Axis-Angle Rotation and Quaternions

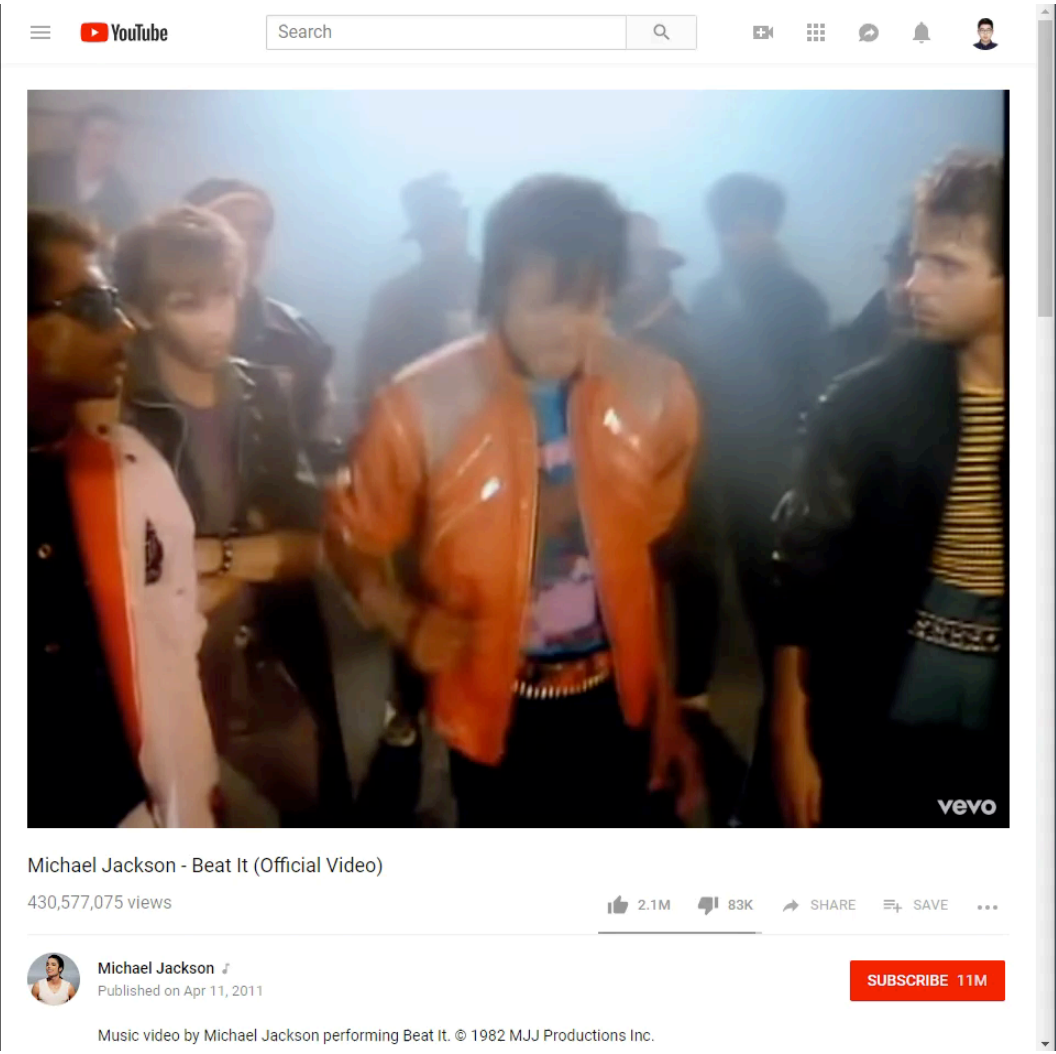
EECS 367
Intro. to Autonomous Robotics

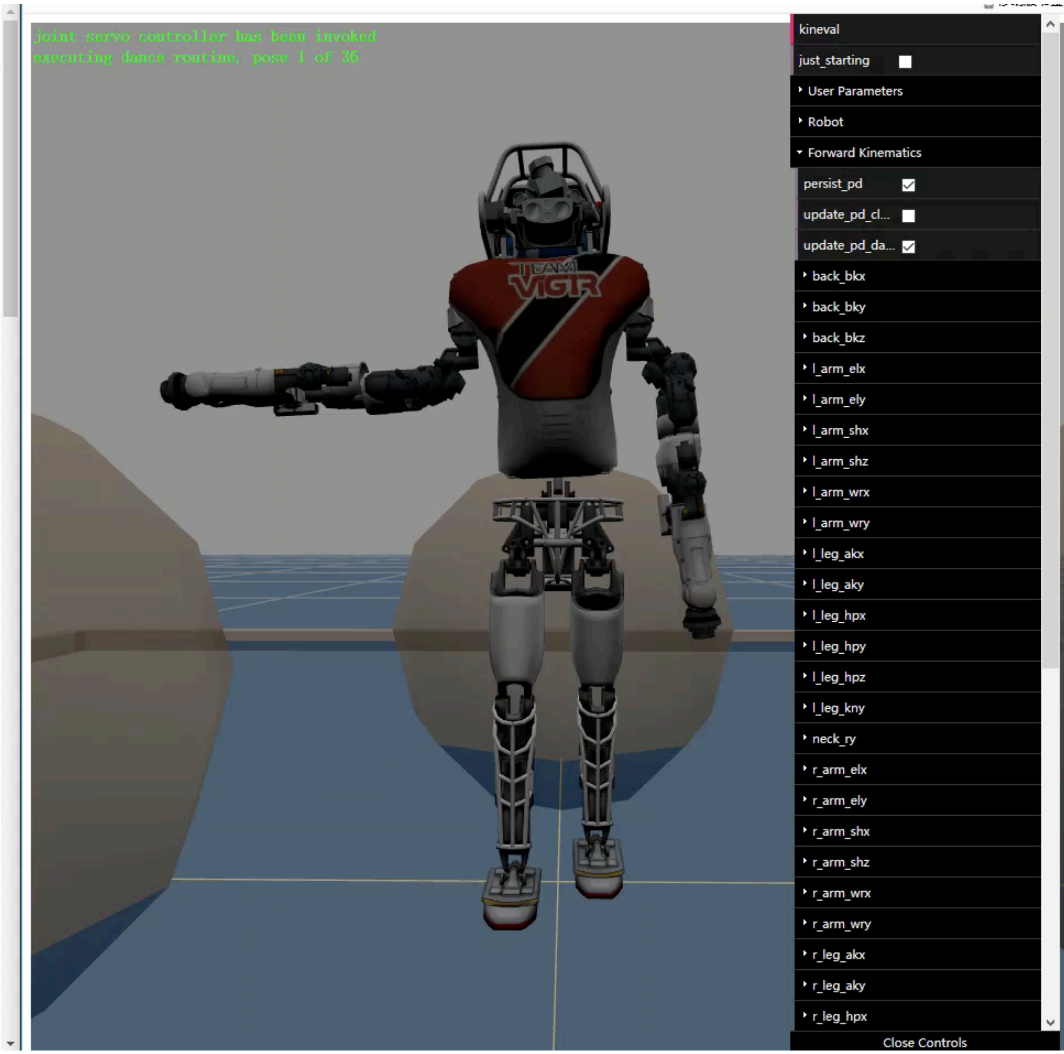
ROB 511
Robot Operating Systems

Fall 2020

autorob.org

FK Stylin’



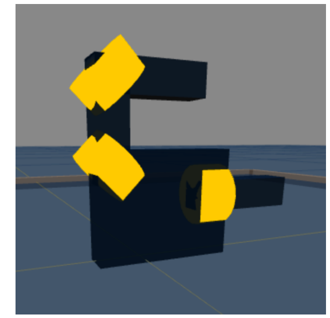


yeyangf Fall 2018

Can your robot do better?

(Assignment 4 is coming…)

Forward Kinematics



Infer: pose of each joint and link in a common world workspace

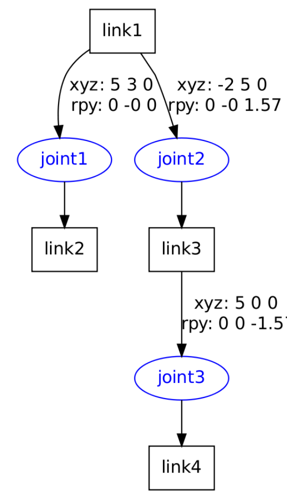
LECTURE 7 ☑ 1) How to represent homogeneous transforms?

☑ 2) How to compute transform to endeffector?

Assuming as given the:

- ☑ geometry of each link
- ☑ robot's kinematic definition

- **current state of all joints** THIS LECTURE
- reactive control for choreography NEXT LECTURE



Hierarchies of Transforms

```
<robot name="test_robot">
  <link name="link1" />
  <link name="link2" />
  <link name="link3" />
  <link name="link4" />

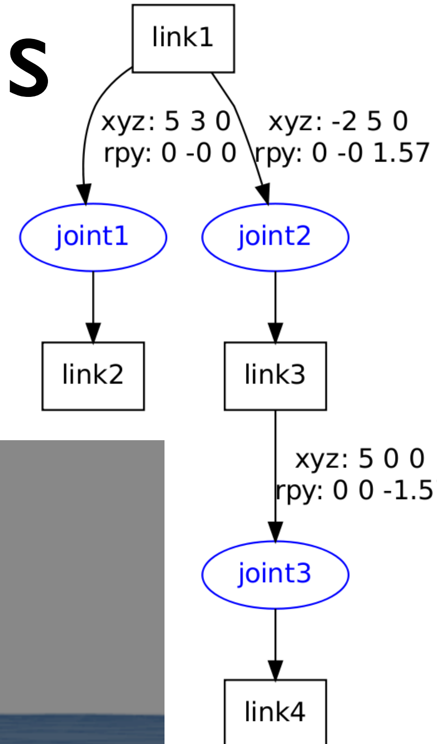
  <joint name="joint1" type="continuous">
    <parent link="link1"/>
    <child link="link2"/>
    <origin xyz="5 3 0" rpy="0 0 0" />
    <axis xyz="-0.9 0.15 0" />
  </joint>

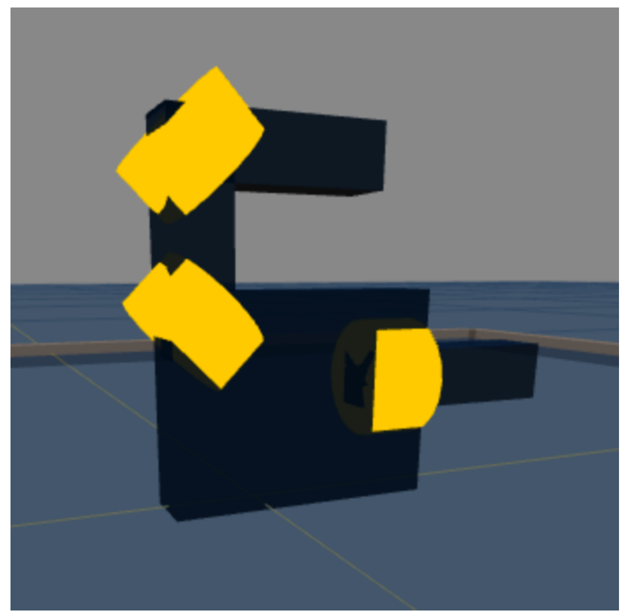
  <joint name="joint2" type="continuous">
    <parent link="link1"/>
    <child link="link3"/>
    <origin xyz="-2 5 0" rpy="0 0 1.57" />
    <axis xyz="-0.707 0.707 0" />
  </joint>

  <joint name="joint3" type="continuous">
    <parent link="link3"/>
    <child link="link4"/>
    <origin xyz="5 0 0" rpy="0 0 -1.57" />
    <axis xyz="0.707 -0.707 0" />
  </joint>
</robot>
```

URDF DEFINES KINEMATICS OF A ROBOT

INCLUDES AXIS FOR EACH JOINT





Hierarchies of Transforms

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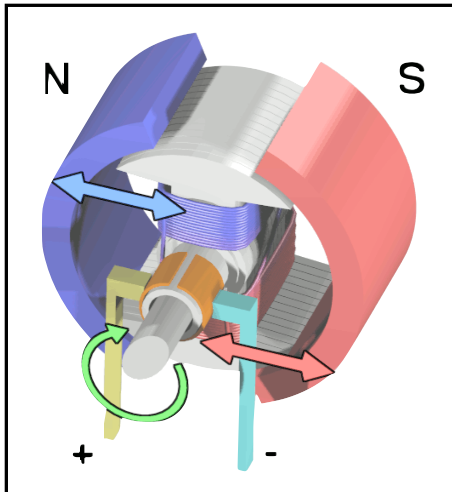
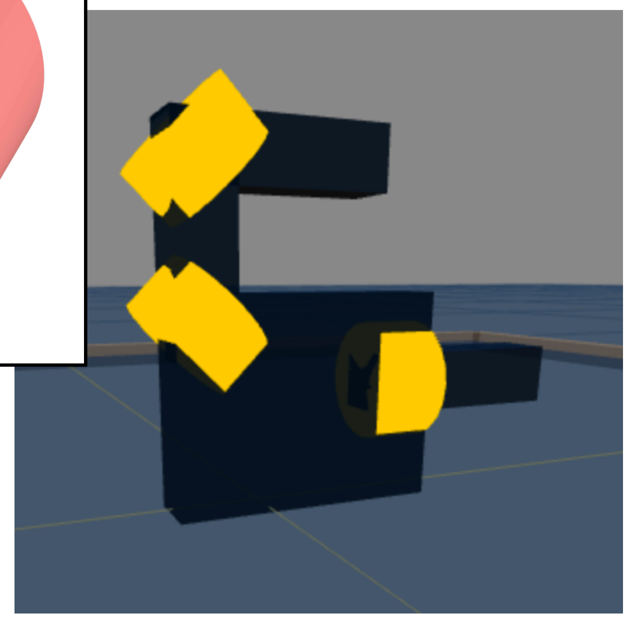
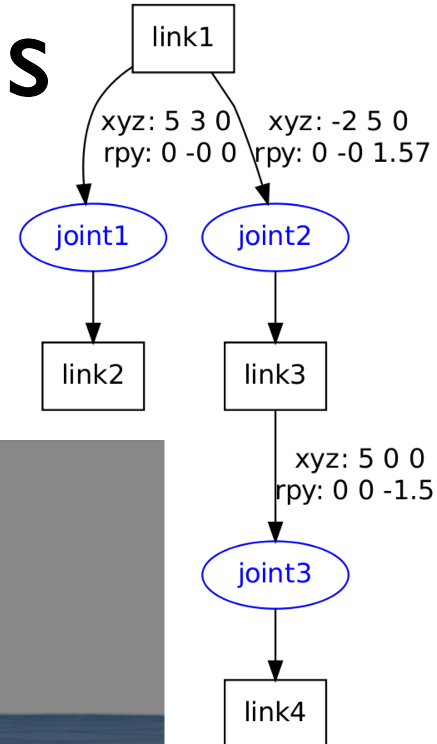
    <origin xyz="5 0 0" rpy="0 0 -1.57" />
    <axis xyz="0.707 -0.707 0" />
  </joint>
</robot>
```

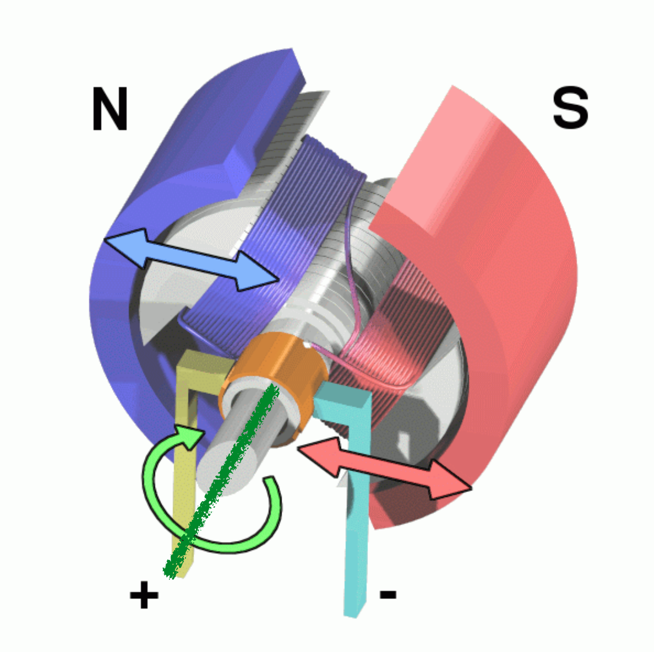
EACH AXIS IS A DOF THAT CAN BE MOVED BY A MOTOR

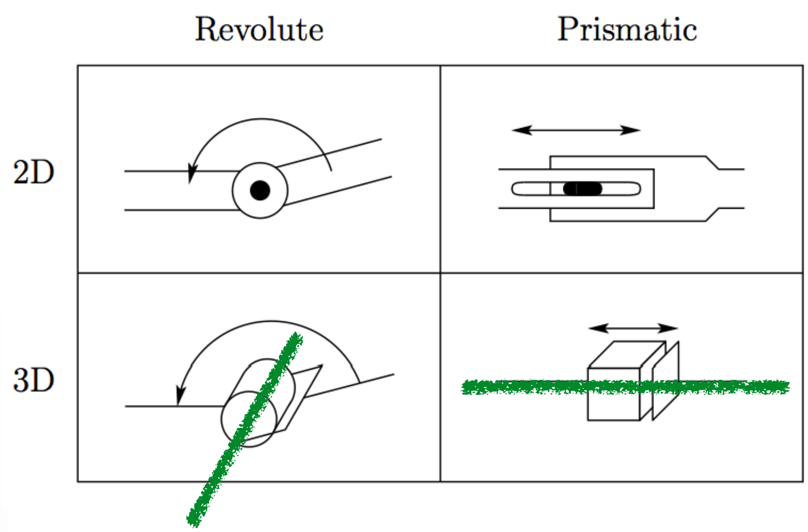






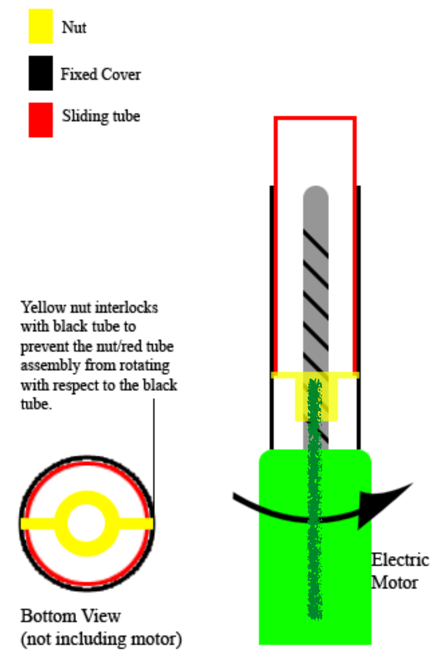
Brushed DC Motor

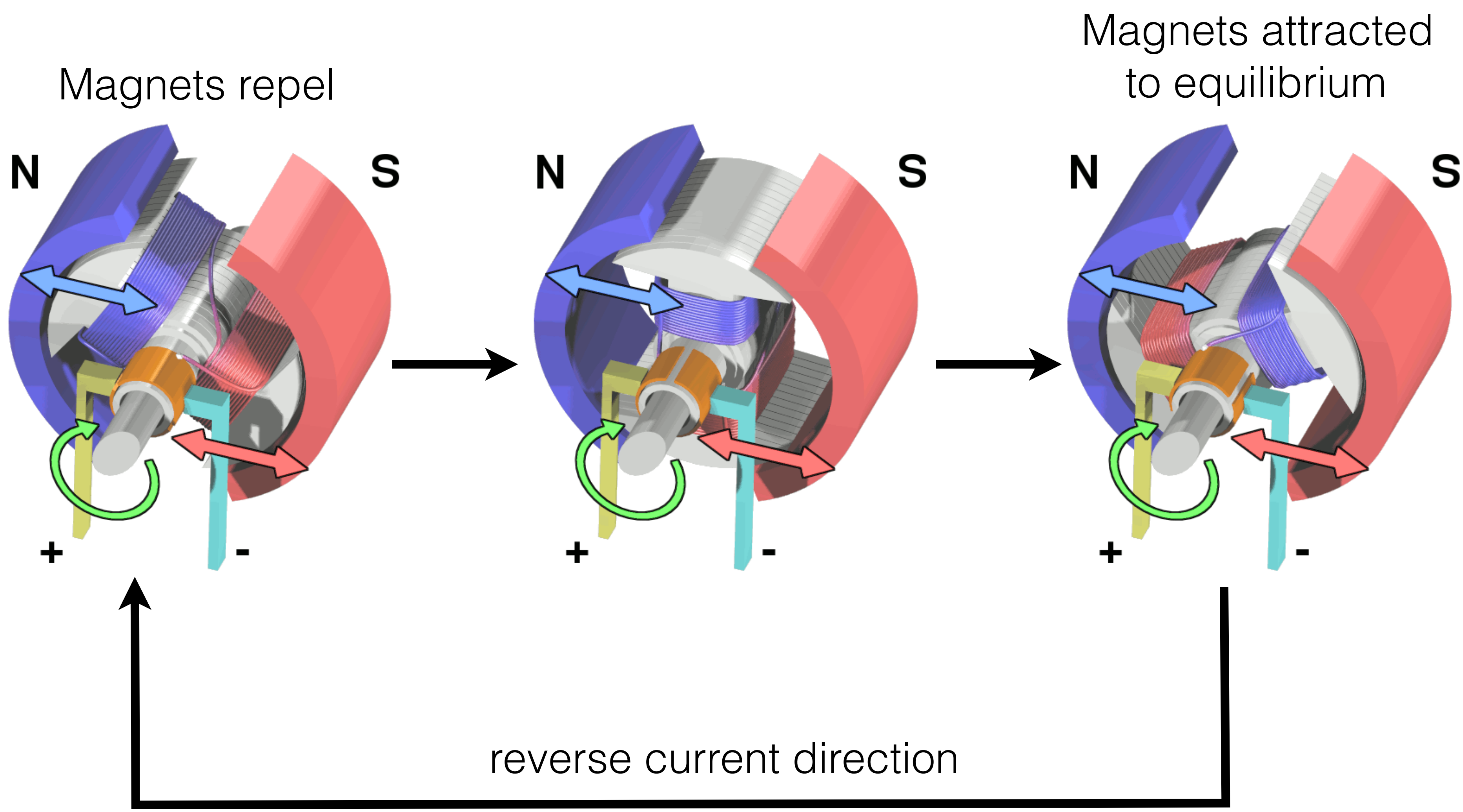




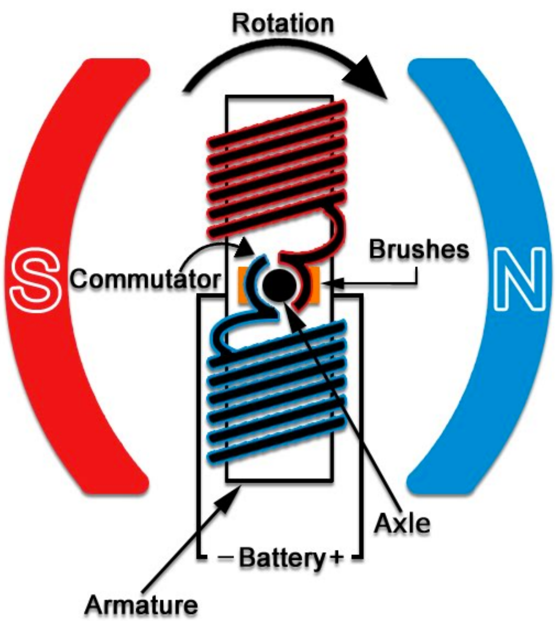
Motor axis in 3D could be placed in any direction

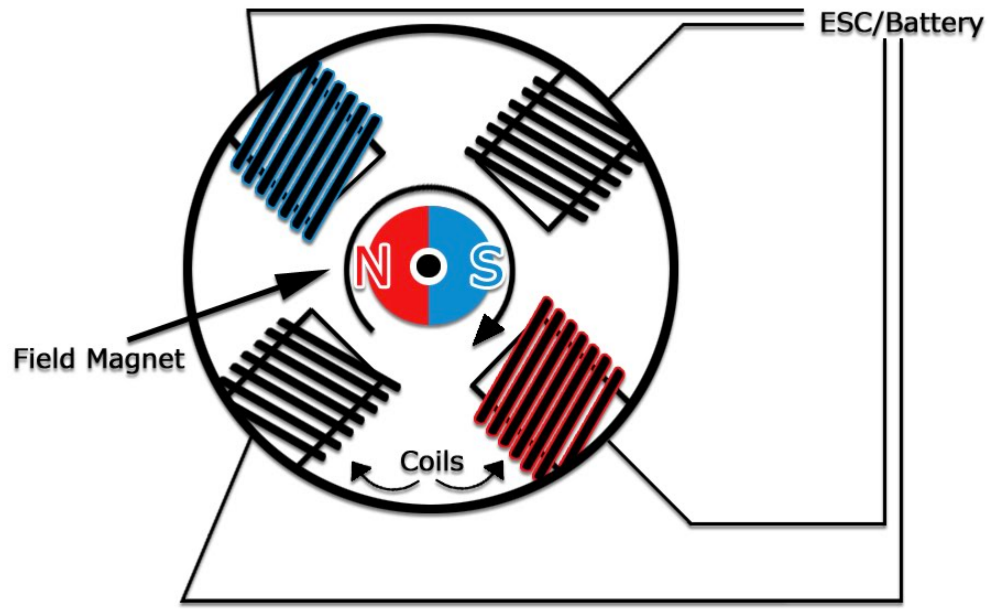
Linear actuator with rotational motor




Magnets repel
Magnets attracted to equilibrium
N
S
N
S
N
S
+
-
+
-
+
-
reverse current direction

Brushed DC Motor VS Brushless DC Motor





https://www.youtube.com/watch?v=RsqHr2cpp4M

How to include joint movement in matrix stack?

How to rotate about an axis?

