Chapter 3 Solutions

Exercise 3.1.

(a) The three frames are:

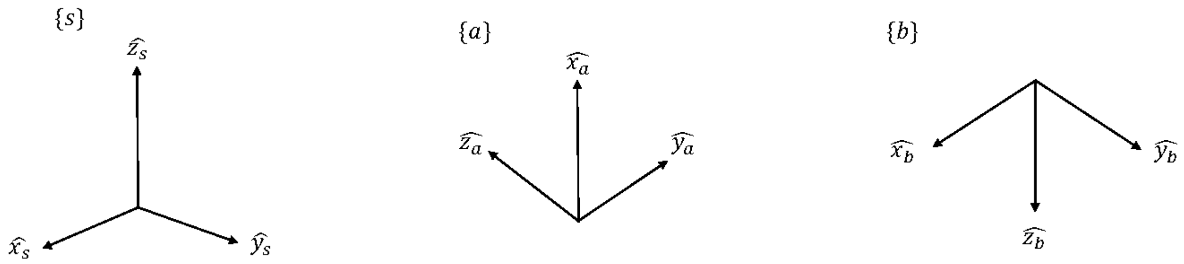

Figure 3.1

(b) The rotation matrices are:

$$R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \qquad R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

(c)

$$R_{sb}^{-1} = R_{bs} = R_{sb}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

(d)

$$R_{ab} = R_{as}R_{sb} = R_{sa}^{\mathrm{T}}R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(e)

$$R_1 = R_{sa}R = R_{sa}R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

R_1 corresponds to rotating R_{sa} by $-90°$ about the body-fixed $\hat{x}_a$ axis.

$$R_2 = RR_{sa} = R_{sb}R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R_2 corresponds to rotating R_{sa} by $-90°$ about the world-fixed $\hat{x}_s$ axis.

(f) $p_s = R_{sb}p_b = (1, 3, -2)^T$.

(g)

$$p' = R_{sb}p_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \implies \text{location transformation}$$

$$p'' = R_{sb}^{\mathrm{T}}p_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \implies \text{coordinate change}$$

(h)

$$R_{as}\omega_s = \omega_a \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

(i)

$$R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Trace R_{sa} is equal to 0, so we are in the third condition of the algorithm. Therefore $\theta = \cos^{-1}(-1/2) = \frac{2\pi}{3}$. By definition

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(\bar{R} - \bar{R}^{\mathrm{T}}) = \frac{\sqrt{3}}{3}(R_{sa} - R_{as})$$

$$\implies [\hat{\omega}] = \frac{\sqrt{3}}{2} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\implies \hat{\omega} = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{(to make it a unit vector we have to divide the vector by its norm)}$$

(j)

$$\hat{\omega}\theta = \sqrt{5} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$\implies [\hat{\omega}] = \begin{bmatrix} 0 & 0 & \frac{2}{\sqrt{5}} \\ 0 & 0 & -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{bmatrix}, \text{ with } \theta = \sqrt{5}$$

By definition $R = \mathrm{e}^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$,

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0.704 \\ 0 & 0 & -0.352 \\ -0.704 & 0.352 & 0 \end{bmatrix} + \begin{bmatrix} -1.294 & 0.647 & 0 \\ 0.647 & -0.324 & 0 \\ 0 & 0 & -1.6173 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2938 & 0.6469 & 0.7037 \\ 0.6469 & 0.6765 & -0.3518 \\ -0.7037 & 0.3518 & -0.6173 \end{bmatrix}$$

Exercise 3.2.
Let point p' be the new position after rotation, and denote the corresponding rotation matrix as R.

(a) Since the rotation is with respect to the fixed frame,

$$
\begin{aligned}
p' &= \mathrm{Rot}(\hat{\mathrm{z}}, -120^\circ)\mathrm{Rot}(\hat{\mathrm{y}}, 135^\circ)\mathrm{Rot}(\hat{\mathrm{x}}, 30^\circ)p \\
&= \begin{bmatrix} \cos(-120^\circ) & -\sin(-120^\circ) & 0 \\ \sin(-120^\circ) & \cos(-120^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 135^\circ & 0 & \sin 135^\circ \\ 0 & 1 & 0 \\ -\sin 135^\circ & 0 & \cos 135^\circ \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & -\sin 30^\circ \\ 0 & \sin 30^\circ & \cos 30^\circ \end{bmatrix} p \\
&= \begin{bmatrix} -0.5526 \\ 0.4571 \\ -0.6969 \end{bmatrix}.
\end{aligned}
$$

(b) From (a),

$$
R = \mathrm{Rot}(\hat{\mathrm{z}}, -120^\circ)\mathrm{Rot}(\hat{\mathrm{y}}, 135^\circ)\mathrm{Rot}(\hat{\mathrm{x}}, 30^\circ) = \begin{bmatrix} 0.3536 & 0.5732 & -0.7392 \\ 0.6124 & -0.7392 & -0.2803 \\ -0.7071 & -0.3536 & -0.6124 \end{bmatrix}.
$$

Exercise 3.3.
From $Rp = q$,

$$
\begin{aligned}
R \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 2 & -1 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\sqrt{2} \\ 2 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & -2 \end{bmatrix} \\
\Leftrightarrow\ R = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\sqrt{2} \\ 2 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & -2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 2 & -1 & 0 \end{bmatrix}^{-1} &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.
\end{aligned}
$$

Exercise 3.4.
Let R_{ab} and R_{bc} be

$$
R_{ab} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, R_{bc} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}.
$$

We then have

$$
\begin{aligned}
\begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix} &= \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\
&= \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} R_{ab} \\
\begin{bmatrix} \hat{x}_c & \hat{y}_c & \hat{z}_c \end{bmatrix} &= \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \\
&= \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix} R_{bc} \\
&= \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} R_{ab} R_{bc}.
\end{aligned}
$$

Since

$$
\begin{bmatrix} \hat{x}_c & \hat{y}_c & \hat{z}_c \end{bmatrix} = \begin{bmatrix} \hat{x}_a & \hat{y}_a & \hat{z}_a \end{bmatrix} R_{ac},
$$

we can conclude that

$$
R_{ac} = R_{ab} R_{bc}.
$$

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Exercise 3.5.

$$\hat{\omega}\theta = \begin{pmatrix} \frac{2\pi}{3\sqrt{3}} \\ -\frac{2\pi}{3\sqrt{3}} \\ \frac{2\pi}{3\sqrt{3}} \end{pmatrix}$$

Exercise 3.6.

$R = \mathrm{Rot}(\hat{x}, \pi/2)\mathrm{Rot}(\hat{z}, \pi)$ which gives us the matrix:

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

In order to find $R = e^{[\hat{\omega}]\theta}$ we have to perform matrix logarithm. The trace of the above matrix is -1, and thus we are in the first condition. Either the first or the second equation can be used to calculate the unit vector $\hat{\omega}$ and angle of rotation by definition is $\theta = \pi$.

$$\hat{\omega} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Exercise 3.7.

(a) From $\mathrm{tr}(R) = -1 = 1 + 2\cos\theta$ it follows that $\theta = \pi$.

$$\begin{aligned} R &= I + 2[\hat{\omega}]^2 \\ &= \begin{bmatrix} 1-2(\omega_2^2+\omega_3^2) & 2\omega_1\omega_2 & 2\omega_1\omega_3 \\ 2\omega_1\omega_2 & 1-2(\omega_1^2+\omega_3^2) & 2\omega_2\omega_3 \\ 2\omega_1\omega_3 & 2\omega_2\omega_3 & 1-2(\omega_1^2+\omega_2^2) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

$\hat{\omega} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^T$ or $(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})^T$.

(b) From the exponential formula

$$e^{[\hat{\omega}]\theta} = I + \frac{\sin\theta}{3}\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} + \frac{1-\cos\theta}{9}\begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix}.$$

Substituting the above equation into $v_2 = Rv_1$ yields

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2\sin\theta - (1 - cos\theta) \\ -\sin\theta + 2(1-\cos\theta) \\ -2\sin\theta - 2(1-\cos\theta) \end{bmatrix} \Leftrightarrow \theta = -\frac{\pi}{2}.$$

Exercise 3.8.

(a) Using the fact that $\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2$,

$$\begin{aligned} r_{11} &= 1 - 2(\hat{\omega}_2^2 + \hat{\omega}_3^2) = 1 - 2(1-\hat{\omega}_1^2) = 2\hat{\omega}_1^2 - 1 \\ r_{22} &= 1 - 2(\hat{\omega}_1^2 + \hat{\omega}_3^2) = 1 - 2(1-\hat{\omega}_2^2) = 2\hat{\omega}_2^2 - 1 \\ r_{33} &= 1 - 2(\hat{\omega}_1^2 + \hat{\omega}_2^2) = 1 - 2(1-\hat{\omega}_3^2) = 2\hat{\omega}_3^2 - 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \hat{\omega}_1^2 &= \tfrac{r_{11}+1}{2} \\ \hat{\omega}_2^2 &= \tfrac{r_{22}+1}{2} \\ \hat{\omega}_3^2 &= \tfrac{r_{33}+1}{2}. \end{aligned}$$

Hence $\hat{\omega}_1 = \pm\sqrt{\frac{r_{11}+1}{2}}, \hat{\omega}_2 = \pm\sqrt{\frac{r_{22}+1}{2}}$, and $\hat{\omega}_3 = \pm\sqrt{\frac{r_{33}+1}{2}}$. Depending on the sign of the off-diagonal terms in the $[\hat{\omega}]^2$ matrix, we can get two combinations of signs of the $\hat{\omega}$ elements. However, the given solution shows only the magnitude of the elements and is incorrect.

