# Chapter 8 Solutions

**Exercise 8.1.**
First, the inertia matrix of a thin rectangular plate perpendicular to the z-axis can be derived as follows:

$$\begin{aligned} I_{xx} &= \begin{bmatrix} \frac{1}{12}Mw^2 & 0 & 0 \end{bmatrix}, \\ I_{yy} &= \begin{bmatrix} 0 & \frac{1}{12}Ml^2 & 0 \end{bmatrix}, \\ I_{zz} &= \begin{bmatrix} 0 & 0 & \frac{1}{12}M(w^2+l^2) \end{bmatrix}, \end{aligned}$$

where the reference frame is located at the center of mass, $M$ denotes the mass of the plate, and $w$ and $l$ are the length of the edges along the x-axis and y-axis, respectively. Similarly, the inertia matrix of a circular plate can be derived as follows:

$$\begin{aligned} I_{xx} &= \begin{bmatrix} \frac{1}{4}Mr^2 & 0 & 0 \end{bmatrix}, \\ I_{yy} &= \begin{bmatrix} 0 & \frac{1}{4}Mr^2 & 0 \end{bmatrix}, \\ I_{zz} &= \begin{bmatrix} 0 & 0 & \frac{1}{2}Mr^2 \end{bmatrix}, \end{aligned}$$

where the reference frame is located at the center of mass with z-axis perpendicular to the plate, $M$ denotes the mass of the plate, and $r$ is the radius of the plate.

(a) Rectangular parallelepiped:

$$I_{xx} = \int \frac{1}{12}(w^2+h^2)dm = \frac{1}{12}(w^2+h^2)\int 1\ dm = \frac{1}{12}(w^2+h^2)M,$$

where $dm$ is the infinitesimal mass of the thin plate between $x$ and $x+dx$. By symmetry, it is straightfoward to derive the remaining inertia components as $I_{yy} = \frac{1}{12}(l^2+h^2)M, I_{zz} = \frac{1}{12}(w^2+l^2)M$.

(b) Circular cylinder:
Following (a), using a thin circulate plate of infinitesimal mass $dm$ between $z$ and $z+dz$, it is straightfoward to compute $I_{zz} = \int \frac{1}{2}r^2 dm = \frac{1}{2}r^2 \int dm = \frac{1}{2}mr^2$. To derive $I_{xx}$, use a thin rectangular plate of an infinitesimal mass $dm_x$ between $x$ and $x+dx$ as follows:

$$I_{xx} = \int \frac{1}{12}(4(r^2-x^2)+h^2)\ dm_x = \int_{-r}^{r} \frac{1}{6}(4(r^2-x^2)+h^2)\rho\sqrt{r^2-x^2}h\ dx,$$

where the infinitesimal mass is $dm_x = 2\rho\sqrt{r^2-x^2}h\ dx$ and $\rho$ is the density. Integrating by substituting $dx = r\tan\theta$ and $m = \int dm_x = \int_{-r}^{r} 2\rho\sqrt{r^2-x^2}h\ dx$, we obtain $I_{xx} = \frac{m(3r^2+h^2)}{12}$ and $I_{yy} = \frac{m(3r^2+h^2)}{12}$.

(c) Ellipsoid:
Note that the inertia $I_{cm}$ of a sphere about the axis through the center of mass is $I_{cm} = \frac{2}{5}MR^2$ where $M$ is the mass and $R$ is the radius. Since $I = \int r^2 dm$, we can compute the inertia terms of an ellipsoid using the following change of coordinates:

$$\begin{aligned} I_{zz} &= \int_{ellipsoid} x^2 + y^2 dm \\ &= \int_{sphere} (\frac{a}{r}x)^2 + (\frac{b}{r}y)^2 dm' \\ &= (\frac{a}{r})^2 \int_{sphere} x^2 dm' + (\frac{b}{r})^2 \int_{sphere} y^2 dm' \\ &= \frac{1}{5}m(a^2+b^2) \end{aligned}$$

where $dm$ is an infinitesimal mass of ellipsoid, $dm'$ is an infinitesimal mass of sphere, and also the fact that $\int_{sphere} x^2 dm' = \frac{1}{2}\int_{sphere} x^2+y^2 dm' = \frac{1}{5}mr^2$. Similarly, $I_{xx} = \frac{1}{5}m(c^2+b^2), I_{yy} = \frac{1}{5}m(a^2+c^2)$.

