Chapter 5 Solutions

Exercise 5.1.

(a) Given a rolling rate $\omega = 1$ and radius $r = 1$, $v = r\omega = 1$, so that the rotation angle around the $-\hat{\mathrm{z}}_{\mathrm{s}}$-axis is $\theta = \omega t = t$. Therefore

$$\begin{aligned}
T_{sb} &= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos t & \sin t & 0 & t + \cos t \\ -\sin t & \cos t & 0 & 1 - \sin t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
[\mathcal{V}_s(t)] &= \dot{T}_{sb} T_{sb}^{-1} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathcal{V}_s(t) &= \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ t \\ 0 \end{bmatrix}.
\end{aligned}$$

(b) Position of the $\{b\}$ frame origin: $p_{sb} = (t + \cos t, 1 - \sin t, 0)^T$.
Linear velocity of the $\{b\}$ frame origin: $\dot{p}_{sb} = (1 - \sin t, -\cos t, 0)^T$.

Exercise 5.2.

(a) The planar space Jacobian $J_s(\theta)$ can be computed and written in matrix form as follows:

$$J_s(\theta) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & -1 & -1 - 1/\sqrt{2} \end{bmatrix}.$$

The planar wrench expressed in the space frame, $\mathcal{F}_s$:

$$\mathcal{F}_s = \begin{bmatrix} m_s \\ f_s \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \in \mathbb{R}^3.$$

So the set of joint torques τ can be obtained as

$$\tau = J_s^T(\theta)\mathcal{F}_s = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 1/\sqrt{2} & -1 - 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -5/\sqrt{2} \\ -5/\sqrt{2} \\ 0 \end{bmatrix}.$$

(b) Similarly to (a), the set of joint torques τ can be obtained as

$$\tau = J_s^T(\theta)\mathcal{F}_s = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 1/\sqrt{2} & -1 - 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 + 5/\sqrt{2} \\ 5 + 5/\sqrt{2} \\ 5 \end{bmatrix}.$$

Exercise 5.3.

(a) By inspection $M \in SE(2)$ can be obtained as

$$M = \begin{bmatrix} 1 & 0 & L_1 + L_2 + L_3 + L_4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The screw axes $\mathcal{S}_i = (\omega_i, v_i) \in \mathbb{R}^3$ are listed in the following table:

i	ω_i	v_i
1	1	(0, 0)
2	1	(0, $-L_1$)
3	1	(0, $-L_1 - L_2$)
4	1	(0, $-L_1 - L_2 - L_3$)

(b) The planar body Jacobian $J_b(\theta)$ can be computed by either inspection in the body frame or tranforming from the planar space Jacobian. The answer can be written in matrix form as follows:

$$J_b(\theta) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ L_3 s_4 + L_2 s_{34} + L_1 s_{234} & L_3 s_4 + L_2 s_{34} & L_3 s_4 & 0 \\ L_4 + L_3 c_4 + L_2 c_{34} + L_1 c_{234} & L_4 + L_3 c_4 + L_2 c_{34} & L_4 + L_3 c_4 & L_4 \end{bmatrix}.$$

(c) The space Jacobian $J_s(\theta)$ in the the configuration $\theta_1 = \theta_2 = 0, \theta_3 = \frac{\pi}{2}, \theta_4 = -\frac{\pi}{2}$ can be computed and written in matrix form as follows:

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & L_3 \\ 0 & -L_1 & -L_1 - L_2 & -L_1 - L_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The wrench expressed in the space frame, $\mathcal{F}_s$:

$$\mathcal{F}_s = \begin{bmatrix} m_s \\ f_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \\ 10 \\ 10 \\ 0 \end{bmatrix}.$$

So the set of joint torques τ can be obtained as

$$\tau = J_s^T(\theta)\mathcal{F}_s = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -L_1 & 0 \\ 0 & 0 & 1 & 0 & -L_1 - L_2 & 0 \\ 0 & 0 & 1 & L_3 & -L_1 - L_2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \\ 10 \\ 10 \\ 0 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 1 - L_1 \\ 1 - L_1 - L_2 \\ 1 - L_1 - L_2 + L_3 \end{bmatrix}.$$

(d) Similarly to (a), the set of joint torques τ can be obtained as

$$\tau = J_s^T(\theta)\mathcal{F}_s = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -L_1 & 0 \\ 0 & 0 & 1 & 0 & -L_1 - L_2 & 0 \\ 0 & 0 & 1 & L_3 & -L_1 - L_2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -10 \\ -10 \\ 10 \\ 0 \end{bmatrix} = -10 \begin{bmatrix} 1 \\ 1 + L_1 \\ 1 + L_1 + L_2 \\ 1 + L_1 + L_2 + L_3 \end{bmatrix}.$$

Go to the table of contents.

(e) Mathematically a singular posture is one in which the Jacobian $J(\theta)$ fails to be of maximal rank. In this case, based on $J_s(\theta)$, $\mathcal{S}_1$ is linearly independent of $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$. So at any singularity, $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ should be linear dependent, which leads to $\theta_2 = \theta_3 = \theta_4 = k\pi, k \in \mathbb{Z}$.

Exercise 5.4.

(a) The displacements from the contact frames to the can frame are given by

$$\begin{aligned} T_{b_1 b} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_{b_1 b} & p_{b_1 b} \\ 0 & 1 \end{bmatrix} \\ T_{b_2 b} &= \begin{bmatrix} 0 & -1 & 0 & L \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_{b_2 b} & p_{b_2 b} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The total spatial force applied to the can expressed in $\{b\}$ coordinates is

$$\begin{aligned} \mathcal{F}_b &= [\mathrm{Ad}_{T_{b_1 b}}]^T \mathcal{F}_{b_1} + [\mathrm{Ad}_{T_{b_2 b}}]^T \mathcal{F}_{b_2} \\ &= \begin{bmatrix} R^T_{b_1 b} & R^T_{b_1 b}[p_{b_1 b}]^T \\ 0 & R^T_{b_1 b} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_{b_1 x} \\ f_{b_1 y} \\ f_{b_1 z} \end{bmatrix} + \begin{bmatrix} R^T_{b_2 b} & R^T_{b_2 b}[p_{b_2 b}]^T \\ 0 & R^T_{b_2 b} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_{b_2 x} \\ f_{b_2 y} \\ f_{b_2 z} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -L \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_{b_1 x} \\ f_{b_1 y} \\ f_{b_1 z} \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & L \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -L & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_{b_2 x} \\ f_{b_2 y} \\ f_{b_2 z} \end{bmatrix} \\ &= \begin{bmatrix} -Lf_{b_1 z} + Lf_{b_2 z} \\ 0 \\ Lf_{b_1 x} - Lf_{b_2 y} \\ f_{b_1 x} + f_{b_2 y} \\ f_{b_1 y} - f_{b_2 x} \\ f_{b_1 z} + f_{b_2 z} \end{bmatrix}. \end{aligned}$$

(b) From the above it can be seen that the second element of $\mathcal{F}_b$ is 0. Therefore, moments about the y-axis cannot be resisted.

Exercise 5.5.

(a) Since $\hat{\mathrm{x}}_\mathrm{b} = (\cos\theta)\hat{\mathrm{x}}_\mathrm{s} + (\sin\theta)\hat{\mathrm{y}}_\mathrm{s}$ and $\hat{\mathrm{y}}_\mathrm{b} = (-\sin\theta)\hat{\mathrm{x}}_\mathrm{s} + (\cos\theta)\hat{\mathrm{y}}_\mathrm{s}$,

$$p_P = L\hat{\mathrm{x}}_\mathrm{s} + L\hat{\mathrm{y}}_\mathrm{s} - d\hat{\mathrm{y}}_\mathrm{b} = (L + d\sin\theta)\hat{\mathrm{x}}_\mathrm{s} + (L - d\cos\theta)\hat{\mathrm{y}}_\mathrm{s}.$$

(b) $\dot{p}_P = (d\dot{\theta}\cos\theta, d\dot{\theta}\sin\theta, 0)^T$.

Go to the table of contents.

