Linear Algebra Refresher

EECS 398
Intro. to Autonomous Robotics

ME/EECS 567 ROB 510
Robot Modeling and Control

Fall 2018

autorob.org

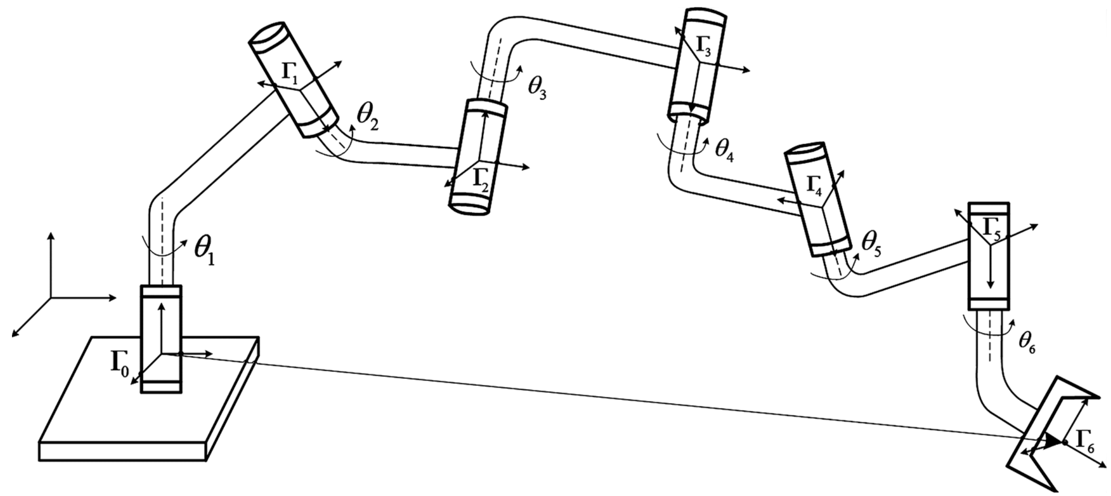
Objective

Goal: Given the structure of a robot arm, compute

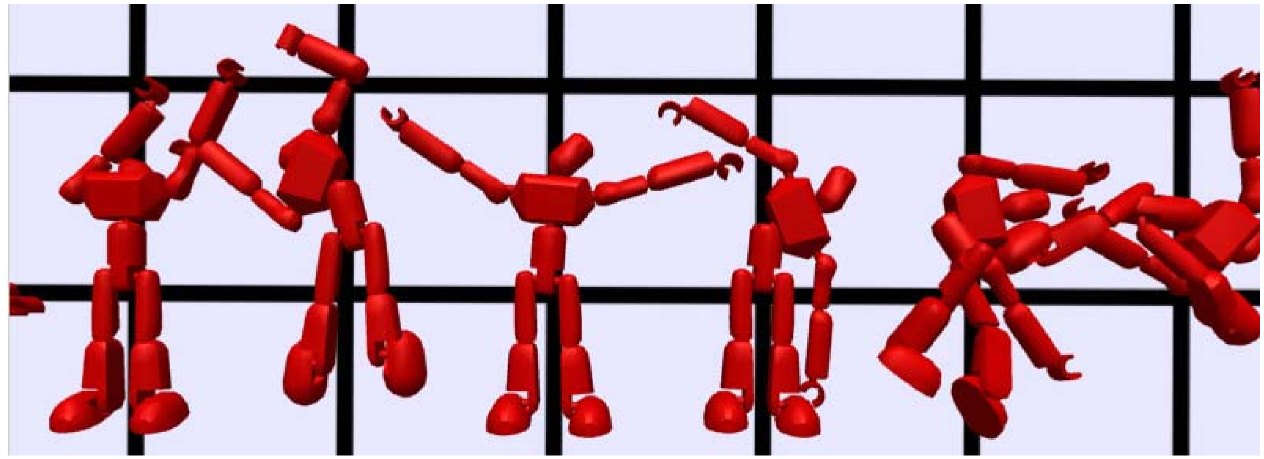
– **Forward kinematics**: inferring the pose of the end-effector, given the state (angle) of each joint.

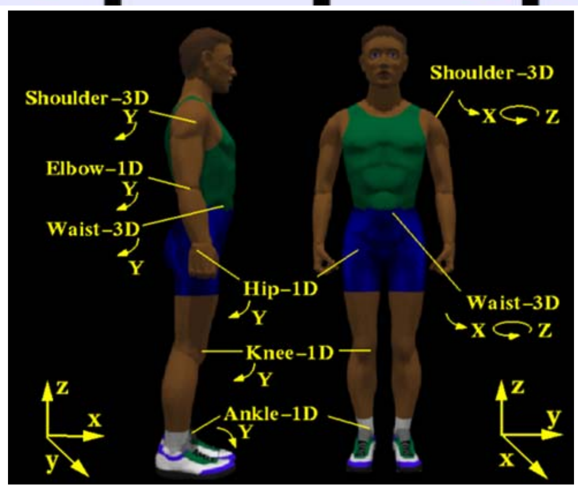
– **Inverse kinematics**: inferring the joint states necessary to reach a desired end-effector pose.

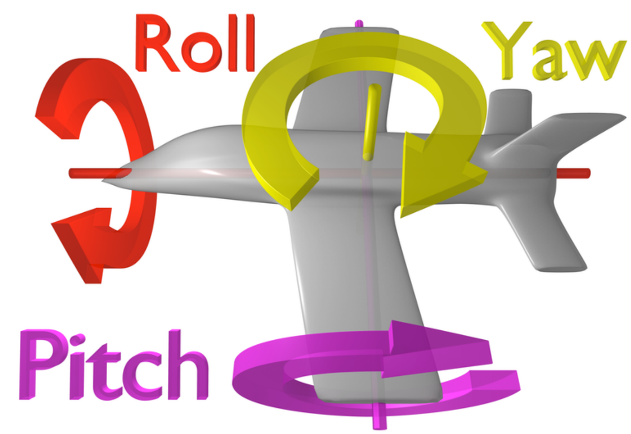
But, we need to start with a linear algebra refresher



Reset: Kinematics



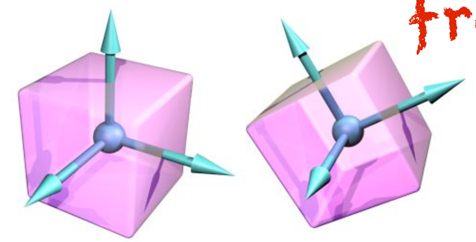




- State comprised of degrees-of-freedom (DOFs)
- DOFs describe translation and rotation axes of system

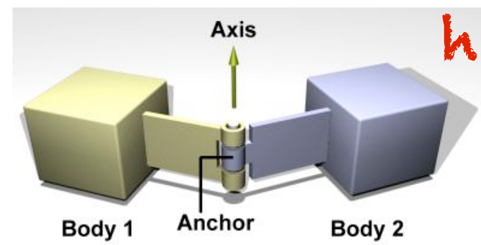
DOFs and Coordinate Spaces

frames



- Each <u>body</u> has its own frame
- Joints connect two links (rigid bodies)
  - e.g., hinge, prismatic, ball-socket
- A motor exerts force on a DOF axis

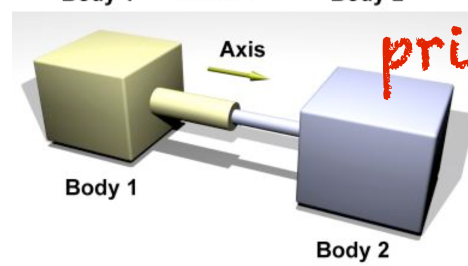
hinge



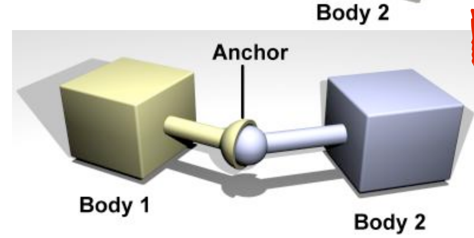
Linear algebra

- Matrix transformations used to relate coordinate systems of bodies and joints
- Spatial geometry attached to each link, but does not affect the body's coordinate frame