**Exercise 8.2.**

(a) Aligning the z axis along the length of the dumbell gives $\mathcal{I}_{\text{cylinder}} = \text{diag}\{0.00647, 0.00647, 0.000377\}$ and $\mathcal{I}_{\text{sphere},1} = \mathcal{I}_{\text{sphere},2} = \text{diag}\{0.126, 0.126, 0.126\}$.
Using the parallel axis theorem gives $\mathcal{I}_{\text{dumbell}} = \text{diag}\{2.771, 2.771, 0.252\}$.
(b) The total mass of the dumbbell is $m_{\text{dumbbell}} = 64.72$ kg. The spatial inertia matrix is $\mathcal{G}_b = \text{diag}\{\mathcal{I}_b, \mathfrak{m}I\}$, so $\mathcal{G}_b = \text{diag}\{2.771, 2.771, 0.252, 64.72, 64.72, 64.72\}$

**Exercise 8.3.**

(a) The relation between $\mathcal{G}_a$ and $\mathcal{G}_b$ can be rewritten as

$$\begin{aligned}
\mathcal{G}_a &= [\text{Ad}_{T_{ba}}]^T \mathcal{G}_b [\text{Ad}_{T_{ba}}] \\
\begin{bmatrix} \mathcal{I}_a & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} &= \begin{bmatrix} {R_{ba}}^T & R_{ba}^T [p_{ba}]^T \\ 0 & {R_{ba}}^T \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix} \begin{bmatrix} R_{ba} & 0 \\ [p_{ba}]R_{ba} & R_{ba} \end{bmatrix}.
\end{aligned}$$

So we get

$$\mathcal{I}_a = {R_{ba}}^T \mathcal{I}_b R_{ba} + \mathfrak{m} R_{ba}^T [p_{ba}]^T [p_{ba}] R_{ba}.$$

When frame {a} is aligned with frame {b}, $R_{ba} = I$. Substituting this into the formula yields

$$\mathcal{I}_a = \mathcal{I}_b + \mathfrak{m}[p_{ba}]^T [p_{ba}].$$

Assume $p_{ba} = q = (q_x, q_y, q_z)$, we finally get

$$\begin{aligned}
\mathcal{I}_a &= \mathcal{I}_b + \mathfrak{m}[q]^T[q] \\
&= \mathcal{I}_b + \mathfrak{m} \begin{bmatrix} {q_y}^2 + {q_z}^2 & -q_x q_y & -q_x q_z \\ -q_y q_x & {q_x}^2 + {q_z}^2 & -q_y q_z \\ -q_z q_x & -q_z q_y & {q_x}^2 + {q_y}^2 \end{bmatrix} \\
&= \mathcal{I}_b + \mathfrak{m}(q^T q I - q q^T).
\end{aligned}$$

So Steiner's theorem is a special case of the given equation and in turn that equation is a generalization of Steiner's theorem.

(b) Given the dynamic equations for a single rigid body, we have

$$\begin{aligned}
\mathcal{F}_a &= [\text{Ad}_{T_{ba}}]^T \mathcal{F}_b \\
&= [\text{Ad}_{T_{ba}}]^T \mathcal{G}_b \dot{\mathcal{V}}_b - [\text{Ad}_{T_{ba}}]^T [\text{ad}_{\mathcal{V}_b}]^T \mathcal{G}_b \mathcal{V}_b \\
&= [\text{Ad}_{T_{ba}}]^T \mathcal{G}_b [\text{Ad}_{T_{ba}}] \dot{\mathcal{V}}_a - [\text{Ad}_{T_{ba}}]^T [\text{ad}_{\mathcal{V}_b}]^T \mathcal{G}_b [\text{Ad}_{T_{ba}}] \mathcal{V}_a \\
&= \mathcal{G}_a \dot{\mathcal{V}}_a - [\text{ad}_{\mathcal{V}_a}]^T \mathcal{G}_a \mathcal{V}_a
\end{aligned}$$

**Exercise 8.4.**

Figure 8.1 shows the rotational inverted pendulum for arbitrary $\theta_1$ and $\theta_2$.

