autorob.org

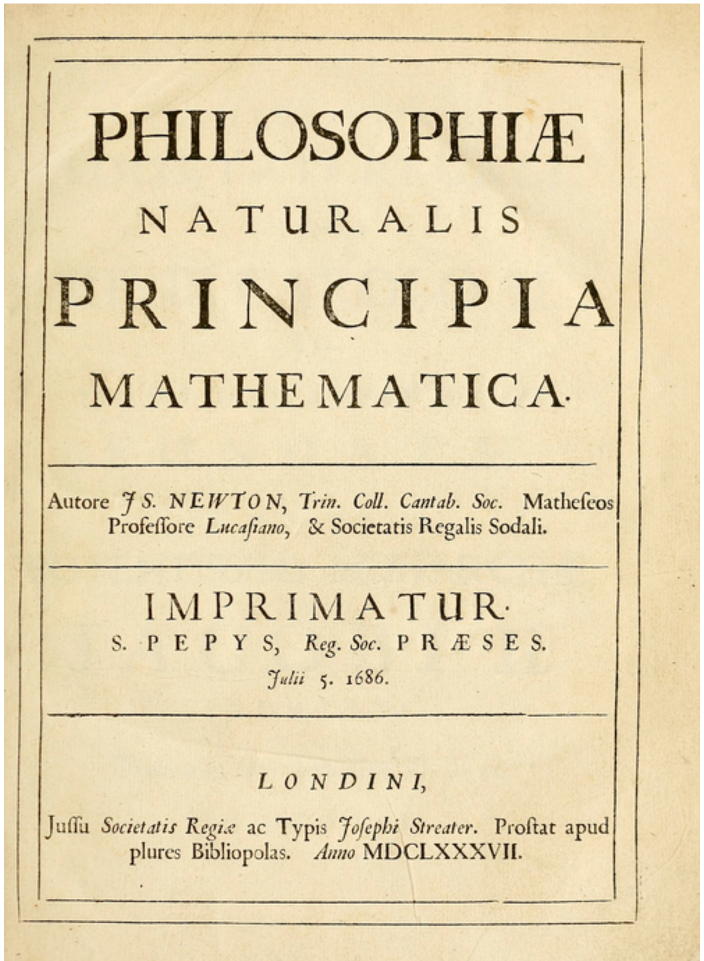
Dynamics and Numerical Integration

EECS 367
Intro. to Autonomous Robotics

ME/EECS 567 ROB 510
Robot Modeling and Control

Fall 2019

1687



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J S. NEWTON, *Trin. Coll. Cantab. Soc.* Mathefeos Profeffore *Lucafiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

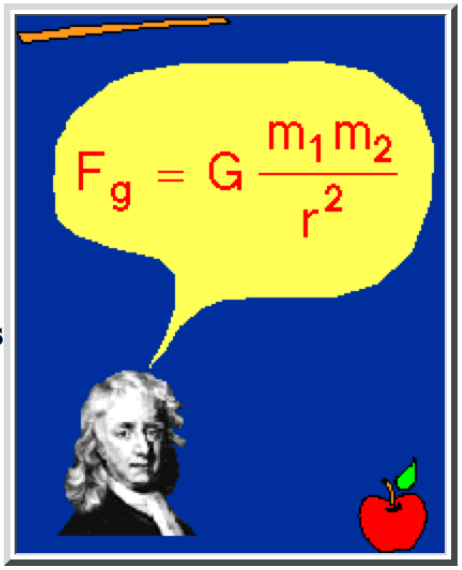
LONDINI,

Juffu *Societatis Regiæ* ac Typis *Jofephi Streater*. Proftat apud plures Bibliopolas. *Anno* MDCLXXXVII.

2012

http://drawception.com/viewgame/azF19y6c8s/edible-science/

There is a popular story that Newton was sitting under an apple tree, an apple fell on his head, and he suddenly thought of the Universal Law of Gravitation. As in all such legends, this is almost certainly not true in its details, but the story contains elements of what actually happened.



What Really Happened with the Apple?

Probably the more correct version of the story is that Newton, upon observing an apple fall from a tree, began to think along the following lines: The apple is accelerated, since its velocity changes from zero as it is hanging on the tree and moves toward the ground. Thus, by Newton's 2nd Law there must be a force that acts on the apple to cause this acceleration. Let's call this force "gravity", and the associated acceleration the "accleration due to gravity". Then imagine the apple tree is twice as high. Again, we expect the apple to be accelerated toward the ground, so this suggests that this force that we call gravity reaches to the top of the tallest apple tree.

http://csep10.phys.utk.edu/astr161/lect/history/newtongrav.html

Administrivia

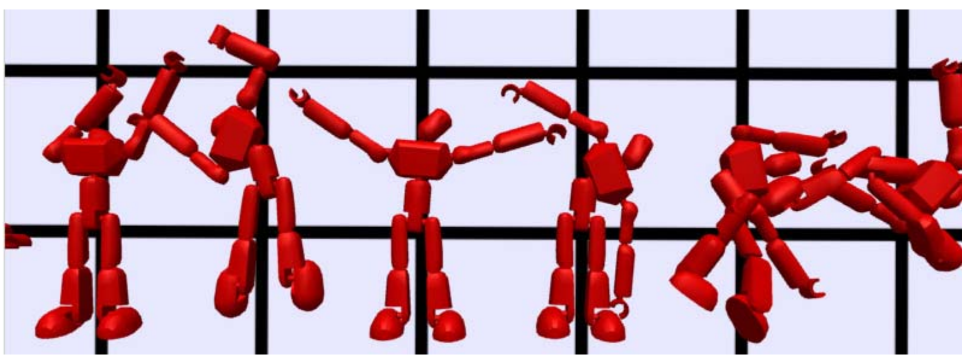
- Project 2 (“Pendularm”) due next Wednesday (Oct 2)
- If you are not on course slack group, let me know ASAP
    - Thanks for the repo links… and the pictures have been great!
- Enrollment: anyone still looking for permission to enroll?

Classical mechanics

physical laws describing the motion of bodies under the action of a system of forces

Kinematics

kin•e•mat•ics | ˌkinəˈmatiks |
plural noun [usu. treated as sing.]
the branch of mechanics concerned with the motion of objects without reference to the forces that cause the motion. Compare with



Dynamics

dy•nam•ics | dīˈnamiks |
plural noun
1 [treated as sing.] the branch of mechanics concerned with the motion of bodies under the action of forces. Compare with STATICS.
• [usu. with modifier] the branch of any science in which forces or changes are considered: *chemical dynamics*.

$$F = ma$$

Classical mechanics

physical laws describing the motion of bodies under the action of a system of forces

Kinematics

“the geometry of motion”

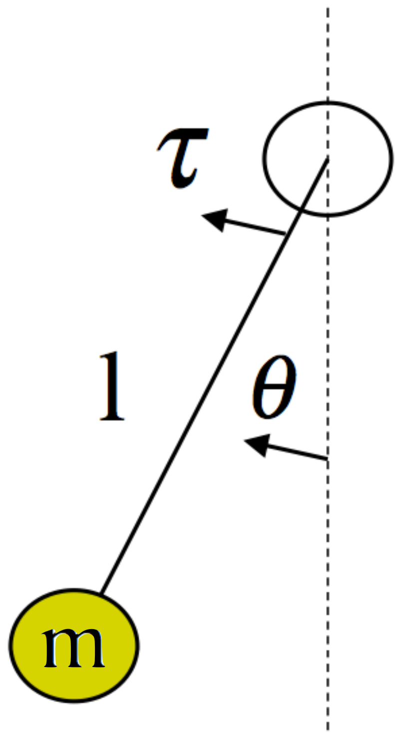
expresses range of possible motion
(or states of the robot)

Dynamics

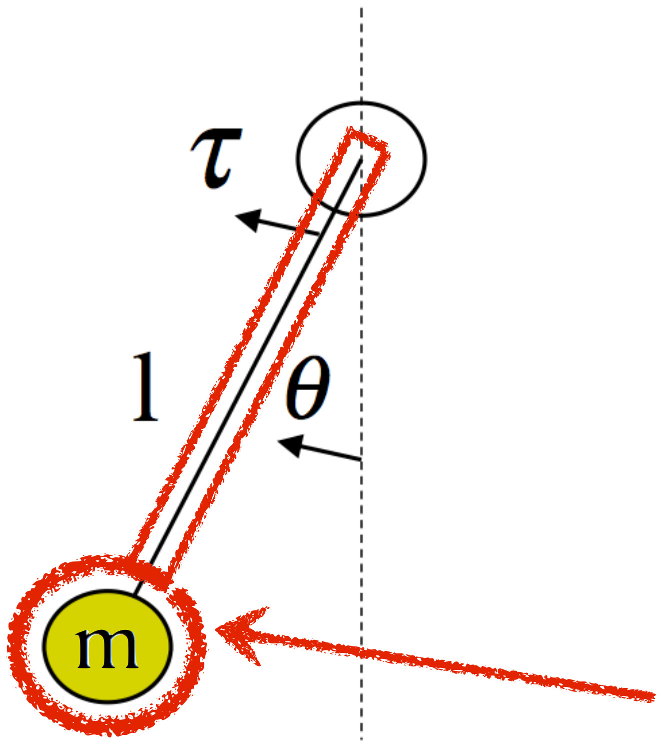
“physical motion over time”

expresses motion over time with
respect to Newton’s laws
(evolution of state over time)

Example: Motorized Pendulum

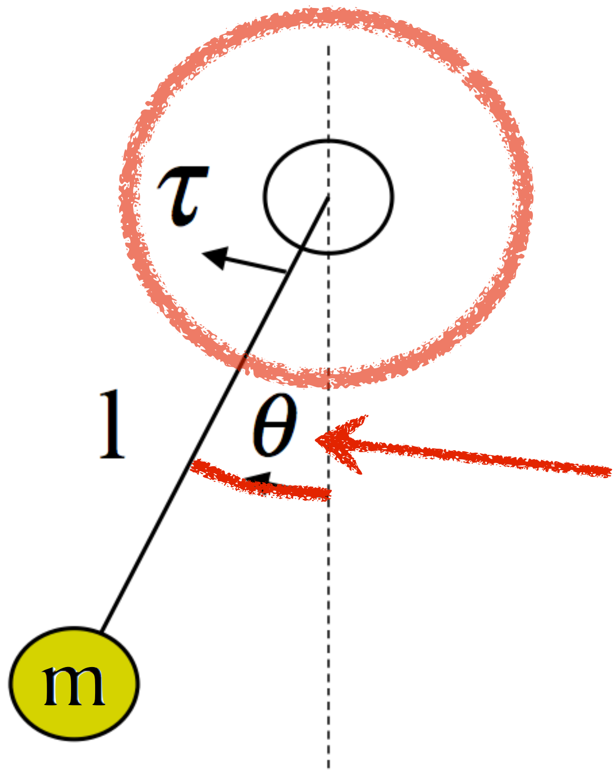


Example: Motorized Pendulum



Pendulum of length *l* with point mass *m*

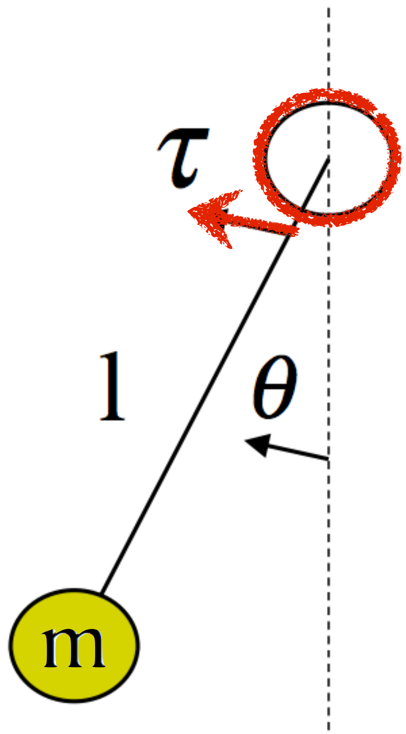
Example: Motorized Pendulum



Angle expresses pendulum range of motion

Pendulum of length *l* with point mass *m*

Example: Motorized Pendulum



Motor produces torque (angular force)

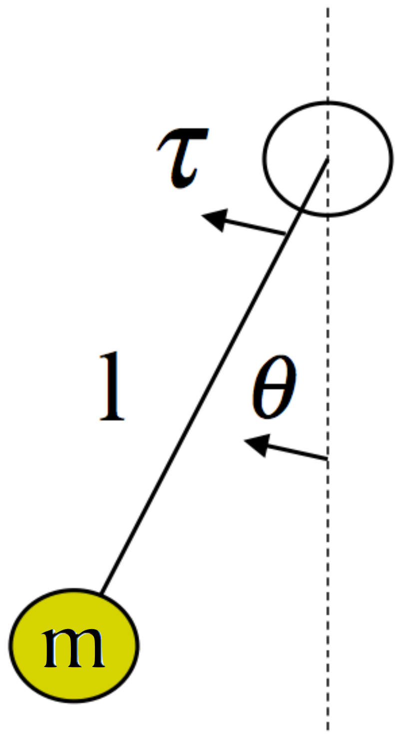
Angle expresses pendulum range of motion

Pendulum of length *l* with point mass *m*

Example: Motorized Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$



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Motorized Pendulum

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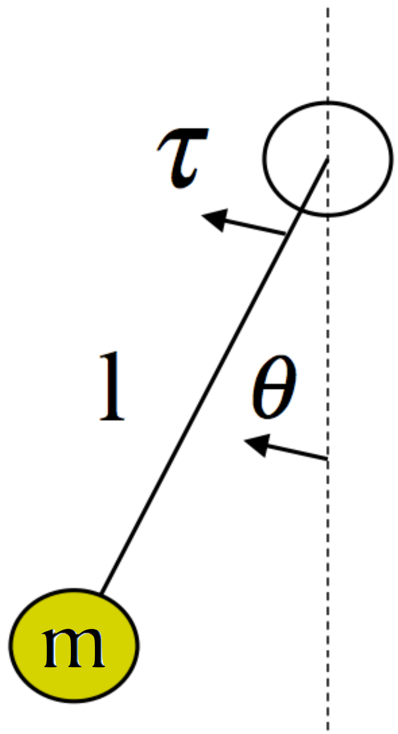


Controls

Motor produces torque
(angular force)

State (or Configuration)

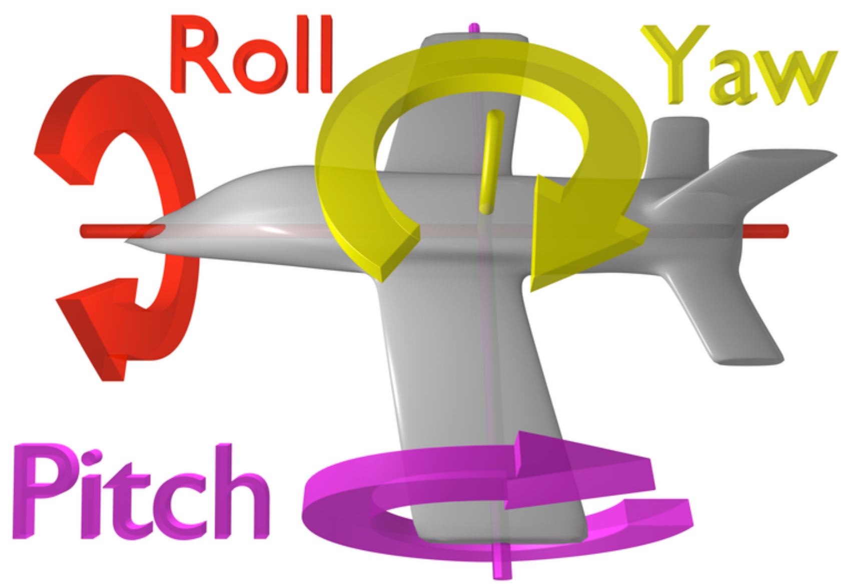
Angle expresses pendulum
range of motion

System

Pendulum of length *l* with
point mass *m*

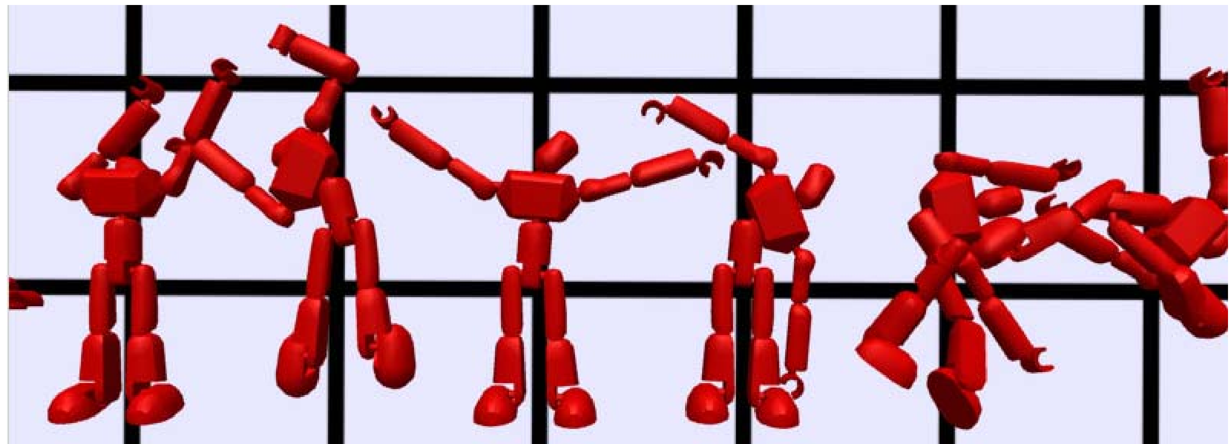
Defining State

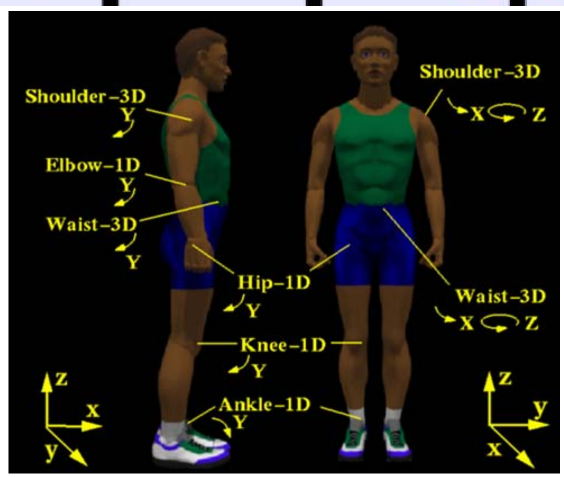
- State comprised of degrees-of-freedom (DOFs)
- DOFs describe translational and rotational axes for joints
- How many DOFs in the example pendulum?
- Airplane DOFs?

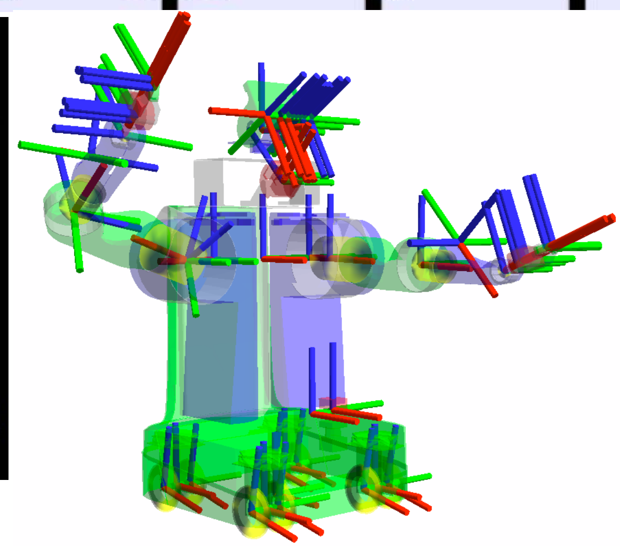


Defining State

- State comprised of degrees-of-freedom (DOFs)
- DOFs describe translational and rotational axes of system
- Humanoid DOFs?
  - joint angles
  - global positioning





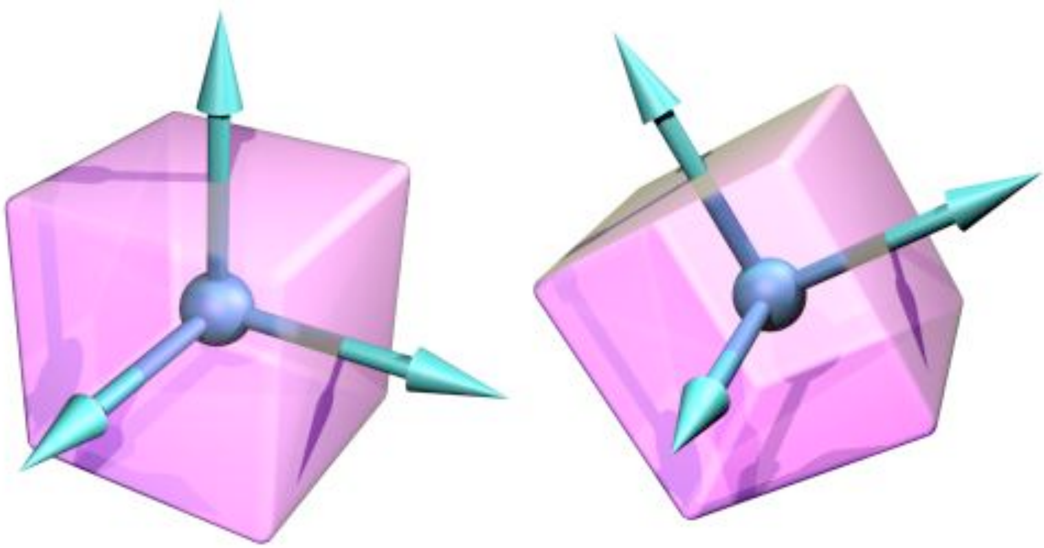


DOFs and Coordinate Spaces

- Each <u>body</u> has its own <u>coordinate system</u>

LINK

FRAME



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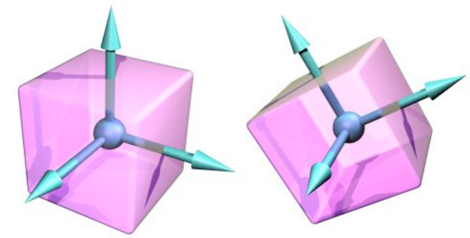
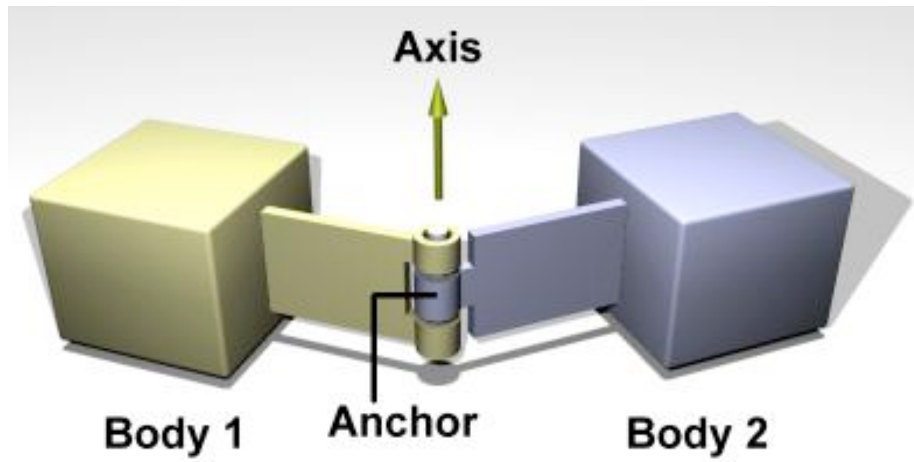
LINK FRAME



- Joints connect two links (rigid bodies)
  - Hinge (1 rotational DOF)

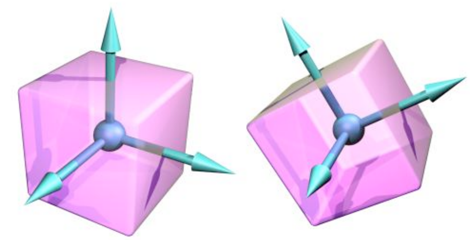


HINGE JOINT

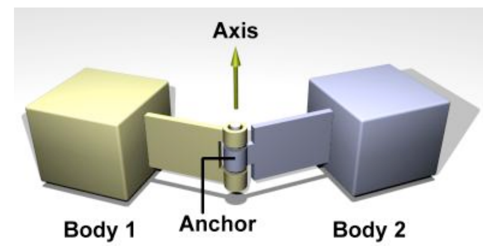
DOFs and Coordinate Spaces

- Each <u>body</u> has its own <u>coordinate system</u>

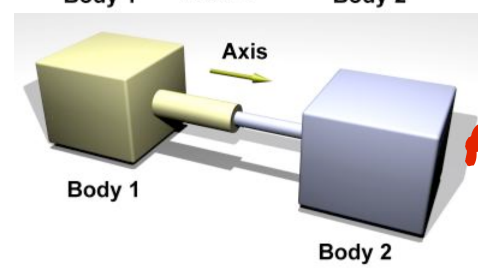
LINK FRAME



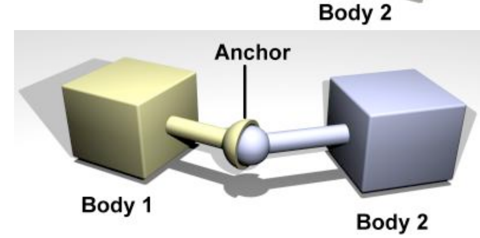
- Joints connect two links (rigid bodies)
  - Hinge (1 rotational DOF)
  - Prismatic (1 translational DOF)
  - Ball-socket (3 DOFs)



HINGE



PRISMATIC

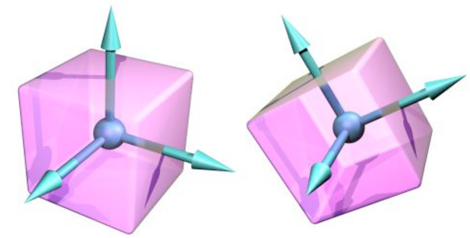


BALL

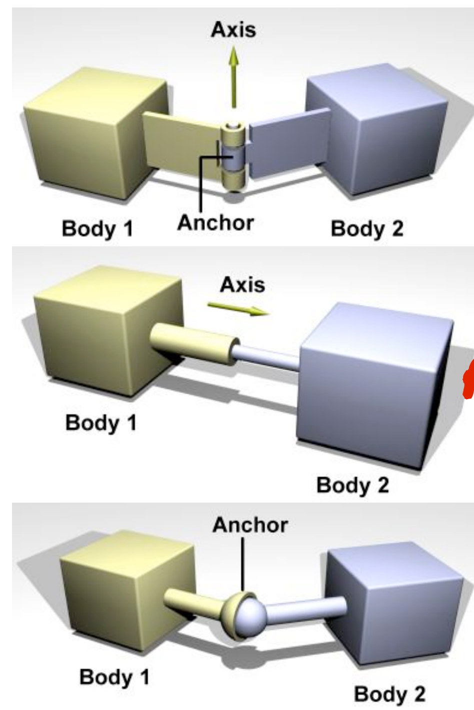
DOFs and Coordinate Spaces

- Each body has its own coordinate system

LINK FRAME



- Joints connect two links (rigid bodies)
  - Hinge (1 rotational DOF)
  - Prismatic (1 translational DOF)
  - Ball-socket (3 DOFs)
- A motor exerts force along a DOF axis



HINGE

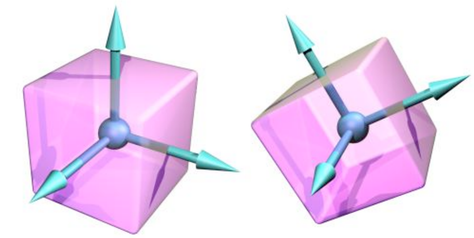
PRISMATIC

BALL

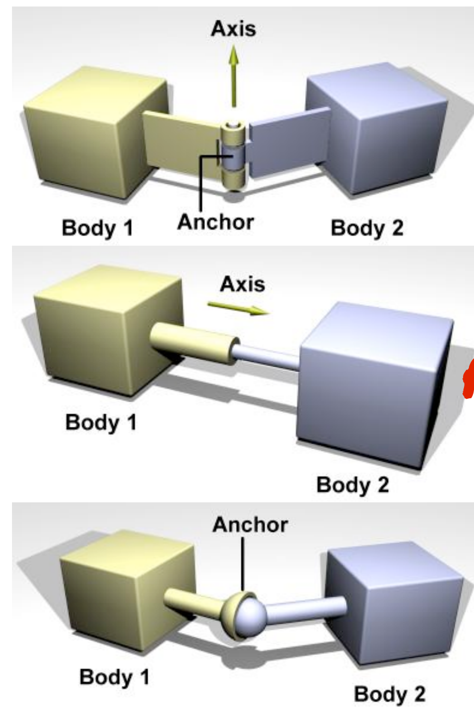
DOFs and Coordinate Spaces

- Each body has its own coordinate system

LINK FRAME



- Joints connect two links (rigid bodies)
  - Hinge (1 rotational DOF)
  - Prismatic (1 translational DOF)
  - Ball-socket (3 DOFs)
- A motor exerts force along a DOF axis
- Linear transformations used to relate coordinate frames of robot links and joints
- Spatial geometry attached to each link, but does not affect the body's coordinate frame



HINGE

PRISMATIC

BALL

Robotic machines are comprised of *N* joints and *N+1* links

Joints and links form a tree hierarchy of articulated motion

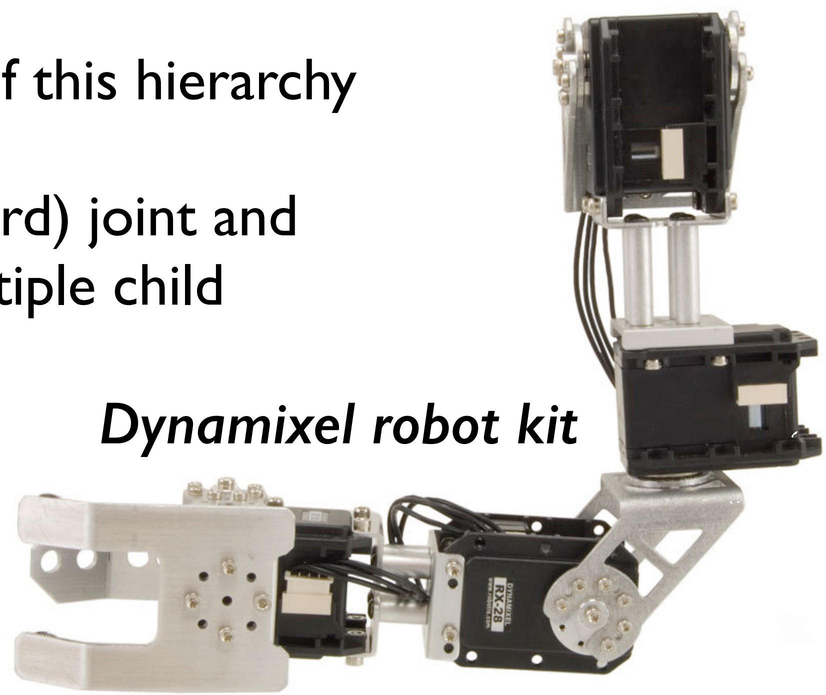
The “base” is the root link of this hierarchy