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(b)

$$[\hat{\omega}](R+I) = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} \begin{bmatrix} r_{11}+1 & r_{12} & r_{13} \\ r_{21} & r_{22}+1 & r_{23} \\ r_{31} & r_{32} & r_{33}+1 \end{bmatrix} = 0$$

$$\frac{\hat{\omega}_3}{\hat{\omega}_2} = \frac{r_{31}}{r_{21}} = \frac{r_{32}}{r_{22}+1} = \frac{r_{33}+1}{r_{23}} \tag{3.1}$$

$$\frac{\hat{\omega}_3}{\hat{\omega}_1} = \frac{r_{31}}{r_{11}+1} = \frac{r_{32}}{r_{12}} = \frac{r_{33}+1}{r_{13}} \tag{3.2}$$

Using Equations (3.1) and (3.2) and the fact that $\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$,

$$\begin{aligned} \left(\frac{r_{13}}{r_{33}+1} \cdot \hat{\omega}_3\right)^2 + \left(\frac{r_{23}}{r_{33}+1} \cdot \hat{\omega}_3\right)^2 + \hat{\omega}_3^2 = 1 \\ \hat{\omega}_3^2 \cdot \left\{\frac{r_{13}^2 + r_{23}^2 + r_{33}^2 + 2r_{33} + 1}{(r_{33}+1)^2}\right\} = 1 \\ \hat{\omega}_3^2 = \frac{r_{33}+1}{2} \end{aligned}$$

since $r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$. In the same way we have

$$\begin{aligned} \hat{\omega}_1^2 &= \frac{r_{11}+1}{2} \\ \hat{\omega}_2^2 &= \frac{r_{22}+1}{2}. \end{aligned}$$

It can be verified that the result is the same as (a).

Exercise 3.9.

To multiply two arbitrary 3×3 matrices, 27 multiplication and 18 addition operations are required. However for rotation matrices, once the first and second columns of $R_1 \times R_2$ — let's call them u and v — are obtained, the third column can be obtained simply by taking the cross-product $u \times v$ (which requires only 6 multiplications and 3 additions). It is possible to save 3 multiplications and 3 additions with this procedure.

Exercise 3.10.

Since A, $R \in SO(3)$ and $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for any $X, Y \in \mathbb{R}^{3\times3}$, the objective function can be equivalently written as

$$\begin{aligned} \min_{R\in SO(3)} \mathrm{tr}((A-R)(A-R)^{\mathsf{T}}) \quad &\Rightarrow \quad \min_{R\in SO(3)} \mathrm{tr}(AA^{\mathsf{T}} + RR^{\mathsf{T}} - 2R^{\mathsf{T}}A) \\ &\Rightarrow \quad \max_{R\in SO(3)} \mathrm{tr}(R^{\mathsf{T}}A). \end{aligned}$$

Note that the constant terms in the objective function can be ignored. The optimization can therefore be formulated as follows:

$$\begin{aligned} \max_{R\in SO(3)} \quad & \sum_{i=1}^{3} r_i^{\mathsf{T}} a_i \\ \text{subject to} \quad & r_i^{\mathsf{T}} r_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \\ & r_3 - [r_1] r_2 = 0, \end{aligned}$$

where a_i is the i-th column of A and r_i is the i-th column of R.

An alternative solution is also possible based on the singular value decomposition (SVD) of a matrix (consult any linear algebra textbook for a discussion of SVD). Letting $A = U\Sigma V^\intercal$ be the SVD of A (and in practice A may not necessarily be a rotation matrix due to measurement errors) the equivalent problem is

$$\max_{R\in SO(3)} \mathrm{tr}(R^\intercal U\Sigma V^\intercal) = \mathrm{tr}(S\Sigma V^\intercal),$$

where U, V as obtained from the SVD are orthonormal (i.e., $U^\intercal U = V^\intercal V = I$) and $S = R^\intercal U$ is also orthonormal. Let s_i, v_i respectively denote the i-th column of U, V, and σ_i be the (i,i) component of Σ (where $\sigma_i \geq 0$). Then

$$\mathrm{tr}(S\Sigma V^\intercal) = \sum_{i=1}^{3} \sigma_i s_i^\intercal v_i.$$

Note that $s_i^\intercal v_i \leq 1$ and equality is achieved when $s_i = v_i$. Consequently $S = V$ is the optimal solution, leading to

$$\begin{aligned} S &= R^\intercal U = V \\ R &= UV^\intercal. \end{aligned}$$

Exercise 3.11.

(a) Expanding $e^A e^B$ and e^{A+B},

$$\begin{aligned} e^A e^B &= (I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots)(I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots) \\ &= I + (A+B) + (\frac{1}{2!}A^2 + AB + \frac{1}{2!}B^2) + (\frac{1}{3!}A^3 + \frac{1}{2!}A^2B + \frac{1}{2!}AB^2 + \frac{1}{3!}B^3) + \cdots \\ &= I + (A+B) + \frac{1}{2!}(A^2 + 2AB + B^2) + \frac{1}{3!}(A^3 + 3A^2B + 3AB^2 + B^3) + \cdots \\ e^{A+B} &= I + (A+B) + \frac{1}{2!}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \cdots . \end{aligned}$$

We can see that $e^A e^B = e^{A+B}$ holds if

$$\begin{aligned} &(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2 \\ &(A+B)^3 = A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3 = A^3 + 3A^2B + 3AB^2 + B^3 \end{aligned}$$

and so forth. Thus, the necessary condition is $AB = BA$.

(b) From (a), the necessary condition is $[\mathcal{V}_a][\mathcal{V}_b] = [\mathcal{V}_b][\mathcal{V}_a]$. Under this condition,

$$\begin{aligned} [\mathcal{V}_a][\mathcal{V}_b] = \begin{bmatrix} [\omega_a] & v_a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\omega_a] & v_a \\ 0 & 0 \end{bmatrix} = [\mathcal{V}_b][\mathcal{V}_a] \\ \begin{bmatrix} [\omega_a][\omega_b] & [\omega_a]v_b \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} [\omega_b][\omega_a] & [\omega_b]v_a \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

That is, the following two equations hold:

$$[\omega_a][\omega_b] - [\omega_b][\omega_a] = 0 \tag{3.3}$$

$$[\omega_a]v_b - [\omega_b]v_a = 0 \tag{3.4}$$

Equation (3.3) implies that cross product of ω_a and ω_b is zero, and so two screw rotation axes are parallel. Let ω_b be $c\omega_a$ where $c \in \mathbb{R}$ is constant. Then, substituting $v_i = -\omega_i \times q_i + h_i\omega_i$ $(i = a, b)$ and $\omega_b = c\omega_a$ into Equation (3.4),

$$\begin{aligned} [\omega_a]v_b - [\omega_b]v_a &= \omega_a \times v_b - \omega_b \times v_a \\ &= \omega_a \times v_b - c\omega_a \times v_a \\ &= \omega_a \times (-c\omega_a \times q_b + ch_b\omega_a) - c\omega_a \times (-\omega_a \times q_a + h_a\omega_a) \\ &= \omega_a \times (-c\omega_a \times q_b) - c\omega_a \times (-\omega_a \times q_a) \\ &= 0 \end{aligned}$$

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Therefore, $\omega_a \times (\omega_a \times q_b) = \omega_a \times (\omega_a \times q_a)$, and so we can conclude that the two screws are collinear.

Exercise 3.12.
Let $A, B \in SO(3)$ have rotation axes and angles such that

$$\begin{aligned} A &= \mathrm{Rot}(\hat{\omega}_a, \alpha) \\ B &= \mathrm{Rot}(\hat{\omega}_b, \beta). \end{aligned}$$

If there exists $R \in SO(3)$ satisfying $AR = RB$, then $A = RBR^T$. We get then

$$\begin{aligned} e^{[\hat{\omega}_a]\alpha} &= Re^{[\hat{\omega}_b]\beta} R^T \\ &= e^{[R\hat{\omega}_b]\beta}. \end{aligned}$$

R therefore satisfies $\hat{\omega}_a \alpha = R\hat{\omega}_b \beta$.

(a) If $\hat{\omega}_a = \hat{\omega}_b = \hat{\mathrm{z}}$ and $\alpha = \beta$, then $\hat{\mathrm{z}} = R\hat{\mathrm{z}}$. The matrix R is therefore any $\hat{\mathrm{z}}$-axis rotation.

(b) If $\hat{\omega}_a = \hat{\omega}_b = \hat{\mathrm{z}}$ and $\alpha \neq \beta$, then two cases arise:
Case 1. $|\alpha| \neq |\beta|$: No solutions.
Case 2. $|\alpha| = |\beta|$: From $\alpha = -\beta$, we get $\hat{\mathrm{z}} = -R\hat{\mathrm{z}}$.

$$R = \mathrm{Rot}(\hat{\omega}_t, \theta),$$

where $\hat{\omega}_t$ is a unit vector lying on the x-y plane and $\theta = \pm\pi$.

(c) If $\hat{\omega}_a = \hat{\omega}_b$, then the problem can be solved in the same way as (a) and (b).
If $\hat{\omega}_a \neq \hat{\omega}_b$, then we separate this problem into three cases:
Case 1. $|\alpha| \neq |\beta|$: No solutions.
Case 2. $\alpha = \beta$: From $\hat{\omega}_a = R\hat{\omega}_b$,

$$R = \mathrm{Rot}(\hat{\omega}_t, \theta),$$

where $\hat{\omega}_t$ is a unit vector with direction $\pm(\hat{\omega}_a + \hat{\omega}_b)$, and $\theta = \pm\pi$.
Case 3. $\alpha = -\beta$: The direction of $\hat{\omega}_t$ in R should be $\pm(\hat{\omega}_a - \hat{\omega}_b)$.

Exercise 3.13.