(a) The dynamic equations are derived via the Lagrangian formulation. Assume that frame $\{b_1\}$ is the end-effector frame. Then the body Jacobian $J_{b_1}$ with respect to frame $\{b_1\}$ is derived as follows:

| $i$ | $w_i$ | $q_i$ | $v_i$ |
|---|---|---|---|
| 1 | $(0, 0, 1)$ | $(-L_1, 0, 0)$ | $(0, L_1, 0)$ |

where $v_i = -w_i \times q_i$. Therefore

$$J_{b_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ L_1 \\ 0 \end{bmatrix}.$$

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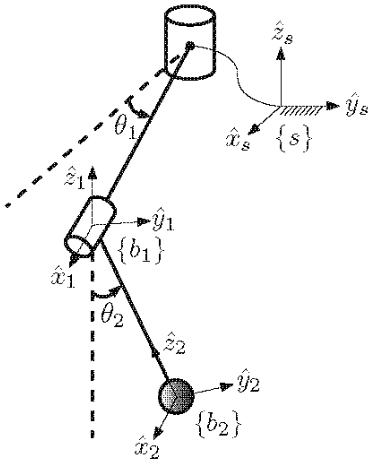


Figure 8.1

The spatial velocity of link 1 expressed in frame $\{b_1\}$ coordinates is

$$\mathcal{V}_{b_1} = J_{b_1}\dot{\theta} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ L_1 \\ 0 \end{bmatrix} \dot{\theta}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \\ 0 \\ L_1\dot{\theta}_1 \\ 0 \end{bmatrix}.$$

Now assume that frame $\{b_2\}$ is the end-effector frame. Then the body Jacobian $J_{b_2}$ with respect to frame $\{b_2\}$ is derived as follows:

| $i$ | $w_i$ | $q_i$ | $v_i$ |
|---|---|---|---|
| 1 | $(0, \sin\theta_2, \cos\theta_2)$ | $(-L_1, 0, L_2)$ | $(-L_2\sin\theta_2, L_1\cos\theta_2, -L_1\sin\theta_2)$ |
| 2 | $(1, 0, 0)$ | $(0, 0, L_2)$ | $(0, L_2, 0)$ |

where $v_i = -w_i \times q_i$. Therefore

$$J_{b_2} = \begin{bmatrix} 0 & 1 \\ \sin\theta_2 & 0 \\ \cos\theta_2 & 0 \\ -L_2\sin\theta_2 & 0 \\ L_1\cos\theta_2 & L_2 \\ -L_1\sin\theta_2 & 0 \end{bmatrix}.$$

The spatial velocity of link 2 expressed in frame $\{b_2\}$ coordinates is

$$\mathcal{V}_{b_2} = J_{b_2}\dot{\theta} = \begin{bmatrix} 0 & 1 \\ \sin\theta_2 & 0 \\ \cos\theta_2 & 0 \\ -L_2\sin\theta_2 & 0 \\ L_1\cos\theta_2 & L_2 \\ -L_1\sin\theta_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_2 \\ \dot{\theta}_1\sin\theta_2 \\ \dot{\theta}_1\cos\theta_2 \\ -L_2\dot{\theta}_1\sin\theta_2 \\ L_1\dot{\theta}_1\cos\theta_2 + L_2\dot{\theta}_2 \\ -L_1\dot{\theta}_1\sin\theta_2 \end{bmatrix}.$$

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Substituting $L_1 = L_2 = 1$,

$$\mathcal{V}_{b_1} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \\ 0 \\ \dot\theta_1 \\ 0 \end{bmatrix}, \quad \mathcal{V}_{b_2} = \begin{bmatrix} \dot\theta_2 \\ \dot\theta_1 \sin\theta_2 \\ \dot\theta_1 \cos\theta_2 \\ -\dot\theta_1 \sin\theta_2 \\ \dot\theta_1 \cos\theta_2 + \dot\theta_2 \\ -\dot\theta_1 \sin\theta_2 \end{bmatrix}.$$

