantaneously compensated by finite actuation forces. *Therefore, when we talk about STLC for second-order systems, we implicitly mean STLC at zero velocity.*

For linear systems of the form $\dot{x} = Ax + Bu$, there is a single notion of controllability (see appendix J). For nonlinear systems, such as those we study, there are a number of notions of controllability, including the four we have defined here. Akey point is that the linearizations of systems of interest to us are generally not controllable, meaning that their controllability is inherently a nonlinear phenomenon.

12.3.1 Local Accessibility and Controllability

Of the controllability properties, STLA can be checked by studying the Lie algebra of the vector fields $g_0, \ldots, g_m$.

THEOREM 12.3.1

The system (12.6) is STLA from *x* if (and only if for analytic vector fields) it satisfies the *Lie algebra rank condition (LARC)*—the Lie algebra of the vector fields, evaluated at *x*, is the tangent space at *x*, or $\overline{\text{Lie}}(\{g_0, \ldots, g_m\})(x) = T_x\mathcal{M}$. This holds for any $\mathcal{U} \in \mathcal{U}_{+}$. If the system is symmetric (drift-free and $\mathcal{U} \in \mathcal{U}_{\pm}$), then the LARC also implies small-time local controllability.

An early version of this result is due to W.-L. Chow [112], and it is sometimes called *Chow's theorem*.

EXAMPLE 12.3.2: Unicycle (cont.)

As shown previously, the rank of the unicycle Lie algebra is three at all states, so the LARC is satisfied. Therefore, for both $\mathcal{U} \in \mathcal{U}_+$ and $\mathcal{U} \in \mathcal{U}_\pm$, the unicycle is STLA. For a control set $\mathcal{U} \in \mathcal{U}_\pm$, the system is also STLC everywhere, and therefore controllable because of the connectedness of its state manifold. It is also true that the unicycle is controllable (but not STLC) for any $\mathcal{U} \in \mathcal{U}_+$, though this cannot be shown by theorem 12.3.1. (The reader may wish to verify controllability by describing a constructive procedure to drive the unicycle to any goal location in an obstacle-free space.)

If we eliminate one vector field from the unicycle example, allowing it only to roll forward and backward ($g^{uni}{}_1$) or spin in place ($g^{uni}{}_2$), the unicycle is confined to an integral curve of the vector field, and none of the controllability properties is satisfied.

Second-order systems with nonzero drift, such as the PBWT, are not symmetric for any control set. The system may still be STLC at zero velocity states, however, since symmetry plus the LARC is sufficient but not necessary for STLC. Sussmann [401] provided a more general sufficient condition for STLC that includes the symmetric case ($g_0 = 0$ and $\mathcal{U} \in \mathcal{U}_\pm$) as a special case. To understand it, we first define a Lie product term to be a *bad bracket* if the drift term g_0 appears an odd number of times in the product and each control vector field g_i, $i = 1 \ldots m$, appears an even number of times (including zero). A *good bracket* is any Lie product that is not bad. For example, $[g_1, [g_0, g_1]]$ is a bad bracket and $[g_2, [g_1, [g_0, g_1]]]$ and $[g_1, [g_2, [g_1, g_2]]]$ are good brackets. With these definitions, we can state a version of Sussmann's theorem:

THEOREM 12.3.3

The system (12.6) is STLC at x if

1. $g_0(x) = 0$,
2. $\mathcal{U} \in \mathcal{U}_{\pm}$,
3. the LARC is satisfied by good Lie bracket terms up to degree k, and
4. any bad bracket of degree $j \leq k$ can be expressed as a linear combination of good brackets of degree less than j.