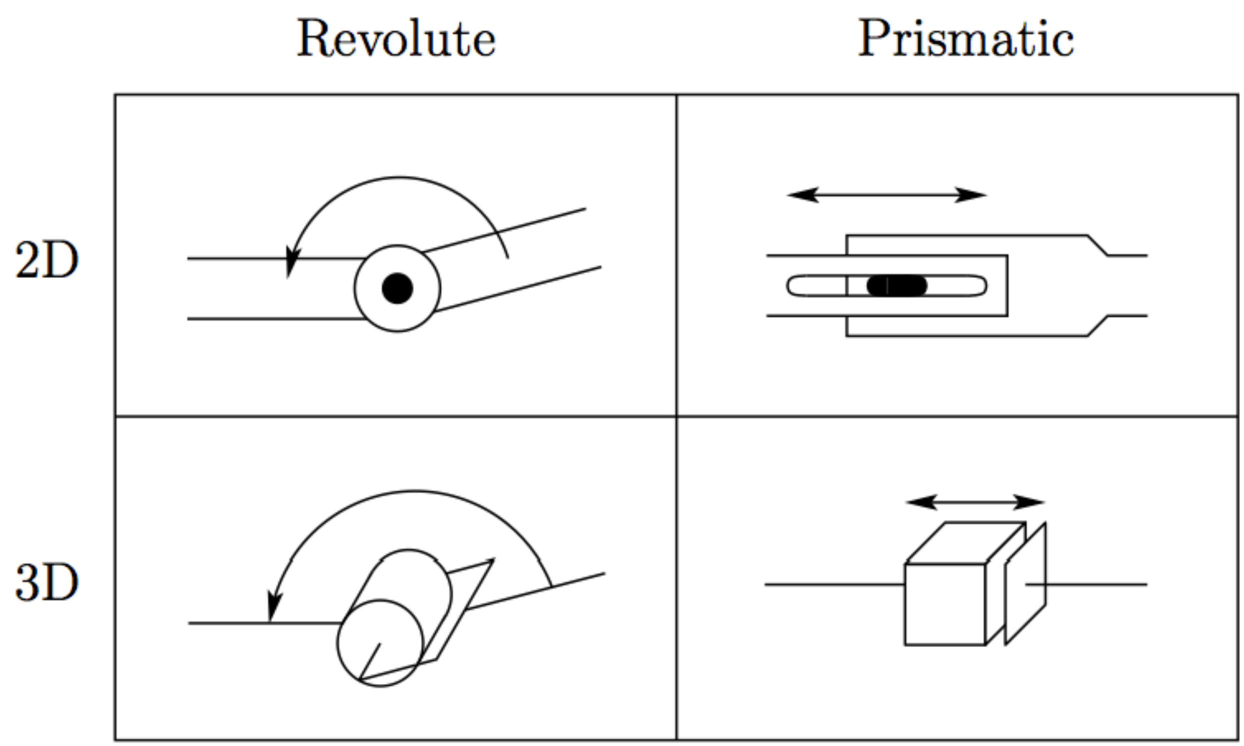
prismatic



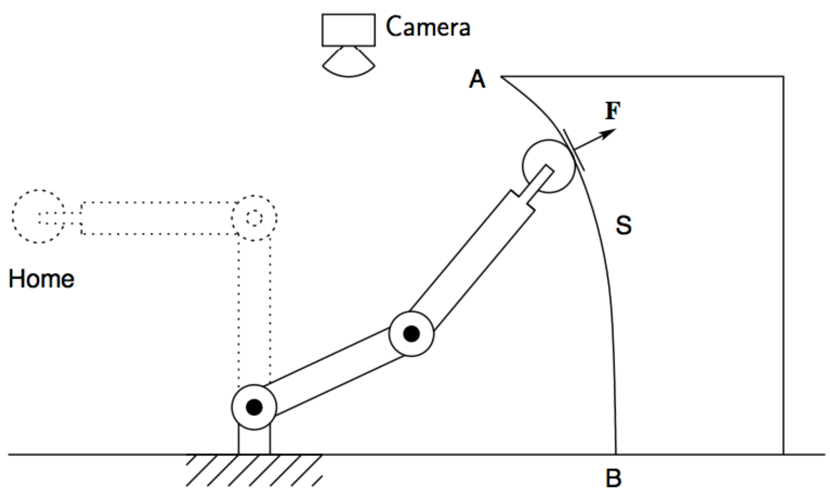
ball

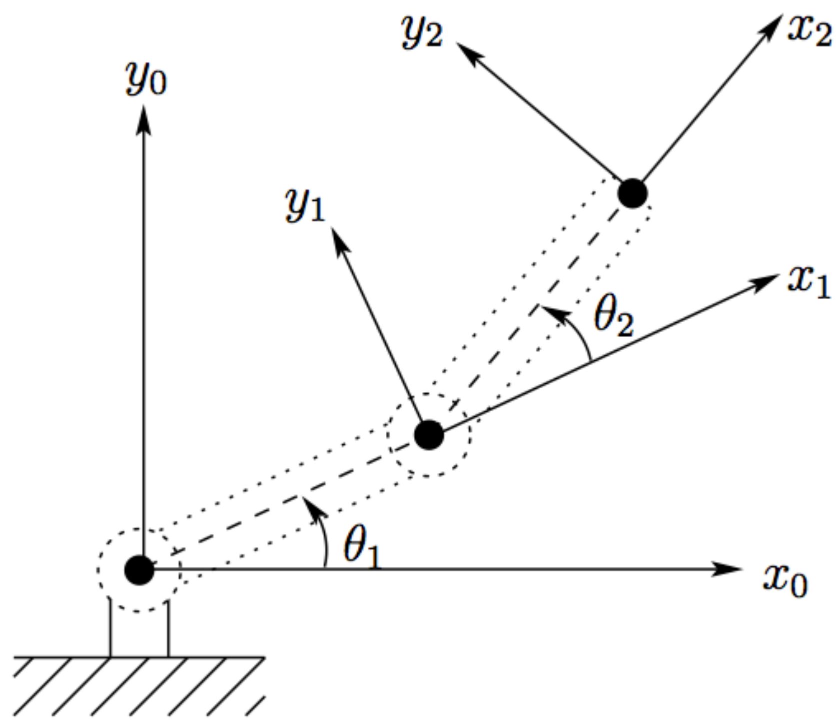


Notation



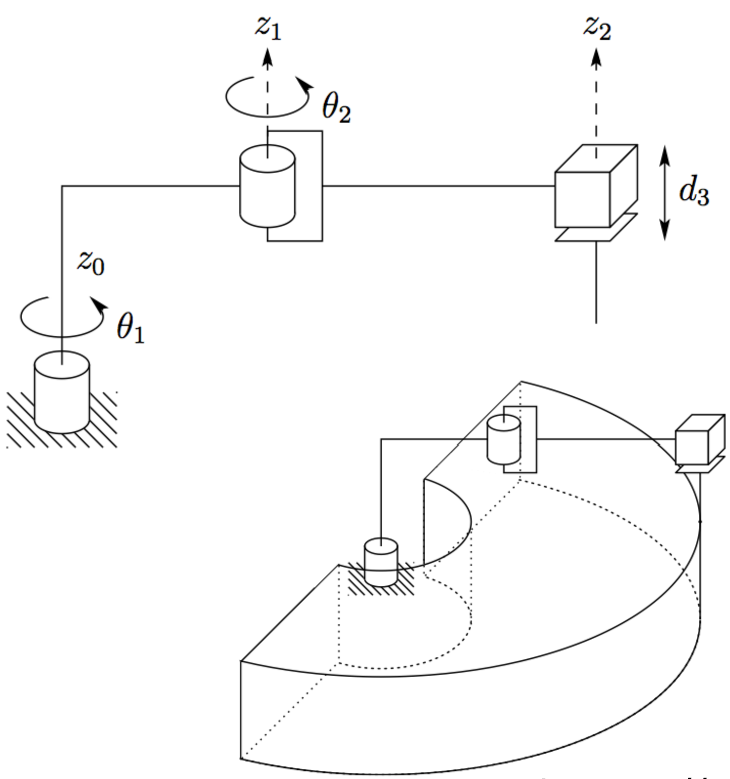
Planar 2-link Arm





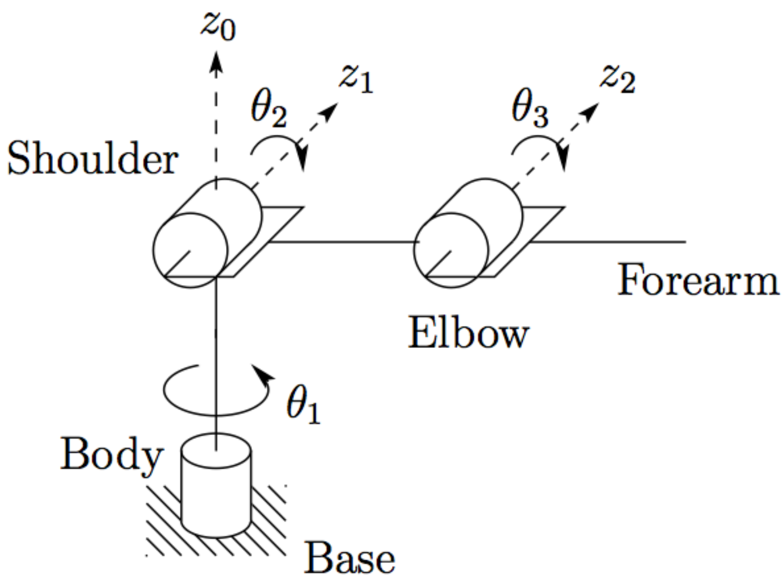
SCARA Arm

Selective Compliance Assembly Robot Arm



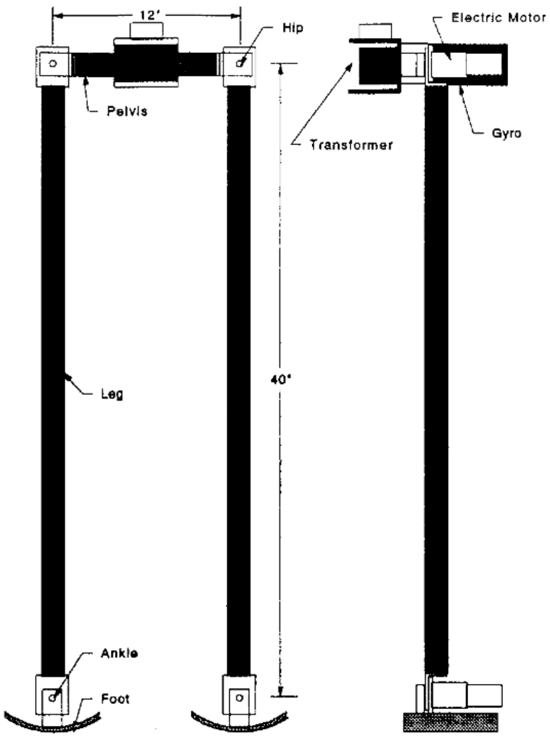
https://youtu.be/7X5Nmk85kQo

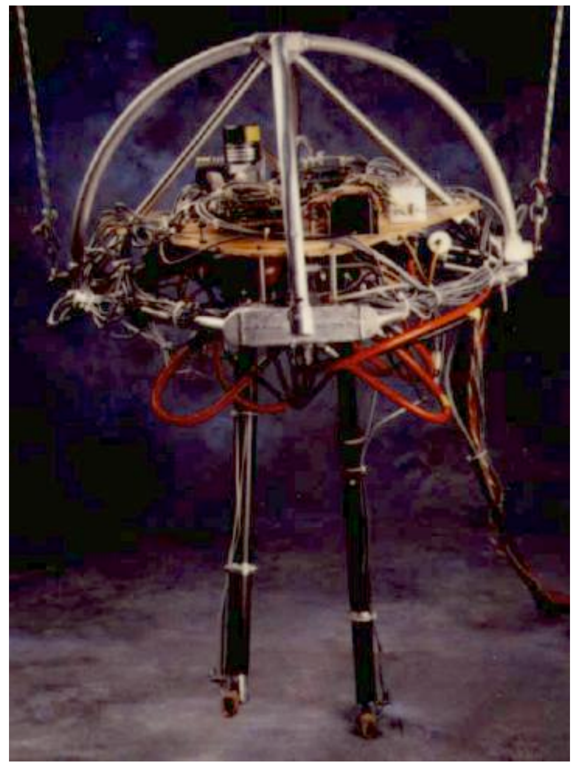
Motoman SK16

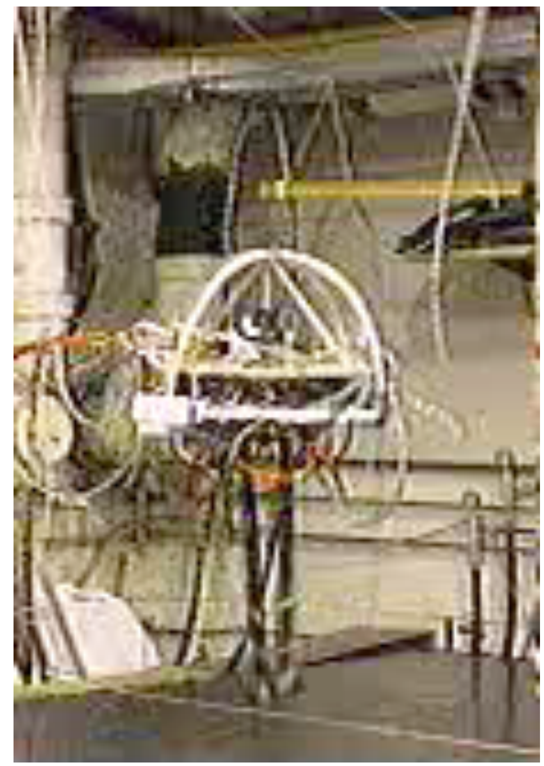


https://youtu.be/Wj17z5iSzEQ

Biped Hopper (MIT Leg Lab)



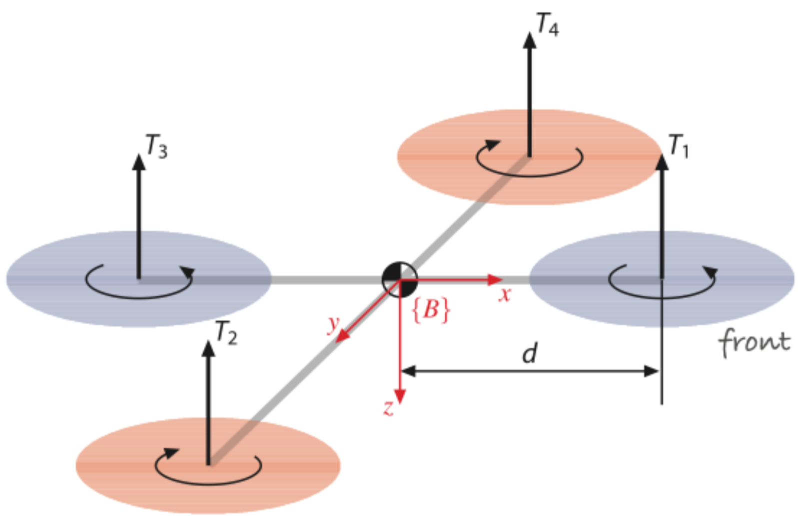




http://www.ai.mit.edu/projects/leglab/robots/robots.html

Big Dog (BDI)