(a) There exists two different ways to find the eigenvalues of R.
Method 1. For $R = e^{[\hat{\omega}]\phi} \in SO(3)$, $R\hat{\omega} = \hat{\omega}$. This implies that one of its eigenvectors is $\hat{\omega}$ and its corresponding eigenvalue is 1. Since the equation $\det(R - \lambda I) = 0$ is a third-order real polynomial, two roots should be a complex conjugate pair. Observe also that

$$\det(R) = \lambda_1\lambda_2\lambda_3 = \lambda_2\lambda_3 = 1,$$

from which it follows that

$$\begin{aligned} \lambda_2 &= \cos\theta + i\sin\theta = e^{i\theta} \\ \lambda_3 &= \cos\theta - i\sin\theta = e^{-i\theta}. \end{aligned}$$

Method 2. For $R \in SO(3)$, the relation between the eigenvector and eigenvalue of R can be written as

$$Rx = \lambda x, \tag{3.5}$$

where $x \in \mathbb{R}^3$ is the eigenvector and λ is its corresponding eigenvalue. Multiplying the complex conjugate transpose of each term in Equation (3.5) to both sides of Equation (3.5),

$$\begin{aligned} (Rx)^* Rx &= (\lambda x)^* \lambda x \\ \Leftrightarrow \quad x^* R^* Rx &= x^* \lambda^* \lambda x \\ \Leftrightarrow \quad x^* x &= \lambda^* \lambda x^* x. \end{aligned}$$

We can thus conclude that λ is a complex number with magnitude 1. Hence, the eigenvalues of R can be written as 1, $e^{i\theta}$, and $e^{-i\theta}$.

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(b) Denote the eigenvector associated with the eigenvalue $\mu + i\nu$ by $x + iy$ for $x, y \in \mathbb{R}^3$. The relation can be written as

$$R(x+iy) = (\mu + i\nu)(x+iy), \tag{3.6}$$

and its complex conjugate can also be written as

$$\begin{aligned} \overline{R(x+iy)} &= \overline{(\mu+i\nu)(x+iy)} \\ \Leftrightarrow \quad R(x-iy) &= (\mu - i\nu)(x-iy). \end{aligned} \tag{3.7}$$

From Equations (3.6) and (3.7),

$$\begin{aligned} Rx &= \mu x - \nu y \\ Ry &= \nu x + \mu y, \end{aligned}$$

and $Rz = z$ from the eigenvalue whose value is 1. We can thus write

$$R = A \begin{bmatrix} \mu & \nu & 0 \\ -\nu & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1},$$

where $A = \begin{bmatrix} x & y & z \end{bmatrix}$. From Equation (3.6),

$$\begin{aligned} (x^T + iy^T)R^T R(x+iy) &= (\mu + i\nu)^2 (x^T + iy^T)(x+iy) \\ \Leftrightarrow \quad (x^T + iy^T)(x+iy) &= (\mu^2 - \nu^2 + 2i\mu\nu)(x^T + iy^T)(x+iy) \\ \Leftrightarrow \quad (x^T x - y^T y + 2i(x^T y)) &= (\mu^2 - \nu^2 + 2i\mu\nu)(x^T x - y^T y + 2i(x^T y)). \end{aligned}$$

Since $(\mu^2 - \nu^2 + 2i\mu\nu)$ cannot be 1,

$$\begin{aligned} x^T x - y^T y &= 0 \\ \Leftrightarrow \quad x^T y &= 0. \end{aligned}$$

which means x and y are orthogonal. From the relation $Rz = z$, the following holds:

$$\begin{aligned} z^T R^T R(x+iy) &= (\mu + i\nu) z^T (x+iy) \\ \Leftrightarrow \quad (z^T x + i z^T y) &= (\mu + i\nu)(z^T x + i z^T y), \end{aligned}$$

from which it follows that $z^T x = 0$ and $z^T y = 0$ (since $\mu + i\nu \neq 1$). x, y, and z are therefore orthogonal. Taking x, y, and z as unit vectors satisfying $\det(A) = 1$ yields $A \in SO(3)$.

Exercise 3.14.

Express the inverse as a quadratic matrix polynomial in $[\omega]$, and multiply it with the original polynomial:

$$(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)(\alpha I + \beta[\omega] + \gamma[\omega]^2) = I.$$

Using the identity $[\omega]^3 = -[\omega]$, the above leads to the following set of simultaneous equations:

$$\begin{cases} \theta\alpha = 1 \\ (1-\cos\theta)\alpha + (\sin\theta)\beta - (1-\cos\theta)\gamma = 0 \\ (\theta - \sin\theta)\alpha + (1-\cos\theta)\beta + (\sin\theta)\gamma = 0. \end{cases}$$

Solving the equations,

$$\alpha = \frac{1}{\theta}, \ \beta = -\frac{1}{2}, \ \gamma = \frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}.$$

Exercise 3.15.

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(a) As a sequence of rotations about the axes of the fixed frame, rotation matrices should be multiplied on the left. Beginning with $R_{01} = \mathrm{I}$,
Step 1. $R_{02} = \mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha) R_{01} = \mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha)$.
Step 2. $R_{03} = \mathrm{Rot}(\hat{\mathrm{y}}_0, \beta) R_{02} = \mathrm{Rot}(\hat{\mathrm{y}}_0, \beta)\mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha)$.
Step 3. $R_{04} = \mathrm{Rot}(\hat{\mathrm{z}}_0, \gamma) R_{03} = \mathrm{Rot}(\hat{\mathrm{z}}_0, \gamma)\mathrm{Rot}(\hat{\mathrm{y}}_0, \beta)\mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha)$.

$$R_{04} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}.$$

(b) As this rotation is about the axes of the moving frame, the rotation matrix should be multiplied on the right:

$$\begin{aligned} R_{04} &= R_{03}\mathrm{Rot}(\hat{\mathrm{z}}_3, \gamma) \\ &= \mathrm{Rot}(\hat{\mathrm{y}}_0, \beta)\mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha)\mathrm{Rot}(\hat{\mathrm{z}}_3, \gamma) \\ &= \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

(c)

$$\begin{aligned} T_{ca} &= T_{cb} T_{ba} = T_{cb} T_{ab}^{-1} = T_{cb} \begin{bmatrix} R_{ab}^T & -R_{ab}^T p_{ab} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} - \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 - \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} - \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Exercise 3.16.
(a) The three frames are shown in Figure 3.2.

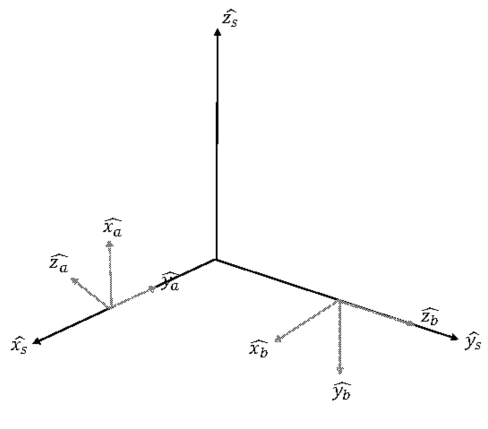

Figure 3.2

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(b)

$$R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$T_{sa} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_{sb} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$T_{sb}^{-1} = \begin{bmatrix} R_{sb} & p_b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R_{sb}^{\mathrm{T}} & -R_{sb}^{\mathrm{T}} p_b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d)

$$T_{ab} = T_{sa}^{-1} T_{sb}$$

$$T_{sa}^{-1} = \begin{bmatrix} R_{sa}^{\mathrm{T}} & -R_{sa}^{\mathrm{T}} p_a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{ab} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e)

$$T_1 = T_{sa} T = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

this corresponds to a transformation in a body frame

$$T_2 = T T_{sa} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

this corresponds to a transformation in s world frame

(f)

$$p_s = T_{sb} p_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \\ 1 \end{bmatrix} \implies p_s = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$

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(g)

$$p' = T_{sb}p_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \\ 1 \end{bmatrix} \implies \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$
$$p'' = T_{sb}^{-1}p_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \implies \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

Therefore p' is a change in location and p'' is change in reference frame.

(h)

$$\mathcal{V}_a = [\mathrm{Ad}_{T_{as}}]\mathcal{V}_s$$
$$T_{as} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\implies [\mathrm{Ad}_{T_{as}}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 & -1 & 0 \end{bmatrix}$$
$$\mathbb{V}_a = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \\ -9 \\ 1 \\ -1 \end{bmatrix}$$

(i)

$$R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore the trace of $\mathrm{tr}(R_{sa}) = 0$ which means we are condition (iii) and when we solve for θ we get

$$\theta = \cos^{-1}((0-1)/2) = 2.094$$

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Then we have

$$
\begin{aligned}
[\omega] &= 1/(2\sin\theta)(R - R^{\mathrm{T}}) = 1/\sqrt{3}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \\
G^{-1}(\theta) &= (1/\theta)I - (1/2)[\omega] + ((1/\theta) - (1/2)(\cot(\theta/2))[\omega]^2 \\
&= \begin{bmatrix} 0.352 & 0.226 & 0.352 \\ -0.352 & 0.352 & 0.2257 \\ -0.2257 & -0.352 & 0.352 \end{bmatrix} \\
v &= G^{-1}(\theta)p = [1.0548 \ -1.0548 \ -0.6772]^{\mathrm{T}} \\
[s] &= \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -0.5774 & -0.5774 & 1.0548 \\ 0.5774 & 0 & -0.5774 & -1.0548 \\ 0.5774 & 0.5774 & 0 & -0.6772 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
s &= [0.5774 \ -0.5774 \ 0.5774 \ 1.0548 \ -1.0548 \ -0.6772]^{\mathrm{T}} = \omega/||\omega|| \qquad \implies \dot{\theta} = 1 \\
h &= \hat{\omega}^{\mathrm{T}} v/\dot{\theta} = 0.827 \\
v &= -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \implies q = [-1 \ 1 \ 0]^{\mathrm{T}}
\end{aligned}
$$

(j)

$$
\begin{aligned}
S\theta &= [0 \ 1 \ 2 \ 3 \ 0 \ 0]^{\mathrm{T}} = [\omega\theta, v\theta]^{\mathrm{T}} \\
\implies \omega\theta &= [0 \ 1 \ 2]^{\mathrm{T}} \implies \theta = \sqrt{5}
\end{aligned}
$$

We can find the matrix exponential by:

$$
\begin{aligned}
\mathrm{e}^{[s]\theta} &= \begin{bmatrix} \mathrm{e}^{[\omega]\theta} & (I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v \\ 0 & 1 \end{bmatrix} \\
[\omega_s] &= \begin{bmatrix} 0 & -0.8944 & 0.4472 \\ 0.8944 & 0 & 0 \\ -0.4472 & 0 & 0 \end{bmatrix} \\
\mathrm{e}^{[\omega]\theta} &= \begin{bmatrix} -0.6173 & -0.7037 & 0.3518 \\ 0.7037 & -0.2938 & 0.6469 \\ -0.3518 & 0.6469 & 0.6765 \end{bmatrix} \\
G(\theta)v &= \begin{bmatrix} 1.0555 \\ 1.9407 \\ -0.9704 \end{bmatrix} \ e^{[s]\theta} = \begin{bmatrix} -0.6173 & -0.7037 & 0.3518 & 1.0555 \\ 0.7037 & -0.2938 & 0.6469 & 1.9407 \\ -0.3518 & 0.6469 & 0.6765 & -0.9704 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix}
\end{aligned}
$$

Exercise 3.17.