```
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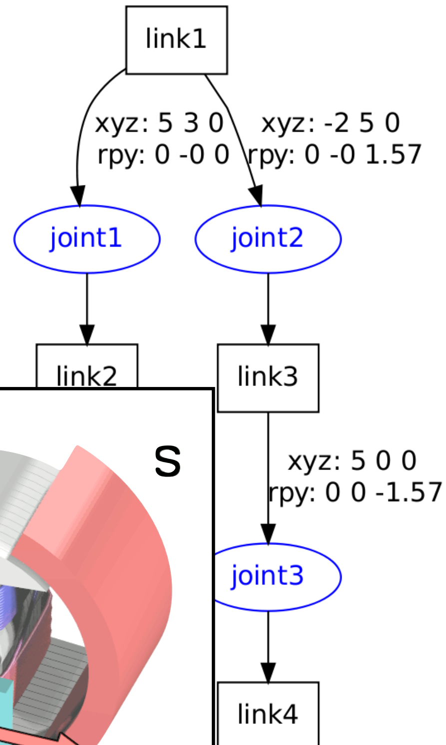
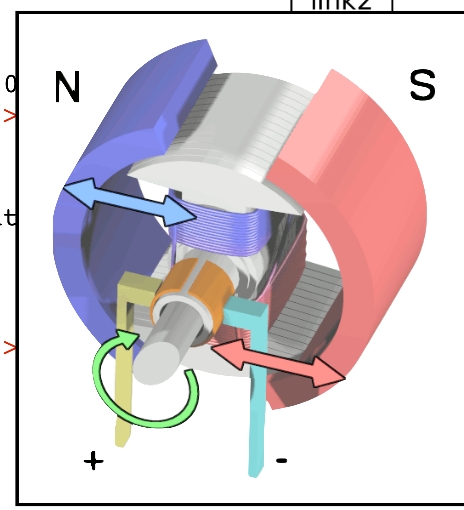
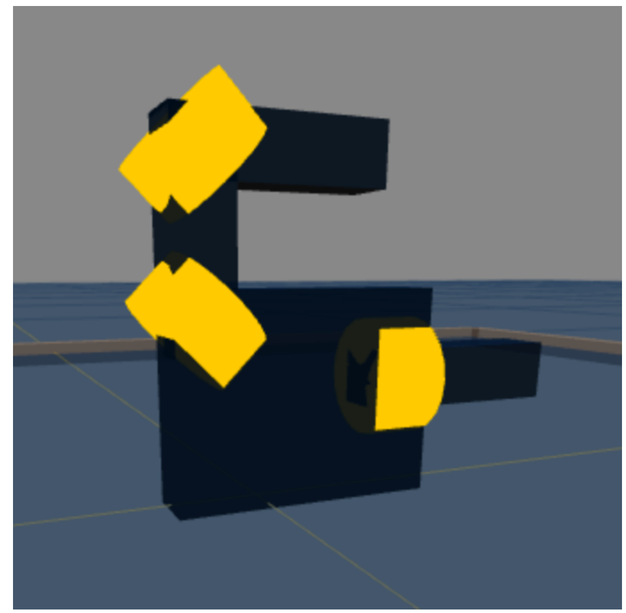
    <axis xyz="-0.707 0.707 0" />
  </joint>

  <joint name="joint3" type="cont
    <parent link="link3"/>
    <child link="link4"/>
    <origin xyz="5 0 0" rpy="0 0
    <axis xyz="0.707 -0.707 0" />
  </joint>
</robot>
```

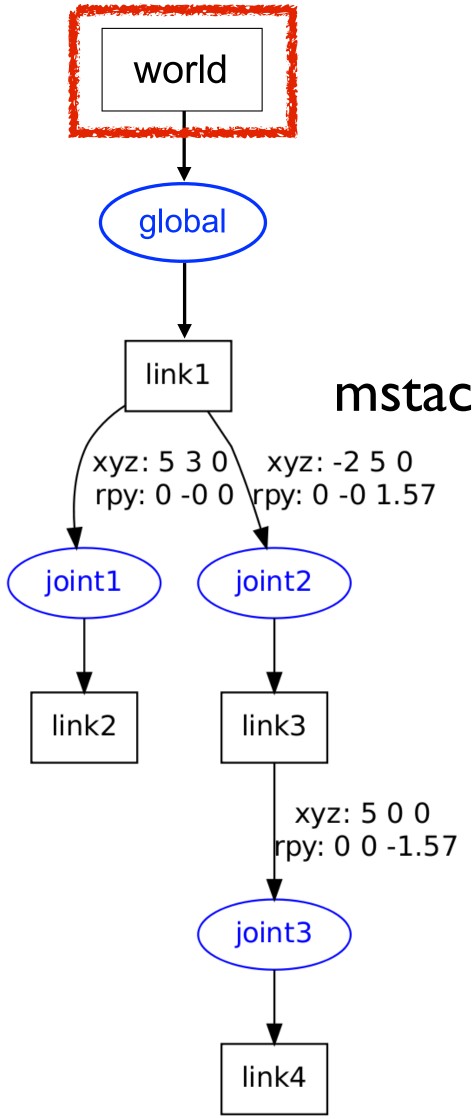




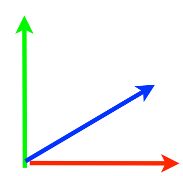


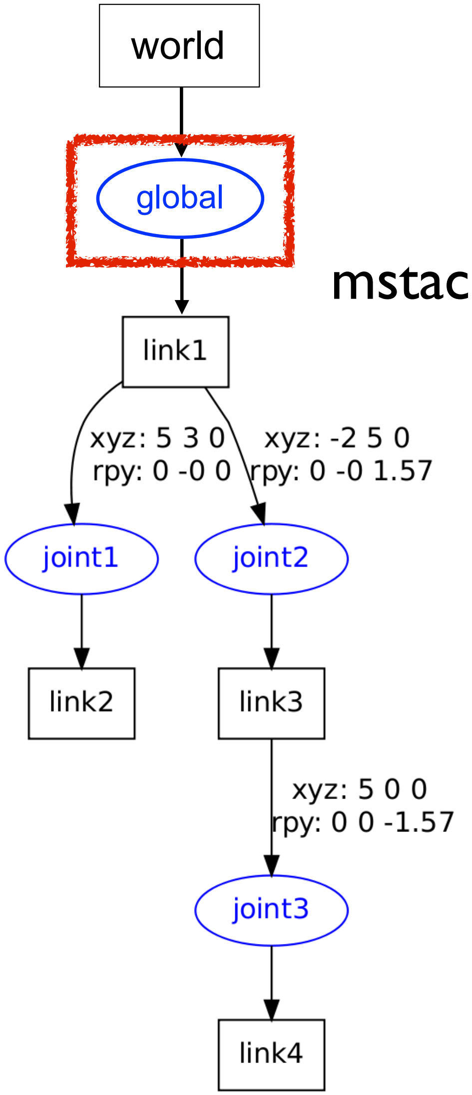
Matrix Stack Reloaded

Matrix Stack Reloaded



mstack= | I |



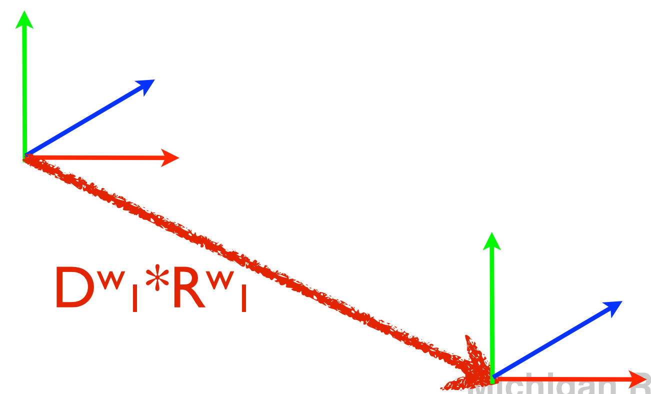


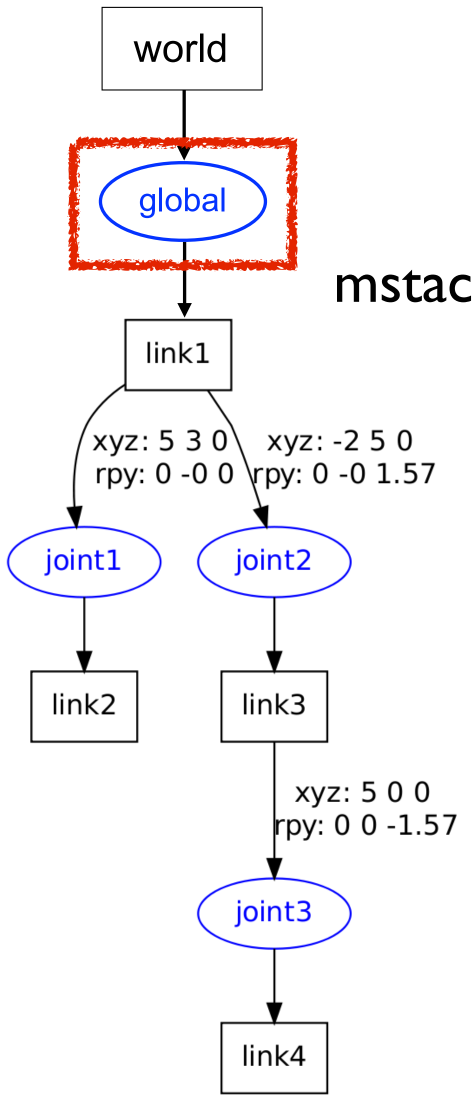
Matrix Stack Reloaded

mstack=

I
I

Push top of matrix stack up one level



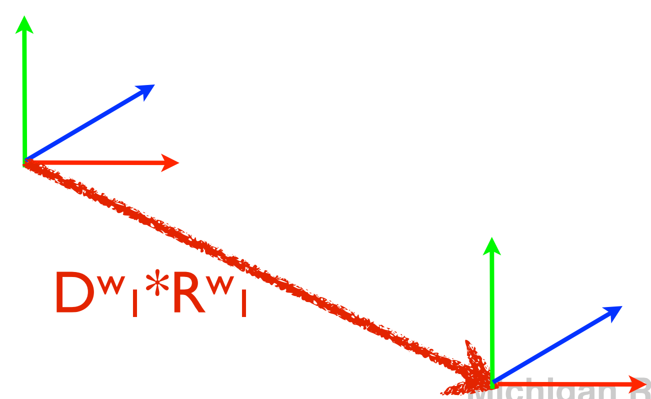


Matrix Stack Reloaded

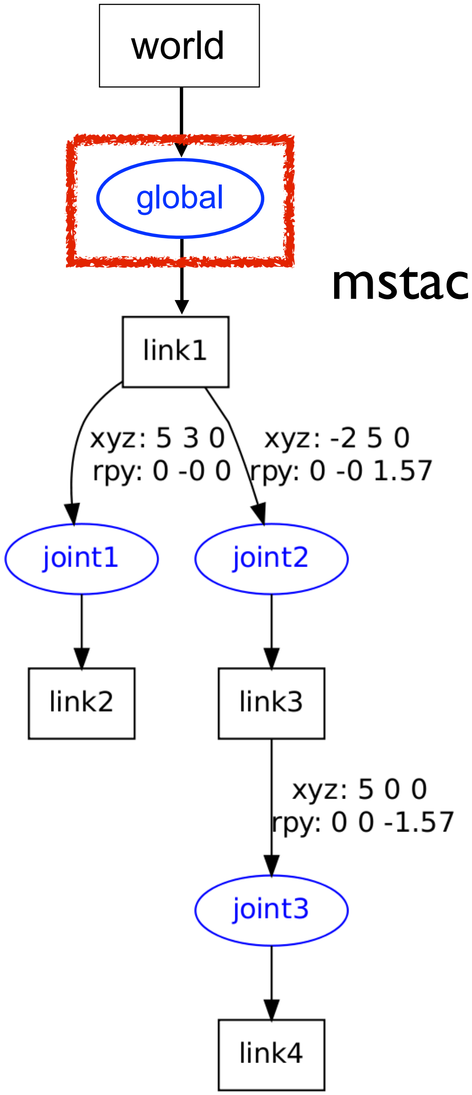
mstack=

$I * D^w{}_l * R^w{}_l$
I

Multiply by transform of base frame wrt. world frame



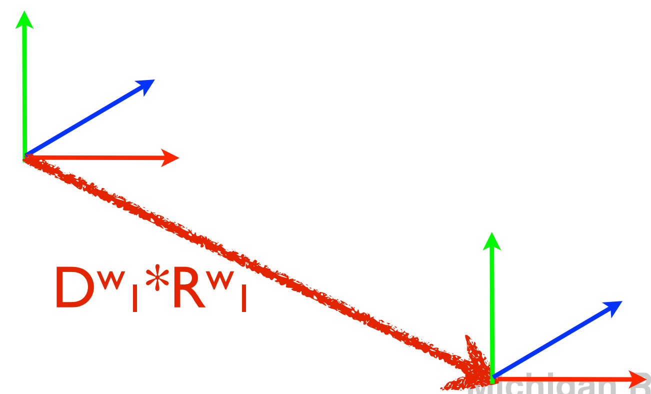
Matrix Stack Reloaded

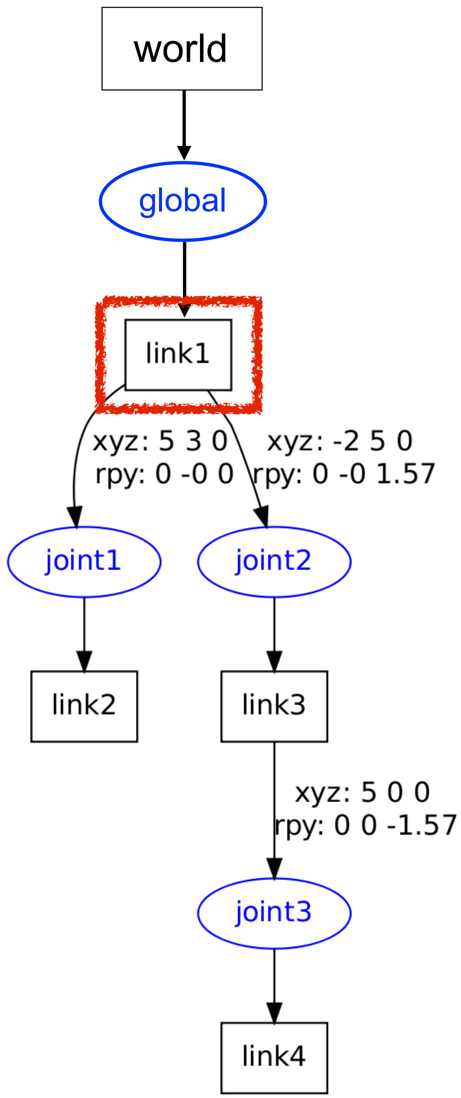


mstack=

$D^w{}_l * R^w{}_l$
I

Top of matrix stack is now base frame posed wrt. the world frame

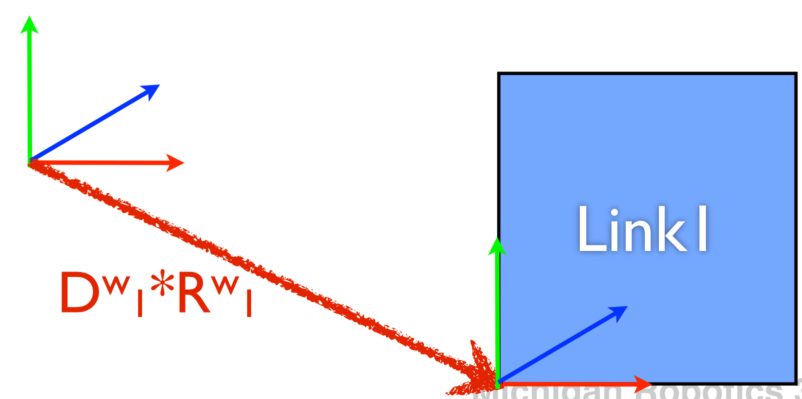


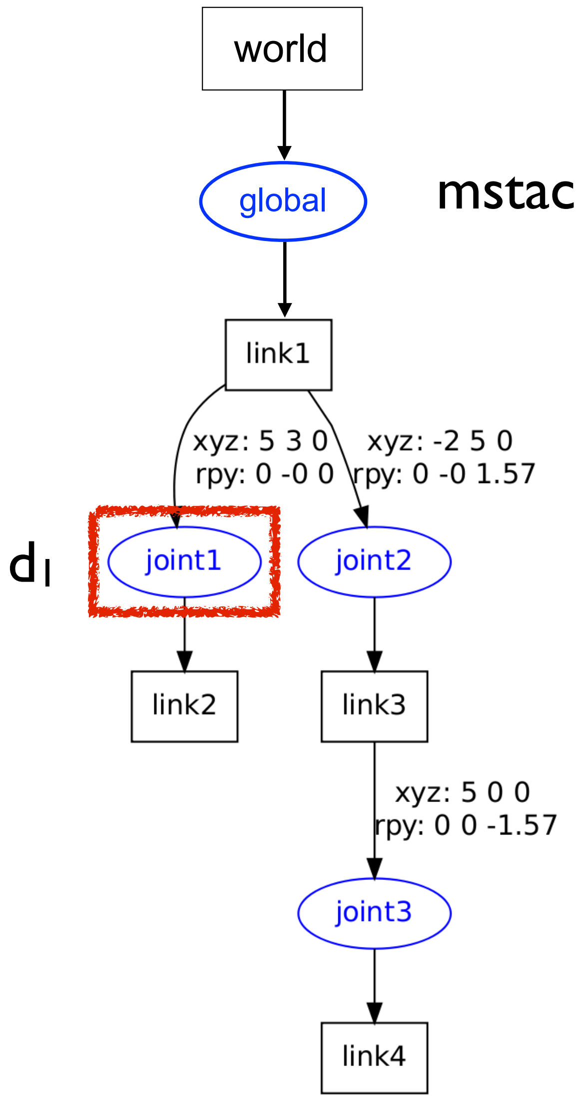


Matrix Stack Reloaded

$D^w{}_l * R^w{}_l$
I

Geometry vertices of link1 can now be transformed into pose in the world frame

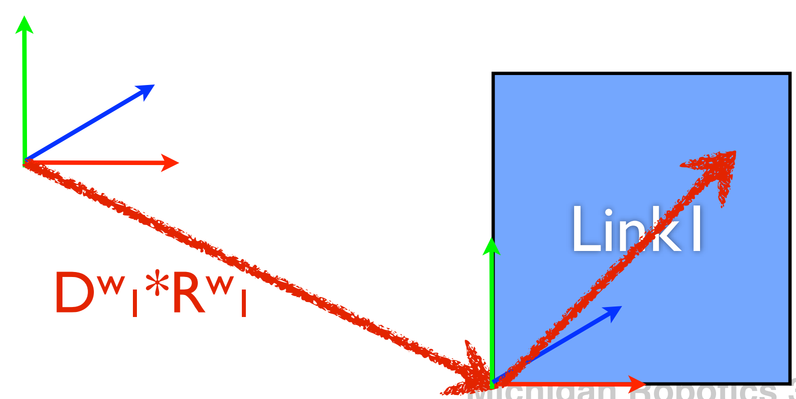


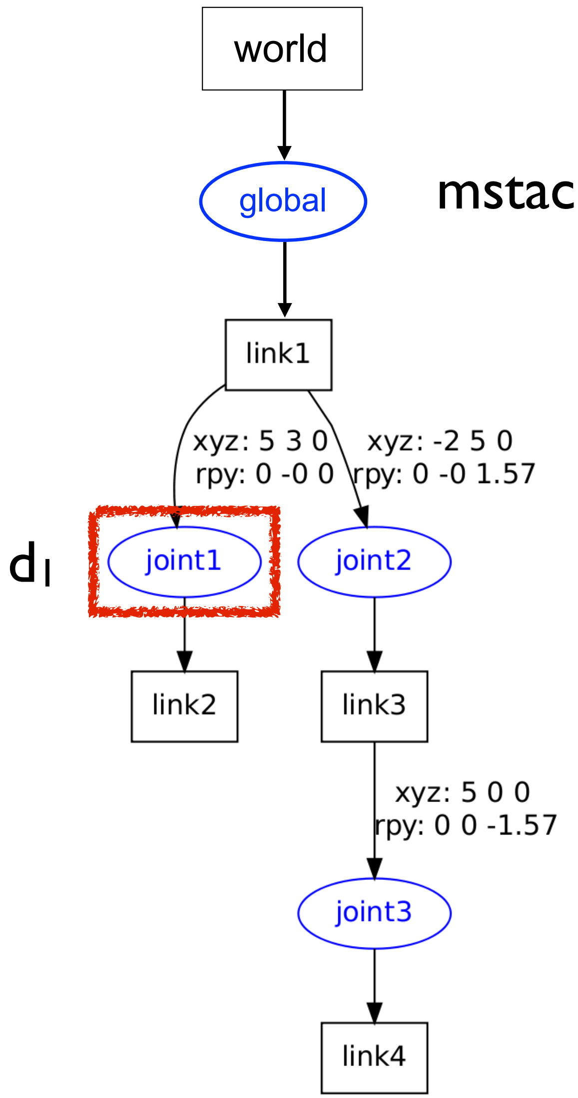


mstack=

$D^w{}_l * R^w{}_l * D^l{}_2 * R^l{}_2$
$D^w{}_l * R^w{}_l$
I

Traverse first child joint (joint1) of link1.
Push top of matrix stack one level.
Multiply by transform from base to joint1 (link2).

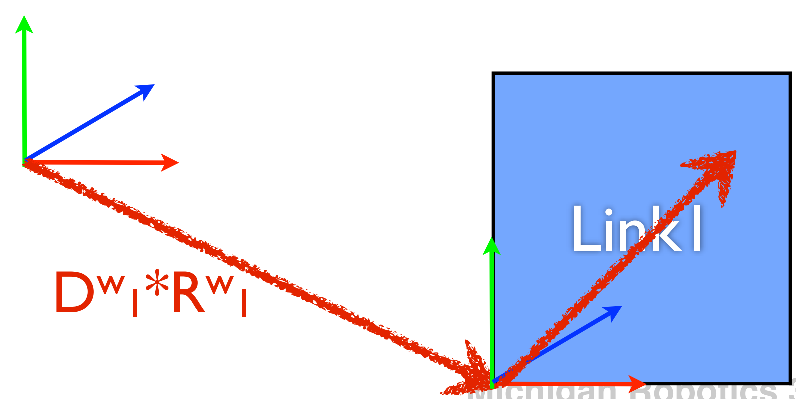


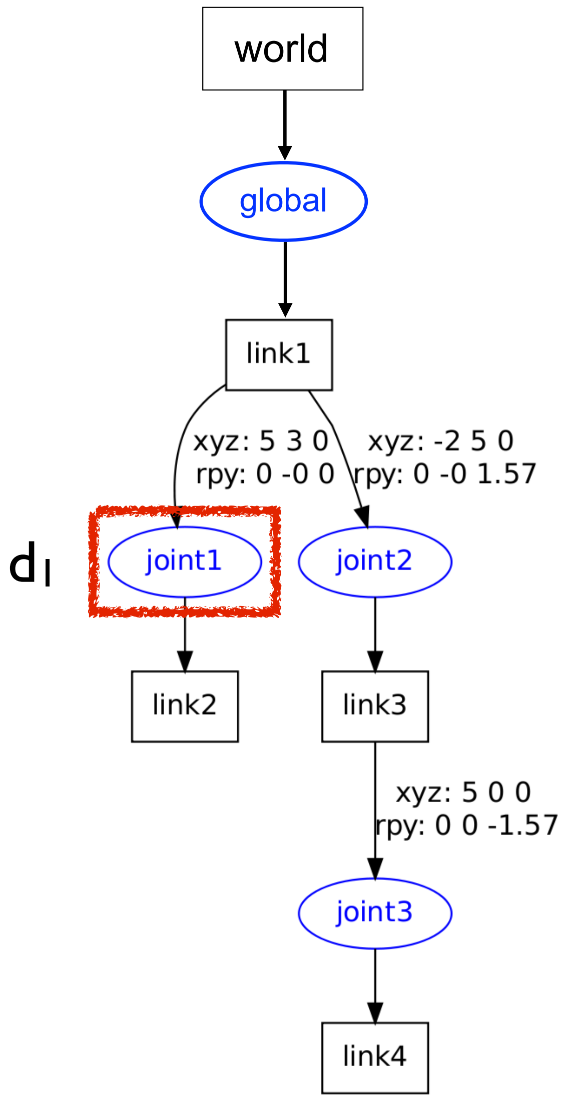


mstack=

$D^w_1 * R^w_1 * D^1_2 * R^1_2$
$D^w_1 * R^w_1$
I

Recursively, call a function to process joint

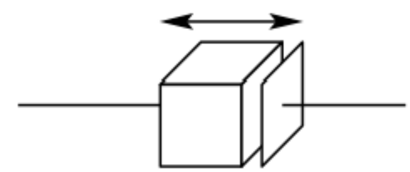


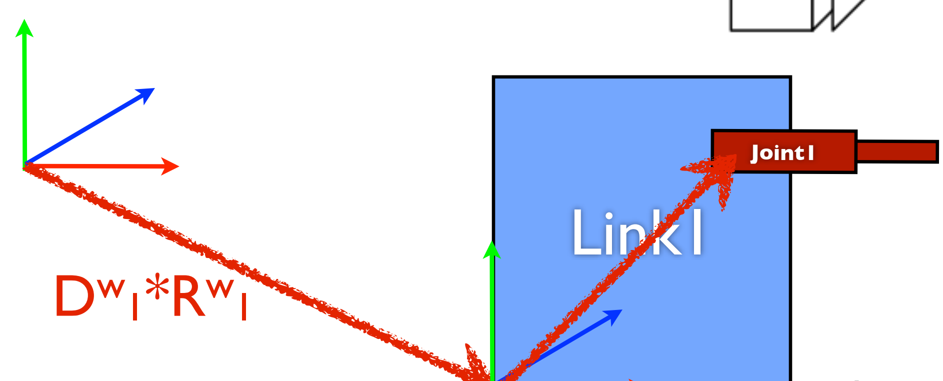


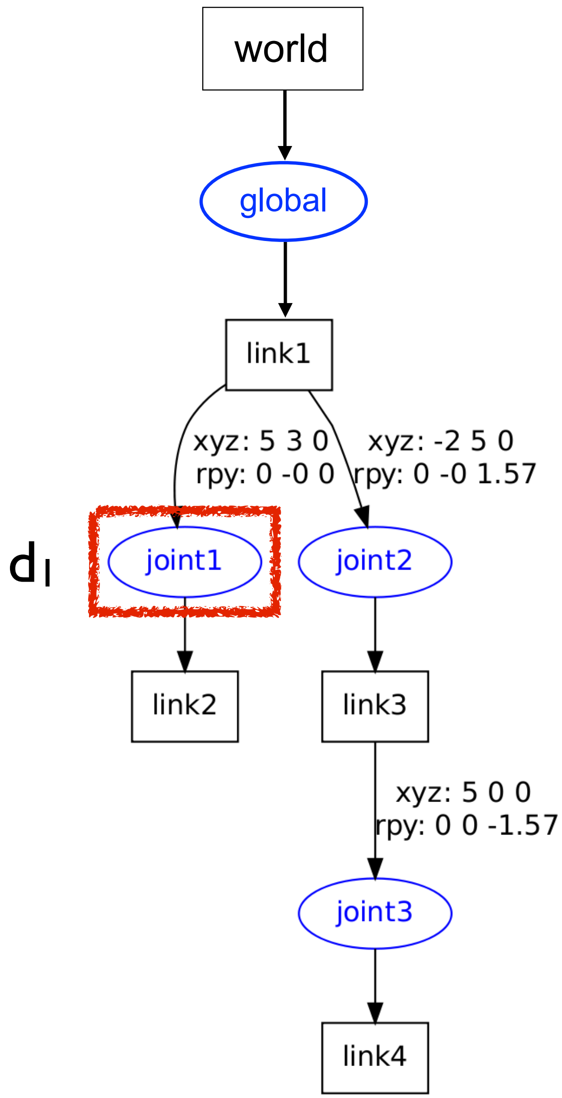
Assume joint1 is prismatic

$D^w_1 * R^w_1 * D^1_2 * R^1_2$
$D^w_1 * R^w_1$
I

How can we account for joint1's motion?





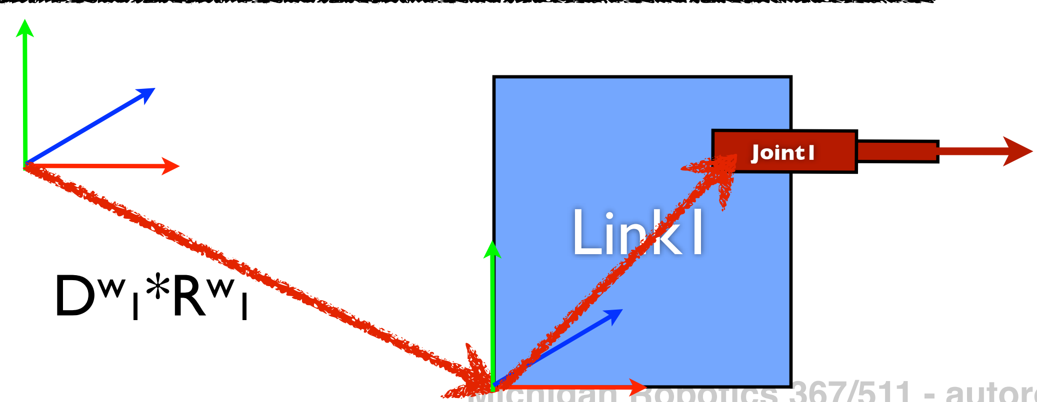


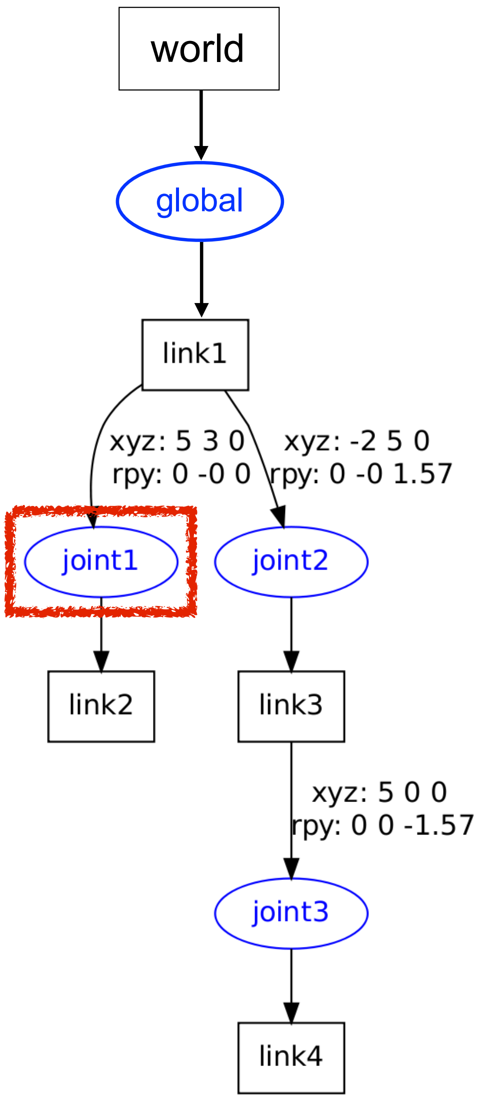
also push transform due to motor DOF

$D^w_1 * R^w_1 * D^1_2 * R^1_2$????
$D^w_1 * R^w_1$
I

What transform can account for joint1's motion?