A link has one parent (inboard) joint and potentially zero, one, or multiple child (outboard) joints

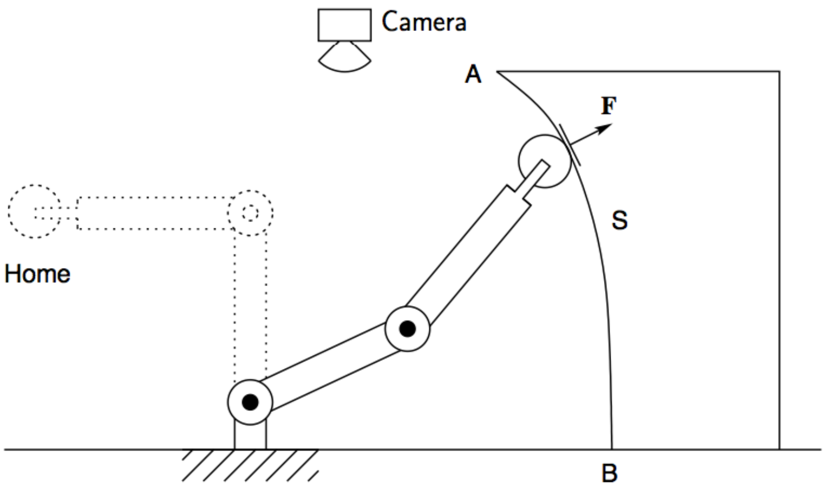
A serial chain is a robot where every link has only one child joint

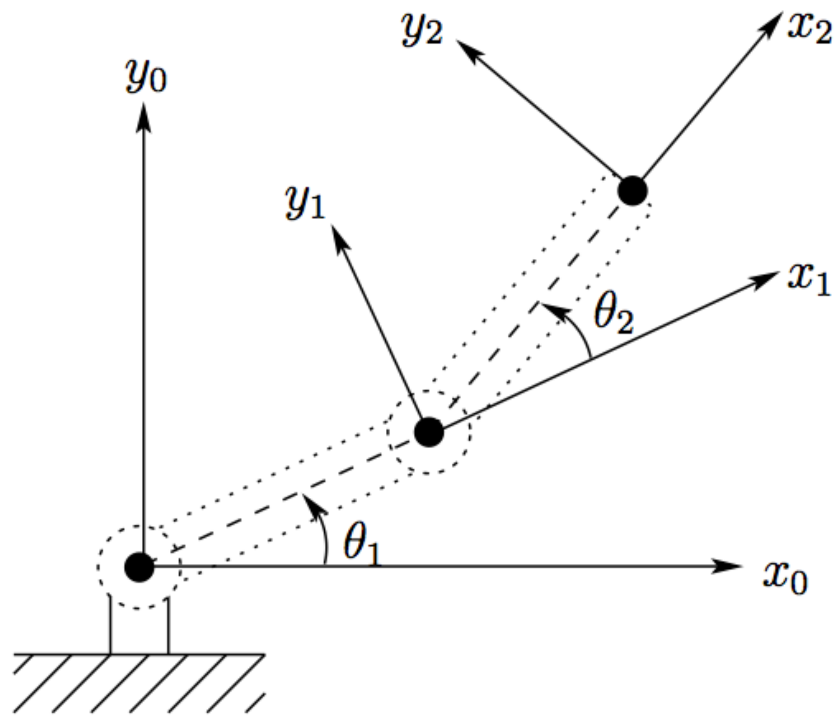
DLR Lightweight arm

Dynamixel robot kit



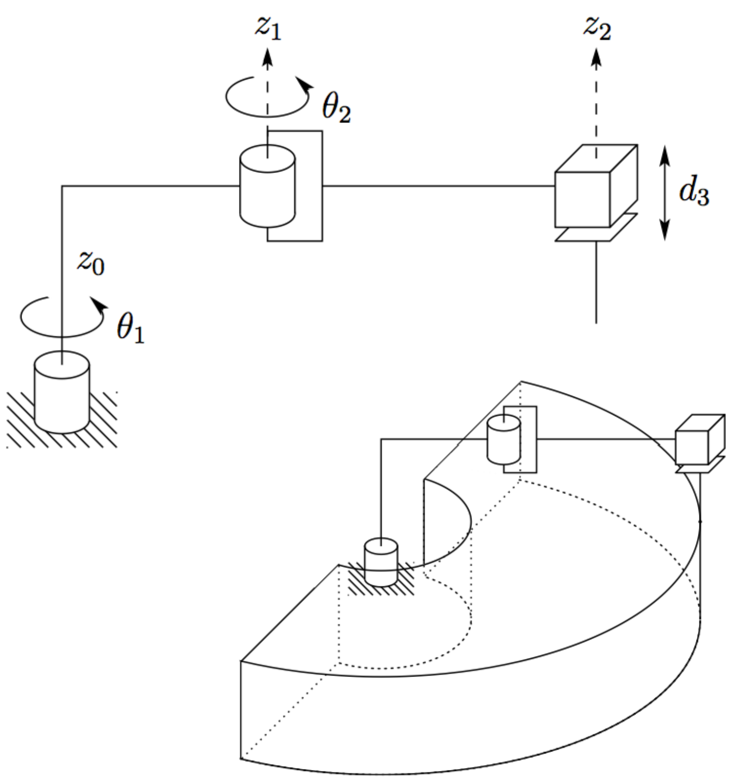
Planar 2-DOF 2-link Arm



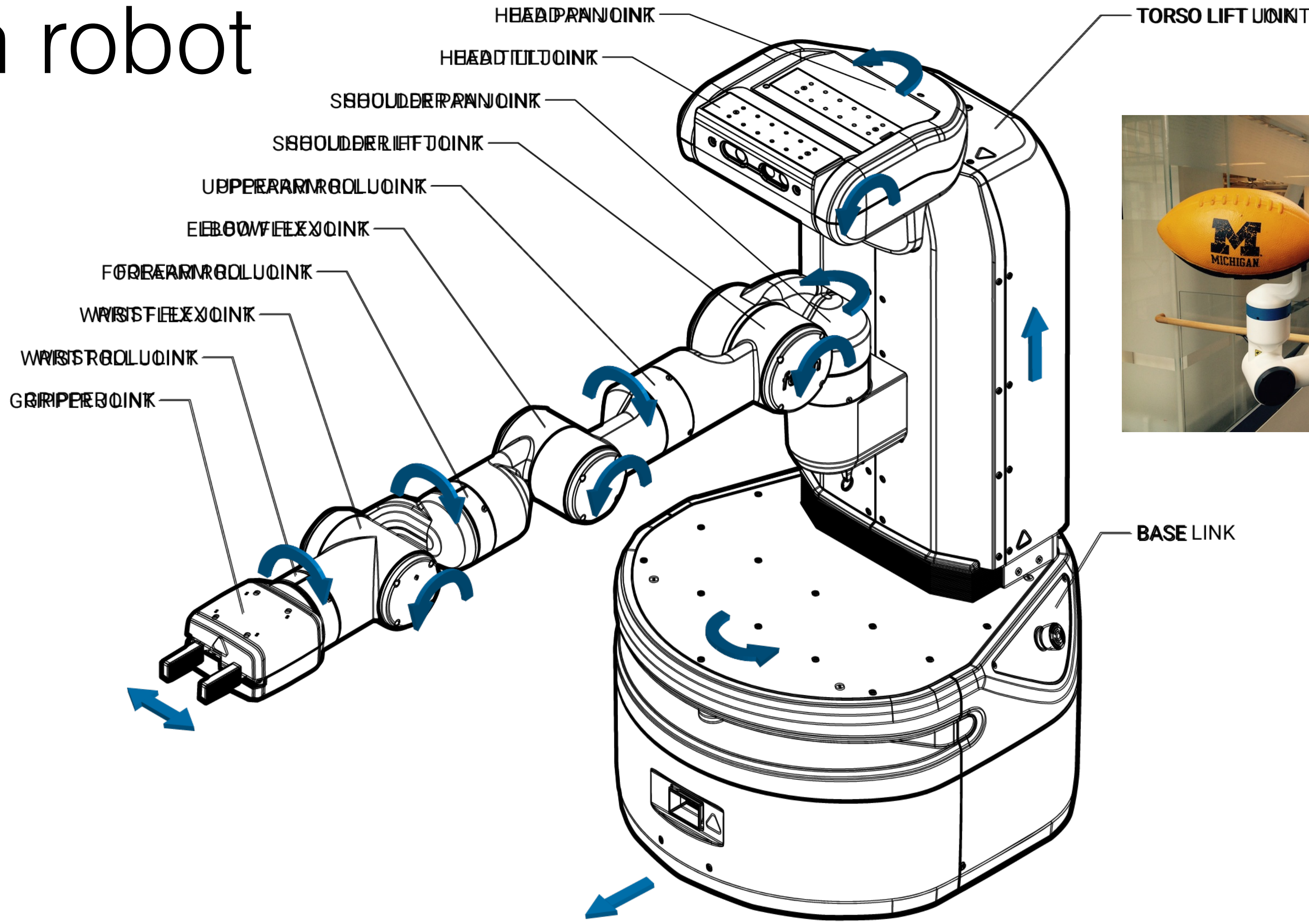


SCARA Arm

Selective Compliance Assembly Robot Arm



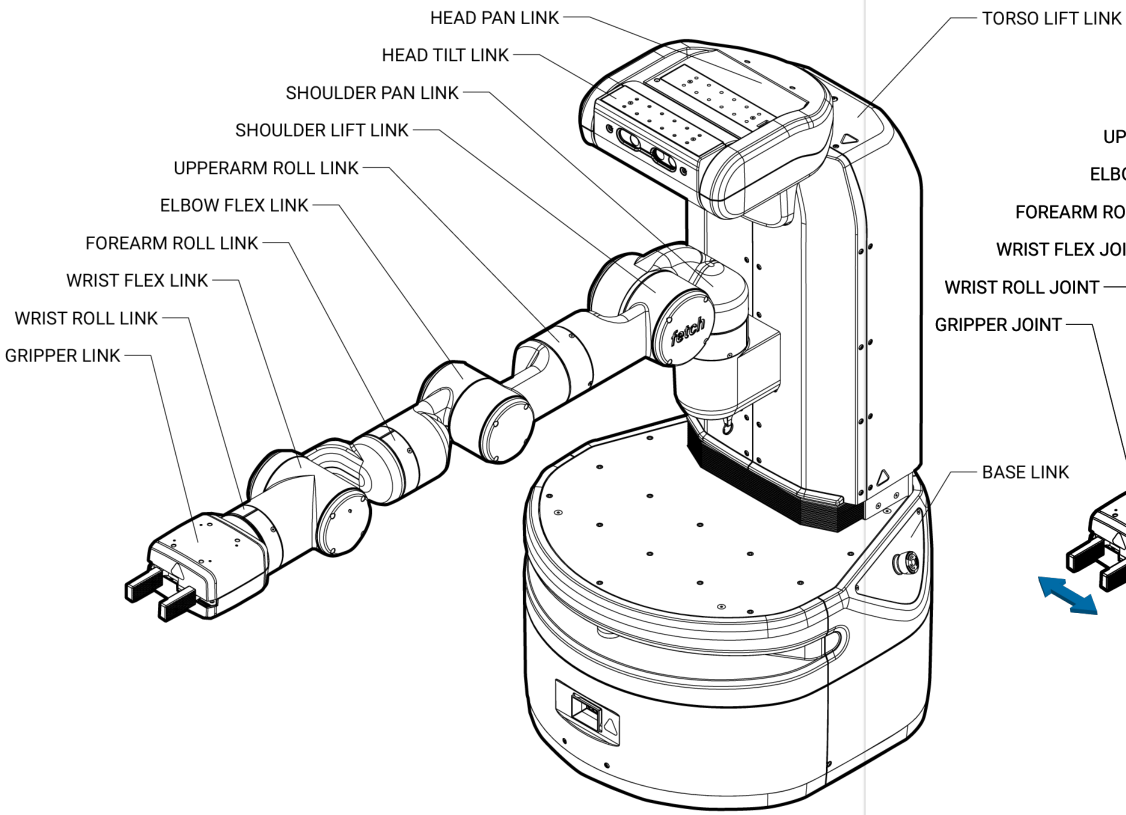
Fetch robot



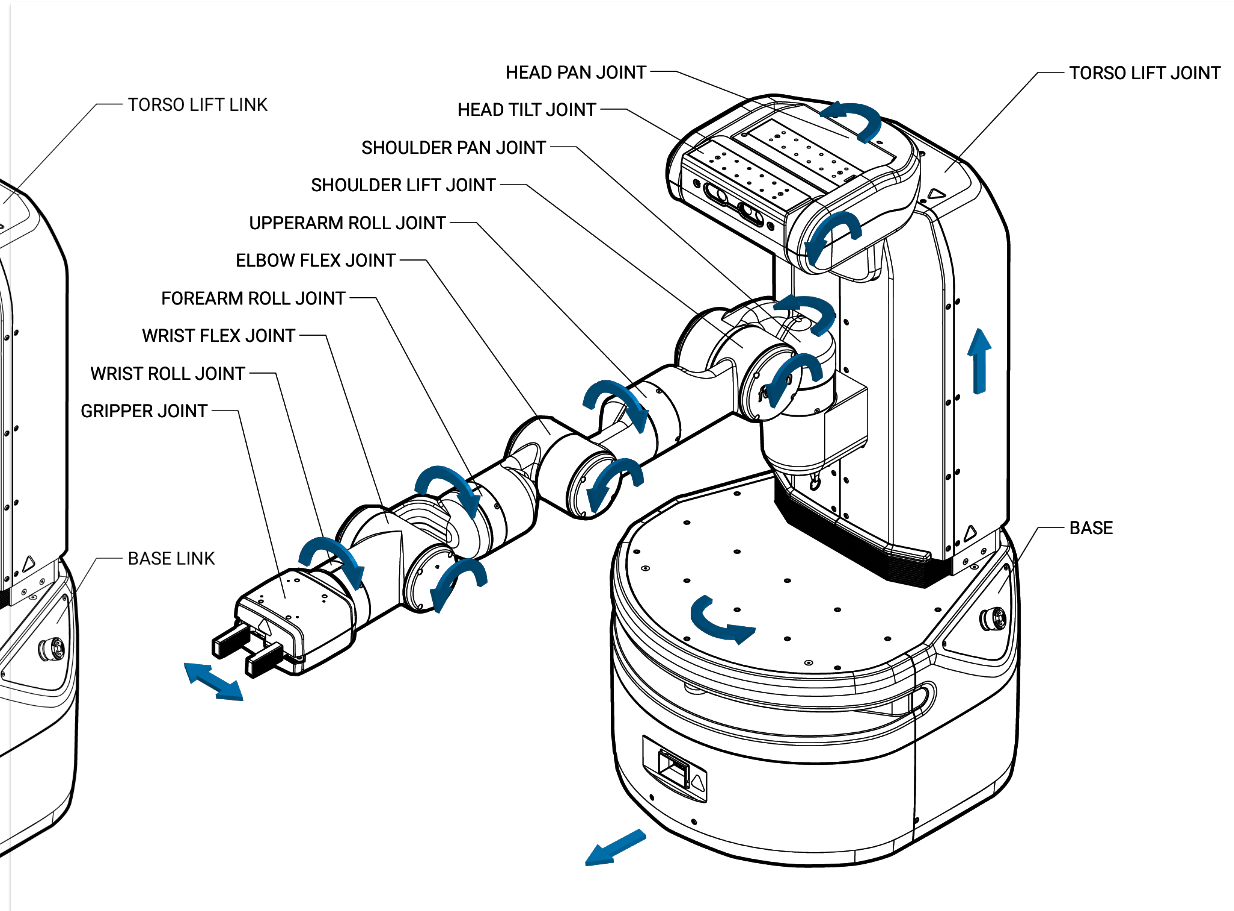
HEAD PAN JOINT
HEAD TILT JOINT
SHOULDER PAN JOINT
SHOULDER LIFT JOINT
UPPERARM ROLL JOINT
ELBOW FLEX JOINT
FOREARM ROLL JOINT
WRIST FLEX JOINT
WRIST ROLL JOINT
GRIPPER JOINT
TORSO LIFT JOINT
BASE LINK

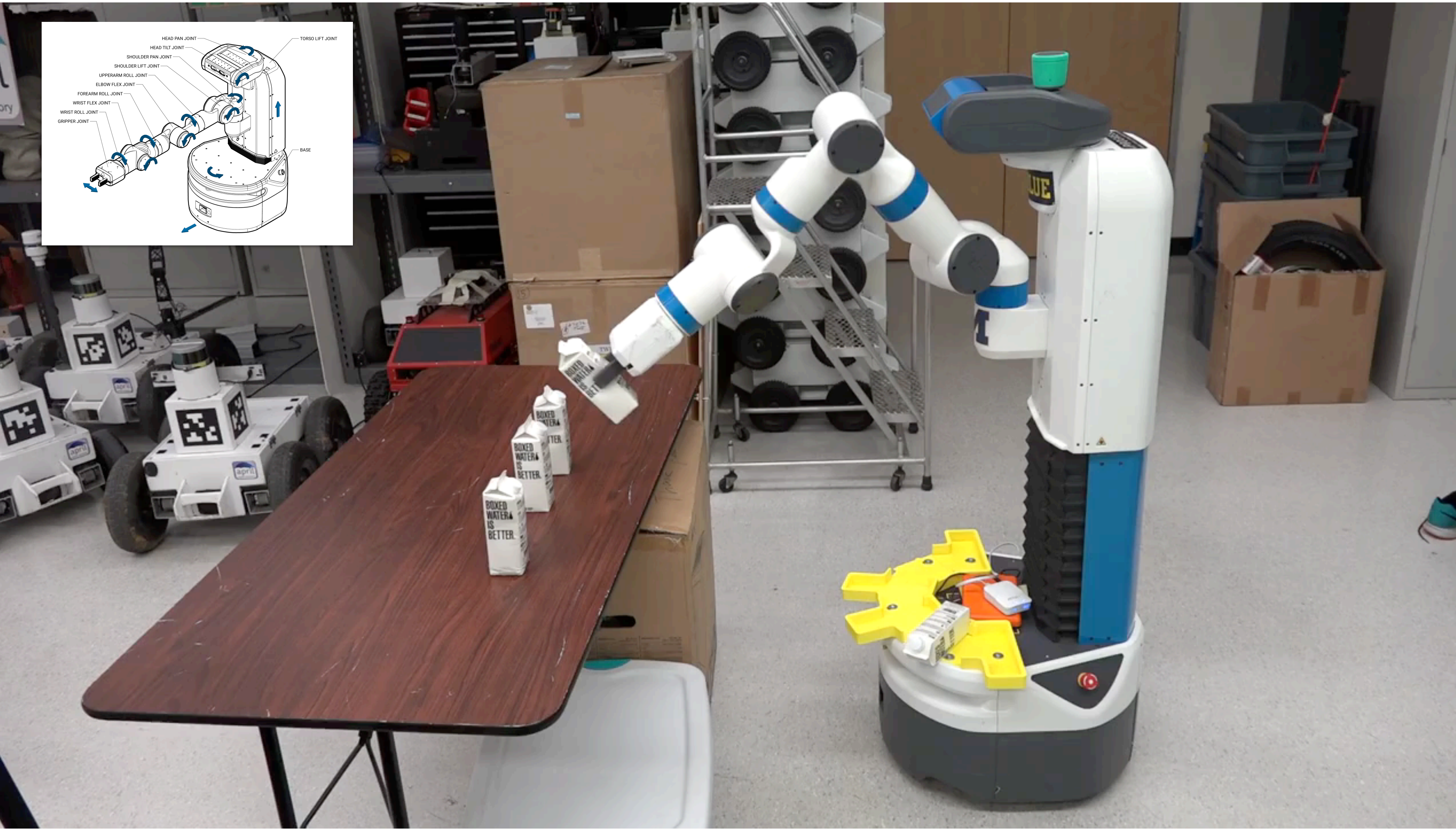
MICHIGAN
fetch

Fetch links

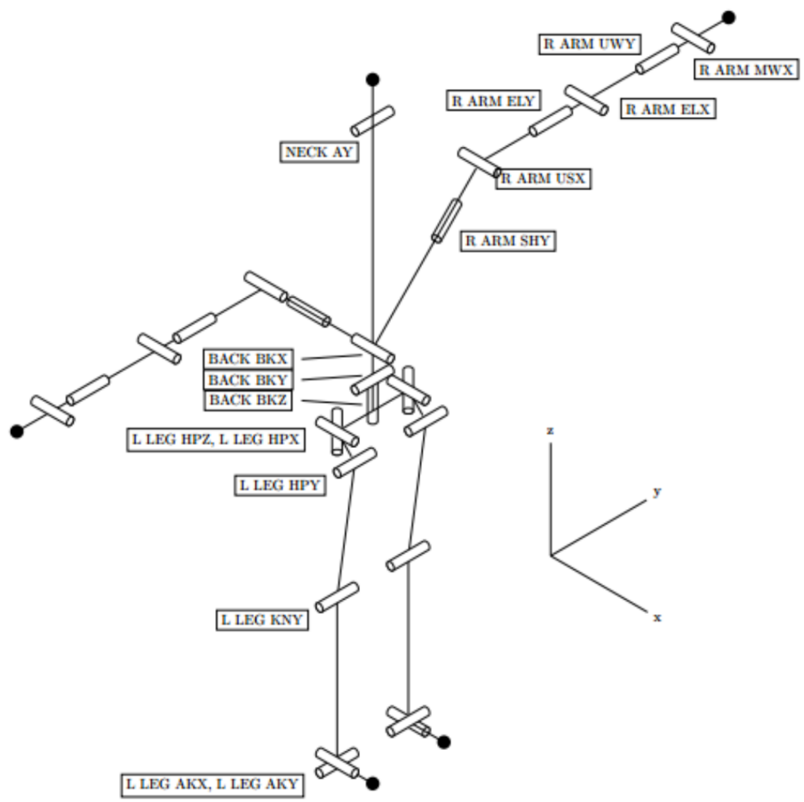


Fetch joints



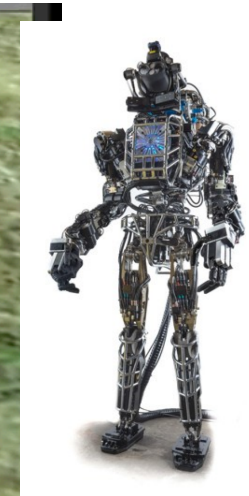

HEAD PAN JOINT
HEAD TILT JOINT
SHOULDER PAN JOINT
SHOULDER LIFT JOINT
UPPERARM ROLL JOINT
ELBOW FLEX JOINT
FOREARM ROLL JOINT
WRIST FLEX JOINT
WRIST ROLL JOINT
GRIPPER JOINT
TORSO LIFT JOINT
BASE
BOXED WATER IS BETTER.

Atlas robot



WPI

Physical Simulation (Gazebo)



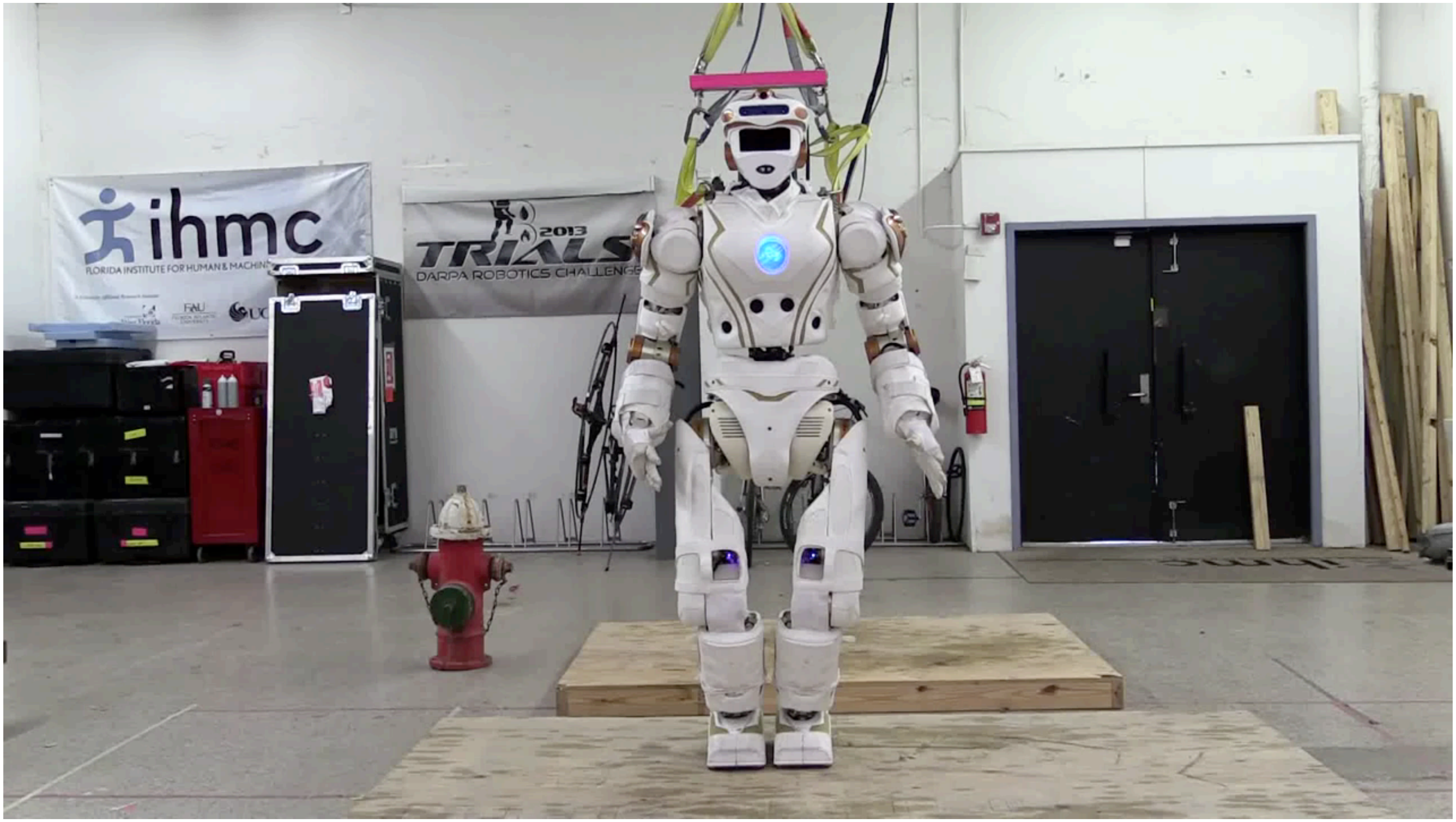
Why simulate robots?

Why simulate robots?

- **Real robots are expensive**
  - Improper controllers can physically break robots
  - Inexpensive to experiment and test robot controllers in simulation
- **Predictive model of dynamics**
  - Necessary for some types of control (e.g., optimal control)

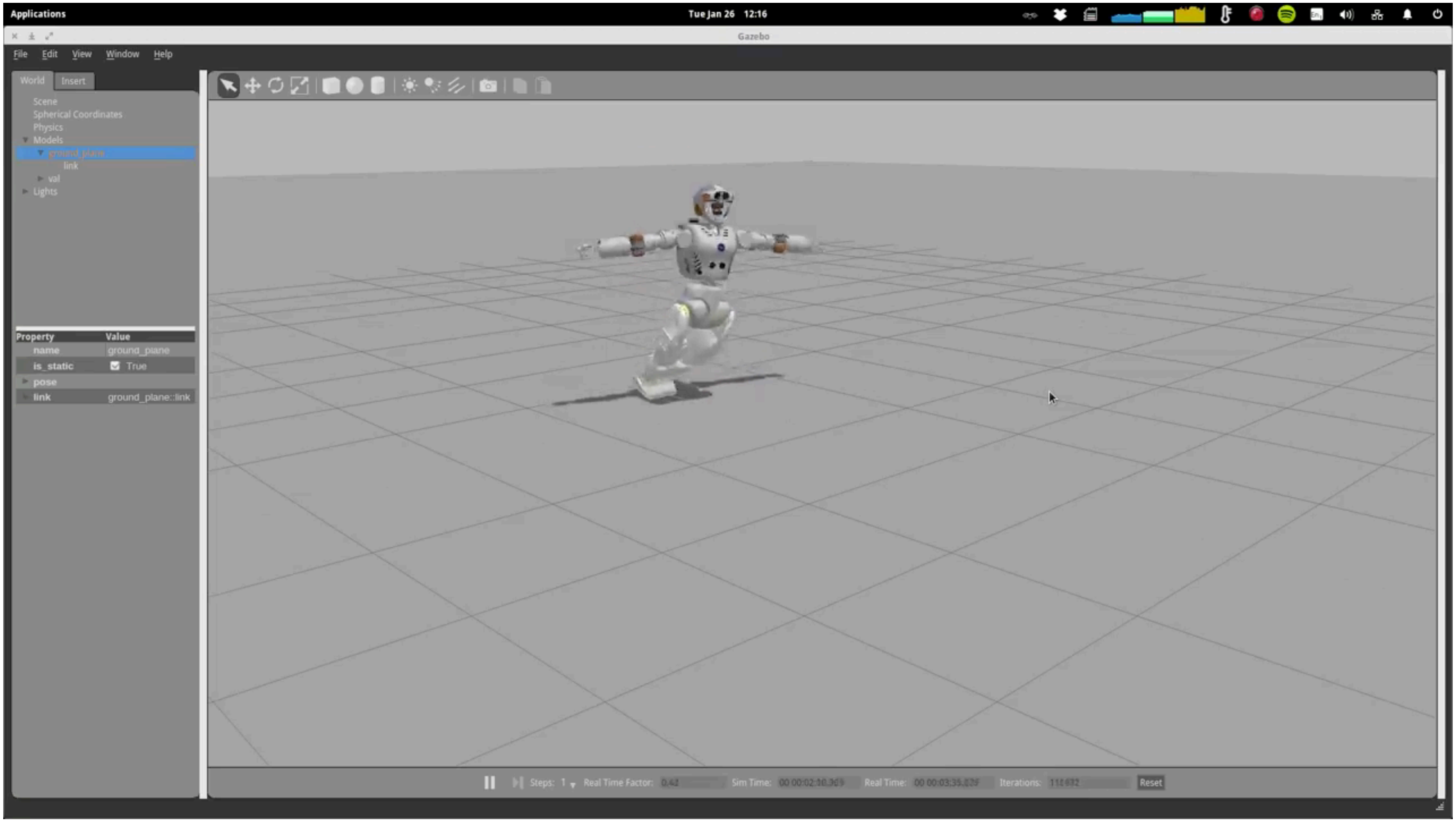
$$J = \Phi\left[\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f\right] + \boxed{\int_{t_0}^{t_f} \mathcal{L}\left[\mathbf{x}(t), \mathbf{u}(t), t\right] \mathrm{d}\, t}$$

For example,
NASA Valkyrie…



Valkyrie for real - https://youtu.be/5Ee5u2ekE8c

NASA Valkyrie…
development in simulation



Valkyrie in simulation - https://youtu.be/tLCpJvqgtRQ

Our first robot:

Pendularm!