(a)

$$
T_{ad} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ T_{cd} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

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(b) $T_{ab}T_{bc}T_{cd} = T_{ad}$. Thus $T_{ab} = T_{ad}(T_{bc}T_{cd})^{-1}$.

$$T_{bc}T_{cd} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(T_{bc}T_{cd})^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{ab} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Exercise 3.18.

(a) $T_{ea}T_{ar} = T_{es}T_{sr}$ or $T_{sr} = (T_{es})^{-1}T_{ea}T_{ar}$. Therefore $T_{rs} = (T_{ar})^{-1}(T_{ea})^{-1}T_{es}$.
(b) $P_{rs} = T_{re}P_{es}$ and

$$P_{es} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T_{re} = T_{er}^{-1} = \begin{bmatrix} R_{er}^T & -R_{er}^T p_{er} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} p_{rs} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Exercise 3.19.

(a) The transformation from the fixed frame to the satellite can be described as follows: Rotate about the {0} frame x̂-axis to align the ŷ-axis with the satellite's ŷ-axis. Then, rotate about the body frame ŷ-axis to align the x̂-axis with the satellite's x̂-axis. Finally, translate about the body frame x̂-axis by

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R. Defining $\theta_1 = \frac{v_1 t}{R_1}$, $\theta_2 = \frac{v_2 t}{R_2}$,

$$\begin{aligned}
T_{01} &= \mathrm{Rot}(\hat{\mathrm{x}}, 120^\circ)\mathrm{Rot}(\hat{\mathrm{y}}, -90^\circ + \theta_1)\mathrm{Trans}(\hat{\mathrm{x}}, -R_1) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 120^\circ & -\sin 120^\circ & 0 \\ 0 & \sin 120^\circ & \cos 120^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-90^\circ + \theta_1) & 0 & \sin(-90^\circ + \theta_1) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-90^\circ + \theta_1) & 0 & \cos(-90^\circ + \theta_1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -R_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta_1 & 0 & -\cos\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -R_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \sin\theta_1 & 0 & -\cos\theta_1 & -R_1 \sin\theta_1 \\ -\frac{\sqrt{3}}{2}\cos\theta_1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2}\sin\theta_1 & \frac{\sqrt{3}}{2} R_1 \cos\theta_1 \\ -\frac{1}{2}\cos\theta_1 & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\theta_1 & \frac{1}{2} R_1 \cos\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
T_{02} &= \mathrm{Rot}(\hat{\mathrm{x}}, 90^\circ)\mathrm{Rot}(\hat{\mathrm{y}}, -90^\circ + \theta_2)\mathrm{Trans}(\hat{\mathrm{x}}, -R_2) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ & 0 \\ 0 & \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-90^\circ + \theta_2) & 0 & \sin(-90^\circ + \theta_2) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-90^\circ + \theta_2) & 0 & \cos(-90^\circ + \theta_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -R_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta_2 & 0 & -\cos\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -R_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \sin\theta_2 & 0 & -\cos\theta_2 & -R_2 \sin\theta_2 \\ -\cos\theta_2 & 0 & -\sin\theta_2 & R_2 \cos\theta_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

(b)

$$\begin{aligned}
T_{21} &= T_{02}^{-1} T_{01} \\
&= \begin{bmatrix} R_{02}^T & -R_{02}^T P_{02} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{01} & P_{01} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \sin\theta_2 & -\cos\theta_2 & 0 & R_2 \\ 0 & 0 & 1 & 0 \\ -\cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta_1 & 0 & -\cos\theta_1 & -R_1 \sin\theta_1 \\ -\frac{\sqrt{3}}{2}\cos\theta_1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2}\sin\theta_1 & \frac{\sqrt{3}}{2} R_1 \cos\theta_1 \\ -\frac{1}{2}\cos\theta_1 & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\theta_1 & \frac{1}{2} R_1 \cos\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & \frac{1}{2}\cos\theta_2 & a_{13} & a_{14} \\ -\frac{1}{2}\cos\theta_1 & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\theta_1 & \frac{1}{2} R_1 \cos\theta_1 \\ a_{31} & \frac{1}{2}\sin\theta_2 & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned}$$

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where

$$\begin{aligned}
a_{11} &= \sin\theta_1 \sin\theta_2 + \frac{\sqrt{3}}{2}\cos\theta_1\cos\theta_2 \\
a_{13} &= -\cos\theta_1 \sin\theta_2 + \frac{\sqrt{3}}{2}\sin\theta_1\cos\theta_2 \\
a_{14} &= -R_1\sin\theta_1\sin\theta_2 - \frac{\sqrt{3}}{2}R_1\cos\theta_1\cos\theta_2 + R_2 \\
a_{31} &= -\sin\theta_1\cos\theta_2 + \frac{\sqrt{3}}{2}\cos\theta_1\sin\theta_2 \\
a_{33} &= \cos\theta_1\cos\theta_2 + \frac{\sqrt{3}}{2}\sin\theta_1\sin\theta_2 \\
a_{34} &= R_1\sin\theta_1\cos\theta_2 - \frac{\sqrt{3}}{2}R_1\cos\theta_1\sin\theta_2.
\end{aligned}$$

Exercise 3.20.
Since the two wheels roll the same distance, we can represent the rotation angle of the two wheels with one variable θ:

$$\begin{aligned}
T_{ab} &= \begin{bmatrix} I & \begin{matrix} 0 \\ L\sin\theta \\ L\cos\theta \end{matrix} \\ 0 & 1 \end{bmatrix} \\
T_{bc} &= \begin{bmatrix} I & \begin{matrix} 0 \\ 2r \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} \\
T_{ac} &= T_{ab}T_{bc} \\
&= \begin{bmatrix} I & \begin{matrix} 0 \\ L\sin\theta + 2r \\ L\cos\theta \end{matrix} \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Exercise 3.21.

(a) From the given T_{ab}, $q_{ab} = (-100, 300, 500)^T$ and

$$R_{ab} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore $r_a = p_a - q_{ab}$, which leads to

$$r_b = R_{ab}^T r_a = R_{ab}^T(p_a - q_{ab}) = (500, -100, -500)^T.$$

(b) From the given $p_{ac} = p_a$ and $R_{ac} = Rot(x, 30°)$, T_{ac} can be derived as follows:

$$T_{ac} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 800 \\ 0 & \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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T_{bc} can then be computed as follows:

$$T_{bc} = T_{ab}^{-1} T_{ac} = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 500 \\ -1 & 0 & 0 & -100 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & -500 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Exercise 3.22.

(a) Since the platform is rising vertically, $p_{01} = (0, 0, vt)^\intercal$, with the $\hat{\mathrm{y}}_2$-axis of the laser frame pointing at the target. Thus $R_{12} = \mathrm{Rot}(\hat{\mathrm{z}}, -\alpha)\mathrm{Rot}(\hat{\mathrm{x}}, -\gamma)$, where α and γ satisfy

$$\tan\alpha = \frac{1 - \cos\theta}{1 - \sin\theta}, \quad \tan\gamma = \frac{Lt}{\sqrt{(1-\cos\theta)^2 + (1 - \sin\theta)^2}}.$$

The target is rotating about its $\hat{\mathrm{z}}$-axis, or $R_{03} = \mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t)$. T_{01}, T_{12}, T_{03} can be therefore be as follows:

$$T_{01} = \begin{bmatrix} I & \begin{matrix} 0 \\ 0 \\ vt \end{matrix} \\ 0 & 1 \end{bmatrix}$$

$$T_{12} = \begin{bmatrix} \mathrm{Rot}(\hat{\mathrm{z}}, -\alpha)\mathrm{Rot}(\hat{\mathrm{x}}, -\gamma) & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix}$$

$$T_{03} = \begin{bmatrix} \mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t) & \begin{matrix} 1 - \cos\omega t \\ 1 - \sin\omega t \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix}.$$

(b)

$$\begin{aligned} T_{23} &= (T_{02})^{-1} T_{03} \\ &= \begin{bmatrix} \mathrm{Rot}(\hat{\mathrm{z}}, -\alpha)\mathrm{Rot}(\hat{\mathrm{x}}, -\gamma) & \begin{matrix} 0 \\ 0 \\ vt \end{matrix} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t) & \begin{matrix} 1 - \cos\omega t \\ 1 - \sin\omega t \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Rot}(\hat{\mathrm{x}}, \gamma)\mathrm{Rot}(\hat{\mathrm{z}}, \alpha) & \begin{matrix} 0 \\ vt\sin\gamma \\ -vt\cos\gamma \end{matrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t) & \begin{matrix} 1 - \cos\omega t \\ 1 - \sin\omega t \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Rot}(\hat{\mathrm{x}}, \gamma)\mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t + \alpha) & \begin{matrix} \cos\alpha(1 - \cos\omega t) - \sin\alpha(1 - \sin\omega t) \\ \cos\gamma(\sin\alpha(1 - \cos\omega t) + \cos\alpha(1 - \sin\omega t)) + vt\sin\gamma \\ \sin\gamma(\sin\alpha(1 - \cos\omega t) + \cos\alpha(1 - \sin\omega t)) - vt\cos\gamma \end{matrix} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Exercise 3.23.