```
// joint axis in parent frame
robot.joints["joint1"].axis = [-0.9 0.15 0];
```

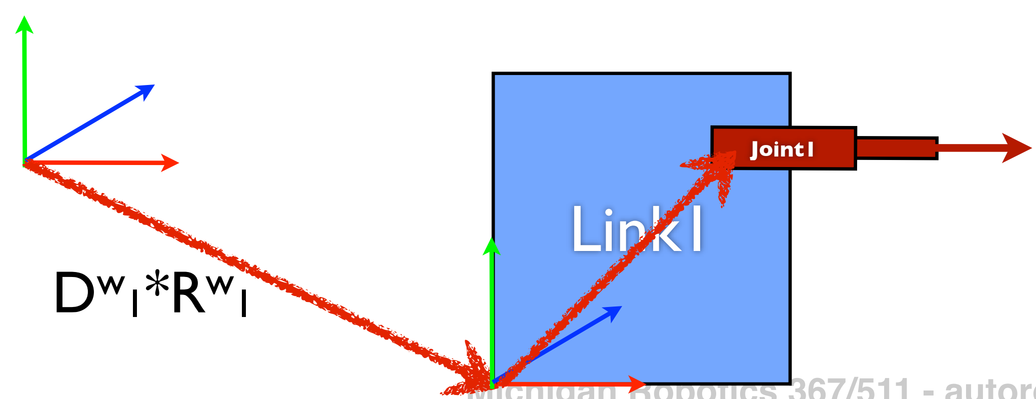


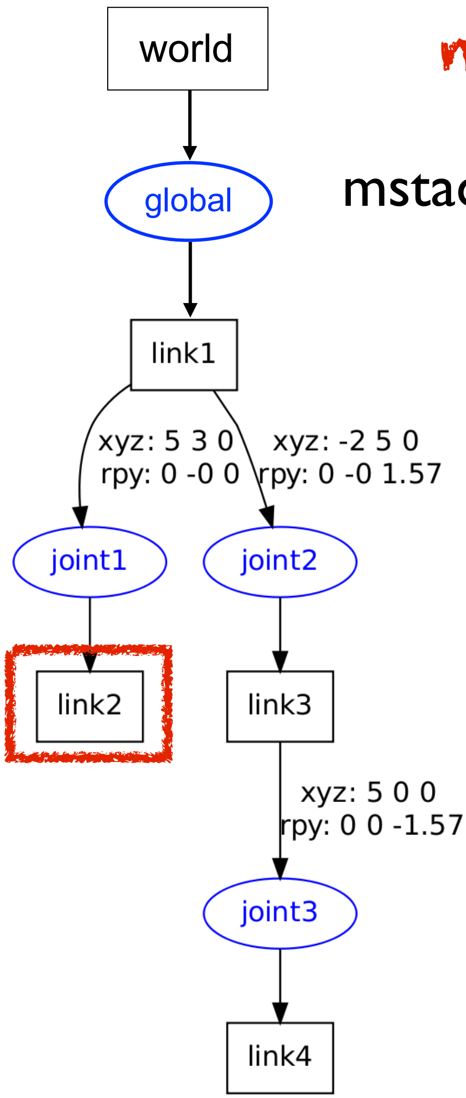


$D^{w}{}_{1}*R^{w}{}_{1}*D^{1}{}_{2}*R^{1}{}_{2}*D_{u1}(q_1)$
$D^{w}{}_{1}*R^{w}{}_{1}$
I

translation on unit joint axis u_1 scaled by joint state q_1

```
// transform of joint wrt. world
robot.joints["joint1"].xform = //this matrix
```



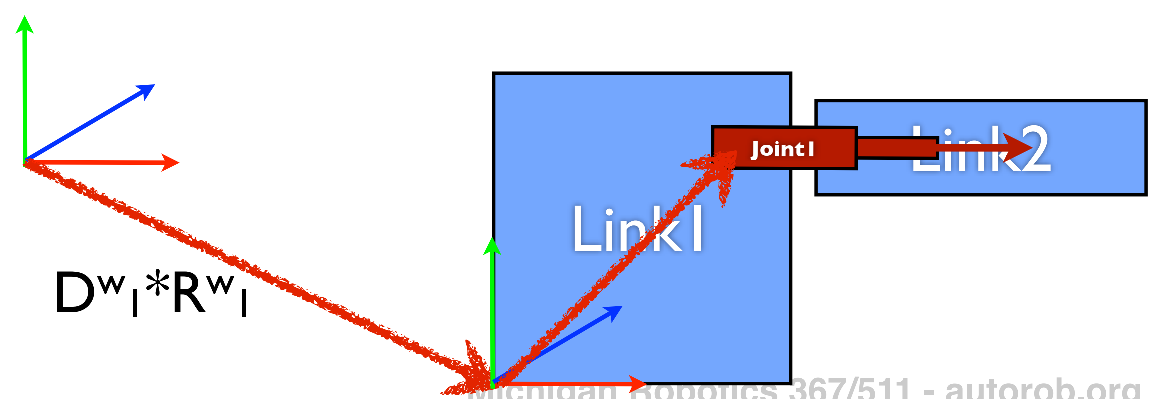


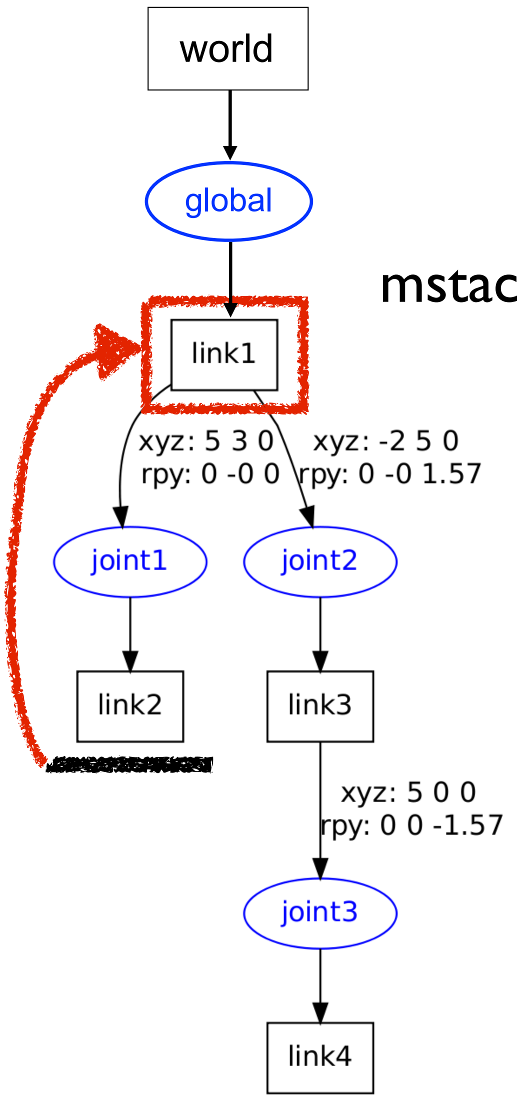
motor transform affects outboard chain

mstack=

$D^{w}{}_{1} * R^{w}{}_{1} * D^{1}{}_{2} * R^{1}{}_{2} * D_{u1}(q_1)$
$D^{w}{}_{1} * R^{w}{}_{1}$
I

$$Link_2^{world} = mstack * Link_2^{link2}$$
$$= (D^{w}{}_{1} * R^{w}{}_{1} * D^{1}{}_{2} * R^{1}{}_{2} * D_{u1}(q_1)) * Link_2^{link2}$$



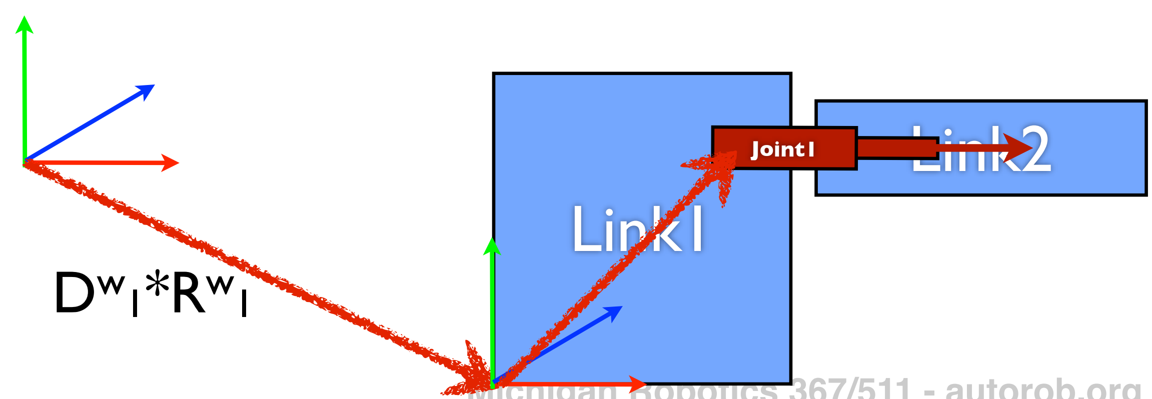


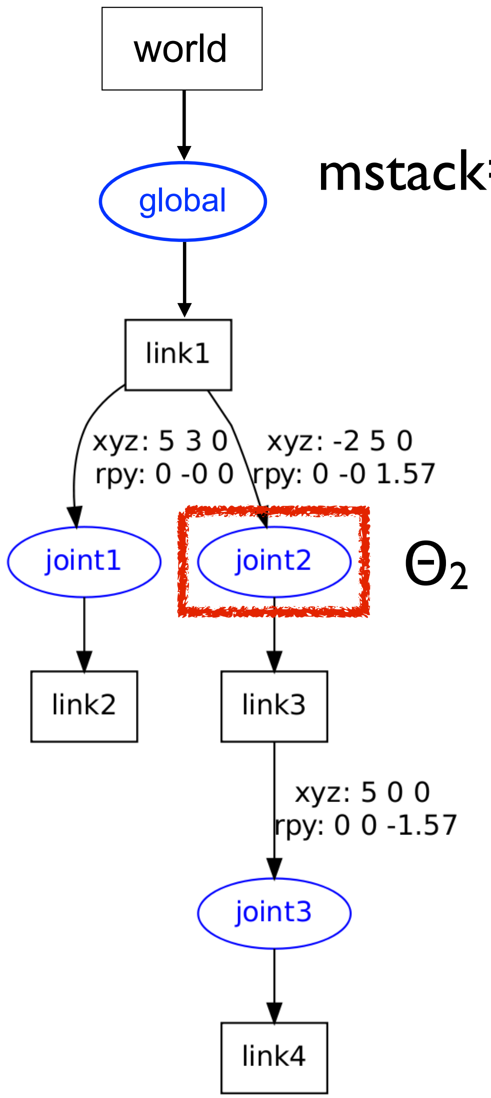
pop!

mstack=

$D^w{}_1 * R^w{}_1$
I

Pop off top level of matrix stack.
Recursion: pop implicit via function return

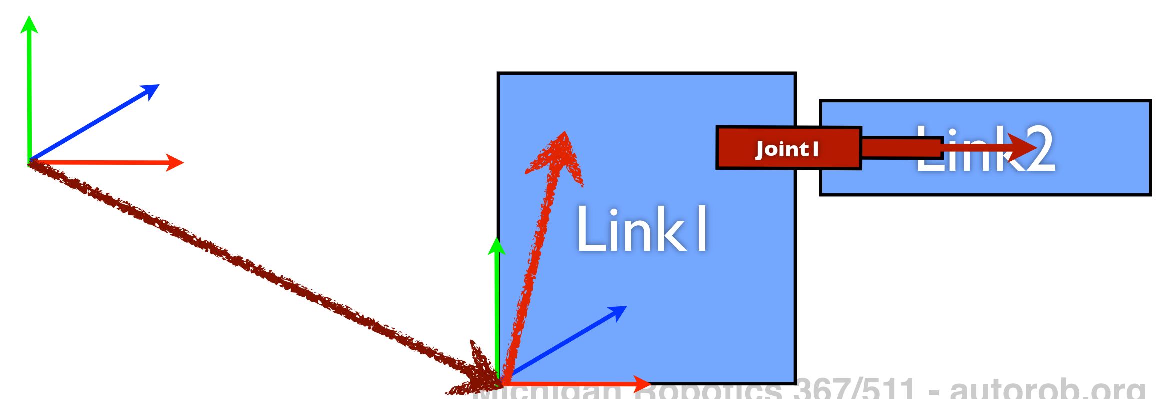


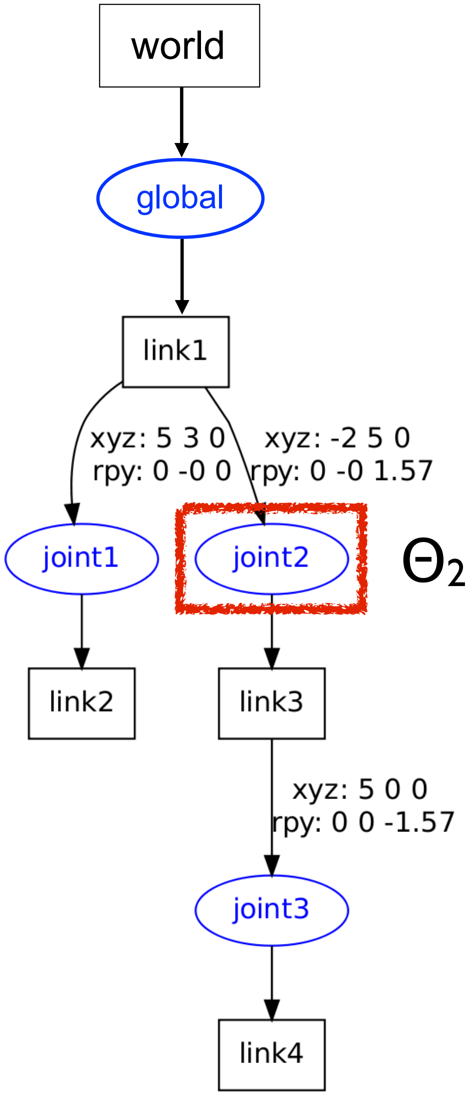


mstack=

$D^w_1 * R^w_1 * D^1_3 * R^1_3$
$D^w_1 * R^w_1$
I

Traverse second child joint (joint2) of link1.

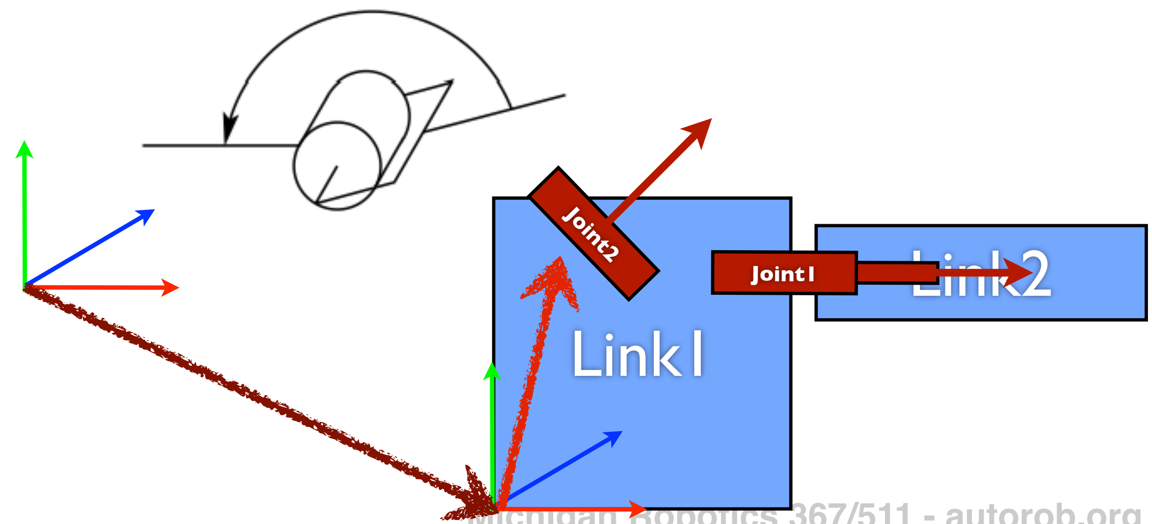


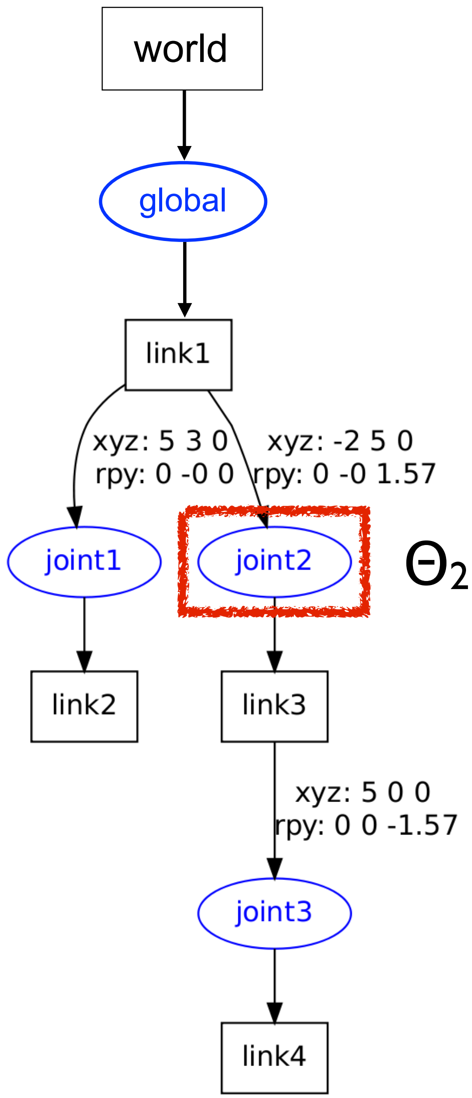


joint2 is revolute

$D^w_1 * R^w_1 * D^1_3 * R^1_3$????
$D^w_1 * R^w_1$
I

How can we account for joint2's motion?



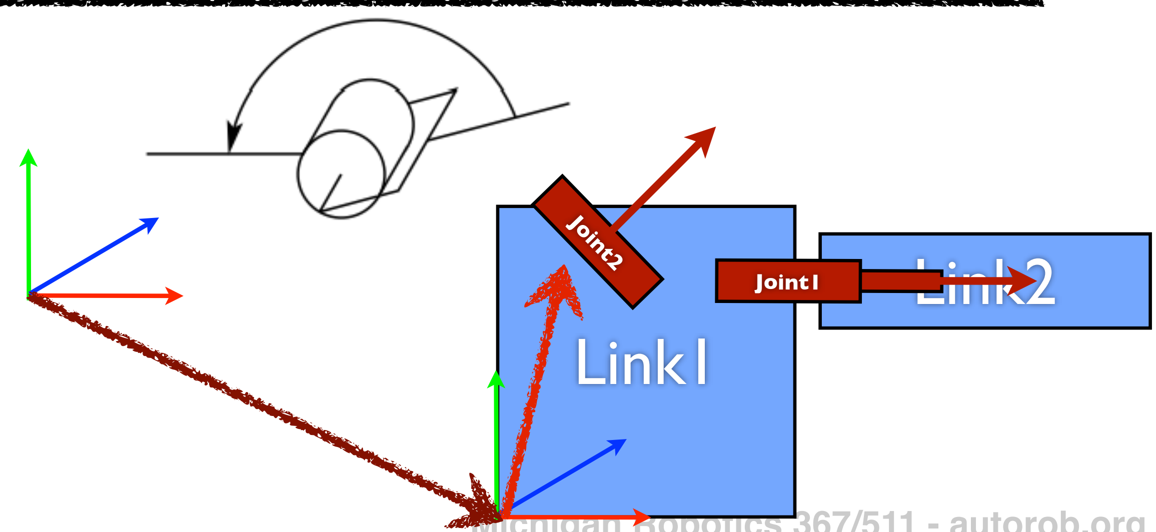


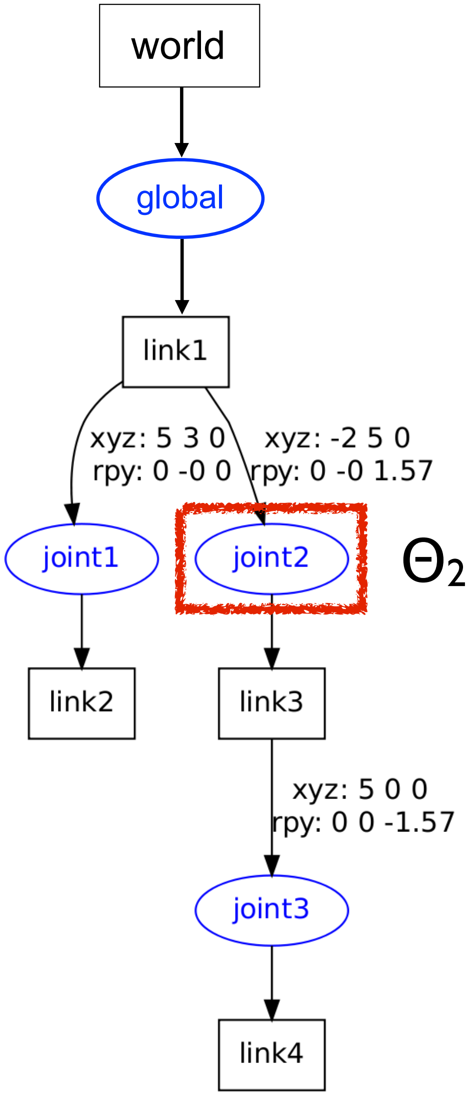
joint2 is revolute

$D^{w}_{1} * R^{w}_{1} * D^{1}_{3} * R^{1}_{3} * R_{u2}(q_2)$
$D^{w}_{1} * R^{w}_{1}$
I

rotation about unit joint axis u_2 by joint state q_2

```
//joint motor rotation axis
robot.joints["joint2"].axis = [0.707,0.0, 0.707]
```

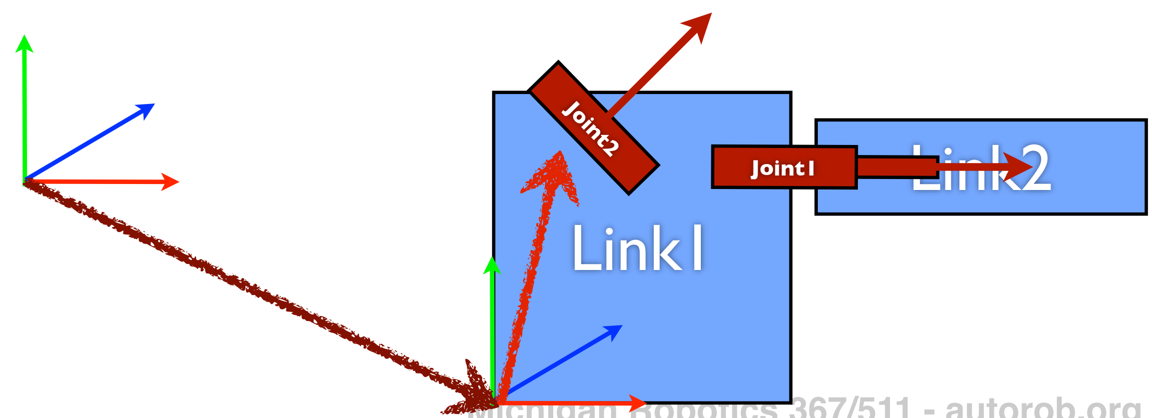




$D^{w}{}_{1} * R^{w}{}_{1} * D^{1}{}_{3} * R^{1}{}_{3} * R_{u2}(q_2)$
$D^{w}{}_{1} * R^{w}{}_{1}$
I