The intuition behind the theorem is the following. Bad brackets are called bad because, after generating the net motion obtained by following the Lie bracket motion prescription, we find that the controls u_i only appear in the net motion with even exponents, meaning that the vector field can only be followed in one direction. In this sense, a bad bracket is similar to a drift field, and we must be able to compensate for it. Since motions in Lie product directions of high degree are essentially "slower" than those in directions with a lower degree, we should only try to compensate for bad bracket motions by good bracket motions of lower degree. If a bad bracket of degree j can be expressed as a linear combination of good brackets of degree less than j, the good brackets are said to *neutralize* the bad bracket. For the bad bracket of degree 1 (the drift vector field g_0) there are no lower degree brackets that can be used to neutralize it, so we require $g_0(x) = 0$. Therefore, this result only holds at states x where the drift vanishes, i.e., equilibrium states.

EXAMPLE 12.3.4: PBWT (cont.)

Assume that the PBWT moves in a horizontal plane, so $a_g = 0$. As before, we define $g^{pbwt}{}_3 = [g^{pbwt}{}_0, g_1{}^{pbwt}]$, $g^{pbwt}{}_4 = [g^{pbwt}{}_0, g^{pbwt}{}_2]$, $g^{pbwt}{}_5 = [g^{pbwt}{}_1, [g^{pbwt}{}_0, g^{pbwt}{}_2]]$, and $g^{pbwt}{}_6 = [g^{pbwt}{}_0, [g^{pbwt}{}_1, [g^{pbwt}{}_0,$

$g^{pbwt}{}_2$]]]. Again as before, a computation shows that

$$\det\left[g_1^{\mathrm{pbwt}}\ g_2^{\mathrm{pbwt}}\ g_3^{\mathrm{pbwt}}\ g_4^{\mathrm{pbwt}}\ g_5^{\mathrm{pbwt}}\ g_6^{\mathrm{pbwt}}\right] = d^4.$$

The LARC is satisfied, so the system is STLA at all states for either control set. If $\mathcal{U} \in \mathcal{U}_{\pm}$, we would like to know if the system satisfies Sussmann's sufficient condition for STLC at equilibrium states $x = [q_1, q_2, q_3, 0, 0, 0]^T$, where $g^{pbwt}{}_0(x) = 0$. Because we use bracket terms up to degree 4 to demonstrate LARC, we must be able to neutralize all bad bracket terms of degree 4 or less. The only such bad bracket terms are the degree 3 terms

$$[g_1^{\mathrm{pbwt}}, [g_0^{\mathrm{pbwt}}, g_1^{\mathrm{pbwt}}]] = [0, 0, 0, 0, 0, 0]^T$$
$$[g_2^{\mathrm{pbwt}}, [g_0^{\mathrm{pbwt}}, g_2^{\mathrm{pbwt}}]] = [0, 0, 0, 2d\cos x_3, 2d\sin x_3, 0]^T = 2dg_1^{\mathrm{pbwt}}.$$

The second term is neutralized by $g^{pbwt}{}_1$. Therefore, by Sussmann's theorem, the system is STLC at equilibrium states.

Note that in gravity, $a_g \neq 0$, so $g^{pbwt}{}_0(x) \neq 0$ at any state and Sussmann's theorem does not allow us to prove or disprove STLC.

Now consider the case where the PBWT is equipped with a single thruster. If the single thruster corresponds to the vector field $g^{pbwt}{}_1$, the thrust always passes through the body center of mass, and the angular velocity of the body cannot be changed. The system is not accessible. If the single thruster corresponds to the vector field $g^{pbwt}{}_2$, however, we can define the vector fields

$$[g_0^{\mathrm{pbwt}}, g_2^{\mathrm{pbwt}}],\quad [g_2^{\mathrm{pbwt}}, [g_0^{\mathrm{pbwt}}, g_2^{\mathrm{pbwt}}]],\quad [g_2^{\mathrm{pbwt}}, [g_0^{\mathrm{pbwt}}, [g_0^{\mathrm{pbwt}}, g_2^{\mathrm{pbwt}}]]],$$
$$[g_2^{\mathrm{pbwt}}, [g_2^{\mathrm{pbwt}}, [g_0^{\mathrm{pbwt}}, [g_0^{\mathrm{pbwt}}, g_2^{\mathrm{pbwt}}]]]],$$
$$[g_0^{\mathrm{pbwt}}, [g_2^{\mathrm{pbwt}}, [g_2^{\mathrm{pbwt}}, [g_0^{\mathrm{pbwt}}, [g_0^{\mathrm{pbwt}}, g_2^{\mathrm{pbwt}}]]]]]$$

