Adaptive Composite AMG Solvers

Utilizing Graph Modularity Matching

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Overview

Problem Statement

$$Ax = b$$

A is s.p.d.

Goal: Adaptively construct a 'realistic' stationary linear iteration with arbitrary convergence factor...

Adaptive AMG is not new...

- A. Brandt, *Multi-level adaptive solutions to boundary-value problems*, Mathematics of Computation **31** (1977), 333–390
- A Brandt, *Multiscale scientific computation: Review 2001*, Multiscale and Multiresolution Methods: Theory and Applications, T. J. Barth, T. F. Chan, and R. Haimes, eds., Springer, Heidelberg (2001), 1–96
- A. Brandt, J. Brannick, K. Kahl, and I. Livshits, *Bootstrap AMG*, SIAM Journal on Scientific Computing **33-2** (2011), 612–632
- M. Brezina, R. Falgout, S. MacLachlan, T. Manteuffel, S. McCormick, and J. Ruge, *Adaptive algebraic multigrid*, SIAM Journal on Scientific Computing **27-4** (2006), 1261–1286
- P. D'Ambra and P.S. Vassilevski, *Adaptive AMG with coarsening based on compatible weighted matching.*, Computing and Visualization in Science **16** (2013), 59–76

Adaptivity Algorithm

$$I - B^{-1}A = (I - B_1^{-T}A)(I - B_0^{-1}A)(I - B_1^{-1}A). \qquad (1)$$

Data: Matrix A, desired convergence factor ρ, max components m, smoother type B

Result: Adaptive Solver $\overline{B}$

```
B̄ ← CreateSolver(B, A)
i, cf ← 1
while ρ < cf and i < m do
    w, cf ← TestHomogeneous(A, B̄)
    w = w/‖w‖₂
    B_new ← AdaptiveMLSolver(B, A, w)
    B̄ ← SymmetricComposition(B̄, B_new) (1)
    i ← i + 1
```

Algebraically Smooth Error is Near-nullspace of A

$$Ax = 0, \text{ gives } \quad B(\mathbf{x}_k - \mathbf{x}_{k-1}) = -A\mathbf{x}_{k-1} \tag{2}$$

Theorem

Let B define an s.p.d. A-convergent iterative method such that $\frac{\mathbf{v}^T A\mathbf{v}}{\mathbf{v}^T B\mathbf{v}} < 1$ and $\|B\| \simeq \|A\|$, i.e., $\|B\| \le c_0 \|A\|$ for a constant $c_0 \ge 1$. Consider any vector $\mathbf{w}$ such that the iteration process (2) with B stalls for it, i.e.,

$$1 \ge \frac{\|(I - B^{-1}A)\mathbf{w}\|_A^2}{\|\mathbf{w}\|_A^2} \ge 1 - \delta, \tag{3}$$

for some small $\delta \in (0, 1)$. Then, the following estimate holds $\|A\mathbf{w}\|^2 \le c_0 \|A\| \; \delta \|\mathbf{w}\|_A^2$.

Strength of Connectivity Graph

Since $A\mathbf{w} \approx 0$ component-wise by construction, we have for each i

$$0 \approx w_i \sum_j a_{ij} w_j,$$

or equivalently

$$0 \leq a_{ii} w_i^2 \approx \sum_{j \neq i} (-w_i a_{ij} w_j).$$

Then, $\overline{A} = (\overline{a}_{ij})$ with non-zero entries $\overline{a}_{ij} = -w_i a_{ij} w_j$, $(i \neq j)$ has positive row-sums.

$\overline{A}$ is the sparse adjacency matrix associated with the connectivity strength graph G.

Modularity Matching (Coarsening) for AMG Hierarchy

Let $\mathbf{1} = (1) \in \mathbb{R}^n$ be the unity constant vector, $\mathbf{r} = A\mathbf{1}$, and $T = \sum_i r_i = \mathbf{1}^T A \mathbf{1}$.