Quad Rotor Helicopter



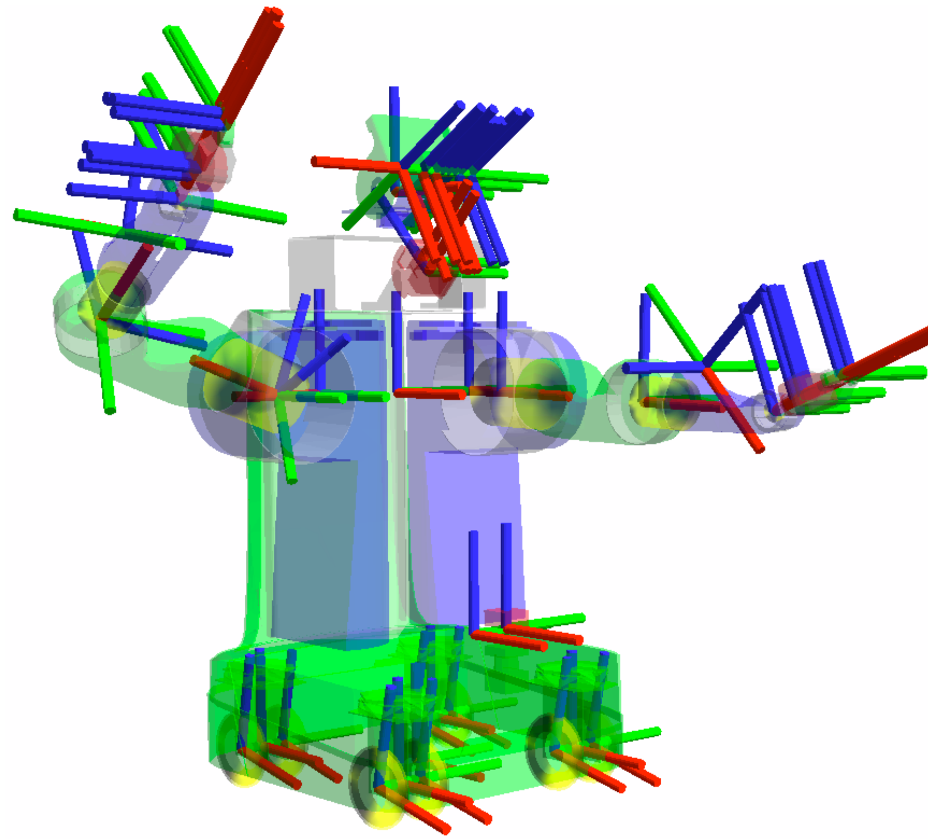
Safety is most important

https://youtu.be/0mDiH_ajStQ

https://www.youtube.com/watch?v=XxFZ-VStApo

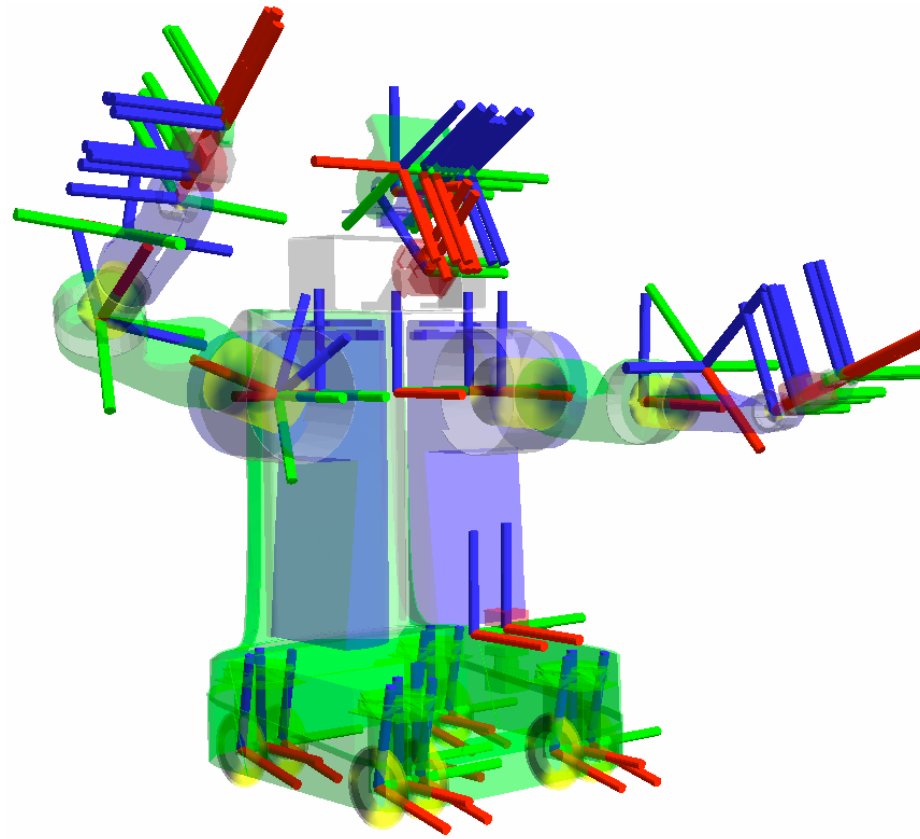
ETH-Zurich

PR2



How to express kinematics as the state of an articulated system?

PR2



How to express kinematics as the state of an articulated system?

We need some math first.

Algebra

From Wikipedia, the free encyclopedia

Algebra (from Arabic *"al-jabr"* meaning "reunion of broken parts"[1]) is one of the broad parts of mathematics, together with number theory, geometry and analysis. In its most general form, algebra is the study of mathematical symbols and the rules for manipulating these symbols;[2] it is a unifying thread of almost all of mathematics.[3] As such, it includes everything from elementary equation solving to the study of abstractions such as groups, rings, and fields. The more basic parts of algebra are

What does algebra provide beyond arithmetic?

Algebra

From Wikipedia, the free encyclopedia

- Arithmetic applies to addition and multiplication of known numbers
- Algebra includes **abstractions as variables**
    - Unknown numbers or expressions that can take on many values
- An algebra supports addition and multiplication of variables and numbers.
    - For example, from: $x^2 = 5x - 6$
    - we get: $(x - 2)(x - 3) = 0$
    - and thus: $x = 2$ or $x = 3$.

Linear algebra

From Wikipedia, the free encyclopedia

Linear algebra is the branch of mathematics concerning vector spaces and linear mappings between such spaces. Such an investigation is initially motivated by a system of linear equations containing several unknowns. Such equations are naturally represented using the formalism of matrices and vectors.[1]

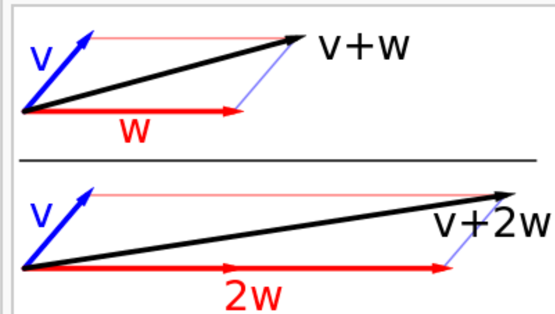
What does is linear algebra provide beyond algebra?

Vector space

From Wikipedia, the free encyclopedia

This article is about linear (vector) spaces. For the structure in incidence geometry, see Linear space (geometry).

A **vector space** (also called a **linear space**) is a collection of objects called **vectors**, which may be added together and multiplied ("scaled") by numbers, called *scalars* in this context. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by complex numbers, rational numbers, or generally any field. The operations of vector addition and scalar multiplication must satisfy certain requirements, called *axioms*, listed below.



Vector addition and scalar multiplication: a vector $\mathbf{v}$ (blue) is added to another vector $\mathbf{w}$ (red, upper illustration). Below, $\mathbf{w}$ is stretched by a factor of 2, yielding the sum $\mathbf{v} + 2\mathbf{w}$.

- Describes spaces where vector operations are closed with respect to:
  - a[redacted]
  - s[redacted]

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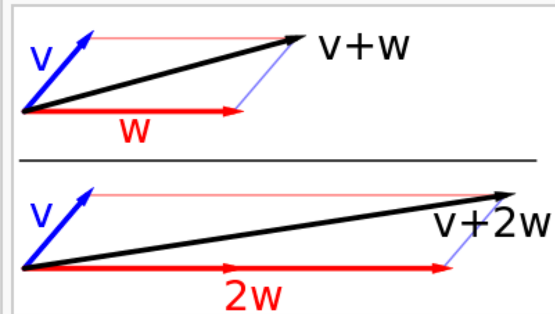
- Describes spaces where vector operations are closed with respect to:
  - addition
  - scalar multiplication



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	Arithmetic	Algebra	Linear Algebra
Abstraction		$x = 3$	$x = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$
Addition	$3 + 2 = 5$	$x + 2 = 5$	$x + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$
Scalar multiplication	$3 \times 2 = 6$	$2x = 6$	$2x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$

Linear algebra

From Wikipedia, the free encyclopedia

- Many important complex systems are described by collections of linear equations.
- An algebra of scalars, vectors, and matrices helps us work with these systems, keeping track of the complexity.
  - Manipulate groups of known and unknown parameters, just like manipulating numbers.
- Linear algebra is essential for representing frames of reference, rotation, translation, and general 3D homogeneous transforms.

Linear Algebra (Rough) Breakdown

- Geometry of Linear Algebra ← primary focus for AutoRob
  - Vectors, matrices, basic operations, lines, planes, homogeneous coordinates, transformations
- Solving Linear Systems ← needed for iterative IK
  - Gaussian Elimination, LU and Cholesky decomposition, over-determined systems, calculus and linear algebra, non-linear least squares, regression
- The Spectral Story
  - Eigensystems, singular value decomposition, principle component analysis, spectral clustering

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$$\begin{alignedat}{4} 3x &+{}& 2y &-{}& z &={}& 1 \\ 2x &-{}& 2y &+{}& 4z &={}& -2 \\ -x &+{}& \tfrac{1}{2}y &-{}& z &={}& 0 \end{alignedat}$$

is solved by [redacted]

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linear systems expressed in general matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

as

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

M linear equations

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$$A \quad \mathbf{x} = \mathbf{b}$$

each equation yields a hyperplane in N-D

vector of N unknowns to be found

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If #unknowns > #equations, [redacted]

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If #unknowns = #equations,

M linear equations

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If #unknowns > #equations, underdetermined system, usually with infinite solutions

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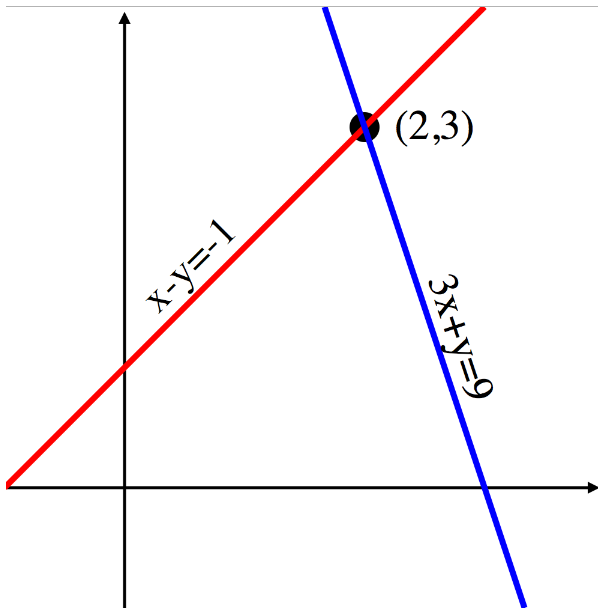
If #unknowns = #equations, usually has a unique solution