(c)

$$T_{sb} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & L + d\sin\theta \\ \sin\theta & \cos\theta & 0 & L - d\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(d)

$$\dot{T}_{sb} = \begin{bmatrix} -\dot\theta\sin\theta & -\dot\theta\cos\theta & 0 & d\dot\theta\cos\theta \\ \dot\theta\cos\theta & -\dot\theta\sin\theta & 0 & d\dot\theta\sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; T_{sb}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & -L(\cos\theta + \sin\theta) \\ -\sin\theta & \cos\theta & 0 & -L(\cos\theta - \sin\theta) + d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$[\mathcal{V}_b] = T_{sb}^{-1}\dot{T}_{sb} = \begin{bmatrix} 0 & -\dot\theta & 0 & d\dot\theta \\ \dot\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{V}_b = \begin{bmatrix} 0 \\ 0 \\ \dot\theta \\ d\dot\theta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}.$$

(e)

$$[\mathcal{V}_s] = \dot{T}_{sb}T_{sb}^{-1} = \begin{bmatrix} 0 & -\dot\theta & 0 & L\dot\theta \\ \dot\theta & 0 & 0 & -L\dot\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{V}_s = \begin{bmatrix} 0 \\ 0 \\ \dot\theta \\ L\dot\theta \\ -L\dot\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix}.$$

(f) $[\mathcal{V}_s] = T_{sb}[\mathcal{V}_b]T_{sb}^{-1}$.
(g) Since $\dot p_P = \dot p_{sb}$, $R_{sb}^{-1}\dot p_P = v_b$.
(h) From $\dot R_{sb}R_{sb}^{-1} = \omega_s$, $-\omega_s p_P + \dot p_P = v_s$.

Exercise 5.6.

(a) Given $\theta_1 = t$, $\theta_2 = t$, $\dot\theta_1 = \dot\theta_2 = 1$, the problem asks for the linear velocity and angular velocity expressed in $\{b\}$ frame coordinates. First find the spatial body velocity V_b as follows:

$$\mathcal{V}_b = J_b(\theta)\dot\theta$$

$$J_b(\theta) = \begin{bmatrix} \mathcal{V}_{b1}(\theta) & \mathcal{V}_{b2}(\theta) \end{bmatrix}$$

where

$$\mathcal{V}_{b1}(\theta) = \begin{bmatrix} \sin\theta_2 \\ \cos\theta_2 \\ 0 \\ -20\cos\theta_2 \\ 20\sin\theta_2 \\ -10\cos\theta_2 \end{bmatrix}, \mathcal{V}_{b2}(\theta) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$

$$\therefore \mathcal{V}_b = J_b(\theta)\dot\theta = \begin{bmatrix} \sin t & 0 \\ \cos t & 0 \\ 0 & 1 \\ -20\cos t & 0 \\ 20\sin t & 10 \\ -10\cos t & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin t \\ \cos t \\ 1 \\ -20\cos t \\ 20\sin t + 10 \\ -10\cos t \end{bmatrix} = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}.$$

Go to the table of contents.

Therefore

$$\omega_b = \begin{bmatrix} \sin t \\ \cos t \\ 1 \end{bmatrix} \quad v_b = \begin{bmatrix} -20\cos t \\ 20\sin t + 10 \\ -10\cos t \end{bmatrix}.$$

(b) The linear velocity of the rider in the fixed frame $\{s\}$ coordinates is $\dot{p}$. Then $\dot{p} = R_{sb}v_b$, which expanded becomes

$$\begin{aligned} \dot{p}(t) &= \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix} \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -20\cos t \\ 20\sin t + 10 \\ -10\cos t \end{bmatrix} \\ &= \begin{bmatrix} -20\cos t - 20\cos t\sin t \\ 10\cos t \\ 20\sin t + 10\sin^2 t - 10\cos^2 t \end{bmatrix}. \end{aligned}$$

Exercise 5.7.

(a) The forward kinematics of the RRP robot is expressed as $T(\theta) = e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}e^{[\mathcal{S}_3]\theta_3}M$ with the screw axes in the space frame:

$$\begin{aligned} \mathcal{S}_1: &\quad \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathcal{S}_2: &\quad \omega_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \\ \mathcal{S}_3: &\quad \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

$$M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

When $\theta = (90^\circ, 90^\circ, 1)$, the forward kinematics can be evaluated as follows:

$$\begin{aligned} T(\theta) &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

In this configuration, the arm and the end-effector frame are shown in Figure 5.1. In order to obtain

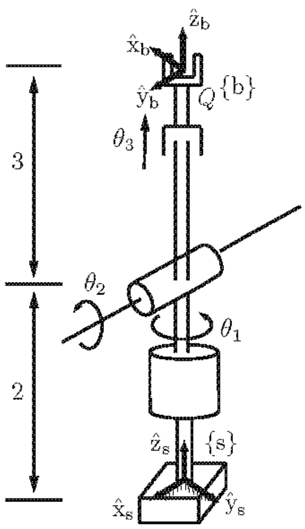

Figure 5.1

the space Jacobian $J_s(\theta)$ at this configuration,

$$\mathcal{V}_{s1}(\theta): \quad \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{V}_{s2}(\theta): \quad \omega_{s2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, v_{s2} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{V}_{s3}(\theta): \quad \omega_{s3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) The forward kinematics of the RRP robot is expressed as $T(\theta) = Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}e^{[\mathcal{B}_3]\theta_3}$ with the screw axes in the end-effector body frame.

$$\mathcal{B}_1: \quad \omega_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, v_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{B}_2: \quad \omega_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$\mathcal{B}_3: \quad \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Go to the table of contents.

$$M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

When $\theta = (90°, 90°, 1)$, the forward kinematics can be evaluated as follows:

$$\begin{aligned} T(\theta) &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which are the same as the results obtained in (a).
In order to obtain the space Jacobian J_b at this configuration,

$$\begin{aligned} \mathcal{V}_{b1}(\theta): \quad & \omega_{b1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{b1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathcal{V}_{b2}(\theta): \quad & \omega_{b2} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, v_{b2} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \\ \mathcal{V}_{b3}(\theta): \quad & \omega_{b3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{b3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$\therefore J_b(\theta) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 5.8.

(a) The space Jacobian $J_s(\theta)$ can be computed and written in matrix form as follows:

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & \sin\theta_1 \\ 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 \\ 0 & 0 & (2L+\theta_2)\cos\theta_1 \\ 0 & 1 & 0 \\ 0 & 0 & -(2L+\theta_2)\sin\theta_1 \end{bmatrix}.$$

(b) In the zero position, The body Jacobian $J_b(\theta)$ can be computed and written in matrix form as follows:

$$J_b(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -L \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Go to the table of contents.

Suppose an external force $f = (f_x, f_y, f_z)^T \in \mathbb{R}^3$, which is applied to the {b} frame origin, can be resisted by the manipulator. The the set of joint torques τ should satisfy:

$$\tau = J_b^T(\theta)\mathcal{F}_b = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -L & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 0 \\ f_y \\ -Lf_x \end{bmatrix}.$$

In this case, there is zero torques from the manipulator, which means $f_y = f_x = 0$. So the external force can be obtained as $f = (0, 0, f_z)^T$, which means force along the $\hat{z}$-axis can be resisted by the manipulator with zero torques.

