Exercise 1

Consider a 3R planar robot with links of equal length $\ell = 1$ [m]. The primary task for this robot is to execute an instantaneous Cartesian velocity $\boldsymbol{v} \in \mathbb{R}^2$ with its end-effector. Denote the associated task Jacobian as $\boldsymbol{J}(\boldsymbol{q})$.

- When the robot is in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & \pi/3 & -2\pi/3 \end{pmatrix}^T$, use the Reduced Gradient (RG) method to determine the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ that realizes a desired Cartesian velocity $\boldsymbol{v} = \begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}^T$ [m/s] while maximizing the objective function

$$H(\boldsymbol{q}) = \sin^2 q_2 + \sin^2 q_3. \tag{7}$$

- As an auxiliary task, the robot should move so as to always keep the position $\boldsymbol{p}_2 = (x_2, y_2)$ of the endpoint of its second link on the circle defined by

$$x_2^2 + (y_2 - 1.5)^2 = 0.75. \tag{8}$$

  Determine the Jacobian $\boldsymbol{J}_a(\boldsymbol{q})$ associated to this auxiliary task. When the robot is in the configuration $\boldsymbol{q}_0$ defined above, is the robot in an algorithmic singularity? Can the two requested primary and auxiliary tasks be executed at the same time?

Exercise 1

Reduced Gradient

The Jacobian of the primary task is similar to that of the previous exercise, see (11)

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\,(s_1 + s_{12} + s_{123}) & -\ell\,(s_{12} + s_{123}) & -\ell s_{123} \\ \ell\,(c_1 + c_{12} + c_{123}) & \ell\,(c_{12} + c_{123}) & \ell c_{123} \end{pmatrix}. \tag{12}$$

When evaluated at $\boldsymbol{q} = \boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & \pi/3 & -2\pi/3 \end{pmatrix}^T$ and for $\ell = 1$ [m], the Jacobian (12) becomes

$$\boldsymbol{J} := \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} -2 & -1 & -0.5 \\ 0 & 0 & \sqrt{3}/2 \end{pmatrix}$$

and is clearly full row rank. However, for the purpose of designing a RG solution, we need to extract from $\boldsymbol{J}$ a non-singular 2×2 submatrix $\boldsymbol{J}_a$, and not every selection will work. In fact, the three possible alternatives (i.e., deleting respectively column 1, 2, or 3) yield

$$\det \boldsymbol{J}_{-1} = -.8660, \qquad \det \boldsymbol{J}_{-2} = -1.7321, \qquad \det \boldsymbol{J}_{-3} = 0 \text{ (singular!)}.$$

We will choose as $\boldsymbol{J}_a$ the minor with the largest determinant (presumably, the best conditioned solution). Thus, $\boldsymbol{q}_a = (q_1, q_3)$, $\boldsymbol{q}_b = q_2$. Accordingly, after a reordering of variables obtained by the unitary matrix $\boldsymbol{T}$ (with $\boldsymbol{T}^{-1} = \boldsymbol{T}^T$)

$$\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{q} \to \boldsymbol{T}\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_3 \\ \hline \dot{q}_2 \end{pmatrix} = \begin{pmatrix} \dot{\boldsymbol{q}}_a \\ \dot{\boldsymbol{q}}_b \end{pmatrix}, \quad \boldsymbol{J} \to \boldsymbol{J}\boldsymbol{T} = \left(\boldsymbol{J}_1 \;\; \boldsymbol{J}_3 \,\middle|\, \boldsymbol{J}_2 \right) = \left(\boldsymbol{J}_a \;\; \boldsymbol{J}_b \right),$$

we have

$$\boldsymbol{J}_a = \begin{pmatrix} -2 & -0.5 \\ 0 & \sqrt{3}/2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}_a^{-1} = \begin{pmatrix} -0.5 & -0.2887 \\ 0 & 1.1547 \end{pmatrix}, \qquad \boldsymbol{J}_b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The gradient of the objective function H in (7) evaluated at $\boldsymbol{q} = \boldsymbol{q}_0$ is

$$\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = \begin{pmatrix} 0 \\ 2\sin q_2 \cos q_2 \\ 2\sin q_3 \cos q_3 \end{pmatrix} \qquad \Rightarrow \qquad \nabla_{\boldsymbol{q}} H := \nabla_{\boldsymbol{q}} H(\boldsymbol{q}_0) = \begin{pmatrix} 0 \\ 0.8660 \\ 0.8660 \end{pmatrix}.$$