2D Example

only single point
satisfies both lines

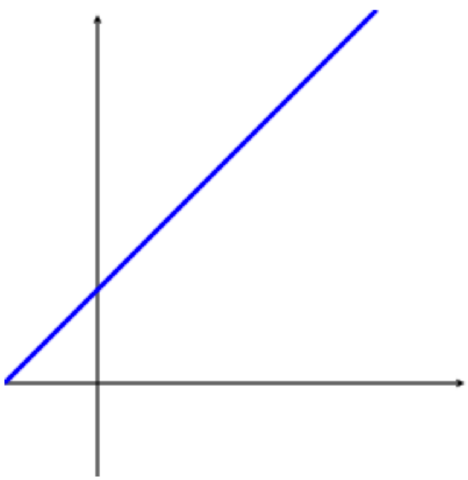


2D Example

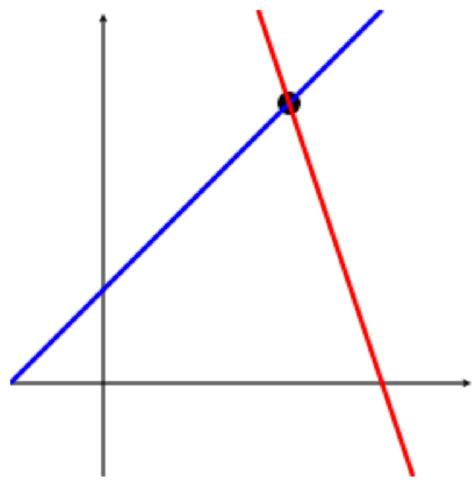
any point on the line satisfies

only single point satisfies both lines

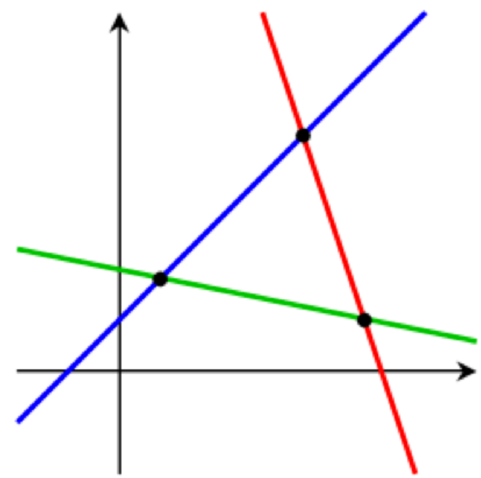
no point satisfies all three lines



One equation



Two equations

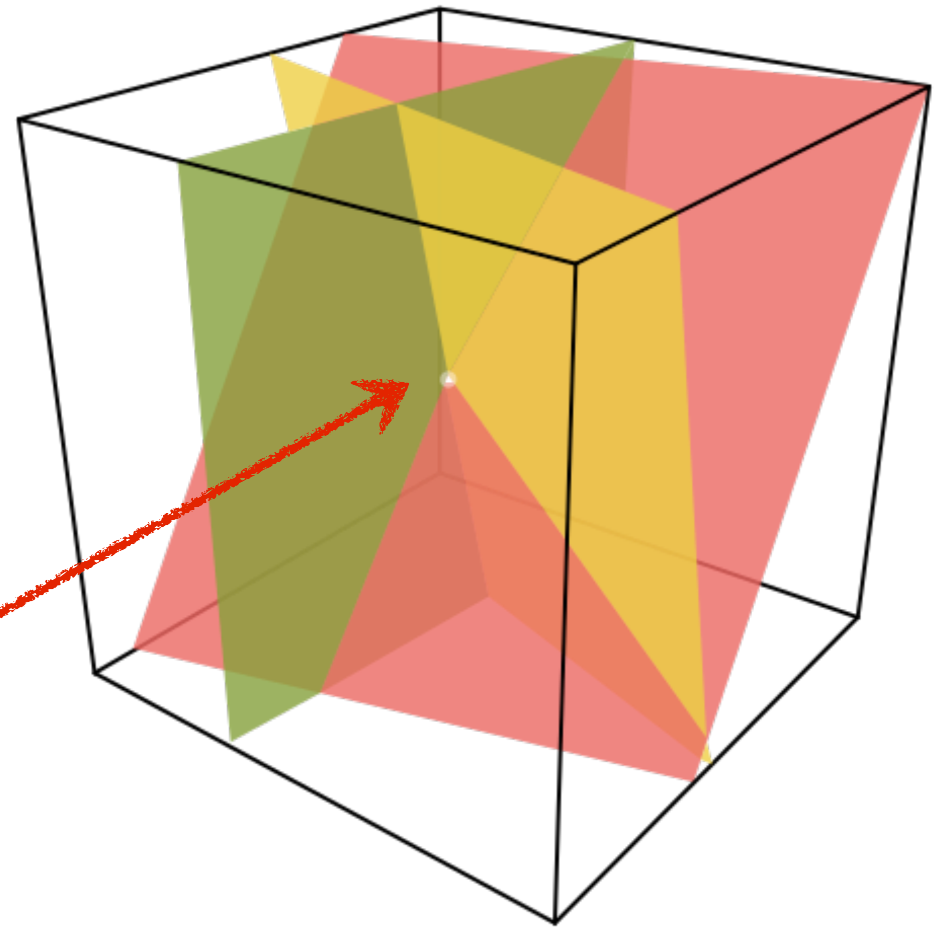


Three equations

3D Example

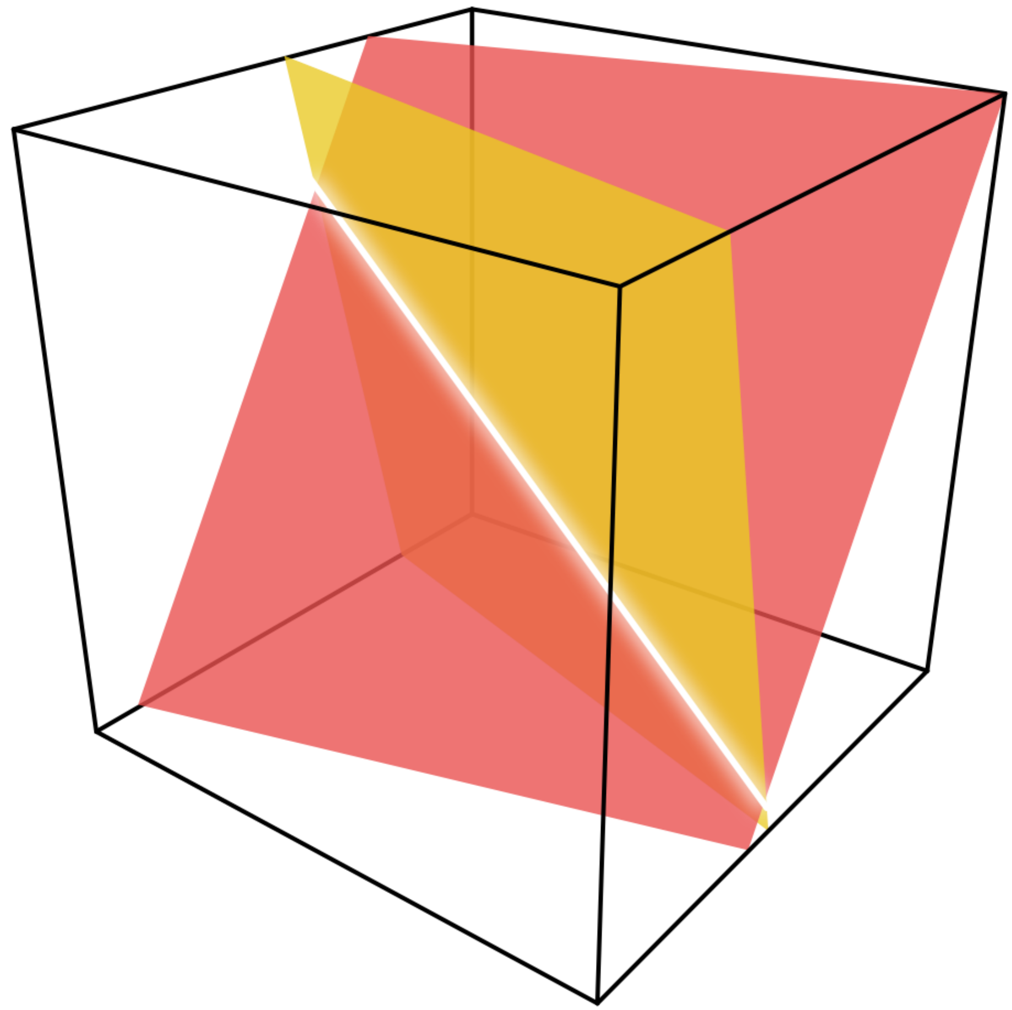
Each equation yields a 2D plane in 3D space

A single point satisfies all equations



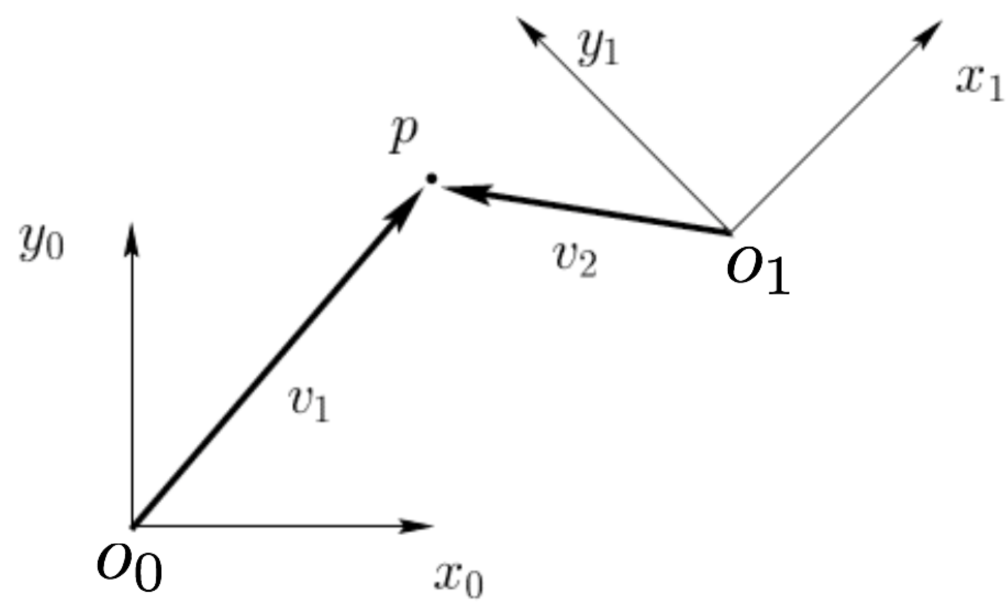
3D Example

How many solutions?



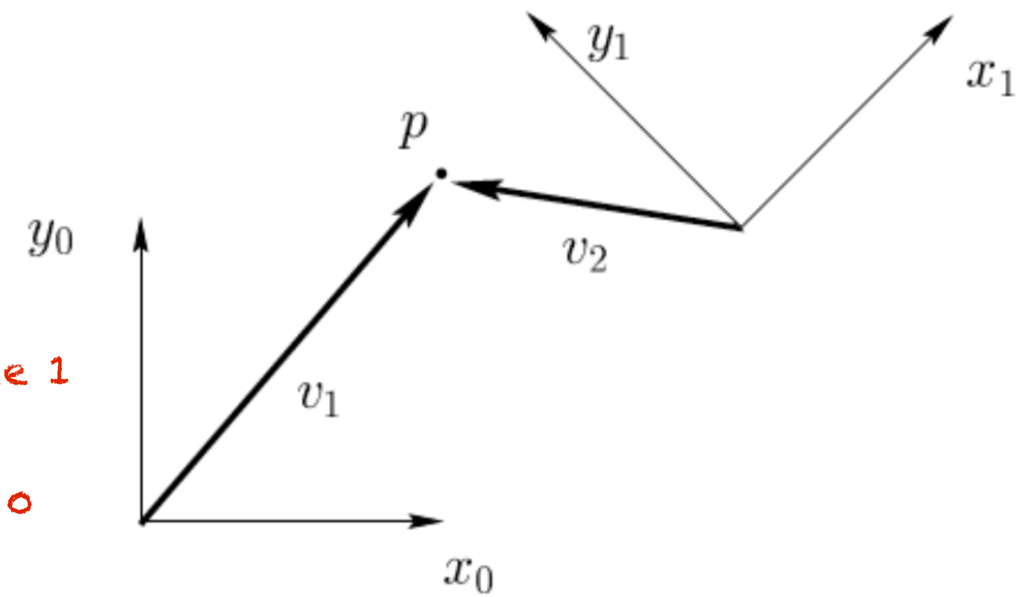
Coordinate Spaces (2D)

- Two coordinate frames $o_0x_0y_0$ and $o_1x_1y_1$, and a point p.
- The location of point p can be described with respect to either coordinate frame: $p^0 = [5, 6]^T$ and $p^1 = [-2.8, 4.2]^T$.
- The vector v_1 is direction and magnitude from o_0 to p, and v_2 is from o_1 to p.



Coordinate Spaces (2D)

- Point *p* has a location.
- Vectors v_1 and v_2 have directions and magnitudes.
- $v_1^0 = [5, 6]^T$ vector 1 in frame 0
- $v_1^1 = [7.77, 0.8]^T$ vector 1 in frame 1
- $v_2^0 = [-5.1, 1]^T$ vector 2 in frame 0
- $v_2^1 = [-2.8, 4.2]^T$ vector 2 in frame 1



Note: Vectors can only be added when they are in the same coordinate frame.