Exercise 5.9.
We consider only the rotation component; in this case $J_s(\theta) = \dot{R}R^T$:

$$\begin{aligned}
\mathcal{V}_{s1}(\theta): &\quad \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathcal{V}_{s2}(\theta): &\quad \omega_{s2} = Rot(\hat{z}, \theta_1) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\cos\theta_1}{\sqrt{2}} \\ \frac{\sin\theta_1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\
\mathcal{V}_{s3}(\theta): &\quad \omega_{s3} = Rot(\hat{z}, \theta_1) e^{[\hat{\omega}_2]\theta_2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\cos\theta_1}{2} + \frac{\cos\theta_1\cos\theta_2}{2} - \frac{\sin\theta_1\sin\theta_2}{\sqrt{2}} \\ \frac{\sin\theta_1}{2} + \frac{\sin\theta_1\cos\theta_2}{2} + \frac{\cos\theta_1\sin\theta_2}{\sqrt{2}} \\ \frac{1}{2} - \frac{\cos\theta_2}{2} \end{bmatrix}
\end{aligned}$$

where

$$e^{[\hat{\omega}_2]\theta_2} = I + \sin\theta_2[\hat{\omega}_2] + (1 - \cos\theta_2)[\hat{\omega}_2]^2.$$

From above the space Jacobian J_s can be expressed as follows:

$$J_s(\theta) = \begin{bmatrix} 0 & \frac{\cos\theta_1}{\sqrt{2}} & \frac{\cos\theta_1}{2} + \frac{\cos\theta_1\cos\theta_2}{2} - \frac{\sin\theta_1\sin\theta_2}{\sqrt{2}} \\ 0 & \frac{\sin\theta_1}{\sqrt{2}} & \frac{\sin\theta_1}{2} + \frac{\sin\theta_1\cos\theta_2}{2} + \frac{\cos\theta_1\sin\theta_2}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{2} - \frac{\cos\theta_2}{2} \end{bmatrix}$$

$$\begin{aligned}
\det(J_s) &= 0 \\
\therefore \frac{\sin\theta_2}{2} &= 0.
\end{aligned}$$

Therefore when θ_2 is 0, $\pm\pi$, a singularity arises. Another approach: Intuitively, when θ_2 is 0, three axes lie in the same plane and intersect at a single point, resulting in a singularity. When θ_2 is $\pm\pi$, two axes are collinear, also resulting in a singularity.

Exercise 5.10.
(a) By Taylor expansion with respect to h,

$$\begin{aligned}
\frac{d}{dt}e^{A(t)} &= \lim_{h\to 0}\frac{1}{h}(e^{A(t+h)} - e^{A(t)}) \\
&= \lim_{h\to 0}\frac{1}{h}(e^{A(t)+h\dot{A}(t)+O(h^2)} - e^{A(t)}).
\end{aligned}$$

Go to the table of contents.

The $O(h^2)$ terms go to zero in the limit, and hence can be ignored.
The above equation then reduces to

$$\begin{aligned}\frac{d}{dt}e^{A(t)} &= \lim_{h\to 0}\frac{1}{h}(e^{A(t)} + h\frac{d}{dh}e^{A(t)+h\dot{A}(t)+O(h^2)} - e^{A(t)}) \\ &= \lim_{h\to 0}\frac{d}{dh}e^{A(t)+h\dot{A}(t)}.\end{aligned}$$

Now if A, B are matrices and ϵ, t are scalars, it can be shown that $\frac{d}{d\epsilon}e^{(A+\epsilon B)t}|_{\epsilon=0} = \int_0^t e^{As}Be^{A(t-s)}ds$.
From this result it follows that

$$\frac{d}{dt}e^{A(t)} = e^{A(t)}\int_0^1 e^{-A(t)s}\dot{A}(t)e^{A(t)s}ds.$$

Therefore

$$X^{-1}\dot{X} = \int_0^1 e^{-A(t)s}\dot{A}(t)e^{A(t)s}ds.$$

Similarly, $\dot{X}X^{-1}$ can be derived using same manner.

(b) By using the result of (a) and the relation $R\,[\omega]\,R^T = [R\omega]$ for $R \in SO(3)$ and $\omega \in \mathbb{R}^3$,

$$\begin{aligned}[\omega_b] &= R^T R \\ &= \int_0^1 e^{-[r(t)]s}[\dot{r}(t)]e^{[r(t)]s}ds \\ &= \int_0^1 \left[e^{-[r(t)]s}\dot{r}(t)\right] ds \\ &= \left[\int_0^1 e^{-[r(t)]s}ds \cdot \dot{r}(t)\right].\end{aligned}$$

Let $\int_0^1 e^{-[r(t)]s}ds$ be $A(r)$. The characteristic polynomial of $[r(t)]$ is $s^3 + ||r(t)||^2 s$, and by the Cayley-Hamilton theorem we have $[r(t)]^3 = -||r(t)||^2[r(t)]$. From this result $A(r)$ can be obtained as follows:

$$\begin{aligned}A(r) &= \int_0^1 e^{-[r(t)]s}ds \\ &= \int_0^1 \left(I - [r(t)]s + [r(t)]^2\frac{s^2}{2!} - [r(t)]^3\frac{s^3}{3!} + \cdots\right) ds \\ &= I - \frac{[r(t)]}{2!} + \frac{[r(t)]^2}{3!} - \frac{[r(t)]^3}{4!} + \cdots \\ &= I - \frac{1}{||r||^2}\left(\frac{||r||^2}{2!} - \frac{||r||^4}{4!} + \frac{||r||^6}{6!} - \cdots\right)[r(t)] + \frac{1}{||r||^3}\left(\frac{||r||^3}{3!} - \frac{||r||^5}{5!} + \frac{||r||^7}{7!} - \cdots\right)[r(t)]^2 \\ &= I - \frac{1-\cos||r||}{||r||^2}[r(t)] + \frac{||r|| - \sin||r||}{||r||^3}[r(t)]^2.\end{aligned}$$

Thus we obtain

$$\omega_b = A(r)\dot{r}$$

where

$$A(r) = I - \frac{1-\cos||r||}{||r||^2}[r(t)] + \frac{||r|| - \sin||r||}{||r||^3}[r(t)]^2.$$

(c) Applying the similar method used in (b) to the angular velocity in the space frame, $[\omega_s] = \dot{R}R^T$, we can obtain

$$\omega_s = A(r)\dot{r}$$

Go to the table of contents.

where

$$A(r) = I + \frac{1-\cos||r||}{||r||^2}[r(t)] + \frac{||r||-\sin||r||}{||r||^3}[r(t)]^2.$$

Exercise 5.11.

(a)

$$\begin{aligned}
J_b(0) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ -L & 0 & 0 \\ 2L & 0 & 0 \\ 0 & 2L & L \end{bmatrix} \\
v_b &= R_{bs} v_{tip} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \; (\because R_{bs} = I) \\
v_b &= J_b(0)\dot{\theta} \\
\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -L & 0 & 0 \\ 2L & 0 & 0 \\ 0 & 2L & L \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \Rightarrow \begin{array}{r} 10 = -L\dot{\theta}_1 \\ 0 = 2L\dot{\theta}_1 \end{array}
\end{aligned}$$

Therefore no solution exists.

(b) At the configuration $\theta = (0°, 45°, -45°)$,

$$J_b(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ -L & -\frac{\sqrt{2}}{2}L & 0 \\ (1+\frac{\sqrt{2}}{2})L & 0 & 0 \\ 0 & (1+\frac{\sqrt{2}}{2})L & L \end{bmatrix}, \quad F_b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
\therefore \tau &= J_b^T(\theta) F_b \\
&= \begin{bmatrix} -10L \\ -5\sqrt{2}L \\ 0 \end{bmatrix}.
\end{aligned}$$

(c) $J_b(\theta)$ is the same as obtained in (b):

$$\begin{aligned}
F_b &= \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\therefore \tau &= J_b^T(\theta) F_b \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Go to the table of contents.

(d)

$$J_b(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ -L & 0 & 0 \\ 2L & 0 & 0 \\ 0 & 2L & L \end{bmatrix}, \quad F_b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_x \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} &= J_b^T(0)F_b \\ &= \begin{bmatrix} -Lf_x \\ 0 \\ 0 \end{bmatrix} \\ \therefore \|f_x\| &= \frac{\|\tau_1\|}{L} \leq \frac{10}{L}. \end{aligned}$$

Exercise 5.12.
(a) Given $\dot{\theta} = (0, 0, \frac{\pi}{2}, L)^T$,

$$J_b(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -L & 0 & -L & 0 \\ L & 0 & 0 & 1 \\ 0 & L & 0 & 0 \end{bmatrix}.$$

(b) Given $\dot{\theta} = (0, 0, \frac{\pi}{2}, L)^T$,

$$\mathcal{V}_b = J_b(\theta)\dot{\theta} = \begin{bmatrix} 0 \\ 0 \\ \frac{\pi}{2} \\ -\frac{\pi L}{2} \\ L \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}.$$

$$R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \dot{p} = R_{sb}v_b = \begin{bmatrix} -L \\ -\frac{\pi L}{2} \\ 0 \end{bmatrix}.$$

Given $\dot{\theta} = (1, 1, 1, 1)^T$,

$$\mathcal{V}_b = J_b(\theta)\dot{\theta} = \begin{bmatrix} 0 \\ -1 \\ 2 \\ -2L \\ L+1 \\ L \end{bmatrix} = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}.$$

$$R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \dot{p} = R_{sb}v_b = \begin{bmatrix} -L-1 \\ -2L \\ L \end{bmatrix}.$$

Go to the table of contents.