```
//joint motor rotation axis
robot.joints["joint2"].axis = [0.707,0.0, 0.707]
```

how to perform this rotation?



Euler Angles

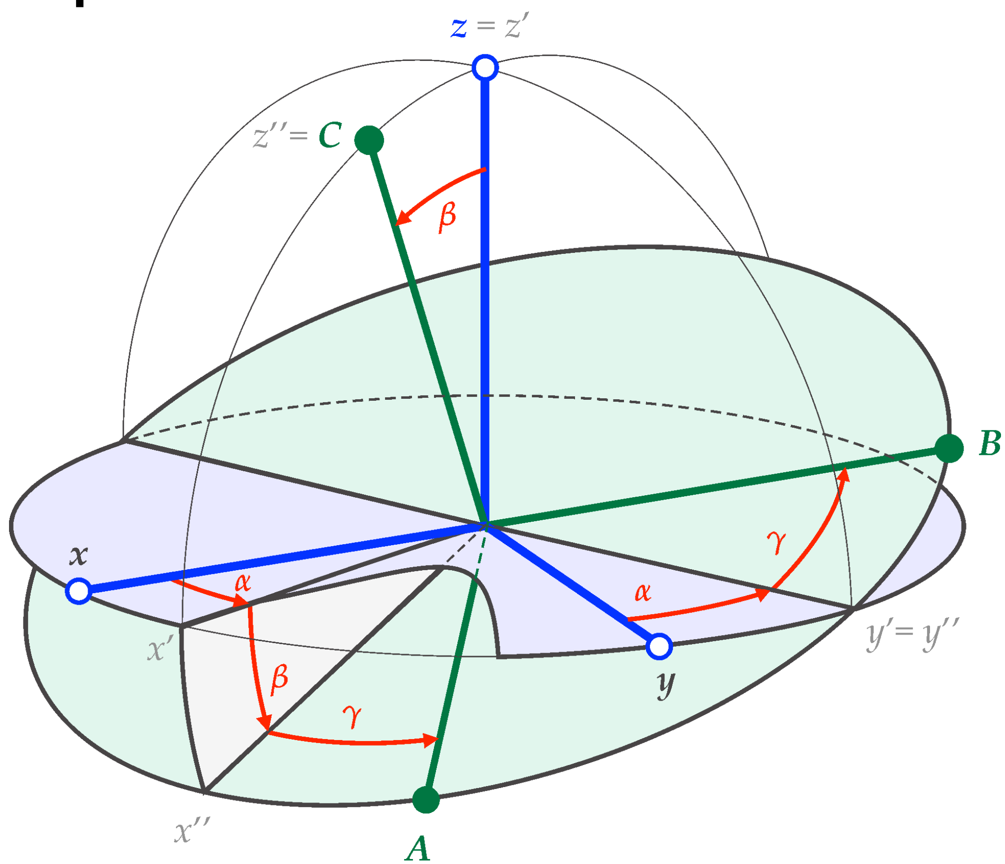
- Rotate about each axis in chosen order: $R = R_x(\Theta_x)\ R_y(\Theta_y)\ R_z(\Theta_z)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- 24 different choices for rotation ordering
- $R_x(\Theta_x)$: roll, $R_y(\Theta_y)$: pitch, $R_z(\Theta_z)$: yaw
- Matrix rotation not commutative across different axes

AutoRob uses XYZ order:
$R_z\ R_y\ R_x$ (X then Y then Z)

Example: ZYZ Euler angles



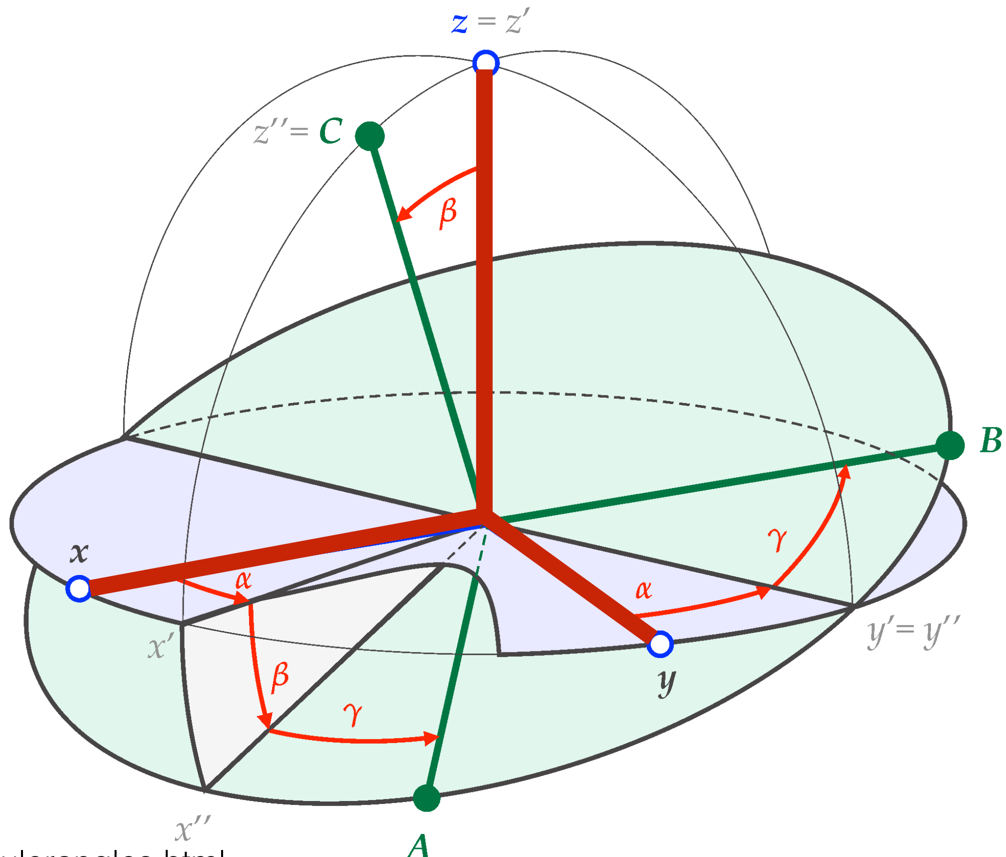
http://easyspin.org/documentation/eulerangles.html

Example: ZYZ Euler angles

Rotate xyz counterclockwise around its z axis by α to give x'y'z'.

Rotate x'y'z' counterclockwise around its y' axis by β to give x''y''z''.

Rotate x''y''z'' counterclockwise around its z'' axis by γ to give the final ABC.



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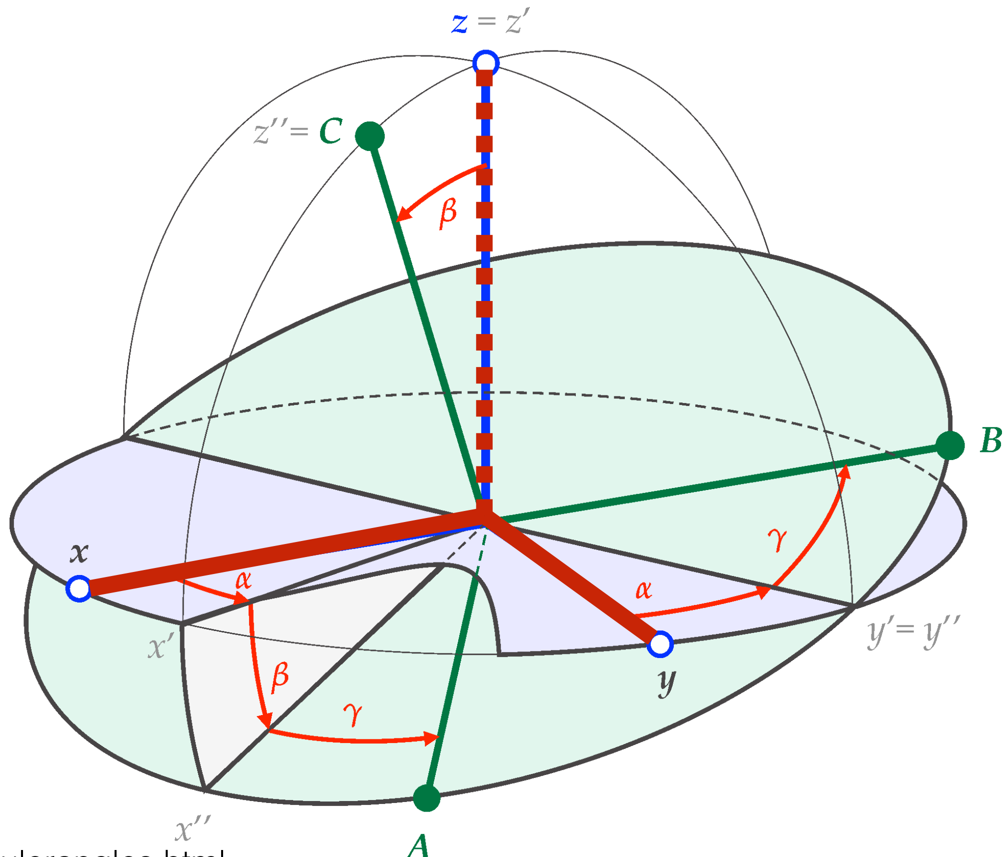

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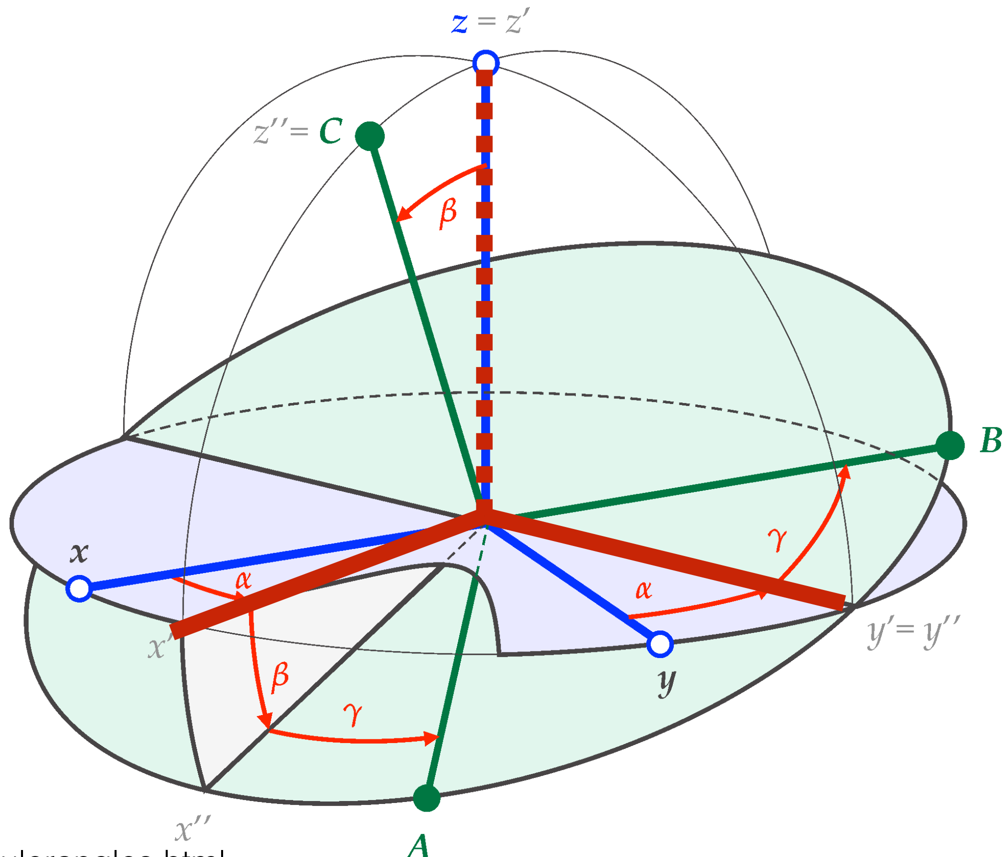

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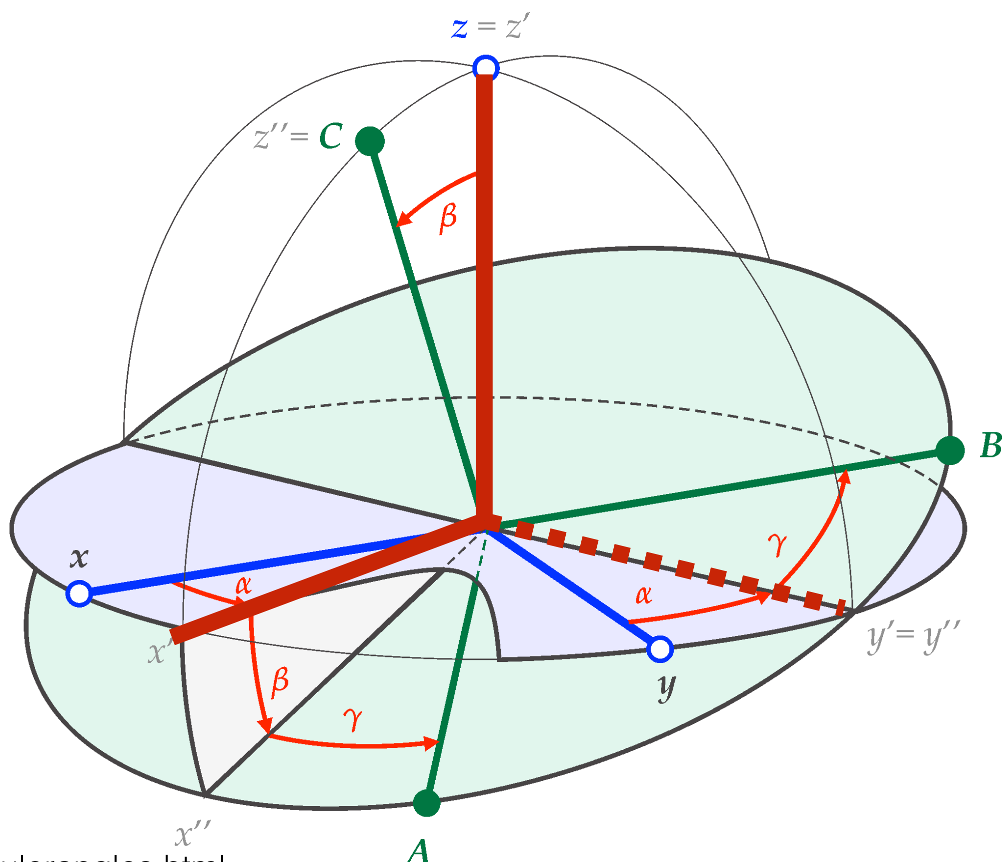

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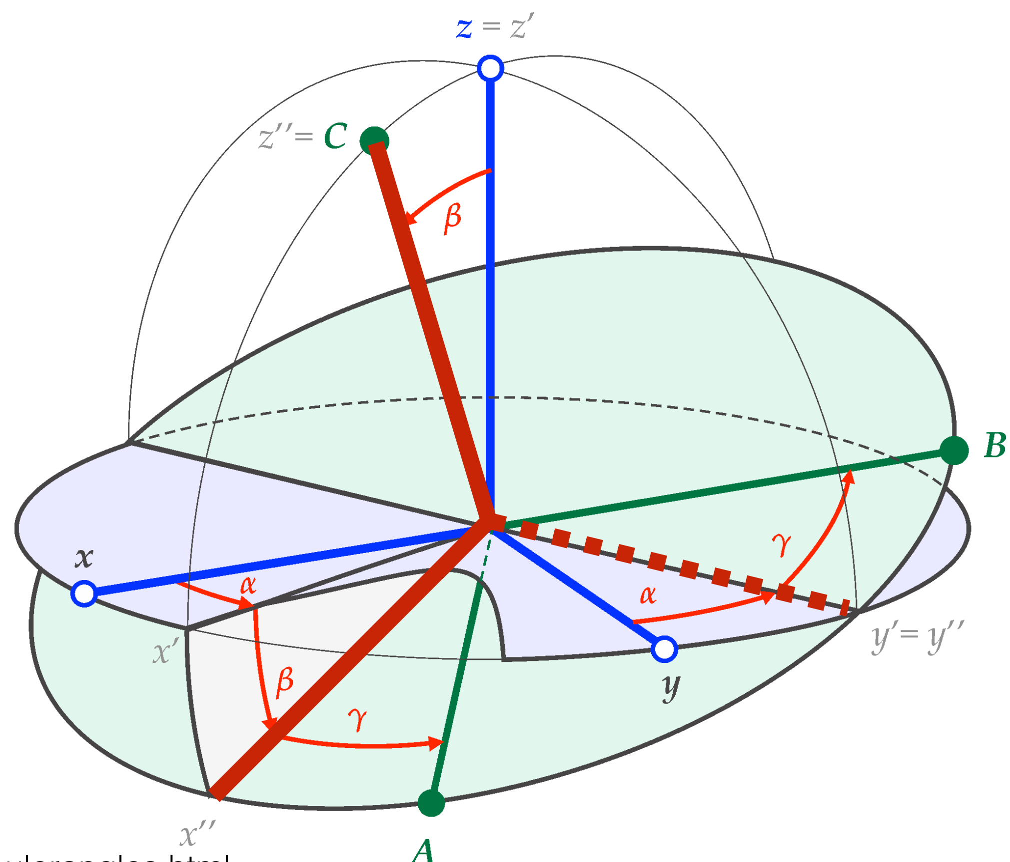

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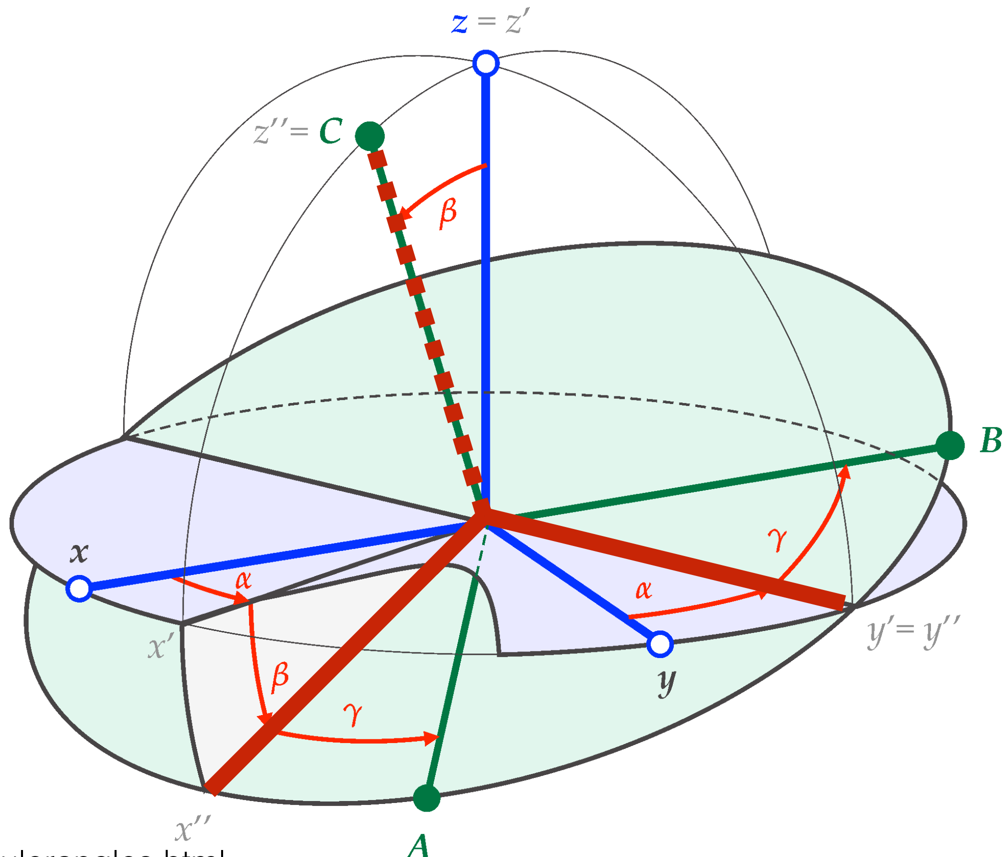

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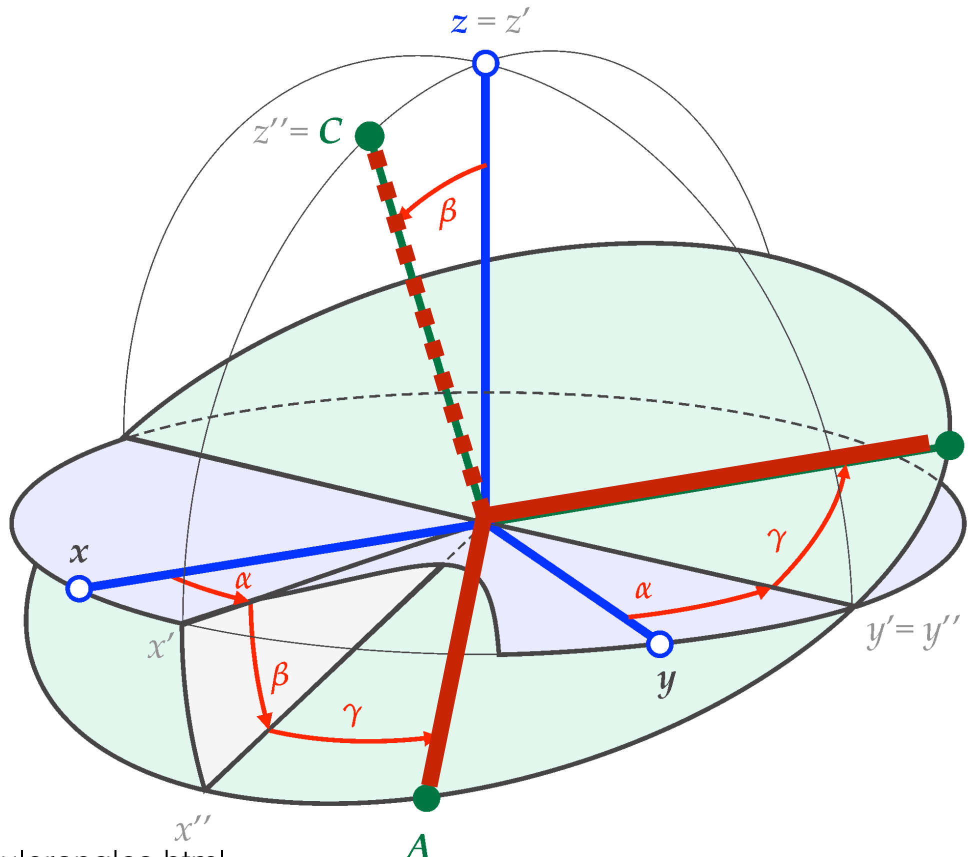
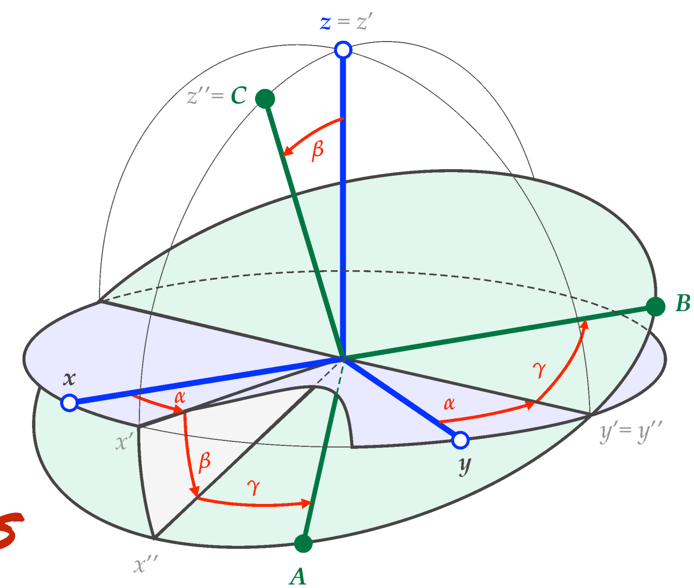


http://easyspin.org/documentation/eulerangles.html

Example: ZYZ Euler angles

$$
\begin{aligned}
R &= R_{z''}(\gamma) \cdot R_{y'}(\beta) \cdot R_z(\alpha) \\
&= \begin{pmatrix} c\gamma & s\gamma & 0 \\ -s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{pmatrix} \cdot \begin{pmatrix} c\alpha & s\alpha & 0 \\ -s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c\gamma c\beta c\alpha - s\gamma s\alpha & c\gamma c\beta s\alpha + s\gamma c\alpha & -c\gamma s\beta \\ -s\gamma c\beta c\alpha - c\gamma s\alpha & -s\gamma c\beta s\alpha + c\gamma c\alpha & s\gamma s\beta \\ s\beta c\alpha & s\beta s\alpha & c\beta \end{pmatrix}
\end{aligned}
$$



http://easyspin.org/documentation/eulerangles.html

EACH ROTATION CHANGES THE NON-ROTATED AXES

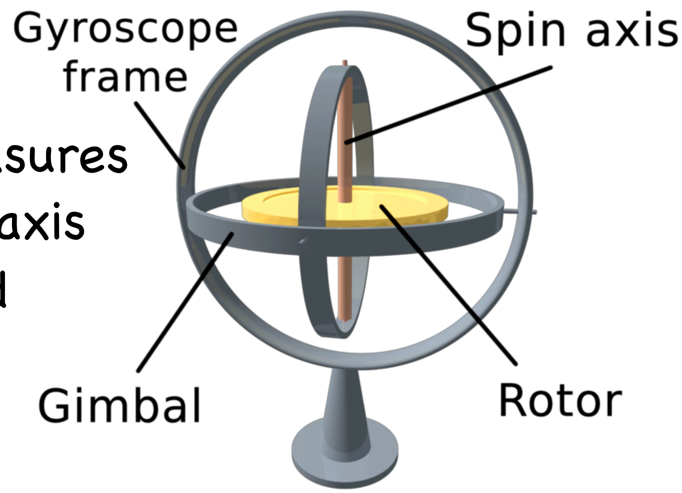
RESULTS IN A NEW FRAME FOR THE NEXT ROTATION

Why not rotate about each axis?

Why not rotate about each axis?

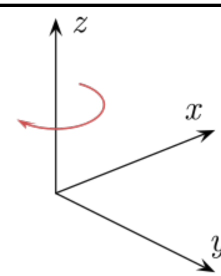
Consider gyroscope

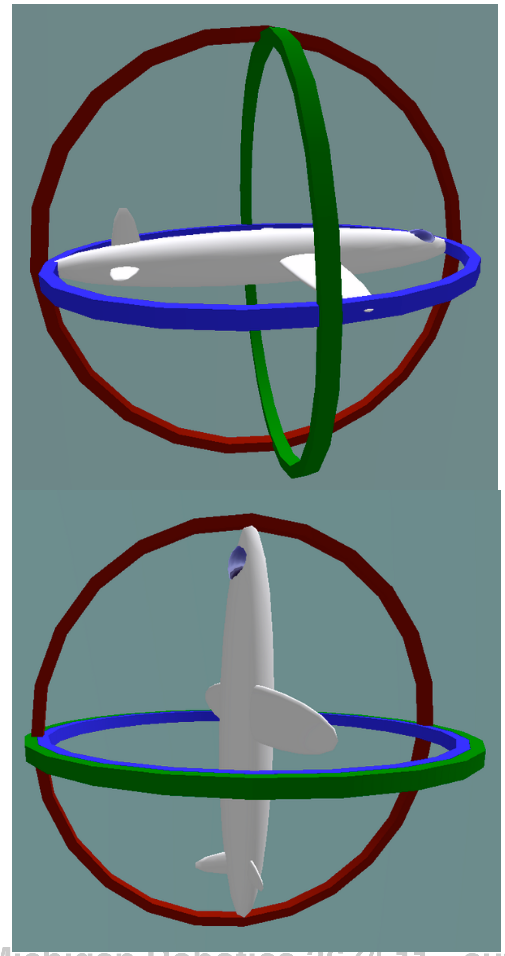
Gyroscope measures angle on each axis (wrt frame and wheel)



Rotate about each axis in order

$R = R_x(\Theta_x)\, R_y(\Theta_y)\, R_z(\Theta_z)$





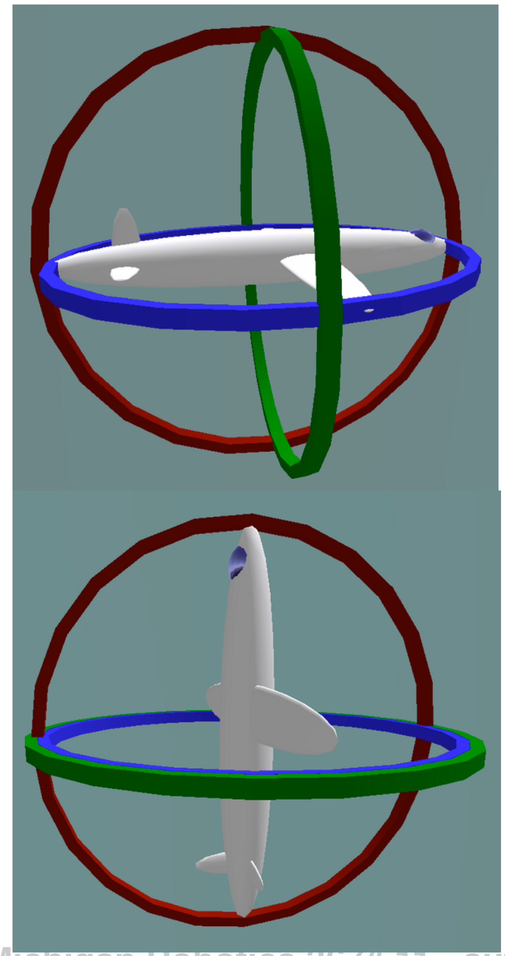
Gimbal Lock

Gimbal lock occurs when two axes are rotated into alignment

Reduces 3 DOFs to 2 based on axis order.

Why is gimbal lock a problem for rotation?

[redacted]



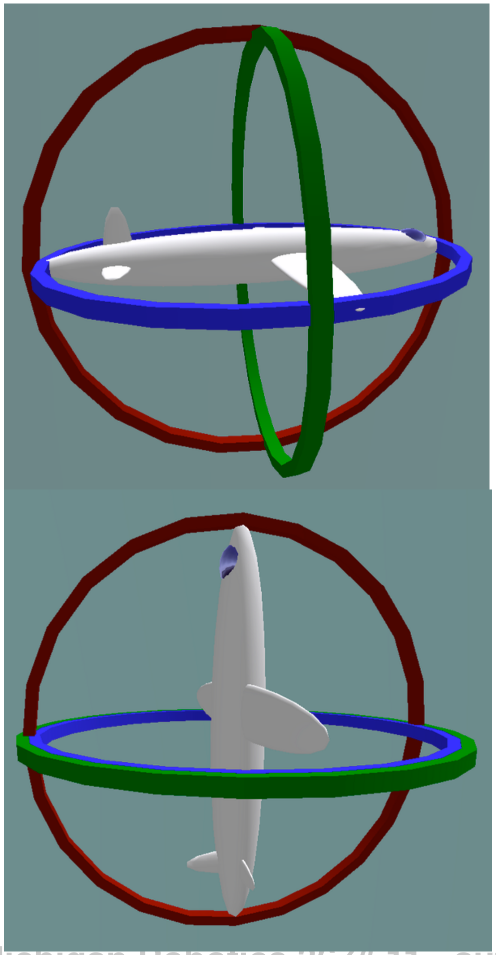
Gimbal Lock

Gimbal lock occurs when two axes are rotated into alignment

Reduces 3 DOFs to 2 based on axis order.

Why is gimbal lock a problem for rotation?

How many linearly independent axes are available when gimbal lock occurs?



Consider a few examples
(on your own)

Consider rotation with this order: $R = R_x(\Theta_x)\ R_y(\Theta_y)\ R_z(\Theta_z)$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assume second rotation (beta) is pi/2

$R =$ [redacted]

Consider rotation with this order: $R = R_x(\Theta_x)\ R_y(\Theta_y)\ R_z(\Theta_z)$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assume second rotation (beta) is pi/2

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider rotation with this order: $R = R_x(\Theta_x)\ R_y(\Theta_y)\ R_z(\Theta_z)$

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Rotation now only occurs about z-axis

$$R = \begin{bmatrix} 0 & 0 & 1 \\ \sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & \sin\alpha & 0 \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \sin\alpha\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\gamma & 0 \\ -\cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \cos\alpha\sin\gamma + \sin\alpha\cos\gamma & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{bmatrix}$$

try multiplying by a vector

beta must change from pi/2 in order for alpha and gamma to have proper effect

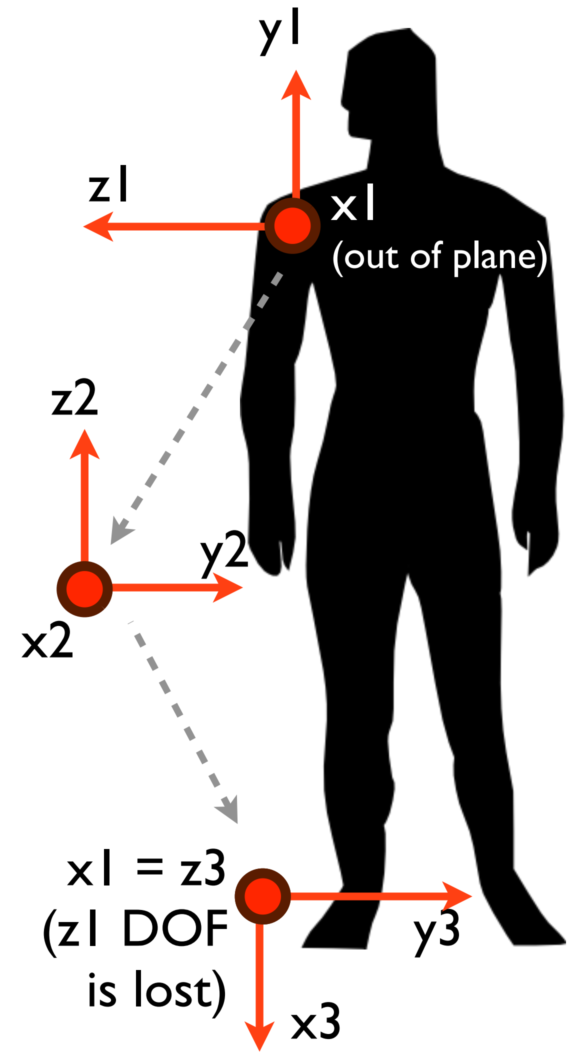
Gimbal lock example

rotation order: X then Y then Z

- Consider: $R_z(90°)\ R_y(90°)\ R_x(90°)$
- Rotate your arm upward 90 degrees about initial x-axis
- Rotate 90 degrees downward about new y-axis

  gimbal lock occurs: current z-axis aligns with initial x-axis
- Rotate 90 degrees about new z-axis

  rotation occurs about initial x-axis
  return to approximately original pose
- Remember: rotations axes move with rotations

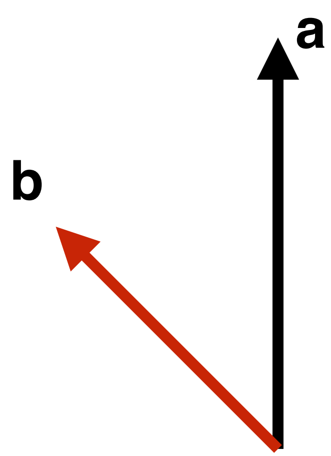


Let’s try rotating about an axis

Rodrigues Axis-Angle Rotation

Benjamin Olinde Rodrigues
1795-1851

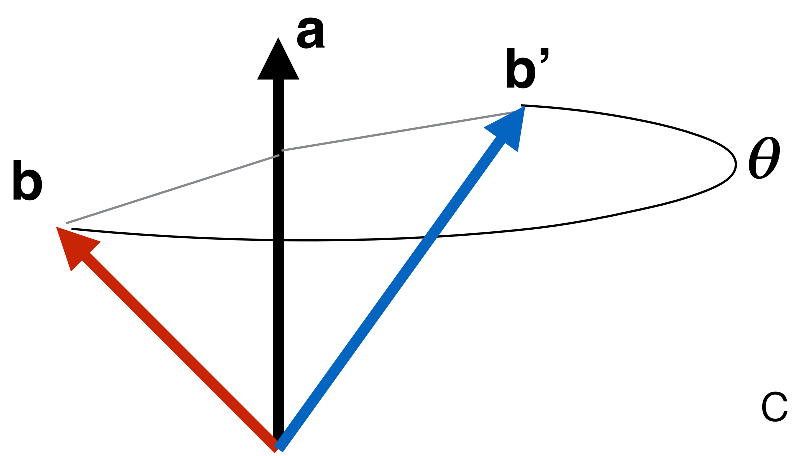
Rodrigues Axis-Angle Rotation



Given two vectors **a** and **b**,

Assume **a** is unit length

Rodrigues Axis-Angle Rotation

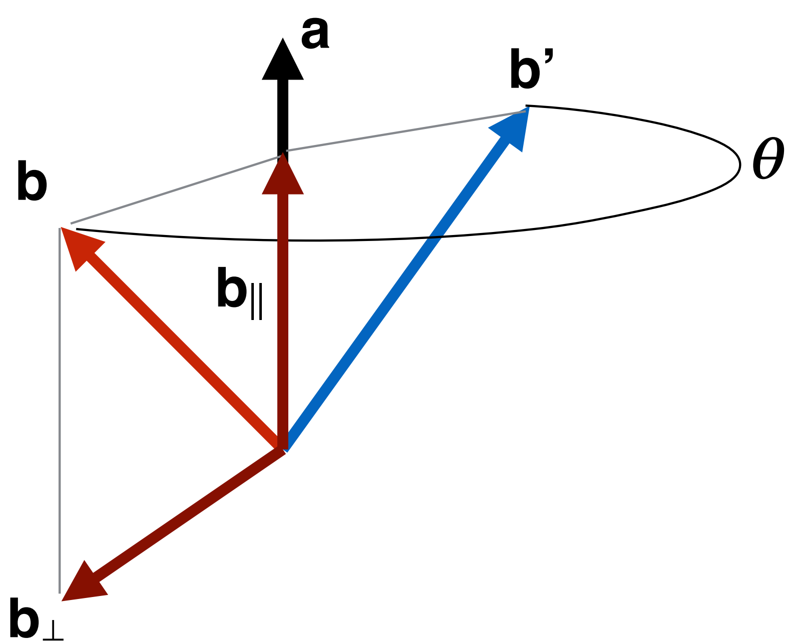


Given two vectors **a** and **b**,

compute **b'** as rotation of **b** around **a** by θ

Assume **a** is unit length

Rodrigues Axis-Angle Rotation



b can be broken down into two vectors:

$\mathbf{b}_{||}$ parallel to **a**

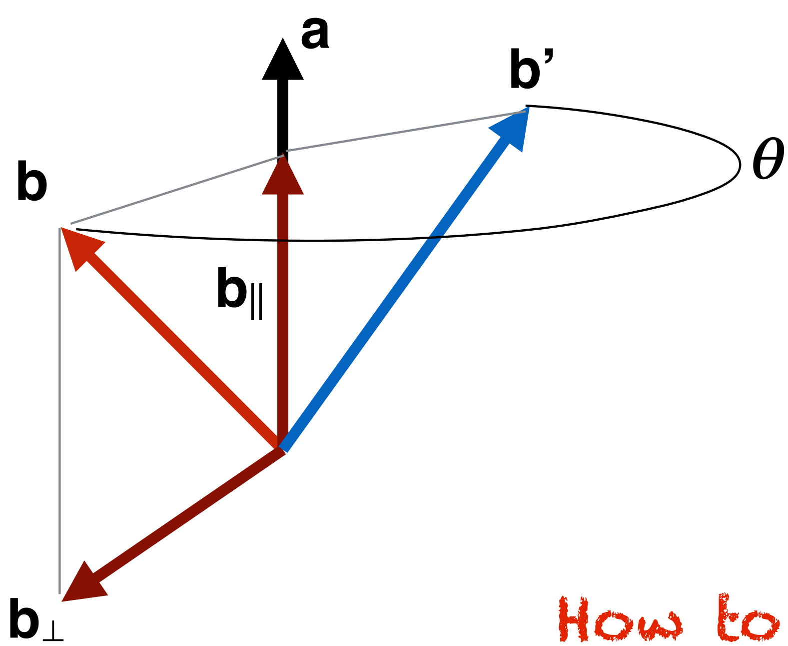
and

$\mathbf{b}_{\perp}$ orthogonal to **a**

such that

$\mathbf{b} = \mathbf{b}_{||} + \mathbf{b}_{\perp}$

Rodrigues Axis-Angle Rotation



b can be broken down into two vectors:

$\mathbf{b}_{||}$ parallel to **a**

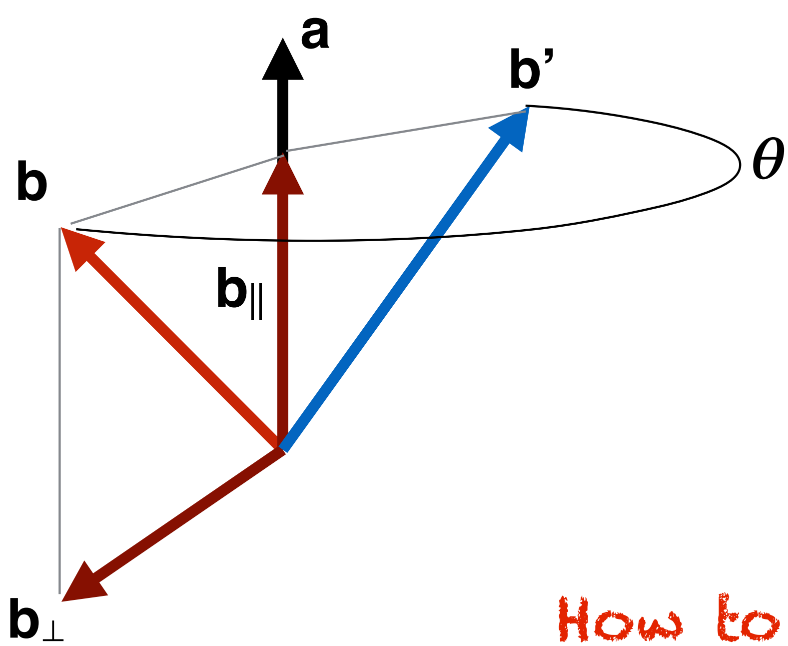
Operator to compute $b_{||}$?

and $\mathbf{b}_{\perp}$ orthogonal to **a**

How to express $b_{\perp}$ with cross products?

such that $\mathbf{b} = \mathbf{b}_{||} + \mathbf{b}_{\perp}$

Rodrigues Axis-Angle Rotation



b can be broken down into two vectors:

$\mathbf{b}_{\|} = \mathbf{a}(\mathbf{ab})$ parallel to **a**

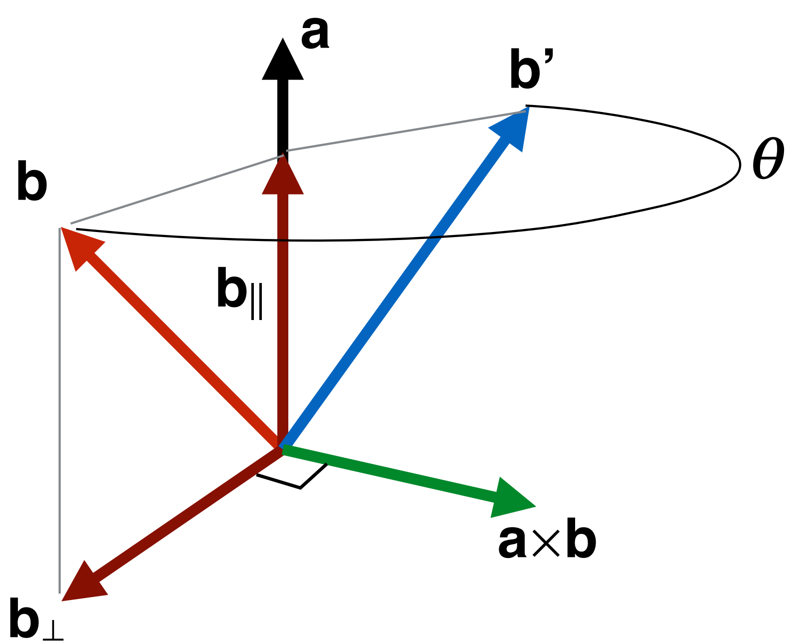
vector projection

and $\mathbf{b}_{\perp}$ orthogonal to **a**

How to express $b_{\perp}$ with cross products?

such that $\mathbf{b} = \mathbf{b}_{\|} + \mathbf{b}_{\perp}$

Rodrigues Axis-Angle Rotation



b can be broken down into two vectors:

$\mathbf{b}_{||} = \mathbf{a}(\mathbf{ab})$ parallel to **a**

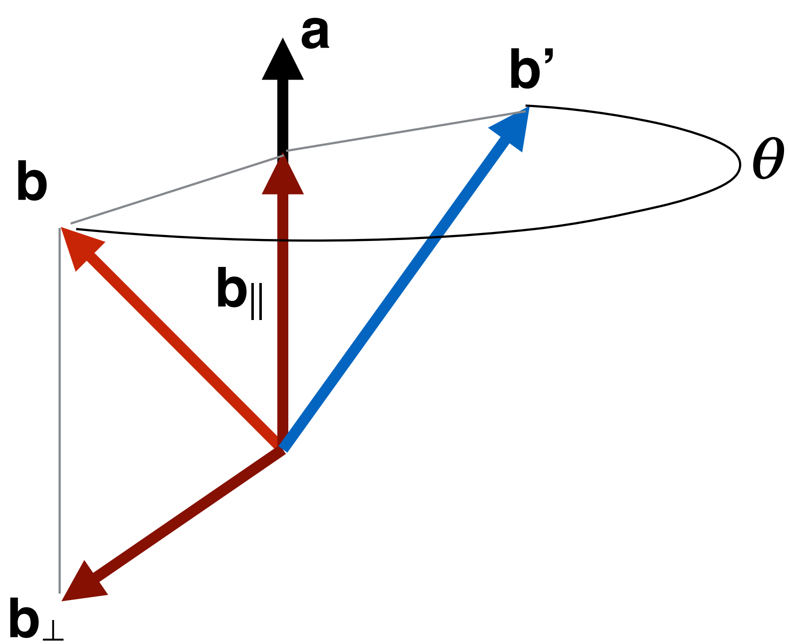
vector projection

and $\mathbf{b}_{\perp}$ orthogonal to **a**

$\boxed{\mathbf{b}_{\perp} = -\mathbf{a}\times(\mathbf{a}\times\mathbf{b})}$

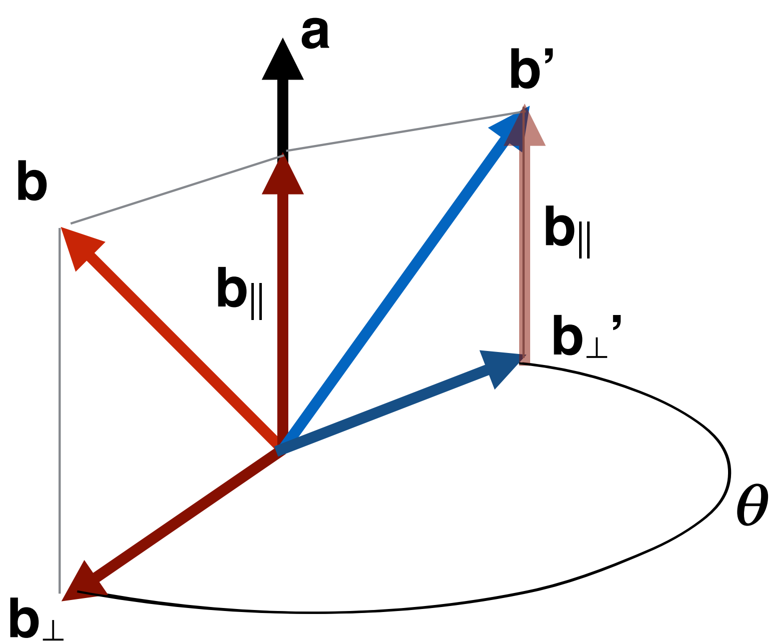
such that $\mathbf{b} = \mathbf{b}_{||} + \mathbf{b}_{\perp}$

Rodrigues Axis-Angle Rotation



$\mathbf{b}_{||}$ is not affected by rotation around **a**, only $\mathbf{b}_{\perp}$ is rotated

Rodrigues Axis-Angle Rotation



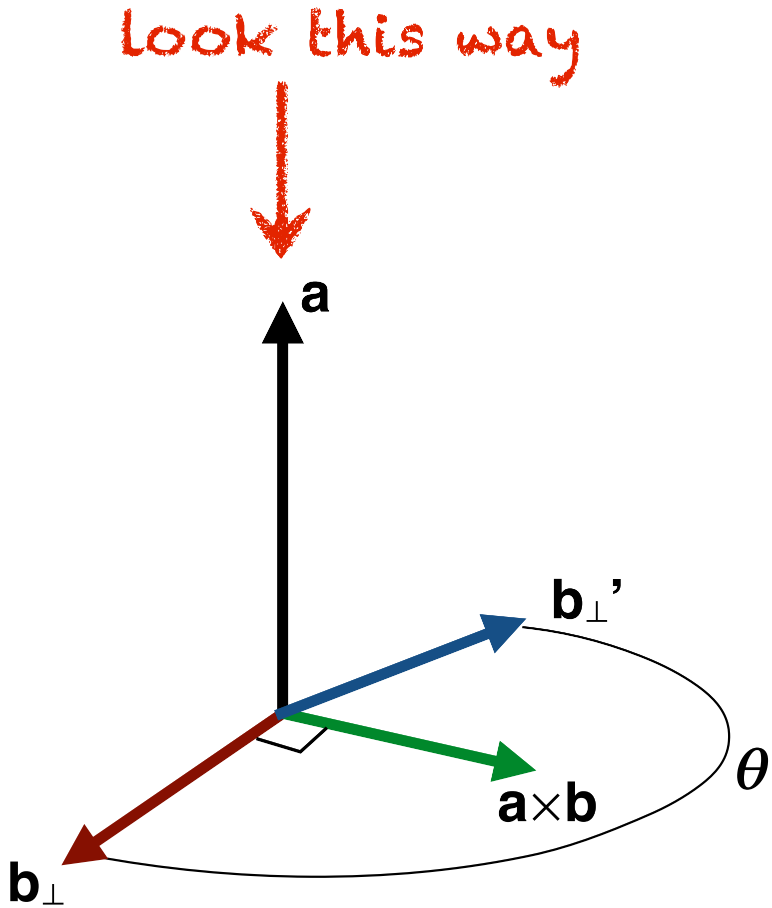
$\mathbf{b}_{\|}$ is not affected by rotation around **a**, only $\mathbf{b}_{\perp}$ is rotated

If we can rotate $\mathbf{b}_{\perp}$ around **a** by θ

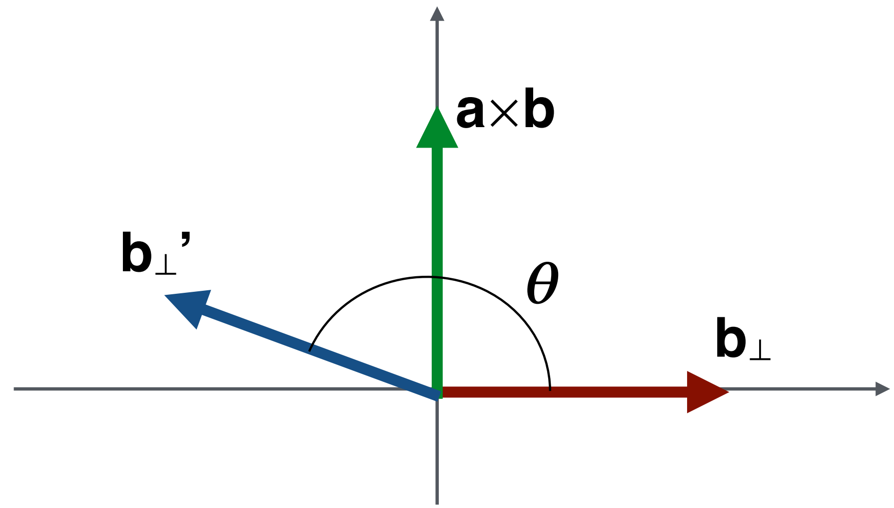
to produce $\mathbf{b}_{\perp}$'

then rotation of **b** is $\mathbf{b}_{\|}+\mathbf{b}_{\perp}$'

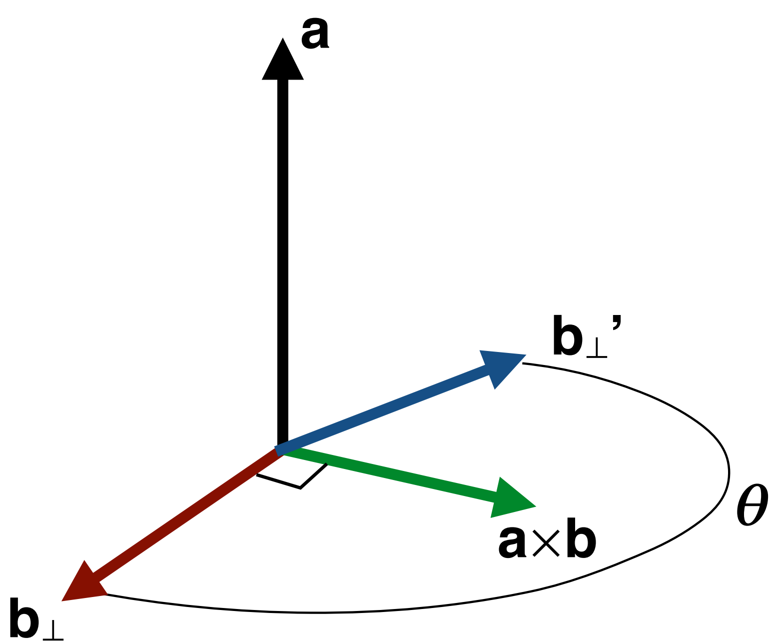
What makes us think we can rotate $\mathbf{b}_{\perp}$ around **a**?



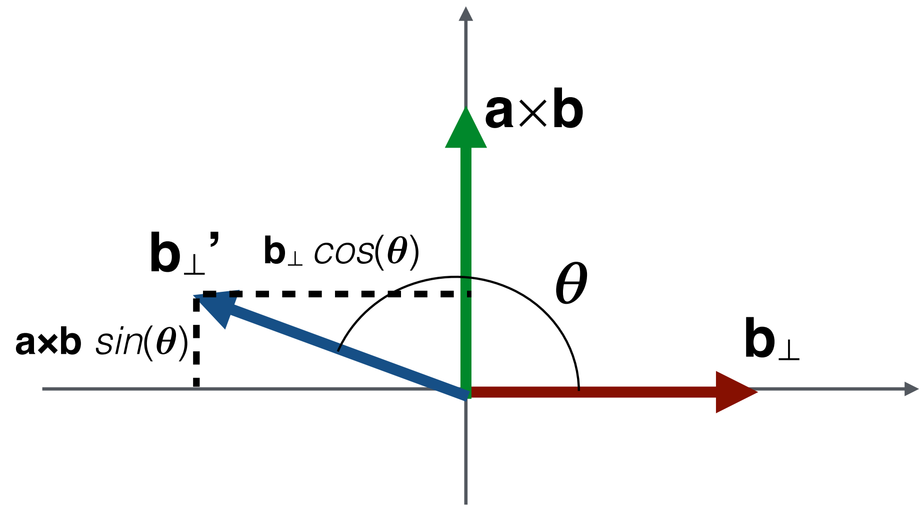
plane orthogonal to **a** defined by
b$_\perp$ and **a**×**b**



assume **b**$_\perp$ aligned with x-axis **e**$_1$
and **a**×**b** aligned with y-axis **e**$_2$



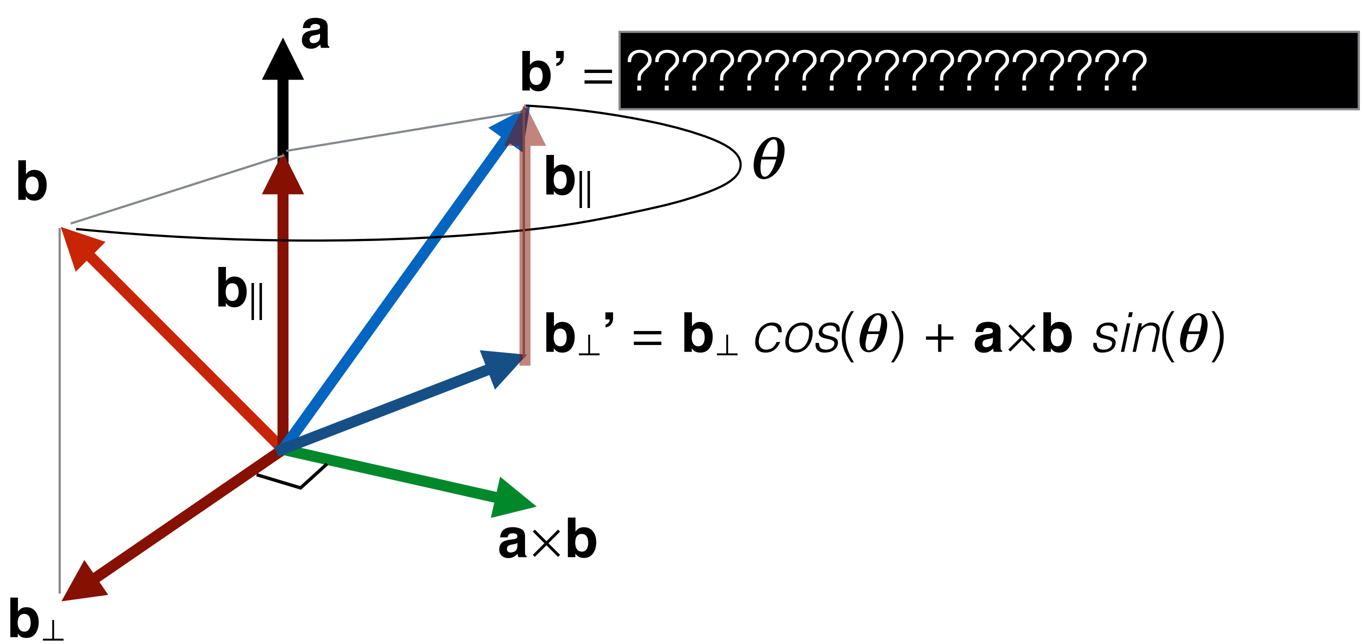
plane orthogonal to **a** defined by $\mathbf{b}_\perp$ and $\mathbf{a}\times\mathbf{b}$

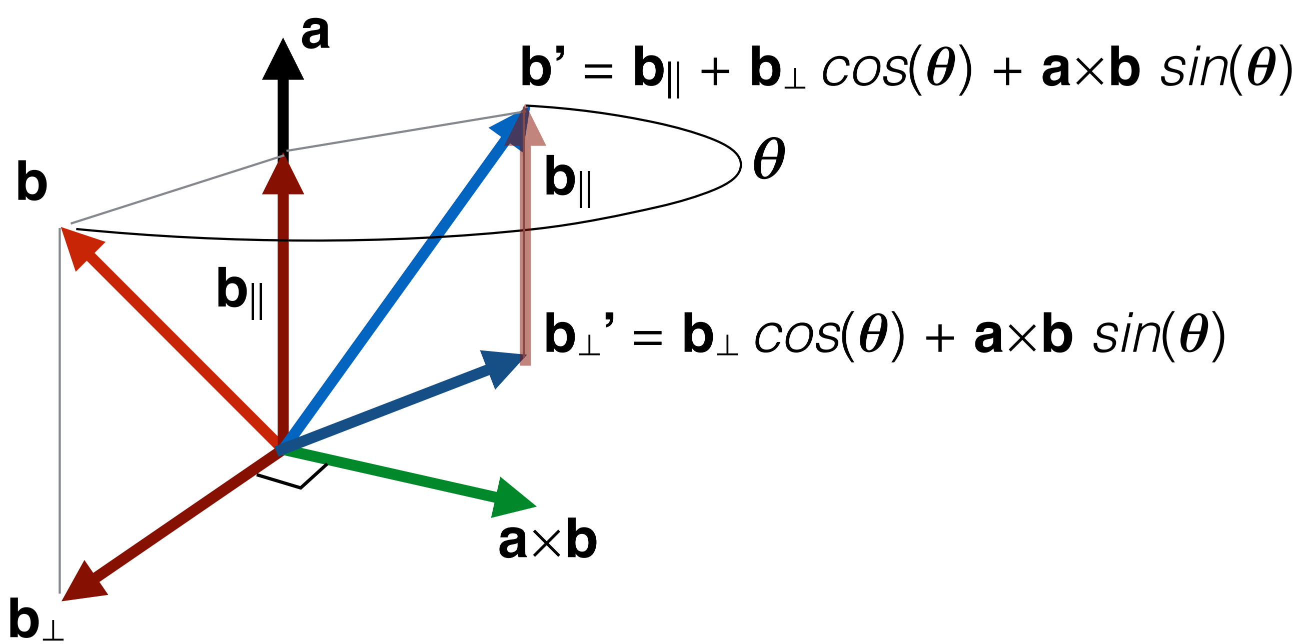


rotation of $\mathbf{b}_\perp$ by θ is then

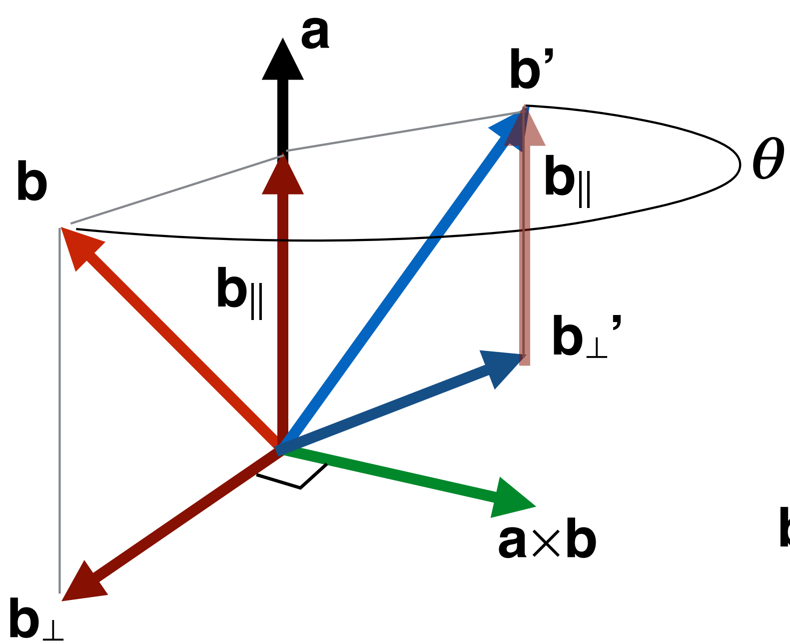
$$\mathbf{b}_\perp' = \mathbf{e}_1\, cos(\theta) + \mathbf{e}_2\, sin(\theta)$$

$$= \mathbf{b}_\perp\, cos(\theta) + \mathbf{a}\times\mathbf{b}\, sin(\theta)$$


a
b' = ???????????????????
θ
b
b∥
b∥
b⊥' = b⊥ cos(θ) + a×b sin(θ)
a×b
b⊥


a
b
$\mathbf{b}' = \mathbf{b}_{\parallel} + \mathbf{b}_{\perp}\, cos(\theta) + \mathbf{a}\times\mathbf{b}\; sin(\theta)$
θ
$\mathbf{b}_{\parallel}$
$\mathbf{b}_{\parallel}$
$\mathbf{b}_{\perp}' = \mathbf{b}_{\perp}\, cos(\theta) + \mathbf{a}\times\mathbf{b}\; sin(\theta)$
$\mathbf{a}\times\mathbf{b}$
$\mathbf{b}_{\perp}$

Rodrigues Rotation Formula



$$\mathbf{b}' = \mathbf{b}_{\|} + \mathbf{b}_{\perp}\, cos(\theta) + \mathbf{a}\times\mathbf{b}\, sin(\theta)$$

substitute out $b_{\perp}$

$$\mathbf{b}' = \mathbf{b}_{\|} + (\mathbf{b}-\mathbf{b}_{\|})\, cos(\theta) + \mathbf{a}\times\mathbf{b}\, sin(\theta)$$

group $b_{\|}$ terms

$$\mathbf{b}' = (1-cos(\theta))\, \mathbf{b}_{\|} + \mathbf{b}\, cos(\theta) + \mathbf{a}\times\mathbf{b}\, sin(\theta)$$

substitute out $b_{\|}$

$$\boxed{\mathbf{b}' = (1-cos(\theta))(\mathbf{ab})\mathbf{a} + \mathbf{b}\, cos(\theta) + \mathbf{a}\times\mathbf{b}\, sin(\theta)}$$

Rodrigues Rotation Matrix

$$R = \cos\theta \mathbf{I} + \sin\theta[\mathbf{u}]_\times + (1 - \cos\theta)\mathbf{u} \otimes \mathbf{u}$$

skew symmetric matrix
of vector u

$$[\mathbf{u}]_\times = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

cross product is multiplication
with skew symmetric matrix

$$\begin{bmatrix} (\mathbf{k} \times \mathbf{v})_x \\ (\mathbf{k} \times \mathbf{v})_y \\ (\mathbf{k} \times \mathbf{v})_z \end{bmatrix} = \begin{bmatrix} k_y v_z - k_z v_y \\ k_z v_x - k_x v_z \\ k_x v_y - k_y v_x \end{bmatrix} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}.$$

Rodrigues Rotation Matrix

$$R = \cos\theta\mathbf{I} + \sin\theta[\mathbf{u}]_\times + (1-\cos\theta)\mathbf{u}\otimes\mathbf{u}$$

skew symmetric matrix
of vector u

outer product

$$[\mathbf{u}]_\times = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

$$\mathbf{u}\otimes\mathbf{u} = \begin{bmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{bmatrix}$$

Rodrigues Rotation Matrix

$$R = \cos\theta \mathbf{I} + \sin\theta[\mathbf{u}]_\times + (1-\cos\theta)\mathbf{u}\otimes\mathbf{u}$$

skew symmetric matrix of vector u

outer product

$$[\mathbf{u}]_\times = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

$$\mathbf{u}\otimes\mathbf{u} = \begin{bmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta + u_x^2(1-\cos\theta) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_y u_x(1-\cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1-\cos\theta) & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_z u_x(1-\cos\theta) - u_y\sin\theta & u_z u_y(1-\cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1-\cos\theta) \end{bmatrix}$$

resulting rotation matrix

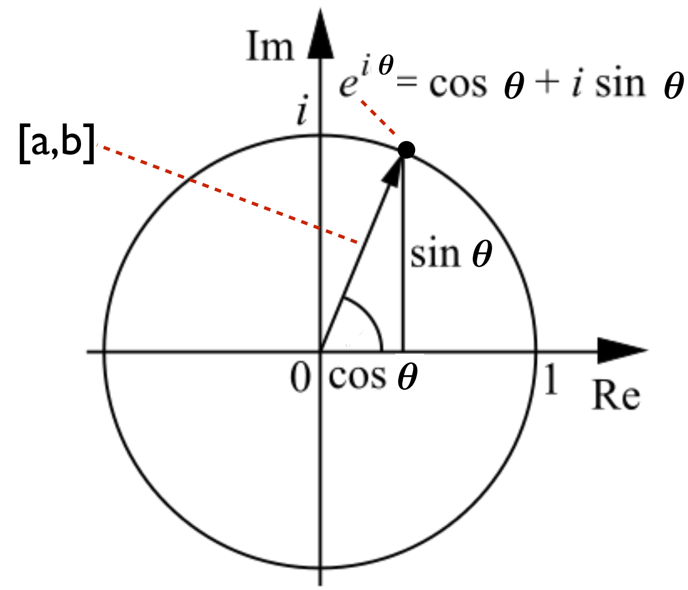
Is there a cleaner expression of axis-angle rotation?

Is there a cleaner expression of axis-angle rotation?

Rotation by complex numbers

Rotation by complex numbers

- Complex number: *a* + *b**i***, ***i*** = *sqrt(-1)*
- *[a,b]* is unit vector in 2D real/imaginary space, and Θ is rotation angle
- Additional 2D rotation can be performed as a complex multiplication, with polar coordinates: $a_i = \cos(\Theta_i)$ and $b_i = \sin(\Theta_i)$
- Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$
- Multiplication of two complex numbers (z and *w*) composes rotation