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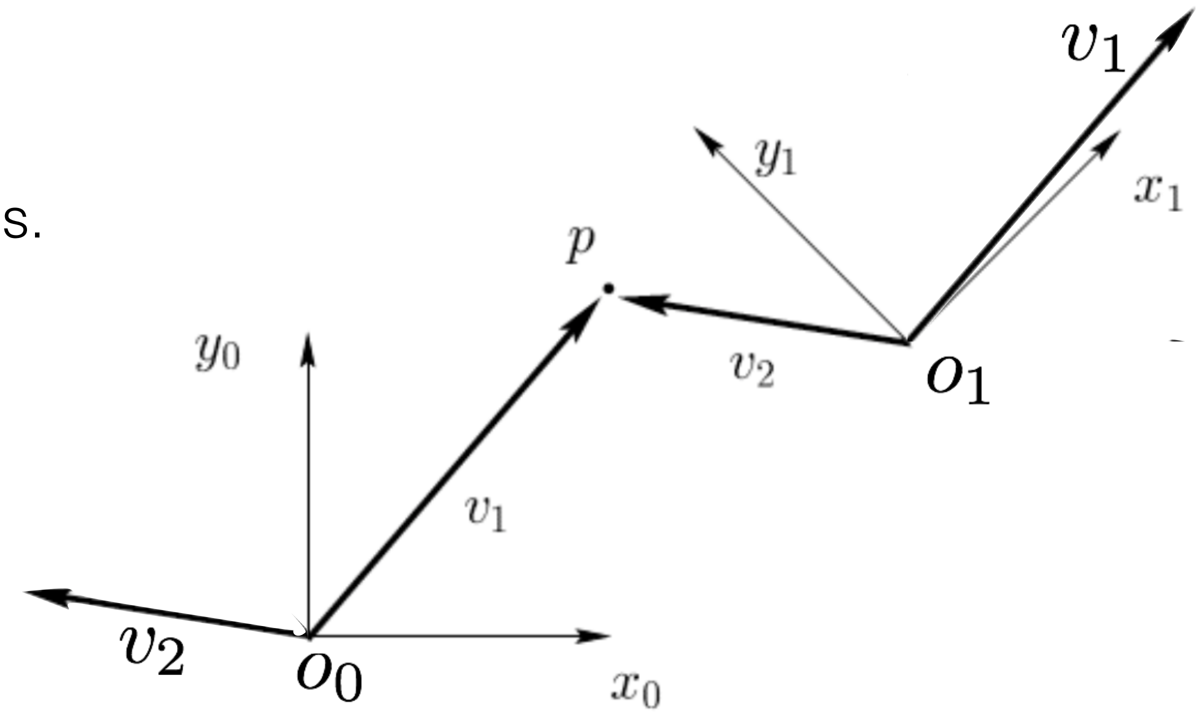

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Vectors and Matrices

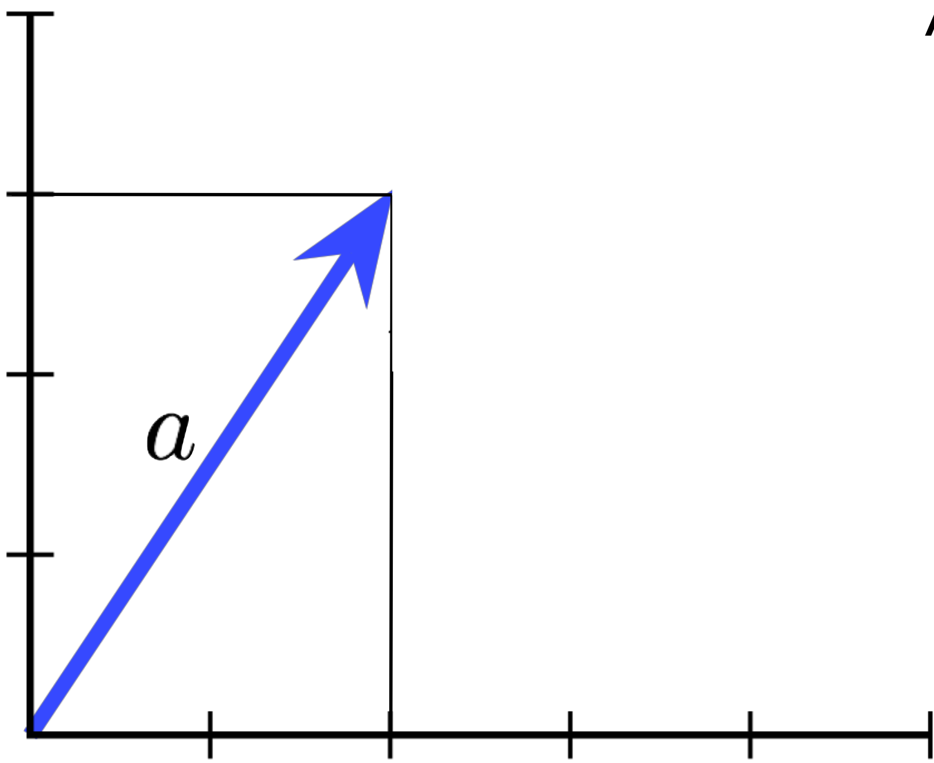
N-dimensional vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

M-by-N matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

2D Vector

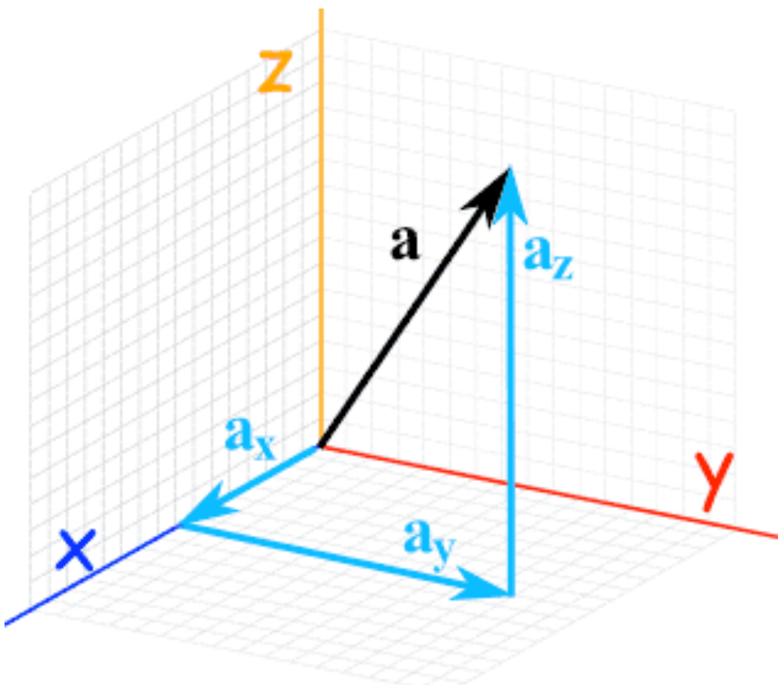


A vector is a motion in space

$$a = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

```
var a = [ [2],
          [3] ];
```

3D Vector



$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

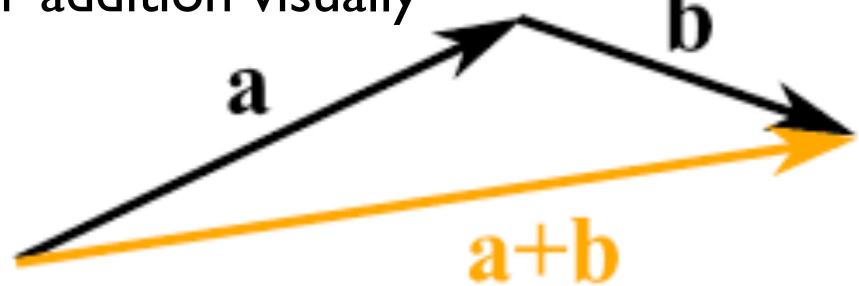
```
var a = [ [ax],
          [ay],
          [az] ];
```

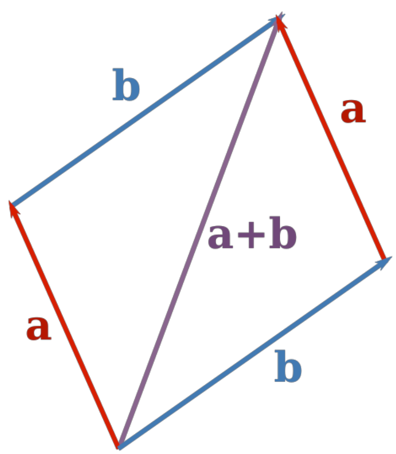
Vector Addition and Subtraction

$$a + b = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix}$$

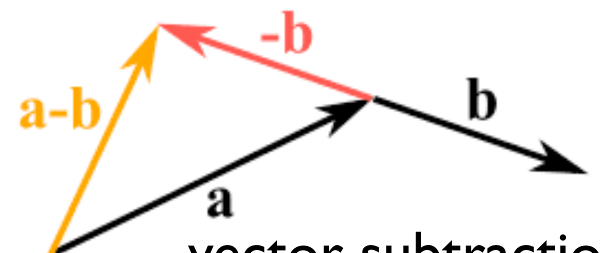
vector result

vector addition visually





vector addition is order independent



vector subtraction is addition with negated vector

Magnitude and Unit Vector

The magnitude of a vector is the square root of the sum of squares of its components

$$||a|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

A unit vector has a magnitude of one. Normalization scales a vector to unit length.

$$\hat{a} = \frac{a}{||a||}$$

A vector can be multiplied by a scalar

$$sa = \begin{bmatrix} sa_x \\ sa_y \\ sa_z \end{bmatrix}$$

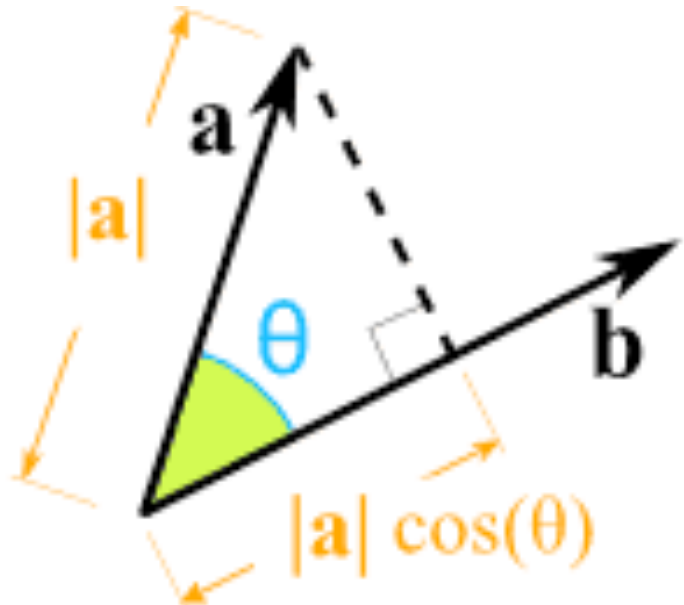
scalar
result

Dot Product

$$a \bullet b = a_x b_x + a_y b_y + a_z b_z$$
$$= ||a|| ||b|| cos(\theta)$$

Measures the similarity in direction of two vectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2 * 3 + 1 * 2 = 8$$



Projections

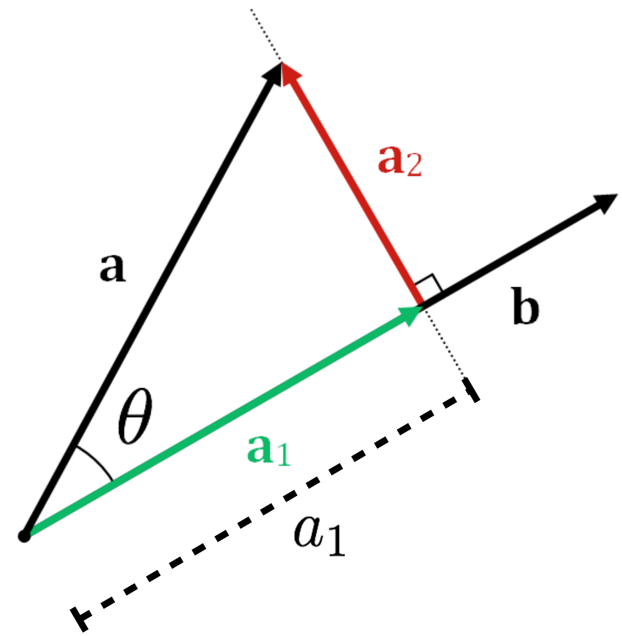
Dot products related to projections onto vectors.

Scalar projection of one vector onto another

$$a_1 = |\mathbf{a}| \cos\theta = \mathbf{a} \cdot \hat{\mathbf{b}} = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|}$$

Vector projection

$$\mathbf{a}_1 = a_1 \hat{\mathbf{b}}$$



$\hat{\mathbf{b}}$ is unit length

Checkpoint

- What is the dot product of a vector with itself?