Exercise 5.13.
(a)

$$\mathcal{V}_{s1}(\theta): \qquad \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{V}_{s2}(\theta): \qquad \omega_{s2} = e^{[\hat{z}]\theta_1}\omega_2 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{V}_{s3}(\theta): \qquad \omega_{s3} = e^{[\hat{z}]\theta_1}e^{[\hat{y}]\theta_2}\omega_3 = \begin{bmatrix} -c_1c_2 \\ -s_1c_2 \\ s_2 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{V}_{s4}(\theta): \qquad \omega_{s4} = e^{[\hat{z}]\theta_1}e^{[\hat{y}]\theta_2}e^{-[\hat{x}]\theta_3}\omega_4 = \omega_{s3},\ q_4 = e^{[\hat{z}]\theta_1}e^{[\hat{y}]\theta_2}e^{-[\hat{x}]\theta_3}\begin{bmatrix} 0 \\ L \\ 0 \end{bmatrix} = \begin{bmatrix} -L(s_1c_3 + c_1s_2s_3) \\ L(c_1c_3 - s_1s_2s_3) \\ -Lc_2s_3 \end{bmatrix},$$

$$v_{s4} = -\omega_{s4} \times q_4 = \begin{bmatrix} -L(s_1c_2^2s_3 - c_1s_2c_3 + s_1s_2^2s_3 \\ L(s_1s_2c_3 + c_1s_2^2s_3 + c_1c_2^2s_3) \\ Lc_2c_3(c_1^2 + s_1^2) \end{bmatrix}$$

$$\mathcal{V}_{s5}(\theta): \qquad \omega_{s5} = \omega_{s4}, q_5 = q_4 + e^{[\hat{z}]\theta_1}e^{[\hat{y}]\theta_2}e^{-[\hat{x}]\theta_3}e^{-[\hat{x}]\theta_4}\begin{bmatrix} 0 \\ L \\ 0 \end{bmatrix}, v_{s5} = -\omega_{s5} \times q_5$$

$$\mathcal{V}_{s6}(\theta): \qquad \omega_{s6} = e^{[\hat{z}]\theta_1}e^{[\hat{y}]\theta_2}e^{-[\hat{x}](\theta_3+\theta_4+\theta_5)}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -s_1c_{345} - c_1s_2s_{345} \\ c_1c_{345} - s_1s_2s_{345} \\ -c_2s_{345} \end{bmatrix}, q_6 = q_5, v_{s6} = -\omega_{s6} \times q_6$$

$$\therefore J_s(\theta) = \begin{bmatrix} \mathcal{V}_{s1}(\theta) & \mathcal{V}_{s2}(\theta) & \mathcal{V}_{s3}(\theta) & \mathcal{V}_{s4}(\theta) & \mathcal{V}_{s5}(\theta) & \mathcal{V}_{s6}(\theta) \end{bmatrix}$$

(b) Case 1. Two collinear revolute joint axes: When $\theta_2 = \frac{\pi}{2}$ and $\frac{3}{2}\pi$, axis 1 and axis 3 are collinear, and z-translation becomes impossible.
Case 2. Three parallel coplanar revolute joint axes: When $\theta_4 = 0$ and π, axis 3, axis 4 and axis 5 are coplanar, and y-translation becomes impossible.
Case 3. Four intersecting revolute joint axes: Axis 1, axis 2 and axis 3 always intersect. When axis 6 also intersects these three axes at a single point, z-rotation becomes impossible.

Exercise 5.14.
Suppose that the the prismatic joint axis is coincident with the z-axis of the fixed frame, and the two revolute joint axes are parallel to the x-axis of the fixed frame as shown in Figure 5.2. Then

$$\mathcal{V}_{s1}(\theta): \qquad \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{V}_{s2}(\theta): \qquad \omega_{s2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ L_1 \\ 0 \end{bmatrix}, v_{s2} = \begin{bmatrix} 0 \\ 0 \\ -L_1 \end{bmatrix}$$

$$\mathcal{V}_{s3}(\theta): \qquad \omega_{s3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ L_1 + L_2 \\ 0 \end{bmatrix}, v_{s3} = \begin{bmatrix} 0 \\ 0 \\ -L_1 - L_2 \end{bmatrix}$$

Go to the table of contents.

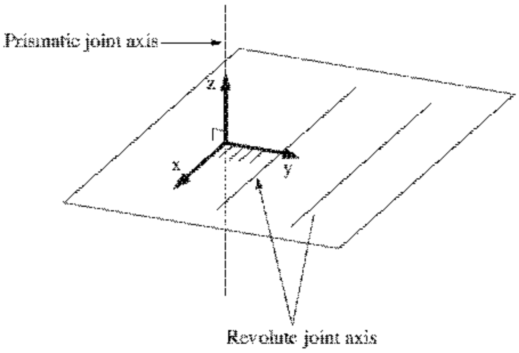

Figure 5.2

$$\therefore J_s(\theta) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -L_1 & -L_1 - L_2 \end{bmatrix}.$$

This is a singularity because the rank of $J_s(\theta)$ is less than 3.

Exercise 5.15.

(a)

$$\begin{aligned}
\mathcal{V}_{s1}(\theta): \quad & \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s2}(\theta): \quad & \omega_{s2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ \theta_1 \\ 0 \end{bmatrix}, v_{s2} = \begin{bmatrix} \theta_1 \\ 0 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s3}(\theta): \quad & \omega_{s3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ \theta_1 \\ 0 \end{bmatrix} + Rot(\hat{z}, \theta_2) \begin{bmatrix} -L \\ L \\ 0 \end{bmatrix} = \begin{bmatrix} -L\cos\theta_2 - L\sin\theta_2 \\ \theta_1 - L\sin\theta_2 + L\cos\theta_2 \\ 0 \end{bmatrix}, \\
& v_{s3} = \begin{bmatrix} \theta_1 + L(\cos\theta_2 - \sin\theta_2) \\ L(\cos\theta_2 + \sin\theta_2) \\ 0 \end{bmatrix}.
\end{aligned}$$

(b) From (a), the first three columns of the space Jacobian $J_s(\theta)$ can be expressed as follows:

$$\begin{aligned}
J_s(\theta) &= \begin{bmatrix} \mathcal{V}_{s1}(\theta) & \mathcal{V}_{s2}(\theta) & \mathcal{V}_{s3}(\theta) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \theta_1 & \theta_1 + L(\cos\theta_2 - \sin\theta_2) \\ 1 & 0 & L(\cos\theta_2 + \sin\theta_2) \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Go to the table of contents.

$$\begin{aligned}(\mathcal{V}_{s3}(\theta))' &= \mathcal{V}_{s3}(\theta) - L(\cos\theta_2 + \sin\theta_2)\mathcal{V}_{s1}(\theta) \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ \theta_1 + L(\cos\theta_2 - \sin\theta_2) \\ 0 \\ 0 \end{bmatrix}.\end{aligned}$$

For $(\mathcal{V}_{s3}(\theta))'$ and $\mathcal{V}_{s2}(\theta)$ to be equal,

$$\begin{aligned}\theta_1 &= \theta_1 + L(\cos\theta_2 - \sin\theta_2) \\ 0 &= \cos\theta_2 - \sin\theta_2 \\ \therefore \theta_2 &= \frac{\pi}{4}, \frac{5\pi}{4}.\end{aligned}$$

(c) At the configuration $\theta = (0°, 0°, 0°, 90°, 0°, 0°)$,

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & L \\ 1 & 0 & L \\ 0 & 0 & 0 \end{bmatrix}.$$

To obtain the wrench $\mathcal{F}_s$ expressed in the space frame $\{s\}$, use the displacement $T_{bs} \in SE(3)$ of the space frame $\{s\}$ expressed in the body frame $\{b\}$:

$$T_{bs} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -5L \\ 1 & 0 & 0 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned}\mathcal{F}_s &= \mathrm{Ad}^T_{T_{bs}}(\mathcal{F}_b) \\ &= \begin{bmatrix} 0 & 0 & 1 & -5L & 0 & 0 \\ 0 & 1 & 0 & -L & 0 & 0 \\ -1 & 0 & 0 & 0 & -L & -5L \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} -50L \\ -10L \\ -50L \\ 10 \\ 0 \\ -10 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\therefore \tau &= J_s(\theta)^T \mathcal{F}_s \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & L & 0 \end{bmatrix} \begin{bmatrix} -50L \\ -10L \\ -50L \\ 10 \\ 0 \\ -10 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -50L \\ -40L \end{bmatrix}.\end{aligned}$$

Exercise 5.16.