The *Modularity Matrix* [New10]

$$B = A - \frac{1}{T}\mathbf{r}\mathbf{r}^T.$$

By construction, we have that

$$B\mathbf{1} = 0. \tag{4}$$

The *Modularity Functional*

$$Q = \frac{1}{T}\sum_{\mathcal{A}}\sum_{i,\,j\in\mathcal{A}} b_{ij} = \frac{1}{T}\sum_{\mathcal{A}}\sum_{i,\,j\in\mathcal{A}}\left(a_{ij} - \frac{r_i r_j}{T}\right).$$

We solve this optimization problem with parallel greedy matching algorithms: [QV19], [JP93]

Two Interpolation Methods

We design interpolants which reconstruct the relaxed error $\mathbf{w}$ used to build the coarse problem.
This results in restriction operators which preserve these components of the residual on the coarse grid.

- Piecewise Constant Interpolation
  - $P_{ij} = \mathbf{w}_i$ when DOF i is assigned coarse DOF j from the matching algorithm
  - Clearly, $P\mathbf{1}_c = \mathbf{w}$
- Modified "Classical" AMG Interpolation
  A subset of each aggregate $\mathcal{A}$ is selected as the coarse vertices and the interpolation weights are chosen such that

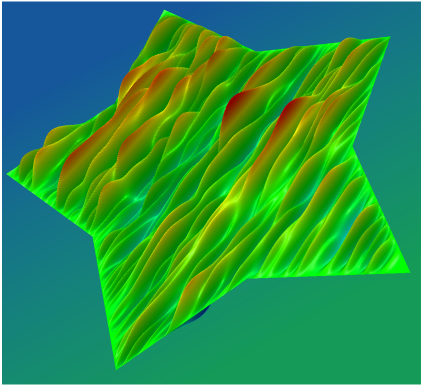
$$P\mathbf{w}_c = \mathbf{w}$$

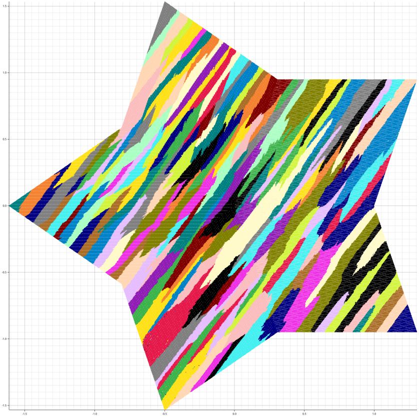
  where $\mathbf{w}_c$ is the restriction of $\mathbf{w}$ to those coarse vertices.

Near Null and Hierarchy Visualization for 2d Anisotropy

Let $\Omega \subset \mathbb{R}^2$ and $\varepsilon = 10^{-3}$.

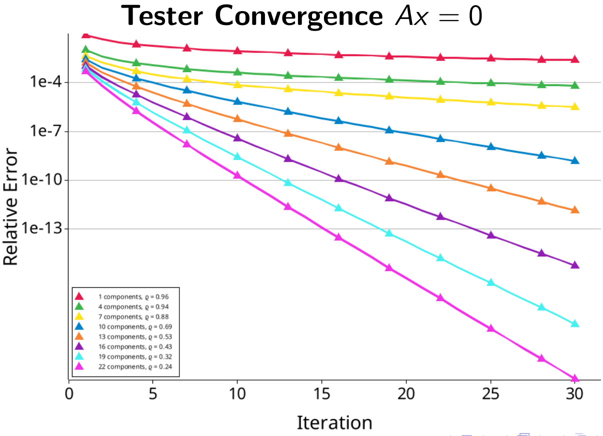
$$\begin{cases} -\nabla \cdot [(\varepsilon I + \boldsymbol{b}\boldsymbol{b}^T)\nabla u] = f & , \text{ on } \Omega \\ u = 0 & , \text{ on } \partial\Omega \end{cases}, \quad \boldsymbol{b} := \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