The kinetic energy of the system is

$$\begin{aligned} \mathcal{K}(\theta, \dot\theta) &= \frac{1}{2}\mathcal{V}_{b_1}^T \mathcal{G}_{b_1} \mathcal{V}_{b_1} + \frac{1}{2}\mathcal{V}_{b_2}^T \mathcal{G}_{b_2} \mathcal{V}_{b_2} \\ &= \frac{1}{2}\mathcal{V}_{b_1}^T \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & m_1 I \end{bmatrix} \mathcal{V}_{b_1} + \frac{1}{2}\mathcal{V}_{b_2}^T \begin{bmatrix} \mathcal{I}_2 & 0 \\ 0 & m_2 I \end{bmatrix} \mathcal{V}_{b_2}, \end{aligned}$$

where $\mathcal{G}_{b_i}$ is the spatial inertia matrix of link $i$ expressed in frame $\{b_i\}$ attached at the center of mass. Substituting $m_1 = m_2 = 2$,

$$\begin{aligned} \mathcal{K}(\theta, \dot\theta) &= \frac{1}{2}\mathcal{V}_{b_1}^T \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & 2I \end{bmatrix} \mathcal{V}_{b_1} + \frac{1}{2}\mathcal{V}_{b_2}^T \begin{bmatrix} \mathcal{I}_2 & 0 \\ 0 & 2I \end{bmatrix} \mathcal{V}_{b_2} \\ &= 3\dot\theta_1^2 + (\dot\theta_1^2 + 3\dot\theta_2 + 3\dot\theta_1^2 \sin^2\theta_2 + 2\dot\theta_1\dot\theta_2\cos\theta_2) \\ &= 4\dot\theta_1^2 + 3\dot\theta_2 + 3\dot\theta_1^2 \sin^2\theta_2 + 2\dot\theta_1\dot\theta_2\cos\theta_2. \end{aligned}$$

Let the potential energy of the zero position be zero. The potential energy of the system is given by

$$\mathcal{P}(\theta) = m_2 g L_2 (1 - \cos\theta_2).$$

Substituting $m_2 = 2$, $g = 10$, and $L_2 = 1$,

$$\mathcal{P}(\theta) = 20 - 20\cos\theta_2.$$

The Lagrangian of the system is then

$$\begin{aligned} \mathcal{L}(\theta, \dot\theta) &= \mathcal{K}(\theta, \dot\theta) - \mathcal{P}(\theta) \\ &= 4\dot\theta_1^2 + 3\dot\theta_2 + 3\dot\theta_1^2 \sin^2\theta_2 + 2\dot\theta_1\dot\theta_2\cos\theta_2 + 20\cos\theta_2 - 20. \end{aligned}$$

Therefore, the Euler-Lagrange equations can be written as

$$\begin{aligned} \tau &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} - \frac{\partial\mathcal{L}}{\partial\theta} \\ &= \frac{d}{dt}\begin{bmatrix} 8\dot\theta_1 + 6\dot\theta_1\sin^2\theta_2 + 2\dot\theta_2\cos\theta_2 \\ 6\dot\theta_2 + 2\dot\theta_1\cos\theta_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 6\dot\theta_1^2 \sin\theta_2\cos\theta_2 - 2\dot\theta_1\dot\theta_2\sin\theta_2 - 20\sin\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} 8\ddot\theta_1 + 6\ddot\theta_1\sin^2\theta_2 + 12\dot\theta_1\dot\theta_2\sin\theta_2\cos\theta_2 + 2\ddot\theta_2\cos\theta_2 - 2\dot\theta_2^2\sin\theta_2 \\ 6\ddot\theta_2 + 2\ddot\theta_1\cos\theta_2 - 6\dot\theta_1^2\sin\theta_2\cos\theta_2 + 20\sin\theta_2 \end{bmatrix}. \end{aligned}$$

Substituting $\theta_1 = \theta_2 = \pi/4$ and $\dot\theta_1 = \dot\theta_2 = \ddot\theta_1 = \ddot\theta_2 = 0$,