Go to the table of contents.

(a)

$$\mathcal{V}_{s1}(\theta): \quad \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{V}_{s2}(\theta): \quad \omega_{s2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{V}_{s3}(\theta): \quad \omega_{s3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s3} = e^{\hat{z}\theta_2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\theta_2 \\ \cos\theta_2 \\ 0 \end{bmatrix}.$$

(b)

$$\mathcal{V}_{b6}(\theta): \quad \omega_{b6} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{b6} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{V}_{b5}(\theta): \quad \omega_{b5} = e^{\hat{x}(-\theta_6)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\theta_6 \\ \cos\theta_6 \end{bmatrix}, q_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{b5} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) At the zero position,

$$J_s(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -L \end{bmatrix}.$$

When $L = 0$, $J_s(0)$ becomes singular.

(d) In the zero position, consider the wrench $\mathcal{F}_s$ expressed in the space frame $\{s\}$:

$$\mathcal{F}_s = \begin{bmatrix} m_s \\ f_s \end{bmatrix} = \begin{bmatrix} p_{sb} \times f_b \\ f_b \end{bmatrix} = \begin{bmatrix} -100L \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \end{bmatrix}$$

where

$$p_{sb} = \begin{bmatrix} 0 \\ L \\ 0 \end{bmatrix}, f_b = \begin{bmatrix} 0 \\ 0 \\ -100 \end{bmatrix}$$

$$\begin{aligned} \therefore \tau &= J_s^T(0)\mathcal{F}_s \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -L \end{bmatrix} \begin{bmatrix} -100L \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \end{bmatrix} = \begin{bmatrix} -100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Exercise 5.17.

Go to the table of contents.

(a) If θ is arbitrary,

$$
\begin{aligned}
\mathcal{V}_{s1}(\theta): \quad & \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathcal{V}_{s2}(\theta): \quad & \omega_{s2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ \theta_1 \end{bmatrix}, v_{s2} = \begin{bmatrix} 0 \\ \theta_1 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s3}(\theta): \quad & \omega_{s3} = e^{[\hat{x}]\theta_2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin\theta_2 \\ \cos\theta_2 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ L\cos\theta_2 \\ \theta_1 + L\sin\theta_2 \end{bmatrix}, v_{s3} = \begin{bmatrix} L + \theta_1 \sin\theta_2 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Therefore, the first three columns of the space Jacobian

$$
J_s(\theta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\sin\theta_2 \\ 0 & 0 & \cos\theta_2 \\ 0 & 0 & L + \theta_1\sin\theta_2 \\ 0 & \theta_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$

(b)

$$
J_s(0) = \begin{bmatrix} 0 & 1 & 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -L & -\frac{L}{\sqrt{2}} & 0 \end{bmatrix}
$$

$$
\begin{aligned}
\therefore \mathcal{V}_s &= J_s(0)\dot{\theta} \\
&= \begin{bmatrix} 0 & 1 & 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -L & -\frac{L}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 + \sqrt{2} \\ \sqrt{2} \\ 1 \\ L \\ 0 \\ 1 + (1 - \sqrt{2})L \end{bmatrix}
\end{aligned}
$$

(c) In the zero position, the space Jacobian $J_s(\theta)$ is singular, since $\mathcal{V}_{s4}(0) + L\mathcal{V}_{s1}(0) = \mathcal{V}_{s2}(0)$.

Exercise 5.18.
(a)

$$
\begin{aligned}
\mathcal{V}_{s1}(\theta): \quad & \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s2}(\theta): \quad & \omega_{s2} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}}\sin\theta_1 \\ \frac{1}{\sqrt{2}}\cos\theta_1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, v_{s2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s3}(\theta): \quad & \omega_{s3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s3} = \begin{bmatrix} -\frac{1}{\sqrt{2}}\sin\theta_1 \\ \frac{1}{\sqrt{2}}\cos\theta_1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}
\end{aligned}
$$

Go to the table of contents.

(b)

$$
\begin{aligned}
J_s(0) &= \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -(3+\frac{1}{\sqrt{2}})L \\
0 & 0 & \frac{1}{\sqrt{2}} & (1+\frac{1}{\sqrt{2}})L & 1 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & -(2+\frac{1}{\sqrt{2}})L & 0 & 0
\end{bmatrix} \\
\mathcal{V}_s &= J_s(0)\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ \frac{1}{\sqrt{2}} - (1+\frac{1}{\sqrt{2}})L + 2 \\ \frac{1}{\sqrt{2}} + (2 + \frac{\sqrt{2}}{)}L \end{bmatrix}
\end{aligned}
$$

(c) There does not exist a kinematic singularity in the zero position, because rank($J_s(0)$)=6.

Exercise 5.19.

(a)

$$
\begin{array}{ll}
\mathcal{V}_{s1}(\theta): & \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathcal{V}_{s2}(\theta): & \omega_{s2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ \theta_1 \end{bmatrix}, v_{s2} = \begin{bmatrix} 0 \\ \theta_1 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s3}(\theta): & \omega_{s3} = \begin{bmatrix} 0 \\ -\sin\theta_2 \\ \cos\theta_2 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ \cos\theta_2 \\ \theta_1 + \sin\theta_2 \end{bmatrix}, v_{s3} = \begin{bmatrix} \theta_1 \sin\theta_2 + 1 \\ 0 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s4}(\theta): & \omega_{s4} = \begin{bmatrix} 0 \\ -\sin\theta_2 \\ \cos\theta_2 \end{bmatrix}, q_4 = \begin{bmatrix} -\cos\theta_3 - \sin\theta_3 \\ \cos\theta_2(\cos\theta_3 - \sin\theta_3) \\ \theta_1 + \sin\theta_2(\cos\theta_3 - \sin\theta_3) \end{bmatrix}, v_{s4} = \begin{bmatrix} \theta_1\sin\theta_2 + \cos\theta_3 - \sin\theta_3 \\ \cos\theta_2(\cos\theta_3 + \sin\theta_3) \\ \sin\theta_2(\cos\theta_3 + \sin\theta_3) \end{bmatrix}
\end{array}
$$

(b) Consider the space Jacobian $J_s(\theta)$ at the zero position:

$$
\begin{aligned}
J_s(0) &= \begin{bmatrix} \mathcal{V}_{s1}(0) & \mathcal{V}_{s2}(0) & \mathcal{V}_{s3}(0) & \mathcal{V}_{s4}(0) & \mathcal{V}_{s5}(0) & \mathcal{V}_{s6}(0) \end{bmatrix} \\
&= \begin{bmatrix}
0 & 1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0
\end{bmatrix}.
\end{aligned}
$$

The space Jacobian $J_s(\theta)$ is singular in the zero position, since $\mathcal{V}_{s4}(0) - \mathcal{V}_{s6}(0) = \mathcal{V}_{s3}(0)$.

Go to the table of contents.

(c) When $\mathcal{F}_s = (0,1,-1,1,0,0)^T$,

$$\begin{aligned}
\tau_{(\mathrm{i})} &= J_s^T(0)\mathcal{F}_s \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}
\end{aligned}$$

When $\mathcal{F}_s = (1,-1,0,1,0,-1)^T$,

$$\begin{aligned}
\tau_{(\mathrm{ii})} &= J_s^T(0)\mathcal{F}_s \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \\ \frac{3}{\sqrt{2}} \\ 0 \end{bmatrix}
\end{aligned}$$

Exercise 5.20.

Go to the table of contents.