Project 2: 1 DOF Pendularm

file:///Users/logan/git_tmp/kineval/pendularm/pendularm1.html

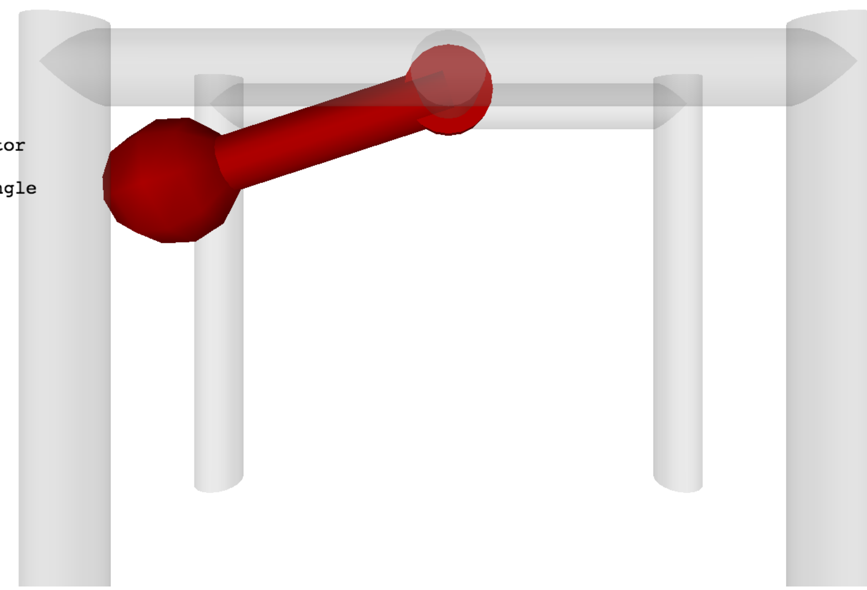
```
System
t = 162.00 dt = 0.05
integrator = velocity verlet
x = -1.26
x_dot = -0.00
x_desired = -1.26

Servo: active
u = -37.32
kp = 1500.00
kd = 15.00
ki = 150.10

Pendulum
mass = 2.00
length = 2.00
gravity = 9.81

Keys
[0-4] - select integrator
a/d - apply user force
q/e - adjust desired angle
c - toggle servo
s - disable servo
```

model, simulate, and control
1 DoF robot arm

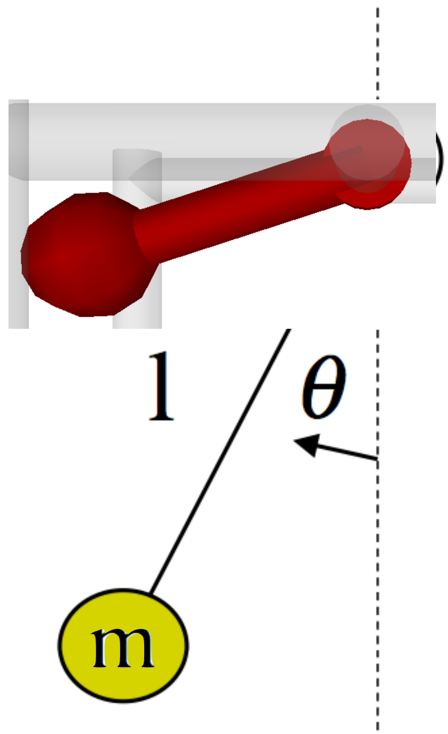


Example: Pendulum

Dynamics

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$



Controls

Motor produces torque
(angular force)

State (or Configuration)

Angle expresses pendulum
range of motion

System

Pendulum of length *l* with
point mass *m*

Example: Pendulum

Dynamics

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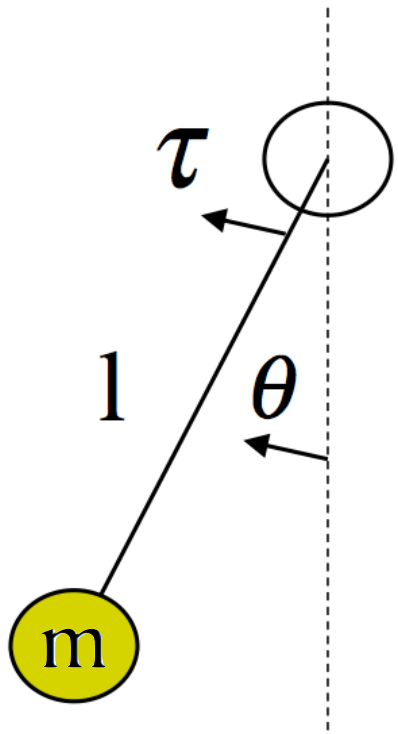

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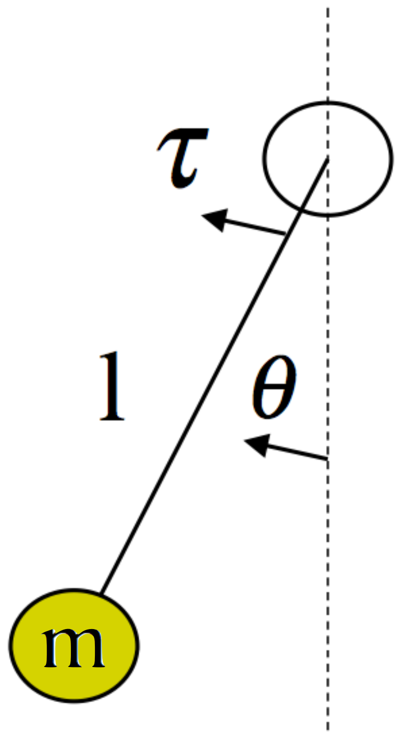
MOTOR
TORQUE



Motor produces torque
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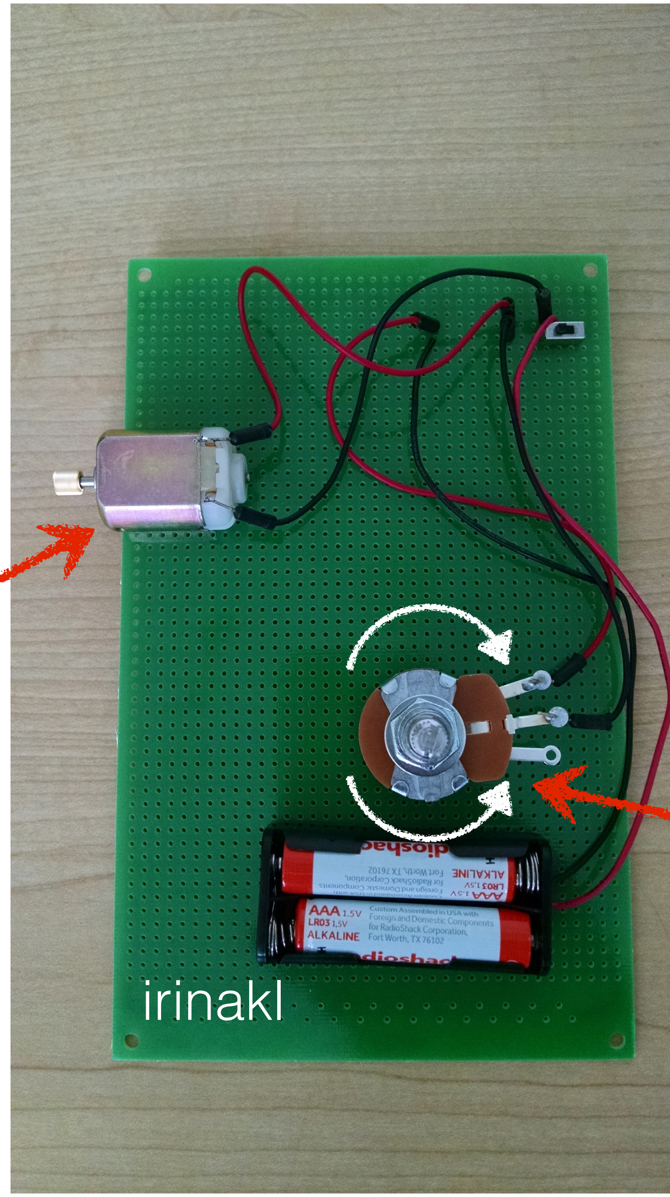
Pendulum of length *l* with
point mass *m*

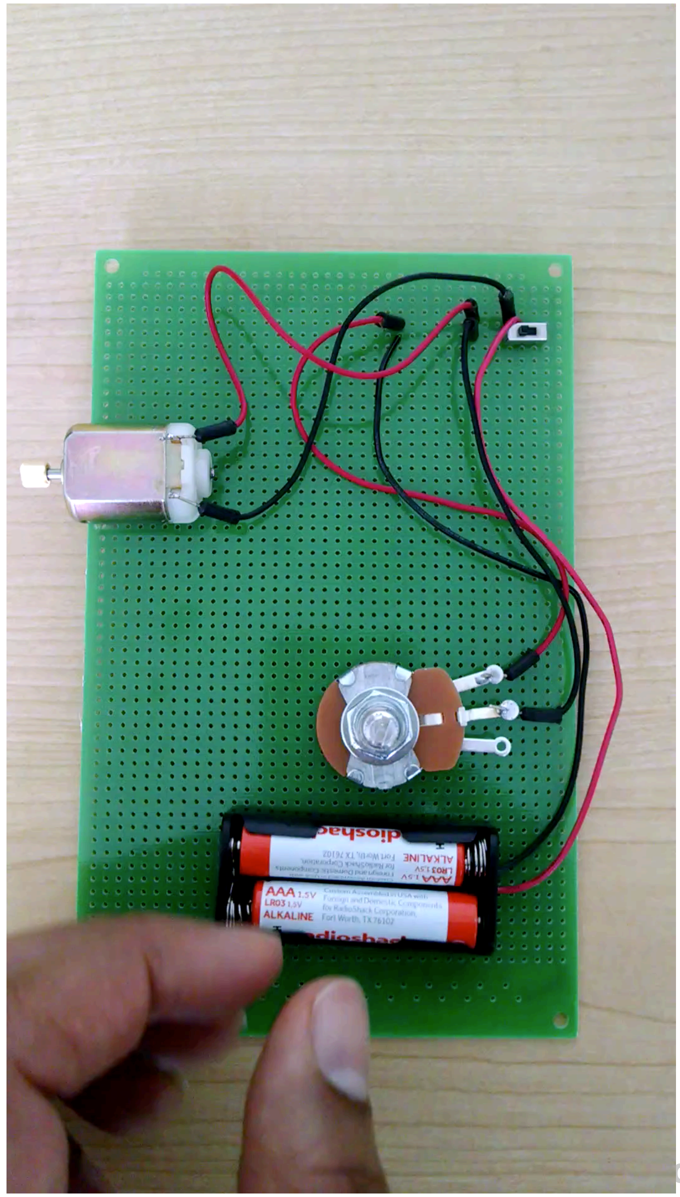
Example device



Control torque of motor as a proportion of current applied

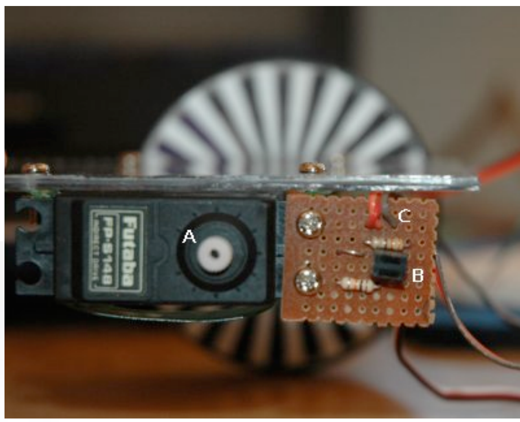
Decrease resistance to increase force

Increase resistance to decrease force

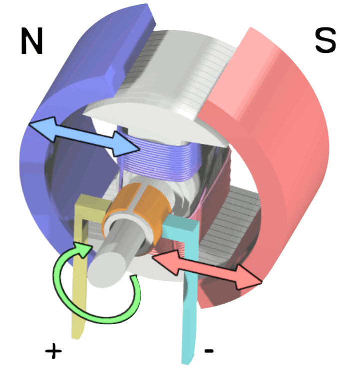

AAA 1.5V
LR03 1.5V
ALKALINE

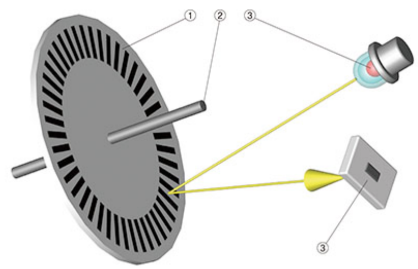
- actuators to produce motion
- proprioception to sense pose
- controller to regulate current to motor

Servo



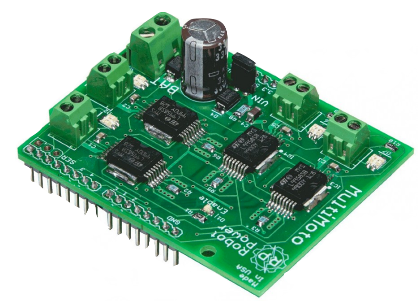
Actuator
(electric)





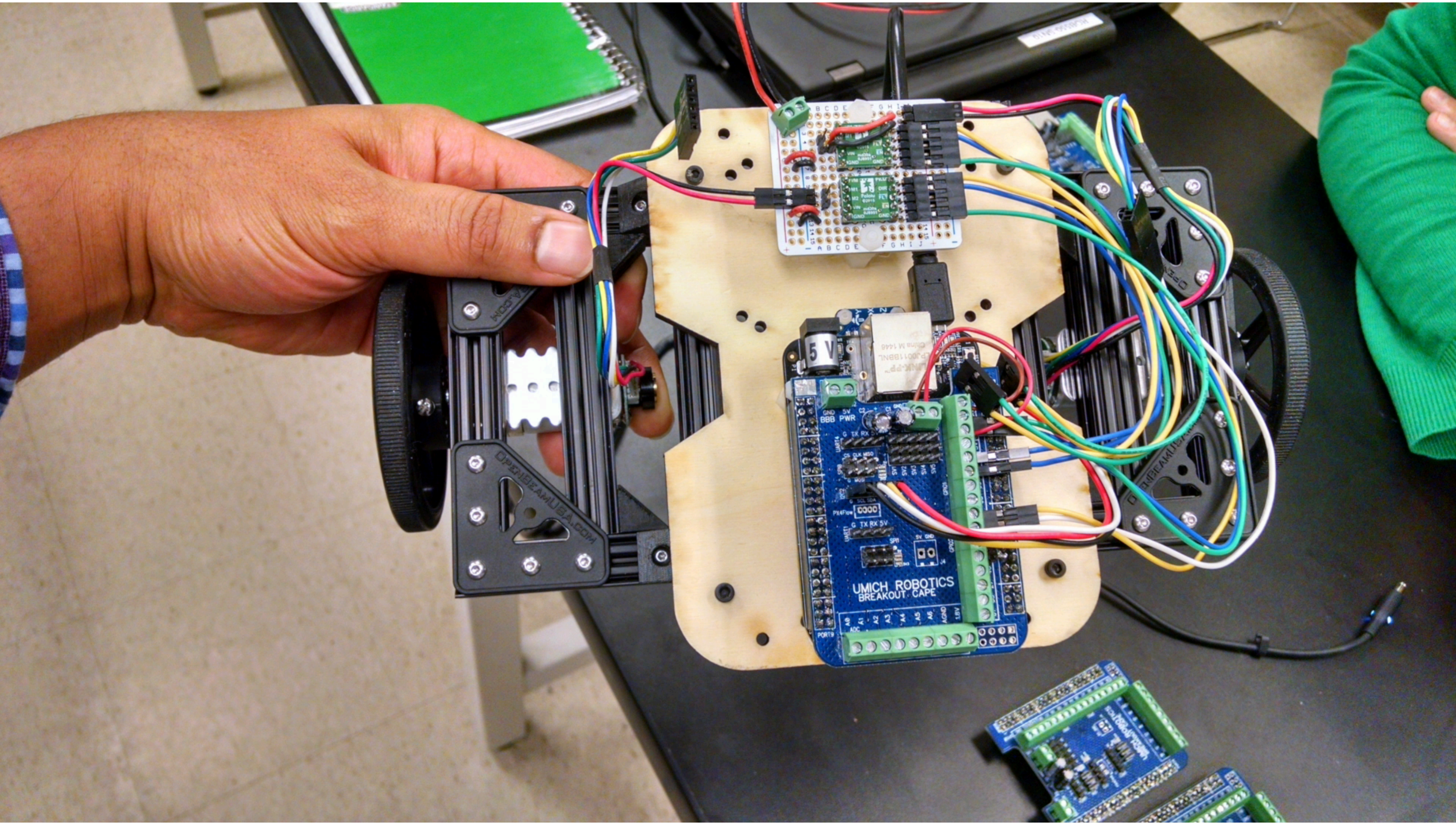
Proprioception
(optical encoder)

Controller
(4 channel H-Bridge)



Consider servos on the wheels of an inverted pendulum

ROB 550 Fall 2016 - https://youtu.be/9QUSexcsSZk


UMICH ROBOTICS
BREAKOUT CAPE
OPENBEAMUSA.COM

What is an equation of motion?

Example: Pendulum

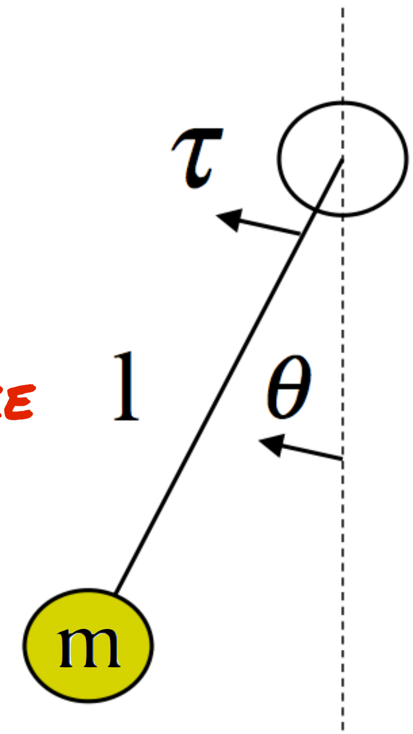
Equation of motion
(with rotational inertia *I*)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

INERTIA

ACCELERATION

NET FORCE



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Example: Pendulum

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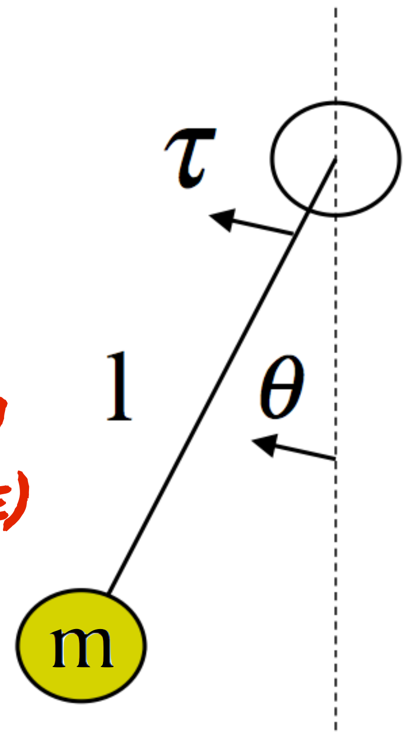
ANGULAR ACCELERATION
(SECOND TIME DERIVATIVE)



Motor produces torque
(angular force)

Angle expresses pendulum
range of motion

Pendulum of length *l* with
point mass *m*

Example: Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

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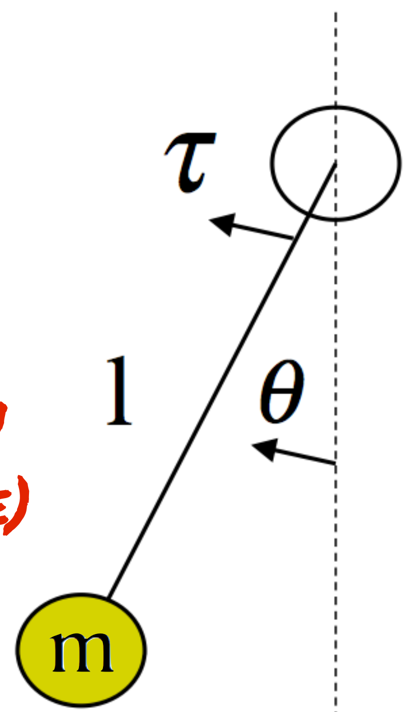
or $\frac{d\dot{\theta}}{dt}$ as the rate of change of the velocity of the pendulum angle



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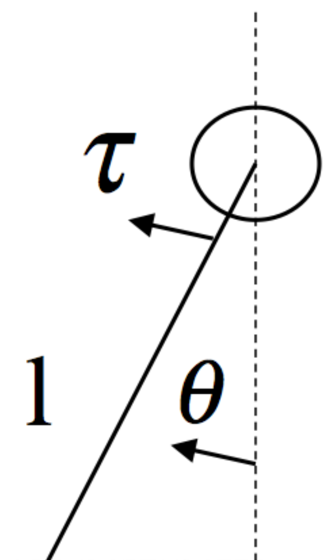
or $\dfrac{d^2\theta}{dt^2}$ as the rate of change of the rate of change of the pendulum angle



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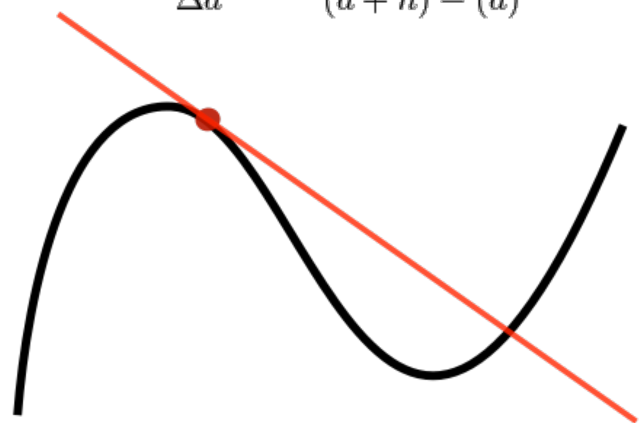
Derivative

From Wikipedia, the free encyclopedia

The **derivative** of a function of a real variable measures the sensitivity to change of a quantity (a function or dependent variable) which is determined by another quantity (the independent variable).

$$m = \frac{\Delta f(a)}{\Delta a} = \frac{f(a+h)-f(a)}{(a+h)-(a)} = \frac{f(a+h)-f(a)}{h}.$$



First derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

with approximation after step length *h*

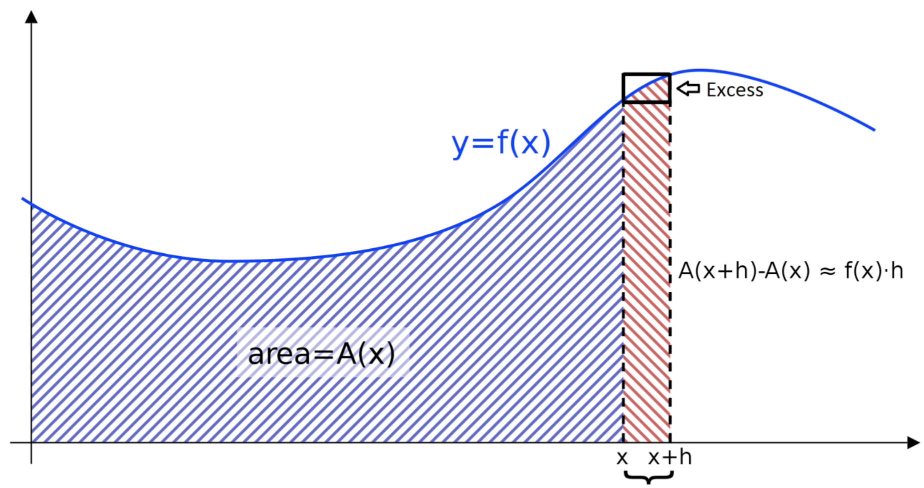
$$f(a+h) \approx f(a) + f'(a)h$$

Second derivative

$$f''(a) = \lim_{h\to 0} \frac{f'(a+h)-f'(a)}{h}$$

Integration inverse operation of differentiation

$$\frac{d}{dx} f(x) = g(x), \int g(x)\,dx = f(x) + C$$



Differential equation

From Wikipedia, the free encyclopedia

A **differential equation** is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

Power rule

$$\frac{d}{dx}x^n = nx^{n-1}, \qquad n \neq 0.$$

Time derivatives (velocity, acceleration)

$$\frac{dx}{dt} \text{ or } \dot{x} \qquad \frac{d^2x}{dt^2} \text{ or } \ddot{x}$$

Product rule

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

Partial derivative

$$f'_x,\ f_x,\ \partial_x f,\ \frac{\partial}{\partial x}f,\ \text{or}\ \frac{\partial f}{\partial x}.$$

Chain rule

$$[f(g(x))]' = f'(g(x))g'(x)$$

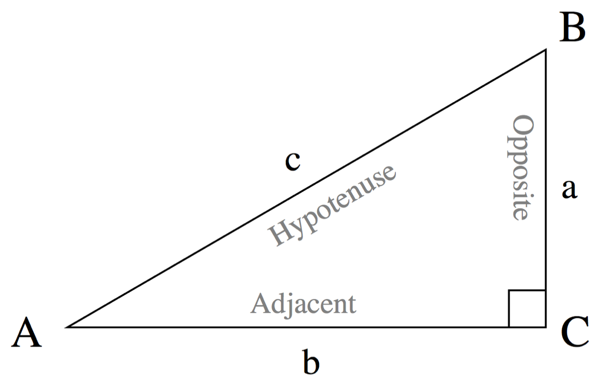
Total derivative

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Trigonometry

From Wikipedia, the free encyclopedia

Trigonometry (from Greek *trigōnon*, "triangle" and *metron*, "measure"[1]) is a branch of mathematics that studies relationships involving lengths and angles of triangles. The field emerged during the 3rd century BC from applications of geometry to astronomical studies.[2]



Angles A, B, C
Lengths a, b, c

Math.sin(A)

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}.$$

Math.cos(A)

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}.$$

Math.tan(A)

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} = \frac{\sin A}{\cos A}$$

Math.atan2(a,b)

$$\operatorname{atan2}(y, x) = \begin{cases} \arctan \frac{y}{x} & x > 0 \\ \arctan \frac{y}{x} + \pi & y \geq 0, x < 0 \\ \arctan \frac{y}{x} - \pi & y < 0, x < 0 \\ +\frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

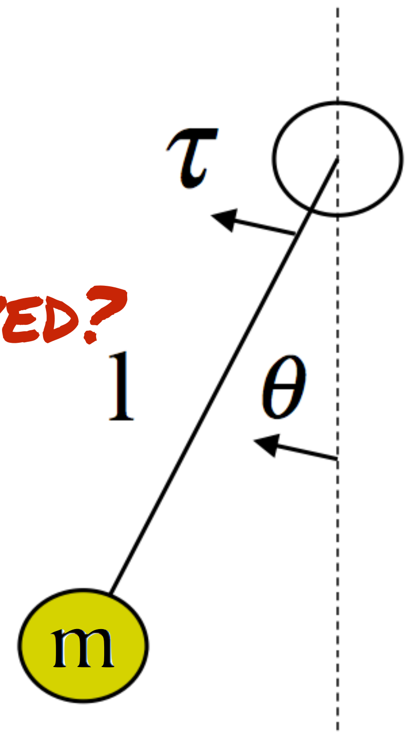
How was the equation of motion derived?

Example: Pendulum

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

HOW WAS THIS DERIVED?