The moving frame $\{t\}$ can be placed on the table as shown in Figure 3.3, with the origin at the center of the table and whose orientation is $\mathrm{Rot}(\hat{\mathrm{z}}, \theta)$ with respect to the $\{0\}$ frame. Set the $\{t_1\}$ frame to $\theta = \frac{v_1 t}{r}$, and the $\{t_2\}$ frame to $\theta = 45°$.

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Top View

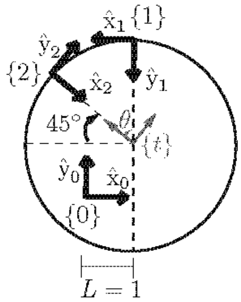

Figure 3.3

(a) Compute and substitute the rotation matrices R_{ij} and the position vectors p_{ij} as follows:

$$T_{0t} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & L \\ \sin\theta & \cos\theta & 0 & L \\ 0 & 0 & 1 & H \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ T_{t_1 1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & R \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ T_{t_2 2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & R - v_2 t \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Substitute $\theta = \frac{v_1 t}{r}$ for the $\{t_1\}$ frame and $\theta = 45°$ for the $\{t_2\}$ frame into T_{ot}. T_{01}, T_{02} can then be obtained as functions of t:

$$T_{01} = T_{0t_1} T_{t_1 1} = \begin{bmatrix} -\cos\frac{v_1 t}{2} & \sin\frac{v_1 t}{2} & 0 & 1 - 2\sin\frac{v_1 t}{2} \\ -\sin\frac{v_1 t}{2} & -\cos\frac{v_1 t}{2} & 0 & 1 + 2\cos\frac{v_1 t}{2} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{02} = T_{0t_2} T_{t_2 2} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1 - \sqrt{2} + \frac{\sqrt{2}}{2} v_2 t \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1 + \sqrt{2} - \frac{\sqrt{2}}{2} v_2 t \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Substituting the result obtained from (a),

$$\begin{aligned} T_{12} &= T_{01}^{-1} T_{02} \\ &= \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} R_{02} & p_{02} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_{01}^T & -R_{01} p_{01} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{02} & p_{02} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{01}^T R_{02} & R_{01}^T p_{02} - R_{01} p_{01} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\cos\frac{v_1 t}{2} & -\sin\frac{v_1 t}{2} & 0 & \sin\frac{v_1 t}{2} + \cos\frac{v_1 t}{2} \\ \sin\frac{v_1 t}{2} & -\cos\frac{v_1 t}{2} & 0 & 2 - \sin\frac{v_1 t}{2} + \cos\frac{v_1 t}{2} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1 - \sqrt{2} + \frac{\sqrt{2}}{2} v_2 t \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1 + \sqrt{2} - \frac{\sqrt{2}}{2} v_2 t \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Exercise 3.24.
Calculate the configuration of the robot at $t = 4$ sec, $(\theta_1, \theta_2, \theta_3) = (\pi, \frac{\pi}{2}, -\pi)$. Since $\theta_1 = \pi$ and the pitch

$h = 2$, we have $L_1(t = 4) = 10 + 1 = 11$.

$$\begin{aligned} T &= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \\ \text{where} \quad R &= Rot(\hat{z}, \pi) Rot(-\hat{y}, \frac{\pi}{2}) Rot(-\hat{z}, -\pi), \\ p &= Rot(\hat{z}, \pi) \begin{bmatrix} -5 \\ 2 \\ 11 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 11 \end{bmatrix}. \end{aligned}$$

To sum up, the transformation matrix of the tip T can be computed as follows:

$$T = \begin{bmatrix} 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Exercise 3.25.

(a) From $AX = XB$, we have $B = X^{-1}AX$. Substitute the given A, X into B:

$$B = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Suppose that both $\mathrm{tr}(R_A)$ and $\mathrm{tr}(R_B)$ are not equal to -1, so that α and β are uniquely defined. Rewriting $R_A R_X = R_X R_A$ as $e^{[\alpha]} R_X = R_X e^{[\beta]}$ and using the property $R[w]R^T = [Rw]$ and $Re^{[w]}R^T$,

$$e^{[\alpha]} = R_X e^{[\beta]} R_X^T = e^{R_X [\beta] R_X^T} = e^{[R_X \beta]}.$$

This leads to the necessary and sufficient condition.

$$\alpha = R_X \beta. \tag{3.8}$$

Since R_X is orthogonal, a solution exists if and only if $\|\alpha\| = \|\beta\|$. The condition $\|\alpha\| = \|\beta\|$ also holds when R_A and R_B have trace -1, since then $\|\alpha\| = \|\beta\| = \pi$.

(c) From Equation (3.8) we can construct a set of equations $\alpha_i = R_X \beta_i$ $i = 1, \ldots, k$. Suppose *two* pairs of (A_i, B_i) are obtained in a physical setting where a solution $X \in SE(3)$ is known to exist. If $\alpha_1 \times \alpha_2 \neq 0$ and $\beta_1 \times \beta_2 \neq 0$, a unique solution to Equation (3.8) is

$$R_X = \mathcal{A}\mathcal{B}^{-1},$$

where $\mathcal{A}$ and $\mathcal{B}$ are matrices whose columns are the vectors $\alpha_1, \alpha_2, \alpha_1 \times \alpha_2$, and $\beta_1, \beta_2, \beta_1 \times \beta_2$, respectively:

$$\begin{aligned} \mathcal{A} &= [\alpha_1, \alpha_2, \alpha_1 \times \alpha_2] \\ \mathcal{B} &= [\beta_1, \beta_2, \beta_1 \times \beta_2] \end{aligned}$$

The translational components of $A_i X = X B_i$ can be written

$$\begin{aligned} R_{A_i} p_X + p_A &= R_X p_{B_i} + p_X \\ [R_{A_i} - I] p_X &= R_X p_{B_i} - p_{A_i}. \end{aligned}$$

The matrix $(R_{A_i} - I)$ has rank two and its null space is spanned by $\{\alpha_i\}$, so p_X cannot be determined uniquely yet. Satisfying our assumption (i.e. $\alpha_1 \times \alpha_2 \neq 0$), a *unique* solution for p_X can be found by solving the following augmented matrix equation:

$$\begin{bmatrix} R_{A_1} - I \\ R_{A_2} - I \end{bmatrix} p_X = \begin{bmatrix} R_X p_{B_1} - p_{A_1} \\ R_X p_{B_2} - p_{A_2} \end{bmatrix}.$$

The minimum number of k is 2 for a unique solution $X \in SE(3)$ to exist.

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Exercise 3.26.

$$\mathcal{V}\dot{\theta} = \mathcal{S} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$$

Drawn as Figure 3.4.

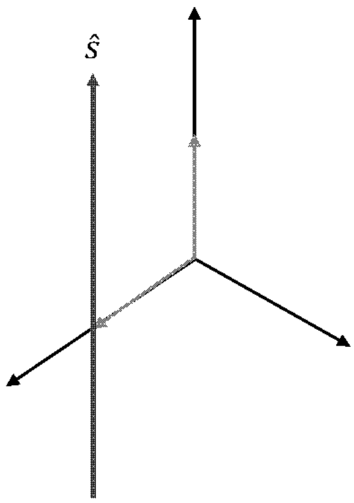

Figure 3.4

Exercise 3.27.

$$\begin{aligned}
&\mathbb{V} = [\omega, v] = [0\ 2\ 2\ 4\ 0\ 0]^T \\
&\omega = [0\ 2\ 2]^T \quad v = [4\ 0\ 0]^T \\
&\hat{s} = \omega/||\omega|| = [0\ 1/\sqrt{2}\ 1/\sqrt{2}] \\
&v = v = [\sqrt{2}\ 0\ 0]^T \\
&h = \hat{\omega}^T v/\dot{\theta} = 0 \\
&q = [0\ 1\ \ -1]^T
\end{aligned}$$

Drawn as Figure 3.5.

Exercise 3.28.

$$R_{bs} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \qquad \Longrightarrow R_{bs}\hat{\omega}_s = \hat{\omega}_b \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

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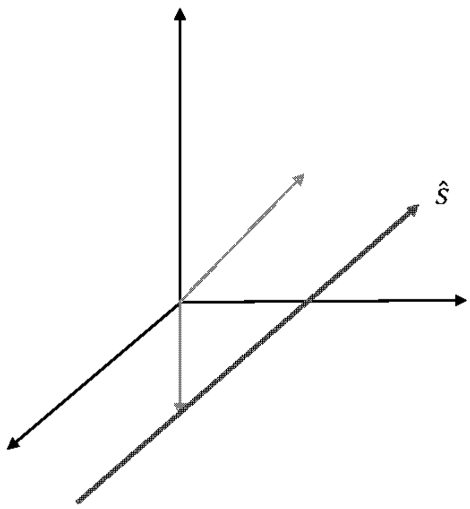

Figure 3.5

Exercise 3.29.
Given two frames {a} and {b} attached to a rigid body, the representations of a twist with respect to these two frames are related by

$$\mathcal{S}_a = [Ad_{T_{ab}}]\mathcal{S}_b \quad (3.9)$$
$$w_a = R_{ab}w_b \quad (3.10)$$
$$v_a = [p_{ab}]R_{ab}w_b + R_{ab}v_b. \quad (3.11)$$

Transforming the twists into coordinates of the space frame {s},

$$\mathcal{S}_a^s = [Ad_{T_{sa}}]\mathcal{S}_a \quad (3.12)$$
$$\mathcal{S}_b^s = [Ad_{T_{sb}}]\mathcal{S}_b \quad (3.13)$$
$$w_a^s = R_{sa}w_a \quad (3.14)$$
$$w_b^s = R_{sb}w_b \quad (3.15)$$
$$v_a^s = [p_{sa}]R_{sa}w_a + R_{sa}v_a \quad (3.16)$$
$$v_b^s = [p_{sb}]R_{sb}w_b + R_{sb}v_b. \quad (3.17)$$

where $(\cdot)_a^s$ is $(\cdot)_a$ expressed in the {s} frame, and $(\cdot)_b^s$ is $(\cdot)_b$ expressed in the {s} frame. Thus, if we substitute Equations (3.10) and (3.11) into (3.14) and (3.16),

$$\begin{aligned} w_a^s &= R_{sa}R_{ab}w_b = R_{sb}w_b = w_b^s \\ v_a^s &= [p_{sa}]R_{sa}R_{ab}w_b + R_{sa}([p_{ab}]R_{ab}w_b + R_{ab}v_b) \\ &= [p_{sa} + R_{sa}p_{ab}]R_{sb}w_b + R_{sb}v_b = [p_{sb}]R_{sb}w_b + R_{sb}v_b \\ &= v_b^s. \end{aligned}$$

In summary, the twist associated with the motion of a rigid body is always the same when expressed in the space frame, regardless of where the body frame is attached to the body:

$$\mathcal{S}_a^s = \mathcal{S}_b^s.$$

Exercise 3.30.