(a)

$$\begin{aligned}
\mathcal{S}_1: &\quad \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\mathcal{S}_2: &\quad \omega_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ L_0 - L_1 \end{bmatrix}, v_2 = \begin{bmatrix} L_0 - L_1 \\ 0 \\ 0 \end{bmatrix} \\
\mathcal{S}_3: &\quad \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathcal{S}_4: &\quad \omega_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q_4 = \begin{bmatrix} 0 \\ 0 \\ L_0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ L_0 \\ 0 \end{bmatrix} \\
\mathcal{S}_5: &\quad \omega_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_5 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix} \\
\mathcal{S}_6: &\quad \omega_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q_6 = \begin{bmatrix} 0 \\ 0 \\ L_0 - L_1 \end{bmatrix}, v_6 = \begin{bmatrix} 0 \\ L_0 - L_1 \\ 0 \end{bmatrix}
\end{aligned}$$

$$M = \begin{bmatrix} 0 & 1 & 0 & L_1 + L_2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & L_0 - L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{aligned}
\mathcal{V}_{s1}(\theta): &\quad \omega_{s1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s2}(\theta): &\quad \omega_{s2} = e^{[\hat{z}]\theta_1} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\theta_1 \\ -\cos\theta_1 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 0 \\ L_0 - L_1 \end{bmatrix}, v_{s2} = \begin{bmatrix} (L_0 - L_1)\cos\theta_1 \\ (L_0 - L_1)\sin\theta_1 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s3}(\theta): &\quad \omega_{s3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s3} = e^{[\hat{z}]\theta_1} e^{[-\hat{y}]\theta_2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\cos\theta_1 \sin\theta_2 \\ -\sin\theta_1 \sin\theta_2 \\ \cos\theta_2 \end{bmatrix}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathcal{V}_{s'1}(\theta): &\quad \omega_{s'1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ -L_1 - L_2 \\ 0 \end{bmatrix}, v_{s'1} = \begin{bmatrix} -L_1 - L_2 \\ 0 \\ 0 \end{bmatrix} \\
\mathcal{V}_{s'2}(\theta): &\quad \omega_{s'2} = e^{[\hat{z}]\theta_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ -L_1 - L_2 \\ 0 \end{bmatrix}, v_{s'2} = \begin{bmatrix} 0 \\ 0 \\ (L_1 + L_2)\cos\theta_1 \end{bmatrix} \\
\mathcal{V}_{s'3}(\theta): &\quad \omega_{s'3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s'3} = e^{[\hat{z}]\theta_1} e^{[\hat{x}]\theta_2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta_1 \sin\theta_2 \\ -\cos\theta_1 \sin\theta_2 \\ \cos\theta_2 \end{bmatrix}
\end{aligned}$$

Go to the table of contents.

(d) Consider the space Jacobian $J_s(\theta)$ at the zero position:

$$\begin{aligned} J_s(0) &= \begin{bmatrix} \mathcal{V}_{s1}(0) & \mathcal{V}_{s2}(0) & \mathcal{V}_{s3}(0) & \mathcal{V}_{s4}(0) & \mathcal{V}_{s5}(0) & \mathcal{V}_{s6}(0) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & L_0 - L_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_0 & -L_1 & L_0 - L_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The space Jacobian $J_s(\theta)$ is singular in the zero position, since $\mathcal{V}_{s1}(0) - \mathcal{V}_{s5}(0) = \mathcal{V}_{s4}(0) - \mathcal{V}_{s6}(0)$. Geometrically, this singularity corresponds to four coplanar revolute joint screw axes.

Exercise 5.21.

(a) We can express the forward kinematics of the manipulator in the following form:

$$T_{0t} = e^{[\mathcal{S}_1]}\theta_1 e^{[\mathcal{S}_2]}\theta_2 e^{[\mathcal{S}_3]}\theta_3 e^{[\mathcal{S}_4]}\theta_4 e^{[\mathcal{S}_5]}\theta_5 e^{[\mathcal{S}_6]}\theta_6 M$$

where $\mathcal{S}_i$ $(i = 1, 2, ..., 6)$ is the screw for the i^{th} link relative to frame $\{0\}$, and M is the displacement from frame $\{0\}$ to frame $\{t\}$ when the manipulator is in its zero position. We further know that

$$Te^{[\mathcal{S}]}T^{-1} = e^{[\mathcal{S}']} \ (\mathcal{S}' = [\mathrm{Ad}_T]\mathcal{S}).$$

Using the above formula, the forward kinematics can be modified to the following form:

$$\begin{aligned} T_{0t} &= e^{[\mathcal{S}_1]}\theta_1 e^{[\mathcal{S}_2]}\theta_2 e^{[\mathcal{S}_3]}\theta_3 e^{[\mathcal{S}_4]}\theta_4 e^{[\mathcal{S}_5]}\theta_5 M e^{[\mathcal{S'}_6]}\theta_6 \quad (\mathcal{S'}_6 \triangleq [\mathrm{Ad}_{M^{-1}}]\mathcal{S}_6) \\ &= e^{[\mathcal{S}_1]}\theta_1 e^{[\mathcal{S}_2]}\theta_2 e^{[\mathcal{S}_3]}\theta_3 e^{[\mathcal{S}_4]}\theta_4 M e^{[\mathcal{S'}_5]}\theta_5 e^{[\mathcal{S'}_6]}\theta_6 \quad (\mathcal{S'}_5 \triangleq [\mathrm{Ad}_{M^{-1}}]\mathcal{S}_5) \\ &= e^{[\mathcal{S}_1]}\theta_1 e^{[\mathcal{S}_2]}\theta_2 e^{[\mathcal{S}_3]}\theta_3 e^{[\mathcal{S}_4]}\theta_4 M_{0c} M_{ct} e^{[\mathcal{S'}_5]}\theta_5 e^{[\mathcal{S'}_6]}\theta_6 \quad (M = M_{0c}M_{ct}) \\ &= e^{[\mathcal{S}_1]}\theta_1 e^{[\mathcal{S}_2]}\theta_2 e^{[\mathcal{S}_3]}\theta_3 M_{0c} e^{[\mathcal{S'}_4]}\theta_4 M_{ct} e^{[\mathcal{S'}_5]}\theta_5 e^{[\mathcal{S'}_6]}\theta_6 \quad (\mathcal{S'}_4 \triangleq [\mathrm{Ad}_{M_{0c}^{-1}}]\mathcal{S}_4) \\ &= e^{[\mathcal{S}_1]}\theta_1 e^{[\mathcal{S}_2]}\theta_2 M_{0c} e^{[\mathcal{S'}_3]}\theta_3 e^{[\mathcal{S'}_4]}\theta_4 M_{ct} e^{[\mathcal{S'}_5]}\theta_5 e^{[\mathcal{S'}_6]}\theta_6 \quad (\mathcal{S'}_3 \triangleq [\mathrm{Ad}_{M_{0c}^{-1}}]\mathcal{S}_3) \\ &= e^{[\mathcal{A}_1]}\theta_1 e^{[\mathcal{A}_2]}\theta_2 M_{0c} e^{[\mathcal{A}_3]}\theta_3 e^{[\mathcal{A}_4]}\theta_4 M_{ct} e^{[\mathcal{A}_5]}\theta_5 e^{[\mathcal{A}_6]}\theta_6 \end{aligned}$$

$\therefore \mathcal{A}_1 = \mathcal{S}_1$, $\mathcal{A}_2 = \mathcal{S}_2$, and $\mathcal{A}_i = \mathcal{S'}_i$ $(i = 3, 4, 5, 6)$

One difficulty with the above is that it is very complex to calculate. Calculating $\mathcal{A}_i$ based on the physical meaning of screws is easier. The adjoint transformation is used to change the relative coordinate frame: $\mathcal{S}_1$ and $\mathcal{S}_2$ are screws relative to frame $\{0\}$, $\mathcal{S'}_3$ and $\mathcal{S'}_4$ are screws relative to frame $\{c\}$, and finally $\mathcal{S'}_5$ and $\mathcal{S'}_6$ are screws relative to frame $\{t\}$:

$$\begin{aligned} \mathcal{S}_1 :& \quad \omega = (0,0,1),\ q = (L,0,0),\ v = (0,-L,0) \\ \mathcal{S}_2 :& \quad \omega = (0,1,0),\ q = (L,L,-L),\ v = (L,0,L) \\ \mathcal{S'}_3 :& \quad \omega = (0,0,0),\ v = (1,0,0), \\ \mathcal{S'}_4 :& \quad \omega = (1,0,0),\ q = (0,L,0),\ v = (0,0,-L) \\ \mathcal{S'}_5 :& \quad \omega = (0,1,0),\ q = (-2L,0,-L),\ v = (L,0,-2L) \\ \mathcal{S'}_6 :& \quad \omega = (0,0,-1),\ q = (-L,0,-L),\ v = (0,-L,0) \end{aligned}$$

Also, $M_{0c} = \begin{bmatrix} 1 & 0 & 0 & 2L \\ 0 & 1 & 0 & -L \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $M_{ct} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Go to the table of contents.