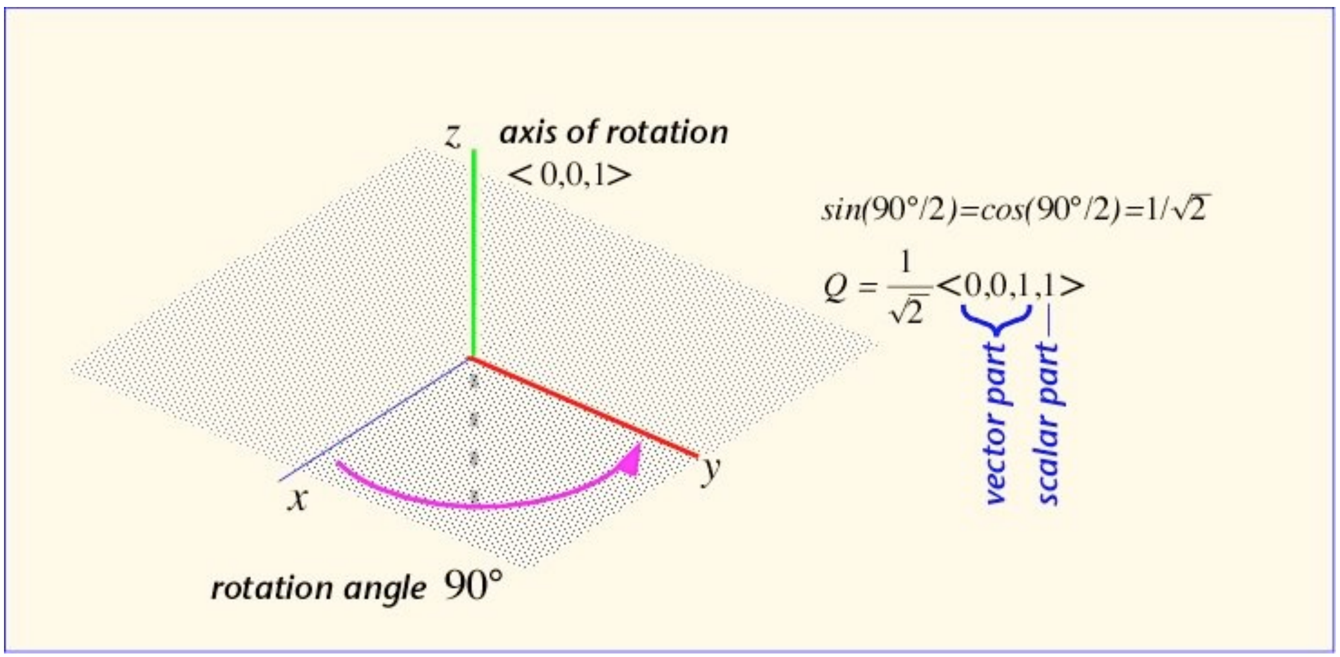
$$zw = (a+bi)(c+di) = e^{\mathrm{i}\phi} r e^{\mathrm{i}\theta} = r e^{\mathrm{i}(\theta+\phi)}$$

Rotation by Quaternion

(No axis order, No gimbal lock)

Quaternions can perform Rodrigues axis-angle 3D rotation

Provide a clean mathematical expression for rotation composition and interpolation



Quaternion

Quaternion plaque on Brougham (Broom) Bridge, Dublin

Sir William Rowan Hamilton
(1805-1865)

Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication $i^2 = j^2 = k^2 = ijk = -1$ & cut it on a stone of this bridge

Quaternions in 3D

- Uses three imaginary numbers ($\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$) to provide a basis that satisfies
  - $\boldsymbol{i}^2 = -1$, $\boldsymbol{j}^2 = -1$, $\boldsymbol{k}^2 = -1$
  - $\boldsymbol{ij} = \boldsymbol{k}$, $\boldsymbol{jk} = \boldsymbol{i}$, $\boldsymbol{ki} = \boldsymbol{j}$, $\boldsymbol{ji} = -\boldsymbol{k}$, $\boldsymbol{kj} = -\boldsymbol{i}$, $\boldsymbol{ik} = -\boldsymbol{j}$
  - Forms a real 3D basis indicated by cross product relations

$$\boldsymbol{i}\times\boldsymbol{j} = -\boldsymbol{j}\times\boldsymbol{i} = \boldsymbol{k}$$
$$\boldsymbol{k}\times\boldsymbol{i} = -\boldsymbol{i}\times\boldsymbol{k} = \boldsymbol{j}$$
$$\boldsymbol{j}\times\boldsymbol{k} = -\boldsymbol{k}\times\boldsymbol{j} = \boldsymbol{i}$$

- Quaternion defined as $\boldsymbol{q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$
  - where a, b, c, d are scalars
  - breaks down into real scalar and imaginary vector: $\boldsymbol{q} = (a, [b,c,d]) = (r, \boldsymbol{v})$

Note: q is typically configuration, but will be used temporarily as a quaternion

Quaternions in 3D

- Set of quaternions is a vector space and has three operations
  - Addition $(r_1,\ \vec{v}_1) + (r_2,\ \vec{v}_2) = (r_1 + r_2,\ \vec{v}_1 + \vec{v}_2)$

$$\boxed{\boldsymbol{q}_1+\boldsymbol{q}_2 = (a+b\boldsymbol{i}+c\boldsymbol{j}+d\boldsymbol{k})(e+f\boldsymbol{i}+g\boldsymbol{j}+h\boldsymbol{k}) = (a+e)+(b+f)\boldsymbol{i}+(c+g)\boldsymbol{j}+(d+h)\boldsymbol{k}}$$

  - Scalar multiplication $\boxed{s\boldsymbol{q}_1 = (sa)+(sb)\boldsymbol{i}+(sc)\boldsymbol{j}+(sd)\boldsymbol{k}}$
  - Quaternion multiplication $(r_1,\ \vec{v}_1)(r_2,\ \vec{v}_2) = (r_1 r_2 - \vec{v}_1 \cdot \vec{v}_2, r_1\vec{v}_2 + r_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2)$

$$\boxed{\begin{aligned}\boldsymbol{q}_1\boldsymbol{q}_2 &= (a+b\boldsymbol{i}+c\boldsymbol{j}+d\boldsymbol{k})(e+f\boldsymbol{i}+g\boldsymbol{j}+h\boldsymbol{k}) \\ &= (ae-bf-cg-dh)+(af+be+ch-dg)\boldsymbol{i} +(ag-bh+ce+df)\boldsymbol{j}+(ah+bg-cf+de)\boldsymbol{k}\end{aligned}}$$

- Not commutative: $q_1q_2 \neq q_2q_1$ Why?

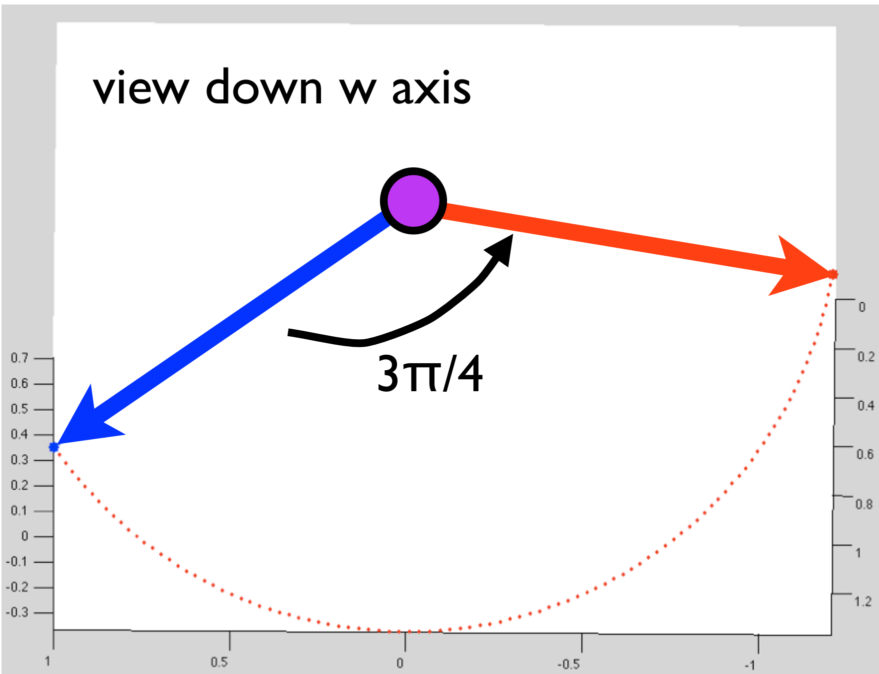
Quaternion Properties

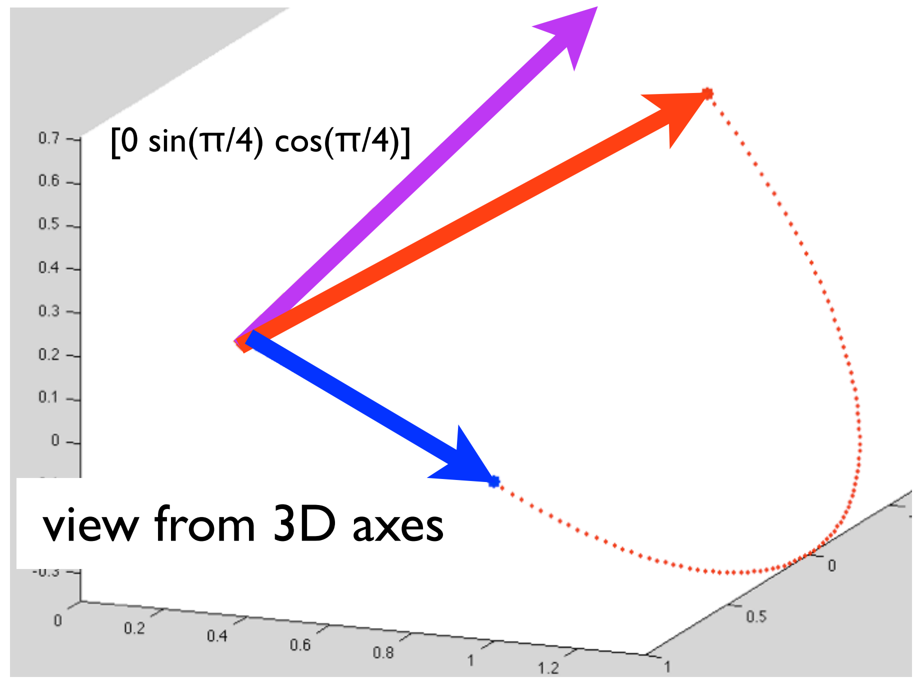
- Norm: $|\boldsymbol{q}|^2 = a^2 + b^2 + c^2 + d^2$
- Conjugate quaternion: $\overline{\boldsymbol{q}} = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k} = (a, -[b,c,d]) = (r, -\boldsymbol{v})$
- Inverse quaternion: $\boldsymbol{q}^{-1} = \overline{\boldsymbol{q}} \,/\, |\boldsymbol{q}|^2$
- Unit quaternion: $|\boldsymbol{q}| = 1$
- Inverse of unit quaternion: $\boldsymbol{q}^{-1} = \overline{\boldsymbol{q}}$

Rotation by Quaternion

- Rotations are represented by unit quaternions
    - quaternion is point on 4D unit sphere geometrically
- Quaternion $\boldsymbol{q} = (a,\boldsymbol{u}) = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} = (\cos(\Theta/2), \boldsymbol{u} \sin(\Theta/2))$
  $= [\cos(\Theta/2), u_x \sin(\Theta/2), u_y \sin(\Theta/2), u_z \sin(\Theta/2)]$
    - $\boldsymbol{u} = [u_x,u_y,u_z]$ is rotation axis, Θ rotation angle
- Rotating a 3D point $\boldsymbol{p}$ by unit quaternion $\boldsymbol{q}$ is performed by conjugation of $\boldsymbol{v}$ by $\boldsymbol{q}$
    - $\boldsymbol{v}' = \boldsymbol{q}\boldsymbol{v}\boldsymbol{q}^{-1}$, where $\boldsymbol{q}^{-1} = a - \boldsymbol{u}$,
    - quaternion $\boldsymbol{v}$ is constructed from point $\boldsymbol{p}$ as $\boldsymbol{v} = 0 + \boldsymbol{p} = 0 + p_x\boldsymbol{i} + p_y\boldsymbol{j} + p_z\boldsymbol{k}$
    - rotated point $\mathbf{p'} = [\mathbf{v'}_x\ \mathbf{v'}_y\ \mathbf{v'}_z]$ is pulled from quaternion resulting from conjugation

Example





Rotation of point $v = 0 + [1\ 1\ 0]$ by

quaternion $w = 3\pi/4 + [0\ \sin(\pi/4)\ \cos(\pi/4)]$

Checkpoint

- What is the unit quaternion for …
  - no rotation? [redacted]
  - rotation 180 degrees about the z axis? [redacted]
  - rotation 90 degrees about the y axis? [redacted]
  - rotation -90 degrees about the x axis? [redacted]

Checkpoint

- What is the unit quaternion for …
  - no rotation? the identity quaternion (1, [0 0 0])
  - rotation 180 degrees about the z axis? [redacted]
  - rotation 90 degrees about the y axis? [redacted]
  - rotation -90 degrees about the x axis? [redacted]

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    - no rotation? the identity quaternion (1, [0 0 0])
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    - rotation 90 degrees about the y axis? [redacted]
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  - no rotation? the identity quaternion (1, [0 0 0])
  - rotation 180 degrees about the z axis? (0, [0 0 1])
  - rotation 90 degrees about the y axis? (sqrt(0.5), [0 sqrt(0.5) 0])
  - rotation -90 degrees about the x axis? [redacted]

Checkpoint

- What is the unit quaternion for …
  - no rotation? the identity quaternion (1, [0 0 0])
  - rotation 180 degrees about the z axis? (0, [0 0 1])
  - rotation 90 degrees about the y axis? (sqrt(0.5), [0 sqrt(0.5) 0])
  - rotation -90 degrees about the x axis? (sqrt(0.5), [-sqrt(0.5) 0 0])

Restating

- Quaternions $\boldsymbol{q}$ and $-\boldsymbol{q}$ give the same rotation
- Composition of rotations $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ equals $\boldsymbol{q}_3 = \boldsymbol{q}_2\boldsymbol{q}_1$
- Remember: 3D rotations do not commute

Rodrigues and Quaternion Equivalency

$$
\begin{aligned}
qpq^{-1} &= qpq^* \\
&= \left(\cos\frac{\alpha}{2} + \hat{a}\sin\frac{\alpha}{2}\right)\vec{b}\left(\cos\frac{\alpha}{2} + \hat{a}\sin\frac{\alpha}{2}\right)^* \\
&= \left(\cos\frac{\alpha}{2} + \hat{a}\sin\frac{\alpha}{2}\right)\vec{b}\left(\cos\frac{\alpha}{2} - \hat{a}\sin\frac{\alpha}{2}\right) \\
&= \left(\vec{b}\cos\frac{\alpha}{2} + \hat{a}\vec{b}\sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2} - \hat{a}\sin\frac{\alpha}{2}\right) \\
&= \vec{b}\cos^2\frac{\alpha}{2} - \vec{b}\hat{a}\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} + \hat{a}\vec{b}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \hat{a}\vec{b}\hat{a}\sin^2\frac{\alpha}{2} \\
&= \vec{b}\cos^2\frac{\alpha}{2} + (\hat{a}\vec{b} - \vec{b}\hat{a})\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \hat{a}\vec{b}\hat{a}\sin^2\frac{\alpha}{2} \\
&= \vec{b}\cos^2\frac{\alpha}{2} + 2(\hat{a}\times\vec{b})\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \left(\vec{b}(\hat{a}\cdot\hat{a}) - 2\hat{a}(\hat{a}\cdot\vec{b})\right)\sin^2\frac{\alpha}{2} \\
&= \vec{b}\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) + (\hat{a}\times\vec{b})2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \hat{a}(\hat{a}\cdot\vec{b})\left(2\sin^2\frac{\alpha}{2}\right) \\
&= \vec{b}\cos\alpha + (\hat{a}\times\vec{b})\sin\alpha + \hat{a}(\hat{a}\cdot\vec{b})(1-\cos\alpha) \\
qpq^{-1} &= (1-\cos\alpha)(\hat{a}\cdot\vec{b})\hat{a} + \vec{b}\cos\alpha + (\hat{a}\times\vec{b})\sin\alpha
\end{aligned}
$$

Quaternion to Rotation Matrix

- Inhomogeneous conversion to 3D rotation matrix of $\mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T$

$$\begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2-q_0q_3) & 2(q_0q_2+q_1q_3) \\ 2(q_1q_2+q_0q_3) & 1-2(q_1^2+q_3^2) & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_0q_1+q_2q_3) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

or equivalently, homogeneous conversion

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_0q_2+q_1q_3) \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_0q_1+q_2q_3) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

- Rotation matrix to quaternion can also be performed

1) form unit quaternion from axis and motor angle

$$q = [\cos(\Theta/2), u_x \sin(\Theta/2), u_y \sin(\Theta/2), u_z \sin(\Theta/2)]$$

2) convert quaternion to rotation matrix

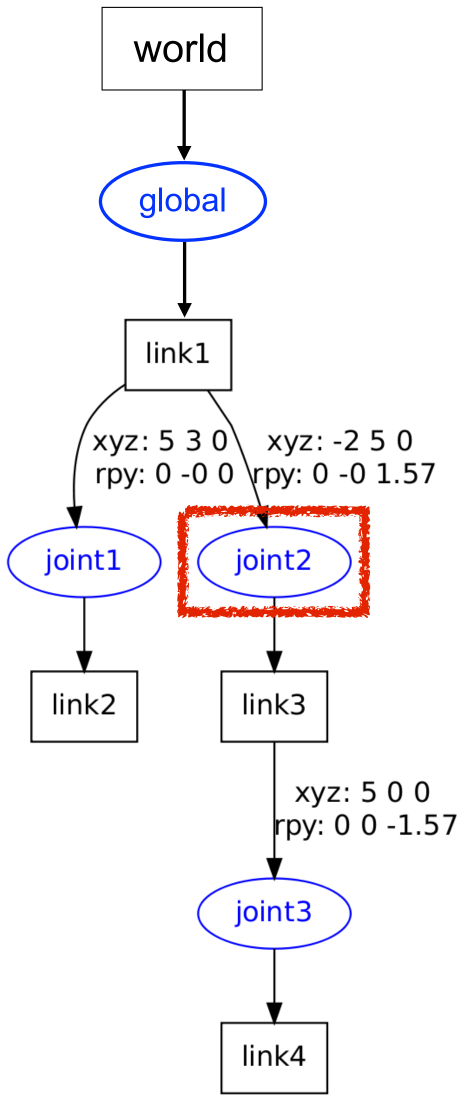
- Inhomogeneous conversion to 3D rotation matrix of $\mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T$

$$\begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2-q_0q_3) & 2(q_0q_2+q_1q_3) \\ 2(q_1q_2+q_0q_3) & 1-2(q_1^2+q_3^2) & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_0q_1+q_2q_3) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

or equivalently, homogeneous conversion

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_0q_2+q_1q_3) \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_0q_1+q_2q_3) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

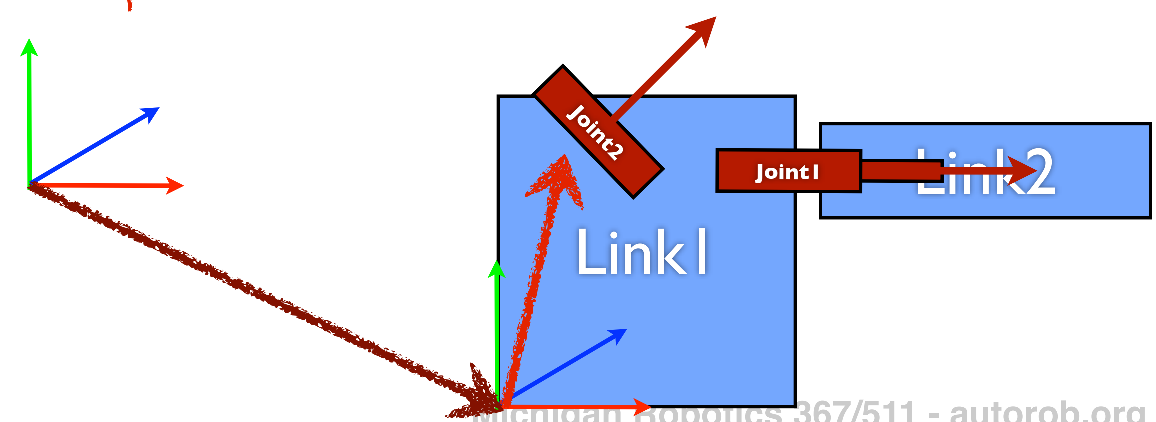
- Rotation matrix to quaternion can also be performed

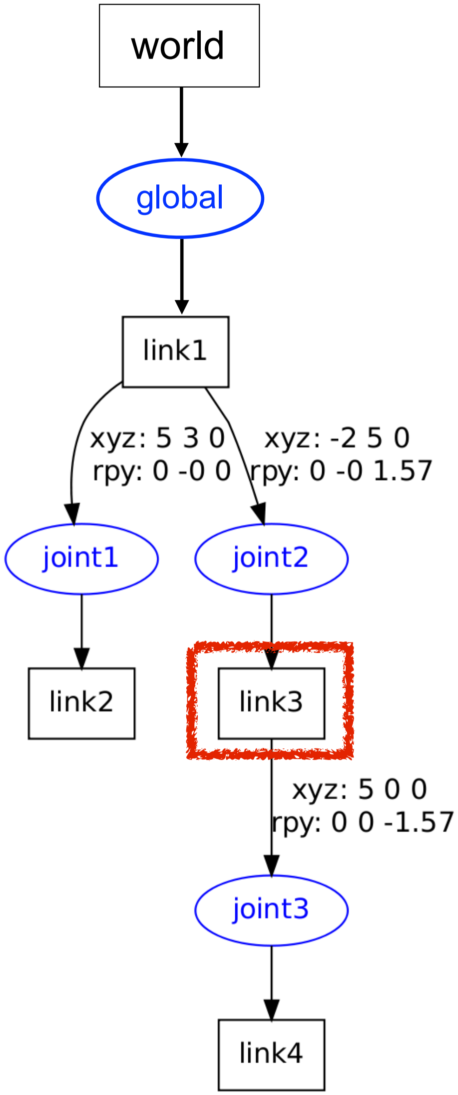


$D^{w}_{1}*R^{w}_{1}*D^{1}_{3}*R^{1}_{3}*R_{u2}(q_2)$
$D^{w}_{1}*R^{w}_{1}$
I

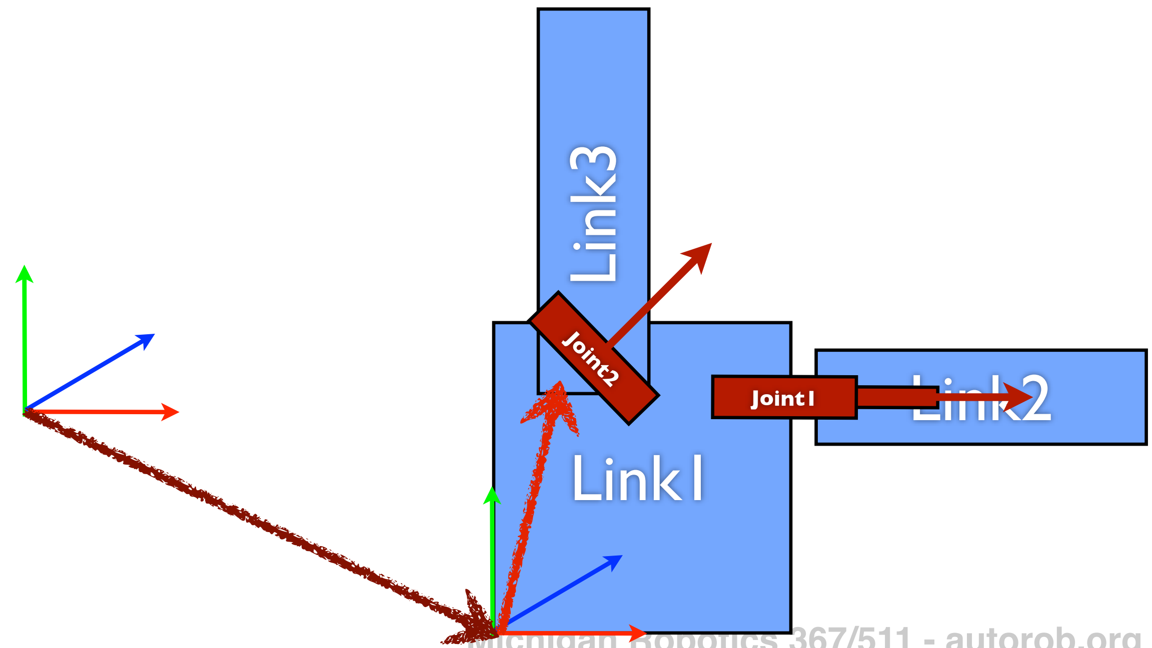
```
//joint motor rotation axis
robot.joints["joint2"].axis = {0.707,0.0, 0.707}
```

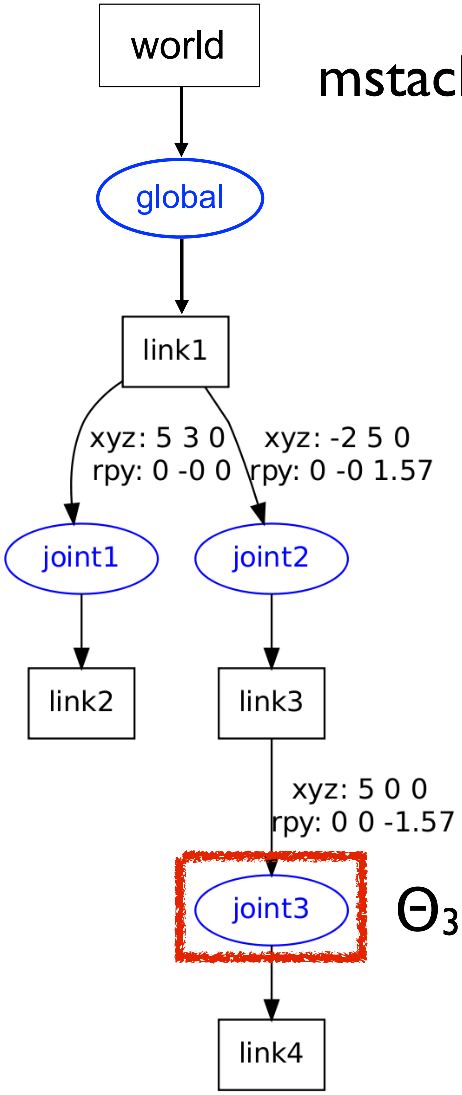
1) form unit quaternion from axis and motor angle
2) convert quaternion to rotation matrix





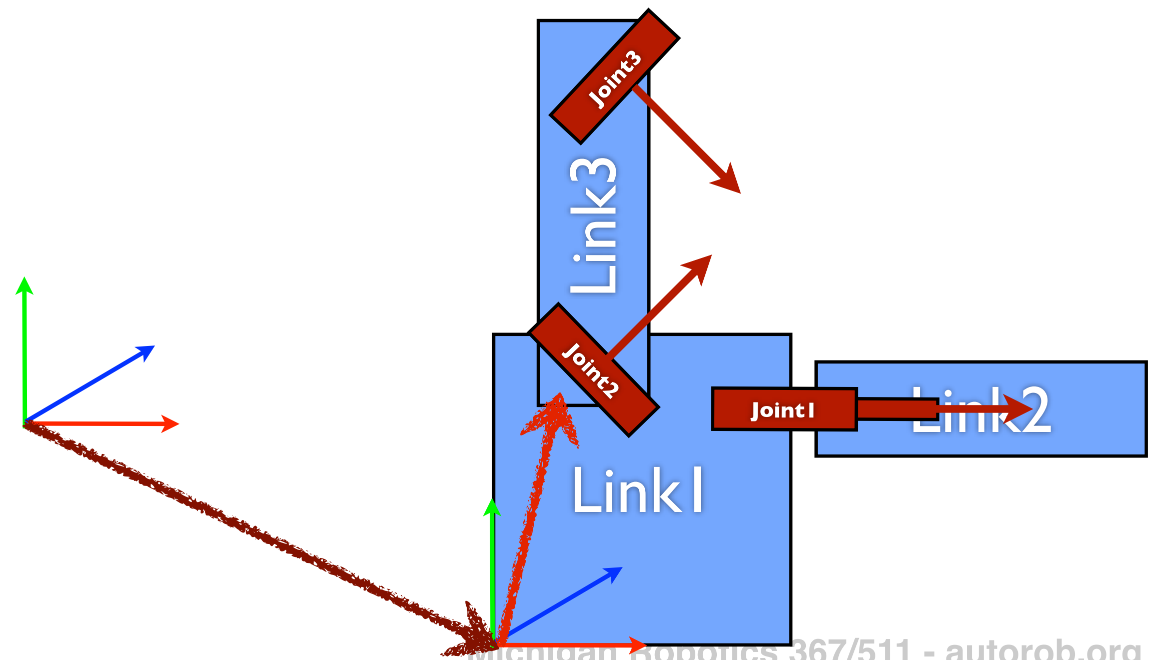
$D^{w}_{1} * R^{w}_{1} * D^{1}_{3} * R^{1}_{3} * R_{u2}(q_2)$
$D^{w}_{1} * R^{w}_{1}$
I

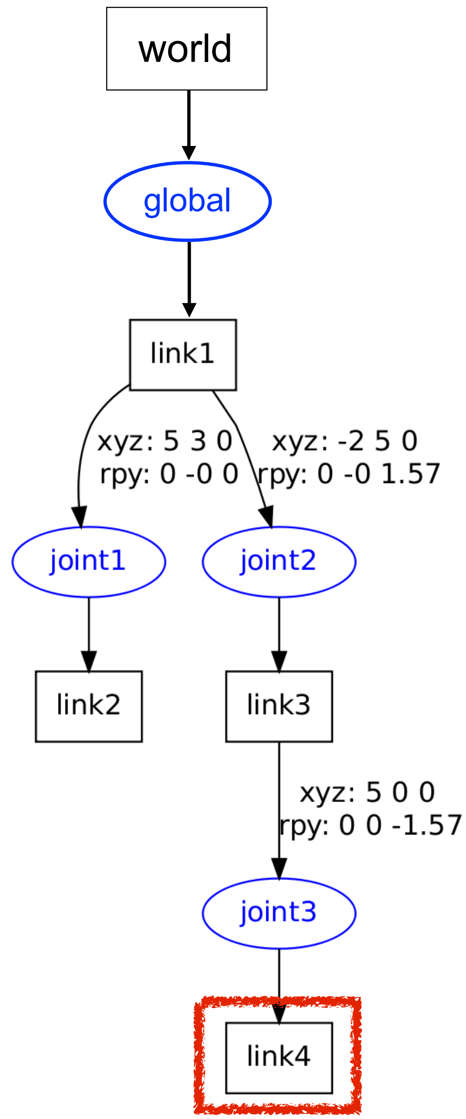




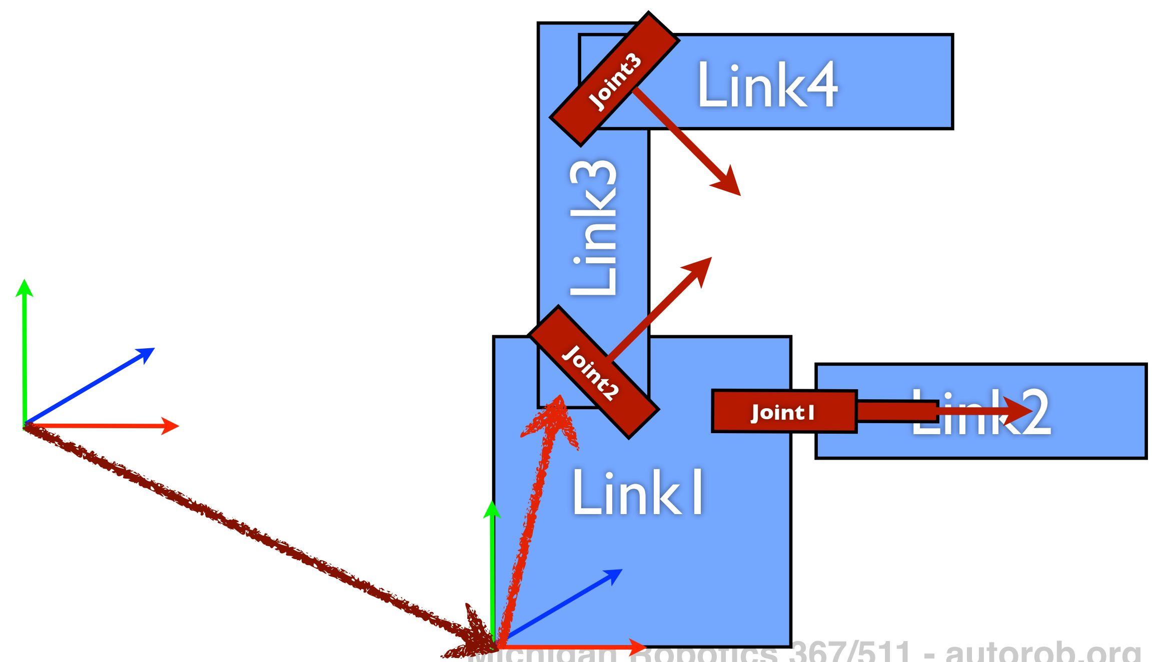
mstack=

$D^{w}_{1} * R^{w}_{1} * D^{1}_{3} * R^{1}_{3} * R_{u2}(q_2) * D^{3}_{4} * R^{3}_{4} * R_{u3}(q_3)$
$D^{w}_{1} * R^{w}_{1} * D^{1}_{3} * R^{1}_{3} * R_{u2}(q_2)$
$D^{w}_{1} * R^{w}_{1}$
I

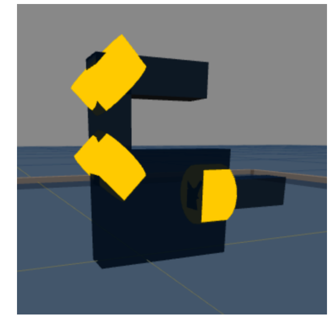




$D^{w}_{1} * R^{w}_{1} * D^{1}_{3} * R^{1}_{3} * R_{u2}(q_2) * D^{3}_{4} * R^{3}_{4} * R_{u3}(q_3)$
$D^{w}_{1} * R^{w}_{1} * D^{1}_{3} * R^{1}_{3} * R_{u2}(q_2)$
$D^{w}_{1} * R^{w}_{1}$
I



Forward Kinematics



Infer: pose of each joint and link in a common world workspace

Lecture 7 ☑ 1) How to represent homogeneous transforms?

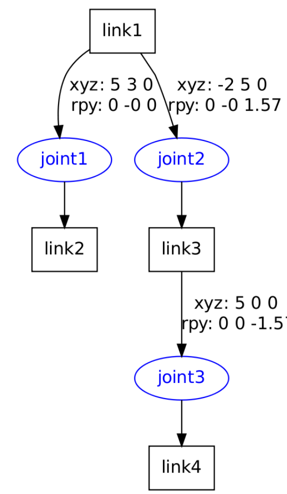
☑ 2) How to compute transform to endeffector?

Assuming as given the:

☑ geometry of each link

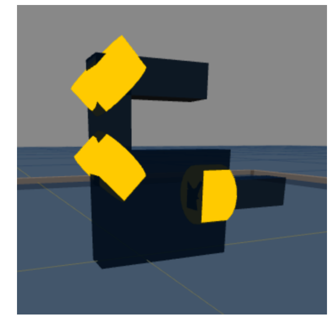
☑ robot's kinematic definition

☑ **current state of all joints** This lecture



- reactive control for choreography Lecture 9

Forward Kinematics



Infer: pose of each joint and link in a common world workspace

Lecture 7 ☑ 1) How to represent homogeneous transforms?
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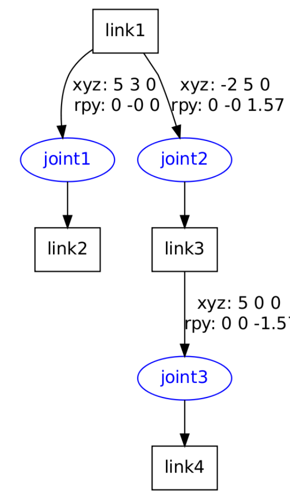
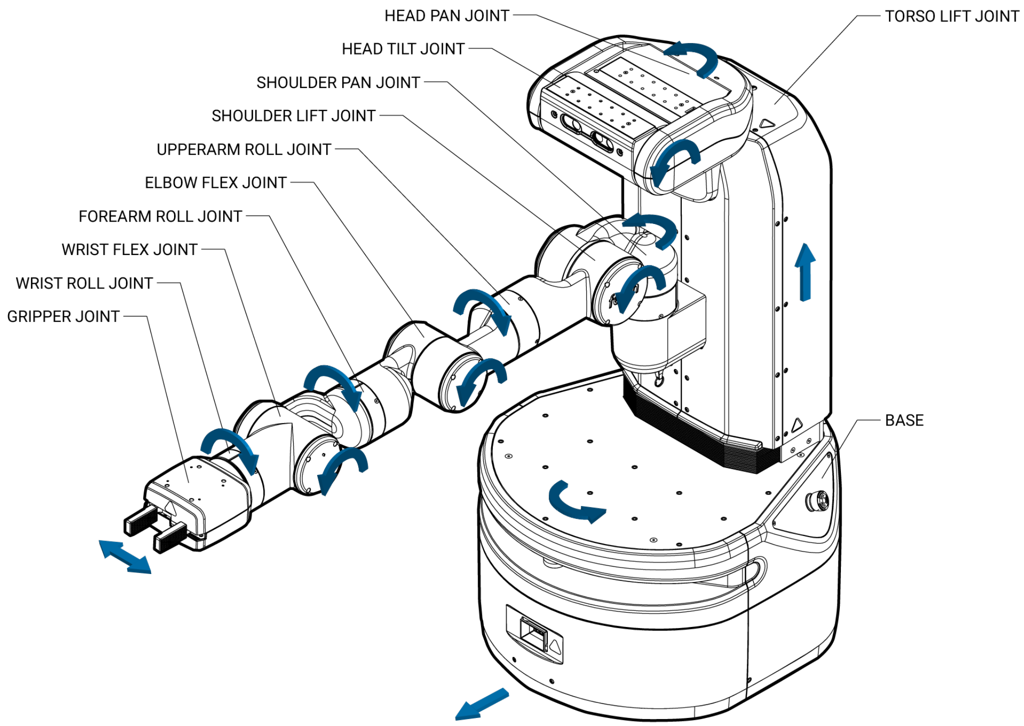


Assuming as given the:

- ☑ geometry of each link
- ☑ robot's kinematic definition
- ☑ current state of all joints   This lecture
- reactive control for choreography   Lecture 9

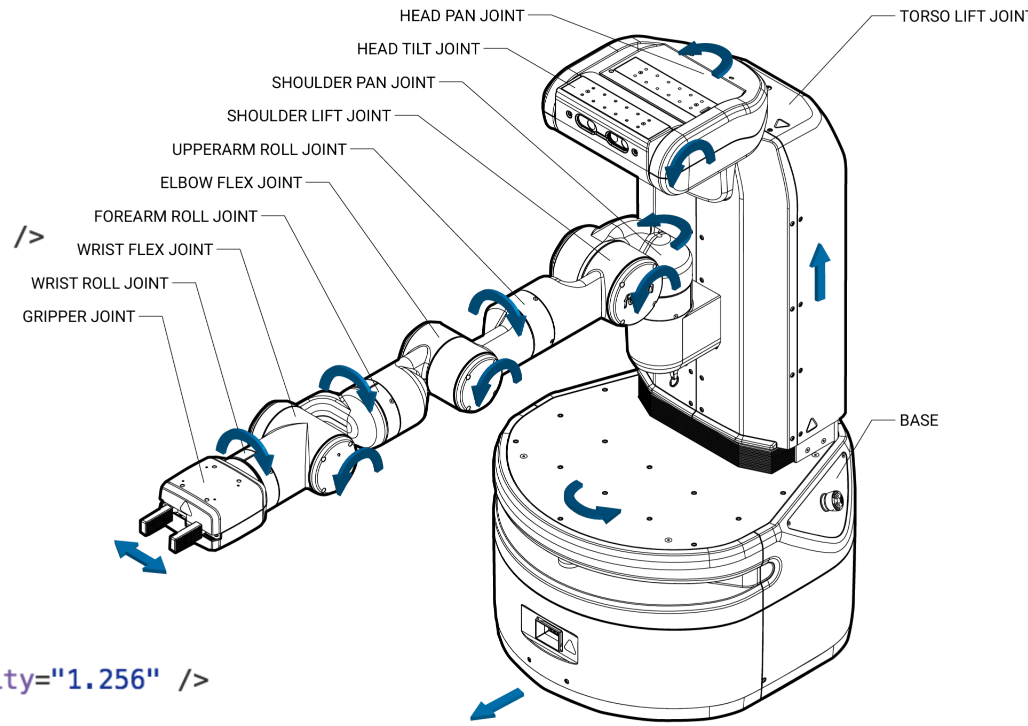
Can a joint move infinitely far?



Joint Limits

Prismatic joint description

