Exercise 2

The 4R planar robot with all links of equal length ℓ in Fig. 3 needs to realize a motion task defined by a desired linear velocity $\boldsymbol{v}_d$ for its end-effector position $\boldsymbol{p}_e$ and by a desired angular velocity $\dot{\phi}_d$ for the orientation ϕ of its end-effector frame. Characterize first all the singular configurations of the robot for this specific task.

Assume then $\ell = 0.5$ [m], $\boldsymbol{q} = (0\ 0\ \pi/2\ 0)$, $\boldsymbol{v}_d = (1\ 0)$ [m/s], and $\dot{\phi}_d = 0.5$ [rad/s]. Moreover, the joints have limited motion range, i.e., $q_i \in [-2, 2]$ [rad], for $i = 1, \ldots, 4$. Determine the joint velocity $\dot{\boldsymbol{q}}$ that realizes the desired task while decreasing instantaneously the objective function that measures the distance from the midpoint of the joint ranges, i.e., in the form

$$H_{range}(\boldsymbol{q}) = \frac{1}{2N} \sum_{i=1}^{N} \left(\frac{q_i - \bar{q}_i}{q_{M,i} - q_{m,i}} \right)^2 .$$

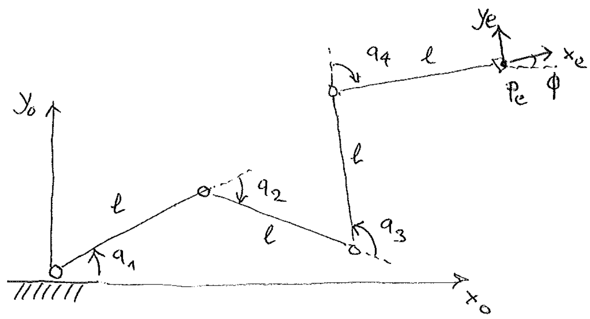

Figure 1: The kinematic skeleton of a planar 4R robot

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The task vector for this 4R planar robot is defined as

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p}_e \\ \phi \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} \ell\,(c_1 + c_{12} + c_{123} + c_{1234}) \\ \ell\,(s_1 + s_{12} + s_{123} + s_{1234}) \\ q_1 + q_2 + q_3 + q_4 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

Differentiating $\boldsymbol{r}$ w.r.t. to time yields

$$\dot{\boldsymbol{r}} = \begin{pmatrix} \boldsymbol{v} \\ \dot{\phi} \end{pmatrix} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the task Jacobian given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\,(s_1 + s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{123} + s_{1234}) & -\ell\, s_{1234} \\ \ell\,(c_1 + c_{12} + c_{123} + c_{1234}) & \ell\,(c_{12} + c_{123} + c_{1234}) & \ell\,(c_{123} + c_{1234}) & \ell\, c_{1234} \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{5}$$

For the purpose of singularity analysis, the matrix $\boldsymbol{J}(\boldsymbol{q})$ can be rewritten as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\, s_1 & -\ell\, s_{12} & -\ell\, s_{123} & -\ell\, s_{1234} \\ \ell\, c_1 & \ell\, c_{12} & \ell\, c_{123} & \ell\, c_{1234} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \boldsymbol{J}_a(\boldsymbol{q})\,\boldsymbol{T},$$

where the square matrix $\boldsymbol{T}$ is clearly nonsingular. Thus, $\boldsymbol{J}$ and $\boldsymbol{J}_a$ have always the same rank. In particular, the Jacobian $\boldsymbol{J}$ will be full (row) rank if and only if the 2×3 upper left block of matrix $\boldsymbol{J}_a$ will have rank equal to 2. This matrix block corresponds to the well-known Jacobian of a planar 3R robot (with equal links of length ℓ) performing a positional task with its end-effector. The singularities of the 4R arm for the given task occur then if and only if

$$q_2 = \{0, \pi\} \;\cap\; q_3 = \{0, \pi\}\,,$$

namely when its *first three* links are stretched or folded along a single direction.

Plugging the link length $\ell = 0.5$ [m] and the given configuration $\boldsymbol{q} = (0\;\;0\;\;\pi/2\;\;0)$ in (5) provides

$$\boldsymbol{J} = \begin{pmatrix} -1 & -1 & -1 & -0.5 \\ 1 & 0.5 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

whose pseudoinverse is computed (by hand or using Matlab) as

$$\boldsymbol{J}^{\#} = \boldsymbol{J}^T \left(\boldsymbol{J}\boldsymbol{J}^T\right)^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0.5 & 1 \\ -1 & 0 & 1 \\ -0.5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3.25 & -1.5 & -3.5 \\ -1.5 & 1.25 & 1.5 \\ -3.5 & 1.5 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 1 & 1/6 \\ -2/3 & 0 & -1/3 \\ -5/3 & -1 & -5/6 \\ 2 & 0 & 2 \end{pmatrix}.$$

The desired velocity task is specified by

$$\dot{\boldsymbol{r}}_d = \begin{pmatrix} \boldsymbol{v}_d \\ \phi_d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.5 \end{pmatrix}.$$

In view of the separability of the objective function $H_{range}(\boldsymbol{q}) = \sum_{i=1}^{N} H_{range,i}(q_i)$ that measures the distance from the midpoint of the joint ranges, its gradient takes the form

$$\nabla_{\boldsymbol{q}} H_{range}(\boldsymbol{q}) = \left(\frac{\partial H_{range}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T, \qquad \text{with} \quad \frac{\partial H_{range}(\boldsymbol{q})}{\partial q_i} = \frac{\partial H_{range,i}(q_i)}{\partial q_i} = \frac{1}{N} \frac{q_i - \bar{q}_i}{\left(q_{M,i} - q_{m,i}\right)^2}.$$

With the data $N = 4$, $q_{M,i} = -q_{m,i} = 2$, and thus $\bar{q}_i = 0$, for $i = 1, \ldots, 4$, the gradient at the given configuration $\boldsymbol{q} = (0\ 0\ \pi/2\ 0)$ is

$$\nabla_{\boldsymbol{q}} H_{range} = \frac{1}{64} \begin{pmatrix} 0 \\ 0 \\ \pi/2 \\ 0 \end{pmatrix}$$

The joint velocity solution that realizes the desired task while *decreasing* instantaneously the objective function H_{range} is evaluated then as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#} \dot{\boldsymbol{r}}_d - \left(\boldsymbol{I} - \boldsymbol{J}^{\#} \boldsymbol{J}\right) \nabla_{\boldsymbol{q}} H_{range} = -\nabla_{\boldsymbol{q}} H_{range} + \boldsymbol{J}^{\#} \left(\dot{\boldsymbol{r}}_d + \boldsymbol{J}\, \nabla_{\boldsymbol{q}} H_{range}\right) = \begin{pmatrix} 0.4126 \\ -0.8252 \\ -2.0874 \\ 3 \end{pmatrix} \ [\text{rad/s}].$$

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