(b) The space Jacobian when $\theta_2 = 90°$ and all the other joint variables are set to zero is given by

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & L & 0 & 0 & 0 & 2L \\ -L & 0 & 0 & L & -3L & L \\ 0 & L & -1 & 0 & L & 0 \end{bmatrix}.$$

Therefore the spatial velocity $\mathcal{V}_s$ is

$$\mathcal{V}_s = J_s(\theta) \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 2L & 0 & -1 \end{bmatrix}^T.$$

(c) First, intuitively it is clear that if $\theta_2 = 90°$, the axes of joint 1 and joint 4 are colinear. For six degree of freedom open chains, two collinear revolute joint axes correspond to a kinematic singularity. We now show this more rigorously. From part (c), the space Jacobian when $\theta_2 = 90°$ and all the other joint variables are at zero is

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & L & 0 & 0 & 0 & 2L \\ -L & 0 & 0 & L & -3L & L \\ 0 & L & -1 & 0 & L & 0 \end{bmatrix}.$$

From above, if we denote the first column by J_1 and the fourth column by J_4, we can write $J_1 = -J_4$. The rank of J_S must therefore be less than six. It follows that this configuration is a kinematic singularity.

(d) Denote the i-th column of J_s as J_i, $i = 1, \ldots, 6$. The spatial forces that the robot is generating are $(-\mathcal{F}_{elbow})$ and $(-\mathcal{F}_{tip})$, respectively (action and reaction). Joint torques τ_5 and τ_6 are affected only by $\mathcal{F}_{tip}$:

$$\begin{aligned} \tau_5 &= -J_5^T \mathcal{F}_{tip} = 3L \\ \tau_6 &= -J_6^T \mathcal{F}_{tip} = -3L \end{aligned}$$

Joint torques τ_1, τ_2, τ_3 and τ_4 are affected by $\mathcal{F}_{elbow}$ and $\mathcal{F}_{tip}$:

$$\begin{aligned} \tau_1 &= -J_1^T(\mathcal{F}_{elbow} + \mathcal{F}_{tip}) = L \\ \tau_2 &= -J_2^T(\mathcal{F}_{elbow} + \mathcal{F}_{tip}) = -1 - 2L \\ \tau_3 &= -J_3^T(\mathcal{F}_{elbow} + \mathcal{F}_{tip}) = 1 \\ \tau_4 &= -J_4^T(\mathcal{F}_{elbow} + \mathcal{F}_{tip}) = -L. \end{aligned}$$

Exercise 5.22.

(a) For $\mathcal{V}_2$,

$$\begin{aligned} T_{02} &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} M_2 \\ \dot{T}_{02} &= [\mathcal{S}_1] e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} M_2 \dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2] e^{[\mathcal{S}_2]\theta_2} M_2 \dot{\theta}_2 \\ T_{02}^{-1} &= M_2^{-1} e^{-[\mathcal{S}_2]\theta_2} e^{-[\mathcal{S}_1]\theta_1} \end{aligned}$$

$$\begin{aligned} [\mathcal{V}_2] &= \dot{T}_{02} T_{02}^{-1} \\ &= [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2] e^{-[\mathcal{S}_1]\theta_1} \dot{\theta}_2 \\ \mathcal{V}_2 &= \mathcal{S}_1 \dot{\theta}_1 + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)\dot{\theta}_2 \end{aligned}$$

Go to the table of contents.

For $\mathcal{V}_3$,

$$\begin{aligned}
T_{03} &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M_3 \\
\dot{T}_{03} &= [\mathcal{S}_1] e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M_3 \dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2] e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M_3 \dot{\theta}_2 + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} [\mathcal{S}_3] e^{[\mathcal{S}_3]\theta_3} M_3 \dot{\theta}_3 \\
T_{03}^{-1} &= M_3^{-1} e^{-[\mathcal{S}_3]\theta_3} e^{-[\mathcal{S}_2]\theta_2} e^{-[\mathcal{S}_1]\theta_1}
\end{aligned}$$

$$\begin{aligned}
[\mathcal{V}_3] &= \dot{T}_{03} T_{03}^{-1} \\
&= [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2] e^{-[\mathcal{S}_1]\theta_1} \dot{\theta}_2 + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} [\mathcal{S}_3] e^{-[\mathcal{S}_2]\theta_2} e^{-[\mathcal{S}_1]\theta_1} \dot{\theta}_3 \\
\mathcal{V}_3 &= \mathcal{S}_1 \dot{\theta}_1 + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)\dot{\theta}_2 + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)\dot{\theta}_3
\end{aligned}$$

(b) Based on the result obtained in (a), we can construct a recursive formula for $\mathcal{V}_{k+1}$ as follows:

$$\begin{aligned}
\mathcal{V}_{k+1} &= \mathcal{S}_1 \dot{\theta}_1 + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)\dot{\theta}_2 + \cdots + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_k]\theta_k}}(\mathcal{S}_{k+1})\dot{\theta}_{k+1} \\
&= \mathcal{V}_k + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_k]\theta_k}}(\mathcal{S}_{k+1})\dot{\theta}_{k+1} \\
&= \begin{bmatrix} \mathcal{V}_{s1}(\theta) & \mathcal{V}_{s2}(\theta) & \cdots & \mathcal{V}_{s,k+1}(\theta) \end{bmatrix} \dot{\theta}
\end{aligned}$$

Exercise 5.23.

(a) The plot of the arm and its manipulability ellipse is shown in Figure 5.3. Based on the ratio between

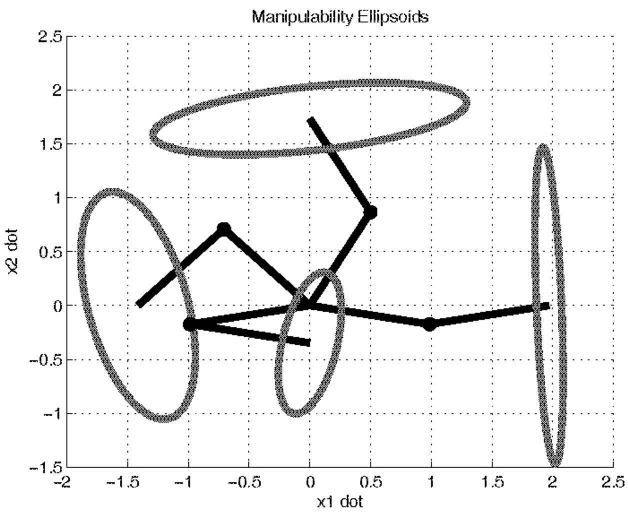

Figure 5.3

the longest and shortest semi-axes of the manipulability ellipsoid,

$$\mu_1(A) = \frac{\sqrt{\lambda_{\max}(A)}}{\sqrt{\lambda_{\min}(A)}} = \sqrt{\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}} \geq 1,$$

where $A = JJ^T$. When $\mu_1(A)$ is low, close to one, then the manipulability ellipsoid is nearly spherical or isotropic. At the configuration $(135°, 90°)$, $\mu_1(A) = 2.618$ comes to be the lowest. So the arm appears most isotropic at the configuration $(135°, 90°)$.