```
<joint name="torso_lift_joint" type="prismatic">
  <origin rpy="-6.123E-17 0 0" xyz="-0.086875 0 0.37743" />
  <parent link="base_link" />
  <child link="torso_lift_link" />
  <axis xyz="0 0 1" />
  <limit effort="450.0" lower="0" upper="0.4" velocity="0.1" />
<dynamics damping="100.0" /></joint>
```



Revolute joint description

```
<joint name="shoulder_pan_joint" type="revolute">
  <origin rpy="0 0 0" xyz="0.119525 0 0.34858" />
  <parent link="torso_lift_link" />
  <child link="shoulder_pan_link" />
  <axis xyz="0 0 1" />
  <dynamics damping="1.0" />
  <limit effort="33.82" lower="-1.6056" upper="1.6056" velocity="1.256" />
</joint>
```

Continuous joints have no limits

Joint Limits

Prismatic joint description

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<joint name="torso_lift_joint" type="prismatic">
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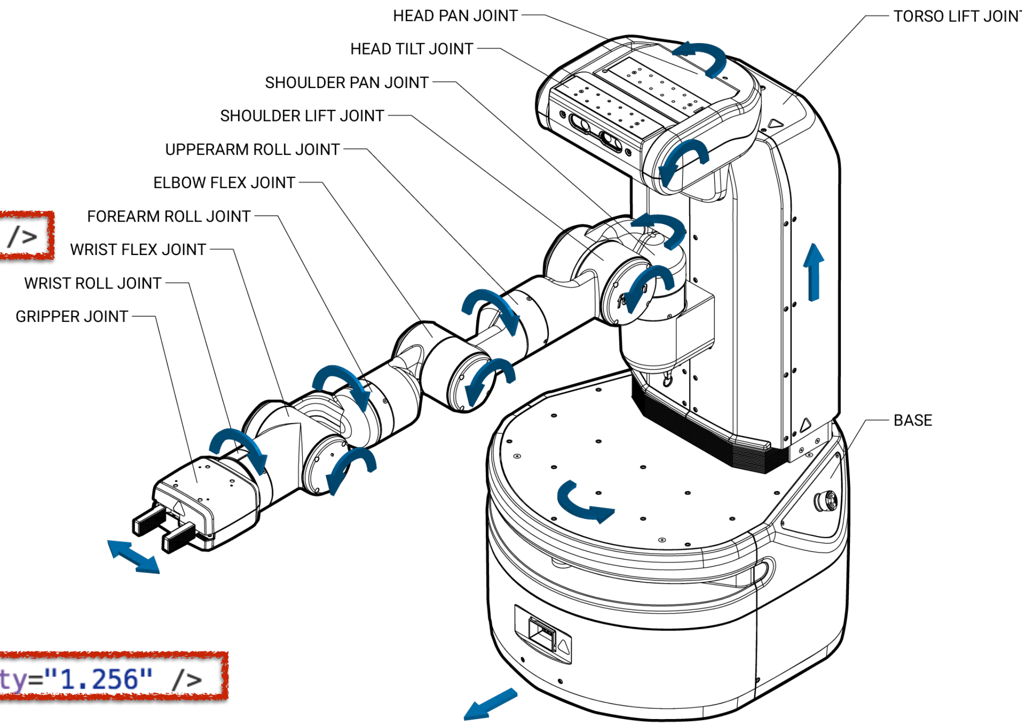
  <limit effort="33.82" lower="-1.6056" upper="1.6056" velocity="1.256" />
</joint>
```



```
robot.joints.torso_lift_joint = {parent:"base_link", child:"torso_lift_link"};
robot.joints.torso_lift_joint.axis = [0,0,1];
robot.joints.torso_lift_joint.type = "prismatic";
robot.joints.torso_lift_joint.origin = {xyz: [-0.086875,0,0.37743], rpy:[-6.123E-17,0,0]};
robot.joints.torso_lift_joint.limit = {lower:0, upper:0.4};
```