[redacted]

- What is the dot product of two orthogonal vectors?

[redacted]

Checkpoint

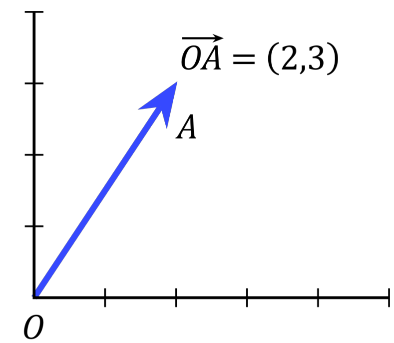
- What is the dot product of a vector with itself?
    - the square of the vector magnitude
- What is the dot product of two orthogonal vectors?

[redacted]

Checkpoint

- What is the dot product of a vector with itself?
  - the square of the vector magnitude
- What is the dot product of two orthogonal vectors?
  - 0

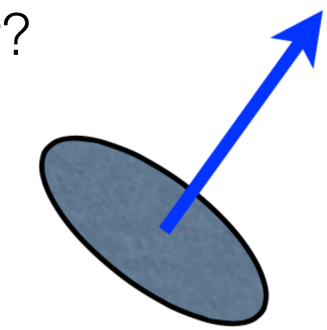
Checkpoint



- How many unit vectors are perpendicular to a 2D vector?

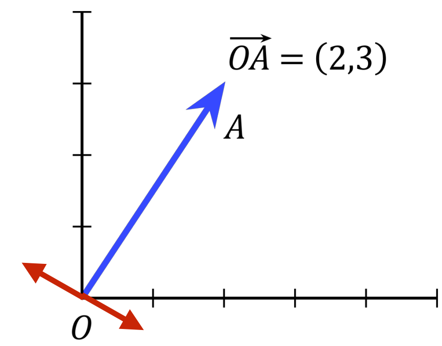
[redacted]

- How many unit vectors are perpendicular to a 3D vector?



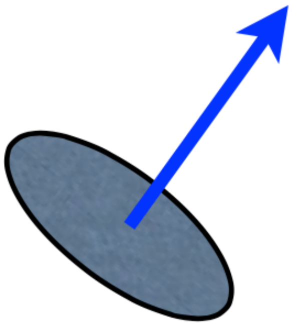
[redacted]

Checkpoint

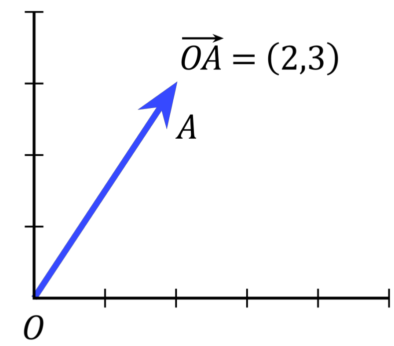


- How many unit vectors are perpendicular to a 2D vector?
    - 2 (positive and negative)
- How many unit vectors are perpendicular to a 3D vector?

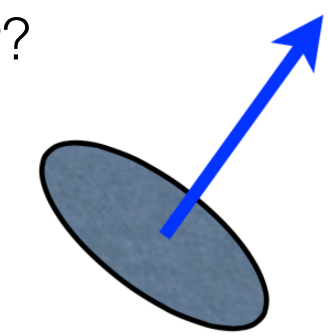
[redacted]



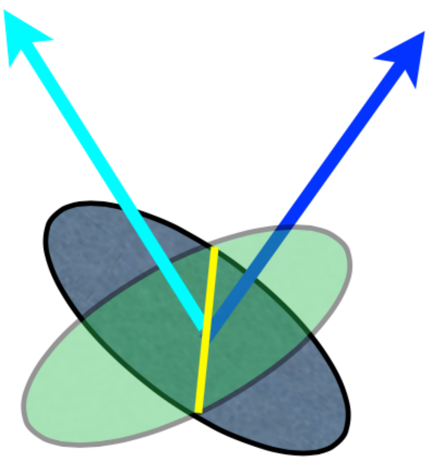
Checkpoint



- How many unit vectors are perpendicular to a 2D vector?
    - 2 (positive and negative)
- How many unit vectors are perpendicular to a 3D vector?
    - Infinite and lie in plane



Given two vectors, how to compute a vector orthogonal to both?



Cross Product

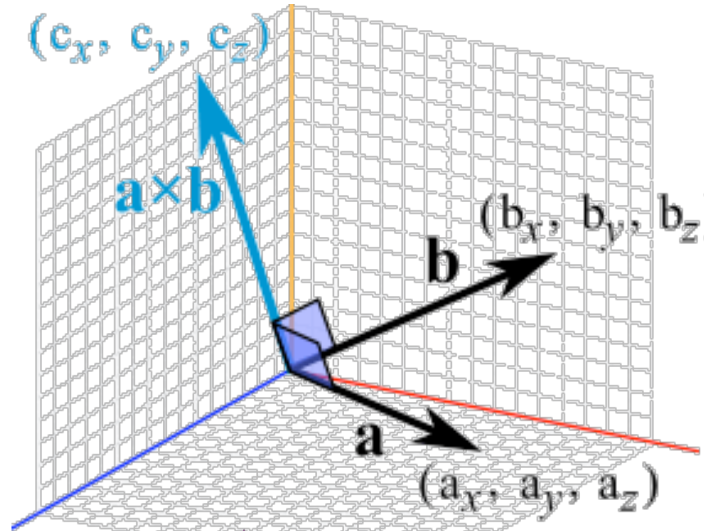
$$c_x = a_y b_z - a_z b_y$$
$$c_y = a_z b_x - a_x b_z$$
$$c_z = a_x b_y - a_y b_x$$

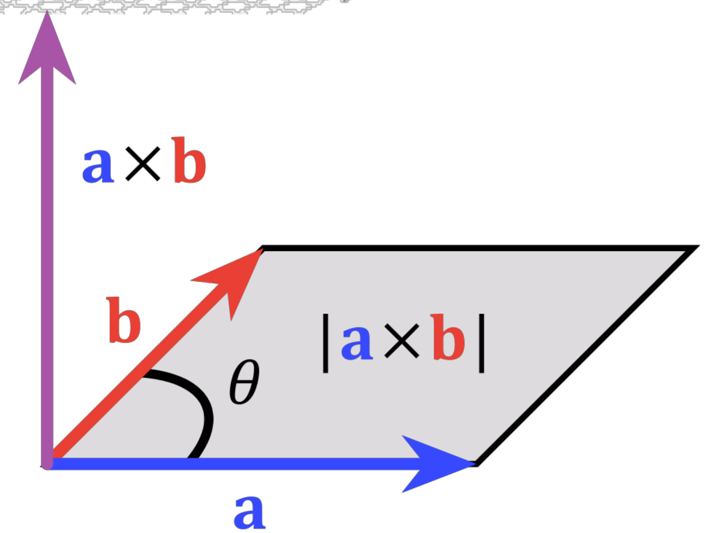
Results in new vector c orthogonal to both original vectors a and b

Length of vector c is equal to area of parallelogram formed by a and b

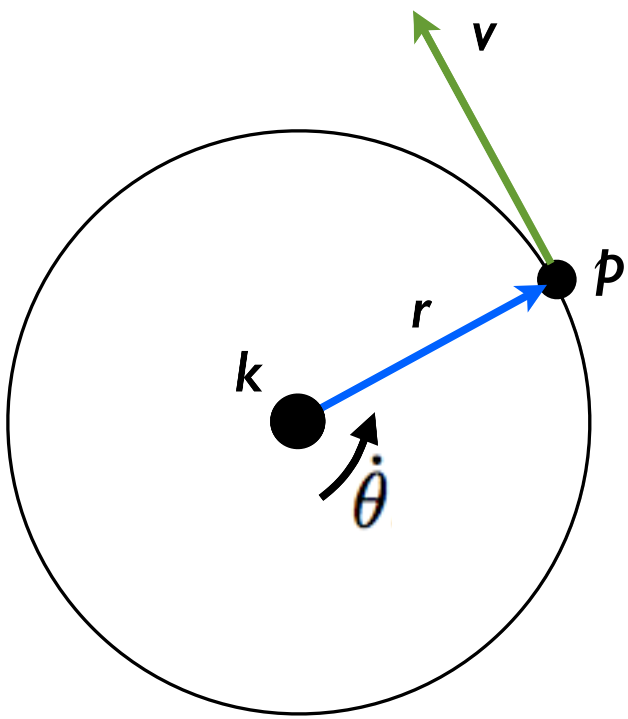
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$$

Assumes **a** and **b** are in same frame





Relating linear and angular velocity



vector to
a point p

$$\boldsymbol{v} = \dot{\theta}\boldsymbol{k} \times \boldsymbol{r}$$

linear velocity
of point p

angular velocity
for rotation of p
about vector k

Matrices

- A Matrix is a rectangular array of numbers

```
var mat = [
    [1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 0],
    [0, 0, 0, 1] ];
```

What is this matrix?

Matrix-vector multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$$

For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} =$$ [redacted]

For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

Matrix-vector multiplication

(two interpretations)

1) **Row story**: dot product of each matrix row

$$\begin{bmatrix} \boxed{a \quad b \quad c} \\ d \quad e \quad f \\ g \quad h \quad i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} \boxed{aj + bk + cl} \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$$

2) **Column story**: linear combination of matrix columns

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix} \qquad \begin{bmatrix} a \\ d \\ g \end{bmatrix} j + \begin{bmatrix} b \\ e \\ h \end{bmatrix} k + \begin{bmatrix} c \\ f \\ i \end{bmatrix} l$$

Revisiting the cross product: Skew-symmetric matrices

A given 3D vector $\mathbf{a} = (a_1 \ a_2 \ a_3)^{\mathrm{T}}$

can be expressed as a skew-symmetric matrix

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

such that the cross product with another vector is a matrix multiplication

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b}$$

Linear Systems

We can use a variable instead of a vector, which gives us a linear system.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Enabling the general form: $A\mathbf{x} = \mathbf{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{aligned} a_{11}x_1 + \; a_{12}x_2 + \cdots + \; a_{1n}x_n &= b_1 \\ a_{21}x_1 + \; a_{22}x_2 + \cdots + \; a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Matrices

- A Matrix is a rectangular array of numbers

```
var mat = [
    [1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 0],
    [0, 0, 0, 1] ];
```

```
var mat = [
    [1, 0, 0, tx],
    [0, 1, 0, ty],
    [0, 0, 1, tz],
    [0, 0, 0, 1] ];
```

What is this matrix?

Translation matrix example

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} =$$ [redacted]

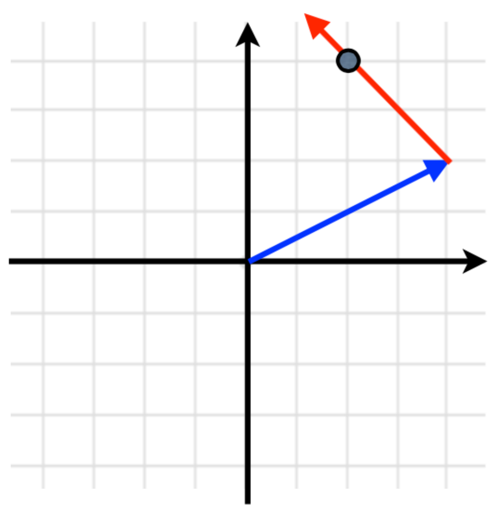
jsmat for Assignment 3

Translation matrix example

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$

jsmat for Assignment 3

Matrix Geometry: Column Story



- Each column can be interpreted as a vector
  - How far do we go in each direction?