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(a) First, compute T_{12} for both cases:

$$\text{Case 1:} \quad T_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Case 2:} \quad T_{12} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Cases 1 and 2 can be thought as a screw motion that translates and rotates along the vertical screw shown in Figure 3.6. Let

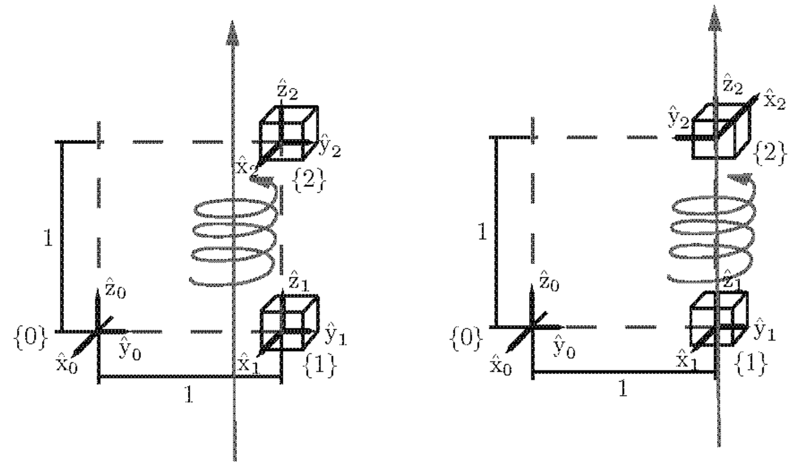

Figure 3.6

$$[S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\hat{\omega}] & \hat{v} \\ 0 & 0 \end{bmatrix} \theta = [\hat{S}]\theta$$

$$e^{[S]} = \begin{bmatrix} e^{[\hat{w}]\theta} & G(\theta)\hat{v} \\ 0 & 1 \end{bmatrix}.$$

Case 1: Based on the physical meaning of a screw, we can compute $\hat{S}$ as $\hat{\omega} = (0,0,1)^T$, $\hat{v} = (v_1, 0, \frac{1}{2n\pi})^T$, where $v_1 = \|r \times \hat{w}\|$ depends on the location of the axis r. However, the orientation is not changed and $\theta = 2n\pi$. From $[S] = [\hat{S}]\theta$, the exponential coordinates can be derived as follows:

$$\omega = (0,0,2n\pi)^{\intercal}, v = (2n\pi v_1, 0, 1)^{\intercal}.$$

Case 2: Similarly, the orientation is rotated about the $\hat{z}$ axis by a half-circle, thus $\theta = 2n\pi + \pi$. Therefore $\hat{\omega} = (0,0,1)^{\intercal}$ and $\hat{v} = (v_1', 0, \frac{1}{2n\pi+\pi})^{\intercal}$, where v_1' also depends on the axis. From $[S] = [\hat{S}]\theta$, the exponential coordinates can be derived as follows:

$$\omega = (0,0,(2n+1)\pi)^T, v = ((2n+1)\pi v_1', 0, 1)^T.$$

(b) Restrict the norm of w as $\|w\| \leq \pi$. From the answers obtained in (a),
Case 1: To satisfy the restriction, $\omega = (0,0,0)^T$, otherwise the norm of ω exceeds π. Thus, the joint can be considered as a prismatic joint. From $[S] = [\hat{S}]\theta$, the exponential coordinates can be derived as follows:

$$\omega = (0,0,0)^T, v = (0,0,1)^T.$$

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Case 2: To satisfy the restriction, $\theta = \pm\pi$. The screw can be computed as $\hat{w} = (0,0,1)^\intercal$ and $\hat{v} = (\pm 1, 0, \frac{1}{\pm\pi})^\intercal$. From $[S] = [\hat{S}]\theta$, the exponential coordinates can be derived as follows:

$$\omega = (0,0,\pm\pi)^T, v = (\pi, 0, 1)^T.$$

Exercise 3.31.
Since the force is applied at the center of mass there is no torque, therefore the wrench vector in the effector frame can be written as:

$$\mathcal{F}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

From the example we get the transformation matrix of the object relative to the robot hand to be:

$$T_{ce} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -260/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 130/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By defintion $\mathcal{F}_c = [\mathrm{Ad}T_{ec}]^\mathrm{T}\mathcal{F}_e$, and $T_{ec} = T_{ce}^{-1}$, which gives us the matrices:

$$T_{ec} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & -65 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 195 \\ 1 & 0 & 0 & 75 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathrm{Ad}T_{ec}]^T = \begin{bmatrix} 0 & 0 & 1.0000 & 195.0000 & 65.0000 & 0 \\ -0.7071 & 0.7071 & 0 & -53.0330 & -53.0330 & 91.9239 \\ -0.7071 & -0.7071 & 0 & 53.0330 & -53.0330 & 183.8478 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 0 & -0.7071 & 0.7071 & 0 \\ 0 & 0 & 0 & -0.7071 & -0.7071 & 0 \end{bmatrix} \implies \mathcal{F}_c = \begin{bmatrix} 0 \\ 919.2 \\ 1838.5 \\ 10 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 3.32.
(a) Let $T_{oa} = (R_{oa}, p_a)$ and $T_{ob} = (R_{ob}, p_b)$. Then

$$\begin{aligned} T_{o'a} &= (R_s R_{oa}, R_s p_a + p_s) \\ &= (R'_{oa}, p'_a), \end{aligned}$$

and

$$\begin{aligned} T_{o'b} &= (R_s R_{ob}, R_s p_b + p_s) \\ &= (R'_{ob}, p'_b). \end{aligned}$$

We can write $R_{ab} = R_{oa}^T R_{ob}$ and $p_{ab} = p_b - p_a$. Also,

$$\begin{aligned} R'_{ab} &= (R'_{oa})^T R'_{ob} \\ &= R_{oa}^T R_s^T R_s R_{ob} \\ &= R_{oa}^T R_{ob} \\ &= R_{ab}, \end{aligned}$$

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and

$$\begin{aligned} p'_{ab} &= p'_b - p'_a \\ &= R_s(p_b - p_a) \\ &= R_s p_{ab}. \end{aligned}$$

Then from the given distance formula,

$$\mathrm{dist}(T_{o'a}, T_{o'b}) = \sqrt{\theta'^2 + \|p'_{ab}\|^2},$$

where $\theta' = \cos^{-1}((\mathrm{tr}(R'_{ab}) - 1)/2)$.

(b) Since $R_{ab} = R'_{ab}$, $\theta' = \theta$. We also have $\|R_s p_{ab}\|^2 = \|p_{ab}\|^2$. Therefore

$$\mathrm{dist}(T_{o'a}, T_{o'b}) = \mathrm{dist}(T_{oa}, T_{ob}),$$

for all $S \in SE(3)$.

Exercise 3.33.

(a) For the matrix exponential, decompose A into PDP^{-1} by obtaining the eigenvalues and eigenvectors of A:

$$\det(Is - A) = (s+2)(s+1) = 0.$$

The eigenvalues of A are -1 and -2. Hence,

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \; P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{(eigenvector matrix)}, \; P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

The general solution can be written as

$$\begin{aligned} x(t) &= e^{At}x(0) = Pe^{Dt}P^{-1}x(0) \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x(0) \\ &= \begin{bmatrix} e^{-2t} & e^{-t} - e^{-2t} \\ 0 & e^{-t} \end{bmatrix} x(0). \end{aligned}$$

Therefore, $\lim_{t\to\infty} x(t) = 0$.

(b)

$$\det(Is - A) = (s-2)^2 + 1 = 0.$$

The eigenvalues of A are $2+i$ and $2-i$. Hence,

$$D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}, \; P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \text{(eigenvector matrix)}, \; P^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}.$$

The general solution can be written as

$$\begin{aligned} x(t) &= e^{At}x(0) = Pe^{Dt}P^{-1}x(0) \\ &= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(2+i)t} & 0 \\ 0 & e^{(2-i)t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} x(0) \\ &= e^{2t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} x(0). \end{aligned}$$

Therefore, $\lim_{n\to\infty} x(t) = \infty$ (increasingly larger oscillations).