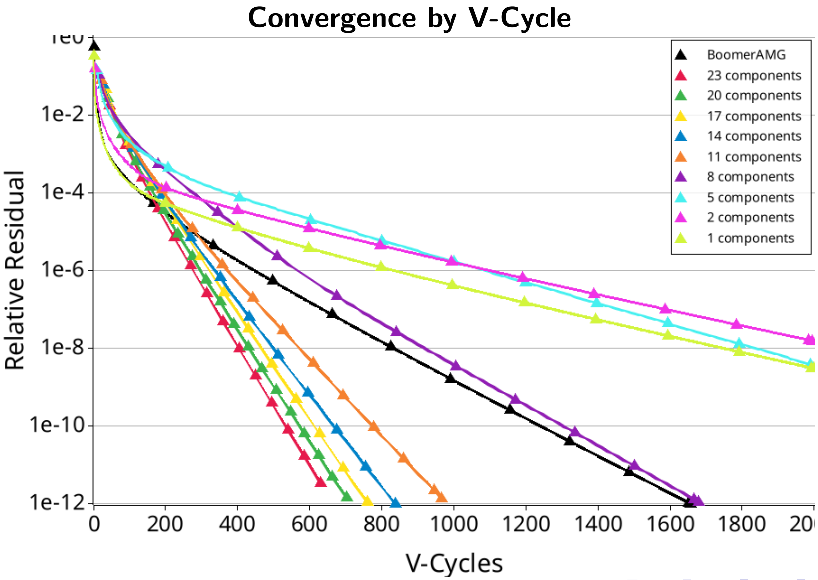


Anisotropic Convergence Experiment

- Matrix details: 326,401 rows, 2,929,951 nnz
- BoomerAMG operator complexity: 1.289138
- Our (Classical) operator complexity: ≈ 1.15 per cycle
- 5 ℓ_1-scaled Jacobi Smoothing steps each level (except coarsest)



Anisotropic Convergence Experiment

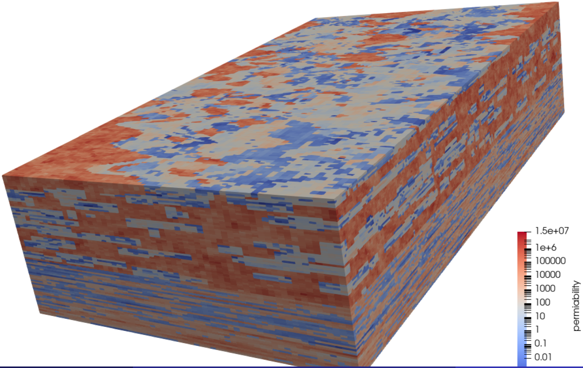


Heterogeneous Coefficients (SPE10)

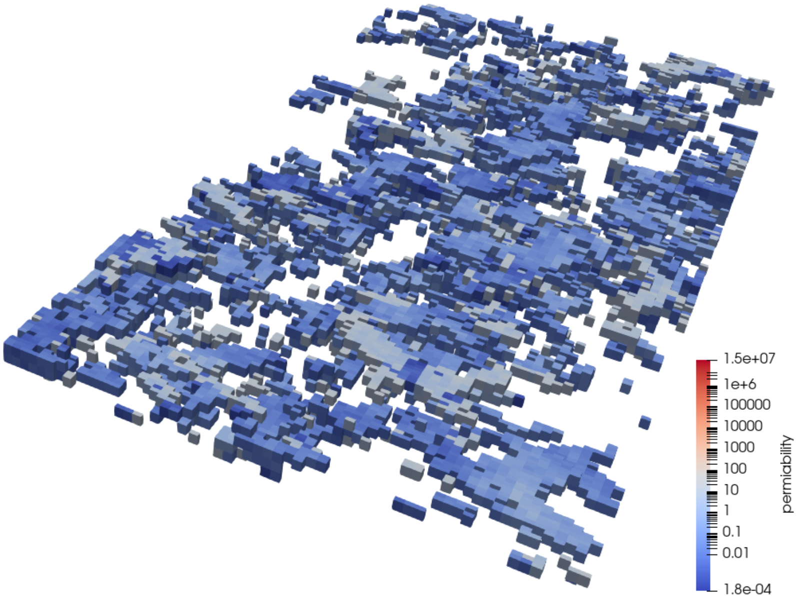
$$\begin{cases} -\nabla \cdot (\beta \nabla u) = f & , \text{ on } \Omega \\ u = 0 & , \text{ on } \partial\Omega \end{cases}$$