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10\sqrt{2} \end{bmatrix}.$$

(b) Substituting $\theta_1 = \theta_2 = \pi/4$ to the dynamic equations derived in (a),

$$\begin{aligned} \tau &= \begin{bmatrix} 11\ddot\theta_1 + 6\dot\theta_1\dot\theta_2 + \sqrt{2}\ddot\theta_2 - \sqrt{2}\dot\theta_2^2 \\ 6\ddot\theta_2 + \sqrt{2}\ddot\theta_1 - 3\dot\theta_1^2 + 10\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 11 & \sqrt{2} \\ \sqrt{2} & 6 \end{bmatrix} \begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \dots. \end{aligned}$$

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Therefore, the mass matrix $M(\theta)$ when $\theta_1 = \theta_2 = \pi/4$ is

$$M(\theta) = \begin{bmatrix} 11 & \sqrt{2} \\ \sqrt{2} & 6 \end{bmatrix}.$$

The eigenvalues of the mass matrix are $\lambda_1 = 5.6277$ and $\lambda_2 = 11.3723$, and the corresponding eigenvectors are $v_1 = (0.2546, -0.9671)^T$ and $v_2 = (-0.9671, -0.2546)^T$. Then the torque ellipsoid for the mass matrix can be drawn as in Figure 8.2.

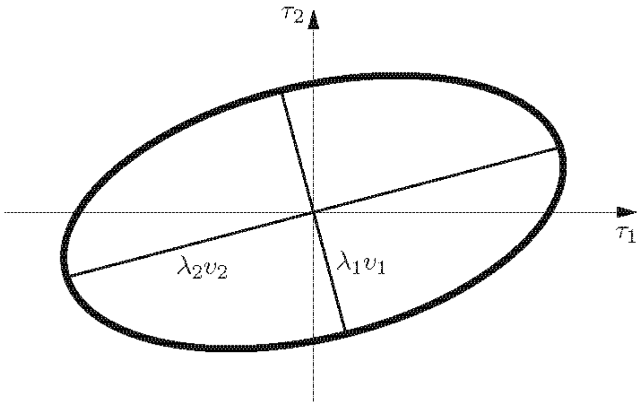


Figure 8.2

**Exercise 8.5.**
For two arbitrary $X, Y \in se(3)$,

$$[\mathrm{ad}_X(Y)] = [X][Y] - [Y][X]$$

holds. Then applying the same computation rule, we get the following identity

$$[\mathrm{ad}_{\mathcal{V}_i}(\mathrm{ad}_{\mathcal{V}_j}(\mathcal{V}_k)] = [\mathcal{V}_i][\mathcal{V}_j][\mathcal{V}_k] - [\mathcal{V}_i][\mathcal{V}_k][\mathcal{V}_j] - [\mathcal{V}_j][\mathcal{V}_k][\mathcal{V}_i] + [\mathcal{V}_k][\mathcal{V}_j][\mathcal{V}_i] \tag{8.1}$$

for arbitrary given $\mathcal{V}_i, \mathcal{V}_j, \mathcal{V}_k \in se(3)$. Summing Equation (8.1) for $(i, j, k) = (1, 2, 3), (3, 1, 2), (2, 3, 1)$ whose left hand side is exactly that of the Jacobi identity, it is straightforward to show that all the terms in the right hand side cancel each other and become zero.

**Exercise 8.6.**
(a) The $i$th column of the space Jacobian $J_s(\theta)$ is

$$J_i = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i),$$

for $i = 2, ..., n$, with the first column $J_1 = \mathcal{S}_1$. Since $J_i$ depends only on $\theta_1, \cdots, \theta_{i-1}$, the partial derivative of the space Jacobian $\frac{\partial J_i}{\partial \theta_j}$ becomes zero for $i \leq j$. Otherwise, for $i > j$, $\frac{\partial J_i}{\partial \theta_j}$ can be derived using $[\cdot]$ notation as follows:

$$\begin{aligned} \frac{\partial [J_i]}{\partial \theta_j} &= \frac{\partial}{\partial \theta_j}(e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{i-1}]\theta_{i-1}})[\mathcal{S}_i]e^{-[\mathcal{S}_{i-1}]\theta_{i-1}} \dots e^{-[\mathcal{S}_1]\theta_1} \\ &+ e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}[\mathcal{S}_i]\frac{\partial}{\partial \theta_j}(e^{-[\mathcal{S}_{i-1}]\theta_{i-1}} \dots e^{-[\mathcal{S}_1]\theta_1}). \end{aligned}$$