The reduced gradient is thus

$$\nabla_{\boldsymbol{q}_b} H' := \nabla_{\boldsymbol{q}_b} H'(\boldsymbol{q}_0) = \begin{pmatrix} -\left(\boldsymbol{J}_a^{-1}\boldsymbol{J}_b\right)^T & 1 \end{pmatrix} \boldsymbol{T} \nabla_{\boldsymbol{q}} H = \begin{pmatrix} -0.5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0.8660 \\ 0.8660 \end{pmatrix} = 0.8660.$$

The solution to the first item is thus

$$\dot{\boldsymbol{q}}_{RG} = \boldsymbol{T}^T \begin{pmatrix} \dot{\boldsymbol{q}}_a \\ \dot{\boldsymbol{q}}_b \end{pmatrix} = \boldsymbol{T}^T \begin{pmatrix} \boldsymbol{J}_a^{-1}\left(\boldsymbol{v} - \boldsymbol{J}_b \nabla_{\boldsymbol{q}_b} H'\right) \\ \nabla_{\boldsymbol{q}_b} H' \end{pmatrix} = \begin{pmatrix} -0.4330 \\ 0.8660 \\ -2.0000 \end{pmatrix} \text{ [rad/s]}.$$

Task augmentation

We consider next the auxiliary task of keeping the endpoint of the second robot link on the circle (8). The endpoint position is

$$\boldsymbol{p}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \ell(c_1 + c_{12}) \\ \ell(s_1 + s_{12}) \end{pmatrix}$$

and its associated Jacobian is

$$\boldsymbol{J}_2(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_2(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\ell\,(s_1 + s_{12}) & -\ell s_{12} & 0 \\ \ell\,(c_1 + c_{12}) & \ell c_{12} & 0 \end{pmatrix}. \tag{13}$$

We first note that when the robot is in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & \pi/3 & -2\pi/3 \end{pmatrix}^T$, the position $\boldsymbol{p}_2$ satisfies already the constraint (8), see Fig. 2. Thus, the auxiliary task should constrain the joint velocities so that $\boldsymbol{p}_2$ (when moving or not) will remain on the assigned circle.

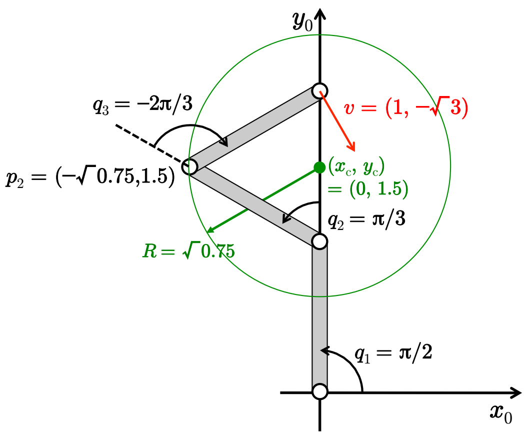

Figure 2: The primary task (in red) and the auxiliary task (in green) for the 3R planar robot.

