Summations

Auriza Akbar*

2017

Abstract

When an algorithm contains an iterative control construct such as a **while** or **for** loop, we can express its running time as the sum of the times spent on each execution of the body of the loop. For example, we found that the jth iteration of insertion sort took time proportional to j in the worst case. By adding up the time spent on each iteration, we obtained the summation (or series)

$$\sum_{j=2}^{n} j$$

When we evaluated this summation, we attained a bound of $\Theta(n^2)$ on the worst-case running time of the algorithm. This example illustrates why you should know how to manipulate and bound summations.

1 Summation Formulas

Given a sequence $a_1, a_2, \ldots, a_n$ of numbers, where n is a nonnegative integer, we can write the finite sum $a_1 + a_2 + \cdots + a_n$ as

$$\sum_{k=1}^{n} a_k$$

If $n = 0$, the value of the summation is defined to be 0. The value of a finite series is always well defined, and we can add its terms in any order.

Given an infinite sequence $a_1, a_2, \ldots$ of numbers, we can write the infinite sum $a_1 + a_2 + \cdots$ as

$$\sum_{k=1}^{\infty} a_k$$

which we interpret to mean

$$\lim_{n \to \infty} \sum_{k=1}^{n} a_k$$

*Taken from: Cormen *et al.* 2009. *Introduction to Algorithms.* pp 1145–1147.

If the limit does not exist, the series ***diverges***; otherwise, it ***converges***. The terms of a convergent series cannot always be added in any order. We can, however, rearrange the terms of an ***absolutely convergent series***, that is, a series $\sum_{k=1}^{\infty} a_k$ for which the series $\sum_{k=1}^{\infty} |a_k|$ also converges.

2 Summation Properties

2.1 Linearity

For any real number c and any finite sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\sum_{k=1}^{n}(ca_k + b_k) = c\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

The linearity property also applies to infinite convergent series.

We can exploit the linearity property to manipulate summations incorporating asymptotic notation. For example,

$$\sum_{k=1}^{n} \Theta(f(k)) = \Theta\left(\sum_{k=1}^{n} f(k)\right)$$

In this equation, the Θ-notation on the left-hand side applies to the variable k, but on the right-hand side, it applies to n. We can also apply such manipulations to infinite convergent series.

2.2 Arithmetic series

The summation

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n$$

is an ***arithmetic series*** and has the value

$$\begin{aligned}\sum_{k=1}^{n} k &= \frac{1}{2}n(n+1) \\ &= \Theta(n^2)\end{aligned} \tag{1}$$

2.3 Sum of squares and cubes

We have the following summations of squares and cubes:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} \tag{3}$$

2.4 Geometric series

For real $x \neq 1$, the summation

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

is a ***geometric*** or ***exponential series*** and has the value

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} \tag{4}$$

When the summation is infinite and $|x| < 1$, we have the infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} \tag{5}$$

2.5 Harmonic series

For positive integers n, the nth ***harmonic number*** is

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{1}{k} \\ &= \ln n + O(1) \end{aligned} \tag{6}$$