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Lagrangian Dynamics

Equation of motion
(with rotational inertia I)

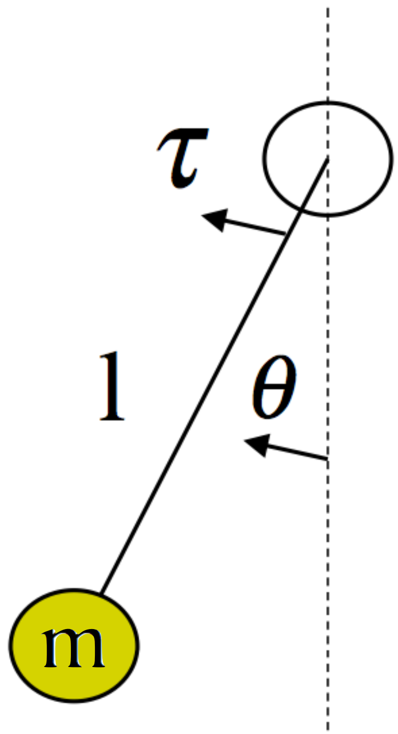
$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$

Lagrangian is kinetic energy minus potential energy

$$L = T - U$$

used to generate equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$



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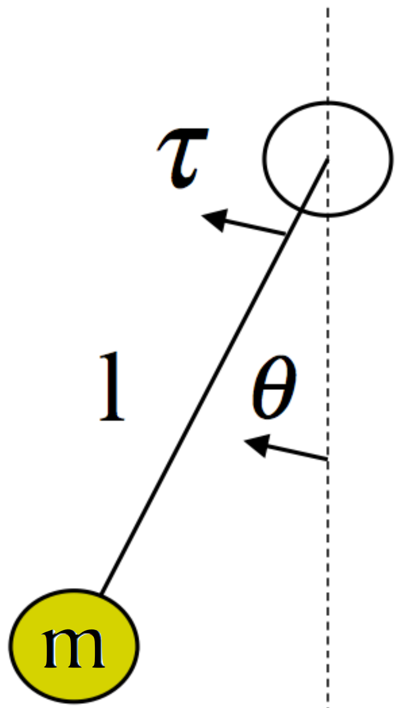
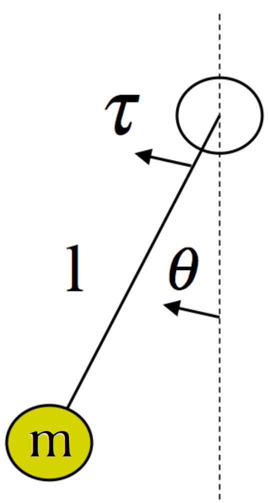


$\frac{\partial L}{\partial \theta_i}$ is the rate of change of the Lagrangian with respect to only the i^{th} degree of freedom

Equation of motion
(with rotational inertia I)

$$I\ddot{\theta} = -mgl\sin(\theta) + \tau$$



Lagrangian Dynamics for Pendulum

- Kinetic Energy $\qquad T = \frac{1}{2} I\dot{\theta}^2$

  $(1/2)mv^2$

- Potential Energy $\qquad U = mgl(1 - \cos\theta)$

  mgh

- Lagrangian $\qquad \frac{1}{2} I\dot{\theta}^2 - mgl(1 - \cos\theta)$

  $L = T - U$

- Equation of motion $\qquad I\ddot{\theta} = -mgl\sin(\theta) + \tau$

  $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)$$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$L = \frac{1}{2} I \dot{\theta}^2 - mgl(1 - \cos\theta)$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \frac{1}{2} I \dot{\theta}^2 - mgl(1 - \cos\theta)$$

- Equation of motion

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$$\boxed{L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)}$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}\overset{0}{\cancel{\frac{1}{2} I\dot{\theta}^2}} - mgl(1-\cos\theta)$$

$\frac{1}{2} I\dot{\theta}^2$ First term is constant wrt. θ

0 differentiates to zero

- Equation of motion

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$\frac{1}{2} I\dot{\theta}^2$ First term is constant wrt. θ

0 differentiates to zero

Second term: $-mgl(1-\cos\theta)$

distributes multiplication: $-mgl + mgl\cos(\theta)$

cosine identity for differentiation: $\frac{d}{dx}\cos(x) = -\sin(x)$

yields: $-mgl\sin(\theta)$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

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Second term is constant wrt. $\dot{\theta}$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$L = \frac{1}{2} I \dot{\theta}^2 - mgl(1 - \cos\theta)$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = -mgl\sin(\theta)$$

- Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{\partial L}{\partial \dot{\theta}_i} = \frac{\partial}{\partial \dot{\theta}_i} \frac{1}{2} I \dot{\theta}^2 - \overset{0}{\cancel{mgl(1 - \cos\theta)}}$$

Second term is constant wrt. $\dot{\theta}$

Apply power rule to first term: $f'(x) = rx^{r-1}$

yielding: $I\dot{\theta} = \frac{\partial}{\partial \dot{\theta}_i} \frac{1}{2} I \dot{\theta}^2$

- Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

$$\boxed{L = \frac{1}{2} I\dot{\theta}^2 - mgl(1-\cos\theta)}$$

- Partial derivative of Lagrangian wrt. pendulum angle

$$\frac{\partial L}{\partial \theta_i} = -mgl\sin(\theta)$$

- Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{\partial L}{\partial \dot{\theta}_i} = I\dot{\theta}$$

- Time derivative of Partial derivative of Lagrangian wrt. pendulum velocity

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = \frac{d}{dt} I\dot{\theta} = I\ddot{\theta}$$

- Equation of motion

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- Equation of motion $I\ddot{\theta} + mgl\sin(\theta) = \tau_i$

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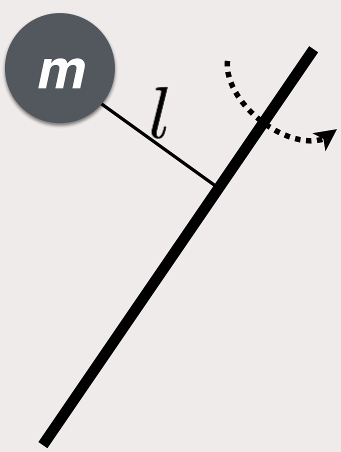
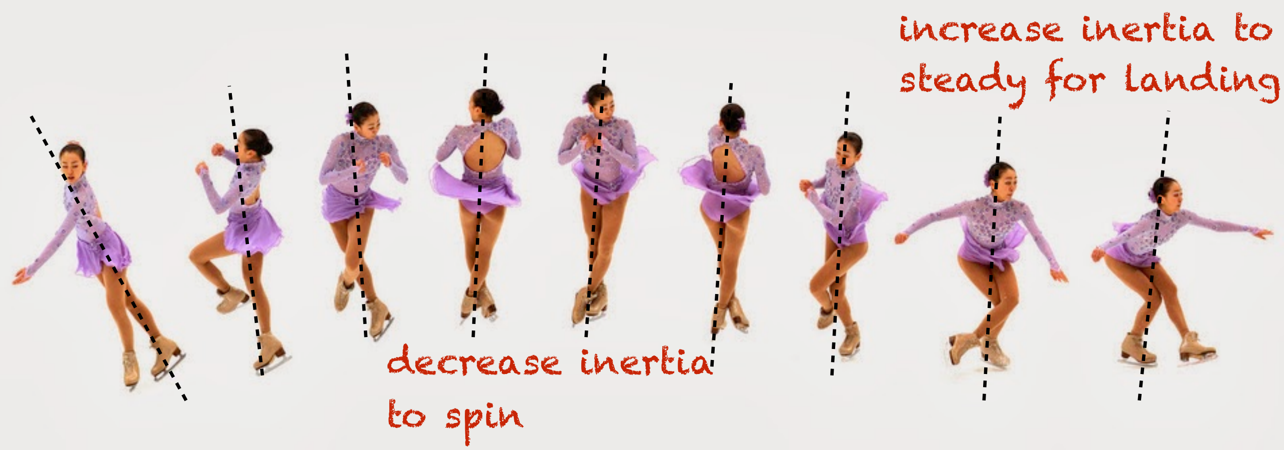
- Equation of motion $$I\ddot{\theta} + mgl\sin(\theta) = \tau_i$$

$$L = \frac{1}{2} I\dot{\theta}^2 - mgl(1 - \cos\theta)$$

- Parallel axis theorem $$I = ml^2$$

Inertia grows quadratically as mass moves further from its axis of rotation





Example: Pendulum

Equation of motion
(with rotational inertia I)

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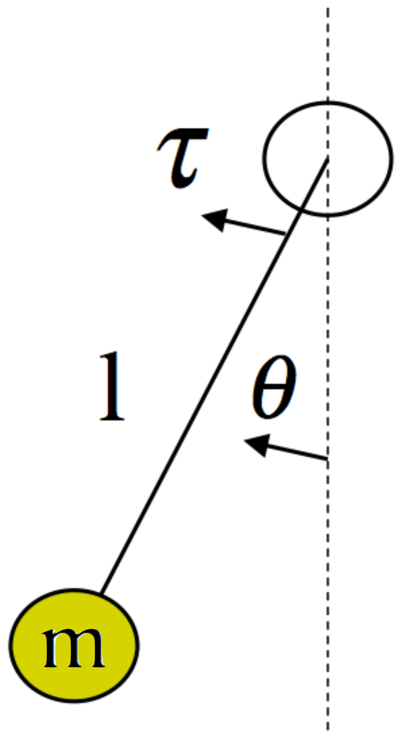
with Parallel Axis Theorem (I=ml^2)

$$I\ddot{\theta} + mgl\sin(\theta) = \tau_i$$



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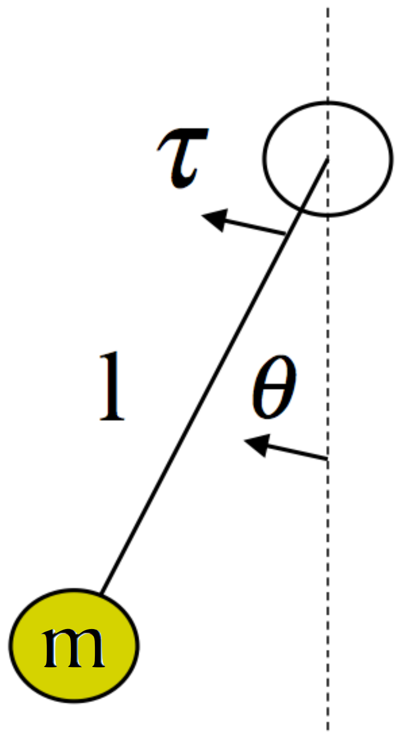
with Parallel Axis Theorem ($I=ml^2$)

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$



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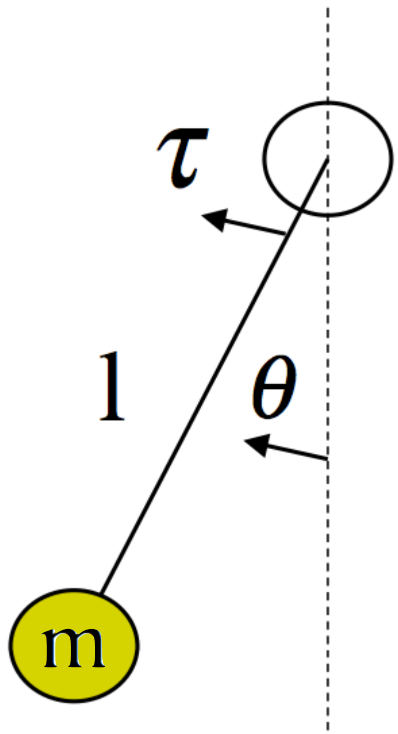
GRAVITY TERM

MOTOR TERM



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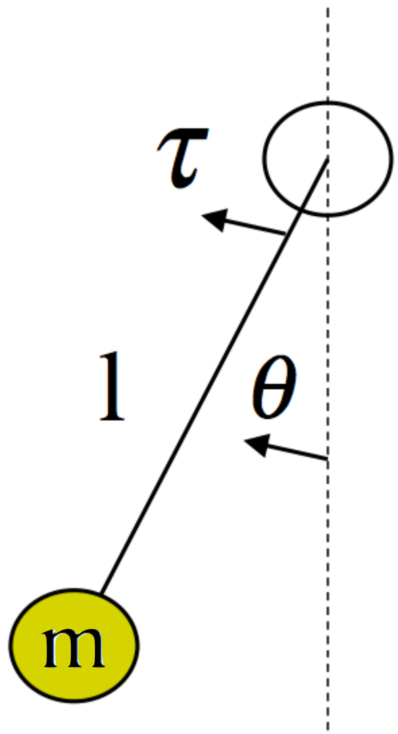
Numerical integration over time

$$\theta_{t+dt} = \theta_t + \dot{\theta}_t dt$$
$$\dot{\theta}_{t+dt} = \dot{\theta}_t + \ddot{\theta}_t dt$$



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What is this?

Numerical integration over time

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Hint

Pendulum of length l with point mass m

Example: Pendulum

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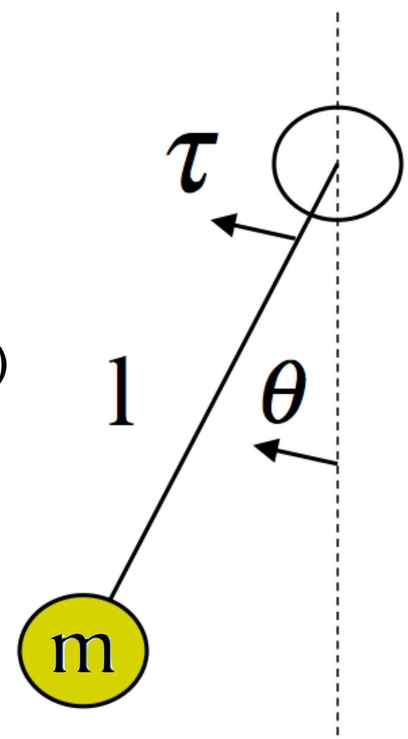

EULER INTEGRATION

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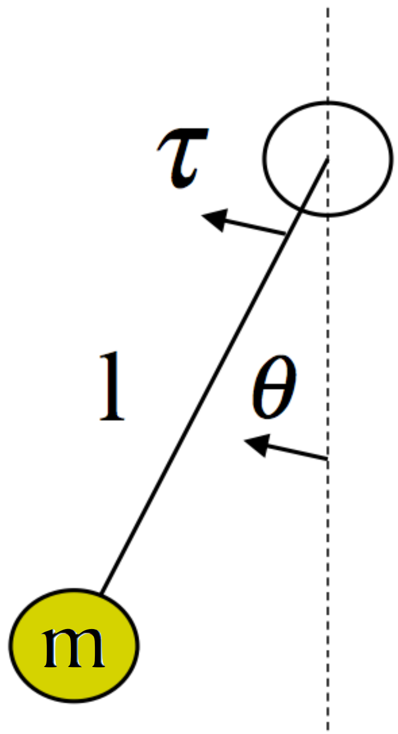

EULER INTEGRATION

Motor produces torque
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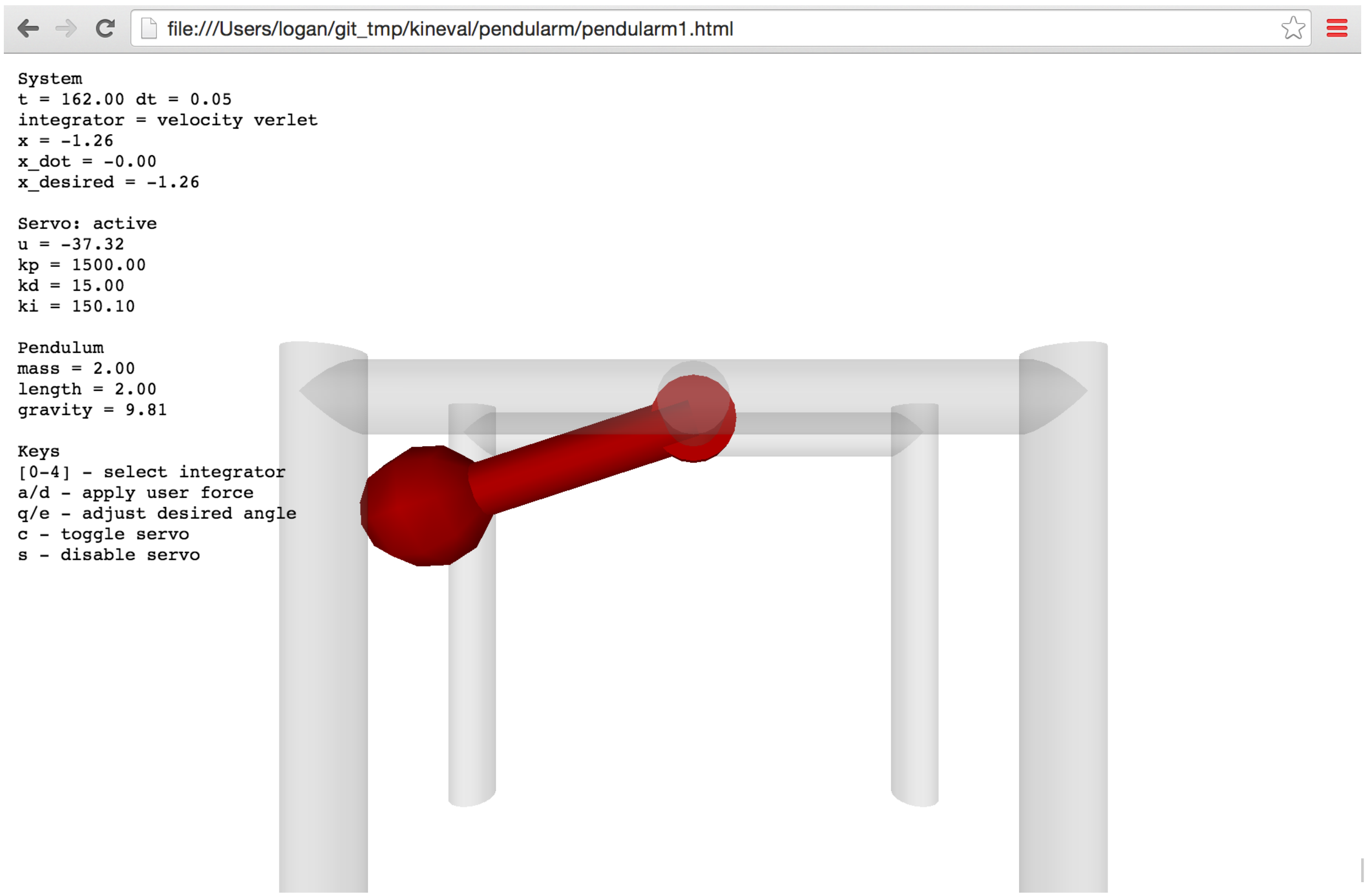
NOTE: STATE IS BOTH PENDULUM ANGLE AND VELOCITY

Pendulum of length *l* with point mass *m*

Let's see what happens



Euler integration would kill this man

Walter Lewin - https://commons.wikimedia.org/wiki/File:Physicsworks.ogg

Why is Euler Integration not the best choice?

Initial value problem

From Wikipedia, the free encyclopedia

In mathematics, in the field of differential equations, an **initial value problem** (also called **the Cauchy problem** by some authors) is an ordinary differential equation together with a specified value, called the **initial condition**, of the unknown function at a given point in the domain of the solution. In physics or other sciences, modeling a system frequently amounts to solving an initial value problem; in this context, the differential equation is an evolution equation specifying how, given initial conditions, the system will evolve with time.

Initial value problem

From Wikipedia, the free encyclopedia

Given initial condition (t_0, y_0) and

Initial value problem

From Wikipedia, the free encyclopedia

Given initial condition (t_0, y_0) and

time

state

Initial value problem

From Wikipedia, the free encyclopedia

Given initial condition (t_0, y_0) and

differential equation $y' = \frac{dy}{dt}$ in the form

$$y'(t) = f(t, y(t))$$

velocity

system dynamics

Initial value problem

From Wikipedia, the free encyclopedia

Given initial condition (t_0, y_0) and

differential equation $y' = \frac{dy}{dt}$ in the form $y'(t) = f(t, y(t))$

how to estimate of future state $y(t + dt)$ at simulation iteration *n*?

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how to estimate of future state $y(t + dt)$ at simulation iteration *n*?

numerical integration over timestep iterations

$$y(t_0 + h) - y(t_0) = \int_{t_0}^{t_0+h} f(t, y(t))\,\mathrm{d}t.$$

next state

current state

integration over timestep

Given initial condition (t_0, y_0) and

differential equation $y' = \frac{dy}{dt}$ in the form $y'(t) = f(t, y(t))$

how to estimate of future state $y(t + dt)$ at simulation iteration *n*?

<u>Euler's method</u> $\int_{t_0}^{t_0+h} f(t, y(t))\,\mathrm{d}t \approx h f(t_0, y(t_0)).$

next state

current state

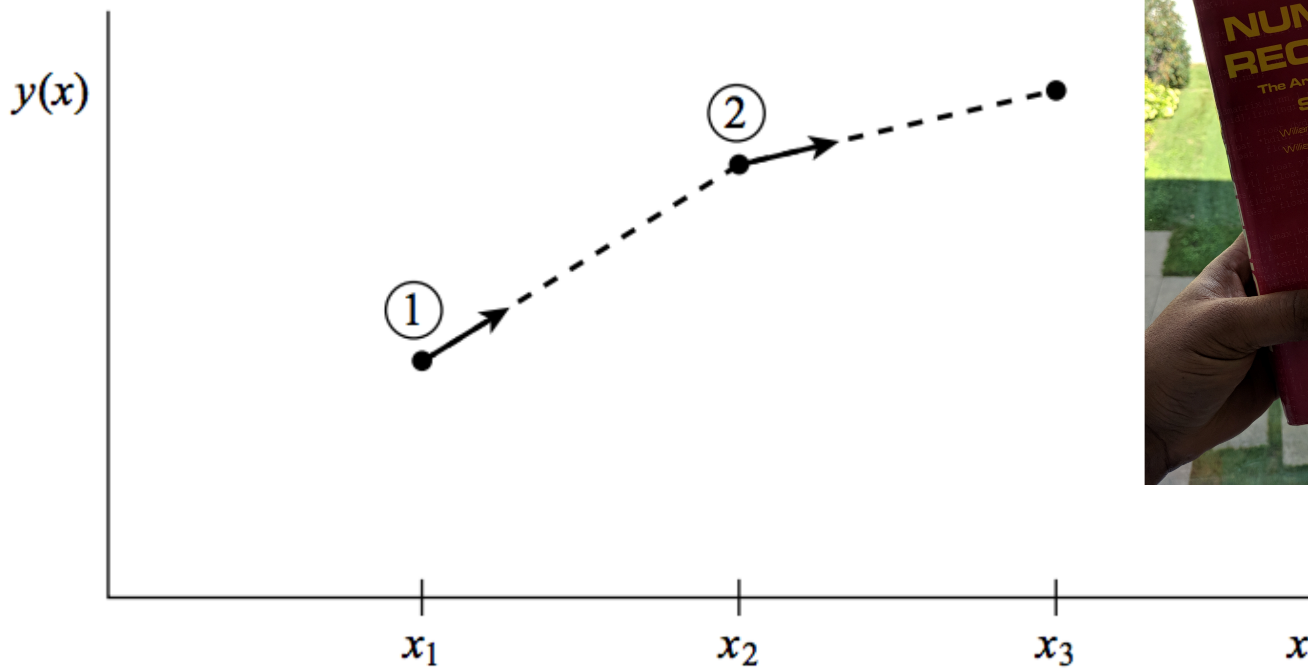
$$y_{n+1} = y_n + h f(t_n, y_n)$$

timestep (dt)

advance time

$$t_{n+1} = t_n + h$$

Example Euler integration of 2D point over 2 time steps



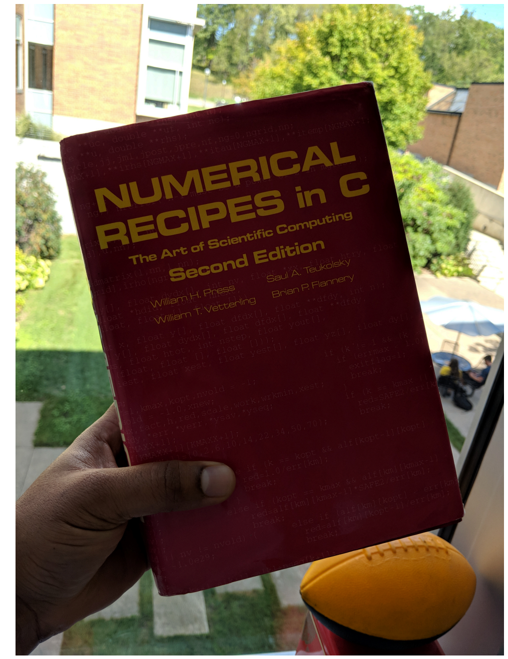


Figure 16.1.1. Euler's method. In this simplest (and least accurate) method for integrating an ODE, the derivative at the starting point of each interval is extrapolated to find the next function value. The method has first-order accuracy.

Second-order state

Reminder:

- State in Newtonian physics has both position (θ) and velocity ($\dot{\theta}$)

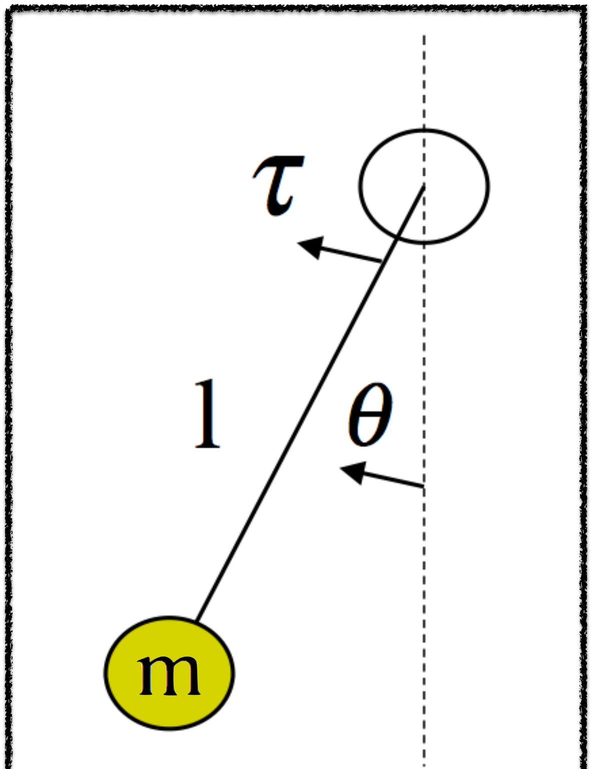
Euler's method

$$y_{n+1} = y_n + \dot{y}_n dt$$

$$\dot{y}_{n+1} = \dot{y}_n + \ddot{y}_n dt$$

acceleration

$$t_{n+1} = t_n + dt$$



$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$

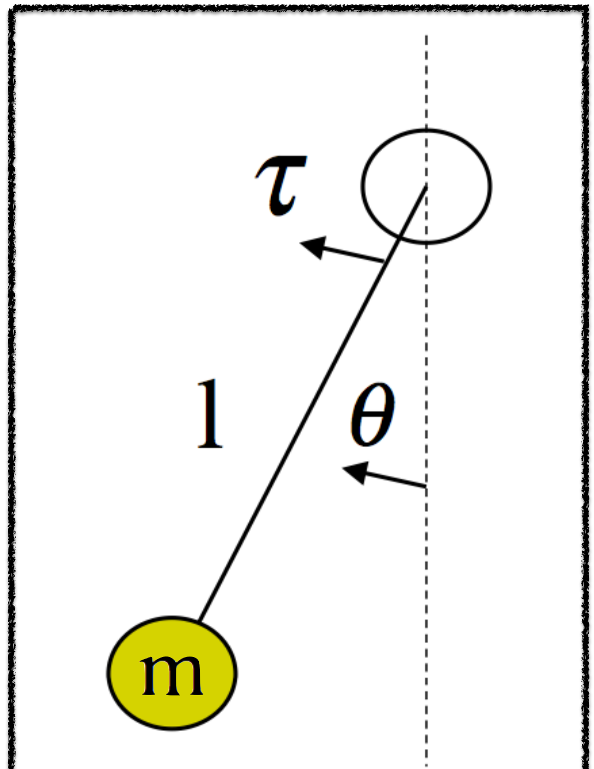
assumes second order system f(y,y',y'') treats velocity y' = f(y) and acceleration y'' = f'(y') individually

Euler's method

$$y_{n+1} = y_n + \dot{y}_n dt$$

$$\dot{y}_{n+1} = \dot{y}_n + \ddot{y}_n dt$$

acceleration



$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\tau}{ml^2}$$

$$t_{n+1} = t_n + dt$$

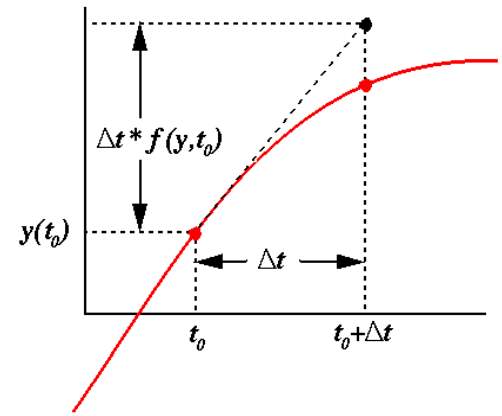
Why is the pendulum going unstable?

Given initial condition (t_0, y_0) and

differential equation $y' = \frac{dy}{dt}$ in the form

$$y'(t) = f(t, y(t))$$

how to estimate of future state $y(t_n)$ at time n?

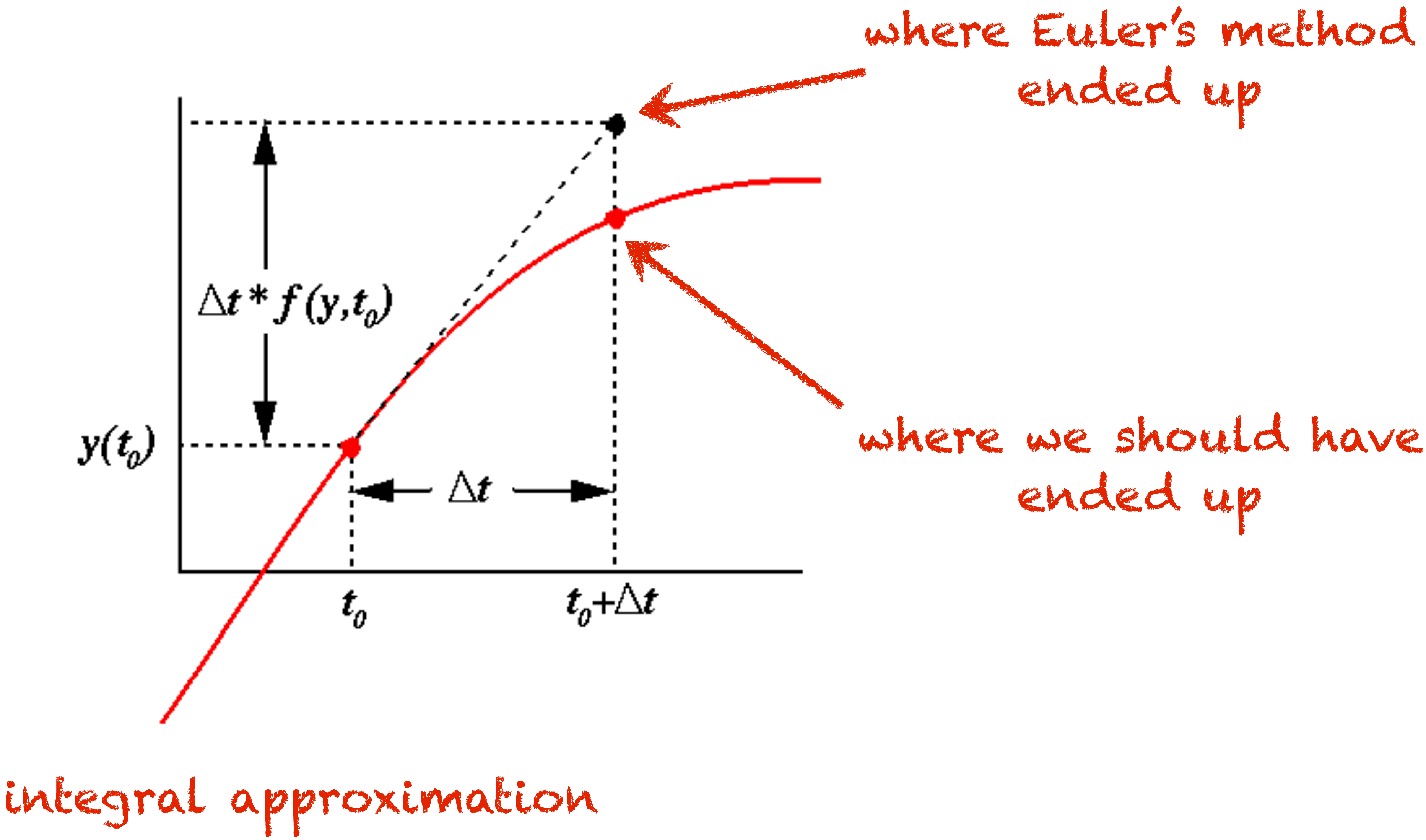


integral approximation

<u>Euler's method</u> $\int_{t_0}^{t_0+h} f(t, y(t))\,\mathrm{d}t \approx h f(t_0, y(t_0)).$

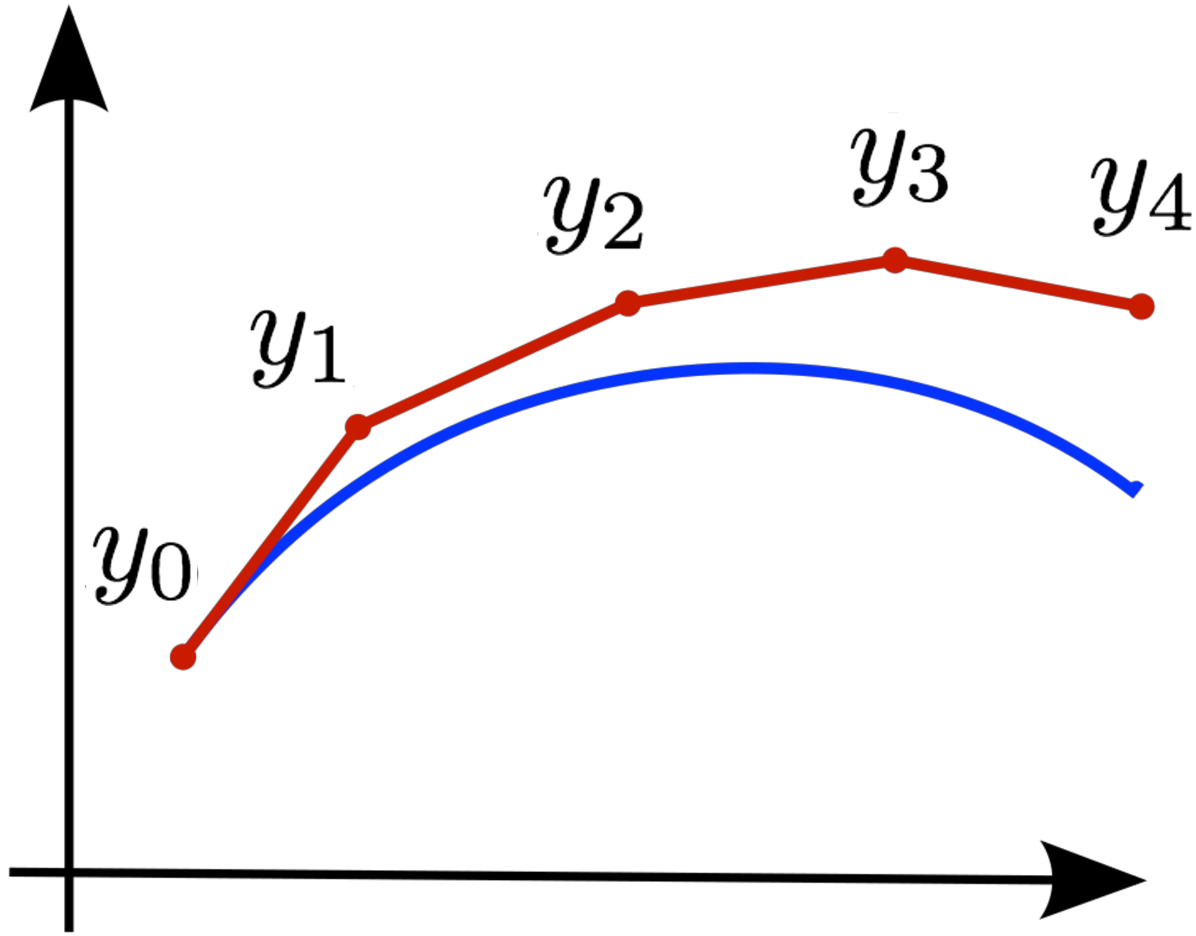
$$y_{n+1} = y_n + h f(t_n, y_n) \qquad t_{n+1} = t_n + h$$

Why is the pendulum going unstable?