(b) The eccentricity of the ellipse depends only on θ_2. θ_1 only effects on the orientation and position of the ellipse. The effect caused by the first joint is related with the distance between its rotation axis and the end-effector, whose magnitude is determined by θ_2. This conclusion can also be drawn from planar body Jacobian shown below, which doesn't include θ_1.

$$J_b(\theta) = \begin{bmatrix} 1 & 1 \\ L_1 \sin\theta_2 & 0 \\ L_2 + L_1 \cos\theta_2 & L_2 \end{bmatrix}.$$

(c) The drawing is shown in Figure 5.4.

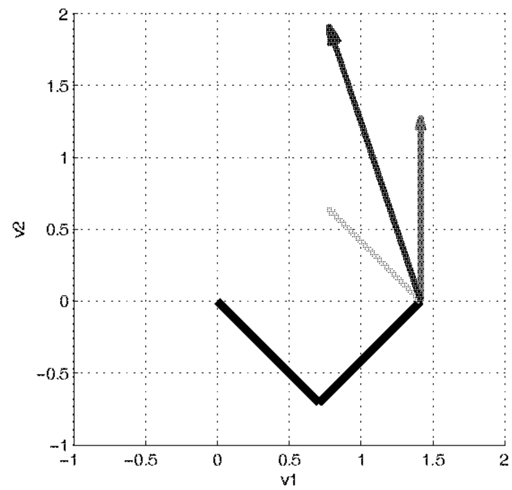

Figure 5.4

Exercise 5.24.
The plot is shown in Figure 5.5.

Exercise 5.25.

(a) Using the software of the textbook and the kinematics of the 6R UR5, the space Jacobian J_s when all joint angles are $\pi/2$ can be obtained as

$$J_s = \begin{bmatrix} 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.336 & -0.297 \\ 0 & -0.089 & 0.336 & 0.336 & 0 & 0.109 \\ 0 & 0 & 0 & -0.392 & -0.109 & 0 \end{bmatrix},$$

which can be separated as

$$J_\omega = \begin{bmatrix} 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$J_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.336 & -0.297 \\ 0 & -0.089 & 0.336 & 0.336 & 0 & 0.109 \\ 0 & 0 & 0 & -0.392 & -0.109 & 0 \end{bmatrix}.$$

(b) Let

$$A_\omega = J_\omega J_\omega^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Go to the table of contents.

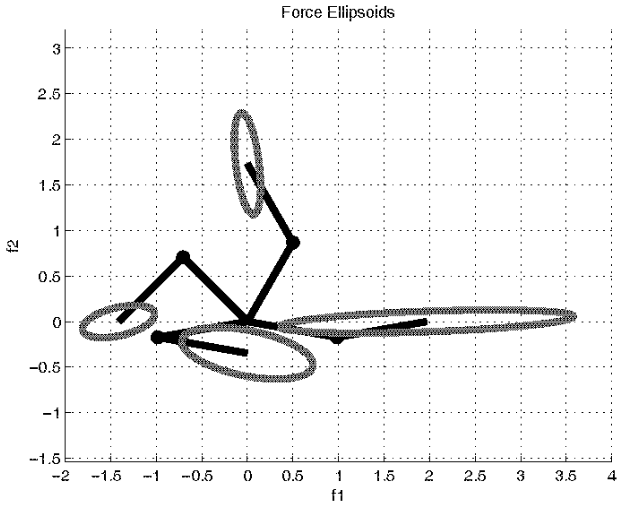

Figure 5.5

The lengths of the principal semi-axes of the angular-velocity manipulability ellipsoid equal to the square roots of its eigenvalues. The eigenvalues are 3, 1, and 2, so the lengths are 1.7321, 1, and 1.4142.
The directions of the principal semi-axes are aligned with its eigenvectors, which are $(1,0,0)^T$, $(0,1,0)^T$, and $(0,0,1)^T$.
Let

$$A_v = J_v J_v^T = \begin{bmatrix} 0.2011 & -0.0324 & -0.0366 \\ -0.0324 & 0.2456 & -0.1317 \\ -0.0366 & -0.1317 & 0.1655 \end{bmatrix}.$$

The lengths of the principal semi-axes of the linear-velocity manipulability ellipsoid equal to the square roots of its eigenvalues, The eigenvalues are 0.0520, 0.2169, and 0.3434, so the lengths are 0.2280, 0.4657, and 0.5860.
The directions of the principal semi-axes are aligned with its eigenvectors, which are
$(0.3106, 0.5696, 0.7609)^T$, $(0.9500, -0.1596, -0.2683)^T$, and $(-0.0314, 0.8062, -0.5907)^T$.

(c) The moment ellipsoid can be obtained by stretching the angular-velocity manipulability ellipsoid along each principal axis i by a factor $1/\lambda_i$, where λ_i is its corresponding eigenvalue. So for current moment ellipsoid, the lengths of the principal semi-axes are 0.5774, 1, and 0.7071. The directions are $(1,0,0)^T$, $(0,1,0)^T$, and $(0,0,1)^T$.
The force ellipsoids can be obtained by stretching the linear-velocity manipulability ellipsoid along each principal axis i by a factor $1/\lambda_i$, where λ_i is its corresponding eigenvalue. So for current force ellipsoid, the lengths of the principal semi-axes are 4.3854, 2.1473, and 1.7066. The directions are $(0.3106, 0.5696, 0.7609)^T$, $(0.9500, -0.1596, -0.2683)^T$, and $(-0.0314, 0.8062, -0.5907)^T$.

Exercise 5.26.

(a) Using the software of the textbook and the kinematics of the 7R WAM, the space Jacobian J_b when all joint angles are $\pi/2$ can be obtained as

$$J_b = \begin{bmatrix} 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -0.105 & 0 & 0.006 & -0.045 & 0 & 0.006 & 0 \\ -0.889 & 0.006 & 0 & -0.844 & 0.006 & 0 & 0 \\ 0 & -0.105 & 0.889 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which can be separated as

$$J_\omega = \begin{bmatrix} 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

and

$$J_v = \begin{bmatrix} -0.105 & 0 & 0.006 & -0.045 & 0 & 0.006 & 0 \\ -0.889 & 0.006 & 0 & -0.844 & 0.006 & 0 & 0 \\ 0 & -0.105 & 0.889 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Let

$$A_\omega = J_\omega J_\omega^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The lengths of the principal semi-axes of the angular-velocity manipulability ellipsoid equal to the square roots of its eigenvalues. The eigenvalues are 2, 2, and 3, so the lengths are 1.4142, 1.4142, and 1.7321.
The directions of the principal semi-axes are aligned with its eigenvectors, which are $(1,0,0)^T$, $(0,1,0)^T$, and $(0,0,1)^T$.
Let

$$A_v = J_v J_v^T = \begin{bmatrix} 0.0131 & 0.1313 & 0.0053 \\ 0.1313 & 1.5027 & -0.0006 \\ 0.0053 & -0.0006 & 0.8013 \end{bmatrix}.$$

The lengths of the principal semi-axes of the linear-velocity manipulability ellipsoid equal to the square roots of its eigenvalues, The eigenvalues are 0.0016, 0.8014, and 1.5142, so the lengths are 0.0400, 0.8952, and 1.2305.
The directions of the principal semi-axes are aligned with its eigenvectors, which are $(-0.9962, 0.0872, 0.0067)^T$, $(0.0067, -0.0004, 1.0000)^T$, and $(-0.0872, -0.9962, 0.0002)^T$.

(c) The moment ellipsoid can be obtained by stretching the angular-velocity manipulability ellipsoid along each principal axis i by a factor $1/\lambda_i$, where λ_i is its corresponding eigenvalue. So for current moment ellipsoid, the lengths of the principal semi-axes are 0.7071, 0.7071, and 0.5774. The directions are $(1,0,0)^T$, $(0,1,0)^T$, and $(0,0,1)^T$.
The force ellipsoids can be obtained by stretching the linear-velocity manipulability ellipsoid along each principal axis i by a factor $1/\lambda_i$, where λ_i is its corresponding eigenvalue. So for current force ellipsoid, the lengths of the principal semi-axes are 25.0246, 1.1171, and 0.8127. The directions are $(-0.9962, 0.0872, 0.0067)^T$, $(0.0067, -0.0004, 1.0000)^T$, and $(-0.0872, -0.9962, 0.0002)^T$.

Exercise 5.27.
Programming assignment.