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For the sake of simplicity, let $e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}} = T_{i-1}$ and $e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{j-1}]\theta_{j-1}} = T_{j-1}$. Then the first term is

$$
\begin{aligned}
& \frac{\partial}{\partial\theta_j}(e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}})[\mathcal{S}_i]e^{-[\mathcal{S}_{i-1}]\theta_{i-1}}\cdots e^{-[\mathcal{S}_1]\theta_1} \\
= \; & T_{j-1}[\mathcal{S}_j]e^{[\mathcal{S}_j]\theta_j}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}[\mathcal{S}_i]T_{i-1}^{-1} \\
= \; & T_{j-1}[\mathcal{S}_j]T_{j-1}^{-1}T_{i-1}[\mathcal{S}_i]T_{i-1}^{-1} \\
= \; & [J_j][J_i],
\end{aligned}
$$

and, in a similar way, the second term is

$$
\begin{aligned}
& e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}[\mathcal{S}_i]\frac{\partial}{\partial\theta_j}(e^{-[\mathcal{S}_{i-1}]\theta_{i-1}}\cdots e^{-[\mathcal{S}_1]\theta_1}) \\
= \; & T_{i-1}[\mathcal{S}_i]e^{-[\mathcal{S}_{i-1}]\theta_{i-1}}\cdots e^{-[\mathcal{S}_j]\theta_j}(-[\mathcal{S}_j])T_{j-1}^{-1} \\
= \; & T_{i-1}[\mathcal{S}_i]T_{i-1}^{-1}T_{j-1}(-[\mathcal{S}_j])T_{j-1}^{-1} \\
= \; & -[J_i][J_j].
\end{aligned}
$$

Therefore, $\frac{\partial[J_i]}{\partial\theta_j}$ for $i > j$ becomes the Lie bracket of $J_j$ and $J_i$,

$$\frac{\partial[J_i]}{\partial\theta_j} = [J_j][J_i] - [J_i][J_j] = [J_j, J_i],$$

which can be written in vector form as

$$\frac{\partial J_i}{\partial\theta_j} = \mathrm{ad}_{J_j}(J_i).$$

(b) The $i$th column of the body Jacobian $J_b(\theta)$ is

$$J_i = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i),$$

for $i = n-1, ..., 1$, with the $n$th column $J_n = \mathcal{B}_n$. Since $J_i$ depends only on $\theta_n, \cdots, \theta_{i+1}$, the partial derivative of the body Jacobian $\frac{\partial J_i}{\partial\theta_j}$ becomes zero for $i \geq j$. Otherwise, for $i < j$, $\frac{\partial J_i}{\partial\theta_j}$ can be derived using $[\cdot]$ notation as follows:

$$
\begin{aligned}
\frac{\partial[J_i]}{\partial\theta_j} \; = \; & \frac{\partial}{\partial\theta_j}(e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}})[\mathcal{B}_i]e^{[\mathcal{B}_{i+1}]\theta_{i+1}}\cdots e^{[\mathcal{B}_n]\theta_n} \\
+ \; & e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}[\mathcal{B}_i]\frac{\partial}{\partial\theta_j}(e^{[\mathcal{B}_{i+1}]\theta_{i+1}}\cdots e^{[\mathcal{B}_n]\theta_n}).
\end{aligned}
$$

For the sake of simplicity, let $e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}} = T'_{i+1}$ and $e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{j+1}]\theta_{j+1}} = T'_{j+1}$. Then the first term is

$$
\begin{aligned}
& \frac{\partial}{\partial\theta_j}(e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}})[\mathcal{B}_i]e^{[\mathcal{B}_{i+1}]\theta_{i+1}}\cdots e^{[\mathcal{B}_n]\theta_n} \\
= \; & T'_{j+1}(-[\mathcal{B}_j])e^{-[\mathcal{B}_j]\theta_j}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}[\mathcal{B}_i](T'_{i+1})^{-1} \\
= \; & T'_{j+1}(-[\mathcal{B}_j])(T'_{j+1})^{-1}T'_{i+1}[\mathcal{B}_i](T'_{i+1})^{-1} \\
= \; & -[J_j][J_i],
\end{aligned}
$$