In this case, β (called the permeability) is a piecewise constant diagonal matrix coefficient (constant on each element).

$\|\beta\|_2$ **on each element**



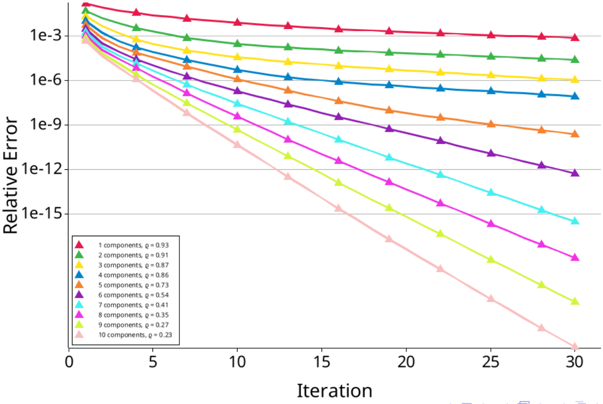
SPE10 Clipped Cross Section (High Permeability)



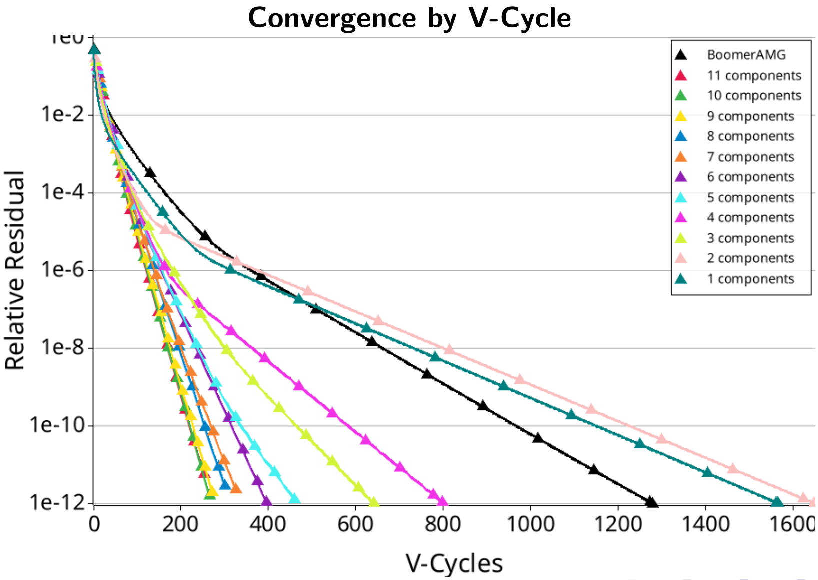
SPE10 Convergence Experiment

- Matrix details: 1,159,366 rows, 30,628,096 nnz
- BoomerAMG operator complexity: 1.115288
- Our (Classical) operator complexity: ≈ 1.45 per cycle
- 5 ℓ_1-scaled Jacobi Smoothing steps each level (except coarsest)