Prismatic joint description

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<joint name="torso_lift_joint" type="prismatic">
  <origin rpy="-6.123E-17 0 0" xyz="-0.086875 0 0.37743" />
  <parent link="base_link" />
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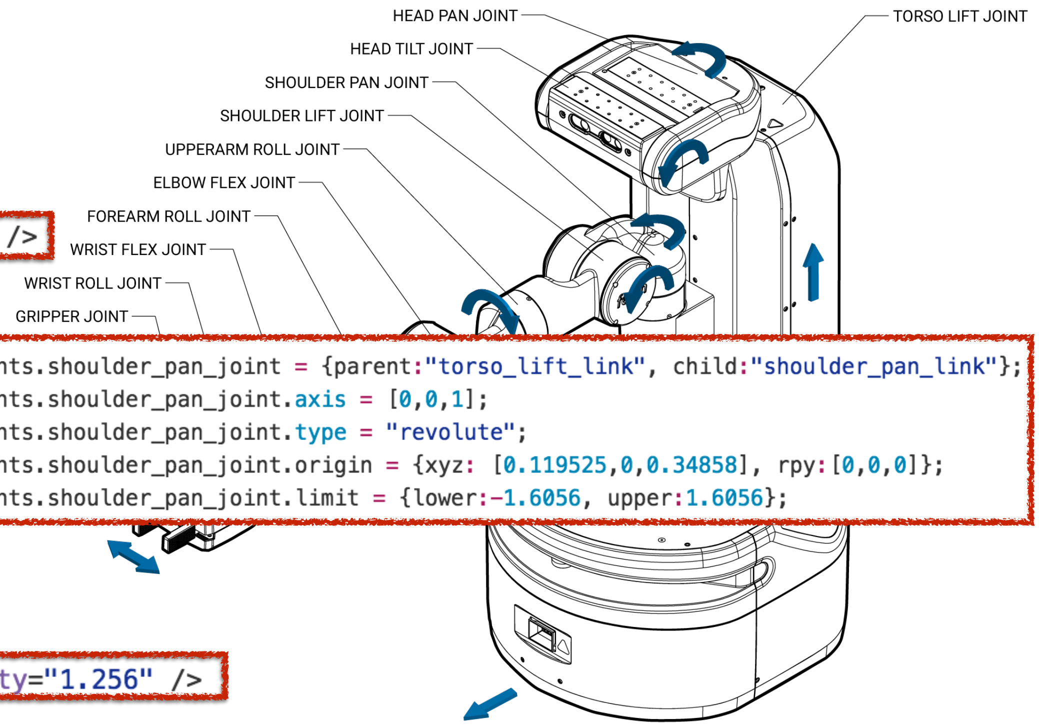
  <axis xyz="0 0 1" />
  <limit effort="450.0" lower="0" upper="0.4" velocity="0.1" />
<dynamics damping="100.0" /></joint>
```



Revolute joint description