Differentiating (8) with respect to time yields

$$2x_2\,\dot{x}_2 + 2(y_2 - 1.5)\,\dot{y}_2 = 0.$$

Rearranging this equation so as to isolate the velocity $\dot{\boldsymbol{p}}_2$ and using (13) leads to

$$\begin{pmatrix} 2x_2 & 2(y_2-1.5) \end{pmatrix} \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2x_2 & 2(y_2-1.5) \end{pmatrix} \boldsymbol{J}_2(\boldsymbol{q})\dot{\boldsymbol{q}} = 0.$$

The auxiliary task Jacobian is then the 1×3 row vector

$$\begin{aligned} \boldsymbol{J}_a(\boldsymbol{q}) &= \begin{pmatrix} 2x_2 & 2(y_2-1.5) \end{pmatrix} \boldsymbol{J}_2(\boldsymbol{q}) = \begin{pmatrix} 2\ell(c_1+c_{12}) & 2(\ell(s_1+s_{12})-1.5) \end{pmatrix} \begin{pmatrix} -\ell\,(s_1+s_{12}) & -\ell s_{12} & 0 \\ \ell\,(c_1+c_{12}) & \ell c_{12} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -3\ell\,(c_1+c_{12}) & -2\ell^2 s_2 - 3\ell c_{12} & 0 \end{pmatrix}. \end{aligned} \tag{14}$$

Augmenting the primary task Jacobian (12) with the auxiliary task Jacobian (14) leads to a square, 3×3 *extended* Jacobian $\boldsymbol{J}_e(\boldsymbol{q})$ and to an extended task vector $\boldsymbol{v}_e \in \mathbb{R}^3$:

$$\boldsymbol{J}_e(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}(\boldsymbol{q}) \\ \boldsymbol{J}_a(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\ell\,(s_1+s_{12}+s_{123}) & -\ell\,(s_{12}+s_{123}) & -\ell s_{123} \\ \ell\,(c_1+c_{12}+c_{123}) & \ell\,(c_{12}+c_{123}) & \ell c_{123} \\ -3\ell\,(c_1+c_{12}) & -2\ell^2 s_2 - 3\ell c_{12} & 0 \end{pmatrix}, \quad \boldsymbol{v}_e = \begin{pmatrix} \boldsymbol{v} \\ 0 \end{pmatrix}, \quad \boldsymbol{J}_e(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{v}_e.$$

At the configuration $\boldsymbol{q}_0$, using also $\ell = 1$, we have

$$\boldsymbol{J}_e := \boldsymbol{J}_e(\boldsymbol{q}_0) = \begin{pmatrix} -2 & -1 & -0.5 \\ 0 & 0 & 0.8660 \\ 2.5981 & 0.8660 & 0 \end{pmatrix}. \tag{15}$$

It is easy to see that $\boldsymbol{q}_0$ is not a singularity for the extended task, and thus in particular not an algorithmic singularity. In fact,

$$\mathrm{rank}(\boldsymbol{J}) = 2, \qquad \mathrm{rank}(\boldsymbol{J}_a) = 1, \qquad \text{and } \mathrm{rank}(\boldsymbol{J}_e) = 3 = 2 + 1.$$

Therefore, in this configuration the robot can realize any generic extended task velocity $\boldsymbol{v}_e \in \mathbb{R}^3$ (in particular, with an arbitrary top part $\boldsymbol{v} \in \mathbb{R}^2$). Therefore, the joint velocity

$$\dot{\boldsymbol{q}}^\dagger = \boldsymbol{J}_e^{-1}\boldsymbol{v}_e = \boldsymbol{J}_e^{-1} \begin{pmatrix} 1 \\ -\sqrt{3} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \text{ [rad/s]}$$

will instantaneously realize both tasks at the same time. For the particular numerical value assigned as $\boldsymbol{v}$, the simple rotation of the third link around its joint axis will realize the primary task. In this case, the endpoint of the second link will remain at rest, thus satisfying in a trivial way the auxiliary task. In general, with a different desired value of $\boldsymbol{v}$, all robot joints will move so as to realize the extended task, instantaneously keeping the endpoint of the second link on the circle (i.e., its velocity will be tangential to the circle in the current position). For instance,

$$\dot{\boldsymbol{q}}^{\dagger\prime} = \boldsymbol{J}_e^{-1}\boldsymbol{v}_e' = \boldsymbol{J}_e^{-1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.5774 \\ -4.7321 \\ 1.1547 \end{pmatrix} \text{ [rad/s]}.$$