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Matrix Multiplication

- Scalar Multiplication

$$\lambda \mathbf{A} = \lambda \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1m} \\ \lambda A_{21} & \lambda A_{22} & \cdots & \lambda A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda A_{n1} & \lambda A_{n2} & \cdots & \lambda A_{nm} \end{pmatrix}.$$

- Multiplication of two matrices

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj} \,.$$

Each entry of product matrix AB is a dot product of a row of A with a column of B

Matrix multiplication

Finger sweeping rule should be second nature!
- Left finger sweeps left to right
- Right finger sweeps top to bottom

$$\overset{\substack{A\\ 3\times 4\text{ matrix}}}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \color{blue}1 & \color{blue}2 & \color{blue}3 & \color{blue}4 \end{bmatrix}}
\overset{\substack{B\\ 4\times 5\text{ matrix}}}{\begin{bmatrix} \cdot & \cdot & \cdot & \color{red}a & \cdot \\ \cdot & \cdot & \cdot & \color{red}b & \cdot \\ \cdot & \cdot & \cdot & \color{red}c & \cdot \\ \cdot & \cdot & \cdot & \color{red}d & \cdot \end{bmatrix}}
=
\overset{\substack{AB\\ 3\times 5\text{ matrix}}}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_{3,4} & \cdot \end{bmatrix}}$$

row 3 of A

col 4 of B

Do this dot product for each row/column combination

$$\begin{aligned} x_{3,4} &= ({\color{blue}1},{\color{blue}2},{\color{blue}3},{\color{blue}4})\cdot({\color{red}a},{\color{red}b},{\color{red}c},{\color{red}d}) \\ &= {\color{blue}1}\times{\color{red}a}+{\color{blue}2}\times{\color{red}b}+{\color{blue}3}\times{\color{red}c}+{\color{blue}4}\times{\color{red}d} \end{aligned}$$

Matrix Multiplication Reminders

- Number of columns of A must match number of rows of B
- Multiplying a (MxK) matrix with a (KxN) matrix will produce an (MxN) matrix
- Matrix multiplication is not commutative: AB != BA

Example

"Dot Product"

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \\ & \end{bmatrix}$$

$(1, 2, 3) \bullet (7, 9, 11) = 1\times7 + 2\times9 + 3\times11 = 58$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ & \end{bmatrix}$$

$(1, 2, 3) \bullet (8, 10, 12) = 1\times8 + 2\times10 + 3\times12 = 64$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

$(4, 5, 6) \bullet (7, 9, 11) = 4\times7 + 5\times9 + 6\times11 = 139$

$(4, 5, 6) \bullet (8, 10, 12) = 4\times8 + 5\times10 + 6\times12 = 154$

For example

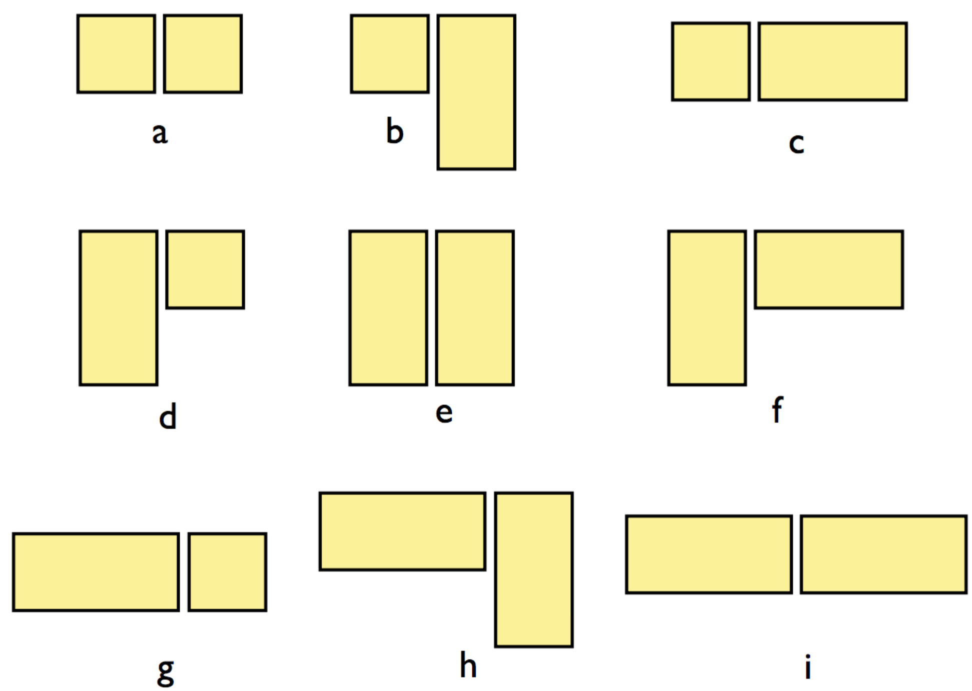
$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix} =$$ [redacted]

For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 2 & 5 \end{bmatrix}$$

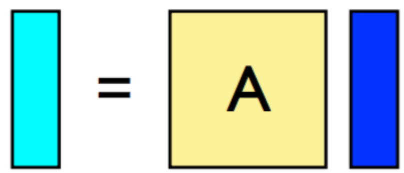
Checkpoint

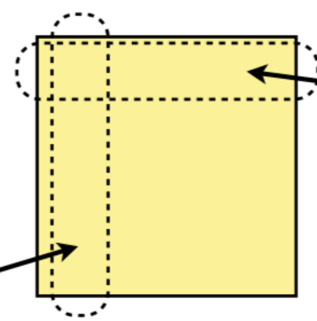
- Which of the following matrix multiplications are valid?



Matrices as projections

- Matrix multiplication projects from one space to another.





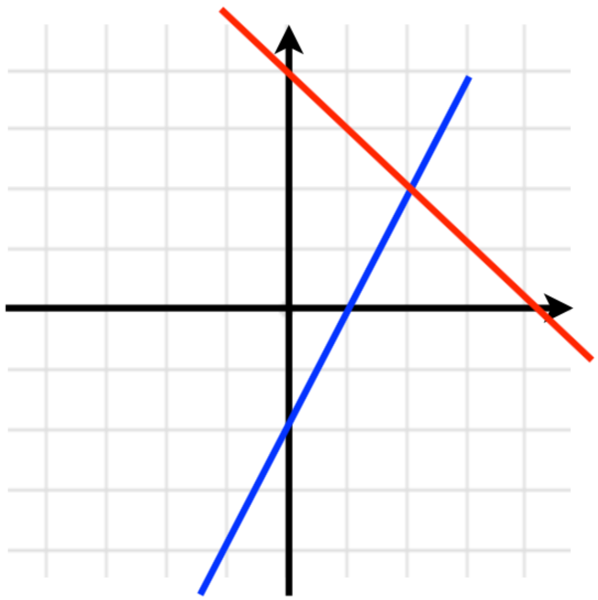
Notable Matrices and Operations

- Matrix identity (I) causes no change: $A = I_m A = A I_n$
  - Diagonal elements $A_{ii} = 1$
  - Off-diagonal elements $A_{ij} = 0$, $i \neq j$
- Matrix inverse (A^{-1}): if $AA^{-1} = A^{-1}A = I$
- Distributing matrix inverse: $(AB)^{-1} = B^{-1}A^{-1}$
- Matrix transpose (A^T): a matrix's reflection about its diagonal
- Distributing matrix transpose: $(AB)^T = B^T A^T$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Matrix Geometry: Row Story



- Each row of a linear system represents a hyperplane. (In 2D, that's also a line!)
- The solution to the system is the intersection of those hyperplanes

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Solving linear systems

What would be the direct way to solve for ***x***? $A\mathbf{x} = \mathbf{b}$

[redacted]

Solving linear systems

What would be the direct way to solve for ***x***?

$$A\mathbf{x} = \mathbf{b}$$

Invert ***A*** and multiply by ***b***

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Matrix rank and inversion

- Let *A* be a square n by n matrix. *A* is invertible if full rank and a matrix *B* exists such that
- Rank of a matrix *A* is the size of the largest collection of linearly independent columns of *A*
- *A* is invertible (nonsingular) if it has full rank
- Gaussian elimination can find matrix inverse
- Singular matrix cannot be inverted this way

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$[I|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

Gaussian Elimination steps

Solution by Decomposition

- In real applications, inverse not computed to solve linear systems
  - Efficiency, numerical precision, etc.
- Matrix decomposed into product of lower and upper triangular matrices
  - LU decomposition $A = LU$ $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$
  - Cholesky decomposition $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathbf{T}}$
- Permits finding solution by forward substitution $\mathbf{L}\mathbf{y} = \mathbf{b}$
  followed by backward substitution $\mathbf{L}^{\mathbf{T}}\mathbf{x} = \mathbf{y}$

LU Decomposition steps

positive definite -> unique Cholesky

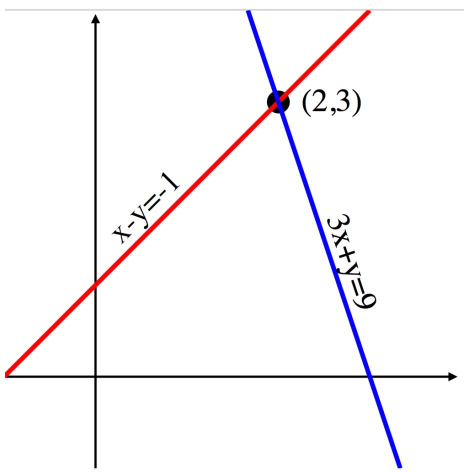
Determinant, condition number

Cramer's rule

From Wikipedia, the free encyclopedia

In linear algebra, **Cramer's rule** is an explicit formula for the solution of a system of linear equations with as many equations as unknowns, valid whenever the system has a unique solution. It expresses the solution in terms of the determinants of the (square) coefficient matrix and of matrices obtained from it by replacing one column by the vector of right-hand-sides of the equations. It is named after Gabriel Cramer (1704–1752), who published the rule for an arbitrary number of unknowns in 1750,[1][2] although Colin Maclaurin also published special cases of the rule in 1748[3] (and possibly knew

Cramer's rule is computationally very i
to elimination methods that have polyn



Explicit formulas for small systems [edit]

Consider the linear system

$$\begin{cases} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{cases}$$

which in matrix format is

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Assume $a_1b_2 - b_1a_2$ nonzero. Then, with help of determinants, x and y can be found with Cramer's rule as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}.$$

Solving linear systems

What would be the direct way to solve for ***x***?

$$A\mathbf{x} = \mathbf{b}$$

Invert ***A*** and multiply by ***b***

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Can this always be done?

[redacted]

Solving linear systems

What would be the direct way to solve for ***x***?

$$A\mathbf{x} = \mathbf{b}$$

Invert ***A*** and multiply by ***b***

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Can this always be done?

No. But, we can approximate. How?

[redacted]

Solving linear systems

What would be the direct way to solve for ***x***? $A\mathbf{x} = \mathbf{b}$

Invert ***A*** and multiply by ***b*** $\mathbf{x} = A^{-1}\mathbf{b}$

Can this always be done?

No. But, we can approximate. How?