```
robot.joints.shoulder_pan_joint = {parent:"torso_lift_link", child:"shoulder_pan_link"};
robot.joints.shoulder_pan_joint.axis = [0,0,1];
robot.joints.shoulder_pan_joint.type = "revolute";
robot.joints.shoulder_pan_joint.origin = {xyz: [0.119525,0,0.34858], rpy:[0,0,0]};
robot.joints.shoulder_pan_joint.limit = {lower:-1.6056, upper:1.6056};
```

```
<joint name="shoulder_pan_joint" type="revolute">
  <origin rpy="0 0 0" xyz="0.119525 0 0.34858" />
  <parent link="torso_lift_link" />
  <child link="shoulder_pan_link" />
  <axis xyz="0 0 1" />
  <dynamics damping="1.0" />
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</joint>
```

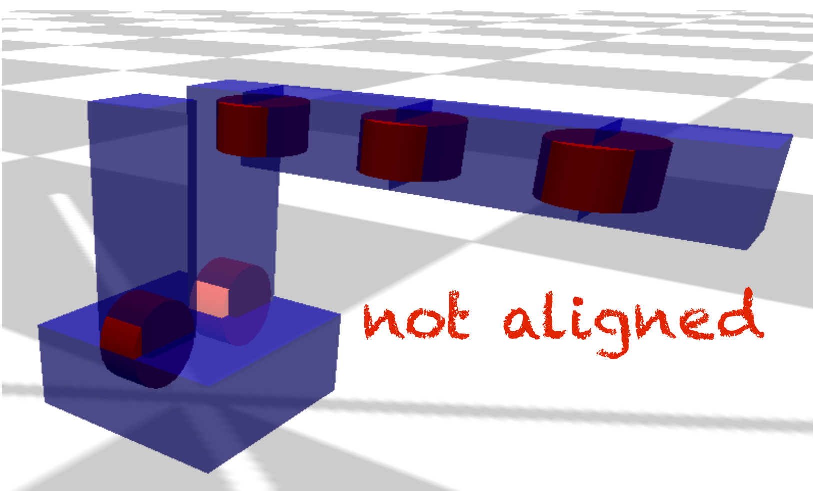
Important notes

Important notes

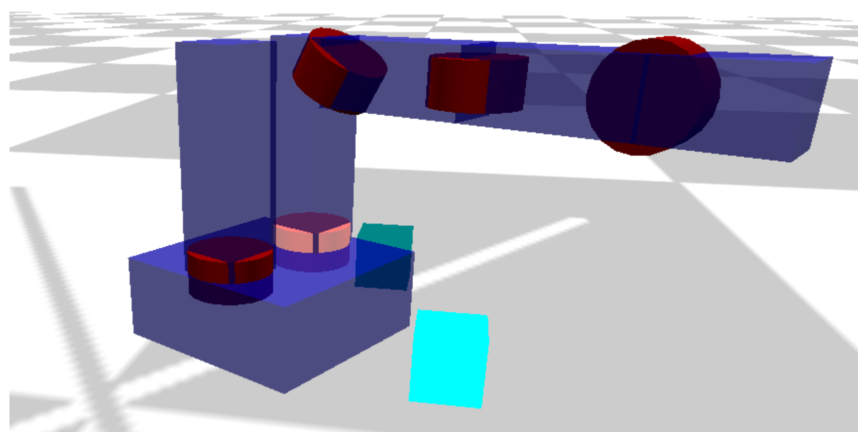
- Rotation order I use: **XYZ** ($R_zR_yR_x$)
- vector_cross(): code stencil tests for and uses this function
- Base controls: must be implemented for interactive control
- The ".origin" field of links and joints are used to store transforms without consideration of joint motion (provided only for debugging)
- A joint and its child link will share the same coordinate frame

KinEval joint cylinder rendering

- threejs creates cylinders with axes aligned along y-axis
- you need to implement vector_cross() for KinEval to render joint cylinders properly along joint axis



aligned



Global controls for base

- Assume we have a base that is holonomic wrt. ground plane
  - holonomic: can move in any direction
- kineval_userinput.js assumes:

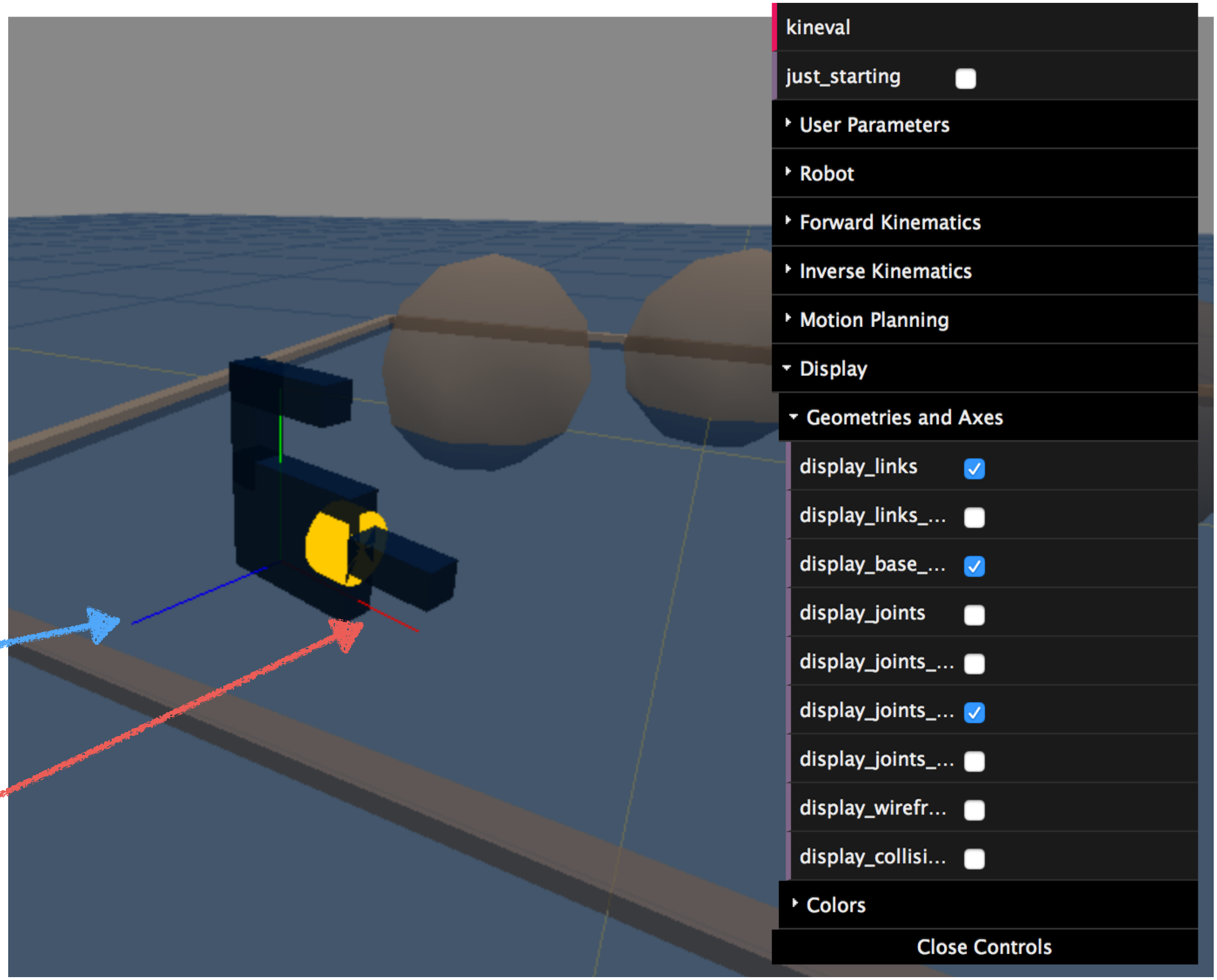
How to perform this base movement?

Transform vectors for heading (local z-axis) and lateral (local x-axis) of robot base into world coordinates

Store transformed vectors in variables “robot_heading” and “robot lateral”

Forward heading of the robot

Lateral heading of the robot



How do we define the kinematics of a robot?

(revisited)

How do we define the kinematics of a robot?

(revisited)

Alternatives for FK

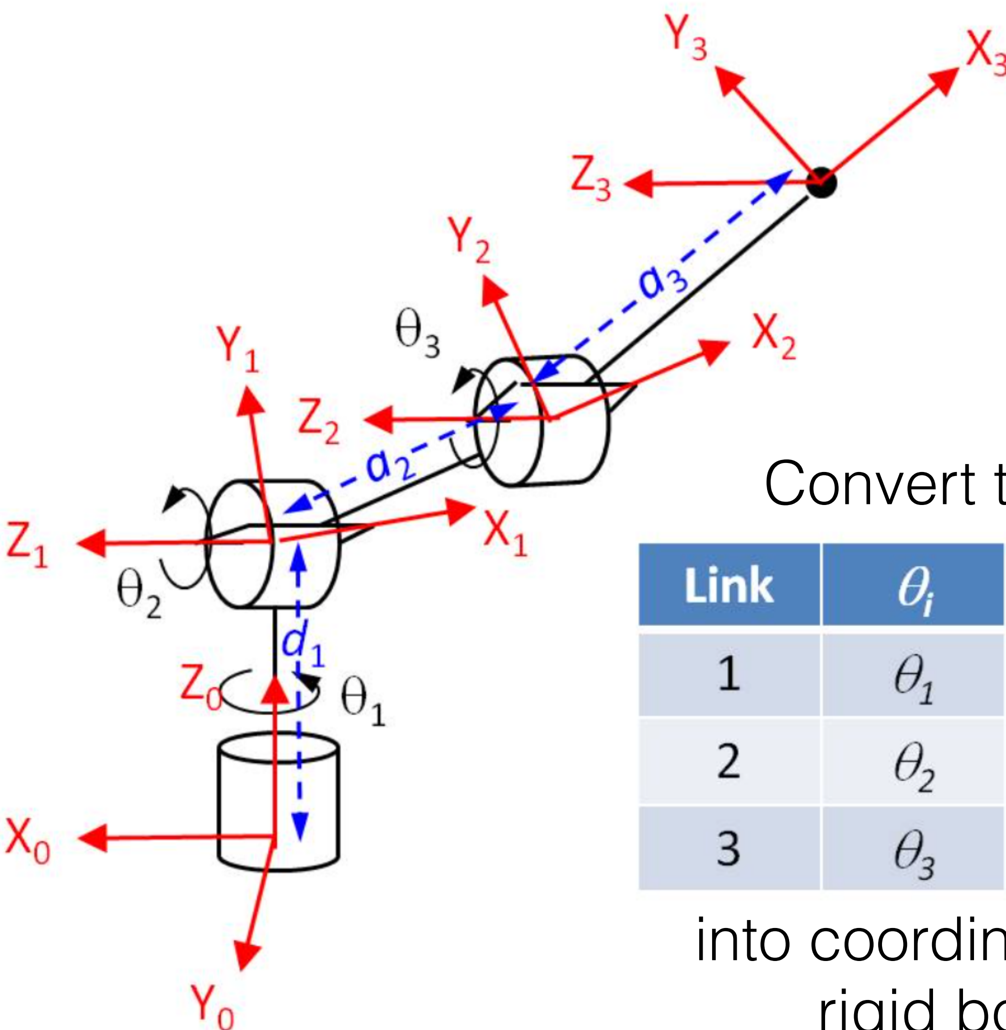
Alternatives for FK

- **Denavit-Hartenberg Convention** ROB 550
- Product of Exponentials with Matrix Stack AutoRob
- Screw vectors as Dual Quaternions
  [Kenwright 2012, Daniilidis 1999]

Denavit-Hartenberg Transform

$$A_i \quad = \quad Rot_{z,\theta_i} \mathrm{Trans}_{z,d_i} \mathrm{Trans}_{x,a_i} Rot_{x,\alpha_i}$$

- $\boldsymbol{A_i}$ represents the transform between a link *i* and its parent *i-1*
    - z-axis of parent link *i-1* aligned with DoF axis of joint *i*
- 4 parameters for joint angle (θ_i), link offset (d_i), link length(a_i), link twist (α_i)
    - Only one parameter will be variable, depending on joint type
    - Each parameter applies a rotation or a translation
- Composed into one homogeneous transformation by matrix multiplication



DH example for individual exploration: 3 DoF robot arm

Convert these parameters

Link	θ_i	d_i	a_i	α_i
1	θ_1	d_1	0	90°
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0

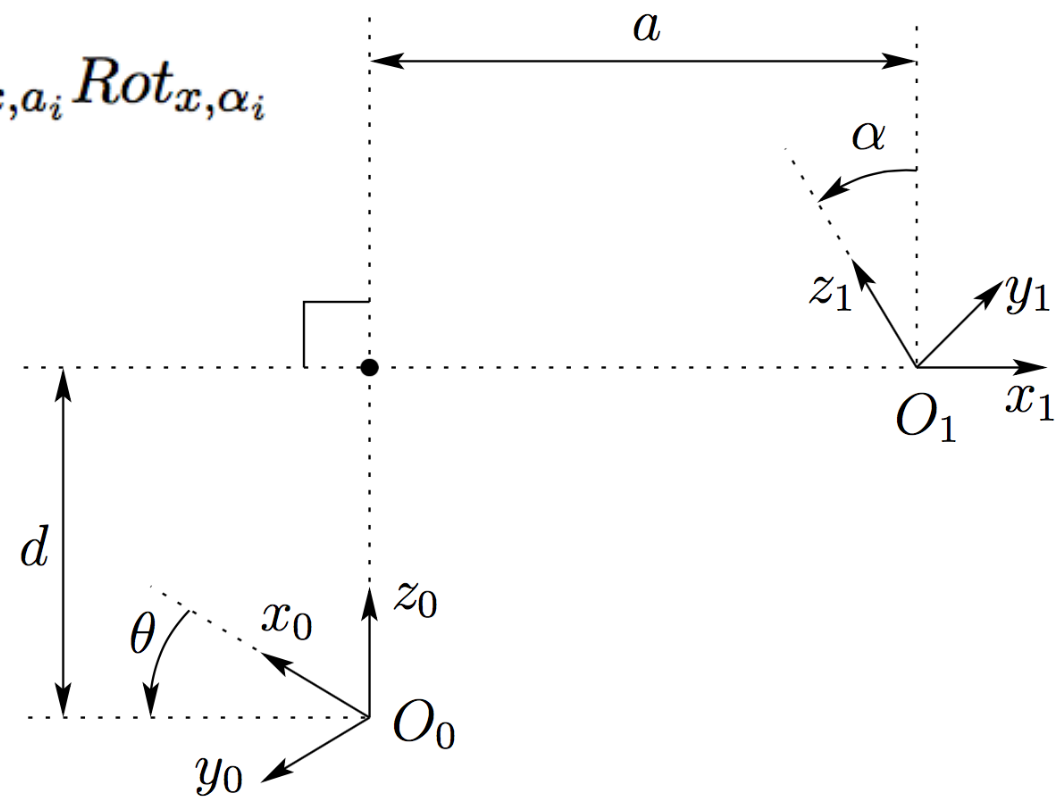
into coordinate frames for each rigid body of the robot

DH Properties

$$A_i = Rot_{z,\theta_i} \text{Trans}_{z,d_i} \text{Trans}_{x,a_i} Rot_{x,\alpha_i}$$

- **DH1:** The z-axis of a link and x-axis of its child link are *perpendicular*

- **DH2:** The z-axis of a link and x-axis of its child link *intersect*



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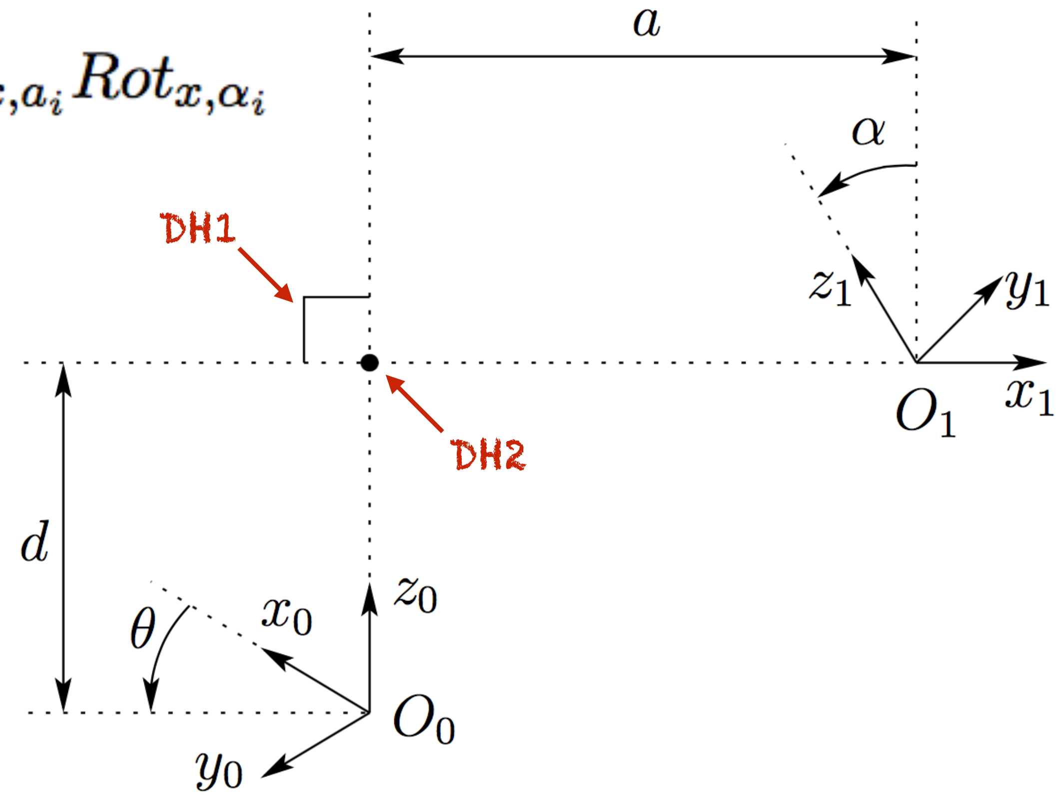
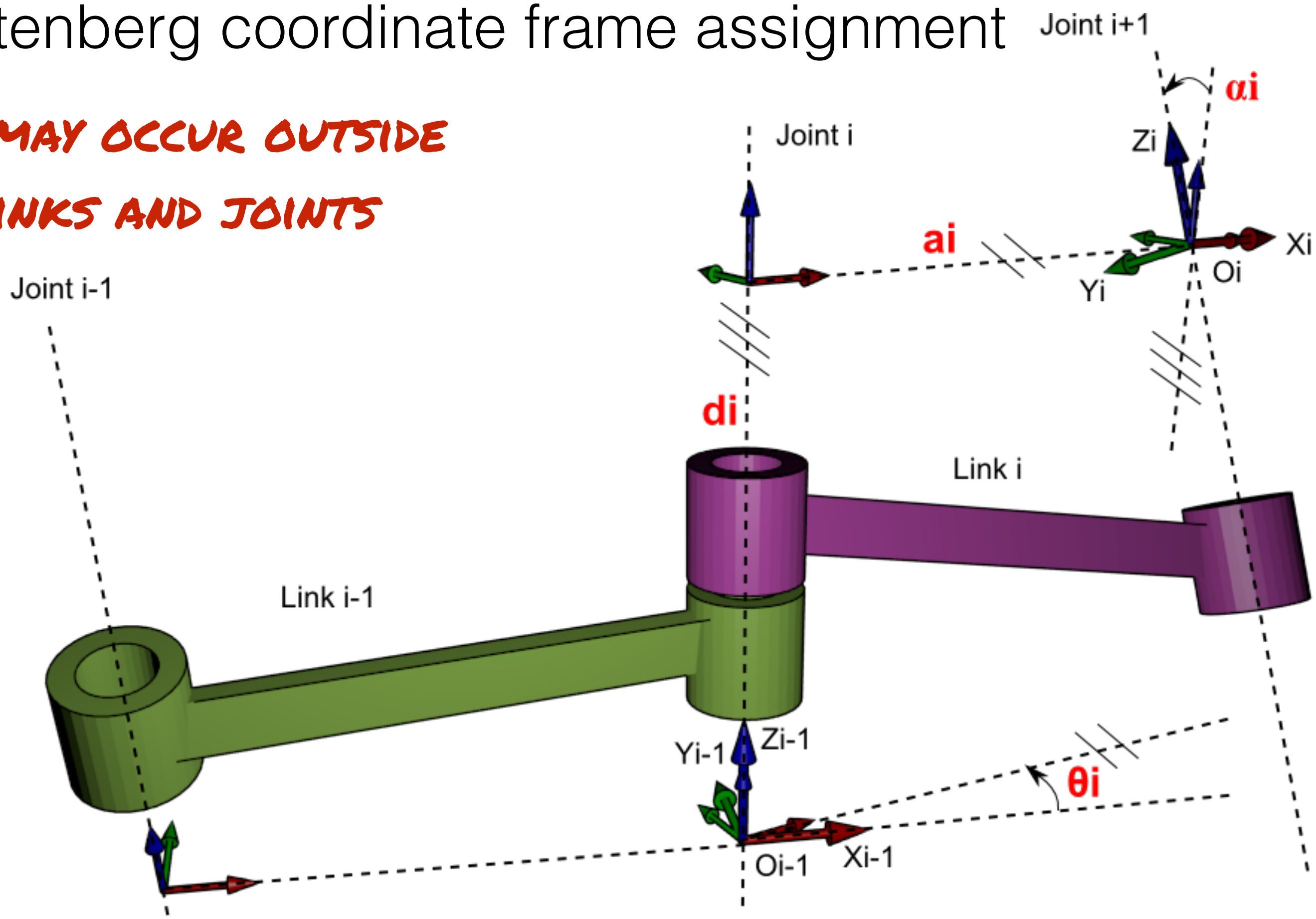


Denavit-Hartenberg coordinate frame assignment

FRAMES MAY OCCUR OUTSIDE THE LINKS AND JOINTS



Coordinate frames for each body align the z-axis with axis of joint rotation and the x-axis with the direction of the link

D-H versus Matrix stack

- 4 parameters to transform between links
- 4 matrices to compose per joint
- Uniformity in frame selection
- Only link frames are necessary for computations
- Less intuitive

- 10 parameters to transform between links
- 3 matrices to compose per joint
- Flexibility in frame selection
- Child link shares frame with parent joint
- More bookkeeping

Approaches to FK

- Denavit-Hartenberg Convention ROB 550
- Product of Exponentials with Matrix Stack AutoRob
- **Screw vectors as Dual Quaternions**

[Kenwright 2012, Daniilidis 1999]

Dual Quaternion

- Dual number: $\check{z} = a + \epsilon b \quad \text{with} \quad \epsilon^2 = 0$
- Dual quaternion: $\check{\boldsymbol{q}} = \boldsymbol{q} + \epsilon \boldsymbol{q}'$
  - comprised of a real quaternion $\boldsymbol{q}$ and dual quaternion $\boldsymbol{q}'$
- Operations include addition, scalar multiplication, and multiplication:

$$\check{\boldsymbol{q}}_1 + \check{\boldsymbol{q}}_2 = (\check{s}_1 + \check{s}_2, \check{\vec{\boldsymbol{q}}}_1 + \check{\vec{\boldsymbol{q}}}_2),$$

$$\check{\lambda}(\check{s}, \check{\vec{\boldsymbol{q}}}) = (\check{\lambda}\check{s}, \lambda\check{\vec{\boldsymbol{q}}}),$$

$$\check{\boldsymbol{q}}_1 \check{\boldsymbol{q}}_2 = (\check{s}_1 \check{s}_2 - \check{\boldsymbol{q}}_1^T \check{\boldsymbol{q}}_2, \check{s}_1 \check{\vec{\boldsymbol{q}}}_2 + \check{s}_2 \check{\vec{\boldsymbol{q}}}_1 + \check{\vec{\boldsymbol{q}}}_1 \times \check{\vec{\boldsymbol{q}}}_2)$$

Screw motion

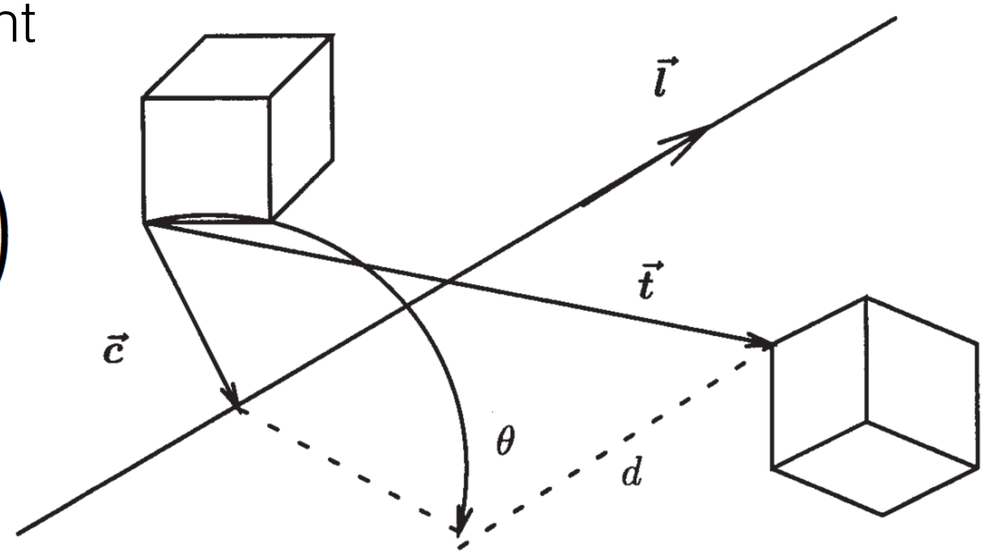
- Every rigid motion can be modeled as a rotation with angle $\boldsymbol{\theta}$ about axis ***c*** with direction ***l*** and subsequent translation ***d*** along the axis

- Dual quaternion can represent screw motion

$$\check{\boldsymbol{q}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\vec{\boldsymbol{l}} \end{pmatrix} + \epsilon \begin{pmatrix} -\frac{d}{2}\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2}\vec{\boldsymbol{m}} + \frac{d}{2}\cos\frac{\theta}{2}\vec{\boldsymbol{l}} \end{pmatrix}$$

- Rigid transformation of a line by a dual quaternion

$$\check{\boldsymbol{l}}_a = \check{\boldsymbol{q}}\check{\boldsymbol{l}}_b\bar{\check{\boldsymbol{q}}}$$



Daniilidis 1999

HRP-2 Aizu-Bandaisan
https://youtu.be/6hLcz-c1Y-M

Next lecture: Finite State Machines