and see that the determinant of the matrix formed by these columns is $-16d^8$, indicating that the system is STLA for either $\mathcal{U} \in \mathcal{U}_+$ or $\mathcal{U} \in \mathcal{U}_{\pm}$. Bad brackets cannot be neutralized, so theorem 12.3.3 cannot be used to show STLC. Note, however, that reducing to a single control vector field does not reduce the dimension of the reachable space, as it did for the kinematic unicycle case. This is because the second-order system provides a drift field with which

Lie bracket terms can be generated.

Finally, the PBWT with the single control vector field $g^{pbwt}{}_2$, a control set $\mathcal{U} \in \mathcal{U}_{\pm}$, and a_g = 0 turns out to be (globally) controllable — any state is reachable in finite time from any other state [303]. Thus the PBWT in zero gravity provides a simple example of different controllability properties (figure 12.14). If we equip it with three independent control vector fields, e.g., a control for each coordinate, the PBWT is a linear system of three double-integrators and it is controllable by linear control theory (see appendix J). If we equip it with the two control vector fields $g^{pbwt}{}_1$ and $g^{pbwt}{}_2$, it is no longer linearly controllable, but remains STLC at zero velocity. If we equip it with just the single control vector field $g^{pbwt}{}_2$, it is no longer STLC at zero velocity, but remains STLA and globally controllable.

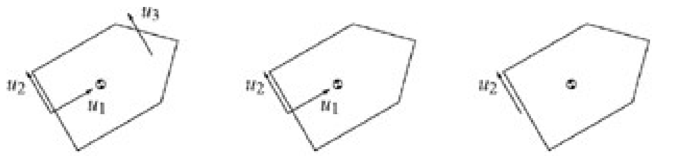

Figure 12.14: The PBWT in zero gravity with different numbers of thrusters. The PBWT on the left has three thrusters that can generate any force and torque combination, and it is controllable by linear control theory. Eliminating one thruster, we get the PBWT in the middle, which is no longer linearly controllable but is STLC at zero velocity. Finally, reducing the thruster count to one, we get the PBWT on the right, which is no longer STLC but remains STLA and controllable in a global sense. (Note that the PBWT with only u_1 thrust is not STLA.) All thrusters are bidirectional.

12.3.2 Global Controllability

For kinematic systems that are STLC everywhere on a connected manifold, (global) controllability follows easily. In general, however, controllability is not easy to decide, as it may depend on nonlocal features of the control system. In the special case of a control system (12.6) with $\mathcal{U} \in \mathcal{U}_{\pm}$ and a drift vector field that repeatedly returns the system to a neighborhood of its initial state, however, demonstrating controllability is as easy as demonstrating the LARC.

First, some definitions. Consider the flow ϕ^{g_0} of the drift vector field. A point $x \in \mathcal{M}$ is called *positively Poisson stable* (PPS) for g_0 if for all $T > 0$ and any neighborhood V of x, there exists a time $t > T$ such that the flow of the vector field returns the system to V, i.e., $\phi^{g_0}{}_T(x) \in V$. The drift vector field g_0 is called positively Poisson stable if the set of PPS points for g_0 is dense in $\mathcal{M}$.

A point $x \in \mathcal{M}$ is called a *nonwandering point* of g_0 if for all time $T > 0$ and any neighborhood V of x there exists a time $t > T$ such that $\phi^{g_0}{}_T(V) \cap V \neq$, where $\phi^{g_0}{}_T(V) = \{\phi^{g_0}{}_T(x) \mid x \in V\}$.(A positively Poisson stable point is necessarily a nonwandering point.) The *nonwandering set* of g_0 is the set of all nonwandering points of g_0. Finally, we say that the drift vector field g_0 is *weakly positively Poisson stable* (WPPS) if its nonwandering set is $\mathcal{M}$.