Pseudoinverse least-squares approximation $\mathbf{x} = A^{+}_{\text{left}}\mathbf{b}$

Pseudoinverse

- For matrix A with dimensions N x M with full rank
- Find solution that minimizes squared error: $\|Ax - b\|_2$
- Left pseudoinverse, for when N > M, (i.e., "tall")

$$A_{\text{left}}^{-1} = \left(A^T A\right)^{-1} A^T \qquad \text{s.t.} \qquad A_{\text{left}}^{-1} A = I_n$$

- Right pseudoinverse, for when N < M, (i.e., "wide")

$$A_{\text{right}}^{-1} = A^T \left(A A^T\right)^{-1} \qquad \text{s.t.} \qquad A A_{\text{right}}^{-1} = I_m$$

Polynomial Regression

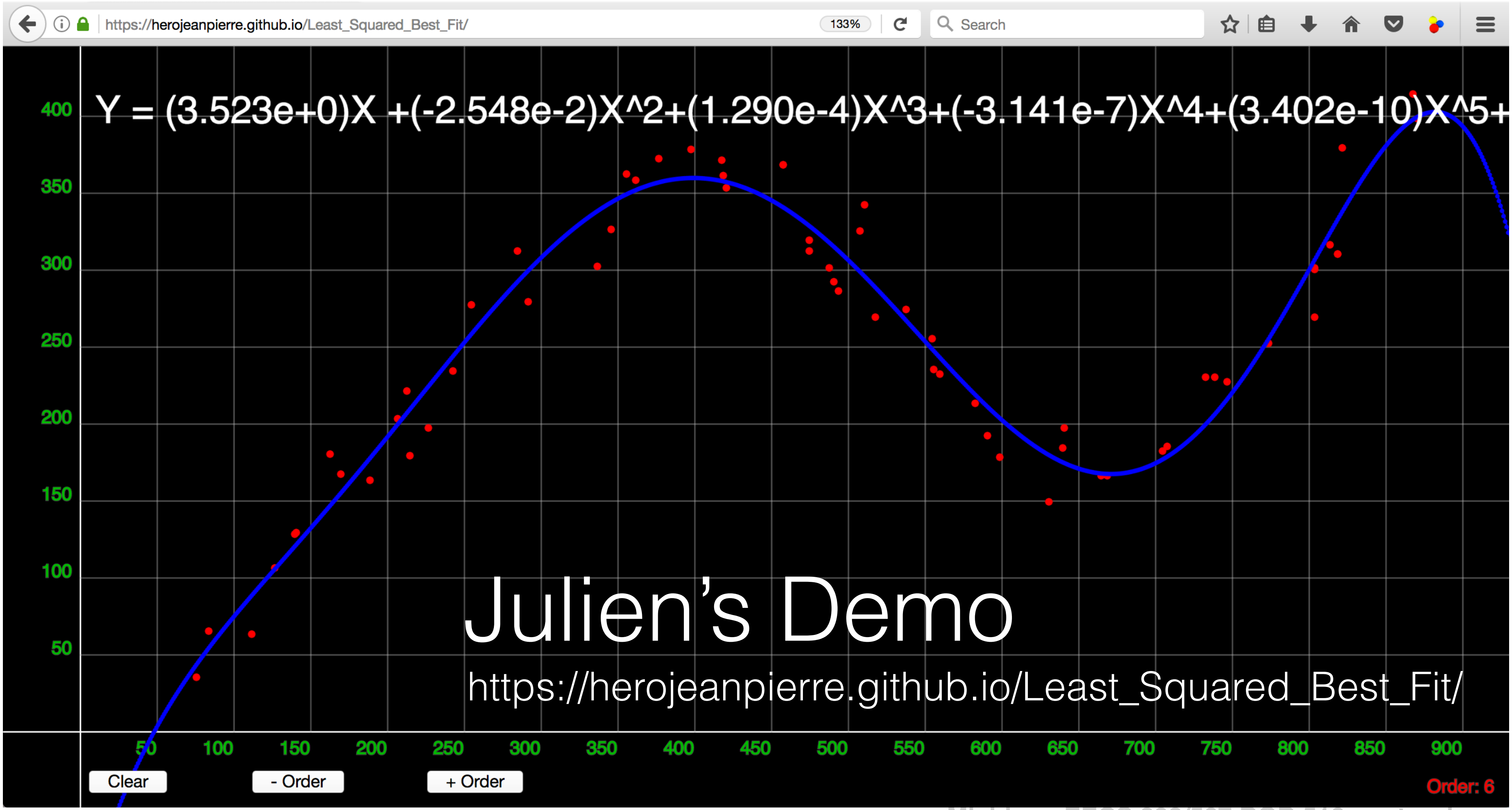
- Given n data points as input-output (x_i, y_i), estimate parameters β of best fitting m-order polynomial:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_m x_i^m \qquad (i = 1, 2, \dots, n)$$

- Model in matrix form:
  - each data point forms a row

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^m \\ 1 & x_2 & x_2^2 & \dots & x_2^m \\ 1 & x_3 & x_3^2 & \dots & x_3^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^m \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$$

- Solve for least squares best fit: $\widehat{\vec{\beta}} = (\mathbf{X}^\mathsf{T}\mathbf{X})^{-1}\, \mathbf{X}^\mathsf{T}\vec{y},$


https://herojeanpierre.github.io/Least_Squared_Best_Fit/
133%
Search
Y = (3.523e+0)X +(-2.548e-2)X^2+(1.290e-4)X^3+(-3.141e-7)X^4+(3.402e-10)X^5+
400
350
300
250
200
150
100
50
50 100 150 200 250 300 350 400 450 500 550 600 650 700 750 800 850 900
Clear
- Order
+ Order
Order: 6
Julien's Demo
https://herojeanpierre.github.io/Least_Squared_Best_Fit/

Eigenvalues and Eigenvectors

Determinant

- The determinant *det(A)* (or $|A|$) of a matrix *A* is a scalar value
- Scaling factor of the linear transformation described by matrix *A*
- For *2*-by-*2* matrix: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- For *3*-by-*3* matrix: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$
- For *N*-by-*N* matrix: $\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i} \right)$

Eigenvalues and Eigenvectors

- Vector ***v*** and scalar λ are an eigenvector and eigenvalue for matrix *A* if *(A-λI)****v*** *=0*
- *(A-λI)=0* has a non-zero solution ***v*** if and only if *det(A-λI)=0*
- *det(A-λI)* produces characteristic polynomial, whose roots are the eigenvalues

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

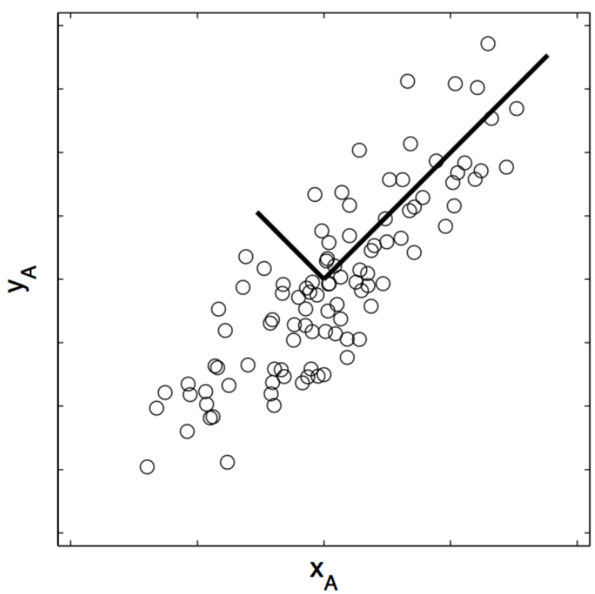
$$|M - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2$$

$$v_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

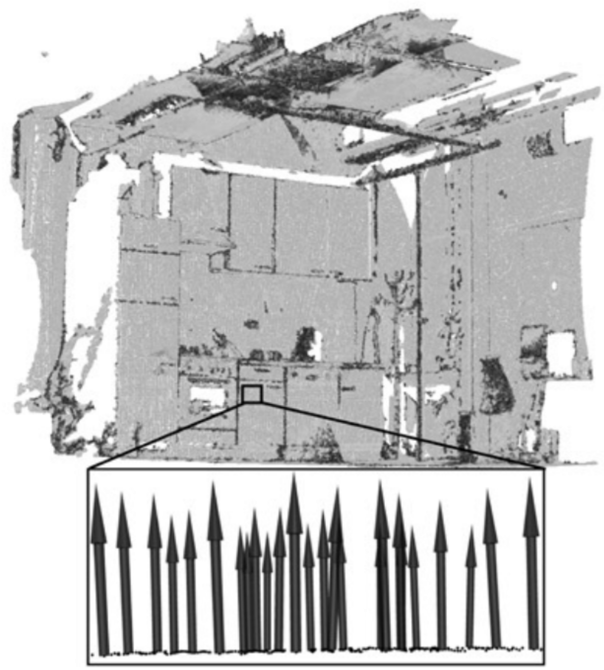
$$v_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

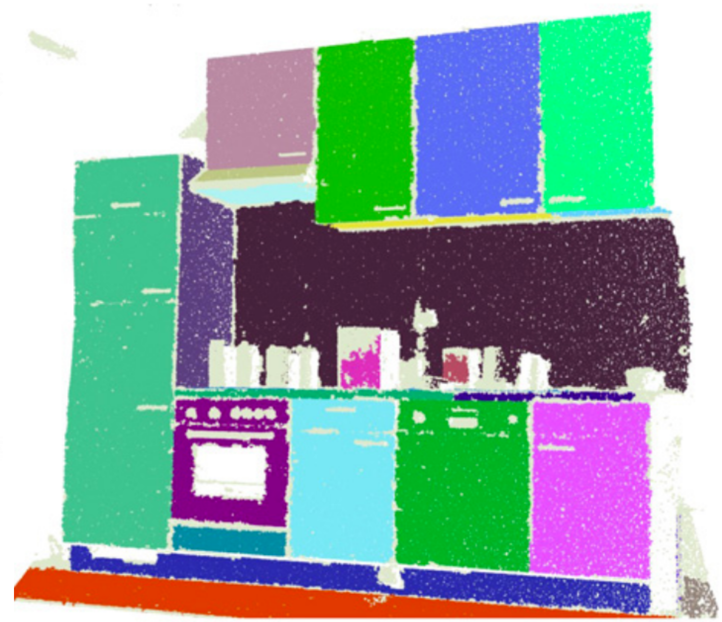
Principal Components

- The eigenvectors of the D-by-D covariance matrix XX^T of N data points observed in a D dimensional space, scaled by the eigenvalues



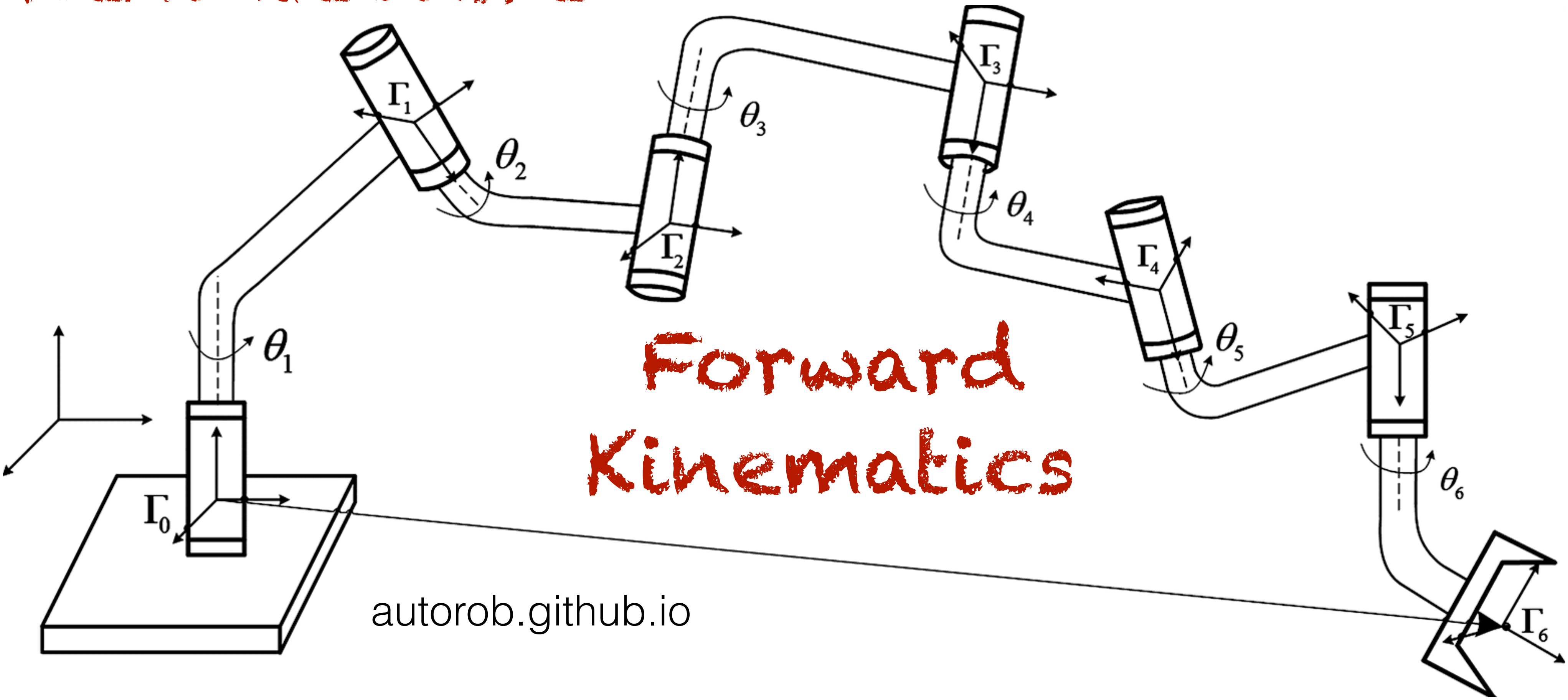
Shlens 2005





PCA-based flat surface estimator - Rusu et al. 2008

Next Lecture



Forward Kinematics

autorob.github.io