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Exercise 3.34.
From the linear differential equation $\dot{x}(t) = Ax(t)$,

$$\begin{cases} \begin{bmatrix} -3e^{-3t} \\ 9e^{-3t} \end{bmatrix} = A \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}, \\ \\ \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} = A \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}. \end{cases}$$

From above one can obtain

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}.$$

The eigenvalues and eigenvectors of A are

$$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -3 \end{cases} \Rightarrow \begin{matrix} v_1 = (1,1) \\ v_2 = (1,-3) \end{matrix}.$$

Therefore

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 3e^{t} + e^{-3t} & e^{t} - e^{-3t} \\ 3e^{t} - 3e^{-3t} & e^{t} + 3e^{-3t} \end{bmatrix}. \end{aligned}$$

Exercise 3.35.
Set $z(t) = e^{-At}x(t)$ and evaluate $\dot{z}(t)$:

$$\begin{aligned} \dot{z}(t) &= -Ae^{-At}x(t) + e^{-At}\dot{x}(t) \\ &= -Ae^{-At}x(t) + e^{-At}(Ax(t) + f(t)) \\ \therefore \dot{z}(t) &= e^{-At}f(t). \end{aligned}$$

Integrate both sides from 0 to t:

$$\int_0^t \dot{z}(s)\, ds = \int_0^t e^{-As}f(s)\, ds.$$

Express the left side of the equation in terms of $x(t)$:

$$\begin{aligned} \int_0^t \dot{z}(s)\, ds &= z(t) - z(0) \\ &= e^{-At}x(t) - e^{-A0}x(0) \\ \therefore \int_0^t \dot{z}(s)\, ds &= e^{-At}x(t) - x(0). \end{aligned}$$

Multiply e^{At} on the left to both sides of the previous equation,

$$\begin{aligned} e^{At}\left(e^{-At}x(t) - x(0)\right) &= e^{At}\int_0^t e^{-As}f(s)\, ds \\ x(t) - e^{At}x(0) &= e^{At}\int_0^t e^{-As}f(s)\, ds \end{aligned}$$

$$\therefore x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)\, ds.$$

Exercise 3.36.

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(a) The rotation matrix corresponding to the ZXZ Euler angles can be represented as

$$\begin{aligned}
R &= \mathrm{Rot}(\hat{z},\alpha)\mathrm{Rot}(\hat{x},\beta)\mathrm{Rot}(\hat{z},\gamma)\\
&= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\gamma\cos\beta & -\cos\alpha\sin\gamma - \sin\alpha\cos\gamma\cos\beta & \sin\alpha\sin\beta \\ \sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma & -\cos\alpha\sin\beta \\ \sin\beta\sin\gamma & \sin\beta\cos\gamma & \cos\beta \end{bmatrix}.
\end{aligned}$$

Since $\beta = \tan^{-1}\frac{\sin\beta}{\cos\beta} = \tan^{-1}\frac{\pm\sqrt{r_{13}^2+r_{23}^2}}{r_{33}}$, we consider two possible cases:
Case 1. $0 \le \beta \le \pi$,

$$\begin{aligned}
\beta &= \mathrm{atan2}(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}),\\
\alpha &= \mathrm{atan2}(r_{13}, -r_{23}),\\
\gamma &= \mathrm{atan2}(r_{31}, r_{32}).
\end{aligned}$$

Case 2. $-\pi \le \beta \le 0$,

$$\begin{aligned}
\beta &= \mathrm{atan2}(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}),\\
\alpha &= \mathrm{atan2}(-r_{13}, r_{23}),\\
\gamma &= \mathrm{atan2}(-r_{31}, -r_{32}).
\end{aligned}$$

(b) From the result obtained in (a), we can find two sets of Euler angles:
Case 1. $0 \le \beta \le \pi$,

$$\alpha = \pi,\ \beta = \frac{\pi}{4},\ \gamma = \frac{\pi}{4}.$$

Case 2. $-\pi \le \beta \le 0$,

$$\alpha = 0,\ \beta = -\frac{\pi}{4},\ \gamma = -\frac{3\pi}{4}.$$

Exercise 3.37.
R is calculated as follows:

$$R = e^{[\hat{\omega}_1]\theta_1}e^{[\hat{\omega}_2]\theta_2} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta_2 & \frac{\sin\theta_2}{\sqrt{2}} & \frac{\sin\theta_2}{\sqrt{2}} \\ -\frac{\sin\theta_2}{\sqrt{2}} & \frac{\cos\theta_2+1}{2} & \frac{\cos\theta_2-1}{2} \\ -\frac{\sin\theta_2}{\sqrt{2}} & \frac{\cos\theta_2-1}{2} & \frac{\cos\theta_2+1}{2} \end{bmatrix}.$$

Then, the third row of R is the same as the third row of $e^{[\hat{\omega}_2]\theta_2}$. By comparing $(-\frac{\sin\theta_2}{\sqrt{2}}, \frac{\cos\theta_2-1}{2}, \frac{\cos\theta_2+1}{2})$ with the third row of the given R, which is $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, it can be seen that there is no θ_2 which satisfies the condition. Therefore, the given orientation R is not reachable.

Exercise 3.38.

$$\begin{aligned}
R &= \mathrm{Rot}(\hat{x},\phi)\mathrm{Rot}(\hat{z},\theta)\\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \end{bmatrix}.
\end{aligned}$$

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Both ϕ and θ range in value over a 2π interval.

Exercise 3.39.
(a) $R = \mathrm{Rot}(\hat{z}_0, \alpha)\mathrm{Rot}(\hat{y}_0, \beta)\mathrm{Rot}(\hat{\omega}, \gamma)$, where $\hat{\omega} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Hence,

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} Rot(\hat{w}, \gamma),$$

where

$$\begin{aligned} \mathrm{Rot}(\hat{\omega}, \gamma) &= I + \sin\gamma[\hat{\omega}] + (1 - \cos\gamma)[\hat{\omega}]^2 \\ &= \begin{bmatrix} \frac{1}{2}(1+\cos\gamma) & \frac{1}{2}(1-\cos\gamma) & \frac{1}{\sqrt{2}}\sin\gamma \\ \frac{1}{2}(1-\cos\gamma) & \frac{1}{2}(1+\cos\gamma) & -\frac{1}{\sqrt{2}}\sin\gamma \\ -\frac{1}{\sqrt{2}}\sin\gamma & \frac{1}{\sqrt{2}}\sin\gamma & \cos\gamma \end{bmatrix}. \end{aligned}$$

(b) (i) From Figure 3.7, $(\alpha, \beta, \gamma) = (0, 0, \pi)$ or $(-\frac{\pi}{2}, \pi, 0)$.

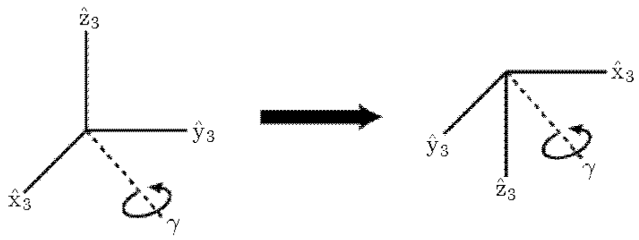

Figure 3.7

(ii) By comparing R_{03} derived in (a) with $e^{[\hat{\omega}]\frac{\pi}{2}}$, (α, β, γ) can be calculated. Denoting the (i, j)th element of R_{03} by r_{ij},

$$r_{11} + r_{12} = \cos\alpha\cos\beta - \sin\alpha = -\frac{2}{\sqrt{5}} \tag{3.18}$$

$$r_{21} + r_{22} = \sin\alpha\cos\beta + \cos\alpha = \frac{2}{\sqrt{5}} + \frac{1}{5} \tag{3.19}$$

$$r_{31} + r_{32} = -\sin\beta = -\frac{1}{\sqrt{5}} + \frac{2}{5}. \tag{3.20}$$

From Equation (3.20), $\beta = 2.71°$ or $\beta = 177.29°$.
Case 1. $\beta = 2.71°$.
By Equations (3.18) and (3.19), $\alpha = 84.23°$. Substituting β into r_{31} and r_{32}, $\gamma = 34.91°$.
Case 2. $\beta = 177.29°$.
By Equations (3.18) and (3.19), $\alpha = 354°$. Substituting β into r_{31} and r_{32}, $\gamma = 219°$.

Exercise 3.40.
(a) Compute the unit quaternion $q \in \mathbb{R}^4$ from the corresponding rotation matrix $R \in SO(3)$. Following the definition and using the Rodrigues formula,

$$\begin{aligned} q &= [\eta, \varepsilon^{\mathsf{T}}] = [\cos(\theta/2), \hat{w}\sin(\theta/2)]^{\mathsf{T}} && (3.21) \\ R &= \mathrm{e}^{[\hat{w}]}\theta = I + \sin\theta[\hat{w}] + (1-\cos\theta)[\hat{w}]^2, && (3.22) \end{aligned}$$

where $\hat{w}$ is the unit vector in the direction of the rotation axis, and θ is the angle of rotation. First obtain θ and $\hat{w}$ from Equation (3.22), and compute the quaternion q with θ and $\hat{w}$ obtained from

Equation (3.21). Useful formulas for a rotation matrix are

$$\begin{aligned} tr(R) &= 2\cos\theta + 1 &&= 4\cos^2(\theta/2) - 2 \\ \frac{R - R^{\intercal}}{2} &= \sin\theta[\hat{w}] &&= 2\sin(\theta/2)\cos(\theta/2)\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}. \end{aligned}$$

Using these formulas, we can derive the quaternion as follows:

$$\begin{aligned} q_0 &= \eta &&= \frac{1}{2}\sqrt{1 + tr(R)} = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} \\ \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} &= \varepsilon &&= \frac{1}{4q_0}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}. \end{aligned}$$

(b) We now compute a rotation matrix from the corresponding quaternion. Following the definitions given in Equations (3.21) and (3.22),

$$\begin{aligned} \theta &= 2\cos^{-1} q_0 \\ [\varepsilon] &= \frac{R - R^{\intercal}}{4q_0}. \end{aligned}$$

We can derive the rotation matrix from the Rodrigues formula of Equation (3.22) as follows:

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

(c) We now derive the formula for the product of the quaternions. Let $R_q, R_p \in SO(3)$ denote two rotation matrices corresponding to unit quaternions q, p, respectively:

$$\begin{aligned} R_q &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \\ R_p &= \begin{bmatrix} p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2(p_1p_2 - p_0p_3) & 2(p_0p_2 + p_1p_3) \\ 2(p_1p_2 + p_0p_3) & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2(p_2p_3 - p_0p_1) \\ 2(p_1p_3 - p_0p_2) & 2(p_0p_1 + p_2p_3) & p_0^2 - p_1^2 - p_2^2 + p_3^2 \end{bmatrix}. \end{aligned}$$

Denote the products qp by n and R_qR_p by R_n. After some calculation, n can be obtained as follows:

$$\begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\ q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2 \\ q_0p_2 + q_2p_0 - q_1p_3 + q_3p_1 \\ q_0p_3 + q_1p_2 - q_2p_1 - q_3p_0 \end{bmatrix}.$$

The product formula for two unit quaternions is therefore given by

$$qp = n = \begin{bmatrix} \eta_n \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} \eta_q\eta_p - \varepsilon_q^{\intercal}\varepsilon_p \\ \eta_q\varepsilon_p + \eta_p\varepsilon_q + \varepsilon_q \times \varepsilon_p \end{bmatrix}.$$

Exercise 3.41.