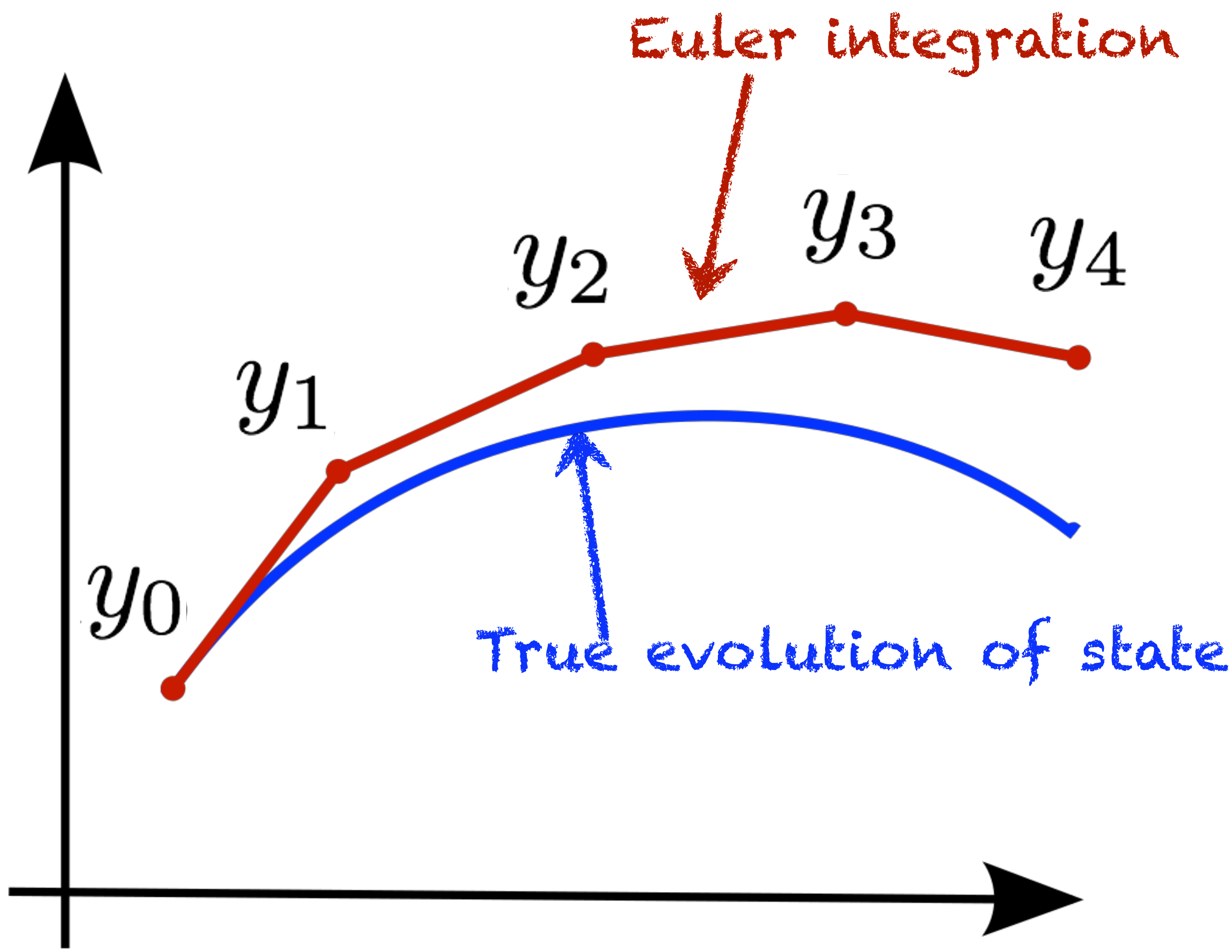


where Euler's method ended up
$\Delta t * f(y,t_0)$
$y(t_0)$
Δt
t_0
$t_0+\Delta t$
where we should have ended up
integral approximation

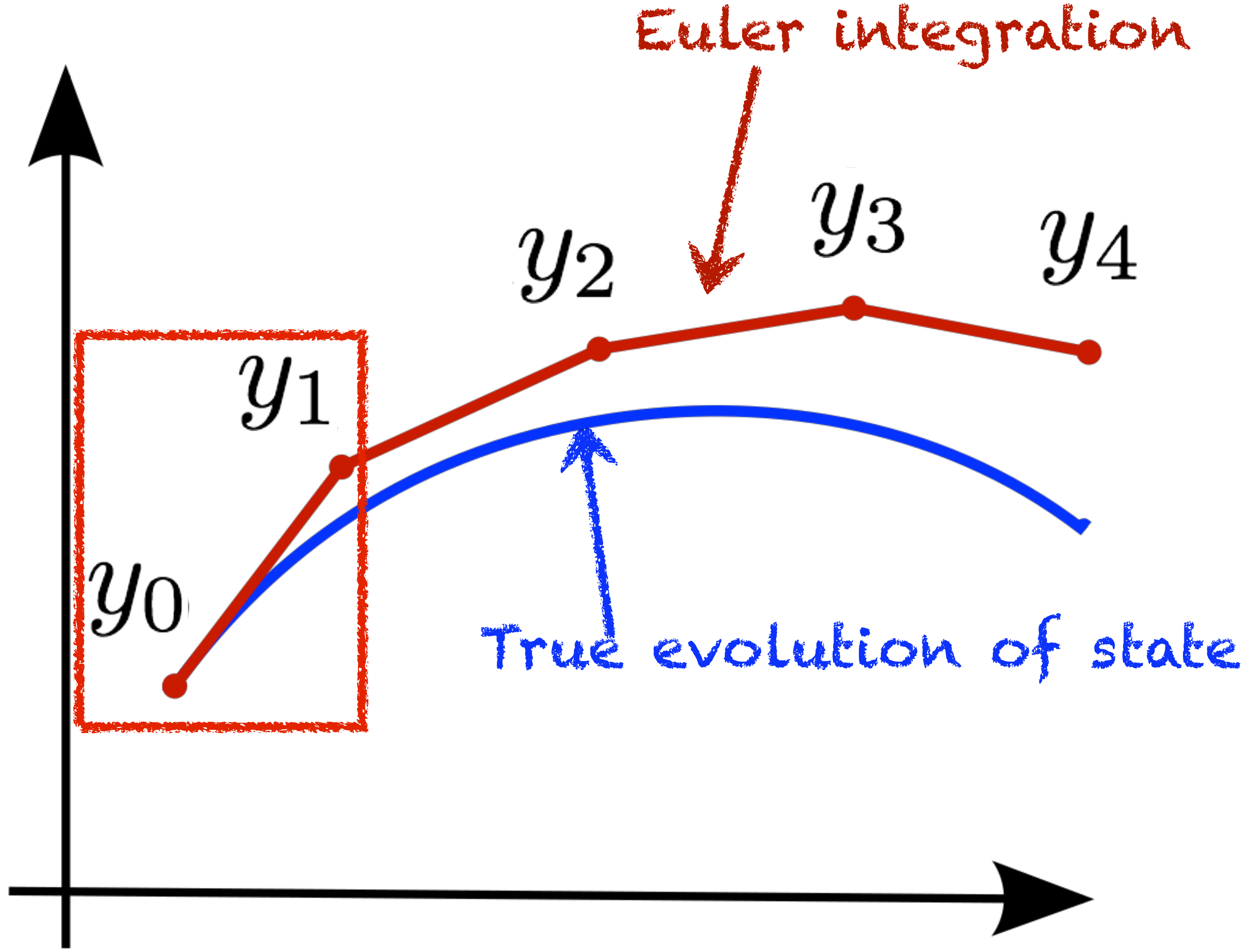
Example Euler integration of 2D point over 4 time steps

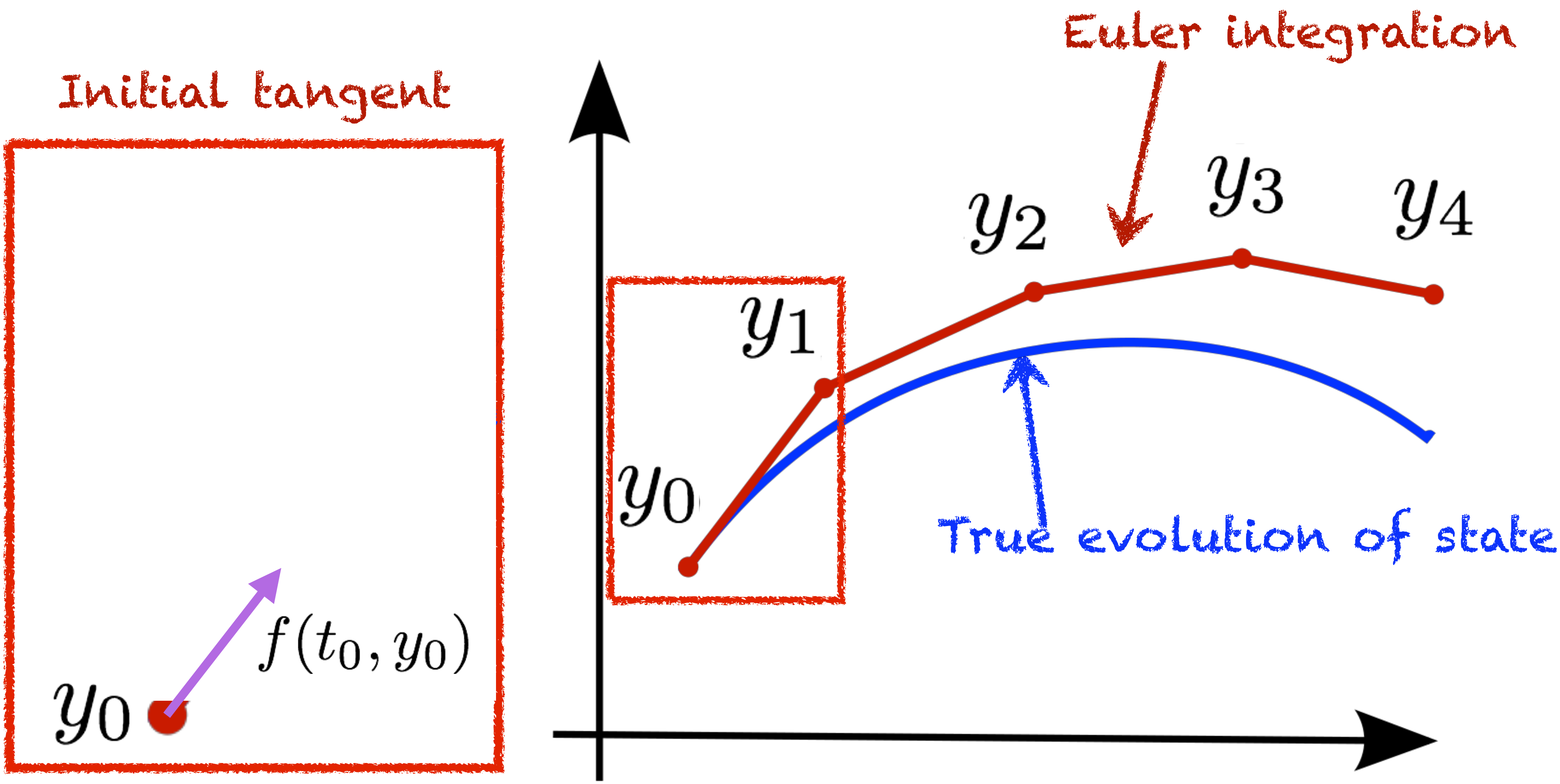


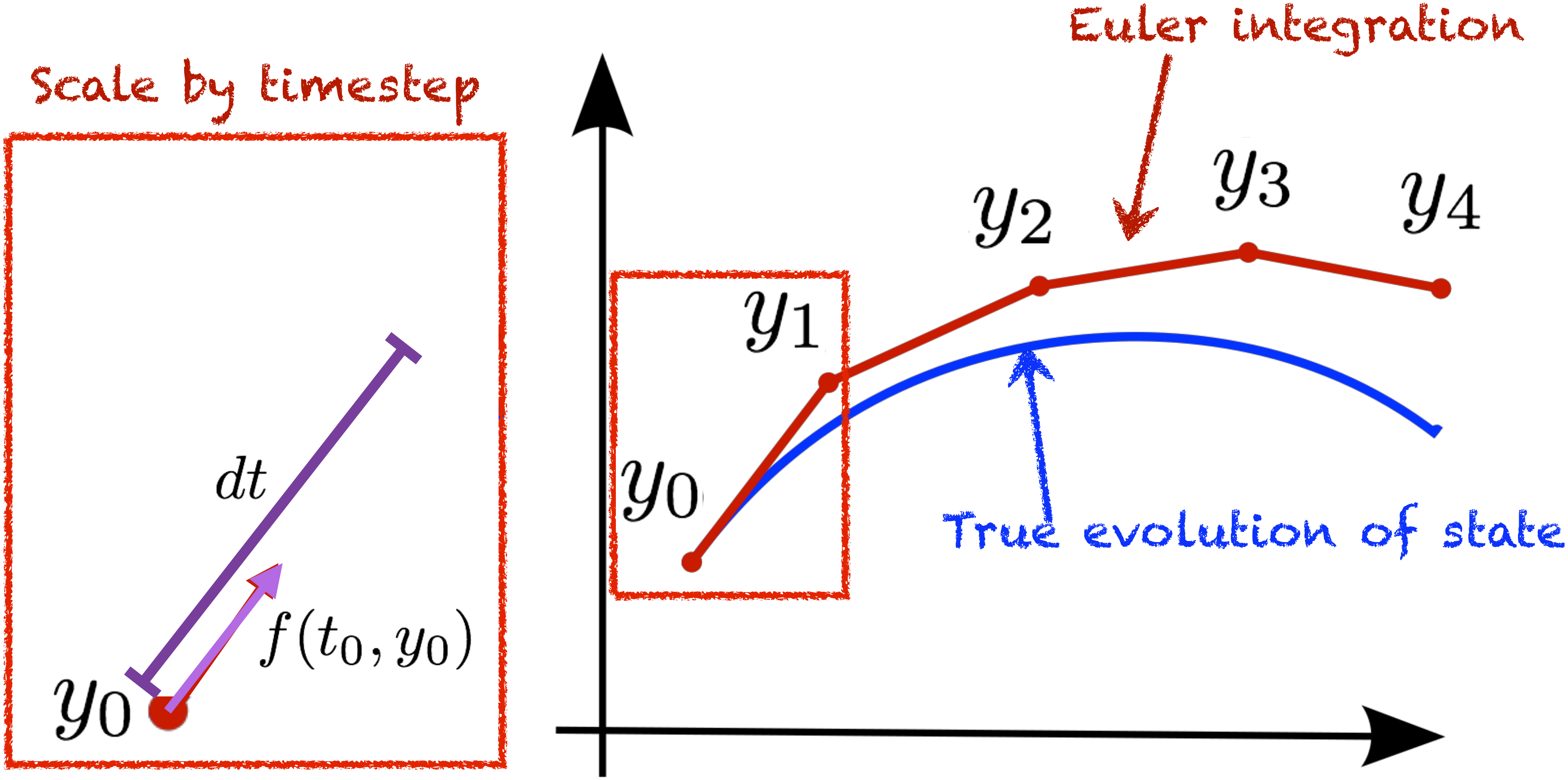
Example Euler integration of 2D point over 4 time steps



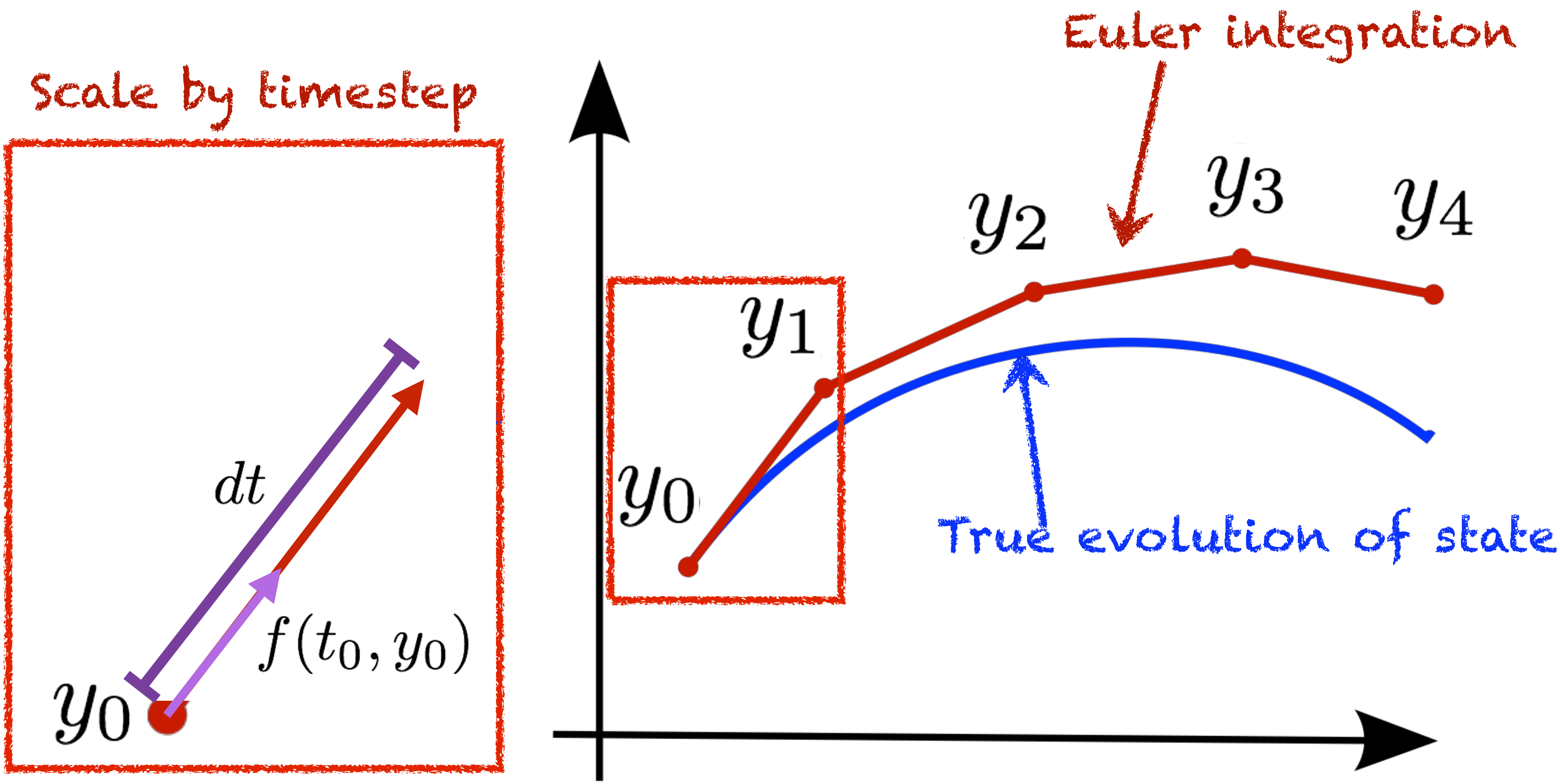
Example Euler integration of 2D point over 4 time steps



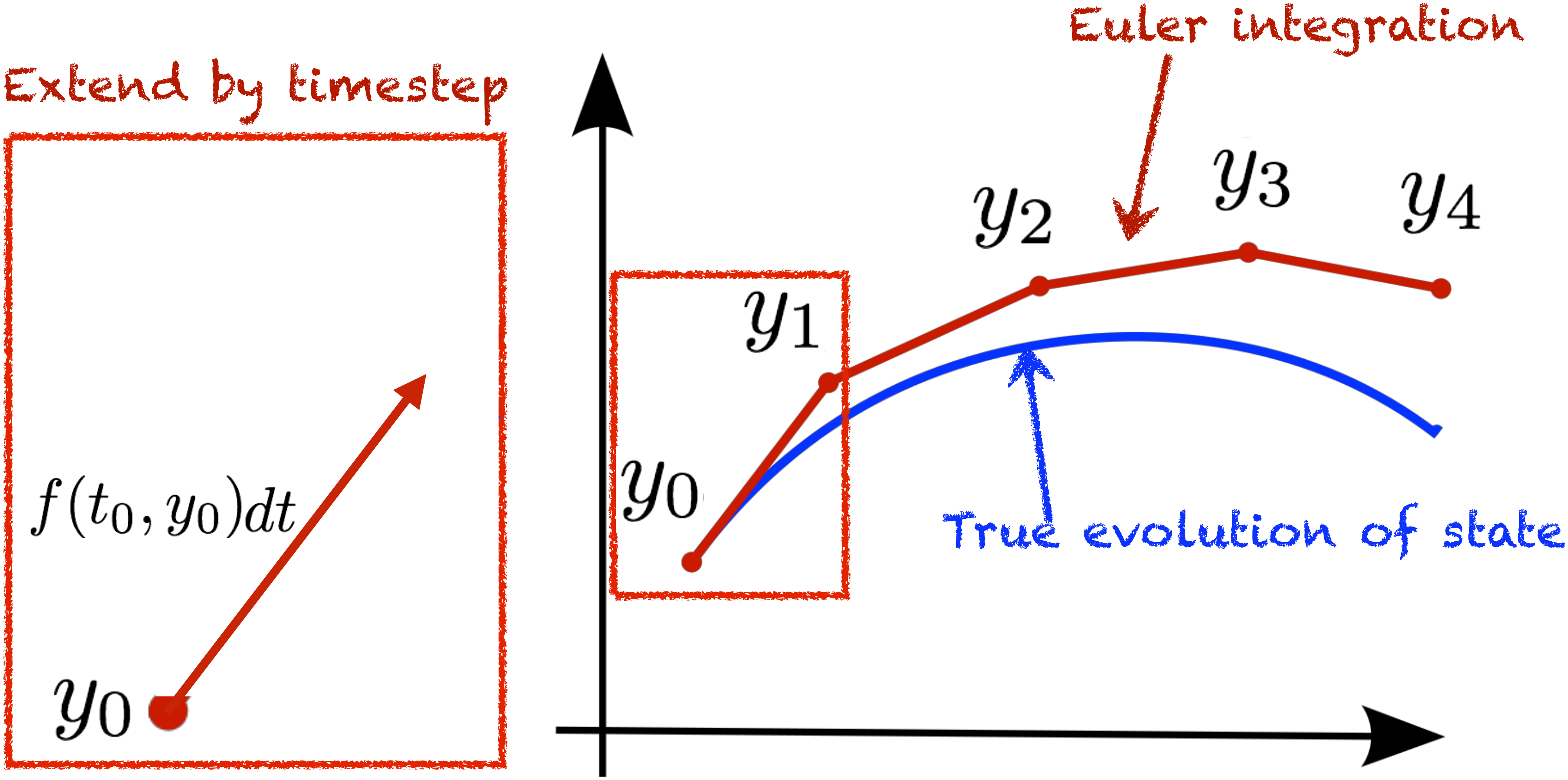

Initial tangent
y_0
$f(t_0, y_0)$
Euler integration
y_0
y_1
y_2
y_3
y_4
True evolution of state

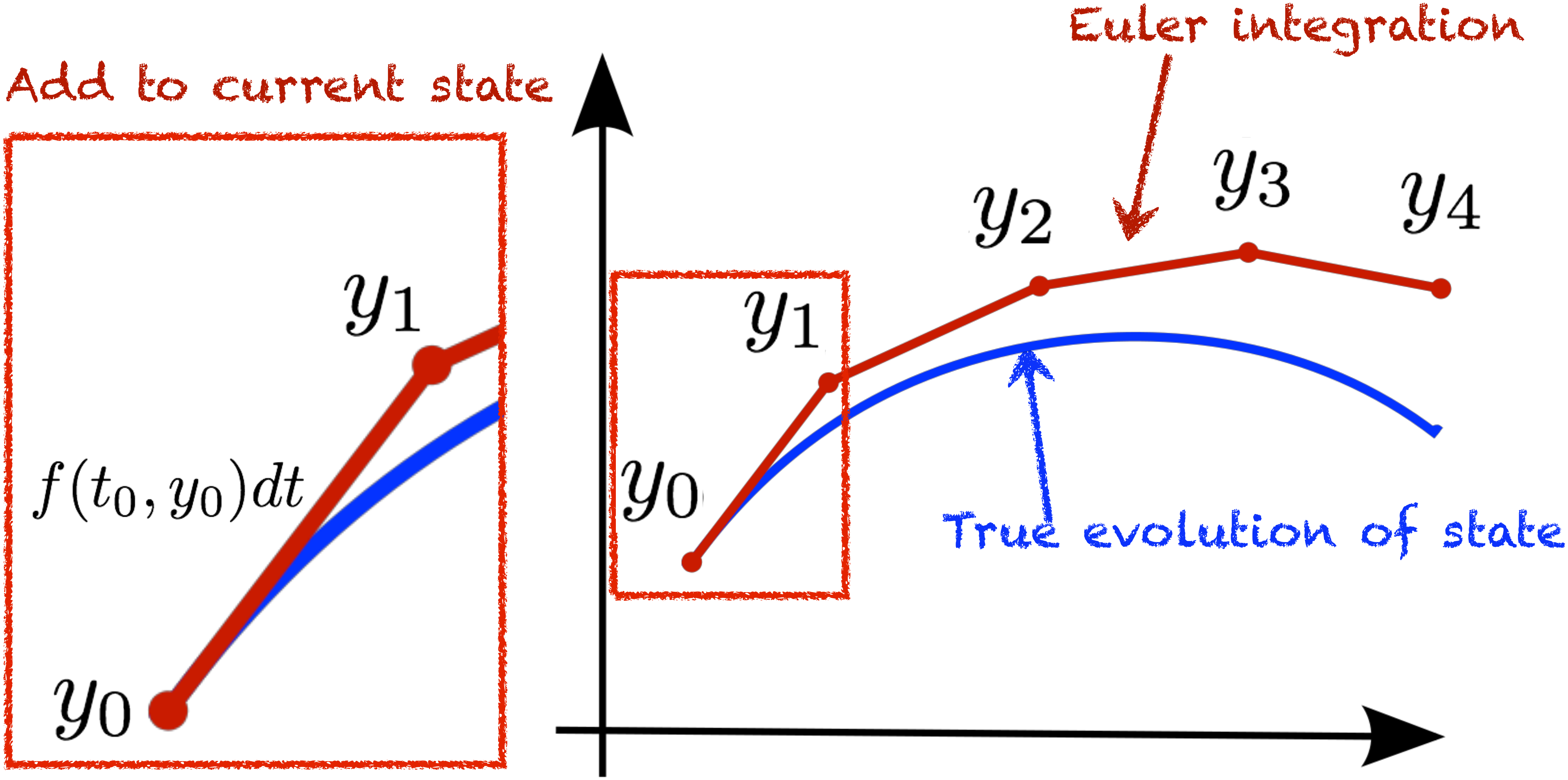


Scale by timestep
dt
$f(t_0, y_0)$
y_0
Euler integration
y_1
y_2
y_3
y_4
y_0
True evolution of state