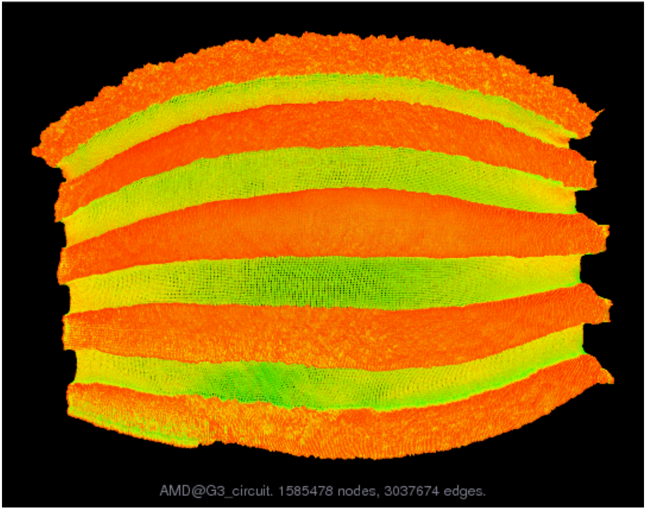
Tester Convergence $Ax = 0$



SPE10 Convergence Experiment

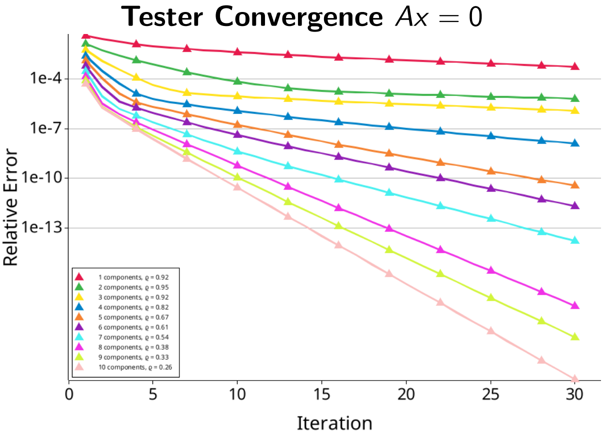


G3 Circuit Simulation Matrix from AMD

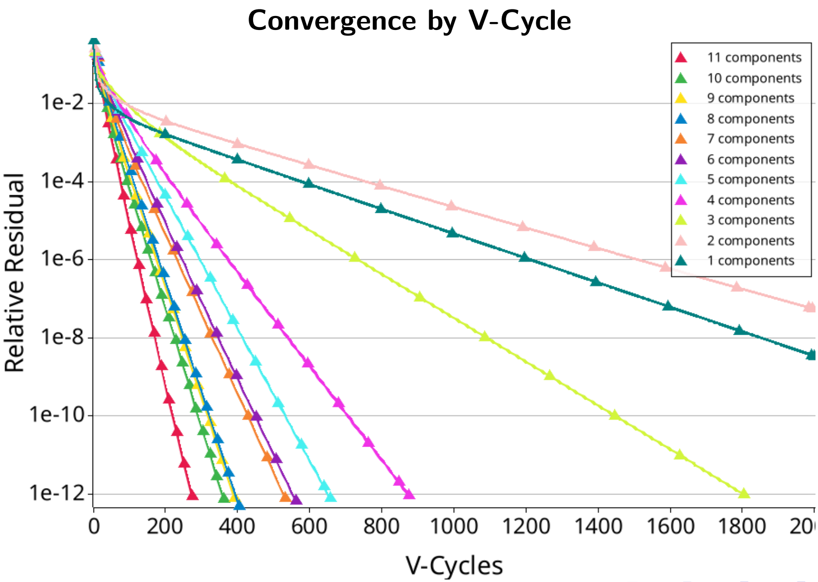


G3 Circuit (AMD) Convergence Experiment

- Matrix details: 1,585,478 rows, 7,660,826 nnz
- Our (Classical) operator complexity: ≈ 1.8 per cycle
- 5 ℓ_1-scaled Jacobi Smoothing steps each level (except coarsest)



G3 Circuit (AMD) Convergence Experiment



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References II

- P. D'Ambra and P.S. Vassilevski, *Adaptive AMG with coarsening based on compatible weighted matching.*, Computing and Visualization in Science **16** (2013), 59–76.
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- M.E.J. Newman, *Networks. an introduction*, Oxford University Press, New York, 2010.
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