We now state the main theorem, taken from Lian, Wang, and Fu [289]. Related results can be found in (Jurdjevic and Sussmann [212]; Lobry [295]; Brockett [65]; Bonnard [58]; and Jurdjevic [211]).

THEOREM 12.3.5

Assume that the drift vector field g_0 is WPPS. Then the system (12.6) with $\mathcal{U} \in \mathcal{U}_{\pm}$ is controllable on $\mathcal{M}$ if the LARC is satisfied.

As an example, consider the system on $\mathbb{R}^2$ described by $\dot{x} = g_0(x) + g_1(x)u_1,\ u_1 \in [-1, 1] \in \mathcal{U}_{\pm}$ where $g_0(x) = \frac{1}{2}[-x_2, x_1]^T$ and $g_1(x) = [1, 0]^T$. The drift vector field (shown in figure 12.15) is WPPS, as its orbits are closed. We find that $[g_0, g_1] = [0, -\frac{1}{2}]^T$ and $\det[g_1\ [g_0, g_1]] = -\frac{1}{2}$, so the LARC is satisfied at all x. By theorem 12.3.5, every state is reachable from every other state. Intuitively, u_1 is used to control x_1 and (waiting) time is used to "control" x_2. (In fact, in this example, it is not hard to see that controllability also holds for $\in \mathcal{U} \in \mathcal{U}_+$. This system is only STLC at the origin.

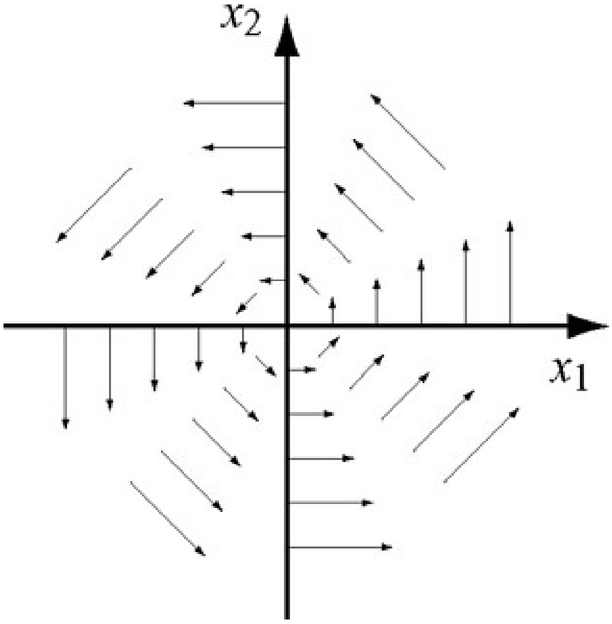

Figure 12.15: The integral curves of the WPPS drift vector field $\frac{1}{2}[-x_2, x_1]^T$ are closed (circles).

THEOREM 12.3.5 is a powerful tool for establishing the global controllability of systems with drift. Systems with periodic natural unforced dynamics (such as an undamped planar pendulum or the example of figure 12.15) or energy-conserving drift on compact configuration spaces are examples of systems with WPPS drift vector fields. The latter follows from an application of Poincaré's recurrence theorem; see, e.g., the discussion by Arnold [26]. As an example, a rotating satellite moves on the compact configuration space *SO*(3), and its natural unforced motion conserves energy. Therefore, the drift is WPPS. The LARC can be satisfied by a single body- fixed control torque, meaning that the satellite can be driven to any orientation and angular velocity with a single control vector field.

For systems with non-WPPS drift, it may be possible to construct feedback laws that always keep the system in a periodic orbit. If the system is always controllable about these periodic trajectories, i.e., if the system can reach neighborhoods of the controlled periodic trajectories, then similar reasoning can be used to demonstrate controllability of the system [87,304].