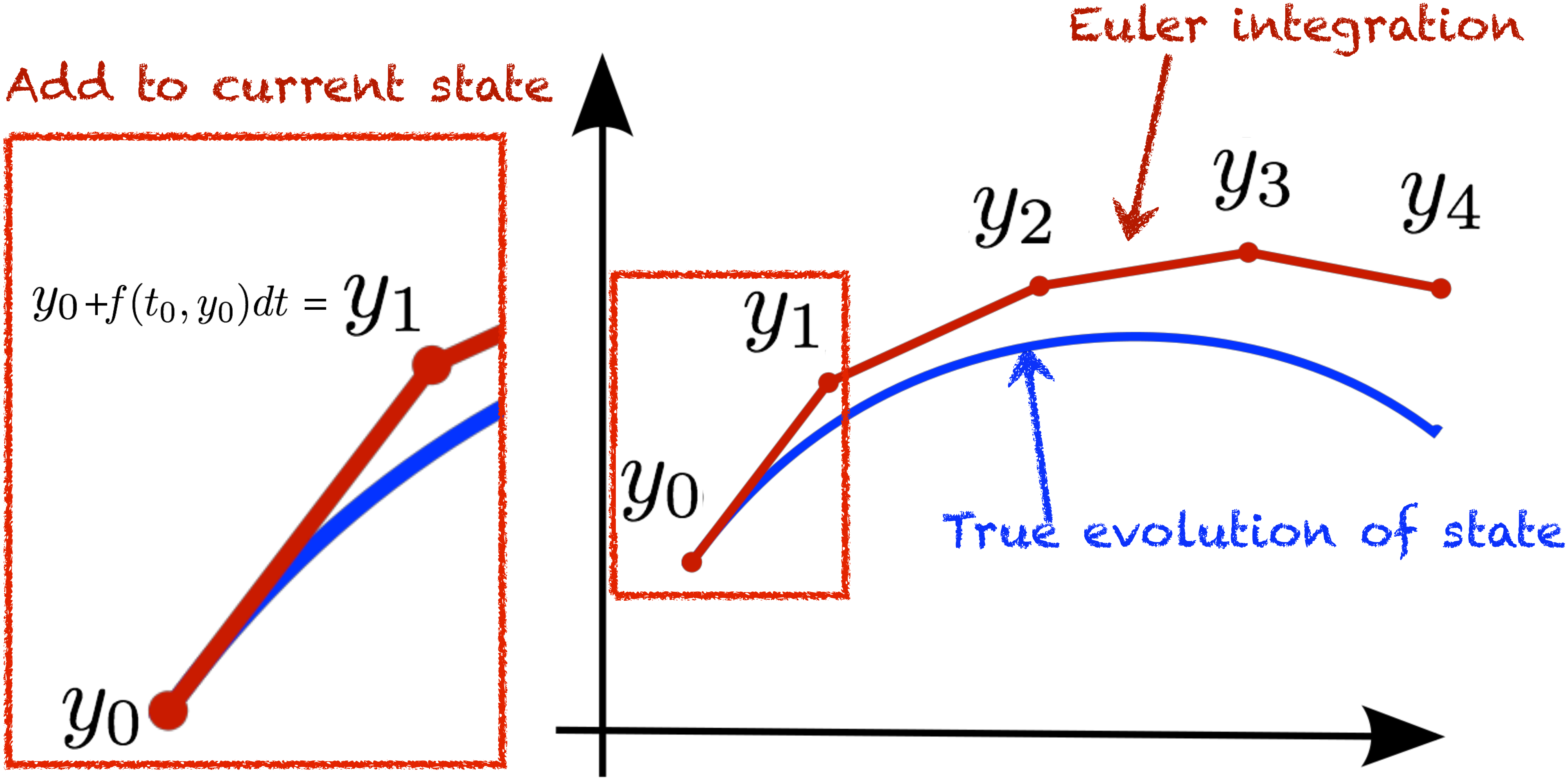


Scale by timestep
dt
$f(t_0, y_0)$
y_0
Euler integration
y_0
y_1
y_2
y_3
y_4
True evolution of state

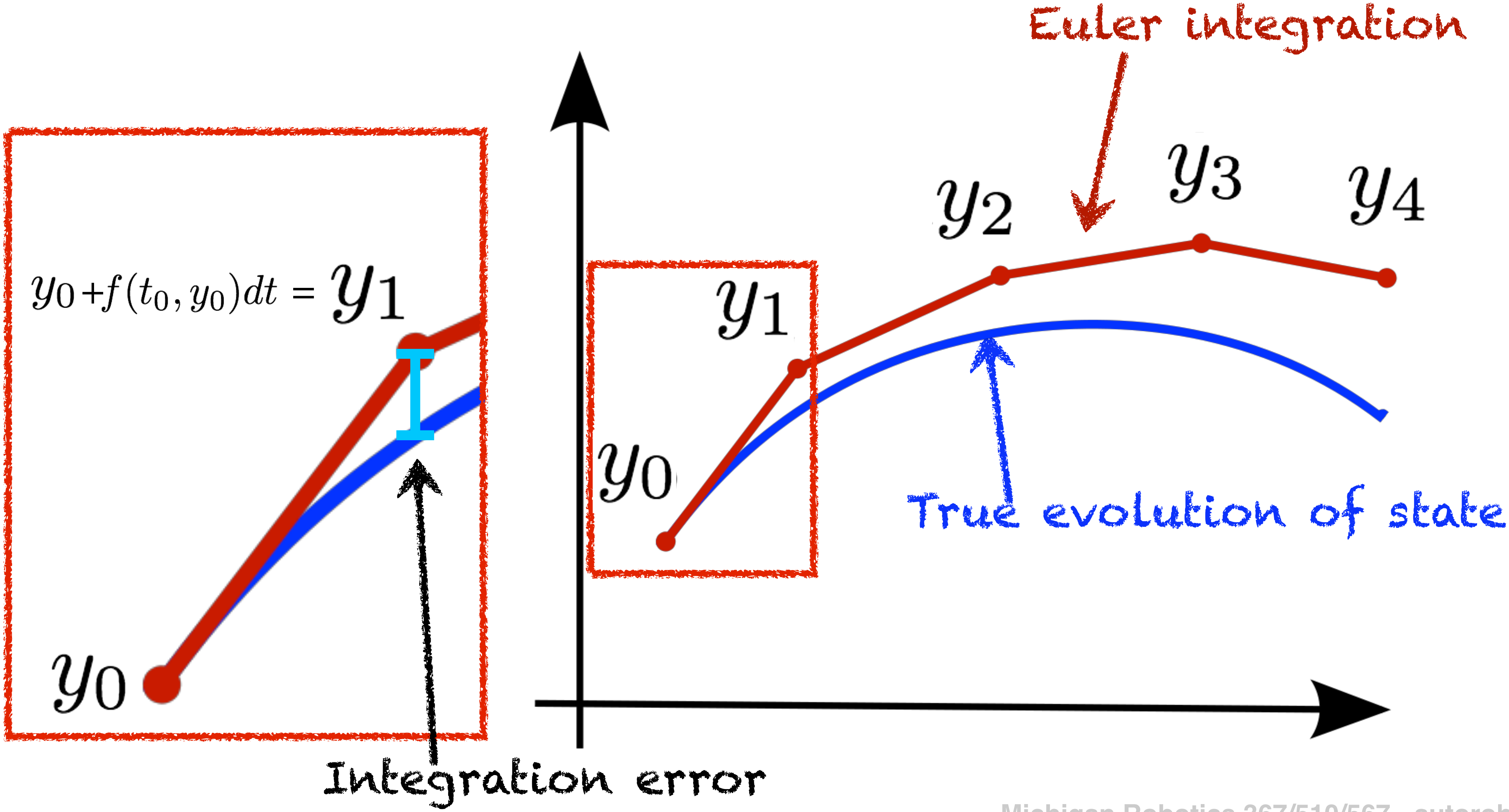

Extend by timestep
$f(t_0, y_0)dt$
y_0
Euler integration
y_0
y_1
y_2
y_3
y_4
True evolution of state



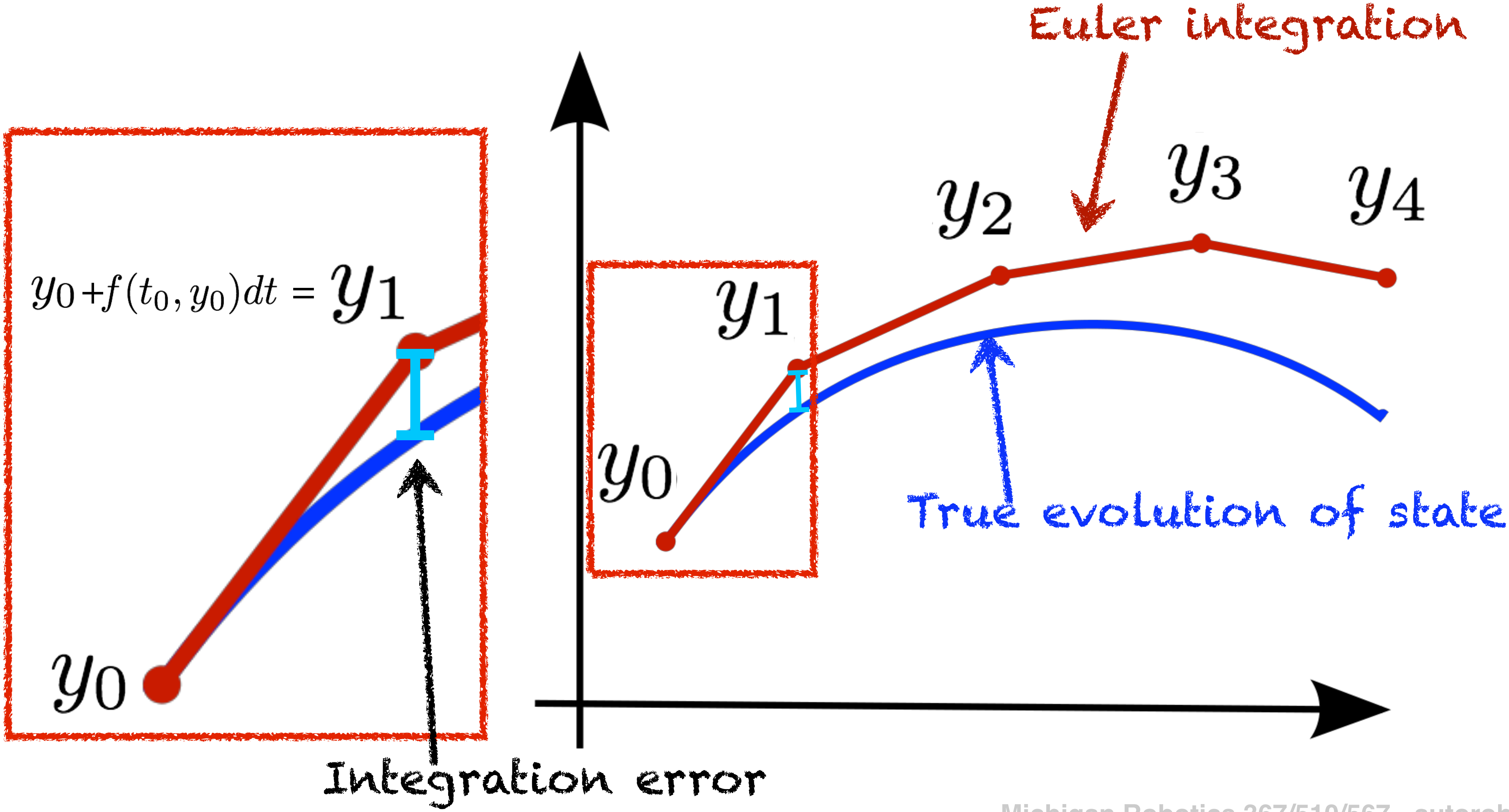
Add to current state
y_1
$f(t_0, y_0)dt$
y_0
Euler integration
y_1
y_2
y_3
y_4
y_0
True evolution of state



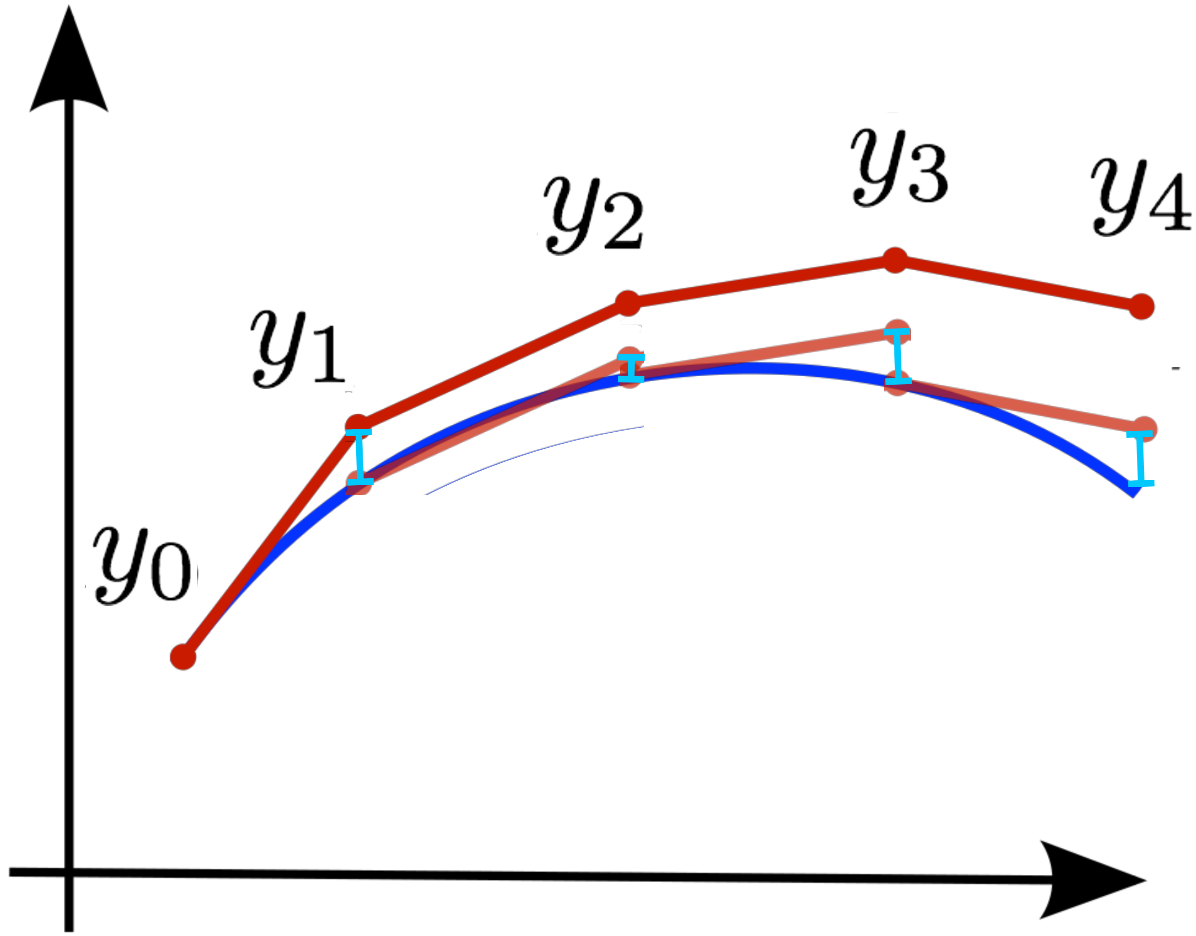
Add to current state
$y_0 + f(t_0, y_0)dt = y_1$
y_0
Euler integration
y_2
y_3
y_4
y_1
y_0
True evolution of state



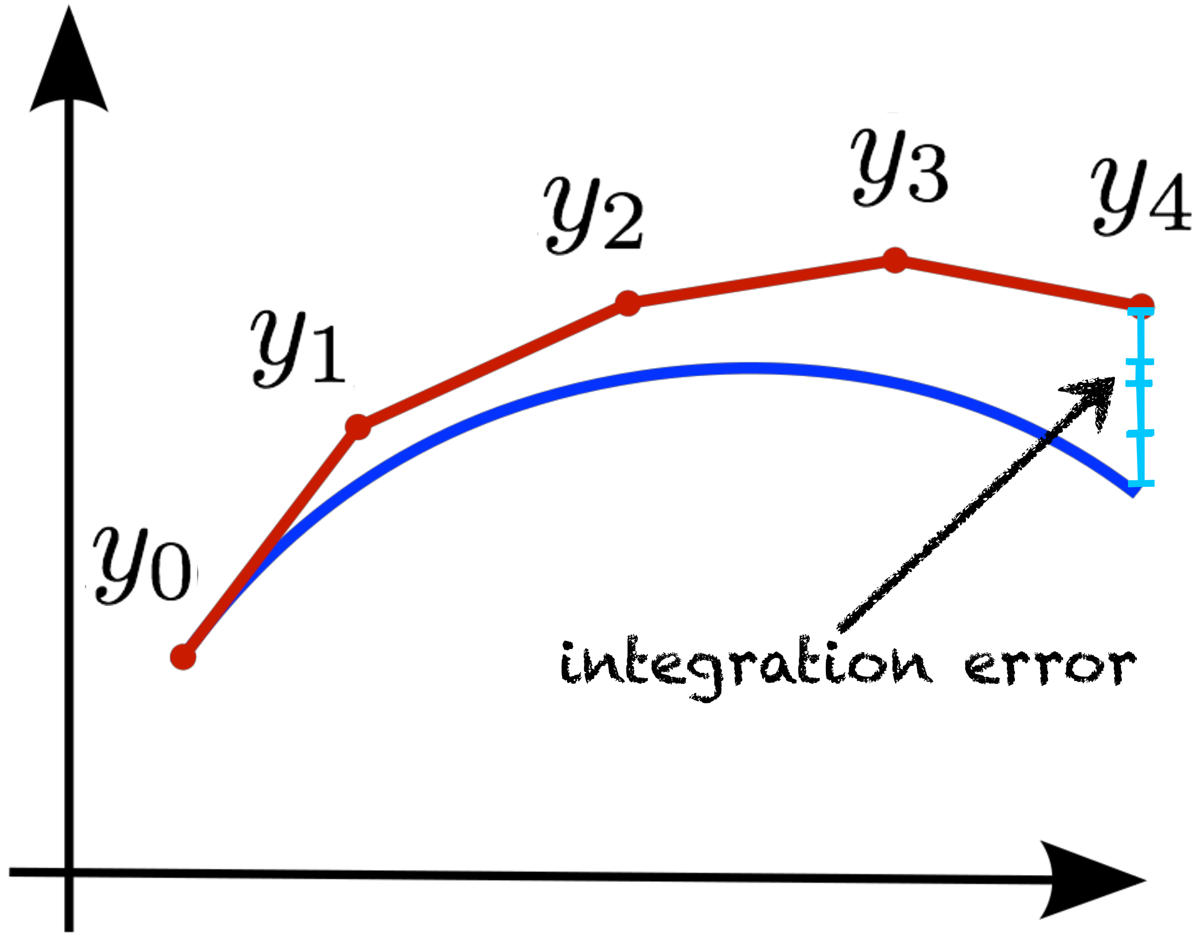
Euler integration
$y_0+f(t_0,y_0)dt = y_1$
y_0
y_1
y_2
y_3
y_4
True evolution of state
Integration error



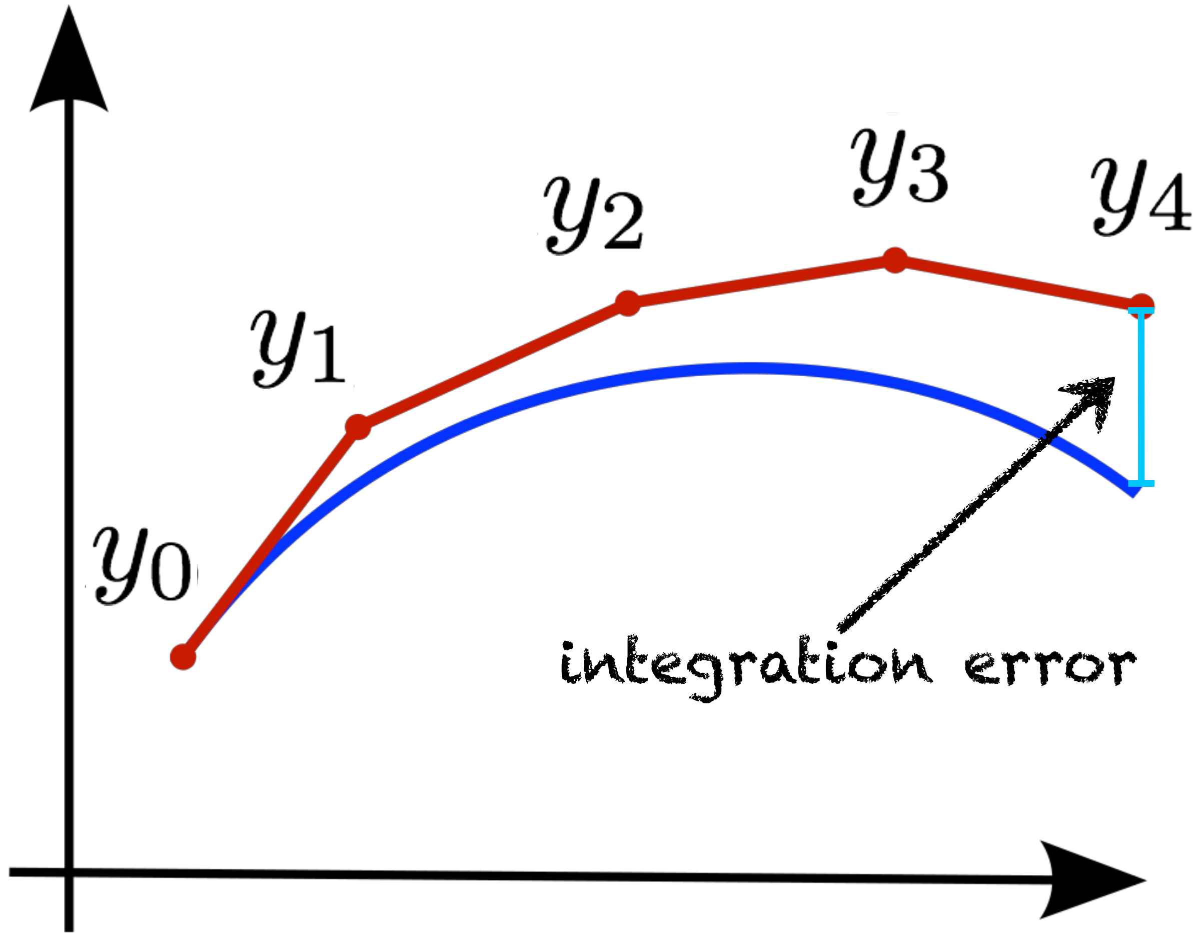
Euler integration
$y_0 + f(t_0, y_0)dt = y_1$
y_0
y_1
y_2
y_3
y_4
y_0
True evolution of state
Integration error



y_0
y_1
y_2
y_3
y_4


y_0
y_1
y_2
y_3
y_4
integration error

Can we improve this integration over time?



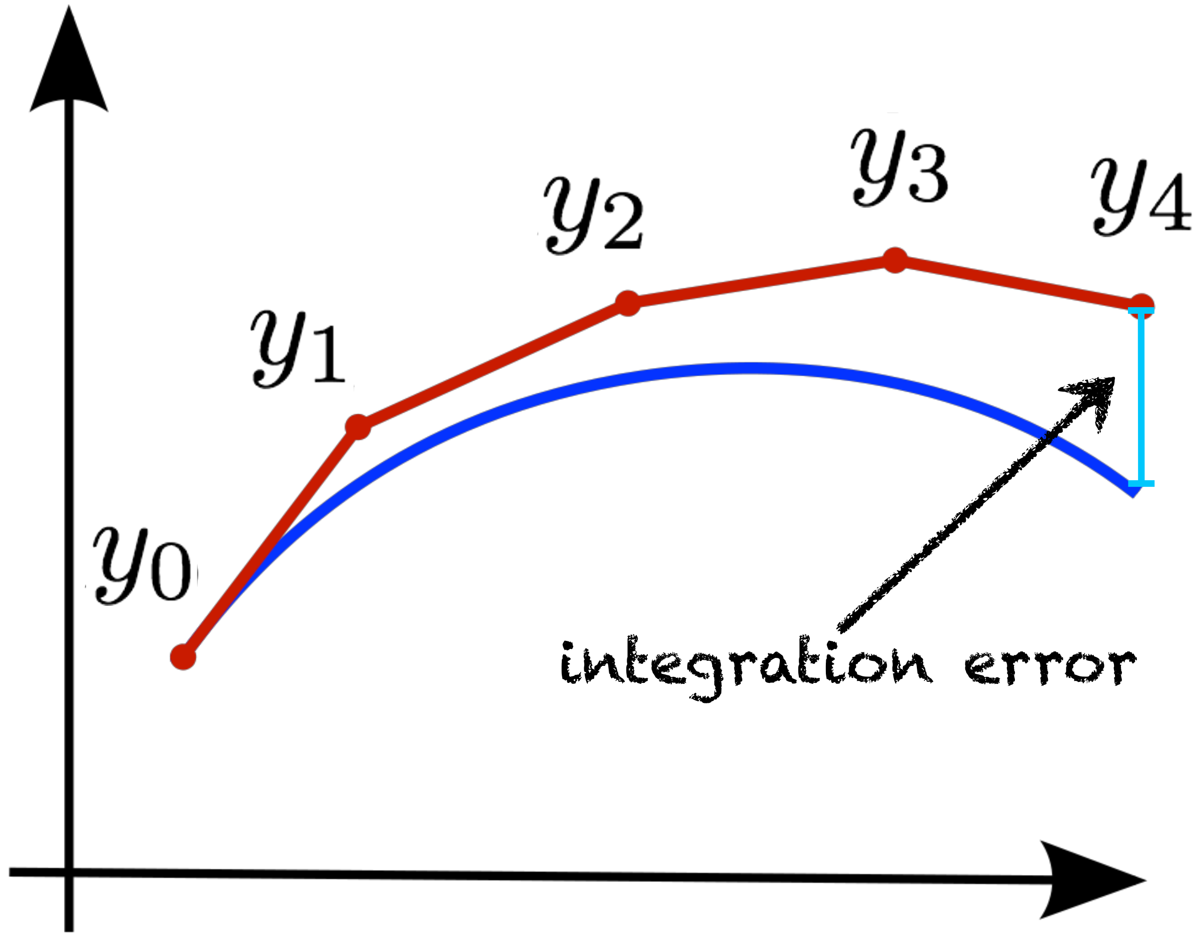
Can we improve this integration over time?

Option 1:

[redacted]

Option 2:

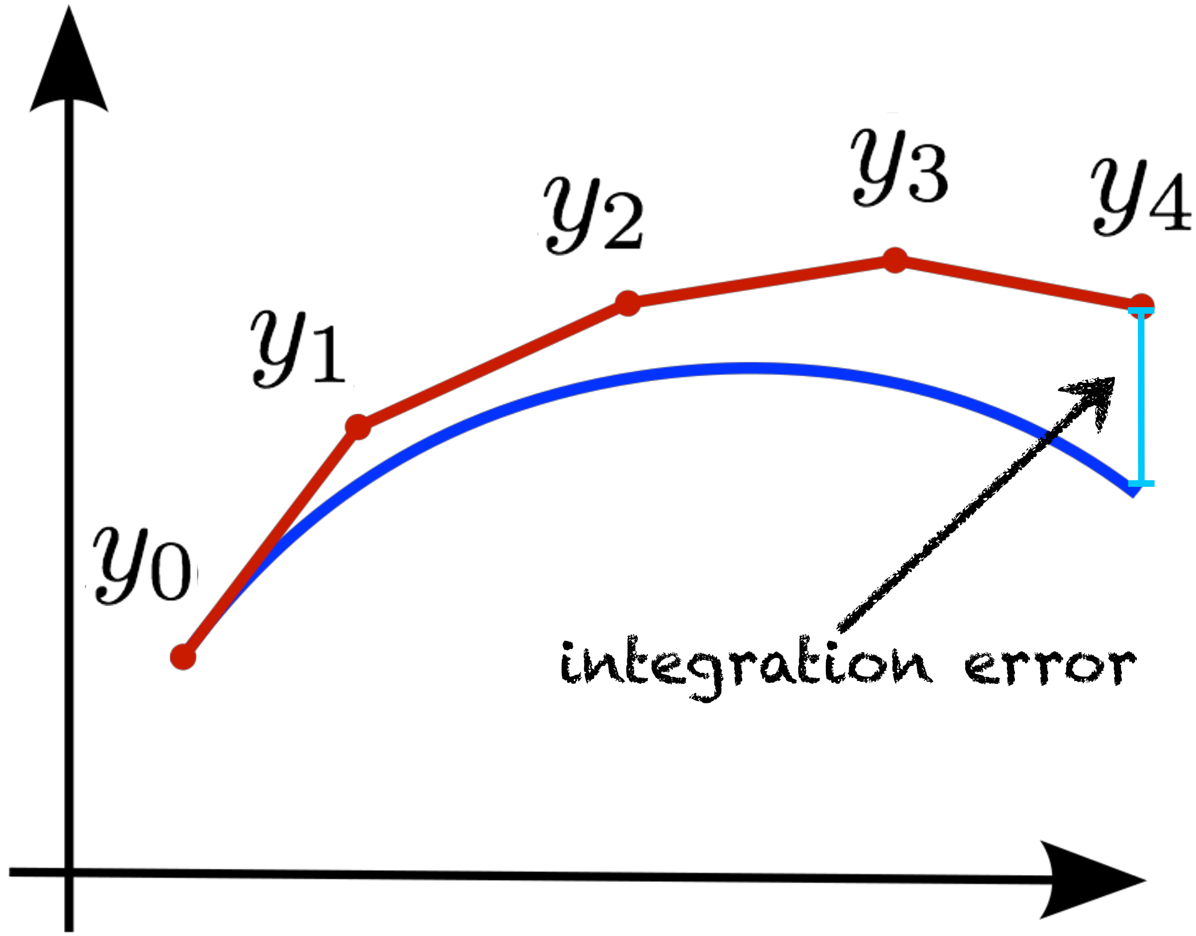
[redacted]



Can we improve this integration over time?