and, in a similar way, the second term is

$$
\begin{aligned}
& e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}[\mathcal{B}_i]\frac{\partial}{\partial\theta_j}(e^{[\mathcal{B}_{i+1}]\theta_{i+1}}\cdots e^{[\mathcal{B}_n]\theta_n}) \\
= \; & T'_{i+1}[\mathcal{B}_i]e^{[\mathcal{B}_{i+1}]\theta_{i+1}}\cdots e^{[\mathcal{B}_j]\theta_j}[\mathcal{B}_j](T'_{j+1})^{-1} \\
= \; & T'_{i+1}[\mathcal{B}_i](T'_{i+1})^{-1}T'_{j+1}[\mathcal{B}_j](T'_{j+1})^{-1} \\
= \; & [J_i][J_j].
\end{aligned}
$$

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Therefore, $\frac{\partial [J_i]}{\partial \theta_j}$ for $i < j$ becomes the Lie bracket of $J_i$ and $J_j$,

$$\frac{\partial [J_i]}{\partial \theta_j} = [J_i][J_j] - [J_j][J_i] = [J_i, J_j],$$

which can be written in vector form as

$$\frac{\partial J_i}{\partial \theta_j} = \mathrm{ad}_{J_i}(J_j).$$

**Exercise 8.7.**
From the closed form dynamics formulation derived in chapter 8.4, the mass matrix is expressed as follows:

$$M(\theta) = \mathcal{A}^T \mathcal{L}^T(\theta) \mathcal{G} \mathcal{L}(\theta) \mathcal{A}.$$

Since $\mathcal{A}$ and $\mathcal{G}$ are time-invariant, taking the derivative of both sides leads to

$$\dot{M}(\theta) = \mathcal{A}^T \frac{d}{dt} \mathcal{L}^T(\theta) \mathcal{G} \mathcal{L}(\theta) \mathcal{A} + \mathcal{A}^T \mathcal{L}^T(\theta) \mathcal{G} \frac{d}{dt} \mathcal{L}(\theta) \mathcal{A}.$$

The time derivative of $\mathcal{L}(\theta)$ is required:

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(\theta) &= \frac{d}{dt}\left((I - \mathcal{W}(\theta))^{-1}\right) \\ &= -(I - \mathcal{W}(\theta))^{-1}\left(\frac{d}{dt}(I - \mathcal{W}(\theta))\right)(I - \mathcal{W}(\theta))^{-1} \\ &= \mathcal{L}(\theta)\left(\frac{d}{dt}\mathcal{W}(\theta)\right)\mathcal{L}(\theta). \end{aligned}$$

The time derivative of the adjoint is derived in Chapter 8.3.1:

$$\begin{aligned} \frac{d}{dt}\left([\mathrm{Ad}_{T_{i,i-1}}]\right) &= -\left[\mathrm{ad}_{\mathcal{A}_i \dot{\theta}_i}\right]\left[\mathrm{Ad}_{T_{i,i-1}}\right] \\ \therefore \frac{d}{dt}\mathcal{W}(\theta) &= -\left[\mathrm{ad}_{\mathcal{A}\dot{\theta}}\right]\mathcal{W}(\theta). \end{aligned}$$

Therefore the time derivative of $\mathcal{L}(\theta)$ is,

$$\frac{d}{dt}\mathcal{L}(\theta) = -\mathcal{L}(\theta)\left[\mathrm{ad}_{\mathcal{A}\dot{\theta}}\right]\mathcal{W}(\theta)\mathcal{L}(\theta).$$