(a) First, calculate $(I + [r])^{-1}$:

$$\begin{aligned} (I + [r])^{-1} &= \frac{1}{1 + r^Tr}\begin{bmatrix} 1 + r_1^2 & r_3 + r_1r_2 & -r_2 + r_1r_3 \\ -r_3 + r_1r_2 & 1 + r_2^2 & r_1 + r_2r_3 \\ r_2 + r_1r_3 & -r_1 + r_2r_3 & 1 + r_3^2 \end{bmatrix} \\ &= \frac{1}{1 + r^Tr}(I + rr^T - [r]), \end{aligned}$$

where $r = [r_1\ r_2\ r_3]^T$. Then,

$$\begin{aligned} R &= (I-[r])^2(I+[r])^{-2} \\ &= (I-2[r]+[r]^2)\left(\frac{1}{1+r^Tr}(I+rr^T-[r])\right)^2 \\ &= \left(\frac{1}{1+r^Tr}\right)^2(I-2[r]+[r]^2)(I+rr^T-[r])^2 \\ &= \left(\frac{1}{1+r^Tr}\right)^2(I-2[r]+[r]^2)(I+(rr^T)^2+[r]^2+2rr^T-2[r]-rr^T[r]-[r]rr^T). \end{aligned}$$

Since $rr^T[r] = [r]rr^T = 0$,

$$\begin{aligned} R &= \left(\frac{1}{1+r^Tr}\right)^2(I-2[r]+[r]^2)(I+(rr^T)^2+[r]^2+2rr^T-2[r]) \\ &= \left(\frac{1}{1+r^Tr}\right)^2(I+2rr^T+(rr^T)^2-4[r]+6[r]^2-4[r]^3+[r]^4) \\ &= \left(\frac{1}{1+r^Tr}\right)^2((I+rr^T)^2-4[r]+6[r]^2-4[r]^3+[r]^4). \end{aligned}$$

The following equations can be derived from straightforward calculation: $[r]^2 = rr^T - r^TrI$, $[r]^3 = -r^Tr[r]$, $[r]^4 = -r^Tr[r]^2$. Substituting these into the equations,

$$\begin{aligned} R &= \left(\frac{1}{1+r^Tr}\right)^2(((1+r^Tr)I+[r]^2)^2-4[r]+6[r]^2-4[r]^3+[r]^4) \\ &= \left(\frac{1}{1+r^Tr}\right)^2((1+r^Tr)^2I+2(1+r^Tr)[r]^2+[r]^4-4[r]+6[r]^2-4[r]^3+[r]^4) \\ &= \left(\frac{1}{1+r^Tr}\right)^2((1+r^Tr)^2I-4[r]+(8+2r^Tr)[r]^2-4[r]^3+2[r]^4) \\ &= \left(\frac{1}{1+r^Tr}\right)^2((1+r^Tr)^2I-4[r]+(8+2r^Tr)[r]^2+4r^Tr[r]-2r^Tr[r]^2) \\ &= I+\left(\frac{1}{1+r^Tr}\right)^2(-4(1-r^Tr)[r]+8[r]^2) \\ &= I-4\frac{1-r^Tr}{(1+r^Tr)^2}[r]+\frac{8}{(1+r^Tr)^2}[r]^2. \end{aligned}$$

(b) Substituting $r = -\hat{w}\tan\frac{\theta}{4}$ into the formula in (a),

$$\begin{aligned} R &= I-4\frac{1-r^Tr}{(1+r^Tr)^2}[r]+\frac{8}{(1+r^Tr)^2}[r]^2 \\ &= I-4\frac{1-\tan^2(\theta/4)}{(1+\tan^2(\theta/4))^2}(-\tan\frac{\theta}{4})[\hat{w}]+\frac{8}{(1+\tan^2(\theta/4))^2}\tan^2\frac{\theta}{4}[\hat{w}]^2 \\ &= I+4\frac{\cos(\theta/2)}{\cos^2(\theta/4)}\cos^4\frac{\theta}{4}\tan\frac{\theta}{4}[\hat{w}]+8\cos^4\frac{\theta}{4}\tan^2\frac{\theta}{4}[\hat{w}]^2 \\ &= I+4\cos\frac{\theta}{2}\cos\frac{\theta}{4}\sin\frac{\theta}{4}[\hat{w}]+8\left(\cos\frac{\theta}{4}\sin\frac{\theta}{4}\right)^2[\hat{w}]^2 \\ &= I+2\cos\frac{\theta}{2}\sin\frac{\theta}{2}[\hat{w}]+8\left(\frac{1}{2}\sin\frac{\theta}{2}\right)^2[\hat{w}]^2 \\ &= I+\sin\theta[\hat{w}]+(1-\cos\theta)[\hat{w}]^2 \\ &= e^{[\hat{w}]\theta}. \end{aligned}$$

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The given r therefore satisfies the formula. This solution is not unique; another solution is given by

$$r = \frac{\hat{w}}{\tan(\theta/4)}.$$

(c) Let the angular velocity in the body-fixed frame be ω. Then ω satisfies the following equation:

$$[\omega] = R^T \dot{R},$$

where $[\omega]$ is the skew-symmetric representation of ω. Substituting the R derived in (a),

$$\dot{r} = \frac{1}{4}\{(1 - r^T r)I + 2[r] + 2rr^T\}\omega.$$

For a detailed derivation, see [I.G. Kang and F.C. Park, Cubic spline algorithms for orientation interpolation, *Int. J. Numerical Methods in Engineering*, vol. 46 (1999): 45-64.].

(d) One of the advantages of the modifed Cayley-Rodrigues parameters is that the singularity at π is now relocated to 2π; rotations up to 2π are now possible. Referring to the radius π solid ball picture of $SO(3)$, the modified Cayley-Rodrigues parameters can be obtained by "stretching" the solid ball of radius 2π (as opposed to π for the standard Cayley-Rodrigues parameters corresponding to the $k = 1$ case) to infinity. However, one now loses the one-to-one correspondence between $\mathbb{R}^3$ and $SO(3)$ that exists for the standard Cayley-Rodrigues parameters. Moreover, (i) the formulas for the angular velocity and acceleration become more complicated, (ii) one cannot obtain r from R by a simple rational expression as in the case of the standard Cayley-Rodrigues parameters, and (iii) multiplication of two rotation matrices in the modified parameters does not admit a simple rational expression like the standard parameters. Going to higher order, for the case $k = 4$ it can be shown that the corresponding r is given by

$$r = \hat{w} \tan \frac{\theta}{8}.$$

As k increases, one obtains successively closer approximation (up to constant scaling factor) to the canonical coordinates—the nonlinear warping effect caused by the tangent function becomes less severe. As expected, however, the formulas are no longer simple rational expressions, but become increasingly complicated expressions involving transcendental functions.

(e) Given two rotation matrices R_1 and R_2, let $R_3 = R_1 R_2$, which is also a rotation matrix.

(i) Multiplying two rotation matrices: By simple calculation, it can be seen that 27 multiplications and 18 additions are needed. Therefore, a total of 45 arithmetic operations are needed. Note that by calculating the third column of R_3 as the cross-product of its first column and second column, 6 operations can be reduced.

(ii) Multiplying two unit quaternions: Let $q_i = (\eta_i, \varepsilon_i) \in \mathbb{R}^4$, $(i = 1, 2, 3)$ denote the unit quaternion parameters for R_i, $(i = 1, 2, 3)$, where $\eta_i \in \mathbb{R}$, $(i = 1, 2, 3)$ and $\varepsilon_i \in \mathbb{R}^3$, $(i = 1, 2, 3)$. Then q_3 is calculated as follows:

$$q_3 = (\eta_1\eta_2 - \varepsilon_1^T \varepsilon_2, \eta_1\varepsilon_2 + \eta_2\varepsilon_1 + (\varepsilon_1 \times \varepsilon_2)).$$

It can be seen that 16 multiplications and 12 additions are needed. Therefore, a total of 28 arithmetic operations are needed.

(iii) Multiplying two Cayley-Rodrigues vectors: Let $r_i =\in \mathbb{R}^3$, $(i = 1, 2, 3)$ denote the Cayley-Rodrigues parameters for R_i, $(i = 1, 2, 3)$. r_3 is then calculated as follows:

$$r_3 = \frac{r_1 + r_2 + (r_1 \times r_2)}{1 - r_1^T r_2}.$$

It can be seen that 9 multiplications and 15 additions are needed. Therefore, a total of 24 arithmetic operations are needed.

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Exercise 3.42.
Programming assignment.

Exercise 3.43.
Programming assignment. Could check how close $R^T R - I$ is to the zero matrix, and whether $\det(R) \approx 1$.

Exercise 3.44.
Programming assignment.

Exercise 3.45.
Programming assignment.

Exercise 3.46.
Programming assignment.

Exercise 3.47.
Programming assignment.

Exercise 3.48.
Programming assignment.

Exercise 3.49.
Programming assignment.

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