Option 1:
Reduce timestep

Option 2:
Use a better integrator



Verlet Integration

Verlet Integration

For a differential equation of second order of the type $\ddot{\vec{x}}(t) = A(\vec{x}(t))$ with initial conditions $\vec{x}(t_0) = \vec{x}_0$ and $\dot{\vec{x}}(t_0) = \vec{v}_0$, an approximate numerical solution $\vec{x}_n \approx \vec{x}(t_n)$ at the times $t_n = t_0 + n\,\Delta t$ with step size $\Delta t > 0$ can be obtained by the following method:

Iterate position $$\vec{x}_{n+1} = 2\vec{x}_n - \vec{x}_{n-1} + A(\vec{x}_n)\,\Delta t^2$$

update notation: x is used as state, and n for time

Verlet Integration

For a differential equation of second order of the type $\ddot{\vec{x}}(t) = A(\vec{x}(t))$ with initial conditions $\vec{x}(t_0) = \vec{x}_0$ and $\dot{\vec{x}}(t_0) = \vec{v}_0$, an approximate numerical solution $\vec{x}_n \approx \vec{x}(t_n)$ at the times $t_n = t_0 + n\,\Delta t$ with step size $\Delta t > 0$ can be obtained by the following method:

Iterate position $$\vec{x}_{n+1} = 2\vec{x}_n - \vec{x}_{n-1} + A(\vec{x}_n)\,\Delta t^2$$

How was this derived?

update notation: x is used as state, and n for time

Verlet Integration

$$A(\vec{x}_n) = \frac{\Delta^2 \vec{x}_n}{\Delta t^2} = \frac{v_n - v_{n-1}}{\Delta t}$$

Start with discrete time
approximation of acceleration
at time n

Verlet Integration

$$A(\vec{x}_n) = \frac{\Delta^2 \vec{x}_n}{\Delta t^2} = \frac{v_n - v_{n-1}}{\Delta t} = \frac{\frac{\vec{x}_{n+1} - \vec{x}_n}{\Delta t} - \frac{\vec{x}_n - \vec{x}_{n-1}}{\Delta t}}{\Delta t}$$

Discrete time
approximation of velocity
at times n and n-1

Verlet Integration

$$A(\vec{x}_n) = \frac{\Delta^2 \vec{x}_n}{\Delta t^2} = \frac{v_n - v_{n-1}}{\Delta t} = \frac{\frac{\vec{x}_{n+1} - \vec{x}_n}{\Delta t} - \frac{\vec{x}_n - \vec{x}_{n-1}}{\Delta t}}{\Delta t}$$

$$= \frac{\vec{x}_{n+1} - 2\vec{x}_n + \vec{x}_{n-1}}{\Delta t^2}$$

Do some algebra

Verlet Integration

$$A(\vec{x}_n) = \frac{\Delta^2 \vec{x}_n}{\Delta t^2} = \frac{v_n - v_{n-1}}{\Delta t} = \frac{\frac{\vec{x}_{n+1} - \vec{x}_n}{\Delta t} - \frac{\vec{x}_n - \vec{x}_{n-1}}{\Delta t}}{\Delta t}$$

$$a_n = \frac{\vec{x}_{n+1} - 2\vec{x}_n + \vec{x}_{n-1}}{\Delta t^2}$$

$$\boxed{\vec{x}_{n+1} = 2\vec{x}_n - \vec{x}_{n-1} + \vec{a}_n \, \Delta t^2}$$

Verlet Integration

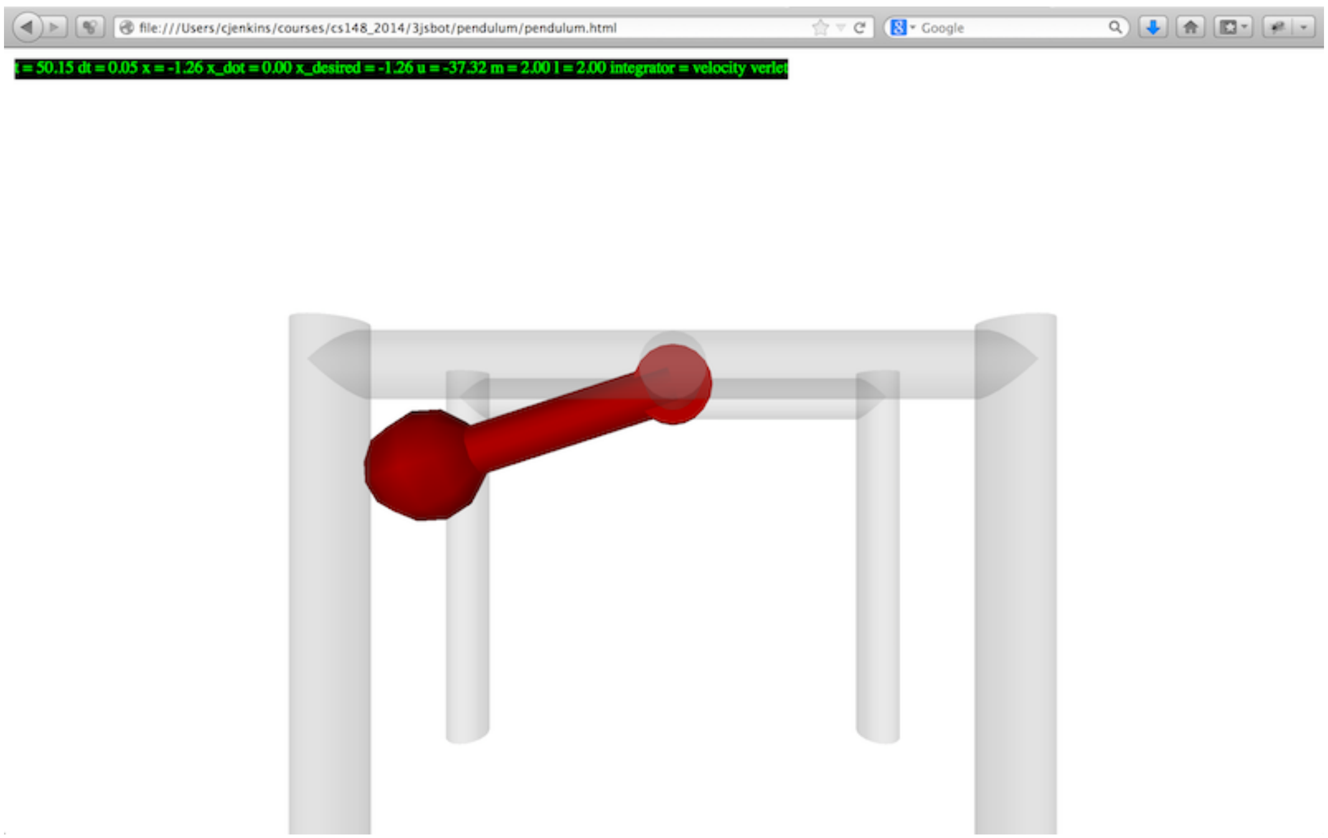
For a differential equation of second order of the type $\ddot{\vec{x}}(t) = A(\vec{x}(t))$ with initial conditions $\vec{x}(t_0) = \vec{x}_0$ and $\dot{\vec{x}}(t_0) = \vec{v}_0$, an approximate numerical solution $\vec{x}_n \approx \vec{x}(t_n)$ at the times $t_n = t_0 + n\,\Delta t$ with step size $\Delta t > 0$ can be obtained by the following method:

Initialize $$\vec{x}_1 = \vec{x}_0 + \vec{v}_0\,\Delta t + \frac{1}{2}A(\vec{x}_0)\,\Delta t^2$$

Iterate position $$\vec{x}_{n+1} = 2\vec{x}_n - \vec{x}_{n-1} + A(\vec{x}_n)\,\Delta t^2$$

Do not forget to initialize

Let's see what happens



Verlet Integration

For a differential equation of second order of the type $\ddot{\vec{x}}(t) = A(\vec{x}(t))$ with initial conditions $\vec{x}(t_0) = \vec{x}_0$ and $\dot{\vec{x}}(t_0) = \vec{v}_0$, an approximate numerical solution $\vec{x}_n \approx \vec{x}(t_n)$ at the times $t_n = t_0 + n\,\Delta t$ with step size $\Delta t > 0$ can be obtained by the following method:

Initialize

$$\vec{x}_1 = \vec{x}_0 + \vec{v}_0\,\Delta t + \frac{1}{2}A(\vec{x}_0)\,\Delta t^2$$

Iterate position

$$\vec{x}_{n+1} = 2\vec{x}_n - \vec{x}_{n-1} + A(\vec{x}_n)\,\Delta t^2$$

Iterate velocity (optional)

$$\vec{v}(t) = \frac{\vec{x}(t+\Delta t) - \vec{x}(t-\Delta t)}{2\Delta t} + \mathcal{O}(\Delta t^2)$$

Velocity Verlet

$$\vec{x}(t+\Delta t) = \vec{x}(t) + \vec{v}(t)\,\Delta t + \frac{1}{2}\,\vec{a}(t)\Delta t^2$$

$$\vec{v}(t+\Delta t) = \vec{v}(t) + \frac{\vec{a}(t) + \vec{a}(t+\Delta t)}{2}\Delta t$$

assumes that acceleration $\vec{a}(t+\Delta t)$ only depends on position $\vec{x}(t+\Delta t)$

Runge-Kutta

Runge-Kutta

$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i,$$

where

think of f() as callable function for dy/dt

$$\begin{aligned}
k_1 &= hf(t_n, y_n), \\
k_2 &= hf(t_n + c_2 h, y_n + a_{21} k_1), \\
k_3 &= hf(t_n + c_3 h, y_n + a_{31} k_1 + a_{32} k_2), \\
&\vdots \\
k_s &= hf(t_n + c_s h, y_n + a_{s1} k_1 + a_{s2} k_2 + \cdots + a_{s,s-1} k_{s-1})
\end{aligned}$$

Runge-Kutta

$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i,$$

where

$$\begin{aligned}
k_1 &= hf(t_n, y_n), \\
k_2 &= hf(t_n + c_2 h, y_n + a_{21} k_1), \\
k_3 &= hf(t_n + c_3 h, y_n + a_{31} k_1 + a_{32} k_2), \\
&\vdots \\
k_s &= hf(t_n + c_s h, y_n + a_{s1} k_1 + a_{s2} k_2 + \cdots + a_{s,s-1} k_{s-1})
\end{aligned}$$

s : order of RK formula

Butcher tableau

$$\boxed{\begin{array}{c|cccc}
0 & & & & \\
c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
\vdots & \vdots & & \ddots & \\
c_s & a_{s1} & a_{s2} & \ldots & a_{s,s-1} \\
\hline
 & b_1 & b_2 & \ldots & b_{s-1} \quad b_s
\end{array}}$$

describes family of RK methods

Runge-Kutta 1

$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i,$$

Euler's Method when $s=1$ and $b_1=1$

$$\boxed{\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}}$$

where

$$k_1 = hf(t_n, y_n),$$

An RK 2: The Midpoint Method

$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i,$$

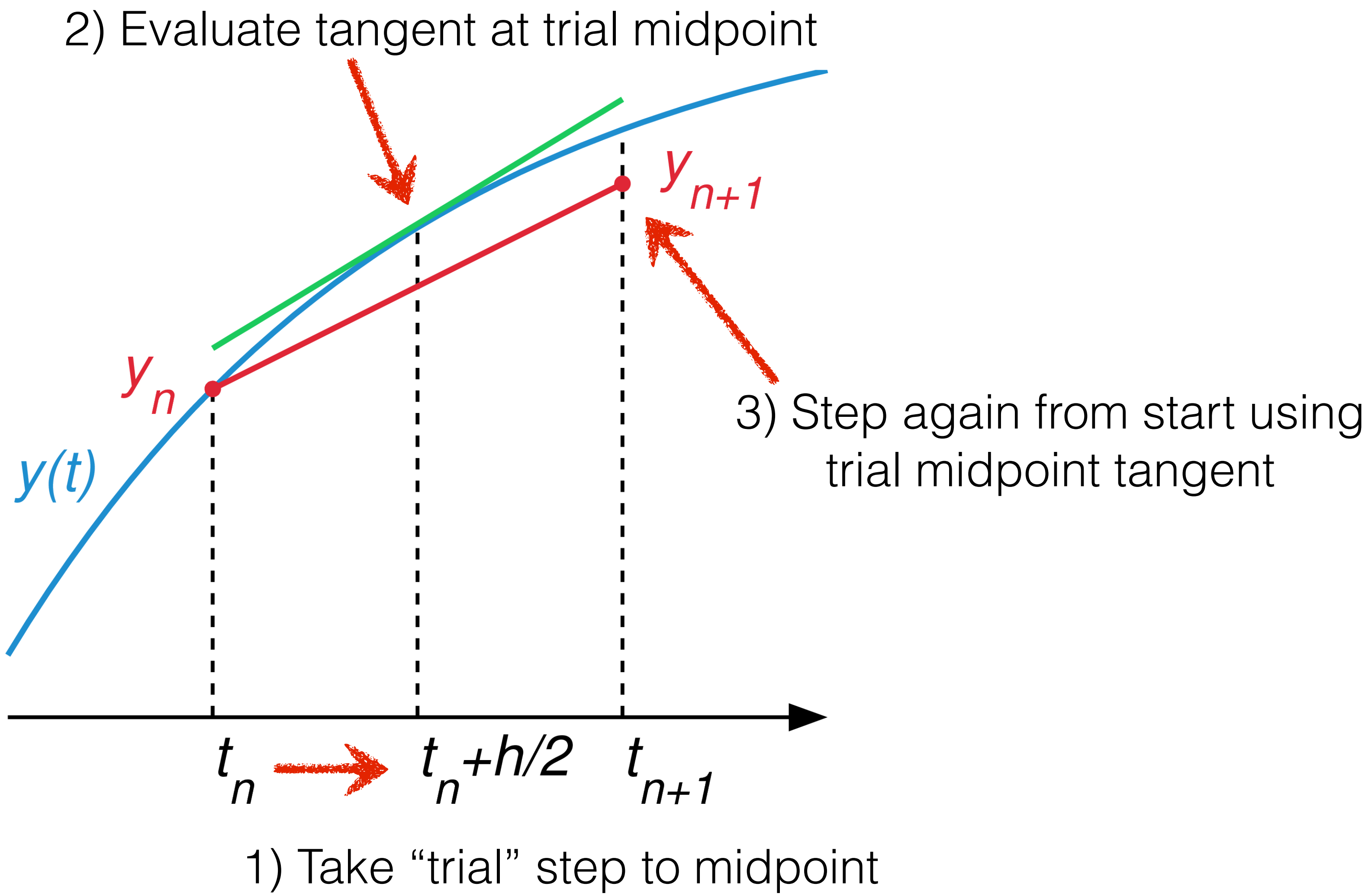
where

$$s=2 \quad \begin{aligned} k_1 &= hf(t_n, y_n), \quad b_1=0 \\ k_2 &= hf(t_n + c_2 h, y_n + a_{21} k_1), \quad b_2=1 \end{aligned}$$

$c_2=0.5$ $\quad$ $a_{21}=0.5$

Midpoint Method

$$\begin{array}{cc|cc} c_1 & 0 & & \\ & & a_{21} & \\ c_2 & 1/2 & 1/2 & \\ \hline & & 0 & 1 \\ & & b_1 & b_2 \end{array}$$


2) Evaluate tangent at trial midpoint
y(t)
y n
y n+1
3) Step again from start using trial midpoint tangent
t n
t n +h/2
t n+1
1) Take "trial" step to midpoint

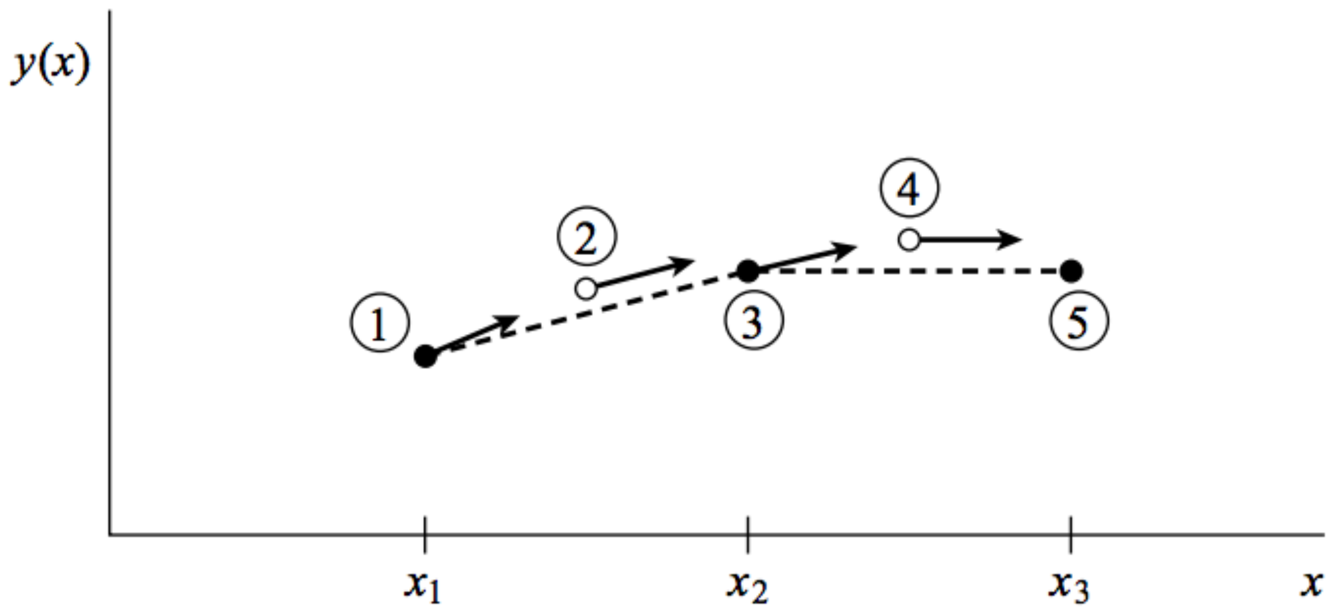
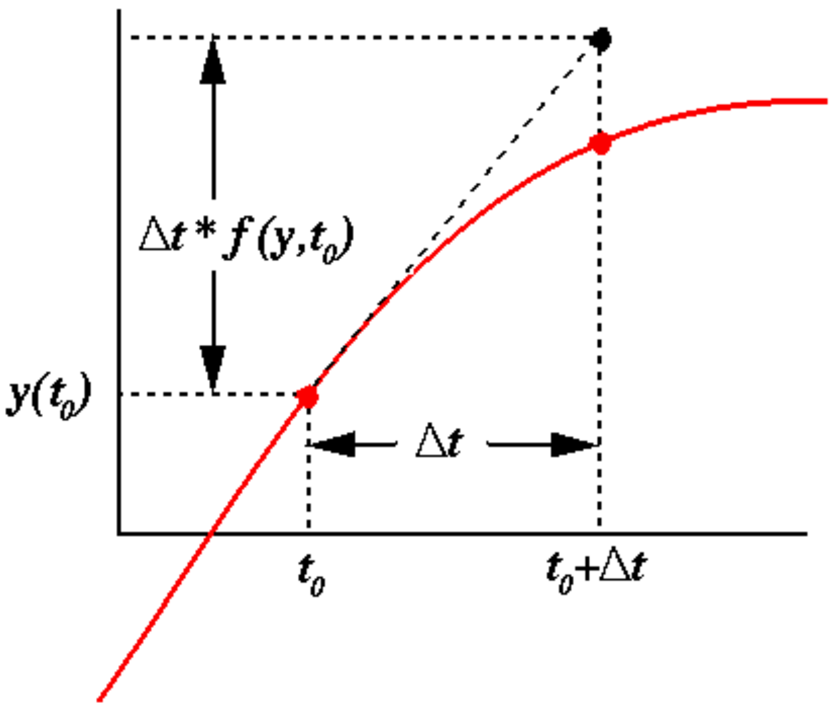
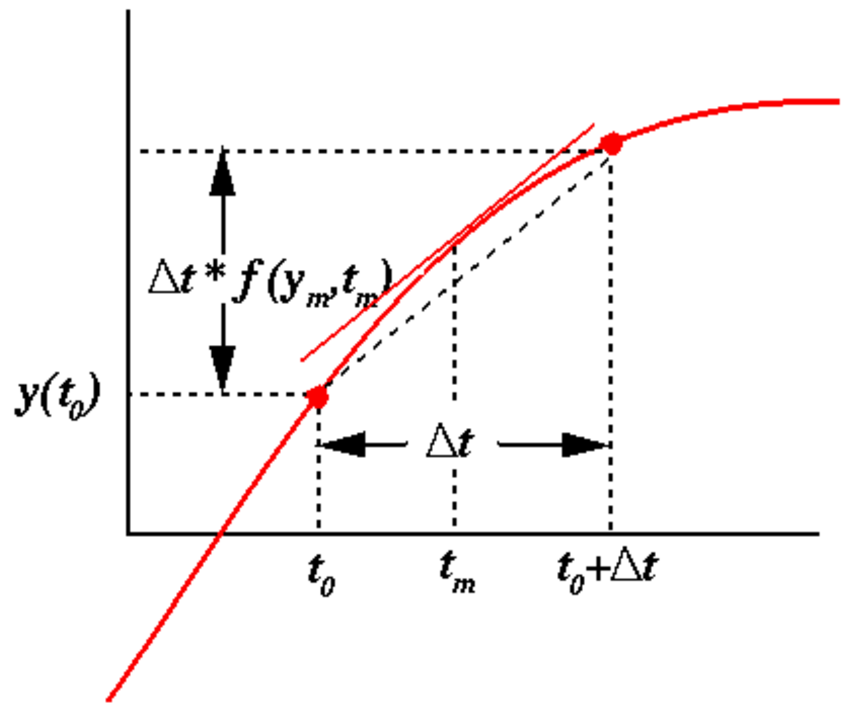


Figure 16.1.2. Midpoint method. Second-order accuracy is obtained by using the initial derivative at each step to find a point halfway across the interval, then using the midpoint derivative across the full width of the interval. In the figure, filled dots represent final function values, while open dots represent function values that are discarded once their derivatives have been calculated and used.



Euler



Midpoint

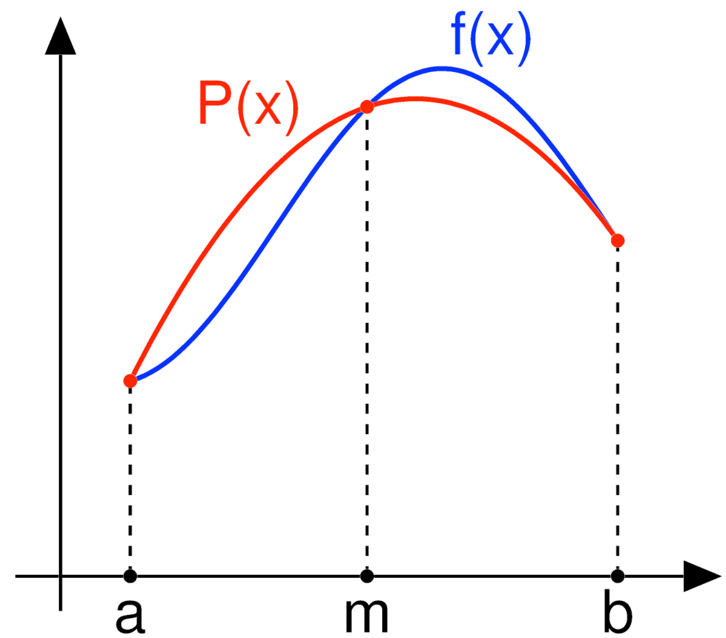
http://astro.physics.ncsu.edu/urca/course_files/Lesson15/index.html

The "RK4": Simpson's Rule

a widely used version of fourth order Runge-Kutta (known as "RK4") implements Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

numerical approximation of definite integrals by evaluating integral extents and midpoint



$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i,$$

where

$$k_1 = hf(t_n, y_n),$$
$$k_2 = hf(t_n + c_2 h, y_n + a_{21} k_1),$$
$$k_3 = hf(t_n + c_3 h, y_n + a_{31} k_1 + a_{32} k_2),$$
$$\vdots$$
$$k_s = hf(t_n + c_s h, y_n + a_{s1} k_1 + a_{s2} k_2 + \cdots + a_{s,s-1} k_{s-1})$$

RK4 uses s=4

$a_{21}=1/2$, $a_{32}=1/2$, $a_{43}=1$,

$c_1=0$, $c_2=1/2$, $c_3=1/2$, $c_4=1$

$b_1=1/6$, $b_2=1/3$, $b_3=1/3$, $b_4=1/6$

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	1/3	1/3	1/6

Runge-Kutta: RK4

$$y_{n+1} = y_n + \tfrac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$t_{n+1} = t_n + h$$

for $n = 0, 1, 2, 3, \ldots,$ using

$$k_1 = f(t_n, y_n),$$ Slope at beginning of interval

$$k_2 = f(t_n + \tfrac{1}{2}h, y_n + \tfrac{h}{2}k_1),$$ Slope at trial midpoint $hk_1/2$

$$k_3 = f(t_n + \tfrac{1}{2}h, y_n + \tfrac{h}{2}k_2),$$ Slope at trial midpoint $hk_2/2$

$$k_4 = f(t_n + h, y_n + hk_3).$$ Slope at trial end of interval

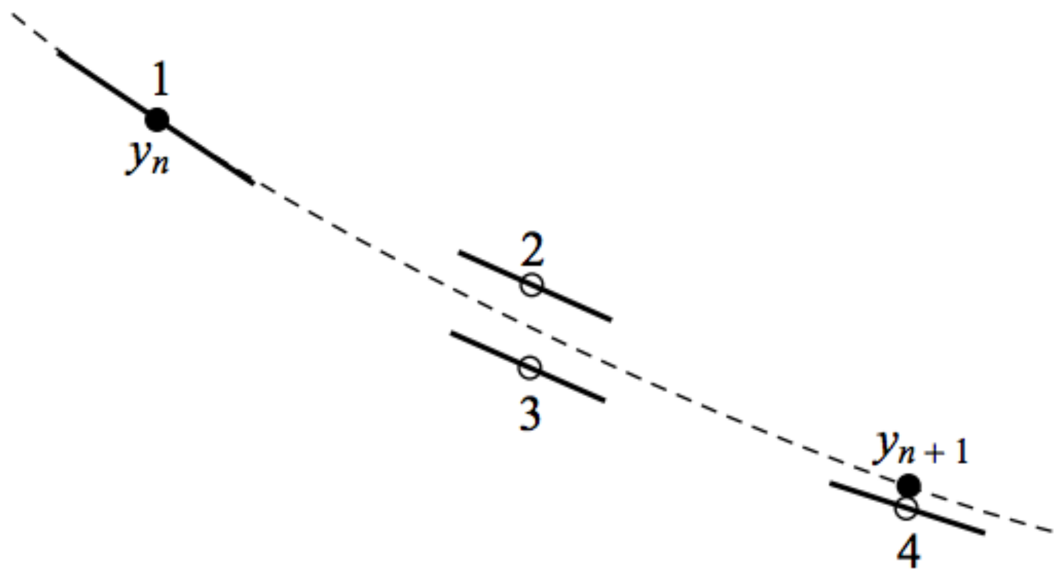


Figure 16.1.3. Fourth-order Runge-Kutta method. In each step the derivative is evaluated four times: once at the initial point, twice at trial midpoints, and once at a trial endpoint. From these derivatives the final function value (shown as a filled dot) is calculated. (See text for details.)

$k(x1) = x_t + 0$

$k(v1) = v_t + 0$ $\leftarrow$ $c_1=0$

remember: 2^{nd} order ODE can represented as 1^{st} order

$$y_{n+1} = y_n + \dot{y}_n \Delta t$$

$$\dot{y}_{n+1} = \dot{y}_n + \ddot{y}_n \Delta t$$

$$k_{(x1)} = x_t + 0$$

$$k_{(v1)} = v_t + 0$$

$$k_{(x2)} = k_{(x1)} + (a_{21} * k_{(v1)} * h)$$

integrate position from t to trial midpoint at $t+c_2h$ using $k_{(v1)}$ ($c_2=1/2$)

$a_{21}=1/2$

$$k_{(x1)} = x_t + 0$$

$$k_{(v1)} = v_t + 0$$

$$k_{(x2)} = k_{(x1)} + (a_{21} * k_{(v1)} * h)$$

$$k_{(v2)} = k_{(v1)} + (a_{21} * \mathrm{accel}(k_{(x1)}) * h)$$

integrate velocity from t to trial midpoint at $t+c_2h$

acceleration depends on system state (i.e., position of pendulum)

$$k_{(x1)} = x_t + 0$$
$$k_{(v1)} = v_t + 0$$
$$k_{(x2)} = k_{(x1)} + (a_{21} * k_{(v1)} * h)$$
$$k_{(v2)} = k_{(v1)} + (a_{21} * accel(k_{(x1)}) * h)$$
$$k_{(x3)} = k_{(x1)} + (a_{32} * k_{(v2)} * h)$$