Similarly, the time derivative of $\mathcal{L}^T(\theta)$ can be derived as follows:

$$\mathcal{L}^T(\theta) = \left((I - \mathcal{W}(\theta))^{-1}\right)^T = \left(I - \mathcal{W}^T(\theta)\right)^{-1}$$

$$\begin{aligned} \frac{d}{dt}\mathcal{L}^T(\theta) &= \frac{d}{dt}\left(I - \mathcal{W}^T(\theta)\right)^{-1} \\ &= -\left(I - \mathcal{W}^T(\theta)\right)^{-1}\left(\frac{d}{dt}\left(I - \mathcal{W}^T(\theta)\right)\right)\left(I - \mathcal{W}^T(\theta)\right)^{-1} \\ &= \mathcal{L}^T(\theta)\left(\frac{d}{dt}\mathcal{W}^T(\theta)\right)\mathcal{L}^T(\theta) = \mathcal{L}^T(\theta)\left(\frac{d}{dt}\mathcal{W}(\theta)\right)^T\mathcal{L}^T(\theta) \\ &= -\mathcal{L}^T(\theta)\mathcal{W}^T(\theta)\left[\mathrm{ad}_{\mathcal{A}\dot{\theta}}\right]^T\mathcal{L}^T(\theta) \end{aligned}$$

Therefore,

$$\dot{M}(\theta) = -\left(\mathcal{A}^T\mathcal{L}^T\mathcal{W}^T\left[\mathrm{ad}_{\mathcal{A}\dot{\theta}}\right]^T\mathcal{L}^T\mathcal{G}\mathcal{L}\mathcal{A} + \mathcal{A}^T\mathcal{L}^T\mathcal{G}\mathcal{L}\left[\mathrm{ad}_{\mathcal{A}\dot{\theta}}\right]\mathcal{W}\mathcal{L}\mathcal{A}\right).$$

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**Exercise 8.8.**
First consider the bold ellipse in the top right plot (force ellipse corresponding to an acceleration circle when the robot is at $(0°, 90°)$). Accelerating the end-effector in the $x$-direction feels only the mass $\mathfrak{m}_2$; the mass $\mathfrak{m}_1$ does not accelerate instantaneously. Hence the required force is just $\mathfrak{m}_2 a$. Accelerating in the $y$-direction feels both masses equally, and the required force is $(\mathfrak{m}_1 + \mathfrak{m}_2)a$.

For the bold ellipse in the bottom right plot, accelerating the end-effector at a 60° angle feels only $\mathfrak{m}_2$, as $\mathfrak{m}_1$ does not accelerate instantaneously. Accelerating at 150° requires the force $\mathfrak{m}_2 a$ to accelerate $\mathfrak{m}_2$ plus an additional force larger than $\mathfrak{m}_1 a$, because only some of this additional force accelerates $\mathfrak{m}_1$ while the rest acts against the constraint of joint 1. This constraint force occurs because link 1 is not orthogonal to the direction of endpoint acceleration, as it is in the top right plot for an acceleration in the $y$-direction.

**Exercise 8.9.**

$$\begin{aligned} d\ddot{\theta}/dG = \frac{\tau_m}{G^2\mathcal{I}_{\text{rotor}} + \mathcal{I}_{\text{link}}} - \frac{2G^2\tau_m\mathcal{I}_{\text{rotor}}}{(G^2\mathcal{I}_{\text{rotor}} + \mathcal{I}_{\text{link}})^2} &= 0 \\ \frac{\tau_m\left(\mathcal{I}_{\text{link}} - G^2\mathcal{I}_{\text{rotor}}\right)}{(G^2\mathcal{I}_{\text{rotor}} + \mathcal{I}_{\text{link}})^2} &= 0 \\ \sqrt{\mathcal{I}_{\text{link}}/\mathcal{I}_{\text{rotor}}} = G \end{aligned}$$

**Exercise 8.10.**
$P\tau = P(M\ddot{\theta} + h)$
$P\tau = PM\ddot{\theta} + Ph$
$P(\tau - h) = PM\ddot{\theta}$
$M^{-1}P(\tau - h) = M^{-1}PM\ddot{\theta}$
$P_{\ddot{\theta}}M^{-1}(\tau - h) = P_{\ddot{\theta}}\ddot{\theta}$, where $P_{\ddot{\theta}} = M^{-1}PM$.

**Exercise 8.11.**
Programming assignment.

**Exercise 8.12.**
Programming assignment.

**Exercise 8.13.**
Programming assignment.

**Exercise 8.14.**
Programming assignment.

**Exercise 8.15.**
Programming assignment.

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