$a_{32}=1/2$

integrate position from t to second trial midpoint at $t+c_3h$ using $k_{(v2)}$ $(c_3=1/2)$

$$k_{(x1)} = x_t + 0$$
$$k_{(v1)} = v_t + 0$$
$$k_{(x2)} = k_{(x1)} + (a_{21} * k_{(v1)} * h)$$
$$k_{(v2)} = k_{(v1)} + (a_{21} * accel(k_{(x1)}) * h)$$
$$k_{(x3)} = k_{(x1)} + (a_{32} * k_{(v2)} * h)$$
$$k_{(v3)} = k_{(v1)} + (a_{32} * accel(k_{(x2)}) * h)$$

acceleration depends on system state

$$k_{(x1)} = x_t + 0$$

$$k_{(v1)} = v_t + 0$$

$$k_{(x2)} = k_{(x1)} + (a_{21} * k_{(v1)} * h)$$

$$k_{(v2)} = k_{(v1)} + (a_{21} * accel(k_{(x1)}) * h)$$

$$k_{(x3)} = k_{(x1)} + (a_{32} * k_{(v2)} * h)$$

$$k_{(v3)} = k_{(v1)} + (a_{32} * accel(k_{(x2)}) * h)$$

$$k_{(x4)} = k_{(x1)} + (a_{43} * k_{(v3)} * h)$$

$a_{43}=1$

integrate position from t to trial endpoint at $t+c_4h$ using $k_{(v3)}$ ($c_4=1$)

$$k_{(x1)} = x_t + 0$$
$$k_{(v1)} = v_t + 0$$
$$k_{(x2)} = k_{(x1)} + (a_{21} * k_{(v1)} * h)$$
$$k_{(v2)} = k_{(v1)} + (a_{21} * accel(k_{(x1)}) * h)$$
$$k_{(x3)} = k_{(x1)} + (a_{32} * k_{(v2)} * h)$$
$$k_{(v3)} = k_{(v1)} + (a_{32} * accel(k_{(x2)}) * h)$$
$$k_{(x4)} = k_{(x1)} + (a_{43} * k_{(v3)} * h)$$
$$k_{(v4)} = k_{(v1)} + (a_{43} * accel(k_{(x3)}) * h)$$

$$k_{(x1)} = x_t + 0$$

$$k_{(v1)} = v_t + 0$$

$$k_{(x2)} = k_{(x1)} + (a_{21} * k_{(v1)} * h)$$

$$k_{(v2)} = k_{(v1)} + (a_{21} * accel(k_{(x1)}) * h)$$

$$k_{(x3)} = k_{(x1)} + (a_{32} * k_{(v2)} * h)$$

$$k_{(v3)} = k_{(v1)} + (a_{32} * accel(k_{(x2)}) * h)$$

$$k_{(x4)} = k_{(x1)} + (a_{43} * k_{(v3)} * h)$$

$$k_{(v4)} = k_{(v1)} + (a_{43} * accel(k_{(x3)}) * h)$$

$b_1=1/6,\ b_2=1/3,\ b_3=1/3,\ b_4=1/6$

$$x_{t+h} = x_t + h*(b_1*k_{(v1)}+b_2*k_{(v2)}+b_3*k_{(v3)}+b_4*k_{(v4)})$$

$$k_{(x1)} = x_t + 0$$

velocity depends on acceleration a(x)

$$k_{(v1)} = v_t + 0$$

$$k_{(x2)} = k_{(x1)} + (a_{21} * k_{(v1)} * h)$$

$$k_{(v2)} = k_{(v1)} + (a_{21} * accel(k_{(x1)}) * h)$$

$$k_{(x3)} = k_{(x1)} + (a_{32} * k_{(v2)} * h)$$

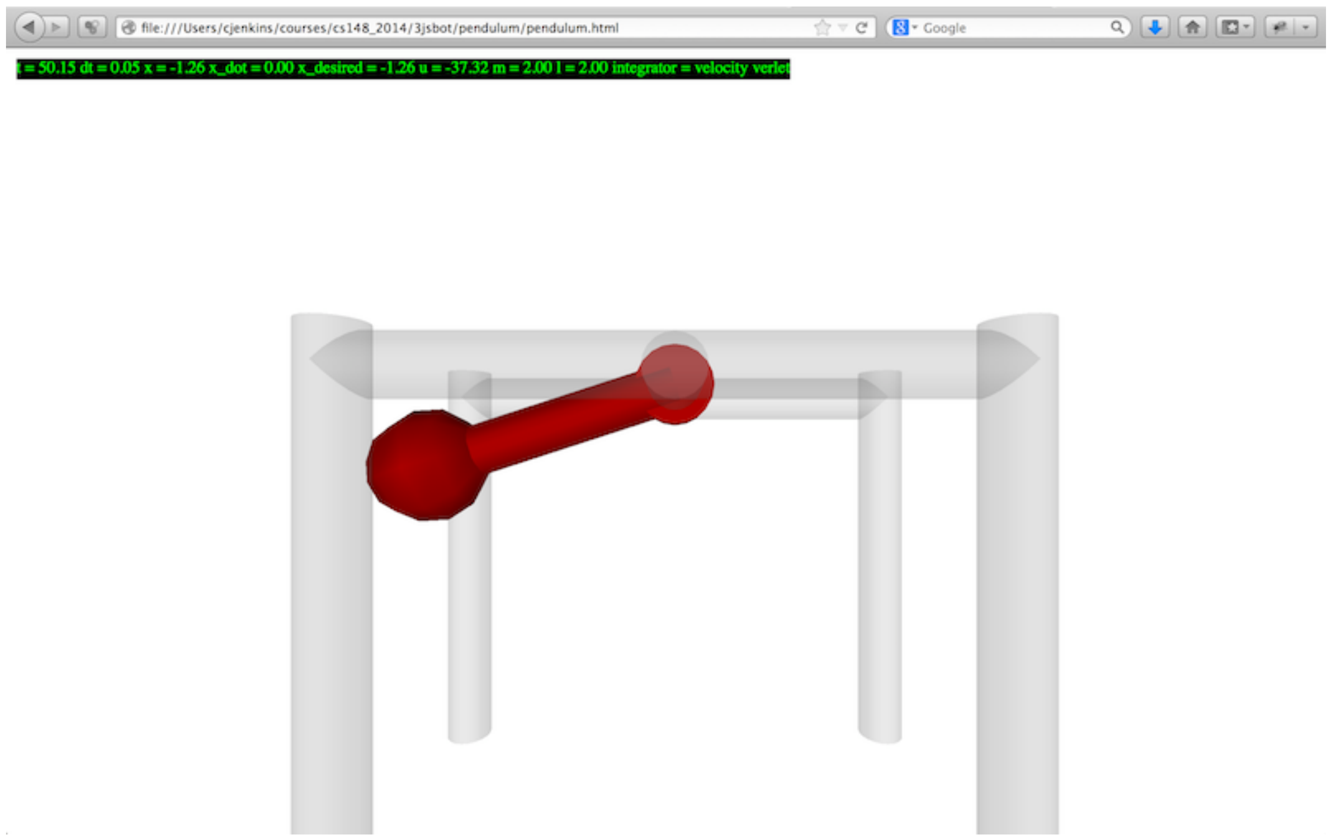
$$k_{(v3)} = k_{(v1)} + (a_{32} * accel(k_{(x2)}) * h)$$

$$k_{(x4)} = k_{(x1)} + (a_{43} * k_{(v3)} * h)$$

$$k_{(v4)} = k_{(v1)} + (a_{43} * accel(k_{(x3)}) * h)$$

$$v_{t+h} = v_t + h*[b_1*a(k_{(x1)})+b_2*a(k_{(x2)})+b_3*a(k_{(x3)}) +b_4*a(k_{(x4)})]$$

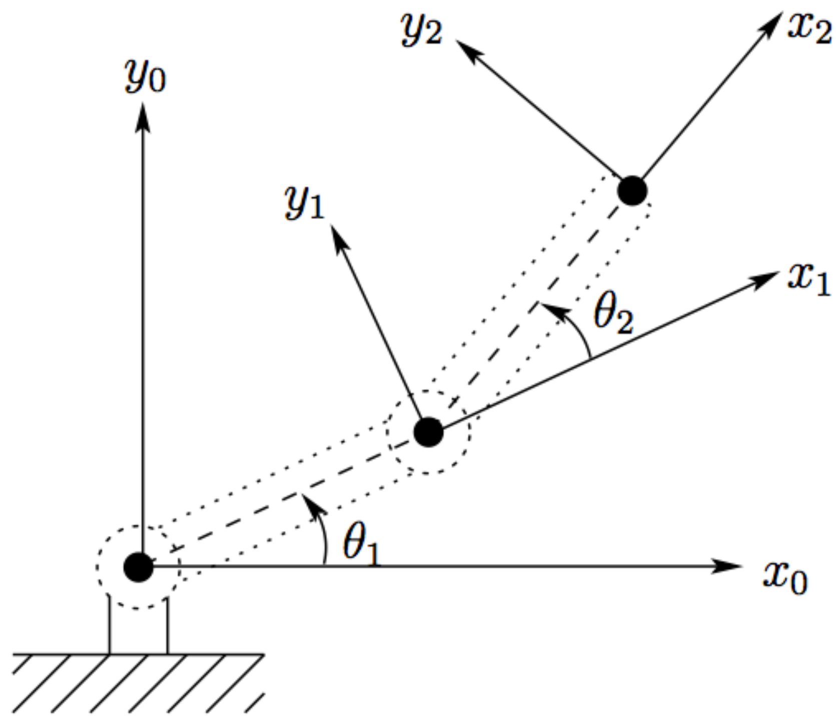
Let's see what happens



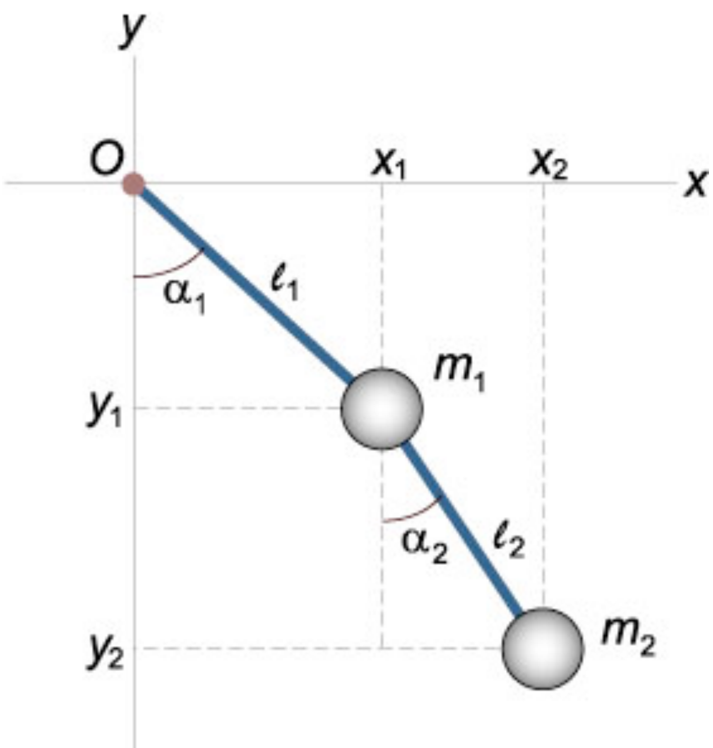
Let's revisit the Planar 2-DOF 2-link Arm

Planar 2-DOF 2-link Arm

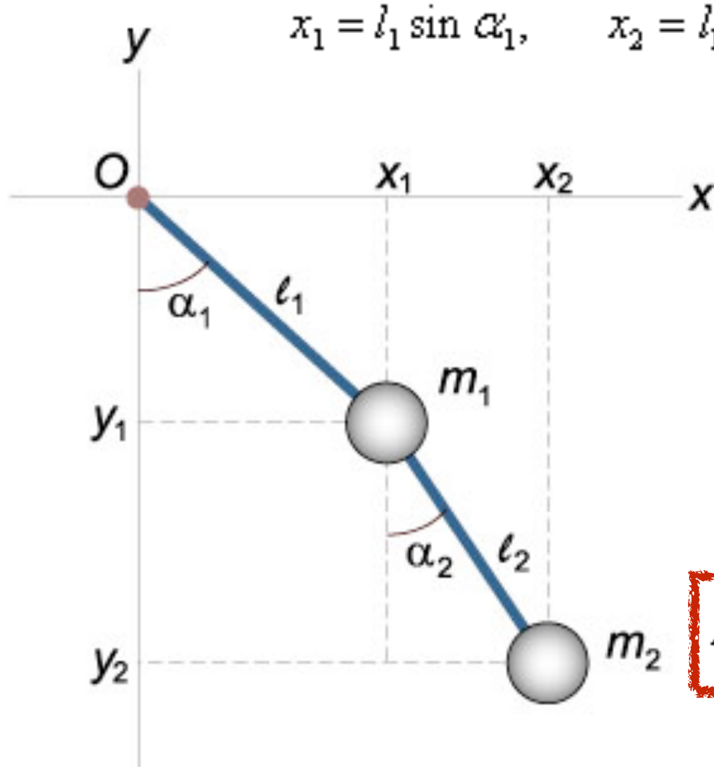
Beyster 3753
Wed Sep 12 2018



Can we add another link?



- Double pendulum



Locations of pendulum bobs

$$x_1 = l_1 \sin\alpha_1, \quad x_2 = l_1 \sin\alpha_1 + l_2 \sin\alpha_2, \quad y_1 = -l_1\cos\alpha_1, \quad y_2 = -l_1\cos\alpha_1 - l_2\cos\alpha_2.$$

Lagrangian of pendulum bob positions

$$T = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right)}{2} + \frac{m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)}{2}, \qquad V = m_1 g y_1 + m_2 g y_2.$$

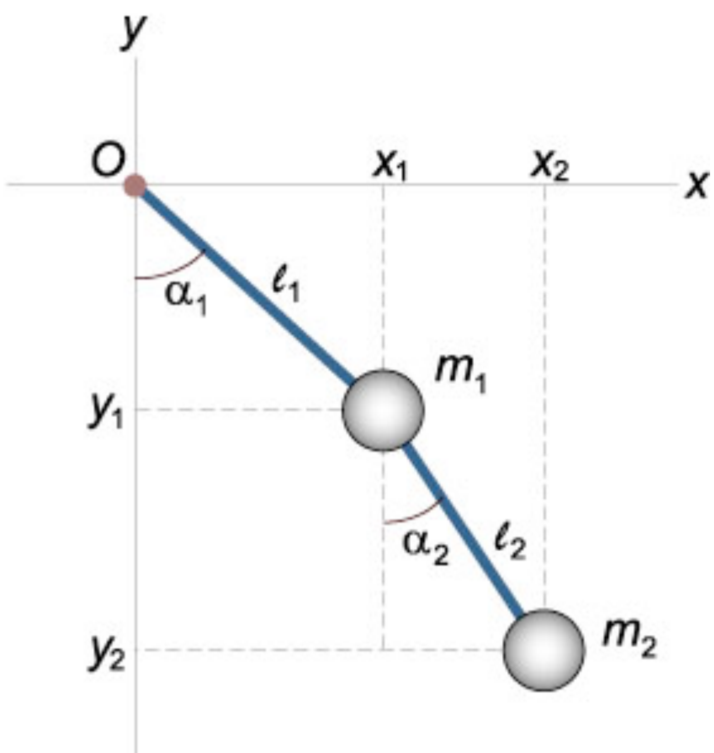
$$\boxed{L = T - V = T_1 + T_2 - (V_1 + V_2) = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) - m_1 g y_1 - m_2 g y_2.}$$

http://www.math24.net/double-pendulum.html

Lagrangian in generalized coordinates (joint angles)

$$\dot{x}_1 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1, \qquad \dot{x}_2 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1 + l_2 \cos\alpha_2 \cdot \dot{\alpha}_2,$$

$$\dot{y}_1 = l_1 \sin\alpha_1 \cdot \dot{\alpha}_1, \qquad \dot{y}_2 = l_1 \sin\alpha_1 \cdot \dot{\alpha}_1 + l_2 \sin\alpha_2 \cdot \dot{\alpha}_2.$$



$$T_1 = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) = \frac{m_1}{2}\left(l_1^2\dot{\alpha}_1^2\cos^2\alpha_1 + l_1^2\dot{\alpha}_1^2\sin^2\alpha_1\right) = \frac{m_1}{2}l_1^2\dot{\alpha}_1^2,$$

$$T_2 = \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) = \frac{m_2}{2}\left[\left(l_1\dot{\alpha}_1\cos\alpha_1 + l_2\dot{\alpha}_2\cos\alpha_2\right)^2 + \left(l_1\dot{\alpha}_1\sin\alpha_1 + l_2\dot{\alpha}_2\sin\alpha_2\right)^2\right]$$

$$= \frac{m_2}{2}\left[l_1^2\dot{\alpha}_1^2\cos^2\alpha_1 + l_2^2\dot{\alpha}_2^2\cos^2\alpha_2 + 2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\cos\alpha_1\cos\alpha_2 + l_1^2\dot{\alpha}_1^2\sin^2\alpha_1 + l_2^2\dot{\alpha}_2^2\sin^2\alpha_2 + 2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\sin\alpha_1\sin\alpha_2\right]$$

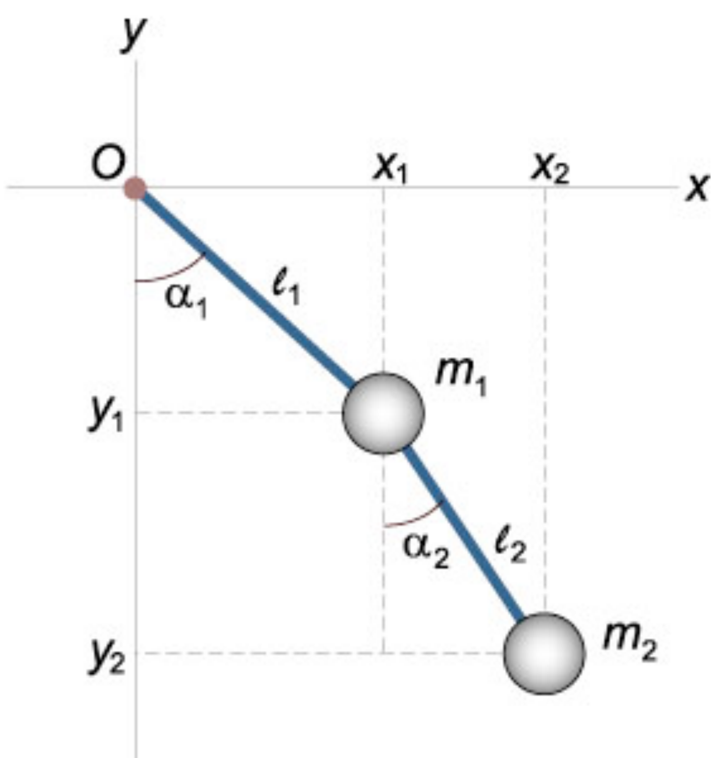
$$= \frac{m_2}{2}\left[l_1^2\dot{\alpha}_1^2 + l_2^2\dot{\alpha}_2^2 + 2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\cos\left(\alpha_1 - \alpha_2\right)\right],$$

$$V_1 = m_1 g y_1 = -m_1 g l_1 \cos\alpha_1,$$

$$V_2 = m_2 g y_2 = -m_2 g\left(l_1\cos\alpha_1 + l_2\cos\alpha_2\right).$$

$$\boxed{\begin{aligned} L &= T - V = T_1 + T_2 - \left(V_1 + V_2\right) = \\ &= \left(\frac{m_1}{2} + \frac{m_2}{2}\right)l_1^2\dot{\alpha}_1^2 + \frac{m_2}{2}l_2^2\dot{\alpha}_2^2 + m_2 l_1 l_2\dot{\alpha}_1\dot{\alpha}_2\cos\left(\alpha_1 - \alpha_2\right) + \left(m_1 + m_2\right)g l_1\cos\alpha_1 + m_2 g l_2\cos\alpha_2. \end{aligned}}$$

http://www.math24.net/double-pendulum.html



http://www.math24.net/double-pendulum.html

Lagrangian equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_i} - \frac{\partial L}{\partial \alpha_i} = 0, \qquad i = 1,2.$$

$$\frac{\partial L}{\partial \dot{\alpha}_1} = (m_1+m_2) l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1-\alpha_2), \qquad \frac{\partial L}{\partial \alpha_1} = -m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1-\alpha_2) - (m_1+m_2) g l_1 \sin\alpha_1,$$

$$\frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1 \cos(\alpha_1-\alpha_2), \qquad \frac{\partial L}{\partial \alpha_2} = m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1-\alpha_2) - m_2 g l_2 \sin\alpha_2.$$

Lagrangian EOM for i=1

$$\frac{d}{dt}\left[(m_1+m_2) l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1-\alpha_2)\right] + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1-\alpha_2) + (m_1+m_2) g l_1 \sin\alpha_1 = 0,$$

$$\Rightarrow (m_1+m_2) l_1^2 \ddot{\alpha}_1 + m_2 l_1 l_2 \left[\ddot{\alpha}_2 \cos(\alpha_1-\alpha_2) - \dot{\alpha}_2 \sin(\alpha_1-\alpha_2)\cdot(\dot{\alpha}_1-\dot{\alpha}_2)\right]$$

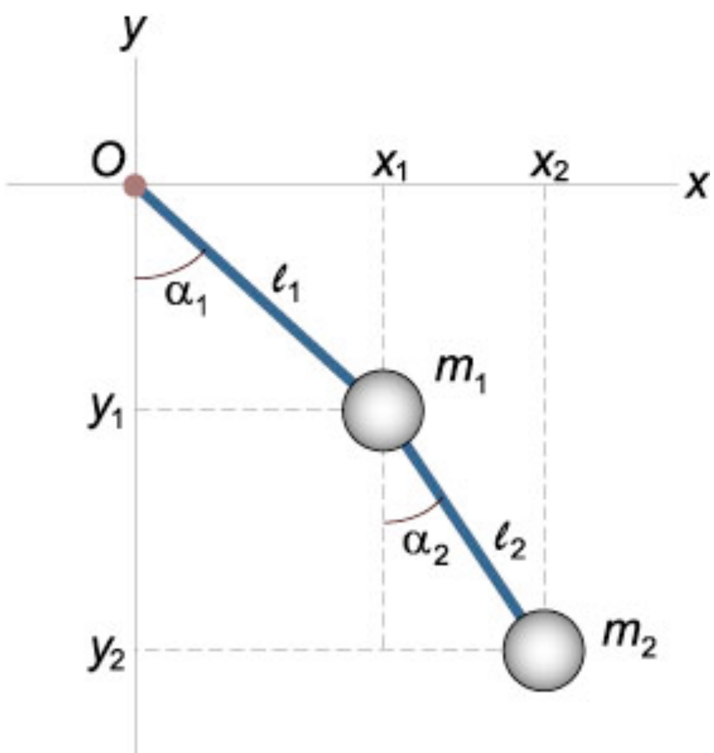
$$+ m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1-\alpha_2) + (m_1+m_2) g l_1 \sin\alpha_1 = 0,$$

$$\Rightarrow (m_1+m_2) l_1^2 \ddot{\alpha}_1 + m_2 l_1 l_2 \ddot{\alpha}_2 \cos(\alpha_1-\alpha_2) - \cancel{m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1-\alpha_2)} + m_2 l_1 l_2 \dot{\alpha}_2^2 \sin(\alpha_1-\alpha_2)$$

$$+ \cancel{m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1-\alpha_2)} + (m_1+m_2) g l_1 \sin\alpha_1 = 0,$$

$$\Rightarrow (m_1+m_2) l_1^2 \ddot{\alpha}_1 + m_2 l_1 l_2 \ddot{\alpha}_2 \cos(\alpha_1-\alpha_2) + m_2 l_1 l_2 \dot{\alpha}_2^2 \sin(\alpha_1-\alpha_2) + (m_1+m_2) g l_1 \sin\alpha_1 = 0.$$

$$\boxed{(m_1+m_2) l_1 \ddot{\alpha}_1 + m_2 l_2 \ddot{\alpha}_2 \cos(\alpha_1-\alpha_2) + m_2 l_2 \dot{\alpha}_2^2 \sin(\alpha_1-\alpha_2) + (m_1+m_2) g \sin\alpha_1 = 0.}$$



http://www.math24.net/double-pendulum.html

Lagrangian equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_i} - \frac{\partial L}{\partial \alpha_i} = 0, \qquad i = 1,2.$$

$$\frac{\partial L}{\partial \dot{\alpha}_1} = (m_1+m_2) l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2), \qquad \frac{\partial L}{\partial \alpha_1} = -m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - (m_1+m_2) g l_1 \sin\alpha_1,$$

$$\frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2), \qquad \frac{\partial L}{\partial \alpha_2} = m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - m_2 g l_2 \sin\alpha_2.$$

Lagrangian EOM for i=2

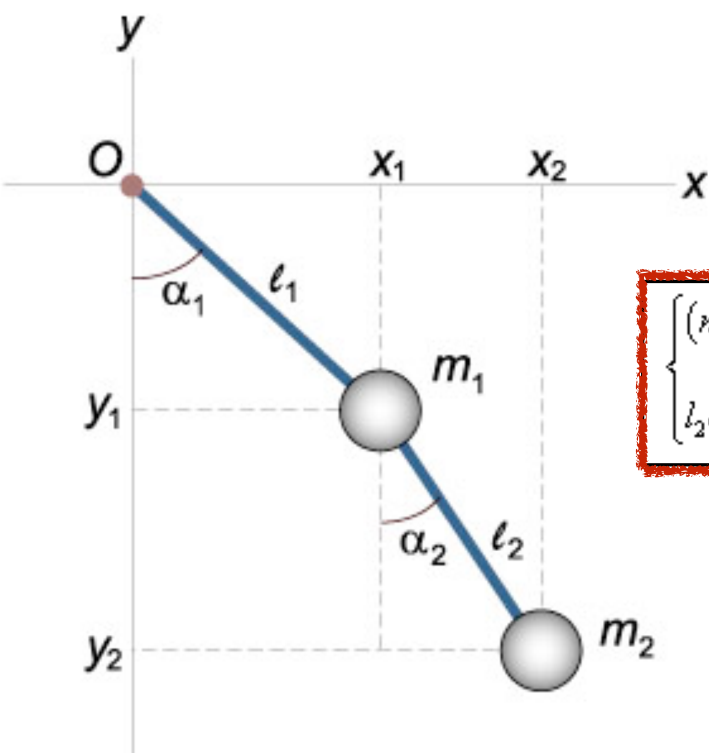
$$\frac{d}{dt}\left[m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2)\right] - m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) + m_2 g l_2 \sin\alpha_2 = 0,$$

$$\Rightarrow m_2 l_2^2 \ddot{\alpha}_2 + m_2 l_1 l_2 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - m_2 l_1 l_2 \dot{\alpha}_1 \sin(\alpha_1 - \alpha_2) \cdot (\dot{\alpha}_1 - \dot{\alpha}_2) - m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)$$
$$+ m_2 g l_2 \sin\alpha_2 = 0,$$

$$\Rightarrow m_2 l_2^2 \ddot{\alpha}_2 + m_2 l_1 l_2 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - m_2 l_1 l_2 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) + \cancel{m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)}$$
$$- \cancel{m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)} + m_2 g l_2 \sin\alpha_2 = 0,$$

$$\Rightarrow m_2 l_2^2 \ddot{\alpha}_2 + m_2 l_1 l_2 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - m_2 l_1 l_2 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) + m_2 g l_2 \sin\alpha_2 = 0.$$

$$\boxed{l_2 \ddot{\alpha}_2 + l_1 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - l_1 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) + g \sin\alpha_2 = 0.}$$



http://www.math24.net/double-pendulum.html

The 2 equations of motion
for double pendulum

$$\begin{cases} (m_1+m_2)l_1\ddot{\alpha}_1 + m_2 l_2 \ddot{\alpha}_2 \cos(\alpha_1-\alpha_2) + m_2 l_2 \dot{\alpha}_2^2 \sin(\alpha_1-\alpha_2) + (m_1+m_2)\, g \sin\alpha_1 = 0 \\ l_2\ddot{\alpha}_2 + l_1\ddot{\alpha}_1 \cos(\alpha_1-\alpha_2) - l_1\dot{\alpha}_1^2 \sin(\alpha_1-\alpha_2) + g\sin\alpha_2 = 0 \end{cases}$$

Substitute equations into each other to solve for $\ddot{\alpha}_1$ and $\ddot{\alpha}_2$

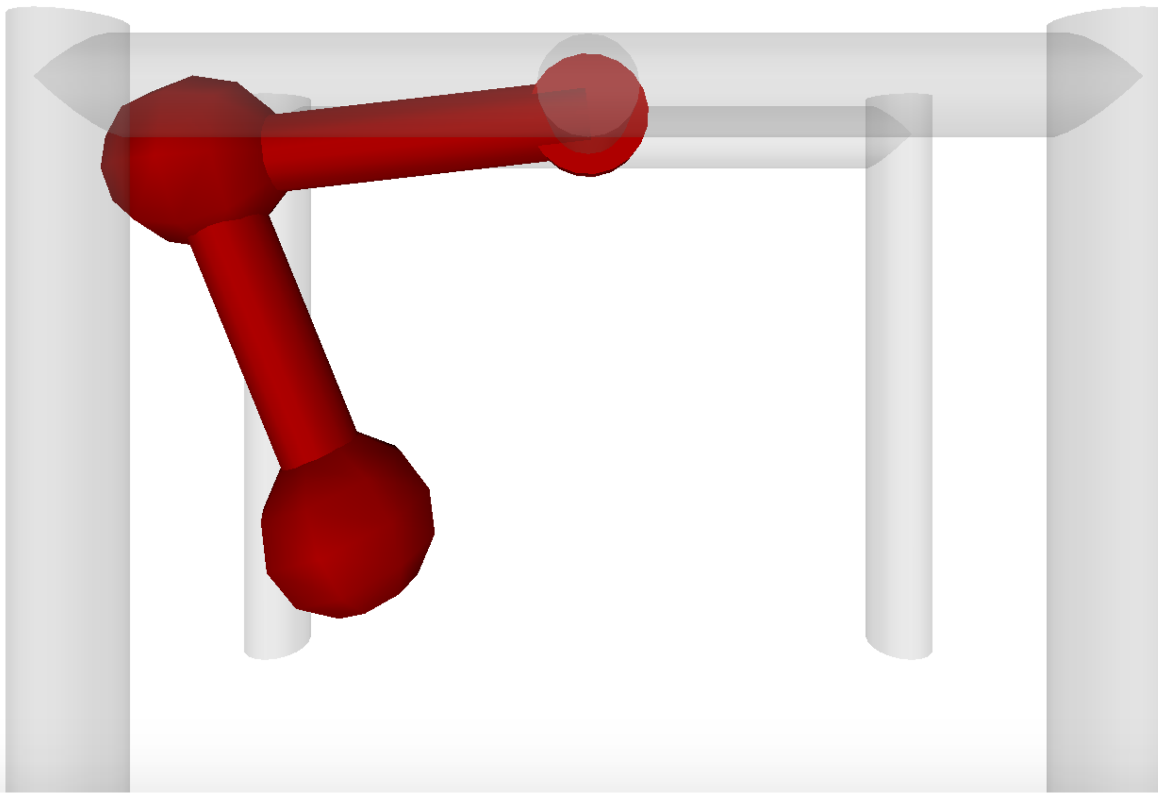
What happened to the torques?

```
System
t = 6119.20 dt = 0.05
integrator = runge-kutta
x1 = -1.46
x1_dot = -0.00
x2 = 1.84
x2_dot = -0.00
x1_desired = -1.46
x2_desired = 1.84
```

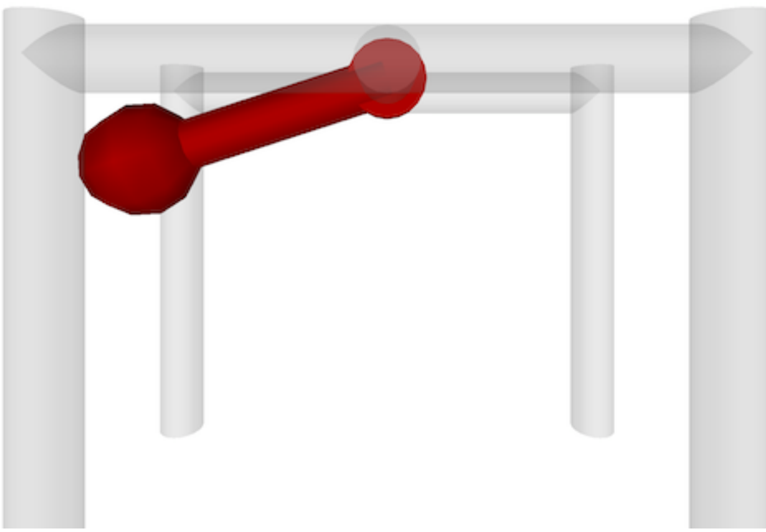
Pendularm2 by oharib

```
Pendulum
mass = 2.00
length = 2.00
gravity = 9.81

Keys
a/d - apply user force
1/2 - adjust desired angle 1
3/4 - adjust desired angle 2
c|x - toggle servo
s - disable servo
```



Project 2: Pendularm



https://raw.githubusercontent.com/autorob/kineval-stencil/master/project_pendularm/pendularm1.html

```
function init() {...
    pendulum = { // pendulum object
        length:2.0,
        mass:2.0,
        angle:Math.PI/2,
        angle_dot:0.0};

 gravity = 9.81;  // Earth gravity
 t = 0; dt = 0.05; // init time
 pendulum.control = 0; // motor

 // next lecture: PID control
 pendulum.desired = -Math.PI/2.5;
 pendulum.desired_dot = 0;
 pendulum.servo =
     {kp:0, kd:0, ki:0};
...}
```

```
function animate() { ...
if (numerical_integrator === "euler") {
    // STENCIL: Euler integrator }
    else if (numerical_integrator === "verlet") {
    // STENCIL: basic Verlet integration }
    else if (numerical_integrator === "velocity verlet") {
    // STENCIL: velocity Verlet }
    else if (numerical_integrator === "runge-kutta") {
    // STENCIL: Runge-Kutta 4 integrator }
    else {   }
    // set the angle of the pendulum
    pendulum.geom.rotation.y = pendulum.angle;
    t = t + dt;  // advance time
...}

function pendulum_acceleration(p,g) {
   // STENCIL: return acceleration from equations of motion }
```


Next up:
Control
(although no Janet Jackson)